$21 = 6+6+6+3$
$6+6+5+4$
$6+5+5+5$

$22 = 6+6+6+4$
$6+6+5+5$

(a) (probability that the total after rolling 4 fair dice is 21) $\underline{>}$ (probability that the total after rolling 4 fair dice is 22)

(b) (probability that a random 2 letter word is a palindrome[1]) ____ (probability that a random 3 letter word is a palindrome)

$\left.\begin{matrix} aa \\ bb \\ \vdots \\ zz \end{matrix}\right\} 26$

Total combinations: $26 \cdot 26$

$Pr[\text{2 letter word is palindrome}] = \frac{26}{26 \cdot 26} = \frac{1}{26}$

aba or aaa
$\vdots$ $\quad \vdots$
zzz

Sequence where two letters form word like aba

$\Rightarrow \underbrace{\binom{26}{2}}_{\text{choose 2 letters}} \cdot \underbrace{2}_{\text{two ways to arrange letters}} = \frac{26!}{2!(26-2)!} \cdot 2 = \frac{26!}{(26-2)!}$
$= 25 \cdot 26$

Total 3 letter combinations: 26^3

$Pr[\text{3 letter word is palindrome}] = \frac{25 \cdot 26 + 26}{26^3} = \frac{25+1}{26^2} = \frac{26}{26^2} = \frac{1}{26}$

b) Ans: $\frac{1}{26} \Rightarrow Pr[\text{3 letter word palindrome}] = Pr[\text{2 letter word palindrome}]$

2) 2. A random 5 card poker hand is dealt from a standard deck of cards. Find the probability of each of the following (in terms of binomial coefficients).

(a) A flush (all 5 cards being of the same suit; do not count a royal flush, which is a flush with an Ace, King, Queen, Jack, and 10)

(b) Two pair (e.g., two 3's, two 7's, and an Ace)

a) Each card has a suit and a rank.
Represent a card as a tuple: (rank, suit)

$(\text{rank}, \text{suit})_1, (\text{rank}, \text{suit})_2, \ldots, (\text{rank}, \text{suit})_5$

$(Q, \heartsuit), (4, \heartsuit), (5, \heartsuit) \ldots (K, \heartsuit)$

We can represent a hand as
$(\text{rank}_1, \ldots, \text{rank}_5, \text{suit})$

$\Rightarrow$ no. of possible flushes: $\binom{13}{5} \cdot 4$ (including royal flush)
no. of royal flushes: 4
no. of flushes without royal flush: $\binom{13}{5} \cdot 4 - 4$

Total no. of hands: $\binom{52}{5}$

$Pr[\text{flush without royal flush}] = \frac{\binom{13}{5}4 - 4}{\binom{52}{5}}$

b) Represent a hand with two pair as

$$\underbrace{(\text{rank}, \text{suit}, \text{suit})}_{\text{1st pair}}, \underbrace{(\text{rank}, \text{suit}, \text{suit})}_{\text{2nd pair}}, \underbrace{(\text{rank}, \text{suit})}_{\text{5th card}}$$

No. of combinations of 1st pair: $13 \cdot \binom{4}{2}$

No of combinations of 2nd pair: $12 \cdot \binom{4}{2}$

No. of combinations of 5th card: $11 \cdot 4$

same K, Q, A same

A, K, Q

same Q, K, A

K, A, Q

A, Q, K same

Q, A, K

2 to 1 mapping from sequence to 2 pair hands. Hence,

No. of two pairs: $13 \cdot 12 \cdot 11 \cdot \binom{4}{2}\binom{4}{2} \cdot 4 \cdot \frac{1}{2}$

$$= \frac{13!}{(13-3)!}\binom{4}{2}\binom{4}{2} 2$$

$$= \frac{13!}{10!}\binom{4}{2}^2 \cdot 2 \quad //$$

$$\underbrace{K, Q, A}$$

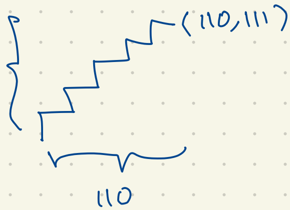
$$\frac{\frac{13!}{10!}\binom{4}{2}^2 2}{\binom{52}{5}}$$

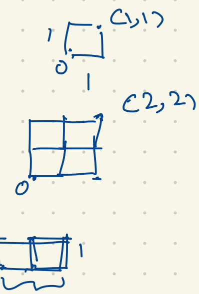
3)

3. (a) How many paths are there from the point $(0,0)$ to the point $(110,111)$ in the plane such that each step either consists of going one unit up or one unit to the right?

