6A-16) Suppose $u, v \in V$ are such that

$$\|u\| = 3, \ \|u+v\| = 4, \ \|u-v\| = 6$$

What does $\|v\|$ equal?

$$\|u+v\|^2 = 16.$$
$$\|u-v\|^2 = 36$$

Using the parallelogram equality,

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$$
$$16 + 36 = 2(9 + \|v\|^2)$$
$$52 = 2(9 + \|v\|^2)$$
$$26 = 9 + \|v\|^2$$
$$17 = \|v\|^2$$
$$\|v\| = \sqrt{17} \ //$$

6B.5) On $P_2(\mathbb{R})$,

$$\langle p, q \rangle = \int_0^1 p(x) q(x) dx$$

Using the basis $1, x, x^2$, produce an orthonormal basis for $P_2(\mathbb{R})$.

Orthonormal $\Rightarrow \langle p, q \rangle = 0$.

Let the basis be e_1, e_2, e_3.

$e_1 = 1 \quad \langle e_1, e_1 \rangle = \int_0^1 1 dx = x|_0^1 = 1.$

$\Rightarrow 1$ is orthonormal.

Using Gram Schmidt Decomposition,

$$e_2 = \frac{x - \langle x, e_1 \rangle e_1}{\|x - \langle x, e_1 \rangle e_1\|}$$

$$\langle x, e_1 \rangle = \int_0^1 x \, dx = \frac{1}{2}x^2 \Big|_0^1 = \frac{1}{2}.$$

$$\|x - \langle x, e_1 \rangle e_1\|$$
$$= \|x - \tfrac{1}{2}\|$$
$$= \sqrt{\langle x - \tfrac{1}{2}, x - \tfrac{1}{2} \rangle}$$
$$= \sqrt{\int_0^1 (x - \tfrac{1}{2})^2 dx}$$
$$= \sqrt{\int_0^1 x^2 - x + \tfrac{1}{4} dx}$$
$$= \sqrt{\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 + \tfrac{1}{4}x \Big|_0^1}$$
$$= \sqrt{(\tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{4})}$$
$$= \sqrt{\tfrac{1}{12}}$$
$$= \frac{1}{2\sqrt{3}} = \frac{1}{6}\sqrt{3}$$

$$\therefore e_2 = \frac{6}{\sqrt{3}}(x - \tfrac{1}{2})$$
$$= \frac{6}{\sqrt{3}}x - \frac{3}{\sqrt{3}} = 2\sqrt{3}x - \sqrt{3}$$
$$= \sqrt{3}(2x - 1)$$

$$e_2 = \sqrt{3}(2x - 1)$$

$$e_3 = \frac{x^2 - \langle x^2, e_1 \rangle e_1 - \langle x^2, e_2 \rangle e_2}{\|\ldots\|}$$

$$\langle x^2, e_1 \rangle = \int_0^1 x^2 dx = \tfrac{1}{3}x^3 \Big|_0^1 = \tfrac{1}{3}$$

$$\langle x^2, e_2 \rangle = \int_0^1 x^2 [\sqrt{3}(2x-1)] dx$$
$$= \sqrt{3}\int_0^1 2x^3 - x^2 dx$$
$$= \sqrt{3}\left[\tfrac{1}{2}x^4 - \tfrac{1}{3}x^3\right]_0^1$$
$$= \sqrt{3}\left[\tfrac{1}{2} - \tfrac{1}{3}\right] = \sqrt{3}/6$$

$\| x^2 - \frac{1}{3} - \sqrt{3}/6 \, (\sqrt{3}(2x-1)) \|$

$= \| x^2 - \frac{1}{3} - \frac{1}{2}(2x-1) \|$

$= \| x^2 - \frac{1}{3} - x + \frac{1}{2} \|$

$= \| x^2 + \frac{1}{6} - x \|$

$= \sqrt{\int_0^1 (x^2 + \frac{1}{6} - x)^2 \, dx}$

$= \sqrt{\int_0^1 x^4 + \frac{1}{6}x^2 - x^3 + \frac{1}{6}x^2 + \frac{1}{36} - \frac{x}{6} - x^3 - \frac{1}{6}x + x^2 \, dx}$

$= \sqrt{\int_0^1 x^4 - 2x^3 + \frac{4}{3}x^2 - \frac{1}{3}x + \frac{1}{36} \, dx}$

$= \sqrt{\left(\frac{1}{5} - \frac{1}{2} + \frac{4}{9} - \frac{1}{6} + \frac{1}{36} \right)} = \sqrt{\frac{1}{180}} = \frac{1}{3\sqrt{20}} = \frac{1}{6\sqrt{5}} = \frac{\sqrt{5}}{30}$

