SINGAPORE POLYTECHNIC
2021/2022 SEMESTER ONE END-SEMESTER TEST

EP0604 FURTHER MATHEMATICS
Time Allowed: 1 hour 30 minutes

Instructions to Candidates
1. The Singapore Polytechnic examination rules are to be complied with.
2. This examination paper consists of FOUR printed pages.
3. Answer **ALL** the questions.
4. Give all non-exact answers to 3 significant figures.
5. A mathematical formulae and tables card is provided for reference.

Additional Formulae

Absolute value Inequalities: (i) $|x-a|<k$ is equivalent to $-k<x-a<k$

(ii) $|x-a|>k$ is equivalent to $x-a>k$ or $x-a<-k$

VECTOR EQUATION OF A LINE

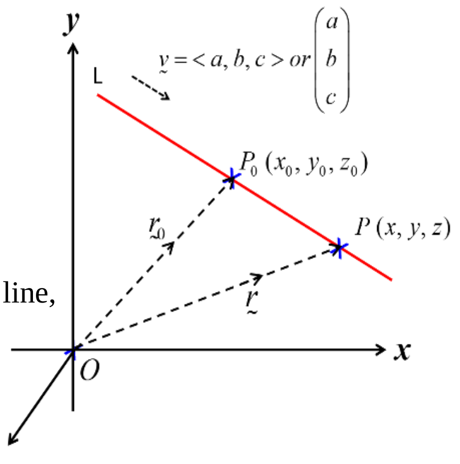
$$\underset{\sim}{r}=\underset{\sim}{r}_0+\lambda\underset{\sim}{v} \quad , \quad \lambda\in\mathbb{R}$$

where

$\underset{\sim}{r} = <x,y,z>$ is the position vector of any point on the line,

$\underset{\sim}{r}_0 = <x_0,y_0,z_0>$ is the position vector of a known point on the line,

$\underset{\sim}{v} = <a,b,c>$ is a non-zero vector parallel to the line.



PARAMETRIC EQUATIONS OF A LINE

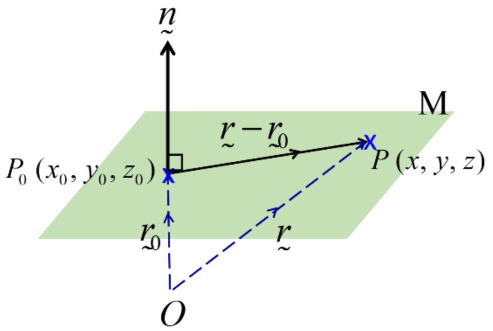
$$x=x_0+\lambda a, \quad y=y_0+\lambda b, \quad z=z_0+\lambda c \qquad \text{where } \lambda\in\mathbb{R}$$

EQUATION OF A PLANE

The plane in $\mathbb{R}^3$ that passes through the point $P_0(x_0,y_0,z_0)$ and is normal to the non-zero vector $\underset{\sim}{n}=\langle a,b,c\rangle=a\underset{\sim}{i}+b\underset{\sim}{j}+c\underset{\sim}{k}$ has equations:

In vector form: $\underset{\sim}{n}\bullet\overrightarrow{P_0P}=0$ or $\underset{\sim}{r}\bullet\underset{\sim}{n}=\underset{\sim}{r}_0\bullet\underset{\sim}{n}$

In point-normal form: $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$



1. The sum of a geometric progression is given by

$$a + ar^1 + ar^2 + \ldots + ar^{n-1} = \frac{a\left(1-r^n\right)}{1-r} \quad \text{for } n \in \mathbb{Z}^+ .$$

Prove the above formula using mathematical induction. (15 marks)

2. (a) The area of an object is given by $A = \int_0^1 x^5 \ln\left(x^3+1\right) dx$.

