Problem 1. [24 points]

Translate the following sentences from English to predicate logic. The domain that you are working over is X, the set of people. You may use the functions $S(x)$, meaning that "x has been a student of 6.042," $A(x)$, meaning that "x has gotten an 'A' in 6.042," $T(x)$, meaning that "x is a TA of 6.042," and $E(x, y)$, meaning that "x and y are the same person."

(a) [6 pts] There are people who have taken 6.042 and have gotten A's in 6.042

(b) [6 pts] All people who are 6.042 TA's and have taken 6.042 got A's in 6.042

(c) [6 pts] There are no people who are 6.042 TA's who did not get A's in 6.042.

(d) [6 pts] There are at least three people who are TA's in 6.042 and have not taken 6.042

a) $\exists x \in X : S(x) \wedge A(x)$.

b) $\forall x \in X : T(x) \wedge S(x) \Rightarrow A(x)$.

c) $\nexists x \in X : T(x) \wedge \neg A(x)$.

d) $\exists x, y, z \in X : \neg E(x,y) \wedge \neg E(y,z) \wedge \neg \exists(x,z) \wedge T(x) \wedge \neg S(x) \wedge$
$T(y) \wedge \neg S(y) \wedge T(z) \wedge \neg S(z)$.

Problem 2. [24 points]

Use a truth table to prove or disprove the following statements:

(a) [12 pts]

$$\neg(P \vee (Q \wedge R)) = (\neg P) \wedge (\neg Q \vee \neg R)$$

(b) [12 pts]

$$\neg(P \wedge (Q \vee R)) = \neg P \vee (\neg Q \vee \neg R)$$

P	Q	R	$\neg P$	$\neg Q$	$\neg R$	$Q \wedge R$	$P \vee (Q \wedge R)$	$\neg Q \vee \neg R$	$\neg P \wedge (\neg Q \vee \neg R)$	$\neg(P \vee (Q \wedge R))$	
T	T	T	F	F	F	T	T	F	F	F	✓
T	T	F	F	F	T	F	T	T	F	F	✓
T	F	T	F	T	F	F	T	T	F	F	✓.
T	F	F	F	T	T	F	T	T	F	F	✓
F	T	T	T	F	F	T	T	F	F	F	✓
F	T	F	T	F	T	F	F	T	T	T	✓
F	F	T	T	T	F	F	F	T	T	T	✓
F	F	F	T	T	T	F	F	T	T	T	✓.

$\Rightarrow \neg(P \vee (Q \wedge R)) = (\neg P) \wedge (\neg Q \vee \neg R)$.

Q.

3)

Problem 3. [24 points]

The binary logical connectives $\wedge$ (*and*), $\vee$ (*or*), and $\Rightarrow$ (*implies*) appear often in not only computer programs, but also everyday speech. In computer chip designs, however, it is considerably easier to construct these out of another operation, `nand`, which is simpler to represent in a circuit. Here is the truth table for `nand`:

P	Q	P nand Q
`true`	`true`	`false`
`true`	`false`	`true`
`false`	`true`	`true`
`false`	`false`	`true`

(a) [12 pts] For each of the following expressions, find an equivalent expression using only `nand` and $\neg$ (*not*), as well as grouping parentheses to specify the order in which the operations apply. You may use A, B, and the operators any number of times.

(i) $A \wedge B$

(ii) $A \vee B$

(iii) $A \Rightarrow B$

(b) [4 pts] It is actually possible to express each of the above using only `nand`, without needing to use $\neg$. Find an equivalent expression for $(\neg A)$ using only `nand` and grouping parentheses.

(c) [8 pts] The constants `true` and `false` themselves may be expressed using only `nand`. Construct an expression using an arbitrary statement A and `nand` that evaluates to `true` regardless of whether A is `true` or `false`. Construct a second expression that always evaluates to `false`. Do not use the constants `true` and `false` themselves in your statements.

i) $\neg$ (A nand B)

ii) $\neg$A nand $\neg$B

iii) $A \Rightarrow B$:

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

$A \Rightarrow B \equiv$ A nand $\neg$B

b) $\neg A \equiv$ A nand A

c) true: A nand $\neg$A = A nand (A nand A)

false: true nand true = [A nand (A nand A)] nand [A nand (A nand A)].

Problem 4. [10 points] You have 12 coins and a balance scale, one of which is fake. All the real coins weigh the same, but the fake coin weighs less than the rest. All the coins visually appear the same, and the difference in weight is imperceptible to your senses. In at most 3 weighings, give a strategy that detects the fake coin. (Note: the scale in this problem is a scale with two dishes, which tips toward the side that is heavier. For clarification, do an image search for "balance scale").

Problem 5. [6 points] Prove the following statement by proving its contrapositive: if r is irrational, then $r^{1/5}$ is irrational. (Be sure to *state* the contrapositive explicitly.)

Problem 6. [12 points] Suppose that $w^2 + x^2 + y^2 = z^2$, where w, x, y, and z always denote positive integers. (Hint: It may be helpful to represent even integers as $2i$ and odd integers as $2j + 1$, where i and j are integers)

Prove the proposition: z is even if and only if w, x, and y are even. Do this by considering all the cases of w, x, y being odd or even.

5) By contraposition,

assume $r^{1/5}$ is not irrational,

then $r^{1/5} = \frac{a}{b}$, where $a, b \in \mathbb{Z}$. $a \nmid b$.

$\Rightarrow r = \frac{a^5}{b^5} \in \mathbb{Q}$

$\Rightarrow r$ is rational.

Hence if $r^{1/5}$ is rational, r must be rational.

Therefore, if $r^{1/5}$ is irrational, r must be irrational. $\square$.

6) By cases,