$(x^2 - x + \frac{1}{6})^2$

$\Rightarrow e_3 = \frac{30}{\sqrt{5}} (x^2 - x + \frac{1}{6})$

$\therefore e_1 = 1, \; e_2 = \sqrt{3}(2x-1), \; e_3 = \frac{30}{\sqrt{5}}(x^2 - x + \frac{1}{6})$

$\frac{30}{\sqrt{5}}(x^2 - x + \frac{1}{6}) = \frac{5}{\sqrt{5}}(6x^2 - 6x + 1)$

$= \sqrt{5}(6x^2 - 6x + 1)$

Orthonormal pf: $\langle e_1, e_2 \rangle = \int_0^1 \sqrt{3}(2x-1) \, dx = \int_0^1 \sqrt{3}(x^2 - x) \, dx$

$= \sqrt{3}(1-1) dx$

$= 0 \; \checkmark.$

$\langle e_1, e_3 \rangle = \int_0^1 \frac{30}{\sqrt{5}} (x^2 - x + \frac{1}{6}) \, dx$

$= \frac{30}{\sqrt{5}} \left(\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{6} \right)$

$= \frac{30}{\sqrt{5}} \left(-\frac{1}{6} + \frac{1}{6} \right) = 0 \; \checkmark.$

$\langle e_2, e_3 \rangle = \int_0^1 \frac{30}{\sqrt{5}} (2x-1)(x^2 - x + \frac{1}{6}) \, dx$

$= \int_0^1 \frac{30}{\sqrt{5}} \left(2x^3 - 2x^2 + \frac{1}{3}x - x^2 + x - \frac{1}{6} \right) dx$

$= \int_0^1 \frac{30}{\sqrt{5}} \left(2x^3 - 3x^2 + \frac{4}{3}x - \frac{1}{6} \right) dx$

$= \frac{30}{\sqrt{5}} \left(\frac{1}{2}x^4 - x^3 + \frac{2}{3}x^2 - \frac{1}{6}x \right) \Big|_0^1$

$= \frac{30}{\sqrt{5}} \left(\frac{1}{2} - 1 + \frac{2}{3} - \frac{1}{6} \right) = 0. \; \checkmark.$

$\therefore e_1, e_2, e_3$ are orthonormal to each other.

$\square$.

6B.7) Find polynomial $q \in \mathcal{P}_2(\mathbb{R})$ s.t

$$p(\tfrac{1}{2}) = \int_0^1 p(x)q(x)\,dx$$

for every $p \in \mathcal{P}_2(\mathbb{R})$.

From 6B.5, we established that

$$\langle p, q \rangle = \int_0^1 p(x)q(x)\,dx$$

$$p(\tfrac{1}{2}) = \langle p, q \rangle \text{ for all } p.$$

Let $\varphi \in \mathcal{L}(\mathcal{P}_2(\mathbb{R}), \mathbb{R})$ where

$$\varphi(p) = p(\tfrac{1}{2}) \text{ for all } p \in \mathcal{P}_2(\mathbb{R})$$

Then by Riesz representational thm, there exists a unique $q \in \mathcal{P}_2(\mathbb{R})$ s.t

$$\varphi(p) = \langle p, q \rangle$$

for all $p \in \mathcal{P}_2(\mathbb{R})$.

From 6.43, given orthonormal basis e_1, e_2, e_3,

$$q = \overline{\varphi(e_1)}e_1 + \dots + \overline{\varphi(e_3)}e_3$$

From 6B.5, we established that

$e_1 = 1$, $e_2 = \sqrt{3}(2x-1)$, $e_3 = \frac{30}{\sqrt{5}}(x^2 - x + \frac{1}{6})$

form an orthonormal basis.

$\varphi(e_1) = 1$. $\quad \varphi(e_2) = \sqrt{3}(0) = 0$ $\quad \varphi(e_3) = \frac{30}{\sqrt{5}}(\frac{1}{4} - \frac{1}{2} + \frac{1}{6}) = \frac{30}{\sqrt{5}}(-\frac{1}{12}) = -\frac{5}{2\sqrt{5}} = -\frac{1}{2}\sqrt{5}$

$$\Rightarrow q = 1 - \tfrac{1}{2}\cancel{\sqrt{5}}\left(\tfrac{30}{\cancel{\sqrt{5}}}(x^2 - x + \tfrac{1}{6})\right) = 1 - 15(x^2 - x + \tfrac{1}{6})$$

$$q = -15x^2 + 15x - \tfrac{3}{2}$$

Pf: $\langle ax^2 + bx + c, -15x^2 + 15x - \frac{3}{2} \rangle$

$= \int_0^1 (ax^2 + bx + c)(-15x^2 + 15x - \frac{3}{2})\,dx$

$= \int_0^1 -15ax^4 + 15ax^3 - \frac{3}{2}ax^2 - 15bx^3 + 15bx^2 - \frac{3}{2}bx - 15cx^2 + 15cx - \frac{3}{2}c\,dx$