(b) How many paths are there from $(0,0)$ to $(210,211)$, where each step consists of going one unit up or one unit to the right, and the path has to go through $(110,111)$?

a) There must be a total of
110 right moves,
111 up moves,
for each path.





We have total of 221 moves.

Let a path be represented as
10110..., where 1 represents up move
0 represents right move.

How many sequences of length 221 with
110 0
111 1

equivalent to: Given 221 objects, how many ways can we pick 110?

Ans: $\binom{221}{110}$ //

$210 - 110 = 100$
$211 - 111 = 100$

b) $\binom{221}{110}$ ways to get to $(110, 111)$

$\binom{200}{100}$ ways to get from $(110, 111) \to (210, 211)$

Ans: $\binom{221}{110} \cdot \binom{200}{100}$

4. A *norepeatword* is a sequence of at least one (and possibly all) of the usual 26 letters a,b,c,…,z, with repetitions not allowed. For example, "course" is a norepeatword, but "statistics" is not. Order matters, e.g., "course" is not the same as "source".

A norepeatword is chosen randomly, with all norepeatwords equally likely. Show that the probability that it uses all 26 letters is very close to $1/e$.

4. Total no. of norepeatwords with 26 letters : $26!$

Total no. of norepeatwords of length $1, \dots, 26$: $1 + 2! + \dots + 26!$

$$\Pr[\text{ all 26 letters nonrepeatwords }] = \frac{26!}{1+2!+\dots+26!}$$

$$\frac{1}{\Pr[\dots]} = \frac{1+\dots+26!}{26!}$$

$$= \frac{1}{26!} + \dots + \frac{1}{2!} + 1$$

$$\approx e$$

$$\Rightarrow \frac{1}{\Pr[\dots]} \approx e \qquad \Pr[\dots] \approx \frac{1}{e} \quad //$$

2. 5. Give a story proof that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n}$$

Let there be n people.
How many ways can form groups of n people?
$\Leftrightarrow$ How many subsets are there of n items?

We let 1 indicate if an item in inside the set
0 if item not in the set.

Let a binary sequence of length n represent whether an item is in the set or not.
e.g 0…1 $\Rightarrow$ all items except 1st item is in the set.

There are 2^n different binary sequences of length n.

$\square$.

6) $$\frac{(2n)!}{2^n \cdot n!} = (2n-1)(2n-3)\cdots 3 \cdot 1.$$ Story proof.

Given 2n people, find how many ways to form n partnerships
Line up in row,
Pair adjacent people e.g $(2n, 2n-1), (2n-2, 2n-3), \dots, (2, 1)$
Order within pair doesn't matter. Order between pairs dont matter.
$\Rightarrow 2^n \cdot n!$ duplicates. $\Rightarrow \frac{(2n)!}{2^n \cdot n!}$

7. Show that for all positive integers n and k with $n \geq k$,

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k},$$

doing this in two ways: (a) algebraically and (b) with a "story", giving an interpretation for why both sides count the same thing.

Hint for the "story" proof: imagine $n+1$ people, with one of them pre-designated as "president".

$$\binom{n}{k} + \binom{n}{k-1}$$

$$\frac{(n+1)!}{k!\,(n+1-k)!}$$

$$\binom{n}{k-1} +$$

a) $$\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!\,(n-k)!} + \frac{n!}{(k-1)!\,(n-k+1)!}$$

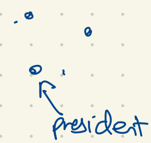
$$= \frac{n!}{k!}\left(\frac{1}{(n-k)!} + \frac{k}{(n-k+1)!}\right)$$

$$= \frac{n!}{k!}\left(\frac{n-k+1}{(n-k+1)!} + \frac{k}{(n-k+1)!}\right)$$

$$= \frac{n!}{k!}\left(\frac{(n+1)}{(n-k+1)!}\right)$$

$$= \frac{(n+1)!}{k!\,(n+1-k)!}$$

$\square$



b) Let there be $n+1$ people. Let 1 person be president.

