(a) I have N dollars.
Bought N dollars worth of 1 stock.

Let A be the random variable representing the amt of money after 1 time step.
Let P be the random variable representing the price of purchased stock after 1 time step.

$A = NP$.
$Ex[A] = N\,Ex[P]$.

$Ex[P] = 0 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = 1$.

$\Rightarrow Ex[A] = N \cdot 1 = N$ //

$Var[A] = Ex[(A - Ex[A])^2]$
$= Ex[A^2] - Ex^2[A]$

$A = NP$. $A^2 = N^2 P^2$
$Ex[A^2] = N^2\, Ex[P^2]$
$Ex[P^2] = 0 \cdot \frac{1}{2} + 4 \cdot \frac{1}{2} = 2$.

$\Rightarrow Ex[A^2] = N^2 \cdot 2 = 2N^2$

$\Rightarrow Var[A] = 2N^2 - N^2 = N^2$ //

$\boxed{Ex[A] = N,\ Var[A] = N^2}$

b) Let S_i be the expected price of stock i.

$A = S_1 + S_2 + \ldots + S_N$.
$Ex[A] = Ex[S_1 + S_2 + \ldots + S_n]$
$= Ex[S_1] + Ex[S_2] + \ldots + Ex[S_n]$ (linearity of expectation)
$= 1 + 1 + \ldots + 1$
$= N$.

$Var[A] = Var[S_1 + S_2 + \ldots + S_n]$
$= Var[S_1] + Var[S_2] + \ldots + Var[S_n]$ (parwise indepent additivity of variance)

$Var[S_i] = Ex[(S_i - Ex[S_i])^2]$
$= Ex[(S_i - 1)^2]$
$= \sum_{\omega \in S} Pr[\omega] \cdot [S_i(\omega) - 1]^2$

$= 0.5 \cdot [0 - 1]^2 + 0.5 \cdot [2 - 1]^2$
$= 0.5 + 0.5$
$= 1$

$\therefore Var[A] = 1 + 1 + \ldots + 1$
$= N$.

$\boxed{Ex[A] = N,\ Var[A] = N}$

c) Take strategy (b). The variance is smaller so the risk is lower.

d) Let D be the random variable of a six-sided dice.
Let S be the set of all outcomes from rolling a dice once.

$$Ex[D] = \sum_{w \in S} Pr(w) \cdot D(w)$$
$$= 1 \cdot \tfrac{1}{6} + 2 \cdot \tfrac{1}{6} + 3 \cdot \tfrac{1}{6} + \dots + 6 \cdot \tfrac{1}{6}$$
$$= \tfrac{1}{6}(1+2+3+\dots+6)$$
$$= \tfrac{1}{6} \cdot \tfrac{6 \cdot 7}{2}$$
$$= \tfrac{7}{2} = 3.5$$

$$Ex[D^2]$$
$$= 1 \cdot \tfrac{1}{6} + 4 \cdot \tfrac{1}{6} + 9 \cdot \tfrac{1}{6} + 16 \cdot \tfrac{1}{6} + 25 \cdot \tfrac{1}{6} + 36 \cdot \tfrac{1}{6}$$
$$= \tfrac{1}{6}(1+4+9+16+25+36)$$
$$= 15\tfrac{1}{6}$$

$$Var[D] = Ex[(D - Ex[D])^2]$$
$$= Ex[(D-3.5)^2]$$
$$= Ex[D^2] - Ex^2[D]$$
$$= 15\tfrac{1}{6} - (3.5)^2$$
$$= 2\tfrac{11}{12}$$

$$\boxed{\text{Ans: } Ex[D] = 3.5,\ Var[D] = 2\tfrac{11}{12} = 2.92}$$

e) Let $T = D^3$

$$Ex[T] = Ex[D^3]$$
$$= \tfrac{1}{6}(1+8+\dots+6^3) = \tfrac{1}{6}\left(\frac{6^2(7)^2}{4}\right) = 73.5$$

$$Var[T] = Ex[T^2] - Ex^2[T]$$
$$= Ex[D^6] - (73.5)^2$$
$$= \tfrac{1}{6}(1+2^6+3^6+\dots+6^6) - (73.5)^2$$
$$= 5792.92$$

$$\boxed{\text{Ans: } Ex[T] = 73.5,\ Var[T] = 5792.92}$$

2) a) 7 propositions.

