6C.4) $U = \text{span}((1,2,3,-4),(-5,4,3,2))$.

Use gram schmidt on U for an orthonormal basis for U
Apply gram schmidt to (1,0,0,0), (0,1,0,0) with the orthonormal basis built previously to get $U^\perp$.

$$e_1 = (1,2,3,-4)\cdot\frac{1}{\sqrt{1+4+9+16}} = \frac{1}{\sqrt{30}}(1,2,3,-4)$$

$$u = (-5,4,3,2) - \langle (-5,4,3,2), \tfrac{1}{\sqrt{30}}(1,2,3,-4)\rangle \tfrac{1}{\sqrt{30}}(1,2,3,4)$$

$$e_2 = \frac{u}{\|u\|}$$

$$u = (-5,4,3,2) - \tfrac{1}{\sqrt{30}}(-5+8+9-8)\cdot\tfrac{1}{\sqrt{30}}(1,2,3,-4)$$
$$= (-5,4,3,2) - \tfrac{1}{30}(4)(1,2,3,-4)$$
$$= (-5,4,3,2) - \tfrac{2}{15}(1,2,3,-4)$$
$$= \tfrac{1}{15}(-77,56,39,38)$$
$$\|u\| = \tfrac{1}{15}\sqrt{12030} = \sqrt{\tfrac{802}{15}}$$

$$e_2 = \frac{\sqrt{15}}{\sqrt{802}}\cdot\frac{1}{15}(-77,56,39,38)$$
$$= \frac{1}{\sqrt{12030}}(-77,56,39,38)$$

$$e_1 = \frac{1}{\sqrt{30}}(1,2,3,-4) \qquad e_2 = \frac{1}{\sqrt{12030}}(-77,56,39,38)$$

$$(1,0,0,0) - \tfrac{1}{30}(1,2,3,-4) - \tfrac{1}{12030}(-77,56,39,38)$$

$$e_3 = \frac{\left(\frac{1951}{2005}, -\frac{143}{2005}, -\frac{207}{2005}, \frac{261}{2005}\right)}{\sqrt{\frac{1964}{2005}}}$$

e_4

lazy workout e_1, e_2, e_3, e_4.

6C.6) $P_U P_W = 0 \iff \langle u, w \rangle = 0$ for all $u \in U$ and all $w \in W$.

Suppose $P_U P_W = 0$.

$\iff P_U P_W v = 0$

Let $P_W v = w \in W$ for all $v \in V$.

$P_U w = 0$

$\iff w \perp U$ $\therefore$ For all $v \in V$, $P_W v \perp U$

$\iff W \perp U$

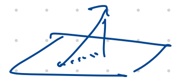
$\iff \langle u, w \rangle = 0$ for all $u \in U, w \in W$.

6C.11)

$U = \operatorname{span}((1,1,0,0),(1,1,1,2))$

To minimize $\|u - (1,2,3,4)\|$, $u - (1,2,3,4) \in U^\perp$

Hence $u = P_U(1,2,3,4)$



Let e_1, e_2 be an orthonormal basis for U.

$$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0, 0\right), \left(0, 0, \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right)$$

$$\Rightarrow u = \langle (1,2,3,4), e_1 \rangle e_1 + \langle (1,2,3,4), e_2 \rangle e_2$$

$$= \left(\frac{\sqrt{2}}{2} + \sqrt{2}\right)\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0, 0\right) + \left(\frac{3\sqrt{5}}{5} + \frac{8\sqrt{5}}{5}\right)\left(0, 0, \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right)$$

$$= \left(\frac{3}{2}, \frac{3}{2}, 0, 0\right) + \left(0, 0, \frac{11}{5}, \frac{22}{5}\right)$$

$$= \left(\frac{3}{2}, \frac{3}{2}, \frac{11}{5}, \frac{22}{5}\right) \quad //$$

7.A.1) $T(z_1,\dots,z_n) = (0, z_1, \dots, z_{n-1})$

$T^*(z_1,\dots,z_n)$? $\quad \langle Tv, w\rangle = \langle T(z_1,\dots,z_n), (x_1,\dots,x_n)\rangle$
$= \langle z_1,\dots,z_n, T^*(x_1,\dots,x_n)\rangle$

$\langle T(z_1,\dots,z_n), (x_1,\dots,x_n)\rangle = \langle (0,z_1,\dots,z_{n-1}), (x_1,\dots,x_n)\rangle$
$= z_1x_2 + \dots + z_{n-1}x_n.$

$\Rightarrow T^*(x_1,\dots,x_n) = (x_2,\dots,x_n,0)$ //

7A.2) $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$.
Prove that λ is eigenvalue of T iff $\bar{\lambda}$ is eigenvalue of T^*.

Pf:

$\langle Tv, w\rangle = \langle v, T^*w\rangle$ for $v, w \in V$.

Let λ be an eigenvalue of T, with corresponding e'vector v.

