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Immersed Boundary Condition Method for solving steady state conductive heat flow in a slot bounded by corrugated walls

by:

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1. Analytical solution for Laplace equation:

We want to find the temperature distribution of a slot bounded with two straight walls. We assume that the temperature distribution is periodic in x-direction with wave length λ. The governing equation for this problem is Laplace equation which is an elliptic partial differential equation. Consider that the upper wall is kept at constant temperature $\theta_u = 0$ while the lower wall has temperature distribution as a periodic function of x (Figure 1).

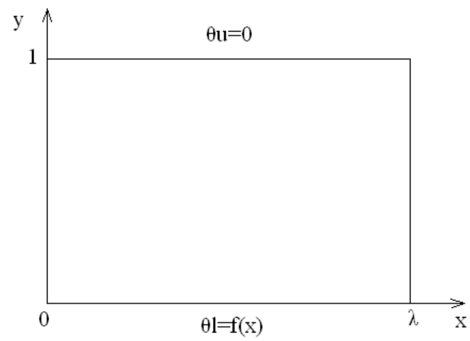

Figure 1. Geometry of the problem

Field equation:

$$\nabla^2 \theta = 0$$

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0 \tag{1}$$

Boundary conditions:

$$\theta(x, y = 1) = 0 \tag{2-a}$$

$$\theta(x, y = 0) = f(x) = \sum_{n=1}^{\infty} L_n \cos(n\alpha x) + K_n \sin(n\alpha x) \tag{2-b}$$

There are two methods for finding the analytical solution for Laplace equation:

- Separation of variables
- Alternative method

1.1. Separation of variables

In this method we assume that the solution has the following form:

$$\theta(x, y) = F(x)G(y)$$

Then substitute above relation into the field equation (1) we get:

$$\frac{\partial^2 F}{\partial x^2} G + \frac{\partial^2 G}{\partial y^2} F = 0$$

Then divide the above equation by *GF*:

$$\frac{\partial^2 F}{\partial x^2}\frac{1}{F} + \frac{\partial^2 G}{\partial y^2}\frac{1}{G} = 0 \tag{3}$$

Now the first term is a function of x only, while the second one is a function of y. It implies that both of these terms should be constant:

$$\frac{\partial^2 F}{\partial x^2}\frac{1}{F} = -\frac{\partial^2 G}{\partial y^2}\frac{1}{G} = \text{constant}$$

There are three possible cases for constant: it can be positive, negative or zero.

a) First case: constant is positive i.e. γ^2

$$\frac{\partial^2 F}{\partial x^2} - \gamma^2 F = 0 \;\rightarrow\; F(x) = c_1 e^{\gamma x} + c_2 e^{-\gamma x}$$

In this case $F(x)$ is not periodic.

b) Second case: constant is zero

$$F(x) = c_1 x + c_2$$

In this case $F(x)$ is not periodic.

c) Third case: constant is negative i.e. $-\gamma^2$

$$\frac{\partial^2 F}{\partial x^2} + \gamma^2 F = 0 \;\rightarrow\; F(x) = c_1 \cos(\gamma x) + c_2 \sin(\gamma x)$$

In this case $F(x)$ is periodic.

In conclusion, the constant should be negative in order to have periodic function for $F(x)$.
For function $G(y)$ we have:

$$\frac{\partial^2 G}{\partial y^2} - \gamma^2 G = 0 \;\rightarrow\; G(y) = c_3 e^{-\gamma y} + c_4 e^{\gamma y}$$

We should choose γ in such way that satisfies our periodicity in the x-direction. The period for sine and cosine functions is 2π, therefore:

$$\cos(\gamma(x+\lambda)) = \cos(\gamma x + 2\pi n) \rightarrow \gamma\lambda = 2\pi n \rightarrow \gamma = n\frac{2\pi}{\lambda}$$

Substitute the above relation for γ into our solution for F:

$$F_n(x) = A_n \cos(n\frac{2\pi}{\lambda}x) + B_n \sin(n\frac{2\pi}{\lambda}x)$$

With using wave number ($\alpha = \frac{2\pi}{\lambda}$) in the above equation we have:

$$\gamma = n\alpha$$

$$\boxed{F_n(x) = A_n \cos(n\alpha x) + B_n \sin(n\alpha x)} \qquad (4)$$

And for G we find:

$$G_n(y) = C'_n e^{-n\alpha y} + D'_n e^{n\alpha y} \quad \text{or} \quad \boxed{G_n(y) = C_n \cosh(n\alpha y) + D_n \sinh(n\alpha y)} \qquad (5)$$

Now let us apply the boundary conditions:
Temperature is zero at the upper wall:

$$\theta(x, y=1) = 0$$

$F_n(x)G_n(1)=0 \quad \rightarrow \quad G_n(1)=0$

$\rightarrow C_n\cosh(n\alpha)+D_n\sinh(n\alpha)=0 \ \rightarrow \ C_n=-D_n\tanh(n\alpha)$

Substitute in equation (5) we get:

$G_n(y)=D_n\left[\sinh(n\alpha y)-\tanh(n\alpha)\cosh(n\alpha y)\right]$

The general solution is the linear combination of all possible solutions:

$$\theta(x,y)=\sum_{n=1}^{\infty}\left[\sinh(n\alpha y)-\tanh(n\alpha)\cosh(n\alpha y)\right]\left[A_nD_n\cos(n\alpha x)+B_nD_n\sin(n\alpha x)\right]$$

This solution should satisfy the other boundary condition $\theta(x,y=0)=f(x)$ as well:

$$\sum_{n=1}^{\infty}\left[-\tanh(n\alpha)\right]\left[A_nD_n\cos(n\alpha x)+B_nD_n\sin(n\alpha x)\right]=\sum_{n=1}^{\infty}L_n\cos(n\alpha x)+K_n\sin(n\alpha x)$$

By equating the coefficients of sine and cosine functions we obtain:

$$L_n=A_nD_n\left[-\tanh(n\alpha)\right] \ \rightarrow \ A_nD_n=-\frac{L_n}{\tanh(n\alpha)}$$

$$K_n=B_nD_n\left[-\tanh(n\alpha)\right] \ \rightarrow \ B_nD_n=-\frac{K_n}{\tanh(n\alpha)}$$

Hence, the total solution is:

$$\boxed{\theta(x,y)=\sum_{n=1}^{\infty}\frac{-1}{\tanh(n\alpha)}\left[\sinh(n\alpha y)-\tanh(n\alpha)\cosh(n\alpha y)\right]\left[L_n\cos(n\alpha x)+K_n\sin(n\alpha x)\right]} \quad (6)$$

