Space Optimal Vertex Cover in Dynamic Streams
(with an overview of graph streaming)

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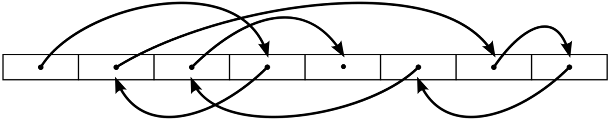
Joint work with Vihan Shah (Rutgers University)

1. Introduction
2. Streaming Models
3. Matchings in Graph Streams
4. Space Optimal Vertex Cover

Classic Setting

Assumption

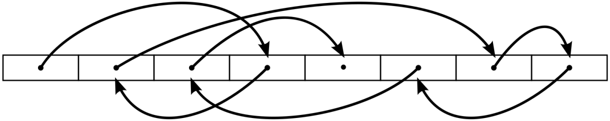
Classical algorithms rely on the assumption that they have a random access to the input of the algorithm



Dafyd Jones (flikr)

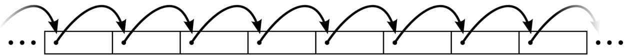
Assumption (Infeasible)

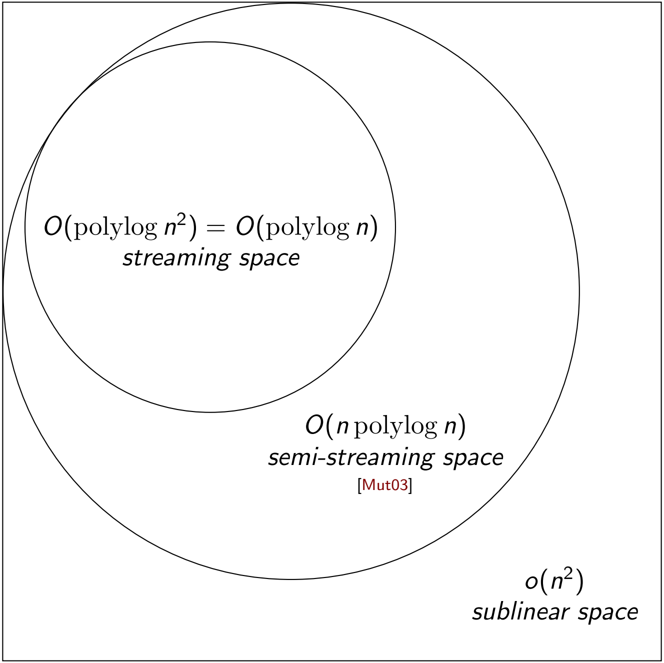
Classical algorithms rely on the assumption that they have a random access to the input of the algorithm



Definition (Graph Streaming)

A n-vertex graph is presented as a sequence of edges to an algorithm uses space **sublinear** in the size of the input ($o(n^2)$).

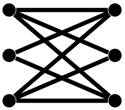




$O(\mathrm{polylog}\, n^2) = O(\mathrm{polylog}\, n)$
streaming space
$O(n\, \mathrm{polylog}\, n)$
semi-streaming space
[Mut03]
$o(n^2)$
sublinear space

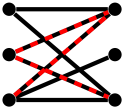
Insertion-Only [FKM+04] (finite stream)

e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8



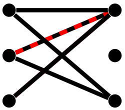
Sliding Window [CMS13] (infinite stream)

