Chapter 5 POLYNOMIAL INTERPOLATION & CLUSTERED GRIDS

Suppose that we wish to work on $[-1, 1]$ with non-periodic functions. One approach would be to pretend that the functions were periodic & use trigonometric interpolation in equispaced points. It is a method that works fine for problems whose solutions are exponentially close to zero (or a constant) near the boundaries. In general, this approach sacrifices the accuracy advantages of spectral methods.

The first idea we might have is to use polynomial interpolation in equispaced points. Now this, as it turns out, is catastrophically bad in general. A problem known as the Runge phenomenon is encountered that is more extreme than the Gibbs phenomenon. When smooth functions are interpolated by polynomials in $N+1$ equally spaced points, the approximations not only fail to converge in general as $N \to \infty$, but they get worse at a rate that may be as great as 2^N.

The right idea is polynomial interpolation in unevenly spaced points. Various different sets of points are effective, but they all share a common property. Asymptotically as $N \to \infty$, the points are distributed with density (per unit length)

$$\boxed{\text{density} \sim \frac{N}{\pi\sqrt{1-x^2}}}$$

This implies that the average spacing between points is $O(N^{-2})$ for $x \simeq \pm 1$ and $O(N^{-1})$ in the interior, with the average spacing between adjacent points near $x = 0$ asymptotic to π/N.

Chebyshev points (Chebyshev extreme points)

$$\boxed{x_j = \cos\left(\frac{j\pi}{N}\right) \qquad j = 0, 1, \ldots, N}$$

Fuller names: Chebyshev–Lobatto points, Gauss–Chebyshev–Lobatto points,

* Potential Theory

- A monic polynomial p of degree N

$$p(z) = \prod_{k=1}^{N}(z - z_k) = (z-z_1)\cdots(z-z_N),$$

$\{z_k\}$ are the roots, counted with multiplicity, which might be complex.

$$\Rightarrow \quad |p(z)| = \left|\prod_{k=1}^{N}(z-z_k)\right| = \prod_{k=1}^{N}|z-z_k|$$

$$\Rightarrow \quad \log|p(z)| = \log\left\{\prod_{k=1}^{N}|z-z_k|\right\} = \sum_{k=1}^{N}\log|z-z_k|$$

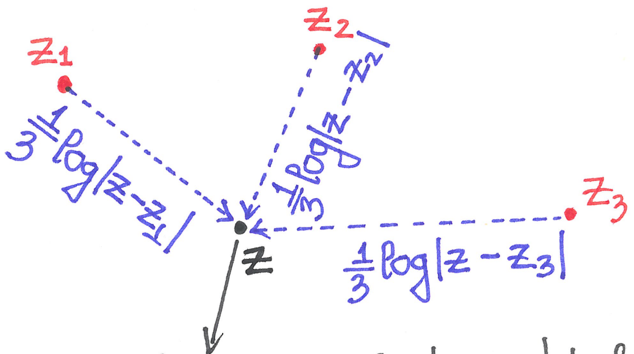
- We consider now

$$\Phi_N(z) = N^{-1}\sum_{k=1}^{N}\log|z-z_k|$$

This function is harmonic in the complex plane except at $\{z_k\}$: It satisfies Laplace's equation except at (x_k, y_k).

- Consider $\Phi_N(z) = \frac{1}{N} \sum_{k=1}^{N} \log|z - z_k|$

$\phi_N(z)$ is the potential at z due to charges at $\{z_k\}$ each with potential $\frac{1}{N}\log|z - z_k|$



$$\frac{1}{3}\left(\log|z-z_1| + \log|z-z_2| + \log|z-z_3|\right)$$

$$\Rightarrow N\phi_N(z) = \sum_{k=1}^{N} \log|z - z_k| \Rightarrow e^{N\phi_N(z)} = \exp\left\{\log \prod_{k=1}^{N} |z - z_k|\right\}$$

$$= \prod_{k=1}^{N} |z - z_k| = |p(z)|$$

There is a correspondence between the size of $p(z)$ and the value $\phi_N(z)$: $|p(z)| = e^{N\phi_N(z)}$

Repeat Correspondence between the size of $p(z)$ and the value of $\Phi_N(z)$

$$|p(z)| = e^{N\Phi_N(z)}$$

If $\Phi_N(z)$ is approximately constant for $z \in [-1, 1]$, then $p(z)$ is approximately constant there too.

If $\Phi_N(z)$ varies along $[-1, 1]$, on the other hand, the effect on $|p(z)|$ will be variations that grow exponentially with N.

