1. Two fair six-sided dice are rolled (one green and one orange), with outcomes X and Y respectively for the green and the orange.

   (a) Compute the covariance of $X+Y$ and $X-Y$.

   (b) Are $X+Y$ and $X-Y$ independent? Show that they are, or that they aren't (whichever is true).

$$\begin{aligned}
\mathrm{Cov}(X+Y, X-Y) &= E((X+Y)(X-Y)) - E(X+Y)E(X-Y) \\
&= E(X^2-Y^2) - (E(X)+E(Y))(E(X)-E(Y)) \\
&= E(X^2) - E(Y^2) - [E(X)^2 - E(Y)^2] \\
&= E(X^2) - E(X)^2 - [E(Y^2) - E(Y)^2] \\
&= \mathrm{Var}(X) - \mathrm{Var}(Y) \\
&= 0
\end{aligned}$$

b) Not independent. Suppose $X+Y=12$,

$$P(X-Y=0 \mid X+Y=12) = 1 \neq P(X-Y=0)$$

2. A chicken lays a Poisson(λ) number N of eggs. Each egg, independently, hatches a chick with probability p. Let X be the number which hatch, so $X|N \sim \mathrm{Bin}(N,p)$.

   Find the correlation between N (the number of eggs) and X (the number of eggs which hatch). Simplify; your final answer should work out to a simple function of p (the λ should cancel out).

$E(X) = \sum x P(X=x)$

$N \sim \mathrm{Pois}(\lambda)$

$X|N \sim \mathrm{Bin}(N,p)$

$\frac{\lambda^k}{k!}$

$$\mathrm{Corr}(N,X) = \frac{\mathrm{Cov}(N,X)}{SD(N)SD(X)}$$

$\mathrm{Var}(X) = E(X^2) - E(X)^2$

$=$

$SD(N) = \sqrt{\lambda}$ $\quad SD(X) = \sqrt{\mathrm{Var}(X)} = \sqrt{Np(1-p)}$

$$\begin{aligned}
\mathrm{Cov}(N,X) &= E((N-E(N))(X-E(X))) \\
&= E(NX)^2 - E(N)E(X)
\end{aligned}$$

$\sum_{x=} x P(X=x)$

$E(N) = \lambda$

$E(X|N=n) = np$

$$\begin{aligned}
E(X) &= \sum_{i=0}^{\infty} E(X|N=i)P(N=i) \\
&= \sum_{i=0}^{\infty} ipe^{-\lambda}\frac{\lambda^i}{i!} \\
&= pe^{-\lambda}\sum_{i=0}^{\infty} i\frac{\lambda^i}{i!} \\
&= pe^{-\lambda}\sum_{i=1}^{\infty} \frac{\lambda^i}{(i-1)!}
\end{aligned}$$

$$\begin{aligned}
E(NX) &= \sum_{n=0}^{\infty}\left[nP(N=n)\sum_{x=0}^{n} xP(X=x)\right] \\
&= \sum_{n=0}^{\infty}[nP(N=n)np] \\
&= \sum_{n=0}^{\infty}\left[n^2pe^{-\lambda}\frac{\lambda^n}{n!}\right] \\
&= pe^{-\lambda}\sum_{n=0}^{\infty} n^2\frac{\lambda^n}{n!} \\
&= pe^{-\lambda}\sum_{n=1}^{\infty} n\lambda\frac{\lambda^{n-1}}{(n-1)!}
\end{aligned}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$= \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

$$xe^x = \sum_{i=0}^{\infty} \frac{x^{i+1}}{i!}$$

$$= \sum_{i=1}^{\infty} \frac{x^i}{(i-1)!} \quad \text{— (1)}$$

$$\Rightarrow E(X) = pe^{-\lambda} \sum_{i=1}^{\infty} \frac{\lambda^i}{(i-1)!}$$

$$= pe^{-\lambda}(\lambda e^{\lambda})$$

$$= \lambda p$$

$$= \lambda p e^{-\lambda} \sum_{n=1}^{\infty} n \frac{\lambda^{n-1}}{(n-1)!}$$

From (1), $xe^x = \sum_{i=0}^{\infty} \frac{x^{i+1}}{i!}$

$$(xe^x)\frac{d}{dx} = \sum_{i=0}^{\infty} (i+1) \frac{x^i}{i!}$$

$$e^x + xe^x = \sum_{i=1}^{\infty} i \frac{x^{i-1}}{(i-1)!}$$

$$\Rightarrow E(NX) = \lambda p e^{-\lambda} \sum_{n=1}^{\infty} n\lambda \frac{\lambda^{n-1}}{(n-1)!}$$

