1) $Ax+By+Cz=D$

$x=1-t$
$y=1+2t$
$z=2-3t$
$P=(-1,1,2)$

$dx=-dt$
$dy=2dt$
$dz=-3dt.$

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$D=-A+B+2C.$

$Ax+By+Cz=-A+B+2C.$

$A(x+1)+B(y-1)+C(z-2)=0.$

At $t=0$, $x=1$, $y=1$, $z=2$.
At $t=1$, $x=0$, $y=3$, $z=-1$
At $t=-1$, $x=2$, $y=-1$, $z=5$.
At $t=2$, $x=-1$, $y=5$, $z=-4$.
At $t=3$, $x=-2$, $y=7$, $z=-7$.

$2A=0$
$A+2B-3C=0.$
$3A-2B+3C=0.$

$2-9=-7$

$B(y-1)+C(z-2)=0.$

$2B-3C=0$
$-2B+3C=0.$
$4B-6C=0$
$6B-9C$

$2B=3C.$
$B=\frac{3}{2}C$

$\frac{3}{2}C(y-1)+C(z-2)=0$

$\frac{3}{2}(y-1)+z-2=0$

$\frac{3}{2}y-\frac{3}{2}+z-2=0.$

$\frac{3}{2}y+z=2+\frac{3}{2}$
$=\frac{4}{2}+\frac{3}{2}=\frac{7}{2}.$

$3y+2z=7.$

Ans: $3y+2z=7$ //

b) $2x+y+z=4.$

normal: $\frac{\langle 2,1,1\rangle}{\sqrt{4+2}}$.
$=\frac{\langle 2,1,1\rangle}{\sqrt{6}}$

L direction: $-\langle 1,1-3,2-(-1)\rangle$
$=-\langle 1,-2,3\rangle$
$=\langle -1,2,-3\rangle$

unit direction $=\frac{\langle -1,2,-3\rangle}{\sqrt{1+4+9}}=\frac{\langle -1,2,-3\rangle}{\sqrt{14}}$

$\begin{array}{r} 14 \\ \times\ 6 \\ \hline 84 \end{array}$

projection: $\frac{\langle 2,1,1\rangle\cdot\langle -1,2,-3\rangle}{\sqrt{6}\times\sqrt{14}}=\frac{-2+2-3}{\sqrt{84}}$

$=-\frac{3}{\sqrt{84}}$

$=-\frac{3}{2\sqrt{21}}$ //

$84=6\cdot 14$
$=2\cdot 3\cdot 7\cdot 2$
$=4\cdot 3\cdot 7$

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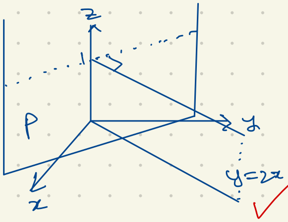
2) $y=2x$



5

$x=t \quad y=2t \quad z=1.$

b) $\langle 1,2,0\rangle\cdot\langle x,y,z\rangle = 0.$ 3

c) i) Substitution into parametric equation of ℓ to get P^*

ii) Find parametric value for point P, and use the negative of it. Since P^* and P are symmetric across the z axis.

iii) Same value as (ii) since P is $\perp$ to ℓ $\Rightarrow$ all points are reflection. 4

3) $A_a = \begin{vmatrix} 1 & 0 & 3 \\ -2 & 1 & -1 \\ -1 & 1 & a \end{vmatrix}$

$$|A_2| = \begin{vmatrix} 1 & 0 & 3 \\ -2 & 1 & -1 \\ -1 & 1 & 2 \end{vmatrix} = (2+1)+3(-2+1) = 3+3(-1) = 0$$

b) $A_2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$

$$\text{RREF}(A_2): \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 5 \\ 0 & 1 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = c\begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}$$

Line of solutions: $c\begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}$

c) $A_1 = \begin{bmatrix} 1 & 0 & 3 \\ -2 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$

