SOLUTIONS

SINGAPORE POLYTECHNIC
EST 2021 / 2022 Semester 1

Module Name: Further Mathematics **Module Code: EP0604**

No.	SOLUTION
1	Let P_n be the statement $a + ar^1 + ar^2 + \ldots + ar^{n-1} = \dfrac{a\left(1-r^n\right)}{1-r}$ Step 1: Prove that P_1 is true. When n = 1, LHS = a RHS $= a(1-r)/(1-r) = a$ = LHS Hence, P_1 is true. Step 2: Assume that P_n is true for an arbitrary $n \in \mathbb{Z}^+$. $P_n :\ a + ar^1 + ar^2 + \ldots + ar^{n-1} = \dfrac{a\left(1-r^n\right)}{1-r}$ Step 3: Prove that P_{n+1} is true $P_{n+1}:\ a + ar^1 + ar^2 + \ldots + ar^{n-1} + ar^n = \dfrac{a\left(1-r^{n+1}\right)}{1-r}$ $LHS = a + ar^1 + ar^2 + \ldots + ar^{n-1} + ar^n$ $= \dfrac{a\left(1-r^n\right)}{1-r} + ar^n$ $= \dfrac{a\left(1-r^n\right) + ar^n\left(1-r\right)}{1-r}$ $= \dfrac{a - ar^n + ar^n - ar^{n+1}}{1-r}$ $= \dfrac{a\left(1-r^{n+1}\right)}{1-r} = RHS$ Hence P_n is true implies P_{n+1} is true. Since P_1 is true, it follows by the principle of mathematical induction that P_n is true for all $n \in \mathbb{Z}^+$.

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2(a) (i)	$u = x^3 + 1$ $\dfrac{du}{3} = x^2 dx$ $x^3 = u - 1$ $u = x^3 + 1$ $x = 0, u = 1$ $x = 1, u = 2$

No.	SOLUTION
(ii)	$A=\int_1^2 (u-1)\ln(u)\frac{du}{3}$ $=\frac{1}{3}\int_1^2 (u-1)\ln(u)\,du$ $A=\frac{1}{3}\int_1^2 (u-1)\ln u\ du$ $=\frac{1}{3}\left[\left(\left(\frac{u^2}{2}-u\right)\ln u\right)_1^2-\int_1^2\left(\frac{u}{2}-1\right)\ du\right]$ $=\frac{1}{3}\left(\left(\frac{u^2}{2}-u\right)\ln u-\frac{u^2}{4}+u\right)_1^2$ $=\frac{1}{3}\left(1-\frac{3}{4}\right)=\frac{1}{12}$
2(b)	$u=2^x+1\Rightarrow\frac{du}{\ln 2}=2^x dx$ $\int\frac{2^x}{\left(2^x+1\right)^2}dx=\frac{1}{\ln 2}\int\frac{1}{u^2}du$ $=\frac{1}{\ln 2}\frac{u^{-1}}{-1}+C$ $=-\frac{1}{\ln 2\left(2^x+1\right)}+C$

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3a(i)	$2-e^{-x}=x$
(ii)	$\int_0^a (2-e^{-x}-x)\ dx$
(iii)	$\int_0^a (2-e^{-x}-x)dx$ $=\left[2x+e^{-x}-\frac{x^2}{2}\right]_0^a$ $=2a+\frac{1}{e^a}-\frac{a^2}{2}-1$

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3(b)	$\int_0^6 \pi \left(\frac{x}{12}\sqrt{36-x^2}\right)^2 \, dx$ $= \int_0^6 \pi \, \frac{x^2}{144}(36-x^2)dx$ $= \pi\left[\frac{36}{144\times 3}x^3 - \frac{x^5}{144\times 5}\right]_0^6$ $= \pi\left(\frac{36}{144\times 3}6^3 - \frac{6^5}{144\times 5}\right)$ $= \pi\left(18 - \frac{54}{5}\right)$ $= 7.2\pi = 22.6 \, unit^3$

