EM 314 – NUMERICAL METHODS

ASSIGNMENT 02

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E/15/220
SEMESTER 04
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01) Bisection method satisfies,

$$e_k = |x_k - x_*| \leq \frac{b-a}{2^{k+1}}$$

for $e_k < T$

$$\frac{b-a}{2^{k+1}} < T$$

$$\frac{b-a}{T} < 2^{k+1}$$

$$\log_2\left(\frac{b-a}{T}\right) < k+1$$

$$k > \log_2\left(\frac{b-a}{T}\right) - 1$$

02) a) $g(x) = e^{-x}$

$g'(x) = -e^{-x} \leq 0$

$\therefore g(x)$ is a monotonically decreasing function,

$$|g'(x)| = e^{-x} \leq e^{-\ln 1.1} = \frac{1}{1.1} = 0.9091 < 1$$

$\therefore g$ is a contraction on G

b) $G = [\ln 1.1, \ln 3] = [0.0953, 1.0986]$

$g[\ln(1.1)] = e^{-\ln 1.1} = 0.9091$

$g[\ln(3)] = e^{-\ln 3} = 0.3333$

Also $g(x)$ is monotonically decreasing function

~~$\therefore g$~~

$\therefore g : G \rightarrow [g(\ln 3), g(\ln 1.1)] \subset G$

c) G is a closed interval
g is a contraction that maps G in to itself.
so, according to Banach fixed point theorem
G contains a unique fixed point x_* for
any $x_0 \in G$ (Existence and uniqueness)

d) $x_1 = g(x_0) = e^{-0.5} = 0.60653$

$x_2 = g(x_1) = e^{-0.60653} = 0.15976$

Using Mean value theorem:

e) $|x_k - x_*| = |g(x_{k-1}) - g(x_*)|$

$= |g'(\xi)|\,|x_{k-1} - x_*| \quad , \xi \in (\ln 1.1, \ln 3)$

$\leq L\,|x_{k-1} - x_*|$

so,

$\cancel{|x_k - x_*| \leq L\, x_{k-1}, x_*}$

$|x_k - x_*| \leq L\,|x_{k-1} - x_*| \leq L^2\,|x_{k-2} - x_*|$

$\leq L^3\,|x_{k-3} - x_*|$

$\cdots$

$\leq L^n\,|x_0 - x_*|$

$\lim_{n \to \infty} |x_n - x_*| \leq \lim_{n \to \infty} L^n\,|x_0 - x_*|$

$= 0$ // $[\because 0 < L < 1, \lim_{n \to \infty} L^n = 0]$

That mean

$x_{k+1} = g(x_k)$ converge to the unique fixed
Point $x_* \in G$ for any $x_0 \in G$

Q3) a) $g(x) = \tan^{-1}(2x)$

$g'(x) = \frac{2}{1+4x^2}$

Let $x_k \in [-\frac{1}{2}, \frac{1}{2}]$

$$e_{k+1} = |x_{k+1} - 0|$$
$$= |g(x_k) - g(0)|$$
$$= |g'(\xi)|\,|x_k - 0|$$
$$= |g'(\xi)|\, e_k$$

Let $\xi \in (-0.5, 0.5)$

for these interval $|g'(x)| > 1$, $\forall x \in (-0.5, 0.5)$

Because of that $e_{k+1} > e_k$

Therefore the fixed point iteration will not converge to $x_* = 0$.

b) i)

$\cancel{x_0 = 2}$

$\cancel{x_1 = g(x_0) = \tan^{-1}(4) =}$

$$\boxed{\begin{gathered} f(x) = g(x) - x = \tan^{-1}(2x) - x \\ x = \tan^{-1}(2x) \end{gathered}}$$

