Chapter 8 CHEBYSHEV SERIES AND THE FFT

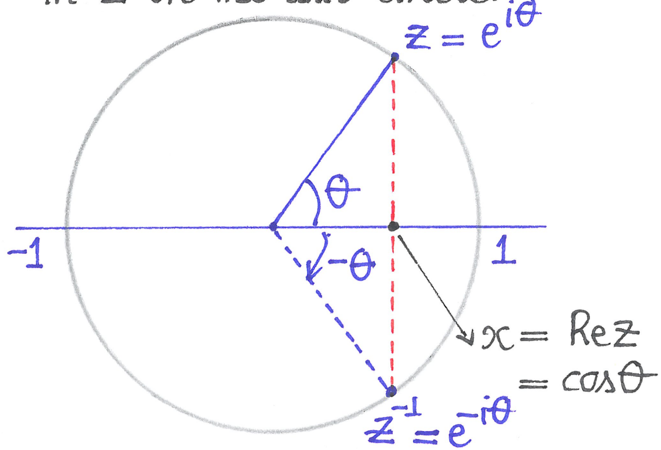
The crucial task of this chapter: to show how Chebyshev spectral methods can be implemented by the FFT, which provides a crucial speed-up for some calculations.

Mathematical idea that underlies this technique: the equivalence of

(i) Chebyshev series in $x \in [-1, 1]$,

(ii) Fourier series in $\theta \in \mathbb{R}$,

(iii) Laurent series in z on the unit circle.



- Let z be a complex number on the unit circle: $|z| = 1$.
- Let θ be the argument of z, a real number that is determined up to multiples of 2π.
- Let $x = \cos\theta = \mathrm{Re}(z)$

Example $n=0$: $\quad \mathrm{Re}\, z^0 = \mathrm{Re}\,(e^{i\theta})^0 = 1 \Rightarrow T_0(x) = 1.$

$n=1$: $\quad \mathrm{Re}\, z = \underbrace{\tfrac{1}{2}(z+z^{-1})}_{x} \Rightarrow T_1(x) = x.$

$n=2$: $\quad \mathrm{Re}\, z^2 = \tfrac{1}{2}(z^2+z^{-2}) = \underbrace{\tfrac{1}{2}(z+z^{-1})^2}_{2\left[\frac{1}{2}(z+z^{-1})\right]^2} - \underbrace{2\,\tfrac{1}{2}\,z z^{-1}}_{1} \Rightarrow T_2(x) = 2x^2 - 1.$

$n=3$: $\quad \mathrm{Re}\, z^3 = \tfrac{1}{2}(z^3+z^{-3}) = \underbrace{\tfrac{1}{2}(z+z^{-1})^3}_{4\left[\frac{1}{2}(z+z^{-1})\right]^2} - \tfrac{3}{2}(z+z^{-1}) \Rightarrow T_3(x) = 4x^3 - 3x.$

Or alternatively

$n=0$: $\quad \cos n\theta = 1 \Rightarrow T_0(x) = 1.$

$n=1$: $\quad \cos n\theta = \cos\theta \Longrightarrow T_1(x) = x.$

$n=2$: $\quad \cos n\theta = \cos 2\theta = 2\cos^2\theta - 1 \Longrightarrow T_2(x) = 2x^2 - 1$

For each $x \in [-1, 1]$, there are two complex conjugate values of z, and we have

$$x = \operatorname{Re} z = \frac{1}{2}(z + z^{-1}) = \cos\theta \in [-1, 1]$$

- If $z \in$ unit circle, we may express z as

$$z = \cos\theta + i\sin\theta, \qquad \theta \in \mathbb{R}.$$

$$\Updownarrow$$

$$|z| = 1.$$

$\Rightarrow$ complex conjugate of z : $\bar{z} = \cos\theta - i\sin\theta = \dfrac{1}{\cos\theta + i\sin\theta} = \dfrac{1}{z}$

- The n^{th} Chebyshev polynomial:

$$T_n(x) = \operatorname{Re} z^n = \frac{1}{2}(z^n + z^{-n}) = \cos n\theta$$

$$z = \cos\theta + i\sin\theta = e^{i\theta} \Rightarrow z^n = e^{in\theta} = \cos n\theta + i\sin n\theta$$

$$\Rightarrow \operatorname{Re} z^n = \cos n\theta = \frac{1}{2}(e^{in\theta} + e^{-in\theta}) = \frac{1}{2}(z^n + z^{-n})$$

Exercise Prove that $T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x)$.

By induction, we deduce that T_n is a polynomial of degree exactly n for each $n \geqslant 0$, with leading coefficient 2^{n-1} for each $n \geqslant 1$.