1.2. Alternative method

There is another method for finding the analytical solution. As it is obvious from equation (6) we have a Fourier expansion in x-direction. So, we can assume that the solution has the following form:

$$\theta(x,y)=\sum_{n=-\infty}^{\infty}\phi^{\langle n\rangle}(y)e^{in\alpha x} \quad (7)$$

We can reformulate the lower boundary condition function using complex exponentials version of Fourier series:

$$\theta(x,y=0)=f(x)=\sum_{n=1}^{\infty}L_n\cos(n\alpha x)+K_n\sin(n\alpha x)=\sum_{n=-\infty}^{\infty}P^{\langle n\rangle}e^{in\alpha x}$$

where:

$$\begin{cases} L_n=P^{\langle n\rangle}+P^{\langle -n\rangle} & , \ n=1,2,\ldots,\infty \\ K_n=i(P^{\langle n\rangle}+P^{\langle -n\rangle}) & , \ n=1,2,\ldots,\infty \\ P^{(0)}=0 \end{cases}$$

Now, we substitute (7) into the field equation:

$$\sum_{n=-\infty}^{\infty}\left(\frac{d^2\phi^{\langle n\rangle}(y)}{dy^2}-(n\alpha)^2\phi^{\langle n\rangle}(y)\right)e^{in\alpha x}=0$$

For each mode, inside bracket should be zero:

$$\frac{d^2\phi^{\langle n\rangle}(y)}{dy^2}-(n\alpha)^2\phi^{\langle n\rangle}(y)=0$$

$$\phi^{\langle n\rangle}(y)=c_1e^{n\alpha y}+c_2e^{-n\alpha y}\qquad \text{or}\qquad \phi^{\langle n\rangle}(y)=R_n\cosh(n\alpha y)+S_n\sinh(n\alpha y) \tag{8}$$

Then substitute (8) in (7):

$$\theta(x,y)=\sum_{n=-\infty}^{\infty}\left[R_n\cosh(n\alpha y)+S_n\sinh(n\alpha y)\right]e^{in\alpha x} \tag{9}$$

Now, we apply the upper boundary condition:

$$\theta(x,y=1)=0=\sum_{n=-\infty}^{\infty}\left[R_n\cosh(n\alpha)+S_n\sinh(n\alpha)\right]e^{in\alpha x}$$

$$\rightarrow\ R_n\cosh(n\alpha)+S_n\sinh(n\alpha)=0\ \rightarrow\ R_n=-S_n\tanh(n\alpha)$$

By considering above relation for general solution we have:

$$\theta(x,y)=\sum_{n=-\infty}^{\infty}S_n\left[\sinh(n\alpha y)-\tanh(n\alpha)\cosh(n\alpha y)\right]e^{in\alpha x}$$

Then let us apply the lower boundary condition:

$$\theta(x,y=0)=\sum_{n=-\infty}^{\infty}P^{\langle n\rangle}e^{in\alpha x}\ \rightarrow\ \sum_{n=-\infty}^{\infty}S_n\left[-\tanh(n\alpha)\right]e^{in\alpha x}=\sum_{n=-\infty}^{\infty}P^{\langle n\rangle}e^{in\alpha x}$$

It implies that:

$$S_n=-\frac{P^{\langle n\rangle}}{\tanh(n\alpha)}$$

Therefore, the total solution is:

$$\boxed{\theta(x,y)=\sum_{n=-\infty}^{\infty}-\frac{P^{\langle n\rangle}}{\tanh(n\alpha)}\left[\sinh(n\alpha y)-\tanh(n\alpha)\cosh(n\alpha y)\right]e^{in\alpha x}} \tag{10}$$

This is the same thing that we have found in equation (6).

2. Galerkin method for steady state heat conduction through a rectangular plate with straight walled boundaries

Consider we have a plate with straight walled boundaries (Figure 2) and we want to find the temperature distribution of this plate.

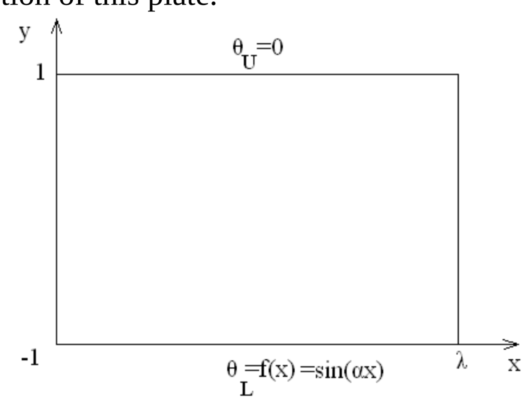

Figure 2

Again the field equation is:

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=0 \tag{11}$$

Boundary conditions:

$$\begin{cases}\theta(x,y=1)=\theta_U=0\\ \theta(x,y=-1)=\theta_L=\sin(\alpha x)\end{cases}$$

2.1. Numerical Solution

We assume that the temperature distribution is periodic in x direction with wave length λ, hence the form of our solution can be defined by Fourier expansion in x-direction:

$$\theta(x,y)=\sum_{n=-\infty}^{\infty}\phi^{\langle n\rangle}(y)e^{in\alpha x} \tag{12}$$

Substitute (12) into (11):

$$\sum_{n=-\infty}^{\infty}-(n\alpha)^2\phi^{\langle n\rangle}(y)e^{in\alpha x}+\sum_{n=-\infty}^{\infty}\frac{d^2\phi^{\langle n\rangle}(y)}{dy^2}e^{in\alpha x}=0$$

$$\sum_{n=-\infty}^{\infty}\left[-(n\alpha)^2\phi^{\langle n\rangle}(y)+\frac{d^2\phi^{\langle n\rangle}(y)}{dy^2}\right]e^{in\alpha x}=0$$

Expanding the above equation leads to:

$$-(n\alpha)^2\phi^{\langle n\rangle}(y)+\frac{d^2\phi^{\langle n\rangle}(y)}{dy^2}=0 \tag{13}$$

Boundary conditions:

$$\begin{cases}\theta(x,y=1)=\theta_U=0 & \rightarrow \quad \sum_{n=-\infty}^{\infty}\phi^{\langle n\rangle}(1)e^{in\alpha x}=0\\ \theta(x,y=-1)=\theta_L=\sin(\alpha x) & \rightarrow \quad \sum_{n=-\infty}^{\infty}\phi^{\langle n\rangle}(-1)e^{in\alpha x}=\sin(\alpha x)\end{cases}$$

