Basic Concepts in Differential Geometry of R^n

Introduction. Ever since Descartes, analytic geometry appeals to representing a geometric object Ω by a collection of functions that define the coordinates of its points, say $X(P) = (x_1(P), x_2(P)..., x_n(P)).$ The Euclidean space $\mathfrak{R}^{\mathbf{n}}$ is intuitively considered the set of such n-tuples or *n-dimensional vectors*. The reference to n-dimensionality of the Euclidean space in this context is intuitive and by the virtue of the following facts: First, the basic building blocks of the space are subsets $B = \{(x_1, x_2..., x_n) : a_k < x_1 < b_k \ ; k = 1, 2, ..., n\}$ that can be described by n inequalities that are logically *independent of one another*; second, that the entire $\mathfrak{R}^{\mathbf{n}}$ can be constructed either by an increasing sequence of such building blocks, or by overlapping or "gluing" of a collection of such building blocks. The term '*independent of one another'* above carries a significant conceptual burden and its clarification in each context calls for a rigorous treatment of numerous subsequent ancillary analytic and geometric properties. The simplest definition of independence is algebraic, that of *linear independence*. In this context, the coordinate space $\mathbb{R}^n$ is necessarily also a vector space, and the operations of vector addition and multiplication of a vector by a scalar lie at the very heart of how the geometry of the space will be defined step-by-step. Thus, at any point in space, the algebraic description of the geometry of the points in its neighborhood relies on *n* conditions that are logically unrelated and without any implications on each other. On the other hand, imposing any metric relationship among points that are near each other could be accomplished by a notion of *distance*. The more versatile concept of an inner product provides a mathematically satisfactory and algebraically elegant approach to axiomatize the analytic geometry in Euclidean space. This is the historical path that has led to successful applications of analytic geometry to problems of mechanics and astronomy, among other physical sciences. On the other hand, such an idealistic way of imposing the geometry on an n-dimensional space is not free from its limitations and shortcomings, as we shall see below.

What are some of consequences of the simple and direct algebraic definition of n-dimensionality? First and foremost, algebraic structure of a vector space necessitates a *homogenous structure* on the space $\mathbb{R}^n$: the geometric properties in each building block $B = \{(x_1, x_2 ..., x_n) : a_k < x_1 < b_k \ ; k = 1, 2, ..., n\}$, say in the small piece of space surrounding a neighborhood of the origin $(0, 0..., 0) \in \mathbb{R}^n$ is the same as any of translates of B to any other point of the space $\mathbb{R}^n$. This assumption may be intuitively appealing when we consider an empty space, that is, the space void of any physical entity that could influence its geometry and physics according to vicinity of various material objects. The natural environment surrounding us, if it is considered a space, is surely unlike such a homogenous void! In general relativity, a space that has massive stars and other heavenly bodies is not homogenous in its geometric properties. In fact, the geometric non-homogeneity defines curvature, and consequently the fundamental physical method for discovery of massive objects in space. As a result, we must be mindful of the limitations of theories that strictly must rely on such a limited notion of *linear independence* to define *dimensionality* of space.

What other options do we have to define *dimensionality* of the coordinate space without imposing unconditional *homogeneity* at all points of space? Reality of our natural environment calls for a definition based on observations of natural phenomena. This implies, in particular, that the notions of *nearness* of two points in space in terms of having *similar properties*, or *homogeneity* in terms of sharing properties that vary according to certain continuity restrictions and only especially allowed translations (transformations) could well be natural conditions that result from our observations of a part of the environment. These observations could vary from place to place and from time to time, in order to accommodate the richness of the environmental phenomena and observations. As a result, we must seek the notions of geometry for a space that share the fundamental intrinsic properties of space (such as its continuity, appropriate variants of connectedness, bounded or unbounded features, finiteness,…) with the n-dimensional Euclidean space. Yet, the desired concepts should not impose undue restrictions such as the extreme homogeneity resulting from algebraic definition of independence and algebraic n-dimensionality. As we see below, there are ways to circumvent undue limitations of algebraic structure of a vector space on the coordinate space while still maintaining much of the qualitative properties of geometry of the n-dimensional space and its continuity. On approach to this problem is through abstract formulation of the notion of distance and neighborhood, that is, the concept of a topological space and that of the structure of a topology on a space in lieu of a algebraic (or analytic) distance metric.

