1. Tunjukkan dengan detail bahwa elemen matriks kerapatan adalah rata-rata perkalian dari koefisien ekspansi fungsi basis dengan konjugatnya

$$\rho_{uv} = \langle \phi_u | \rho | \phi_v \rangle = \sum_j P_j c_u^{(j)} \left(c_v^{(j)} \right) = \overline{c_u c_v^*}$$

Jawab:

$$\hat{M} = \sum_{ij} M_{ij} |v_i\rangle\langle v_j|$$

$$M_{ij} = \langle v_i | \hat{M} | v_j \rangle$$

$$\rho_{ij} = \langle \phi_i | \rho | \phi_j \rangle$$

$$\rho_{ij} = \sum_k P_k c_i^{(K)} \left(c_j^{(K)} \right)$$

$$\rho_{ij} = \sum_k \langle \phi_i | P_k | \phi_j \rangle c_i^{(k)^*} c_j^{(k)}$$

$$\rho_{ij} = \sum_k P_k^* \langle \phi_i | \phi_i \rangle \, c_i^{(k)^*} c_j^{(k)}$$

$$\rho_{ij} = \sum_k P_k^* \delta_{ij} c_i^{(k)^*} c_j^{(k)}$$

$dimana\ \delta_{ij}\ adalah\ delta\ kroenecker\ dengan\ i = j\ dan\ i \neq j$

$jika\ i = j, maka\ \delta_{ij} = 1$

$$\rho_{ij} = \sum_k P_k c_i^{(k)^*} c_j^{(k)}$$

$jika\ i \neq j, maka\ \delta_{ij} = 0$

$\rho_{ij} = 0$

Jadi, elemen-elemen matriks kerapatan di mana indeks i dan j berbeda selalu nol, yang sesuai dengan sifat matriks kerapatan. Oleh karena itu, dapat disimpulkan bahwa elemen matriks kerapatan ρ_{uv} adalah rata-rata perkalian konjugat dari koefisien ekspansi fungsi basis

2. Tunjukkan dengan detail bahwa nilai rata-rata ensemble untuk operator $\hat{A}$ yang bekerja pada keadaan campuran adalah trace dari matriks kerapatan dikalikan operator $\hat{A}$

$$\overline{\langle A \rangle} = Tr\left(\rho \hat{A}\right)$$

Jawab:

$$\rho \hat{A} = \sum_{uv} \left(\sum_j P_j c_u^{(j)} \left(c_v^{(j)} \right)^* \right) |\phi_u\rangle\langle \phi_v | \hat{A}$$

$$\langle\phi_q|\rho\hat{A}|\phi_q\rangle = \sum_{uv}\left(\sum_j P_j c_u^{(j)}\left(c_v^{(j)}\right)^*\right)\langle\phi_q|\phi_u\rangle\langle\phi_v|\hat{A}|\phi_q\rangle$$

$$= \sum_{uv}\left(\sum_j P_j c_u^{(j)}\left(c_v^{(j)}\right)^*\right)\delta_{qu}\langle\phi_v|\hat{A}|\phi_q\rangle$$

$$= \sum_v\sum_j P_j c_q^{(j)}\left(c_v^{(j)}\right)^*\langle\phi_v|\hat{A}|\phi_q\rangle$$

$$\sum_q\langle\phi_q|\rho\hat{A}|\phi_q\ \rangle = \sum_j P_j\left(\sum_v\left(c_v^{(j)}\right)^*\langle\phi_v|\right)\hat{A}\left(\sum_q c_q^{(j)}|\phi_q\rangle\right)$$

$$\overline{\langle A\rangle} = Tr(\rho\hat{A})$$

Ensemble average : $\overline{\langle A\rangle} = \sum_j P_j\langle\phi_j|\hat{A}|\phi_j\rangle$

Untuk mixed state, probabilitas P_j dapat di ekspresikan : $P_j = Tr\left(\rho\ |\phi_j\rangle\langle\phi_j|\right)$, dimana ρ adalah density matrix dan Tr adalah trace.

