Gowers Limit Problem

ImpartialDerivatives

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Problem

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that for every $\theta > 0$, the sequence $f(\theta), f(2\theta), f(3\theta), \ldots$ tends to 0. Prove that $f(x)$ tends to 0 as $x \to \infty$.

Proof

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that $f(x)$ does not tend to 0 as $x \to \infty$. That is, there exists $\varepsilon > 0$ s.t. for all $y \in \mathbb{R}$, there exists $x > y$ s.t. $|f(x)| \geq \varepsilon$. Fix this ε. We can construct (using transfinite recursion with choice) a sequence $x_1, x_2, x_3, \ldots$ of positive reals such that for any $i \in \mathbb{Z}^+$, $|f(x_i)| \geq \varepsilon$ and $x_{i+1} \geq x_i + 1$. (The last condition ensures that the sequence tends to ∞.) Because f is continuous, for any x_i there exists some $\delta > 0$ s.t. for all x with $|x - x_i| < \delta$, $|f(x) - f(x_i)| < \varepsilon/2$. For such an x, $|f(x) - f(x_i)| < \varepsilon/2$ implies $|f(x)| > \varepsilon/2$. So for any x in the closed interval $[x_i - \delta/2, x_i + \delta/2]$, $|f(x)| > \varepsilon/2$. This means we can construct (using the axiom of choice) a sequence $\delta_1, \delta_2, \delta_3, \ldots$ of positive reals such that for any $i \in \mathbb{Z}^+$ and any $x \in [x_i - \delta_i, x_i + \delta_i]$, $|f(x)| > \varepsilon/2$. We have constructed a set of closed intervals within which $|f(x)| > \varepsilon/2$ and which contain arbitrarily large x. (It's not hard to get rid of the axiom of choice here, but it would be a little annoying to write.)

Consider any real a, b with $0 < a < b$. For any integer n greater than $a/(b-a)$, $n > a/(b-a) \Rightarrow n(b-a) > a \Rightarrow nb > na + a \Rightarrow nb > (n+1)a$, so the open intervals (na, nb) and $((n+1)a, (n+1)b)$ overlap. That means that for any integer $N > a/(b-a)$, $\bigcup_{n=N}^{\infty} (na, nb)$ is just the open interval (Na, ∞). The sequence $x_1, x_2, x_3, \ldots$ tends to ∞, so (Na, ∞) contains some x_j. This means there is some integer $n \geq N$ where $x_j \in (na, nb)$. So we have that for any reals a, b with $0 < a < b$ and integer $N > a/(b-a)$, there exists some integer $n \geq N$ and some x_j s.t. $x_j \in (na, nb)$. This is actually true for any positive integer N because lowering N just weakens the condition on n. Thus, for any reals a, b with $0 < a < b$ and positive integer N, there exists some integer $n \geq N$ and some x_j s.t. $x_j \in (na, nb)$.

We construct the sequences $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ of real numbers and $N_0, N_1, N_2, \ldots$ of positive integers inductively as follows. We will construct the sequence so we always have $0 < a_i < b_i$. Start with $a_0 = 1$, $b_0 = 2$, $N_0 = 1$. Now suppose we have constructed a_{i-1}, b_{i-1}, N_{i-1} for some $i \in \mathbb{Z}^+$. By the previous paragraph, there is some integer $n \geq N_{i-1} + 1$ and some x_j s.t. $x_j \in (na_{i-1}, nb_{i-1})$. Let N_i be the smallest integer greater than N_{i-1} s.t. for some x_j, $x_j \in (N_i a_{i-1}, N_i b_{i-1})$. Let x_j be the smallest member of $x_1, x_2, x_3, \ldots$ which is contained in $(N_i a_{i-1}, N_i b_{i-1})$. Because $N_i a_{i-1} < x_j < N_i b_{i-1}$, the intersection of the closed intervals $[N_i a_{i-1}, N_i b_{i-1}]$ and $[x_j - \delta_j, x_j + \delta_j]$ is neither empty nor a single point. (It's also clearly a closed interval.) The same must then be true of $[\frac{N_i a_{i-1}}{N_i}, \frac{N_i b_{i-1}}{N_i}] \cap [\frac{x_j - \delta_j}{N_i}, \frac{x_j + \delta_j}{N_i}]$, which equals $[a_{i-1}, b_{i-1}] \cap [\frac{x_j - \delta_j}{N_i}, \frac{x_j + \delta_j}{N_i}]$. Now define a_i, b_i by

$$[a_i, b_i] := [a_{i-1}, b_{i-1}] \cap \left[\frac{x_j - \delta_j}{N_i}, \frac{x_j + \delta_j}{N_i}\right].$$

We have $0 < a_i < b_i$ as required. Note that for any $x \in [a_i, b_i]$, $x \in [a_i, b_i] \Rightarrow x \in [\frac{x_j - \delta_j}{N_i}, \frac{x_j + \delta_j}{N_i}] \Rightarrow N_i x \in [x_j - \delta_j, x_j + \delta_j] \Rightarrow |f(N_i x)| > \varepsilon/2$.

The sequence $a_0, a_1, a_2, \ldots$ is nondecreasing and bounded above, so $\lim\limits_{n\to\infty} a_n$ exists. $b_0, b_1, b_2, \ldots$ is nonincreasing and bounded below, so $\lim\limits_{n\to\infty} b_n$ exists. $a_i < b_i$ for all i, so $\lim\limits_{n\to\infty} a_n \leq \lim\limits_{n\to\infty} b_n$. Pick any θ with $\lim\limits_{n\to\infty} a_n \leq \theta \leq \lim\limits_{n\to\infty} b_n$. For any $i \in \mathbb{Z}^+$, $\theta \in [a_i, b_i]$, so $|f(N_i\theta)| > \varepsilon/2$. N_i tends to ∞ as $i \to \infty$, so there is no $N \in \mathbb{Z}^+$ s.t. for all integers $n \geq N$, $|f(n\theta)| < \varepsilon/2$. Thus, the sequence $f(\theta), f(2\theta), f(3\theta), \ldots$ does not tend to 0.