A_1^{-1}:
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ -2 & 1 & -1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 2 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 2 & 1 & 0 \\ 0 & 1 & 4 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 2 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 3 \\ 0 & 1 & 0 & -3 & -4 & 5 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$$

Ans: -4

32

4) $r(t) = \langle \cos(e^t), \sin(e^t), e^t \rangle$

$T(t) = \dfrac{r'(t)}{|r'(t)|}$ $\quad r'(t) = \langle -\sin(e^t)\cdot e^t, \cos(e^t)\cdot e^t, e^t \rangle$

$|r'(t)| = \sqrt{(\sin^2(e^t)e^{2t} + \cos^2(e^t)e^{2t} + e^{2t})}$

$= \sqrt{(e^{2t} + e^{2t})}$

$= \sqrt{2e^{2t}} = \sqrt{2}e^t$

$T(t) = \frac{1}{\sqrt{2}}\langle -\sin(e^t), \cos(e^t), 1 \rangle$ //

b) $T'(t) = \frac{1}{\sqrt{2}}\langle -\cos(e^t)\cdot e^t, -\sin(e^t)\cdot e^t, 0 \rangle$.

$= -\frac{e^t}{\sqrt{2}}\langle \cos(e^t), \sin(e^t), 0 \rangle$. //

5) $F = z\sqrt{x^2+y} + 2\frac{y}{z}$.

$z\sqrt{x^2+y} + 2\frac{y}{z} = 7$. $\quad P_0 : (1, 3, 2)$

$\nabla F = \langle \frac{z}{2}(x^2+y)^{-\frac{1}{2}}(2x), \frac{z}{2}(x^2+y)^{-\frac{1}{2}} + \frac{2}{z}, \sqrt{x^2+y} - 2\frac{y}{z^2} \rangle$

$= \langle zx(x^2+y)^{-\frac{1}{2}}, \frac{z}{2}(x^2+y)^{-\frac{1}{2}} + \frac{2}{z}, \sqrt{x^2+y} - \frac{4}{z^2} \rangle$.

$\nabla F_{P_0} = \langle 2(1+3)^{-\frac{1}{2}}, (1+3)^{-\frac{1}{2}} + 1, \sqrt{1+3} - \frac{3}{4} \rangle$

$= \langle 1, \frac{1}{2}+1, 2 - \frac{3}{4} \rangle$

$= \langle 1, \frac{3}{2}, \frac{5}{4} \rangle$.

$4x + 6y + 5z = c$.

$4 + 6\cdot 3 + 5\cdot 2 = C$.

$4 + 18 + 10 = C$

$4 + 28 = C$

$C = 32$

6

$4x + 6y + 5z = 32$ //

b) $\Delta F \approx \Delta x + \frac{3}{2}\Delta y + \frac{5}{4}\Delta z$.

largest change is in y direction.

$0.1 \cdot 1.5 = 0.15$.

$\Delta F = 0.15$ //

c) $\hat{u} = \langle -2, 2, 1 \rangle \cdot \frac{1}{\sqrt{4+4+1}} = \langle -2, 2, 1 \rangle \cdot \frac{1}{3}$.

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$\left.\frac{dF}{dt}\right|_{\hat{u}, P_0} \hat{u}\cdot\nabla F = \langle -2, 2, 1 \rangle \cdot \langle 1, \frac{3}{2}, \frac{5}{4} \rangle \cdot \frac{1}{3} = (-2 + 3 + \frac{5}{4})\cdot\frac{1}{3} = (1 + \frac{5}{4})\cdot\frac{1}{3}$

$= \frac{9}{4}\cdot\frac{1}{3} = \frac{3}{4}$.

