Problem 1. [25 points] Find Θ bounds for the following divide-and-conquer recurrences. Assume $T(1) = 1$ in all cases. Show your work.

(a) [5 pts] $T(n) = 8T(\lfloor n/2 \rfloor) + n$

(b) [5 pts] $T(n) = 2T(\lfloor n/8 \rfloor + 1/n) + n$

(c) [5 pts] $T(n) = 7T(\lfloor n/20 \rfloor) + 2T(\lfloor n/8 \rfloor) + n$

(d) [5 pts] $T(n) = 2T(\lfloor n/4 \rfloor + 1) + n^{1/2}$

(e) [5 pts] $T(n) = 3T(\lfloor n/9 + n^{1/9} \rfloor) + 1$

$T(n) = 8T(\lfloor n/2 \rfloor) + n.$

Akra-Bazzi $\Rightarrow \sum_{i=1}^{k} a_i b_i^p = 1.$ $\quad T(n) = \Theta\left(n^p\left(1 + \int_1^n \frac{g(u)}{u^{p+1}} du\right)\right)$

$8 \cdot \left(\frac{1}{2}\right)^p = 1.$

$\left(\frac{1}{2}\right)^p = \frac{1}{8} \quad 8 = 2^p$

$\Rightarrow p = 3.$

$n^3\left(1 + \int_1^n \frac{g(u)}{u^4} du\right) = n^3\left(1 + \int_1^n \frac{n}{u^4} du\right) = n^3\left(1 - \left[\frac{1}{3} n u^{-3}\right]_1^n\right)$

$= n^3\left(1 - \left(\frac{1}{3} n^{-2} - \frac{1}{3} n\right)\right)$

$= n^3\left(1 + \frac{1}{3} n - \frac{1}{3} n^{-2}\right)$

$= n^3 + \frac{1}{3} n^2 - \frac{1}{3} n$

$T(n) = \Theta\left(n^3 + \frac{1}{3} n^2 - \frac{1}{3} n\right)$

$\Rightarrow T(n) = \Theta(n^3)$

b) $T(n) = 2T\left(\lfloor \frac{n}{8} \rfloor + \frac{1}{n}\right) + n.$

$2 \cdot \left(\frac{1}{8}\right)^p = 1$

$\left(\frac{1}{8}\right)^p = \frac{1}{2}$

$8^p = 2.$

$p = \frac{1}{3}.$

$T(n) = \Theta\left(n^p\left(1 + \int_1^n \frac{g(u)}{u^{p+1}} du\right)\right)$

$= \Theta\left(n^p\left(1 + \int_1^n \frac{u}{u^{p+1}} du\right)\right)$

$= \Theta\left(n^{\frac{1}{3}}\left(1 + \int_1^n \frac{u}{u^{\frac{4}{3}}} du\right)\right)$

$= \Theta\left(n^{\frac{1}{3}}\left(1 + \int_1^n u^{-\frac{1}{3}} du\right)\right)$

$= \Theta\left(n^{\frac{1}{3}}\left(1 + \frac{3}{2}\left[u^{\frac{2}{3}}\right]_1^n\right)\right)$

$= \Theta\left(n^{\frac{1}{3}}\left(1 + \frac{3}{2}\left[n^{\frac{2}{3}} - 1\right]\right)\right)$

$= \Theta\left(n^{\frac{1}{3}} + \frac{3}{2} n - n^{\frac{1}{3}}\right)$

$= \Theta\left(\frac{3}{2} n\right)$

$= \Theta(n)$ //

$-\frac{1}{3} + 1 = -\frac{1}{3} + \frac{3}{3}$

$= \frac{2}{3}$

c) $T(n) = 7T(\frac{n}{20}) + 2T(\frac{n}{8}) + n$

$7 \cdot (\frac{1}{20})^p + 2 \cdot (\frac{1}{8})^p = 1.$ $\quad 7 \cdot (\frac{1}{20})^1 + 2(\frac{1}{8})^1 = \frac{7}{20} + \frac{2}{8} < 1.$