$= \int_0^1 -15ax^4 + x^3(15a - 15b) + x^2(-\frac{3}{2}a + 15b - 15c) + x(-\frac{3}{2}b + 15c) - \frac{3}{2}c\,dx$

$= -3a + \frac{15a - 15b}{4} - \frac{1}{2}a + 5b - 5c - \frac{3}{4}b + \frac{15}{2}c - \frac{3}{2}c$

$= a(-3 + \frac{15}{4} - \frac{1}{2}) + b(-\frac{15}{4} + 5 - \frac{3}{4}) + c(-5 + \frac{15}{2} - \frac{3}{2})$

$= a(\frac{1}{4}) + b(\frac{1}{2}) + c$

$= a(\frac{1}{2})^2 + b(\frac{1}{2}) + c$

$\Rightarrow p = ax^2 + bx + c$, $p(\frac{1}{2})$

$\square$.

6B.8) $\int_0^1 p(x)(\cos\pi x)\,dx = \int_0^1 p(x)q(x)\,dx$
for every $p \in \mathcal{P}_2(\mathbb{R})$.

$\Rightarrow \int_0^1 p(x)(\cos\pi x)\,dx = \langle p(x), q(x) \rangle$.

Let $\varphi \in \mathcal{L}(\mathcal{P}_2(\mathbb{R}), \mathbb{R})$, where
$\varphi(p) = \int_0^1 p(x)(\cos\pi x)\,dx$

$$\begin{aligned}\varphi(p) &= \varphi(\langle p, e_1\rangle e_1 + \langle p, e_2\rangle e_2 + \langle p, e_3\rangle e_3) \\ &= \langle p, e_1\rangle \varphi(e_1) + \langle p, e_2\rangle \varphi(e_2) + \langle p, e_3\rangle \varphi(e_3) \\ &= \langle p, \overline{\varphi(e_1)} e_1\rangle + \langle p, \overline{\varphi(e_2)} e_2\rangle + \langle p, \overline{\varphi(e_3)} e_3\rangle \\ &= \langle p, \overline{\varphi(e_1)} e_1 + \overline{\varphi(e_2)} e_2 + \overline{\varphi(e_3)} e_3\rangle\end{aligned}$$

$\Rightarrow q = \overline{\varphi(e_1)} e_1 + \overline{\varphi(e_2)} e_2 + \overline{\varphi(e_3)} e_3$

From 6B.5, $e_1 = 1$, $e_2 = \sqrt{3}(2x-1)$, $e_3 = \frac{30}{\sqrt{5}}(x^2 - x + \frac{1}{6})$

$$\begin{aligned}\varphi(e_1) &= \int_0^1 \cos\pi x\,dx \\ &= \frac{1}{\pi}\sin\pi x \Big|_0^1 \\ &= 0\end{aligned}$$

$\varphi(e_2) = \sqrt{3}\int_0^1 (2x-1)(\cos\pi x)\,dx$
By parts: Let $u = 2x-1$ $\quad dv = \cos\pi x\,dx$
$du = 2dx$ $\quad v = \frac{1}{\pi}\sin\pi x$

$$\begin{aligned}\sqrt{3}\int_0^1 (2x-1)(\cos\pi x)\,dx &= \sqrt{3}\left[(2x-1)\left(\frac{1}{\pi}\sin\pi x\right)\right]_0^1 - \sqrt{3}\int_0^1 \frac{1}{\pi}\sin\pi x \cdot 2dx \\ &= -\frac{2\sqrt{3}}{\pi}\int_0^1 \sin\pi x\,dx \\ &= \frac{2\sqrt{3}}{\pi^2}\cos\pi x\Big|_0^1 = \frac{2\sqrt{3}}{\pi^2}[-1-1] = -\frac{4\sqrt{3}}{\pi^2}\end{aligned}$$

$\varphi(e_3) = \frac{30}{\sqrt{5}}\int_0^1 (x^2 - x + \frac{1}{6})\cos\pi x\,dx$

Let $u = x^2 - x + \frac{1}{6}$ $\quad dv = \cos\pi x\,dx$
$du = 2x - 1dx$ $\quad v = \frac{1}{\pi}\sin\pi x$

$$\begin{aligned}\frac{30}{\sqrt{5}}\int_0^1 (x^2 - x + \tfrac{1}{6})\cos\pi x\,dx &= \frac{30}{\sqrt{5}\pi}(x^2 - x + \tfrac{1}{6})\sin\pi x\Big|_0^1 - \frac{30}{\pi\sqrt{5}}\int_0^1 (2x-1)\sin\pi x\,dx \\ &= -\frac{30}{\pi\sqrt{5}}\int_0^1 (2x-1)\sin\pi x\,dx\end{aligned}$$

