Q3

i) $\ddot{x} + \dot{x} + kx = 0$

$p(s) = s^2 + s + k$

$$s = \frac{-1 \pm \sqrt{1 - 4(1)(k)}}{2}$$

For critical damping, $\sqrt{1-4k} = 0$

$1 - 4k = 0$

$4k = 1$

$k = \frac{1}{4}$ //

b) For $k > \frac{1}{4}$

$4k > 1$

$4k - 1 > 0$

$1 - 4k < 0$

$\sqrt{1-4k} < 0$

$\Rightarrow$ complex roots

$\Rightarrow$ underdamped //

c) $\ddot{x} + \dot{x} + kx = 0$

$p(s) = s^2 + s + k$

$x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

$$s = \frac{-1 \pm \sqrt{1 - 4(1)(k)}}{2}$$

$$s = -\frac{1}{2} \pm \frac{\sqrt{1-4k}}{2}$$

$$= -\frac{1}{2} \pm i\frac{\sqrt{4k-1}}{2}$$

$x = e^{-\frac{1}{2}t}(A\cos\omega t + B\sin\omega t)$

$\omega = \frac{2\pi}{2} = \pi$

$\frac{\sqrt{4k-1}}{2} = \pi$

$\sqrt{4k-1} = 2\pi$

$4k - 1 = 4\pi^2$

$4k = 4\pi^2 + 1$

$k = \pi^2 + \frac{1}{4}$ //

20

2a) $\ddot{x}+x=5te^{2t}$

$5te^{2t}$

$10te^{2t}+5e^{2t}$

Let trial solution $x = Ate^{2t}+Be^{2t}$

$\dot{x} = 2Ate^{2t}+Ae^{2t}+2Be^{2t}$

$\ddot{x} = 4Ate^{2t}+2Ae^{2t}+2Ae^{2t}+4Be^{2t}$

$\ddot{x}+x = 4Ate^{2t}+4Ae^{2t}+4Be^{2t}+Ate^{2t}+Be^{2t}$

$5Ate^{2t}+4Ae^{2t}+5Be^{2t}=5te^{2t}$

$5Ate^{2t}+e^{2t}(4A+5B)=5te^{2t}$

$5A=5 \qquad 4A+5B=0$

$A=1 \qquad 4+5B=0$

$5B=-4$

$B=-\frac{4}{5}$

$x_p = te^{2t}-\frac{4}{5}e^{2t}$

b) $y(0)=1 \quad \dot{y}(0)=2 \qquad$ At $t=0$,

$p(s)=s^2+1$

$s=i$ or $s=-i$

$x_h=c_1e^{it}+c_2e^{-it}$

$= A\cos t+B\sin t$

$x = x_h+x_p$

$= A\cos t+B\sin t+x_p$

$x(0)=A+x_p(0)=3$

$\dot{x}=-A\sin t+B\cos t+\dot{x}_p$

$\dot{x}(0)=B+\dot{x}_p(0)=5$

$\ddot{y}(0)=-1$

$\ddot{x}=-A\cos t-B\sin t+\ddot{x}_p$

$\ddot{x}(0)=-A+\ddot{x}_p(0)=-3$

$A+x_p(0)=3$

$B+\dot{x}_p(0)=5$

$-A+\ddot{x}_p(0)=-3$

Let $x_p=y$

$A+1=3 \qquad A=2$

$B+2=5 \qquad B=3$

$-A-1=-3$

$x=2\cos t+3\sin t+y$

20

3) $\ddot{x} + b\dot{x} + kx = \cos(\omega t)$

$= Re(e^{i\omega t})$

$x = \dfrac{e^{i\omega t}}{p(i\omega)}$ $\quad p(s) = s^2 + bs + k$

$x = \dfrac{e^{i\omega t}}{(i\omega)^2 + bi\omega + k} = \dfrac{e^{i\omega t}}{-\omega^2 + ib\omega + k}$

$= \dfrac{e^{i\omega t}}{(-\omega^2 + k) + ib\omega}$

To maximize x, we must minimize $|-\omega^2 + ib\omega + k|^2$

$(-\omega^2 + k)^2 + (b\omega)^2 = \omega^4 - 2k\omega^2 + k^2 + b^2\omega^2$

$(\omega^4 - 2k\omega^2 + k^2 + b^2\omega^2)\dfrac{d}{dk} = -2\omega^2 + 2k = 0$

$2k = 2\omega^2$

$k = \omega^2$ //

b) $\dddot{x} - \dot{x} = 0$

$p(r) = r^3 - r = 0$

$r(r^2 - 1) = 0$

$r(r+1)(r-1) = 0$

$r = 0,\ r = 1,\ r = -1$

$x = C_1 e^{0t} + C_2 e^{t} + C_3 e^{-t}$

$= C_1 + C_2 e^{t} + C_3 e^{-t}$ //

20

4) $\ddot{x} + \dot{x} + 6x = 6y$ where y is sinusoidal.

Let $y = \cos\omega t$

$$\ddot{x} + \dot{x} + 6x = 6\cos\omega t$$

$$\ddot{x} + \dot{x} + 6x = \mathrm{Re}\left(6e^{i\omega t}\right)$$

$$p(s) = s^2 + s + 6$$

$$p(i\omega) = -\omega^2 + i\omega + 6$$

$$x = \frac{6e^{i\omega t}}{(6-\omega^2) + i\omega}$$

$$H(\omega) = \frac{1}{(6-\omega^2) + i\omega} \qquad 8$$

At $\omega = 2$, $H(2) = \dfrac{1}{2+2i} = \dfrac{2-2i}{8} = \dfrac{1}{4} - \dfrac{1}{4}i$

Gain: $|H(2)| = \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2}$

$$= \frac{1}{2\sqrt{2}} \qquad 5$$

Phase lag: $\tan\phi = \dfrac{-0.25}{0.25}$

$$\phi = -\frac{\pi}{4} \qquad 3$$

16

5) $m\ddot{x}+b\dot{x}+kx=4\cos(2t)$

$x_p=\frac{1}{2}t\sin(2t)$

a) $x_p=\frac{1}{2}t\sin(2t-1)$ $\checkmark$ 2

b) $x_p=t\sin(2t)$ $\checkmark$ 4

c) $x=\frac{1}{2}t\sin 2t$

$\dot{x}=\frac{1}{2}\sin 2t+t\cos 2t$

$\ddot{x}=\cos 2t-2t\sin 2t+\cos 2t$

$=2\cos 2t-2t\sin 2t$

$2m\cos 2t-2mt\sin 2t+\frac{b}{2}\sin 2t+tb\cos 2t+\frac{k}{2}t\sin 2t=4\cos(2t)$

Let $t=0$

$2m=4$

$m=2$

$t=\frac{\pi}{2}$

$-4-\frac{\pi}{2}b=-4$

$-\frac{\pi}{2}b=0$

$b=0$

Let $t=\frac{\pi}{4}$

$-2\cdot 2\cdot\frac{\pi}{4}+\frac{b}{2}+\frac{k}{2}\frac{\pi}{4}=0$

$-\pi+\frac{b}{2}+\frac{\pi k}{8}=0$

$-\pi+\frac{\pi k}{8}=0$

$\frac{\pi k}{8}=+\pi$

$\frac{k}{8}=+1$

$k=+8$

11

$m=2,\ b=0,\ k=-8$ $\checkmark$

17