CSCI-B 565, Homework 1.

A.

To show that $y^T z = \|z\|^2$, where y is a unit vector, x is another vector, and z is the projection of y onto x.

$$z = \frac{y^T x}{x^T x} x$$

$$y^T z = y^T \left(\frac{y^T x}{x^T x} x \right)$$

$$y^T z = \frac{y^T x}{x^T x} y^T x$$

$$y^T z = \frac{(y^T x)^2}{x^T x}$$

- Now, we will compute the l_2-norm of z, which is denoted by $\|z\|^2$

$$\|z\|^2 = \sqrt{z^T z}$$

Substituting $z = \frac{y^T x}{x^T x} x$:

$$\|z\|^2 = \sqrt{\left(\frac{y^Tx}{x^Tx}x\right)^T\left(\frac{y^Tx}{x^Tx}x\right)}$$

Simplifying :

$$\|z\|^2 = \frac{|y^Tx|\sqrt{x^Tx}}{x^Tx} = \frac{|y^Tx|}{\sqrt{x^Tx}}$$

$$y^Tz = \frac{(y^Tx)^2}{x^Tx}$$

and

$$\|z\|^2 = \frac{|y^Tx|}{\sqrt{x^Tx}}$$

since both expressions are quivalent

$$y^Tz = \|z\|^2.$$

B

$(x-2)^2 + (y-5)^2 = 9$

The Lagrangian function is:

$L(x, y, \lambda) = x^2 + y^2 + \lambda[(x-2)^2 + (y-5)^2 - 9]$

- To find the critical points, we need to take the partial derivatives of the lagrangian function.

$\frac{\partial L}{\partial x} = 2x + \lambda \cdot 2(x-2) = 0$

$= 2x + 2\lambda(x-2) = 0$

$x(1+\lambda) = 2\lambda$ —— 1

$\frac{\partial L}{\partial y} = 2y + \lambda \cdot 2(y-5) = 0$

$2y + 2\lambda(y-5) = 0$

$y(1+\lambda) = 5\lambda$ —— 2

$\frac{\partial L}{\partial \lambda} = (x-2)^2 + (y-5)^2 - 9 = 0$

Solving ① & ② ;

$$x = \frac{2\lambda}{1+\lambda}$$

$$y = \frac{5\lambda}{1+\lambda}$$

Substitute this into constraint equation

$$\left(\frac{2\lambda}{1+\lambda} - 2\right)^2 + \left(\frac{5\lambda}{1+\lambda} - 5\right)^2 = 9$$

$\left(\frac{2\lambda - 2(1+\lambda)}{1+\lambda}\right)^2$	$\left(\frac{5\lambda - 5(1+\lambda)}{1+\lambda}\right)^2$
$= \left(\frac{2\lambda - 2 - 2\lambda}{1+\lambda}\right)^2$	$= \left(\frac{5\lambda - 5 - 5\lambda}{1+\lambda}\right)^2$
$= \left(\frac{-2}{1+\lambda}\right)^2$	$= \left(\frac{-5}{1+\lambda}\right)^2$
$= \frac{4}{(1+\lambda)^2}$	$= \frac{25}{(1+\lambda)^2}$

After substituting;

$$\frac{4}{(1+\lambda)^2} + \frac{25}{(1+\lambda)^2} = 9$$

$$\frac{29}{(1+\lambda)^2} = 9$$

$$(1+\lambda)^2 = \frac{29}{9}$$

$$1+\lambda = \frac{\sqrt{29}}{3}$$

$$\lambda = \frac{\sqrt{29}}{3} - 1$$

substituting lamda to get the value of x & y.

$$x = \frac{2\left(\frac{\sqrt{29}}{3} - 1\right)}{1 + \left(\frac{\sqrt{29}}{3} - 1\right)}$$

$$= 2$$

$$y = \frac{5\left(\frac{\sqrt{29}}{3} - 1\right)}{1 + \left(\frac{\sqrt{29}}{3} - 1\right)} = 5$$

c. $D = \begin{pmatrix} 1 & -1 & 8 \\ 4 & 2 & 1 \\ 0 & 1 & 5 \\ 5 & -2 & -5 \\ -2 & 0 & -7 \\ 3 & 5 & 3 \end{pmatrix}$ $v_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$

$\|v_1\| = \sqrt{0^2+1^2+2^2} = \sqrt{5}$ $\hat{v_1} = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

$\|v_2\| = \sqrt{-1^2+-2^2+1^2} = \sqrt{6}$ $\hat{v_2} = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$

$P = \hat{v_1}\hat{v_1}^T + \hat{v_2}\hat{v_2}^T$

$\hat{v_1}\hat{v_1}^T = \frac{1}{5}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}(0\ 1\ 2)$ $\hat{v_2}\hat{v_2}^T = \frac{1}{6}\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}(-1\ -2\ 1)$

$= \frac{1}{5}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix}$ $= \frac{1}{6}\begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{pmatrix}$

