MATH 801 -
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HOMEWORK DIFF GEOMETRY I

The First Fundamental Form (FFF) and Geometry of Curves on Surfaces

In this homework, we introduce more geometric concepts for parameterized surfaces. Then ask you to use basic linear algebra and differential calculus to compute the geometric invariants. As before, let $X(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v))$ be the equation of a parameterized surface $S \subset \mathbb{R}^3$. In some cases, the equation of a parameterized surface could describe a 2-dimensional surface in a higher dimensional Euclidean space $S \subset \mathbb{R}^n$. You are asked to be mindful of such generalizations and show that the arguments apply equally well for $n \geq 3$. Define as before the two tangent vectors $\{X_u, X_v\} \subset T_pS \subset T_p\mathbb{R}^n \cong \mathbb{R}^n$ at a point $p \in S$ by the formulas below. Assume throughout that all surfaces have a regular parameterization, unless otherwise specified:

$$X_u = \frac{\partial X(u,v)}{\partial u} = (\frac{\partial}{\partial u} x_1(u,v), \frac{\partial}{\partial u} x_2(u,v), \ldots \frac{\partial}{\partial u} x_n(u,v)); \qquad ;$$

$$X_v = \frac{\partial X(u,v)}{\partial v} = (\frac{\partial}{\partial v} x_1(u,v), \frac{\partial}{\partial v} x_2(u,v), \ldots \frac{\partial}{\partial v} x_n(u,v)).$$

Definition. The First Fundamental Form for S at $p \in S$ is defined as the Euclidean inner product of tangent vectors of S as vectors in $\mathbb{R}^n$ using the identification $T_pS \subset \mathbb{R}^n$:

$$I_p : T_pS \times T_pS \to \mathbb{R},\ I_p(v_1, v_2) = v_1.v_2 = \langle v_1, v_2 \rangle_p,. \qquad (0.1)$$

where $\langle v_1, v_2 \rangle_p$ and $v_1.v_2$ both indicate the usual dot product of vectors. Sometimes, the notation $\langle v_1, v_2 \rangle_p$ is used directly for the First Fundamental Form. According to this definition, the First Fundamental Form appears to be dependent on the embedding of the surface by the regular given parameterization in $\mathbb{R}^n$. To the extent that such an embedding is concerned, the inner product on the tangent space of the surface is inherited from that of $\mathbb{R}^n$ (also called the *ambient space*). On the other hand, any two regular parameterization of the same surface yield the same Fundamental Form, as a symmetric bilinear form on the 2-dimensional vector space $T_pS \cong \mathbb{R}^2$. The matrix representation of this form, of course, depends on the choice of basis for $T_pS \cong \mathbb{R}^2$, which in turn is expressed explicitly using one such regular representation. Any other regular representation gives a new basis for $T_pS \cong \mathbb{R}^2$, hence a new matrix representation. As a result, the two matrix representations for the First Fundamental Form satisfy a similarity transformation.

In terms of the given parameterization $X(u,v)$ and the basis $\{X_u, X_v\}$, we have the following coefficients of a 2×2 matrix:

$E(u,v) = \langle X_u, X_v \rangle_p$; $F(u,v) = \langle X_u, X_v \rangle_p = \langle X_v, X_u \rangle_p$ and $G(u,v) = \langle X_v, X_v \rangle_p$ due to symmetry in inner product. The matrix:

$$\begin{pmatrix} E(u,v) & F(u,v) \\ F(u,v) & G(u,v) \end{pmatrix} \tag{0.2}$$

is the local representation of the First Fundamental Form.

<u>Problem 1</u>. In order to calculate with the first fundamental form in local coordinates, sometimes it is useful to write the formula in terms of tangent vectors to two curves $\alpha_1(t)$ and $\alpha_2(t)$ on the surface, such that $\alpha_1(0) = p$ and $\alpha_2(0) = p$. For example, choose the curves $(u_1(t), v_1(t))$ and $(u_2(t), v_2(t))$ in the domain of definition of X and let $X((u_1(t), v_1(t)) = \alpha_1(t) : (-a,a) \to S$ and $X((u_2(t), v_2(t)) = \alpha_2(t) : (-a,a) \to S$ with $v_1 = \frac{d\alpha_1}{dt}|_{t=0}$ and $v_2 = \frac{d\alpha_2}{dt}|_{t=0}$.