$\cancel{x_1 = f(x_0) = \tan^{-1}(4) - 2 =}$

$\cancel{x_0 = 2}$ $\quad \cancel{\tan^{-1}(2\times 2) = 75.964}$

$\cancel{x_1 = 75.96}$ $\quad \cancel{\tan^{-1}(2\times 75.96) = 89.62}$

$x_0 = 2 \quad \tan^{-1}(2\times 2) = 1.3258$

I/ $x_1 = 1.3258 \quad \tan^{-1}(2\times 1.3258) = 1.210$

$e_1 = |x_1 - x_*| = (1.33 - 1.16) = 0.17$

II/ $x_2 = 1.210 \quad \tan^{-1}(2\times 1.210) = 1.179$

$e_2 = |x_2 - x_*| = (1.21 - 1.16) = 0.05$

Using Newton's Method

(ii) $f(x) = \tan^{-1}(2x) - x$

$$f'(x) = \frac{2}{1+4x^2} - 1 = \frac{1-4x^2}{1+4x^2}$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$x_{k+1} = x_k - \frac{\left[\tan^{-1}(2x_k) - x_k\right]\left(1+4x_k^2\right]}{\left[1-4x_k^2\right]}$$

using $x_0 = 2$,

I/ $x_1 = 1.24$/

$e_1 = |x_1 - x_*| = (1.24 - 1.16) = 0.08$

II/ $x_2 = 1.17$/

$e_2 = |x_2 - x_*| = (1.17 - 1.16) = 0.01$

4 (a)

Editor - C:\Users\User\NewtonMethod(c)\newton.m

newton.m | testNewtons.m | bisection.m | testBisection.m

```matlab
function [solution, res, no_iterations] = newton(f, df, x0, tol ,nmax)

    exact_x = 0.8284271247;
    %exact_x = 3;
    x = x0-f(x0)/df(x0);
    iteration_counter = 1;
    fprintf("k        xk          ek         ek/ek-1^2\n");
    disp("---------------------------------------");
    fprintf('%d    %f \t %f \t %f\n',iteration_counter,x,abs(x-exact_x),abs(x-exact_x)/(abs(x0-exact_x)^2));
    while abs(x0-x) > tol && iteration_counter < nmax
        x0=x;
        x=x0-f(x0)/df(x0);
        iteration_counter = iteration_counter + 1;
        fprintf('%d    %f \t %f \t %f\n',iteration_counter,x,abs(x-exact_x),(abs(x-exact_x))/(abs(x0-exact_x)^2));
    end
    % Here, either a solution is found, or too many iterations
    if iteration_counter > nmax
        fprintf("Newtons method stoped without convergence");
        return;
    end
    solution = x;
    res=abs(x0-x);
    no_iterations = iteration_counter;
end
```

Workspace: ans 0.8284

4 (b)

