Maximum Matching via Maximal Matching Queries

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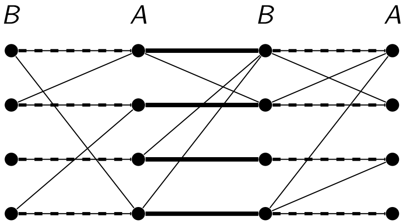
Preliminaries

In this talk, we consider bipartite graphs $G = (A, B, E)$.



Matchings

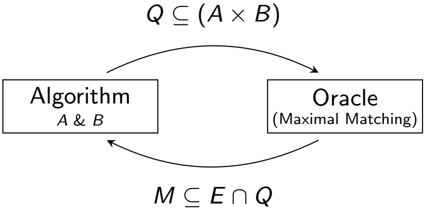
A matching M is a subset of vertex-disjoint edges of a graph.

- **Maximal**: Every edge $e \in E \backslash M$ is incident to M.
- **Maximum**: Largest size, $\mu(G)$.
- Maximal matchings are 0.5-approximations of maximum matchings.

Edge Query Model

Algorithm's Goal

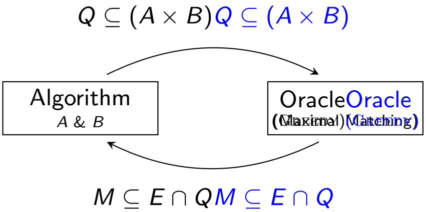
Return a **large matching** of the bipartite input graph $G = (A, B, E)$ using only **deterministic edge queries** to a **maximal matching oracle**.

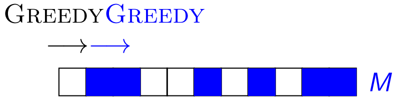


Motivation

Computing a maximal matching is **easy** in various computational models such as **data streaming** and Massively Parallel Computation.

Edge Query Model and Data Streaming





As long as Q can be **specified in $\tilde{O}(n)$ space**, each round can be implemented in **one pass** of the stream using **semi-streaming space**.

Known Algorithms

1. 0.6-approximation **MBM** in 3-passes [KT17] (see also [KMM12, FKM$^+$05]) – state-of-the-art is 0.611-approximation [FS22].
2. $(1-\epsilon)$-approximation **MBM** in $O(\frac{1}{\epsilon^2})$-passes [ALT21] – current state-of-the-art.

Our Results

Algorithm

0.625-approximation algorithm in **3-rounds** of the deterministic edge-query model.

Implies a **3-pass semi-streaming** algorithm for **MBM** (state-of-the-art – improving on 0.611 [FS22]).

Lower Bounds

There **does not exist** a deterministic algorithm for **MM** (even for **MBM**) in the edge query model that achieves a better than

1. 0.5-approximation in **1 round**,
2. $(0.5 + o(1))$-approximation in **2 rounds**, and
3. $(0.625 + o(1))$-approximation in **3 rounds**.

Algorithm is optimal!

Previous Related Work

Vertex Query Model ([bKK20])

$$\text{Algorithm}\ (A \,\&\, B) \xrightarrow{Q \subseteq (A \cup B)} \text{Oracle (Maximal Matching)} \xrightarrow{M \subseteq G[Q]} \text{Algorithm}$$

# Rounds	Vertex Query	Edge Query
1	0.5	0.5
2	0.5	$0.5 + o(1)$
3	0.6	$0.625 + o(1)$

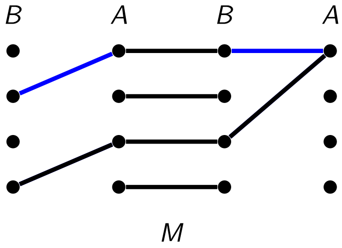
Algorithmic Idea

First Round

Find a maximal matching M in G by querying the complete graph $Q = A \times B$.

Subsequent Rounds

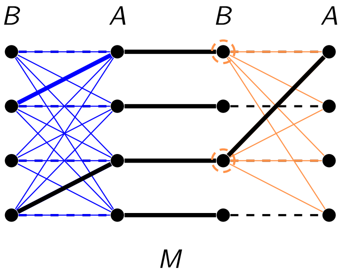
Find vertex-disjoint augmenting paths. **(How?)**



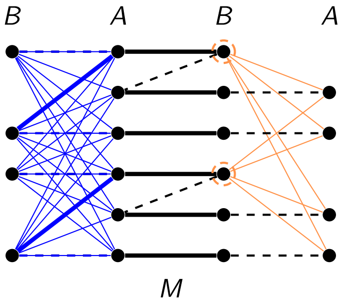
Finding length-3 augmenting paths

Simple Strategy

1. Find left wings
2. Extend with right wings



- $|M| = 0.5 \cdot \mu(G)$
- 0.625-approximation
- Not hard!