We use Chebyshev polynomials to discretize in the y-direction. One of the reasons that we use these polynomials is that they have high resolution near the walls which is desired.

$$\phi^{\langle n\rangle}(y)=\sum_{k=0}^{\infty}G_k^{\langle n\rangle}T_k(y) \tag{14}$$

After that, substitute (14) into (13):

$$-(n\alpha)^2\sum_{k=0}^{\infty}G_k^{\langle n\rangle}T_k(y)+\sum_{k=0}^{\infty}G_k^{\langle n\rangle}\frac{d^2T_k(y)}{dy^2}=R \tag{15}$$

Where R is residual. We should select G_k's in such way (Galerkin method) that the residual becomes zero. We take the inner product of equation (15) by an arbitrary Chebyshev polynomial (T_j). In other words, we want to take the projection of R in each Chebyshev polynomial directions equal to zero.

$$\left\langle T_j, R \right\rangle = -(n\alpha)^2 \sum_{k=0}^{\infty} G_k^{\langle n \rangle} \left\langle T_j, T_k \right\rangle + \sum_{k=0}^{\infty} G_k^{\langle n \rangle} \left\langle T_j, D^2 T_k \right\rangle = 0 \quad , \quad \text{where } D = \frac{d}{dy}$$

Or in compact form:

$$\sum_{k=0}^{\infty} \left[-(n\alpha)^2 \left\langle T_j, T_k \right\rangle + \left\langle T_j, D^2 T_k \right\rangle \right] G_k^{(n)} = 0 \qquad (16)$$

The definition of inner product of two Chebyshev polynomials is:

$$\left\langle T_j, T_k \right\rangle = \int_{-1}^{1} \frac{1}{\sqrt{1-y^2}} T_j(y) T_k(y) dy$$

Because of orthogonality properties of Chebyshev polynomials we have simple solutions for above integral:

$$\left\langle T_j, T_k \right\rangle = \frac{\pi}{2} c_k \delta_{j,k} = \begin{cases} \pi & , \ j = k = 0 \\ \dfrac{\pi}{2} & , \ j = k \geq 1 \\ 0 & , \ j \neq k \end{cases} \quad , \qquad (17)$$

where $\delta_{j,k}$ is Kronecker delta , $c_k = \begin{cases} 2 & , \ k = 0 \\ 1 & , \ k \geq 1 \end{cases}$

There are two ways to compute $\left\langle T_j, D^2 T_k \right\rangle$:

First method: We have following formulas for second derivative of Chebyshev polynomials,

$$D^2 T_0 = 0 \quad , \quad D^2 T_1 = 0 \quad , \quad D^2 T_k = \sum_{r=0}^{k-2} k(k^2 - r^2) \frac{T_r(y)}{c_r} \ \text{ for } k \geq 2 \text{ and } (k-r) = \text{even}$$

Hence,

$$\left\langle T_j, D^2 T_0 \right\rangle = 0$$

$$\left\langle T_j, D^2 T_1 \right\rangle = 0$$

$$\left\langle T_j, D^2 T_k \right\rangle = \sum_{r=0}^{k-2} k(k^2 - r^2) \frac{\left\langle T_j, T_r \right\rangle}{c_r} \quad , \ \text{ for } k \geq 2 \text{ and } (k-r) = \text{even} \qquad (18)$$

By using equation (17) in equation (18) we get:

$$\left\langle T_j, D^2 T_k \right\rangle = \sum_{r=0}^{k-2} k(k^2 - r^2) \frac{\pi}{2} \delta_{j,r} \quad , \ \text{ for } k \geq 2 \text{ and } (k-r) = \text{even}$$

Duo to Kronecker delta property the above summation reduces to one term:

$$\left\langle T_j, D^2 T_k \right\rangle = \begin{cases} \dfrac{\pi}{2} k(k^2 - j^2) & , \ \text{ for } (k-j) = 2, 4, 6, \ldots, \text{even} \\ 0 & , \ \text{ otherwise} \end{cases} \qquad (19)$$

Second method: is based on the following formulas:

$$D^2 T_{2k} = 8k \sum_{r=1}^{k-1} \left(k^2 - r^2 \right) T_{2r} + 4k^3 T_0$$

$$D^2T_{2k+1} = 4(2k+1)\sum_{r=0}^{k-1}\left[(k+r+1)(k-r)\right]T_{2r+1}$$

With these formulas we have:

$$\begin{cases} \left\langle T_j, D^2T_{2k}\right\rangle = 8k\sum_{r=1}^{k-1}\left(k^2-r^2\right)\left\langle T_j, T_{2r}\right\rangle + 4k^3\left\langle T_j, T_0\right\rangle \\ \left\langle T_j, D^2T_{2k+1}\right\rangle = 4(2k+1)\sum_{r=0}^{k-1}\left[(k+r+1)(k-r)\right]\left\langle T_j, T_{2r+1}\right\rangle \end{cases} \tag{20}$$

First method is both easier to program and considerably faster than second method.

Now, we want to apply the boundary conditions:

$$\begin{cases} \theta(x, y=1) = \theta_U = 0 \quad \rightarrow \quad \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty} G_k^{\langle n\rangle} T_k(1) e^{in\alpha x} = \theta_U = 0 \\ \theta(x, y=-1) = \theta_L = \sin(\alpha x) \quad \rightarrow \quad \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty} G_k^{\langle n\rangle} T_k(-1) e^{in\alpha x} = \theta_L = \sin(\alpha x) \end{cases} \tag{21}$$

From Chebyshev properties we know:

$$\begin{cases} T_k(1) = 1 \\ T_k(-1) = (-1)^k \end{cases}$$

By substituting the above properties into (21) and using Fourier expansion for boundary conditions, we get:

$$\begin{cases} \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty} G_k^{\langle n\rangle} e^{in\alpha x} = \theta_U \quad \rightarrow \quad \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty} G_k^{\langle n\rangle} e^{in\alpha x} = \sum_{n=-\infty}^{\infty} Q^{\langle n\rangle} e^{in\alpha x} \\ \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty} G_k^{\langle n\rangle} (-1)^k e^{in\alpha x} = \theta_L \quad \rightarrow \quad \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty} G_k^{\langle n\rangle} (-1)^k e^{in\alpha x} = \sum_{n=-\infty}^{\infty} P^{\langle n\rangle} e^{in\alpha x} \end{cases}$$