$\binom{n+1}{k} \Rightarrow$ In $n+1$ people, pick k to form a group.

The group can include or exclude the president.

$\binom{n}{k} \Rightarrow$ no. of groups without president

$\binom{n}{k-1} \Rightarrow$ no. of groups with president.

$\therefore \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

$\square$

Homework

1. 6 children. 3 boys 3 girls. All birth orders equally likely

Total permutations of birth orders: $6!$
Permutations where first 3 births are girls: $3! \cdot 3!$

$$P(\text{3 eldest children are girls}) = \frac{3!\,3!}{6!} = \frac{1}{20} = 0.05$$

2. 2. (a) How many ways are there to split a dozen people into 3 teams, where one team has 2 people, and the other two teams have 5 people each?

(b) How many ways are there to split a dozen people into 3 teams, where each team has 4 people?

a) $\binom{12}{2}\binom{10}{5} \div 2$ (since order of two teams don't matter)

b) $\binom{12}{4}\binom{8}{4} \div 3$ (since order of three teams don't matter)

3. 3. A college has 10 (non-overlapping) time slots for its courses, and blithely assigns courses to time slots randomly and independently. A student randomly chooses 3 of the courses to enroll in (for the PTP, to avoid getting fined). What is the probability that there is a conflict in the student's schedule?

The 10^3 different sequences are equally likely, not the combinations. Order of time slot matters.

Total ways to pick timeslots: $\binom{12}{3}$
Total ways to pick non-conflicting timeslots: $\binom{10}{3}$

$$P(\text{timeslot has conflict}) = \frac{\binom{10}{3}}{\binom{12}{3}} = \frac{6}{11} = 0.545$$

$$P(\text{no overlap}) = \frac{10}{10} \cdot \frac{9}{10} \cdot \frac{8}{10}$$

$$P(\text{has overlap}) = 1 - \frac{10 \cdot 9 \cdot 8}{10^3} = 0.28$$

4. 4. A city with 6 districts has 6 robberies in a particular week. Assume the robberies are located randomly, with all possibilities for which robbery occurred where equally likely. What is the probability that some district had more than 1 robbery?

Ways to pick districts to rob: $\binom{11}{6}$

P(districts has more than 1 robbery): $1 - \frac{1}{\binom{11}{6}} = \cancel{0.998}$ OR

Total permutations of robbery sequences: 6^6
Permutations of sequences that visited each district: $6!$

$$P(\text{district has more than 1 robbery}): 1 - \frac{6!}{6^6} = 0.985$$

5) 5. Elk dwell in a certain forest. There are N elk, of which a simple random sample of size n are captured and tagged ("simple random sample" means that all $\binom{N}{n}$ sets of n elk are equally likely). The captured elk are returned to the population, and then a new sample is drawn, this time with size m. This is an important method that is widely-used in ecology, known as *capture-recapture*.

What is the probability that exactly k of the m elk in the new sample were previously tagged? (Assume that an elk that was captured before doesn't become more or less likely to be captured again.)

No. of ways to tag n out of N elks: $\binom{N}{n}$
No. of ways to recapture m elks: $\binom{N}{m}$
Combinations of m elks with k tagged elks: $\binom{n}{k} \cdot \binom{N-n}{m-k}$

$$P(\text{m recaptured elks have k tags}) = \frac{\binom{n}{k} \cdot \binom{N-n}{m-k}}{\binom{N}{m}}$$

6)

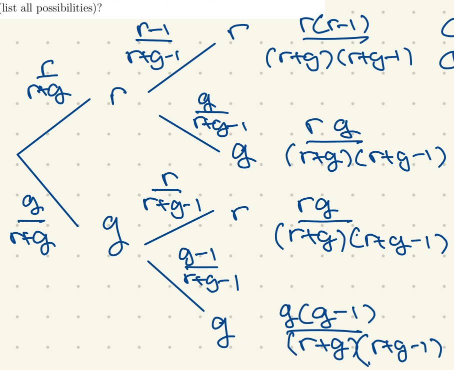
6. A jar contains r red balls and g green balls, where r and g are fixed positive integers. A ball is drawn from the jar randomly (with all possibilities equally likely), and then a second ball is drawn randomly.

