Math 801
(A. Assadi)

Lyapunov Functions and Stability

Linear systems can be analyzed because the behavior over the entire state space can be determined. But nonlinear systems are more complicated, and their behavior can be analyzed only in the neighborhood of equilibrium points. Remember that these are the points for which

$F(x) = 0$. Linear systems are simple to characterize because they have a *single equilibrium point*. The behavior of a system is determined entirely by the eigenvalues and directions of the eigenvectors of the matrix describing system. One of the main reasons that nonlinear systems are not easy to analyze is that they may contain multiple equilibrium points. In these notes we outline how they should be analyzed. As remarked earlier in the lectures, the study of nonlinear systems is best done by characterizing the behavior at a local level around each equilibrium point, and a separate method should be used to patch together the local information into a global overall picture. The key issue for local analysis is stability: what will happen to the state vector when it is displaced from an equilibrium point? Broadly speaking, there are three kinds of behavior. Broadly speaking, there are three kinds of behavior:

• ***Asymptotic stability***: A displaced state vector will eventually return to the equilibrium point.

• ***Instability***: A displaced state vector will continue to move away from the equilibrium point.

• ***Marginal stability***: A displaced state vector will oscillate near the equilibrium point.

We proceed to formalize these concepts of stability as follows. Define a local region about the equilibrium point in terms of a ball of radius R centered at the point. The idea is that the analysis will be confined only for points inside this ball because of nonlinear effects or other nearby equilibrium points. An equilibrium point is *unstable* if for some r, $0 < R_1 < R$, there is a ball of smaller radius r that has the following property: There is a point inside $B(x, r)$ such that when started at that point, the state vector will eventually move outside of $B(x, R_1)$. If an equilibrium point is not unstable, it is said to be stable, but that characterization does not mean a trajectory will tend toward the equilibrium point. For this behavior to happen, we require that the stronger condition of asymp*totic stability* be satisfied. According to this condition, *in addition to being stable*, there is some radius *r'* such that when started inside *B(x, r'),* the trajectory will tend toward the equilibrium point. An equilibrium point is called *marginally stable* if it is *stable* but not asymptotically stable. Stability analysis can proceed in two directions. One way is to resurrect the techniques for linear systems. If the location of an equilibrium point can be determined, then the dynamical equations can be linearized about that point and linear analysis applied. Another way is to find a special function of the state space, called a *Lyapunov function* (after the Russian mathematician Aleksandr Mikhailovich Lyapunov, born 6 June 1857 in Yaroslavl, Russia, and died 3 Nov 1918 in Odessa, Russia). *If the trajectory is such that the value of the Lyapunov function is always decreasing, then the dynamical system will be stable.*

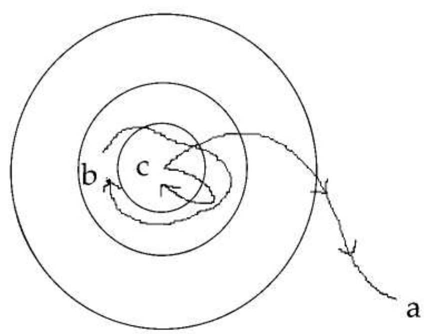

Figure 1. Different kinds of stability for state space trajectories: (a) unstable; *(b)* marginally stable; (c) asymptotically stable.

Let us review the results on equilibrium points for a second-order system. Consider as an example the system dX/dt=AX, and consider the case where A is 2 by 2.. Suppose that the matrix A has two eigenvalues λ_1 and λ_2 that are real and negative and $\lambda_1 \neq \lambda_2$. Call the corresponding eigenvectors v_1 and v_2. Note that they define directions along which a trajectory will move toward the origin, because along these lines $x = v$ and $\dot{v} = Av = \lambda v$.. Away from these lines the trajectories will tend toward the origin. In contrast, if we have a case such that $\lambda_1 \neq \lambda_2$ and both λ_1 and λ_2 are real and positive, then we see that the eigenvectors v_1 and v_2 define directions along which a trajectory will move away from the origin, again because along these lines $x = v$

and $\dot{\boldsymbol{v}} = A\boldsymbol{v} = \lambda\boldsymbol{v}$. Away from these lines the trajectories will move outward but tend toward the eigenvector directions. Now look at the case where λ_1 and λ_2 are complex conjugates, so $\lambda_1 = i\omega$, $\lambda_2 = -i\omega$. In this case the state space trajectory will oscillate around the origin along an ellipse determined by the eigenvector directions.