Therefore,

$$\begin{cases} \sum_{k=0}^{\infty} G_k^{\langle n\rangle} = Q^{\langle n\rangle} \quad , \quad Q^{\langle 0\rangle} = \theta_U = 0 \;\; , \;\; Q^{\langle n\rangle} = 0 \;\text{ for } n \neq 0 \\ \sum_{k=0}^{\infty} G_k^{\langle n\rangle} (-1)^k = P^{\langle n\rangle} \;\; , \; \sin(\alpha x) = \dfrac{e^{i\alpha x} - e^{-i\alpha x}}{2i} \;\rightarrow\; P^{\langle -1\rangle} = \dfrac{-1}{2i} \;, P^{\langle 1\rangle} = \dfrac{1}{2i} \;,\; P^{\langle n\rangle} = 0 \text{ for } n \neq -1\&1 \end{cases}$$

We can not program the above summations for infinite number of terms, so we truncate the summation for Chebyshev polynomials to N_T which is the number of Chebyshev polynomials and for Fourier modes to N_M which is the number of Fourier modes. So equation (16) becomes:

$$\sum_{k=0}^{N_T-1}\left[-(n\alpha)^2\left\langle T_j,T_k\right\rangle+\left\langle T_j,D^2T_k\right\rangle\right]G_k^{\langle n\rangle}=0 \tag{22}$$

Now we can introduce the following matrices:

$$E_{j,k}=\left\langle T_j,T_k\right\rangle \quad , \quad D_{j,k}=\left\langle T_j,D^2T_k\right\rangle$$

Then equation (22) becomes:

$$\sum_{k=0}^{N_T-1}\left[-(n\alpha)^2E_{j,k}+D_{j,k}\right]G_k^{\langle n\rangle}=0 \quad , \qquad j=0,1,2,\ldots,N_T-1 \tag{23}$$

In order to apply our two boundary conditions to the system of equations of each mode and have a closed system of equations, we throw away the last two equations of (23), and use only the first (N_T-2) equations from equations (23) and the two other conditions come from the boundary conditions.
Thus,

$$\begin{cases}\displaystyle\sum_{k=0}^{N_T-1}\left[-(n\alpha)^2E_{j,k}+D_{j,k}\right]G_k^{\langle n\rangle}=0 \quad , \qquad j=0,1,2,...,N_T-3\\ \displaystyle\sum_{k=0}^{N_T-1}G_k^{\langle n\rangle}=Q^{\langle n\rangle}\\ \displaystyle\sum_{k=0}^{N_T-1}G_k^{\langle n\rangle}(-1)^k=P^{\langle n\rangle}\end{cases} \tag{24}$$

The above system of equations in matrix notation for mode $n\in(-N_M,N_M)$ becomes:

$$A^{\langle n\rangle}G^{\langle n\rangle}=B^{\langle n\rangle} \tag{25}$$

Vector $G^{\langle n\rangle}$ has coefficients of Chebyshev polynomials for mode n which are unknowns and matrices $A^{\langle n\rangle},B^{\langle n\rangle}$ are as follows:

$$A^{\langle n\rangle}=\begin{bmatrix}-(n\alpha)^2E_{0,0}+D_{0,0} & -(n\alpha)^2E_{0,1}+D_{0,1} & \cdots & -(n\alpha)^2E_{0,N_T-1}+D_{0,N_T-1}\\ -(n\alpha)^2E_{1,0}+D_{1,0} & -(n\alpha)^2E_{1,1}+D_{1,1} & \cdots & -(n\alpha)^2E_{1,N_T-1}+D_{1,N_T-1}\\ \vdots & \vdots & \vdots & \vdots\\ -(n\alpha)^2E_{N_T-3,0}+D_{N_T-3,0} & -(n\alpha)^2E_{N_T-3,1}+D_{N_T-3,1} & \cdots & -(n\alpha)^2E_{N_T-3,N_T-1}+D_{N_T-3,N_T-1}\\ 1 & 1 & \cdots & 1\\ 1 & -1 & \cdots & (-1)^{N_T-1}\end{bmatrix}$$

$$B^{\langle n\rangle}=\begin{bmatrix}0\\0\\\vdots\\0\\Q^{\langle n\rangle}\\P^{\langle n\rangle}\end{bmatrix}_{N_T\times 1}\qquad \text{where}:\begin{cases}\text{for upper boundary } \theta_U=\sum_{n=-\infty}^{\infty}Q^{\langle n\rangle}e^{in\alpha x}=0\rightarrow Q^{\langle n\rangle}=0\\ \text{for lower boundary } \theta_L=\sum_{n=-\infty}^{\infty}P^{\langle n\rangle}e^{in\alpha x}=\sin(\alpha x)=\dfrac{e^{i\alpha x}-e^{-i\alpha x}}{2i}\\ \rightarrow\ P^{\langle -1\rangle}=\dfrac{-1}{2i}=0.5i\ ,\ P^{\langle 1\rangle}=\dfrac{1}{2i}=-0.5i\ ,\ P^{\langle n\rangle}=0 \text{ for } n\neq -1\,\&\, n\neq 1\end{cases}$$

Now we want to put all matrices $A^{\langle n\rangle}$ in the main diagonal of a big matrix and call this matrix as *M*.

$$M=\begin{bmatrix}A^{\langle -N_M\rangle} & & & 0\\ & A^{\langle -N_M+1\rangle} & & \\ & & \ddots & \\ 0 & & & A^{\langle N_M\rangle}\end{bmatrix}$$

In fact we want to solve a system of equations which contains all of the modes. In this problem which the boundaries are straight walls, all of the modes are decoupled and there is no need to solve them together, but we arrange our matrix equation in this way so it makes our job easier for next step when we want to solve for corrugated boundaries. So the matrix equation becomes:

$$M\begin{bmatrix}G^{(-N_M)}\\G^{(-N_M+1)}\\\vdots\\G^{(N_M)}\end{bmatrix}=\begin{bmatrix}B^{(-N_M)}\\B^{(-N_M+1)}\\\vdots\\B^{(N_M)}\end{bmatrix}\tag{26}$$

2.2. Analytical Solution

We start from equation (9):

$$\theta(x,y)=\sum_{n=-\infty}^{\infty}\left[R_n cosh(n\alpha y)+S_n\sinh(n\alpha y)\right]e^{in\alpha x}$$