$P = \frac{1}{5}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{pmatrix}$

$$MSE = \frac{1}{m} \sum_{i=1}^{m} ||x_i - x_{proj,i}||^2$$

$$v_1 = \begin{pmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}, \quad v_2 = \begin{pmatrix} -1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}$$

$$P = \begin{pmatrix} 0.1667 & 0.3333 & -0.1667 \\ 0.3333 & 0.8667 & 0.0333 \\ -0.1667 & 0.0333 & 0.8333 \end{pmatrix}$$

- After projection, MSE is:

$MSE \simeq 31.21.$

D.

$$L = \begin{pmatrix} -0.42 & -0.002 & -0.10 & +0.89 & -0.08 \\ 0.833 & -0.139 & -0.33 & 0.3 & -0.29 \\ -0.22 & -0.43 & -0.79 & -0.17 & 0.32 \\ -0.10 & -0.89 & 0.33 & -0.05 & -0.40 \\ 0.243 & -0.29 & 0.37 & 0.23 & 0.81 \end{pmatrix}$$

Δ Diag ([11.182, 6.293, 3.60, 2.71])

$$A = \begin{bmatrix} 2 & 0 & -4 & 3 \\ -5 & 1 & 8 & 0 \\ 3 & -3 & 0 & 2 \\ 5 & 1 & 2 & 1 \\ 0 & 2 & 3 & 0 \end{bmatrix}$$

$$A = L \Delta R^T$$

$$\begin{pmatrix} | & | & & | \\ l_1 & l_2 & \cdots & l_r \\ | & | & & | \end{pmatrix} \begin{bmatrix} \delta_1 & 0 & 0 & 0 \\ 0 & \delta_2 & 0 & 0 \\ 0 & 0 & \delta_3 & 0 \\ 0 & 0 & 0 & \delta_4 \end{bmatrix}$$

$$R^T = \frac{L_i^T A}{\Delta} \qquad R^T = \Delta^{-1} L^T A$$

$$\frac{LA}{\Delta} = [-0.42 \quad 0.83 \quad -0.22 \quad -0.10 \quad 0.24]_{1 \times 5}$$

X

$$\begin{bmatrix} 2 & 0 & -4 & 3 \\ -5 & 1 & 8 & 0 \\ 3 & -3 & 0 & 2 \\ 5 & 1 & 2 & 1 \\ 0 & 2 & 3 & 0 \end{bmatrix}_{5\times 4}$$

$= 11.1826$

$R_1^T = [-6.2 \quad \frac{1.88}{11.18} \quad 8.9 \quad -1.8]$

$R_2^T = [-4.8 \quad \frac{-0.27}{6.29} \quad -3.6 \quad -1.7]$

$R_3^T = [0.73 \quad \frac{3.12}{3.60} \quad -0.47 \quad -1.56]$

$R_4^T = [0.67 \quad \frac{1.25}{2.714} \quad -0.27 \quad 2.28]$

$$L^T = \begin{pmatrix} -0.428 & 0.833 & -0.224 & -0.106 & 0.243 \\ -0.502 & -0.139 & -0.430 & -0.842 & -0.291 \\ -0.1034 & -0.333 & -0.793 & 0.331 & 0.37 \\ 0.893 & 0.338 & -0.1703 & -0.052 & 0.233 \\ -0.081 & -0.244 & +0.325 & -0.406 & 0.813 \end{pmatrix}$$

$$U^T A = \begin{pmatrix} -3.143 & -0.647 & 0.163 & -2.555 \\ -3.92 & -0.804 & 4.120 & 2.379 \\ -6.479 & -1.308 & 2.795 & 3.246 \\ 9.602 & 2.567 & -3.768 & 2.747 \\ 4.115 & 0.965 & 2.373 & 0.144 \end{pmatrix}$$

$$\Delta^{-1} = \text{diag}\left(\frac{1}{11.1826}, \frac{1}{6.2933}, \frac{1}{3.6019}, \frac{1}{2.7147}\right)$$

$$\Delta^{-1} = \text{diag}\left(0.0894, 0.1589, 0.2776, 0.3684\right)$$

$$R^T = \begin{pmatrix} -3.14 & -0.64 & 0.16 & -2.55 \\ -3.92 & -0.80 & 4.12 & 2.37 \\ -6.47 & -1.30 & 2.79 & 3.24 \\ 9.60 & 2.56 & -3.76 & 2.74 \\ 4.11 & 0.96 & 2.37 & 0.144 \end{pmatrix} \begin{pmatrix} 0.0894 & 0 & 0 & 0 \\ 0 & 0.1589 & 0 & 0 \\ 0 & 0 & 0.2776 & 0 \\ 0 & 0 & 0 & 0.3684 \end{pmatrix}$$

$$R^T = \begin{pmatrix} -0.2811 & -0.0579 & 0.0146 & -0.2284 \\ -0.6230 & -0.1279 & 0.6546 & 0.3787 \\ -1.7994 & -0.3636 & 0.7751 & 0.9003 \\ 3.5365 & 0.9459 & -1.3871 & 1.0121 \end{pmatrix}$$