$$= \lambda p e^{-\lambda}[e^{\lambda} + \lambda e^{\lambda}]$$

$$= \lambda p(1+\lambda)$$

$$\Rightarrow Cov(N,X) = E(NX)^2 - E(N)E(X)$$

$$= [\lambda p(1+\lambda)]^2 - \lambda^2 p$$

$$= \lambda^2 p^2 (1+\lambda)^2 - \lambda^2 p$$

$$= \lambda^2 p[p(1+\lambda)^2 - 1]$$

$$Corr(N,X) = \frac{\lambda^2 p[p(1+\lambda)^2 - 1]}{\sqrt{\lambda}\sqrt{}}$$

I give up. $Y = N - X$, and X are independent

Let $Y = N - X$, $Y \sim Pois(\lambda(1-p))$, $X \sim Pois(\lambda p)$

$$Cov(N,X) = Cov(X+Y, X)$$

$$= Cov(X,X) + Cov(Y,X)$$

$$= Var(X) + Cov(Y,X)$$

$$= Var(X)$$

$$Corr(N,X) = \frac{Cov(N,X)}{SD(N)SD(X)}$$

$$= \frac{Var(X)}{SD(N)SD(X)} = \frac{SD(X)}{SD(N)} = \frac{SD(X)}{\sqrt{\lambda}}$$

$$Var(X) = \lambda p \Rightarrow SD(X) = \sqrt{\lambda p}$$

$$\Rightarrow Corr(N,X) = \sqrt{p} \quad //$$

4. Let $(X_1, \ldots, X_k)$ be Multinomial with parameters n and $(p_1, \ldots, p_k)$. Use indicator r.v.s to show that $\text{Cov}(X_i, X_j) = -np_ip_j$ for $i \neq j$.

$\text{Cov}(X$

4. $\text{Cov}(X_i, X_j) = E(X_iX_j) - E(X_i)E(X_j)$

WLOG, let $i=1, j=2$.

Let $X_1 = X_1^{(1)} + \ldots + X_n^{(1)}$ where $X_k^{(1)} \sim \text{Bern}(p_1)$
Let $X_2 = X_1^{(2)} + \ldots + X_n^{(2)}$ where $X_k^{(2)} \sim \text{Bern}(p_2)$

$\text{Cov}(X_1, X_2) = \text{Cov}(X_1^{(1)} + \ldots + X_n^{(n)}, X_1^{(2)} + \ldots + X_n^{(2)})$

$= 2\binom{n}{2}\text{Cov}(X_i^{(1)}, X_j^{(2)}) + n\text{Cov}(X_k^{(1)}, X_k^{(2)})$ where $i \neq j$

$= 2\binom{n}{2}[E(X_i^{(1)}X_j^{(2)}) - E(X_i^{(1)})E(X_j^{(2)})]$
$+ n[-E(X_k^{(1)})E(X_k^{(2)})]$

$= \frac{n}{n-2}[p_1p_2 - p_1p_2] - n(p_1p_2)$

$= -np_1p_2$

$\Rightarrow \text{Cov}(X_i, X_j) = -np_ip_j$ //

6. Consider the following method for creating a *bivariate Poisson* (a joint distribution for two r.v.s such that both marginals are Poissons). Let $X = V + W, Y = V + Z$ where V, W, Z are i.i.d. Pois(λ) (the idea is to have something borrowed and something new but not something old or something blue).

(a) Find $\text{Cov}(X, Y)$.

(b) Are X and Y independent? Are they conditionally independent given V?

(c) Find the joint PMF of X, Y (as a sum).