$\Rightarrow p < 1.$

$7 \cdot (\frac{1}{2} \cdot \frac{1}{10})^p + 2(\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2})^p = 1.$

$7 \cdot (\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{5})^p + 2(\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2})^p = 1.$

$(\frac{1}{4})^p [\ 7 \cdot (\frac{1}{5})^p + 2 \cdot (\frac{1}{2})^p\ ] = 1.$

$4^p = 7(\frac{1}{5})^p + 2(\frac{1}{2})^p$

$$\begin{aligned}
T(n) &= \Theta(n^p(1 + \int_1^n \frac{u}{u^{p+1}} du)) \\
&= \Theta(n^p(1 + \int_1^n u^{-p}\ du)) \\
&= \Theta(n^p(1 + \frac{1}{1-p}[u^{-p+1}]_1^n)) \\
&= \Theta(n^p(1 + \frac{1}{1-p}[n^{1-p} - 1])) \\
&= \Theta(n^p + \frac{1}{1-p} n - n^p) \\
&= \Theta(n) \ //
\end{aligned}$$

d) $T(n) = 2T(\frac{n}{4} + 1) + n^{\frac{1}{2}}.$

$2 \cdot (\frac{1}{4})^p = 1.$

$(\frac{1}{4})^p = \frac{1}{2}.$

$p = \frac{1}{2}.$

$$\begin{aligned}
T(n) &= \Theta(n^{\frac{1}{2}}(1 + \int_1^n \frac{u^{\frac{1}{2}}}{u^{\frac{3}{2}}} du)) \\
&= \Theta(n^{\frac{1}{2}}(1 + \int_1^n u^{-1} du)) \\
&= \Theta(n^{\frac{1}{2}}(1 + (\ln u)_1^n)) \\
&= \Theta(n^{\frac{1}{2}}(1 + \ln n)) \\
&= \Theta(n^{\frac{1}{2}} + n^{\frac{1}{2}} \ln n) \\
&= \Theta(n^{\frac{1}{2}} \ln n)
\end{aligned}$$

e) $T(n) = 3T(\frac{n}{9} + n^{\frac{1}{9}}) + 1.$

$\frac{\frac{n}{9}}{n^{\frac{1}{9}}} = \frac{1}{9} \frac{n}{n^{\frac{1}{9}}} = \frac{1}{9} n^{\frac{8}{9}} \to \infty.$

$\Rightarrow$ $\frac{n}{9}$ grows faster than $n^{\frac{1}{9}}$.

$\Rightarrow 3(\frac{1}{9})^p = 1$

$(\frac{1}{9})^p = \frac{1}{3}$

$p = \frac{1}{2}.$

$$\begin{aligned}
T(n) &= \Theta(n^{\frac{1}{2}}(1 + \int_1^n \frac{1}{u^{\frac{3}{2}}} du)) \\
&= \Theta(n^{\frac{1}{2}}(1 - 2[u^{-\frac{1}{2}}]_1^n)) \\
&= \Theta(n^{\frac{1}{2}}(1 - 2(n^{-\frac{1}{2}} - 1))) \\
&= \Theta(n^{\frac{1}{2}} - 2 + n^{\frac{1}{2}}) \\
&= \Theta(n^{\frac{1}{2}}) \ //
\end{aligned}$$

2a) Flaw is that the proof uses asymptotic notation in the proof. $O(n)$ is a statement not on $T(n)$, but the upper bound of $T(n)$ as it approaches infinity. Its validity does not depend on the value of n, hence it cannot be used for proofs.

(b) [10 pts] A simple attempt to prove $T(n) \neq O(n)$ via induction ultimately fails. We assume for sake of contradiction that $T(n) = O(n)$. Then there exists positive integer n_0 and positive real number c such that for all $n \geq n_0$, $T(n) \leq cn$. We then define $P(n)$ as the proposition that $T(n) \leq cn$.

We then proceed with strong induction.