Editor - C:\Users\User\NewtonMethod(c)\testNewtons.m

newton.m | testNewtons.m | bisection.m | testBisection.m

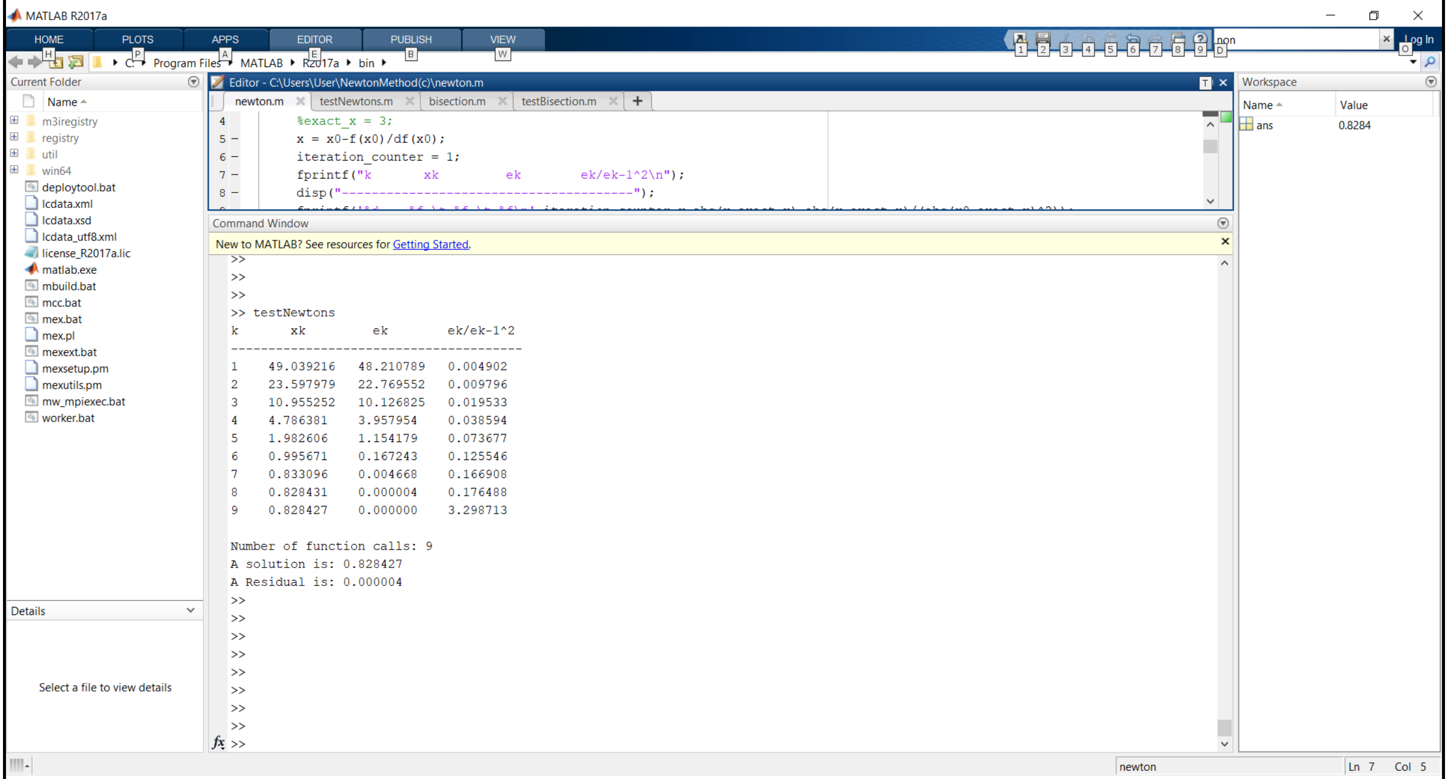
```matlab
function testNewtons()

    f=@(x) x^2+4*x-4;
    df=@(x) 2*x+4;
    %f=@(x) x-0.8*sin(x)-3;
    %df=@(x) 1-0.8*cos(x);
    tol = 1e-5;
    nmax = 100;
    x0 = 100;
    [solution,res,no_iterations] = newton(f, df, x0, tol ,nmax);
    if no_iterations > 0   % Solution found
        fprintf('\nNumber of function calls: %d\n', no_iterations);
        fprintf('A solution is: %f\n', solution)
        fprintf('A Residual is: %f\n', res)
    end
end
```

