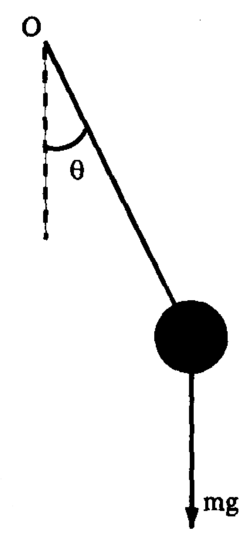
**Math 801 – ODE HW**
**(A. Assadi)**

You need to review/learn the following topics:
Two-dimensional flows: phase plane, stability of fixed points, periodic solutions, and limit cycles. This homework exercises is intended to help you to master the numerical and analytic techniques that are typically needed when you encounter the concepts, regardless of which textbook you have read so far. You would need also a suitable set of commands from MATLAB (or other packages) for modeling and making use of your data.

To summarize the topics that will be needed in the following weeks by the end of semester:
Introduction to bifurcation theory, local and global bifurcations.
Tools for studying global behavior of flows:
    Lyapunov functions, Poincare-Bendixson Theorem, gradient flows.
Three-dimensional flows:
    Lyapunov exponents, Poincare sections, strange attractors, chaos.

**Exercise 1.** In this exercise, we use the MATLAB ODE programs, such as ode45 etc that we have seen earlier. Use the following figure that represents a pendulum and the Newton's laws of motion to formulate the differential equations governing the dynamics of the pendulum's oscillatory motion. The differential equation that you find is one of the most popular and useful models for what is called a "harmonic oscillator", used all the time in physics, chemistry, and engineering. In biological models, of course, such idealized models are unrealistic. Nonetheless, the harmonic oscillator is the starting point for understanding the nonlinear versions of oscillatory dynamics that biological systems exhibit.




Let us consider the frictionless simple pendulum, shown in the opposite figure. The rod is rigid and assumed to be very light and of length $l$. A mass ($m$) is attached to the rod which can rotate without friction about the point 0. Use the notation $\omega^2 = g/l$ where $g$ is Newton's constant for gravitation. Apply Newton's second law on the force perpendicular to the rod to find the following:

$$-mg\sin\theta = ml\frac{d^2\theta}{dt^2}.$$

Conclude that

$$\frac{d^2\theta}{dt^2} + \omega^2\sin\theta = 0.$$

1*a*) Consider the oscillations of small amplitude ($\theta < 1$), and use the Taylor's formula for the approximation $sin\theta \sim \theta$. Use this estimate to simplify the equation above to obtain the ODE for a *simple harmonic motion*: $\frac{d^2\theta}{dt^2} + \omega^2\theta = 0$. Note that $\frac{d^2\theta}{dt^2} + \omega^2\theta = 0$ is the linearization of the nonlinear equation $\frac{d^2\theta}{dt^2} + \omega^2\sin\theta = 0$, so we must take special care to ensure that the linearized equation is a good approximation to the nonlinear one. Use **`ode45`** to study numerically both equations and determine the range where $\frac{d^2\theta}{dt^2} + \omega^2\theta = 0$ is a good approximation, and a couple of examples that show that it would be a poor approximation. Write your MATLAB commands as M-files with parameters that you can control and change, or repeatedly select from an array of values, so that you can graph each case in a separate figure, and decide the good range and poor approximations. As for initial values, use your own judiciously.

1b) If $\theta$ is not small, we must study the full equation of motion which is nonlinear. Do you expect the solutions to be qualitatively different to those of simple harmonic motion? If we push a pendulum hard enough, we should be able to make it swing round and round its point of support, with $\theta$ increasing continuously with *t* (in the absence of friction, as assumed above), so you would like to show that the nonlinear equation has solutions of this type.

**Exercise 2. Analytic Solutions.** In general, nonlinear ordinary differential equations cannot be solved analytically. For equations in which the first derivative, $d\theta/dt$ does not appear explicitly (as in the nonlinear pendulum ODE in Exercise 1), an analytical solution could be constructed. Using the notation $\dot{\theta} = \frac{d\theta}{dt}$, we treat $d\theta/dt$ as a function of $\theta$ instead of *t.* Verify the following calculation and complete the missing steps:

**<u>2a)</u>** $\frac{d^2\theta}{dt^2} = \frac{1}{2}\frac{d}{d\theta}(\dot{\theta})^2$

**<u>2b)</u>** $\frac{1}{2}\frac{d}{d\theta}(\dot{\theta})^2 = -\omega^2 \sin\theta$

**<u>2c)</u>** $\frac{1}{2}(\dot{\theta})^2 = \omega^2 \cos\theta + C$ , where C is a constant.

**<u>2d)</u>** Provide a physical meaning for (2c) by the back-substitution $\omega^2 = g/l$ :

$$\frac{1}{2}ml^2\left(\dot{\theta}\right)^2 - mgl\cos\theta = E.$$

Interpret E as energy, and the formula above as a form of conservation of energy. The first term represents kinetic energy and the second one is gravitational potential energy. Systems that determine a conserved quantity (here energy E) after one integration are called *conservative systems*. If you try to take into account a small amount of friction at the point of suspension of the pendulum, additional terms are needed for the left-hand side of the last formula, such as a term proportional to $d\theta/dt$. In that case, the system will dissipate heat as a result of friction, and no longer would be *conservative*. This seemingly benign change of viewpoint has dramatic consequences for the motion of the pendulum, as we shall study later. Biological systems are prime examples of non-conservative systems!

