

# Untitled

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2020 2 7

## Binomial Option Pricing

### Option Valuation

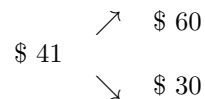
- The binomial option pricing model enables us to determine the price of an option, given the characteristics of the underlying asset.
- The binomial option pricing model assumes that the asset price in each period move only up or down by a specified amount
- The binomial option pricing model is often referred to as the “Cox-Ross-Rubinstein (CRR) pricing model”

### Risk-free hedge portfolio valuation

- Begin by using simple binomial model
  - Consider a European call option on the stock of XYZ, with a \$40 strike price and 1 year to expiration
  - XYZ does not pay dividends, and its current price is \$ 41.
  - The continuously compounded risk-free interest rate is 8%.

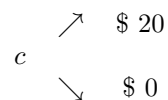
### Risk-free hedge portfolio

- Stock price in 1 year will be \$60 or \$30.

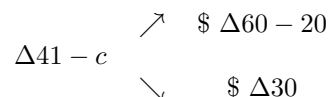


### Risk-free hedge portfolio valuation

- Call values at expiration are known.



- Form portfolio by buying  $\Delta$  units of asset and selling 1 unit of 1-year call options at price  $c$



### Risk-free hedge portfolio valuation

- Set terminal values equal to one another (thereby creating risk-free hedge) and solve for  $\Delta$ .

$$\begin{aligned}\Delta 60 - 20 &= \Delta 30 \\ \Rightarrow \Delta &= \frac{2}{3}\end{aligned}$$

- The value  $2/3$  is the *delta* ( $\Delta$ ) of the option: the number of shares that replicates the option payoff.

### Risk-free hedge portfolio valuation

- Substitute into binomial lattice.

$$\begin{array}{rcl} & \nearrow & \$ 60(2/3) - 20 \\ 41(2/3) - c & & \\ & \searrow & \$ 30(2/3) \end{array}$$

- Portfolio is risk-free because its terminal value (i.e., 20) is invariant to asset price.

$$\begin{array}{rcl} & \nearrow & \$ 20 \\ 41(2/3) - c & & \\ & \searrow & \$ 20 \end{array}$$

### Risk-free hedge portfolio valuation

- $41(2/3) - c$  invested in risk-free bonds would also have terminal value of \$20.
- If risk-free rate over 1-year interval is 8%, no arbitrage implies

$$\begin{aligned}41(2/3) - c &= 20e^{-0.08} \\ \Rightarrow c &= \$8.871\end{aligned}$$

### Risk Neutral Pricing

- Risk-free hedge portfolio was formed without knowing probabilities.
- Implies that call value was derived without knowing investor risk/return preferences
- Since investor preferences are not required to value call, assume investors are risk-neutral
  - Risk-neutral investors are indifferent between a sure thing and a risky bet with an expected payoff equal to the value of the sure thing
  - Under risk-neutrality, all assets have expected return equal to risk-free interest rate.

### Risk Neutral Pricing

- To value call under risk-neutrality, need to specify upstate ( $p^*$ ) and downstate probabilities ( $1 - p^*$ ).

$$\begin{array}{rcl} & \nearrow_{p^*} & \$ 60 \\ 41 & & \\ & \searrow_{1-p^*} & \$ 30 \end{array}$$

- To identify implied risk-neutral probabilities  $p^*$ , equate expressions for terminal asset price and solve for  $p^*$ .

$$\begin{aligned} 41e^{0.08} &= 60p^* + 30(1 - p^*) \\ p^* &= 48.05\% \end{aligned}$$

### Risk Neutral Pricing

- With risk-neutral probabilities known, value call using traditional approach (i.e., take PV of expected future value)

$$c = e^{-rT} E[\tilde{c}_T]$$

- Call values are known

$$E[\tilde{c}_T] = 20(0.4805) + 0(1 - 0.4805) = 9.61$$

- Take PV of expected future value

$$c = e^{-0.08} 9.61 = 8.871$$

### Risk Neutral Pricing

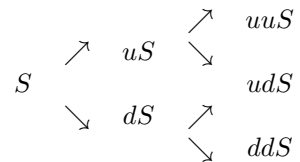
- Two approaches lead to same result.
  - risk-free hedge portfolio valuation
  - risk-neutral valuation
- Since risk-neutral valuation is simplest, use it.