Command Window

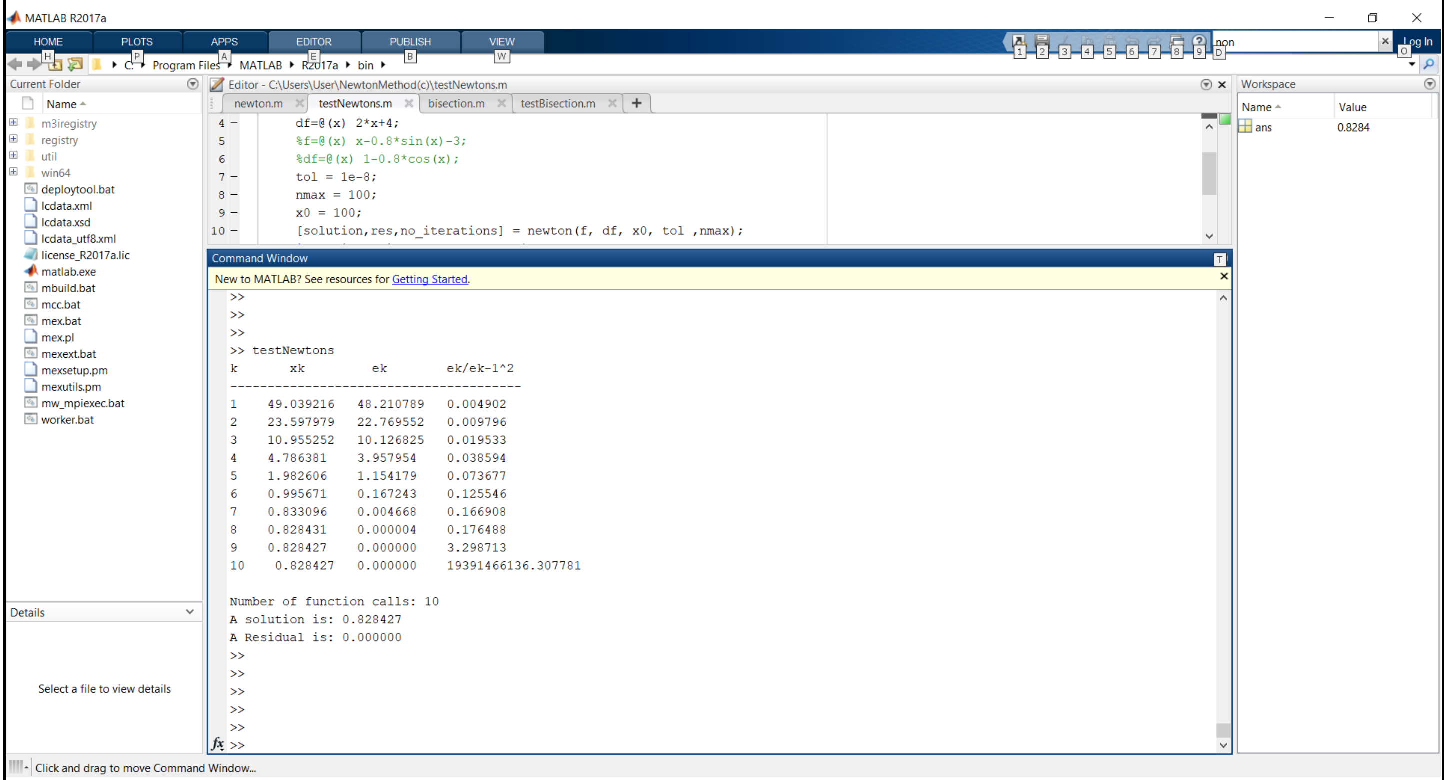
```
Number of function calls: 9
A solution is: 0.828427
A Residual is: 0.000004
>>
>>
>>
>>
>>
>>
>>
>>
>>
>>
```

Workspace: ans 0.8284

4 (c)


MATLAB R2017a
Editor - C:\Users\User\NewtonMethod(c)\newton.m
newton.m
testNewtons.m
bisection.m
testBisection.m
%exact_x = 3;
x = x0-f(x0)/df(x0);
iteration_counter = 1;
fprintf("k   xk   ek   ek/ek-1^2\n");
disp("---------------------------------------");
Command Window
>> testNewtons
k   xk   ek   ek/ek-1^2
1   49.039216   48.210789   0.004902
2   23.597979   22.769552   0.009796
3   10.955252   10.126825   0.019533
4   4.786381   3.957954   0.038594
5   1.982606   1.154179   0.073677
6   0.995671   0.167243   0.125546
7   0.833096   0.004668   0.166908
8   0.828431   0.000004   0.176488
9   0.828427   0.000000   3.298713
Number of function calls: 9
A solution is: 0.828427
A Residual is: 0.000004
Workspace
ans 0.8284

4 (d)