Let E_i be the indicator variable for proposition i. $E_i = 1$ if true, $E_i = 0$ if false.

Let $T = E_1 + E_2 + \dots + E_7$.

$$\Pr[E_i = 1] = 1 - \Pr[E_i = 0]$$
$$= 1 - \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$$
$$= 1 - \frac{1}{8} = \frac{7}{8}.$$

$Ex[T] = 7 \cdot \frac{7}{8} = \frac{49}{8} = 6.125$ (linearity of expectation)

b) Suppose that there does not exist an assignment such that all propositions are true, then $Ex[T] \leq 6$.

However $Ex[T] = 6.125 > 6$

$\Rightarrow$ Contradiction

$\Rightarrow$ Such an assignment exists.

3) a) Let P_i be the time taken to complete problem set i.

$B = P_1 + P_2 + P_3$ $\qquad$ $Ex[P_i] = 1 \cdot \frac{2}{3} + 2 \cdot \frac{1}{3} = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$.

$Ex[B] = 3Ex[P] = 3 \cdot \frac{4}{3} = 4$ //

$\boxed{\text{Ans: } Ex[B] = 4}$

b) Pr[student rolls $\frac{1}{6}$].

$Ex[R] = \frac{1}{(\frac{1}{6})} - 1 = \boxed{5 \text{ (time to failure)}}$.

c) Let R_i be the value of a 6 sided dice roll.

$Ex[R_i] = 3.5$

$Ex[R_1 \cdot R_2] = Ex[R_1] \cdot Ex[R_2]$ since R_i are all mutually independent.

$= 3.5 \cdot 3.5$

$= \boxed{12.25}$

d) D is the number of days a student delays laundry.

$D = \frac{1}{2}E[B] + \frac{1}{3}E[R] + \frac{1}{6}E[U]$. (Law of total expectation)

$= \frac{1}{2}(4) + \frac{1}{3}(5) + \frac{1}{6}(12.25)$

$\approx \boxed{5.708}$

4) Suppose we flip the chosen coin n times.
Let C_1 be the number of heads if we flip the fair coin.
Let C_2 be the number of heads if we flip the unfair coin.

$PDF_{C_1}[k] = \binom{n}{k}\left(\frac{1}{2}\right)^n$ $\quad n=40 \quad 20 = 0.125$

$PDF_{C_2}[k] = \binom{n}{k}\left(\frac{3}{4}\right)^k\left(\frac{1}{4}\right)^{n-k}$

Find n s.t. $Pr[C_1 = k] < 5\%$
$Pr[C_2 = k] > 95\%$ where $\frac{1}{2} < k < \frac{3}{4}$.. Let $k = \left(\frac{1}{2}+\frac{3}{4}\right) \div 2 = \frac{5}{8}$.

$\Rightarrow \binom{n}{k}\left(\frac{1}{2}\right)^n < 5\%$ and $\binom{n}{k}\left(\frac{3}{4}\right)^k\left(\frac{1}{4}\right)^{n-k} > 95\%$

5) a) Let the total score of the true false section be T.
Let score of the i^{th} qn be T_i.

$Ex[T] = Ex[T_1] + Ex[T_2] + \ldots$
$= \left(2 \cdot \frac{3}{4}\right) 10$
$= 15.$

Let value of dice roll be D

$Ex[D_1 + D_2 + 3] = Ex[D_1] + Ex[D_2] + 3$
$= 3.5 + 3.5 + 3$
$= 10.$

$Ex[\text{2nd section}] = 10 \cdot 4 = 40$

$Ex[\text{last qn}] = 12 \cdot \frac{1}{2} + 18 \cdot \frac{1}{2} = 15.$

$\boxed{Ex[\text{Total}] = 15 + 40 + 15 = 70}$

b) $3.5 \cdot 3.5 = 12.25$

$Ex[\text{general impression}] = \frac{4}{10} \cdot 40 + 50 \cdot \frac{3}{10} + \frac{3}{10} \cdot 60$
$= 49$

