Chapter 8 – Definite Integrals and Area Under a Curve

Objectives:

1. *Define and explain definite integral* $\int_a^b f(x)dx = F(b) - F(a)$.
2. *Evaluate definite integrals.*
3. *Explain that* $\int_a^b y\,dx$ *denotes the net-signed area bounded by the curve* $y = f(x)$ *between the ordinates x = a and x = b.*
4. *Find the area under a curve.*
5. *Explain the case where negative area is involved.*

8.1 Introduction

Calculus is traditionally divided into two branches: the differential calculus and the integral calculus. In differential calculus we study the concept of the derivative; in integral calculus the concept of the definite integral. In this lesson we shall learn what definite integral is and see how such a mathematical idea, first used by the great Greek mathematicians more than 2000 years ago, developed into a mathematical tool of great beauty and immense power.

8.2 Define Definite Integrals

If $f(x)$ is continuous on the interval $[a, b]$, then

$$\boxed{\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)}$$

where

$f(x)$ is the integrand;
x is the variable of integration;
a is the lower limit of integration;
b is the upper limit of integration;
$a \le x \le b$ the interval of integration;
and
$\int_a^b f(x)\,dx$ is the integral of the function $f(x)$ over the interval $[a, b]$.

8.3 Evaluate Definite Integrals

To find the value of the **definite integral** $\int_a^b f(x)\,dx$, where $f(x)$ is continuous over the interval [a, b], we first find an anti-derivative of $f(x)$. Call it $F(x)$. Substitute the upper and lower limit b and a into $F(x)$ to obtain the values $F(b)$ and $F(a)$. Then do the subtraction $F(b) - F(a)$.

$$\boxed{\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)}$$

8.4 Properties of the Definite Integral

1. $\int_a^b k\, f(x)\,dx = k\int_a^b f(x)\,dx$, where k is constant

   Example: $\int_0^1 6\cos x\,dx = 6\int_0^1 \cos x\,dx$.

2. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

   Example: $\int_0^1 (x + \sqrt{3x} - e^x)\,dx = \int_0^1 x\,dx + \int_0^1 \sqrt{3x}\,dx - \int_0^1 e^x dx$.

3. Let c be a value inside the interval $[a, b]$

   $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

   It is possible to split the integral $\int_a^b f(x)\,dx$ into several parts. For example, we could write, if there is a need for it,

   $$\int_0^6 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^5 f(x)\,dx + \int_5^6 f(x)\,dx.$$

4. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

5. In $\int_a^b f(x)\,dx$, x is a *dummy* variable. It means that x can be substituted by any symbol without affecting the value of the definite integral. Thus

   $$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

Example 1 : Evaluate:

(a) $\int_1^3 x^3 dx$ (b) $\int_{\pi/3}^{\pi} \cos 2x \, dx$ (c) $\int_0^1 e^{2x} dx$

Example 2 : Evaluate:

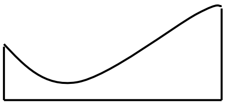
(a) $\int_0^2 (x+2)\, dx$ (b) $\int_{-1}^1 (4u-3)^2 du$

(c) $\int_0^{0.5} \left(3e^{-2t} - \frac{1}{2}\cos \pi t\right) dt$ (d) $\int_2^4 \left(5\sin 3x + \frac{2}{x}\right) dx$

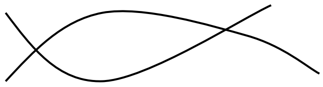
8.5 Applications of the Definite Integrals – Area

Finding areas is an important part of mathematics. In schools, we learned very early how to find the area of squares, of rectangles, of triangles, and many other shapes. We all know that some areas are easy to find and some areas are difficult to find. Consider the following three figures and the areas they enclose.

A_1



A_2



A_3

Area A_1 is easy to find; but areas A_2 and A_3 are difficult to find. The definite integral is a powerful and versatile tool for tackling the problem of areas.

