Problem 1. [15 points] Suppose $\Pr\{\} : \mathcal{S} \to [0, 1]$ is a *probability function* on a sample space, $\mathcal{S}$, and let B be an event such that $\Pr\{B\} > 0$. Define a function $\Pr_B\{\cdot\}$ on outcomes $w \in \mathcal{S}$ by the rule:

$$\Pr_B\{w\} = \begin{cases} \Pr\{w\} / \Pr\{B\} & \text{if } w \in B, \\ 0 & \text{if } w \notin B. \end{cases} \tag{1}$$

(a) [7 pts] Prove that $\Pr_B\{\cdot\}$ is also a probability function on $\mathcal{S}$ according to Definition 14.4.2.

(b) [8 pts] Prove that

$$\Pr_B\{A\} = \frac{\Pr\{A \cap B\}}{\Pr\{B\}}$$

for all $A \subseteq \mathcal{S}$.

a) Definition 14.4.2 states that a probability function of S is such that

$\Pr[\omega] \geq 0$ for all $\omega \in S$ and
$\sum_{\omega \in S} \Pr[\omega] = 1$.

For all $\omega \in B$, $\frac{\Pr\{\omega\}}{\Pr\{B\}} > 0$. For all $\omega \notin B$, $\Pr_B(\omega) = 0$.

$\Rightarrow \forall \omega \in S, \Pr_B\{\omega\} \geq 0$.

By definition on an event probability,

$$\sum_{\omega_i \in B} \omega_i = \Pr\{B\}.$$

$$\sum_{\omega_i \in B} \Pr_B\{\omega_i\} = \frac{1}{\Pr\{B\}} \left[\Pr\{\omega_1\} + \Pr\{\omega_2\} + \dots \right]$$
$$= \frac{\Pr\{B\}}{\Pr\{B\}} = 1$$

$\therefore \Pr_B$ is a probability function.

b) $\Pr_B\{A\} = \frac{\Pr\{A \cap B\}}{\Pr\{B\}}$ For all $A \subseteq S$.

Pf: $\Pr_B\{A\} = \sum_{\omega_i \in A} \Pr_B\{\omega_i\}$

By cases,
If $\omega_i \in A$ and $\omega_i \in B \Rightarrow \omega_i \in A \cap B$, then
$\Pr_B\{\omega_i\} \neq 0$.

If $\omega_i \in A$ and $\omega_i \notin B \Rightarrow \omega_i \notin A \cap B$, then
$\Pr_B\{\omega_i\} = 0$.

$$\therefore \sum_{\omega_i \in A} \Pr_B\{\omega_i\} = \frac{1}{\Pr\{B\}} \sum_{\omega_i \in A \cap B} \Pr\{\omega_i\}$$
$$= \frac{1}{\Pr\{B\}} \cdot \Pr\{A \cap B\}$$

$\square$.

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Problem 2. [20 points]

(a) [10 pts] Here are some handy rules for reasoning about probabilities that all follow directly from the Disjoint Sum Rule. Use Venn Diagrams, or another method, to prove them.

$$\Pr\{A - B\} = \Pr\{A\} - \Pr\{A \cap B\} \qquad \text{(Difference Rule)}$$

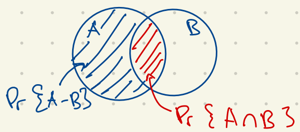
$$\Pr\{\bar{A}\} = 1 - \Pr\{A\} \qquad \text{(Complement Rule)}$$

$$\Pr\{A \cup B\} = \Pr\{A\} + \Pr\{B\} - \Pr\{A \cap B\} \qquad \text{(Inclusion-Exclusion)}$$

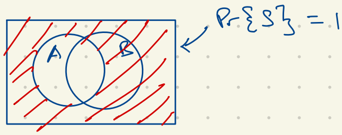
$$\Pr\{A \cup B\} \le \Pr\{A\} + \Pr\{B\}. \qquad \text{(2-event Union Bound)}$$

$$\text{If } A \subseteq B, \text{ then } \Pr\{A\} \le \Pr\{B\}. \qquad \text{(Monotonicity)}$$

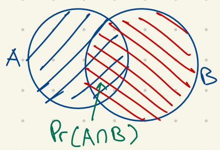
$\Pr\{A-B\} = \Pr\{A\} - \Pr\{A\cap B\}$.



