1 Random Afriat's Index

1.1 Notations, Afriat's Index, and Revealed Preference

Let us fix $e \in [0,1]$ throughout the paper. Consider a data set $\mathcal{O} = \{(\mathbf{x}^t, \mathbf{p}^t)\}_{t \in T}$ where $\mathbf{x}^t \in \mathbb{R}_+^K$ and $\mathbf{p}^t \in \mathbb{R}_{++}^K$ for each $t \in T$. Let $B^t = \mathbb{B}(\mathbf{p}^t, m^t)$ and $B_e^t = \mathbb{B}(\mathbf{p}^t, e\,m^t)$ where $m^t = \mathbf{p}^t \cdot \mathbf{x}^t$. Let $\mathscr{U}$ be the set of all increasing, concave, and continuous utility functions on $\mathbb{R}_+^K$. The data set $\mathcal{D}$ is e-**rational** if there is a utility function $u \in \mathscr{U}$ such that for each $t \in T$,

$$u(\mathbf{x}^t) \geqslant u(\mathbf{x}) \text{ for any } \mathbf{x} \in B_e^t.$$

The **Afriat index** given data set $\mathcal{O}$, denoted by $\mathrm{AI}(\mathcal{O})$, is the largest $e \in [0,1]$ such that $\mathcal{O}$ is e-**rational**.

We say that $\mathbf{x}^t$ is e-**revealed preferred** to $\mathbf{x}^s$, denoted by $\mathbf{x}^t \succsim_e \mathbf{x}^s$, if $\mathbf{x}^s \in B_e^t$. We say that $\mathbf{x}^t$ is **strictly** e-**revealed preferred** to $\mathbf{x}^s$, denoted by $\mathbf{x}^t >_e \mathbf{x}^s$, if $\mathbf{p}^t \mathbf{x}^s < e\,m^t$. We also say that $\mathcal{O}$ satisfies GARP_e if $\succsim_e$ is acyclic. The following simple extension of Afriat's theorem will be useful.

PROPOSITION 1. *A data set* $\mathcal{O} = \{(\mathbf{x}^t, \mathbf{p}^t)\}_{t \in T}$ *is e-rational if and only if it satisfies GARP_e.*

We omit the proof as it is implied by Theorem 1 of Halevy et al. (2018).

1.2 Random utility framework

We now define Afriat's index for the random utility framework. A data set in this framework takes the form

$$\mathcal{D} = \{(\mu^t, \mathbf{p}^t)\}_{t \in T},$$

where μ^t is a probability distribution over B^t. For simplicity, we assume that μ^t has a finite support. Since Afriat's index meant to measure how much the data set deviates from utility maximization, we certainly do not want assume away scenarios where observed consumption bundles are not on budget lines.

We say that the dataset $\mathcal{D}$ is e-**RUM-rational** if there is a probability distribution ρ on $\mathscr{U}$ such that for any $t \in T$ and $\mathbf{x}^t \in B^t$,

$$\mu^t(\mathbf{x}^t) = \rho\big(\{u \in \mathscr{U} : u(\mathbf{x}^t) \geqslant u(\mathbf{x}) \text{ for any } \mathbf{x} \in B_e^t\}\big).$$

The **random Afriat's index** given data set $\mathcal{D}$, denoted by RAI($\mathcal{D}$), is the largest $e \in [0, 1]$ such that $\mathcal{D}$ is e-RUM-rational.

PROPOSITION 2. *For any data set $\mathcal{D}$, RAI($\mathcal{D}$) exists.*

The above result trivially follows from the fact that any data set is 0-RUM-rationalizable and the discreteness of our data set. Indeed, it is not difficult to extend our results to scenarios with continuous distributions.