$\boxed{\text{Ans: } 61.25 \approx 61}$

$61.25 \cdot \frac{2}{7} + 70 \cdot \frac{4}{7} + 84 \cdot \frac{1}{7} = \boxed{69.5}$

$\Pr[X_i = 1] = \left(\frac{n-1}{n}\right)^n$

$\Pr[X_i = 0] = 1 - \left(\frac{n-1}{n}\right)^n$

$$PDF_{X_i}(x) = \begin{cases} x < 0 = 0 \\ 0 \le x < 1 = 1 - \left(\frac{n-1}{n}\right)^n \\ x \geq 1 = \left(\frac{n-1}{n}\right)^n \end{cases}$$

$\Pr[X_i = 1 \mid X_1 = X_2 = \dots = X_n = 0] = 1 \neq \Pr[X_i]$

Since if all the boxes are empty, all the balls must have gone into X_i.

b) $Ex[X_1 + X_2 + \dots + X_n] = Ex[X_1] + Ex[X_2] + \dots + Ex[X_n]$

$= n\left(\frac{n-1}{n}\right)^n$

$$\lim_{n\to\infty} \left(\frac{n-1}{n}\right)^n = \lim_{n\to\infty} \left(\frac{1}{\left(\frac{n}{n-1}\right)^n}\right) = \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^{n+1}}$$

$$= \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n \cdot \left(\frac{n+1}{n}\right)}$$

$$= \frac{1}{e \cdot 1} = \frac{1}{e}$$ //

c) $\Pr[$At least k balls fall into the first box$]$.

Number of ways we can pick k throws into the first box $= \binom{n}{k}$

Probability of at least k balls falling into first box. $\left(\frac{1}{n}\right)^k$

Using Boole's Inequality, $\binom{n}{k}\left(\frac{1}{n}\right)^k$

d) $\Pr[R \geq k] \leq \frac{n}{k!}$ $\qquad \binom{n}{k}\left(\frac{1}{n}\right)^k = \frac{n!}{k!(n-k)!}\left(\frac{1}{n}\right)^k = \frac{n!}{k!(n-k)!\,n^k}$

$\left(\frac{1}{k!}\right) n$

$\Pr[R \geq k] = \Pr[R = k] + \Pr[R = k+1] + \dots + \Pr[R = n]$

$= \binom{n}{k}\left(\frac{1}{n}\right)^k(\dots) + \dots$ $\qquad 1 > \frac{n!}{(n-k)!\,n^k}$

$\Pr[$At least k balls fall into any box$] = n \cdot \binom{n}{k}\left(\frac{1}{n}\right)^k$

$$= n \cdot \frac{n!}{k!(n-k)!\,n^k}$$

$$= \frac{n}{k!} \cdot \frac{n!}{(n-k)!\,n^k}$$

$$\leq \frac{n}{k!} \cdot \frac{n!}{n!} = \frac{n}{k!}$$

$\Rightarrow \Pr[$At least k balls fall into any box$] \leq \frac{n}{k!}$ $\qquad$ D,

e) $\lim\limits_{n\to\infty} \Pr[R \geq n^{\epsilon}] = 0$ for all $\epsilon > 0$.

As n grows large, the no. of boxes that are empty is $\frac{n}{e}$.

$$\Pr[R \geq n^{\epsilon}] \leq \frac{n}{(n^{\epsilon})!}$$

$$\approx \frac{n}{\sqrt{2\pi n^{\epsilon}}\left(\frac{n^{\epsilon}}{e}\right)^{n}}$$

$$= \frac{1}{\sqrt{2\pi}} n^{1-\frac{1}{2}\epsilon} \cdot \left(\frac{n^{\epsilon}}{e}\right)^{-n}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{n^{1-\frac{1}{2}\epsilon - n\epsilon}}{e^{n}}$$

