Chapter 2 UNBOUDED GRIDS: THE SEMIDISCRETE FOURIER TRANSFORM

* Review of the continuous case : Fourier Transform

Fourier Transform $\quad \hat{u}(k) = \int_{-\infty}^{\infty} e^{-ikx} u(x)\,dx \qquad k \in \mathbb{R}$

$\hat{u}(k)$: interpreted as the amplitude density of u at the wavenumber k

Inverse Fourier Transform $\quad u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \hat{u}(k)\,dk \qquad x \in \mathbb{R}$

x ↓ physical variable	k ↓ Fourier variable/wavenumber

⟹ We want to consider x ranging over $h\mathbb{Z}$ rather than $\mathbb{R}$

* Our infinite grid

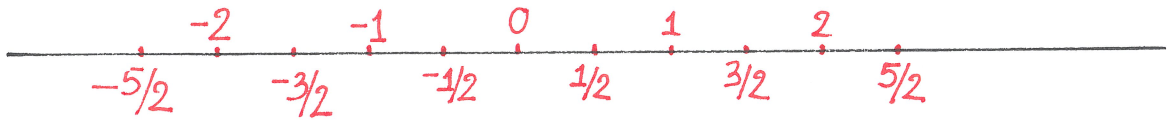
$$h\mathbb{Z} = \{ x_j = hj \ , j \in \mathbb{Z} \} \ , \quad \mathbb{Z}: \text{set of all integers}$$

Our infinite grid: $\quad h\mathbb{Z} = \{x_j = jh, \; j \in \mathbb{Z}\}$

Examples

$h = 1$

$h = 1/2$



Remember $\quad k$ (wavenumber) is bounded because x (physical variable) is discrete

Physical space $\qquad$ discrete, $\qquad$ unbounded : $\quad x \in h\mathbb{Z}$

$\qquad\qquad\qquad\qquad \updownarrow \qquad\qquad\qquad \updownarrow$

Fourier space $\qquad$ bounded, $\qquad$ continuous : $\quad k \in [-\pi/h, \pi/h]$

The reason for these connections is the phenomenon known as aliasing

Aliasing Two complex exponentials $f(x) = \exp(ik_1 x)$ and $g(x) = \exp(ik_2 x)$ are unequal over $\mathbb{R}$ if $k_1 \neq k_2$.

Remind $\exp(i\theta) = e^{i\theta} = \cos\theta + i\sin\theta$

where "i" is the imaginary unit: $\sqrt{-1} = i$, or $i^2 = -1$.

$$\begin{cases} f(x) = e^{ik_1 x} = \cos(k_1 x) + i\sin(k_1 x) \\ g(x) = e^{ik_2 x} = \cos(k_2 x) + i\sin(k_2 x) \end{cases}$$

$$\boxed{\textbf{Recall } \frac{e^a}{e^b} = e^{a-b} \text{ for } a, b \in \mathbb{C}}$$

If we restrict f and g to $h\mathbb{Z}$, however, they take values

$$f_j = \exp(ik_1 x_j) = \cos(k_1 x_j) + i\sin(k_1 x_j) \qquad g_j = \exp(ik_2 x_j) = \cos(k_2 x_j) + i\sin(k_2 x_j)$$

We compute $\frac{f_j}{g_j} = \frac{\exp(ik_1 x_j)}{\exp(ik_2 x_j)} = \exp(i(k_1 - k_2)x_j) = \exp[i\underbrace{(k_1 - k_2)h}_{*}j]$

If $(k_1 - k_2)h$ is the integer multiple of 2π, then $\frac{f_j}{g_j} = 1 \rightarrow f_j = g_j$

If $(k_1-k_2)h$ is an integer multiple of $2\pi \Longleftrightarrow (k_1-k_2)h = N2\pi,\ N\in\mathbb{Z}$

$\Updownarrow$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Updownarrow$

(k_1-k_2) is an integer multiple of $\frac{2\pi}{h} \Longleftrightarrow k_1-k_2 = N\frac{2\pi}{h},\quad N\in\mathbb{Z}$

then $f_j = \exp(ik_1x_j) = \exp(ik_2x_j) = g_j$

$\Longrightarrow$ For any complex exponential $\exp(ikx) = e^{ikx} = \cos(kx) + i\sin(kx)$, there are infinitely many other complex exponentials that match it on the grid $h\mathbb{Z}$. — "aliases" of k

