Problem 1. [20 points] [15] For each of the following, either prove that it is an equivalence relation and state its equivalence classes, or give an example of why it is not an equivalence relation.

(a) [5 pts] $R_n := \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \text{ s.t. } x \equiv y \pmod{n}\}$

(b) [5 pts] $R := \{(x, y) \in P \times P$ s.t. x is taller than $y\}$ where P is the set of all people in the world today.

(c) [5 pts] $R := \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \text{ s.t. } gcd(x, y) = 1\}$

(d) [5 pts] $R_G :=$ the set of $(x, y) \in V \times V$ such that V is the set of vertices of a graph G, and there is a path $x, v_1, \ldots, v_k, y$ from x to y along the edges of G.

a) Transitivity: $x \equiv y \bmod n \Rightarrow n \mid y - x$.
$y \equiv z \bmod n \Rightarrow n \mid z - y$.
If transitivity, then $x \equiv z \bmod n \Rightarrow z - x \mid n$.

$(y - x) = kn$
$y = kn + x$.
$n \mid z - kn - x$
$n \mid z - x - kn \Rightarrow n \mid z - x$
$\Rightarrow x \equiv z \bmod n$
$\Rightarrow$ transitivity applies.

Symmetry: $x \equiv y \bmod n \Rightarrow n \mid y - x$
$(y - x) = kn$
$x - y = -kn$
$\Rightarrow n \mid x - y$
$\Rightarrow$ symmetric.

Reflexivity $x \equiv x \bmod n$
$n \mid x - x$
$\Rightarrow n \mid 0$ ✓.
$\Rightarrow$ reflexivity.

∴ Congruency is an equivalence relation.

b) Reflexivity: x is not taller than themselves $\Rightarrow$ not reflexive.

Symmetry. x is taller than y $\not\Leftrightarrow$ y is taller than x.

c) Symmetry: $gcd(x, y) = gcd(y, x)$ ✓
Reflexivity: $gcd(x, x) \neq 1$.
$\Rightarrow$ not equivalence relation.

d) Reflexivity: If $xRy \Rightarrow x, v_1, v_2, \ldots, v_k, y, v_k, \ldots v_1, x \Rightarrow$ path from x to x ✓.

Symmetric: path from x to y $\Rightarrow$ path from y to x. ✓.

Transitivity: path from $x \to y$ and path from $y \to z \Rightarrow x \to z$
since $x \to y \to z$.
✓.

Problem 2. **[20 points]** Every function has some subset of these properties:

injective surjective bijective

Determine the properties of the functions below, and briefly explain your reasoning.

(a) [5 pts] The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x\sin(x)$.

(b) [5 pts] The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 99x^{99}$.

(c) [5 pts] The function $f : \mathbb{R} \to \mathbb{R}$ defined by $\tan^{-1}(x)$.

(d) [5 pts] The function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) =$ the number of numbers that divide x. For example, $f(6) = 4$ because $1, 2, 3, 6$ all divide 6. *Note: We define here the set $\mathbb{N}$ to be the set of all positive integers $(1, 2, \ldots)$.*

a) $f(x) = x\sin x$.
not injective, since $f(0) = f(2\pi)$.

range of $f(x)$ is $\pm\infty$ $\Rightarrow$ the whole range of $\mathbb{R}$ is covered by $f(x)$
$\Rightarrow$ surjective.

Since $f(x)$ is not injective, $f(x)$ is not bijective.

b) $f(x) = 99x^{99}$. Range of $f(x)$ is $\pm\infty \Rightarrow$ surjective.

$y = 99x^{99}$

$\left(\frac{y}{99}\right)^{\frac{1}{99}} = x$.

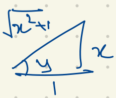
$f'(x) = 99^2 x^{98} \geq 0$
$\Rightarrow f'(x) \geq 0$ for all x.
$f'(x) = 0$ at $x = 0$
$\Rightarrow$ inflection point.
$\Rightarrow$ no repeated values since $f(x)$ has no negative gradient
$\Rightarrow$ injective.

$\left(\frac{y}{99}\right)^{\frac{1}{99}}$ is defined every and range is $\pm\infty$
$\Rightarrow$ total.

Since $y = 99x^{99}$ is surjective, injective, and total, y is bijective. $\square$.

c) $\tan^{-1}(x)$ range is between $\pm\frac{\pi}{2}$ $\not\Rightarrow$ not total and surjective. $\Rightarrow$ not bijective.

