(Q) Each front tire on a particular type of vehicle is supposed to be filled to a pressure of 26 psi. Suppose the actual air-pressure in each tire is a random variable - X for the right tire and Y for the left tire, with joint pdf:

$$f(x,y) = \begin{cases} K(x^2+y^2), & x \in [20,30], \; y \in [20,30] \\ 0, & \text{otherwise.} \end{cases}$$

(a) Calculate value of K

$$K \int_{20}^{30} \int_{20}^{30} (x^2+y^2)\, dx\, dy = 1$$

(b) What is the probability that both tires are underfilled

$= P(X \in [20,26], Y \in [20,26])$

$$= \int_{20}^{26} \int_{20}^{26} (K)(x^2+y^2)\, dx\, dy.$$

(c) Probability that difference in air pressure b/w two tires is at most 2 psi?

$= P(|X-Y| \leq 2)$

(d) Determine the marginal distribution of air pressure in right tire

$$= P_X(x) = \int_{y=20}^{y=30} K(x^2+y^2)\, dy.$$

(e) Compute the covariance b/w X and Y.

$$= E(xy) - E(x) \times E(Y)$$

$$= \int_{20}^{30}\int_{20}^{30} xy\, p(x,y)\, dx dy - \left(\int_{20}^{30} x P_X(x)\right)\left(\int_{20}^{30} y P_Y(y)\right)$$

(f) Compute the correlation cofficient r for this X an Y

$$\text{corr} = \frac{\text{cov}(X,Y)}{\sigma_X \sigma_Y}$$

$$\sigma_X^2 = E(X^2) - (E(X))^2 \rightarrow E(X^2) = \int_{20}^{30} x^2 P_X(x)\, dx$$

$$\sigma_Y^2 = E(Y^2) - (E(Y))^2 \rightarrow E(Y^2) = \int_{20} y^2 P_Y(y)\, dy.$$

(g) Determine the conditional pdf of Y given that $X = x$ and the conditional pdf of X given that $Y = y$.

$$P(X/Y) = \frac{p(x,y)}{P_Y(y)}, \quad P(Y/X) = \frac{p(x,y)}{P_X(x)}.$$

(h) If the pressure in the right tire is found to be 22psi, what is the prob that the left tire has a pressure of atleast 25psi?

$$\therefore P(X/Y) = \frac{p(x,y)}{P_Y(y)} = \frac{P(x,22)}{P_Y(22)} = \frac{P(\geq 25, 22)}{P_Y(22)}$$

$$= \frac{P(25,22)}{P_Y(22)} + \frac{P(26,22)}{P_Y(22)} + \dots \frac{P(30,22)}{P_Y(22)} = \int_{25}^{30} \frac{P(x,22)}{P_Y(22)}\, dx.$$

(ii) If the pressure in right tire is found to be 22 psi, what is the expected pressure in left tire, and also calculate the standard deviation of pressure in this tire.

$$E(Y/X=22) = \int (y)\, \frac{p(x,y)}{P_X(x)} = \int_{20}^{30} y\, \frac{p(22,y)}{P_X(22)}\, dy$$

$$\therefore \sigma^2 = E(X^2) - [E(X)]^2$$
$$= E((Y/X)^2) - (E(Y/X))^2$$

$$\therefore E((Y/X)^2) = \int_{20}^{30} y^2\, \frac{p(22,y)}{P_X(22)}\, dy$$

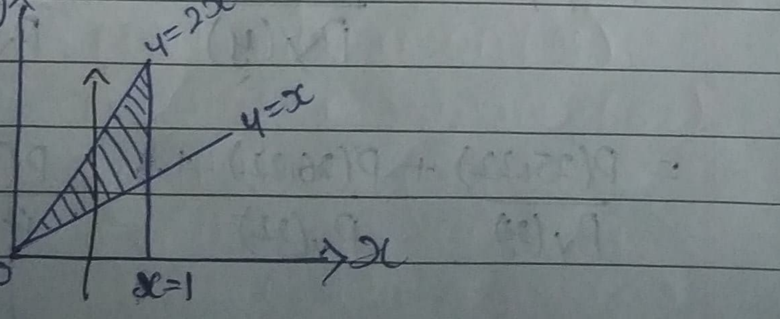
(Q) Let x and y be continuous random variable with joint probability distribution

$$f(x,y) = \begin{cases} 8/3\, xy, & x \in [0,1],\ x \le y \le 2x \\ 0 & \text{otherwise} \end{cases}$$

Calculate $\mathrm{cov}(x,y)$

$\therefore \mathrm{Cov}(x,y) = E(xy) - E(x)\,E(y)$

$$\therefore E(xy) = \int_0^1 \int_x^{2x} (xy)\left(\frac{8}{3} xy\right) dy\, dx.$$



$$\therefore\ f_X(x) = \int_{2x} p(x,y)\,dy$$

$$= \int_{x} (8/3)(xy)\,dy$$

$$\therefore\ E(x) = \int_{0}^{1} (x)\, f_X(x)\,dx$$

$$\therefore\ f_Y(y) = \int_{y} p(x,4)\,dx$$

$$= \int_{y/2} (8/3)(xy)\,dx + \int_{y/2}^{1} (8/3)(xy)\,dx.$$

$$\therefore\ E(y) = \int_{0}^{2} (y)\,(f_Y(y))\,dy$$