7A.11) Let U be a subspace of V with orthogonal complement $U^\perp$ such that

$V = U \oplus U^\perp$

Let $P = P_U$.

Let $v \in V$ such that $v = u + w$ where $u \in U$ and $w \in U^\perp$

$$
\begin{aligned}
\langle Pv, v\rangle &= \langle P(u+w), u+w\rangle \\
&= \langle u, u+w\rangle \\
&= \langle u, u\rangle + \langle u, w\rangle \\
&= \|u\|^2 \in \mathbb{R}
\end{aligned}
$$

$\Rightarrow \langle Pv, v\rangle \in \mathbb{R}$ for every $v \in V$.

This is equivalent to P being self-adjoint.

Thus P is self adjoint.

Conversely, suppose P is self-adjoint, then

$\langle Pv, u\rangle = \langle v, Pu\rangle$

Let $U = \operatorname{range} P$.

Let $W = \operatorname{null} P$

$V = U \oplus W$.

Since P is self adjoint,

$\operatorname{range} P \perp \operatorname{null} P$

$\Rightarrow U \perp W$

$\Rightarrow W = U^\perp$.

Hence $\operatorname{null} P = U^\perp$, $\operatorname{range} P = U$.

Let $v = u + w$ where $u \in U$, $w \in W$.

$Pv = P(u+w) = Pu + Pw = Pu$

$\langle P^2 v, v\rangle = \langle Pv, v\rangle$

$\|Pv\|^2 = \langle Pv, v\rangle$

$\langle Pu, Pu\rangle - \langle Pu, v\rangle = 0$

$\langle Pu, Pu - v\rangle = 0$

$\Rightarrow Pu - v \in W$

$Pu - u - w \in W$

$\Rightarrow Pu - u = 0$

$Pu = u$

$\therefore P(u+w) = u \Rightarrow P = P_U$ $\square$.

12) Suppose T is a normal operator on V such that $3, 4$ are eigenvalues. Prove that $\exists\, v \in V$ such that $\|v\| = \sqrt{2}$ and $\|Tv\| = 5$.

Let $\lambda_1 = 3$, $Tv_1 = \lambda_1 v_1$
Let $\lambda_2 = 4$, $Tv_2 = \lambda_2 v_2$.

From 7.22, $\langle v_1, v_2 \rangle = 0$.

Let $e_1 = \frac{v_1}{\|v_1\|}$, $e_2 = \frac{v_2}{\|v_2\|}$ $\quad Te_1 = \frac{1}{\|v_1\|} Tv_1 = \frac{\lambda_1}{\|v_1\|} v_1$

$\|e_1\| = \|e_2\| = 1$

Let $v = a_1 e_1 + a_2 e_2$ s.t
$\|v\| = \sqrt{2}$.
Since $e_1 \perp e_2$, $\|v\|^2 = |a_1|^2 + |a_2|^2 = 2$ by Pythagoras Thm

$$\begin{aligned} Tv &= T(a_1 e_1 + a_2 e_2) \\ &= a_1 Te_1 + a_2 Te_2 \\ &= a_1 \lambda_1 e_1 + a_2 \lambda_2 e_2 \\ &= 3a_1 e_1 + 4a_2 e_2 \end{aligned}$$

Suppose $3a_1 e_1 + 4a_2 e_2 = Tv$
$\|Tv\|^2 = 9|a_1|^2 + 16|a_2|^2 = 25$ by Pythagoras Thm.

Let $x = |a_1|^2$, $y = |a_2|^2$

$9x + 16y = 25 \quad — (1)$
$x + y = 2 \quad — (2)$

(2) $x = 2 - y \quad — (3)$
sub (3) into (1)
$9(2-y) + 16y = 25$
$18 - 9y + 16y = 25$

$18 + 7y = 25$
$7y = 7$
$y = 1$
$x = 1$
$\Rightarrow |a_1|^2 = 1 \qquad |a_2|^2 = 1$
$a_1 = 1$ or $-1 \qquad a_2 = 1$ or -1
$\therefore$ If $|a_1| = 1$ and $|a_2| = 1$,
$a_1 e_1 + a_2 e_2$
$v = \frac{a_1}{\|v_1\|} v_1 + \frac{a_2}{\|v_2\|} v_2$ such that

$\|v\| = \sqrt{2}$ and
$\|Tv\| = 5$

$\square$.

16) Suppose $T \in \mathcal{L}(V)$ is normal, prove that
range T = range T^*.

Pf: Normal $\Leftrightarrow TT^* = T^*T$.

Let $v \in V$.
Let $u = T^*v$, $u \in$ range T^*
$Tu \in$ range T.
$\Rightarrow$ range $T^*T \subseteq$ range T.

Let $w \in Tv$, $w \in$ range T
$T^*w \in$ range $^*T$
$\Rightarrow$ range $TT^* \subseteq$ range $^*T$.

