4. PARAMETRIC EQUATIONS

4.1 CARTESIAN FORM AND PARAMETRIC FORM

The equation of a curve can be expressed in two forms – Cartesian form and Parametric form.

In Cartesian form, the equation is expressed in terms of the x and y variables.

The equation of a curve can also be expressed in parametric form, where the x and y coordinates are expressed in terms of a variable t, thus

$$x = g(t), \qquad y = h(t)$$

The third variable t is known as the Parameter of the equations.

E.g. The Cartesian equation $y = \frac{x^2}{4}$ and the parametric equations

$x = 2t, \quad y = t^2$ both represent the same curve.

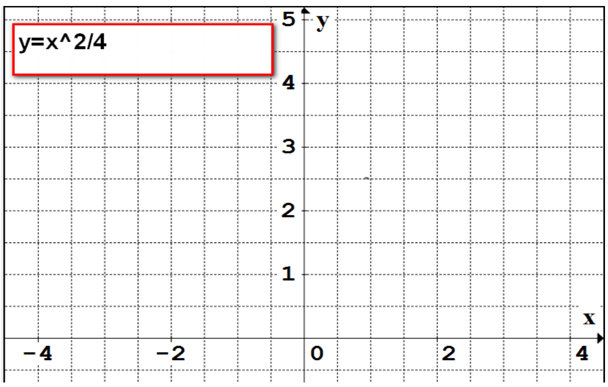
EXAMPLE 1
Fill in the tables and sketch the curves expressed in the

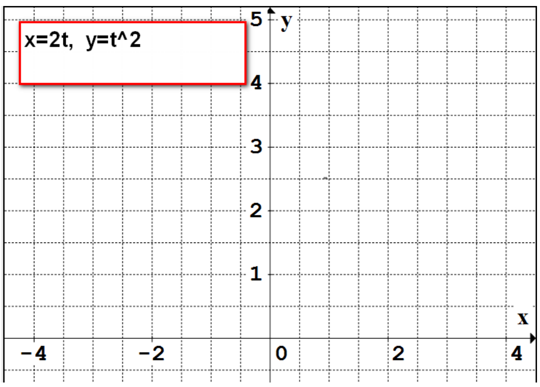
(a) Cartesian equation $y = \frac{x^2}{4}$

(b) Parametric equation $x = 2t, \quad y = t^2$

x	-4	-2	0	2	4
$y = \frac{x^2}{4}$					

t	-2	-1	0	1	2
$x = 2t$					
$y = t^2$					





4.2 CONVERTING: PARAMETRIC TO CARTESIAN

There is no exact method for converting parametric equations for a curve to an equation in x and y only. If we can solve for t in terms of either x or y, we can substitute this for the value of t in one of the equations to get an equation in x and y only.

EXAMPLE 2

Convert the following parametric equation to an equation relating x and y.

(a) $x = 2t, \qquad y = t^2$ (b) $x = 2\sin(t), \qquad y = \cos(t) - 1$

4.3 DERIVATIVES AND PARAMETRIC EQUATIONS

To find $\frac{dy}{dx}$ for a pair of parametric equations

$$x = g(t), \qquad y = h(t).$$

We need to apply the chain rule such that

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{h'(t)}{g'(t)}$$

EXAMPLE 3

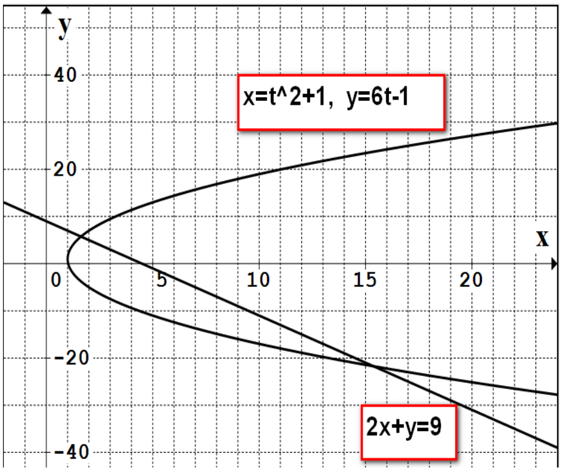
Find the $\frac{dy}{dx}$ of the following parametric equations.

(a) $x = 2t, \qquad y = t^2$ (b) $x = \frac{2-3t}{1+t}, \qquad y = \frac{3+2t}{1+t}$.

EXAMPLE 4

The parametric equations of a curve are $x = t^2 + 1,\ y = 6t - 1$.

