18.06SC Unit 3 Exam

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1 (34 pts.) (a) If a square matrix A has all n of its *singular values* equal to 1 in the SVD, what basic classes of matrices does A belong to? (Singular, symmetric, orthogonal, positive definite or semidefinite, diagonal)

(b) Suppose the (orthonormal) columns of H are eigenvectors of B:

$$H = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \qquad H^{-1} = H^{\mathrm{T}}$$

The eigenvalues of B are $\lambda = 0, 1, 2, 3$. Write B as the product of 3 specific matrices. Write $C = (B + I)^{-1}$ as the product of 3 matrices.

(c) Using the list in question (a), which basic classes of matrices do B and C belong to? (Separate question for B and C)

a) SVD $M = U\Sigma V$

$M = UV$

$M^T = V^T U^T$

$M^T M = V^T U^T U V$

$= I$

$\Rightarrow$ M is orthogonal.

b) $B = H\Lambda H^T = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 2 & \\ & & & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix}$

$C = (B+I)^{-1}$

$(B+I)^{-1} = (B+I)^T$

$B + I = H(\Lambda + I)H^T$

$(B+I)x = \lambda x$

$C = (B+I)^{-1}$

$(B+I-\lambda I)x = 0$

$= H(\Lambda + I)^{-1} H^T$

$(B - (\lambda+1)I)x = 0$

$= \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & \frac{1}{2} & & \\ & & \frac{1}{3} & \\ & & & \frac{1}{4} \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix}$

$B = H\Lambda H^T \quad \Lambda > 0$

$B^T = H\Lambda H^T$

$\Rightarrow B$ is singular symmetric positive semidefinite

$C \Rightarrow$ symmetric positive definite.

2 (33 pts.) (a) Find three eigenvalues of A, and an eigenvector matrix S:

$$A = \begin{bmatrix} -1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Explain why $A^{1001} = A$. Is $A^{1000} = I$? Find the three diagonal entries of e^{At}.

(c) The matrix $A^{\mathrm{T}}A$ (for the same A) is

$$A^{\mathrm{T}}A = \begin{bmatrix} 1 & -2 & -4 \\ -2 & 4 & 8 \\ -4 & 8 & 42 \end{bmatrix}.$$

How many eigenvalues of $A^{\mathrm{T}}A$ are positive? zero? negative? (Don't compute them but explain your answer.) Does $A^{\mathrm{T}}A$ have the same eigenvectors as A?

$A = \begin{bmatrix} -1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 1 \end{bmatrix}$

$(A - \lambda I) = \begin{bmatrix} -1-\lambda & 2 & 4 \\ & -\lambda & 5 \\ & & 1-\lambda \end{bmatrix}$

$\det(A - \lambda I) = \lambda(1+\lambda)(1-\lambda)$

$\lambda_1 = 1, \ \lambda_2 = -1, \ \lambda_3 = 0$

$N(A - I) = \begin{bmatrix} -2 & 2 & 4 \\ & -1 & 5 \\ & & \end{bmatrix} x$ $\qquad N(A + I) = \begin{bmatrix} & 2 & 4 \\ & 1 & 5 \\ & & 2 \end{bmatrix} x$

$x_1 = \begin{bmatrix} 7 \\ 5 \\ 1 \end{bmatrix}$ $\qquad x_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\qquad x_3 = N(A) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$\Lambda = \begin{bmatrix} 1 & & \\ & -1 & \\ & & 0 \end{bmatrix}$ $\quad S = \begin{bmatrix} 7 & 1 & 2 \\ 5 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

b) $A^{1001} = x_1 x^T - x_2 x^T$

$= A$

$A^{1000} = X \Lambda^{1000} X^{-1}$

$= X \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} X^{-1} = [x_1 \ x_2 \ 0] X^{-1}$

$\neq I$

$e^{At} = X e^{\Lambda t} X^{-1}$

$e^{\Lambda t} = \begin{bmatrix} e^t & & \\ & e^{-t} & \\ & & 1 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 1 & -2 & -4 \\ -2 & 4 & 8 \\ -4 & 6 & 42 \end{bmatrix}$$

$$A^TA = Q\Lambda Q^T$$

Since A^TA must have all eigenvalues ≥ 0. Since $\lambda_1 = 0$, A^TA must have 1 0 eigenvalue
0 negative. 1 zero. 2 positive.

No. Different trace.

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3 (33 pts.) Suppose the n by n matrix A has n orthonormal eigenvectors $q_1, \ldots, q_n$ and n positive eigenvalues $\lambda_1, \ldots, \lambda_n$. Thus $Aq_j = \lambda_j q_j$.

(a) What are the eigenvalues and eigenvectors of A^{-1}? *Prove that your answer is correct.*

(b) Any vector b is a combination of the eigenvectors:

$$b = c_1 q_1 + c_2 q_2 + \cdots + c_n q_n \,.$$

What is a quick formula for c_1 using orthogonality of the q's?

(c) The solution to $Ax = b$ is also a combination of the eigenvectors:

$$A^{-1}b = d_1 q_1 + d_2 q_2 + \cdots + d_n q_n \,.$$

What is a quick formula for d_1? You can use the c's even if you didn't answer part (b).

$A = Q\Lambda Q^T$

$A^{-1} = (Q\Lambda Q^T)^{-1}$

$= (Q^T)^{-1}\Lambda^{-1}Q^{-1}$

$= Q\Lambda^{-1}Q^T$

same eigenvectors. Eigenvalues of A^{-1}: $\lambda^{-1} = \underline{\underline{\frac{1}{\lambda}}}$ ✓

b) $b = c_1q_1 + c_2q_2 + \ldots c_nq_n$

$q_1^T b = c_1 \|q_1\|^2$

$c_1 = \underline{\underline{\frac{q_1^T b}{\|q_1\|^2}}}$ ✓

$\|q_1\|^2 = 1$

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c) $A^{-1}b = d_1q_1 + d_2q_2 + \ldots d_nq_n$

$Q\Lambda^{-1}Q^T b = d_1q_1 + \ldots$

$b = Qc$

$Q\Lambda^{-1}Q^TQc = d_1q_1 + \ldots$

$Q\Lambda^{-1}c = \frac{q_1}{\lambda_1}c_1 + \frac{q_2}{\lambda_2}c_2 + \ldots$

$= d_1q_1 + d_2q_2$

$\Rightarrow d_1 = \underline{\underline{\frac{c_1}{\lambda_1}}}$ ✓

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