Problem 1. **[12 points]** Define a *3-chain* to be a (not necessarily contiguous) subsequence of three integers, which is either monotonically increasing or monotonically decreasing. We will show here that any sequence of five distinct integers will contain a *3-chain*. Write the sequence as a_1, a_2, a_3, a_4, a_5. Note that a monotonically increasing sequences is one in which each term is greater than or equal to the previous term. Similarly, a monotonically decreasing sequence is one in which each term is less than or equal to the previous term. Lastly, a subsequence is a sequence derived from the original sequence by deleting some elements without changing the location of the remaining elements.

(a) [4 pts] Assume that $a_1 < a_2$. Show that if there is no *3-chain* in our sequence, then a_3 must be less than a_1. (Hint: consider a_4!)

(b) [2 pts] Using the previous part, show that if $a_1 < a_2$ and there is no *3-chain* in our sequence, then $a_3 < a_4 < a_2$.

(c) [2 pts] Assuming that $a_1 < a_2$ and $a_3 < a_4 < a_2$, show that any value of a_5 must result in a *3-chain*.

(d) [4 pts] Using the previous parts, prove by contradiction that any sequence of five distinct integers must contain a *3-chain*.

a) Assume $a_1 < a_2$. If there is no 3 chain in the sequence,
$a_3 < a_2$ since if $a_3 > a_2$, a 3 chain is constructed.
$a_1 < a_2 > a_3$.
For a_4, $a_2 > a_3 < a_4$ since no 3-chain.
$a_1 < a_2 > a_4$ since no 3-chain.
$\Rightarrow$ $a_3 < a_4$ $a_2 > a_4$.
$a_3 < a_4 < a_2$
For the subsequence a_1, a_3, a_4,
Since $a_3 < a_4$, for the subsequence to not be a 3-chain,
$a_1 > a_3 < a_4$.
$\Rightarrow$ $a_1 > a_3 \Rightarrow a_3 < a_1$
$\square$.

b) $a_3 < a_4 < a_2$ shown in part (i).
c) Let $a_1 < a_2$ and $a_3 < a_4 < a_2$.
$a_5 > a_4 \Rightarrow$ $a_5 > a_4 > a_3$ is 3-chain.
$a_5 < a_4 \Rightarrow$ $a_5 < a_4 < a_2$ is 3-chain.
Since $a_5 > a_4$ can build a 3 chain, and $a_5 < a_4$ also builds a 3 chain.
All values of a_5 can give a 3-chain.
$\square$.

d) By contradiction,
assume that a sequence of 5 distinct integers do not build a 3 chain,

then in the subsequence a_1, a_2, a_3, a_4, there is no 3-chain.
However, we've shown that if there is no 3-chan for a_1, a_2, a_3, a_4, all values of a_5 can produce a 3-chan in part (c).

Hence a contradiction is derived. Thus, a_1, a_2, a_3, a_4, a_5 must have a 3-chain.

2) **Problem 2.** [8 points]

Prove by either the Well Ordering Principle or induction that for all nonnegative integers, n:

$$\sum_{i=0}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2. \quad (1)$$

By Induction, let $p(n) ::= \sum_{i=0}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$.

Base case: $n=1$. $p(1)$: $\sum_{i=0}^{1} 1 = \left(\frac{1(2)}{2}\right)^2 = 1$ ✓.

Inductive step: Assume $p(n)$ is true, then for $n=k+1$, $p(k+1)$: $\sum_{i=0}^{k+1} i^3 = \sum_{i=0}^{k} i^3 + (k+1)^3$

$$= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$$
$$= (k+1)^2\left(\frac{k^2}{4} + k+1\right)$$
$$= (k+1)^2\left(\frac{k^2+4k+4}{4}\right)$$
$$= (k+1)^2\left(\frac{(k+2)^2}{4}\right)$$
$$= \left(\frac{(k+1)(k+2)}{2}\right)^2$$
$$\Rightarrow p(n+1) \text{ is true}$$

Since $p(n)$ is true for $p(1)$ and $p(k+1)$, $k \in \mathbb{Z}$, $p(n)$ is true for all n. D.

3)

We define the perimeter of the system as the total sum of edges that are either between infected and healthy students or between infected students and the border.

Lemma 1: The perimeter either remains constant or decreases at each time step.

Pf: By induction on time step,

Base case: $t=0$. At $t=0$, the perimeter remains the same. $\Rightarrow$ lemma 1 s

Inductive step: Assuming lemma 1 is true for $1 \leq k \leq n$, let $t=k+1$. At $t=k$, the perimeter is less than or equal to the initial as per the inductive hypothesis. For $t=k+1$,

By cases,

assume a healthy student is adjacent to 2 infected. The 2 infected share total 2 edges with the healthy student. After one time step, the healthy student gets infected. The edges previously are now gone as it is now between infected students. However, the newly infected student now has 2 exposed edges. Hence the amount of edges remain the same.

Assume a healthy student is adjacent to 3 infected. $\Rightarrow$ 3 exposed edges between healthy and infected. After infection, only the newly infected student will have an exposed edge. Edges decreased from 3 to 1.

Assume a healthy student is adjacent to 4 infected. $\Rightarrow$ 4 exposed edges between healthy and infected. After infection, there are no more exposed edges by the newly infected student. Edges decreased from 4 to 0.

Therefore, the perimeter can only either remain the same or decrease as time passes.

$\square$

Lemma 2: n initially infected students can have a maximum perimeter of $4n$

Pf: Let all n infected students be arranged s.t none are adjacent to another infected. Then, each infected student has 4 edges and the total sum of all edges is $4n$. From lemma 1, the perimeter can only decrease or remain the same. Hence $4n$ is the maximum perimeter.

$\square$

Theorem proof: Given an $n \times n$ grid, for the grid to be completely infected, the infected students must have a perimeter of $4n$.

If there are fewer than n initially infected students, the maximum perimeter possible is $< 4n$, hence since the perimeter of the grid is $4n$, the grid can never be fully infected.

Thus the theorem is true.

■

Problem 5. **[10 points]** Let the sequence $G_0, G_1, G_2, \ldots$ be defined recursively as follows: $G_0 = 0$, $G_1 = 1$, and $G_n = 5G_{n-1} - 6G_{n-2}$, for every $n \in \mathbb{N}, n \geq 2$.

Prove that for all $n \in \mathbb{N}$, $G_n = 3^n - 2^n$.

By strong induction,

Base case: $n=1$. $G_1 = 3^1 - 2^1 = 1$ ✓.

Inductive step: Assume proposition is true for $1, 2, 3, \ldots, n$,

For $n+1$,

$$
\begin{aligned}
G_{n+1} &= 5G_n - 6G_{n-1} \\
&= 5(3^n - 2^n) - 6(3^{n-1} - 2^{n-1}) \\
&= 5(3^n - 2^n) - (2 \cdot 3^n - 3 \cdot 2^n) \\
&= 3^n(5-2) - 2^n(5-3) \\
&= 3^n \cdot 3 - 2^n \cdot 2 \\
&= 3^{n+1} - 2^{n+1}
\end{aligned}
$$

$\Rightarrow$ proposition is true for $n+1$.

$\therefore$ proposition is true for $\forall n \in \mathbb{N}, n \geq 2$.

$\square$

6) Row move cannot change the order of the tiles.

A row move changes the position from column i to column $i+1$ while still remaining on the same row. Hence, the order still remains the same.

$\square$