$\langle Tv, w\rangle = \lambda\langle v, w\rangle = \langle v, T^*w\rangle$

$\langle v, T^*w\rangle - \lambda\langle v, w\rangle = 0$
$\langle v, T^*w\rangle - \langle v, \bar{\lambda}w\rangle = 0$
$\langle v, T^*w - \bar{\lambda}w\rangle = 0$
$\langle v, (T^*-\bar{\lambda}I)w\rangle = 0$

$\Rightarrow$ range$(T^*-\bar{\lambda}I) \perp$ eigenvector v.

Let $e_1,\dots,e_n$ be a basis for V.
Assuming $e_1 = v$, then $e_1 \notin$ range $(T^*-\bar{\lambda}I)$
$\therefore$ dim range $(T^*-\bar{\lambda}I) <$ dim V.

dim V = dim range$(T^*-\bar{\lambda}I)$ + dim null $(T^*-\bar{\lambda}I)$

dim range $(T^*-\bar{\lambda}I) <$ dim range $(T^*-\bar{\lambda}I)$ + dim null$(T^*-\bar{\lambda}I)$
$\Rightarrow$ dim null$(T^*-\bar{\lambda}I) > 0$
$\therefore \exists\, u \in V;\ u \neq 0$ s.t

$(T^*-\bar{\lambda}I)u = 0$
$\Rightarrow T^*u = \bar{\lambda}u$
$\Rightarrow \bar{\lambda}$ is an eigenvalue of T^*.

Suppose $\bar{\lambda}$ is an eigenvalue of T^*, with eigenvector w, then

$\langle Tv, w\rangle = \langle v, T^*w\rangle = \langle v, \bar{\lambda}w\rangle$
$\langle Tv, w\rangle = \lambda\langle v, w\rangle$
$\langle Tv, w\rangle - \langle \lambda v, w\rangle = \langle (T-\lambda I)v, w\rangle = 0$

Same argument as above
$\Rightarrow \lambda$ is an eigenvalue of T. Thus proposition is true. $\square$

7A.4) Suppose $T \in \mathcal{L}(V,W)$

Prove T is injective $\Leftrightarrow$ T^* is surjective.

From 7.7, $\text{range } T^* = (\text{null } T)^\perp$

$$\begin{aligned} T \text{ is injective} &\Leftrightarrow \text{null } T = \emptyset \\ &\Leftrightarrow (\text{null } T)^\perp = V \\ &\Leftrightarrow \text{range } T^* = V \\ &\Leftrightarrow T^* \text{ is surjective.} \quad \square. \end{aligned}$$

$$\begin{aligned} T \text{ is surjective} &\Leftrightarrow \text{range } T = W \\ &\Leftrightarrow (\text{null } T^*)^\perp = W \\ &\Leftrightarrow \text{null } T^* = \emptyset \\ &\Leftrightarrow T^* \text{ is injective} \quad \square. \end{aligned}$$

1) $(e_1, \dots, e_m)$ is orthonormal in V. Let $v \in V$.

Prove that

$$\|v\|^2 = |\langle v, e_1\rangle|^2 + \dots + |\langle v, e_m\rangle|^2$$

iff $v \in \operatorname{span}(e_1, \dots, e_m)$.

Pf: Suppose $\|v\|^2 = |\langle v, e_1\rangle|^2 + \dots + |\langle v, e_m\rangle|^2$

$\langle v, v\rangle = |\langle v, e_1\rangle|^2 + \dots + |\langle v, e_m\rangle|^2$

Since $e_1, \dots, e_m$ is orthonormal, we can extend the list into an orthonormal basis for V, which we denote as

$$\operatorname{span}(e_1, \dots, e_m, \sigma_1, \dots, \sigma_n) = V.$$

Hence $v = \langle v, e_1\rangle e_1 + \dots + \langle v, e_m\rangle e_m + \langle v, \sigma_1\rangle \sigma_1 + \dots + \langle v, \sigma_n\rangle \sigma_n$

Using Pythagoreas theorem,

$$\|v\|^2 = \|\langle v, e_1\rangle e_1 + \dots + \langle v, \sigma_n\rangle \sigma_n\|^2$$
$$= |\langle v, e_1\rangle|^2 + \dots + |\langle v, e_m\rangle|^2 + |\langle v, \sigma_1\rangle|^2 + \dots + |\langle v, \sigma_n\rangle|^2$$

Since $\|v\|^2 = |\langle v, e_1\rangle|^2 + \dots + |\langle v, e_m\rangle|^2$,

$$|\langle v, \sigma_1\rangle|^2 + \dots + |\langle v, \sigma_n\rangle|^2 = 0$$

lemma 1: $|\langle v, \sigma_1\rangle|^2 + \dots + |\langle v, \sigma_n\rangle|^2 = 0 \Rightarrow \sigma_1 = \dots = \sigma_n = 0$

Pf: Let $v = \sigma_k$.