1.3 Patches and Rational Matrices

Following Kitamura and Stoye (2018), the following definitions are necessary. Let $\mathcal{Y}^e = \{Y_1^e, \ldots, Y_L^e\}$ be the coarsest partition of $\bigcup_{t \in T} B_t$ such that for any $l \leqslant L$ and $t \in T$, Y_l^e is either completely on, completely strictly above, or completely strictly below budget plane B_e^t. Formally, for any $t \in T$, two elements $\mathbf{x}, \mathbf{x}' \in B^t$ are in the same element of the partition if and only if $\mathrm{sign}(\mathbf{p}^s \mathbf{x} - e\, m^s) = \mathrm{sign}(\mathbf{p}^s \mathbf{x}' - e\, m^s)$ for all $s \neq t$. We refer to elements of $\mathcal{Y}^e$ as *patches*.

We say profiles of consumption bundles $\{\mathbf{x}^t\}_{t \in T}$ and $\{\mathbf{y}^t\}_{t \in T}$ are e-**patch-equivalent** if for each $t \in T$, $\mathbf{x}^t$ and $\mathbf{y}^t$ both belong to the same patch of B^t. The following corollary of Proposition 1 will be useful for our main result.

COROLLARY 1. *If $\{\mathbf{x}^t\}_{t \in T}$ and $\{\mathbf{y}^t\}_{t \in T}$ are e-patch-equivalent, then $\{\mathbf{x}^t, \mathbf{p}^t\}_{t \in T}$ is e-rational if and only if $\{\mathbf{y}^t, \mathbf{p}^t\}_{t \in T}$ is e-rational.*

Proof of Corollary 1. By the definition of patches, e-revealed preference relations for $\{\mathbf{x}^t, \mathbf{p}^t\}_{t \in T}$ and $\{\mathbf{y}^t, \mathbf{p}^t\}_{t \in T}$ identical. Hence, by Proposition 1, $\{(\mathbf{x}^t, \mathbf{p}^t)\}_{t \in T}$ is e-rational if and only if $\{(\mathbf{y}^t, \mathbf{p}^t)\}_{t \in T}$ is e-rational.

$\square$

Note that each budget set B^t can be uniquely expressed as a union of patches and so we may write $B^t = \bigcup_{l \in L_t} Y_{l,t}^e$. Let $\pi^e = (\pi_{l,t}^e)_{l \in L_t, t \in T}$ be the vector such that

$$\pi_{l,t}^e = \mu^t(Y_{l,t}^e) \text{ for any } l \in L_t \text{ and } t \in T,$$

i.e., π^e is the vector representation of budget sets given e.

Let A^e be the e-rational demand matrix, which is the $(0,1)$-matrix such that the vector representation of each e-rationalizable nonstochastic demand system is exactly one column of A^e. In other words,

$$A^e = \begin{bmatrix} a^e_{1,1} & \cdots & a_{1,h} & \cdots & a_{1,H} \\ \vdots & & \vdots & & \vdots \\ a^e_{L_1,1} & \cdots & a_{L_1,h} & \cdots & a_{L_1,H} \\ \vdots & & \vdots & & \vdots \\ a^e_{\mathcal{L}_{t-1}+1,1} & \cdots & a_{\mathcal{L}_{t-1}+1,h} & \cdots & a_{\mathcal{L}_{t-1}+1,H} \\ \vdots & & \vdots & & \vdots \\ a^e_{\mathcal{L}_{t-1}+L_t,1} & \cdots & a_{\mathcal{L}_{t-1}+L_t,h} & \cdots & a_{\mathcal{L}_{t-1}+L_t,H} \\ \vdots & & \vdots & & \vdots \\ a^e_{\mathcal{L}_{T-1}+1,1} & \cdots & a_{\mathcal{L}_{T-1}+1,h} & \cdots & a_{\mathcal{L}_{T-1}+1,H} \\ \vdots & & \vdots & & \vdots \\ a^e_{\mathcal{L}_{T-1}+L_T,1} & \cdots & a_{\mathcal{L}_{T-1}+L_T,h} & \cdots & a_{\mathcal{L}_{T-1}+L_T,H} \end{bmatrix}$$

is a $(0,1)$-matrix such that for each $t \in T$ and $h \leq H$, $\sum_{l \in L_t} a_{\mathcal{L}_{t-1}+l,h} = 1$, and for each $h \leq H$, there is $u_h \in \mathscr{U}$ such that

$$a_{\mathcal{L}_{t-1}+l,h} = 1 \text{ iff } \max u_h(Y^e_{l,t}) \geq \max u_h(B^e_t).$$

1.4 Main Result

We now can state our main result.