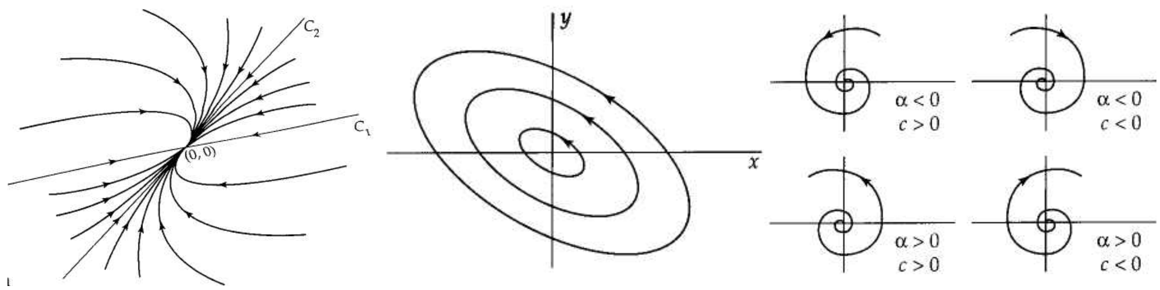

Figure 2. A nonlinear system may be understood by characterizing the behavior of trajectories of the system linearized near equilibrium points, as in this second-order system. This behavior is entirely determined by the eigenvalues and eigenvectors of the linearized system. *(a)* Two negative eigenvalues, *(b)* Two imaginary eigenvalues, (c) Complex eigenvalues whose real part, *a,* is negative or positive, as shown.

The idea of Lyapunov stability is illustrated by the following metaphor. Suppose that you can pick a function *V(x)* whose graph is in the shape of a bowl such that its lowest point is the equilibrium point. If this function is decreasing in value as the trajectory progresses from all starting points near the equilibrium point, then the system is asymptotically stable, because when the trajectory continues to reduce the value of the chosen function, then eventually it must end up at the lowest point, which is the equilibrium point. As a result, the idea of such a function generalizes the intuition that we had to define stability at the beginning of these notes. If the trajectory is always decreasing the value of *V(x),* then the trajectory must always cross the level curves of *V*, moving inward toward the equilibrium point. This intuition is captured by the Lyapunov stability theorem:

The Lyapunov Stability Theorem. If there exists a Lyapunov function $V(x)$ for an equilibrium point x_0 and the region $X = B(x_0, R)$, then the equilibrium point is stable. Furthermore, if $\dot{V}(x) < 0$ for $x \neq x_0$, that is, if it is strictly negative everywhere except the equilibrium point itself, then the stability is asymptotic.

Formally this requirement is equivalent to the following condition: a function *V(x)* is a *Lyapunov function* if over a region Ω of the state space that contains an equilibrium point *x,* (a) V is continuous and has continuous partial derivatives; (b) *V* has a single minimum at the equilibrium point; (c) $\nabla V . F(x) \leq 0$ along all trajectories within the region Ω .

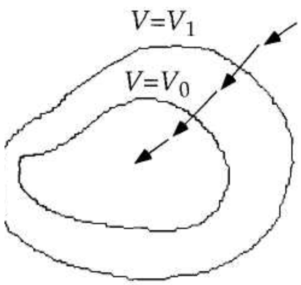

Figure 3. The idea behind Lyapunov stability. The state space trajectory always crosses level contours of *V*.

We can elaborate on this theme as follows: Using the chain rule for differentiation,

$$\frac{dV(X)}{dt} = \sum_j \frac{\partial V}{\partial x_j} \frac{dx_j}{dt}$$

Recall that $\nabla V = \sum_j \frac{\partial V}{\partial x_j}$ and $\frac{dX}{dt} = (\frac{dx_1}{dt}, \frac{dx_2}{dt}, ... \frac{dx_n}{dt})$, so the decrease of V(x) along trajectories is described by the gradient condition $\nabla V . \frac{dX}{dt} \leq 0$, and since the ODE system is $\frac{dX}{dt} = (F_1(x_1, x_2, ...x_n), ..., F_n(x_1, x_2, ...x_n))$, the Lyapunov stability condition could be also written as $\sum_j \frac{\partial V}{\partial x_j} F_j(x_1, x_2, ...x_n) \leq 0$, or more compactly in terms of the dot product above.

Exercise 1. A second-order system has a dynamics described by

$$\frac{dX}{dt} = (F_1(x_1, x_2), F_2(x_1, x_2))$$

in which $F_1(x_1, x_2) = -3x_1 + x_2$ and $F_2(x_1, x_2) = -6x_1 + 12x_2$. Find the points of equilibria and check if the equilibrium stable or unstable.

Exercise 2. In the dynamical system whose ODE system is given by $\frac{dX}{dt} = AX$, assume that A is symmetric and positive definite. Show that the quadratic function $V(X) = -X^T AX$ is a Lyapunov function for the system. How does this question relate to the eigenvalues of A?

Exercise 3. 9. Consider a pendulum of subtended mass *m* and length L that is displaced by an angle θ. The dynamics of this system is governed by the equations: $mL\frac{d^2\theta}{dt^2} + mb\frac{d\theta}{dt} + mg\sin\theta = 0$. Transform this to a first order system by $\frac{d\theta}{dt} = \omega$ and show that $V(\theta,\omega) = \frac{1}{2}mL^2\omega^2 + mgL(1-\cos\theta)$ is a Lyapunov function for this system.

Remark. An excellent short treatment of the qualitative theory of ODE systems are by Witold Hurewicz, (Lectures on Ordinary Differential Equations, MIT Press1966), and by Vladimir Arnold (Qualitative Theory of Ordinary Differential Equations, Springer) which are also the reference for the figures and some of materials above.