CSE 6643 Homework 1 Solutions

Schäfer, Spring 2023

Deadline: Jan. 19 Thursday, 8:00 am

- There are 2 sections in grade scope: Homework 1 and Homework 1 Programming. Submit your answers as a PDF file to Homework 1 (report the results that you obtain using programming by using plots, tables, and a description of your implementation like you would when writing a paper.) and also submit your code in a zip file to Homework 1 Programming.
- Programming questions are posted in Julia. You are allowed to use basic library functions like sorting, plotting, matrix-vector products etc, but nothing that renders the problem itself trivial. Please use your common sense and ask the instructors if you are unsure. You should never add additional packages to the environment.
- Late homework incurs a penalty of 20% for every 24 hours that it is late. Thus, right after the deadline, it will only be worth 80% credit, and after four days, it will not be worth any credit.
- We recommend the use of LaTeX for typing up your solutions. No credit will be given to unreadable handwriting.
- List explicitly with whom in the class you discussed which problem, if any. Cite all external resources that you were using to complete the homework. For details, consult the collaboration policy in the class syllabus on canvas.

1 Basics [25 pts] <Anshuman Sinha>

(a) [5 pts]

Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ is a basis of the vector space $V \subset \mathbb{R}^n$. Prove that the list

$$\boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{v}_2 + \boldsymbol{v}_3, \boldsymbol{v}_3 + \boldsymbol{v}_4, \boldsymbol{v}_4 \tag{1}$$

is also a basis of V.

Solution. It is given that the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ form basis of the vector space $V \subset \mathbb{R}^n$. Thus, we know that:

1. Vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ are linearly independent. ...(a)
2. The span of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ is V . ...(b)

Now, we need to show that the set of vectors $\boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{v}_2 + \boldsymbol{v}_3, \boldsymbol{v}_3 + \boldsymbol{v}_4, \boldsymbol{v}_4$ is also a basis. Thus, similar to above, we need to prove the following 2 properties:

1. Vectors $\boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{v}_2 + \boldsymbol{v}_3, \boldsymbol{v}_3 + \boldsymbol{v}_4, \boldsymbol{v}_4$ are linearly independent. ...(c)
2. The span of vectors $\boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{v}_2 + \boldsymbol{v}_3, \boldsymbol{v}_3 + \boldsymbol{v}_4, \boldsymbol{v}_4$ is V . ...(d)

We first prove (c): To prove (c), we take a linear combination of vectors $\boldsymbol{v}_1+\boldsymbol{v}_2, \boldsymbol{v}_2+\boldsymbol{v}_3, \boldsymbol{v}_3+\boldsymbol{v}_4, \boldsymbol{v}_4$ with coefficients α_1, α_2, α_3, α_4 respectively such that the linear combination results in a $\mathbf{0}$ vector.

$$\alpha_1 * (\boldsymbol{v}_1 + \boldsymbol{v}_2) + \alpha_2 * (\boldsymbol{v}_2 + \boldsymbol{v}_3) + \alpha_3 * (\boldsymbol{v}_3 + \boldsymbol{v}_4) + \alpha_4 * (\boldsymbol{v}_4) = \mathbf{0} \tag{2}$$

$$\boldsymbol{v}_1 * (\alpha_1) + \boldsymbol{v}_2 * (\alpha_1 + \alpha_2) + \boldsymbol{v}_3 * (\alpha_2 + \alpha_3) + \boldsymbol{v}_4 * (\alpha_3 + \alpha_3) = \mathbf{0} \tag{3}$$

Carefully, looking at the above derived equation (3); we find that it actually represents linear combination of the original basis vectors. Hence, by (a), we know that (α_1),$(\alpha_1 + \alpha_2)$,$(\alpha_2 + \alpha_3)$,$(\alpha_3 + \alpha_4)$ must all be 0. Thus can only be true when, α_1, α_2, α_3, α_4 are all equal to 0.

$$\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0, \alpha_4 = 0 \tag{4}$$

Therefor, from 3 and 4 we can conclude that the set of vectors $\boldsymbol{v}_1+\boldsymbol{v}_2, \boldsymbol{v}_2+\boldsymbol{v}_3, \boldsymbol{v}_3+\boldsymbol{v}_4, \boldsymbol{v}_4$ forms an independent set.