Upper wall boundary condition:

$$\theta(x,y=1)=0\ \ \rightarrow\sum_{n=-\infty}^{\infty}\left[R_n cosh(n\alpha)+S_n\sinh(n\alpha)\right]e^{in\alpha x}=0$$

$$\rightarrow R_n=-S_n\tanh(n\alpha)$$

Substitution the above relation into the general solution gives:

$$\theta(x,y)=\sum_{n=-\infty}^{\infty}S_n\left[\sinh(n\alpha y)-tanh(n\alpha)cosh(n\alpha y)\right]e^{in\alpha x}$$

Lower wall boundary condition:

$$\theta(x, y=-1) = \sin(\alpha x)$$

$$\rightarrow \sum_{n=-\infty}^{\infty} P^{\langle n\rangle} e^{in\alpha x} = \sum_{n=-\infty}^{\infty} S_n \left[\sinh(-n\alpha) - tanh(n\alpha)cosh(-n\alpha)\right] e^{in\alpha x}$$

$$P^{\langle n\rangle} = S_n \left[-\sinh(n\alpha) - tanh(n\alpha)cosh(n\alpha)\right]$$

$$P^{\langle n\rangle} = S_n \left[-2\sinh(n\alpha)\right] \;\; \rightarrow \boxed{S_n = -\frac{P^{\langle n\rangle}}{2\sinh(n\alpha)}}$$

Finally, the analytical solution for this problem is:

$$\boxed{\theta(x,y) = \sum_{n=-\infty}^{\infty} -\frac{P^{\langle n\rangle}}{2\sinh(n\alpha)}\left[\sinh(n\alpha y) - tanh(n\alpha)cosh(n\alpha y)\right] e^{in\alpha x}} \tag{27}$$

Where:

$P_1 = -0.5i$, $P_{-1} = 0.5i$, $P_n = 0$ for other values of n

2.3. Numerical Results

The temperature distribution for a 2-D steady state heat conduction problem for $N_T = 70$, $N_M = 17$ and $\alpha = 2$ is shown in Figure 3:

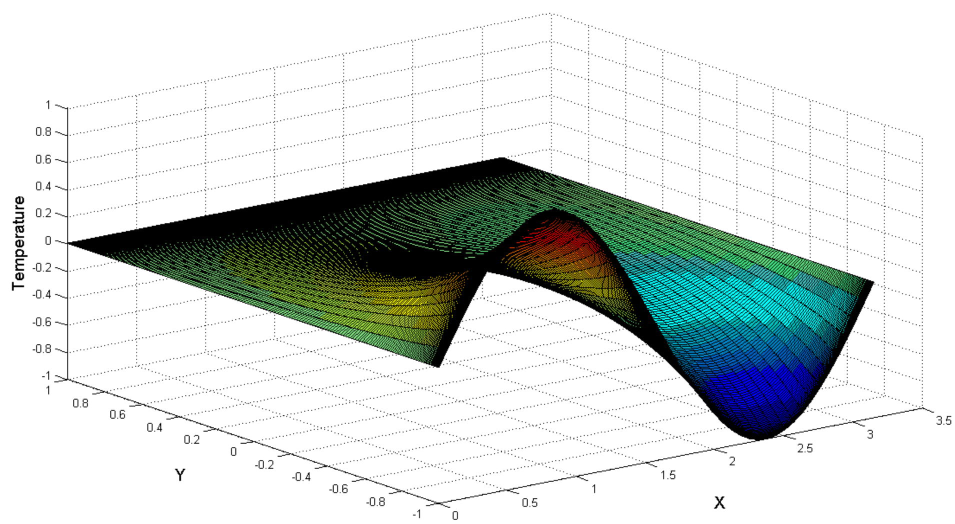

Figure 3. Temperature distribution.

With these values for N_T , N_M the error of numerical solution in comparison with analytical solution is:

error =4.440892098500626e-016

time = 4.53224378372633

3. Immersed boundary condition method for steady state heat conduction through a rectangular plate with corrugated walled boundaries

Immersed boundary condition method uses a fixed computational domain with the physical domain immersed inside the computational domain. This method obtained its name because this submersion of boundary conditions inside computational domain. The advantage of this method is that there is no need to conform grid for boundaries. In this method the physical boundary conditions are forced as constraints on the field equations [1].

The Chebyshev polynomials can be applied for discretization in y-direction, but the domain should be [-1,1]. In cases that we have corrugated walls, all of our domain may not lie within [-1,1], we should transform our domain in such way that it lies into [-1,1]. In other words, we should map from physical coordinates (x, y) to computational coordinates $(x, \hat{y})$ with the following transformation:

$$\hat{y} = 2\left[\frac{y-(1+y_t)}{y_t + y_b + 2}\right]+1 \tag{28}$$

where y_t, y_b are extremities of domain for top and bottom walls respectively (Figure 4).

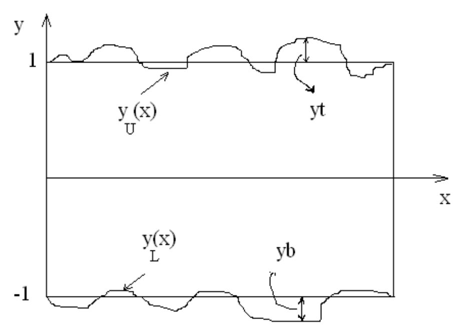

Figure 4. Physical domain with corrugated boundaries.

Now with transformation (28) we have $-1 < \hat{y} < 1$.
Again the governing equation for this problem is:

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=0$$

Boundary conditions:

$$\begin{cases} \theta(x, y_U(x)) = \theta_U = \text{const.} \\ \theta(x, y_L(x)) = \theta_L = \text{const.} \end{cases} \tag{29}$$

Walls geometries:

$$\begin{cases} y_U(x) = 1 + \sum_{n=-\infty}^{\infty} H_U^{\langle n\rangle} e^{in\alpha x} \\ y_L(x) = -1 + \sum_{n=-\infty}^{\infty} H_L^{\langle n\rangle} e^{in\alpha x} \end{cases} \tag{30}$$

Now, we should express the field equation in $(x,\hat{y})$ system:

$$\frac{\partial \hat{y}}{\partial y}=\frac{2}{y_t+y_b+2}=\Gamma$$

$$\frac{\partial \theta}{\partial y}=\frac{\partial \theta}{\partial \hat{y}}\frac{\partial \hat{y}}{\partial y}=\Gamma\frac{\partial \theta}{\partial \hat{y}}$$

$$\frac{\partial^2 \theta}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial \theta}{\partial y}\right)=\frac{\partial}{\partial \hat{y}}\frac{\partial \hat{y}}{\partial y}\left(\Gamma\frac{\partial \theta}{\partial \hat{y}}\right)=\Gamma^2\frac{\partial^2 \theta}{\partial \hat{y}^2}$$

