Math-801

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Parallelism: Analytic Theory

In this lecture note, we prove the basic theorem that allows us to generalize the basic Parallel Postulate of Euclidean plane geometry: Given a straight line L and a point P outside L, there is a straight line Λ parallel to L. Furthermore, the choice of P and L provides a unique such Λ. In Riemannian geometry, for instance given a surface $S \subset \mathbb{R}^3$ and a geodesic $L \subset S$, we might translate the first assertion as the problem of constructing a geodesics Λ that passes through the point $P \in S$ and "parallel to the given geodesic L" and then investigate the number of such solutions. Indeed Euclid's Parallel Postulate turned out to be close to being a characterization of the very special type of geometry in which the space has the property that all its geodesics have zero curvature in all directions. Since nonzero curvature for geodesics and nonlinearity go hand in hand, uniqueness of a geodesic passing through a given point and parallel to given geodesic need not be part of the generalization of parallelism.

Differential geometry takes advantage of approximation of infinitesimal variations of geometric structures and quantities. In generalizing parallelism from the Euclidean plane to curved surfaces, we could first formulate the properties and concepts at the infinitesimal level. To extend the results to a "macroscopic scale", we integrate the infinitesimal quantities in the appropriate range. At a point $P \in C$, the tangent vector to a smooth curve $\xi \in T_P C$ is infinitesimal linearization of C near P. The tangent vector field along C is the global object that organizes all such infinitesimal linearization across a whole arc of the curve. The unit-speed parameterization of a curve is a unique way to suppress the variation in the length of the tangent and put the emphasis on the direction of the tangent line. The unit-speed parameterization of a straight line in the Euclidean plane shows that the tangent vector field is a constant vector, so its variation is zero. This property should be re-examined from the viewpoint of intrinsic geometry of the plane and its geodesics. On a surface $S \subset \mathbb{R}^3$, the unit speed parameterization of a curve C shows that the tangent vector field along the curve has constant unit length, but its direction could change no matter which curve is selected on the surface. For example, any smooth curve that lies on the surface of a sphere has the property that its tangent vector field in the unit speed parameterization has constant length but it cannot be a constant. As we mentioned earlier, the directional derivative of the tangent vector field V of C, for example with respect to , is typically a vector field W along $C \subset \mathbb{R}^3$, and it could have a nonzero component in the direction normal to the surface. From the view point of intrinsic geometry of the surface, only the tangential component could contribute to variation of the tangent vector field $V \subset C$. In other words, the normal component of $W = D_V V$ could be nonzero, and the vanishing of its tangential component would suffice to make the curve into a geodesic. The tangential component of $W = D_V V$ is called the covariant derivative of $V \subset C$ with respect to V, and is denoted by $\nabla_V V$. Earlier we introduced the definition of a geodesic of S to be a smooth curve whose acceleration vector field is normal to the surface. According to the discussion above, the covariant derivative $D_V V$ is along the normal vector field to the surface precisely when the covariant derivative vanishes: $\nabla_V V = 0$. Therefore, we have the following characterization of a geodesic on a surface:

Theorem 1. A regular curve $\gamma(t), \gamma : (-a, a) \to S$ is a geodesic if and only if $\nabla_V V = 0$.

We can now turn around this definition, and interpret the property $\nabla_V V = 0$ as a generalization of the tangent vector field being a constant, or all tangent vectors being parallel to a constant direction. More generally, if Y is any tangent vector field of S along a smooth curve $C \subset S$, we could still test if Y is constant by taking the directional derivative with respect to the tangent vector field of C along the curve. From the view point of intrinsic geometry of the surface, we need to test if $\nabla_V Y = 0$. This more general observation embodies the germ of the idea of generalizing parallelism from the plane to any smooth surface, and it is called Levi-Civita parallelism. Therefore we have the following:

Definition. A tangent vector field Y of S is parallel along a smooth regular curve $C \subset S$ if the covariant derivative of Y with respect to the tangent vector field of C is zero. Given two points $P \in C$ and $Q \in C$, two vectors $\xi \in T_P(S)$ and $\zeta \in T_Q(S)$ are called Levi-Civita parallel along $C \subset S$ if there is a tangent vector field Y that is parallel along C and such that $Y(P) = \xi$ and $Y(Q) = \zeta$.