MATLAB R2017a
Editor - C:\Users\User\NewtonMethod(c)\testNewtons.m
newton.m
testNewtons.m
bisection.m
testBisection.m
df=@(x) 2*x+4;
%f=@(x) x-0.8*sin(x)-3;
%df=@(x) 1-0.8*cos(x);
tol = 1e-8;
nmax = 100;
x0 = 100;
[solution,res,no_iterations] = newton(f, df, x0, tol ,nmax);
Command Window
>> testNewtons
k   xk   ek   ek/ek-1^2
1   49.039216   48.210789   0.004902
2   23.597979   22.769552   0.009796
3   10.955252   10.126825   0.019533
4   4.786381   3.957954   0.038594
5   1.982606   1.154179   0.073677
6   0.995671   0.167243   0.125546
7   0.833096   0.004668   0.166908
8   0.828431   0.000004   0.176488
9   0.828427   0.000000   3.298713
10   0.828427   0.000000   19391466136.307781
Number of function calls: 10
A solution is: 0.828427
A Residual is: 0.000000
Workspace
ans 0.8284

05) $M = E - e \sin E \quad , \quad 0 < |e| < 1$

$$f(E) = E - e \sin E - M$$

Here,

$e = 0.8$

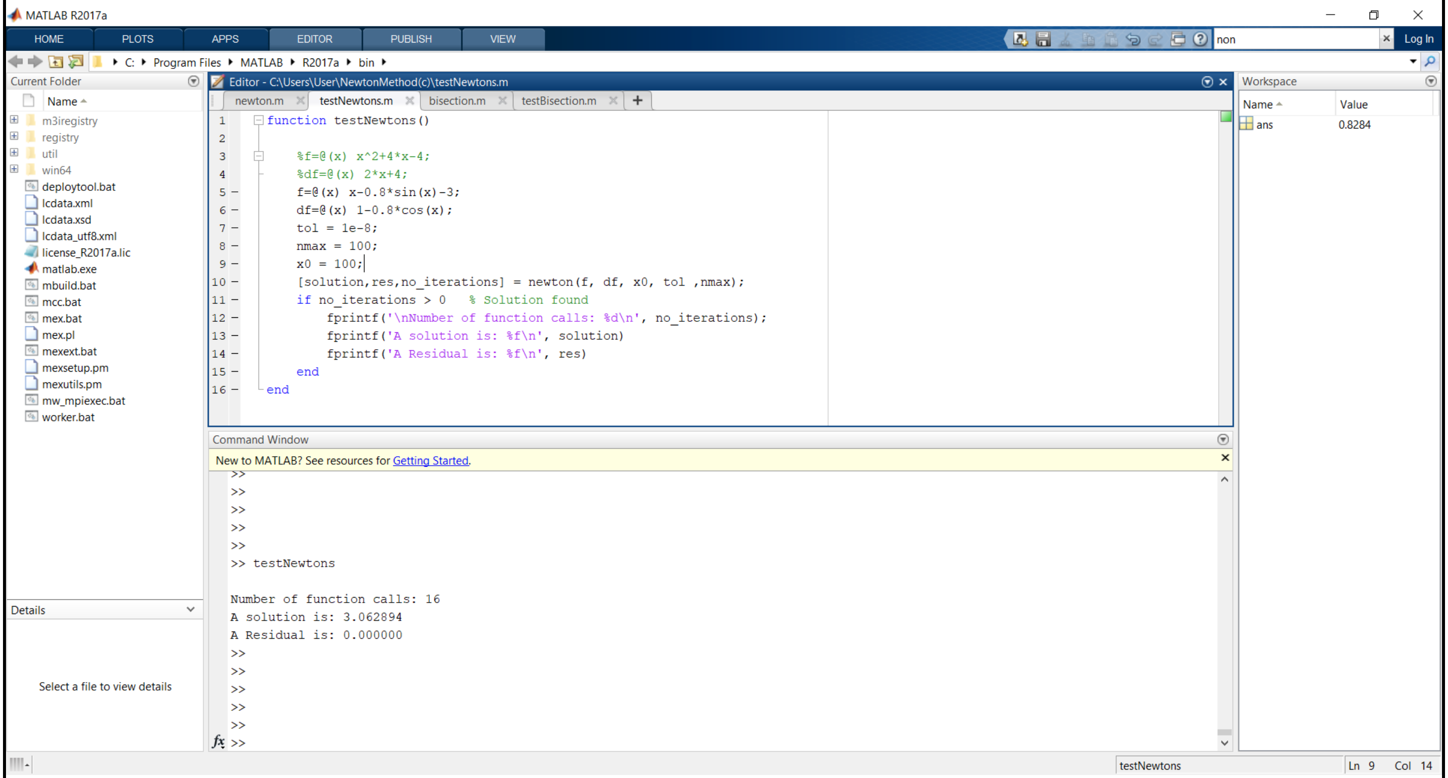
$M = 3$

$T = 10^{-8}$

Using Newton's method to solve,

$$f(E) = E - 0.8 \sin E - 3$$

$$f'(E) = 1 - 0.8 \cos E$$


MATLAB R2017a
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EDITOR