$$\overline{\langle A\rangle} = \sum_j Tr\left(\rho\ |\phi_j\rangle\ \langle\phi_j|\hat{A}|\phi_j\rangle\right)$$

$$\overline{\langle A\rangle} = Tr\left(\ \rho\sum_j|\phi_j\rangle\ \langle\phi_j|\hat{A}\right)$$

$$\overline{\langle A\rangle} = Tr\left(\rho\sum_j\sum_k|\phi_k\rangle\ c_j^{(k)}\langle\phi_k|\hat{A}|\phi_i\rangle\langle\phi_j|\right)$$

$$\overline{\langle A\rangle} = Tr\left(\rho\sum_j\sum_k\sum_l|\phi_k\rangle c_j^{(k)}\langle\phi_k|\phi_l\rangle\langle\phi_l|\phi_j\rangle c_j^{(k)^*}\right)$$

$$|\phi_j\rangle\langle\phi_j| = \left(\sum_k c_j^{(k)^*}|\phi_k\rangle\right)\left(\sum_l c_j^{(l)^*}|\phi_l\rangle\right)^+$$

Dapat mengekspansi braket tanpa menubah hasil akhir, dengan menggunakan sifat trace:

$$\overline{\langle A\rangle} = Tr\left(\rho\hat{A}\sum_j\sum_k\sum_l\left(\sum_m c_j^{(m)^*}\ |\phi_k\rangle\langle\phi_k|\hat{A}|\phi_l\rangle\langle\phi_l|c_j^{(k)^*}\right)^+\right)$$

$$\overline{\langle A\rangle} = Tr\left(\rho\hat{A}\sum_k\sum_l\sum_m c_m^*|\phi_k\rangle\langle\phi_k|\hat{A}|\phi_l\rangle\langle\phi_l|c_m\right)$$

$$\overline{\langle A\rangle} = Tr\left(\rho\hat{A}\sum_m\sum_k\sum_l c_m^*|\phi_k\rangle\langle\phi_k|\hat{A}|\phi_l\rangle\langle\phi_l|c_m\right)$$

$$\overline{\langle A\rangle} = Tr\left(\rho\hat{A}\sum_m c_m|\phi_k\rangle\langle\phi_l|\hat{A}|\phi_l\rangle\langle\phi_k|c_m\right)$$

$$\overline{\langle A\rangle} = Tr\left(\rho\hat{A}\sum_m c_m|\phi_k\rangle\langle\phi_k|\hat{A}|\phi_l\rangle\langle\phi_l|c_m\right)$$

Dengan menganggati $\sum_k|\phi_k\rangle\langle\phi_k|$ dengan I, maka dapat disederhanakan menjadi $\overline{\langle A\rangle} = Tr(\rho\hat{A})$. Sehingga dapat dibuktikan bahwa nilai ensemble average untuk oprator $\hat{A}$ pada mixied state adalah trace dari matrix kerapatan dikalikan dengan operator $\hat{A}$

3.

14.3.1 Suppose we have a set of photons in a mixed state, with probabilities $P_1 = 0.2$ and $P_2 = 0.8$ respectively of being in the two different pure states

$$|\psi_1\rangle = |\psi_H\rangle \text{ and } |\psi_2\rangle = \frac{3}{5}|\psi_H\rangle + \frac{4i}{5}|\psi_V\rangle$$

where $|\psi_H\rangle$ and $|\psi_V\rangle$ are the normalized and orthogonal basis states representing horizontal and vertical polarization respectively. ($|\psi_1\rangle$ therefore is a horizontally polarized state, and $|\psi_2\rangle$ is an elliptically polarized state.) Write the density matrix for this state, in the $|\psi_H\rangle$ and $|\psi_V\rangle$ basis, with $\langle\psi_H|\rho|\psi_H\rangle$ as the top left element.

Jawab:

$$\rho = P_1|\psi_1\rangle\langle\psi_1| + P_2|\psi_2\rangle\langle\psi_2|$$

$$\rho = \frac{1}{5}|\psi_1\rangle\langle\psi_1| + \frac{4}{5}|\psi_2\rangle\langle\psi_2|$$

$$\rho = \frac{1}{5}|\psi_H\rangle\langle\psi_H| + \frac{4}{5}\left(\frac{3}{5}|\psi_H\rangle + \frac{4i}{5}|\psi_V\rangle\right)\left(\frac{3}{5}\langle\psi_H| - \frac{4i}{5}\langle\psi_V|\right)$$

$$\rho = \frac{1}{5}|\psi_H\rangle\langle\psi_H| + \frac{4}{5}\left(\frac{9}{25}|\psi_H\rangle\langle\psi_H| + \frac{12i}{25}|\psi_V\rangle\langle\psi_H| - |\psi_H\rangle\langle\psi_V| + \frac{16}{25}|\psi_V\rangle\langle\psi_V|\right)$$

$$\rho = \frac{1}{5}|\psi_H\rangle\langle\psi_H| + \frac{36}{125}|\psi_H\rangle\langle\psi_H| + \frac{48i}{125}|\psi_V\rangle\langle\psi_H| - |\psi_H\rangle\langle\psi_V| + \frac{64}{125}|\psi_V\rangle\langle\psi_V|$$

$$\rho = \frac{1}{125}\left(61|\psi_H\rangle\langle\psi_H| + 48i\,|\psi_V\rangle\langle\psi_H| - |\psi_H\rangle\langle\psi_V| + 64|\psi_V\rangle\langle\psi_V|\right)$$