$\Longrightarrow$ It suffices to measure wavenumbers for the grid in an interval of length $\frac{2\pi}{h}$, and for reasons of symmetry, we choose the interval

$$\left[-\frac{\pi}{h},\ \frac{\pi}{h}\right]$$

The argument is valid for $f(x) = \cos(kx)$ or $f(x) = \sin(kx)$

Examples: $\quad h=1 \Rightarrow k\in[-\pi;\pi] \qquad h=\frac{1}{2} \Rightarrow k\in[-2\pi,2\pi]$

$\qquad\qquad h=2 \Rightarrow k\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Semidiscrete Fourier Transform

$$\hat{v}(k) = h \sum_{j=-\infty}^{\infty} e^{-ikx_j} v_j \qquad k \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]$$

for function v defined on $h\mathbb{Z}$ with value v_j at x_j: $v(x_j) = v_j$

Inverse semidiscrete Fourier Transform

$$v_j = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} e^{ikx_j} \hat{v}(k)\, dk \qquad j \in \mathbb{Z}$$

Remark (i) The formula $\hat{v}(k) = h \sum_{j=-\infty}^{\infty} e^{-ikx_j} v_j$ approximates the integral $\hat{u}(k) = \int_{-\infty}^{\infty} e^{-ikx} u(x)\, dx$ by a trapezoid rule.

(ii) The formula $v_j = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} e^{ikx_j} \hat{v}(k)\, dk$ approximates the integral $u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \hat{u}(k)\, dk$ by truncating $\mathbb{R}$ to $\left[-\frac{\pi}{h}, \frac{\pi}{h}\right]$.

As $h \to 0$, the two pairs of formula converges

Digression & some mathematical background

If the expression "Semidiscrete Fourier Transform" is unfamiliar that may be because we have given a new name to an old concept. A Fourier series represents a function on a bounded interval as a sum of complex exponentials at discrete wavenumbers.

Recall Fourier series $\quad f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx} \qquad x \in [-\pi, \pi]$

We have used the term "semidiscrete Fourier transform" to emphasize that our concern here is the inverse problem. It is the "space" variable that is discrete and the "Fourier" variable that is a bounded interval. Mathematically, there is no difference from the theory of Fourier series, which is presented in numerous books. and is one of the most extensively worked branches of mathematics.

Interpolant for spectral differentiation

After determining $\hat{v}$, we define our interpolant as follows

$$p(x) = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{ikx}\hat{v}(k)\,dk \qquad x \in \mathbb{R}. \quad (\star)$$

$$\Longrightarrow \begin{cases} p(x) \text{ is analytic function of } x \\ p(x_j) = v_j \quad \text{for each } j \end{cases}$$

The Fourier transform $\hat{p}$ is

$$\hat{p}(k) = \begin{cases} \hat{v}(k) & k \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right] \\ 0 & \text{otherwise} \end{cases}$$

$$\Longrightarrow \begin{cases} \hat{p} \text{ has compack support in } \left[-\frac{\pi}{h}, \frac{\pi}{h}\right] \\ p \text{ is the band-limited interpolant of } v \end{cases}$$

$\{v_j\}_{j=-\infty}^{\infty} \Longrightarrow$ compute $\hat{v}(k) \Longrightarrow$ construct interpolant $p(x)$

Recap

* We have values v_j of the "hidden" function $v(x)$ on the grid points $h\mathbb{Z}$. We can compute $\hat{v}(k)$ for all $k \in [-\pi/h, \pi/h]$

$\Longrightarrow$ Then we define the interpolant $p(x)$ that approximates the hidden function $v(x)$ according to $(\star)$

(A) Spectral differentiation of v defined on $h\mathbb{Z}$: option 1.

- Given v, determine its band-limited interpolant p
- Set $w_j = p'(x_j)$

(B) Spectral differentiation of v defined on $h\mathbb{Z}$: option 2.

