Problem 1. [20 points] Recall that a tree is a connected acyclic graph. In particular, a single vertex is a tree. We define a *Splitting Binary Tree*, or *SBTree* for short, as either the lone vertex, or a tree with the following properties:

1. exactly one node of degree 2 (called the root).
2. every other node is of degree 3 or 1 (called internal nodes and leaves, respectively).

For the case of one single vertex (see above), that vertex is considered to be a leaf. It is easier to understand the definition visually, so an example is shown in Figure 1. An example of a tree which is not an SBTree is shown in Figure 2.

(a) [10 pts] Show if an SBTree has more than one vertex, then the induced subgraph obtained by removing the unique root consists of two disconnected SBTrees. You may assume that by removing the root you obtain two separate connected componenents, so all you need to prove is that those two components are SBTrees.

(b) [10 pts] Prove that two SBTrees with the same number of leaves must also have the same total number of nodes. *Hint: As a conjecture, guess an expression for the total number of nodes in terms of the number of leaves $N(l)$. Then use induction to prove that it holds for all trees with the same l*

(a) Let $P(n)$ be the proposition that for an SBTree $G = (V, E)$, $|V| = n > 1$, the induced subgraph obtained by removing the unique root consists of two disconnected SBTrees.

By contradiction, assume $P(n)$ is false, such that the induced subgraph obtained by removing the unique root of G does not consist of exactly two SBTrees

By the assumed proposition, at least one of the induced subgraph must have a property that invalidates it from an SBTree. This means that the original graph G also must have the same property, which invalidates G. However this derives a contradiction as G is an SBTree.

Since a contradiction is derived, $P(n)$ must be true.

$\square$

b) Let $P(n)$ be the proposition that two SBTrees with equal number of n leaves must have the same number of nodes.

By contradiction, assume that $P(n)$ is false, such that there exists two SBTrees with equal number of n leaves with different number of nodes.
We collate all counterexamples of $P(n)$ into a nonempty set C.

$$C ::= \{ n \in \mathbb{N} \mid \neg P(n) \}$$

By well ordering principle, $\exists k \in C$ such that k is the smallest number of leaves where there exists two SBTrees with k leaves that have different number of nodes.

Let $G = (V, E)$, $L ::= \{ v \in V \mid v \text{ is a leaf node of } G \}$
$G' = (V', E')$ $L' ::= \{ v \in V' \mid v \text{ is a leaf node of } G' \}$

$$|L| = |L'| = k$$
$$|V| \neq |V'|$$

G and G' are SBTrees.

Since G and G' are SBTrees, we can remove a subgraph of 3 nodes, 2 leaves and the parent of the two leaves from both G and G', to form two subgraphs S and S'

$$S = (V_S, E_S) \quad , |L_S| = |L| - 2, \; |V_S| = |V| - 3$$
$$S' = (V_{S'}, E_{S'}) \; , |L_{S'}| = |L'| - 2, \; |V_{S'}| = |V'| - 3$$

$$|L_s| = |L_{s'}|$$
$$|V_s| \neq |V_{s'}|$$

Thus there exists two graphs with equal leaves of $k-2$, such that the amount of vertices are different. However this derives a contradiction as we defined k to be the smallest number of leaves with a counterexample to $P(n)$.

Since a contradiction is derived, $P(n)$ must be true.

$\square$

2)

Lemma 1: Let $G=(V,E)$ be an $N \times M$ grid, and let $v_{i,j} \in V$ be a vertex at row i column j, then G is bipartite with partitions $L \subset V$, $R \subset V$ such that

$$L ::= \{v_{i,j} \in V \mid i+j \text{ is odd}\}$$
$$R ::= \{v_{i,j} \in V \mid i+j \text{ is even}\}$$

Proof:

For a vertice $v_{i,j} \in V$ $v_{i,j}$ must be adjacent to at most 4 other vertices, $v_{i\pm1,j}$ or $v_{i,j\pm1}$. Let adjacent vertices be $v_{i',j'} \in V$.

Without loss of generality, assume $i+j$ is even, then $i'+j'$ must be odd. Let L, R be sets such that

$$L ::= \{v_{i,j} \in V \mid i+j \text{ is odd}\}$$
$$R ::= \{v_{i,j} \in V \mid i+j \text{ is even}\}$$

$\forall v_L \in L$ are adjacent to only $v_R \in R$ as proven above. Thus L and R can form the partitions of a bipartite graph.

$\square$

Lemma 2: Given an $N \times M$ undirected grid, if N and M are both odd, the grid cannot be Hamilton

Pf: Let $G=(V,E)$ be an undirected grid of $N \times M$, where N and M is odd. Thus $|V| = N \times M$ is also odd. By lemma 1, let there be sets L, R, the partitions of a bipartite graph

$$L ::= \{v_{i,j} \in V,\ i+j \text{ is odd}\}$$
$$R ::= \{v_{i,j} \in V,\ i+j \text{ is even}\}$$

Since there are odd vertices, $|L| = |R| + 1$.