- Diubah menjadi matrix

$$\langle\psi_V| = (0 \quad 1) \qquad |\psi_V\rangle = \begin{pmatrix}0\\1\end{pmatrix} \qquad \langle\psi_H| = (1 \quad 0) \qquad |\psi_H\rangle = \begin{pmatrix}1\\0\end{pmatrix}$$

$$\rho = \frac{1}{125}\left(61\begin{pmatrix}1\\0\end{pmatrix}(1 \quad 0) + 48i\begin{pmatrix}0\\1\end{pmatrix}(1 \quad 0) - 48i\begin{pmatrix}1\\0\end{pmatrix}(0 \quad 1) + 64\begin{pmatrix}0\\1\end{pmatrix}(0 \quad 1)\right)$$

$$\rho = \frac{1}{125}\left(61\begin{vmatrix}1 & 0\\0 & 0\end{vmatrix} + 48i\begin{vmatrix}0 & 0\\1 & 0\end{vmatrix} - 48i\begin{vmatrix}0 & 1\\0 & 0\end{vmatrix} + 64\begin{vmatrix}0 & 0\\0 & 1\end{vmatrix}\right)$$

$$\rho = \frac{1}{125}\begin{vmatrix} 61 & -48i \\ 48i & 64 \end{vmatrix}$$

$$\langle\psi_H|\rho|\psi_H\rangle = \langle\psi_H|\frac{1}{125}\,(61|\psi_H\rangle\langle\psi_H| + 48i|\psi_V\rangle\langle\psi_H| - |\psi_H\rangle\langle\psi_V| \;+ 64|\psi_V\rangle\langle\psi_V|)\,|\psi_H\rangle$$

$$= \frac{61}{125}\langle\psi_H|\psi_H\rangle\langle\psi_H|\psi_H\rangle$$

$$\rho_{HH} = \frac{61}{125}$$

4. 14.3.2 Consider the mixed spin state, with equal probabilities of the electrons being in the pure state $|s_x\rangle$ and the pure state $|s_y\rangle$. Here $|s_x\rangle$ and $|s_y\rangle$ are respectively spin states oriented along the $+x$ and $+y$ directions. (See Problem 14.1.1)

(i) Evaluate the density operator ρ on the z spin basis (i.e., $|\uparrow\rangle$ and $|\downarrow\rangle$)

(ii) Now write this density operator as a density matrix, with the term in $|\uparrow\rangle\langle\uparrow|$ in the top left element.

(iii) Taking the spin magnetic dipole moment operator to be $\hat{\boldsymbol{\mu}}_e = g\mu_B\hat{\boldsymbol{\sigma}}$, evaluate $\hat{\boldsymbol{\mu}}_e$ as a matrix on the same z spin basis (i.e., $|\uparrow\rangle$ and $|\downarrow\rangle$, with the element $\langle\uparrow|\hat{\boldsymbol{\mu}}_e|\uparrow\rangle$ in the top left corner.

(iv) Using the expression of the form $\overline{\langle A\rangle} = Tr\left(\rho\hat{A}\right)$, evaluate the ensemble average expectation value for the spin magnetic dipole moment in this mixed state. [Hint: the answer should be the same as that for Problem 14.1.1 (ii)(a).]

Jawab:

w/

$$|S_x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \qquad \langle S_x| = \frac{1}{\sqrt{2}}(\langle 0| + \langle 1|) = \frac{1}{\sqrt{2}}(\langle\uparrow| + \langle\downarrow|)$$

$$|S_y\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) = \frac{1}{\sqrt{2}}(|\uparrow\rangle + i|\downarrow\rangle) \qquad \langle S_y| = \frac{1}{\sqrt{2}}(\langle 0| - i\langle 1|) = \frac{1}{\sqrt{2}}(\langle\uparrow| - i\langle\downarrow|)$$

$$|\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix} \qquad \langle\uparrow| = (1 \quad 0) \qquad |\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix} \qquad \langle\downarrow| = (0 \quad 1)$$

