MATH 801
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Geometry of Curves on Surfaces, Foliations and Bundles

The differential geometry of curves that lie on a surface plays a key role in understanding the geometry of a surface. In return, the geometry of a surface exerts constraints on the geometric properties of curves, such as their curvature, torsion and length. This situation could be the basis for a powerful set of techniques to study and explore geometry and dynamics in Riemannian geometry, differential topology and applied areas such as optimal control, mechanics and so on. In this notes, I will provide the outline of steps that explain the methods in differential geometry that use the concepts of foliation and vector bundles. The background to do these exercises are covered in what we have done so far, and many more powerful concepts in Riemannian geometry could be covered using the theories of vector bundles and foliations in the remaining weeks this semester. The basic ideas are explained intuitively, and some concrete exercises are designed to help you understand the theoretical concepts rigorously and computationally using differential geometry of curves and surfaces. The purpose of these exercises is to introduce you to basic geometric concepts that arise in this context in the limited circumstances of study of n-dimensional hypersurfaces in the (n+1)-dimensional Euclidean space.

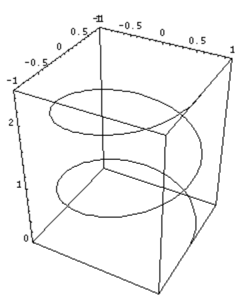
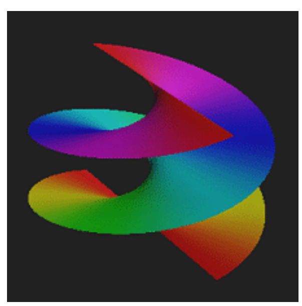
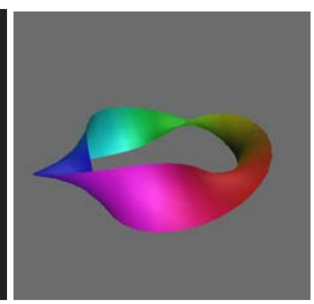
Instructions. Each problem is preceded by some discussion to help clarify the context and sometimes, motivation for what is asked to be done. A number of exercises ask you to prove certain statements. Often, the proof would follow from direct calculations. When in doubt, use a specific example of a surface and carry out all steps for that one, then see if you could generalize the successful solution. In each case that the formulas for the surfaces and curves could be explicitly constructed, do use the concrete representation and carry out the calculations. Your emphasis must be on trying to construct your examples explicitly or numerically, and compute with them. I would suggest that you write MATLAB or Maple code to do explicit calculations in cases that are challenging to do by hand.

Required Background. You need to know: (1) the definition and properties of parametric representation of surfaces and n-dimensional hypersurfaces in the Euclidean space; (2) how to define the tangent space and the normal space to such a surface at a given point; (3) working with vector fields on surfaces; (4) in a small number of circumstances, you must know the inverse function theorem and the implicit function theorem; (5) in a couple of cases, knowledge of the fundamental theorem of ODE theory is very useful.

Cylindrical Surfaces and Foliations of $\mathbb{R}^3$

Example 1. A cylindrical surface **S** in $\mathbb{R}^3$ is typically constructed by taking a *plane curve* **C** in $\mathbb{R}^3$, say $C(s)=(x_1(s),x_2(s),x_3(s))$ parameterized by arc-length (i.e. unit speed) and a direction vector $v=(a,b,c)$ that determines the family of all lines $L(s)\subset\mathbb{R}^3$ that are parameterized by the same parameter s, typically as $L(s)(t)=C(s)+tv$. The curve **C** is called the base curve, and the line L is called a generator for the cylinder.

Remark. If **C** is not a plane curve, or the vector $v=(a(s),b(s),c(s))$ would also vary from point-to-point with the base curve, then **S** is usually not called a cylinder. In such more general circumstances, we have a curve that we call a cross-section curve, and a family of straight lines that are parameterized in any way, passing through points on the cross-section curve. In Figure 1, the curve with equation $\alpha(u)=(R\cos(u),R\sin(u),a.u),\ -\pi\le u\le -\pi$ defines a space curve (a helix) and the surface generated (to its right) is a helicoids. The third figure is a ruled surface constructed from parameterized the family of lines that twist three times when the (angle) parameter describes the circle as a simple closed curve in space.

