3F.15) Suppose W is finite dimensional and $T \in \mathcal{L}(V,W)$.
Prove that $T' = 0$ i.f.f $T = 0$.

Pf: Let $T \in \mathcal{L}(V,W)$, where $T' = 0$.

By definition of T', $T' \in \mathcal{L}(W', V')$ such that

$$T'(\varphi) = \varphi \circ T \text{ for } \varphi \in W'$$

$T' = 0 \Rightarrow$ For all other dual maps $S' \in \mathcal{L}(W', V')$

$$S' + T' = S'$$

$$(S'+T') \in \mathcal{L}(W', V') \Rightarrow (S'+T')\varphi = S'\varphi$$

$$(S'+T')\varphi \Rightarrow \varphi \circ (S+T) \qquad S'\varphi = \varphi \circ S$$

$$\therefore \varphi \circ (S+T) = \varphi \circ S$$

For some $v \in V$,

$$\varphi \circ (S+T)v = (\varphi \circ S)v$$
$$\varphi((S+T)v) = \varphi(Sv)$$
$$\varphi(Sv + Tv) = \varphi Sv$$
$$\varphi(Sv) + \varphi(Tv) = \varphi Sv$$

$$\Rightarrow \varphi Tv = 0$$

Since φ can be any element in W' and v can be any element in V,
for $\varphi Tv = 0$ to satisfy all φ and v,

$$T = 0.$$

$$\therefore T' = 0 \Rightarrow T = 0$$

Now we prove implication the other way.

Suppose $T = 0$, then for T' and some $\varphi \in W'$

$$T'(\varphi) = \varphi \circ T, \text{ where } \varphi \circ T \in V'$$

Let $S \in \mathcal{L}(V,W)$, then $(S'+T') \in \mathcal{L}(W', V')$

$$(S'+T')\varphi = \varphi \circ (S+T)$$

For some $v \in V$, $[\varphi \circ (S+T)]v$
$$= \varphi[(S+T)v]$$
$$= \varphi(Sv + Tv)$$
$$= \varphi(Sv + 0)$$
$$= \varphi Sv$$
$$= (\varphi \circ S)v$$

$$\Rightarrow \varphi \circ (S+T) = \varphi \circ S$$
$$\Rightarrow (S'+T')\varphi = S'\varphi$$
$$\Rightarrow T'\varphi = 0$$
$$\Rightarrow T' = 0$$

$$\therefore T' = 0 \iff T = 0$$

D.

5A.20) $T \in \mathcal{L}(\mathbb{F}^\infty)$

$$T(z_1, z_2, z_3, \ldots) = (z_2, z_3, \ldots)$$

Eigenvector, eigenvalue $\Rightarrow \exists v \in \mathbb{F}^\infty, \lambda \in \mathbb{F}, v \neq 0, \lambda \neq 0$, s.t $Tv = \lambda v$.

$v = b(1, a, a^2, \ldots)$ for $a, b \in \mathbb{F}$.
$\lambda = \frac{1}{a}$ //

Include a proof that the proposition is true.

5A.22) Suppose $T \in \mathcal{L}(V)$ and there exists nonzero vectors $v, w \in V$ s.t $Tv = 3w$ and $Tw = 3v$.
Prove that 3 or -3 is an eigenvalue of T.

Pf: For 3 to be an eigenvalue, there must exist $v \in V$ such that
$$Tv = 3v.$$

Since $Tv = 3w$ and $Tw = 3v$,
$$Tv + Tw = 3w + 3v$$
$$T(v+w) = 3(v+w)$$
$\Rightarrow$ 3 is an eigenvalue, with $v+w$ as its eigenvector.

Similarly for -3 to be an eigenvalue, $\exists u \in V$ such that

$$Tu = -3u$$
$$Tv - Tw = 3w - 3v$$
$$T(v-w) = 3(w-v)$$
$$T(v-w) = -3(v-w)$$
$\Rightarrow \lambda = -3, u = v - w$. $\square$

5A.30
Suppose $T \in \mathcal{L}(\mathbb{R}^3)$ and $-4, 5, \sqrt{7}$ are eigenvalues, prove that $\exists x \in \mathbb{R}^3$ s.t $Tx - 9x = (-4, 5, \sqrt{7})$

Pf:

Let v_1, v_2, v_3 be eigenvalues of T s.t
$Tv_1 = -4v_1$
$Tv_2 = 5v_2$
$Tv_3 = \sqrt{7}v_3$, where v_1, v_2, v_3 form a basis of $\mathbb{R}^3$.

