Applied Time Series Econometrics

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Rank estimation: idea

- Hence, the rank of matrix Π is crucial to determine the number of cointegration vectors.
- Johansen's procedure (Johansen, 1988) is based on a fact that the rank of a matrix is equal to the number of its non-zero eigenvalues.
- If an estimate of Π is known, it is possible to find its eigenvalues $\lambda_i \quad i = 1, \cdots, n$ and order them: $\lambda_1 > \lambda_2 > \cdots > \lambda_n$.

Rank estimation: trace test

Johansen suggests two tests that aim at determining the number of non-zero eigenvalues.

Trace test

$$H_0 : rank(\Pi) \leq r$$
$$H_1 : rank(\Pi) > r$$

Test statistics:

$$\lambda_{trace}(r) = -T \sum_{i=r+1}^{n} \ln(1 - \hat{\lambda}_i),$$

where T is a number of observations and $\hat{\lambda}_i$ are eigenvalues estimates.

Rank estimation: maximum eigenvalue test

Max eigenvalue test

$$H_0 : rank(\Pi) \leq r$$
$$H_1 : rank(\Pi) = r + 1$$

Test statistics:

$$\lambda_{max}(r, r+1) = -T \ln(1 - \hat{\lambda}_{r+1}),$$

where T is a number of observations and $\hat{\lambda}_i$ are eigenvalues estimates.

In both cases the distributions are non-standard and depend on the the number of non-stationary terms and the type of included deterministic components. Johansen in his paper (Johansen, 1994) prefered the trace test.

Alternative VECM specifications(1)

1. Constants and trends both in the short run and the long run

$$\Delta y_t = \delta_0 + \delta_1 t + \alpha(\beta_0' + \beta_1' t + \beta' y_{t-1}) + \sum_{j=1}^{p-1} \Gamma_j \Delta y_{t-j} + \upsilon_t$$

2. No trends in the short run $\delta_1 = 0$

$$\Delta y_t = \delta_0 + \alpha(\beta_0' + \beta_1' t + \beta' y_{t-1}) + \sum_{j=1}^{p-1} \Gamma_j \Delta y_{t-j} + \upsilon_t$$

3. No trends either in the short run or in the long run $(\delta_1 = 0, \beta_1 = 0)$

$$\Delta y_t = \delta_0 + \alpha(\beta_0' + \beta' y_{t-1}) + \sum_{j=1}^{p-1} \Gamma_j \Delta y_{t-j} + \upsilon_t$$

Alternative VECM specifications(2)

4. No trends either in the short run or in the long run, no constant in the short run ($\delta_1 = 0, \beta_1 = 0, \delta_0 = 0$)

$$\Delta y_t = \alpha(\beta_0' + \beta' y_{t-1}) + \sum_{j=1}^{p-1} \Gamma_j \Delta y_{t-j} + \upsilon_t$$

5. No trends either in the short run or in the long run, no constant either in the short run or in the long run ($\delta_1 = 0, \beta_1 = 0, \delta_0 = 0, \beta_0 = 0$)

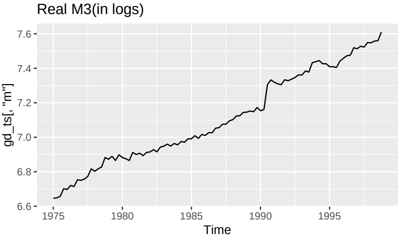
$$\Delta y_t = \alpha\beta' y_{t-1} + \sum_{j=1}^{p-1} \Gamma_j \Delta y_{t-j} + \upsilon_t$$

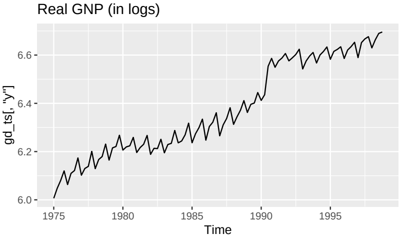
Example of using coinegration analysis

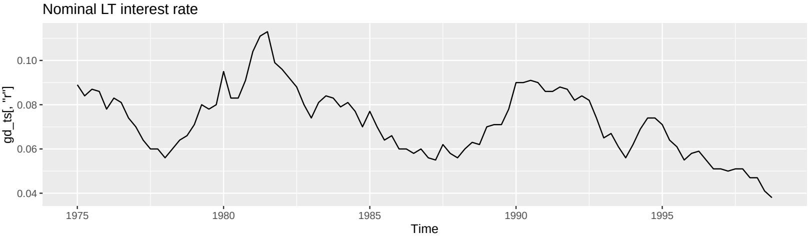
- The goal is to estimate the money demand for Germany and a VEC model (VECM).
- The dataset contains quaterly data on:
- nominal long term interest rate (r)
- log of real GNP (y)
- log of real M3 monetary aggregate (m)

