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2020 2 7

Binomial Option Pricing

Option Valuation

- The binomial option pricing model enables us to determine the price of an option, given the characteristics of the underlying asset.
- The binomial option pricing model assumes that the asset price in each period move only up or down by a specified amount
- The binomial option pricing model is often referred to as the "Cox-Ross-Rubinstein (CRR) pricing model"

Risk-free hedge portfolio valuation

- Begin by using simple binomial model
  - Consider a European call option on the stock of XYZ, with a $40 strike price and 1 year to expiration
  - XYZ does not pay dividends, and its current price is $ 41.
  - The continuously compounded risk-free interest rate is 8%.

Risk-free hedge portfolio

- Stock price in 1 year will be $60 or $30.

$$\$\ 41 \begin{array}{ll} \nearrow & \$\ 60 \\ \searrow & \$\ 30 \end{array}$$

Risk-free hedge portfolio valuation

- Call values at expiration are known.

$$c \begin{array}{ll} \nearrow & \$\ 20 \\ \searrow & \$\ 0 \end{array}$$

- Form portfolio by buying Δ units of asset and selling 1 unit of 1-year call options at price c

$$\Delta 41 - c \begin{array}{ll} \nearrow & \$\ \Delta 60 - 20 \\ \searrow & \$\ \Delta 30 \end{array}$$

Risk-free hedge portfolio valuation

- Set terminal values equal to one another (thereby creating risk-free hedge) and solve for Δ.

$$\begin{aligned} \Delta 60 - 20 &= \Delta 30 \\ \Rightarrow \Delta &= \frac{2}{3} \end{aligned}$$

- The value 2/3 is the *delta* (Δ) of the option: the number of shares that replicates the option payoff.

Risk-free hedge portfolio valuation

- Substitute into binomial lattice.

$$41(2/3) - c \quad \begin{array}{ll} \nearrow & \$\ 60(2/3) - 20 \\ \searrow & \$\ 30(2/3) \end{array}$$

- Portfolio is risk-free because its terminal value (i.e., 20) is invariant to asset price.

$$41(2/3) - c \quad \begin{array}{ll} \nearrow & \$\ 20 \\ \searrow & \$\ 20 \end{array}$$

Risk-free hedge portfolio valuation

- $41(2/3) - c$ invested in risk-free bonds would also have terminal value of \$20.
- If risk-free rate over 1-year interval is 8%, no arbitrage implies

$$\begin{aligned} 41(2/3) - c &= 20e^{-0.08} \\ \Rightarrow c &= \$8.871 \end{aligned}$$

Risk Neutral Pricing

- Risk-free hedge portfolio was formed without knowing probabilities.
- Implies that call value was derived without knowing investor risk/return preferences
- Since investor preferences are not required to value call, assume investors are risk-neutral
  - Risk-neutral investors are indifferent between a sure thing and a risky bet with an expected payoff equal to the value of the sure thing
  - Under risk-neutrality, all assets have expected return equal to risk-free interest rate.

Risk Neutral Pricing

- To value call under risk-neutrality, need to specify upstate (p^*) and downstate probabilities ($1 - p^*$).

$$\begin{array}{ccc} & \nearrow_{p^*} & \$\ 60 \\ 41 & & \\ & \searrow_{1-p^*} & \$\ 30 \end{array}$$

- To identify implied risk-neutral probabilities p^*, equate expressions for terminal asset price and solve for p^*.

$$\begin{aligned} 41e^{0.08} &= 60p^* + 30(1 - p^*) \\ p^* &= 48.05\% \end{aligned}$$

Risk Neutral Pricing

- With risk-neutral probabilities known, value call using traditional approach (i.e., take PV of expected future value)

$$c = e^{-rT} E[\tilde{c}_T]$$

- Call values are known

$$E[\tilde{c}_T] = 20(0.4805) + 0(1 - 0.4805) = 9.61$$

- Take PV of expected future value

$$c = e^{-0.08} 9.61 = 8.871$$

Risk Neutral Pricing

- Two approaches lead to same result.
  - risk-free hedge portfolio valuation
  - risk-neutral valuation
- Since risk-neutral valuation is simplest, use it.

