A Learning Theoretic Approach to Differential and Perceptual Geometry
Part I: Curvature and Torsion are the Independent Components of Space Curves

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ABSTRACT

In standard differential geometry, the Fundamental Theorem of Space Curves states the that two differential invariants of a curve, namely curvature and torsion, determine its geometry, or equivalently, the isometry class of the curve up to rigid motions in the Euclidean three-dimensional space. Consider a physical model of a space curve made from a sufficiently thin, yet visible rigid wire, and the problem of perceptual identification (by a human observer or a robot) of two given physical model curves. In a previous paper (perceptual geometry) we have emphasized a learning theoretic approach to construct a perceptual geometry of the surfaces in the environment. In particular, we have described a computational method for mathematical representation of objects in the perceptual geometry inspired by the ecological theory of Gibson, and adhering to the principles of Gestalt in perceptual organization of vision. In this paper, we continue our learning theoretic treatment of perceptual geometry of objects, focusing on the case of physical models of space curves. In particular, we address the question of perceptually distinguishing two possibly novel space curves based on observer's prior visual experience of physical models of curves in the environment. The Fundamental Theorem of Space Curves inspires an analogous result in perceptual geometry as follows. We apply learning theory to the statistics of a sufficiently rich collection of physical models of curves, to derive two statistically independent local functions, that we call by analogy, the curvature and torsion. This pair of invariants distinguish physical models of curves in the sense of perceptual geometry. That is, in an appropriate resolution, an observer can distinguish two perceptually identical physical models in different locations. If these pairs of functions are approximately the same for two given space curves, then after possibly some changes of viewing planes, the observer confirms the two are the same.

Keywords: Independent Component Analysis (ICA), Learning, Perceptual Geometry, Gestalt, Curvature, Torsion.

INTRODUCTION

Learning theoretic approach to perceptual geometry aims at elucidating the process by which an intelligent system extracts features and discovers rules regarding the static and phenomenon and objects in the physical world. By an intelligent system we mean a system that is capable of retaining information regarding samples of data that characterize the physical state of the world. Such intelligent system represent information in what we call data space and is capable of performing statistics in this data space in order to extract feature and discover associations (rules).

In previous articles [Assadi 99(2), Assadi 99(3)] we described the Gestalt theory of surfaces in perceptual geometry. In this paper we demonstrate how to extend this theory to what may be called a perceptual geometry of curves. In perceptual geometry any object is represented by a sample of data in the data space.

For simplicity we adopt a vector space equipped with a metric (the perceptual metric) as a representation for such a data space. Roughly speaking, the perceptual metric represents the resolution and precision of the measurements that the intelligent system can afford. Again, for simplicity of exposition, we assume that a Euclidean (Positive definite matrix) could be used to approximate the perceptual metric. Therefore, the task of perceptual geometry is to derive statistical features of data with respect to the perceptual metric and discover associations.

Suppose that a data set D has local the principal component subspace that is d-dimensional (within the assumed scale and error tolerance) [Eghbalnia 99], Then we may assume that the object represented by D affords d degrees of freedom by the intelligent systems; thus we may call it a d-dimensional object (A surface with a very complex color, texture and shading, could have a Gestalt that has a high-dimensional representation). Suppressing certain attributes of the object is equivalent to projecting the data in to a lower dimensional principle subspace. An example of a perceptual curve is what may be called (perceptually) a straight line L. The data set representing an exemplar of L could be projected generically to a data set with 1 or more principal components with the exception of the singular case where the line is perpendicular to view plane of the visual observer. More generally, according the above paradigm, a perceptual curve is represented by a data set D such that all its projections except possibly one can be represented with at least one principal component.

What features could be extracted from the statistics of the sample S representing a perceptual curve? Consider the simplest possible case where the projection of principal components of S can be parameterized (possibly nonlinearly) by a single variable s. This corresponds to visual attention being aimed at following the position and direction of an object in space regardless of its attributes such as color, shading; what may be called the Gestalt of a curve.

In this paper we show that curvature and torsion ,which are geometric invariant of curves, are statistically independent components of a curve. Therefore, invariant features representing a perceptual curve can be extracted using a statistical algorithm. We begin with a basic review of relevant facts from differential geometry of curves. A short review of Independent Component Analysis is followed by computational results and discussion.

CURVATURE, TORSION AND FRENET FRAMES

Let $\beta(s)$ be a curve parameterized by arc length. The vector $T(s) = \beta'(s)$ is the unit **tangent vector** (i.e. of unit length) to the curve at s. Since $T \circ T = 1$, differentiation yields $2T \circ T' = 0$; thus T' is orthogonal to T. Write

$$T'(s) = \kappa N(s)$$

where κ is a scalar and N is the unit vector orthogonal to T – called the **unit normal** to the curve. The scalar κ is called **curvature**. Note that T' is nonzero only if the direction of the unit tangent T is changing. Since κ is simply the norm of T', namely $\kappa = \left\|T'(s)\right\|$, it measures the rate of change of tangent vector direction.

