Applied Time Series Econometrics

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Structural VAR models: introduction

- VARs have the status of "reduced form models" and therefore are merely vehicles to summarize the dynamic properties of the data (Cooley and Le Roy, 1985).
- The reduced-form VAR parameters are not related to "deep" structural parameters characterizing preferences, technologies, and optimization behaviour, the parameters do not have an economic meaning and are subject to Lucas critique.
- Structural (identified) VARs explicitely reflect the implied structure among variables in the system.

Specification of a SVAR model

A structural VAR model may be represented as follows:

$$B_0 y_t = \lambda + B_1 y_{t-1} + \cdots + B_p y_{t-p} + \varepsilon_t, \qquad \varepsilon_t \sim iidN(0, \Sigma)$$

B_0 is a matrix of contemporanous structural relations. Covariance matrix Σ is diagonal. An example with bivariate model:

$$b_{0,1,1} y_{1,t} + b_{0,1,2} y_{2,t} = \lambda_1 + b_{1,1,1} y_{1,t-1} + b_{1,1,2} y_{2,t-1} + \varepsilon_{1,t}$$
$$b_{0,2,1} y_{1,t} + b_{0,2,2} y_{2,t} = \lambda_2 + b_{1,2,1} y_{1,t-1} + b_{1,2,2} y_{2,t-1} + \varepsilon_{2,t}$$

We want to determine $n(1 + pn + n)$ parameters of B_i matrices and n parameters of Σ.

SVAR estimation

- We cannot estimate SVAR directly because of contemporenous relations.
- We need to estimate a reduced-form VAR and then "restore" the parameters of the structural form.

$$
\begin{aligned}
B_0 y_t &= \lambda + B_1 y_{t-1} + \cdots + B_p y_{t-p} + \varepsilon_t, \qquad \varepsilon_t \sim iidN(0, \Sigma) \\
y_t &= B_0^{-1}\lambda + B_0^{-1} B_1 y_{t-1} + \cdots + B_0^{-1} B_p y_{t-p} + B_0^{-1}\varepsilon_t \\
y_t &= \mu + \Phi_1 y_{t-1} + \cdots + \Phi_p y_{t-p} + \upsilon_t, \qquad \upsilon_t \sim iidN(0, \Omega)
\end{aligned}
$$

where $\mu = B_0^{-1}\lambda, \quad \Phi_i = B_0^{-1} B_i, \quad \upsilon_t = B_0^{-1}\varepsilon_t$

We have $n(1+pn)$ elements in $\hat{\Phi}_i$ and $\frac{(n+1)n}{2}$ unique elements in $\hat{\Omega}$.

Restrictions

$$\underbrace{(n(1+pn+n)+n)}_{\text{parameters of structural form to estimate}} - \underbrace{\left(n(1+pn)+\frac{(n+1)n}{2}\right)}_{\text{estimates of reduced form}} = \frac{n(n+1)}{2}$$

- So, we need $\frac{n(n+1)}{2}$ additional restrictions to have an exactly identified VAR.
- What kind of restrictions exist in economic applcations?
  - recursive scheme restrictions (Choleski identification)
  - short-run restrictions
  - long-run restrictions
  - sign restrictions
  - explicit prior distributions in Bayesian econometrics
  - identification through heteroscedasticity

Recursive identification: idea (1)

Square matrix B_0 of dimension $(n \times n)$ contains $\frac{n(n-1)}{2}$ elements above the main diagonal.

$$B_0 = \begin{pmatrix} b_{0,1,1} & b_{0,1,2} & \cdots & b_{0,1,n-1} & b_{0,1,n} \\ b_{0,2,1} & b_{0,2,2} & \cdots & b_{0,2,n-1} & b_{0,2,n} \\ \vdots & \ddots & \vdots & & \\ b_{0,n-1,1} & b_{0,n-1,2} & \cdots & b_{0,n-1,n-1} & b_{0,n-1,n} \\ b_{0,n,1} & b_{0,n,2} & \cdots & b_{0,n,n-1} & b_{0,n,n} \end{pmatrix}$$