(a) Explain intuitively why the probability of the second ball being green is the same as the probability of the first ball being green.

(b) Define notation for the sample space of the problem, and use this to compute the probabilities from (a) and show that they are the same.

(c) Suppose that there are 16 balls in total, and that the probability that the two balls are the same color is the same as the probability that they are different colors. What are r and g (list all possibilities)?

a) b)



Choices for first green ball: gg, gr

Choices for second green ball: gg, rg

c) $P(rr) + P(gg) = P(rg) + P(gr)$

$$P(rr) + P(gg) - P(rg) - P(gr) = \frac{1}{(r+g)(r+g-1)}\left[r(r-1) + g(g-1) - rg - rg\right] = 0$$

$$r^2 - r + g^2 - g - 2rg = 0$$

$-b \pm$

$$r^2 - r - 2rg + g^2 - g = 0$$

$$r^2 - r(1+2g) + g^2 - g = 0$$

$$r = \frac{1+2g \pm \sqrt{(1+2g)^2 - 4(g^2-g)}}{2} = \frac{1+2g \pm \sqrt{1+4g+4g^2-4g^2+4g}}{2}$$

$$r = \frac{1+2g \pm \sqrt{1+8g}}{2}$$

$r + g = 16$

$r = 16 - g$

$$2(16-g) = 1 + 2g \pm \sqrt{1+8g}$$

$$32 - 2g = 1 + 2g \pm \sqrt{1+8g}$$

$$31 - 4g = \sqrt{1+8g}$$

$$(31-4g)^2 = 1+8g$$

$$961 - 248g + 16g^2 - 1 - 8g = 0$$

$$960 - 256g + 16g^2 = 0$$

$$60 - 16g + g^2 = 0$$

$$(g-10)(g-6) = 0$$

$$g = 10 \text{ or } g = 6$$

//

6)

7. (a) Show using a story proof that

$$\binom{k}{k}+\binom{k+1}{k}+\binom{k+2}{k}+\cdots+\binom{n}{k}=\binom{n+1}{k+1},$$

where n and k are positive integers with $n \geq k$.
Hint: imagine arranging a group of people by age, and then think about the oldest person in a chosen subgroup.

(b) Suppose that a large pack of Haribo gummi bears can have anywhere between 30 and 50 gummi bears. There are 5 delicious flavors: pineapple (clear), raspberry (red), orange (orange), strawberry (green, mysteriously), and lemon (yellow). There are 0 non-delicious flavors. How many possibilities are there for the composition of such a pack of gummi bears? You can leave your answer in terms of a couple binomial coefficients, but not a sum of lots of binomial coefficients.

a) Let there be a group of n+1 people. We want to pick k+1 people out of n+1 people. In each group of k+1 people, there must be an oldest person.

Suppose we arrange the n+1 people by age from youngest to oldest.
For the people from idx $1, \dots, k$, they cannot be the oldest in a group of k+1 people.
For the person at idx k+1, there are $\binom{k}{k}$ different choices of people younger to form a group of size k+1.

For pearson at idx $i+1$, where $i+1 > k$, there are $\binom{i}{k}$ ways to choose k younger people to form a size k+1 group.

Thus all possible groups are $\binom{k}{k}+\dots+\binom{i+1}{k}+\dots+\binom{n}{k}$

b) We got 5 groups of Haribo gummy flavors.
We must pick n gummies from 5 groups of flavors.

No. of combinations are $\binom{5+n-1}{n}=\binom{5+n-1}{5-1}$

$\binom{30+5-1}{4}+\binom{31+5-1}{4}+\dots+\binom{50+5-1}{4}$

$=\binom{34}{4}+\binom{35}{4}+\dots+\binom{54}{4}$

$\binom{4}{4}+\dots+\binom{33}{4}=\binom{34}{5}$

$\binom{4}{4}+\dots+\binom{54}{4}=\binom{55}{5}$

$\binom{34}{4}+\dots+\binom{54}{4}=\binom{55}{5}-\binom{34}{5}$ //