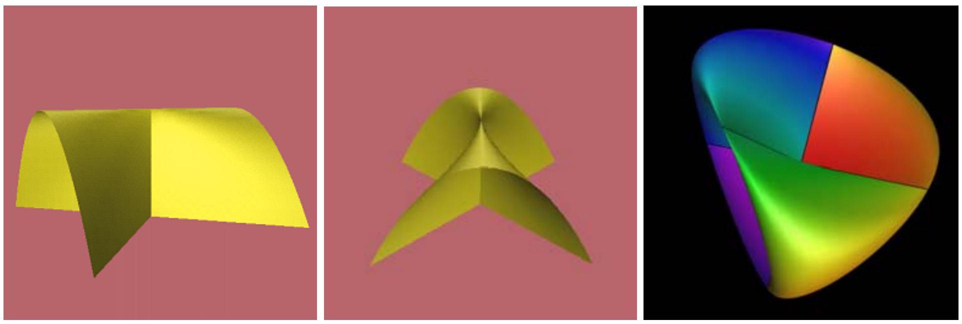
Surfaces such as the helicoids and the triple-twisted band above belong to a more general family of surfaces, called *ruled surfaces* that include all cylinders. The description of a surface as a ruled surface by specifying the cross-section curve and the family of lines is called a *ruling* for the surface. Ruling is an example of a geometric structure on a surface as we see below, and existence of such a structure has major implications for its geometric invariants and dynamics on the surface.

To describe a cylindrical surface, we choose a new parameter t in the direction of the lines $L(s)$ can be now combined with previous parameter s, and we have the following formula to give a parametric representation of **S**:

$$
\begin{aligned}
S(s,t) &= (x_1(s), x_2(s), x_3(s)) + t(a,b,c) \\
&= (x_1(s) + at, x_2(s) + bt, x_3(s) + ct)
\end{aligned}
\tag{1.1}
$$

Exercise 1. (a) Write the parametric equation of a cylinder S whose base curve is the circle of radius R in the (y, z)-plane and center at the point $(0, \lambda R, 0)$ with a fixed $0 < \lambda < 1$, and the direction vector is the unit vector along the x-axis. (b) Generalize part (a) by choosing a curve C with a quadratic equation--that is, parabola, ellipse (including circle), hyperbola--which lies in the (x, y)-plane, and define the corresponding surfaces parametrically. (c) Generalize (a) further by choosing C to be a cubic curve that is given explicitly by a cubic polynomial, and choose the simplest representation of the cubic by dividing into cases that enumerate the roots of the cubic. (d) Choose an implicit equation $f(x, y) = 0$ whose zero set is a cubic curve C, for example $y^2 - x(x^2 - 1) = 0$, and describe parametrically the cylindrical surface generated by these. For this example, use the **Implicit Function Theorem** (and/or the Inverse Function Theorem) to show that the surface is *regular* (i.e. the tangent space is well-defined) at those points of the curve that the tangent-line to the curve is defined and the velocity vector is not zero (i.e. a *regular parameterization* for the curve). (e) There is a great deal in common between a cylindrical surface and the more general members of the family of ruled surfaces. In this part of the exercise, consider the helix mentioned above as the base curve for a ruled surface S such as the helicoids. Define the *generalized curvilinear coordinates* on the surface of S, and establish a *diffeomorphism* $h : (-\pi, \pi) \times \mathbb{R} \to S$ that preserves distances (i.e. an isometry), so that you can essentially relate the geometry in the Euclidean strip $(-\pi, \pi) \times \mathbb{R}$ to the surface helicoid. Is it possible to define helicoid coordinates for $\mathbb{R}^3$ in analogy with cylindrical coordinates? (f) (optional) Now explore generalization of part (d) by studying the family of cubic curves $y^2 - x(x^2 - \lambda) = 0$ in which each value of λ provides a cylinder, possibly with a singularity. Determine the parts of the surface where the tangent plane is defined by specifying the much smaller number of (singular) points in which the cylinder is no longer a regular surface. Your answer that specifies the set of singular points of the cylinder generally depends on λ. This gives a subset of $\mathbb{R}^3$ that has a structure given by parameterization by λ. Describe this set and its parameterization by λ.