$\frac{3}{4} \cdot t = 0.1$

$\frac{3}{4}t = \frac{1}{10}$

$t = \frac{4}{30} = \frac{2}{15}$

Distance: $\frac{2}{15}$ //

6) $f = x + 4y + \frac{2}{xy} = x + 4y + 2x^{-1}y^{-1}$

$f_x = 1 - 2x^{-2}y^{-1} = 1 - \frac{2}{yx^2}$ $\quad f_y = 4 - 2x^{-1}y^{-2} = 4 - \frac{2}{xy^2}$.

$f_{xy} = \frac{2}{(xy)^2}$. $\quad f_{xx} = \frac{4}{yx^3}$ $\quad f_{yy} = \frac{4}{xy^3}$.

$f_x = 0$ $\qquad f_y = 0$

$\frac{2}{x^2y} = 1.$ $\qquad \frac{2}{xy^2} = 4$

$2 = x^2y.$ $\qquad 1 = 2xy^2.$

$y = \frac{2}{x^2}.$

$1 = 2x\left(\frac{2}{x^2}\right)^2$ $\qquad y = \frac{2}{4} = \frac{1}{2}.$

$1 = 2x\left(\frac{4}{x^4}\right)$

$= 8\frac{x}{x^4}$

$1 = \frac{8}{x^3}$

$x^3 = 8$

$x = 2.$

Critical point at $(2, \frac{1}{2})$.

$H = f_{xx}f_{yy} - f_{xy}^2 = -\frac{2}{(xy)^2} + \frac{16}{(xy)^4}$

$H_{(2,\frac{1}{2})} = 16 - 2 = 14 \Rightarrow H > 0$

$f_{xx} = \frac{4}{yx^3} = \frac{4}{\frac{1}{2}(2)^3} = 4. \Rightarrow f_{xx} > 0 \Rightarrow$ Minimum point //

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7) $Ax+By+Cz=D$

normal to plane = $\dfrac{\langle A,B,C\rangle}{\sqrt{A^2+B^2+C^2}}$.

line that intersects P_0 and plane perpendicularly: $x=At+x_0$
$y=Bt+y_0$
$z=Ct+z_0$

$\nabla f=\lambda\nabla g$. $\quad g=(x-x_0)^2+(y-y_0)^2+(z-z_0)^2$.
$\nabla g=\langle 2(x-x_0),2(y-y_0),2(z-z_0)\rangle$.

$\nabla f=\langle A,B,C\rangle$.

$A=2\lambda(x-x_0)\quad B=2\lambda(y-y_0)\quad C=2\lambda(z-z_0)$. //

8) $\nabla F(1,-1,\sqrt{2})=\langle 1,2,-2\rangle$

$\dfrac{\partial F}{\partial\phi}$ at $(\rho,\phi,\theta)=(2,\frac{\pi}{4},-\frac{\pi}{4})$ where $x=\rho\sin\phi\cos\theta$, $y=\rho\sin\phi\sin\theta$, $z=\rho\cos\phi$

$df=f_x\,dx+f_y\,dy+f_z\,dz$.

At $(\rho,\frac{\pi}{4},-\frac{\pi}{4})$, $x=2\sin\frac{\pi}{4}\cos\frac{\pi}{4}=1$ $\quad y=2\cdot\frac{1}{\sqrt{2}}\cdot-\frac{1}{\sqrt{2}}=-1$. $\quad z=2\cdot\frac{1}{\sqrt{2}}=\sqrt{2}$.

$df=f_x\,dx+f_y\,dy+f_z\,dz$

$\dfrac{\partial f}{\partial\phi}=f_x\dfrac{\partial x}{\partial\phi}+f_y\dfrac{\partial y}{\partial\phi}+f_z\dfrac{\partial z}{\partial\phi}$

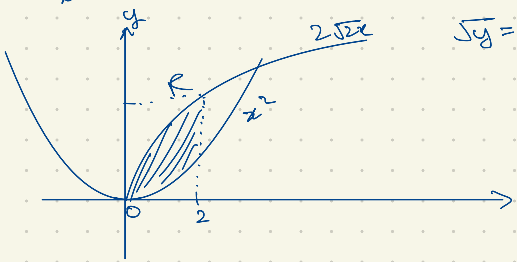
$=\rho\cos\phi\cos\theta+2\rho\cos\phi\sin\theta+2\rho\sin\phi$ //

b) By contradiction,
If gradient $\nabla f=\langle -y,x\rangle$.

potential fn: $f=xy$.

