Bayes Filter

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July 2020

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1 Introduction

Bayes filter is an algorithm which is used for calculating belief distribution *bel* from the control given and the measurements collected by various sensors.

2 Algorithm

1 def Bayes_Filter(bel(x_{t-1}), u_t, z_t):
2   for x in x:
3         $\overline{bel}(\mathrm{x}) = \int p(x|u_t, \mathrm{bel}(x_{t-1}))$
4         $\mathrm{bel}(x) = \eta\ p(z_t|x)\overline{\mathrm{bel}(x)}$

2.1 Explanation

This is **one iteration** of the Bayes Filter algorithm at time step t. The filter takes as input the belief at time step $t-1$, bel(x_{t-1}), the controls at time t, u_t and the measurements made at time t, z_t.

In the line 3. of the algorithm, it calculates $\overline{bel}(\mathrm{x}_t)$ using only the controls given at time t and the belief of previous time step. The line basically is calculating the probability of getting some state x in x_t given the believed state from the last time step and the controls given at this time step. It is just approximating how the controls might have changed the state of the system.

In the line 4. of the algorithm, It incorporates the measurements taken by the sensors. It calculates the belief based on the measurements and the $\overline{bel}$ calculated in line 3 (probability of getting the measurement z if the system is in state x times the probability of the system being in the state x).

3 Derivation

To show the correctness of the Bayes Filter, we need to show that it correctly calculates $p(x_t|z_{1:t}, u_{1:t})$ from the posterior from the previous time step $p(x_t|z_{1:t-1}, u_{1:t-1})$ (Induction).
From the Bayes Rule,

$$\begin{aligned} p(x_t|z_{1:t}, u_{1:t}) &= \frac{p(z_t|x_t, z_{1:t-1}, u_{1:t})p(x_t|z_{1:t-1}, u_{1:t})}{p(z_t|z_{1:t-1}, u_{1:t})} \\ &= \eta p(z_t|x_t, z_{1:t-1}, u_{1:t})p(x_t|z_{1:t-1}, u_{1:t}) \end{aligned} \tag{1}$$

Here, we make the assumption of x_t being a complete state *(Markov assumption)*.

Definition 1. ***Complete state:*** *A state x_t is said to be complete if no variables prior to x_t may influence the stochastic evolution of future states.*

Therefore, if we were to know the measurements z_t and we knew the state x_t, we don't need any other variables.

$$\implies p(z_t|x_t, z_{1:t-1}, u_{1:t}) = p(z_t|x_t)$$

Therefore, equation **(1)** becomes,

$$p(x_t|z_{1:t}, u_{1:t}) = \eta p(z_t|x_t)p(x_t|z_{1:t-1}, u_{1:t}) \tag{2}$$

$p(x_t|z_{1:t}, u_{1:t})$ is the $bel(x_t)$ and $p(x_t|z_{1:t-1}, u_{1:t})$ is the $\overline{bel}(x_t)$ (because it doesn't use z_t). Therefore,

$$bel(x_t) = \eta p(z_t|x_t)\overline{bel}(x_t) \tag{3}$$

equation **(3)** is implemented in line 4. of the algorithm.
Now, let us calculate $\overline{bel}(x_t)$,

$$\begin{aligned} \overline{bel}(x_t) &= p(x_t|z_{1:t-1}, u_{1:t}) \\ &= \int p(x_t|x_{t-1}, z_{1:t-1}, u_{1:t})p(x_{t-1}|z_{1:t-1}, u_{1:t})dx_{t-1} \end{aligned} \tag{4}$$

Now, since our state is complete, the controls $u_{1:t-1}$ are of no use to us in knowing the state x_t. Also, the measurements don't play any role in evolution of the states. Therefore,

$$p(x_t|x_{t-1}, z_{1:t-1}, u_{1:t}) = p(x_t|x_{t-1}, u_t) \tag{5}$$

Therefore, **(5)** becomes,

$$\overline{bel}(x_t) = \int p(x_t|x_{t-1}, u_t)p(x_{t-1}|z_{1:t-1}, u_{1:t})dx_{t-1} \tag{6}$$

This equation is implemented in line 3. of the algorithm.