Remark. Study of singularities of mappings, such those that occur in level-surfaces has many applications to science, biology and engineering. The subject called catastrophe theory is the study of singularities of maps in rather special circumstances. Consider for example the beautiful geometric object known as the Whitney umbrella which has singularities as a surface. The following is direct quote from the Geometry Center (http://www.geom.uiuc.edu/zoo/features/whitney/) web pages. “Whitney's Umbrella may be pictured as a self-intersecting rectangle in 3 dimensions. It demonstrates a *pinch point,* which occurs at the top endpoint of the segment of self-intersection. Every neighborhood of the pinch point intersects itself. Pinch points are also called *Whitney singularities* or *branch points*. In Alfred Gray’s (our text) you find many interesting formulas such as the those in the figure below for the Whitney umbrella: $f(u,\ v)\ =\ (\ u.v,\ u,\ v^2\ )$ for $\{-1 \le u \le 1,\ -1 \le v \le 1\}$ and the veronese surface (the first to the right.)



Example 2. Let us list some basic properties of a cylinder that are immediately apparent from its construction, and ask ourselves if we can generalize the construction of the cylinder by generalizing these properties.

First, every point $P \in S$ could be described more naturally in the cylindrical coordinates! That is, $P \in S$ could be given coordinates in terms of parameterization of C and a single fixed line L , that we often choose to be the line passing through the origin and parallel to the generators of the cylinder: $L = t(v/\|v\|)$. In this way, the coordinates of $P \in S$ are written as $P = (\vec{p}, \vec{q})$, where $\vec{p} \in C,\ \vec{q} \in L$ could be defined in any manner that is convenient and appropriate for the problem. Note that the cylindrical coordinates in this way generalize the standard cylindrical coordinates that are typically mentioned in vector calculus. The advantages of the cylindrical coordinates go beyond what you have encountered in calculus. For example, they make it clear that the structure of a cylinder is simply that of a Cartesian product $C \times L$ of two lower-dimensional geometric structures. Moreover, the mapping $F : S \to C \times L$ defined by the inverse of the assignment of $P = (\vec{p}, \vec{q})$ to the previously defined $P = S(s(P), t(P)) = (x_1(s) + at, x_2(s) + bt, x_3(s) + ct)$ is a *diffeomorphism*; that is, F is smooth and has a smooth inverse F^{-1} .

Second, consider a family of simple closed plane curves C whose members shrink to the origin and expand indefinitely to pass through any point in the plane containing C. Then we can define the *cylindrical coordinates* for the entire 3-dimensional Euclidean space $\mathbb{R}^3$. This means that we can fill all of $\mathbb{R}^3$ *except the line* $\boldsymbol{L}$ with a family of cylinders that are disjoint, and this *family varies smoothly*. Even the mechanism that shows how the cylinders approach the line L is better understood by projecting the cylinders to the plane containing the curves, and observing how the base curves shrink towards the origin. This is an example of a structure on the Euclidean space $\mathbb{R}^3$ called a *foliation*, where the cylinders are called the *leaves of the foliation*, with distinguishing the line $\boldsymbol{L}$ as a *singular* (or degenerate) *leaf*, and any other one called a *regular leaf.* Any time a 3-dimensional geometric structure can be described by a *foliation*, it is said to have a foliated structure, such as the foliated structure of the Euclidean space $\mathbb{R}^3$. Foliation by surfaces generalizes flows of curves in a dynamical system in the following sense. Consider the family of all lines positioned on all the cylinders in the cylindrical coordinates parallel to L, then we have a simple flow whose ordinary differential equation (ODE) describes the velocity vectors at each point of the particles moving on the lines. When a space is filled with flow lines (including singularities, or flow lines that degenerate to a point), then we say it is foliated by curves, and the flow lines are called leaves of the foliation (including singular leaves that are points, that is, of lower dimension.) The equations of the ODE system are not unique: they depend on how the lines are parameterized. Nonetheless, the flow lines fill the Euclidean space by straight lines. This means that the structure of the space can be studied by "*following*" the flow lines, and we could use any appropriate parameterization that is suitable for the problem as long as the solution of the ODE for any given initial point is a line parallel to L and passing through that line.