(1a) Verify the following calculations:

$$\begin{aligned} I_p(v_1, v_2) &= I_p\left(\frac{d\alpha_1}{dt}|_{t=0}, \frac{d\alpha_2}{dt}|_{t=0}\right) \\ &= I_p\left(X_u \frac{du_1}{dt} + X_v \frac{dv_1}{dt}, X_u \frac{du_2}{dt} + X_v \frac{dv_2}{dt}\right) \\ &= I_p(X_u, X_u)\frac{du_1}{dt}\frac{du_2}{dt} + I_p(X_u, X_v)\left(\frac{du_1}{dt}\frac{dv_2}{dt} + \frac{du_2}{dt}\frac{dv_1}{dt}\right) + I_p(X_v, X_v)\frac{dv_1}{dt}\frac{dv_2}{dt}. \end{aligned}$$

(1b) In the (u,v)-plane, the vectors $\begin{pmatrix} \frac{du_1}{dt} \\ \frac{dv_1}{dt} \end{pmatrix}$ and $\begin{pmatrix} \frac{du_2}{dt} \\ \frac{dv_2}{dt} \end{pmatrix}$ are mapped to the tangent vectors $\frac{d\alpha_1}{dt}|_{t=0}$ and $\frac{d\alpha_2}{dt}|_{t=0}$. Verify that:

$$I_p(\frac{d\alpha_1}{dt}|_{t=0}, \frac{d\alpha_2}{dt}|_{t=0}) = \begin{pmatrix} \frac{du_1}{dt} \\ \frac{dv_1}{dt} \end{pmatrix}^t \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \frac{du_2}{dt} \\ \frac{dv_2}{dt} \end{pmatrix}.$$

(1c) Compute the First Fundamental Form of the 2-dimensional plane Π that passes through the three unit vectors on the first three axes in $\mathbb{R}^n$. Repeat your calculations using a parameterization that results in the polar coordinates on Π.

(1d) Compute the First Fundamental Form of the 2-dimensional sphere of radius R in standard spherical coordinates in $\mathbb{R}^n$. When n=3, use the notation (φ,θ) for the angles, and show that: $E(\varphi,\theta)=\sin^2\theta,\ F(\varphi,\theta)=0,\ G(\varphi,\theta)=1$. Further, prove the following property of the determinant of the matrix for the First Fundamental Form $EG-F^2>0$. Is this property true in general for any regular representation of a surface S? Why?

Problem 2. Use the FFF to calculate the length of curves on a surface. We continue to use the notation above for the surface S, and curves on them are called $\alpha_1(t)$, $\alpha_2(t)$.. or $\alpha(t)$ when only one curve is considered.

(2a) Prove that the length of the curve $\alpha(t)$ is expressed by the formula:

$$s(t)=\int_{t_0}^{t}[E(u(\tau),v(\tau))(\frac{du}{d\tau})^2+2F(u(\tau),v(\tau))\frac{du}{d\tau}\frac{dv}{d\tau}+\frac{du_2}{dt}\frac{dv_1}{dt})+G(u(\tau),v(\tau))(\frac{dv}{d\tau})^2]^{\frac{1}{2}}d\tau$$

(2b) The formula above has a standard notation in many older books in differential geometry and physics as $ds^2=Edu^2+2Fdudv+Gdv^2$, and ds is called the element of arc length (also called the line element) for the surface S. This agrees with the well-known formula $ds^2 = du^2 + dv^2$ as the infinitesimal form of the Pythagorean Theorem. Compute the line element in the most convenient local coordinates for the surfaces in Problem **(1c,d).**

(2c) The general equations for a cylinder S over a plane curve C generated by a straight line in the direction of a unit vector ξ in $\mathbb{R}^n$ is defined earlier. Compute the line element for any such cylinder and prove that it has the same equation as the line element for the standard Euclidean plane when we regard that as a cylinder over the curve C which is the first axis and the unit vector ξ is on the second axis in $\mathbb{R}^n$. This demonstrates that the metric geometry of curves on all cylinders is the same as the Euclidean plane.