$\Pr\{\bar{A}\} = 1 - \Pr\{A\}$



$\Pr\{A\cup B\} = \Pr\{A\} + \Pr\{B\} - \Pr\{A\cap B\}$.



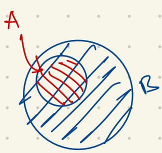
$\Pr(A) + \Pr(B) = \Pr(A\cup B) + \Pr(A\cap B)$
$\Rightarrow \Pr(A\cup B) = \Pr(A) + \Pr(B) - \Pr(A\cap B)$.

$\Pr\{A\cup B\} \le \Pr(A) + \Pr(B)$.

<u>Pf</u>: Using the Inclusion exclusion rule,
$\Pr\{A\cup B\} = \Pr(A) + \Pr(B) - \Pr\{A\cap B\}$

Using the definition of $\Pr\{\}$, $\Pr\{A\cap B\} > 0$,
$\Rightarrow \Pr\{A\} + \Pr\{B\} \ge \Pr\{A\} + \Pr\{B\} - \Pr\{A\cup B\}$
$\ge \Pr\{A\cup B\}$

D.



$\Pr\{B\} = \Pr\{B\cap\bar{A}\} + \Pr\{A\}$
$\Rightarrow \Pr\{B\} \ge \Pr\{A\}$

D.

(b) [10 pts] Prove the following probabilistic identity, referred to as the **Union Bound**. You may assume the theorem that the probability of a union of *disjoint* sets is the sum of their probabilities.

Theorem. *Let $A_1, \ldots, A_n$ be a collection of events on some sample space. Then*

$$\Pr(A_1 \cup A_2 \cup \cdots \cup A_n) \le \sum_{i=1}^{n} \Pr(A_i).$$

$$\Pr(A_1 \cup A_2 \cup \ldots \cup A_n) \le \sum_{i=1}^{n} \Pr(A_i)$$

Pf:

By induction on the number of sets $A_1, A_2, \ldots A_n$,
For $n=1$,

$\Pr(A_1) = \Pr(A_1) \Rightarrow$ proposition holds.

Assuming proposition holds for $n=k$, $1<k\le n$, then for $k+1$,

$\Pr(A_1 \cup A_2 \cup \ldots \cup A_k \cup A_{k+1})$: Let $B = A_1 \cup A_2 \cup \ldots \cup A_k$.
By cases on how independent A_{k+1} and B are,
If B is disjoint from A_{k+1}, then

$$\Pr(B \cup A_{k+1}) = \Pr(B) + \Pr(A_{k+1})$$
$$\le \sum_{i=1}^{k} \Pr(A_i) + \Pr(A_{k+1}) = \sum_{i=1}^{k+1} \Pr(A_i)$$

$\Rightarrow$ Proposition holds if A_{k+1} is disjoint from B.

If $\exists B \cap A_{k+1}$ and $A_{k+1} \not\subseteq B$, then

$$\Pr(B \cup A_{k+1}) = \Pr(B) + \Pr(A_{k+1}) - \Pr(B \cap A_{k+1}) \text{ By inclusion exclusion}$$
$$\le \sum_{i=1}^{k+1} \Pr(A_i)$$

$\Rightarrow$ Proposition holds.

If $A_{k+1} \subseteq B$, then $\Pr(B \cup A_{k+1}) = \Pr(B) \le \sum_{i=1}^{k+1} \Pr(A_i)$

$\Rightarrow$ Proposition holds.

$\therefore$ Proposition holds for all cases in the inductive step
$\Rightarrow$ Proposition holds for all n.

$\square$.

Problem 3. [15 points] Recall the strange dice from lecture:

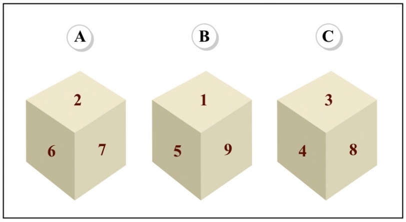


Image by MIT OpenCourseWare.

