Math-801

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Homework

Parallel Transport

In this homework, we apply covariant derivative to construct parallel transport of vectors. The notation and definitions are as in Homework #6 on covariant differentiation of vector fields. Further:

(i) S is a surface given defined either as the graph of a function $h : \mathbb{R}^2 \to \mathbb{R}$ (or merely defined on an open subset of $\mathbb{R}^2$) or by a regular parameterization $X : \Omega \to \mathbb{R}^3$ using the same notation conventions. Of HW #6 . Use the parameters (u, v) for the domain in $\mathbb{R}^2$ and the coordinate functions (x_1, x_2, x_3).

(ii) We have smooth curve $C \subset S$ that is parameterized via $\gamma(t) : (-\varepsilon, \varepsilon) \to S$, its tangent vector field $V(\gamma(t)) \equiv \dot{\gamma}(t) = \dfrac{d\gamma}{dt}$, a point $P \in C$, and the tangent vector $\xi \in T_P S$ that we wish to extend to a parallel vector field $W(\gamma(t))$ along C.

We will keep this notation throughout the discussion below. Recall the basic definitions and properties of parallelism.

Definition. Let V be the tangent vector field of a smooth regular curve $C \subset S$ with unit-speed parameterization. A tangent vector field Y on S is parallel along C if the covariant derivative of Y with respect to the tangent vector field of C is zero, that is $\nabla_V Y = 0$.

Theorem 1. A regular curve $\gamma(t), \gamma : (-a, a) \to S$ is a geodesic if and only if $\nabla_V V = 0$. That is, the tangent vector field of a geodesic is parallel along the geodesic curve itself (also called auto-parallel.)

Theorem 2. Assume that we have a regular curve C that is a geodesic and it is given by the unit-speed parameterization $\gamma(t), \gamma:(-a,a)\to S$. Suppose that the vector field W is Levi-Civita parallel along C. Then the inner product $\varphi(t)=\langle W(\gamma(t)), V(\gamma(t))\rangle$ (as a function of t) is a constant along C. That is, the length of W and the angle that it makes with $\gamma(t)$ are constant throughout the curve.

Corollary 3. If we have a geodesic $\gamma(t), \gamma:(-a,a)\to S$ with unit speed parameterization and passing through a point P, and a vector $\xi\in T_P S$, then we can construct the vector field W that is Levi-Civita parallel along C by choosing the appropriate vector in $T_{\gamma(t)}S$ that has length equal to the length of ξ, that is, $\|W(\gamma(t))\|=\|\xi\|$ and $\cos^{-1}(\langle V(\gamma(0)),\xi\rangle/\|\xi\|)=\cos^{-1}(\langle V(\gamma(t)),W(\gamma(t))\rangle/\|W(\gamma(t))\|)$ are constant along C.

Explicit Construction of Parallel Vector Fields

Theorem 2 and Corollary 3 describe a direct method that can be turned into an algorithm for construction of parallel transport along a **geodesic**. When the curve in question is not a geodesic, we must actually solve the ODE system that we shall describe below. The problem of constructing parallel vector fields requires the steps below. We are given a vector $\xi\in T_P S$ that we wish to extend to a vector field W such that W is Levi-Civita parallel along C. If such a vector field would exist, then it should satisfy the following. The covariant derivative of W vector field with respect to the tangent vector field V vanishes, $\nabla_V W=0$ and $W(\gamma(0))=\xi$. We proceed by making the latter conditions explicit. The directional derivative $D_V W=V(W(\gamma(t)))$ is decomposed at every point $\gamma(t)$ into the normal component $\langle V(W(\gamma(t))), N(\gamma(t))\rangle N(\gamma(t))$ and its tangential component $\nabla_V W=V(W(\gamma(t)))-\langle V(W(\gamma(t))), N(\gamma(t))\rangle N(\gamma(t))$. For W to be Levi-Civita parallel along C, it is necessary and sufficient that the tangential component of $\nabla_V W$ be zero, so $V(W(\gamma(t)))-\langle V(W(\gamma(t))), N(\gamma(t))\rangle N(\gamma(t))=0$, which is a differential equation with the initial value condition $W(\gamma(0))=\xi$. As in the case of construction of geodesics, we proceed to compute the expression in terms of local coordinates for the Euclidean space.

