Math 415 – Spring 2005
(A. Assadi) **Week of May 1, 2005**

Bifurcation and Stability

Nonlinear systems often have complicated global behavior that can greatly vary depending on the parameters that determine the position and qualities of their *equilibrium points*. In most real-world models, systems of ODE that describe a particular phenomena, for instance spread of disease epidemics or predator-prey models, are related to each other by sharing a rather general parametric form of the equation. Specific instances of these models are then obtained by assigning various values of the parameters. For example, models of epidemics could be given by a general ODE system that depends on factors such as density of population under study. This parameter is vastly different in densely populated urban areas versus sparseness of the populations in most rural areas. Similarly, the position of an equilibrium point may depend upon a parameter. For instance, the numbers infected people when an epidemic reaches equilibrium might be different for different rates of infection. As a result, the modelers would anticipate different qualitative behavior from the same family of parametric equations when the value of the parameter is allowed to vary. Such parameters are sometimes referred to as *control parameters* to distinguish them from any other parameters which remain fixed.

How does the variation of a control parameter affect equilibrium in a dynamical system? The brief notes below provide a basic introduction and some well-known examples of a phenomenon called bifurcation that emerge as a result of varying control parameters in a system.

Example.

Consider the non-linear dynamical system below that is constrained to the interval $\pi < x < \pi$, and for values of the control parameter $\xi \geq 0$.

$$(1.1) \qquad \begin{aligned} \dot{x} &= 2y \\ \dot{y} &= \xi \sin x - x. \end{aligned} ,$$

The equilibrium points are found by solving the simultaneous system:

$$\begin{aligned} 2y &= 0, \\ \xi \sin x - x &= 0. \end{aligned} \tag{1.2}$$

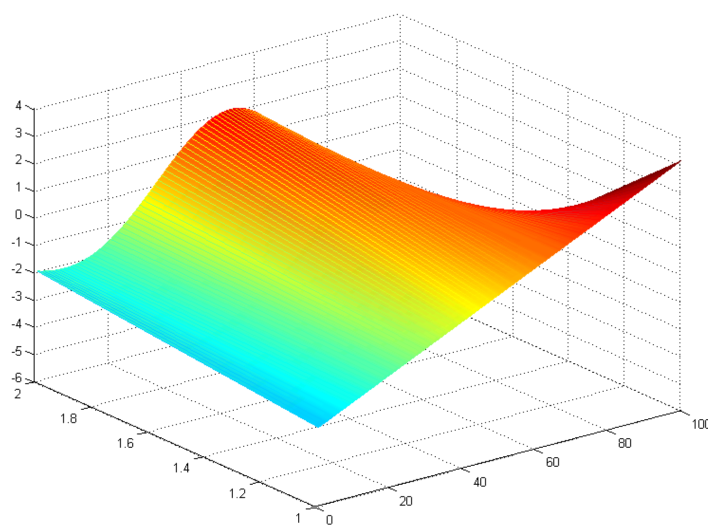
Directly or by inspection, we find {x = 0 , y=0} as one solution. To find other solutions, we further study the behavior of the function $F(x,\xi) = \xi \sin x - x$ with varying values of x and ξ. Notice that the if a value $x = x_1$ satisfies the equation $\xi \sin x - x = 0$, so does $x = -x_1$. Next, the derivative of $\xi \sin x - x$ with respect to x is $\xi \cos x - 1$, which is always negative if $0 \le \xi < 1$. Therefore, while $0 \le \xi < 1$, the function $F(x,\xi) = \xi \sin x - x$ (which is zero at x = 0) decreases steadily as x increases from 0. This shows that $F(x,\xi) = \xi \sin x - x$ cannot vanish in this range, and that we have found all the solutions for $0 \le \xi < 1$. Similarly, the derivative $F'(x,\xi) = \xi \cos x - 1$ is positive for $\xi > 1$ at x = 0. Further, increasing x from zero keeps the function $\xi \cos x - 1$ positive until it eventually crosses zero and remains negative until $x = \pi$. We could also visualize the values of the function $\xi \cos x - 1$ using MATLAB:

```
x=linspace(-pi,pi,100);

control=linspace(1,5,100);

F=cat(1,x,control.*cos(x)-
ones(size(x)));

figure, mesh(F)
```



