6. Let $X_n \sim \text{Bin}(n, p_n)$ for all $n \geq 1$, where np_n is a constant $\lambda > 0$ for all n (so $p_n = \lambda/n$). Let $X \sim \text{Pois}(\lambda)$. Show that the MGF of X_n converges to the MGF of X (this gives another way to see that the $\text{Bin}(n,p)$ distribution can be well-approximated by the $\text{Pois}(\lambda)$ when n is large, p is small, and $\lambda = np$ is moderate).

$$P(X_n = k) = \binom{n}{k} p_n^k (1-p_n)^{n-k}$$

$$\lim_{n\to\infty} np$$

$$P(X=k) = e^{-np}\frac{(np)^k}{k!}$$

$$M_X(t) = E(e^{Xt})$$

$$np = \lambda$$
$$p = \frac{\lambda}{n}$$

$$E(e^{\text{Bern}(p_n)t}) = p_n \cdot e^t + (1-p_n)$$

$$M_{X_n}(t) = (p_n e^t + 1 - p_n)^n$$

$$e^{-\lambda}\frac{\lambda^k}{k!}$$

$$\lim_{n\to\infty} M_{X_n}(t) = \left(\frac{\lambda}{n}e^t + 1 - \frac{\lambda}{n}\right)^n$$
$$= \left(\frac{\lambda}{n}(e^t - 1) + 1\right)^n$$
$$= \left(\frac{\lambda(e^t-1)}{n} + 1\right)^n$$
$$= e^{\lambda(e^t-1)}$$
$$= M_X(t)$$

$$E(e^{tX})$$
$$= \sum e^{tx} e^{-\lambda}\frac{\lambda^x}{x!}$$
$$= e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^t)^x}{x!}$$
$$= e^{-\lambda}\cdot e^{\lambda e^t}$$
$$= e^{\lambda(e^t-1)}$$