$$|\langle \sigma_k, \sigma_1\rangle|^2 + \dots + |\langle \sigma_k, \sigma_k\rangle|^2 + \dots + |\langle \sigma_k, \sigma_n\rangle|^2 = 0$$

Since $\sigma_1, \dots, \sigma_n$ are orthonormal, $\langle \sigma_k, \sigma_1\rangle = \langle \sigma_k, \sigma_2\rangle = \dots = 0$

$$|\langle \sigma_k, \sigma_k\rangle|^2 = 0$$
$$\Rightarrow \sigma_k = 0. \quad \square.$$

Hence $\sigma_1, \dots, \sigma_n = 0 \Rightarrow \operatorname{span}(e_1, \dots, e_m) = V$. $\square$

Now suppose $\operatorname{span}(e_1, \dots, e_m) = V$, then

$$v = \langle v, e_1\rangle e_1 + \dots + \langle v, e_m\rangle e_m \text{ for } v \in V$$

By pythagoras thm, $\|v\|^2 = |\langle v, e_1\rangle|^2 + \dots + |\langle v, e_m\rangle|^2$ $\square$.

■

2) Inverse $\Leftrightarrow (T+I)^{-1}(a_1, a_1+a_2, \dots) = (a_1, a_2, \dots)$

$(I-T)$ $\qquad (a_1, a_2, a_3-a_1, a_4-a_2, a_5-a_3, \dots)$

$(I-T)+T^2$ $\qquad a_1, a_2, a_3, a_4+a_1, a_5+a_2, \dots)$

$I-T+T^2-T^3+T^4-\dots(-T)^k+\dots =$

$\Rightarrow (T+I)^{-1} = \sum_{r=0}^{\infty} (-T)^r$

b) $(T-\lambda I)(a_1, a_2, a_3, \dots) = (-\lambda a_1, a_1-\lambda a_2, a_2-\lambda a_3, \dots)$

Let $S(a_1, a_2, a_3, \dots) = (-\frac{1}{\lambda}a_1, -\frac{1}{\lambda^2}a_1 - \frac{1}{\lambda}a_2, -\frac{1}{\lambda^3}a_1 - \frac{1}{\lambda^2}a_2 - \frac{1}{\lambda}a_3, \dots)$

We will now prove that for
$(T-\lambda I)(a_1, a_2, a_3, \dots) = (-\lambda a_1, a_1-\lambda a_2, a_2-\lambda a_3, \dots)$
$= (b_1, b_2, b_3, \dots)$

$S(b_1, b_2, b_3, \dots) = (c_1, c_2, c_3, \dots) = (a_1, a_2, a_3, \dots)$.

By induction on c_i,
$i=1$: $(c_1, \dots) = S(b_1, \dots) = S(-\lambda a_1, \dots) = (-\frac{1}{\lambda}(-\lambda a_1), \dots) = (a_1, \dots)$ ✓.

I.H: Assuming that for $i=k$, $(\dots, c_k, \dots) = (\dots, -\frac{1}{\lambda^k}b_1 - \frac{1}{\lambda^{k-1}}b_2 - \dots - \frac{1}{\lambda}b_k, \dots)$
$= (\dots, a_k, \dots)$

For c_{k+1}, $\quad (\dots, c_{k+1}, \dots) = S(\dots, b_{k+1}, \dots)$
$= (\dots -\frac{1}{\lambda^{k+1}}b_1 - \frac{1}{\lambda^k}b_2 - \dots - \frac{1}{\lambda}b_{k+1}, \dots)$
$= (\dots, -\frac{1}{\lambda}[a_k - b_{k+1}], \dots)$

$= (\dots, -\frac{1}{\lambda}[a_k - (a_k - \lambda a_{k+1})], \dots)$
$= (\dots, -\frac{1}{\lambda}(-\lambda a_{k+1}), \dots)$
$= (\dots, a_{k+1}, \dots)$

$\square$.

$\therefore S = (T-\lambda I)^{-1}$.

However $S = T^{-1}$?
$ST(a_1, a_2, \dots) = S(0, a_1, a_2, \dots)$
$= (-\frac{1}{0}(0), -\frac{1}{0^2}a_1 - \frac{1}{0}a_2, \dots)$
$\neq (a_1, a_2, \dots)$

$\Rightarrow (T-(0)I)$ is not invertible
$\therefore \lambda = 0$

$\square$.

c) No eigenvalue.

$\therefore$ $T \cdot (a_1, a_2, a_3, \dots) = (0, a_1, a_2, a_3, \dots) = \lambda (a_1, a_2, a_3, \dots)$

$\Rightarrow \lambda a_1 = 0 \Rightarrow \lambda = 0$

Contradiction as

$0 \cdot (a_1, a_2, a_3, \dots) = \vec{0} \neq (0, a_1, a_2, a_3, \dots)$

$\Rightarrow$ no eigen value.

d). Discrepancy as $T - \lambda I$ not being invertible means there exist eigenvalue of λ.
But (c) show that no eigenvalue can exist.