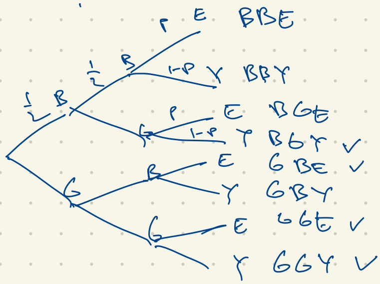
1) Sample Space: (elder sex, younger sex, age open)

a) $T = \{(G,G,E),(G,G,Y)\}$

$O = \{(G,B,E),(G,G,E),(B,G,Y),(G,G,Y)\}$ ✓

b) $Pr(T \mid O) = \dfrac{Pr(T \cap O)}{Pr(O)}$



$T \cap O = T \Rightarrow Pr(T \mid O) = \dfrac{Pr(T)}{Pr(O)}$

$Pr(T) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$.

Let probability that the elder opens the door be p.

Then $Pr[\{(B,G,Y),(G,B,E)\}]$
$= \frac{1}{2}\cdot\frac{1}{2}\cdot(1-p) + \frac{1}{2}\cdot\frac{1}{2}\cdot p$
$= \frac{1}{2}\cdot\frac{1}{2}(1-p+p)$
$= \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$

$\Rightarrow Pr[O] = \frac{1}{4}+\frac{1}{4} = \frac{1}{2}$.
$\Rightarrow Pr[T \mid O] = \frac{1}{4} \div \frac{1}{2} = \frac{1}{4}\cdot 2 = \frac{1}{2}$ ✓

$\boxed{\text{Ans: } \frac{1}{2}}$

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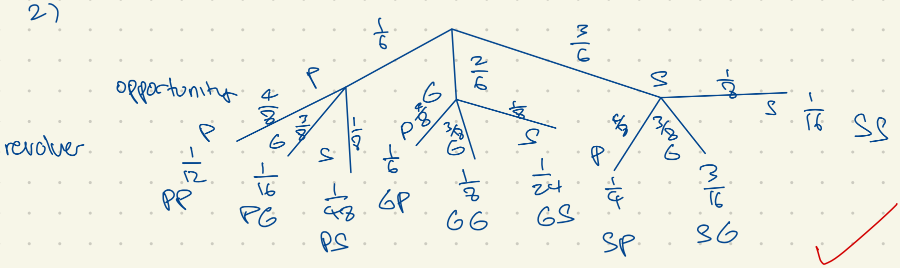
c) Mistake is that $Pr(T \mid \text{at least one girl in household}) \neq Pr(T \mid O)$ since there can be a girl in household but boy opens the door. ✓

//

2)



✓

b) $E = \{(P,P),(G,G),(S,S)\}$.
$Pr[E] = \frac{1}{12}+\frac{1}{8}+\frac{1}{16}$
$= \frac{13}{48}$ // ✓

c) $Pr[\{(G,G),(P,P)\}] = \frac{1}{12}+\frac{1}{8} = \frac{5}{24}$ // ✗

$Pr[\text{shot} \mid \text{S no revolver}]$
$= \dfrac{\frac{5}{24}}{\cdots} = \frac{5}{21}$.

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d) $Pr[(G,G) \mid E] = \dfrac{Pr[(G,G)]}{Pr[E]}$

$= \frac{1}{8} \div \frac{13}{48} = \frac{6}{13}$ // ✓

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3) a) Probability that m has all different birthdays $= \left(N \cdot (N-1)(N-2)\ldots(N-m+1)\right) \div N^m$

$$= \frac{N!}{(N-m)!\,N^m}$$

Probability that k-m does not have same birthday as m $= \left(1-\frac{m}{N}\right)^{k-m}$