Now consider the vector $B = T \times N$, the so-called binormal. The triplet (T, N, B) forms an orthonormal set. In particular, we have $T \circ B = 0$ and $B \circ B = 1$. Then,

$$B' \circ T + B \circ T' = 0 \text{, and}$$

$$B \circ B' = 0 \quad \Rightarrow \quad B' \circ T = -T' \circ B = \kappa N \circ B = 0$$

Therefore B' is orthogonal to the (T, B)-plane and proportional to N – that is

$$B' = \tau N(s).$$

The quantity τ is called the **torsion** and it measure the rate of change of the binormal. κ and τ can be written explicitly in terms of beta as follows:

$$\kappa = \left\| \beta''(s) \right\|$$

$$\tau = \beta' \circ (\beta'' \times \beta''') \Big/ \beta'' \circ \beta''$$

The Fundamental Theorem of differential geometry of curves asserts that:

Theorem: Let $\kappa(s)$ and $\tau(s)$ ($s > 0$) be any two continuous functions. Then, there exists a unique curve (up to rigid motion) for which *s* is the arc length, $\kappa(s)$ is the curvature and $\tau(s)$ is its torsion.

A great deal of insight can be gained by examining the case for which $\beta(s)$ has a Taylor expansion about the point s (say $\beta(s)$ is three or more times differentiable.) Consider the Taylor expansion of beta at s = 0.

$$\beta(s) = \beta(0) + \beta'(0)s + \frac{\beta''(0)}{2!}s^2 + \frac{\beta'''(0)}{3!}s^3 + \cdots$$

Since *s* is the arc length parameter we can write

$$T(0) = \beta'(0)$$

$$(\kappa N)(0) = \beta''(0)$$

$$(-\kappa^2 T + \kappa N + \kappa\tau B)(0) = \beta'''(0)$$

Keeping only the lowest terms in T, N and B gives a first order Frenet frame approximation to the curve beta.

$$\beta(s) = const. + T(0)s + \frac{1}{2}(\kappa N)(0)s^2 + \frac{1}{6}\kappa(0)\tau(0)B(0)s^3 + \cdots$$

A linear approximation to the curve beta can be obtained by the first two terms of this expression. An approximation of the curve by a parabola in (T, N)-plane (the so-called the osculating plane) is obtained from the first three terms. Note that when torsion vanishes, the curve is locally a plane curve which lies in the osculating plane. Thus, the curvature measures the deviation of a curve from being a straight line while the torsion measures the deviation of a curve from lying in a plane.

PCA AND ICA

Consider a finite sample $(x_1, \cdots, x_n)$ of d-vectors representing a set of measurements of the "state of nature". It is often the case that extracting the desired features (knowledge) from this data part does not need all the d dimensions. This could be due to redundancy, noise or most likely, relationship between the coordinates from which the sample is taken. Therefore, when data vectors are represented in terms of the standard basis for $\mathbb{R}^n$, the true structure of the data is often hidden because of sub-optimality of the coordinate system used to represent the data.

Principal Components Analysis (PCA) is a well-known technique for finding a transformation of a data set such that the variance in the direction of each basis vector in the new coordinate space is inductively maximized [Jotliffe(87)] The computation of principal components can be simply formulated as the computation of eigenvalues and eigenvectors of the covariance matrix of the data set. PCA minimizes the first order dependencies among the given in the data set. Moreover, the maximization of the variance in the principal directions is equivalent to minimization of the reconstruction error.

Independent Components Analysis (ICA) builds on the basic idea of PCA by looking for components with higher order statistical independence. The key computational ingredient of Independent Components Analysis (ICA) [Cardoso(97), Comon(94)] is to find a linear map that transforms the observed multivariate data into a new collection of statistically independent components. In the present context of geometry of space curves, the goal would be to find independent features of a set of data sampled from curves. There are a number of different, mostly equivalent, ways to formulate Independent Components Analysis dpending on the circumstances:

1) Minimizing the Kullback-Leibler divergence between the joint probability and marginal probabilities of the output signals, in the so-called semi-parametric case where some estimates of pdfs are available Amari et al. 1996).
2) Finding a set of directions that factorize the joint probabilities.
3) Find a set of directions with minimum mutual information.

Consider the following system

$$X = AY$$

where X is a random vector whose components $[X_1, X_2, \cdots, X_m]^T$ are linear combinations of m unknown statistically independent sources given by the random vector $Y = [Y_1, Y_2, \cdots, Y_m]^T$. These two random vectors are related by the m- by-m matrix A. Furthermore, this unknown matrix (the so-called **mixing matrix**) is assumed to be of full rank for simplicity of discussion. In the over-determined case, i.e. when the number of samples is greater than their dimension, one must first perform PCA to reduce the number of samples in the data set to equal the dimension of the samples. ICA attempts to estimate either Y or A from X subject to the constraint that the components of Y are as statistically independent as possible.