Problem 1. Carry out the following calculation symbolically for a cylindrical surface. Use a domain Ω that is a rectangle in $\mathbb{R}^2$, such as $\Omega = [0, 2\pi) \times (-z, z)$, where z is some value that will be determined each time by data. Introduce the cylinder $S \subset \mathbb{R}^3$ with base curve $C \subset \mathbb{R}^2, C(s) = (x_1(s), x_2(s), 0)$ such that $\| C'(s) \| = 1$) and a direction vector $v = (c_1, c_2, c_3)$ of unit length ($\| v \| = 1$). The equation of the surface is $L(s,t) = C(s) + tv$ **.** Give the equation for the curve on the surface $\gamma(\tau), \gamma : (-a, a) \to S$ described by a pair of functions $(s(\tau), t(\tau))$ in the parameter domain of the surface as follows.

$$\begin{aligned}\gamma(\tau) &= L(s(\tau), t(\tau)) \\ &= (x_1(s(\tau)) + c_1 t(\tau), x_2(s(\tau)) + c_2 t(\tau), c_3 t(\tau)).\end{aligned}$$

STEP 0 (SPECIAL CASE FOR WARM-UP). Describe the special geodesics of the surface S that are parallel to the generating curve and parallel to the base of the cylinder. Use Theorem 2 and Corollary 3 describe by a direct construction of parallel transport along these geodesics. Take a mesh for the surface whose coordinate curves are the geodesics and describe an array that contains the coordinates of the vector field transported along the mesh, once using the geodesics that are parallel to the base, then using the geodesics that are parallel to the generating axes. Graph the resulting vector field using MATLAB.

STEP I. In terms of coordinates in $\mathbb{R}^3$, compute the following: the tangent space at a point $P = L(s_0, t_0) = C(s_0) + t_0 v$ generated by the unit vectors $L_s = C'(s), L_t = v$, and the normal vector $N(s,t) = C'(s) \times v$ has also unit length. The ordered triple $\{C'(s), v, N(s,t)\}$ is the oriented basis that we select locally for $T_{\gamma(\tau)}\mathbb{R}^3$. Describe the given vector $\xi \in T_P S$ in terms of the basis for $T_P S$ by the formula below:

$$\xi = (ax'_1(s_0) + bc_1, ax'_2(s_0) + bc_2, bc_3).$$

The equation of the normal for a general cylindrical surface is calculated by the formula below:

$$N(s,t) == (c_3 x'_2(s_0), -c_3 x'_1(s_0), c_2 x'_1(s_0) - c_1 x'_2(s_0)).$$

$$N(\gamma(\tau)) = (c_3 \frac{dx_2(s(\tau))}{ds}, -c_3 \frac{dx_1(s(\tau))}{ds}, c_2 \frac{dx_1(s(\tau))}{ds} - c_1 \frac{dx_2(s(\tau))}{ds})$$

STEP II. Compute $D_{\dot{\gamma}(\tau)} W = \operatorname{span}\{N(\gamma(\tau))\}$ as follows (or use the HW MATLAB function for covariant derivative):

$$V(\gamma(\tau)) = \frac{d\gamma}{d\tau} = (\frac{dx_1}{ds}\frac{ds}{d\tau} + c_1 \frac{dt}{d\tau}, \frac{dx_2}{ds}\frac{ds}{d\tau} + c_2 \frac{dt}{d\tau}, c_3 \frac{dt}{d\tau});$$

$$\begin{aligned} D_V W(\gamma(\tau)) &= (D_V W_1, D_V W_2, D_V W_3) \\ &= (\langle V, gradW_1 \rangle, \langle V, gradW_2 \rangle, \langle V, gradW_3 \rangle) \\ &= (\langle V, gradW_1(\gamma(\tau)) \rangle, \langle V, gradW_2(\gamma(\tau)) \rangle, \langle V, gradW_3(\gamma(\tau)) \rangle). \end{aligned}$$