Onne of the great advantages in realization of the homogenous structure of space $\mathbb{R}^n$ has been the discovery of making this concept abstract without sacrificing its intuitive appeal. This was accomplished by Henri Poincaré who introduced the concept of *topology* as a far reaching generalization of the notion of distance when continuity properties of space are to be preserved. A topology on a set X is a choice of the family of subsets of X, say $\Theta = \{U_\alpha :\ U_\alpha \subseteq X;\ \alpha \in I\}$ subject to the basic axioms that the two obvious members X and $\varnothing$ are members of Θ, and that Θ contains intersections of any pair of its members as well as unions of an arbitrary number of its members. The subsets U_α are called open subsets (or open neighborhoods) of the space X, and essentially they dictate our preferences for how the various points in X are considered "near each other" or share "similarly defined" continuous properties. In a topological space X, the choice of distance is completely bypassed, and homogeneity is not necessary to be defined in algebraic terms!

as those subsets whose coordinates are bounded by finite numbers An important simplification of analytic description of Ω occurs when we use an auxiliary set of points in . The coordinates of any point Pcollection of coordinate functions (In differential calculus, we study *local properties* of differentiable functions and use derivatives (small changes in values of functions due to arbitrarily small variations of the parameters in a neighborhood of a given point $P \in \Omega$. The great advantage of using sufficiently small variations of the parameters is that we can approximate complicated formulas that describe the changes in values of the function by much simpler linear quantities. A prime example of this is the definition of the tangent line to a curve Ω at a point P whose slope in plane (or more generally, its direction in space) is calculated using the rate of change of values of projections of the small pieces of the curve onto various coordinate axes.

When we think of geometry, usually we mean the spatial properties of objects lying in the environment surrounding us. In mathematics, objects (and even the environment itself is made into an abstract mathematical structure that we denote by $\mathfrak{R}^3$) are idealized to the extent that the coordinates of a point P, say the triple of numbers (*a* , *b*, *c*), signify the most primitive geometric object in space, namely the ideal dimensionless point P situated in space, and such that it lies a distance from the origin **O** according to the measurements *a* , *b*, *c* units along the three perpendicular *x*-, *y*- and *z*-axis that are drawn at **O**. Taking this idealization a step further, all geometric objects are *subsets of* $\mathfrak{R}^3$. Now suppose we have one of these subsets that we call **S**, for instance a straight line passing through P and in the direction parallel to the *x*-axis. The set of all points Q that lie near P, say within a prescribed distance δ is called a neighborhood of P. Usually, we must be more specific about a neighborhood of P depending on having any other constraints on such nearby points Q, or simply all points of $\mathfrak{R}^3$ that are within distance δ. If the points Q must also belong to **S**, then we refer to a neighborhood of P in **S**. The main focus of local differential geometry of a *mathematical structure* **S** *in* $\mathfrak{R}^3$ is to elucidate the many relationships and logical properties that are implicit in subsets of $\mathfrak{R}^3$ that contain P, or are mapped to such by means of a careful study of the properties of infinitesimal neighborhood of P in **S** points (i.e. arbitrarily nearby).

Differential Calculus in $\mathfrak{R}^n$

Suppose that we are given a geometric object Ω, such a curve, surface, polyhedron or any other object of interest in the Euclidean space and we wish to study the geometry of Ω using methods of differential and integral calculus. Without loss of generality, we can assume that Ω is connected. In other words, any pair of points on Ω can be joined together by a continuous curve that lies entirely on Ω. Given any *reasonable* geometric object in the Euclidean space, we can regard it as a disjoint union of pieces that are connected (as defined above). Such pieces are called *connected components* of Ω if Ω happens to be disconnected. The reason for this step is that we can give more succinct mathematical arguments when we deal with a connected Ω, and we need not each time consider what happens to continuity properties of functions if Ω consists of several pieces that cannot be joined by continuous curves. Moreover, we will see later that many global properties of geometric objects in dimensions two and higher can be studied one connected component at a time.

Definition. Suppose Ω is a subset of the n-dimensional Euclidean space, and we have selected a collection of points $\{P_\alpha \in \Omega : \alpha \in I\}$ and a corresponding collection of open subsets of the Euclidean space U_α such that $\{P_\alpha \in \Omega,\ P_\alpha \in U_\alpha : \alpha \in I\}$ and that $\Omega \subset \bigcup_{\alpha \in I} U_\alpha$. Then the collection $\{U_\alpha : \alpha \in I\}$ is called an open cover of Ω.