$\nabla f=\langle y,x\rangle\neq\langle -y,x\rangle\Rightarrow$ contradiction
$\Rightarrow$ no such potential fn exists //

9) $\int_0^2\int_{x^2}^{2\sqrt{2x}}dy\,dx$ $\quad y=2\sqrt{2x}\quad y=x^2.\quad y=2\sqrt{2x}$



$\sqrt{y}=x$.

$\frac{y}{2}=\sqrt{2x}$

$\frac{y^2}{4}=2x$

$\frac{y^2}{8}=x$.

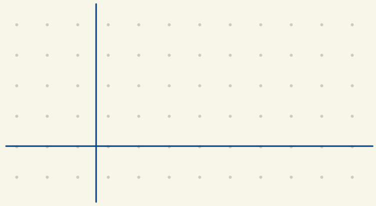
$x=\frac{y^2}{8}$.

$2\sqrt{2(2)}=2(2)=4$.

32

$$\int_0^4 \int_{\frac{y^2}{8}}^{\sqrt{y}} f(x,y)\, dx\, dy \quad //$$

10) $\iint_R f(x,y)\,dA =$



$x^2y=4$. $\quad u = x^2y$
$x^2y=9$ $\quad v = \frac{y}{x}$.
$\frac{y}{x}=1$
$\frac{y}{x}=2$

$u=4 \quad v=1$
$u=9 \quad v=2$.

$x^2y=u \qquad y=\frac{u}{x^2}$.
$x^2=\frac{u}{y}$
$x=\sqrt{\frac{u}{y}}$. $\qquad y_u=\frac{1}{x^2}$.
$x_u=\frac{1}{2}\left(\frac{u}{y}\right)^{-\frac{1}{2}}$
$=\frac{1}{2}\left(\frac{y}{u}\right)^{\frac{1}{2}}$.

$v=\frac{y}{x} \qquad vx=y$
$x=\frac{y}{v} \qquad y_v=x$.
$x_v=-\frac{y}{v^2}$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{3}u^{-\frac{2}{3}}v^{-\frac{1}{3}} & -\frac{1}{3}u^{\frac{1}{3}}v^{-\frac{4}{3}} \\ \frac{1}{3}u^{-\frac{2}{3}}v^{\frac{2}{3}} & \frac{2}{3}u^{\frac{1}{3}}v^{-\frac{1}{3}} \end{vmatrix}$$

$$= \frac{2}{9}u^{-\frac{1}{3}}v^{-\frac{2}{3}} + \frac{1}{9}u^{-\frac{1}{3}}v^{-\frac{2}{3}}$$

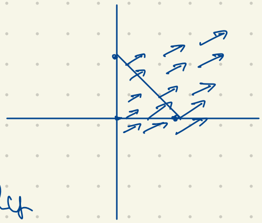
$$= \frac{1}{3}u^{-\frac{1}{3}}v^{-\frac{2}{3}}.$$

$$\int_1^2 \int_4^9 \frac{1}{3}u^{-\frac{1}{3}}v^{-\frac{2}{3}}\, du\, dv \quad //$$

7

11) $F = \langle x, x \rangle$

Positive.



$y=-x+1$
$dy=-dx$.

$y=-t+1$
$dy=-dt$.