Hence, the governing equation becomes:

$$\frac{\partial^2 \theta}{\partial x^2}+\Gamma^2\frac{\partial^2 \theta}{\partial \hat{y}^2}=0 \qquad (31)$$

We shall use Fourier expansion in x-direction (because we assume periodicity in the direction of x):

$$\theta(x,\hat{y})=\sum_{n=-\infty}^{\infty}\phi^{\langle n\rangle}(\hat{y})e^{in\alpha x} \qquad (32)$$

Then, substitute (32) into (31) we get:

$$\sum_{n=-\infty}^{\infty}-(n\alpha)^2\phi^{\langle n\rangle}(\hat{y})e^{in\alpha x}+\Gamma^2\sum_{n=-\infty}^{\infty}\frac{d^2\phi^{\langle n\rangle}(\hat{y})}{d\hat{y}^2}e^{in\alpha x}=0$$

$$\sum_{n=-\infty}^{\infty}\left(\Gamma^2 D^2-(n\alpha)^2\right)\phi^{\langle n\rangle}(\hat{y})e^{in\alpha x}=0 \quad , \quad \text{where: } D=\frac{d}{d\hat{y}}$$

which yields to:

$$\left[\Gamma^2 D^2-(n\alpha)^2\right]\phi^{\langle n\rangle}=0 \qquad (33)$$

Now we can use Chebyshev polynomials for discretization in $\hat{y}$-direction because now $\hat{y}\in(-1,1)$:

$$\phi^{\langle n\rangle}(\hat{y})=\sum_{k=0}^{\infty}G_k^{\langle n\rangle}T_k(\hat{y}) \qquad (34)$$

Then substitute (34) into (33):

$$\left[\Gamma^2 D^2-(n\alpha)^2\right]\sum_{k=0}^{\infty}G_k^{\langle n\rangle}T_k(\hat{y})=R \quad , \qquad (35)$$

where R is residual. We should select G_k 's in such way that the residual becomes zero.

We take the inner product of equation (35) by an arbitrary Chebyshev polynomial (T_j). In other words, we want to take the projection of R in each Chebyshev polynomial directions equal to zero and as the result, enforce R to be zero.

$$\sum_{k=0}^{\infty}\left[-(n\alpha)^2\left\langle T_j,T_k\right\rangle+\Gamma^2\left\langle T_j,D^2T_k\right\rangle\right]G_k^{\langle n\rangle}=\left\langle T_j,R\right\rangle=0 \qquad (36)$$

In order to evaluate the above sum numerically we should truncate the sum to finite number of terms. We use N_T number of Chebyshev polynomials to evaluate (36).

$$\sum_{k=0}^{N_T-1}\left[-(n\alpha)^2\left\langle T_j,T_k\right\rangle+\Gamma^2\left\langle T_j,D^2T_k\right\rangle\right]G_k^{\langle n\rangle}=0 \tag{37}$$

We should define the corrugated boundaries in the $(x,\hat{y})$ plane. For this purpose we substitute equations (30) into (28) that results in:

$$\begin{cases}\hat{y}_U=1-\Gamma y_t+\Gamma\sum_{n=-\infty}^{\infty}H_U^{\langle n\rangle}e^{in\alpha x}\\ \hat{y}_L=1-\Gamma(2+y_t)+\Gamma\sum_{n=-\infty}^{\infty}H_L^{\langle n\rangle}e^{in\alpha x}\end{cases} \tag{38}$$

We can reformulate the above equations in a more compact form as follows:

$$\begin{cases}\hat{y}_U=\sum_{n=-\infty}^{\infty}A_U^{\langle n\rangle}e^{in\alpha x} \quad , \quad \text{where} \quad A_U^{\langle 0\rangle}=1+\Gamma\left[-y_t+H_U^{\langle 0\rangle}\right] \quad , \quad A_U^{\langle n\rangle}=\Gamma H_U^{\langle n\rangle} \ \text{ for } n\neq 0\\ \hat{y}_L=\sum_{n=-\infty}^{\infty}A_L^{\langle n\rangle}e^{in\alpha x} \quad , \quad \text{where} \quad A_L^{\langle 0\rangle}=1+\Gamma\left[-2-y_t+H_L^{\langle 0\rangle}\right] \ , \ A_L^{\langle 0\rangle}=\Gamma H_L^{\langle n\rangle} \ \text{ for } n\neq 0\end{cases} \tag{39}$$

Now we want to use our assumed form of θ at boundaries. Consider a boundary which is a periodic function of x. So we can use Fourier expansions for this function:

$$\hat{y}=f(x)=\sum_{n=-\infty}^{\infty}A^{\langle n\rangle}e^{in\alpha x} \tag{40}$$

Then we use the assumed form of θ at this boundary:

$$\theta(x,f(x))=\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}G_k^{\langle n\rangle}T_k(f(x))e^{in\alpha x} \tag{41}$$

We should evaluate Chebyshev polynomials at the boundary, but the boundary is a periodic function in x-direction, so the values of Chebyshev polynomials are periodic and we can use Fourier expansion for them:

$$T_k(f(x))=\sum_{j=-\infty}^{\infty}w_k^{\langle j\rangle}e^{ij\alpha x} \tag{42}$$

Now we should evaluate $w_k^{\langle j\rangle}$ for different Chebyshev polynomials:

$$T_0=1\rightarrow\sum_{j=-\infty}^{\infty}w_0^{\langle j\rangle}e^{ij\alpha x}=1\rightarrow\begin{cases}w_0^{\langle 0\rangle}=1\\ w_0^{\langle j\rangle}=0\end{cases}$$

$$T_1(f(x))=f(x)\rightarrow\sum_{j=-\infty}^{\infty}w_1^{\langle j\rangle}e^{ij\alpha x}=\sum_{j=-\infty}^{\infty}A^{\langle j\rangle}e^{ij\alpha x} \ \Rightarrow w_1^{\langle j\rangle}=A^{\langle j\rangle}$$

For finding the remaining $w_k^{\langle j\rangle}$ we use the following Chebyshev polynomials recursive formula:

$$T_{k+1}(f(x))=2f(x)T_k(f(x))-T_{k-1}(f(x)) \tag{43}$$

Substituting (42) and (40) into (43):

$$\sum_{j=-\infty}^{\infty} w_{k+1}^{\langle j\rangle} e^{ij\alpha x} = 2\sum_{n=-\infty}^{\infty}\sum_{s=-\infty}^{\infty} A^{\langle n\rangle} w_k^{\langle s\rangle} e^{i\langle n+s\rangle \alpha x} - \sum_{j=-\infty}^{\infty} w_{k-1}^{\langle j\rangle} e^{ij\alpha x}$$