Now we need to prove that this independent set also spans V. Which would mean that, any vector $\boldsymbol{u}$ in vector space V can be written as a linear combination of the set of these independent vectors $\boldsymbol{v}_1+\boldsymbol{v}_2, \boldsymbol{v}_2+\boldsymbol{v}_3, \boldsymbol{v}_3+\boldsymbol{v}_4, \boldsymbol{v}_4$.

Since, vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ form a basis of the vector space V (given in the problem), they can be used to express the arbitrary vector $\boldsymbol{u}$ as their linear combination with some scalars β_1, β_2, β_3, β_4 .

$$\beta_1\boldsymbol{v}_1 + \beta_2\boldsymbol{v}_2 + \beta_3\boldsymbol{v}_3 + \beta_4\boldsymbol{v}_4 = \boldsymbol{u} \tag{5}$$

Similarly, our aim is to find some non-zero scalars δ_1, δ_2, δ_3, δ_4 such that a similar linear combination can be made with our new independent set of vectors $\boldsymbol{v}_1+\boldsymbol{v}_2, \boldsymbol{v}_2+\boldsymbol{v}_3, \boldsymbol{v}_3+\boldsymbol{v}_4, \boldsymbol{v}_4$ such that.

$$\delta_1 * (\boldsymbol{v}_1 + \boldsymbol{v}_2) + \delta_2 * (\boldsymbol{v}_2 + \boldsymbol{v}_3) + \delta_3 * (\boldsymbol{v}_3 + \boldsymbol{v}_4) + \delta_4 * (\boldsymbol{v}_4) = \boldsymbol{u} \tag{6}$$

$$\boldsymbol{v}_1 * (\delta_1) + \boldsymbol{v}_2 * (\delta_1 + \delta_2) + \boldsymbol{v}_3 * (\delta_2 + \delta_3) + \boldsymbol{v}_4 * (\delta_3 + \delta_3) = \boldsymbol{u} \tag{7}$$

Just like before, we re-arrange equation 6 to form equation 7, so that we can use our inference from equation 5. From above equations (6 & 7) we see that, $\delta_1 = \beta_1$, $\delta_1 + \delta_2 = \beta_2$, $\delta_2 + \delta_3 = \beta_3$, $\delta_3 + \delta_4 = \beta_4$. Rearranging the terms we obtain, $\delta_1 = \beta_1$, $\delta_2 = \beta_2 - \beta_1$, $\delta_3 = \beta_3 - \beta_2$, $\delta_4 = \beta_4 - \beta_3$. Since, β_1, β_2, β_3, β_4 are all unique constants which are not all zero, we conclude that δ_1, δ_2, δ_3, δ_4 are also unique constants which can't all be zero together.

□

(b) [10 pts]

For U a subspace of the vector space $V \subset \mathbb{R}^n$ with $\dim(U) = \dim(V)$. Prove that $U = V$.

Solution. It is given in the question that U a subspace of the vector space V, Which means that U is a subset of V and a vector space as well. Along with that it is also given that $\dim(U) = \dim(V)$, which would mean the number of vectors which form the basis of U and V are the same.

In order to prove that $U = V$, i.e. both are identical vector spaces; we would require to prove the following two statements to close the problem:

1. $\forall u_i \in U$, implies $u_i \in V$ i.e. for every vector u_i in U , it also belongs to V.
2. $\forall v_i \in V$, implies $v_i \in U$ i.e. for every vector u_i in V , it also belongs to U.

Statement 1 is easily proved, as it is already given in the question that, U a subspace of the vector space V. This would mean that any vector in U would be already present in V.

In order to prove Statement 2; we assume a vector $x \in V$, note that the vector x can be written in terms of linear combination of the n basis vectors of vector space V. Let the basis vectors of V be $\boldsymbol{v}_1, \boldsymbol{v}_2, ...\boldsymbol{v}_n$

$$\boldsymbol{x} = \beta_1 \boldsymbol{v}_1 + \beta_2 \boldsymbol{v}_2 ... + \beta_n \boldsymbol{v}_n \tag{8}$$

Since U a subspace of the vector space V then the basis vectors of U $(\boldsymbol{u}_1, \boldsymbol{u}_2, ...\boldsymbol{u}_k)$ can always be extended to span the whole vector space V. Which would imply that the vector x can now be written with the help of basis vectors of U along with some extension vectors.