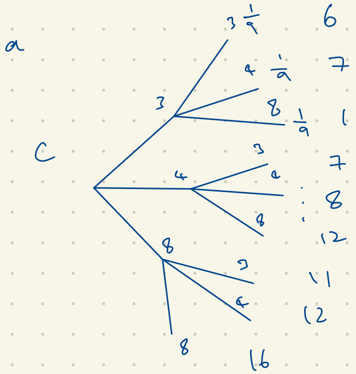
In the book we proved that if we roll each die once, then die A beats B more often, die B beats die C more often, and die C beats die A more often. Thus, contrary to our intuition, the "beats" relation $>$ is not transitive. That is, we have $A > B > C > A$.

We then looked at what happens if we roll each die twice, and add the result. In this problem we will show that the "beats" relation reverses in this game, that is, $A < B < C < A$, which is very counterintuitive!

(a) [5 pts] Show that rolling die C twice is more likely to win than rolling die B twice.

(b) [5 pts] Show that rolling die A twice is more likely to win that rolling die C twice.

(c) [5 pts] Show that rolling die B twice is more likely to win than rolling die A twice.



$3 \cdot 3 \cdot 3 \cdot 3 -$

Let (c_1, c_2) be an outcome of two rolls of C, such that one roll is c_1 and the other is c_2.

$\forall c_1, c_2$, $\Pr[(c_1, c_2)] = \frac{1}{9}$ since the probability of getting each number is $\frac{1}{3}$.

Similarly for B, $\Pr[(b_1, b_2)] = \frac{1}{9}$.

$\Rightarrow$ Probability for all two number sequences is uniform.

All possible sums of B are 6, 10, 14, 2, 18
Possible sums of C are 7, 11, 12, 6, 8, 16.

For B,

sum	P[sum]
6	2/9
10	3/9
14	2/9
2	1/9
18	1/9

For C,

sum	P[sum]
7	2/9
11	2/9
12	2/9
6	1/9
8	1/9
16	1/9.
15	

For all sums of C, the probability that a C beats B = sum probability · event where outcomes are less than C sum.

$$\Rightarrow \overset{(6)}{\tfrac{1}{9}\left(\tfrac{1}{9}\right)} + \overset{(7)}{\tfrac{2}{9}\left(\tfrac{2}{9}+\tfrac{1}{9}\right)} + \overset{(8)}{\tfrac{1}{9}\left(\tfrac{2}{9}+\tfrac{1}{9}\right)} + \overset{(11)}{\tfrac{2}{9}\left(\tfrac{2}{9}+\tfrac{3}{9}+\tfrac{1}{9}\right)} + \overset{(12)}{\tfrac{2}{9}\left(\tfrac{1}{9}+\tfrac{2}{9}+\tfrac{3}{9}\right)}$$

$$\overset{(16)}{\tfrac{1}{9}\left(\tfrac{1}{9}+\tfrac{2}{9}+\tfrac{3}{9}+\tfrac{2}{9}\right)} = \tfrac{1}{9}\left(\tfrac{1}{9}\right) + \tfrac{2}{9}\left(\tfrac{3}{9}\right) + \tfrac{1}{9}\left(\tfrac{3}{9}\right) + \tfrac{2}{9}\left(\tfrac{6}{9}\right) + \tfrac{2}{9}\left(\tfrac{6}{9}\right) + \tfrac{1}{9}\left(\tfrac{8}{9}\right)$$

$$= \tfrac{1}{9}\left(\tfrac{12}{9}\right) + \tfrac{2}{9}\left(\tfrac{15}{9}\right)$$

$$= \tfrac{14}{27} = \tfrac{42}{81} = 0.519$$

$$\Rightarrow \Pr(B \text{ beats } C) = 1 - 0.519 < 0.519 < \Pr(C \text{ beats } B)$$

$\therefore$ C is more likely to beat B.

$\square$.

A :

Sum.	Pr(sum)
4	1/9
8	2/9
9	2/9
12	1/9
13	2/9
14	1/9

$Pr(A \text{ beats } C) = \frac{2}{9}\left(\frac{3}{9}\right) + \frac{2}{9}\left(\frac{4}{9}\right) + \frac{1}{9}\left(\frac{6}{9}\right) + \frac{2}{9}\left(\frac{8}{9}\right)$
$+ \frac{1}{9}\left(\frac{8}{9}\right) = \frac{44}{81}$