Changing the dummy indices as follows, give:

$n+s=j \rightarrow s=j-n$

$j-n=-\infty \ , \ j=n-\infty=-\infty$

$$\sum_{j=-\infty}^{\infty} w_{k+1}^{\langle j\rangle} e^{ij\alpha x} = 2\sum_{n=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} A^{\langle n\rangle} w_k^{\langle j-n\rangle} e^{ij\alpha x} - \sum_{j=-\infty}^{\infty} w_{k-1}^{\langle j\rangle} e^{ij\alpha x}$$

Then by expanding the above sum, we end up with the following recursive formula:

$$w_{k+1}^{\langle j\rangle} = 2\sum_{n=-\infty}^{\infty} A^{\langle n\rangle} w_k^{\langle j-n\rangle} - w_{k-1}^{\langle j\rangle} \tag{44}$$

Now we substitute (42) in (41) and again play with the dummy indices:

$$\theta_{\text{boundary}} = \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty} G_k^{\langle n\rangle}\left[\sum_{s=-\infty}^{\infty} w_k^{\langle s\rangle} e^{i(n+s)\alpha x}\right]$$

$n+s=j \rightarrow s=j-n$

$j-n=\infty \ , \ j=n-\infty=-\infty$

And finally we have the following formula for temperature distribution at boundary:

$$\theta_{\text{boundary}} = \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\sum_{j=-\infty}^{\infty} G_k^{\langle n\rangle} w_k^{\langle j-n\rangle} e^{ij\alpha x} \tag{45}$$

On the other hand, we can use a Fourier expansion for temperature distribution at boundary:

$$\theta_{\text{boundary}} = \sum_{j=-\infty}^{\infty} \Psi^{\langle j\rangle} e^{ij\alpha x} \tag{46}$$

Then, by equating (46) and (45) we have:

$$\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\sum_{j=-\infty}^{\infty} G_k^{\langle n\rangle} w_k^{\langle j-n\rangle} e^{ij\alpha x} = \sum_{j=-\infty}^{\infty} \Psi^{\langle j\rangle} e^{ij\alpha x}$$

$$\rightarrow \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty} G_k^{\langle n\rangle} w_k^{\langle j-n\rangle} = \Psi^{\langle j\rangle}$$

then rename $n \leftrightarrow j$:

$$\sum_{j=-\infty}^{\infty}\sum_{k=0}^{\infty} G_k^{\langle j\rangle} w_k^{\langle n-j\rangle} = \Psi^{\langle n\rangle}$$

For constant temperatures at boundaries:

$\Psi^{\langle 0\rangle} = \theta_{\text{boundary}} \ , \ \Psi^{\langle n\rangle} = 0 \ \text{ if } \ n \neq 0$

$$\rightarrow \sum_{j=-\infty}^{\infty}\sum_{k=0}^{\infty} G_k^{\langle j\rangle} w_k^{\langle n-j\rangle} = \begin{cases} \theta_{\text{boundary}} & if \quad n=0 \\ 0 & if \quad n\neq 0 \end{cases} \tag{47}$$

In matrix format:

We replace two equations per Fourier mode by boundary equations as follows:

$$\sum_{k=0}^{N_T-1}\left[-(n\alpha)^2 E_{j,k}+\Gamma^2 D_{j,k}\right]G_k^{\langle n\rangle}=0 \qquad j=0,1,2,........,N_T-3 \tag{48}$$

$$\sum_{j=-N_m}^{N_m}\sum_{k=0}^{N_T-1}G_k^{\langle j\rangle}w_k^{\langle n-j\rangle}=\begin{cases}\theta_U & if \quad n=0\\ 0 & if \quad n\neq 0\end{cases} \tag{49}$$

$$\sum_{j=-N_m}^{N_m}\sum_{k=0}^{N_T-1}G_k^{\langle j\rangle}w_k^{\langle n-j\rangle}=\begin{cases}\theta_L & if \quad n=0\\ 0 & if \quad n\neq 0\end{cases} \tag{50}$$

As it is obvious from (49) and (50) the system of equations are coupled. So we should solve for all modes simultaneously and construct a big matrix that contains the coefficients for all modes of $G_k^{\langle n\rangle}$.

Numerical result:

1)

I used these following values:

$N_T = 17$, $N_M = 7$, $\alpha = 2$

upper wall geometry: $y_U(x) = 1 + 0.01\sin(\alpha x)$

lower wall geometry: $y_L(x) = -1 + 0.01\cos(\alpha x)$

temperature at upper boundary: $\theta_U = 0$

temperature at upper boundary: $\theta_L = 1$

Figure 5 is the temperature distribution for $0 < x < 4\pi$ and $-1 < \hat{y} < 1$. It shows a linear temperature difference between two walls which is what we have expected.

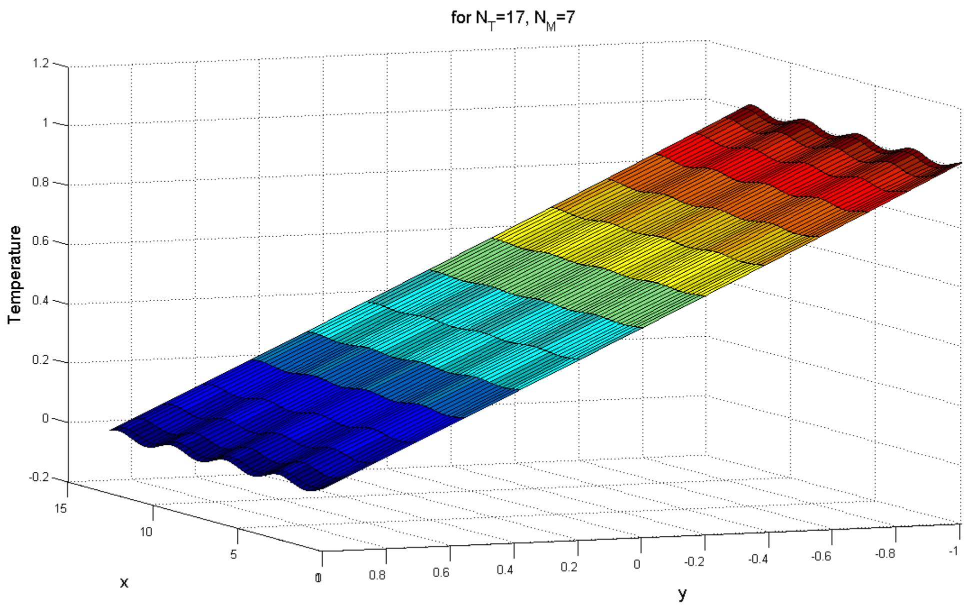

Figure 5. Temperature distribution

2)