(i) Use the substitution $u = x^3 + 1$ to show that

$$A = \frac{1}{3}\int_1^2 (u-1)\ln u \; du .$$ (4 marks)

(ii) By using integration by parts, or otherwise, find the value of A. (6 marks)

(b) Find $\int \frac{2^x}{\left(2^x+1\right)^2} dx$. (5 marks)

3. (a) Figure 1 shows sketches of the graphs of $y = 2 - e^{-x}$ and $y = x$. These graphs intersect at $x = a$.

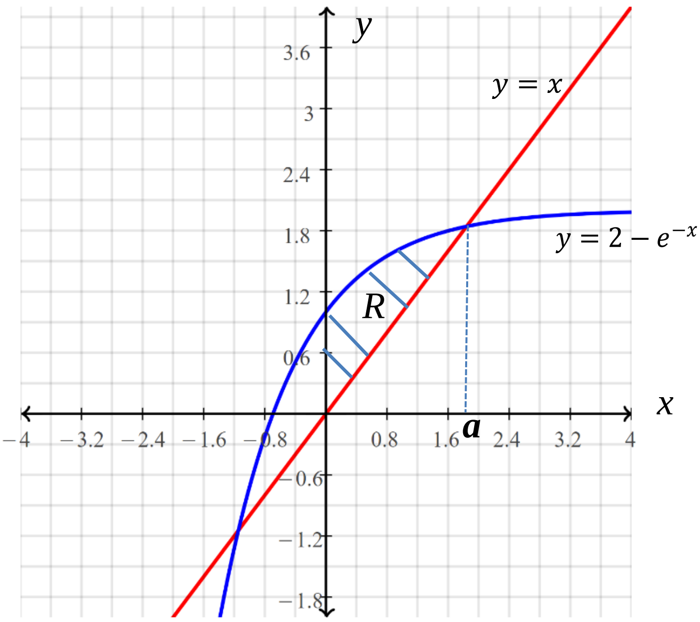


Figure 1

(i) Write down an equation satisfied by $x = a$.
Do not attempt to solve the equation. (2 marks)

(ii) Write down an integral which is equal to the area of the shaded region R. (3 marks)

(iii) Show that the result in part (ii) is $2a + \frac{1}{e^a} - \frac{a^2}{2} - 1$. (5 marks)

3. (b) A wine bottle stopper is modeled by the function $y = \frac{x}{12}\sqrt{36 - x^2}$. Find the volume of the stopper when it is rotated 2π radians about the x-axis between x = 0 and x = 6.

(10 marks)

4. (a) Given $\overrightarrow{PQ} = \underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$, find

(i) $\left|\overrightarrow{PQ}\right|$. (2 marks)

(ii) A unit vector in the direction of $\overrightarrow{PQ}$. (2 marks)

(iii) Find the force $\overrightarrow{F_1}$ which has a magnitude of 9N in the direction of $\overrightarrow{PQ}$. (2 marks)

(iv) Given another force $\overrightarrow{F_2} = a\underset{\sim}{i} + 15\underset{\sim}{j}$ and both $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$ together move an object from point P to point Q with a total work done of 51 J. Find the value of *a*.

(4 marks)

(b) The line *L* passes through the points $P(1, -2,4)$ and $Q(3,2,10)$.

(i) Find $\overrightarrow{PQ}$ (2 marks)

(ii) Find the parametric equations of the line through *P* and *Q*. (3 marks)

(iii) Determine whether $R(4,4,13)$ lies on the line through *P* and *Q*. (5 marks)

(Please Turn Over)

5. A plane $\prod$ contains a line $L_1: \underset{\sim}{r} = (\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}) + \lambda(\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k})$ and a point $P(4,-4,3)$, as shown in figure 2.