If u is a differentiable function with Fourier transform $\hat{u}$; then the Fourier transform of u' is $ik\hat{u}(k)$

$$\widehat{u'}(k) = ik\hat{u}(k)$$

$\Longrightarrow$ Procedure equivalent to spectral differentiation (A)

- Given v, compute its semidiscrete Fourier transform $\hat{v}$
$$\hat{v}(k) = h\sum_{j=-\infty}^{\infty} e^{-ikx_j} v_j \qquad k \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]$$
- Define $\hat{w}(k) = ik\hat{v}(k)$
- Compute w from $\hat{w}$ by $\quad w_j = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{ikx_j}\hat{w}(k)\,dk \qquad j \in \mathbb{Z}.$

Kronecker delta function δ

$$\delta_j = \begin{cases} 1 & j=0 \\ 0 & j\neq 0 \end{cases}$$

* A general grid function can be written

$$v_j = \sum_{m=-\infty}^{\infty} v_m \delta_{j-m}$$

Linearity of the semidiscrete Fourier transform (and of course of inverse transform) implies interpolant of v is a linear combination of the interpolants of the Kronecker delta function δ_{j-m}.

$\Rightarrow$ Task We look for band-limited interpolants of δ_{j-m}.

If $q(x)$ is the interpolant of δ_j, then the interpolant of δ_{j-m} is $q(x-x_m)$

$\Rightarrow$ Task We look for band-limited interpolant of δ

Kronecker delta function δ^0

$$\delta^0_j = \begin{cases} 1 & j=0 \\ 0 & j\neq 0 \end{cases}$$

Kronecker delta function δ^m

$$\delta^m_j = \begin{cases} 1 & j=m \\ 0 & j\neq m \end{cases}$$

* A general grid function v on $h\mathbb{Z}$ can be written

$$v_j = \sum_{m=-\infty}^{\infty} v_m \delta^m_j \iff v = \sum_{m=-\infty}^{\infty} v_m \delta^m$$

Linearity of the semidiscrete Fourier transform (and of course of inverse transform) implies interpolant of v is a linear combination of the interpolants of the Kronecker delta function δ^m

$\Longrightarrow$ Task We look for band-limited interpolants of δ^m

If $q(x)$ is the interpolant of δ^0, then the interpolant of δ^m is $q(x-x_m)$

$\Longrightarrow$ Task We look for band-limited interpolants of δ^0

Derive band-limited interpolant of δ^0

- Compute the semidiscrete Fourier transform:

$$\hat{\delta^0}(k) = h\sum_{j=-\infty}^{\infty} e^{-ikx_j}\delta_j^0 = h\underbrace{e^{-ikx_0}}_{1} = h$$

- Compute the band-limited of δ^0:

(i) $$p(x) = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{ikx}\hat{\delta^0}(k)\,dk = \frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} e^{ikx}\,dk = \frac{h}{2\pi}\left[\frac{\exp(ikx)}{ik}\right]_{k=-\pi/h}^{k=\pi/h}$$

$$= \frac{h}{2\pi}\left\{\frac{\exp\left(i\frac{\pi}{h}x\right)}{i\pi/h} - \frac{\exp\left(-i\frac{\pi}{h}x\right)}{-i\pi/h}\right\} = \frac{\sin(x\pi/h)}{\pi x/h} \quad \forall x \neq 0.$$

(ii) $$p(0) = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^0\hat{\delta^0}(k)\,dk = \frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} dk = \frac{h}{2\pi}\left(\frac{2\pi}{h}\right) = 1.$$

Because $\lim_{x\to 0} S_h(x) = 1$

$$\Longrightarrow p(x) = \begin{cases} S_h(x) = \dfrac{\sin(\pi x/h)}{\pi x/h} & x \neq 0 \\ 1 & x = 0 \end{cases} \Longrightarrow p(x) = S_h(x) = \frac{\sin(\pi x/h)}{\pi x/h}$$

(Sinc-function)

The interpolant of δ^0 is $S_h(x) = \dfrac{\sin(\pi x/h)}{\pi x/h}$ $[S_h(x_0) = 1]$

$\updownarrow$

The interpolant of δ^m is $S_h(x - x_m) = \dfrac{\sin[\pi(x - x_m)/h]}{\pi(x - x_m)/h}$

$[S_h(x_m - x_m) = S_h(x_0) = 1]$

Repeat: $v = \sum_{m=-\infty}^{\infty} v_m \delta^m$

Interpolant of v is a linear combination of translated Sinc-function

$$p(x) = \sum_{m=-\infty}^{\infty} v_m \underbrace{S_h(x - x_m)}_{\text{Interpolant of } \delta^m}$$