**<u>2e)</u>** The equation in (2d) could be rewritten as $\left(\dot{\theta}\right)^2 = 2mgl\cos\theta / ml^2 + 2E/ml^2$. Use $\theta_0$ as the angle of the pendulum when $t = 0$, so that the two constants of integration become physically meaningful quantities $E$ and $\theta_0$ , the initial energy and angle of the pendulum.

Integrate $dt/d\theta$ in terms of functions of $\theta$ as follows: $\dot{\theta} = \sqrt{2(g/l)\cos\theta + 2E/ml^2}$ ,

$t = \int_{\theta_0}^{\theta_1} \frac{d\theta}{\sqrt{(2g/l)\cos\theta + 2E/ml^2}}$. The latter equation is an analytic representation of the solution. In closed form, the integral can be written in terms of a new class of functions called Jacobian elliptic functions that play important roles in many areas of mathematics and physics. If you are interested to learn more about these functions, the Encyclopedia Britannica as well as the MIT Press' Encyclopedia of Mathematics have readable accounts of these topics and further reading guides.

**Exercise 3**. The analytic equation of Ex. 2 provide a description of oscillatory solutions for small initial angles and kinetic energies, and solutions with $\theta$increasing with *t* for large enough initial energies. Most readers would find that the dynamic behavior of pendulum as observed, such as in the preceding scenario, is not readily obvious from the analytic formula. What we have is a *quantitative* expression for the solution, but we do not get the real intuition if we are more interested in the *qualitative* nature of the solution. This exercise is meant to show you how a *qualitative* exploration of the pendulum dynamics could be done.

***3a)*** Go back to the drawing board, and start over with $\left(\dot{\theta}\right)^2 = 2mgl\cos\theta / ml^2 + 2E / ml^2$

and graph $(d\theta/dt)^2$ as a function of $\theta$ for different values of the energy $E$. Your graphs should provide adequate details in order to observe that:

- For $E > mgl$ the curves lie completely above the *θ-axis.*
- For $-mgl < E < mgl$ the curves intersect the *θ-axis.*
- For $E < -mgl$ the curves lie completely below the *θ-axis* .

***3b)*** Determine $d\theta/dt$ as a function of $\theta$ by taking the square root of the curves in graphs of (3a) to obtain the new curves. You must take both the positive and negative square root. Your new graphs should provide sufficient details to convince the viewer that:

- For $E > mgl,$ the curves lie either fully above or fully below the $\theta$-axis.
- For $-mgl < E < mgl,$ only finite portions of the graph of $(d\theta/dt)^2$ lie above the *θ-axis,* so the square root gives finite, closed curves.

- For $E < -mgl$, there is no real solution. This corresponds to the fact that the pendulum always has a gravitational potential energy of at least $-mgl$.

***3c)*** How do $\theta$ and $d\theta/dt$ vary along these solution curves as $t$ increases? Show that when $d\theta/dt$ is positive, $\theta$ increases with $t$, and vice versa. (Recall $d\theta/dt$ is the rate of change of $\theta$ as function of $t$). Use such observations to add arrows to your figure in the preceding graph, and indicate the directions in which the solution changes with time. In turn, use these observations to construct the *phase portrait* for the nonlinear ordinary differential equation of the pendulum. The $(\theta, \dot{\theta})$-plane is the *phase plane*. Each of the solution curves represents a possible solution of the nonlinear pendulum equation, which we refer to as an *integral path* or *trajectory*.

***3d)*** Your graphs should provide you with three qualitatively different types of integral paths as follows.

*Equilibrium solutions.* Which integral paths are the equilibrium solutions? You should identify the two equilibrium solutions that physically correspond to the equilibrium with the pendulum vertically downward and the equilibrium with the pendulum vertically upward. Points close to the first should lie on small closed trajectories close to it. Argue that this is a *stable equilibrium point*, since a small change in the state of the system away from equilibrium leads to solutions that remain close to equilibrium. If you cough on a pendulum hanging downwards, you will only excite a small oscillation. In contrast, points close to second equilibrium lie on trajectories that take the solution far away from it, which indicates that this is an *unstable equilibrium point.* The slightest physical perturbation to a pendulum balanced precariously above its point of support, is sufficient to make it fall. In practice, a minimum amount friction is needed or else it is impossible to balance a pendulum in this way, because the equilibrium is unstable and reaching such precision to achieve the exact equilibrium is almost always beyond the ordinary physical means.