But while extending the vector space, we can use the fact that $\dim(U) = \dim(V)$; which mean we don't require to add any extension vector (and k=n). Hence we can write

$$\boldsymbol{x} = \gamma_1 \boldsymbol{u}_1 + \gamma_2 \boldsymbol{u}_2 ... + \gamma_n \boldsymbol{u}_n \tag{9}$$

Hence, any vector x can be mapped by linear combination of basis vectors of space U.

□

(c) [10 pts]

Show that the subspaces of $\mathbb{R}^3$ are precisely $\{0\}$, $\mathbb{R}^3$, all lines in $\mathbb{R}^3$ through the origin, and all planes in $\mathbb{R}^3$ through the origin.

Solution. In order to prove the given question, again we follow the same solution strategy. We prove the following two statements to close the problem.

1. $\{0\}$, $\mathbb{R}^3$, all lines in $\mathbb{R}^3$ through the origin, and all planes in $\mathbb{R}^3$ through the origin are each sub-spaces of $\mathbb{R}^3$. ... (a)
2. Any subspace of $\mathbb{R}^3$ would already lie in one of the sub-spaces which are mentioned in the previous point. ... (b)

In order to construct the proofs of our problem, we first look at the conditions which would make any subset a subspace. So, any subset of a vector space is subspace of that vector space when the following three conditions hold

1. additive identity: $0 \in U$
2. closure under addition: $u, v \in U \implies u + v \in U$
3. closure under scalar multiplication: $u \in U$, then for any $\alpha \in \mathbb{R}, \alpha u \in U$

Let's first prove (a):

- 0 : $0 \subseteq \mathbb{R}^3$ and all the three conditions of subspace are easily satisfied.
- $\mathbb{R}^3 : \mathbb{R}^3 = \mathbb{R}^3$ hence no further proof required.
- all lines in $\mathbb{R}^3$ through the origin : Equation of a vector that lies on the line that passes through origin and has direction along the vector v is $0 + tv = tv$, where 0 is the position vector of origin, which is a zero vector. Now consider any two vector u_1 and u_2 on the line with having equations $t_1 u$ and $t_2 u$; their sum $(t_1 + t_2)u$ is also on a line passing through origin; so is the scalar product $zt_1 u$ where $z \in \mathbb{R}$
- all planes in $\mathbb{R}^3$ through the origin : Equation of the plane which contains a general vector v that lies on the plane passing through origin is $v \cdot n$, where n is the normal vector of the plane. Now consider any two vector u_1 and u_2 in this plane; their sum u_1 and u_2 will also satisfy the equation of the plane as $(u_1 + u_2) \cdot n = 0$; so is the scalar product $zu_1 \cdot n$ where $z \in \mathbb{R}$.

Now we move towards proving (b):

As we are given the original vector space is $\mathbb{R}^3$, which means it has a dim = 3. i.e. a set of 3 basis vectors will span the entire vector space. Any vector space U in order to satisfy the criteria of sub-spaces must have $\mathbb{R}^3$ must have $\dim(U) \leq \dim(R^3)$ and must satisfy the properties of sub-spaces which are mentioned above.

Which brings us to all the vector spaces which can either have dimensions $0, 1, 2, 3$, but not all the vector space will satisfy the subspace criteria.

- Vector space with dim = 0: A vector space with zero dimension and satisfying the properties of sub-spaces, i.e. containing origin is (closed under scalar multiplication) will only be 0.
- Vector space with dim = 1: A vector space with 1 dimension and satisfying the sub-space properties, i.e. containing the origin (closed under scalar multiplication) must pass through the origin. Let, v_1 be the basis vector of this vector space; then the equation for all such vectors can be parameterised as $\alpha v_1 = 0$ which is the parametric equation of a line passing through origin.
- Vector space with dim = 1: A vector space with 2 dimension and satisfying the sub-space properties, i.e. containing the origin (closed under scalar multiplication) must pass through the origin. Now it should also be closed under addition, i.e. let's take any two general vector's v_1 and v_2 of this vector space, now any linear combination (using property 1 and 2 of sub spaces together) $\alpha v_1 + \beta v_2$ must also lie in the sub-space itself. Which represents the equation of a plane, as we know the subspace must also contain the zero vector, thus it shows all possible plane through origin.