By contradiction, assume that G is Hamilton, then let C be the hamilton cycle.

$$C ::= v_1, v_2, \ldots, v_k, \ldots v_{N\times M}, v_1$$

If k is odd, let $v_k \in L$, and if k is even, $v_k \in R$. $\{v_k, v_{k+1}\} \in E$ must be an edge incident with L and R since G is Hamilton.

Thus since C is Hamilton, $\{v_{N\times M}, v_1\} \in E$ to complete the cycle. However $N \times M$ is odd, and 1 is odd. $v_{N\times M}, v_1 \in L$. By definition of a bipartite graph, $\{v_{N\times M}, v_1\} \notin E$, which is a contradiction.

Since a contradiction is derived the lemma must be true.

$\square$

b) Let $G=(V,E)$ be an $N \times M$ undirected grid. Let $v_{i,j} \in V$ be a vertex at row i column j. Without loss of generality, let N be even and $M > 1$.

We can prove there exists a hamilton path starting from $v_{1,1}$. At $v_{1,1}$, we traverse down the row to $v_{1,M}$. We move down 1 row to $v_{2,M}$, the traverse to $v_{2,2}$.

We repeat row traversals in the following fashion. From $v_{i,2}$, we move down the row to $v_{i,M}$. Then move down the column to $v_{i+1,M}$ and head back to the second column at $v_{i+1,2}$. This step allows us to traverse 2 rows at a time finishing at the second column.

algorithm must be exact with eqn.

Since N is even, we will be able to traverse all N rows accordingly and end up at $v_{N,2}$. Now we head to $v_{N,1}$ and head up the column to $v_{1,1}$, completing the hamilton cycle. □

This is handwavy -

b2) By part a) if N and M are both odd, then $N \times M$ is odd, and $N \times M$ is not hamilton. □

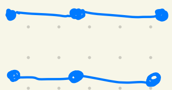
If N is odd, by the algorithm in 1) we will end up at $v_{N,M}$ and be unable to return back to $v_{1,1}$ without revisiting nodes.

b3) Yes they will survive. The grid is 20×7. Thus the grid is Hamilton by pt 1). There exists a hamilton cycle starting and ending at any vertex. Hence it does not matter where they are placed. Since grid is 20×7, there are 140 street corners. They'll take $140+2=142$ minutes to visit all corners and defuse the bomb, which is the exact duration before the bomb explodes.

3)

a) If we partition G into two pieces with the first piece being $\lceil \frac{n}{3} \rceil$, the second piece must be $\lfloor \frac{2n}{3} \rfloor$. However the definition of a tangled graph does not state that a $\lfloor \frac{2n}{3} \rfloor$ subgraph is tangled. Hence we cannot use the inductive hypothesis on the $\lfloor \frac{2n}{3} \rfloor$ subgraph. Thus the proof is false.

b)



c) By contradiction, assume $G = (V, E)$ is mangled but is not fully connected. This implies that there exists at least 1 subgraph in G not connected to vertices outside this subgraph.

Without loss of generality, let $A = (V_A, E_A)$ be a subgraph of G. Let $B = (V_B, E_B)$ be the other subgraph of G such that A is not connected to B. $|V_A| + |V_B| = |V|$.

If $1 \leq |V_A| \leq \lceil \frac{|V|}{2} \rceil$, $|V_A| \leq \lceil \frac{|V|}{2} \rceil$ hence A must have a connection with B by definition of a mangled graph. Thus $|V_A| > \lceil \frac{|V|}{2} \rceil$ for A to not be connected to B.

If $|V_A| > \lceil \frac{|V|}{2} \rceil$, $|V_B| < \lceil \frac{|V|}{2} \rceil$ since $|V_A| + |V_B| = |V|$. Now that $V_B < \lceil \frac{|V|}{2} \rceil$, for B to not be connected to A, $|V_B| > \lceil \frac{|V|}{2} \rceil$. However this contradicts the previous statement that $|V_B| < \lceil \frac{|V|}{2} \rceil$. Hence since a contradiction is derived, the proposition must be true.

$\square$

4)

Let $G = (V, E)$. G is a simple connected graph with $|V| = n$.
By corollary 5.5.6, every connected graph with v vertices has at least $v-1$ edges.
Hence G must have at least $n-1$ edges to be connected.

When $|E| = n-1$, G has no cycles. This is proven through contradiction. If G has a cycle with $|E| = n-1$, let C be the set of edges in the cycle:

$$C ::= \{ \{v_0, v_1\}, \{v_1, v_2\}, \ldots \{v_k, v_0\} \}$$

WLOG, we remove $\{v_k, v_0\}$ from C. However there still exists a path between v_0 to v_k. Hence C is still connected, and G is still connected. However $|E| = n-1-1 = n-2$, which contradicts our initial statement that G must have at least $n-1$ edges to be connected.