$= 0.5432$

$\Rightarrow P(A \text{ beats } C) > P(C \text{ beats } A)$ //

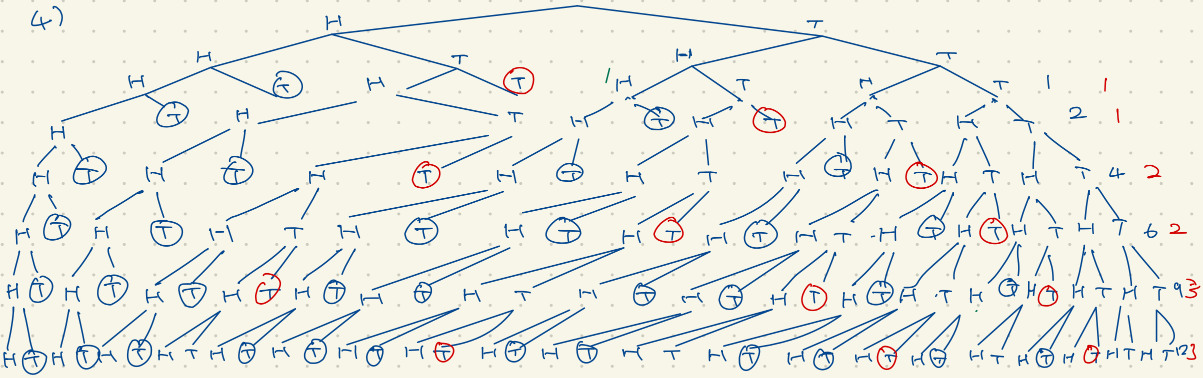
$Pr(B \text{ beats } A) = \frac{2}{9}\left(\frac{1}{9}\right) + \frac{2}{9}\left(\frac{5}{9}\right) + \frac{2}{9}\left(\frac{8}{9}\right) + \frac{1}{9}\left(\frac{9}{9}\right) = 42$

$\frac{42}{81} = 0.5185$

$\Rightarrow Pr(B \text{ beats } A) > Pr(A \text{ beats } B)$

D.

(4)



1 2 3 4 5 6 7 8 9 10 11 12

1 1 2 2 3 3 4 4 5 5 6 6 . . .

1 2 4 6 9 12 16 20 25 30 36 42 . . .

×1 ×2 ×2 ×3 ×3 ×4 ×4 ×5 ×5 ×6 ×6 ×7

8 12 18 24 32 40

$\underbrace{\;}_{4}\ \underbrace{\;}_{6}\ \underbrace{\;}_{6}\ \underbrace{\;}_{8}\ \underbrace{\;}_{8}$ $2\left(1+2+3+\dots+\lceil \frac{n}{2} \rceil\right)$

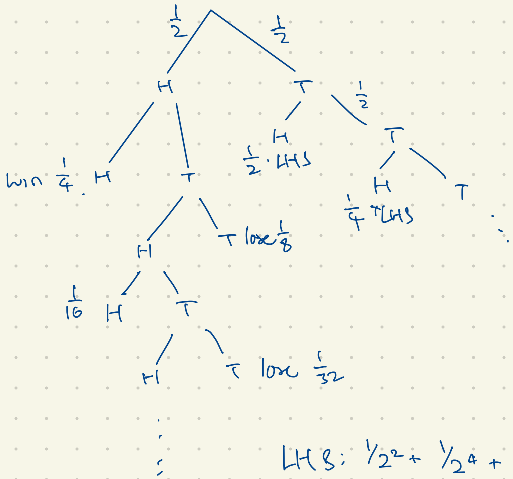
$\frac{1}{8} + \frac{2}{12} + \frac{4}{18} + \frac{6}{24} + \frac{9}{32} + \frac{12}{40}$

$\frac{1}{8} \quad \frac{1}{6} \quad \frac{2}{9} \quad \frac{1}{4} \quad \frac{9}{32} \quad \frac{3}{16}$

$\frac{1}{8} + \frac{6}{8}\cdot\frac{2}{12} + \frac{9}{12}\cdot\frac{4}{18} + \frac{12}{18}\cdot\frac{6}{24} + \frac{16}{24}\cdot\frac{9}{32}$

$\frac{1}{8} + \frac{6}{8}\cdot\frac{2}{12} + \frac{6}{8}\cdot\frac{9}{12}\cdot\frac{4}{18} + \frac{6}{8}\cdot\frac{9}{12}\cdot\frac{12}{18}\cdot\frac{6}{24}$