(2d) Generalize the preceding problem to cylinders over any regular space curve C in $\mathbb{R}^n$ to show that the metric geometry of such surfaces are also the same as the Euclidean plane.

Problem 3. Write the general equation for a cone K on a plane curve C and vertex P which is any point in $\mathbb{R}^n$ that does not belong to C. You can define K as the set of all half-lines joining P to each point of C and continuing indefinitely.

(3a) Suppose C has a regular parameterization as a simple closed curve. Show that K has a regular parameterization as smooth surface at all points except its vertex P. Compute the line element for K at all points outside its vertex, and show that it is the same as the Euclidean plane.

(3b) Generalize **(3a)** to include arbitrary piecewise smooth space curves and determine where the surface has a regular parameterization, then estimate the line element in the most convenient coordinate system.

Problem 4. Prove that the expression for the arc length of a curve in a surface, given locally by the line element, depends only on the curve and not on the choice of coordinates. Equivalently, compute how a line element transforms under a change of coordinates and show the invariance of its local expression.

Problem 5. (5a) Compute the line element for surfaces that are given as the graph of a smooth real-valued of two variables in $\mathbb{R}^3$, $f: U \to \mathbb{R},\ U \subseteq \mathbb{R}^2$ an open set with coordinates (u,v) as

$$ds^2 = (1+(\frac{\partial f}{\partial u})^2)du^2 + 2(\frac{\partial f}{\partial u}\frac{\partial f}{\partial v})dudv + (1+(\frac{\partial f}{\partial v})^2)dv^2 .$$

(5b) Verify your formula for the upper hemisphere in $\mathbb{R}^3$ by computing the line element in two different ways. First, in terms of spherical coordinates (Problem **(1d)** above) and then as the graph of the function $f(u,v)=\sqrt{R^2-u^2-v^2}$ as in **(5a).**

Problem 6. We can also compute the angle between two curves $\alpha_1(t)$ and $\alpha_2(t)$ on the surface, such that $\alpha_1(0)=p$ and $\alpha_2(0)=p$ using the FFF of the surface. The angle between two curves intersecting at a point is defined in terms of tangent vectors to the two curves at the point of intersection. Verify that the cosine of the angle between the two curves is given by the formula $\cos\theta=\dfrac{I_p\left(\frac{d\alpha_1}{dt}|_{t=0},\frac{d\alpha_2}{dt}|_{t=0}\right)}{\sqrt{I_p\left(\frac{d\alpha_1}{dt}|_{t=0},\frac{d\alpha_1}{dt}|_{t=0}\right).I_p\left(\frac{d\alpha_2}{dt}|_{t=0},\frac{d\alpha_2}{dt}|_{t=0}\right)}}$.

(6b) Verify that two coordinate curves are orthogonal to each other at their point of intersection if and only if $F(u,v)=0$. If this hold for all points in the local coordinate neighborhood, then we call $X(u,v)$ an *orthogonal parameterization.*

Problem 7. We can also compute the area of a figure on the surface S using its FFF. Let B be a bounded region on the surface S, and let $A=X^{-1}(B)$ be its pre-image in the (u,v)-plane. Define the area of B by the formula $area(B)=\iint_A\sqrt{E(u,v)G(u,v)-F(u,v)^2}\,dudv$.

(7a) Verify that the area of B remains the same quantity when computed in two different regular parameterizations. That is, the area of a region is independent of the coordinate system that we use to compute it. Hence, area on a surface, just like the line element and angle between two curves is a geometric invariant.