$y = \tan^{-1}(x)$.
$\tan y = x$



$\cos y = \frac{1}{\sqrt{x^2+1}}$
$\sec^2 y = x^2 + 1$.

$\sec^2 y \frac{dy}{dx} = 1$.

$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{x^2+1} > 0$ for $\forall x$.

$\Rightarrow \frac{dy}{dx} > 0$ for all x
$\Rightarrow$ y is constantly increasing
$\Rightarrow$ $\nexists a, b,\ a \neq b : \tan^{-1}(a) = \tan^{-1}(b)$
$\Rightarrow$ injective

$\Rightarrow$ $\tan^{-1} x$ is injective and not surjective and bijective.

d) $\forall x \in \mathbb{N}, f(x) ::= \{ y \in \mathbb{N} \mid y \mid x \}$.

Surjective since, Let $\exists a, b \in \mathbb{N}$, $a \cdot b = c \in \mathbb{N}$.

Hence for all values of a and b, there exists a larger value c that is a multiple of a and b.

Thus, the whole range of $\mathbb{N}$ can be present from the result of $f(x)$.

Not injective since 1 is mapped as the result for all elements in $\mathbb{N}$.

Since $f(x)$ is not injective, $f(x)$ is not bijective.

$\square$.

(a) [7 pts] Label the given sequence of $(n-1)(m-1)+1$ integers $a_1, a_2, \ldots, a_{(n-1)(m-1)+1}$. Show the following relation $\preceq$ on $\{1, 2, 3, \ldots, (n-1)(m-1)+1\}$ is a weak poset: $i \preceq j$ if and only if $i \leq j$ and $a_i \leq a_j$ (as integers).

For the next part, we will need to use Dilworth's theorem, as covered in lecture. Recall that Dilworth's theorem states that if $(X, \preceq)$ is any poset whose longest chain has length n, then X can be partitioned into n disjoint antichains.

(b) [7 pts] Show that in any sequence of $(n-1)(m-1)+1$ integers, either there is a non-decreasing subsequence of length n or a decreasing subsequence of length m.

(c) [6 pts] Construct a sequence of $(n-1)(m-1)$ integers, for arbitrary n and m, that has no non-decreasing subsequence of length n and no decreasing subsequence of length m. Thus in general, the result you obtained in the previous part is best-possible.

a) Weak poset $\Rightarrow$ antisymmetric, reflexive, transitive.

Proof that the relation R on the given set, $iRj \Leftrightarrow i \leq j$ and $a_i \leq a_j$ is weak poset.

Pf:

Let $S ::= \{1, 2, 3, \ldots, (n-1)(m-1)+1\}$.

Reflexivity: $a \in S$. Since $a \leq a$ and $a_i \leq a_i$, $aRa \Rightarrow$ relation is reflexive.

Antisymmetry: Let $a, b \in S$, $S_i = a$, $S_j = b$, $j > i$. and $b > a \Rightarrow iRj$.

By contradiction, suppose symmetry, then $iRj \Rightarrow jRi$.

If jRi, then $j \leq i$ and $S_j \leq S_i$. But this contradicts our initial conditions that $j > i$ and $b > a$.

Hence the relation cannot be symmetric.

Since relation is also reflexive the relation is antisymmetric.

Transitivity: Suppose $i \preceq j$ and $j \preceq k$, then $a_i \leq a_j$ and $i \leq j$.

$a_j \leq a_k$ and $j \leq k$.

$\Rightarrow a_i \leq a_j \leq a_k$ and $i \leq j \leq k$

$\Rightarrow a_i \leq a_k$ and $i \leq k$

$\Rightarrow i \preceq k$

$\Rightarrow$ transitive.

Since relation is antisymmetric and transitive, relation is a weak poset.

$\square$.

b) Let $M = \{a_1, a_2, \ldots a_{(n-1)(m-1)+1}\}$, where $a_1, a_2, \ldots a_{(n-1)(m-1)+1}$ is a sequence of integers.
Let $(M, \preceq)$ be a poset defined by the relation $i \preceq j \iff i \leq j$ and $a_i \leq a_j$

By Dilworth's Theorem on $(M, \preceq)$, let ℓ be the length of the longest chain in the poset.
If $\ell \geq n$, then there exists a non-decreasing subsequence of length n.

