Math-801

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Covariant Derivative on Surfaces

For a smooth function in the Euclidean space $f(x_1, x_2, \ldots x_{n+1}): \mathbb{R}^{n+1} \to \mathbb{R}$ and its regular value $c \in \mathbb{R}$, we have defined the level surfaces $M^n = f^{-1}(c) \subset \mathbb{R}^{n+1}$ to be n-dimensional hypersurfaces of the Euclidean space that are also smooth manifolds. That is, the tangent space $T_P M^n$ at every point $P \in M$ is isomorphic to the n-dimensional vector space $\mathbb{R}^n$ and the gradient of the function $grad_P\, f$ at $P \in M$ is perpendicular to $T_P M^n$. This means that at every point $P \in M$, the tangent space to the Euclidean space has the orthogonal direct sum decomposition $\mathbb{R}^{n+1} \cong T_P\mathbb{R}^{n+1} \cong T_P M \oplus span\{grad_P\, f\} \cong \mathbb{R}^n \oplus \mathbb{R}$. The notion of the directional derivative $D_U\, f(P)$ of a function at a point $P \in \mathbb{R}^{n+1}$ in a given direction that is prescribed by a unit vector $\vec{U} = (a_1, a_2, \ldots a_{n+1})$ takes advantage of this decomposition of tangent vectors of $\mathbb{R}^{n+1}$ by providing a simple formula for $D_U\, f(P)$. The direct approach defines $D_U\, f(P)$ as the linear estimation to infinitesimal changes of the function $f(x_1, x_2, \ldots x_{n+1})$ as the values of the average change $\frac{f(x_1 + ta_1, x_2 + ta_2, \ldots x_{n+1} + ta_{n+1}) - f(x_1, x_2, \ldots x_{n+1})}{t}$ approaches its limit by shrinking $t \to 0$.

The direct definition then is computed in terms of the dot product in $\mathbb{R}^{n+1}$ to show $D_U\, f(P) = \vec{U} \bullet grad_P\, f$. If a vector $\vec{V}$ other than a unit vector is specified, then we simply define the operation of *differentiation along* $\vec{V}$ at $P \in \mathbb{R}^{n+1}$ by mimicking the same formula: $D_V\, f(P) = \vec{V} \bullet grad_P\, f$ (but no longer call it the directional derivative.)

A powerful generalization of differentiation along a vector is differentiation of $f(x_1, x_2, \ldots x_{n+1})$ along a vector field:

$V(x_1, x_2, \ldots x_{n+1}) = ((x_1, x_2, \ldots x_{n+1}); (h_1(x_1, x_2, \ldots x_{n+1}), \ldots h_{n+1}(x_1, x_2, \ldots x_{n+1})))$ by a formula that is pointwise the same expression $D_V f(P) = \vec{V} \bullet grad_P f$. This formula globalizes the local notion of $D_V f(P) = \vec{V} \bullet grad_P f$ for smooth functions and vector fields that are globally defined on $\mathbb{R}^{n+1}$.

In this homework, we study the same ideas except that we focus on functions and the vector fields that are defined on the surface $S \subset \mathbb{R}^3$ (or the hypersurface $M^n = f^{-1}(c) \subset \mathbb{R}^{n+1}$), and wish to compute the values from the view point of the intrinsic surface geometry (as opposed to quantities that depend crucially on the position of the surface in its ambient space. This requires defining the operations in such a way that they would still make sense in terms of geometric structures that are do not depend on, for example, transforming S in $\mathbb{R}^3$ by a rigid motion $A: \mathbb{R}^3 \to \mathbb{R}^3$ and having another position for the surface $S \subset \mathbb{R}^3$ but keeping the geometry of curves on the surface all the same.

One approach to gain insight into this question is to take advantage of the direct definition of functions and vector fields in $\mathbb{R}^3$, but always remember that they are constrained to S, in analogy to Lagrange's Multiplier method for finding constrained maxima and minima of functions $f: \mathbb{R}^3 \to \mathbb{R}$. Therefore, for the time being, assume that the functions, vector fields etc are defined on S by restricting them from the same kind of objects defined on $\mathbb{R}^3$. Now suppose we have such a vector field $V(x_1, x_2, x_3) = ((h_1(x_1, x_2, x_3), h_2(x_1, x_2, x_3), h_3(x_1, x_2, x_3))$ but at every point $P \in S$, $V(P) \in T_P S$, so $V(P) \perp N(P)$. Here $N(P)$ is the unit normal to the surface. For instance, when $S = f^{-1}(c)$ is the level surface of a regular value of a smooth function $f: \mathbb{R}^3 \to \mathbb{R}$, $N(P) = \dfrac{1}{\| grad_P f \|} grad_P f$ and $V(P) \perp grad_P f$ in terms of functions defining S. Given a smooth function $g: S \to \mathbb{R}$ and a vector field V tangent to S, the operation $D_{V(x_1, x_2, x_3)} g(x_1, x_2, x_3) = \vec{V}(x_1, x_2, x_3) \bullet grad_{(x_1, x_2, x_3)} g$ defines the operation of tangent vector fields of the surface S on functions defined on S, and it is simply denoted by $V(g) = V(x_1, x_2, x_3)$. When we replace $g: S \to \mathbb{R}$ by a tangent vector field $W(x_1, x_2, x_3) = (g_1(x_1, x_2, x_3), g_2(x_1, x_2, x_3), g_3(x_1, x_2, x_3))$, we compute the formula component by component as:

$$D_{V(x_1,x_2,x_3)}W(x_1,x_2,x_3) = (\vec{V}(x_1,x_2,x_3)\bullet grad_{(x_1,x_2,x_3)}g_1, \vec{V}\bullet grad_{(x_1,x_2,x_3)}g_2, \vec{V}\bullet grad_{(x_1,x_2,x_3)}g_3).$$

