Homework 9.

Date No.

1) $V = \mathbb{R}^2$. $T \in \mathcal{L}(V)$.

$Tv = 2v$ and $Tw = -w$.

Find $\det(T^4 + T)$.

v, w are eigenvectors of T with eigenvalues 2 and -1.

Let $\Lambda^2 V = \{a(v \wedge w) \mid a \in \mathbb{F}\}$.

$\det(T^4+T) \Leftrightarrow \quad (T^4+T)_*(v \wedge w) = \det(T^4+T)(v \wedge w)$

$(T^4+T)_*(v \wedge w)$

$= (T^4+T)v \wedge (T^4+T)w$

$= T^4v + Tv \wedge T^4w + Tw$

$= (2^4v + 2v) \wedge (-1)^4 w - w$

$= 18v \wedge 0 = 0$ //

2) characteristic polynomial: $(z-2)(z-3)^3$.

3) $T \in \mathcal{L}(V)$ with $\dim V = n$.

$\chi_T(x)$ is the characteristic polynomial of T.

I is true. II is false.

Pf for I: Suppose $\chi_T(x)$ has n distinct roots.

Let $\lambda_1, \dots, \lambda_n$ be the roots of $\chi_T(x)$.

By the Casey-Hamilton Theorem,

$\chi_T(T) = (T-\lambda_1 I)(T-\lambda_2 I)\dots(T-\lambda_n I) = 0$

For some eigenvalue a with eigenvector v_k,

$\chi_T(T)v_k = (T-\lambda_1 I)(T-\lambda_2 I)\dots(T-\lambda_n I)v_k$

$= (Tv_k - \lambda_1 v_k)\dots(Tv_k - \lambda_n v_k)$

$= (av_k - \lambda_1 v_k)\dots(av_k - \lambda_n v_k) = 0$

Hence $\exists \lambda_k$ s.t. $a = \lambda_k$.

Thus the roots of $\chi_T(x)$ are the eigenvalues of T.

Since there are n distinct roots of $\chi_T(x)$,

there are n distinct eigenvalues

Since $\dim V = n$, the eigenvalues form a basis of V.

Thus $M(T)$ w.r.t eigenvalues of T is diagonal

and ~~M~~ hence T is diagonalizable

$\square$.

Besform

Counterexample for II

Let $V = \mathbb{R}^2$. Let $M(T) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. $\Rightarrow$ $T = I$.

I is diagonalizable.

Characteristic polynomial of I : $(x-1)^2$.

$\Rightarrow$ $\chi_T(z)$ does not have 2 distinct roots

$\square$

4. Suppose T is an isometry, hence

$\|Tv\| = \|v\|$

$\langle Tv, Tv \rangle = \langle v, v \rangle$

From 7.43, there exists an orthonormal basis of V of eigenvectors of T with corresponding eigenvalues with ~~value~~ absolute value 1

Let $v_1, \ldots, v_n$ be eigenvector basis of V with eigenvalues $\lambda_1, \ldots, \lambda_n$

$v_1 \wedge \ldots \wedge v_n \in \Lambda^n V$

$$\begin{aligned} T_*(v_1 \wedge \ldots \wedge v_n) &= Tv_1 \wedge \ldots \wedge Tv_n \\ &= \lambda_1 v_1 \wedge \ldots \wedge \lambda_n v_n \\ &= \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n \, v_1 \wedge \ldots \wedge v_n \end{aligned}$$

$\Rightarrow$ $\det T = \lambda_1 \lambda_2 \lambda_3 \ldots \lambda_n$.

$$\begin{aligned} |\det T| &= \sqrt{(\lambda_1 \ldots \lambda_n \cdot \bar{\lambda}_1 \ldots \bar{\lambda}_n)} \\ &= \sqrt{(\lambda_1 \bar{\lambda}_1 \lambda_2 \bar{\lambda}_2 \ldots \lambda_n \bar{\lambda}_n)} \\ &= \sqrt{(1 \cdot 1 \cdot \ldots \cdot 1)} \\ &= 1. \end{aligned}$$

$\therefore$ $|\det T| = 1$.

$\square$

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5) $\det T^* = \overline{\det T}$ where $T \in \mathcal{L}(V)$ and V is finite dimensional.

Pf: By Schur's Theorem, there exist an orthonormal basis of V such that T has an upper-triangular matrix w.r.t this basis. Let the orthonormal basis be $e_1, \dots, e_n$ for $\dim V = n$.

Hence $Te_j \in \operatorname{span}(e_1, \dots, e_j)$

Let $\lambda_1, \dots, \lambda_n$ be the values along the diagonals of $\mathcal{M}(T)$.

Then $\det(T) = \lambda_1 \cdot \dots \cdot \lambda_n$

$= \prod_{i=1}^{n} \lambda_i$.

$\mathcal{M}(T^*)$ is the conjugate transpose of $\mathcal{M}(T)$.

Thus $T^* e_1 \in \operatorname{span}(e_1, \dots, e_n)$

$\vdots$

$T^* e_n \in \operatorname{span}(e_n)$

$$T^*_* (e_1 \wedge \dots e_n) = T^* e_1 \wedge \dots \wedge T^* e_n$$
$$= \bar{\lambda}_1 e_1 + v_1 \wedge \bar{\lambda}_2 e_2 + v_2 \wedge \dots \wedge \bar{\lambda}_n e_n$$

where $v_k \in \operatorname{span}(e_k, \dots, e_n)$

Thus $\bar{\lambda}_1 e_1 + v_1 \wedge \dots \wedge \bar{\lambda}_n e_n$

$= \bar{\lambda}_1 e_1 \wedge \dots + \bar{\lambda}_n e_n + v_1 \wedge \dots \wedge \bar{\lambda}_n e_n$

$= \bar{\lambda}_1 e_1 \wedge \bar{\lambda}_2 e_n + v_2 \wedge \dots \wedge \bar{\lambda}_n e_n + 0$

$\vdots$

$= \bar{\lambda}_1 e_1 \wedge \bar{\lambda}_2 e_2 \wedge \dots \wedge \bar{\lambda}_k e_k + v_k \wedge \dots \wedge \bar{\lambda}_n e_n$

$= \bar{\lambda}_1 e_1 \wedge \bar{\lambda}_2 e_2 \wedge \dots \wedge \bar{\lambda}_n e_n$

$= \bar{\lambda}_1 \bar{\lambda}_2 \dots \bar{\lambda}_n \, e_1 \wedge e_2 \wedge \dots \wedge e_n$

$\Rightarrow \det T^* = \bar{\lambda}_1 \bar{\lambda}_2 \dots \bar{\lambda}_n$

$= \overline{\lambda_1 \lambda_2 \dots \lambda_n}$

$= \overline{\det T}$.

$\therefore \det T^* = \overline{\det T}$

$\square$.