18.06SC Unit 2 Exam

1 (24 pts.) Suppose q_1, q_2, q_3 are orthonormal vectors in $\mathbb{R}^3$. Find **all possible values** for these 3 by 3 determinants and explain your thinking in 1 sentence each.

(a) $\det \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} =$

(b) $\det \begin{bmatrix} q_1 + q_2 & q_2 + q_3 & q_3 + q_1 \end{bmatrix} =$

(c) $\det \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}$ times $\det \begin{bmatrix} q_2 & q_3 & q_1 \end{bmatrix} =$

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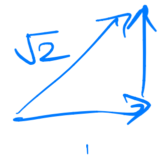

a) $|q_1\ q_2\ q_3| = \pm 1$ volume of unit cube

b) $\|q_1 + q_2\| = \sqrt{2}$

$(\sqrt{2})^3 = 2\sqrt{2}$ volume of cube with length $\sqrt{2}$.

c) 1.

$|q_1 \quad q_2 + q_3 \quad q_3 + q_1|$

$+ |q_2 \quad q_2 + q_3 \quad q_3 + q_1|$

$= q_1 \ q_2$

−10

2 (24 pts.) Suppose we take measurements at the 21 equally spaced times $t = -10, -9, \dots, 9, 10$. All measurements are $b_i = 0$ except that $b_{11} = 1$ at the middle time $t = 0$.

(a) Using least squares, what are the best $\widehat{C}$ and $\widehat{D}$ to fit those 21 points by a straight line $C + Dt$?

(b) You are projecting the vector b onto what subspace? (*Give a basis.*) Find a nonzero vector perpendicular to that subspace.

2a)

$p \in C(A)$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix}, \begin{bmatrix} -10 \\ -9 \\ \vdots \\ 0 \\ 1 \\ 2 \\ \vdots \\ 10 \end{bmatrix}$$

A		x	b	
1	-10		0	b_1
1	-9		0	b_2
1	-8	$\begin{bmatrix} C \\ D \end{bmatrix} =$	0	
⋮	⋮		⋮	
1	0		1	b_{11}
⋮			⋮	
1	10		0	

$2(10^2 + 9^2 + 8^2 + \dots + 1^2)$

$=$

$C(A) \perp N(A^T)$

$$\begin{bmatrix} 1 & 1 & \\ -10 & -9 & \end{bmatrix} x = 0$$

$$x = \begin{bmatrix} 1 \\ \vdots \\ -20 \\ 1 \\ \vdots \end{bmatrix} \begin{matrix} \}10 \\ \\ \}10 \end{matrix}$$

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$e = b - p$

$Pb = p$

$A\hat{x} = p$

$A^Te = 0$

$A^T(b-p)$

$A^Tb = A^TA\hat{x}$

$A^Tb = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$A^TA = \begin{matrix} 10 & 0 \\ 0 & k \end{matrix}$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix} C + \begin{bmatrix} 0 \\ k \end{bmatrix} D$$

$\hat{C} = 1/10$, $\hat{D} = 0$ #

1

-4

3 (9 + 12 + 9 pts.) The Gram-Schmidt method produces orthonormal vectors q_1, q_2, q_3 from independent vectors a_1, a_2, a_3 in $\mathbb{R}^5$. Put those vectors into the columns of 5 by 3 matrices Q and A.

(a) Give formulas using Q and A for the projection matrices P_Q and P_A onto the column spaces of Q and A.

(b) ***Is $P_Q = P_A$ and why? What is P_Q times Q? What is $\det P_Q$?***

(c) Suppose a_4 is a new vector and a_1, a_2, a_3, a_4 are independent. Which of these (if any) is the new Gram-Schmidt vector q_4? (P_A and P_Q from above)

$$\mathbf{1.}\ \frac{P_Q a_4}{\|P_Q a_4\|} \qquad \mathbf{2.}\ \frac{a_4 - \frac{a_4^T a_1}{a_1^T a_1}a_1 - \frac{a_4^T a_2}{a_2^T a_2}a_2 - \frac{a_4^T a_3}{a_3^T a_3}a_3}{\|\text{ norm of that vector }\|} \qquad \mathbf{3.}\ \frac{a_4 - P_A a_4}{\|a_4 - P_A a_4\|}$$

a) $P_Q = Q(Q^TQ)^{-1}Q^T$ ✓ $\quad P_A = A(A^TA)^{-1}A^T$ ✓

$= Q I Q^T$

$= QQ^T$

b) Yes. $C(Q) = C(A) \Leftrightarrow P_Q = P_A$ ✓

$P_Q Q = Q$. Since Q is already in $C(Q)$, the projection of a matrix onto itself will just return itself.

$\det P_Q = 1$

c) 2 ✗

-9

4 (22 pts.) Suppose a 4 by 4 matrix has the same entry $\times$ throughout its first row and column. The other 9 numbers could be anything like $1, 5, 7, 2, 3, 99, \pi, e, 4$.

$$A = \begin{bmatrix} \times & \times \quad \times \quad \times \\ \times & \text{any numbers} \\ \times & \text{any numbers} \\ \times & \text{any numbers} \end{bmatrix}$$

$$\begin{bmatrix} 0 & + & x & x \\ x & & & \\ x & & & \\ x & & & \end{bmatrix}$$

(a) The determinant of A is a polynomial in $\times$. What is the largest possible degree of that polynomial? **Explain your answer**.

(b) If those 9 numbers give the identity matrix I, what is $\det A$? Which values of $\times$ give $\det A = 0$?

$$A = \begin{bmatrix} \times & \times & \times & \times \\ \times & 1 & 0 & 0 \\ \times & 0 & 1 & 0 \\ \times & 0 & 0 & 1 \end{bmatrix}$$

a) x^4. $\prod \lambda = \det A$ If $\lambda_1 = \cdot$ $\lambda_3 = \lambda_4 = x^4$.

b) $\det A = x - x \begin{vmatrix} x & 0 & 0 \\ x & 1 & 0 \\ x & 0 & 1 \end{vmatrix} + x \begin{vmatrix} x & 1 & 0 \\ x & 0 & 0 \\ x & 0 & 1 \end{vmatrix}$

$- x \begin{vmatrix} x & 1 & 0 \\ x & 0 & 1 \\ x & 0 & 0 \end{vmatrix}$ $\quad -6$

$= x - x^2 + x(-x) - x(x)$

$\det A = x - 3x^2$

$0 = x - 3x^2$

$= x(1 - 3x)$

$x = \frac{1}{3}$ or

$x = 0$.