Probability that m has all diff bdays an N-m has no same bday $\doteq \frac{N!}{m!\,N^m} \cdot \left(1-\frac{m}{N}\right)^{k-m}$

$$= e^{-\frac{m^2}{2N}} \cdot \left(e^{-\frac{m}{N}}\right)^{k-m}$$

$$= e^{-\frac{m^2}{2N}} \cdot e^{\frac{-m\cdot(k-m)}{N}}$$

$$= e^{-\frac{m^2}{2N} + \frac{2m^2-2mk}{2N}} = e^{\frac{1}{2N}(m^2-2mk)} = e^{\frac{m(m-2k)}{2N}}$$ //

b) For half chance of match, $1-Pr[A] = \frac{1}{2}$.

$$Pr[A] = \frac{1}{2}$$

$$\Rightarrow e^{\frac{m(m-2k)}{2N}} = \frac{1}{2}$$

$$\frac{m(m-2k)}{2N} = \ln\left(\frac{1}{2}\right)$$

$$m(m-2k) = 2N\ln\left(\frac{1}{2}\right)$$
$$m^2-2km-2N\ln\left(\frac{1}{2}\right) = 0$$
$$m^2-2km+2N\ln 2 = 0$$

$$m = \frac{2k \pm \sqrt{4k^2-4(2N\ln 2)}}{2}$$
$$= k \pm \sqrt{k^2-2N\ln 2}$$

Since $m \leq k$, $m = k-\sqrt{k^2-2N\ln 2}$ for $Pr\{A\} = \frac{1}{2}$.

$$\frac{m}{k} = 1-\sqrt{1-\frac{2N\ln 2}{k^2}}$$

$$\approx 1-\left(1-\frac{N\ln 2}{k^2}\right)$$

$$\frac{m}{k} = \frac{N\ln 2}{k^2} \Rightarrow m = \frac{N\ln 2}{k}$$

$$\therefore m \sim \frac{N\ln 2}{k}$$ //

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1. Flip the coin twice.

2. Based on the results:

   - $TH \Rightarrow$ you win $[W]$, and the game terminates.
   - $HT \Rightarrow$ Professor Leighton wins $[L]$, and the game terminates.
   - $(HH \vee TT) \Rightarrow$ discard the result and flip again.

3. If at the end of N rounds nobody has won, declare a tie.

As an example, for $N = 3$, an outcome of HT would mean the game ends early and you lose, $HHTH$ would mean the game ends early and you win, and $HHTTTT$ would mean you play the full N rounds and result in a tie.

(a) [5 pts] Assume the flips are mutually independent. Show that $\Pr\{W\} = \Pr\{L\}$.

(b) [5 pts] Show that, if $p < 1$, the probability of a tie goes to 0 as N goes to infinity.

Let probability of heads be p.

$\Pr[(TH)] = p\cdot(1-p)$ $\quad \Pr[(HT)] = p(1-p)$.

$\Pr[(TT)] = (1-p)^2$ $\quad \Pr[(HH)] = p^2$.

$\Pr[\{(TT),(HH)\}] = p^2 + (1-p)^2 = p^2+1-2p+p^2 = 2p^2-2p+1$.

Win in N $= \Pr[\{(TH), (Tie, TH), (Tie, Tie, TH), \ldots (Tie, Tie, \ldots, Tie, TH)\}]$

$= p(1-p) + (2p^2-2p+1)p(1-p) + (2p^2-2p+1)^2 p(1-p) + \ldots + (2p^2-2p+1)^{N-1} p(1-p)$

$= p(1-p)[1 + 2p^2-2p+1 + (2p^2-2p+1)^2 + \ldots + (2p^2-2p+1)^{N-1}]$

$= p(1-p)\left[\dfrac{1-(2p^2-2p+1)^N}{1-2p^2+2p-1}\right] = p(1-p)\left[\dfrac{1-(2p^2-2p+1)^N}{2p-2p^2}\right]$

Lose in N $= p(1-p)[\ldots] =$ Win in N

$\therefore \Pr[W] = \Pr[L]$ //

Tie $= 1 - 2p(1-p)\left[\dfrac{1-(2p^2-2p+1)^N}{2p-2p^2}\right]$

$= 1 - p(1-p)\left[\dfrac{1-(2p^2-2p+1)^N}{p(1-p)}\right]$

$= 1 - [1-(2p^2-2p+1)^N]$

$= (2p^2-2p+1)^N$

At $p=1$, $2p^2-2p+1 = 1$.