Let $x \in \mathbb{R}^3$. Let $y \in \mathbb{R}^3$ such that $Tx = y$.

$$Tx - 9x = (-4, 5, \sqrt{7})$$
$$y - 9x = (-4, 5, \sqrt{7})$$

$x = c_1v_1 + c_2v_2 + c_3v_3$. since $\text{span}(v_1, v_2, v_3) = \mathbb{R}^3$.
$$Tx = c_1Tv_1 + c_2Tv_2 + c_3Tv_3$$
$$= -4c_1v_1 + 5c_2v_2 + \sqrt{7}c_3v_3$$

$$Tx - ax = -4c_1v_1 + 5c_2v_2 + \sqrt{7}c_3v_3 - 9c_1v_1 - 9c_2v_2 - 9c_3v_3$$
$$= c_1v_1(-13) + c_2v_2(-4) + c_3v_3(\sqrt{7} - 9)$$
$$= -13c_1v_1 - 4c_2v_2 + (\sqrt{7} - 9)c_3v_3$$

Let $a_1v_1 + a_2v_2 + a_3v_3 = (-4, 5, \sqrt{7})$

For $Tx - 9x = (-4, 5, \sqrt{7})$,

$a_1 = -13c_1$
$a_2 = -4c_2$
$a_3 = (\sqrt{7} - 9)c_3$

$\therefore \exists x \in \mathbb{R}^3$ s.t $Tx - 9x = (-4, 5, \sqrt{7})$

$\square$.

5B.2) Suppose $T \in \mathcal{L}(V)$ and $(T-2I)(T-3I)(T-4I) = 0$. Suppose λ is an eigenvalue of T.

Prove that $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$.

<u>Pf</u>: $(T-2I)(T-3I)(T-4I) = 0$
$\Rightarrow (T-2I)(T-3I)(T-4I)v = 0$

$(T^2 - 3T - 2T + 6I)(T - 4I)$
$= (T^2 - 5T + 6I)(T - 4I)$
$= (T^3 - 4T^2 - 5T^2 + 20T + 6T - 24I) = 0$

$\Rightarrow T^3 - 20T^2 + 26T - 24I = 0$
$\Rightarrow (T^3 - 20T^2 + 26T - 24I)v = 0$

If v is an eigenvector,

$T^3v - 20T^2v + 26Tv - 24v = 0$
$\lambda^3 v - 20\lambda^2 v + 26\lambda v - 24v = 0$
$v(\lambda^3 - 20\lambda^2 + 26\lambda - 24) = 0$
$v(\lambda - 2)(\lambda - 3)(\lambda - 4) = 0$
Since $v \neq 0$, then
$\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$ for equation to be true

$\therefore \lambda = 2$ or $\lambda = 3$ or $\lambda = 4$

$\square$

Question 1: Suppose U is a subspace of V s.t $\dim V/U = 1$.
Prove that there exists a linear functional $f \in V'$ such that

$$\text{null } f = U$$

Pf: $V/U = \{ v + U : v \in V \}$ $\quad V' = \mathcal{L}(V, \mathbb{F})$

Since $\dim V/U = 1$,

$\dim V/U = \dim V - \dim U$

$1 = \dim V - \dim U$

$\dim U = \dim V - 1$

Let $f \in \mathcal{L}(V, \mathbb{F})$.

$\dim V = \dim \text{range } f + \dim \text{null } f$

$= 1 + \dim \text{null } f$

$\dim \text{null } f = \dim V - 1$.

Since null f is a subspace with dimension $\dim V - 1$, and U is also a subspace with dimension $\dim V - 1$, null f and U are isomorphic.

Since null f and U are both subspaces of V, we can let

$$\text{null } f = U$$

which still satisfies the properties of f as stated by the fundamental thm of linear maps.

Thus $\exists f \in V'$ s.t null $f = U$.

$\square$.

But proof feels kinda weak.

Question 2

Let $C^\infty(\mathbb{R})$ be the vector space of infinitely differentiable real valued functions $f: \mathbb{R} \to \mathbb{R}$.

a) Let U be the subspace of $C^\infty(\mathbb{R})$ consisting of functions that vanish at 42 and π.

$$U = \{ f \in C^\infty(\mathbb{R}) \mid f(42) = 0,\ f(\pi) = 0 \}$$

Prove that $C^\infty(\mathbb{R})/U$ is finite-dimensional.