Remark. It is natural to ask if there is a mathematical structure to describe the structure of the space *foliated by curves that are specified by their tangent lines*. Such a structure is more flexible than a vector field (hence its description will be less constrained to choices of individual tangent vectors that are prescribed by the ODE system.) A smooth vector field on S is simply a smooth mapping $X : S \to \mathbb{R}^{n+1}$. When each $X(p)$ is tangent to S at every point, then X is called a *tangent vector field*. When each vector $X(p)$ is orthogonal to the tangent space of S at every point $p \in S$, then it is called a normal vector field. The mathematical structure that is suitable to generalize a vector field to parameterized families of lines is called a *line field* (in analogy to a vector field), or a *one-dimensional distribution*. Thus we have the corresponding of tangent line fields and normal line fields. The differential geometric terminology *distribution* is NOT at all related to the similar (linguistically the same) terminologies in probability theory or functional analysis! We shall define later the notion of a k-dimensional distribution and how it relates to a foliation. It suffices for our purposes to define smoothness for a line field on an n-dimensional hypersurface $S \subset \mathbb{R}^{n+1}$ is a description of a family of lines such that the domain of parameterization is S itself, and for any smooth curve $\alpha(t)$ that lies on *S*, the family restricted to $\alpha(t)$ is a family of lines that varies smoothly with the curve parameter *t*. In other words, the line field describes a *smooth ruled surface* in $\mathbb{R}^{n+1}$.

Exercise 2. This exercise aims to makes the remarks above more concrete. (a) Prove that if **X** is a smooth vector field defined on a hypersurface $S \subset \mathbb{R}^{n+1}$, then there is smooth tangent line field **L** on S such that the line at $P \in S$ contains the vector **X**(P). Furthermore, **L** is non-singular (i.e. no line degenerates to a point) if and only if the vector field **X** has no zeros. (b) Consider an n-dimensional hypersurface $S \subset \mathbb{R}^{n+1}, n \geq 3$ and two non-singular smooth tangent line fields **X** and **Y** on S. Construct a smooth 2-dimensional tangent plane field Π on **S** (also called a 2-dimensional distribution) that contains both line fields **X** and **Y.** (c) Prove that when $n \geq 3$, construct a smooth tangent line field **Z** on S that is perpendicular to both line fields **X** and **Y.** When n=3, there is a unique line field **Z** that is perpendicular to Π, and describe a method to construct it. (d) Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be a smooth function such that at the level surface $S = f^{-1}(c)$ for a regular value *c*. Describe a smooth non-singular line field **L** on S such that **L(P)** is normal to S at every point.