In this framework we may take the limit $N \to \infty$ and think in terms of points $\{x_j\}$ distributed in $[-1, 1]$ according to density function $\rho(x)$ with $\int_{-1}^{1} \rho(x)\, dx = 1$. Such a function gives the number of grid points in an interval $[a, b]$ by the integral value

$$N \int_a^b \rho(x)\, dx$$

Remark For finite $N < +\infty$, ρ must be the sum of Dirac delta functions of amplitude N^{-1} but in the limit $N \to +\infty$, we "may" take it to be smooth.

- For equidistant/equispaced points: the limit of ρ is

$$\rho(x) = \frac{1}{2} \qquad x \in [-1, 1] \quad \text{(UNIFORM DENSITY)}$$

The corresponding potential ϕ is given by the integral

$$\phi(z) = \int_{-1}^{1} \rho(x) \log|z-x|\, dx$$

$\Longrightarrow$ The potential for equispaced points in the limit $N \to \infty$ is

$$\phi(z) = -1 + \underbrace{\operatorname{Re}}_{\text{Denote the real part}} \left\{ \frac{z+1}{2} \log(z+1) + \frac{z-1}{2} \log(z-1) \right\}$$

$\Longrightarrow \phi(0) = -1, \quad \phi(\pm 1) = -1 + \log 2$

Conclusion If a polynomial p has roots equally spaced in $[-1, 1]$, then it will take values about 2^N times larger near $x = \pm 1$ than near $x = 0$.

$$|p(x)| \simeq e^{N\phi(x)} = \begin{cases} (2/e)^N & \text{near } x = \pm 1 \\ (1/e)^N & \text{near } x = 0 \end{cases}$$

• For Chebyshev points:

$$\rho(x) = \frac{1}{\pi\sqrt{1-x^2}} \qquad x \in [-1, 1]$$

The corresponding potential ϕ is

$$\phi(z) = \int_{-1}^{1} \rho(x) \log|z-x|\,dx$$

$$= \log \frac{|z - \sqrt{z^2-1}|}{2}$$

Conclusion If a monic polynomial p has N roots spaced according to the Chebyshev distribution in $[-1, 1]$, then it will oscillate between values of comparable size on the order of 2^{-N} throughout $[-1,1]$: $|p(x)| \simeq e^{N\phi(x)} = 2^{-N}$.

$$\Rightarrow \quad |p(x)| \simeq \exp(N\phi(x)) = 2^{-N}, \quad x \in [-1, 1] \Longrightarrow \textbf{Explanation}$$

$\ast$ The level curves of $\phi(z)$ are the ellipses with foci ± 1, and the value of $\phi(z)$ along any such ellipse is the logarithm of half the sum of the semi-major and semi-minor axes. In particular, the degenerate ellipse $[-1, 1]$ is a level curve where $\phi(z)$ takes constant value $-\log 2$.

$\longrightarrow$ **Conclusion** : See above

Theorem Accuracy of polynomial interpolation

Given a function u and a sequence of sets of interpolation points $\{x_j\}_N$, $N = 1, 2, \ldots$ that converge to a density function ρ as $n \to \infty$ with corresponding potential ϕ given by

$$\phi(z) = \int_{-1}^{1} \rho(x) \log |z - x| \, dx ,$$

define

$$\phi_{[-1,1]} = \sup_{x \in [-1,1]} \phi(x).$$

For each N construct the polynomial p_N of degree $\leqslant N$ that interpolates u at the points $\{x_j\}_N$. If there exists a constant $\phi_u > \phi_{[-1,1]}$ such that u is analytic throughout the closed region

$$\{ z \in \mathbb{C} : \phi(z) \leqslant \phi_u \},$$

then there exists a constant $C > 0$ such that for all $x \in [-1, 1]$ and all N,

$$|u(x) - p_N(x)| \leqslant C e^{-N(\phi_u - \phi_{[-1,1]})}.$$

Theorem Accuracy of polynomial interpolation (continued)

The same estimate holds, though with a new constant C (still independent of x and N), for the difference of the ν^{th} derivatives, namely $u^{(\nu)} - p_N^{(\nu)}$, for any $\nu \geq 1$.

Theorem Accuracy of Chebyshev spectral differentiation

Suppose u is analytic on and inside the ellipse with foci ± 1 on which the Chebyshev potential takes the value ϕ_f, that is, the ellipse whose semi-major and semi-minor axis lengths sum to $K = e^{\phi_f + \log 2}$. Let w be the νth Chebyshev spectral derivative of u. $(\nu \geq 1)$. Then

$$|w_j - u^{(\nu)}(x_j)| = O(e^{-N(\phi_f + \log 2)}) = O(K^{-N})$$

as $N \to +\infty$.