Chapter 13: Applications of Second Order Differential Equations

Objectives :
1. Solve application problems using auxiliary equation and Laplace transform method.

13.1 Modelling a Mass Spring Damper System

The differential equation is given by Newton's second law of motion:

mass $\times$ acceleration = sum of forces acting on the mass

$$m\frac{d^2x}{dt^2} = F_g + F_s + F_d + F(t)$$

where:

m is the mass

$\frac{d^2x}{dt^2}$ is the acceleration (i.e. rate of change of velocity $\frac{dv}{dt}$)

F_g is the force due to the earth's gravity

F_s is the tension in the spring

F_d is the damping force due to air resistance etc.

$F(t)$ is the external applied force

Since

- $F_g = mg$, where g is the acceleration due to gravity,
- $F_s = -k(L_0 + x)$, where k is the spring constant and L_0 is the extension of the spring at equilibrium position. [Hooke's law states that the tension in the spring is proportional to the extension of the spring],
- $F_d = -c\frac{dx}{dt}$, where c is the damping constant, in which the minus sign indicates that the damping force is always opposite to the direction of motion, and
- $mg = kL_0$ in equilibrium,

the equation becomes:

$$\begin{aligned} m\frac{d^2x}{dt^2} &= mg - k(L_0 + x) - c\frac{dx}{dt} + F(t) \\ &= mg - kL_0 - kx - c\frac{dx}{dt} + F(t) \\ &= -kx - c\frac{dx}{dt} + F(t) \end{aligned}$$

$$\therefore \quad m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$$

Hence, the mechanical vibration of a spring-mass-damper system is described by the 2nd order ODE

$$\boxed{m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=F(t)}$$

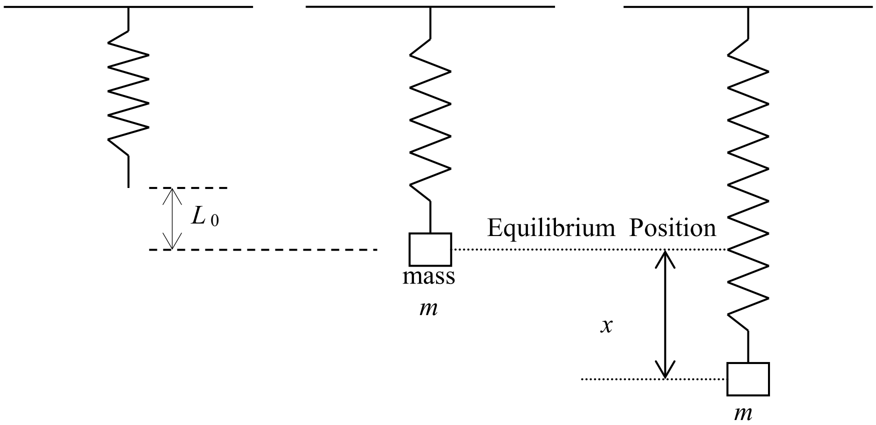
where:

x (m) is the displacement of the mass m (kg) at any time t (s) from equilibrium position,

c = damping coefficient

k = stiffness of spring (N/m)

$F(t)$ = external force



The value of x is measured $\begin{cases}\text{positive, if it is below the equilibrium position.}\\ \text{negative, if it is above the equilibrium position.}\end{cases}$

13.1.1 Mass Spring Damper System: Without External Force, i.e. Free Oscillations

There is no driving force $F(t)$, i.e. $F(t)=0$. Differential equation of motion becomes

$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=0$$

(a) No Damping ($c=0$)

In undamped free vibration (which is known as Simple Harmonic Motion), there is no damping force, i.e. $c = 0$.

The differential equation becomes: $m\frac{d^2x}{dt^2}+kx=0$.

Example 1: A 1 kg mass is suspended from a spring of stiffness 25 N/m. The mass is pulled below the equilibrium position and released. Assuming no frictional force and air resistance,

(a) set up the differential equation that describes the motion of the mass;

(b) find the position of the mass at any time t.

(b) With Damping ($c \neq 0$)

There is a damping force, that is, $c \neq 0$.