The sample goes from 1975Q1 (start of monetary targeting policy in Western Germany) 1998Q4 (last quarter before the start of euro). The data represent West German data until 1990Q2, and all of Germany aferwards.

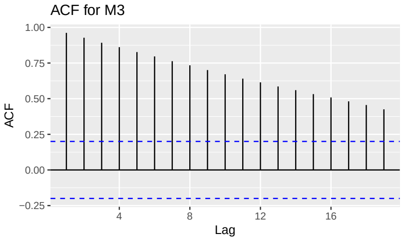
Data graphs

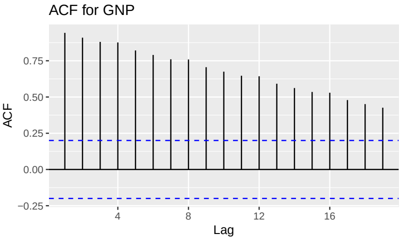

Real M3(in logs)
7.6
7.4
7.2
7.0
6.8
6.6
gd_ts[, "m"]
1975
1980
1985
1990
1995
Time

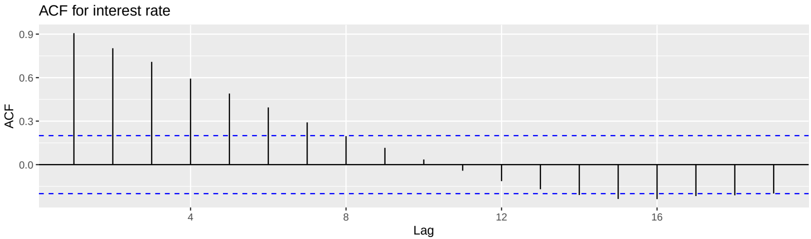

Real GNP (in logs)
6.6
6.4
6.2
6.0
gd_ts[, "y"]
1975
1980
1985
1990
1995
Time


Nominal LT interest rate
0.10
0.08
0.06
0.04
gd_ts[, "r"]
1975
1980
1985
1990
1995
Time

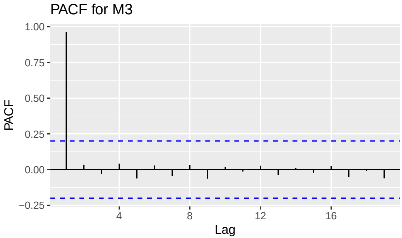
Autocorrelation functions

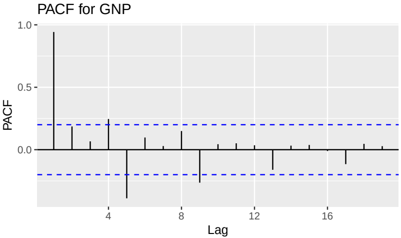


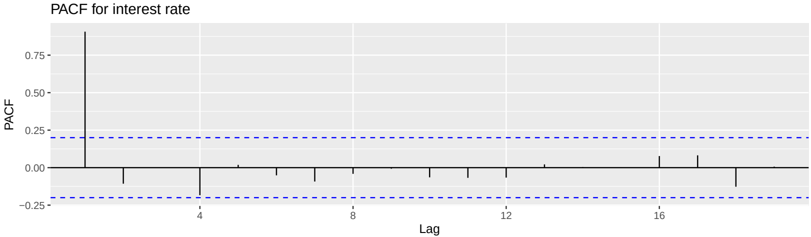




Partial autocorrelation functions







ADF tests for variables in levels and differences

variable	test stat.	1pct	5pct	10pct
m	0.28	-3.51	-2.89	-2.58
Δm	-2.81	-2.6	-1.95	-1.61
y	-0.36	-3.51	-2.89	-2.58
Δy	-2.55	-2.6	-1.95	-1.61
r	-1.35	-3.51	-2.89	-2.58
Δr	-6.31	-2.6	-1.95	-1.61

We can consider all variables in the dataset as I(1).