Diagonal matrix Σ of dimension $(n \times n)$ contains n non-zero elements on the main diagonal.

$$\frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}$$

Recursive identification: idea (2)

Let us put zeros above the main diagnal in B_0 matrix and ones on the main diagonal of Σ matrix.

$$B_0 = \begin{pmatrix} b_{0,1,1} & 0 & \cdots & 0 & 0 \\ b_{0,2,1} & b_{0,2,2} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & & \\ b_{0,n-1,1} & b_{0,n-1,2} & \cdots & b_{0,n-1,n-1} & 0 \\ b_{0,n,1} & b_{0,n,2} & \cdots & b_{0,n,n-1} & b_{0,n,n} \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & & \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

Recursive identification: idea(3)

$$b_{0,1,1}y_{1,t} = \sum_{i=1}^{p} b_{i,1,1}y_{1,t-i} + \ldots + \sum_{i=1}^{p} b_{i,1,n}y_{n,t-i} + \lambda_1 + \varepsilon_{1,t}$$

$$b_{0,2,2}y_{2,t} = -b_{0,2,1}y_{1,t} + \sum_{i=1}^{p} b_{i,2,1}y_{1,t-i} + \ldots + \sum_{i=1}^{p} b_{i,2,n}y_{n,t-i} + \\ + \lambda_2 + \varepsilon_{2,t}$$

$$b_{0,3,3}y_{3,t} = -b_{0,2,1}y_{1,t} - b_{0,3,2}y_{2,t} + \sum_{i=1}^{p} b_{i,3,1}y_{1,t-i} + \ldots + \\ + \sum_{i=1}^{p} b_{i,3,n}y_{n,t-i} + \lambda_3 + \varepsilon_{3,t}$$

$$\cdots = \cdots\cdots$$

$$b_{0,n,n}y_{n,t} = -b_{0,n,1}y_{1,t} - b_{0,n,2}y_{2,t} - \ldots - b_{0,n,n-1}y_{n-1,t} + \\ + \sum_{i=1}^{p} b_{i,n,1}y_{1,t-i} + \ldots + \sum_{i=1}^{p} b_{i,n,n}y_{n,t-i} + \lambda_n + \varepsilon_{n,t}$$

Recursive identification: computation

- The recursive scheme means that each variable can be written now as a function of all contemporanious values of variables that are put before this variable in the list of variables, and all lags of all variables.
- To rationalize this scheme, we need to rank our variables from the "most exogenous" to the "most endogenous" (even if all variables are considered to be endogenous).
- The necessary condition (order condition) for exact identification is respected.

$$\upsilon_t = B_0^{-1}\varepsilon_t$$

$$E(\upsilon_t \upsilon_t') = B_0^{-1} E(\varepsilon_t \varepsilon_t')(B_0^{-1})'$$

$$\Omega = B_0^{-1}(B_0^{-1})'$$

! Inverse of a lower triangular matrix is also a lower triangular matrix.

Recursive identification: computation

Choleski decomposition: a decomposition of a symmetric positive definite matrix A as:

$$A = LL',$$

where L is a lower triangular matrix with positive elements on the main diagonal. Choleski decomposition always exists and it is unique.

$$\hat{B}_0^{-1} = chol(\hat{\Omega}) \qquad \hat{B}_i = \hat{B}_0 \hat{\Phi}_i \qquad \hat{\lambda} = \hat{B}_0 \hat{\mu},$$

where *chol* is a Choleski decomposition operator

Computation

```
Omega_hat <- summary(var3)$covres
chol(Omega_hat)%>%t()
```

```
##             consumption   income
## consumption   0.6304155 0.000000
## income        0.3111407 0.796349
```

```
Psi(var3)[,,1]
```

```
##           [,1]     [,2]
## [1,] 0.6304155 0.000000
## [2,] 0.3111407 0.796349
```

Identification: next steps

When we have estimates of the structural form, we can compute:

1. Impulse response functions
2. Forward error variance decomposition

Impulse response functions: idea

- Impulse response functions show the effect of a shock on endogenous variables of the system.
- To calculate impulse response functions we need to make two steps:
  1. represent VAR(p) process as $VMA(\infty)$ process following Wold theorem.
  2. replace reduced form errors with structural shocks.

