Chapter 6 CHEBYSHEV DIFFERENTIATION MATRICES

* Chebyshev spectral differentiation

We shall use the Chebyshev points to construct Chebyshev differentiation matrices. Given a grid function v defined on Chebyshev points, we obtain a discrete derivative w:

- Let p be the unique polynomial of degree $\leq N$ with $p(x_j) = v_j$, $0 \leq j \leq N$
- Set $w_j = p'(x_j)$

$\Longrightarrow$ This operation is linear: $\underline{w} = \underline{D}_N \underline{v}$

$\hookrightarrow$ $(N+1)\times(N+1)$ matrix.

N = positive integer, even or odd.

$\Longrightarrow$ Task: We need to establish the closed-form formula to compute $\underline{D}_N$

Example 1: $N=1 \Rightarrow x_0 = 1,\ x_1 = -1 \Rightarrow$ interpolating polynomial

$$p(x) = \frac{1}{2}(1+x)v_0 + \frac{1}{2}(1-x)v_1 \quad \text{(Lagrange interpolation)}$$

Example 1 $N=1 \Rightarrow x_0 = 1, \; x_1 = -1$

interpolating polynomial: $p(x) = \frac{1}{2}(1+x)v_0 + \frac{1}{2}(1-x)v_1$

derivative: $p'(x) = \frac{1}{2}v_0 - \frac{1}{2}v_1$

differentiation matrix: $\underline{D}_1 = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix}$

Example 2 $N=2 \Rightarrow x_0 = 1, \; x_1 = 0, \; x_2 = -1$

interpolant: $p(x) = \frac{1}{2}x(1+x)v_0 + (1+x)(1-x)v_1 + \frac{1}{2}x(x-1)v_2$

derivative: $p'(x) = \left(x+\frac{1}{2}\right)v_0 - 2xv_1 + \left(x-\frac{1}{2}\right)v_2$

differentiation matrix: the matrix whose column is obtained by sampling the j^{th} term of this expression at $x = 1, 0, -1$.

$$\underline{D}_2 = \begin{bmatrix} 3/2 & -2 & 1/2 \\ \hdashline 1/2 & 0 & -1/2 \\ \hdashline -1/2 & 2 & -3/2 \end{bmatrix}$$

→ coefficients for 1-sided approximations

→ coefficients for a centered 3-point FD

but these will be based on uneven grid with $N \geqslant 3$

Exercise Derive the Chebyshev differentiation matrix with $N=3$; namely $\underline{D}_3$ following the procedure as above.

- Define the collocation grid points.
- Derive the interpolant / interpolating polynomial
- Compute the derivative of the interpolant & construct the matrix

CHEBYSHEV DIFFERENTIATION MATRIX

For each $N \geq 1$, let the rows & columns of the $(N+1)\times(N+1)$ Chebyshev spectral differentiation matrix D_N be indexed from 0 to N. The entries of this matrix are.

$$(D_N)_{00} = \frac{2N^2+1}{6} \qquad (D_N)_{NN} = -\frac{2N^2+1}{6}$$

$$(D_N)_{jj} = \frac{-x_j}{2(1-x_j^2)} \qquad j = 1, \dots, N-1$$

$$(D_N)_{ij} = \frac{c_i}{c_j} \frac{(-1)^{i+j}}{(x_i - x_j)} \qquad i \neq j, \quad i,j = 1, \dots, N-1 \qquad c_i = \begin{cases} 2 & i = 0, N \\ 1 & \text{otherwise} \end{cases}$$

$$
D_N = \left[\begin{array}{c|c|c}
\dfrac{2N^2+1}{6} & 2\dfrac{(-1)^j}{1-x_j} & \dfrac{1}{2}(-1)^N \\[2ex]
\hline
 & \dfrac{(-1)^{i+j}}{x_i - x_j} & \\[2ex]
-\dfrac{1}{2}\dfrac{(-1)^i}{1-x_i} & \dfrac{-x_j}{2(1-x_j^2)} & \dfrac{1}{2}\dfrac{(-1)^{N+i}}{1+x_i} \\[2ex]
\hline
-\dfrac{1}{2}(-1)^N & -2\dfrac{(-1)^{i+j}}{1+x_j} & -\dfrac{2N^2+1}{6}
\end{array}\right]
$$

The j^{th} column of D_N contains the derivative of the degree N polynomial interpolant $p_j(x)$ to the delta function supported at x_j, sampled at the grid points $\{x_i\}$

REMARK It has been discovered that in the presence of rounding errors it is better to program $(D_N)_{ii}$ according to.

$$(D_N)_{ii} = -\sum_{\substack{j=0 \\ j \neq i}}^{N} (D_N)_{ij}$$

to have better stability properties.

SUMMARY The entries of the Chebyshev differentiation matrix $\underline{D}_N$ can be computed by explicit formulas, which can be conviniently collected in a short MATLAB function. More general explicit formula can be used to construct the differentiation matrix for an arbitrarily prescribed set of distinct points $\{x_j\}$.

Exercise If $x_0, x_1, \ldots, x_N \in \mathbb{R}$ are distinct, then the cardinal function $p_j(x)$ defined by

$$p_j(x) = \frac{1}{a_j} \prod_{\substack{k=0 \\ k \neq j}}^{N} (x - x_k) \qquad a_j = \prod_{\substack{k=0 \\ k \neq j}}^{N} (x_j - x_k)$$

is the unique polynomial interpolant of degree N to the values 1 at x_j and 0 at x_k, $k \neq j$. Take the logarithm & differentiate to obtain

$$p_j'(x) = p_j(x) \sum_{\substack{k=0 \\ k \neq j}}^{N} \frac{1}{(x - x_k)}$$

and from this derive the formulas

$$D_{ij} = \frac{1}{a_j} \prod_{\substack{k=0 \\ k \neq i,j}}^{N} (x_i - x_k) = \frac{a_i}{a_j (x_i - x_j)} \quad (i \neq j) \quad \& \quad D_{jj} = \sum_{\substack{k=0 \\ k \neq j}}^{N} (x_j - x_k)^{-1}$$

for the entries of the $N \times N$ differentiation matrix associated with the points $\{x_j\}$.

Exercise Derive the Chebyshev differentiation matrix in the lecture/theory from the formulas

$$D_{ij} = \frac{1}{a_j} \prod_{\substack{k=0 \\ k \neq j}}^{N} (x_i - x_k) = \frac{a_i}{a_j (x_i - x_j)} \qquad (i \neq j)$$

$$D_{jj} = \sum_{\substack{k=0 \\ k \neq j}}^{N} (x_j - x_k)^{-1}.$$