Problem 1. [15 points] Let $G = (V, E)$ be a graph. A *matching* in G is a set $M \subset E$ such that no two edges in M are incident on a common vertex.

Let M_1, M_2 be two matchings of G. Consider the new graph $G' = (V, M_1 \cup M_2)$ (i.e. on the same vertex set, whose edges consist of all the edges that appear in either M_1 or M_2). Show that G' is bipartite.

Helpful definition: A *connected component* is a subgraph of a graph consisting of some vertex and every node and edge that is connected to that vertex.

By induction on vertices in G,

Let $P(n) ::=$ For a graph $G=(V,E)$, $|V|=n$, with matchings M_1, M_2. The graph $G' = (V, M_1 \cup M_2)$ is bipartite.

Base case, $P(1)$: G has only 1 vertex which is bipartite. $P(1)$ is true.

Inductive step. Assume $P(n)$ is true, for $n+1$,
let the new vertex be v_i. Let graph be $G=(V,E)$, $|V|=n+1$.
We remove v_i from G, to produce $G^* = (V^*, E^*)$ $|V|=n$. Let M_1^*, M_2^* be matchings of G^*. By the induction hypothesis, $G^{*\prime} = (V^*, M_1^* \cup M_2^*)$ is bipartite.

We now analyse cases in which v_i can be added into G^* to get back G.

If $\deg(v_i) = 0$, then v_i can be added into one of the partitions of $G^{*\prime}$ without changing the edges, hence $G' = (V, M_1^* \cup M_2^*)$ which is bipartite.

If $\deg(v_i) \neq 0$,

For case $M_1 = M_1^* \cup \{\{v_i, v_k\}\}$ where $v_k \in V^*$,
$M_2 = M_2^*$.

Since $G^{*\prime}$ is bipartite, $G = (V, M_1 \cup M_2)$ is also bipartite as v_i can be placed in the set opposite to v_k. This also applies to the case where

$$M_2 = M_2^* \cup \{\{v_i, v_k\}\}$$
$$M_1 = M_1^*.$$

For case where $M_2 = M_2^* \cup \{\{v_i, v_j\}\}$ $M_1 = M_1^* \cup \{\{v_i, v_k\}\}$, $v_j, v_k \in V^*$.

For $\{v_i, v_j\} \in M_2$, v_j should not be incident to any edge in M_2^*. Similarly, v_k should not be incident to any edge in M_1^*. Hence we can arrange v_j, v_k to be in the same set opposing v_i, and we get a bipartite graph.

Hence in all cases of v_i being added to G^*, the graph G' will be a bipartite graph. $\square$

Problem 2. [20 points] Let $G = (V, E)$ be a graph. Recall that the *degree* of a vertex $v \in V$, denoted d_v, is the number of vertices w such that there is an edge between v and w.

(a) [10 pts] Prove that

$$2|E| = \sum_{v \in V} d_v.$$

(b) [5 pts] At a 6.042 ice cream study session (where the ice cream is plentiful and it helps you study too) 111 students showed up. During the session, some students shook hands with each other (everybody being happy and content with the ice-cream and all). Turns out that the University of Chicago did another spectacular study here, and counted that each student shook hands with exactly 17 other students. Can you debunk this too?

(c) [5 pts] And on a more dull note, how many edges does K_n, the complete graph on n vertices, have?

a) $2|E| = \sum_{v \in V} d_v$.

By induction on the edges,

Let $p(n) ::= 2|E| = \sum_{v \in V} d_v$, where $|E| = n$.

let $n=1$, $p(1)$: $2|1| = \sum_{v \in V} d_v$. Since there is only one edge, the edge must connect two vertices, both with degree 1. Rest of the vertices has degree 0. Thus $\sum_{v \in V} d_v$ is $2 = 2|E|$. $p(1)$ holds.

Assume $p(n)$ is true for n,

$p(n+1) \Rightarrow |E| = n+1 = 2(n+1) = 2n+2$

Since a new edge is added, two vertices with have 1 extra degree

$$\Rightarrow \sum_{v \in V} d_v = 2|E| + 2$$
$$= 2(|E|+1)$$
$$\Rightarrow p(n+1) \text{ is true.}$$

b) By contradiction,

Assume each student shook hands exactly 17 times.