Vector Bundles on Smooth Curves

Definition of the tangent space and normal line fields to a hypersurface makes it clear that somehow linear algebra done within a tangent space (or a normal line) at a point $P \in S$ could be done as usual without encountering any difficulties as long as we acknowledge the constraint that the vector operations avoid mixing vectors that are tangent or normal at different points. What we seek here is a language along the lines of differential calculus on hypersurfaces to define all the linear algebra operations on tangent spaces, normal line fields, or more generally on any k-dimensional distribution. One way to develop this language is to appeal to construction of new differentiable manifolds from hypersurfaces that allow this flexibility, and the operations of vector addition and multiplication by scalars become ordinary smooth mapping. The abstract theory for such constructions is called the theory of vector bundles (indeed more generally, called the theory of fiber bundles to accommodate nonlinear forms of tangent spaces etc.) In what follows, we construct such manifolds and operations in elementary ways, and attempt to relate their intuitive properties to elementary differential calculus in the Euclidean space. One approach to this issue is to regard the natural vector space operations on the tangent vector spaces and/or smooth vector fields as operations defined in *generalized cylindrical coordinates* defined by the hypersurface and its tangent space or normal line. First, we show that certain families of lines or planes that are parameterized by a curve could be positioned in a Euclidean space so that they could be accommodated without intersecting each other in space. Such a positioning in a higher-dimensional space is called an embedding, and if the geometry of the surface does not change (that is, distances are all preserved under such an embedding), then the embedding is called isometric. The important fact that we must remember is the following: If an (n-1)-dimensional hypersurface S surface in $\mathbb{R}^n$ is obtained as a level surface of a smooth function $f:\mathbb{R}^n \to \mathbb{R}$ for a regular value $c \in \mathbb{R}$, then embedded surface in $\mathbb{R}^{n+m}$ under the smooth mapping $F:\mathbb{R}^n \to \mathbb{R}^{n+m}$ could be locally parameterized, using the Implicit Function Theorem. Also, the tangent space T_P of the hypersurface S in $\mathbb{R}^n$ is mapped in a 1-1 manner to a subspace of $\mathbb{R}^{n+m}$ by the differential $dF_P:\mathbb{R}^n \to \mathbb{R}^{n+m}$ when we translate T_P to the origin and identify it with the (n-1)-dimensional subspace of $\mathbb{R}^n$ that is parallel to it. Depending how complex the smooth mapping F could be, the image of S could twist and turn in the larger Euclidean space[1]. Therefore, there are many ways that a hypersurface could be positioned in a higher-dimensional space while preserving its geometry (i.e. by isometric embedding, preserving distances). As a result, there arises the natural question if we could study the intrinsic geometry of a hypersurface (or any smooth manifold) without necessarily complicating the matters by positioning them in a Euclidean space. The answer to this question is positive, and intrinsic Riemannian geometry will provide the means to do that.

Exercise 3. Define a cylindrical n-dimensional hypersurface in $\mathbb{R}^{n+1}$ as a surface S that could be parameterized by a family of lines $L(\vec{p})$, where $\vec{p}$ is an arbitrary point on a regular (n-1)-dimensional submanifold B of $\mathbb{R}^{n+1}$. In this case, we refer to B as the base and the family lines as the generators. When $n=2$, we are back in the circumstances of Example 1 and Exercise 1. (a) Generalize the concept of *cylindrical coordinates for* $\mathbb{R}^{n+1}$ to $n \geq 3$ by describing the conditions on the family of base sub manifolds, and give examples in dimensions n>3 to illustrate your generalized cylindrical coordinates. (b) Let n=2 as in Exercise 1 (b), select a base curve C, a vector v of length one and the corresponding cylindrical surface S that has a regular parameterization. Write down the equation of the tangent surface $\mathbf{T_P}$ to S at an arbitrary point $P \in S$. Show that the family of planes given by the tangent spaces at all points of S have the property that they all contain a line parallel to L and a line

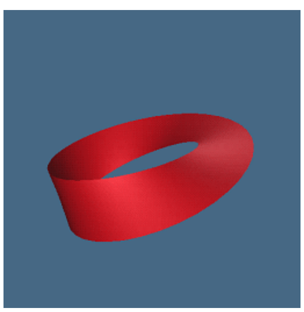
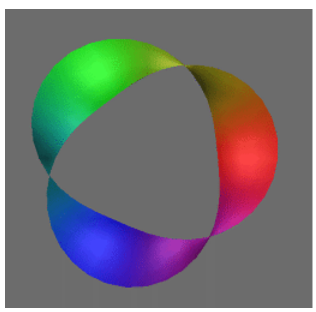
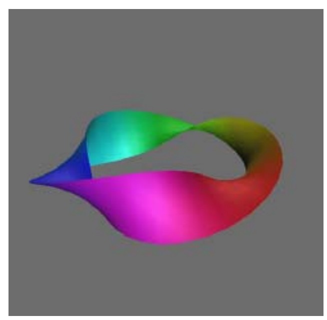
[1] A simple example of this phenomenon is

perpendicular to L. (c) Ordinarily, given two points $P,Q \in S$, the tangent planes T_P and T_Q intersect in the Euclidean space $\mathbb{R}^3$ in a line. Prove this fact rigorously and compute the intersection line in terms of the surface parameterization. Use the generalized cylindrical coordinates or any other method to describe this family of lines as a smooth line field.