- Vector space with dim $= 3$: This will always be the same $\mathbb{R}^3$ vector space. and as $\mathbb{R}^3 = \mathbb{R}^3$ thus it always a subspace of itself.

$\square$

2 Norm Equivalencies [25 pts]

In a finite-dimensional space, all norms are equivalent. In this problem, you will be asked to verify this theorem for some special norms. Prove the following inequalities.

Let $\boldsymbol{x} \in \mathbb{R}^n$ be an n-dimensional vector. Let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ be an $m \times n$ matrix. Then:

(a) [10 pts]

$$\|\boldsymbol{x}\|_\infty \leq \|x\|_2 \leq \sqrt{n}\|\boldsymbol{x}\|_\infty.$$

Solution.

$$\|x\|_\infty = \max_{1\leq i\leq n} |x_i| = \left(\max_{1\leq i\leq n} |x_i|^2\right)^{1/2} \leq \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} = \|x\|_2,$$

$$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} \leq \left(\sum_{i=1}^{n} \max_{1\leq i\leq n} |x_i|^2\right)^{1/2} = \sqrt{n}\left(\max_{1\leq i\leq n} |x_i|^2\right)^{1/2} = \sqrt{n}\|x\|_\infty.$$

$\square$

(b) [7.5 pts]

$$\|\boldsymbol{A}\|_\infty \leq \sqrt{n}\|\boldsymbol{A}\|_2.$$

Solution.

$$\|A\|_\infty = \sup_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} \leq \sup_{x\neq 0} \frac{\|Ax\|_2}{\frac{1}{\sqrt{n}}\|x\|_2} = \sqrt{n}\|A\|_2.$$

$\square$

(c) [7.5 pts]

$$\|\boldsymbol{A}\|_2 \leq \sqrt{m}\|\boldsymbol{A}\|_\infty.$$

Solution.

$$\|A\|_2 = \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} \leq \sup_{x\neq 0} \frac{\sqrt{m}\|Ax\|_\infty}{\|x\|_\infty} = \sqrt{m}\|A\|_\infty.$$

$\square$

3 Perturbing [25 pts]

For $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{K}^m$, the matrix $\boldsymbol{A} := \mathbf{I} + \boldsymbol{u}\boldsymbol{v}^*$ is called a *rank-one* perturbation of the identity.

(a) [15 pts]

Show that if $\boldsymbol{A}$ is nonsingular, then its inverse has the form $\boldsymbol{A}^{-1} = \mathbf{I} + \alpha \boldsymbol{u}\boldsymbol{v}^*$ for some scalar α, and give an expression for α.

Solution. We first try solving the equation for α:

$$(\mathbf{I} + \boldsymbol{u}\boldsymbol{v}^*)(\mathbf{I} + \alpha \boldsymbol{u}\boldsymbol{v}^*) = \mathbf{I},$$

that is,

$$(1 + \alpha(1 + \boldsymbol{v}^*\boldsymbol{u}))\boldsymbol{u}\boldsymbol{v}^* = 0$$

Thus, if $\boldsymbol{u}\boldsymbol{v}^* = 0$ (or equivalently $\boldsymbol{u} = \boldsymbol{v} = 0$), then α can be arbitrary value in $\mathbb{K}$; if $\boldsymbol{u}\boldsymbol{v}^* \neq 0$, then $\alpha = -1/(1 + \boldsymbol{v}^*\boldsymbol{u})$. But in this case, we should have $1 + \boldsymbol{v}^*\boldsymbol{u} \neq 0$ as a prerequisite. In the following, we show that if A is nonsingular, $1 + \boldsymbol{v}^*\boldsymbol{u} \neq 0$ automatically holds.

