5C.2) Proposition: For $T \in \mathcal{L}(V)$, if $V = \text{null } T \oplus \text{range } T$, then T must be diagonalizable.

Counterexample: If V is not finite dimension, then T cannot be diagonalizable.

Let $V = \mathbb{F}^\infty$. Let the standard basis for V be $e_1, e_2, \ldots$ e.g. $e_1 = (1, 0, 0, \ldots)$

Let $v \in V$: $v = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + \ldots$

Let $Tv = a_3 e_3 + a_4 e_4 + \ldots$

Hence $\text{null } T = \text{span}(e_1, e_2)$

$\text{range } T = \text{span}(e_3, e_4, e_5, \ldots)$

$\text{null } T \oplus \text{range } T = V$.

But range T is infinite dimensional $\Rightarrow$ V is infinite dimensional

$\Rightarrow$ T cannot be diagonalized.

$\square$.

5C.1) Suppose $T \in \mathcal{L}(V)$ is diagonalizable

Then let $e_1, \ldots, e_n$ be the basis such that $M(T)$ is diagonal.

Hence $e_1, \ldots, e_n$ are eigenvectors of T.

$$Te_k = \lambda_k e_k \text{ for all } k = 1, \ldots, n.$$

We know that $e_1, \ldots, e_n$ are linearly independent.

For the vectors $Te_k = \lambda_k e_k$ where $\lambda_k \neq 0$, $e_k \in \text{range } T$.

Alternatively, for vectors $Te_k = \lambda_k e_k$ where $\lambda_k = 0$, $e_k \in \text{null } T$.

Hence all vectors $e_1, \ldots, e_n$ can be partitioned into two sets: range T and null T.

No vector in $e_1, \ldots, e_n$ exists in both range T and null T.

Since $e_1, \ldots, e_n$ form a basis in V, this also means that

$\text{range } T \cap \text{null } T = \{0\}$.

Hence we showed that $e_1, \ldots, e_n$ which forms a basis of V can be partitioned into range T and null T, with $\text{range } T \cap \text{null } T = \{0\}$

$$\Rightarrow V = \text{null } T \oplus \text{range } T.$$

$\square$.

5C.8) $T \in \mathcal{L}(\mathbb{F}^5)$

$\dim E(8, T) = 4 \Rightarrow$ there exists a linearly independent list of vectors v_1, v_2, v_3, v_4 such that

$$Tv_k = 8v_k \text{ for } k = 1, \dots, 4.$$

Suppose $T - 2I$ is not invertible, then 2 is an eigenvalue. Let v_5 be the eigenvector corresponding to 2.

Hence v_5 is independent of $v_1, \dots, v_4$, thus $v_1, \dots, v_5$ form a basis of $\mathbb{F}^5$.

Using the fundamental thm of linear maps,

$$\dim \mathbb{F}^5 = \dim \operatorname{range} T + \dim \operatorname{null} T$$
$$5 = \dim \operatorname{range} T + \dim \operatorname{null} T.$$

Since $v_1, \dots, v_5$ are all eigenvectors of T with non-zero eigenvalues, and $v_1, \dots, v_5$ also forms the basis of $\mathbb{F}^5$,

$$\operatorname{null} T = \{0\}.$$

Hence $V = E(8, T) + E(2, T)$

$\Rightarrow$ T does not have eigenvalue with value 6

and $T - 6I$ is injective.

The same argument applies for the case where 6 is an eigenvalue of T, instead of 2.

Hence either $T - 2I$ or $T - 6I$ is invertible

$\square$.

5C.14) $T \in \mathcal{L}(\mathbb{C}^3)$ s.t $\exists v_1, v_2$ where $Tv_1 = 6v_1$, $Tv_2 = 7v_2$

and T does not have a diagonal matrix.

Let $T(x, y, z) = (6x, 7y, 6z + y)$

$$M(T) = \begin{bmatrix} 6 & & \\ & 7 & 1 \\ & & 6 \end{bmatrix}$$

$Tv = \lambda v$

$6x = \lambda x \Rightarrow$ 6 can be an eigenvalue

$7y = \lambda y \Rightarrow$ 7 can be an eigenvalue.

$6z + y = \lambda z$

$y = (\lambda - 6)z$

T is not diagonalizable since there are only 2 eigenvectors of T.

$\square$.

8C.8) $T \in \mathcal{L}(V)$. T is invertible iff constant term in minimal polynomial of T is non-zero.

Pf: Assuming that T is invertible,

By lemma 5.27, T must have an upper triangular matrix w.r.t some basis,
Since T is invertible, by lemma 5.30, the entries along the diagonals of the matrix must be non-zero.

Hence there must exist dim V eigenvectors for T, since by lemma 5.32, the entries in the diagonals of the upper triangular representation of T are the eigenvalues.

By lemma 8.49, the eigenvalues must be the roots of the minimal polynomial.

Since none of the eigenvalues of V are non-zero, zero cannot be a root of the minimal polynomial.
Thus there must exist a constant term in the minimal polynomial so zero cannot be a root.

$\square$.

Now, assume that the minimal polynomial has a non-zero constant term.
Then by the fundamental theorem of algebra, there exists a root in the minimal polynomial.

This root cannot be 0 since the minimal polynomial has a nonzero constant term.
Since the roots of the minimal polynomial are the eigen values of T, and 0 is not an eigenvalue of T, null $T = \{0\}$ and T is invertible

$\square$.

