b) For fixed integer n, k, how many non-negative integer solutions are there to the eqn

$$\sum_{i=0}^{k} x_i = n.$$

n zeroes, k ones.
$\Rightarrow$ string of length $n+k$.
$\Rightarrow \binom{n+k}{k}$ ways to choose to put k ones.

Ans: $\binom{n+k}{k} = \dfrac{(n+k)!}{k!\,n!}$

c) $\sum_{i=0}^{k} x_i \leq n$ $\quad \binom{n+k}{k} + \binom{n+k-1}{k} + \binom{n+k-2}{k} + \dots + \binom{n+k-n}{k}$

$$= \frac{(n+k)!}{k!\,n!} + \frac{(n+k-1)!}{(n-1)!\,k!} + \dots + \frac{k!}{k!\,(0!)}$$

$$= \frac{1}{k!}\left[\frac{(n+k)!}{n!} + \frac{(n+k-1)!}{(n-1)!} + \dots + k!\right]$$

$$= \frac{1}{k!}\left[(n+1)(n+2)\dots(n+k) + (n)(n+1)\dots(n+k-1) + (n-1)(n)(n+1)\dots + (1)(2)(3)\dots(k)\right]$$

$$= \sum_{i=0}^{n} \binom{n+k-i}{k} =$$

$$\sum_{i=0}^{k} x_i \leq n \Rightarrow \sum_{i=0}^{k} x_i - x_{k+1}$$

$$x_1 + x_2 + \dots + x_k - x_{k+1}$$

$$\sum_{i=0}^{k+1} x_i = n$$

$n=1$ 0 1

2 1

d) How many simple undirected graph with n vertices?

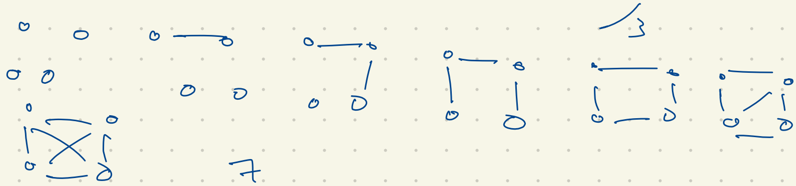
$n=2$

$n=3$

4 2

3

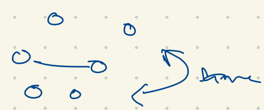
$n=4$



7

) 4

0	1	1
1	0	1
1	1	0



same

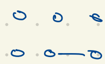
$\frac{2 \cdot 3}{2} = 3$

11

6 bit 3 ones.

$1 + \frac{n(n-1)}{2}$ //

111000 →
110100 →
11



$2^{\binom{n}{2}}$ $2^{\binom{2}{2}}$ $2^{\binom{3}{2}}$.

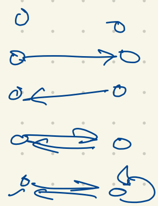
by → $\frac{(n-1)\cdot n}{2} = \binom{n}{2}$

$= \frac{n!}{2!(n-2)!}$

$2^4 = 16$

$\frac{2\cdot 3}{2} = 4$

Directed graphs with n vertices (self loops allowed).



$2^n (2^{\binom{n}{2}})(2^{\binom{n}{2}})$

$= 2^n \cdot 2^{\binom{n}{2}\binom{n}{2}} = 2^n \cdot 2^{\frac{(n-1)(n)}{2} + \frac{(n-1)(n)}{2}} = 2^n \cdot 2^{(n-1)n}$

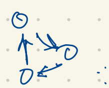
$= 2^n \cdot 2^{n^2-n}$

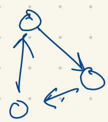
$= 2^{n^2}$ //

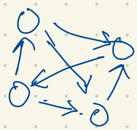
Tournament graph with n vertices.

no. of edges: $\frac{(n-1)\cdot n}{2}$.

$2^{\frac{(n-1)n}{2}} = 2^{\binom{n}{2}}$



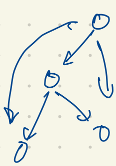


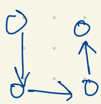


acyclic tournament graph with n vertices.

2 colorable. -

⇒ numb of combinations of trees?.





⇒ corresponding acyclic

if there exists a directed path to node, then there exists an edge between.

If v_1 is directly connected to v_2, then there is an edge between $v_1 \to v_2$.

fully connected bipartite with n-1 edges $\xleftrightarrow{}$ Undirected tree $\to$ directed tree $\xrightarrow{\text{bijection}}$ directed acyclic tournament graph.

bijection

n-1 edges. in undirected tree.