Exercise. (a) Suppose Ω is a piecewise smooth made of finitely many arcs in the n-dimensional Euclidean space, and that $\{U_\alpha : \alpha \in I\}$ is an open cover of Ω such that each U_α is connected. Prove that Ω is connected if and only if we can select finitely many U_α and index them as $\{U_k : k = 1, 2, \ldots m\}$ such that $\{U_k \cap U_{k+1} \neq \emptyset : k = 1, 2, \ldots m-1\}$. In other words, every such connected curve can be covered with a linearly ordered chain of *connected* open subsets that must have a non-empty overlap; conversely, in case any covering by arbitrary connected open subsets of the Euclidean space that are linearly ordered as above must necessarily have nonempty consecutive overlaps, then Ω is connected. (b) Generalize part (a) to higher dimensional geometric objects of the

Euclidean space, for example, to the class of geometric objects that are made of a finite number of subsets $\{\Omega_k : k = 1,...,m\}$ that are reasonably like piecewise smooth curves. For example, we can assume that $\{\Omega_k : k = 1,...,m\}$ are images of differentiable mappings whose domains are open subsets of a fixed Euclidean space of dimension *s*; or in terms of formulas:
$\{\Phi_k : S_k \to \mathbb{R}^n :\ S_k \text{ open subset of } \mathbb{R}^s,\ k = 1,...,m\}$.

In the exercise above, the restriction that all pieces have the same dimension is rather artificial and only for simplicity and could be avoided. The point of this exercise is to show you that study of geometric properties of fairly general objects can be reduced to study of much simpler building blocks $\{\Omega_k : k = 1,...,m\}$ that satisfy some mild regularity conditions such as being parametrically definable by differentiable functions. As usual, there are pathological examples that show the subtleties that must be observed in generalizing an "obvious geometric property" beyond what meets the eyes and could be captured by intuition alone! Logical reasoning in terms of set-theoretic properties of subsets of the Euclidean space are essentially the only way to prove connectedness for general subsets that do not necessarily lend themselves to parametric description by differentiable functions. The study of the properties such as connectedness and other phenomena in terms of open coverings (especially if the number of open sets is not finite and the open sets are fairly general in nature) constitute a branch of mathematics called *point-set topology*.

Now suppose that we have a geometric object Ω, e.g. a connected differentiable curve whose curvature and torsion are smooth function $\kappa : \Omega \to \mathbb{R}$ and $\tau : \Omega \to \mathbb{R}$. More generally, let $\varphi : \Omega \to \mathbb{R}$ be a differentiable (actually even continuous) function that describes the type of geometric property of Ω that could be discovered locally using differential calculus. That is, suppose that we have succeeded in construction of $\varphi_\alpha : \Omega_\alpha \to \mathbb{R}$ on open subsets of Ω using a collection of differentiable functions $f_\alpha : U_\alpha \to \mathbb{R}$ that are defined locally on connected open sets $U_\alpha \subset \mathbb{R}^n$ that cover our geometric object Ω, and $\Omega_\alpha = \Omega \cap U_\alpha$ are conveniently parameterized (e.g. as in using unit-speed parameterization of curves in order to have a more direct definition of curvature, torsion and the Frenet moving frames along space curves. As above, we reorganize the covering and index the appropriate open sets as $\{U_k \ : k = 1,2,...\ m\}$ such that $\{U_k \cap U_{k+1} \neq \varnothing \ : k = 1,2,...\ m-1\}$ and $\varphi_k : \Omega_k \to \mathbb{R}$ are differentiable (or merely continuous if $\varphi : \Omega \to \mathbb{R}$ is assumed to be only continuous.) Are we at liberty in selecting *any* coordinate system for each $f_k : U_k \to \mathbb{R}$ and the resulting description of $\varphi_k : \Omega_k \to \mathbb{R}$, or are there constraints on selection of parameterization and coordinate systems on $\{U_k \ : k = 1,2,... \ m\}$ and $\{\Omega_k \ : k = 1,2,... \ m\}$? The answer is that continuously varying properties of connected curves, such as $\varphi : \Omega \to \mathbb{R}$, require an important compatibility condition to be satisfied when we construct piecewise definition of functions $\varphi_\alpha : \Omega_\alpha \to \mathbb{R}$. Namely, on overlaps $\{U_\alpha \cap U_\beta \neq \emptyset \ : \alpha, \beta \in I\}$ the locally defined functions $\varphi_\alpha : \Omega_\alpha \to \mathbb{R}$ and $\varphi_\beta : \Omega_\beta \to \mathbb{R}$ must agree, at least after suitable choices of parameters for Ω_α and Ω_β.

