Problem 1. [20 points] Recall that a tree is a connected acyclic graph. In particular, a single vertex is a tree. We define a *Splitting Binary Tree*, or *SBTree* for short, as either the lone vertex, or a tree with the following properties:

1. exactly one node of degree 2 (called the root).
2. every other node is of degree 3 or 1 (called internal nodes and leaves, respectively).

For the case of one single vertex (see above), that vertex is considered to be a leaf. It is easier to understand the definition visually, so an example is shown in Figure 1. An example of a tree which is not an SBTree is shown in Figure 2.

(a) [10 pts] Show if an SBTree has more than one vertex, then the induced subgraph obtained by removing the unique root consists of two disconnected SBTrees. You may assume that by removing the root you obtain two separate connected componenents, so all you need to prove is that those two components are SBTrees.

(b) [10 pts] Prove that two SBTrees with the same number of leaves must also have the same total number of nodes. *Hint: As a conjecture, guess an expression for the total number of nodes in terms of the number of leaves N (l). Then use induction to prove that it holds for all trees with the same l*

a) Lemma 1: Every subgraph of an SBTree is an SBTree.

By contradiction,

Suppose that there exists a subgraph of an SBTree that is not an SBTree.
This means that the subgraph either has a node, v, that isn't the root with degree 2,
or it has a node with degree $\neq 1$ or degree $\neq 3$.

Since the subgraph is connected to the main graph, this means that v is also connected to the main graph. Hence the main graph has a node that does not satisfy the SBTree property,

This means that if the maingraph cannot be an SBTree, which is a contradiction.

Hence, lemma 1 must be true.

To prove the theorem, we first recognise that the two induced subgraphs are subgraphs of an SBTree. Then, by lemma 1, the subgraphs must be an SBTree.

$\square$

b) $2n-1$. Induction. Base case $n=1$.

Inductive step - add 2 vertices to a leaf. leaf $+1$ but total vertices $+2$.

2) a) The 2x2 grid is 2 colorable.
Let each vertex have a label (i, j), where i is the row number and j is the column.

Bipartite coz 2 colorable.
If N, M is odd, then there are odd no of vertices, $\Rightarrow$

Bipartite $\Rightarrow$ cycle of even length $\Rightarrow$ cycle only can visit even no of vertices
Since total odd vertices, hamilton cycle is odd
$\Rightarrow$ not possible

b) $N \times M$ if one or another is even $\Rightarrow$ total even vertices.

4a) The subset of cardinality $\lceil \frac{n}{3} \rceil$ does not satisfy requirements for the inductive hypothesis. Hence there is no proof that the subset is connected.

Thus while the pieces are connected by an edge, the vertices in the piece might not be connected.

By contradiction,
suppose $\exists$ mangled graph that is not fully connected.

Let $G = (V, E)$ be such a graph. Let $M = (V_m, E_m)$ be one connected component of the graph.
Let $N = (V_n, E_n)$ be the remaining graph s.t. no edge connects M to N.

Let $n = |V|$. Suppose $|V_m| \geq \lceil \frac{n}{2} \rceil$. Then $|V_n| = n - \lceil \frac{n}{2} \rceil \leq \lceil \frac{n}{2} \rceil$.
This is sufficient for all component sizes of M, N since if $|V_m| \leq \lceil \frac{n}{2} \rceil$, then $\lceil V_n \rceil \geq \lceil \frac{n}{2} \rceil$.

$\Rightarrow$ edge leaves N and must enter M $\Rightarrow$ M connected to N.

contradiction as we defined M and N to have no edge between them

b) Each time we leave v, we consume one edge of v.
Each time we return to v, we consume another edge of v.

To leave and return v, we consume 2 edges.

To leave and return to v n times, we consume $2n$ edges.

Suppose $\text{degree}(v) = 2n+1$. We can leave & return to v $2n$ times.
However, this is not the longest walk as there is still 1 untraversed edge.
After traversing the edge, there is no edge left to return to v.

Hence we cannot return to v.
Thus $v \neq w$.

b) Suppose w is even.