If $\ell < n$, then, there is no non-decreasing subsequence of length n.
Then, there must be less than n disjoint subchains.

min no. of elements in a decreasing subsequence: $\frac{(n-1)(m-1)+1}{\ell}$

The maximum length of antichain if the elements are all delegated into partitions as evenly as posible
is $\left\lceil \frac{(n-1)(m-1)+1}{\ell} \right\rceil$ and the minimum length of a disjointed antichain would be $\left\lfloor \frac{(n-1)(m-1)+1}{\ell} \right\rfloor$.
Hence, there will exist an antichain of $\left\lceil \frac{(n-1)(m-1)+1}{\ell} \right\rceil$ for all values of n and m.

Since $\ell < n$, $\left\lceil \frac{(n-1)(m-1)+1}{\ell} \right\rceil > \left\lceil \frac{(n-1)(m-1)}{n} + \frac{1}{n} \right\rceil$

$$\left\lceil \frac{(n-1)(m-1)+1}{n} \right\rceil = \left\lceil \frac{nm-n-m+2}{n} \right\rceil = \left\lceil m-1-\frac{m}{n}+\frac{2}{n} \right\rceil$$
$$= m-1+\left\lceil \frac{2}{n}-\frac{m}{n} \right\rceil$$

Suppose $n > m$, WLOG since n, m are such that they are interchangeable.

$$\frac{2-m}{n} > 0 \text{ if } m < 2.$$

$$-1 < \frac{2-m}{n} < 0 \text{ if } m > 2.$$

If $m < 2$, $\left\lceil \frac{(n-1)(m-1)+1}{n} \right\rceil = m-1+1 = m \Rightarrow$ there is an antichain of size m
$\Rightarrow$ there is a decreasing sequence of size m.

If $m > 2$, $\left\lceil \frac{(n-1)(m-1)+1}{n} \right\rceil = m-1$. Since $\ell < n$,
there is an antichain of size bigger than $m-1 \Rightarrow$ size m
$\Rightarrow$ there is a decreasing sequence of size m.

Hence, if the length of a non-degree subsequence is g.e.q than n, there will be a subsequence equals n.

If longest chain is less than n, then there will be a decreasing sequence of size m, as shown
Thus, the proposition holds. $\square$

We define a subsequence for $1 \le i \le n-1$ B_i: $n(m-1), (n-1)(m-1)+m-1, (n-1)(m-1)+m-2, (n-1)(m-1)+m-3, \dots, (n-1)(m-1)+1$.

B_i is a decreasing sequence of length $m-1$.

We build a sequence $B_1, B_2, B_3, \dots, B_{n-1}$. This sequence has length $(n-1)(m-1)$.

There are maximum decreasing length of $m-1$. And a maximum increasing length of $n-1$. This a sequence is constructed.

□

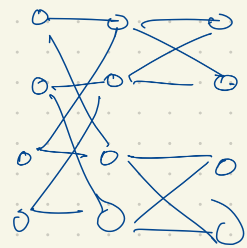
Problem 4. **[20 points]** Louis Reasoner figures that, wonderful as the Beneš network may be, the butterfly network has a few advantages, namely: fewer switches, smaller diameter, and an easy way to route packets through it. So Louis designs an N-input/output network he modestly calls a *Reasoner-net* with the aim of combining the best features of both the butterfly and Beneš nets:

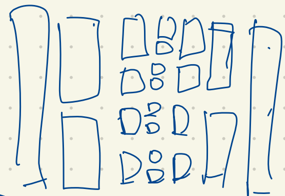
> The ith input switch in a Reasoner-net connects to two switches, a_i and b_i, and likewise, the jth output switch has two switches, y_j and z_j, connected to it. Then the Reasoner-net has an N-input Beneš network connected using the a_i switches as input switches and the y_j switches as its output switches. The Reasoner-net also has an N-input butterfly net connected using the b_i switches as inputs and¡ the z_j switches as outputs.