Since range $TT^* \subseteq$ range T^*,
range $T^*T \subseteq$ range T,
and $T^*T = TT^*$,

range $T^* \supseteq TT^* = T^*T \subseteq$ range T
$\Rightarrow$ range T^* = range T

$\square$.

7B.17 True.

Let $T \in \mathcal{L}(\mathbb{R}^3)$
Let v_1, v_2, v_3 be eigenvectors of T with distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$
and v_1, v_2, v_3 are not orthogonal.

Let $v = a_1 v_1 + a_2 v_2 + a_3 v_3$

$$\langle Tv, v \rangle \neq \langle v, Tv \rangle$$

$$\langle Tv, v_1 \rangle = \langle a_1 \lambda_1 v_1 + a_2 \lambda_2 v_2 + a_3 \lambda_3 v_3, v_1 \rangle$$
$$= a_1 \lambda_1 \|v_1\|^2 + a_2 \lambda_2 \langle v_2, v_1 \rangle + a_3 \lambda_3 \langle v_3, v_1 \rangle$$

$$\langle v, Tv_1 \rangle = \langle a_1 v_1 + a_2 v_2 + a_3 v_3, \lambda_1 v_1 \rangle$$
$$= \lambda_1 a_1 \|v_1\|^2 + a_2 \lambda_1 \langle v_2, v_1 \rangle + a_3 \lambda_1 \langle v_3, v_1 \rangle$$

$$\langle Tv, v_1 \rangle - \langle v, Tv_1 \rangle = a_2 \langle v_2, v_1 \rangle (\lambda_2 - \lambda_1) + a_3 \langle v_3, v_1 \rangle (\lambda_3 - \lambda_1)$$
$\neq 0$ since v_1, v_2, v_3 are not orthogonal and $\lambda_1, \lambda_2, \lambda_3$ are distinct

$\Rightarrow \langle Tv, v_1 \rangle \neq \langle v, Tv_1 \rangle$

$\Rightarrow$ T is not self-adjoint

$\therefore$ We have built an operator with eigenvector basis that is not self-adjoint $\square$.

7B.2) T is self-adjoint such that 2, 3 are the only eigenvalues/eigenvectors of T

Let $u_1, \dots, u_n$ be the eigenvectors of T with eigenvalue 2. Let $\operatorname{span}(u_1, \dots, u_n) = U$
Let $w_1, \dots, w_m$ be the eigenvectors of T with eigenvalue 3. $\operatorname{span}(w_1, \dots, w_m) = W$.

Since T is self-adjoint, T is also normal, hence by the Spectral Theorem, there exists an eigenvector basis of T for V, with eigenvalues 2 and 3.

Since T has only two eigenvalues,
For $u \in U$,

$$(T^2 - 5T + 6I) \cdot u = (\lambda_1^2 - 5\lambda_1 + 6) \cdot u$$
$$= (4 - 10 + 6) u$$
$$= 0$$
$$\Rightarrow \quad u \in \operatorname{null}(T^2 - 5T + 6I)$$

For $w \in W$

$$(T^2 - 5T + 6I) w = (\lambda_2^2 - 5\lambda_2 + 6) w$$
$$= (9 - 15 + 6) w$$
$$= 0$$
$$\Rightarrow \quad w \in \operatorname{null}(T^2 - 5T + 6I).$$

Since $E(2, T) \oplus E(3, T) = V$,
$v = u + w$.

Hence $(T - 5T + 6I)(u + w)$
$= (T - 5T + 6I) u + (T - 5T + 6I) w$
$= 0$

$\therefore \; T - 5T + 6I = 0$.

$\square$

7B.7) Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$ is normal opr such that $T^9 = T^8$.
Prove that T is self-adjoint and $T^2 = T$.

Pf: By the Complex Spectral Theorem,
since $V = \mathbb{C}$ and T is normal, then
T has $\dim V$ orthonormal eigenvectors.
Let $e_1, \ldots, e_n$ be the orthonormal eigenvector basis of V with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$.

For $j = 1, \ldots, n$,

$\langle T^9 v, e_j \rangle = \langle T^8 v, T^* e_j \rangle = \langle T^8 v, e_j \rangle$ (by given definition)

$\langle T^8 v, \bar{\lambda}_j e_j \rangle = \langle T^8 v, e_j \rangle$ since T is normal & T^* has

$\lambda_j \langle T^8 v, e_j \rangle = \langle T^8 v, e_j \rangle$ same eigenvectors as T.