Alternatively, observe that if n > 1, the derivative is positive at x = 0. As x increases in its interval of definition, the derivative $F'(x,\xi) = \xi \cos x - 1$ decreases, passes through zero and then stays negative up to $x = \pi$. Therefore, as x increases, $\xi \sin x - 1$ first increases from 0 to some positive maximum where $F'(x, \xi) = 0$, or its maximum value that is attained for $\xi \cos x = 1$. Then $\xi \sin x - 1$ decreases steadily to $x = \pi$, where $\xi \sin x - 1$ is negative . The Intermediate Value Theorem and continuity of the functions under study show that there is a zero at $x = x_1$ and that $x_1 \neq 0$ when $\xi > 1$. Earlier, we argued that $\xi \cos x < 1$ for $x = x_1$. As noted above, $x = x_1$ and $x = -x_1$ are both solutions. To summarize, we have determined one equilibrium point for $0 \leq \xi < 1$ and the three equilibriums for $\xi > 1$.

We proceed to analyze the qualitative nature of these equilibrium points using the *Linearization Methods* that we discussed earlier. For example, for points (*u* , *v*) near the equilibrium point (0 , 0), the nonlinear function *sin*(*u*) and *cos*(*u*) are approximated by their Taylor's expansion, where only the first terms are needed. The linear system that approximates or nonlinear system is:

$$\begin{aligned} \dot{u} &= 2v \\ \dot{v} &= \xi u - u. \end{aligned} \tag{1.3}$$

In matrix form, let us use $X = (u, v)^t$, so that we have:

$$\dot{X} = \begin{pmatrix} 0 & 2 \\ \xi - 1 & 0 \end{pmatrix} X \quad . \tag{1.4}$$

We compute the eigenvalues of the coefficient matrix by solving the characteristic polynomial:

$$\lambda^2 - 2(\xi - 1) = 0.$$

Depending on the values of the control parameter ξ , we have different circumstances. When $\xi < 1$, the two eigenvalues are pure imaginary and complex conjugate, so the equilibrium point is a center and nearby trajectories are circles or more generally, ellipses. When $\xi > 1$, on the other hand, the two eigenvalues are real with different signs, making the determinant of the coefficient negative, so that the equilibrium point is a saddle point. On the other hand, for $\xi > 1$, we have two other equilibriums, namely $x = x_1$ and $x = -x_1$. Let the nearby points be denoted by $x = x_1 + u$ and $y = v$, assuming small values for (u, v). Using Taylor's expansion again, the equations become

$$\begin{aligned} \dot{u} &= 2v \\ \dot{v} &= (\xi \cos x_1 - 1)v. \end{aligned} \tag{1.5}$$

We have essentially a the same analysis to do with the linear system

$$\dot{X} = \begin{pmatrix} 0 & 2 \\ \xi \cos x_1 - 1 & 0 \end{pmatrix} X . \tag{1.6}$$

The characteristic polynomial is calculated to be

$$\lambda^2 - 2(\xi \cos x_1 - 1) = 0.$$

Note that according to our previous discussion, $\xi \cos x_1 - 1 < 0$, so the equilibrium is a center. The case $x = -x_1$ gives also the same conclusion.

Discussion. The above-mentioned example has several interesting features. First, the position of the equilibrium point $x = 0$ does not change as the control parameter is altered but its character switches as ξ passes through 1, however, its behavior is changing from a stable regime to an unstable one. This phenomenon was called "*interchange of stabilities*" by Poincare'. Furthermore, at the switching stage extra equilibrium points are born. As we saw, their position varies with the control parameter but their qualitative behavior remain the same. The changing of the behavior of an equilibrium point and/or the creation of extra equilibriums by alteration of a control parameter is called *bifurcation*, and the value of ξ where bifurcation occurs is called a *bifurcation point*.

Exercises

1. Examine the bifurcation properties of the dynamical system below for various values of the constant c.

$$\dot{x} = \xi x(2y-1)$$
$$\dot{y} = \xi - y(2x+c)$$

Show that for c=1, this system has an equilibrium point, which is a stable node, for $0.5 < \xi < 1$ and a stable focus for $\xi > 1$, and that $\xi = 0.5$ is a bifurcation point.

2. As in Problem 1, analyze the bifurcation properties of the following system for several values of the constant c.

$$\dot{x} = x(y-c)$$
$$\dot{y} = \xi - y(x+c)$$

(2a) Show that for c=1, the system has an equilibrium point Q which is a stable node.

(2b) When $\xi > 1$ there is an additional equilibrium point P which is a stable node.

(2c) For $\xi < 1$, Q becomes a saddle-point as ξ passes through the bifurcation point $\xi = 1$.