1.2.2 no rational number r such that $2^r = 3$

By contradiction, suppose $\exists r \in \mathbb{Q}$ s.t $2^r = 3$,

Suppose $\exists a, b \in \mathbb{N}$ s.t $r = \frac{a}{b}$

$$2^{\frac{a}{b}} = 3$$

$$(2^a)^{\frac{1}{b}} = 3$$

$$2^a = 3^b$$

$\Rightarrow 3^b$ must be even.

However, $b \in \mathbb{Z}$ hence 3^b cannot be even

$\Rightarrow$ contradiction

□.

1.2.4. **Exercise 1.2.4.** Produce an infinite collection of sets $A_1, A_2, A_3, \ldots$ with the property that every A_i has an infinite number of elements, $A_i \cap A_j = \emptyset$ for all $i \neq j$, and $\bigcup_{i=1}^{\infty} A_i = \mathbf{N}$.

$|A_i| = \infty$.

Exercise 1.2.2. Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

(a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all sets containing an infinite number of elements, then the intersection $\cap_{n=1}^{\infty} A_n$ is infinite as well.

(b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \cdots$ are all finite, nonempty sets of real numbers, then the intersection $\cap_{n=1}^{\infty} A_n$ is finite and nonempty.

(c) $A \cap (B \cup C) = (A \cap B) \cup C$.

(d) $A \cap (B \cap C) = (A \cap B) \cap C$.

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

a) By contraposition

we will prove that if $\cap_{n=1}^{\infty} A_n$ is not finite, then for $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$, there exists a set A_i where A_i is finite, while the rest are infinite.

Since $\cap_{n=1}^{\infty} A_n$ is not finite, there must be ... ~~True~~ False

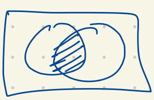
b) True.

1.2.3) **Exercise 1.2.3 (De Morgan's Laws).** Let A and B be subsets of $\mathbf{R}$.

(a) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subseteq A^c \cup B^c$.

(b) Prove the reverse inclusion $(A \cap B)^c \supseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$.

(c) Show $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.



If $x \in (A \cap B)^c$, then $x \notin A \cap B$

$\Rightarrow x \in A^c$ and $x \in B^c$

$\Rightarrow x \in A^c \cup B^c$

$\therefore (A \cap B)^c \subseteq A^c \cup B^c$.

If $x \in A^c \cup B^c$, then $x \in A^c$ and $x \in B^c$

$\Rightarrow x \notin A \cap B$

$\Rightarrow x \in (A \cap B)^c$

$\Rightarrow (A \cap B)^c \supseteq A^c \cup B^c$

Since $(A \cap B)^c \subseteq A^c \cup B^c$ and $(A \cap B)^c \supseteq A^c \cup B^c$,

$(A \cap B)^c = A^c \cup B^c$ //

1.2.5

a) $|a-b| \leq |a-c| + |c-b|$ (By Triangle Inequality)

Let $c=0$

$\Rightarrow |a-b| \leq |a| + |-b|$

$= |a| + |-1 \cdot b|$

$= |a| + |-1| \cdot |b|$

$= |a| + |b|$

$\therefore |a-b| \leq |a| + |b|$ $\square$

1.2.8

Exercise 1.2.8. Form the logical negation of each claim. One way to do this is to simply add "It is not the case that..." in front of each assertion, but for each statement, try to embed the word "not" as deeply into the resulting sentence as possible (or avoid using it altogether).

(a) For all real numbers satisfying $a < b$, there exists an $n \in \mathbf{N}$ such that $a + 1/n < b$.

(b) Between every two distinct real numbers, there is a rational number.

(c) For all natural numbers $n \in \mathbf{N}$, $\sqrt{n}$ is either a natural number or an irrational number.

(d) Given any real number $x \in \mathbf{R}$, there exists $n \in \mathbf{N}$ satisfying $n > x$.

a) $\exists a, b \in \mathbb{R}$ s.t $a < b$, where $\forall n \in \mathbb{N}$, $a + \frac{1}{n} \geq b$.

b) $\exists a, b \in \mathbb{R} : a \neq b$ s.t for all $a \leq x \leq b$, x is irrational.

c) $\exists n \in \mathbb{N}$, $\sqrt{n}$ is not a natural number and $\sqrt{n}$ is not an irrational number.

d) $\exists x \in \mathbb{R}$, s.t $\forall n \in \mathbb{N}$, $n \leq x$.

Exercise 1.2.7. Given a function $f : D \to \mathbf{R}$ and a subset $B \subseteq \mathbf{R}$, let $f^{-1}(B)$ be the set of all points from the domain D that get mapped into B; that is, $f^{-1}(B) = \{x \in D : f(x) \in B\}$. This set is called the *preimage* of B.

(a) Let $f(x) = x^2$. If A is the closed interval $[0, 4]$ and B is the closed interval $[-1, 1]$, find $f^{-1}(A)$ and $f^{-1}(B)$. Does $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ in this case? Does $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$?

(b) The good behavior of preimages demonstrated in (a) is completely general. Show that for an arbitrary function $g : \mathbf{R} \to \mathbf{R}$, it is always true that $g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B)$ and $g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$ for all sets $A, B \subseteq \mathbf{R}$.

a) $f(x) = x^2$. $A = [0, 4]$, $B = [-1, 1]$.

$f^{-1}(A) = [-2, 2]$.

$f^{-1}(B) =$ $[-i, i]$ where $f^{-1}(B)$ is imaginary and $[-1, 1]$

$A \cap B = [0, 1] \Rightarrow f^{-1}(A \cap B) = [0, 1]$

$\Rightarrow f^{-1}(A) \cap f^{-1}(B)$ ✓.

$A \cup B = [-1, 4] \Rightarrow f^{-1}(A \cup B) = [-i, i], [-2, 2]$

$= f^{-1}(A) \cup f^{-1}(B)$

b) $g : \mathbb{R} \to \mathbb{R}$ Let $g^{-1}(A) = M$ $g(M) = A$

$g^{-1}(B) = N$ $g(N) = B$

Let $y \in A \cap B$. $g^{-1}(y) = x$ $g(x) = y$.

Let $x \in g^{-1}(A \cap B) \Leftrightarrow g(x) \in A \cap B$

$\Leftrightarrow g(x) \in A$ and $g(x) \in B$

$\Leftrightarrow x \in g^{-1}(A)$ and $x \in g^{-1}(B)$

$\Leftrightarrow x \in g^{-1}(A) \cap g^{-1}(B)$.

$g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$

Let $x = g^{-1}(A \cup B)$

$\Leftrightarrow g(x) = A \cup B$

$\Leftrightarrow g(x) \in A$ or $g(x) \in B$.

$\Leftrightarrow x \in g^{-1}(A)$ or $x \in g^{-1}(B)$

$\Leftrightarrow x \in g^{-1}(A) \cup g^{-1}(B)$.