6a) $X = V + W, Y = V + Z$

$\text{Cov}(X, Y) = \text{Cov}(V+W, V+Z)$
$= \text{Cov}(V, V) + \text{Cov}(V, Z) + \text{Cov}(W, V) + \text{Cov}(W, Z)$
$= \text{Var}(V) + 0$
$= \lambda$ //

b) Not independent since $\text{Cov}(X, Y) \neq 0$
Conditionally independent given V.

c) $P(X=x, Y=y) = \sum_{v=0}^{\infty} P(X=x, Y=y | V=v)P(V=v)$

$= \sum_{v=0}^{\infty} P(W=x-v, Z=y-v)P(V=v)$

$= \sum_{v=0}^{\infty} P(W=x-v)P(Z=y-v)P(V=v)$

$= \sum_{v=0}^{\infty} e^{-\lambda}\frac{\lambda^{x-v}}{(x-v)!} e^{-\lambda}\frac{\lambda^{y-v}}{(y-v)!} e^{-\lambda}\frac{\lambda^{v}}{v!}$

$= e^{-3\lambda}\lambda^{x+y}\sum_{v=0}^{\infty}\frac{\lambda^{-v}}{(x-v)!(y-v)!v!}$ //

1. Let $X \sim \text{Unif}(0,1)$. Find the PDFs of X^2 and $\sqrt{X}$.

$X \sim Unif(0,1)$ Let $Y = X^2$

$= g(X)$ whee $g(x) = x^2$

$f_X(x) = 1$

$y = g(x)$

$y = x^2$

$\Rightarrow x = \sqrt{y}$ or $x = -\sqrt{y}$ (rej since $X \sim Unif(0,1)$)

$x = \sqrt{y}$

$\frac{dx}{dy} = \frac{1}{2}y^{-\frac{1}{2}}$

$\Rightarrow f_Y(y) = \frac{1}{2\sqrt{y}}$ //

Homework 8

1) Let X be a r.v for the no. of distinct birthdays in a group of 110 people.

Let $X = X_1 + \dots + X_{365}$ where $X_i \sim \text{Bern}(p)$ s.t $X_i = 1$ if a someone has a birthday on day i.

$$P(\text{someone has birthday on day } i) = 1 - P(\text{no one has a birthday on day } i)$$
$$= 1 - \left(\frac{364}{365}\right)^{110}$$
$$\Rightarrow p = 1 - \left(\frac{364}{365}\right)^{110}$$

$$E(X) = E(X_1 + \dots + X_{365})$$
$$= E(X_1) + \dots + E(X_{365})$$
$$= \left[1 - \left(\frac{364}{365}\right)^{110}\right] \cdot 365$$
$$= 365 - \frac{364^{110}}{365^{109}} \quad //$$

$$E(g(X)) = \sum g(x) P(X = x)$$

$$Var(X) = Var(X_1 + \dots + X_{365})$$
$$= Var(X_1) + \dots + Var(X_{365}) \text{ since } X_1, \dots, X_{365} \text{ are i.i.d.}$$

not indep.

$$Var(X_i) = E(X_i^2) - E(X_i)^2$$
$$= p - p^2$$
$$= p(1-p)$$

$$Var(X) = 365p(1-p)$$
$$= 365\left(1 - \left(\frac{364}{365}\right)^{110}\right)\left(\frac{364}{365}\right)^{110} \quad //$$

$$Var(X) = Var(X_1 + \dots + X_{365})$$
$$= 365 Var(X_i) + 2\binom{365}{2} Cov(X_i, X_j)$$

$$Cov(X_i, X_j) = E(X_i X_j) - E(X_i)E(X_j)$$
$$= 1 - P(\text{no one has birthdays on day } i \text{ and } j) - p^2$$
$$= 1 - \left(\frac{363}{365}\right)^{110} - p^2$$

$$Var(X) = 365p(1-p) + 2\binom{365}{2}\left[1 - \left(\frac{363}{365}\right)^{110} - p^2\right] \quad //$$

2) Let $X, Y \overset{iid}{\sim} N(0,1)$

Let $L = \max(X, Y)$ (larger)
$S = \min(X, Y)$ (smaller)

$\Phi \quad \varphi$

Find $\mathrm{Corr}(L,S)$.