By parts
Let $u = 2x-1$ $\quad dv = \sin\pi x\,dx$
$du = 2dx$ $\quad v = -\frac{1}{\pi}\cos\pi x$

$$\begin{aligned}&\frac{30}{\pi^2\sqrt{5}}(\cos\pi x)(2x-1)\Big|_0^1 - \frac{60}{\pi^2\sqrt{5}}\int_0^1 \cos\pi x\,dx \\ &= \frac{30}{\pi^2\sqrt{5}}[-(2-1)-(-1)] - \frac{60}{\pi^3\sqrt{5}}\sin\pi x\Big|_0^1 \\ &= \frac{30}{\pi^2\sqrt{5}}[-1+1] = 0\end{aligned}$$

$\Rightarrow \varphi(e_1) = 0$, $\varphi(e_2) = \frac{4\sqrt{3}}{\pi^2}$, $\varphi(e_3) = 0$
$\Rightarrow q = -\frac{4\sqrt{3}}{\pi^2}(\sqrt{3}(2x-1)) = -\frac{12}{\pi^2}(2x-1)$

Verify: $\langle ax^2+bx+c, -\frac{12}{\pi^2}(2x-1)\rangle = -\frac{12}{\pi^2}\int_0^1 (2x-1)(ax^2+bx+c)\,dx$

$$= -\frac{12}{\pi^2}\int_0^1 2ax^3+2bx^2+2cx-ax^2-bx-c\,dx$$

$$= -\frac{12}{\pi^2}\int_0^1 2ax^3+x^2(2b-a)+x(2c-b)-c\,dx$$

$$= -\frac{12}{\pi^2}\left(\frac{1}{2}a+\frac{2b-a}{3}+\frac{2c-b}{2}-c\right)$$

$$= -\frac{12}{\pi^2}\left(a\left(\frac{1}{2}-\frac{1}{3}\right)+b\left(\frac{2}{3}-\frac{1}{2}\right)\right)$$

$$= -\frac{12}{\pi^2}\left(\frac{1}{6}a+\frac{1}{6}b\right) = -\frac{2}{\pi^2}(a+b)$$

$$\int_0^1 (ax^2+bx+c)(\cos\pi x)\,dx$$

By parts: $u = ax^2+bx+c$ $\quad dv = \cos\pi x\,dx$

$du = 2ax+b\,dx$ $\quad v = \frac{1}{\pi}\sin\pi x$

$$(ax^2+bx+c)\left(\frac{1}{\pi}\sin\pi x\right)\Big|_0^1 - \int_0^1 \left(\frac{1}{\pi}\sin\pi x\right)(2ax+b)\,dx$$

$$= -\int_0^1 \left(\frac{1}{\pi}\sin\pi x\right)(2ax+b)\,dx$$

By parts: $u = 2ax+b$ $\quad dv = \sin\pi x\,dx$

$du = 2a$ $\quad v = -\frac{1}{\pi}\cos\pi x$

$$\frac{1}{\pi^2}\cos\pi x\,(2ax+b)\Big|_0^1 - \int_0^1 \frac{1}{\pi}\cos\pi x\cdot 2a\,dx$$

$$= \frac{1}{\pi^2}(2a+b)\cos\pi - \frac{1}{\pi^2}b\cos 0 + \frac{2a}{\pi^2}\sin\pi x\Big|_0^1$$

$$= -\frac{2a+b}{\pi^2} - \frac{1}{\pi^2}b = \frac{1}{\pi^2}(-2a-2b)$$

$$= -\frac{2}{\pi^2}(a+b)$$

$$= \langle ax^2+bx+c, -\frac{12}{\pi^2}(2x-1)\rangle.$$

$\square$.

17) a) Pf: Let $\lambda \in \mathbb{C}$ be an eigenvalue of T, with a corresponding eigenvector $v \in V$.

Since $V = U \oplus W$, $v = u + w$ where $u \in U, w \in W$.

$$Tv = T(u+w) = \lambda(u+w)$$
$$Tu + Tw = \lambda u + \lambda w$$

Since U and W are invariant under T,
$Tu \in U$ and $Tw \in W$.
Hence, for $Tu + Tw = \lambda u + \lambda w$,
$Tu = \lambda u$ and $Tw = \lambda w$.

If $u \neq 0$ and $w \neq 0$, then $v = u+w$ is an eigenvector of T.
Thus λ is an eigenvector of $T|_U$ since $Tu = \lambda u$
and λ is an eigenvector of $T|_W$, since $Tw = \lambda w$.

If $u \neq 0$, and $w = 0$, then $v = u \Rightarrow Tv = Tu = \lambda u$
$\Rightarrow \lambda$ is an eigenvalue of $T|_U$.

If $u = 0$ and $w \neq 0$, then $v = w \Rightarrow Tv = Tw = \lambda w$
$\Rightarrow \lambda$ is an eigenvalue of $T|_W$.

Hence λ is an eigenvalue of $T|_U$, $T|_W$, or both.