VMA representation of a VAR(1) process

If $p = 1$ then the VMA representation is derived easily:

$$\begin{aligned}
y_t &= \mu + \Phi_1 y_{t-1} + \upsilon_t = \\
&= \mu + \Phi_1(\mu + \Phi_1 y_{t-2} + \upsilon_{t-1}) + \upsilon_t = \\
&= \mu + \Phi_1\mu + \Phi_1^2(\mu + \Phi_1 y_{t-3} + \upsilon_{t-2}) + \upsilon_t + \Phi_1\upsilon_{t-1} = \\
&\ldots\ldots\ldots \\
&= (I + \Phi_1 + \Phi_1^2 + \ldots)\mu + \upsilon_t + \Phi_1\upsilon_{t-1} + \Phi_1^2\upsilon_{t-2} + \ldots = \\
&= (I - \Phi_1)^{-1}\mu + \upsilon_t + \Phi_1\upsilon_{t-1} + \Phi_1^2\upsilon_{t-2} + \ldots
\end{aligned}$$

as:

$$(I - \Phi_1)(I + \Phi_1 + \Phi_1^2 + \ldots) = I$$

#see also Lecture Notes 6

VMA representation of a VAR(p) process(1)

If $p > 1$ then we can proceed in two ways:

1. We can rewrite the $VAR(p)$ model as $VAR(1)$ making use of a companion matrix:

$$y_t = \mu + \Phi_1 y_{t-1} + \Phi_2 y_{t-2} + \ldots + \Phi_p y_{t-p} + \upsilon_t$$

$$\begin{pmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-p+2} \\ y_{t-p+1} \end{pmatrix} = \begin{pmatrix} \mu_t \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \Phi_1 & \Phi_2 & \cdots & \Phi_{p-1} & \Phi_p \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{pmatrix} \begin{pmatrix} y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \\ y_{t-p} \end{pmatrix} + \begin{pmatrix} \upsilon_t \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

$$Y_t = M + \Phi Y_{t-1} + U_t$$

where Y_t, M, and U_t have dimension $(np \times 1)$ and Φ has dimension of $(np \times np)$

VMA representation of a VAR(p) process(2)

2. An alternative way is to rewrite the model with a lag matrix polinomial.

$$y_t = \mu + \Phi_1 y_{t-1} + \Phi_2 y_{t-2} + \ldots + \Phi_p y_{t-p} + \upsilon_t$$
$$\Phi(L) y_t = \mu + \upsilon_t,$$

where $\Phi(L) = I - \Phi_1 L - \cdots - \Phi_p L^p$.

Let $C(L) = C_0 + C_1 L^1 + C_2 L^2 + \ldots$ be an operator such that:

$$C(L)\Phi(L) = I$$
$$C(L)\Phi(L) y_t = C(L)\mu + C(L)\upsilon_t$$
$$y_t = (C_0 + C_1 L^1 + C_2 L^2 + \ldots)\mu + (C_0 + C_1 L^1 + C_2 L^2 + \ldots)\upsilon_t$$

because $L_i \mu = \mu \quad \forall i$ and $L_i \upsilon_t = \upsilon_{t-i}$.

VMA representation of a VAR(p) process(2)

$$y_t = \left(\sum_{i=0}^{\infty} C_i\right)\mu + C_0 \upsilon_t + C_1 \upsilon_{t-1} + C_2 \upsilon_{t-2} + \cdots =$$
$$= \tilde{\mu} + C_0 \upsilon_t + C_1 \upsilon_{t-1} + C_2 \upsilon_{t-2} + \cdots ,$$

where $\tilde{\mu} = \left(\sum_{i=0}^{\infty} C_i\right)\mu$

The coefficient matrices C_i can by obtain from definition:

$$I = (C_0 + C_1 L + C_2 L^2 + \cdots)(I - \Phi L - \cdots \Phi_p L_p) =$$
$$= C_0 + (C_1 - C_0 \Phi_1) L + (C_2 - C_1 \Phi_1 - C_0 \Phi_2) L^2 + \cdots$$
$$+ (C_i - C_{i-1}\Phi_1 - C_{i-2}\Phi_2 - \ldots - C_0 \Phi_i) L^i + \cdots$$