Base Case, $n = n_0$: $P(n_0)$ is true, by assumption.

Inductive Step: We assume $P(n_0)$, $P(n_0 + 1)$, ..., $P(n-1)$ true.

Fill in the rest of this proof attempt, and explain why it doesn't work.

Note: As this problem was updated so late, the graders will be instructed to be exceedingly lenient when grading this.

(c) [5 pts] Using Akra-Bazzi theorem, find the correct asymptotic behavior of this recurrence.

(d) [10 pts] We have now seen several recurrences of the form $T(n) = aT(\lfloor n/b \rfloor) + n$. Some of them give a runtime that is $O(n)$, and some don't. Find the relationship between a and b that yields $T(n) = O(n)$, and prove that this is sufficient.

$T(n) = 4T(\frac{n}{2}) + n$

By contradiction, suppose that $T(n) = O(n)$,

$\Rightarrow \exists n_0 \in \mathbb{Z}^+$ and $\exists c \in \mathbb{R}^+$ s.t for all $n \geq n_0$, $T(n) \leq cn$.

By strong induction,

For $P(n+1)$, $T(n+1) = 4T(\frac{n+1}{2}) + (n+1)$

$$\begin{aligned} &\leq 4c(\tfrac{n+1}{2}) + (n+1) \\ &= 2c(n+1) + (n+1) \\ &= (n+1)(2c+1) \\ &> c(n+1) \end{aligned}$$

$\Rightarrow$ contradiction since we defined $T(n) \leq cn$.

c) Using Akra-Bazzi theorem,

$$4(\tfrac{1}{2})^p = 1.$$
$$(\tfrac{1}{2})^p = \tfrac{1}{4}$$
$$p = 2.$$

$$n^2\left(1 + \int_1^n \frac{u}{u^3} du\right)$$
$$\begin{aligned} &= n^2\left(1 + \int_1^n \frac{1}{u^2} du\right) \\ &= n^2\left(1 - [u^{-1}]_1^n\right) \\ &= n^2\left(1 - [\tfrac{1}{n} - 1]\right) \\ &= n^2 - n + n^2 \\ &= 2n^2 - n \end{aligned}$$
$$\Rightarrow T(n) = \Theta(n^2)$$

d) $a(\frac{1}{b})^p = 1.$

$p = 1. \Rightarrow a = b.$

$aT(\frac{n}{a}) + n.$

$$\begin{aligned} &n\left(1 + \int_1^n \frac{u}{u^2} du\right) \\ &= n\left(1 + \int_1^n u^{-1} du\right) \\ &= n\left(1 + \ln u \big|_1^n\right) \\ &= n(1 + \ln n) \\ &= n + n\ln n. \end{aligned}$$

$p < 1.$
$a(\frac{1}{b})^p = 1.$

$$\begin{aligned} &n^p\left(1 + \int_1^n \frac{u}{u^{p+1}} du\right) \\ &= n^p\left(1 + \int_1^n u^{-p} du\right) \\ &= n^p\left(1 + \tfrac{1}{1-p}[u^{1-p}]_1^n\right) \\ &= n^p\left(1 + \tfrac{1}{1-p}(n^{1-p} - 1)\right) \\ &= n^p + \tfrac{1}{1-p} n^1 - \tfrac{1}{1-p} n^p \end{aligned}$$
$\Rightarrow O(n^p).$

$p \neq 1$ since if $p = 1$, then $T(n) = O(n\ln n)$

Hence $p < 1 \Rightarrow O(n)$.

If $p < 1$,

$a(\frac{1}{b})^p = 1$

$(\frac{1}{b})^p = \frac{1}{a}.$

$a = b^p$ when $p < 1$

$\Rightarrow a < b.$

3)

Lemma 1: $A_{n+1} \leq \frac{1}{2}(\sqrt{2}+\frac{1}{2^n}) + (\sqrt{2}+\frac{1}{2^n})^{-1}$, where $A_{n+1} = \frac{A_n}{2}+\frac{1}{A_n}$, $A_n \leq \sqrt{2}+\frac{1}{2^n}$.