Determining the cointegration rank

The hypothesis that Π matrix has reduced rank is tested with the trace and maximum eigenvalue statistics. Seasonal dummis and one unification dummy are included in the model.

```
library(urca);
unif_dum = rep(0, 96)
unif_dum[63] = 1
coint_h1 <- ca.jo(gd_ts, type = "trace", K = 2,
                  spec = "transitory", season = 4,
                  dumvar = unif_dum)
coint_h2 <- ca.jo(gd_ts, type = "eigen", K = 2,
                  spec = "transitory", season = 4,
                  dumvar = unif_dum)
summary(coint_h1)
summary(coint_h2)
```

Long-run versus transitory specification

The same VAR model in reduced form:

$$y_t = \mu + \Phi_1 y_{t-1} + \Phi_2 y_{t-2} + \cdots + \Phi_p y_{t-p} + \nu_t$$

can be rewritten in *transitory form*:

$$\Delta y_t = \mu + \Pi y_{t-1} + \Gamma_1 \Delta y_{t-1} + \cdots + \Gamma_{p-1} \Delta y_{t-p+1} + \upsilon_t,$$

where $\Gamma_i = -(\Phi_{i+1} + \cdots + \Phi_p)$ for $i = 1, \ldots, p-1$.

or in *long-run form*

$$\Delta y_t = \mu + \Pi y_{t-p} + \Gamma_1 \Delta y_{t-1} + \cdots + \Gamma_{p-1} \Delta y_{t-p+1} + \upsilon_t,$$

where $\Gamma_i = -(I - \Phi_1 - \cdots - \Phi_i)$ for $i = 1, \ldots, p-1$

$\Pi = -(I - \Phi_1 - \cdots - \Phi_p)$ for both specifications.

Results: interpretation of the trace test

Eigenvalues (lambda): 0.20244 0.06928 0.00054

Values of teststatistic and critical values of test:

variable	test stat.	10pct	5pct	1pct
$r \leq 2$	0.05	6.5	8.18	11.65
$r \leq 1$	6.8	15.66	17.95	23.52
$r = 0$	28.06	28.71	31.52	37.22

The hypothesis of no cointegration cannot be rejected at 5% level.

Results: interpretation of the max eigenvalue test

Eigenvalues (lambda): 0.20244 0.06928 0.00054

Values of teststatistic and critical values of test:

variable	test stat.	10pct	5pct	1pct
$r \leq 2$	0.05	6.5	8.18	11.65
$r \leq 1$	6.75	12.91	14.9	19.19
$r = 0$	21.26	18.9	21.07	25.75

The maximum eigenvalue statistics indicates a cointegration space of $r = 1$ given a 5% significance level.

Eigenvectors and cointegration relations

Eigenvectors, normalised to first column: (These are the cointegration relations)

```
coint_h2@V
```

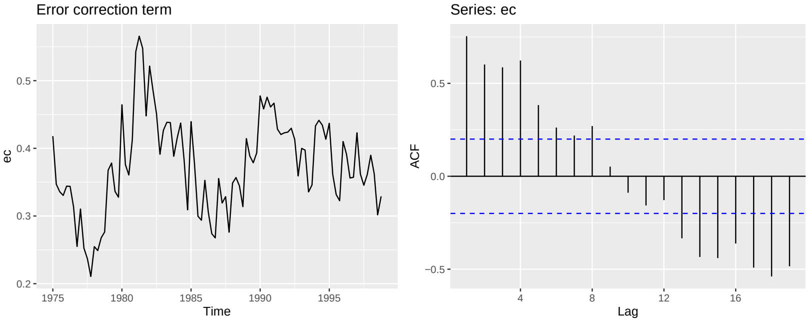
```
##          m.l1    y.l1   r.l1
## m.l1  1.000  1.0000  1.000
## y.l1 -1.119 -1.4387 -1.035
## r.l1  5.530  0.6194 -1.412
```

Normalization is necessary because loading matrix and cointegration vector are not unique.