(i) Since T_n is of exact degree n for each n, any degree N polynomial can be written uniquely as a linear combination of Chebyshev polynomials

$$p(x) = \sum_{n=0}^{N} a_n T_n(x) \qquad x \in [-1, 1]$$

(ii) Corresponding to this is a degree N Laurent polynomial in z and z^{-1}

$$\tilde{p}(x) = \frac{1}{2} \sum_{n=0}^{N} a_n (z^n + z^{-n}) \qquad |z| = 1$$

(iii) Corresponding to these is a degree N 2π-periodic trigonometric polynomial that is even, that is $P(\theta) = P(-\theta)$

$$P(\theta) = \sum_{n=0}^{N} a_n \cos n\theta \qquad \theta \in \mathbb{R}$$

* The functions p, $\tilde{p}$ an P are equivalent: $p(x) = \tilde{p}(z) = P(\theta)$ with $x = \operatorname{Re} z = \cos\theta$

Repeat (i) $p(x) = \sum_{n=0}^{N} a_n T_n(x)$, $x \in [-1, 1]$, (ii) $\tilde{p}(z) = \frac{1}{2}\sum_{n=0}^{N} a_n (z^n + z^{-n})$, $|z| = 1$

and (iii) $P(\theta) = \sum_{n=0}^{N} a_n \cos n\theta$ are equivalent in the sense

$$p(x) = \tilde{p}(z) = P(\theta)$$

when x, z and θ are related by $x = \mathrm{Re}\, z = \cos\theta$.

* **Question** Given a function f defined on $[-1, 1]$, can we form the corresponding functions $\tilde{f}$ and F as above?

equivalent

Answer From an arbitrary function $f(x)$ defined for $x \in [-1, 1]$, we can form a self-reciprocal function $\tilde{f}(z)$ defined on the unit circle and a periodic function $F(\theta)$ defined on $\mathbb{R}$ as follows:

$$\tilde{f}(z) = f\left(\frac{z + z^{-1}}{2}\right) \qquad F(\theta) = f(\cos\theta)$$

For spectral collocation methods, we mainly deal with $p(x)$, $\tilde{p}(z)$, $P(\theta)$ as interpolants of function f, $\tilde{f}$, and F, respectively.

Equivalences

(i) The interpolation points are:

(a) $\theta_j = j\pi/N$

(b) $z_j = e^{i\theta_j}$ with $0 \leq j \leq N$

(c) $x_j = \cos\theta_j = \operatorname{Re} z_j$

(ii) We have the equivalences:

$P(\theta)$ interpolates $F(\theta)$ (even and 2π-periodic) in the equispaced points $\{\theta_j\}$

$$\Updownarrow$$

$\tilde{p}(z)$ interpolates $\tilde{f}(z)$ (self-reciprocal) in the roots of unity $\{z_j\}$

$$\Updownarrow$$

$p(x)$ interpolates $f(x)$ (arbitrary) in the Chebyshev points $\{x_j\}$

$\Longrightarrow$ Connection & Task: The key point is that the polynomial interpolant q of f can be differentiated by finding a trigonometric polynomial interpolant Q of F, differentiating in Fourier space, and transforming back to the x variable. $\Longrightarrow$ Take advantage of the DFT/ FFT

⁎ Chebyshev spectral derivative via FFT

- Given data $v_0, \dots, v_N$ at Chebyshev points $x_0 = 1, \dots, x_N = -1$, extend this data to a vector V of length $2N$ with $V_{2N-j} = v_j$, $j = \overline{1, N-1}$.
- Using the FFT, calculate

$$\widehat{V}_k = \frac{\pi}{N} \sum_{j=1}^{2\pi} e^{-ik\theta_j} V_j \qquad k = -N+1, \dots, N$$

- Define $\widehat{W}_k = ik\widehat{V}_k$, except $\widehat{W}_N = 0$
- Compute the derivative of the trigometric interpolant Q on the equispaced grid by the inverse FFT:

$$W_j = \frac{1}{2\pi} \sum_{k=-N+1}^{N} e^{ik\theta_j} \widehat{W}_k \qquad j = 1, \dots, 2N.$$

- Calculate the derivative of the algebraic polynomial interpolant q on the interior grid points by

$$w_j = -W_j / \sqrt{1 - x_j^2} \qquad j = 1, \dots N-1.$$

with special formulas: $w_0 = \frac{1}{2\pi} \sum_{n=0}^{N}{}' n^2 \hat{v}_n$, $w_N = \frac{1}{2\pi} \sum_{n=0}^{N}{}' (-1)^{n+1} n^2 \hat{v}_n$.