Note: Definite Integral vs Area

Before we begin to apply definite integrals to the problem of areas, it is important to note that although the *definite integral* $\int_a^b f(x)dx$ is often illustrated by the *area* under a curve, these two are entirely different concepts. $\int_a^b f(x)dx$ must be understood as the 'limit of a sum' whereas the area is a quantity associated with a plane figure like rectangle or a circle. Furthermore note also that $\int_a^b f(x)dx$ is a number which can be positive, negative or zero; the area, however, is non-negative (always positive, can be zero, but never negative) by definition.

8.5.1 The Classic Case: Area Under a Curve

Consider the area of the region bounded by a curve $y = f(x)$, two vertical lines, and the x-axis as illustrated in Figure 2. Note that in this "classic case", $f(x) \geq 0$ throughout the interval $[a, b]$. In other words, the curve does not cross to the negative side of the y-axis in the interval $[a, b]$. This type of area is generally known as the 'area under a curve'.

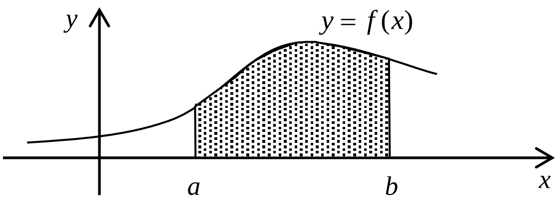


Figure 2

1st Step: Identify the quantity we are interested in. It is the area A.

2nd Step: Find an expression for ΔA
Slice the area vertically into many small areas as shown in Figure 3.
ΔA, the small area, can be approximated by a rectangle of height y_i and width Δx.

$$\Delta A \approx y_i \cdot \Delta x = f(x_i) \cdot \Delta x$$

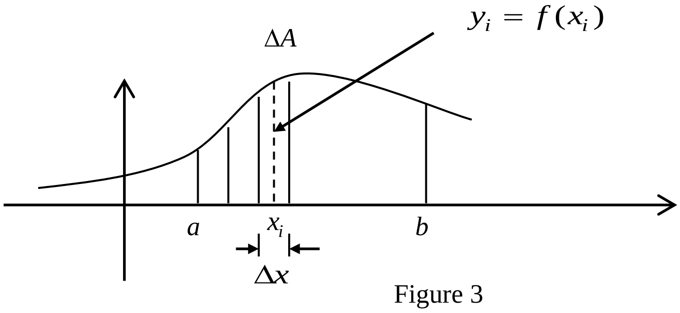


Figure 3

3rd Step: Sum them up.
We now have the approximate area.

$$A = \sum \Delta A \approx \sum f(x_i) \cdot \Delta x$$

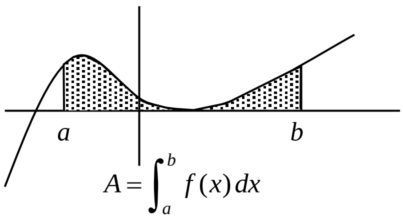
4th Step: Carry out the limiting operation.
By carrying out the limiting operation $n \to \infty$, the approximation becomes an equality (an equation) and we have

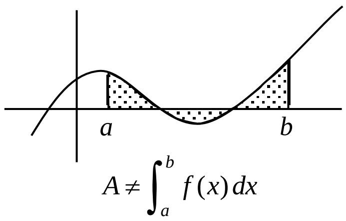
$$A = \lim_{n\to\infty} \sum_{x=a}^{x=b} f(x_i) \cdot \Delta x = \int_a^b f(x)\,dx\ .$$

$$A = \int_a^b f(x)\,dx \qquad f(x) \geq 0 \quad \text{in} \quad [a, b]$$

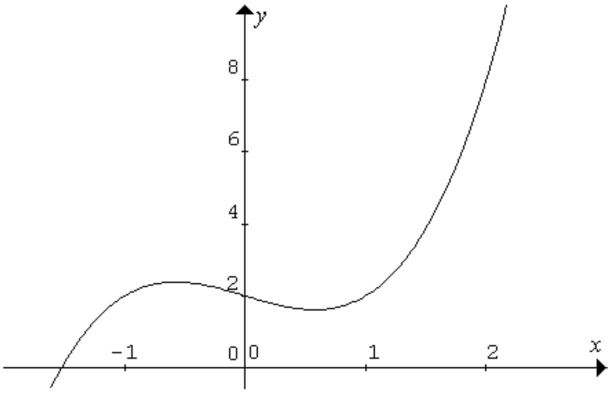
Important

This formula is applicable only to the standard 'area under the curve' type of areas. And it is valid only under the assumption that the function $f(x)$ is non-negative over the interval [a, b]. It is very easy to check this condition visually; just make sure that the curve is above the x-axis (touching the x-axis is all right) and that it does not cross the x-axis to the other side in the interval [a, b].