Q1) Example of $T \in \mathcal{L}(\mathbb{C}^3)$ whose minimal polynomial is $(x+2)^2$.

$$T(x,y,z) = (-2x, -2y, -2y-2z)$$

$$\begin{aligned}
(T+2)^2 &= (T+2)(T+2)v \\
&= (T+2)(Tv+2v) \\
&= (T+2)(Tv + (2x,2y,2z)) \\
&= (T+2)[(-2x,-2y,-2y-2z)+(2x,2y,2z)] \\
&= (T+2)(0,0,-2y) \\
&= (0,0,4y)+(0,0,-4y) \\
&= (0,0,0).
\end{aligned}$$

$\Rightarrow$ $T(x,y,z) = (-2x,-2y,-2y-2z)$ has the minimal polynomial $(x+2)^2$ $\square$.

b) Minimal polynomial : $(x^2+1)(x-3)^2$
$\Rightarrow i, -i, 3$ are eigenvalues.

$$T = \begin{bmatrix} i &&&& \\ & -i &&& \\ && 3 && \\ &&& 3 & \\ &&& x & 3 \end{bmatrix} \quad T^2 = \begin{bmatrix} -1 &&&& \\ & -1 &&& \\ && 9 && \\ &&& 9 & \\ &&& 6x & 9 \end{bmatrix} \quad T^2+1 = \begin{bmatrix} 0 &&&& \\ & 0 &&& \\ && 10 && \\ &&& 10 & \\ &&& 6x & 10 \end{bmatrix}$$

$$T-3 = \begin{bmatrix} i-3 &&&& \\ & -i-3 &&& \\ && 0 && \\ &&& 0 & \\ &&& x & 0 \end{bmatrix} \quad (T-3)^2 = \begin{bmatrix} -1-6i+9 &&&& \\ & -1+6i+9 &&& \\ && 0 && \\ &&& 0 & \\ &&&& 0 \end{bmatrix} = \begin{bmatrix} 8-6i &&&& \\ & 8+6i &&& \\ && 0 && \\ &&& 0 & \\ &&&& 0 \end{bmatrix}$$

$$(T-3)^2(T^2+1) = \begin{bmatrix} 8-6i &&& \\ & 8+6i && \\ && \ddots & \end{bmatrix} \begin{bmatrix} \ddots &&& \\ & 10 && \\ && 10 & \\ && 6x & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 0 && \\ & \ddots & \\ && \end{bmatrix}$$

$x=1 \Rightarrow T = \begin{bmatrix} i &&&& \\ & -i &&& \\ && 3 && \\ &&& 3 & \\ &&& 1 & 3 \end{bmatrix}$ has minimal polynomial $(x^2+1)(x-3)^2$

$\Rightarrow T(x,y,z,a,b) = (ix, -iy, 3z, 3a, a+3b)$

Suppose $T(x,y,z,a,b) = (ix,-iy,3z,3a,a+3b)$, prove that $(x^2+1)(x-3)^2$ is a minimal polynomial.

Pf: $T^2(x,y,z,a,b) = (-x,-y,9z,9a,3a+3a+9b)$
$= (-x,-y,9z,9a,6a+9b)$

$(T^2+1)(x,y,z,a,b) = (0,0,10z,10a,6a+10b)$

$(T-3)(x,y,z,a,b) = (ix-3x, -iy-3y, 0,0,a)$

$(T-3)^2(x,y,z,a,b) = (i(ix-3x)-3(ix-3x), \dots, 0,0,0)$

$(T-3)^2(T^2+1)(x,y,z,a,b) = (0,0,0,0,0)$
$\Rightarrow (x-3)^2(x^2+1)$ is a minimal polynomial.

c) a) eigenvalues: -2
b) $i, -i, 3$

2)
$$T = \begin{bmatrix} 2 & & & \\ & 3 & & 1 \\ & & 3 & \\ & & & 3 \end{bmatrix}$$

$$Tv = \lambda v$$

$$T\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2a \\ 3b + d \\ 3c \\ 3d \end{bmatrix} = \begin{bmatrix} \lambda a \\ \lambda b \\ \lambda c \\ \lambda d \end{bmatrix}$$

$$2a = \lambda a$$
$$3b + d = \lambda b$$
$$3c = \lambda c \quad \Rightarrow \text{eigenvalues}: (2, 3, 3, 3)$$
$$3d = \lambda d$$

Characteristic polynomial: $(x-2)(x-3)^3 = 0$

Factors of $(x-2)(x-3)^3$:

$(x-2) \Rightarrow T - 2I \Rightarrow \begin{bmatrix} 0 & & & \\ & 1 & & 1 \\ & & 1 & \\ & & & 1 \end{bmatrix}$ X

$(x-3) \Rightarrow T - 3I \Rightarrow \begin{bmatrix} -1 & & & \\ & 0 & & 1 \\ & & 0 & \\ & & & 0 \end{bmatrix}$ X

$(x-2)(x-3) \Rightarrow \begin{bmatrix} 0 & & & \\ & 1 & & 1 \\ & & 1 & \\ & & & 1 \end{bmatrix}\begin{bmatrix} -1 & & & \\ & 0 & & 1 \\ & & 0 & \\ & & & 0 \end{bmatrix} = \begin{bmatrix} 0 & & & \\ & 0 & & 1 \\ & & 0 & \\ & & & 0 \end{bmatrix}$ X

$(x-3)^2 \Rightarrow \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$ X

$(x-3)^2(x-2) \Rightarrow \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} \Rightarrow (x-3)^2(x-2)$ is minimal polynomial

$\square$.