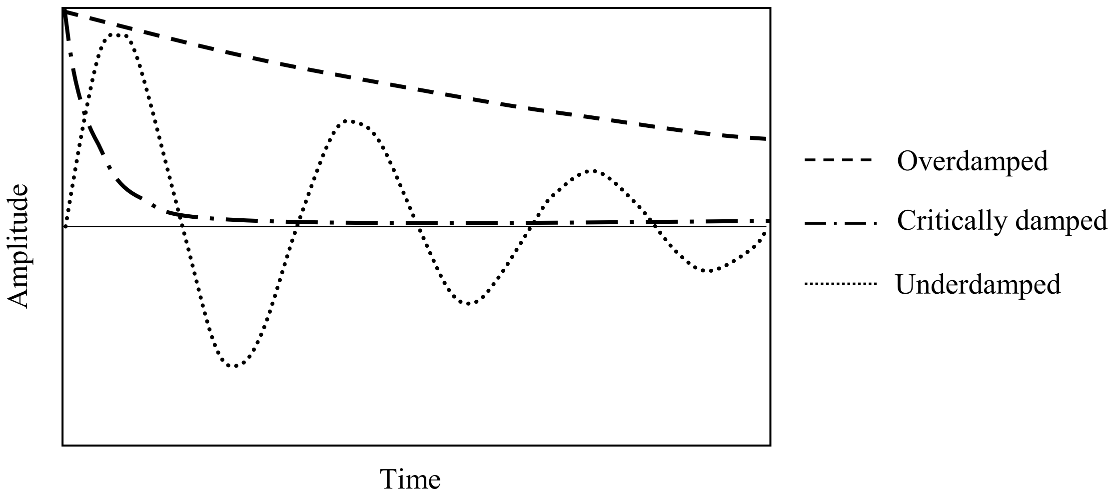
The differential equation becomes: $m\dfrac{d^2x}{dt^2} + c\dfrac{dx}{dt} + kx = 0$

The characteristic equation is: $m\lambda^2 + c\lambda + k = 0$

with roots: $\lambda = \dfrac{-c \pm \sqrt{c^2 - 4km}}{2m}$

There are 3 cases:

ROOTS	DISCRIMINANT	CONDITION	SOLUTION
2 real roots: λ_1 and λ_2	$c^2 - 4km > 0$	***Overdamped***	$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$
2 equal roots: $\lambda_1 = \lambda_2 = \lambda$	$c^2 - 4km = 0$	***Critically damped***	$x(t) = e^{\lambda t}\left(At + B\right)$
2 complex roots: $\lambda = \alpha \pm j\beta$	$c^2 - 4km < 0$	***Underdamped***	$x(t) = e^{\alpha t}\left(A\cos\beta t + B\sin\beta t\right)$



Example 2: A 2 kg mass is suspended from a spring of stiffness 20 N/m. The mass is being pulled 1m below the equilibrium position with no initial velocity and released. If the air resistance is numerically equal to $4V$, where V is the instantaneous velocity in m/s, find the position of the mass at time t and determine whether the motion is one of over damping, critical damping or under damping.

Ans: $x(t) = e^{-t}[\cos(3t) + \frac{1}{3}\sin(3t)]$, underdamped

13.1.2 Mass Spring Damper System: With External Force

Recall that the Mechanical Spring Vibration is described by the 2nd order ODE:

$$\boxed{m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)}$$

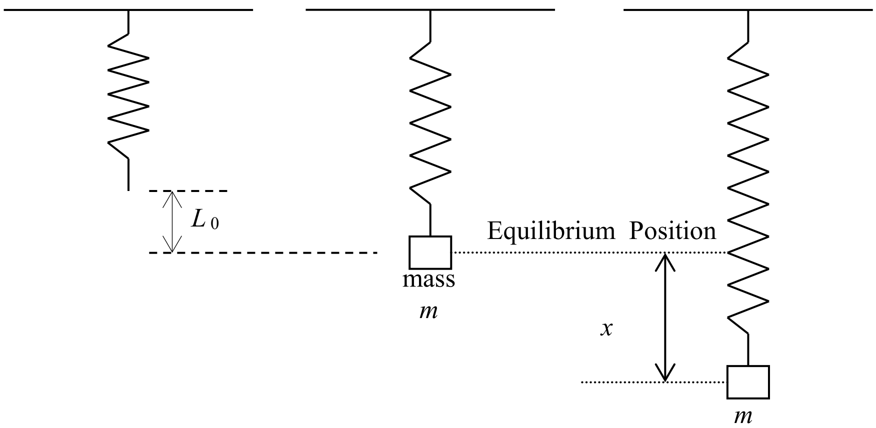
where:

x (m) is the displacement of the mass m (kg) at any time t (s) from equilibrium position,

c = damping coefficient

k = stiffness of spring (N/m)

$F(t)$ = external force



The value of x is measured $\begin{cases} \text{positive, if it is below the equilibrium position.} \\ \text{negative, if it is above the equilibrium position.} \end{cases}$

The solution is $x = \boxed{\text{Solution of motion due to applied force}} + \boxed{\text{Solution of motion due to damped oscillation}}$

steady state solution transient state solution

The damped motion in the solution is called the **transient solution** which diminishes with time.
The undamped motion in the solution is called the **steady-state solution** which remains throughout.