VMA representation

$$
\begin{aligned}
I &= C_0 \\
0 &= C_1 - C_0\Phi_1 \\
0 &= C_2 - C_1\Phi_1 - C_0\Phi_2 \\
&\dots \\
0 &= C_i - C_{i-1}\Phi_1 - C_{i-2}\Phi_2 - \ldots - C_0\Phi_i \\
&\dots
\end{aligned}
$$

where $\Phi_j = 0$ for $j > p$.

Hence, the C_i can be computed recursively using:

$$
\begin{aligned}
C_0 &= I \\
C_i &= C_{i-1}\Phi_1 + C_{i-2}\Phi_2 + \ldots + C_0\Phi_i \qquad i = 1, 2, \ldots
\end{aligned}
$$

Example of VMA representation

For VAR(2) these recursions result in

$$\begin{aligned}
C_0 &= I \\
C_1 &= \Phi_1 \\
C_2 &= C_1\Phi_1 + \Phi_2 = \Phi_1^2 + \Phi_2 \\
C_3 &= C_2\Phi_1 + C_1\Phi_2 = \Phi_1^3 + \Phi_2\Phi_1 + \Phi_1\Phi_2 \\
&\dots \\
C_i &= C_{i-1}\Phi_1 + C_{i-2}\Phi_2
\end{aligned}$$

Orthogonal impulse response functions(1)

Hense, we can write:

$$y_t = \tilde{\mu} + C_0 \upsilon_t + C_1 \upsilon_{t-1} + C_2 \upsilon_{t-2} \dots$$

- For the inpulse response analysis, orthogonal innovations are needed because economic shocks are commonly assumed to be independent.

In case of Choleski decomposition:

$$\upsilon_t = B_0^{-1} \varepsilon_t$$
$$y_t = \tilde{\mu} + C_0 B_0^{-1} \varepsilon_t + C_1 B_0^{-1} \varepsilon_{t-1} + C_2 B_0^{-1} \varepsilon_{t-2} \dots$$
$$y_t = \tilde{\mu} + \Psi_0 \varepsilon_t + \Psi_1 \varepsilon_{t-1} + \Psi_2 \varepsilon_{t-2} \dots,$$

where $\Psi_i = C_i B_0^{-1}$

Orthogonal impulse response functions(2)

$$y_t = \tilde{\mu} + \Psi_0 \varepsilon_t + \Psi_1 \varepsilon_{t-1} + \Psi_2 \varepsilon_{t-2} \dots$$

- The elements of Ψ_s represent the impulse responses of the components of y_t with respect to the ε_t innovations.
- The $(i, j)th$ elements of the matrices Ψ_s, regarded as a function of s, trace out the expected response of $y_{i,t+s}$ to a unit change in ε_{jt}.
- In the presently considered I(0) case $\Psi_s \to 0$ as $s \to \infty$. Hense, the effect of an impulse is transitory as it vanishes over time.

Accumulated Impulse Responses

Occasionally, interest centers on the accumulated effects of the impulses. They are easily obtained by adding up the Ψ matrices and using the VMA representation of VAR(p):

$$
\begin{aligned}
\Psi &= \Psi_0 + \Psi_1 + \Psi_2 + \cdots = (C_0 + C_1 + C_2 + \cdots)B_0^{-1} = \sum C_i B_0^{-1} \\
I &= C_0 \\
0 &= C_1 - C_0\Phi_1 \\
0 &= C_2 - C_1\Phi_1 - C_0\Phi_2 \\
&\dots
\end{aligned}
$$

Let us sum all LHS and RHS of equatons:

$$
\begin{aligned}
I &= (\sum C_i - \Phi_1 \sum C_i - \cdots - \Phi_p \sum C_i) = \\
&= \sum C_i (I - \Phi_1 - \cdots - \Phi_p) \\
\sum C_i &= (I - \Phi_1 - \cdots - \Phi_p)^{-1} \\
\Psi &= (I - \Phi_1 - \cdots - \Phi_p)^{-1} B_0^{-1}
\end{aligned}
$$