Cointegration relation: $ec_t = m_t$ -1.12 $y_t +$ 5.53 r_t

Error correction term

```
ec = gd_ts[,"m"] + coint_h2@V[2,1]*gd_ts[,"y"] +
  coint_h2@V[3,1]*gd_ts[,"r"]
plot_ec <- autoplot(ec) + ggtitle("Error correction term")
acf_ec <- ggAcf(ec)
grid.arrange(plot_ec, acf_ec, ncol = 2 )
```



Unit root and stationarity tests for EC_t

```
adf_ec <- ur.df(ec, type = "drift",lags = 8,
                selectlags = "BIC")
kpss_ec <- ur.kpss(ec, type = "mu",lags = "short")
```

test	test stat.	1pct	5pct	10pct
ADF	-3.03	-3.51	-2.89	-2.58
KPSS	0.23	0.74	0.46	0.35

According to ADF test, the hypothesis of a unit root can be rejected at 5% significance level. According to KPSS test, the hypothesis of stationarity cannot be rejected at 5% significance level.

Matrix decomposition

$$\Pi y_{t-1} = \alpha\beta' y_{t-1} = \begin{pmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{31} \end{pmatrix} \begin{pmatrix} \beta_{11} & \beta_{21} & \beta_{31} \end{pmatrix} \begin{pmatrix} y_{1,t-1} \\ y_{2,t-1} \\ y_{3,t-1} \end{pmatrix} =$$

$$= \begin{pmatrix} \alpha_{11} cr_{1,t-1} \\ \alpha_{21} cr_{1,t-1} \\ \alpha_{31} cr_{1,t-1} \end{pmatrix}$$

```
alpha_m <- coint_h2@W[,1] # Loading matrix
alpha_m
beta_m <- coint_h2@V[,1] # Cointegration matrix
beta_m
pi_m <- alpha_m %*% t(beta_m)
pi_m
library(matrixcalc)
matrix.rank(pi_m)
```

Matrix decomposition(2)

```
##     m.d    y.d    r.d
## -0.062 -0.011 -0.015

## m.l1 y.l1 r.l1
##  1.0 -1.1  5.5

##        m.l1  y.l1   r.l1
## [1,] -0.062 0.069 -0.343
## [2,] -0.011 0.013 -0.063
## [3,] -0.015 0.017 -0.086

## [1] 1
```

VECM estimation

When the number of cointegration relations is determined, the VECM can be estimated with the `cajorls()` function.

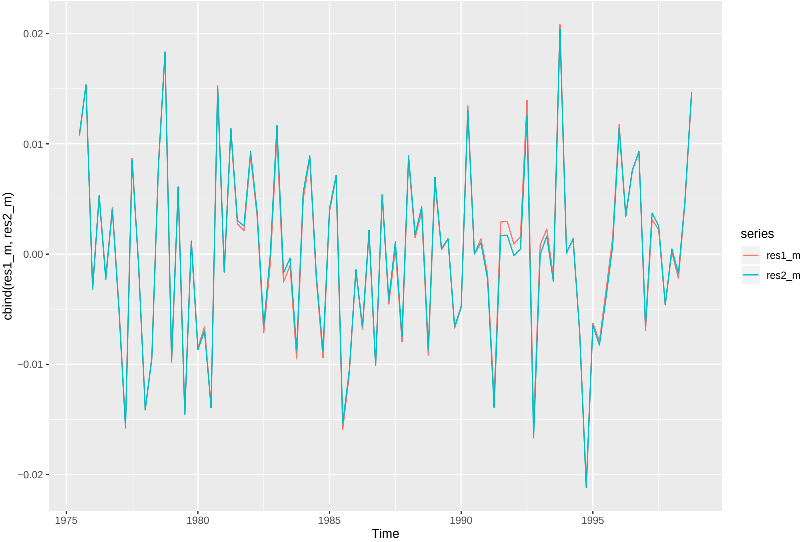
```
gd_vecm <- cajorls(coint_h2, r = 1)
```

Alternatively we can transform VECM to its level VAR representation with the `vec2var()` function from the `vars` package.