An explanation for the Chebyshev spectral derivative via FFT

- First, the trigonometric interpolant of the extended $\{v_j\}$ data is given by evaluating the inverse DFT at arbitrary θ.

$$\mathcal{l}(\theta) = \frac{1}{2\pi} \sum_{k=-N+1}^{N} e^{ik\theta} \hat{v}_k$$

Using the a_n coefficients we find that

$$\mathcal{l}(\theta) = \sum_{n=0}^{N} a_n \cos n\theta = \frac{1}{2\pi} \sum_{k=-N+1}^{N} e^{ik\theta} \hat{v}_k$$

- The algebraic polynomial interpolant of the $\{v_j\}$ data is $p(x) = \mathcal{l}(\theta)$, where $x = \cos\theta$ & the derivative is

$$q'(x) = \frac{\mathcal{l}'(\theta)}{dx/d\theta} = \frac{-\sum_{n=0}^{N} n a_n \sin n\theta}{-\sin\theta} = \frac{\sum_{n=0}^{N} n a_n \sin\theta}{\sqrt{1-x^2}}$$

- As for the special formulas for w_0 and w_N, we determine the value of $q'(x)$ at $x = \pm 1$ by l'Hôpital's rule.

$$q'(1) = \lim_{x\to 1} q'(x) = \lim_{\theta\to 0} q'(\cos\theta) = \lim_{\theta\to 0} \frac{\sum_{n=0}^{N} n a_n \sin n\theta}{\sin\theta}$$

$$= \lim_{\theta\to 0} \frac{\sum_{n=0}^{N} n^2 a_n \cos n\theta}{\cos\theta} = \sum_{n=0}^{N} n^2 a_n$$

$$q'(-1) = \lim_{x\to -1} q'(x) = \lim_{\theta\to \pi} q'(\cos\theta) = \lim_{\theta\to \pi} \frac{\sum_{n=0}^{N} n^2 a_n \cos n\theta}{\cos\theta} = \sum_{n=0}^{N} n^2 (-1)^{n+1} a_n$$

$$\Longrightarrow \begin{cases} q'(1) = \sum_{n=0}^{N} n^2 a_n \\ q'(-1) = \sum_{n=0}^{N} n^2 (-1)^{n+1} a_n \end{cases}$$

It is straightforward to generalize the method for higher derivatives. At the stage of differentiation in Fourier space we multiply by $(ik)^\nu$ to calculate the ν^{th} derivative, and if ν is odd, we set $\widehat{W}_N = 0$

Exercise In order to compute the derivative of a polynomial interpolant using the FFT, the appropriate factors need to be calculated for converting between derivatives on the equispaced grid and on the Chebyshev grid, that is, derivatives in the θ and x variables.

- Prove that, the second derivatives are related by

$$q''(x) = \frac{-x}{(1-x^2)^{3/2}} Q'(\theta) + \frac{1}{1-x^2} Q''(\theta).$$

If W_j and $W_j^{(2)}$ are the 1^{st} and 2^{nd} derivatives on the equispaced grid, respectively, then the second derivative on the Chebyshev grid is

$$w_j^{(2)} = \frac{-x_j}{(1-x_j^2)^{3/2}} W_j + \frac{1}{1-x_j^2} W_j^{(2)} \qquad 1 \leq j \leq N-1.$$

- Also, derive the special formulas needed for $j = 0$ and N.

* Application of the Chebyshev differentiation to solving a PDE.

Problem 1

Consider the wave equation

$$u_{tt} = u_{xx}, \quad -1<x<1, \quad t>0, \quad u(\pm 1) = 0.$$

- We use leap frog formula in t and Chebyshev spectral differentiation in x.
- We need two initial conditions. For the finite difference scheme, we need conditions on u at $t=0$ and at $t=-\Delta t$, the previous time step. Basically, we need to use another time stepping method to obtain the solution at $t=-\Delta t$ & thus have enough information for the entire scheme. In our example, our choice at $t=-\Delta t$ is initial data corresponding to a left-moving Gaussian pulse and interpret that this choice actually comes from some appropriate initial conditions

$$u(x, t=0) = u_0(x) \quad \& \quad u_t(x, t=0) = v_0(x).$$

Problem 2 Solve the wave equation in two space dimensions

$$u_{tt} = u_{xx} + u_{yy} \qquad -1 < x, y < 1, \ t > 0, \ u = 0 \text{ on the boundary},$$

with initial data

$$\begin{cases} u(x,y,0) = \exp\{-40[(x-\frac{2}{5})^2 + y^2]\} \\ u_t(x,y,0) = 0 \end{cases}$$

Exercise Write a program to perform a study of the relative efficiencies of cheb and chebfft as a function of N. Do not count the time taken by to form D_N, just the time taken to multiply D_N by a vector.

Exercise Write a code chebfft2.m for second-order differentiation by the FFT, and show by examples that it matches the results obtained by matrices, apart from rounding errors.

Exercise Write a code chebdct.m that computes the same result as chebfft.m, but makes use of the function DCT from MATLAB's Signal Processing Toolbox. The code will be restricted to real data. What gain of efficiency do you observe?