Pf: Let $T: C^\infty(\mathbb{R}) \to \mathbb{R}^2$ where $T = \begin{bmatrix} f(\pi) \\ f(42) \end{bmatrix}$

$\text{null } T = U.$

$$\dim C^\infty(\mathbb{R}) = 2 + \dim \text{null } T.$$
$$= 2 + \dim U$$
$$\dim C^\infty(\mathbb{R}) - \dim U = 2.$$

Using the fundamental thm of linear maps,

$$\dim C^\infty(\mathbb{R}) = \dim C^\infty(\mathbb{R})/U + \dim U$$
$$\dim C^\infty(\mathbb{R}) - \dim U = \dim C^\infty(\mathbb{R})/U$$
$$\Rightarrow \dim C^\infty(\mathbb{R})/U = 2$$

$\square$.

b) $W = \{ f \in C^\infty(\mathbb{R}) \mid f(0), f'(0) = 0, f''(0) = 0 \}.$

Let $T: C^\infty(\mathbb{R}) \to \mathbb{R}^3$, where $T = \langle f(0), f'(0), f''(0) \rangle$.

$\text{null } T = W$

$$\dim C^\infty(\mathbb{R}) = \dim \text{null } T + 3$$
$$\dim C^\infty(\mathbb{R}) - \dim \text{null } T = 3.$$

Let π be the quotient map of W.

$$\dim C^\infty(\mathbb{R}) = \dim C^\infty(\mathbb{R})/W + \dim W$$
$$\dim C^\infty(\mathbb{R}) - \dim W = \dim C^\infty(\mathbb{R})/W$$
$$\dim C^\infty(\mathbb{R}) - \dim \text{null } T = \dim C^\infty(\mathbb{R})/W$$

$$\Rightarrow \dim C^\infty(\mathbb{R})/W = 3$$

$\Rightarrow$ quotient space is finite-dimensional.

Since $\text{null } T = W$, $C^\infty(\mathbb{R})/W$ is isomorphic to range T

$\Rightarrow$ finding a $f_1, f_2, f_3 \in C^\infty(\mathbb{R})$ s.t.

$$\tilde{T}(f_1 + W) = \langle 1, 0, 0 \rangle,$$
$$\tilde{T}(f_2 + W) = \langle 0, 1, 0 \rangle,$$
$$\tilde{T}(f_3 + W) = \langle 0, 0, 1 \rangle$$

$\Rightarrow f_1 + W, f_2 + W, f_3 + W$ form a basis for $C^\infty(\mathbb{R})/W$

Question 3

a) For $\sin x$ and $\cos x$ to lie in V,
the functions must map from $\mathbb{R}$ to $\mathbb{C}$, and satisfy

$$f'' = -f.$$

First, since $\mathbb{R} \subset \mathbb{C}$, and $\sin x, \cos x$ under a domain of $\mathbb{R}$ maps to a range $[-1, 1] \subset \mathbb{R}$,
$\sin x, \cos x \in C^\infty(\mathbb{R}, \mathbb{C})$.

Next, $\sin x \frac{d^2}{dx^2} = \cos x \frac{d}{dx} = -\sin x \Rightarrow (\sin x)'' = -\sin x$.
$\cos x \frac{d^2}{dx^2} = -\sin x \frac{d}{dx} = -\cos x \Rightarrow (\cos x)'' = -\cos x$

$$\Rightarrow \text{both } \sin x, \cos x \in V.$$

Now, we prove that $\sin x, \cos x$ form a basis for V.

For $\sin x, \cos x$ to be a basis, they must first be linearly independent.
For them to be linearly independent, null span$(\sin x, \cos x) = \{0\}$.

$$c_1 \sin x + c_2 \cos x = 0.$$
$$\sqrt{c_1^2 + c_2^2}\, \sin\left(x + \tan^{-1}\left(\frac{c_2}{c_1}\right)\right) = 0$$

either $\sqrt{c_1^2 + c_2^2} = 0$ or $\sin(x + \phi) = 0$ where $\phi = \tan^{-1}\left(\frac{c_2}{c_1}\right)$
Since $\sin(x + \phi) \neq 0$ for all x,

$$\sqrt{c_1^2 + c_2^2} = 0$$
$$c_1^2 + c_2^2 = 0$$
$$c_1^2 = -c_2^2$$
$$c_1 = \pm\sqrt{-c_2^2} = \pm i c_2$$
$$\Rightarrow c_1 = i c_2 \quad \text{or} \quad c_1 = -i c_2$$