$\square$.

b) $f(T|_U) = 0$ in $\mathcal{L}(U)$
$g(T|_W) = 0$ in $\mathcal{L}(W)$.

Let $v = u + w$, $u \in U, w \in W$. since $V = U \oplus W$.

$$\begin{aligned} &f(T)g(T)v \\ &= f(T)g(T)(u+w) \\ &= f(T)g(T)u + f(T)g(T)w \\ &= f(Tu)g(Tu) + f(Tw)g(Tw) \\ &= 0 \cdot g(Tu) + f(Tw) \cdot 0 \end{aligned}$$

since $Tu \in U \Rightarrow f(Tu) = 0$
$Tw \in W \Rightarrow g(Tw) = 0$ as U, W are invariant under T.

Thus $f(T)g(T)v = 0$ for $v = u+w$, $u \in U, w \in W$, $u \neq 0, w \neq 0$.

For $v = u$,
$$\begin{aligned} f(T)g(T)v &= f(T)g(T)u \\ &= f(Tu)g(Tu) \\ &= 0 \end{aligned}$$

For $v = w$,
$$\begin{aligned} f(T)g(T)v &= f(T)g(T)w \\ &= f(Tw)g(Tw) \\ &= 0. \end{aligned}$$

$\therefore f(T)g(T) = 0$ in $\mathcal{L}(V)$.

$\square$.

c) Prove that if $f(x), g(x)$ has no shared roots, then $f(x)g(x)$ is a minimal polynomial of T.

<u>Pf</u>: Suppose $f(x), g(x)$ have no shared roots,

Let $f(x) = (x-\lambda_1)\dots(x-\lambda_n)$
where $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ are the roots of $f(x)$.
Let $g(x) = (x-\sigma_1)\dots(x-\sigma_m)$
where $\sigma_1, \dots, \sigma_m \in \mathbb{C}$ are the roots of $g(x)$.

Since $f(x)$ has $\lambda_1, \dots, \lambda_n$ roots, there must be $u_1, \dots, u_n$ corresponding eigenvectors.
Similarly, $g(x)$ has $\sigma_1, \dots, \sigma_m$ roots, thus $w_1, \dots, w_m$ corresponding eigenvectors.

Let $p(x)$ be the minimal polynomial of T.
We will now prove that $p(x) = g(x)f(x)$.

From 8.46, polynomials $q(T)=0 \Rightarrow q(T)$ is a multiple of the minimal polynomial
Hence $f(x)g(x)$ must be a polynomial multiple of $p(x)$.

$p(x)d(x) = f(x)g(x)$ where $d \in P(\mathbb{C})$.

Hence $\dfrac{f(x)g(x)}{d(x)} = p(x)$

$\Rightarrow d(x)$ is a factor of $f(x)g(x)$ such that

$$\frac{f(T)g(T)}{d(T)} = 0$$

<u>Lemma</u>: $d(x)$ cannot be a non-constant polynomial factor of $f(x)$ or $g(x)$.
<u>Pf</u>: We prove the case for $f(x)$. $g(x)$ trivially follows.
By contradiction,
assume $d(x)$ is a non-constant polynomial factor of $f(x)$.
Let $d(x) = x - \lambda_k$, $1 \le k \le n$.

WLOG, let $k=1$.
$d(x) = x - \lambda_1$.

$$p(x) = \frac{f(x)g(x)}{x-\lambda_1} = (x-\lambda_2)\dots(x-\lambda_n)(x-\sigma_1)\dots(x-\sigma_m)$$

$$\begin{aligned} p(T)u_1 &= (T-\lambda_2 I)\dots(T-\lambda_n I)(T-\sigma_1 I)\dots(T-\sigma_m I)u_1 \\ &= (\lambda_1 u_1 - \lambda_2 u_1)\dots(\lambda_1 u_1 - \lambda_n u_1)(\lambda_1 u_1 - \sigma_1 u_1)\dots(\lambda_1 u_1 - \sigma_m u_1) \\ &= (\lambda_1-\lambda_2)\dots(\lambda_1-\lambda_n)(\lambda_1-\sigma_1)\dots(\lambda_1-\sigma_m)u_1 \\ &\neq 0 \end{aligned}$$

since all eigenvalues $\lambda_1, \dots, \lambda_n, \sigma_1, \dots, \sigma_m$ are unique.

Hence $d(x)$ cannot be a non-constant polynomial factor of $f(x)$.

This applies similarly to $g(x)$. Hence $d(x)$ cannot be a non constant polynomial of $f(x)$ or $g(x)$.
$\square$.

From lemma 1, $d(x)$ can only hence be a constant multiple of $f(x), g(x)$.
Since $f(x), g(x)$ are minimal polynomials,
$f(x), g(x)$ are monic,
thus $d(x) = 1$.

$$\therefore \quad p(x) = \frac{f(x)g(x)}{d(x)}$$

$$p(x) = f(x)g(x)$$

$\Rightarrow f(x)g(x)$ is the minimal polynomial of T. $\blacksquare$

d) Suppose $f(x), g(x)$ have shared roots.

