CSE 6643 Homework 1

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1 Basics [25 pts]

(a) [5 pts]

Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ is a basis of the vector space $V \subset \mathbb{R}^n$. Prove that the list

$$\boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{v}_2 + \boldsymbol{v}_3, \boldsymbol{v}_3 + \boldsymbol{v}_4, \boldsymbol{v}_4 \tag{1}$$

is also a basis of V.

1. Since $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ is a basis of the vector space $V \subset \mathbb{R}^n$, for all $\boldsymbol{u} \in V$,

$$\boldsymbol{u} = a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + a_3\boldsymbol{v}_3 + a_4\boldsymbol{v}_4, \;\; a_i \in \mathbb{R}.$$

2. Rearranging terms:

$$\begin{aligned}
\boldsymbol{u} &= a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + a_3\boldsymbol{v}_3 + a_4\boldsymbol{v}_4 && (2)\\
&= a_1\boldsymbol{v}_1 + 2a_2\boldsymbol{v}_2 - a_2\boldsymbol{v}_2 + 2a_3\boldsymbol{v}_3 - a_3\boldsymbol{v}_3 + 2a_4\boldsymbol{v}_4 - a_4\boldsymbol{v}_4 && (3)\\
&= a_1(\boldsymbol{v}_1 + \frac{2a_2}{a_1}\boldsymbol{v}_2) + (-a_2)(\boldsymbol{v}_2 + \boldsymbol{v}_3) + ()(\boldsymbol{v}_3 + \boldsymbol{v}_4) + (-a_4)\boldsymbol{v}_4 && (4)\\
& && (5)
\end{aligned}$$

(b) [10 pts]

For U a subspace of the vector space $V \subset \mathbb{R}^n$ with $\dim(U) = \dim(V)$. Prove that $U = V$.

(c) [10 pts]

Show that the subspaces of $\mathbb{R}^3$ are precisely $\{0\}$, $\mathbb{R}^3$, all lines in $\mathbb{R}^3$ through the origin, and all planes in $\mathbb{R}^3$ through the origin.

2 Norm Equivalencies [25 pts]

In a finite-dimensional space, all norms are equivalent. In this problem, you will be asked to verify this theorem for some special norms. Prove the following inequalities.

Let $\boldsymbol{x} \in \mathbb{R}^n$ be an n-dimensional vector. Let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ be an $m \times n$ matrix. Then:

(a) [10 pts]

$$\|\boldsymbol{x}\|_\infty \le \|x\|_2 \le \sqrt{n}\|\boldsymbol{x}\|_\infty.$$

(b) [7.5 pts]

$$\|\boldsymbol{A}\|_\infty \le \sqrt{n}\|\boldsymbol{A}\|_2.$$

(c) [7.5 pts]

$$\|\boldsymbol{A}\|_2 \leq \sqrt{m}\|\boldsymbol{A}\|_\infty.$$

3 Perturbing [25 pts]

For $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{K}^m$, the matrix $\boldsymbol{A} \coloneqq \mathbf{I} + \boldsymbol{u}\boldsymbol{v}^*$ is called a *rank-one* perturbation of the identity.

(a) [15 pts]

Show that if $\boldsymbol{A}$ is nonsingular, then its inverse has the form $\boldsymbol{A}^{-1} = \mathbf{I} + \alpha\boldsymbol{u}\boldsymbol{v}^*$ for some scalar α, and give an expression for α.

(b) [5 pts]

For what $\boldsymbol{u}$ and $\boldsymbol{v}$ is $\boldsymbol{A}$ nonsingular?

(c) [5 pts]

If $\boldsymbol{A}$ is singular, what is $\mathrm{null}(\boldsymbol{A})$?