MEASURES OF INDEPENDENCE

The components of a random vector are independent if the probability density function of the vector can be written as the product of the densities of its components. Namely that we can write

$$f_y(Y) = \prod_{i=1}^{m} f_{y_i}(Y_i) \qquad (1)$$

For a given Y, we can measure how close its components are from being independent by measuring what is known as the Kullback-Leibler (KL) divergence of the right and left hand sides of equation (1). We denote the KL divergence as

$$I(Y) = \int f_y(Y) \log \left(\frac{f_y(Y)}{\prod_{i=1}^{m} f_{y_i}(Y_i)} \right) dY \qquad (2)$$

When the probability density function of a random vector can be written as in equation (1), we see that the KL divergence takes its minimum value of zero.

It is worth noting that implementations of the ICA vary greatly and each implementation is designed to achieve the specific goals of data analysis dictated by the underlying data. For example, the Bell and Sejnowski infomax [Bell& Sjenowski(95)] algorithm works rather well with fMRI data. However, it does not perform well when applied to EEG and MEG. Comon's implementation of [Comon(94)] is to achieve statistical independence up to 4-th order moments (kurtosis), however, in both simulated and experimental data, one finds that this is not sufficient grounds to assume near independence of the distributions.

LEARNING GEOMETRY OF CURVES

As discussed above, curvature and torsion are reflected in the variation of two orthonormal components that locally determine the curve. Our goal is to show how the statistics of the sampled data can be used to extract curvature and torsion as independent components. Moreover, this process uses only the information available from two-dimensional projections, potentially, what is available to the visual system, or in applications to computer vision, the cameras record from the environment. We start by considering the discrete version of a curve described parametrically as C(t) = [x(t), y(t), z(t)]. We denote the discrete version as the set [$c(t_1)$, $c(t_2)$, …, $c(t_n)$] where $c(t_j) = [x(t_j)\ y(t_i)\ \ z(t_i)]$. The first and second order differences give the discrete version of the first and second derivative. For example, the first order difference is obtained as:

$$[c(t_2) - c(t_1),\ c(t_3) - c(t_2),\ \ldots,\ c(t_n) - c(t_{n-1})]$$

The first and second order differences can be used to construct approximations to the normal and binormal vectors using the formulas described in the discussion of curvature and torsion. However, we wish to use information available in two-dimensional projections. We first project the vectors onto a two-dimensional coordinate plane and use the information contained in the two-dimensional projection only. To obtain samples from a given curve we simulated looking at the curve from multiple views by applying a set of randomly drawn orthonormal transformation to the curve. For each transformation we obtained three projections for each vector giving a total of six values. Notice that various noises are naturally present in the process described. The discretization process and the difference formation as well as projection of transformed vectors onto various coordinated planes are naturally noisy due to various truncation errors.

The set of n six-dimensional vectors obtained above can be plotted in as 2n three-dimensional vectors as depicted to the right. Two sets of points are shown where the points in the lighter shading represent the torsion samples and the darker set the curvature samples. If the curvature and torsion are independent components, under proper conditions for the ICA algorithm, it should be possible to obtain a transformation of the point set such that the two 3-vectors defining the torsion and curvature can be separated. We investigate this question as well as the question of robustness of this approach in the presence of noise. The noise model is based on possible error that can come from sampling error. A good intuitive model to keep in mind is sampling from a thickened curve like that depicted in figure 2.

There are a number of ICA Algorithms available. We tested variations of two algorithms based on the fourth order cumulant methods [cardoso 97] and gradient decent methods [Bell& Sjenowski(95)]. We found the fourth order cumulant methods most suitable to our goal. In fact, the gradient decent approach showed some inconsistency across trials. The basic method of the algorithm is to form the covariance matrix of the data and perform a singular value decomposition on this matrix. The original data is then projected onto the Eigenvectors of the SVD decomposition such that the projections have unit norm. A weighed covariance matrix of this data $E\{Y^2YY^T\}$ is formed and diagonalized to give the independent components.

RESULTS AND DISCUSSION

A family of spirals was used for constructing the synthetic data. Spirals were parameterized so that a large sample of torsion and curvatures were available for selection. The point of interest for sampling was selected at random and then fixed throughout the experiment. Each point and its neighborhood were sampled at 150 different orientations with three projection samples at each orientation. The samples then were analyzed using The ICA algorithm. A sample of the results of analysis for the case of no added noise is illustrated in figure 2. In each case, the ICA algorithm performs a transformation that clearly separates the projected components of curvature and torsion. Each column depicts a curve with the independent components shown in two shades (blue and red in color and light gray and dark gray in grayscale). The last row is the original data before the ICA transformation.