$\oint \vec{F}\cdot\hat{n}\,ds = \int x\,dx + \int x\,dy$
$\int_0^1 t\,dt - \int_0^1 t\,dt = \int_0^1 0$.
$\int_0^1 x\,dx = \int_0^1 t\,dt = \frac{1}{2}t^2\Big|_0^1 = \frac{1}{2} //$

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$$\int_0^1\int_0^{-x+1} \operatorname{div} F \, dA = \int_0^1\int_0^{-x+1} dA = \int_0^1 -x+1 \, dx$$

$$= -\frac{1}{2}x^2 + x \Big|_0^1 = -\frac{1}{2}+1 = \frac{1}{2} //$$

12) $z = 2\sqrt{x^2+y^2}$ and $z=2$.

$\delta = z$.

$$\iiint z \, dV$$

$$= \iiint \rho\cos\phi \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$z = 2\sqrt{x^2+y^2} = 2r = 2\rho\sin\phi$

$\rho\cos\phi = 2\rho\sin\phi$ $\qquad$ $\rho\cos\phi = 2.$

$\cos\phi = 2\sin\phi.$ $\qquad$ $\rho = \frac{2}{\cos\phi}$

$1 = 2\tan\phi$ $\qquad\qquad$ $\tan\phi = \frac{1}{2}$

$= \phi.$

$\sqrt{5}$, 1, 2

$$\int_0^{2\pi}\int_0^{\arctan(\frac{1}{2})}\int_0^{2\sec\phi} \rho^3 \sin\phi\cos\phi \, d\rho \, d\phi \, d\theta$$

$$\int_0^{2\pi}\int_0^1\int_{2r}^2 z \cdot dz \, r \, dr \, d\theta = \int_0^{2\pi}\int_0^1 \frac{1}{2}z^2 \Big|_{2r}^2 \, r \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(\frac{1}{2}\cdot 4 - \frac{1}{2}(2r)^2\right) r \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^1 (2-2r^2) r \, dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^1 2r - 2r^3 \, dr \, d\theta$$

$$= \int_0^{2\pi} r^2 - \frac{1}{2}r^4 \Big|_0^1 \, d\theta$$

$$= \int_0^{2\pi} 1 - \frac{1}{2} \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} \, d\theta$$

$$= \frac{1}{2}\cdot 2\pi = \pi //$$

Ans: π

$$\frac{1}{\pi}\int_0^{2\pi}\int_0^1\int_{2r}^2 z \cdot z \cdot r \, dz \, dr \, d\theta \; // \qquad \cancel{\int_0^{2\pi}\int_0^{\arctan(\frac{1}{2})}\int_0^{2\sec\phi} \rho^4 \sin\phi\cos^2\phi \, d\rho \, d\phi \, d\theta} \; //$$

18.

(3) $F(x,y,z) = \langle y+y^2z,\ x-z+2xyz,\ -y+xy^2 \rangle$.

$F = \langle M, N, Q \rangle$.

$M_y = 1+2yz \quad N_x = 1+2yz \Rightarrow M_y = N_x$.
$M_z = y^2 \quad Q_x = y^2 \Rightarrow Q_x = M_z$.
$N_z = -1+2xy \quad Q_y = -1+2xy \Rightarrow N_z = Q_y$.

$\therefore$ $M_y = N_x$, $Q_x = M_z$, $N_z = Q_y$ $\Rightarrow$ exact, gradient field //

b) Let c_1 be line from $(0,0,0) \to (x_0,0,0)$
c_2 " " " $(x_0,0,0) \to (x_0,y_0,0)$
c_3 " " " $(x_0,y_0,0) \to (x_0,y_0,z_0)$.

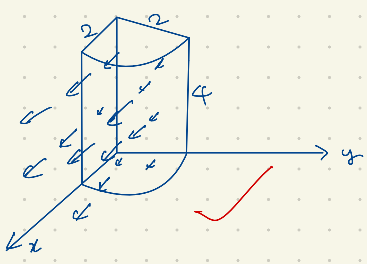
$$\int_{c_1+c_2+c_3} F\,d\vec{r} = \int_0^{x_0} y+y^2z\,dx + \int_0^{y_0} x_0-z+2x_0yz\,dy + \int_0^{z_0} -y+xy^2\,dz$$
$$= 0 + xy - yz + xy^2z$$
$$\therefore f = xy - yz + xy^2z.$$

c) $(2,2,1) \to (1,-1,2)$.