At $p=0$, $2p^2-2p+1 = 1$

$(2p^2-2p+1)\frac{d}{dp} = 4p-2$.

$4p-2=0$
$4p=2$
$p=\frac{1}{2}$.

At $p=\frac{1}{2}$, $2(0.5)^2-2(0.5)+1 = 0.5$.

$\Rightarrow$ between $0<p<1$, $\frac{1}{2} < 2p^2-2p+1 < 1$

$\therefore \lim_{N\to\infty}(2p^2-2p+1)^N = 0$ for $0<p<1$

$\Rightarrow \lim_{N\to\infty}\Pr[Tie] = 0$.

D.

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Problem 5. [20 points]

(a) [5 pts] Suppose A and B are *disjoint* events. Prove that A and B are *not independent*, unless $\Pr(A)$ or $\Pr(B)$ is zero.

(b) [5 pts] If A and B are independent, prove that A and $\bar{B}$ are also independent.
Hint: $\Pr\left(A \cap \bar{B}\right) = \Pr(A) - \Pr(A \cap B)$.

(c) [5 pts] Give an example of events A,B, C such that A is independent of B, A is independent of C, but A is not independent of $B \cup C$.

(d) [5 pts] Prove that if C is independent of A, and C is independent of B, and C is independent of $A \cap B$, then C is independent of $A \cup B$.
Hint: Calculate $\Pr(A \cup B \mid C)$.

a) Disjoint $\Rightarrow$ $A \cap B = \{0\}$.
Independence $\Leftrightarrow$ $\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$.
$\Rightarrow$ $\Pr[B|A] = \Pr[B]$.

Since they are disjoint, $\Pr[A \cap B] = 0$
$\Rightarrow$ doesn't satisfy the equivalence condition for independence. since $\Pr[A] \cdot \Pr[B] \neq 0$.
$\square$.

Alternatively, $\Pr[B|A] = 0 \neq \Pr[B] \Rightarrow$ not independent
$\square$.

b) Suppose A and B are independent, then for A and $\bar{B}$ to be independent,

$\Pr[A \cap B] = \Pr[A]\Pr[B]$.
$\Pr[A \cap \bar{B}] = \Pr[A] - \Pr[A \cap B]$
$= \Pr[A] - \Pr[A]\Pr[B]$
$= \Pr[A](1 - \Pr[B])$
$= \Pr[A]\Pr[\bar{B}]$
$\Leftrightarrow$ A and $\bar{B}$ are independent. $\square$.

c) A independent of B. A independent of C.
A not independent of $B \cup C$ $\Leftrightarrow$ A not independent of $B \cap C$.

Let A be the presence of NaCl in a solution.
Let B be the presence of HCl in the solution.
Let C be the presence of NaOH in the solution.

d) $\Pr(A \cup B \mid C) = \Pr(A|C) + \Pr(B|C) - \Pr(A \cap B|C)$ (inclusion-exclusion rule)
$= \Pr(A) + \Pr(B) - \Pr(A \cap B)$
$= \Pr(A \cup B)$
$\Rightarrow$ $A \cup B$ is independent of C.
$\square$.

6 a) The DNA markers are mutually independent.

b) Pairwise independence $\Rightarrow$ $Pr(A \cap B \mid C) \neq Pr(A \cap B)$.

$Pr(A \cap C \mid B) \neq Pr(A \cap C)$

$Pr(B \cap C \mid A) \neq Pr(B \cap C)$.

$A = \frac{1}{1000}$ $B = \frac{1}{3000}$ $C = \frac{1}{5000}$

$$Pr(A \cap B \cap C) \leq Pr(B \cap C) = \frac{1}{5000} \cdot \frac{1}{3000} = \frac{1}{15000000}$$

$\Rightarrow$ 1 in 15 million. No.

No independence between markers $\Rightarrow$ $Pr(A \cap B \cap C) \leq \frac{1}{5000}$.