$$\mathrm{Corr}(L,S) = \frac{\mathrm{Cov}(L,S)}{SD(L)SD(S)}$$

$E(L+S) = E(X+Y)$
$E(L) = E(X) + E(Y) - E(S)$
$E(L) = -E(S)$

$E(L-S) = E(|X-Y|) = \frac{2}{\sqrt{\pi}}$

$E(L) - E(S) = \frac{2}{\sqrt{\pi}}$

$E(L) = \frac{2}{\sqrt{\pi}} + E(S)$

$\frac{2}{\sqrt{\pi}} + E(S) = -E(S)$

$E(S) = -\frac{1}{\sqrt{\pi}} \Rightarrow E(L) = \frac{1}{\sqrt{\pi}}$

$$\begin{aligned} \mathrm{Cov}(L,S) &= E(LS) - E(L)E(S) \\ &= E(XY) + E(L)^2 \\ &= E(XY) + \frac{1}{\pi} \\ &= \frac{1}{\pi} \end{aligned}$$

$\mathrm{Var}(L) = E(L^2) - E(L)^2$

$\mathrm{Var}(L+S) = \mathrm{Var}(L) + \mathrm{Var}(S) + 2\mathrm{Cov}(L,S)$
$\mathrm{Var}(L-S) = \mathrm{Var}(L) + \mathrm{Var}(S) - 2\mathrm{Cov}(L,S)$

$\mathrm{Var}(L+S) = \mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) = 2$

$$\begin{aligned} \mathrm{Var}(L-S) = \mathrm{Var}(|X-Y|) &= E(|X-Y|^2) - E(|X-Y|)^2 \\ &= E((X-Y)^2) - E(|X-Y|)^2 \\ &= E(X^2) - 2E(XY) + E(Y^2) - \frac{4}{\pi} \\ &= -\frac{4}{\pi} + 2E(X^2) = \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(L) + \mathrm{Var}(S) + \frac{2}{\pi} &= 2 \\ \mathrm{Var}(L) + \mathrm{Var}(S) &= 2 - \frac{2}{\pi} \\ &= \frac{2\pi - 2}{\pi} \\ \mathrm{Var}(L) = \mathrm{Var}(S) &= \frac{\pi - 1}{\pi} \end{aligned}$$

$$\begin{aligned} \Rightarrow \mathrm{Corr}(L,S) &= \left(\frac{1}{\pi}\right) \div \left(\frac{\pi-1}{\pi}\right) \\ &= \frac{1}{\pi} \times \frac{\pi}{\pi - 1} = \frac{1}{\pi - 1} \mathbin{/\!/} \end{aligned}$$

3. Let R be a r.v for the no. of records made.

$R = R_1 + \dots + R_n$ where $R_i \sim \text{Bern}(p_i)$ s.t $R_i = 1$ if jumper i sets a record.

$$p_i = \frac{(i-1)!}{i!} = \frac{1}{i}$$

$$\text{Var}(R) = \text{Var}(R_1) + \dots + \text{Var}(R_n) + \sum_{j=i+1}^{n} \sum_{i=1}^{n-1} \text{Cov}(R_i, R_j)$$

$$\text{Cov}(R_i, R_j) = E(R_i R_j) - E(R_i)E(R_j)$$
$$= P(R_i = 1, R_j = 1) - \frac{1}{ij}$$

$$P(R_i = 1, R_j = 1) = P(R_i = 1 \mid R_j = 1) \cdot P(R_j = 1)$$
$$= \frac{1}{ij}$$

$$\Rightarrow \text{Cov}(R_i, R_j) = 0$$

$$\Rightarrow \text{Var}(R) = \sum_{i=1}^{n} \frac{1}{i}\left(1 - \frac{1}{i}\right)$$
$$= \sum_{i=1}^{n} \frac{1}{i} - \frac{1}{i^2} \quad //$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i} - \frac{1}{i^2} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i} - \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i^2}$$
$$\to \infty - c$$
$$= \infty \quad //$$

d) Let $D_2, \dots, D_n \overset{\text{i.i.d}}{\sim} \text{Bern}(p_i)$ s.t $D_i = 1$ if i and $i+1$ jumper both set records

$$D = D_2 + \dots + D_n$$

$$E(D) = E(D_2) + \dots + E(D_n)$$

$E(D_i) = P($both i and $i-1$ set records$)$ where $i > 1$

$$= \frac{(i-2)!}{i!} = \frac{1}{i(i-1)}$$

$$\frac{1}{1 \cdot \left(1 - \frac{1}{2}\right)}$$

$$E(D) = \sum_{i=2}^{n} \frac{1}{i(i-1)} = \left(\sum_{i=2}^{n} \frac{1}{i-1} - \frac{1}{i}\right)$$
$$= \left(1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \dots + \frac{1}{n-1} - \frac{1}{n}\right)$$
$$= 1 - \frac{1}{n} \quad //$$

mean $\to 1$ // when $n \to \infty$

4) $Z \sim N(0,1)$. Find PDF of Z^4.

