Chapter 6 : Summary of Integration Techniques

Objectives :
1. Discuss and apply the techniques of integration to various integrals.

6.1 Introduction

In finding the derivative of a function, it is obvious which differentiation rules must be applied. But it may not be at all obvious which technique should be used to integrate a given function.

Since the problems in the previous chapters have focused on the method of that chapter, it has usually been clear which method is to be used on a given integral.

Here in this chapter we will try to consolidate and have a strategy to apply Standard Formula, Substitution, Partial Fractions or By Parts method to any given integral.

6.2 Strategy for Introduction

Step 1. Use the Standard Formula.
An important prerequisite and must-know knowledge is Integration by Standard Formula.

The list below is **NOT given** in the Maths Formula card, you might want to memorise them. $(n \neq -1)$

1. $\int (ax+b)^n \, dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + C$
2. $\int \dfrac{1}{ax+b} dx = \dfrac{1}{a} \ln | \, ax+b \, | + C$
3. $\int e^{ax+b} \, dx = \dfrac{1}{a} e^{ax+b} + C$
4. $\int \sin(ax+b) dx = -\dfrac{1}{a} \cos(ax+b) + C$
5. $\int \cos(ax+b) dx = \dfrac{1}{a} \sin(ax+b) + C$
6. $\int [f(x)]^n \cdot f'(x) \, dx = \dfrac{[f(x)]^{n+1}}{n+1} + C$
7. $\int \dfrac{f'(x)}{f(x)} dx = \ln |f(x)| + C$
8. $\int e^{f(x)} f'(x) \, dx = e^{f(x)} + C$

The list below is **GIVEN** in the Maths Formula Card. $(n \neq -1)$

$\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + C$

$\int \dfrac{1}{x} dx = \ln|x| + C$

$\int a^x \, dx = \dfrac{a^x}{\ln a} + C$

$\int e^x \, dx = e^x + C$

$\int \sin x \, dx = -\cos x + C$

$\int \cos x \, dx = \sin x + C$

$\int \tan x \, dx = -\ln|\cos x| + C$

$\int \cot x \, dx = \ln|\sin x| + C$

$\int \sec x \, dx = \ln|\sec x + \tan x| + C$

$\int \csc x \, dx = -\ln|\csc x + \cot x| + C$

$\int \sec^2 x \, dx = \tan x + C$

$\int \csc^2 x \, dx = -\cot x + C$

$\int \sec x \tan x \, dx = \sec x + C$

$\int \csc x \cot x \, dx = -\csc x + C$

$\int \dfrac{1}{x^2+a^2} dx = \dfrac{1}{a} \tan^{-1} \dfrac{x}{a} + C$

$\int \dfrac{1}{x^2-a^2} dx = \dfrac{1}{2a} \ln \left| \dfrac{x-a}{x+a} \right| + C$

$\int \dfrac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1} \dfrac{x}{a} + C$

$\int \dfrac{1}{\sqrt{x^2+a^2}} dx = \ln\left|x + \sqrt{x^2+a^2}\right| + C$

$\int \dfrac{1}{\sqrt{x^2-a^2}} dx = \ln\left|x + \sqrt{x^2-a^2}\right| + C$

However sometimes we need to simplify the integral first before applying any one of the above standard formula.

For Examples 1 and 2 below, use the Laws of Indices to **simplify first** then use the following properties.

- $\int k \cdot f(x)dx = k\int f(x)dx$, k is a constant.
- $\int [f(x) \pm g(x)]dx = \int f(x)\,dx \pm \int g(x)\,dx$

Example 1: Find $\int x^2\left(\sqrt{x}+2x^3\right)dx$

Solution:

$\int x^2\left(\sqrt{x}+2x^3\right)dx$ Open up the brackets, before integrating

$=\int\left(x^{\frac{5}{2}}+2x^5\right)dx$ $a^x \cdot a^y = a^{x+y}$, $\sqrt{a}=a^{\frac{1}{2}}$

$=\frac{x^{\frac{7}{2}}}{\frac{7}{2}}+2\cdot\frac{x^6}{6}+C \;\; \underline{=\frac{2}{7}x^{\frac{7}{2}}+\frac{1}{3}x^6+C}$

Example 2: Find $\int \frac{4x^3-\sqrt{x}}{2x}dx$

Solution:

Note that there is only one term $2x$ in the **denominator** of $\int \frac{4x^3-\sqrt{x}}{2x}dx$