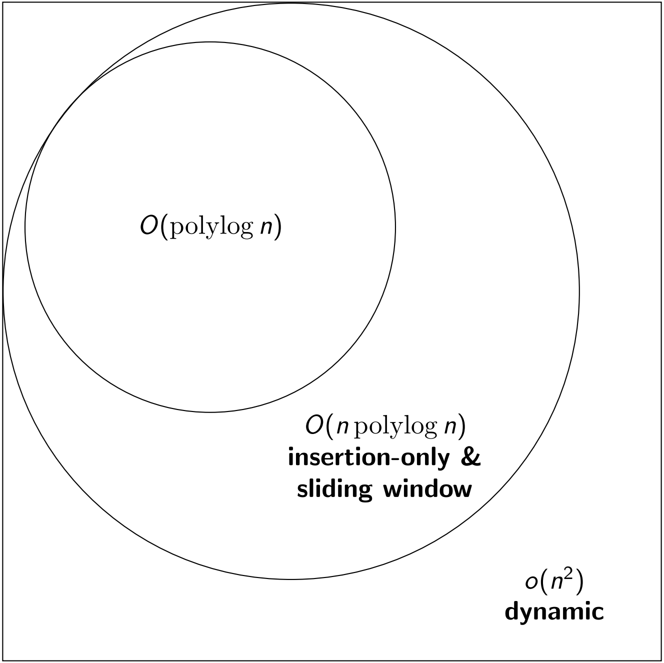
e_1	e_2	e_3	e_4	e_5	e_6	e_7	$\cdots$



Dynamic [AGM12] (finite stream)

e_1	e_2	e_3	$\overline{e_1}$	$\overline{e_3}$	e_4	e_5	e_1



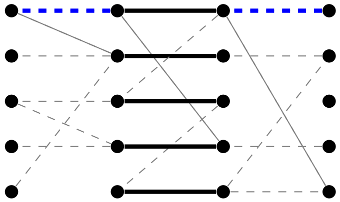

$O(\mathrm{polylog}\, n)$
$O(n\, \mathrm{polylog}\, n)$
insertion-only & sliding window
$o(n^2)$
dynamic

Simple and Powerful

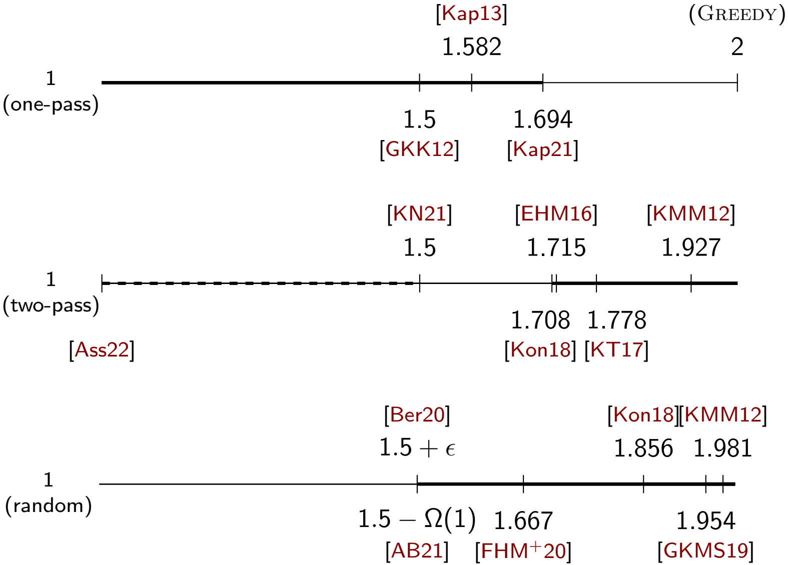
GREEDY Matching:

1. Add edge if neither endpoint is matched

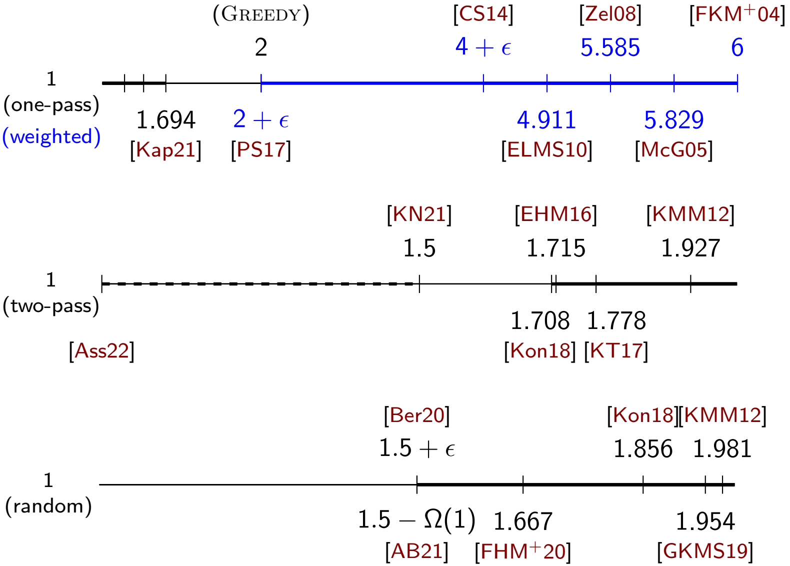
- Maximal
- 2-approximation



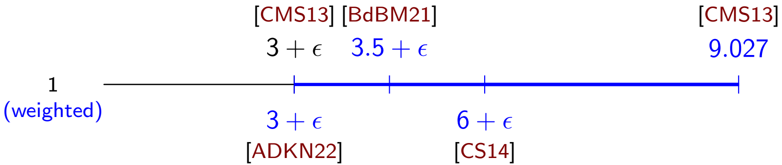
Insertion-Only



Insertion-Only



Sliding Window



Dynamic (Insertion-Deletion)

Goal:

- Find an α-approximation

[AKLY16] $\Omega(\frac{n^{2-o(1)}}{\alpha^3})$

[AS22] $O(\frac{n^2}{\alpha^3})$

[Kon15] $O(\frac{n^2}{\alpha^2}\log^4 n)$

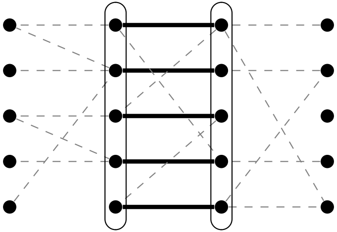
$\Omega(\frac{n^{1.5}}{\alpha^4})$ [Kon15]

$\Omega(\frac{n^2}{\alpha^3})$ [DK20]

$O(\frac{n^2}{\alpha^3}\log^4 n)$ [CCE+16, AKLY16]

GREEDY
MatchingVertex Cover:

- Maximal
- 2-approximation



Relation to Vertex Cover (Results)

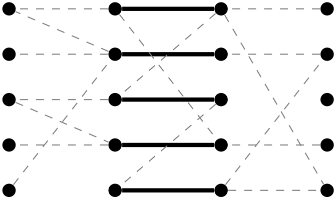
	matching	**vertex cover**
insertion-only	[1.694, 2]	[1, 2] GREEDY
sliding window	$[1,\ 3+\epsilon]$	$[1,\ 3+\epsilon]$ [Sub21]
dynamic	$\Theta(\frac{n^2}{\alpha^3})$	$\Theta(\frac{n^2}{\alpha^2})$ [DK20] & this work

All UBs use GREEDY in some way!

What makes dynamic graph streams hard?

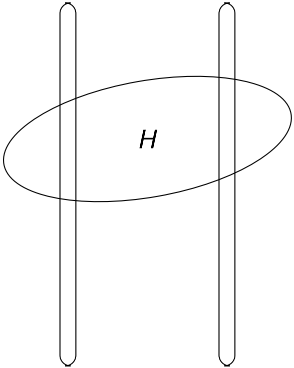
GREEDY Matching:

- ~~**Not**~~ Maximal
- ~~**0**-approximation~~
  2-approximation



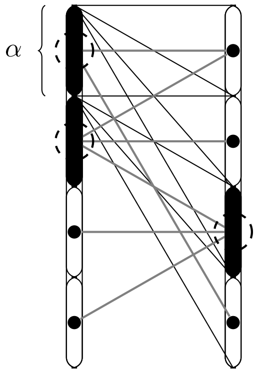
Note: Deterministically returning a single edge requires $\Omega(n^2)$ bits of space for dense graphs.