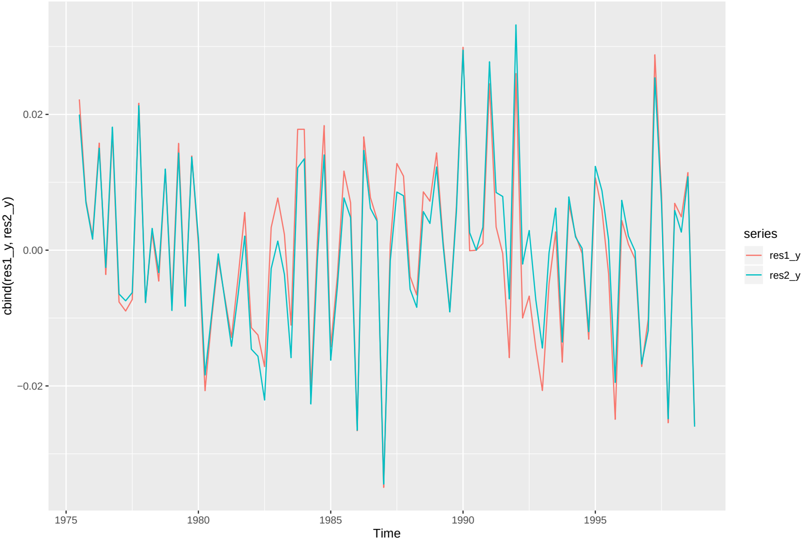
```
gd_var <- vec2var(coint_h2, r = 1)
gd_var2 <- VAR(gd_ts, p = 2, type = "const",
           season = 4, exogen = unif_dum)
```

The biggest advantage of VECM is that it has a clear interpretation with long-run and short-run terms. But essentially the model is the same.

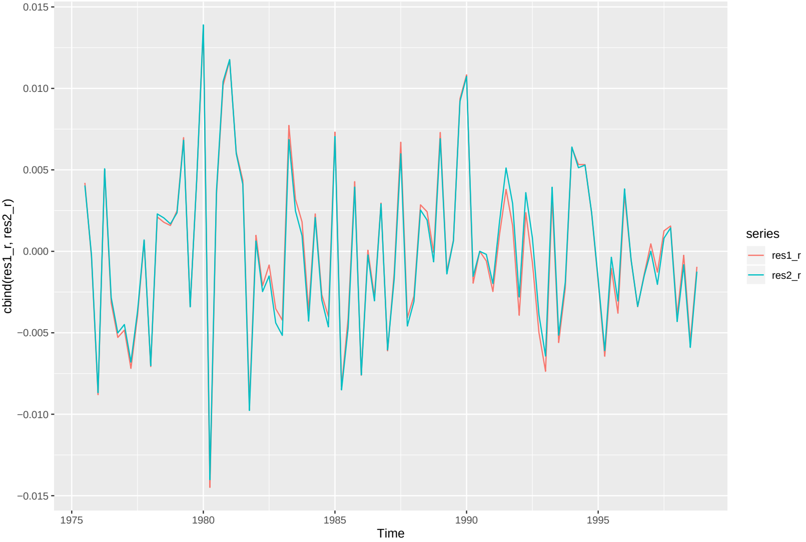
Graph of residuals: m - equation



Graph of residuals: y - equation



Graph of residuals: r - equation



VECM: discussion

- The difference in fitted values and residuals is slight but it exists. In principle, when we determine a cointegration rank and estimate VECM, we restrict some coefficients of VAR model.
- The advantage of VECM with respect to VAR in levels (estimated without paying attention to cointegration relations) is that the VAR model coming from VECM representation has more efficient coefficient estimates if the VECM is correctly specified.

Forecasting with VECM

- As a VECM is a form of representation of a VAR model, there is no special functions for forecasting with VECMs.
- If the research interest centers in forecasting with a VECM, then the algorithm should be as follows 1. estimate a VEC model 2. transform it into a VAR model 3. forecast using the VAR model

Literature

1. Lütkepohl, H., and Krätzig, M. (2004) Applied Time Series Econometrics, New York, Cambridge University Press
2. Martin, V., Hurn, S., and HarriS,D. (2013) Econometric Modelling with Time Series: Specification, Estimating and Testing, New York, Cambridge University Press
3. Pfaff, B. (2008) Analysis of Integrated and Cointegrated Time Series with R, New York, Springer