$\Rightarrow \lambda_j \langle T^8 v, e_j \rangle - \langle T^8 v, e_j \rangle = 0$

$(\lambda_j - 1) \langle T^8 v, e_j \rangle = 0$

$\Rightarrow \lambda_j = 1$ or $\langle T^8 v, e_j \rangle = 0$

For $\langle T^8 v, e_j \rangle = 0$

Let $v = a_1 e_1 + \ldots + a_n e_n$

$\langle T^8 v, e_j \rangle = \langle a_1 \lambda_1^8 e_1 + \ldots + a_n \lambda_n^8 e_n, e_j \rangle$

$= a_1 \lambda_j^8 = 0$

$\Rightarrow \lambda_j = 0$

Hence $\lambda_j = 1$ or $\lambda_j = 0$

Thus T has all real eigenvalues

$\Rightarrow T$ is self-adjoint.

Let $U = \operatorname{span}(u_1, \ldots, u_k)$ where for u_j, $j = 1, \ldots, k$,
$Tu_j = u_j$.

Let $W = \operatorname{span}(w_1, \ldots, w_m)$ where for w_j, $j = 1, \ldots, m$,
$Tw_j = 0$.

$U \oplus W = V$ since $U = E(1, T)$ and $W = E(0, T)$.

Hence all $v \in V$ are
$v = u + w$ where $u \in U$ and $w \in W$.

$Tv = T(u + w)$
$= Tu + Tw$
$= u$.

$Tu = u$.

$\therefore Tv = u$, $T^2 v = T(u) = u$.

$\Rightarrow Tv = T^2 v$

$\Rightarrow T = T^2$

$\square$.

1) $S \in \mathcal{L}(\mathbb{R}^2)$ $T \in \mathcal{L}(\mathbb{R}^2)$ where S, T are self-adjoint, but ST is not self-adjoint.

Let $S(x,y) = (x+y, x+y)$ $\quad T(x,y) = (0,y)$

$$M(S) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad M(T) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Let $v = (x,y)$ $u = (a,b)$

$$\begin{aligned} \langle Sv, u \rangle &= (x+y, x+y) \cdot (a,b) \\ &= a(x+y) + b(x+y) \\ &= (x+y)(a+b) \\ \langle u, Sv \rangle &= (x,y) \cdot (a+b, a+b) \\ &= (a+b)x + y(a+b) \\ &= (x+y)(a+b) \end{aligned}$$

$\Rightarrow \langle Sv, u \rangle = \langle u, Sv \rangle$

$\Rightarrow S$ is self-adjoint

$$\begin{aligned} \langle Tv, u \rangle &= (0,y)(a,b) \\ &= by \\ \langle v, Tu \rangle &= (x,y)(0,b) \\ &= by \end{aligned}$$

$\Rightarrow T$ is self-adjoint.

$ST(x,y) = S(0,y) = (y,y)$

$\Rightarrow ST(x,y) = (y,y).$

$$\begin{aligned} \langle STv, u \rangle &= (y,y)(a,b) \\ &= ay + by \end{aligned} \qquad \begin{aligned} \langle v, STu \rangle &= (x,y) \cdot (b,b) \\ &= bx + by \end{aligned}$$

$\Rightarrow \langle STv, u \rangle \neq \langle v, STu \rangle$

$\Rightarrow ST$ is not self-adjoint.

$\square$

b) Let V be finite-dimensional inner-product space. Let $S, T \in \mathcal{L}(V)$ be self-adjoint.

Proposition: $ST + TS$ is self-adjoint.

Pf: $\langle (ST+TS)v, u \rangle$ where $v, u \in V$

$$\begin{aligned} &= \langle STv, u \rangle + \langle TSv, u \rangle \\ &= \langle Tv, Su \rangle + \langle Sv, Tu \rangle \quad \text{since } S, T \text{ are self-adjoint} \\ &= \langle v, TSu \rangle + \langle v, STu \rangle \\ &= \langle v, STu \rangle + \langle v, TSu \rangle \\ &= \langle v, STu + TSu \rangle \\ &= \langle v, (ST+TS)u \rangle \end{aligned}$$

$\Rightarrow \langle (ST+TS)v, u \rangle = \langle v, (ST+TS)u \rangle$

$\Rightarrow ST + TS$ is self adjoint.

$\square$.

c) Let $v, u \in V$.

$$\langle STv, u\rangle$$
$$= \langle Tv, Su\rangle$$
$$= \langle v, TSu\rangle$$

If $ST = TS$, then $\langle v, TSu\rangle = \langle v, STu\rangle$
$\Rightarrow \langle STv, u\rangle = \langle v, STu\rangle$
$\Rightarrow ST$ is self-adjoint.

Conversely, suppose ST is self-adjoint,

$$\langle STv, u\rangle = \langle v, STu\rangle$$
$$= \langle Sv, Tu\rangle$$
$$= \langle TSv, u\rangle$$
$\Rightarrow \quad \langle STv, u\rangle - \langle TSv, u\rangle = 0$
$\langle (ST-TS)v, u\rangle = 0$
$\Rightarrow \quad ST - TS = 0$
$ST = TS$

$\square$.