Exercise (a). Suppose we have a regular surface that has the following property: Every smooth geodesic can be indefinitely extended in both directions. Such surfaces are called *geodesically complete.* Surfaces that have missing point, for example $\mathbb{R}^2 - \{0\}$, are not geodesically complete. In the example $\mathbb{R}^2 - \{0\}$, lines that normally pass through origin cannot be extended past the missing point. Show

Exercise (b). Suppose we have a regular surface that has the following property: Every smooth geodesic $C \subset S$ can be indefinitely extended in both directions without self-intersection. Such surfaces geodesically complete and have also a kind of "unboundedness" in their global geometry. Suppose, we have a point $P \in C$ and a unit tangent vector $\xi \in T_P C$, with the property that all tangent lines to C (that is one of their direction vector of unit length) are either Levi-Civita parallel to ξ or they all pass through P. What can you say about geometry of S using its geodesics and Levi-Civita parallelism of its geodesics? For instance, given a pair of geodesic curves C_1 and C_2, suppose that they have two tangent vectors $\xi_1 \in T_P C_1$ and $\xi_2 \in T_P C_2$ that are Levi-Civita parallel. Would such geodesic curves C_1 and C_2 ever intersect? Would the property of two geodesically that never intersect imply that they have two tangent vectors that are Levi-Civita parallel?

Construction of Parallel Vector Fields

As before, we assume that $S \subset \mathbb{R}^3$ is a smooth surface given as a level surface of a smooth function $S = f^{-1}(c)$ or as a regularly parameterized surface $X : \Omega \to S \subset \mathbb{R}^3$. We have also a smooth curve C that has a parameterization (unit speed if necessary) $\gamma(t), \gamma : (-a,a) \to S$ such that $\gamma(0) = P \in S$. The normal vector field $N(\gamma(t)) \perp T_{\gamma(t)} S$ consists of unit normal vectors and restricted to the curve. The tangent vector field $\frac{d\gamma(t)}{dt} \equiv \dot{\gamma}(t) \in T_{\gamma(t)}S$ is denoted by V for simplicity of notation. We are given a vector $\xi \in T_P S$ that we wish to extend to a vector field W such that W is Levi-Civita parallel along C. If such a vector field would exist, then it should satisfy the following. The covariant derivative of W vector field with respect to the tangent vector field V vanishes, $\nabla_V W = 0$ and $W(\gamma(0)) = \xi$. We proceed by making the latter conditions explicit in terms of coordinate description of the surface, the curve and the desired parallel vector field. The directional derivative $D_V W = V(W(\gamma(t)))$ is decomposed at every point $\gamma(t)$ into the normal component $\langle V(W(\gamma(t))), N(\gamma(t))\rangle N(\gamma(t))$ and its tangential component $\nabla_V W = V(W(\gamma(t))) - \langle V(W(\gamma(t))), N(\gamma(t))\rangle N(\gamma(t))$. For W to be Levi-Civita parallel along C, it is necessary and sufficient that the tangential component of $\nabla_V W$ be zero, so $V(W(\gamma(t))) - \langle V(W(\gamma(t))), N(\gamma(t))\rangle N(\gamma(t)) = 0$, which is a differential equation with the initial value condition $W(\gamma(0)) = \xi$. As in the case of construction of geodesics, we proceed to compute the expression in terms of local coordinates for the Euclidean space. We work out an example to illustrate the steps in the general case that would be valid for arbitrary regular surfaces in space.

Example 1. Consider a cylinder $S \subset \mathbb{R}^3$ with base curve in $C \subset \mathbb{R}^2, C(s) = (x_1(s), x_2(s), 0)$ (for example a circle of radius 1) such that $\| C'(s) \| = 1$) and a direction vector $v = (c_1, c_2, c_3)$ of unit length ($\| v \| = 1$) , (in the special case, choose the unit vector along the third coordinate axis). Let $\gamma(\tau), \gamma : (-a, a) \to S$ be the equation of a curve on the surface described by a pair of functions $(s(\tau), t(\tau))$ in the parameter domain of the surface. In terms of coordinates for $\gamma(\tau) \subset \mathbb{R}^3$, we compute:

$$\begin{aligned}\gamma(\tau) &= L(s(\tau), t(\tau)) \\ &= (x_1(s(\tau)) + c_1 t(\tau), x_2(s(\tau)) + c_2 t(\tau), c_3 t(\tau)).\end{aligned}$$

The equation of the surface is $L(s,t) = C(s) + tv$ and as before, the tangent space at a point $P = L(s_0, t_0) = C(s_0) + t_0 v$ is generated by the unit vectors $L_s = C'(s), L_t = v$, and the normal vector $N(s,t) = C'(s) \times v$ has also unit length. The ordered triple $\{C'(s), v, N(s,t)\}$ is the oriented basis that we select locally for $T_{\gamma(\tau)}\mathbb{R}^3$. The given vector $\xi \in T_P S$ is described in terms of the basis for $T_P S$:

$$
\begin{aligned}
\xi &= aL_s + bL_t \\
&= aC'(s_0) + bv \\
&= a(x'_1(s_0), x'_2(s_0), 0) + bv \\
&= (ax'_1(s_0) + bc_1, ax'_2(s_0) + bc_2, bc_3).
\end{aligned}
$$