By contradiction, assume that

$$\frac{A_n}{2}+\frac{1}{A_n} > \frac{1}{2}(\sqrt{2}+\frac{1}{2^n}) + (\sqrt{2}+\frac{1}{2^n})^{-1} \quad —(1)$$

Let $x = \sqrt{2}+\frac{1}{2^n}$.

$A_n \leq x$.

$$(1) \quad \frac{A_n^2+1}{2A_n} > \frac{x^2+1}{2x}.$$

Let $f(x) = \frac{x^2+1}{2x} = \frac{x}{2}+\frac{1}{x}$.

$$f'(x) = \frac{1}{2}-\frac{1}{x^2}.$$

At $n=0$, $x = \sqrt{2}+1$

$$\Rightarrow f'(x) = \frac{1}{2} - \frac{1}{(\sqrt{2}+1)^2} > 0.$$

For all $x > \sqrt{2}$, $f'(x) > 0$.

$\Rightarrow$ $f(x)$ is increasing for all $x > \sqrt{2}$.

Since $A_n \leq x$

$f(A_n) \leq f(x)$

but this contradicts (1).

Since contradiction is derived, proposition must be true.

By induction,

Base case, $n=0$. $A_0 = 2 < \sqrt{2}+1$ ✓.

Inductive step. Assume $A_n \leq \sqrt{2}+\frac{1}{2^n}$ is true for $0 \leq k \leq n$,

For $A_{n+1} = \frac{A_n}{2}+\frac{1}{A_n}$, $A_{n+1} = \frac{1}{2}A_n + A_n^{-1}$

$$\leq \frac{1}{2}(\sqrt{2}+\frac{1}{2^n}) + (\sqrt{2}+\frac{1}{2^n})^{-1} \text{ by lemma 1.}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{2^{n+1}} + \frac{2^n}{2^n\sqrt{2}+1}$$

$$= \frac{2^n}{2^n\sqrt{2}+1} + \frac{1}{\sqrt{2}} + \frac{1}{2^{n+1}}.$$

For $A_{n+1} \leq \sqrt{2}+\frac{1}{2^{n+1}}$, $\frac{2^n}{2^n\sqrt{2}+1}+\frac{1}{\sqrt{2}} \leq \sqrt{2}$.

Let $x = 2^n \Rightarrow \frac{x}{x\sqrt{2}+1}+\frac{1}{\sqrt{2}} \leq \sqrt{2}$ for $x \geq 1$.

Let $f(x) = \frac{x}{x\sqrt{2}+1}+\frac{1}{\sqrt{2}}$

$f(0) = \frac{1}{\sqrt{2}}$.

$$f'(x) = \frac{1}{x\sqrt{2}+1} - \frac{\sqrt{2}x}{(x\sqrt{2}+1)^2}$$

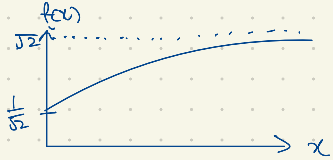
$$= \frac{1}{(x\sqrt{2}+1)^2}(x\sqrt{2}+1-\sqrt{2}x)$$

$$= \frac{1}{(x\sqrt{2}+1)^2} \Rightarrow f'(x) > 0 \text{ for all } x \geq 0.$$

$\Rightarrow$ $f(x)$ is strictly increasing for all $x \geq 0$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{\sqrt{2}+\frac{1}{x}}+\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}} = \sqrt{2}.$$

$\Rightarrow$ $f(x)$ has horizontal asymptote at $\sqrt{2}$.



$\Rightarrow f(x) \leq \sqrt{2}$ for all $\infty \geq x \geq 0$

$\Rightarrow A_{n+1} \leq \sqrt{2} + \frac{1}{2^{n+1}}$

$\square$.