***Example 3*:** A mass of 500 g is suspended from a spring of stiffness 100N/m. The mass is pulled down 20 cm below its equilibrium position and released from rest. A periodic force $F(t) = 3\cos t$ is simultaneously applied to the mass. Assuming that the damping force is negligible.

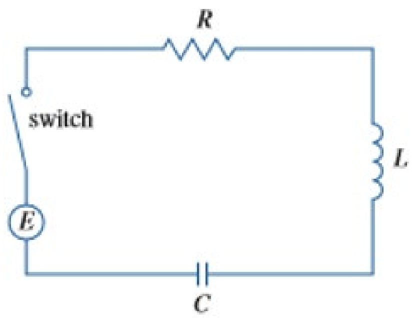
(a) Set up the differential equation to model the displacement $x(t)$, and indicate clearly the initial conditions.

(b) Determine the position and the velocity of the mass 3 seconds after it is released. *Ans*: – 2.7 cm, 2.4 m/s

13.2 Electrical Circuits

In section 9.3 of chapter 9, we were able to use first-order differential equation to analyze RL and RC series circuits that contain resistor, inductor and capacitor. Now that we know how to solve second-order differential equations, we are able to extend the analysis to electrical circuit shown in the figure below.

In the RLC circuit below, it contains an electromotive force E (supplied by a battery or generator), a resistor R, an inductor L, and a capacitor C, in series. If the charge on the capacitor at time t is $q = q(t)$, then the current i is the rate of change of q with respect to t: $i = \frac{dq}{dt}$.



RLC circuit

It is known from physics that the voltage drops across the resistor, inductor, and capacitor are respectively,

$$v_R = iR \qquad v_L = L\frac{di}{dt} \qquad v_C = \frac{q}{C}$$

Kirchhoff's voltage law says that the sum of these voltage drops is equal to the supplied voltage:

$$L\frac{di}{dt} + Ri + \frac{q}{C} = E(t) \quad \cdots\cdots\cdots\cdots(1)$$

Since $i = \frac{dq}{dt}$, the above equation (1) becomes:

$$\boxed{L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)} \quad \cdots\cdots\cdots(2)$$

which is a second-order linear differential equation with constant coefficients. If the charge q_0 and the current i_0 are known at time $t = 0$, then we have the initial conditions

$$q(0) = q_0 \qquad q'(0) = i(0) = i_0$$

and the initial-value problem can be solved by the Laplace Transform method.

A differential equation for the current can be obtained by differentiating equation (1) with respect to t and remembering that $i = \frac{dq}{dt}$:

$$\boxed{L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = E'(t)} \quad \cdots\cdots\cdots(3)$$

Example 4: Find the charge on the capacitor at time t in the *RLC* series circuit given on page 6, if $L = 0.1\text{H}$, $R = 2\Omega$, $C = 0.1\text{F}$, $E(t) = 0\text{V}$, with initial conditions $q(0) = 0$ and $q'(0) = 1$.

Example 5: In the RLC circuit shown on page 6, it is known that $R = 2\Omega$, $L = 1\text{H}$, $C = 0.5\,\text{F}$, and $E(t) = 2\cos 2t\,\text{V}$. If initially there is no current flowing thru the circuit, and the rate of change of the current is zero,

(a) Set up the differential equation to model the current $i(t)$ flowing through the circuit, and indicate clearly the initial conditions.

(b) Find the current at time t in the circuit. Hence, find the transient and steady-state current.