What techniques are used?



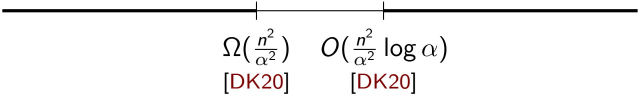
For a subgraph H of G with m edges:

1. Check if H is empty
   - A counter using $\Theta(\log m)$ bits
2. Retrieve an edge if one is present
   - An L_0-sampler using $\Theta(\log^3 n)$ bits
   - Neighbourhood sampler



1. Vertex groups of size α
   - about $\frac{n}{\alpha}$ groups
2. Check if there is an edge between each pair of groups
   - about $\frac{n^2}{\alpha^2}$ pairs
3. Return vertices of the group-level vertex cover

Space: $O(\frac{n^2}{\alpha^2})$ counters, each using $O(\log \alpha)$ bits. Hence, $O(\frac{n^2}{\alpha^2} \log \alpha)$ bits.



What's the issue?

Problem:

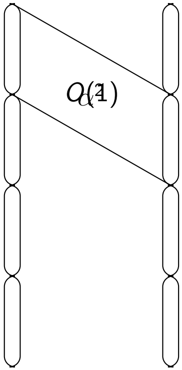
- Counters use $O(\log \alpha)$ bits.
- Each counter counts upto α^2 edges.

Goal:

- Counters to use $O(1)$ bits.
- Counters to count upto constant many edges

How?

- Sparse graph (GREEDY!)
- Randomly partition (easy)

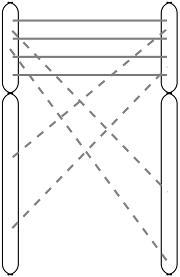


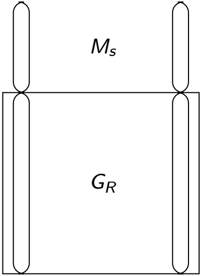
Sparsness properties of GREEDY

Lemma

Let G be a n-vertex graph with m edges and let M_s be a GREEDY *matching on $s \leq m$ uniform randomly sampled edges. Then, for $G_R = G[V \backslash V(M_s)]$,*

$$\Delta(G_R) \leq \frac{C \cdot m \cdot \log n}{s}.$$





Proof (sketch).

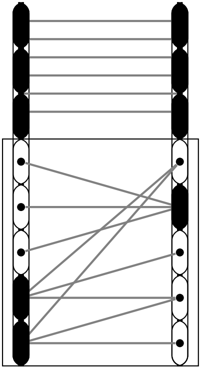
For any vertex $v \in G$,

- $v \in G_R$
- $\deg_{G_R}(v) > \frac{C \cdot m \cdot \log n}{s}$

both do not occur w.h.p. □

Algorithm

1. Randomly partition vertices ($\frac{n}{\alpha}$ groups)
2. Sample $O(\frac{n^2}{\alpha^2 \log^3 n})$ edges randomly and run $\textsc{Greedy}$ on them to get M
3. Check if an edge is present between pairs and compute group-level vertex cover
4. Return vertices of the covering groups including those with matched vertices



Limitation:

- Spareseness properties are only sufficient for $\alpha \ll n^{\frac{1}{3.5}}$

Extension:

- Non-uniform sampling instead of uniform sampling, i.e., using neighbourhood edge sampling [AS22]
- Results are an average degree bound instead of max degree bound
- Full range, i.e., $\alpha \ll n$

[this work]

$O(\frac{n^2}{\alpha^2})$

$\Omega(\frac{n^2}{\alpha^2})$ [DK20] $\quad$ $O(\frac{n^2}{\alpha^2} \log \alpha)$ [DK20]

Open Questions:

- Deterministic techniques or LB instead
- Other problems like dominating set and spectral sparsification

Thank You

Questions?

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