Exercise 4. Use the notation and assumptions of the preceding exercise. (a) Write down the equations of a smooth mapping $F: S \to \mathbb{R}^{3+m}$ with the following properties: (i) the image of *F* is diffeomorphism to S, so S can be parameterized by a composition of its original parameterization in $\mathbb{R}^3$ followed by the mapping *F*; (ii) the tangent surfaces to S in $\mathbb{R}^{3+m}$ do NOT intersect at all. The mapping F is called an *embedding* of S in $\mathbb{R}^{3+m}$. Among all embeddings, we are usually interested in having the smallest integer *m*, so you should also find the mapping such that (iii) the integer *m* is the smallest such integer with respect to all mappings that satisfy conditions (i) and (ii).
(b) Choose *F* such that in addition, the differential *dF* is an *isometry* of tangent spaces at each point. Such a mapping is called an *isometric embedding* of S in $\mathbb{R}^{3+m}$. In finding isometric embeddings, the smallest *m* in (b) is usually preferred. Suggestion - You could use the intuition from part (c) in Exercise (3) to determine the conditions that the mapping F must fulfill, and then come up with a general method for finding such mappings. This suggestion is particularly useful when trying to find embeddings that satisfy more than just being an isometry. (c) Consider the differential of the mapping in (b) above $dF_P: T_P \to \mathbb{R}^{3+m}$ and observe that for each $P \in S$, $dF_P(T_P)$ is a plane tangent to S that contains two distinct lines: The first line describes the tangents to the cylinder's generating lines passing through P, and the second line is tangent to the curves that are parallel to the base curve C. Call such a line K_P. Write the equation of a curve $\alpha(u)$ that describe the motion of a point $P \in S$ as a surface isometrically embedded in $\mathbb{R}^{3+m}$ as in (b) above in such a way that $\alpha(u)$ always intersects the generating line of the cylindrical description at an angle other than $0, \pi$. Use $\alpha(u)$ to describe the parametric embedding of a new 2-dimensional surface $M \subset \mathbb{R}^{3+m}$ that is generated by the lines $dF_P(K_P)$ using $dF_P: T_P \to \mathbb{R}^{3+m}$. If we use any curve parallel to the base curve C for $\alpha(u)$, then the surface M is a differentiable manifold that is called the *tangent bundle* of $\alpha(u)$. The cylindrical structure of the surface M provides us with two types of information. First, we can do vector space algebra in the direction of tangent spaces to $\alpha(u)$ pointwise, and such operations are readily described as smooth mappings on various manifolds constructed from the

tangent bundle. For example, multiplication of a tangent vector by a scalar that varies from point to point $c(\alpha(u))$ (using the parametric description of $\alpha(u)$) is given by a function defined by $c(\alpha(u)) = \varphi(u)$. In terms of M, we can define this operation by $\Phi : M \to M$ using the intuitive formula $\Phi((\alpha(u), w)) = \varphi(u).w$ in the cylindrical coordinates provided by the surface itself, as explained above. Addition of two tangent vectors $(\alpha(u), w_1)$ and $(\alpha(u), w_2)$ in the cylindrical coordinates has the very appealing intuitive formula $(\alpha(u), w_1) + (\alpha(u), w_2) = (\alpha(u), w_1 + w_2)$. In terms of M itself, we proceed as follows: The subset of $M \times M$ called by V(M) defined via $V(M) = \{(q, w_1, w_2) : q = \alpha(u),\ w_1, w_2 \in T_{\alpha(u)}\}$ is a smooth 3-dimensional surface in $\mathbb{R}^{3+m}$ and we can define a smooth mapping $\psi : V(M) \to M$ that precisely is described in the above-mentioned cylindrical coordinates as vector addition by $\psi(q, w_1, w_2) = (q, w_1 + w_2)$.