(i) The density operator

$$\rho = 0.5|S_x\rangle\langle S_x| + 0.5|S_y\rangle\langle S_y|$$

$$\rho = \frac{1}{2}\left(\left(\frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)\right)\left(\frac{1}{\sqrt{2}}(\langle\uparrow| + \langle\downarrow|)\right)\right) + \frac{1}{2}\left(\left(\frac{1}{\sqrt{2}}(|\uparrow\rangle + i|\downarrow\rangle)\right)\left(\frac{1}{\sqrt{2}}(\langle\uparrow| - i\langle\downarrow|)\right)\right)$$

$$\rho = \frac{1}{2}\left(\frac{1}{2}(|\uparrow\rangle + |\downarrow\rangle)(\langle\uparrow| + \langle\downarrow|) + \frac{1}{2}(|\uparrow\rangle + i|\downarrow\rangle)(\langle\uparrow| - i\langle\downarrow|)\right)$$

$$\rho = \frac{1}{4}(|\uparrow\rangle + |\downarrow\rangle)(\langle\uparrow| + \langle\downarrow|) + \frac{1}{4}(|\uparrow\rangle + i|\downarrow\rangle)(\langle\uparrow| - i\langle\downarrow|)$$

$$\rho = \frac{1}{4}\left(|\uparrow\rangle\langle\uparrow| + |\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow| + |\uparrow\rangle\langle\uparrow| - |\uparrow\rangle\langle\downarrow| + i|\downarrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\right)$$

$$\rho = \frac{1}{4}\left(2(|\uparrow\rangle\langle\uparrow|) + (|\downarrow\rangle\langle\downarrow|) + (1+i)|\downarrow\rangle\langle\uparrow| + (1-i)|\uparrow\rangle\langle\downarrow|\right)$$

(ii) Diubah menjadi matrix

$$\rho = \frac{1}{4}\left(2\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} + (1+i)\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + (1-i)\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}\right)$$

$$\rho = \frac{1}{4}\left(2\left(\begin{vmatrix}1 & 0\\0 & 0\end{vmatrix} + \begin{vmatrix}0 & 0\\0 & 1\end{vmatrix}\right) + (1+i)\begin{vmatrix}0 & 0\\1 & 0\end{vmatrix} + (1-i)\begin{vmatrix}0 & 1\\0 & 0\end{vmatrix}\right)$$

$$\rho = \frac{1}{4}\begin{vmatrix}2 & (1-i)\\(1+i) & 2\end{vmatrix}$$

(iii) Spin magnetic dipole moment menjadi matrix

$$\hat{\mu}_e = g\mu_B\left(\sigma_x\hat{x} + \sigma_y\hat{y} + \sigma_z\hat{z}\right)$$

$$\hat{\mu}_e = g\mu_B\left(\begin{vmatrix}0 & 1\\1 & 0\end{vmatrix}\hat{x} + \begin{vmatrix}0 & -i\\i & 0\end{vmatrix}\hat{y} + \begin{vmatrix}1 & 0\\0 & -1\end{vmatrix}\hat{z}\right)$$

$$\hat{\mu}_e = g\mu_B\begin{vmatrix}\hat{z} & \hat{x} - i\hat{y}\\\hat{x} + i\hat{y} & -\hat{z}\end{vmatrix}$$

(iv) Hasil trace

$$\overline{\langle\hat{\mu}_e\rangle} = Tr(\rho\hat{\mu}_e)$$

$$\overline{\langle\hat{\mu}_e\rangle} = Tr\left(\frac{g\mu_B}{4}\begin{vmatrix}2 & 1-i\\1+i & 2\end{vmatrix}\begin{vmatrix}\hat{z} & \hat{x} - i\hat{y}\\\hat{x} + i\hat{y} & -\hat{z}\end{vmatrix}\right)$$

$$\overline{\langle\hat{\mu}_e\rangle} = Tr\left(\frac{g\mu_B}{4}\begin{vmatrix}2\hat{z} - (1-i)(\hat{x} - i\hat{y}) & 2(\hat{x} - i\hat{y}) + (1-i)\hat{z}\\(1+i)\hat{z} + 2(\hat{x} + i\hat{y}) & (1+i)(\hat{x} + i\hat{y}) - 2\hat{z}\end{vmatrix}\right)$$

$$\overline{\langle\hat{\mu}_e\rangle} = \frac{g\mu_B}{4}(2, 2, 0)$$

$$\overline{\langle\hat{\mu}_e\rangle} = \frac{g\mu_B}{4}(1, 1, 0)$$

$$\overline{\langle\hat{\mu}_e\rangle} = \frac{g\mu_B}{\sqrt{2}}$$