Local-to-Global Constructions

In differential calculus, we study *local properties* of differentiable functions and use derivatives (small changes in values of functions due to arbitrarily small variations of the parameters in a neighborhood of a given point $P \in \Omega$. The great advantage of using sufficiently small variations of the parameters is that we can approximate complicated formulas that describe the changes in values of the function by much simpler linear quantities. A prime example of this is the definition of the tangent line to a curve Ω at a point P whose slope in plane (or more generally, its direction in space) is calculated using the rate of change of values of projections of the small pieces of the curve onto various coordinate axes. As soon as we define local (infinitesimal) approximation of the curve Ω at a point P by the tangent line to the curve, say denoted by $T_P(\Omega)$, we discover the key geometric invariant of a curve that is captured by studying the local variation of the tangent line $T_Q(\Omega)$ for nearby points Q that are in a sufficiently neighborhood of P (all nearby points Q must also lie on the curve). Measuring how $T_P(\Omega)$ changes to $T_Q(\Omega)$ leads to definition of curvature of the curve $\kappa_\Omega(P)$. Similarly, we define the normal $N_P(\Omega)$ and binormal $B_P(\Omega)$ to the curve, and the second basic geometric invariant of a space curve, namely its torsion $\tau_\Omega(P)$ that is the rate of variation of the plane spanned by $\{T_P(\Omega), N_P(\Omega)\}$.

In integral calculus, we piece together (or sum up) such tiny linear approximations, having in mind that definition of each such linear quantity depends on the points that we have used their neighborhoods to estimate the linearization. Global properties of geometric objects are then revealed through these two complementary operations. For example, consider the geometric properties of a simple closed curve in by $\mathfrak{R}^3$. Some of these properties are local, such as how a curve is bent in space, or how it twists. These are local properties and can be quantified using differential calculus. the total curvature of a simple closed curve Ω is defined as the integral of its curvature function over the curve. This philosophy essentially leads us to try and do our differential calculations completely locally in terms of all differentiable functions that we have at our disposal definable in the particular neighborhoods of points on the geometric structure under study, or a collection of appropriate points $\{P_\alpha \in \Omega: \ \alpha \in I\}$. The basic assumption then is that we are given a class of real-valued *smooth functions* $f_\alpha : U_\alpha \to \mathbb{R}$ defined on open subsets of the (n+1)-dimensional space $\mathbb{R}^{n+1}$, that is, all partial derivatives of such functions are defined and are continuous; or in symbols: $\{f_\alpha : U_\alpha \to \mathbb{R}: \ U_\alpha \text{ open subset of } \mathbb{R}^{n+1}, \ \alpha \in I\}$. Further, the choices of $\{P_\alpha \in \Omega, \ P_\alpha \in U_\alpha \ : \alpha \in I\}$ are such that we cover Ω. The fundamental question that we wish to explore in this course is the following so-called Local-to-Global Problem:

Problem. Suppose the functions φ_α *implicitly* describe a particular *metric property* of a geometric object Ω in $\mathbb{R}^n$, and that the union of domains of definition of these functions cover Ω. How can we piece together geometrically useful information that we extract from φ_α and their infinitesimal variations?

This question, in particular, requires us to be particularly careful about definition of all globally continuous functions $\psi : \Omega \to \mathbb{R}$ that define geometric properties of our object of interest Ω. Namely, we use differential calculus to define pieces of ψ in a neighborhood of the points $\{P_\alpha \in \Omega: \ \alpha \in I\}$ as $\psi_\alpha : U_\alpha \to \mathbb{R}$ from $f_\alpha : U_\alpha \to \mathbb{R}$. Then we put together the small pieces $\psi_\alpha : U_\alpha \to \mathbb{R}$ in order to define $\psi : \Omega \to \mathbb{R}$.