Binomial Option Pricing

Risk Neutral Pricing

Binomial Method

- Step 1: Create asset price lattice.
  - Assume underlying asset price has discrete proportional jumps through option's life.
  - $u(>1)$ is up-step coefficient and $d(<1)$ is down-step coefficient

$$\begin{array}{ccccc} & & & \nearrow & uuS \\ & \nearrow & uS & \searrow & \\ S & & & & udS \\ & \searrow & dS & \nearrow & \\ & & & \searrow & ddS \end{array}$$

Binomial Method

- Step 2: Decide on length of time increment h.
  - Setting time increment to one year : $h = 1$
  - Setting time increment to one day : $h = 1/365$
  - Setting time increment to one hour : $h = 1/(365 \times 24)$

$$\begin{array}{ccccc} & & & \nearrow & uuS \\ & \nearrow & uS & \searrow & \\ S & & & & udS \\ & \searrow & dS & \nearrow & \\ & & & \searrow & ddS \\ \multicolumn{3}{c}{\longleftarrow h \longrightarrow} & & \end{array}$$

Binomial Method

- Step 3: Value option at expiration.
  - $c_u = \max(uS - K, 0)$ and $c_d = \max(dS - K, 0)$
  - $c_{uu} = \max(u^2 S - K, 0), c_{ud} = \max(udS - K, 0)$, and $c_{dd} = \max(d^2 S - K, 0)$

$$\begin{array}{ccccc} & & & \nearrow & c_{uu} \\ & \nearrow & c_u & \searrow & \\ c & & & & c_{ud} \\ & \searrow & c_d & \nearrow & \\ & & & \searrow & c_{dd} \end{array}$$

Binomial Method

- Step 4: Value option one step back in time by taking present value of expected future values.
  - $c_u = e^{-rh}[p^* c_{uu} + (1 - p^*) c_{ud}]$
  - $c_d = e^{-rh}[p^* c_{ud} + (1 - p^*) c_{dd}]$
  - $c = e^{-rh}[p^* c_u + (1 - p^*) c_d]$

$$\begin{array}{ccccc} & & & \nearrow & c_{uu} \\ & \nearrow & c_u & \searrow & \\ c & & & & c_{ud} \\ & \searrow & c_d & \nearrow & \\ & & & \searrow & c_{dd} \end{array}$$

$$\xleftarrow{\text{Backward Induction}}$$

Binomial Method

- Risk-Neutral probabilities are identified by equating expressions for expected asset price

$$\begin{aligned} Se^{\alpha h} &= p^* uS + (1-p^*)dS \\ \Rightarrow p^* &= \frac{e^{\alpha h} - d}{u - d} \end{aligned}$$

  - $\alpha = r$: non-dividend paying stock
  - $\alpha = r - \delta$: dividend paying stock with dividend rate δ
  - $\alpha = r - \delta$: currency with foreign risk-free rate δ.
  - $\alpha = 0$: futures contract

Choosing u and d

- Coefficients and probabilities used in lattice are not unique.
  - Cox-Ross-Rubinstein(1979) binomial tree

$$u = e^{\sigma\sqrt{h}} \quad d = e^{-\sigma\sqrt{h}} \quad p^* = \frac{1}{2} + \frac{1}{2}\left(\frac{\alpha - 0.5\sigma^2}{\sigma}\right)h$$

  - Jarrow-Rudd(1983) lognormal binomial tree

$$u = e^{(\alpha - 0.5\sigma^2)h + \sigma\sqrt{h}} \quad d = e^{(\alpha - 0.5\sigma^2)h - \sigma\sqrt{h}} \quad p^* = \frac{1}{2}$$

Finding Δ

- Consider portfolio of buying Δ shares of asset and selling 1 unit of call options at price c. The value of the portfolio at time h is

$$\Delta S - c \quad \begin{matrix} \nearrow & \Delta e^{\delta h} uS - c_u \\ \\ \searrow & \Delta e^{\delta h} dS - c_d \end{matrix}$$