Thus when $|E| = n-1$, G has no cycles.

To conclude, for a connected graph $G = (V, E)$, $|V| = n$, $E = n-1$, G must have no cycles. Since G is connected acyclic, G is a tree.

Now we must prove that if G is a tree, G must have exactly n-1 edges.

By theorem 5.7.4, the number of vertices in a tree is one larger than the number of edges.

$$|E|+1 = |V| = n$$
$$|E| = n-1$$

$\square$

b) By induction,

$P(n) ::=$ A connected graph $G=(V,E)$, $|V|=n$, has a spanning tree.

Base case:

$n=1$. A graph with 1 vertex is connected and acyclic. $n=1$ has a spanning tree.

Inductive step:

Assume $P(n)$ is true. Let $G=(V,E)$, $|V|=n+1$ be a connected graph. Suppose we remove a vertex from G such that the resultant graph is connected.

Let $v_i \in V$ denote the removed vertex, and

$$E_i ::= \{e \in E \mid e \text{ is incident to } v_i\}$$

Let $G'=(V',E')$ be the resultant graph without v_i and E_i. Since $|V'|=n$, G' has a spanning tree.

Let $S=(V', E_s)$ be the spanning tree within G', where $|V'|=n$, $|E_s|=n-1$.

We add back v_i into G' to get back G.

Similarly to S, we can create a subgraph $A=(V, E_A)$

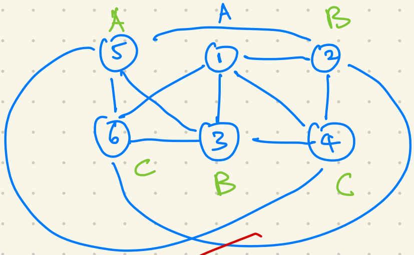
$$V = V \cup \{v_i\}$$
$$e_i \in E_i, \; E_A = E_s \cup \{e_i\}$$
$$|V| = n+1, \quad |E_A| = n-1+1 = n.$$
$$E_i \subseteq E \land E_s \subseteq E \Rightarrow E_A \subseteq E$$

Since A has the same n+1 vertices as G, and E_A is a subset of E with n edges. A must be connected acyclic, equivalent to a spanning tree. Thus proving $P(n)$.

$\square$

5)

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$



b) 2. Each node has degree 4.
Each node has only 1 node which cannot be accessed on the first step. By symmetry, the nodes that are not adjacent to each other are adjacent to everything else. Hence only 2 steps maximum are required to visit all vertexes.

□

c) 1, 2, 4, 3, 5, 6, 1. A cycle is a closed walk involving all different vertices. There are only 6 vertices hence maximum length is 6.

d) Each node is adjacent to 4 nodes, and not adjacent to 1. The two nodes not adjacent to each other can be colored the same, and the remaining nodes colored differently. Thus 5-colorable.

6) Let $G = (V, E)$, $v, w \in V$, v has odd degree. For the longest walk starting at v and ending at w without repeating edges, $w \neq v$.

By contradiction,
suppose that the longest walk from v, which has an odd degree, ends at v. Let E_v be the set of all edges incident to v. $|E_v|$ is odd.

Since the walk can start and finish at v, v must have an ingoing edge for every outgoing edge to ensure that v always has an edge for the walk to return to. Thus,

$$k \in \mathbb{N}, |E_v| = 2k$$

where k is the number of pairs of ingoing and outgoing edges of v.
This shows that for the walk to finish at v, $|E_v|$ must be even, which contradicts our definition that $|E_v|$ is odd.

Since a contradiction is derived, $w \neq v$.

$\square$

b) By contradiction, suppose w has an even degree and the walk ends at w. Let E_w be the edges incident to w. $|E_w|$ is even. Since the walk ended at w, the walk must have traversed $2k+1$ edges in E_w, where k is the number of times the walk entered and left w. However since $|E_w|$ is even, and the walk only traversed $2k+1$ edges, which is odd, there still is at least one edge in E_w left untraversed, contradicting our statement that the walk is the longest walk. Thus, w must be odd.

$\square$

Proof: By lemma 5.2.1, the sum of degrees of the vertices of a graph is equal to twice the number of edges.

By contradiction, suppose there exists a graph where only 1 vertex has an odd degree. Let S be the sum of degrees. S must therefore be odd. By lemma 5.2.1, the sum of degrees is twice the no. of edges, hence the sum must be even, which contradicts our derivation of S if only 1 vertex is odd. Hence there cannot exist a graph with only 1 odd degree vertex. Since there must be > 1 odd degree vertex connected, this proves the theorem

$\square$