Undirected graphs

We shall mostly be interested in the case where Γ is undirected, without loops or multiple edges. For A this means that it is symmetric ($A = A^\top$), has zero diagonal ($a_{xx} = 0$), and is a 0-1 matrix ($a_{xy} \in \{0,1\}$).

A number θ is eigenvalue of A if and only if it is a zero of the polynomial p_A. Since A is real and symmetric, all its eigenvalues are real. Also, for each eigenvalue θ, its algebraic multiplicity (that is, its multiplicity as a root of the polynomial p_A) coincides with its geometric multiplicity (that is, the dimension of the eigenspace $V_\theta = \{v \mid Av = \theta v\}$), so that we may omit the adjective and just speak about `multiplicity'. Conjugate algebraic integers have the same multiplicity.

Two very useful properties of this matrix representation of a graph are:

(i) Γ is regular (of degree k) if and only if $AJ = kJ$ (where J is the matrix with all entries equal to 1);

(ii) $(A^h)_{xy}$ is the number of paths of length h from x to y. In particular, $(A^2)_{xx}$ is the degree of the vertex x, and $\mathrm{tr}\, A^2$ equals twice the number of edges of Γ; similarly, $\mathrm{tr}\, A^3$ is six times the number of triangles in Γ.

By Perron-Frobenius theory, we know something about the largest eigenvalue of a connected graph. (Indeed, A is irreducible if and only if Γ is connected.)

Proposition 1.4.2 *Let Γ be a connected graph with largest eigenvalue θ_1 and with minimum, maximum and average degree $k_{\min}$, $k_{\max}$ and $\bar{k}$. Then either $k_{\min} < \bar{k} < \theta_1 < k_{\max}$ or $k_{\min} = \bar{k} = \theta_1 = k_{\max}$.*

Proof: Let $\mathbf{1}$ be the vector with all entries equal to 1. Then $A\mathbf{1} \le k_{\max}\mathbf{1}$, and by (iv) of Perron-Frobenius we have $\theta_1 \le k_{\max}$ with equality if and only if $A\mathbf{1} = \theta_1 \mathbf{1}$, that is, if and only if Γ is regular of degree θ_1. This settles the case where Γ is regular. If Γ is not regular, then $k_{\min} < \bar{k} < k_{\max}$ and we have seen $\theta_1 < k_{\max}$. Finally, let u be any nonzero vector. We show that

$$\theta_1 \geq \frac{u^\top Au}{u^\top u}$$

with equality if and only if $Au = \theta_1 u$. Indeed, let $u_1, \ldots, u_v$ be an orthonormal basis of V consisting of eigenvectors of A, so that $Au_j = \theta_j u_j$ ($1 \leq j \leq v$). Then we can write u as a linear combination of the u_j, say $u = \sum_i a_i u_i$, and then $u^\top Au = \sum_i a_i^2 \theta_i \leq \theta_1 \sum_i a_i^2 = \theta_1 u^\top u$. Applying this with $u = \mathbf{1}$ we find $\theta_1 \geq \frac{\bar{k}v}{v} = \bar{k}$, with equality if and only if Γ is regular. $\square$

Remark Choosing the vector u in the previous proof more subtly, we may obtain better bounds on θ_1.

Exercise Use interlacing (see below) to show that $\theta_1 \geq \sqrt{k_{\max}}$. When does equality hold?

The spectrum of a disconnected graph is easily found from the spectra of its components:

Proposition 1.4.3 *Let Γ be a graph with connected components Γ_i ($1 \leq i \leq s$). Then the spectrum of Γ is the union of the spectra of Γ_i (and multiplicities are added).* $\square$

So, for not necessarily connected graphs, we have $\bar{k} \leq \theta_1 \leq k_{\max}$, and $\bar{k} = \theta_1$ if and only if Γ is regular, but if $\theta_1 = k_{\max}$ then we only know that Γ has a regular component with this valency, but Γ need not be regular itself.

Exercise Show that no (undirected) graph has eigenvalue $\sqrt{2+\sqrt{5}}$.

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Andries Brouwer 2003-09-30