$$f_Z = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$P(Z \le z) = \int_{-\infty}^{z} f_Z(x)\,dx$$

$$P(Z^4 \le z) = P(-(z)^{\frac{1}{4}} \le Z \le z^{\frac{1}{4}})$$

$$= P(Z \le z^{\frac{1}{4}}) - P(Z \le -(z^{\frac{1}{4}}))$$

$$P(Z^4 \le z) = \Phi(z^{\frac{1}{4}}) - \Phi(-z^{\frac{1}{4}})$$

$$P(Z^4 \le z)\frac{d}{dz} = \Phi(z^{\frac{1}{4}}) - \Phi(-z^{\frac{1}{4}})\frac{d}{dz}$$

$$= \varphi(z^{\frac{1}{4}}) \cdot \frac{1}{4} z^{-\frac{3}{4}} - \varphi(-z^{\frac{1}{4}})\left(-\frac{1}{4} z^{-\frac{3}{4}}\right)$$

$$= \varphi(z^{\frac{1}{4}})\frac{1}{4} z^{-\frac{3}{4}} + \varphi(-z^{\frac{1}{4}})\frac{1}{4} z^{-\frac{3}{4}}$$

$$= \frac{1}{4\sqrt{2\pi}} z^{-\frac{3}{4}}\left(e^{-\frac{\sqrt{z}}{2}} + e^{-\frac{\sqrt{z}}{2}}\right)$$

$$= \frac{1}{2\sqrt{2\pi}} z^{-\frac{3}{4}} e^{-\frac{\sqrt{z}}{2}} \quad //$$

5) Let X, Y be r.v with respective PDF f_X, f_Y.

$T = XY$.

a) PDF of T: $\int_0^\infty f_X(x) f_Y(t/x)\,dx$

CDF of T using LOTP:

$$P(T \le t) = \int_{-\infty}^{\infty} P(T \le t \mid X = x) f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} P(XY \le t \mid X = x) f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} P(Y \le \tfrac{t}{x}) f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} F_Y(\tfrac{t}{x}) f_X(x)\,dx$$

$$f_T = \int_{-\infty}^{\infty} F_Y(\tfrac{t}{x}) f_X(x)\,dx \frac{d}{dt}$$

$$= \int_{-\infty}^{\infty} f_Y(\tfrac{t}{x}) \cdot \frac{1}{x} f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{x} f_Y(\tfrac{t}{x}) f_X(x)\,dx \quad //$$

$\log T = \log X + \log Y$ $\qquad\qquad$ $f_Y(y)$

Let $Z = \log T \quad V = \log X \quad W = \log Y$

$Z = V + W$

$\Rightarrow f_Z(z) = \int_{-\infty}^{\infty} f_V(z-w) f_W(w)\, dw$

$Z = \log T \Rightarrow T = e^z$
$X = e^v$
$Y = e^w$

$P(V \le v) = F_V(v)$ $\qquad$ $w = \log Y$

$F_V(v) = P(\log X \le v) = P(X \le e^v)$ $\qquad$ $dw = \frac{1}{y} dy$

$f_V(v) = f_X(e^v) e^v$

$f_W(w) = f_Y(e^w) e^w$

$\Rightarrow f_Z(z) = \int_{-\infty}^{\infty} f_X(e^{z-y}) e^{z-x} f_Y(e^y) e^x \frac{1}{y} dy$

$f_Z(z) = \int_{-\infty}^{\infty} f_X(e^{z-y}) f_Y(e^y) e^z dy \frac{1}{y}$ ✓

$P(Z \le z) = P(\log T \le z)$
$= P(T \le e^z)$ $\qquad$ $f_T(t) = f_Z(\log t)$

$f_Z(z) = f_T(e^z) e^z$

$f_T(e^z) e^z = \int_{-\infty}^{\infty} f_X(e^{z-y}) f_Y(e^y) e^z dy \frac{1}{y}$

