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Midterm

Name: ______________________________

- This quiz is **closed book**, but you may have one $8.5 \times 11''$ sheet with notes in your own handwriting on both sides.
- Calculators are not allowed.
- You may assume all of the results presented in class.
- Please show your work. Partial credit cannot be given for a wrong answer if your work isn't shown.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- Be neat and write legibly. You will be graded not only on the correctness of your answers, but also on the clarity with which you express them.
- If you get stuck on a problem, move on to others. The problems are not arranged in order of difficulty.
- The exam ends at 9:30 PM.

Problem	Points	Grade	Grader
1	10	0	
2	10	10	
3	20	20	
4	15	15	
5	20	20	
6	25	25	
7	10	5	
8	10	8	
Total	120	103	

85.8/100

Problem 1. [10 points]

Consider these two propositions:

$$\texttt{P:} \quad (A \vee B) \Rightarrow C$$

$$\texttt{Q:} \quad (\neg C \Rightarrow \neg A) \vee (\neg C \Rightarrow \neg B)$$

Which of the following best describes the relationship between P and Q? Please circle exactly one answer.

$\neg C \Rightarrow \neg A \vee \neg C \Rightarrow \neg B$

$\neg C \Rightarrow \neg (A \vee B)$

$\neg A \wedge \neg B$

1. P and Q are equivalent
2. $P \Rightarrow Q$
3. $Q \Rightarrow P$
4. All of the above
5. None of the above

F	F	T
F	T	T
T	F	F
T	T	T

Draw a truth table to illustrate your reasoning. You can use as many columns as you need.

A	B	$\neg A$	$\neg B$	C	$\neg C$	$A \vee B \Rightarrow C$ (P)	$\neg C \Rightarrow \neg A$	$\neg C \Rightarrow \neg B$	$(\neg C \Rightarrow \neg A) \vee (\neg C \Rightarrow \neg B)$ (Q)
F	F	T	T	F	T	T	T	T	T
F	F	T	T	T	F	T	T	T	T
F	T	T	F	F	T	F	T	F	T
F	T	T	F	T	F	T	T	T	T
T	F	F	T	F	T	F	F	T	T
T	F	F	T	T	F	T	T	T	T
T	T	F	F	F	T	F	F	F	F
T	T	F	F	T	F	T	T	T	T

$P \Rightarrow Q$	$Q \Rightarrow P$
T	T
T	T
T	F
T	
F T	
T	
T	
T	

−10

Problem 2. [10 points]

Let $G_0 = 1$, $G_1 = 3$, $G_2 = 9$, and define

$$G_n = G_{n-1} + 3G_{n-2} + 3G_{n-3} \tag{1}$$

for $n \geq 3$. Show by induction that $G_n \leq 3^n$ for all $n \geq 0$.

By strong induction,

Let $p(n) ::= G_n = G_{n-1} + 3G_{n-2} + 3G_{n-3}$, $G_n \leq 3^n$ for $\forall n \in \mathbb{Z}^+$. $G_0 = 1, G_1 = 3, G_2 = 9$.

Base case, $n = 3$: $G_3 = G_2 + 3G_1 + 3G_0$

$= 9 + 9 + 3$

$= 21$

$3^3 = 27 > G_3 \Rightarrow p(n)$ is true for $n = 3$.

Inductive step: Assume $p(n)$ is true for $k \in \mathbb{Z}^+$, $0 \leq k \leq n$,

For $p(n+1)$, $G_{n+1} = G_n + 3G_{n-1} + 3G_{n-2}$

$= 3^n + 3 \cdot 3^{n-1} + 3 \cdot 3^{n-2}$

$= 3^n + 3^n + 3^{n-1}$

$= 2 \cdot 3^n + 3^{n-1}$

$= 3^n (2 + \frac{1}{3})$

$= 3^n (\frac{7}{3}) < 3^{n+1}$

$\Rightarrow G_{n+1} < 3^{n+1}$

$\Rightarrow p(n+1)$ is true.