$\Longrightarrow$ Derivative of the interpolant:

$$w(x) = p'(x) = \sum_{m=-\infty}^{\infty} v_m S_h'(x - x_m) \underbrace{\frac{d(x - x_m)}{dx}}_{1} = \sum_{m=-\infty}^{\infty} v_m S_h'(x - x_m)$$

$$\boxed{w_j = w(x_j) = p'(x_j) = \sum_{m=-\infty}^{\infty} v_m S_h'(x_j - x_m)}$$

Differentiation Matrix

We may interprete

$$w_j = \sum_{m=-\infty}^{\infty} v_m S_h'(x_j - x_m)$$

as a matrix equation

$$\underline{w} = \underline{D}\,\underline{v} \iff w_j = D_{jm} v_m$$ (Summation Convention is assumed here)

$$\Rightarrow D_{jm} = S_h'(x_j - x_m)$$

Examples

$D_{j0} = S_h'(x_j)$ is the column $m = 0$

$D_{j1} = S_h'(x_j - x_1)$ is the column $m = 1$ and so on ....

$D_{j(-1)} = S_h'(x_j - x_{-1})$ is the column $m = -1$

$\Longrightarrow \underline{D}$ is a Toeplitz matrix obtained by shifting column $m = 0$ up or down appropriately.

$$S_h'(x_j) = \begin{cases} 0 & j = 0 \\ \dfrac{(-1)^j}{jh} & j \neq 0. \end{cases}$$

$$S_h'(x) = \frac{d}{dx}\left[\frac{\sin(\pi x/h)}{\pi x/h}\right] = \frac{\cos(\pi x/h)\,\pi/h}{\pi x/h} - \frac{\sin(\pi x/h)}{\pi x^2/h}$$

$$= \frac{\pi x \cos(\pi x/h) - h \sin(\pi x/h)}{\pi x^2}$$

$$\Longrightarrow S_h'(x_j) = \frac{\pi x_j \cos(\pi x_j/h) - h\sin(\pi x_j/h)}{\pi x_j^2} \qquad x_j = jh, \quad j \in \mathbb{Z}$$

$$\text{If } j \neq 0: \; S_h'(x_j) = \frac{\pi \cos(\pi j)}{\pi jh} - \frac{h \sin(\pi j)}{\pi (jh)^2} = \frac{(-1)^j}{jh} - 0 = \frac{(-1)^j}{jh}$$

$$\text{If } j = 0: \; S_h'(x_0) = \lim_{x\to 0} S_h'(x) = \lim_{x \to 0} \frac{\pi x \cos(\pi x/h) - h\sin(\pi x/h)}{\pi x^2}$$

$$\underset{\text{L'Hôpital's rule}}{=} \lim_{x\to 0} -\frac{\pi \sin(\pi x/h)}{2h} = 0.$$

$$\Rightarrow S_h'(x_j) = \begin{cases} 0 & j = 0 \\ (-1)^j/jh & j \neq 0. \end{cases}$$

$$
\underline{D} = \frac{1}{h}
\begin{bmatrix}
 & \vdots & \\
\ddots & \frac{1}{3} & \\
\ddots & -\frac{1}{2} & \\
\ddots & 1 & \\
 & 0 & \\
 & -1 & \ddots \\
 & \frac{1}{2} & \ddots \\
 & -\frac{1}{3} & \ddots \\
 & \vdots &
\end{bmatrix}
$$

To find higher-order spectral derivatives, we can differentiate $p(x)$ several times. For example:

$$
S_h''(x_j) = \begin{cases} -\dfrac{\pi^2}{3h^2} & j = 0 \\[2ex] 2\,\dfrac{(-1)^{j+1}}{j^2 h^2} & j \neq 0. \end{cases}
\Rightarrow \underline{D}^2 = \frac{2}{h^2}
\begin{bmatrix}
 & \vdots & \\
\ddots & -\frac{1}{4} & \ddots \\
\ddots & 1 & \ddots \\
\ddots & -\frac{\pi^2}{6} & \ddots \\
\ddots & 1 & \ddots \\
\ddots & \frac{1}{4} & \ddots \\
 & \vdots &
\end{bmatrix}
$$