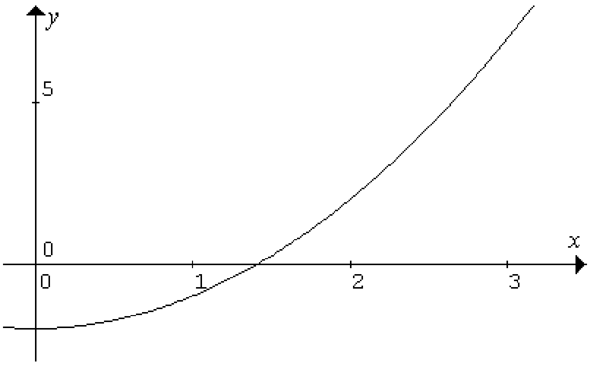




Example 3 : Find the area under the curve $y = x^3 - x + 2$ from $x = -1$ to $x = 2$.

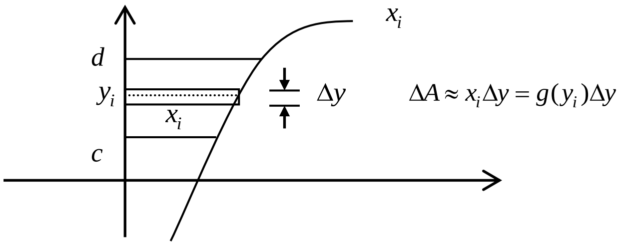


Example 4 : Find the area bounded by the curve $y = x^2 - 2$ and the *x*-axis, from *x* = 0 to $x = 3$.



8.5.2 Reversing the Role of x and y

For certain type of areas, it may be easier to slice the area *horizontally* instead of vertically.



$$\boxed{A = \int_c^d g(y)\,dy}$$

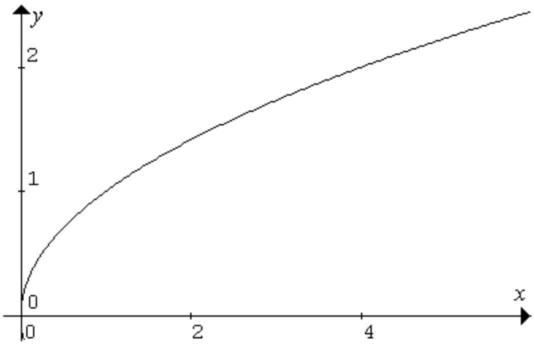
Important

This formula is applicable only to the standard 'area under the curve' type of areas. And it is valid only under the assumption that the function $g(y)$ is non-negative over the interval $[c, d]$.

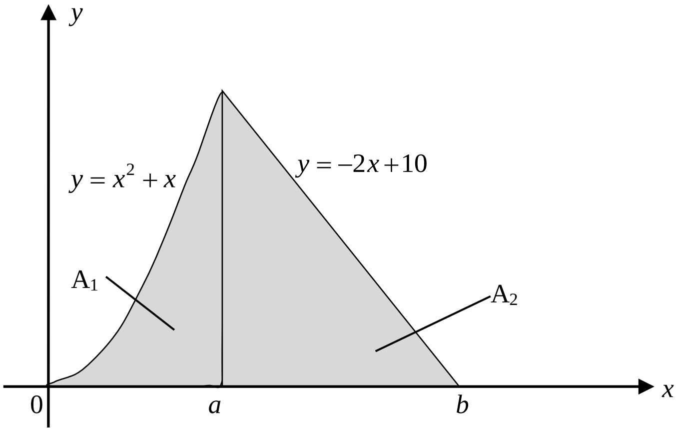
Example 5 **:** Find the area of the region bounded by the curve $x = y^2$,

(i) the y-axis and $y = 2$.