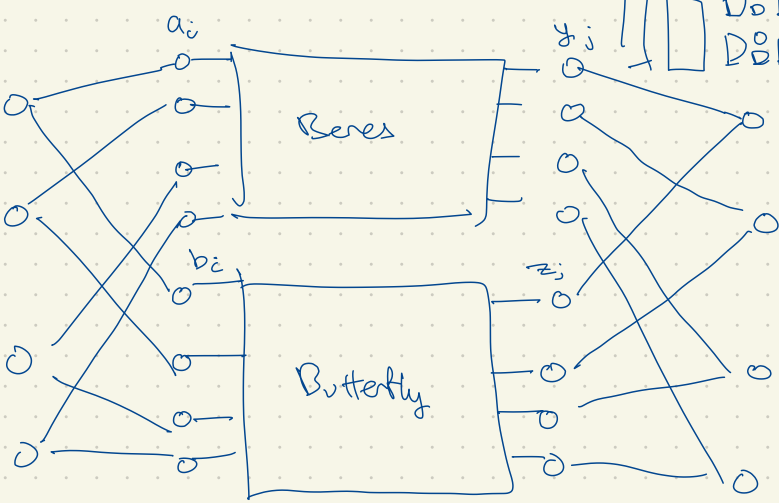
In the Reasoner-net the minimum latency routing does not have minimum congestion. The *latency for min-congestion* (LMC) of a net is the best bound on latency achievable using routings that minimize congestion. Likewise, the *congestion for min-latency* (CML) is the best bound on congestion achievable using routings that minimize latency.

Fill in the following chart for the Reasoner-net and briefly explain your answers.

diameter	switch size(s)	# switches	congestion	LMC	CML







Diameter: $2 +$ Butterfly diameter
$= 2 + \log_2 N + 2$
$= 4 + \log_2 N$ //

Since N nodes, let $N = 2^n$.
diameter & n.

$\log_2 N = n$.
Butterfly diameter is $\log_2 N$.

Switch size : 4 ~~4~~ (2×2)

# of switches Butterfly: $(\log_2 N + 2) N$.

# of switches: 4N + Butterfly + Benes.
$= 4N + 3N\log_2 N + 3N$.

input terminals.

# of switches Benes: $2N\log_2 N - N + 2N$
$= 2N\log_2 N + N$

congestion: 1. Since Benes has 1 congestion and the input-output are connected to a Benes that has sufficient rows to support N input-outputs.

LMC: To minimize congestion, we route all $i \Rightarrow \pi(i)$ through the Benes. Then, the latency is the diameter of the Benes, which is $2\log_2 N - 1 + 4 = 2\log_2 N + 3$.

CML: To minimize latency we route all $i \Rightarrow \pi(i)$ through the butterfly. The butterfly a congestion of $\sqrt{2}$.

$\sqrt{N}$ ★★★.

Problem 5. **[20 points]** Let B_n denote the butterfly network with $N = 2^n$ inputs and N outputs, as defined in Notes 6.3.8. We will show that the congestion of B_n is exactly $\sqrt{N}$ when n is even.

Hints:

- For the butterfly network, there is a unique path from each input to each output, so the congestion is the maximum number of messages passing through a vertex for any matching of inputs to outputs.
- If v is a vertex at level i of the butterfly network, there is a path from exactly 2^i input vertices to v and a path from v to exactly 2^{n-i} output vertices.
- At which level of the butterfly network must the congestion be worst? What is the congestion at the node whose binary representation is all 0s at that level of the network?

(a) [10 pts] Show that the congestion of B_n is at most $\sqrt{N}$ when n is even.

(b) [10 pts] Show that the congestion achieves $\sqrt{N}$ somewhere in the network and conclude that the congestion of B_n is exactly $\sqrt{N}$ when n is even.

a) Congestion at a switch cannot exceed the number of outputs accessible from the switch, since all packets congested at the switch must have a unique output to travel to.

At level i, the switch has access to 2^{n-i} outputs. Similarly, it is connected to 2^i inputs.

The maximum congestion is at some level $0 < i < \log_2 n$ s.t. $2^{n-i} = 2^i$

$$2^{n-i} = 2^i \Rightarrow n - i = i$$
$$n = 2i.$$
$$i = \frac{n}{2}.$$

Now, since we declared that n is even, let $n = 2k, k \in \mathbb{N}$.

$$i = \frac{2k}{2} = k.$$

$\Rightarrow$ maximum congestion is $2^k = 2^{\frac{n}{2}} = \sqrt{2^n} = \sqrt{N}$ $\square$.

b) For the case where we route all inputs with the format: $\underbrace{XX.\dots.X}_{\frac{n}{2}}\ \underbrace{00\dots0}_{\frac{n}{2}}$ to outputs $\underbrace{000\dots0}_{\frac{n}{2}}\ \underbrace{XXX\dots X}_{\frac{n}{2}}$

By the method of routing inputs to outputs, all the routes must pass through the switch at $(\underbrace{000\dots0}_{n}, \frac{n}{2})$ at level n.

This node at $\frac{n}{2}$ will hence have a congestion of $2^{\frac{n}{2}} = \sqrt{2^n}$.

Thus, the congestion must be exactly $\sqrt{N}$.

$\square$.