$\Rightarrow p(n)$ is true for $\forall n \in \mathbb{Z}^+$.

□.

Problem 3. [20 points]

In the game of Squares and Circles, the players (you and your computer) start with a shared finite collection of shapes: some circles and some squares. Players take turns making moves. On each move, a player chooses any two shapes from the collection. These two are replaced with a single one according to the following rule:

A pair of identical shapes is replaced with a square. A pair of different shapes is replaced with a circle.

At the end of the game, when only one shape remains, you are a winner if the remaining shape is a circle. Otherwise, your computer wins.

(a) [5 pts] Prove that the game will end.

Let $S(n)$ be the number of shapes on turn n.

Lemma: For $a, b \in \mathbb{N}$, if $a < b$, then $S(a) > S(b)$.

Pf: If $a < b$, then there are $b-a$ turns made after turn a.

By induction on the number of turns made after a,

Base case: $b-a = 1$.

For one turn, the player picks two shapes. Any two shapes are removed and 1 shape is added. The total number of shapes decrease by $1 \Rightarrow S(a) > Sb$.

Inductive step:

Suppose lemma is true for some $b-a = k$, then for $b-a+1$,

One turn before, there are $S(b)$ shapes. By the IH, $S(b) < S(a)$.

For turn $b+1$, shapes decrease by 1 as shown similarly in lemma 1.

$\Rightarrow S(b+1) < S(b) < S(a)$

$\Rightarrow S(b+1) < S(a)$

$\Rightarrow S(a) > S(b)$ for $\forall a, b \in \mathbb{N}, a < b$ $\square$.

Since $S(b) < S(a)$ for $\forall a, b \in \mathbb{N}, a < b$, $S(n)$ is strictly decreasing each turn

$\Rightarrow$ at some point, $S(n)$ reaches a minimum where either $S(n)$ cannot decrease further

$\Rightarrow$ game ends.

(b) [15 pts] Prove that you will win if and only if the number of circles initially is odd.

Circle   square
odd   doesn't matter

Lemma: If no. of circles is odd initially, there will always be an odd amount of circle for the entire game.

Pf: By cases on methods of making a move.
Let n be the no of circles. n is odd. Let m be the no. of squares.
Case 1: Pick both circles: If 2 circles picked, a square is added. there will be
$n-2$ circles $\Rightarrow$ odd, $m+1$ square
$\Rightarrow$ case 1 is odd.

case 2: pick circle & square; a circle is added.
There are $n-1+1$ circles $\Rightarrow$ n circles $\Rightarrow$ odd
$m-1$ squares.

Case 3: pick both squares. $\Rightarrow$ 1 square removed.
n circles, $m-1$ squares.
$\Rightarrow$ ~~odd~~ odd circles.

$\therefore$ Lemma is true.

At each turn, there will be pieces removed. Since the game must end. The game must end with an odd amt of circles. Thus there will always be 1 circle left if the game starts with even circles.

Problem 4. [15 points]

$11 \cdot 11 = 121$

(a) [8 pts] Find a number $x \in \{0, 1, \ldots, 112\}$ such that $11x \equiv 1 \pmod{113}$.

$11x \equiv 1 \pmod{113}$.

113 is a prime. $\Rightarrow$ $11^{112} \equiv 1 \bmod 113$

$11 \cdot 11^{111} \equiv 1 \bmod 113$ by Fermats little theorem.

$121 - 113 = 8$

$11^2 = 121 \equiv 8 \bmod 113$

$11^4 = 11^2 \cdot 11^2 \equiv 8 \cdot 8$
$= 64$

$11^8 = 11^4 \cdot 11^4 \equiv 64 \cdot 64$
$= 8 \cdot 8 \cdot 8 \cdot 8$
$= 512 \cdot 8$
$\equiv 60 \cdot 8 \bmod 113$
$\equiv 28 \bmod 113$

$11^{16} \equiv 28 \cdot 28 \bmod 113$
$= 2^4 \cdot 7^2 \bmod 113$
$= 224 \cdot 7$
$\equiv 111 \cdot 7$
$\equiv 99 \bmod 113$.