# Binomial Option Pricing

## Risk Neutral Pricing

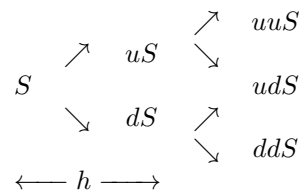
### Binomial Method

- Step 1: Create asset price lattice.
  - Assume underlying asset price has discrete proportional jumps through option's life.
  - $u(> 1)$  is up-step coefficient and  $d(< 1)$  is down-step coefficient



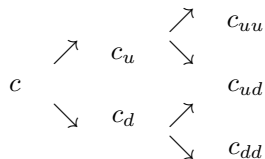
### Binomial Method

- Step 2: Decide on length of time increment  $h$ .
  - Setting time increment to one year :  $h = 1$
  - Setting time increment to one day :  $h = 1/365$
  - Setting time increment to one hour :  $h = 1/(365 \times 24)$



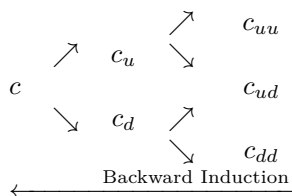
## Binomial Method

- Step 3: Value option at expiration.
  - $c_u = \max(uS - K, 0)$  and  $c_d = \max(dS - K, 0)$
  - $c_{uu} = \max(u^2S - K, 0)$ ,  $c_{ud} = \max(udS - K, 0)$ , and  $c_{dd} = \max(d^2S - K, 0)$



## Binomial Method

- Step 4: Value option one step back in time by taking present value of expected future values.
  - $c_u = e^{-rh}[p^*c_{uu} + (1 - p^*)c_{ud}]$
  - $c_d = e^{-rh}[p^*c_{ud} + (1 - p^*)c_{dd}]$
  - $c = e^{-rh}[p^*c_u + (1 - p^*)c_d]$



## Binomial Method

- Risk-Neutral probabilities are identified by equating expressions for expected asset price

$$\begin{aligned} Se^{\alpha h} &= p^*uS + (1 - p^*)dS \\ \Rightarrow p^* &= \frac{e^{\alpha h} - d}{u - d} \end{aligned}$$

- $\alpha = r$  : non-dividend paying stock
- $\alpha = r - \delta$  : dividend paying stock with dividend rate  $\delta$
- $\alpha = r - \delta$  : currency with foreign risk-free rate  $\delta$ .
- $\alpha = 0$  : futures contract

## Choosing $u$ and $d$

- Coefficients and probabilities used in lattice are not unique.
  - Cox-Ross-Rubinstein(1979) binomial tree

$$u = e^{\sigma\sqrt{h}} \quad d = e^{-\sigma\sqrt{h}} \quad p^* = \frac{1}{2} + \frac{1}{2} \left( \frac{\alpha - 0.5\sigma^2}{\sigma} \right) h$$

- Jarrow-Rudd(1983) lognormal binomial tree

$$u = e^{(\alpha - 0.5\sigma^2)h + \sigma\sqrt{h}} \quad d = e^{(\alpha - 0.5\sigma^2)h - \sigma\sqrt{h}} \quad p^* = \frac{1}{2}$$

### Finding $\Delta$

- Consider portfolio of buying  $\Delta$  shares of asset and selling 1 unit of call options at price  $c$ . The value of the portfolio at time  $h$  is

$$\Delta S - c \quad \begin{array}{l} \nearrow \quad \Delta e^{\delta h} u S - c_u \\ \searrow \quad \Delta e^{\delta h} d S - c_d \end{array}$$

- Solving for  $\Delta$  by setting terminal values to one another,  $\Delta$  is

$$\Delta = e^{-\delta h} \frac{c_u - c_d}{S(u - d)}$$

### Replicating Portfolio

- Plugging  $\Delta$  into either  $\Delta e^{\delta h} u S - c_u$  or  $\Delta e^{\delta h} d S - c_d$  and discounting the value, we have

$$(\Delta S - c) = -e^{-rh} \frac{uc_d - dc_u}{u - d}$$

- If we buy  $\Delta$  shares of stock and buy a dollar amount of  $B$ , the cost for constructing portfolio is  $\Delta S + B$  where

$$B = e^{-rh} \frac{uc_d - dc_u}{u - d}$$

- The cost of creating the option is the net cash flow required to buy the shares and bonds. Thus, the cost of the option is  $\Delta S + B$

$$\Delta S + B = e^{-rh} \left( c_u \frac{e^{(r-\delta)h} - d}{u - d} + c_d \frac{u - e^{(r-\delta)h}}{u - d} \right)$$



