3. Give a proof of the Baker–Hausdorf lemma: for any two operators $\hat{A}$ and $\hat{B}$,

$$e^{i\lambda\hat{A}}\hat{B}\,e^{-i\lambda\hat{A}} = \hat{B} + i\lambda[\hat{A},\hat{B}] + \frac{(i\lambda)^2}{2!}[\hat{A},[\hat{A},\hat{B}]] + \cdots$$

With:

$$e^{A} = I + A + \frac{A^2}{2} + \cdots$$

$$e^{-A} = I - A + \frac{A^2}{2} - \cdots.$$

$$\left(I + A + \frac{A^2}{2} + \cdots\right) B \left(I - A + \frac{A^2}{2} - \cdots\right)$$

$$e^{A}Be^{-A} = B + [B,A] + \frac{1}{2!}\big[B,[B,A]\big] + \cdots$$

$$= B + (BA - AB) + \frac{1}{2}(2ABA + BA^2 + A^2B) + \frac{1}{6}(3ABA^2 - BA^3 + 3A^2BA + A^3B) + \cdots$$

$$= B + [A,B] + \frac{1}{2!}\big[A,[AB]\big] + \frac{1}{3!}[A,[A,AB]]] + \cdots$$

$$e^{A}Be^{-A} = \sum_{n}^{\infty}\frac{[A^n,B]}{n!}, \qquad [A^n,B] \cong \Big[A,\Big[A,\big[\ldots,[A,B]\big]\Big]\Big]$$