$11^{32} \equiv 99^2 = 11^2 \cdot 9^2$
$= 11^2 \cdot 3^2$
$= 8 \cdot 9$
$= 72$

$64 \cdot 9 = 576$

$11^{64} = 72^2$
$= 8^2 \cdot 9^2$
$= 576 \cdot 9$
$\equiv 11 \cdot 9$
$\equiv 99$

$8^2 = 2^6 \cdot 3^4$

$113 \cdot 4 = 452$ $\quad$ $113 \cdot 5 = 565$

$576 - 565 = 11$

$11^{111} = 11^{64+32+8+2+4+1}$
$= 99 \cdot 72 \cdot 28 \cdot 64 \cdot 8 \cdot 11$
$\equiv 99 \cdot 72 \cdot 28 \cdot 60 \cdot 11$
$= 9 \cdot 11 \cdot 9 \cdot 8 \cdot 2^2 \cdot 7 \cdot 2 \cdot 3 \cdot 10 \cdot 11$
$= 9^2 \cdot 11^2 \cdot 2^6 \cdot 7 \cdot 3 \cdot 5 \cdot 2$
$\equiv 72 \cdot 2^6 \cdot 7 \cdot 3 \cdot 5 \cdot 2$
$\equiv 8 \cdot 9 \cdot 7 \cdot 3 \cdot 5 \cdot 2$
$= 2^4 \cdot 9 \cdot 2^6 \cdot 7 \cdot 3 \cdot 5$
$= 2^{10} \cdot 9 \cdot 7 \cdot 3 \cdot 5$
$= 2^{10} \cdot 3^3 \cdot 7 \cdot 5$
$= 2 \cdot 3 \cdot 5 \cdot 2 \cdot 2^8 \cdot 3^2 \cdot 7$
$= 250 \cdot 2^8 \cdot 3^2 \cdot 7$
$= 2^2 \cdot 7$
$= 2^{10} \cdot 3^2 \cdot 7^2$
$\equiv 99 \cdot 2^6 \cdot 3^2$
$\equiv 99 \cdot 11$
$\equiv 11^2 \cdot 9$
$\equiv 8 \cdot 9 = 72$

$64 + 32 = 96$, $111 - 96 = 15$

$7 \cdot 5 = 35$
$35 \cdot 27 \cdot 2^{10}$

24

Ans: 72.

(b) [7 pts] Find a number $y \in \{0, 1, \ldots, 112\}$ such that $11^{112111} \equiv y \pmod{113}$ (*Hint: Note that 113 is a prime.*)

$11^{112111} =$

112111
$= 111 \cdot 111$
$= 111 \cdot 100 + 111 \cdot 11$
$= 11100 + 1221$
$= 22311$

$11 \cdot 111 = 1100 + 111 = 1221$

$\times 5$
111555

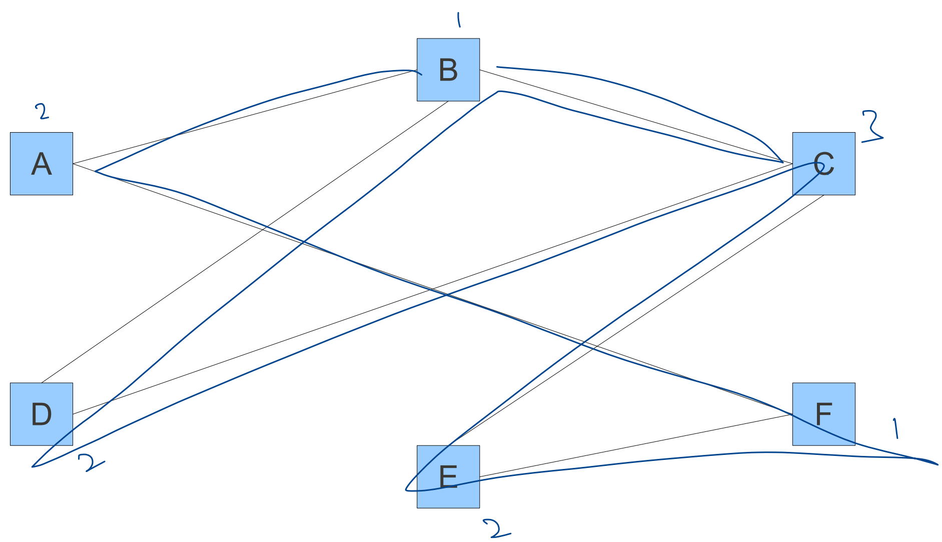
$11^{112000} \cdot 11^{111}$
$= (11^{112})^{1000} \cdot 11^{111}$
$\equiv (1)^{1000} \cdot 11^{111}$
$= 11^{111}$
$\equiv 72$ //