4)

(a) [15 pts] $x_n = 4x_{n-1} - x_{n-2} - 6x_{n-3}$ $\quad (x_0 = 3, x_1 = 4, x_2 = 14)$

(b) [15 pts] $x_n = -x_{n-1} + 2x_{n-2} + n$ $\quad (x_0 = 5, x_1 = -4/9)$

$x_n = 4x_{n-1} - x_{n-2} - 6x_{n-3}$.

Let $x_n = x^n$

$x^n = 4x^{n-1} - x^{n-2} - 6x^{n-3}$

$x^3 = 4x^2 - x - 6$

$x^3 - 4x^2 + x + 6 = 0$.

Let $x = -1$.

$-1 - 4 - 1 + 6 = 0$

$\Rightarrow x = -1$ is a root.

$(x+1)(x^2 + Bx + 6) = 0$.

$1 + B = -4$

$B = -5$

$(x+1)(x^2 - 5x + 6) = 0$

$(x+1)(x$

$x = \frac{5 \pm \sqrt{25 - 4(6)}}{2}$

$= \frac{5 \pm 1}{2} \Rightarrow x = 3$ or $x = 2$.

$(x+1)(x-2)(x-3)$

$\Rightarrow x_n = a(-1)^n + b(2)^n + c(3)^n$

$3 = a + b + c$

$4 = -a + 2b + 3c$

$14 = a + 4b + 9c$.

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 14 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 4 \\ 0 & 3 & 8 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$$

$$4c = 4$$
$$c = 1$$
$$3b + 4 = 7$$
$$3b = 3$$
$$b = 1$$
$$a + 2 = 3$$
$$a = 1.$$

$$\Rightarrow x_n = (-1)^n + 2^n + 3^n \quad //$$

b) $x_n = -x_{n-1} + 2x_{n-2} + n$

Homogenous: $x^n = -x^{n-1} + 2x^{n-2}$
$$x^2 = -x + 2$$
$$x^2 + x - 2 = 0.$$
$$(x-1)(x+2) = 0$$

particular: Let $x_n = an + b$.

$$an + b = -(a(n-1) + b) + 2(a(n-2) + b) + n$$
$$= -(an - a + b) + 2(an - 2a + b) + n$$
$$= -an + a - b + 2an + 4a + 2b + n$$
$$an + b = an \quad 5a + b + n$$
$$5a + n = 0$$

Let $x_n = an^2 + bn + c$.

$$an^2 + bn + c = -(a(n-1)^2 + b(n-1) + c$$
$$\vdots$$
$$n(6a-1) + (-7a + 3b) = 0. \qquad 0 \quad 5 \quad 1 \quad -\frac{4}{9}$$

$$6a - 1 = 0 \qquad -\frac{7}{6} + 3b = 0$$
$$6a = 1 \qquad 3b = \frac{7}{6}$$
$$a = \frac{1}{6} \qquad b = \frac{7}{3}.$$

$$\frac{1}{6}n^2 + \frac{7}{3}n + a + b(-2)^n = x_n.$$

$$a + b = 5$$
$$\frac{1}{6} + \frac{7}{3} + a - 2b = -\frac{4}{9}$$
$$\frac{3 + 46}{18} + a - 2b = -\frac{4}{9}.$$
$$\frac{52}{18} + a - 2b = -\frac{4}{9}$$
$$\frac{26}{9} + a - 2b = -\frac{4}{9} \qquad -4 - 26$$
$$a - 2b = -\frac{30}{9}$$

$a = 5 - b$

$5 - b - 2b = -\frac{30}{9}$

$5 - 3b = -\frac{30}{9}$

$-3b = -\frac{30}{9} - 5$

$-3b = \frac{-30 - 45}{9}$

$-3b = -\frac{75}{9}$

$b = -\frac{25}{9}$ //

$a = 5 + \frac{25}{9}$

$= \frac{45 + 25}{9}$

$a = \frac{70}{9}$ //

$$\begin{array}{r} 30 \\ 45 \\ \hline 75 \end{array}$$

$$\begin{array}{r} 15 \\ 25 \\ 3 \overline{)\ 75} \end{array}$$