Let $f(x) = (x-\lambda_1)\dots(x-\lambda_n)$
where $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ are the roots of $f(x)$.
Let $g(x) = (x-\sigma_1)\dots(x-\sigma_m)$
where $\sigma_1, \dots, \sigma_m \in \mathbb{C}$ are the roots of $g(x)$.
Let $\lambda_1 = \sigma_1$.

Since $f(x)$ has $\lambda_1, \dots, \lambda_n$ roots, there must be
$u_1, \dots, u_n$ corresponding eigenvectors.
Similarly, $g(x)$ has $\sigma_1, \dots, \sigma_m$ roots, thus
$w_1, \dots, w_m$ corresponding eigenvectors.

Since $\lambda_1 = \sigma_1$,

$$f(x)g(x) = (x-\lambda_1)\dots(x-\lambda_n)(x-\sigma_1)\dots(x-\sigma_m)$$
$$= (x-\lambda_1)^2\dots(x-\lambda_n)(x-\sigma_2)\dots(x-\sigma_m)$$

$$f(x)g(x)/(x-\lambda_1) = (x-\lambda_1)\dots(x-\lambda_n)(x-\sigma_2)\dots(x-\sigma_m).$$

<u>Lemma 2:</u> $f(T)g(T)/(T-\lambda_1) = 0$ in $\mathcal{L}(V)$.
<u>Pf:</u> $V = U \oplus W$. U, W are invariant under T.

Let $u \in U$

$$[f(T)g(T)/(T-\lambda_1)]u = [(T-\lambda_1)\dots(T-\lambda_n)(T-\sigma_2)\dots(T-\sigma_m)]u$$
$$= [f(T)(T-\sigma_2)\dots(T-\sigma_m)]u$$
$$= f(Tu)(Tu-\sigma_2 u)\dots(Tu-\sigma_m u)$$
$$= 0 \text{ since } Tu \in U \Rightarrow f(Tu) = 0.$$

Let $w \in W$

$$[f(T)g(T)/(T-\lambda_1)]w = [(T-\lambda_1)\dots(T-\lambda_n)(T-\sigma_2)\dots(T-\sigma_m)]w$$
$$= [(T-\sigma_1)(T-\lambda_2)\dots(T-\lambda_n)(T-\sigma_2)\dots(T-\sigma_m)]w$$
$$= [(T-\lambda_2)\dots(T-\lambda_n)g(T)]w$$
$$= 0 \text{ since } Tw \in W \Rightarrow g(Tw) = 0.$$

Let $v = au + bw$.

$$h(T) = [f(T)g(T)/(T-\lambda_1)]$$
$$h(T)v = h(Tv)$$
$$= h(Tau + Tbw)$$
$$= h(Tau) + h(Tbw)$$
$$= ah(Tu) + bh(Tw)$$
$$= 0.$$

Hence $f(T)g(T)/(T-\lambda_1) = 0$

$\square$.

Thus there exists a polynomial $q(x) = f(x)g(x)/(x-\lambda_1)$ with smaller degree than $f(x)g(x)$ such that $q(T) = 0$

$\therefore$ $f(x)g(x)$ is not the minimal polynomial.

$\square$.

2) Let λ be an eigenvalue of T.
Let v be the corresponding eigenvector.

$$\|Tv\| = \|\lambda v\| = \|v\|$$
$$\Rightarrow \sqrt{\langle \lambda v, \lambda v\rangle}$$
$$= \sqrt{\lambda^2 \langle v, v\rangle}$$
$$\Rightarrow \lambda = 1, -1.$$

$\therefore$ If T were to have eigenvalues, the only possible values are $-1, 1$.
Hence T can have at most 2 eigenvalues.

$\square$.

3) $V = \mathbb{C}^n$ $\quad R: V \to V$ be defined as
$$R(a_1, \dots, a_n) = (a_2, \dots, a_n, a_1)$$
Given $p(x) = x^n - 1$, prove that $p(R) = 0$.

$$p(R) = R^n - I$$

$$R(a_1, \dots, a_n) = (a_2, \dots, a_n, a_1)$$
$$R^2(a_1, \dots, a_n) = (a_3, \dots, a_n, a_1, a_2)$$
$$R^k($$

Lemma 1: $R^k(a_1, \dots, a_n) = (a_{1+k}, \dots, a_n, a_1, \dots, a_k)$ for $1 \leq k < n-1$

Pf: Base case: $k=1$. $\quad R^k(a_1, \dots, a_n) = (a_2, \dots, a_n, a_1)$ ✓.
Assuming lemma is true for $k = i$, $1 < i < n-2$,

Let $k = i+1$, $R^{i+1}(a_1, \dots, a_n) = R(R^i(a_1, \dots, a_n))$
$$= R(a_{1+i}, \dots, a_n, a_1, \dots, a_i)$$
$$= (a_{i+2}, \dots, a_n, a_1, \dots, a_{i+1})$$
$$= (a_{1+(i+1)}, \dots, a_n, a_1, \dots, a_{(i+1)})$$
$\Rightarrow$ lemma is true for $k = i+1$.