The equation of the normal for a general cylindrical surface was calculated by the formula below:

$$N(s,t) == (c_3 x'_2(s_0), -c_3 x'_1(s_0), c_2 x'_1(s_0) - c_1 x'_2(s_0)).$$

$$N(\gamma(\tau)) = (c_3 \frac{dx_2(s(\tau))}{ds}, -c_3 \frac{dx_1(s(\tau))}{ds}, c_2 \frac{dx_1(s(\tau))}{ds} - c_1 \frac{dx_2(s(\tau))}{ds})$$

We must now unravel the condition $D_{\dot{\gamma}(\tau)} W = \mathrm{span}\{N(\gamma(\tau))\}$ as follows:

$$V(\gamma(\tau)) = \frac{d\gamma}{d\tau} = (\frac{dx_1}{ds}\frac{ds}{d\tau} + c_1 \frac{dt}{d\tau}, \frac{dx_2}{ds}\frac{ds}{d\tau} + c_2 \frac{dt}{d\tau}, c_3 \frac{dt}{d\tau});$$

$$
\begin{aligned}
D_V W(\gamma(\tau)) &= (D_V W_1, D_V W_2, D_V W_3) \\
&= (\langle V, gradW_1 \rangle, \langle V, gradW_2 \rangle, \langle V, gradW_3 \rangle) \\
&= (\langle V, gradW_1(\gamma(\tau)) \rangle, \langle V, gradW_2(\gamma(\tau)) \rangle, \langle V, gradW_3(\gamma(\tau)) \rangle).
\end{aligned}
$$

We must compute the individual inner products $\langle V, gradW_k(\gamma(\tau)) \rangle$:

$$gradW_k(x_1, x_2, x_3) = (\frac{\partial W_k}{\partial x_1}, \frac{\partial W_k}{\partial x_2}, \frac{\partial W_k}{\partial x_3}),$$

$$
\begin{aligned}
gradW_k(\gamma(\tau)) \bullet \frac{d\gamma(\tau)}{d\tau} &= \left\langle (\frac{\partial W_k}{\partial x_1}, \frac{\partial W_k}{\partial x_2}, \frac{\partial W_k}{\partial x_3}), \frac{d\gamma(\tau)}{d\tau} \right\rangle \\
&= \sum_{j=1}^{3} \frac{\partial W_k}{\partial x_j}\frac{\partial x_j}{\partial s}\frac{ds}{d\tau} + \frac{\partial W_k}{\partial x_j}\frac{\partial x_j}{\partial t}\frac{dt}{d\tau}.
\end{aligned}
$$

We calculate the resulting directional derivative as a vector field on the curve $\gamma(t) \subset \mathbb{R}^3$ and project the vector field from the 3-dimensional space to the tangent planes of the surface. This requires substitution of the quantities that we just calculated in $D_V W(\gamma(t)) - \langle D_V W(\gamma(t)), N(\gamma(t)) \rangle N(\gamma(t)) = 0$.

The r-th term of $\langle D_V W(\gamma(t)), N(\gamma(t))\rangle N(\gamma(t))$ is:

$$\left(\sum_{k=}^{3}(\sum_{j=1}^{3}\frac{\partial W_k}{\partial x_j}\frac{\partial x_j}{\partial s}\frac{ds}{d\tau}+\frac{\partial W_k}{\partial x_j}\frac{\partial x_j}{\partial t}\frac{dt}{d\tau}).N_k(\gamma(\tau)\right)N_r(\gamma(\tau)$$

So the system has the following r-th equation:

$$\sum_{j=1}^{3}\frac{\partial W_r}{\partial x_j}\frac{\partial x_j}{\partial s}\frac{ds}{d\tau}+\frac{\partial W_r}{\partial x_j}\frac{\partial x_j}{\partial t}\frac{dt}{d\tau}-N_r(\gamma(\tau).\sum_{i=1}^{3}(\sum_{j=1}^{3}\frac{\partial W_i}{\partial x_j}\frac{\partial x_j}{\partial s}\frac{ds}{d\tau}+\frac{\partial W_i}{\partial x_j}\frac{\partial x_j}{\partial t}\frac{dt}{d\tau}).N_i(\gamma(\tau)=0.$$

The initial values for the resulting ODE system are given in terms of the tangent vector to the surface $\xi \in T_P S$ and coordinates of the curve γ at P .

Exercise. Compute the simplified system for a cylinder whose base curve is a circle of radius R and its generator is parallel to the third coordinate.