- Solving for Δ by setting terminal values to one another, Δ is

$$\Delta = e^{-\delta h} \frac{c_u - c_d}{S(u-d)}$$

Replicating Portfolio

- Plugging Δ into either $\Delta e^{\delta h} uS - c_u$ or $\Delta e^{\delta h} dS - c_d$ and discounting the value, we have

$$(\Delta S - c) = -e^{-rh} \frac{uc_d - dc_u}{u-d}$$

- If we buy Δ shares of stock and buy a dollar amount of B, the cost for constructing portfolio is $\Delta S + B$ where

$$B = e^{-rh} \frac{uc_d - dc_u}{u-d}$$

- The cost of creating the option is the net cash flow required to buy the shares and bonds. Thus, the cost of the option is $\Delta S + B$

$$\Delta S + B = e^{-rh} \left(c_u \frac{e^{(r-\delta)h} - d}{u-d} + c_d \frac{u - e^{(r-\delta)h}}{u-d} \right)$$

Illustration

- Asset price (S) : \$ 40
- Volatility rate (σ): 20%
- Dividend rate (δ) : 0 %
- Call option exercise price (K) : 40
- Time to expiration (T) : 1
- Number of steps : 3
- Time increment (h) : 1/3
- Risk-free rate (r) : 6 %

Illustration

- Simple Binomial.xlsx
- Sheet : Binomial Option Pricing.

Illustration

- Compute up-step and down-step CRR coefficients.

$$u = e^{0.2\sqrt{1/3}} = 1.1224 \quad d = e^{-0.2\sqrt{1/3}} = 0.8909$$

- Complete asset price lattice

			56.559
		50.391	
	44.896		44.896
40.000		40.000	
	35.638		35.638
		31.751	
			28.289

Illustration

- Value option with $K = 40$ at expiration

Option Payoff	Asset Price
$\max(56.559 - 40, 0) = 16.559$	56.559
$\max(44.896 - 40, 0) = 4.896$	44.896
$\max(35.638 - 40, 0) = 0$	35.638
$\max(28.289 - 40, 0) = 0$	28.289

Illustration

- Value option one step back in time by taking present value of expected future value
- Risk Neutral probability p^*

$$p^* = \frac{e^{0.06(1/3)} - 0.8909}{1.1224 - 0.8909} = 0.5584$$

$$\begin{array}{lcc}
 & & 16.559 \\
11.183 = e^{-0.06}[(0.5584)16.559 + (1 - 0.5584)4.896] & \swarrow & \\
 & \nwarrow & 4.896 \\
2.680 = e^{-0.06}[(0.5584)4.896 + (1 - 0.5584)0] & \swarrow & \\
 & \nwarrow & 0.000 \\
0.000 & \swarrow & \\
 & \nwarrow & 0.000
\end{array}$$

Illustration

- Continue iterative process.

$$\begin{array}{cccc}
 & & & 16.559 \\
 & & 11.183 & \\
 & 7.282 & & 4.896 \\
4.621 & & 2.680 & \\
 & 1.467 & & 0.000 \\
 & & 0.000 & \\
 & & & 0.000
\end{array}$$

- Call option price with exercise price $K = 40$ at the maturity of 1 year is 4.621

Is the Binomial Model Realistic?

- The binomial model is a form of the random walk model, adapted to modeling stock prices. The random walk model in this section assumes among other things, that
  - Volatility is constant
  - "Large" stock price movements do not occur
  - Returns are independent over time
- All of these assumptions appear to be violated in the data.

Summary

- In order to price an option, we need to know
  - Asset price (S)
  - Strike price (K)
  - Standard deviation of returns on the asset ,i.e., asset volatility (σ)
  - Dividend yield (δ)
  - Risk-free rate (r)
- Using the risk-free rate (r) and volatility (σ) , we can approximate the future distribution of the asset by creating a binomial tree.
- Once we have the binomial tree, is it possible to price the option.

Summary

- Steps of binomial method
  - Step 1 : Create asset price lattice.
  - Step 2 : Decide on length of time increment h.
  - Step 3 : Value option at expiration
  - Step 4 : Step back one time increment (h) and take present value of expected future option value.