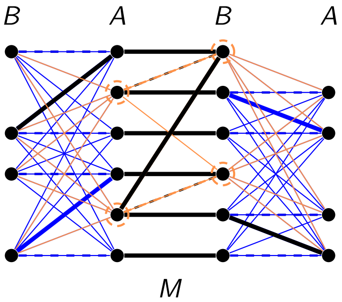


- $|M| = 0.6 \cdot \mu(G)$
- 0.6-approximation
- **Hard instance!**

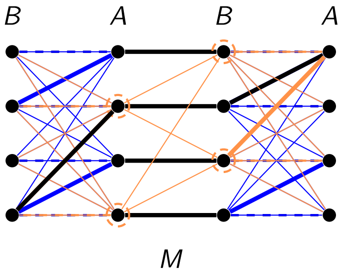
Finding length-3 & length-5 augmenting paths

Our Strategy

1. Find left and right wings
2. Extend paths to either length-3 or length-5 augmenting paths



- $|M| = 0.6 \cdot \mu(G)$
- 0.7-approximation
- Not hard anymore!



- $|M| = 0.5 \cdot \mu(G)$
- 0.625-approximation
- **Hard instance!**

Our Analysis

Main Lemma

Let $|M| = (0.5 + \epsilon) \cdot \mu(G)$ for $\epsilon \geq 0$, then our strategy finds

$$(0.125 - \frac{3}{4}\epsilon) \cdot \mu(G)$$

vertex-disjoint augmenting paths and the large matching found is of size

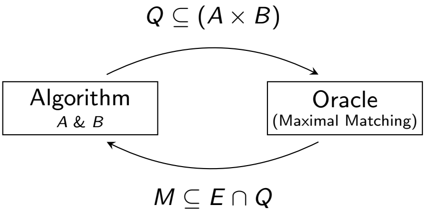
$$(0.625 + \frac{\epsilon}{4}) \cdot \mu(G).$$

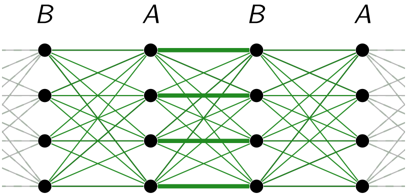
This is tight for our algorithm.

Semi-Streaming

Using Greedy this immediately gives a 3-pass semi-streaming algorithm with the same guarantees.

Lower Bound Idea





Observation

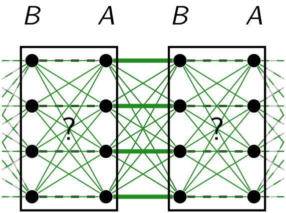
The algorithm learns about **edges** M and **non-edges** N of G.

Main Idea

- Find a hard instance for any sequence of queries $Q_1, Q_2 \ldots$
- For any query Q_i, the information committed is a subset of $\tilde{M}_i$ and $\tilde{N}_i$ (up to isomorphisms)

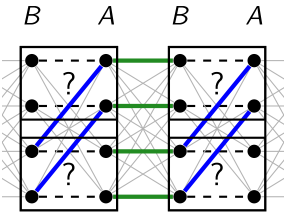
One, Two & Three Query Rounds

One Round



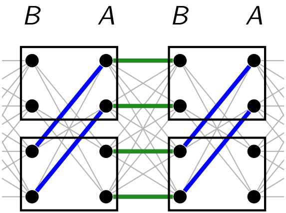
0.5
approx LB

Two Rounds



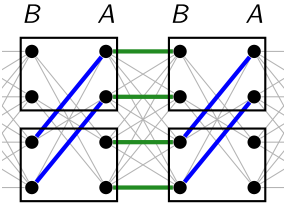
$0.5 + o(1)$
approx LB

Three Rounds

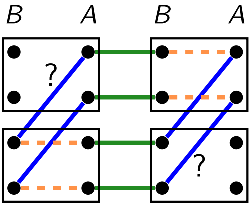


(more involved)

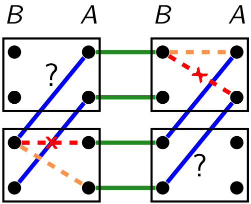
Three Round Proof Sketch I



Part A

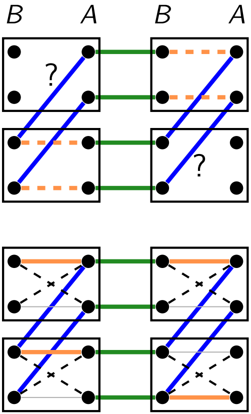


Part B



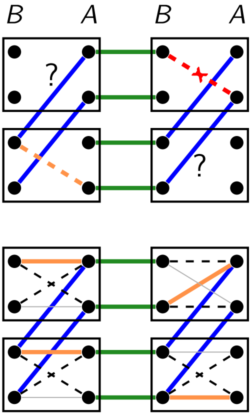
Three Round Proof Sketch II

Part A



$0.625 + o(1)$
approx LB

Part B



$0.625 + o(1)$
approx LB

Summary

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There **does not exist** a deterministic algorithm for **MM** (even for **MBM**) in the edge query model that achieves a better than

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Algorithm

0.625-approximation algorithm in **3-rounds** of the deterministic edge-query model.

Open Questions

Randomisation

Do randomised query algorithms allow us to improve on our results?

Adaptivity

Can we obtain better query algorithms if we allow multiple non-adaptive queries per round?

Semi-Streaming

Is there a 3-pass semi-streaming algorithms for **MBM** that improves on our 0.625-approximation algorithm?

Thank You!

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