$\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{3}{32} +$



LHS: $1/2^2 + 1/2^4 + 1/2^6 + \dots + 1/2^{2n}$

$= 1/4 + 1/4^2 + 1/4^3 + \dots + 1/4^n$

$= \frac{1-(1/4)^n}{1-1/4} - 1 = \frac{1-(1/4)^n}{(3/4)} - 1 = \frac{4}{3} - \frac{4}{3}\cdot\left(\frac{1}{4}\right)^n - 1$

$= \frac{1}{3} - \frac{4}{3}\cdot\left(\frac{1}{4}\right)^n$

$\lim_{n\to\infty} \frac{1}{3} - \frac{4}{3}\left(\frac{1}{4}\right)^n = \frac{1}{3}$

$\Rightarrow$ LHS win probability is $\frac{1}{3}$

RHS win probability: $\left(\frac{1}{3} \cdot \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} - \frac{1}{3}\right)$

$$= \frac{2}{3}\left(1-\left(\frac{1}{2}\right)^n\right) - \frac{1}{3}$$

$$= \frac{2}{3} - \frac{2}{3}\left(\frac{1}{2}\right)^n - \frac{1}{3}$$

$$= \frac{1}{3} - \frac{2}{3}\left(\frac{1}{2}\right)^n$$

$\lim_{n \to \infty}$ RHS win probability $= \frac{1}{3}$.

Total win probability: $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$ //

5) Sample space is all possible sequences of selecting 5 cards from a deck.
The event outcomes are hands with at most 2 different suits.

e.g 4-1, 2-3, 5,0

b) $Pr[\omega]$ of each possible hand is uniform across all outcomes.
All possible hands are $52 \cdot 51 \cdot 50 \cdot 49 \cdot 48$.

The set of all sequences is a 5! to 1 correspondence to the set containing all possible combinations of hands.

$\Rightarrow$ there are $52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \div 5!$ posible combinations of cards.
$= 2598960$.

No. of hands with 4 cards of the same suit

$\{(\text{suit}, \text{rank}, \text{rank}, \text{rank}, \text{rank}), (\text{suit}, \text{rank})\} \div 4!$ (since 4! to 1)
$= (4 \cdot 13 \cdot 12 \cdot 11 \cdot 10) \cdot (3 \cdot 13) \div 4!$
$= 111540$

No. of hands with 5 cards same suit
$\{(\text{suit}, \text{rank}, \text{rank}, \text{rank}, \text{rank})\} \div 5!$
$= (4 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9) \div 5!$
$= 5148$

No. of hands with 2 cards same suit, 3 cards same suit

$\{(\text{suit}, \text{rank}, \text{rank}), (\text{suit}, \text{rank}, \text{rank}, \text{rank})\}$
$= (4 \cdot 13 \cdot 12) \cdot (3 \cdot 13 \cdot 12 \cdot 11) \div 2! \div 3!$
$= 267696$

Total hands with 2 suits: 384384
$Pr[E] = 0.1479$
≈ 0.15

6) Uniform probability distribution for first card.

$\frac{26}{52}$ chance its red, $\frac{26}{52}$ chance its black

$\Rightarrow \frac{1}{2}$ chance you win.

b) Suppose 1st card is red $\Rightarrow$ $\frac{25}{51}$ next card is red, $\frac{26}{51}$ next card is black.

$$\frac{26}{51} \approx 0.51 > \frac{1}{2}.$$

c) r red cards, b black cards. r+b total cards left.
probability of winning is $\frac{b}{r+b}$. since probability of each card is $\frac{1}{r+b}$ and

b black cards $\Rightarrow b\left(\frac{1}{r+b}\right) = \frac{b}{r+b}$. //

d) When $b > r$, choose.

$$\frac{26!}{52 \cdot 51 \cdot 50 \cdot \ldots \cdot 26} \qquad \frac{26!}{\left(\frac{52!}{25!}\right)} = 26! \cdot \frac{25!}{52!}$$

$$= 7.7556 \times 10^{-17}$$

probability that $b \leq r$ for whole game

b r b r b r
b b b r r r b r b r.

No such strategy exists.

By contradiction, suppose such a strategy exists, then