18.06SC Final Exam

83

1 (4+7=11 pts.) Suppose A is 3 by 4, and $Ax = 0$ has exactly 2 special solutions:

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad x_2 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

(a) Remembering that A is 3 by 4, find its row reduced echelon form R.

(b) Find the dimensions of all four fundamental subspaces $C(A)$, $N(A)$, $C(A^{\mathrm{T}})$, $N(A^{\mathrm{T}})$.

You have enough information to find bases for one or more of these subspaces—find those bases.

a) $$\mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \checkmark$$

b) $C(A)$ is 2 dimensional.
$N(A)$ is 2 dimensional.

$C(A^T)$ is 2 dimensional.
$N(A^T)$ is 1 dimensional.

$C(A) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ ✓

$N(A) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ ✓

$C(A^T) = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ ✓

$N(A^T) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ✓

-2

2 (6+3+2=11 pts.) (a) Find the inverse of a 3 by 3 upper triangular matrix U, with **nonzero** entries a, b, c, d, e, f. You could use cofactors and the formula for the inverse. Or possibly Gauss-Jordan elimination.

Find the inverse of $U = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$.

(b) Suppose the columns of U are eigenvectors of a matrix A. Show that A is also upper triangular.

(c) Explain why this U **cannot** be the same matrix as the first factor in the Singular Value Decomposition $A = U\Sigma V^{\mathrm{T}}$.

a) $U^{-1} = \dfrac{C^T}{\det U} = \dfrac{1}{adf}\begin{bmatrix} df & -bf & be-cd \\ 0 & af & -ae \\ 0 & 0 & ad \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{a} & -\frac{b}{ad} & \frac{be}{adf} - \frac{c}{af} \\ 0 & \frac{1}{d} & -\frac{e}{df} \\ 0 & 0 & \frac{1}{f} \end{bmatrix}$

b) $A = U \Lambda U^{-1}$

Since U is upper triangular, and U^{-1} is also upper triangular,

ΛU^{-1} is upper triangular.

$M = \Lambda U^{-1}$, M is upper triangular.

$A = UM \Rightarrow A$ is upper triangular.

c) $U^T U \neq I. \Rightarrow U$ is not orthogonal.

$\Rightarrow U$ cannot be an orthogonal basis for A.

3 (3+3+5=11 pts.) (a) A and B are any matrices with the same number of rows. What can you say (*and explain why it is true*) about the comparison of

rank of A rank of the block matrix $\begin{bmatrix} A & B \end{bmatrix}$

(b) Suppose $B = A^2$. How do those ranks compare? Explain your reasoning.

(c) If A is m by n of rank r, what are the dimensions of these nullspaces?

Nullspace of A Nullspace of $\begin{bmatrix} A & A \end{bmatrix}$

a) $\text{rank}([A\ B]) \geq \text{rank } A.$

$[A\ B]$ must have the same pivots as A. A comes after B, hence A will be decomposed first during LU. Thus $[A\ B]$ must have the same pivots as A. Suppose rank $A \neq m$, then if B is independent from A a column from B will become a pivot, thus $[A\ B] = m$. Hence rank $[A, B] \geq$ rank A.

b) $B = A^2$

A has same rank as A^T.

Hence $B = a_1 a_1^T + a_2 a_2^T + \dots a_n a_n^T$ must have same rank as A, thus rank B = rank A.

−3

c) $N(A): n-r$

$N([A\ A]): n-r+n = 2n-r$

4 (3+4+5=12 pts.) Suppose A is a 5 by 3 matrix and Ax is never zero (except when x is the zero vector).

(a) What can you say about the columns of A?

(b) Show that $A^{\mathrm{T}}Ax$ is also never zero (except when $x = 0$) by explaining this key step:

If $A^{\mathrm{T}}Ax = 0$ then obviously $x^{\mathrm{T}}A^{\mathrm{T}}Ax = 0$ and then (WHY?) $Ax = 0$.

(c) We now know that $A^{\mathrm{T}}A$ is invertible. Explain why $B = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ is a one-sided inverse of A (which side of A?). B is NOT a 2-sided inverse of A (*explain why not*).

a) Columns of A are linearly independent.
Columns of A form a basis of A.

b) $A^TAx = 0$

$x^TA^TAx = 0$

$(Ax)^TAx = 0$

$\|Ax\| = 0$

$\Rightarrow Ax = 0.$

c) $B = (A^TA)^{-1}A^T$

$BA = (A^TA)^{-1}A^TA$

$= I.$

Left inverse of A.