Rewrite the integral, into 2 separate terms:

$$\int \frac{4x^3-\sqrt{x}}{2x}dx = \int\left(\frac{4x^3}{2x}-\frac{x^{\frac{1}{2}}}{2x}\right)dx$$

$$=\int\left(2x^2-\frac{1}{2}x^{-\frac{1}{2}}\right)dx \qquad \frac{a^x}{a^y}=a^{x-y}$$

$$\underline{=\frac{2}{3}x^3-x^{\frac{1}{2}}+C}$$

Example 3: Find $\int \frac{x+1}{x-2}dx$ *Ans:* $x+3\ln|x-2|+C$

Solution:

In this integral $\int \frac{x+1}{x-2}dx$, the fraction is an improper fraction.

An easier method, known as "add a zero method" shown below, is used to simplify the fraction first. Do take note that this can be used only if the degree of the polynomial in the numerator is the same as the degree in the denominator.

$$\int \frac{x+1}{x-2}dx = \int \frac{x-2+2+1}{x-2}dx = \int\left(\frac{x-2}{x-2}+\frac{2+1}{x-2}\right)dx = \int\left(1+\frac{3}{x-2}\right)dx$$

$=$

Step 2. If the integral does not fit into any one of the Standard formula, then use the *<u>Substitution method.</u>*

Try to find some function $u = g(x)$ in the integral whose derivative $du = g'(x)dx$ is also present, apart from a constant factor, as shown in the next few examples. Note that besides the letter *u*, other alphabets can be used too! The transformed integral may now be solved either by ***<u>Standard formula</u>*** or ***<u>Partial Fractions</u>*** (see step 3) or ***<u>By Parts method</u>*** (see step 4).

Example 4: Find $\int \frac{x}{\sqrt{9-x^2}}\,dx$ *Ans*: $-\sqrt{9-x^2}+C$

Solution:

This integral is NOT standard formula.

Try substitute $u = 9 - x^2$ as $du = -2x\,dx$ and $x\,dx$ is present in the given integral.

Similarly in Example 5 below.

Example 5: Find $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$ *Ans*: $-2\cos\sqrt{x}+C$

Solution:

This integral is NOT standard formula.

Try Substitute $p = \sqrt{x}$ as $dp = \frac{1}{2\sqrt{x}}dx$ and $\frac{dx}{\sqrt{x}}$ is present in the given integral.

Example 6: Find $\int \frac{1}{x\sqrt{9-x^2}}\,dx$ *Ans:* $\frac{1}{6}\ln\left|\sqrt{9-x^2}-3\right|-\frac{1}{6}\ln\left|\sqrt{9-x^2}+3\right|+C$

Solution:

Show that if you try substitute $z = 9 - x^2$, you will not be able to integrate. Hence you will need to try another suitable Substitution

Step 3. If the integral is a rational function of the form $\frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials, use *__Partial Fractions__* to decompose the function.

Note that $q(x)$ **must be** factorised completely first!

Example 7: Find $\int \frac{x}{(x-2)(x-1)^2}\,dx$ *Ans*: $2\ln|x-2| - 2\ln|x-1| + \frac{1}{x-1} + C$

Solution: $\frac{x}{(x-2)(x-1)^2} = \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$

$$\int \frac{x}{(x-2)(x-1)^2}\,dx = \int\left(\frac{2}{x-2} - \frac{2}{x-1} - \frac{1}{(x-1)^2}\right)dx$$

$$=$$

Step 4. If the integral is a product of different types of functions, try the *__By Parts method__*.

The different types of functions are as listed in the acronym "LIATE". By Parts method also works on individual functions like $\ln f(x)$.

The By Parts formula : $\int u\,dv = uv - \int v\,du$ (Do not be confused with the u from the Substitution method)

Example 8: Find $\int x\,e^{-2x}\,dx$ *Ans*: $-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$

Solution:

Example 9: Find $\int e^{2x}\cos(e^{x})\,dx$ *Ans:* $e^{x}\sin(e^{x}) + \cos(e^{x}) + C$

Solution:

Step 5. Last step, be observant and persistent!

If a method does not work, be ready to try another and sometimes need several methods to solve. Doing a problem more than one way is good learning experience!

Some of the tutorial problems below may require more than one method to solve.