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C: ▸ Program Files ▸ MATLAB ▸ R2017a ▸ bin ▸
Current Folder
Name
m3iregistry
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util
win64
deploytool.bat
lcdata.xml
lcdata.xsd
lcdata_utf8.xml
license_R2017a.lic
matlab.exe
mbuild.bat
mcc.bat
mex.bat
mex.pl
mexext.bat
mexsetup.pm
mexutils.pm
mw_mpiexec.bat
worker.bat
Details
Select a file to view details
Editor - C:\Users\User\NewtonMethod(c)\testNewtons.m
newton.m
testNewtons.m
bisection.m
testBisection.m
function testNewtons()
%f=@(x) x^2+4*x-4;
%df=@(x) 2*x+4;
f=@(x) x-0.8*sin(x)-3;
df=@(x) 1-0.8*cos(x);
tol = 1e-8;
nmax = 100;
x0 = 100;
[solution,res,no_iterations] = newton(f, df, x0, tol ,nmax);
if no_iterations > 0 % Solution found
fprintf('\nNumber of function calls: %d\n', no_iterations);
fprintf('A solution is: %f\n', solution)
fprintf('A Residual is: %f\n', res)
end
end
Command Window
New to MATLAB? See resources for Getting Started.
>> testNewtons
Number of function calls: 16
A solution is: 3.062894
A Residual is: 0.000000
Workspace
Name
Value
ans
0.8284
testNewtons
Ln 9
Col 14