$\Rightarrow$ lemma is true for $1 \leq k < n-1$.

$\square$

Lemma 2: Let $v = (a_1, \dots, a_n)$. $R^n v = v$.

Pf: From lemma 1, $R^{n-1}v = (a_n, a_1, \dots, a_{n-1})$
$$R(R^{n-1}v) = R(a_n, a_1, \dots, a_{n-1})$$
$$= (a_1, \dots, a_{n-1})$$
$$= v$$
$$\Rightarrow R^n v = v.$$

$\square$.

Since $R^n v = v$, $R^n = I$.
$$\therefore\ p(R) = R^n - I$$
$$= I - I$$
$$= 0$$

$\blacksquare$.

c) $\omega = \cos(\frac{2\pi}{n}) + i\sin(\frac{2\pi}{n})$.

Since $x^n - 1$ is a minimal polynomial, then all the roots $1, \omega, \omega^2, \dots, \omega^{n-1}$ must be eigenvalues for R.

$$R\vec{v} = \vec{v} \Rightarrow \vec{v} = (1, 1, \dots, 1)$$
$$\|\vec{v}\| = \sqrt{1+1+\dots+1} = \sqrt{n}$$
$$\Rightarrow \vec{v}_1 = \frac{1}{\sqrt{n}}(1, \dots, 1).$$

$$R(1, \omega, \omega^2, \dots, \omega^{n-1}) = (\omega, \omega^2, \dots, \omega^{n-1}, 1)$$
$$= \omega(1, \omega, \omega^2, \dots, \omega^{n-1})$$
$\Rightarrow (1, \omega, \omega^2, \dots, \omega^{n-1})$ is an eigenvector with eigenvalue ω

$$\|(1, \omega, \omega^2, \dots, \omega^{n-1})\| = \sqrt{n}$$
$$\Rightarrow \vec{v}_2 = \frac{1}{\sqrt{n}}(1, \omega, \omega^2, \dots, \omega^{n-1})$$

$$R(1, \omega^2, \dots, \omega^{2(n-1)}) = (\omega^2, \omega^4, \dots, \omega^{2(n-1)}, 1)$$
$$= \omega^2(1, \omega^2, \dots, \omega^{2(n-1)}, 1)$$

<u>Lemma 3</u>: For $R(\vec{v}_k) = \omega^k \cdot \vec{v}_k$, $\vec{v}_k = (1, \omega^k, \omega^{2k}, \dots, \omega^{k(n-1)})$ for $0 \leq k \leq n-1$.

<u>Pf</u>: By induction,

<u>Base case</u>: $k = 0$.
$$R(1, 1, \dots, 1) = (1, 1, \dots, 1)$$
$$= 1 \cdot \vec{v}_k \checkmark$$

<u>Inductive step</u>: Assume lemma 3 is true for some i, $0 \leq i \leq k$, then for $k = i+1$,

$$R(\vec{v}_{k+1}) = R(1, \omega^{k+1}, \omega^{2(k+1)}, \dots, \omega^{(k+1)(n-1)})$$
$$= (\omega^{k+1}, \omega^{2(k+1)}, \dots, \omega^{(k+1)(n-1)}, 1)$$
$$= \omega^{k+1}(1, \omega^{(k+1)}, \dots, \omega^{(k+1)(n-2)}, \omega^{(k+1)(n-1)})$$
$\Rightarrow$ lemma 3 is true.

$\therefore \vec{v}_k = (1, \omega^k, \omega^{2k}, \dots, \omega^{k(n-1)})$ for $0 \leq k \leq n-1$ form a basis for $\mathbb{C}^n$.
Since $0 \leq k \leq n-1$ and $\vec{v}_k$ are all linearly independent

$\square$.

d) <u>Proposition</u>: If $\mu \in \mathbb{C}$ satisfies $\mu^n = 1$ but $\mu \neq 1$, then
$$1 + \mu + \mu^2 + \dots + \mu^{n-1} = 0.$$

<u>Pf</u>: $(1+\mu+\mu^2+\dots+\mu^{n-1})(\mu-1)$
$$= \mu + \mu^2 + \dots + \mu^n - (1+\mu+\mu^2+\dots+\mu^{n-1})$$
$$= \mu + \mu^2 + \dots + \mu^{n-1} + 1 - (1+\mu+\mu^2+\dots+\mu^{n-1})$$
$$= 0.$$

