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18.03SC Unit 1 Exam

1. **(a)** In a perfect environment, the population of Norway rat that breeds on the MIT campus increases by a factor of $e \simeq 2.718281828459045\ldots$ each year. Model this natural growth by a differential equation. [8]

What is the growth rate k?

$\frac{dr}{dt} = kr$ $t=0$ $r=r_0$

$C = r_0$

$r = Ce^{kt}$ $r = r_0 e^{kt}$ $k = 1.$

(b) MIT is a limited environment, with a maximal sustainable Norway rat population of $R = 1000$ rats. Write down the logistic equation modeling this. (You may use "k" for the natural growth rate here if you failed to find it in **(a)**.) [4]

$\dot{r} = r\left(1 - \frac{r}{1000}\right)k$

(c) The MIT pest control service intends to control these rats by killing them at a constant rate of a rats per year. If it wants to limit the rat population to 75% of the maximal sustainable population, what rate a it should aim for (in rats per year)? [8]

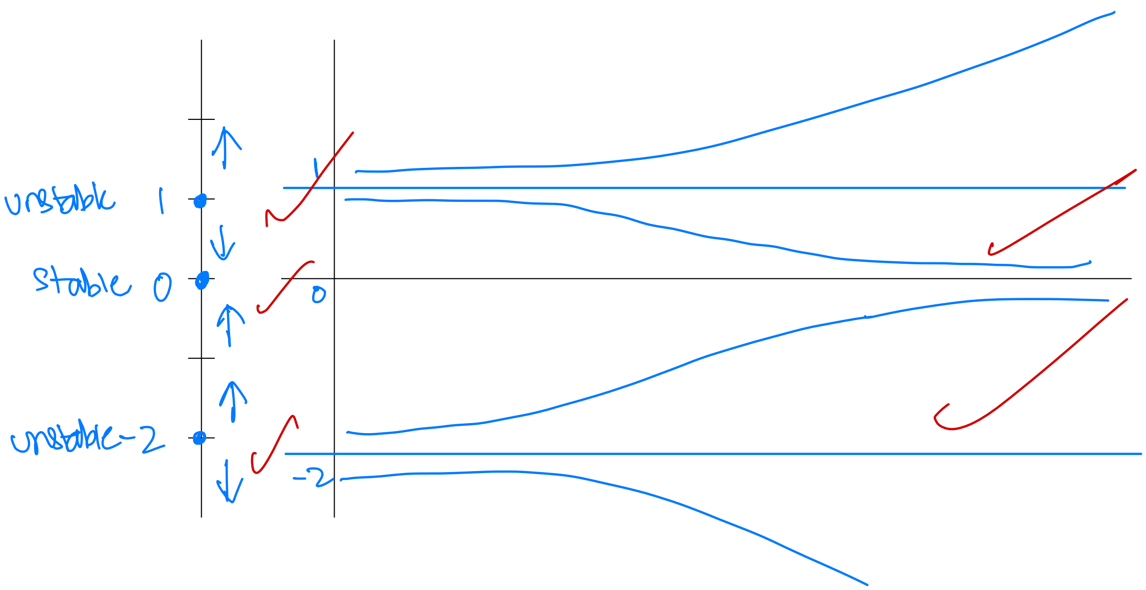
$\dot{r} = r\left(1 - \frac{r}{1000}\right) - a$

-6

2. For the autonomous equation $\dot{x} = x(x-1)(x+2)$, please sketch:

(a) the phase line, identifying the critical points and whether they are stable, unstable, or neither. [4]

(b) at least one solution of each basic type (so that every solution is a time-translate of one you have drawn) [4]



Below is a diagram of a direction field of the differential equation $y' = (1/4)(x - y^2)$. On it please plot and label:

(c) the nullcline [3]

$0 = \frac{1}{4}(x - y^2)$

(d) at least two quite different solutions [3]

$y^2 = \frac{1}{4}x$

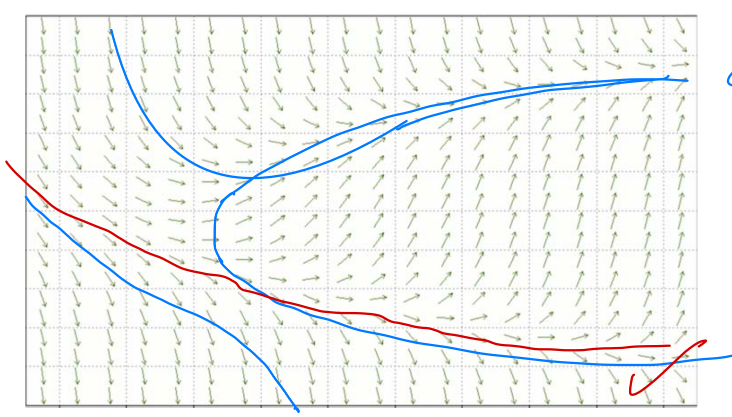
(e) the separatrix (if there is one) [3]