For any $\boldsymbol{x} \in \mathbb{K}^m$, $\boldsymbol{A}\boldsymbol{x} = 0$ is equivalent to $(\mathbf{I} + \boldsymbol{u}\boldsymbol{v}^*)\boldsymbol{x} = 0$, that is,

$$\boldsymbol{x} = -\boldsymbol{u}\boldsymbol{v}^*\boldsymbol{x} = -(\boldsymbol{v}^*\boldsymbol{x})\boldsymbol{u}, \quad \text{(note that } \boldsymbol{v}^*\boldsymbol{x} \text{ is a scalar in } \mathbb{K}).$$

It implies that $\boldsymbol{x} \in \operatorname{span}\{\boldsymbol{u}\}$. Then there exists $c \in \mathbb{K}$ such that $\boldsymbol{x} = c\boldsymbol{u}$. Thus,

$$\boldsymbol{A}\boldsymbol{x} = (\mathbf{I} + \boldsymbol{u}\boldsymbol{v}^*)c\boldsymbol{u} = c(\boldsymbol{u} + \boldsymbol{u}\boldsymbol{v}^*\boldsymbol{u}) = c(1 + \boldsymbol{v}^*\boldsymbol{u})\boldsymbol{u},$$

which implies if $\boldsymbol{A}\boldsymbol{x} = 0$, then

$$c(1 + \boldsymbol{v}^*\boldsymbol{u}) = 0, \quad \text{or} \quad \boldsymbol{u} = 0. \tag{10}$$

Case 1: $c(1 + \boldsymbol{v}^*\boldsymbol{u}) = 0$. If A is nonsingular, $\boldsymbol{x}$ must be 0, that is, there is a unique solution $c = 0$ to $c(1 + \boldsymbol{v}^*\boldsymbol{u}) = 0$. It means that we must have $1 + \boldsymbol{v}^*\boldsymbol{u} \neq 0$.

Case 2: $\boldsymbol{u} = 0$. In this case, $\boldsymbol{A} = \mathbf{I}$ is automatically nonsingular. In addition, $1 + \boldsymbol{v}^*\boldsymbol{u} = 1 \neq 0$.

In all, if $\boldsymbol{A}$ is nonsingular, it holds that $1 + \boldsymbol{v}^*\boldsymbol{u} \neq 0$.

(Alternative proof: use formula $\det(I_m + AB) = \det(I_n + BA)$, where A is a $m \times n$ matrix and B is a $n \times m$ matrix. In our case, $\det(A) = \det(I + uv^*) = \det(1 + v^*u) = 1 + v^*u$. Since A is nonsingular if and only if $\det(A) \neq 0$, we have A is nonsingular if and only if $1 + v^*u \neq 0$.)

Thus we can conclude that if $\boldsymbol{u}\boldsymbol{v}^* = 0$, then α can be arbitrary value in $\mathbb{K}$; if $\boldsymbol{u}\boldsymbol{v}^* \neq 0$, then $\alpha = -1/(1 + \boldsymbol{v}^*\boldsymbol{u})$. □

(b) [5 pts]

For what $\boldsymbol{u}$ and $\boldsymbol{v}$ is $\boldsymbol{A}$ nonsingular?

Solution. Claim: $1 + \boldsymbol{v}^*\boldsymbol{u} \neq 0 \Leftrightarrow A$ is nonsingular.

($\Rightarrow$) By (10), if $1 + \boldsymbol{v}^*\boldsymbol{u} \neq 0$, c must be 0 or $\boldsymbol{u} = 0$. In both cases, $\boldsymbol{x} = 0$ and thus A is nonsingular.

($\Leftarrow$) If $1 + \boldsymbol{v}^*\boldsymbol{u} = 0$, c can be arbitrary value in $\mathbb{K}$ to fulfill (10). It means that there exists a nonzero vector $\boldsymbol{x}$ such that $\boldsymbol{A}\boldsymbol{x} = 0$. Thus, $\boldsymbol{A}$ is singular.

(Or you can just use the alternative proof mentioned in (a) to prove it.) □

(c) [5 pts]

If $\boldsymbol{A}$ is singular, what is $\text{null}(\boldsymbol{A})$?