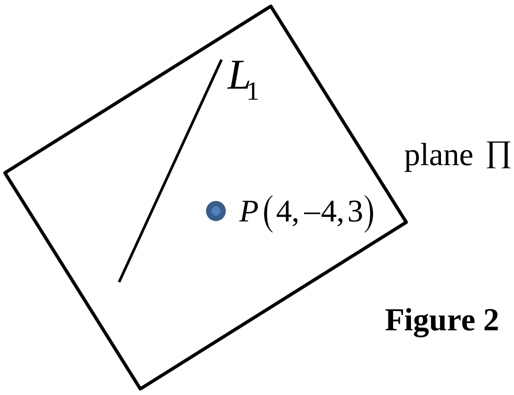


Figure 2

(a) Find a vector which is perpendicular to the plane $\prod$. (5 marks)

(b) Using the vector in (a) and the point $P(4,-4,3)$, find equation of the plane $\prod$. (3 marks)

(c) The vector equation of a line, L_2, is given by $\langle 1,4,3\rangle + \mu\langle 1,3,-2\rangle$.
Find the point of intersection of the plane $\prod$ and the line L_2 (7 marks)

6. (a) Find the range of values of x for which $0 < x^2 + 4x$ and $x^2 + 4x \leq 6x + 3$. (7 marks)

(b) Sketch, on the same diagram, the graphs of $y = |x-2|$ and $y = |2x-3|$ for $0 \leq x \leq 2$.
Hence or otherwise, solve the inequality of $|x-2| \geq |2x-3|$. (8 marks)

~END OF PAPER~

Answers

1 Step 3 need to prove: $a+ar^1+ar^2+\ldots+ar^{n-1}+ar^n=\dfrac{a\left(1-r^{n+1}\right)}{1-r}$

2 (a)(i) change the limits using $u=x^3+1$ (ii) $\dfrac{1}{12}$ (b) $-\dfrac{1}{\ln 2\left(2^x+1\right)}+C$

3 (a)(i) $2-e^{-x}=x$ (ii) $\int_0^a (2-e^{-x}-x)\,dx$

(b) $7.2\pi\ or\ 22.6\ unit^3$

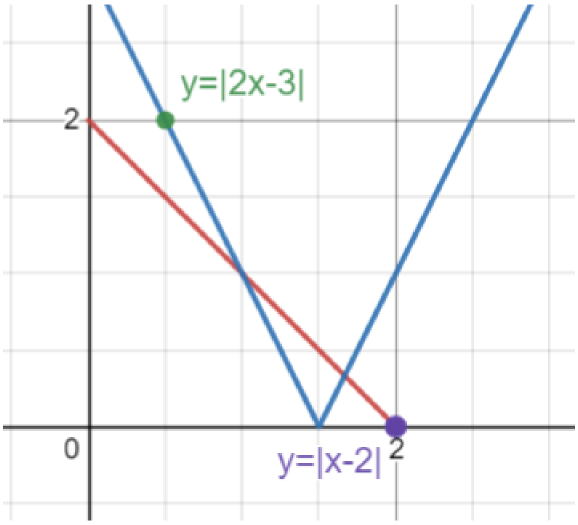
4 (a)(i) 3 (ii) $\frac{1}{3}(\underset{\sim}{i}+2\underset{\sim}{j}+2\underset{\sim}{k})$ (iii) $(3\underset{\sim}{i}+6\underset{\sim}{j}+6\underset{\sim}{k})\ N$ (iv) $a=-6$

(b)(i) $2\underset{\sim}{i}+4\underset{\sim}{j}+6\underset{\sim}{k}$ (ii) $\begin{aligned} x&=1+2\lambda\ldots(1)\\ y&=-2+4\lambda\ldots(2)\\ z&=4+6\lambda\ldots(3)\end{aligned}$ Or $\begin{aligned} x&=3+2\lambda\\ y&=2+4\lambda\\ z&=10+6\lambda\end{aligned}$

(b) (iii) Subst $Q(4,4,13)$ into (1), (2) and (3) give the same $\lambda=\dfrac{3}{2}$. Hence Q lies on the given line.

5 (a) $2\underset{\sim}{i}+\underset{\sim}{j}+3\underset{\sim}{k}$ (b) $2x+y+3z=13$ (c) $(3,10,-1)$

6 (a) $0< x\le 3$



(b) $1\le x\le\dfrac{5}{3}$