(ii) the x-axis, $x = 2$ and $x = 4$.



Example 6 **:** Find the shaded area.



Example 7 **:** In a certain electric circuit, the current i amperes flowing through a capacitor C farad is given by

$$i = 10\sin 100\pi t - 4\cos 200\pi t$$

where t is the time in seconds and the initial voltage v is zero.

Find the mean voltage over the interval $0 \le t \le 2$ given $C = \dfrac{1}{\pi}$ mF.

(Note : $i = C\dfrac{dv}{dt}$)

<u>Tutorial</u> : MCQ

1. Given $\int_{-1}^{k} dx = 1$, then $k =$ _____.

(a) 0 (b) -1 (c) 1 (d) 2

2. Given $w = \int_{-1}^{k} x\,dx$, where k is a positive constant. By carrying out integration and simplification, $w =$ _____.

(a) $0.5(k^2 - 1)$ (b) $0.5(k^2 + 1)$ (c) $k + 1$ (d) $k - 1$

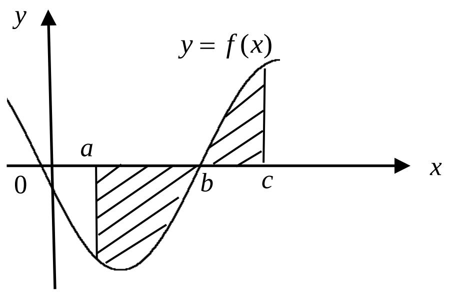
3. Which of the following integrals gives the area of the shaded region?

(a) $\int_{a}^{c} f(x)\,dx$

(b) $\int_{a}^{b} f(x)\,dx + \int_{b}^{c} f(x)\,dx$

(c) $\left| \int_{a}^{b} f(x)\,dx \right| + \int_{b}^{c} f(x)\,dx$

(d) $\int_{a}^{b} f(x)\,dx + \left| \int_{b}^{c} f(x)\,dx \right|$



Tutorial 8

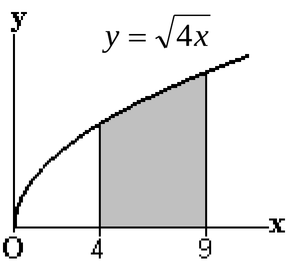
1. Evaluate the following integrals.

(a) $\int_{2}^{3} x\,dx$ (b) $\int_{2}^{5} dx$ (c) $\int_{1}^{4} (x^2 + 3x)\,dx$

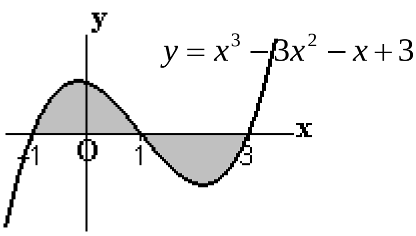
(d) $\int_{1}^{10} \frac{1}{2x}\,dx$ (e) $\int_{-2}^{-1} \left(4e^{-2x} + \frac{3}{x} \right) dx$ (f) $\int_{0}^{1} (5x - \sin 3x)\,dx$

2. Find the shaded areas.

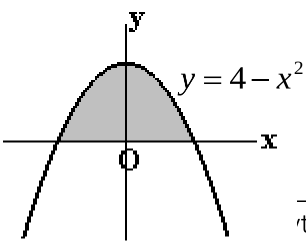
(a)



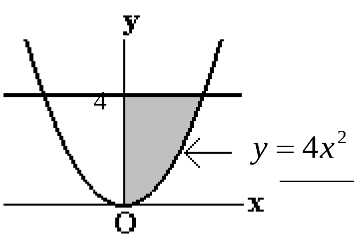
(b)



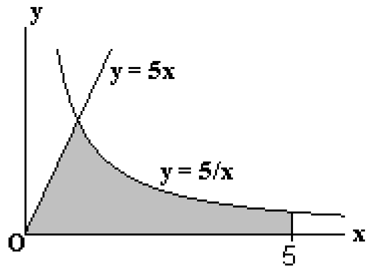
(c)