**Illustration**

- Asset price ( $S$ ) : \$ 40
- Volatility rate ( $\sigma$ ): 20%
- Dividend rate ( $\delta$ ) : 0 %
- Call option exercise price ( $K$ ) : 40
- Time to expiration ( $T$ ) : 1
- Number of steps : 3
- Time increment ( $h$ ) : 1/3
- Risk-free rate ( $r$ ) : 6 %

**Illustration**

- Simple Binomial.xlsx
- Sheet : Binomial Option Pricing.

### Illustration

- Compute up-step and down-step CRR coefficients.

$$u = e^{0.2\sqrt{1/3}} = 1.1224 \quad d = e^{-0.2\sqrt{1/3}} = 0.8909$$

- Complete asset price lattice

|        |        |        |        |
|--------|--------|--------|--------|
|        |        |        | 56.559 |
|        |        | 50.391 |        |
|        | 44.896 |        | 44.896 |
| 40.000 |        | 40.000 |        |
|        | 35.638 |        | 35.638 |
|        |        | 31.751 |        |
|        |        |        | 28.289 |

### Illustration

- Value option with  $K = 40$  at expiration

| Option Payoff                   | Asset Price |
|---------------------------------|-------------|
| $\max(56.559 - 40, 0) = 16.559$ | 56.559      |
| $\max(44.896 - 40, 0) = 4.896$  | 44.896      |
| $\max(35.638 - 40, 0) = 0$      | 35.638      |
| $\max(28.289 - 40, 0) = 0$      | 28.289      |

## Illustration

- Value option one step back in time by taking present value of expected future value
- Risk Neutral probability  $p^*$

$$p^* = \frac{e^{0.06(1/3)} - 0.8909}{1.1224 - 0.8909} = 0.5584$$

$$\begin{array}{rcl}
 11.183 & = & e^{-0.06}[(0.5584)16.559 + (1 - 0.5584)4.896] \\
 2.680 & = & e^{-0.06}[(0.5584)4.896 + (1 - 0.5584)0] \\
 0.000 & & 
 \end{array}
 \begin{array}{c}
 \swarrow \searrow \\
 \swarrow \searrow \\
 \swarrow \searrow \\
 \swarrow \searrow
 \end{array}
 \begin{array}{c}
 16.559 \\
 4.896 \\
 0.000 \\
 0.000
 \end{array}$$

## Illustration

- Continue iterative process.

$$\begin{array}{rcl}
 & & 16.559 \\
 & & 11.183 \\
 & 7.282 & 4.896 \\
 4.621 & 2.680 & \\
 & 1.467 & 0.000 \\
 & 0.000 & \\
 & & 0.000
 \end{array}$$

- Call option price with exercise price  $K = 40$  at the maturity of 1 year is 4.621

### **Is the Binomial Model Realistic?**

- The binomial model is a form of the random walk model, adapted to modeling stock prices. The random walk model in this section assumes among other things, that
  - Volatility is constant
  - "Large" stock price movements do not occur
  - Returns are independent over time
- All of these assumptions appear to be violated in the data.

## Summary

- In order to price an option, we need to know
  - Asset price ( $S$ )
  - Strike price ( $K$ )
  - Standard deviation of returns on the asset ,i.e., asset volatility ( $\sigma$ )
  - Dividend yield ( $\delta$ )
  - Risk-free rate ( $r$ )
- Using the risk-free rate ( $r$ ) and volatility ( $\sigma$ ) , we can approximate the future distribution of the asset by creating a binomial tree.
- Once we have the binomial tree, is it possible to price the option.

## Summary

- Steps of binomial method
  - Step 1 : Create asset price lattice.
  - Step 2 : Decide on length of time increment  $h$ .
  - Step 3 : Value option at expiration
  - Step 4 : Step back one time increment ( $h$ ) and take present value of expected future option value.