Since $\mu \neq 1$, then $\mu - 1 \neq 0$. Thus for
$$(1+\mu+\mu^2+\dots+\mu^{n-1})(\mu-1) = 0,$$
$$1+\mu+\mu^2+\dots+\mu^{n-1} = 0$$
$\square$.

e) For my basis to be orthonormal,
$$\langle v_i, v_j \rangle = 0$$

Let $v_i = (1, \omega^i, \omega^{2i}, \dots, \omega^{(n-1)i})$
$v_j = (1, \omega^j, \omega^{2j}, \dots, \omega^{(n-1)j})$

$$\langle v_i, v_j \rangle = 1 + \omega^i \overline{\omega^j} + \dots + \omega^{(n-1)i}\, \overline{\omega^{(n-1)j}}$$
$$1 + 1 + \dots + 1$$

Suppose $i > j$,
$$\langle v_i, v_j \rangle = 1 + \omega^{i-j}\, \omega^j \overline{\omega^j} + \dots + \omega^{(n-1)i-(n-1)j}\, \omega^{(n-1)j}\, \overline{\omega^{(n-1)j}}$$
Since ω is a root of 1,
$$\|\omega^k\| = 1.$$
$$\therefore \langle v_i, v_j \rangle = 1 + \omega^{i-j} + \dots + \omega^{(n-1)i-(n-1)j}$$
$$= 1 + \omega^{i-j} + \omega^{2(i-j)} + \dots + \omega^{(n-1)(i-j)}$$

$$\omega^{n(i-j)} = (\omega^n)^{(i-j)} = 1$$
but $\omega^{i-j} \neq 1$

Hence by (d),
$$1 + \omega^{i-j} + \omega^{2(i-j)} + \dots + \omega^{(n-1)(i-j)} = 0$$
$\Rightarrow \langle v_i, v_j \rangle = 0$ if $i > j$.

Same argument applies to if $i < j$, hence all vectors $v_1, \dots, v_n$ are orthonormal.

$\square$.

f) Since $v_1, \dots, v_n$ are orthonormal, then

For some orthonormal vector v_k,

$$\langle v, v_k \rangle = \langle b_1 v_1 + \dots + b_k v_k + \dots + b_n v_n, v_k \rangle \frac{1}{\sqrt{n}}$$
$$= \frac{1}{\sqrt{n}} \langle b_1 v_1, v_k \rangle + \dots + \langle b_k v_k, v_k \rangle + \dots + \langle b_n v_n, v_k \rangle$$
$$= \frac{1}{\sqrt{n}} b_k$$

$$\Rightarrow b_i = \langle v, v_i \rangle$$
$$= \langle (a_1, \dots, a_n), (1, \omega^i, \dots, \omega^{i(n-1)}) \rangle$$
$$= a_1 + a_2 \overline{\omega^i} + a_3 \overline{\omega^{2i}} + \dots + a_k \overline{\omega^{(k-1)i}} + \dots + a_n \overline{\omega^{(n-1)i}} \quad \square. \quad \frac{1}{\sqrt{n}}$$

g) Let the standard basis be $e_1, \dots, e_n$.

$$v = a_1 e_1 + \dots + a_n e_n.$$

$$a_i = \langle v, e_i \rangle$$
$$= \langle b_1 v_1 + \dots + b_n v_n, e_i \rangle$$
$$= b_1 \langle v_1, e_i \rangle + \dots + b_n \langle v_n, e_i \rangle$$
$$= b_1 + b_2 \omega^i + b_3 \omega^{2i} + \dots + b_n \omega^{i(n-1)} \quad // \quad \frac{1}{\sqrt{n}}$$

h)
$$|a_1|^2 + \dots + |a_n|^2 = a_1 \overline{a_1} + \dots + a_n \overline{a_n}$$
$$= \langle v, v \rangle$$
$$= \langle b_1 v_1 + \dots + b_n v_n, v \rangle$$
$$= b_1 \langle v_1, v \rangle + \dots + b_n \langle v_n, v \rangle$$

$$\langle v_1, v \rangle$$
$$= \langle v_1, b_1 v_1 + \dots + b_n v_n \rangle$$
$$= \overline{\langle b_1 v_1 + \dots + b_n v_n, v_1 \rangle} = \overline{b_1} \overline{\langle v_1, v_1 \rangle} + \dots + \overline{b_n} \overline{\langle v_n, v_1 \rangle}$$
$$= \overline{b_1}$$

$$\therefore b_1 \langle v_1, v \rangle + \dots + b_n \langle v_n, v \rangle$$
$$= b_1 \overline{b_1} + \dots + b_n \overline{b_n}$$
$$= |b_1|^2 + \dots + |b_n|^2$$
$$\Rightarrow |a_1|^2 + \dots + |a_n|^2 = |b_1|^2 + \dots + |b_n|^2$$

$\square$.