Forecast Error Variance Decomposition

- We can use the VMA representation of VAR(p) model to calculate the forecast.

$$
\begin{aligned}
y_{T+h} &= \tilde{\mu} + C_0 \upsilon_{T+h} + C_1 \upsilon_{T+h-1} + \cdots + C_h \upsilon_T + \cdots \\
y_{T+h|T} &= \tilde{\mu} + C_h \upsilon_T + C_{h+1} \upsilon_{T-1} + \cdots
\end{aligned}
$$

The corresponding forecast error is:

$$y_{T+h} - y_{T+h|T} = C_0 \upsilon_{T+h} + C_1 \upsilon_{T+h-1} + \cdots + C_{h-1} \upsilon_{T+1}$$

Expressing this error in termes of the structural innovarions ε_t using that $\upsilon_t = B_0^{-1} \varepsilon_t$ and $\Psi_i = C_i B_0^{-1}$ gives:

$$y_{T+h} - y_{T+h|T} = \Psi_0 \varepsilon_{T+h} + \Psi_1 \varepsilon_{T+h-1} + \cdots + \Psi_{h-1} \varepsilon_{T+1}$$

Forecast error variance decomposition

Example: VAR(1) model with 2 variables (y_1, y_2), and forecasting horizon $h = 2$

$$\begin{pmatrix} y_{1,T+2} - y_{1,T+2|T} \\ y_{2,T+2} - y_{2,T+2|T} \end{pmatrix} = \begin{pmatrix} \psi_{0,1,1} & \psi_{0,1,2} \\ \psi_{0,2,1} & \psi_{0,2,2} \end{pmatrix} \begin{pmatrix} \varepsilon_{1,T+2} \\ \varepsilon_{2,T+2} \end{pmatrix} + \\ + \begin{pmatrix} \psi_{1,1,1} & \psi_{1,1,2} \\ \psi_{1,2,1} & \psi_{1,2,2} \end{pmatrix} \begin{pmatrix} \varepsilon_{1,T+1} \\ \varepsilon_{2,T+1} \end{pmatrix}$$

$$FEV_1 = (\psi_{0,1,1}^2 + \psi_{1,1,1}^2) + (\psi_{0,1,2}^2 + \psi_{1,1,2}^2)$$
$$FEV_2 = (\psi_{0,2,1}^2 + \psi_{1,2,1}^2) + (\psi_{0,2,2}^2 + \psi_{1,2,2}^2)$$

Forward error variance decomposition

At the forecast in horizon $h = 2$ the forward error variance of y_1 and y_2 is explained by two structural shocks in the following proportions:

	ε_1	ε_2	$\sum$
y_1	$\frac{\psi^2_{0,1,1}+\psi^2_{1,1,1}}{FEV1}$	$\frac{\psi^2_{0,1,2}+\psi^2_{1,1,2}}{FEV1}$	1
y_2	$\frac{\psi^2_{0,2,1}+\psi^2_{1,2,1}}{FEV2}$	$\frac{\psi^2_{0,2,2}+\psi^2_{1,2,2}}{FEV2}$	1

In general case:

$$FEV_i = (\psi^2_{0,i,1} + \cdots + \psi^2_{h-1,i,1}) + (\psi^2_{0,i,2} + \cdots + \psi^2_{h-1,i,2}) + \cdots + \\ + (\psi^2_{0,i,n} + \cdots + \psi^2_{h-1,i,n})$$

The ratio of the forward error variance of the variable i explained by the shock j at forecasting horizon h is equal to:

$$\frac{\psi^2_{0,i,j} + \cdots + \psi^2_{h-1,i,j}}{FEV_i}$$

Literature

1. Martin, V., Hurn, S., and HarriS,D. (2013) Econometric Modelling with Time Series: Specification, Estimating and Testing, New York, Cambridge University Press
2. Lütkepohl, H., and Krätzig, M. (2004) Applied Time Series Econometrics, New York, Cambridge University Press