I used these following values:

$N_T = 71$, $N_M = 15$, $\alpha = 1$

upper wall geometry: $y_U(x) = 1 + 0.3\sin(\alpha x)$

lower wall geometry: $y_L(x) = -1$

temperature at upper boundary: $\theta_U = 0$

temperature at upper boundary: $\theta_L = 1$

total_time = 11.99993822181370

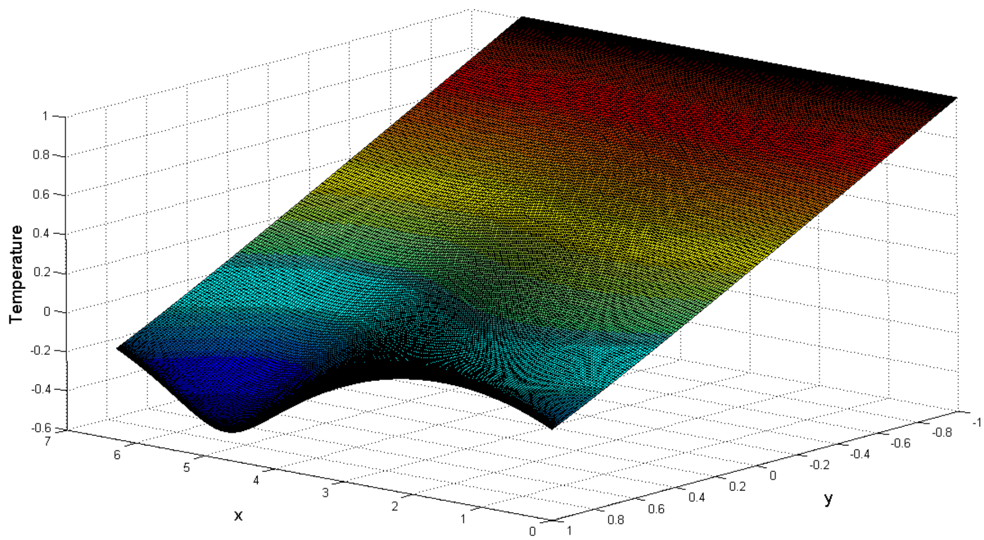

Figure 6- surface of temperature

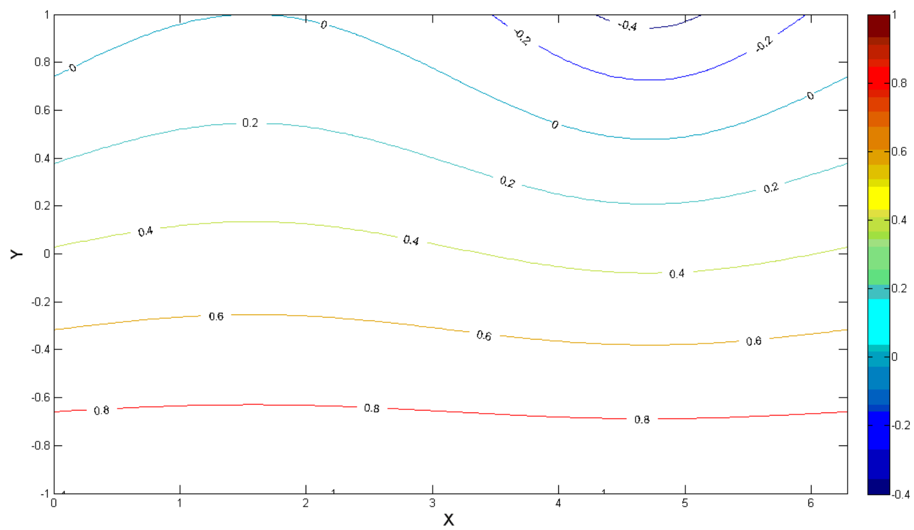

Figure 7- contour of temperature

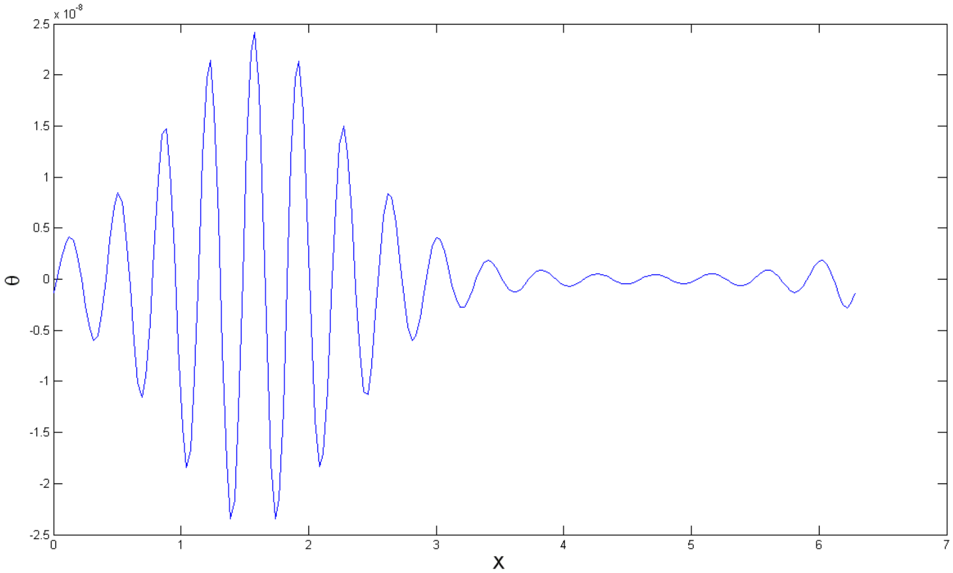

Figure 8- temperature distribution at the upper wall

Reference

[1] S.Z. Husain and J.M. Floryan. Immersed boundary conditions method for unsteady flow problems described by Laplace operator. *Department of Mechanical and Materials Engineering the University of Western Ontario*, 2006.