06) $\left\{\frac{PV^2 + aN^2}{V^2}\right\}(V - Nb) = KNT$

$PV^3 - NbV^2P + aN^2V - abN^3 = KNTV^2$

$f(V) = PV^3 - (NbP + KNT)V^2 + (aN^2)V - abN^3$

here,

$P = 3.5 \times 10^7$ Pa

$a = 0.401$ Pam^6

$T = 300K$

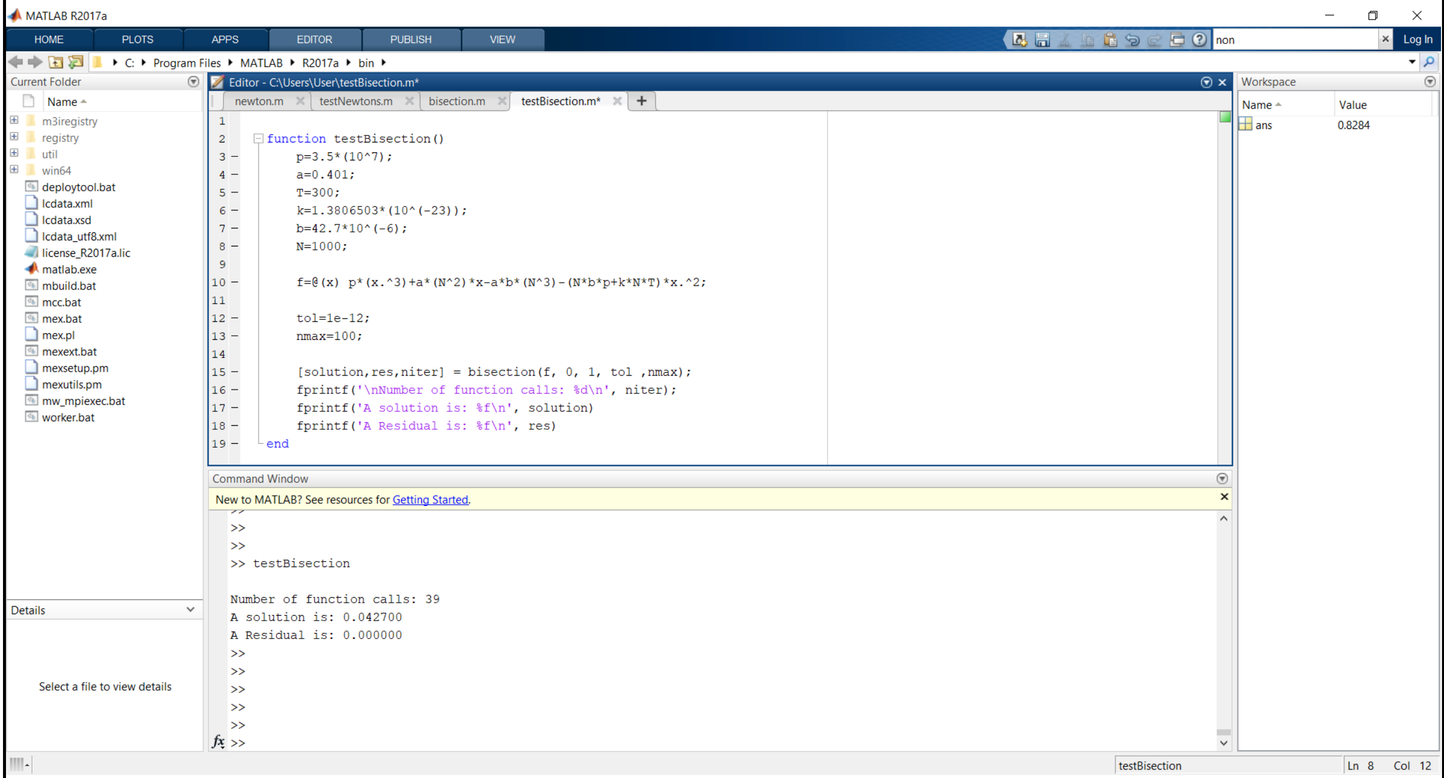
$k = 1.3806503 \times 10^{-23}$ JK^{-1}

$b = 42.7 \times 10^{-6}$ m^3.

$N = 1000$

using bisection method to solve $f(V) = 0$,

The Volume is $= 0.0427 \, m^3$ //


MATLAB R2017a
HOME
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C: ▸ Program Files ▸ MATLAB ▸ R2017a ▸ bin ▸
Current Folder
Name
m3iregistry
registry
util
win64
deploytool.bat
lcdata.xml
lcdata.xsd
lcdata_utf8.xml
license_R2017a.lic
matlab.exe
mbuild.bat
mcc.bat
mex.bat
mex.pl
mexext.bat
mexsetup.pm
mexutils.pm
mw_mpiexec.bat
worker.bat
Details
Select a file to view details
Editor - C:\Users\User\testBisection.m*
newton.m
testNewtons.m
bisection.m
testBisection.m*
function testBisection()
p=3.5*(10^7);
a=0.401;
T=300;
k=1.3806503*(10^(-23));
b=42.7*10^(-6);
N=1000;
f=@(x) p*(x.^3)+a*(N^2)*x-a*b*(N^3)-(N*b*p+k*N*T)*x.^2;
tol=1e-12;
nmax=100;
[solution,res,niter] = bisection(f, 0, 1, tol ,nmax);
fprintf('\nNumber of function calls: %d\n', niter);
fprintf('A solution is: %f\n', solution)
fprintf('A Residual is: %f\n', res)
end
Command Window
New to MATLAB? See resources for Getting Started.
>> testBisection
Number of function calls: 39
A solution is: 0.042700
A Residual is: 0.000000
Workspace
Name
Value
ans
0.8284
testBisection
Ln 8
Col 12