$x = 4y^2$

(f) True or false: If $y(x)$ is a solution with a minimum, then for all large enough x, $y(x) < \sqrt{x}$. (No explanation needed: just circle one.) [3]

$y = \frac{1}{4}\sqrt{x}$

-6



$\frac{1}{4}(x - y^2)$ has min

$y = \pm \frac{1}{2}\sqrt{x}$ at min

$-\frac{1}{2}$

3. (a) Use Euler's method with stepsize $h = 1/2$ to estimate the value at $x = 3/2$ of the solution to $y' = x + y$ such $y(0) = 1$. [10]

x	y	y'	$y'\Delta x$
0	1	1	0.5
0.5	1.5	2	1
1	2.5	3.5	1.75
1.5	4.25		

$y = 4.25$

(b) Find the solution of $t\dot{x} + x = \cos t$ such that $x(\pi) = 1$. [10]

$\dot{xy} = \dot{x}y + \dot{y}x$

$\int (t\dot{x} + x)\, dt = xt$

$xt = \sin t + C$

$x = \frac{\sin t}{t} + \frac{C}{t}$

$x(\pi) = \frac{C}{\pi} = 1$

$C = \pi$

$x = \frac{\sin t}{t} + \frac{\pi}{t}$

4. (a) Find real a, b such that $\frac{1}{3+2i} = a + bi$. [3]

$$\frac{1}{(3+2i)}\frac{(3-2i)}{(3-2i)} = \frac{3-2i}{9+4} = \frac{3}{13} - \frac{2}{13}i$$

$$a = \frac{3}{13},\ b = -\frac{2}{13}$$

(b) Find real r, θ such that $1 - i = re^{i\theta}$. [3]

$$1-i = \sqrt{2}\,e^{-i\frac{\pi}{4}}$$

$$r = \sqrt{2},\ \theta = -\frac{\pi}{4}$$

(c) Find real a, b such that $(1-i)^8 = a + bi$. [3]

$$(1-i)^8 = \left(\sqrt{2}\,e^{-i\frac{\pi}{4}}\right)^8$$

$$= 16e^{-i2\pi}$$

$$= 16(1) = 16$$

$$a = 16,\ b = 0$$

(d) Find real a, b such that $b > 0$ and $a + bi$ is a cube root of -1. [3]

$$b = 0,\ a = -1$$

-3

(e) Find real a, b such that $e^{\ln 2 + i\pi} = a + bi$. [3]

$$e^{\ln 2}e^{i\pi}$$

$$2e^{i\pi} = -2$$

$$a = -2,\ b = 0$$

(f) Write $f(t) = 2\cos(4t) - 2\sin(4t)$ in the form $A\cos(\omega t - \phi)$. [5]

$$= [2, 2i]\begin{bmatrix}\cos \\ i\sin\end{bmatrix}$$

$$= 2\sqrt{2}\,e^{i\frac{\pi}{4}} \cdot e^{i4t}$$

$$= 2\sqrt{2}\,e^{i(\frac{\pi}{4}+4t)} = 2\sqrt{2}\left(\cos\left(\frac{\pi}{4}+4t\right) + i\sin\left(\frac{\pi}{4}+4t\right)\right)$$

$$A = 2\sqrt{2} \quad \omega = 4,\ \phi = -\frac{\pi}{4}$$

5. (a) Find a particular solution to the equation $\dot{x} + 3x = e^{2t}$. [5]

$\dot{x} + 3x = e^{2t}$

$x = Ae^{2t}$

$\dot{x} = 2Ae^{2t}$

$2Ae^{2t} + 3Ae^{2t} = e^{2t}$

$5A = 1$

$A = \frac{1}{5}$

$x = \frac{1}{5}e^{2t}$

f

(b) Find the solution to the same equation such that $x(0) = 1$. [5]

$\dot{x} + 3x = 0$

$\frac{dx}{dt} = -3x$

$\frac{dx}{x} = -3t$

$\ln x = -3t + c$

$x = Ce^{-3t}$

-3

$1 = C$

$x_h = e^{-3t}$

$x = x_p + x_h = \frac{1}{5}e^{2t} + \cancel{e}\,e^{-3t}$

c

(c) Write down a linear equation with exponential right hand side of which $\dot{x} + 3x = \cos(2t)$ is the real part. [5]

$\dot{x} + 3x = e^{i2t}$

(d) Find a particular solution to the equation $\dot{x} + 3x = \cos(2t)$. [5]

$x = Ae^{i2t}$

$\dot{x} = i2Ae^{i2t}$

$$i2Ae^{i2t} + 3Ae^{i2t} = e^{i2t}$$

$$A(3+i2) = 1$$

$$A = \frac{1}{3+i2}$$

$$= \frac{3}{13} - \frac{2}{13}i$$

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$$\mathrm{Re}\left(\left(\frac{3}{13} - \frac{2}{13}i\right)(\cos 2t + i\sin 2t)\right)$$

$$= \frac{3}{13}\cos 2t + \frac{2}{13}\sin 2t$$