$AB = A(A^TA)^{-1}A^T$

A is rectangular with $m>n$.

Let $M = A^TA$

$AB = AMA^T$

$\neq I.$

−2

5 (5+5=10 pts.) If A is 3 by 3 symmetric positive definite, then $Aq_i = \lambda_i q_i$ with positive eigenvalues and orthonormal eigenvectors q_i.

Suppose $x = c_1 q_1 + c_2 q_2 + c_3 q_3$.

(a) Compute $x^{\mathrm{T}} x$ and also $x^{\mathrm{T}} A x$ in terms of the c's and λ's.

(b) Looking at the ratio of $x^{\mathrm{T}} A x$ in part (a) divided by $x^{\mathrm{T}} x$ in part (a), what c's would make that ratio as large as possible? You can assume $\lambda_1 < \lambda_2 < \ldots < \lambda_n$. Conclusion: the ratio $x^{\mathrm{T}} A x / x^{\mathrm{T}} x$ is a maximum when x is ______.

a) $x^T x = c_1^2 + c_2^2 + c_3^2$

$A = Q \Lambda Q^T \qquad x = Qc$

$x^T A x = x^T Q \Lambda Q^T x$

$= c^T Q^T Q \Lambda Q^T Q c$

$= c^T \Lambda c$

$= \lambda_1 c_1^2 + \lambda_2 c_2^2 + \lambda_3 c_3^2$

b) $c_1 = c_2 = \ldots = c_{n-1} = 0$

$c_n = 1$

$x^T A x = \lambda_n$

$x = q_n$

$x = c\, q_n$

-2

6 (4+4+4=12 pts.) (a) Find a linear combination w of the linearly independent vectors v and u that is perpendicular to u.

(b) For the 2-column matrix $A = \begin{bmatrix} u & v \end{bmatrix}$, find Q (orthonormal columns) and R (2 by 2 upper triangular) so that $A = QR$.

(c) In terms of Q only, using $A = QR$, find the projection matrix P onto the plane spanned by u and v.

a) $\omega = v - \frac{u^T v}{u^T u} \cdot \frac{u}{u^T u}$ -2

b) $q_1 = \frac{u}{\|u\|}$ $\quad \omega_2 = v - q_1^T v \cdot q_1$

$q_2 = \frac{\omega_2}{\|\omega_2\|}$

$R = \begin{bmatrix} \|u\| & q_1^T v \\ 0 & \|\omega_2\| \end{bmatrix}$

c) $P = Q(Q^TQ)^{-1}Q^T$

$= QQ^T$

7 (4+3+4=11 pts.) (a) Find the eigenvalues of

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad C^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

(b) Those are both permutation matrices. What are their inverses C^{-1} and $(C^2)^{-1}$?

(c) Find the determinants of C and $C+I$ and $C+2I$.

a) $\begin{bmatrix} -\lambda & & & 1 \\ 1 & -\lambda & & \\ & 1 & -\lambda & \\ & & 1 & -\lambda \end{bmatrix} \rightarrow \lambda_1 = 1, \lambda_2 = -1, \lambda_3 = i, \lambda_4 = -i$

$= \lambda^4 - \begin{vmatrix} 1 & -\lambda & 0 \\ & 1 & -\lambda \\ & & 1 \end{vmatrix}$

$= \lambda^4 - 1$

$= (\lambda^2+1)(\lambda^2-1)$

$= (\lambda^2+1)(\lambda+1)(\lambda-1)$

$C^2 = X\Lambda X^{-1} X \Lambda X^{-1} \Rightarrow \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = -1, \lambda_4 = -1$

$= X\Lambda^2 X^{-1}$

b) $C^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad (C^2)^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

$= C^T \qquad = (C^2)^T$

-1

c) $\det C = -1$

$(C+I)x = (\lambda+1)x$

$\det(C+I) = 0$

$\det(C+2I) = 3 \times 1 \times (i+2) \times (-i+2)$

$= 4(2+i)(2-i)$

$= \overset{3}{4}(4+1)$

$= 20$

15

-1

8 (4+3+4=11 pts.) Suppose a rectangular matrix A has independent columns.

(a) How do you find the best least squares solution $\widehat{x}$ to $Ax = b$? By taking those steps, give me a formula (letters not numbers) for $\widehat{x}$ and also for $p = A\widehat{x}$.