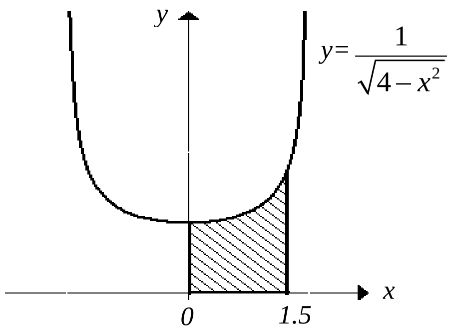
(d)



(e)



(f)

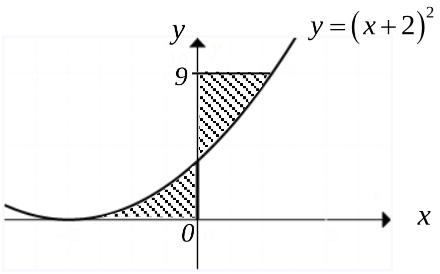


3. Find the area bounded by the given curves:
Note: $x = 0$ is the y-axis and $y = 0$ is the x-axis. Sketch the graphs for each question.

(a) $y=4-x^2$; $x-axis$, from $x=0$ to $x=4$

(b) $y=5\sin 3x$; $x-axis$, from $x=\dfrac{\pi}{4}$ to $x=\dfrac{2\pi}{3}$

4. Find the shaded area.



5. Find the mean current over the interval $0 \le t \le 3$ seconds, given that the current is
$i = 1 + 2\sin t$ amperes.

*6. The voltage V in an electric circuit drops uniformly at a rate of 0.007 volt per second. The initial voltage in the circuit is 100 volts, the resistance being equal to 5 ohms. Find the mean power during the first hour of operation.

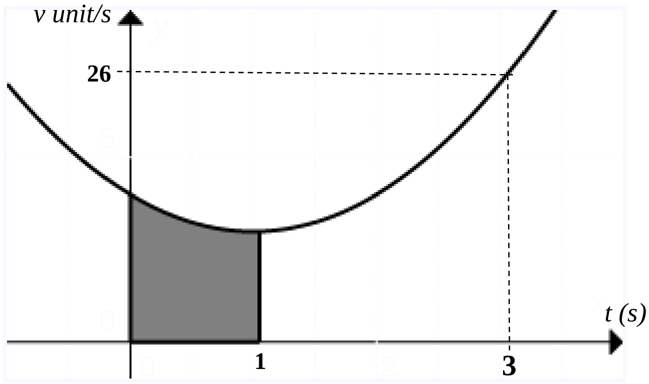
[Note : Power , $P=\dfrac{V^2}{R}$]

*7. Find the mean value of $i=\dfrac{E}{R}+C\sin\omega t$ over **one period** where E, R, C and ω are constants.

[Mean value over 1 period = $\dfrac{1}{T}\int_0^T i\,dt$, where T = period of sinusoidal waveform $=\dfrac{2\pi}{\omega}$]

*8. The velocity v is defined as the rate of change of displacement , that is , $v = \dfrac{ds}{dt}$.

The graph of the velocity function $v = at^2 - bt + c$ as shown below has a minimum point at $t=1$.



Given the shaded area is 4 square units, use Cramer's Rule to find the values of the real constants a, b and c

ANSWERS

Eg 1 : (a) 20 (b) $-\dfrac{\sqrt{3}}{4}$ (c) 3.195

Eg 2 : (a) 6 (b) 86/3 (c) 0.7889 (d) 1.581

Eg 3 : 33/4

Eg 4 : 6.78

Eg 5 : (i) 8/3 (ii) 3.448

Eg 6 : 41/3

Eg 7 : 100 volts

MCQ

1. (a) 2. (a) 3. (c)

Tutorial 8

1. (a) 2.5 (b) 3 (c) 43.5 (d) 1.15 (e) 92.34 (f) 1.84
2. (a) $25\frac{1}{3}$ (b) 8 (c) $\dfrac{32}{3}$ (d) $2\frac{2}{3}$ (e) 10.55 (f) 0.848
3. (a) 16 (b) 3.82
4. 16/3
5. 2.33 amperes
6. 1538.33 Watts
7. $\dfrac{E}{R}$ amperes
8. $a = 6$, $b = 12$, $c = 8$