Compute the individual inner products $\langle V, gradW_k(\gamma(\tau)) \rangle$:

$$gradW_k(x_1, x_2, x_3) = (\frac{\partial W_k}{\partial x_1}, \frac{\partial W_k}{\partial x_2}, \frac{\partial W_k}{\partial x_3}),$$

$$\begin{aligned} gradW_k(\gamma(\tau)) \bullet \frac{d\gamma(\tau)}{d\tau} &= \left\langle (\frac{\partial W_k}{\partial x_1}, \frac{\partial W_k}{\partial x_2}, \frac{\partial W_k}{\partial x_3}), \frac{d\gamma(\tau)}{d\tau} \right\rangle \\ &= \sum_{j=1}^{3} \frac{\partial W_k}{\partial x_j}\frac{\partial x_j}{\partial s}\frac{ds}{d\tau} + \frac{\partial W_k}{\partial x_j}\frac{\partial x_j}{\partial t}\frac{dt}{d\tau}. \end{aligned}$$

STEP III. Compute the ODE system that describes the parallel transport as follows. The resulting directional derivative as a vector field on the curve $\gamma(t) \subset \mathbb{R}^3$ and project the vector field from the 3-dimensional space to the tangent planes of the surface. This requires substitution of the quantities that we just calculated in $D_V W(\gamma(t)) - \langle D_V W(\gamma(t)), N(\gamma(t)) \rangle N(\gamma(t)) = 0$.

The r-th term of $\langle D_V W(\gamma(t)), N(\gamma(t)) \rangle N(\gamma(t))$ is:

$$\left(\sum_{k=}^{3} (\sum_{j=1}^{3} \frac{\partial W_k}{\partial x_j}\frac{\partial x_j}{\partial s}\frac{ds}{d\tau} + \frac{\partial W_k}{\partial x_j}\frac{\partial x_j}{\partial t}\frac{dt}{d\tau}).N_k(\gamma(\tau) \right) N_r(\gamma(\tau)$$

So the system has the following r-th equation:

$$\sum_{j=1}^{3}\frac{\partial W_r}{\partial x_j}\frac{\partial x_j}{\partial s}\frac{ds}{d\tau}+\frac{\partial W_r}{\partial x_j}\frac{\partial x_j}{\partial t}\frac{dt}{d\tau}-N_r(\gamma(\tau).\sum_{i=1}^{3}(\sum_{j=1}^{3}\frac{\partial W_i}{\partial x_j}\frac{\partial x_j}{\partial s}\frac{ds}{d\tau}+\frac{\partial W_i}{\partial x_j}\frac{\partial x_j}{\partial t}\frac{dt}{d\tau}).N_i(\gamma(\tau)=0.$$

The initial values for the resulting ODE system are given in terms of the tangent vector to the surface $\xi \in T_P S$ and coordinates of the curve γ at P.

STEP IV. Compute the simplified system for a cylinder whose base curve is a circle of radius R and its generator is parallel to the third coordinate.

STEP V. Compute the simplified system for a cylinder whose base curve is an ellipse with aspect ratios $\sigma_1 \geq \sigma_2$ and its generator is parallel to the third coordinate.

Use numerical values to solve the ODE system for parallel transport on this cylindrical surface. Assign variables to all numbers so they could be selected by the user of the program.

Problem 2. Repeat the steps of Problem 1 for S that is a surface given defined as the graph of a function $h:\Omega \to \mathbb{R},\ \Omega \subset \mathbb{R}^2$ defined on an open subset of $\mathbb{R}^2$.

Problem 3. Repeat the steps of Problem 1 for the case of a surfaces that is an ellipsoid as in a previous homework (using the same notation conventions as in Problem 1.)