THEOREM 1. *A data set $\mathcal{D}$ is e-RUM-rational if and only if there is a probability distribution $\nu \in \Delta^{H-1}$ such that $\pi^e = A^e \nu$.*

Proof of Theorem 1. We first prove the only if direction. Suppose $\mathcal{D}$ is e-RUM-rational; i.e., there is a probability distribution ρ on $\mathscr{U}$ such that for any $t \in T$ and $\mathbf{x}^t \in B^t$,

$$\mu^t(\mathbf{x}^t) = \rho\big(\{u \in \mathscr{U} : u(\mathbf{x}^t) \geq u(\mathbf{x}) \text{ for any } \mathbf{x} \in B^t_e\}\big).$$

We say that utility functions u and v are e-equivalent, denoted by $u \approx_e v$, if the rankings of patches according u and v are identical. Let $\nu_h = \rho(u \in \mathscr{U} : u_h \approx_e u)$. Then

$$\pi^e_{l,t} = \mu^t(Y^e_{l,t}) = \rho\big(\{u \in \mathscr{U} : \max u(Y^e_{l,t}) \geq \max u(B^t_e)\}\big)$$

3

$$= \sum_{h \in H} \mathbf{1}\{\max u_h(Y_{l,t}^e) \geqslant \max u_h(B_t^e)\} \, \nu_h = \sum_{h \leqslant H} a_{\mathcal{L}_{t-1}+l,h} \, \nu_h.$$

In other words, $\pi^e = A_e \, \nu$.

To prove the if direction, suppose there is a probability distribution ν such that $\pi_e = A_e \, \nu$. In other words,

$$\pi_{l,t}^e = \mu^t(Y_{l,t}^e) = \sum_{h \in H} a_{\mathcal{L}_{t-1}+l,h} \, \nu_h \text{ for any } l, t.$$

For given h and t, let $Y_{h,t}$ be the most preferred patch $Y_{l,t}^e$ in B_t; i.e., $Y_{h,t} = Y_{l,t}^e$ and $\max u_h(Y_{l,t}^e) \geqslant \max u_h(B_t^e)$. We write $X = (\mathbf{x}^t)_{t \in T}$, which is a consumption bundle profile. Let $\mathscr{U}_h = \{v \in \mathscr{U} : u_h \approx_e v\}$. Let $\mathcal{X} = \{X = (\mathbf{x}^t)_{t \in T} : \{\mathbf{x}^t, \mathbf{p}^t\}_{t \in T} \text{ is } e\text{-rational}\}$ is the set of all e-rational consumption bundle profiles. Le $\mathcal{X}_h$ be the set of all e-rational consumption bundle profiles that are rationalized by u_h. By Corollary 1, $\mathcal{X}_h = \prod_{t \in T} Y_{h,t}$ and $\{\mathcal{X}_h\}_{h \in H}$ constitutes a partition of $\mathcal{X}$. We also write $X^{-t} = (\mathbf{x}^s)_{s \neq t}$ and $\mathcal{X}_h^{-t} = \{X^{-t} : X \in \mathcal{X}_h \text{ for some } \mathbf{x}^t \in Y_{h,t}\}$. For every h and $X \in \mathcal{X}_h$, there is $u_{X,h} \in \mathscr{U}_h$ such that $\mathbf{x}^t = \arg\max u_{X,h}(Y_{h,t})$ for each t.