Suppose $c_1 = i c_2$, then
$i c_2 \sin x + c_2 \cos x = 0$
$c_2 [i \sin x + \cos x] = 0$
$c_2 e^{ix} = 0$
$\Rightarrow c_2 = 0$ or $e^{ix} = 0$ for all x.
$\Rightarrow c_2 = 0$ since $e^{ix} \neq 0$ unless $x \to -\infty$.

Suppose $c_1 = -i c_2$, then
$-i c_2 \sin x + c_2 \cos x = 0$
$c_2 [-i \sin x + \cos x] = 0$
$c_2 [e^{i(x - \frac{\pi}{2})}] = 0$
$\Rightarrow c_2 = 0$

$\therefore c_2 = 0$ for $c_1 \sin x + c_2 \cos x = 0$
Since $c_1 = \pm i c_2$,
$c_1 = 0$
$\Rightarrow c_1 = c_2 = 0$ for $c_1 \sin x + c_2 \cos x = 0$
$\Rightarrow$ null span$(\sin x, \cos x) = \{0\}$
$\Rightarrow$ $\sin x, \cos x$ is linearly independent.

Since $\dim V = 2$, and $\sin x, \cos x$ are linearly independent elements in V,
$\sin x, \cos x$ form a basis of V.

$\square$.

b) $D \in \mathcal{L}(C^\infty(\mathbb{R}, \mathbb{C}))$ such that for $f \in C^\infty(\mathbb{R}, \mathbb{C})$, $Df = f'$.

For V to be an invariant subspace for D,

for all $f \in V$, $D(f) \in V$

Since $\operatorname{span}(\sin x, \cos x) = V$, let $f \in V$ be

$$f = c_1 \sin x + c_2 \cos x$$

We'll prove that $D(f) \in V$.

$$\begin{aligned} D(f) &= D(c_1 \sin x + c_2 \cos x) \\ &= c_1 D(\sin x) + c_2 D(\cos x) \\ &= c_1 \cos x - c_2 \sin x \in V. \end{aligned}$$

Since for all $f \in V$, $D(f) \in V$, V is an invariant subspace for D.

D.

c) An eigenvector $f \in V$ for D must be such that

$D(f) = \lambda f$, where $\lambda \in \mathbb{F}$.

$\Rightarrow f' = \lambda f$

Let $y = f(x)$. $\frac{dy}{dx} = f'(x)$.

$$\frac{dy}{dx} = \lambda y$$

$$\frac{dy}{y} = \lambda dx$$

$$\ln y = \lambda x + C$$

$y = ke^{\lambda x}$ where $k \in \mathbb{F}$.

Since $f'' = -f$, $y'' = \lambda^2 k e^{\lambda x} = -ke^{\lambda x}$

$$\Rightarrow \lambda^2 = -1$$

$$\lambda = \pm\sqrt{-1} = \pm i$$

$\Rightarrow e^{ix}, e^{-ix}$ are eigenvectors of V.

For e^{ix}, e^{-ix} to be a basis of V,

$\operatorname{null} \operatorname{span}(e^{ix}, e^{-ix}) = \{0\}$

$$c_1 e^{ix} + c_2 e^{-ix} = 0$$

$$e^{ix}(c_1 + c_2 e^{-2ix}) = 0$$

$$\Rightarrow c_1 + c_2 e^{-2ix} = 0$$

Since c_1, c_2 are constants, and e^{-2ix} is not constant,

for $c_1 + c_2 e^{-2ix} = 0$,

$c_1 = 0$ and $c_2 = 0$

$\Rightarrow \operatorname{null} \operatorname{span}(e^{ix}, e^{-ix}) = \{0\}$

$\Rightarrow e^{ix}, e^{-ix}$ are linearly independent

$\Rightarrow e^{ix}, e^{-ix}$ forms a basis for V. since $\dim V = 2$.

eigenvalues : $f_1 = e^{ix}$, $\lambda_1 = i$
$f_2 = e^{-ix}$, $\lambda_2 = -i$.

$\square$.