$11^{112000} \cdot 11^{111}$
$=$

Problem 5. [20 points]

Consider the simple graph G given in figure 1.

Figure 1: Simple graph G



(a) [4 pts] Give the diameter of G.

3

(b) [4 pts] Give a Hamiltonian Cycle on G.

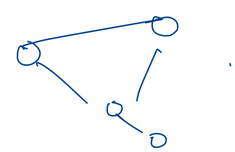
ABDCEFA

(c) [4 pts] Give a coloring on G and show that it uses the smallest possible number of colors.

3 coloring.

A=2. B=1 C=3 E=2 F=1 D=2.

By contradiction, suppose a 2 coloring of G exists.
For C, since C is adjacent to B, D, E, C must be colored x and BDE colored y.
However, B is adjacent to D, hence either B or D must be colored x. If one is colored y.
However this is a contradiction as for C to be colored x, B and D must both be y.
Thus 3 coloring is smallest.

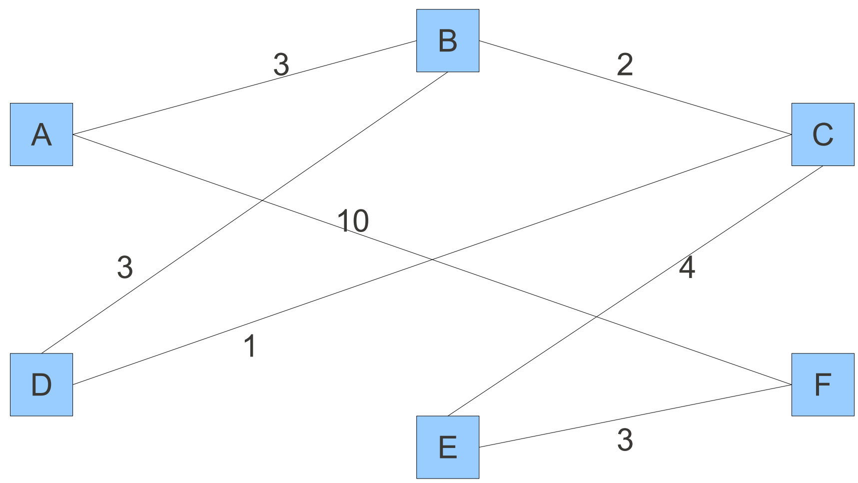


(d) [4 pts] Does G have an Eulerian cycle? Justify your answer.

No. Eulerian implies all vertex must have even degrees. However, there are vertex with odd degree ⇒ not Eulerian.

Now consider graph H, which is like G but with weighted edges, in figure 2:

Figure 2: Weighted graph H



(e) [4 pts] Give a list of edges reflecting the order in which one of the greedy algorithms presented in class (i.e. in lecture, recitation, or the course text) would choose edges when finding an MST on H.

D – C, C – B, B – A, E – F, E – C
1 2 3 3 4

Problem 6. [25 points] Let G be a graph with m edges, n vertices, and k components. Prove that G contains at least $m - n + k$ cycles. (Hint: Prove this by induction on the number of edges, m)

By induction on the number of edges,

Base case: $m=1$.