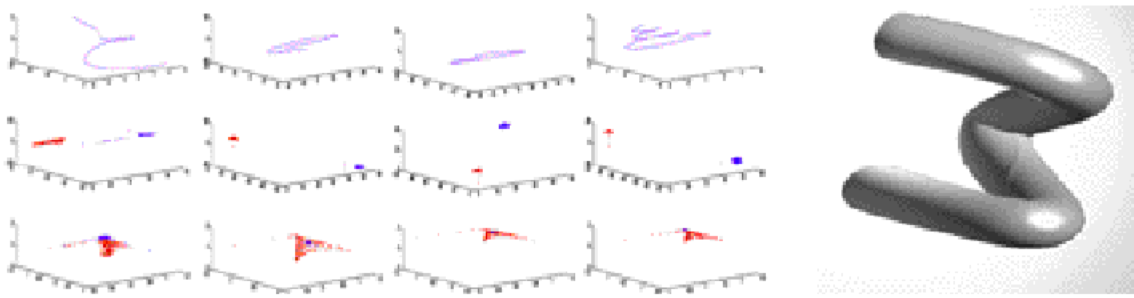

Figure 2. Each column shows the curve, its independent components, and the components before transformation for four different curves. The last column is the noisy example of a curve.

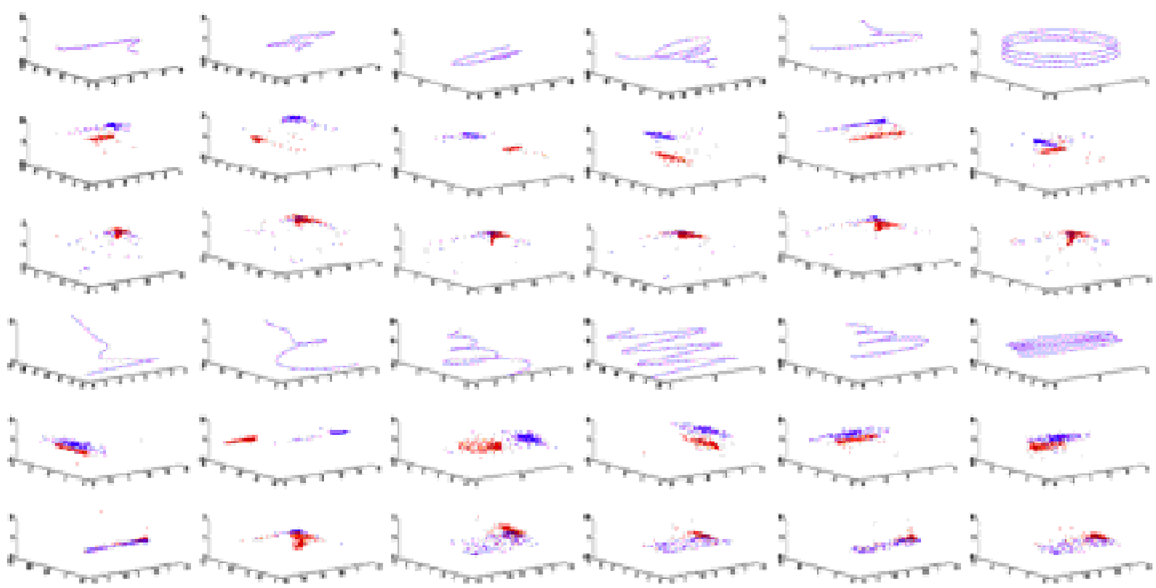

Figure-3 above demonstrates the results with 5% and 30% noise (1/SNR – SNR: signal to noise ration) added to the estimation of coordinates on the curve. The model of noise is based on a sub-Gaussian distribution normalized to provide noise in the specified range. Color codes are the same as Figure 2.

As the figures demonstrate, ICA is able to extract independent components effectively at the 5% noise level. However, we found that at 30% noise level independent components of some curves could not be extracted (see Figure 3 bottom left and right). The right column of Figure 3 shows that a helical curve with slow rise becomes susceptible to noise even at 5%. This phenomenon is computationally plausible. Since in the case of the tight helical curves the torsion is and curvature have small variations, noise becomes the dominant factor.

We have demonstrated that local behavior of curves based on their torsion and curvature can be determined from projections of the torsion and curvature components using ICA. ICA recovers these components as independent components even in the presence of noise levels as high as 30%. A number of possible directions for additional investigation remain. How these findings correlate with cognitive findings may shed light on the psychophysical significance of these findings. Furthermore, we may ask if the independent torsion and curvature components are more readily available without the pre-processing we have performed (forming differences and projections). The latter question may lead to investigation of independent subspaces for discovery of features.

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