$f(2,2,1) - f(1,-1,2)$ By FTLI,
$$= 4-2+2\cdot4-(-1+2+2)$$
$$= 2+8-(3)$$
$$= 7 \,//$$

-3

14)



b) $\iint_S \vec{F}\cdot\hat{n}\,dS$
$\hat{n} = \langle x, y, 0 \rangle$.
$\vec{F} = \langle x, 0, 0 \rangle$

$\vec{F}\cdot\hat{n} = x^2$.
$$\int_0^{3/4}\int_0^4 x^2\, r\, dz\, d\theta$$
$$= \int_0^{\pi/2}\int_0^4 2x^2\,dz\,d\theta$$
$$= \int_0^{\pi/2} 8x^2\,d\theta$$
$$= \int_0^{\pi/2} 8r^2\cos^2\theta\,d\theta$$
$$= \int_0^{\pi/2} 8\cdot4\cos^2\theta\,d\theta$$
$$= 32\int_0^{\pi/2} \tfrac{1}{2}(1+\cos2\theta)\,d\theta$$
$$= 16\int_0^{\pi/2} 1+\cos2\theta\,d\theta$$

-5

$$= 16\left[\theta + \tfrac{1}{2}\sin2\theta\right]_0^{\pi/2}$$
$$= 16\left[\tfrac{\pi}{2}\right] = 8\pi \,//$$

30

c) $\int_0^{\pi/2}\int_0^2\int_0^4 r\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^2 4r\,dr\,d\theta = \int_0^{\pi/2} 2r^2\Big|_0^2 d\theta$

$= \int_0^{\pi/2} 8\,d\theta = 4\pi$ ✓

Flux outward through surface is 4π.

since flux through surface is $8\pi = 4\pi +$ surface flux.

-5

(5) $F = \langle yz, -xz, 1\rangle$. $z = 4 - x^2 - y^2$

$\oint \vec{F}\cdot d\vec{r} = \iint_S \text{curl}F\,dS$ $\quad$ $\text{curl}F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & -xz & 1 \end{vmatrix}$

$= \hat{i}(x) - \hat{j}(-y) + \hat{k}(-z - z)$

$= \langle x, y, -2z\rangle$.

$\hat{n}\,dS = \langle -z_x, -z_y, 1\rangle\,dx\,dy$

$= \langle 2x, 2y, 1\rangle\,dx\,dy$

$\iint \langle 2x, 2y, 1\rangle\langle x, y, -2z\rangle\,dx\,dy$

$= \iint 2x^2 + 2y^2 - 2z\,dx\,dy$

$= \iint 2x^2 + 2y^2 - 8 + 2x^2 + 2y^2\,dx\,dy$

$= \iint 4x^2 + 4y^2 - 8\,dx\,dy$

$= \int_0^{\pi}\int_0^2 (4r^2 - 8)r\,dr\,d\theta$

$= \int_0^{\pi/2}\int_0^2 4r^3 - 8r\,dr\,d\theta = \int_0^{\pi/2} r^4 - 4r^2\Big|_0^2 d\theta$

$= \int_0^{\pi/2} 16 - 4\cdot 4\,d\theta$

$= \int_0^{\pi/2} 0\,d\theta = 0$ // ✓

b) $\int_{C_1}\vec{F}\cdot d\vec{r} + \int_{C_2}\vec{F}\cdot d\vec{r} + \int_{C_3}\vec{F}\cdot d\vec{r}$

$= \int_{C_1} yz\,dx - xz\,dy - \int xz\,dy + dz + \int yz\,dx + dz$.

$= 0 + \int_0^{\sqrt{2}} dz - \int_0^{\sqrt{2}} dz = 0$ // ✓

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