(i) The line $2x + y = 9$ meets the curve at points A and B. Find the coordinates of A and of B.



(ii) Obtain an expression for $\dfrac{dy}{dx}$ in terms of t.

(iii) Find the equation of the tangent to the curve at the point (2, −7).

(iv) Obtain the Cartesian equation of the curve.

TUTORIAL 4
PARAMETRIC EQUATIONS

1 A curve is defined parametrically by the equations $x = \frac{3}{(1+t)^2}$, $y = \frac{2-t}{1+t}$, where $t \neq -1$.

(a) Show that $\frac{dy}{dx} = \frac{1+t}{2}$.

(b) Find the equation of the tangent to the curve at the point where the curve crosses the x-axis.

(c) Find the values of t at a point of the intersection of the curve with the line $4x + 3y = 0$.

2 (a) The parametric equations of the curve are $x = t + \frac{1}{t}$ and $y = t^2 - 2t$ (for $t \neq 0$).

Show that $\frac{dy}{dx} = \frac{2t^2}{1+t}$. Find the co-ordinates of the points at which the tangents to the curve are parallel to the line $y = x - 1$.

(Note: When 2 lines are parallel, they have the same gradient)

(b) A curve has parametric equations $x = 2\cos\theta + 1$ and $y = 6\sin\theta$ for $0 \leq \theta \leq \pi$.

(i) Find the equation of the normal to the curve at the point where $\theta = \frac{\pi}{3}$.

(ii) Obtain the Cartesian equation of the curve.

3 A curve is represented parametrically by the equations $x = \frac{4}{(2+t)^2}$, $y = \frac{10}{2+t}$.

Find

(i) the equation of the chord joining the points P and Q with parameters -1 and 0 respectively,

(ii) $\frac{dy}{dx}$ in terms of t,

(iii) the equation of the tangent to the curve at the point $y = 1$, and

(iv) the Cartesian equation of the curve.

4 The parametric equations of a curve are

$$x = t^2 + 1, \qquad y = 3t^2 - 2t.$$

(a) Obtain an expression for $\frac{dy}{dx}$ in terms of t.

(b) Find the coordinates of the turning point of the curve. Hence find the equation of the normal to the curve at this point.

(c) Show that the equation of the normal to the curve at the point $t = 2$ is

$$5y + 2x = 50.$$

Find the value of t at the point where the normal intersect the curve again.

(d) Show that the Cartesian equation of the curve can be expressed in the form

$$4(x - m) = (3x - y - 3)^n,$$

where m and n are constants.

Miscellaneous Exercises

1 A curve is given parametrically by the equations $x = (1+t)^2$, $y = (1-t)^2$.
Find the equation of the tangent to the curve at the point where $x = y$. (MA1301 0809)

2. Given that $x = \sin t + \cos t,\ y = \ln(\cos t) + t$ $\quad 0 < t < \frac{\pi}{2}$, prove that

$$\frac{dy}{dx} = \sec t. \qquad \text{(MA1301 1112)}$$

3. The parametric equations of a curve are $x = a(\ln t + t),\ y = a(1 + t\ln t)$ where a is a constant and $t > 0$.

(i) Find the equation of the normal to the curve at the point where $t = 1$

(ii) Obtain and simplify an expression for $\frac{d^2y}{dx^2}$. (MA1301 1213)

Note:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

ANSWERS

1 (b) $2y = 3x - 1$ (c) $t = 3$ or -2

2 (a) $(-2\frac{1}{2}, 1\frac{1}{4})$ and $(2, -1)$

(b) (i) $\sqrt{3}y = x + 7$ b(ii) $9x^2 + y^2 - 18x - 27 = 0$

3 (i) $3y = 5x + 10$ (ii) $\frac{dy}{dx} = \frac{5}{4}(2 + t)$ (iii) $y = \frac{25}{2}x + \frac{1}{2}$

(iv) $y^2 = 25x$

4 (a) $3 - \frac{1}{t}$ (b) $(1\frac{1}{9}, -\frac{1}{3})$, equation of normal is $x = 1\frac{1}{9}$

(c) $-1\frac{7}{17}$ (d) $4(x - 1) = (3x - y - 3)^2$

Miscellaneous Exercises

1 $y = -x + 2$

3 (i) $y = -2x + 3a$ (ii) $\frac{d^2y}{dx^2} = \frac{t(2 + \ln t + t)}{a(1+t)^3}$