Homework Set #4 – PHYS 6260
Prof. John Wise
Due Friday, February 9th, 11:59pm (Submit github URL to Canvas; all code on github)

- Your assignment should be uploaded as a **single Jupyter notebook** with all of the problems included.
- Please use the template notebook uploaded on Canvas and github as a starter.
- Comment your code through inline comments with # or markdown blocks, where the latter option is preferred.
- In the problem descriptions, "programs" are referring to single or multiple code blocks in a notebook.
- The materials that you are required to include are indicated at the end of each problem, next to the check symbol: ☑

---

1. **One-dimensional advection (50 points total):** For an incompressible fluid that is subject to no external forces will obey the advection equation,

$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} = 0 \tag{1}$$

$$\frac{\partial q}{\partial t} + \frac{df}{dq}\frac{\partial q}{\partial x} = 0, \tag{2}$$

$$\frac{\partial q}{\partial t} + c\frac{\partial q}{\partial x} = 0, \tag{3}$$

where q is some advected quantity (e.g. density, momentum), and $c \equiv df/dq$ is the wave speed. Fluid advection is the most basic behavior that needs to be modeled before moving onto Euler's equations, incompressible fluids, and Riemann problems.

Consider a one-dimensional system in the domain $x \in [0, 1]$ m with a density $q = 1$ kg/m^3 at $x \in [0.3, 0.6]$ m, $q = 0.1$ kg/m^3 elsewhere, and a wave speed $c = 1$ m/s. Calculate the following problems with $N = 250$ points.

(a) (25 points) Use the FTCS method to evolve the system for a total time of $t_{\rm end} = 0.1$ s with a timestep $dt = 10^{-4}$ s. You should find that the numerical solution, using FTCS, is unstable, regardless of the CFL number. First plot the initial condition of the system. Then create a movie of the system's evolution, using the example code in the template notebook. Describe the numerical artifacts and the reasons they arise.

(b) (25 points) Now solve the same system with the FTUS (Forward Time, Upwind Space) scheme. Here the finite differences are skewed in the "upwind" direction, i.e. the direction from which the advecting flow originates. This method is formally stable for the advection equation, which reads after finite differencing as

$$q_i^{n+1} = q_i^n - \frac{cdt}{dx}\left(q_i^n - q_{i-1}^n\right) \quad (c > 0) \tag{4}$$

or

$$q_i^{n+1} = q_i^n - \frac{cdt}{dx}\left(q_{i+1}^n - q_i^n\right) \quad (c < 0). \tag{5}$$

Here c is the wave speed. With a stable solver, we can now integrate the system longer. Evolve the system for a total time of $t_{\text{end}} = 2$ s with a timestep $dt = 10^{-3}$ s. Because the square wave will interact with the boundary, consider periodic boundary conditions for the system. Create a movie of the system's evolution and describe the evolution of the system.

☑ **For full credit**, include your program with comments along with the evolution movies and any answers to the questions above.

2. **Two-dimensional advection (40 points total):** Now that we have explored the advection equation in one dimension, let's extend our code to two dimensions. The 3D advection equation is given by the continuity equation

$$\frac{\partial q}{\partial t} + \nabla \cdot (q\vec{u}) = 0, \tag{6}$$

where $\vec{u}$ is the velocity. For an incompressible flow, $\nabla \cdot \vec{u} = 0$, giving

$$\frac{\partial q}{\partial t} + \vec{u} \cdot \nabla q = 0. \tag{7}$$

In two dimensions, we can apply the FTUS scheme to it to find

$$q_{i,j}^{n+1} = q_{i,j}^n - \frac{c_x dt}{dx}\left(q_{i,j}^n - q_{i-1,j}^n\right) - \frac{c_y dt}{dx}\left(q_{i,j}^n - q_{i,j-1}^n\right) \tag{8}$$

given that $c_x > 0$ and $c_y > 0$.

Consider a square domain $x \in [0, 1]$ m, $y \in [0, 1]$ m with $N = 64$ points in each dimension. It has an initial density

$$q^0(x, y) = \sin(2\pi x) + \cos(2\pi y). \tag{9}$$

The wave speed points in a diagonal, $c_x = c_y = 1$ m/s. Evolve the system for a total of $t_{\text{end}} = 0.5$ s with a timestep $dt = 10^{-3}$ s and CFL number of 0.5. Just like problem 1(b), you will have to consider periodic boundary conditions. First plot the initial condition of the system and then create a movie of the system's evolution. Describe the evolution of the system and any numerical artifacts that might arise.

☑ **For full credit**, include your program with comments along with the evolution movies and any answers to the questions above.

3. **Application question (10 points):** In a couple of paragraphs (about 250 words), discuss different numerical methods and/or modifications or variations in the methods used in the two problems that would diminish the numerical artifacts seen. How would you quantify the errors and determine their reduction as you change the parameters or numerical methods in your approach? You do not have to provide any code.