Ans:

$$i(t) = \frac{1}{5}\left\{2\sin 2t + 4\cos 2t - 4e^{-t}\left(\cos t + 2\sin t\right)\right\}\text{A},$$

$$i_s(t) = \frac{1}{5}\left(2\sin 2t + 4\cos 2t\right)\text{A} \quad \text{and}$$

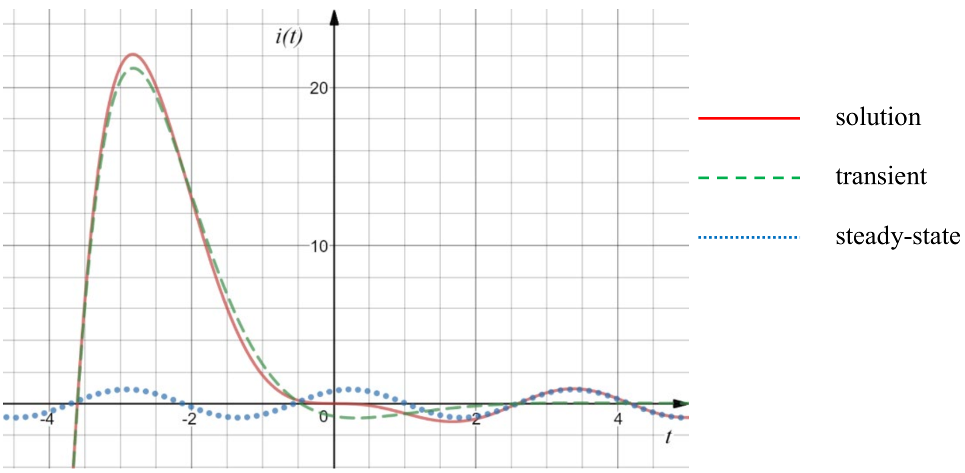
$$i_t(t) = -\frac{4}{5}e^{-t}\left(\cos t + 2\sin t\right)\text{A}$$

Note that:

- The first term of the solution, $i_S(t)=\frac{2}{5}\sin(2t)+\frac{4}{5}\cos(2t)$ is called the **steady state solution**.
- The second term of the solution, $i_t(t)=-\frac{4}{5}e^{-t}[\cos(t)+2\sin(t)]$ is called the **transient state solution**.

Remark:
The transient state of the current decays to zero as $t\to\infty$. Damping is caused by the resistance in the circuit. It is the damping force which caused the free vibration to disappear after some time and the steady state vibration remained which are constantly maintained by the action of the applied force 4cos(2*t*).

The graph is shown below.



Tutorial 13

1. A mass of 0.6 kg is attached to the lower end of a vertical spring of stiffness 200 N/m. The mass is raised 3 cm above the equilibrium position, i.e. $x(0) = -3$ cm, and released from rest, i.e. $v(0) = x'(0) = 0$ cm/s. Assuming no air resistance,
   (a) describe the motion of the mass;
   (b) set up the differential equation to model the displacement $x(t)$, and indicate clearly the initial conditions;
   (c) find the position of the mass 5 seconds after it is released; and
   (d) determine the frequency of the motion.

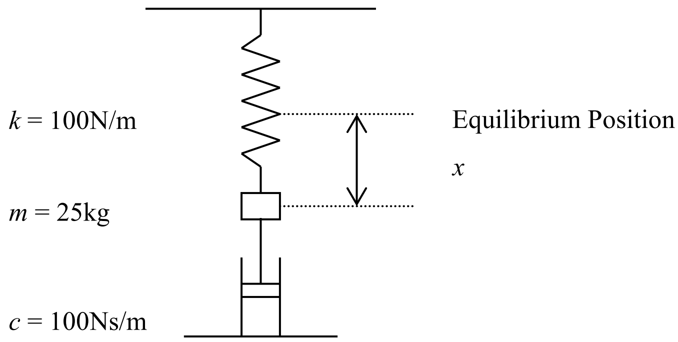
2. A mass of 10 kg is suspended from a spring of spring constant 300 N/m. The mass is pushed up 15 cm above its equilibrium position and released from rest. Assuming there is no damping force,
   (a) set up the differential equation to model the displacement $x(t)$, and indicate clearly the initial conditions ;
   (b) find the position of the mass after 1 second ;
   (c) the amplitude, period and frequency of the vibration.

3. A 1 kg mass is attached to the lower end of a vertical spring of stiffness 25 N/m. The mass is set into motion from rest at the equilibrium position by an external force $F(t) = \sin(5t)$ (N). If the resistance to the motion is numerically equal to $8v$ (N) where v (m/s) is the velocity of the mass at time t (s),
   (a) set up the differential equation to model the displacement $x(t)$, and indicate clearly the initial conditions ;
   (b) find the displacement x (m) of the mass at any time t (s) ,
   (c) indicate the amplitude of the steady-state vibration of the mass ;
   (d) what is the ratio of the displacement in the steady-state motion to that in the transient-state motion when $t = 0.5$s?