Solution. By (a), $\text{null}(\boldsymbol{A}) = \text{span}\{\boldsymbol{u}\}$, where $\boldsymbol{u}$ must satisfy $\boldsymbol{u} \neq 0$ and $1 + \boldsymbol{v}^*\boldsymbol{u} = 0$. □

4 Julia [25 pts]

This class will use the Julia programming language. This first programming assignment is not very involved, and primarily serves to introduce you to Julia. The homework contains a folder `HW1_CODE` with four files `HW1_your_code.jl`, `HW1_driver.jl`, `Manifest.toml`, `Project.toml`. **Only the first file should be modified!** `HW1_driver.jl` is used to execute your code, while `Manifest.toml` and `Project.toml` tell Julia which versions of packages to use, to ensure reproducibility. You should not install additional packages to solve the homework.

(a) Installing Julia [5 pts]

Solution. Install Julia version 1.8 on your computer, and make yourself familiar with its basic functionality. □

(b) Matrix-vector-multiplication [5 pts]

Solution. Complete the function `u_is_A_times_v!(u, A, v)` in `HW1_your_code.jl` that overwrites the input vector `u` with the product of the input matrix `A` and the input vector `v`.

```
# This function takes in a matrix A and a vector v and writes their product into the vector v
function u_is_A_times_v!(u, A, v)
    for i in 1:length(u)
     u[i] = 0
        for j in 1:length(v)
            u[i] += A[i,j] * v[j]
        end
    end
end
```

□

(c) Matrix-matrix-multiplication [5 pts]

Solution. Complete the function `A_is_B_times_C!(A, B, C)` in `HW1_your_code.jl` that overwrites the input matrix `A` with the product of the input matrices `B` and `C`.

```
# This function takes in matrices ABC and writes B times C into the matrix A
function A_is_B_times_C!(A, B, C)
    for i in 1:size(B)[1]
        for j in 1:size(C)[2]
            A[i,j] = 0
            for k in 1:size(B)[2]
                A[i,j] += B[i,k] * C[k,j]
            end
        end
    end
end
```

□

(d) Testing [5 pts]

Solution. From the directory `HW1_CODE`, run the command

```
julia --project=. HW1 driver.jl
```

to test your code. Here, the `--project=.` tells Julia to use `Manifest.toml` and `Project.toml` determine which version (if any) of packages to use. Make sure that your code passes all `@assert` statements, which test your functions against Julia's built-in functions. □

(e) Optimization [5 pts]

Make sure that your code does not allocate any memory, as evidenced by the `@btime` calls in `HW1_driver.jl` returning `(0 allocations: 0 bytes)`. This is important for performance reasons since allocating memory may be orders of magnitudes slower than floating-point arithmetic. Now try reordering the for-loops in your implementations from parts (a) and (b) and observe the resulting timings provided by `@btime`. Which order leads to the best and worst performance? What are the corresponding wall-clock times as measured by `@btime`?

Solution. The exact time that you get may be slightly different depending on your machine and the random variable you use. Below are rough estimates of the value you should be getting.

The matvec function should calculate

$$u_i = \sum_{j=1}^{n} A_{i,j} v_j$$

For the matvec function the loop orientation in part (c) above runs in 4.283ms

The following reordering runs in 826.4 us

```
function u_is_A_times_v!(u, A, v)
    for i in 1:length(u)
        u[i] = 0
    end
    for j in 1:length(v)
        for i in 1:length(u)
            u[i] += A[i,j] * v[j]
        end
    end
end
```

The matmul function computes the following

$$A_{i,j} = \sum_{k=1}^{m} B_{i,k} C_{k,j}$$

Loop Order	**Time**
ijk	2.55s
jik	6.57s
ikj	24.17s
jki	1.80s
kij	2.98s
kji	21.24s

here are the run times depending on the ordering of the i, j, k loops

The following order resutls in the fastest runtime

```
function A_is_B_times_C!(A, B, C)
    for i in 1:size(B,1)
        for j in 1:size(C,2)
            A[i,j] = 0
        end
    end
    for j in 1:size(C,2)
        for k in 1:size(B,2)
            for i in 1:size(B,1)
                A[i,j] += B[i,k] * C[k,j]
            end
        end
    end
end
```

□