Exercise 5. You might have heard of the Möbius band as the one-sided surface, or perhaps you have seen the drawings of Escher making a concrete demonstration of ants walking on the Möbius band. There are hundreds of thousands of web-pages that have something to do with this beautiful, yet simple construction in topology, for example, the Geometry Center (http://www.geom.uiuc.edu/) or http://www.geom.uiuc.edu/zoo/features/mobius/ provides very nice visualization, e.g.:

This exercise makes it possible to write down explicit parameterization of the Möbius band, and explore the structure of certain one-dimensional vector bundles on simple closed curves that are very closely associated to the Möbius band. (a) Let the unit-speed parameterization $\alpha(u)$ define a simple closed curve. For simplicity, use a plane curve, such as the circle in the plane. Construct a ruled surface M by defining a smooth vector field $X : C \to \mathbb{R}^3$ on a smooth simple closed curve C that is normal to C at any place; for example, use the normal vector field from the Frenet frame for the circle in Exercise 1(a) or the standard circle of radius 1 in the (x , y)-plane. (b) Using the parameterization $\alpha(u) : [a,b] \to C \subset \mathbb{R}^3;\ \alpha(a) = \alpha(b)$, we can define $X \circ \alpha : [a,b] \to \mathbb{R}^3$ that gives a parameterization for the vector field. Note that given $X : C \to \mathbb{R}^3$, there is exactly one other vector normal field $Y : C \to \mathbb{R}^3$ such that the pair $\{X(p), Y(p)\}$ is orthogonal at every point of the curve C. This observation generalizes the construction of the pair of normal and binormal vector fields from the Frenet frame. (b) Construct a smooth normal plane field Π to C by defining $\Pi(p) = span\{X(p), Y(p)\}$. We can define a parameterized family of rotations $\rho : C \to \mathbb{R}^3$ for the normal plane filed to C, such that for each $p \in C$, $\rho(p)$ is a rotation of the plane field $\Pi(p)$. If we use the generalized cylindrical coordinates for Π, then at each point $p \in C$, the pair $\{X(p), Y(p)\}$ provides an orthonormal basis for $\Pi(p)$. In this basis, we can define the matrix-valued smooth mapping $\rho : [a,b] \to \mathbb{R}^{2\times 2}$ explicitly by $\rho(u) = \begin{pmatrix} \cos u & -\sin u \\ \sin u & \cos u \end{pmatrix}$. Correspondingly, we define a smooth transformation $G : [a,b] \to \Pi$ of the plane field Π by:

$$G(u)[a_1 X(\alpha(u)), a_2 Y(\alpha(u))]^t = [a_1 \cos u X(\alpha(u)) + a_2 \sin u Y(\alpha(u)), -a_1 \sin u X(\alpha(u)) + a_2 \cos u Y(\alpha(u))]^t$$

Use this transformation to construct explicit formulas for parameterization of ruled surfaces that are one-sided, such as the Möbius band or the triple-twist version of the Möbius band. (c) Fine parametric formulas for the tangent plane field and the normal line field of the surfaces that you construct in the preceding part (b).

Discussion. Given a simple closed curve C, the cylindrical or ruled surfaces M could be defined above such that $C \subset M$ and each generating line $L(p)$ intersects the base curve C at $p \in C$. There is a natural smooth mapping $\pi : M \to C$ that sends each vector in the line $L(p)$ to the point $p \in C$. Clearly, $\pi^{-1}(p) \subset M$ is the line $L(p)$, and we can assume that $L(p) \cap C$ is the origin for the vector space $L(p)$. The surface M is called the total space of the vector bundle , the mapping $\pi : M \to C$ is called the projection, and curve C is called the base (or base space), and the subspace $\pi^{-1}(p) \subset M$ is the fiber over the point $p \in C$. These ingredients together define examples of *vector bundles* over C, and in this case they are called *line bundles*, because for each $p \in C$, the dimension of vector space $M(p)$ is one. Any mapping $\sigma : C \to M$ is called *a section of the bundle* provided that $\pi \circ \sigma = identity$ on C. Smoothness again refers to the differentiability of the corresponding mappings. In this context, if $\pi : M \to C$ is the projection of a tangent line bundle, then a section $\sigma : C \to M$ is the same as having a tangent vector field, and conversely, every tangent vector field arises as a section of the bundle $\pi : M \to C$. This shows that we can indeed define all the notions of differential geometry intrinsically and without any reference to a Euclidean space!