(b) The projection p is in which fundamental subspace associated with A? The error vector $e = b - p$ is in which fundamental subspace?

(c) Find by any method the projection matrix P onto the column space of A:

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 0 \\ 0 & -1 \\ 0 & -3 \end{bmatrix}.$$

$A^T = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & -1 & -3 \end{bmatrix}$

a) $Ax = b$

let solution be $\hat{x}$

$A\hat{x} = p$

$e = b - p$

$e = Ax - A\hat{x}$

$A^T e = 0$

$A^T A x = A^T A \hat{x}$

$\hat{x} = (A^T A)^{-1} A^T b$

$A\hat{x} = p$

$p = A(A^T A)^{-1} A^T b$

b) p is in column space of A.

$e \perp A \Rightarrow e$ in $N(A^T)$.

c) $A^T A = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$

$= 10I$.

$(A^T A)^{-1} = \frac{1}{10} I$.

$P = A \cdot \frac{1}{10} I = \frac{1}{10} A \cdot A^T$

$= \frac{1}{10} \begin{bmatrix} 1 & 3 & 0 & 0 \\ 3 & 9 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 9 \end{bmatrix}$

9 (3+4+4=11 pts.) This question is about the matrices with 3's on the main diagonal, 2's on the diagonal above, 1's on the diagonal below.

$$A_1 = \begin{bmatrix} 3 \end{bmatrix} \quad A_2 = \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix} \quad A_3 = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 3 & 2 \\ 0 & 1 & 3 \end{bmatrix} \quad A_n = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 0 & 1 & 3 & \cdot \\ 0 & 0 & \cdot & \cdot \end{bmatrix}$$

(a) What are the determinants of A_2 and A_3?

(b) The determinant of A_n is D_n. Use cofactors of row 1 and column 1 to find the numbers a and b in the recursive formula for D_n:

$$(*) \qquad D_n = a\,D_{n-1} + b\,D_{n-2}\,.$$

(c) This equation $(*)$ is the same as

$$\begin{bmatrix} D_n \\ D_{n-1} \end{bmatrix} = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \begin{bmatrix} D_{n-1} \\ D_{n-2} \end{bmatrix}.$$

$X \Lambda X^{-1}$ D_{n-1} D_{n-2}

$x c_1 \lambda_1 + x c_2 \lambda_2$

From the eigenvalues of that matrix, how fast do the determinants D_n grow? (If you didn't find a and b, say how you would answer part (c) for any a and b) For 1 point, find D_5.

a) $\det A_2 = 9 - 2 = 7$ $\quad \det A_3 = 21 - 2\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix}$

$= 21 - 6$

$= 15$

b) $D_n = \begin{bmatrix} D_{n-1} & D_{n-2} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$

$a = 3, \ b = 2$ $\quad$ -2

c) $\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$ $\quad \det \begin{vmatrix} 3-\lambda & 2 \\ 1 & -\lambda \end{vmatrix} = -\lambda(3-\lambda) - 2 = 0$

$\lambda^2 - 3\lambda - 2 = 0$

$\begin{matrix} D_n \\ D_{n-1} \end{matrix} = X \Lambda X^{-1} \begin{bmatrix} D_{n-1} \\ D_{n-2} \end{bmatrix}$ $\quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $\quad \lambda_1 = \frac{3 + \sqrt{17}}{2}$ $\quad \lambda_2 = \frac{3 - \sqrt{17}}{2}$

18.06SC Linear Algebra
Fall 2011

$$= X\Lambda^2 X^{-1}\begin{bmatrix} D_{n-2} \\ D_{n-3} \end{bmatrix}$$

$$\vdots$$

$$\begin{bmatrix} D_n \\ D_{n-1} \end{bmatrix} = X\Lambda^{n-2}X^{-1}\begin{bmatrix} D_2 \\ D_1 \end{bmatrix}$$

$n-(n-2)$

$c_1 x_1 \lambda^{n-2}$

$n \to \infty, \quad c_1 x_1 \lambda^{\infty} \to \infty.$

λ^{n-2}

$= [$

2

grows exponentially

$$\begin{bmatrix} D_5 \\ D_4 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} D_4 \\ D_3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} D_3 \\ D_2 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 6 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 15 \\ 7 \end{bmatrix}$$

$$D_5 = \underline{\underline{207}}$$