Let each student be a vertex, and let handshake pairs be edges.

Let student $= v_i \in V$, handshake $= \{v_i, v_j\} \in E$.

If degree $(v_i) = 17$ for $\forall v_i \in V$. $|V| = 111$

$$\sum_{v \in V} \text{degree}(v_i) = 17 \times 111 = 1887.$$

Since $\sum \text{degree}(v_i) = 2|E|$

$$2|E| = 1887$$
$$|E| = 1887/2$$
$$\Rightarrow |E| \notin \mathbb{N}$$

$\Rightarrow$ contradiction since $|E|$ must be countable.

Thus, they cannot have all shooked hands exactly 17 times. ☐

4) a) Suppose $G = (V, E)$ is 2-colorable, and G has maximum degree 2.

Then let G have edge $\{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_3\} \in E$. We let v_1 be color 1. v_2 must be color 2 since $v_1 - v_2$. $v_3 - v_1$ and for $v_3 - v_2 \Rightarrow v_3$ cannot be color 1 or 2 $\Rightarrow$ G is not 2 colorable.

$\square$

4b) $n=2$ cannot satisfy the induction hypothesis since if $n=2$, and graph have maximum degree of 1, then there does not exist a vertex with degree < 1 as both vertices must have degree 1.

Hence $n=2$ is false and does not hold. Thus inductive step for $n > 2$ does not hold and proof is false.

5) False. To disprove, we will first show that $\exists$ a girl, A that is rated worst by at most $n-2$ people.

By contradiction, assume there does not exist a girl rated worst by at most $n-2$ people
$\Rightarrow$ all girls are rated worst by $n-1$ boys, then there must be $n(n-1)$ boys in total, which is not possible since $n \geq 3$ and for $n=3$, $n(n-1) = 6 \neq 3$.

Hence, since a contradiction is derived, there must be a girl A, that is rated worst by at most $n-2$ people.

Now, we prove that for any set of preferences, there will be an unstable matching.

To do this, we will build an unstable matching.

From above, we shown that A will be rated worst by at most $n-2$ boys.
Let A be paired with boy 1, that does not rate her worst, but she rates worst.
There must be another boy, 2, that does not rate A worst.

Let 2 be paired with another girl that he rates worst. However, a rogue couple is built between 2 and A. Hence, in this matching, there is a rogue couple.

Hence, there must be an unstable matching regardless of preferences. $\square$

Problem 6. [20 points]

Let $(s_1, s_2, ..., s_n)$ be an arbitrarily distributed sequence of the number $1, 2, ..., n-1, n$. For instance, for $n = 5$, one arbitrary sequence could be $(5, 3, 4, 2, 1)$.

Define the graph G=(V,E) as follows:

1. $V = \{v_1, v_2, ..., v_n\}$
2. $e = (v_i, v_j) \in E$ if either:
   (a) $j = i + 1$, for $1 \le i \le n - 1$
   (b) $i = s_k$, and $j = s_{k+1}$ for $1 <= k <= n - 1$

(a) [10 pts] Prove that this graph is 4-colorable for any $(s_1, s_2, ..., s_n)$.

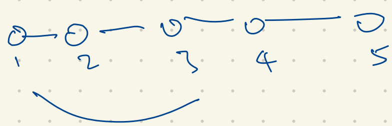
Hint: First show that a line graph is 2-colorable. Note that a line graph is defined as follows: The n-node graph containing $n - 1$ edges in sequence is known as the line graph L_n.

(b) [10 pts] Suppose $(s_1, s_2, ..., s_n) = (1, a_1, 3, a_2, 5, a_3, ...)$ where $a_1, a2....$ is an arbitrary distributed sequence of the even numbers in $1, ..., n-1$. Prove that the resulting graph is 2-colorable.

a) Lemma 1: A line graph is two colorable.
Pf: By induction on vertices n in the graph,

Base case: $n=1$. For 1 vertex, the graph is 1 colorable $<$ 2 colorable $\Rightarrow$ lemma holds for $n=1$.

Inductive step: Assume lemma holds for some k, $0 \le k \le n$, then for line



even odd. all odd 1 color
all even another color.

$(s_1, s_2, \dots, s_n) =$

$i = s_k$, $j = s_{k+1}$ is odd even edge.
$\Rightarrow$ no need addl. third coloring.