$f_T(e^z) = \int_{-\infty}^{\infty} f_X(e^{z-y}) f_Y(e^y)\, dy \frac{1}{y}$

let $z = \log t$

$f_T(t) = \int_{-\infty}^{\infty} f_X\left(\frac{t}{e^y}\right) f_Y(y) \frac{dy}{y}$

$= \int_0^{\infty} f_X\left(\frac{t}{x}\right) f_Y(t) \frac{dy}{y}$ //

6) $f(x,y) = g(x^2+y^2)$

a) $P(R \le r, \theta \le k) =$

$$PDF_{X,Y} = g(x^2+y^2)$$

$$PDF_{r,\theta} = g(r^2\cos^2\theta + r^2\sin^2\theta) \begin{vmatrix} x_r & y_r \\ x_\theta & y_\theta \end{vmatrix}$$

$$= g(r^2) \begin{vmatrix} \cos\theta & \sin\theta \\ -R\sin\theta & R\cos\theta \end{vmatrix}$$

$$= g(r^2)\ R\cos^2\theta + R\sin^2\theta$$

$$= r g(r^2) //$$

b) $PDF_{r,\theta} = r g(r^2)$

If uniform, then $g = 1$

$\Rightarrow PDF_{r,\theta} = \frac{r^3}{\pi r^2} = \frac{r}{\pi}$ where $0 \le r \le 1$.

c) $f(x,y) = g(x^2+y^2)$

$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ $\quad f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$

$$f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}r^2} \cdot r$$

$$= \frac{r}{2\pi} e^{-\frac{1}{2}r^2} //$$

7) a) Given a random selection of k nodes, what is the probability that they form a clique.

$P(\text{clique}) = P(\text{node 1 is connected to } k-1) \cdot P(\text{node 2 connected to } k-2) \cdots P(\text{node } k-1 \text{ connected to node } k)$

$= \left(\frac{1}{2}\right)^{k-1} \cdot \left(\frac{1}{2}\right)^{k-2} \cdot \left(\frac{1}{2}\right)^{k-3} \cdots \left(\frac{1}{2}\right)$

$= \left(\frac{1}{2}\right)^{\frac{(k-1)k}{2}}$

$E(\text{no. of clique}) = \binom{n}{k}\left(\frac{1}{2}\right)^{\frac{(k-1)k}{2}}$ //

b) let T be a r.v for the no. of cliques of size 3 given a n node random network.

$T = T_1 + \dots + T_{\binom{n}{3}}$ where $T_i \sim Bern(p_i)$ s.t $T_i = 1$ if a i^{th} selection of 3 nodes form a clique.

$Var(T) = E(T^2) - E(T)^2$

$= \binom{n}{3} Var(T_i) + 2\binom{\binom{n}{3}}{2} Cov(T_i, T_j)$

$Cov(T_i, T_j) = E(T_i T_j) - E(T_i)E(T_j)$

$E(T_i T_j) = P(T_i = 1, T_j = 1)$

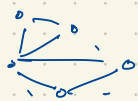
$= P(T_j = 1 \mid T_i = 1) P(T_i = 1)$

$= \frac{1}{8} P(T_j = 1 \mid T_i = 1)$

$\binom{n}{5} + \binom{n}{4} + \binom{n}{6} = \binom{\binom{n}{3}}{2}$

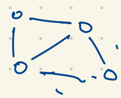
Suppose T_j and T_i share no nodes, then

$E(T_i T_j) = \frac{1}{8} \cdot \frac{1}{8} = \frac{1}{64} \Rightarrow Cov(T_i, T_j) = 0$



Suppose T_i and T_j share one node, then

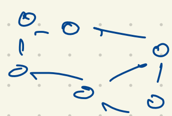
$E(T_i T_j) = \frac{1}{8} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{64} \Rightarrow Cov(T_i, T_j) = 0$



Suppose T_i and T_j share two nodes, then

$E(T_i T_j) = \frac{1}{8} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{32}$

$\Rightarrow Cov(T_i, T_j) = \frac{1}{32} - \frac{1}{64} = \frac{1}{64}$



$Var(T) = \binom{n}{3} \frac{1}{8}\left(1 - \frac{1}{8}\right) \cdot \binom{n}{4} \cdot \binom{4}{2} \frac{1}{64} \cdot 2$

$= \frac{7}{64}\binom{n}{3} + \frac{3}{16} \cdot \binom{n}{4}$ //