$1-n+k = k+1-n$. Since 1 edge only, there is only 1 component

$\Rightarrow k+1=2 \Rightarrow$ there are $2-n$ cycles

$2-n<0$

$\Rightarrow$ proposition is true.

Induction step: $m+1$,

If $m+1$ edges, suppose we remove an edge, then there are $m-n+k_0$ cycles by the inductive hypothesis. Let this graph have m edges, n vertices, k_0 components.

Now we add back the edge. This edge can either be a unique path between 2 components, or it can be an edge between 2 connected components.

If unique path, then k decreases by 1. There are no changes in the no. of cycles.
Let k_1 be the amount of components for the initial graph.
$k_1 = k_0 - 1$. The number of cycles is $m+1-n+k_0-1 = m-n+k_0$.

$\Rightarrow$ new graph has same amt of cycles as old graph

$\Rightarrow$ proposition is true for this case.

If edge between 2 components, then no. of cycles by proposition is $m+1-n+k = m-n+k+1$

$\Rightarrow$ 1 more than before

$\Rightarrow$ true since the addition of this edge must make at least 1 new cycle

$\therefore$ proposition is true for all amt of edges.

$\square$.

Problem 7. [10 points] For the following sum, find an upper and a lower bound that differ by at most 1.

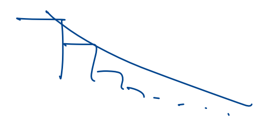
$$\sum_{i=1}^{\infty} \frac{1}{\sqrt{i^3}}$$

$$\int_1^\infty \frac{1}{i^{\frac{3}{2}}}\,di = \int_1^\infty i^{-\frac{3}{2}}\,di = -\frac{1}{2}\, i^{-\frac{1}{2}} \Big|_1^\infty$$

$$= -\frac{1}{2\sqrt{i}} \Big|_1^\infty$$

$$= -\left(0 - \frac{1}{2}\right)$$

$$= \frac{1}{2}.$$



$$\frac{1}{2} < \sum_{i=1}^{\infty} \frac{1}{\sqrt{i^3}} < \frac{3}{2} \quad //$$

-3

Problem 8. [10 points] State whether each of the following claims is True or False and prove your answer.

(a) [2 pts] $x \ln x$ is $O(x)$

$x \ln x = O(x) \Rightarrow \lim_{x \to \infty} \frac{x \ln x}{x} < \infty$

$x = O(x^3)$ $\frac{x}{x^3}$

$\lim_{x \to \infty} \frac{x \ln x}{x} = \lim_{x \to \infty} \ln x = \infty$

$\Rightarrow$ False.

(b) [2 pts] $x/100$ is $o(x)$

$\frac{x}{100} = o(x) \Rightarrow \lim_{x \to \infty} \frac{\left(\frac{x}{100}\right)}{x} = \infty$ 0

$\lim_{x \to \infty} \frac{x}{100} \cdot \frac{1}{x} = \frac{1}{100} \neq \infty$

$\Rightarrow$ false.

-2

(c) [2 pts] x^{n+1} is $\Omega\left(x^n\right)$

$$x^{n+1} = \Omega(x^n) \Rightarrow \lim_{x \to \infty} \frac{x^{n+1}}{x^n} > 0.$$

$$\lim_{x \to \infty} \frac{x^{n+1}}{x^n} = \lim_{x \to \infty} x = \infty$$

$\Rightarrow$ true

(d) [4 pts] $n!$ is $\Theta\left(n^n\right)$.

$$n! = \Theta(n^n) \Rightarrow \lim_{n \to \infty} \frac{n!}{n^n} = c.$$

$$\lim_{n \to \infty} \frac{n!}{n^n} = \frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \cdot \frac{4}{n} \cdot \ldots \cdot \frac{n}{n}.$$

$$= 0 \cdot 0 \cdot 0 \cdot \ldots \cdot \frac{n-1}{n} \cdot 1.$$

$$= 0.$$

$\Rightarrow$ False.