Let ρ be the probability distribution over $\mathscr{U}$ such that for any h and $X \in \mathcal{X}_h$,

$$\rho(u_{X,h}) = \nu_h \prod_{t \in T} \frac{\mu^t(\mathbf{x}^t)}{\mu^t(Y_{h,t})}.$$

Note that, since $\mathcal{X}_h = \prod_{t \in T} Y_{h,t}$,

$$\sum_{h \in H} \sum_{X \in \mathcal{X}_h} \rho_{X,h} = \sum_{h \in H} \sum_{X \in \mathcal{X}_h} \nu_h \prod_{t \in T} \frac{\mu^t(\mathbf{x}^t)}{\mu^t(Y_{h,t})} = \sum_{h \in H} \nu_h \prod_{t \in T} \left(\sum_{\mathbf{x}^t \in Y_{h,t}} \frac{\mu^t(\mathbf{x}^t)}{\mu^t(Y_{h,t})} \right) = \sum_{h \in H} \nu_h = 1.$$

Fix t and $\mathbf{y}^t \in B^t$. Note that there is a unique patch $Y_{l,t}^e$ of B_t where $\mathbf{y}^t \in Y_{l,t}^e$. Then

$$\rho\big(\{u \in \mathscr{U} : u(\mathbf{y}^t) \geqslant u(\mathbf{x}) \text{ for any } \mathbf{x} \in B_e^t\}\big) = \sum_{h \in H} \sum_{X \in \mathcal{X}_h} \rho_{X,h} \mathbf{1}\{\mathbf{x}^t = \mathbf{y}^t\}$$

$$= \sum_{h \in H} \sum_{X \in \mathcal{X}_h} \nu_h \prod_{s \in T} \frac{\mu^s(\mathbf{x}^s)}{\mu^t(Y_{h,s})} \mathbf{1}\{\mathbf{x}^t = \mathbf{y}^t\} = \sum_{h \in H} \nu_h \frac{\mu^t(\mathbf{y}^t)}{\mu^t(Y_{l,t}^e)} \mathbf{1}\{Y_{h,t} = Y_{l,t}^e\} \sum_{X^{-t} \in \mathcal{X}_h^{-t}} \prod_{s \neq t} \frac{\mu^s(\mathbf{x}^s)}{\mu^s(Y_{h,s})}$$

$$= \sum_{h \in H} \nu_h \frac{\mu^t(\mathbf{y}^t)}{\mu^t(Y_{l,t}^e)} \mathbf{1}\{Y_{h,t} = Y_{l,t}^e\} \prod_{s \neq t} \sum_{\mathbf{x}^s \in Y_{h,s}} \frac{\mu^s(\mathbf{x}^s)}{\mu^s(Y_{h,s})} = \sum_{h \in H} \nu_h \frac{\mu^t(\mathbf{y}^t)}{\mu^t(Y_{l,t})} \mathbf{1}\{Y_{h,t} = Y_{l,t}^e\}$$

$$= \frac{\mu^t(\mathbf{y}^t)}{\mu^t(Y_{l,t}^e)} \sum_{h \in H} \nu_h \mathbf{1}\{Y_{h,t} = Y_{l,t}^e\} = \frac{\mu^t(\mathbf{y}^t)}{\mu^t(Y_{l,t}^e)} \sum_{h \in H} \nu_h \, a_{\mathcal{L}_{t-1}+l,h} = \frac{\mu^t(\mathbf{y}^t)}{\mu^t(Y_{l,t}^e)} \mu^t(Y_{l,t}^e) = \mu^t(\mathbf{y}^t).$$

$\square$

2 Two goods

GARP and WARP are equivalent when there are only two goods (see Rose (1958) and Chambers and Echenique (2016)) and it is much simpler to check WARP. However, unfortunately, GARP and WARP are not equivalent in the context of e-rationality. Hence, we need to check GARP.

3 Statistical Testing

Using the methodogy in Kitamura and Stoye (2018), our non-parametric testing for $e-$RUM-rationality can be extended to a stastical testing for $e-$RUM-rationality. Hence, we can find (in principle) the largest e where e-RUM-rationality is not rejected and 95% confidence interval for the random Afriat's index.

4 Application

We apply our methodology to the dataset of Halevy et al. (2018).

References

HALEVY, Y., D. PERSITZ, AND L. ZRILL (2018): "Parametric recoverability of preferences," *Journal of Political Economy*, 126, 1558–1593.

KITAMURA, Y. AND J. STOYE (2018): "Nonparametric analysis of random utility models," *Econometrica*, 86, 1883–1909.