Tutorial 6

<u>Section A</u> : Integration techniques

Find the following:

1. $\displaystyle\int \frac{x^2}{(x+1)^8}\,dx$
2. $\displaystyle\int \frac{\sqrt{\ln x}}{x}\,dx$
3. $\displaystyle\int x^5 e^{-x^3}\,dx$
4. $\displaystyle\int \sin\sqrt{x}\,dx$
5. $\displaystyle\int \frac{1}{-4+4e^x+e^{-x}}\,dx$

<u>Section B</u> : Multiple Choice Questions

1. Which of the following integrals **cannot** be found using the substitution method?

   (a) $\displaystyle\int x^2 e^{x^3}\,dx$ (b) $\displaystyle\int \frac{x}{1+x^2}\,dx$

   (c) $\displaystyle\int \frac{1}{1+x^2}\,dx$ (d) $\displaystyle\int 4\cos^2 x\sin x\,dx$

2. When $\int xe^{-x^2}\,dx$ is integrated by substitution method, which of the following is **NOT** the right substitution?

   (a) Let $u = x^2$ (b) Let $u = e^{x^2}$

   (c) Let $u = xe^{-x^2}$ (d) Let $u = e^{-x^2}$

3. Integrate $\displaystyle\int \frac{e^{\frac{1}{t}}}{t^2}\,dt$ by using the substitution $u = \dfrac{1}{t}$. The integral can be transformed as

   (a) $\int e^u\,du$ (b) $\int e^{-u}\,du$

   (c) $-\int e^u\,du$ (d) $-\int e^{-u}\,du$

4. $\int \cos x\sin^3 x\,dx$ is equal to

   (a) $\dfrac{\sin x\cos^3 x}{3}+C$ (b) $\dfrac{\sin^4 x}{4}+C$

   (c) $-\dfrac{\sin^3 x}{3}+C$ (d) $-\dfrac{\sin x\cos^4 x}{4}+C$

5. The integral $\displaystyle\int 2\sin\frac{\pi}{2}t\cos\frac{\pi}{2}t\,dt$ is equal to

   (a) $\dfrac{1}{\pi}\sin\pi t+C$ (b) $\dfrac{1}{2}\cos\pi t+C$

   (c) $-\dfrac{1}{2}\sin\pi t+C$ (d) $-\dfrac{1}{\pi}\cos\pi t+C$

6. The maximum number of partial fractions that $\dfrac{x^4-16}{(2x+1)^3(x^2-1)}$ can be expressed to is ____________________.

(a) 5 (b) 4 (c) 3 (d) 2

7. The possible partial fractions of the rational function $\dfrac{x^2+3x-1}{\left(x^2-x-2\right)^2}$ are

(a) $\dfrac{A}{x+1}+\dfrac{B}{\left(x+1\right)^2}+\dfrac{C}{x-2}+\dfrac{D}{\left(x-2\right)^2}$ (b) $\dfrac{A}{x^2-x-2}+\dfrac{Bx+C}{x^2-x-2}$

(c) $\dfrac{Ax+B}{x^2-x-2}$ (d) $\dfrac{A}{x^2-x-2}+\dfrac{B}{\left(x^2-x-2\right)^2}$

8. To find $\int x\sec^2(5x)\,dx$ using 'integration by parts', we choose

(a) $u = xdx$ and $dv = \sec^2\left(5x\right)$ (b) $u = \sec^2\left(5x\right)$ and $dv = x\,dx$

(c) $u = x$ and $dv = \sec^2\left(5x\right)dx$ (d) $u = \sec^2\left(5x\right)dx$ and $dv = x$

9. Which of the following method can be used to integrate $\int \ln\left(1+x^2\right)dx$?

(a) Method of substitution by letting $u = 1+x^2$.

(b) Method of substitution by letting $u = \ln\left(1+x^2\right)$.

(c) Method of integration by parts by letting $u = 1$ and $dv = \ln\left(1+x^2\right)dx$.

(d) Method of integration by parts by letting $u = \ln\left(1+x^2\right)$ and $dv = dx$.

Answers

Section A

1. $\dfrac{-1}{5(x+1)^5}+\dfrac{1}{3(x+1)^6}-\dfrac{1}{7(x+1)^7}+C$

2. $\dfrac{2}{3}(\ln|x|)^{3/2}+C$

3. $-\dfrac{x^3}{3}e^{-x^3}-\dfrac{1}{3}e^{-x^3}+C$

4. $-2\sqrt{x}\cos\sqrt{x}+2\sin\sqrt{x}+C$

5. $-\dfrac{1}{2(2e^x-1)}+C$

Section B MCQ

1\. c 2. c 3. c 4. b 5. d 6. a 7. a 8. c 9. d