The formula above defines a vector field on $\mathbb{R}^{n+1}$, so it must be further modified to define a tangent vector field on S, which we know must be perpendicular at the normal to S at every point. Therefore, we subtract the normal component of $D_{V(x_1,x_2,x_3)}W(x_1,x_2,x_3)$ to achieve the latter and also give it a new notation to signify the composition of the two operations:

$$\nabla_{V(x_1,x_2,x_3)}W(x_1,x_2,x_3) = D_{V(x_1,x_2,x_3)}W - (N(x_1,x_2,x_3)\bullet D_{V(x_1,x_2,x_3)}W)N(x_1,x_2,x_3).$$

The operation $\nabla_{V(x_1,x_2,x_3)}(\ldots)$ maps the set of smooth tangent vector fields on the surface to itself, therefore it defines an intrinsic geometric operation for S which is called covariant *differentiation*, and $\nabla_{V(x_1,x_2,x_3)}W(x_1,x_2,x_3)$ is also known as the ***covariant derivative*** of $W(x_1,x_2,x_3)$.

Problem 1. (1a) Compute the general formula for covariant differentiation for vector fields on the surface defined as the graph of a function $h:\mathbb{R}^2 \to \mathbb{R}$ or merely defined on an open subset of $\mathbb{R}^2$. Use the parameters (u,v) for the domain in $\mathbb{R}^2$ and the coordinate functions (x_1,x_2,x_3). **(1b**) Answer the preceding problem for the case of surfaces that are defined by a regular parameterization $X:\Omega \to \mathbb{R}^3$ using the same notation conventions. **(1c**) Generalize the preceding problem for $X:\Omega \to \mathbb{R}^n$ when $n \geq 3$. **(1d**) Generalize the preceding problem for level surfaces of smooth functions $\mathbb{R}^{n+1} \to \mathbb{R}$ when $n \geq 2$.

Problem 2. (2a) Compute the formula for covariant differentiation for vector fields on the ellipsoid in the 3-dimensional space defined as the graph of a function $h:\Omega \to \mathbb{R}$ for an appropriately selected domain of definition of the function. Keep the notation conventions from Problem 1. **(2b**) Compute the formula for covariant differentiation for vector fields on the ellipsoid using the spherical coordinates. **(2c**) Repeat (2a)(2b) above for the special case when the ellipsoid has two equal minor axis, and simplify the answer in spherical coordinates. **(2d**) Do (2c) specializing further to the case when the surface is a sphere and use spherical coordinates.

Problem 3. (3a) Compute the formula for covariant differentiation for vector fields on a cylindrical surface in the 3-dimensional space defined as the surface generated by tracing a circle $C \subset \mathbb{R}^2$ of radius R and the line parallel to the 3rd axis. Keep the notation conventions from Problem 1. **(3b**) Compute the formula for covariant differentiation for vector fields on the cylindrical surface that is generated by tracing a simple closed plane curve C using the appropriate coordinates.

Problem 4. Suppose a tangent vector field V is defined along a piecewise smooth curve $C \subset S$ given by the unit speed parameterization $\alpha(t):(-\varepsilon,\varepsilon) \to S$. Suppose further that V has zero covariant derivative with respect to the vector field $\dot{\alpha}(t) = \dfrac{d\alpha}{dt}$. Such a vector field is called *parallel along* C , or equivalently, that $V(\alpha(t))$ is obtained from parallel transport of the vector $\xi \in T_{\alpha(0)}S$ along $C \subset S$. **(4a**) Suppose a cylindrical surface is given as in the previous problem whose base curve is a circle of radius R, also a tangent vector $\xi \in T_{\alpha(0)}S$. Construct a vector field that is the parallel transport of ξ along $\alpha(t):(-\varepsilon,\varepsilon) \to S$ for various cases of distinguished curves on S . Keep the notation conventions from Problem 1.

Problem 5. Suppose two smooth surfaces S_1, S_2 in $\mathbb{R}^3$ are tangent along the curve $C \subset S_1 \cap S_2$ that is parameterized as $\alpha(t):(-\varepsilon,\varepsilon) \to S$ with $\alpha(0) = P \in S_j,\ j = 1,2$. Choose two arbitrary vectors $\xi_1 \in T_P S_1$ and $\xi_2 \in T_P S_2$. Construct the tangent vector field V that is the parallel transport of $\xi_1 \in T_P S_1$ along C and the tangent vector field W that is the parallel transport of $\xi_2 \in T_P S_2$ also along C . Compute the function $\langle V(\alpha(t)), W(\alpha(t)) \rangle_{\mathbb{R}^3}$ and explain your geometric reasoning behind the calculation.