Why cant

n^{n-2}

2

Numbers in the range [1, 700] divisible by 2, 5, 7 ?

divisible by 2: $700/2 = 350$
divisible by 5: $700/5 = 140$
divisible by 7: $700/7 = 100$
divisible by 10: $700/10 = 70$
divisible by 14: $700/14 = 50$
divisible by 35: $700/35 = 20$
divisible by 70: $700/70 = 10$

$$S_2 \cup S_5 \cup S_7 = S_2 + S_5 + S_7 - S_2 \cap S_5 - S_2 \cap S_7 - S_5 \cap S_7 + S_2 \cap S_5 \cap S_7$$
$$= 300 + 140 + 100 - 70 - 50 - 20 + 10$$
$$= 460$$

i) arrange n books on k bookshelves with order.

sequence $(b_1, b_2, \ldots b_n) = n!$

partition sequence into k $\Rightarrow \binom{n+k-1}{k-1}$

each sequence has $\binom{n+k-1}{k-1}$ arrangements.

$\Rightarrow n! \cdot \binom{n+k-1}{k-1}$ //

3 bookshelves. 7 books.

10000 10100

0 0 0 0 0 1 1 1

0 0 0 1 1 1 0 0

1 0 0 0 1 0 0 1 1 0 0 1 1

0 0 0 0 1 1 1

j) One book on each bookshelf.

Let each bookshelf be represented by the sequence $(1,0)$, since each bookshelf needs at least one book.

Then the number of free books is $n-k$.
Performing the same partitioning of the $n-k$ book gives

$$\binom{n+k-k-1}{k-1} = \binom{n-1}{k-1} \text{ different partitions of books.}$$

$$\Rightarrow \binom{n-1}{k-1} n! \quad //$$

$$k! \binom{n}{k} \cdot (n-k)!$$

$$= k! \frac{n!}{k!(n-k)!} \cdot (n-k)! = n!$$

$n2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}$ using combinatorial proof.

$$\sum_{k=1}^{n} k\binom{n}{k} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \dots + n\binom{n}{n}.$$

In the sequence of $(0,0,\dots 0)$ length n,

$\binom{n}{1}$ ways to put 1 *

$2\binom{n}{2}$ ways to put 2 ones, 1 *.

All sequences must have a *.
The subsequence we get after removing * has $n-1$ 0s and 1s.
The total combinations of this subsequence is 2^{n-1}.
There are n ways we can insert * into this subsequence.
Hence the total sum is $n2^{n-1}$ //

5)

(*Hint: Consider the set of all length-n sequences of 0's, 1's and a single *.*)

Problem 5. [15 points] At a congressional hearing, there are $2n$ members present. Exactly n of them are Democrats and n of them are Republicans. The members want to select a smaller subcommittee of size n from within those present at the hearing. However, since the Democrats currently hold majority, they want there to be more Democrats then Republicans in the committee. In how many ways can you select such a committee? (*Hint: Consider two cases: n odd and n even.*)

For case where n is even,
Let $n = 2k$, $k \in \mathbb{N}$.
There must be democrats $> k$.
$\Rightarrow$ There must be $k+1 \le$ democrats $\le n$

$$\binom{n}{k+1}\cdot\binom{n}{k-1} + \binom{n}{k+2}\binom{n}{k-2} + \dots \binom{n}{n}\binom{n}{0}$$

$$= \binom{n}{k-1}^2 + \binom{n}{k-2}^2 + \dots + \binom{n}{0}^2$$

$$= \binom{n}{0}^2 + \binom{n}{1}^2 + \dots + \binom{n}{k-1}^2$$

$$= \sum_{i=0}^{k-1}\binom{n}{i}^2 = 2n$$

$$\binom{n}{k}^2 + 2\left(\sum_{i=0}^{k-1}\binom{n}{i}^2\right) = \binom{2n}{n}$$

$$\left(\frac{n!}{k!\,k!}\right)^2 + 2\left(\sum_{i=0}^{k-1}\binom{n}{i}^2\right) = \binom{2n}{n}$$

$$2\left(\sum_{i=0}^{k-1}\binom{n}{i}^2\right) = \binom{2n}{n} - \left(\frac{n!}{k!\,k!}\right)^2$$

$$= \frac{(2n)!}{n!\,n!} - \left(\frac{n!}{k!\,k!}\right)^2$$

$$= \frac{(2n)!}{(n!)^2} - \left(\frac{n!}{k!\,k!}\right)^2$$

$$\sum_{i=0}^{k-1}\binom{n}{i}^2 = \frac{1}{2}\left[\binom{2n}{n} - \binom{n}{\frac{n}{2}}^2\right] //$$

For odd n,
Let $n = 2k+1$
then there must be $k+1 \le$ democrats $\le n$.

$$\binom{n}{k+1}\binom{n}{2k+1-k-1} = \binom{n}{k+1}\binom{n}{k}$$

$$\binom{n}{k+1}\binom{n}{k} + \binom{n}{k+2}\binom{n}{k-1} + \dots + \binom{n}{n}\binom{n}{0}$$

$$\binom{2n}{n} = 2\sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i}$$

$$\frac{1}{2}\binom{2n}{n} //$$