4. A spring has a spring constant of 125 Nm^{-1}. A mass of 5 kg is suspended from the spring and, after it has come to equilibrium, is pulled down 20 cm and released from rest. Assuming that there is a damping force numerically equal to $30v$, where v (m/s) is the instantaneous velocity at time t (s),
   (a) set up the differential equation to model the displacement $x(t)$, and indicate clearly the initial conditions ;
   (b) find the position and the velocity of the mass at any time.

5. A 10 kg mass is attached to a spring of stiffness 120 N/m. The mass is set into motion from the equilibrium position with an initial velocity of 1 m/s in the upward direction. If the air resistance is numerically equal to $70v$ N where v is the velocity at time t,
   (a) set up the differential equation to model the displacement $x(t)$, and indicate clearly the initial conditions ;
   (b) find the subsequent motion.

6. Determine the differential equation of motion for the damped vibratory system shown. Indicate the initial conditions clearly as well.



Given that the mass is pushed down 1 cm and released from rest, determine the position of the mass at time $t = 1$ sec.

7. Find the charge on the capacitor in the RLC-series circuit when $L = 0.25$ H, $R = 20\ \Omega$, $C = \dfrac{1}{300}$ F, $E(t) = 0$ V, $q(0) = 4$ C and $q'(0) = 0$ A.

8. In a RLC circuit, it is known that R = 10 ohms, L = 5/3 henry, C = 1/30 farad, and the electromotive force $E(t) = 300$ volts. If initially, there is no current flowing thru the circuit, and the rate of change of the current is 180 amp/sec,
   (a) set up the differential equation to model the current in the circuit, and indicate clearly the initial conditions;
   (b) hence, find the current $i(t)$.

9. In a RLC circuit, it is known that R = 10 ohms, L = 0.5 henry, C = 0.01 farad, and the electromotive force $E(t) = 150$ volts. If initially, the charge on the capacitor is 1 coulomb, and there is no current,
   (a) set up the differential equation to model the charge on the capacitor, and indicate clearly the initial conditions;
   (b) hence, find the charge $q(t)$.

*10. In a RLC circuit, it is known that R = 2 ohms, L = 1 henry, C = 0.25 farad, and the electromotive force $E(t) = 50\cos(t)$ volts.
   (a) set up the differential equation to model the current in the circuit,
   (b) hence, find the current $i(t)$.

*11. Use Laplace transform method to find the current i given the differential equation:

$$Ri + \frac{1}{C}\int_0^t i\,dt - v_C = 0$$

where v_C, R and C are constants.

Multiple Choice Questions

1. If the motion of an engineering system is described by

$y(t) = \frac{1}{2}[e^{-2t}\cos(t) + 3e^{-2t}\sin(t) - e^{-t}]$, the motion is considered ___________.

(a) un-damped (b) under-damped

(c) critically-damped (d) over-damped

Answers

1. (a) Simple Harmonic Motion; (c) 3 cm below equilibrium position (d) 2.9 Hz
2. (a) 10.4 cm above equilibrium position (b) 0.15m., 1.15s, 0.87 Hz
3. (a) $x(t) = -\frac{1}{40}\cos 5t + \frac{1}{40}e^{-4t}[\cos 3t + \frac{4}{3}\sin 3t]$ m (b) $\frac{1}{40}$ m (c) 4.22
4. (a) $x(t) = e^{-3t}[0.2\cos(4t) + 0.15\sin(4t)]$ cm (b) $x'(t) = -1.25\,e^{-3t}\sin(4t)$ cm/s
5. $x(t) = e^{-4t} - e^{-3t}$ m
6. 0.41 cm
7. $q(t) = 6e^{-20t} - 2e^{-60t}$ C
8. $i(t) = 60e^{-3t}\sin 3t$ ampere
9. $q(t) = \frac{3}{2} - \frac{1}{2}e^{-10t}(\cos 10t + \sin 10t)$ coulomb
10. $i_s(t) = \frac{100}{13}\cos t - \frac{150}{13}\sin t$ ampere
11. $i(t) = \frac{v_C}{R}e^{-\frac{t}{RC}}$ ampere

MCQ

1. (b)