$$= \frac{1}{e^{n}\sqrt{2\pi}} n^{1-\epsilon(\frac{1}{2}+n)}$$

$$= \frac{n^{1-\epsilon(\frac{1}{2}+n)}}{e^{n}\sqrt{2\pi}}$$

$$= \frac{n}{n^{\epsilon(\frac{1}{2}+n)} e^{n}\sqrt{2\pi}}$$

$$\lim_{n\to\infty}$$

$$\Pr[R \geq k] \leq \frac{n}{k!}$$

$$\approx \frac{n}{\sqrt{2\pi k}\left(\frac{k}{e}\right)^{k}}$$

$$= \frac{n}{\sqrt{2\pi k}} \cdot \left(\frac{e}{k}\right)^{k}$$

$$= \frac{n e^{k}}{k^{k}\sqrt{2\pi k}}$$

Let $k = n^{\epsilon}$

$$\frac{n e^{n^{\epsilon}}}{n^{\epsilon n^{\epsilon}} \sqrt{2\pi n^{\epsilon}}}$$

$$\frac{e^{n^{\epsilon}}}{n^{\epsilon n^{\epsilon}-1} \sqrt{2\pi} \cdot n^{\frac{1}{2}\epsilon}}$$

$$\frac{e^{n^{\epsilon}}}{n^{\epsilon n^{\epsilon}-1-\frac{1}{2}\epsilon} \sqrt{2\pi}}$$

$$= \frac{e^{n^{\epsilon}}}{n^{\epsilon(n^{\epsilon}-\frac{1}{2})-1} \sqrt{2\pi}}$$

$$\lim_{n\to\infty} \frac{e^{n^{\epsilon}}}{n^{\epsilon(n^{\epsilon}-\frac{1}{2})-1} \sqrt{2\pi}}$$

$$= \lim_{n\to\infty} \frac{e^{n^{\epsilon}}}{n^{\epsilon(n^{\epsilon}-\frac{1}{2})-1}}$$

$$= \lim_{n\to\infty} \frac{e^{n^{\epsilon}}}{e^{\ln(n)[\epsilon(n^{\epsilon}-\frac{1}{2})-1]}}$$

$$= \lim_{n\to\infty} e^{n^{\epsilon} - \ln(n)[\epsilon(n^{\epsilon}-\frac{1}{2})-1]}$$

$$\lim_{n\to\infty} n^{\epsilon} - \ln(n)[\epsilon(n^{\epsilon}-\tfrac{1}{2})-1]$$

$$= \lim_{n\to\infty} n^{\epsilon} - \epsilon n^{\epsilon}\ln(n) + \tfrac{1}{2}\epsilon\ln(n) + \ln(n)$$

$$= \lim_{n\to\infty} n^{\epsilon}(1-\epsilon\ln(n)) + \ln(n)[\tfrac{1}{2}\epsilon+1]$$

$$= \lim_{n\to\infty} n^{\epsilon}(1-\epsilon\ln(n)) + \ln(n)$$

$$\lim_{n\to\infty} \frac{n^{\epsilon}(1-\epsilon\ln(n))}{\ln(n)} = \frac{n^{\epsilon} - \epsilon n^{\epsilon}\ln(n)}{\ln(n)}$$

$$= \frac{n^{\epsilon}}{\ln(n)} - \epsilon n^{\epsilon}$$

$$= n^{\epsilon}\left(\frac{1}{\ln(n)} - \epsilon\right)$$

$$\Rightarrow \lim_{n\to\infty} n^{\epsilon}\left(\frac{1}{\ln(n)} - \epsilon\right)$$

$$= \lim_{n\to\infty} -n^{\epsilon} = -\infty$$

$$\Rightarrow \ln(n) = o\left(n^{\epsilon}(1-\epsilon\ln(n))\right)$$

$$\Rightarrow \lim_{n\to\infty} n^{\epsilon}(1-\epsilon\ln(n)) + \ln(n) = -\infty$$

$$\Rightarrow \lim_{n\to\infty} e^{n^{\epsilon}(1-\epsilon\ln(n)) + \ln(n)} = e^{-\infty}$$

$$= 0.$$