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4a (i)	$\overrightarrow{PQ} = \underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$ $\lvert\overrightarrow{PQ}\rvert = \sqrt{1^2+2^2+2^2} = \sqrt{9} = 3$
(ii)	$\widehat{\overrightarrow{PQ}} = \frac{1}{\lvert\overrightarrow{PQ}\rvert}\overrightarrow{PQ} = \frac{1}{3}(\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k})$
(iii)	$\overrightarrow{F_1} = 9\widehat{\overrightarrow{PQ}} = 9(\frac{1}{3})(\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k})$ $= 3(\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}) = (3\underset{\sim}{i} + 6\underset{\sim}{j} + 6\underset{\sim}{k}) \, N$ Total Work Done $=(\overrightarrow{F_1} + \overrightarrow{F_2}).\overrightarrow{PQ}$ $= \begin{pmatrix} 3+a \\ 15+6 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 51$ $3 + a + 42 + 12 = 51$ Hence, $a = -6$

No.	SOLUTION
4(b)	$P(1,-2,4)$ and $Q(3,2,10)$ $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$ $= (3\underset{\sim}{i} + 2\underset{\sim}{j} + 10\underset{\sim}{k}) - (\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k})$ $= 2\underset{\sim}{i} + 4\underset{\sim}{j} + 6\underset{\sim}{k}$ $L: \underset{\sim}{r} = (\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}) + \lambda(2\underset{\sim}{i} + 4\underset{\sim}{j} + 6\underset{\sim}{k})$ $x = 1 + 2\lambda \ldots (1)$ $y = -2 + 4\lambda \ldots (2)$ $z = 4 + 6\lambda \ldots (3)$ Or $L: \underset{\sim}{r} = (3\underset{\sim}{i} + 2\underset{\sim}{j} + 10\underset{\sim}{k}) + \lambda(2\underset{\sim}{i} + 4\underset{\sim}{j} + 6\underset{\sim}{k})$ $x = 3 + 2\lambda$ $y = 2 + 4\lambda$ $z = 10 + 6\lambda$ $4 = 1 + 2\lambda \ldots (1)$ $4 = -2 + 4\lambda \ldots (2)$ $13 = 4 + 6\lambda \ldots (3)$ Subst $Q(4,4,13)$ into (1), (2) and (3) give the same $\lambda = \frac{3}{2}$ Hence Q lies on the given line.

No.	SOLUTION
5(a)	$L_1: \underset{\sim}{r} = (\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}) + \lambda(\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k})$ $\overrightarrow{OP} = 4\underset{\sim}{i} - 4\underset{\sim}{j} + 3$ From the equation of line L1, the point Q(1,2, 3) also lies on the plane. Hence, $\overrightarrow{PQ}$ is a vector which is // to the plane: $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$ $= < 1, 2, 3 > - < 4, -4, 3 >$ $= -3\underset{\sim}{i} + 6\underset{\sim}{j}$

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To find a normal to the plane:

$\overrightarrow{PQ} \times (\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k})$
$= (-3\underset{\sim}{i} + 6\underset{\sim}{j}) \times (\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k})$
$= (-12 - 0)\underset{\sim}{i} - (6 - 0)\underset{\sim}{j} + (-12 - 6)\underset{\sim}{k}$
$= -12\underset{\sim}{i} - 6\underset{\sim}{j} - 18\underset{\sim}{k}$
$= -6(2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k})$

Hence, a normal vector to the plane
$\underset{\sim}{n} = 2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$

(b)

Hence, equation of the plane
$2(x-4)+1(y+4)+3(z-3)=0$
$2x+y+3z=13$

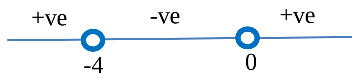
(c)

$L_2: \underset{\sim}{r} = (\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k}) + \mu(\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k})$

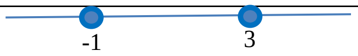
$x = 1 + \mu$
$y = 4 + 3\mu$
$z = 3 - 2\mu$
$2x + y + 3z = 13$

$2(1 + \mu) + (4 + 3\mu) + 3(3 - 2\mu) = 13$
$2 + 2\mu + 4 + 3\mu + 9 - 6\mu = 13$
$\mu = 2$

$x = 1 + 2 = 3$
$y = 4 + 3(2) = 10$
$z = 3 - 2(2) = -1$
Hence, plane π intersects L_2 at $(3, 10, -1)$

No.	SOLUTION
6(a)	$x^2 + 4x = 0 \Rightarrow x(x+4) = 0 \Rightarrow x = 0 \text{ or } -4$  $\therefore x < -4 \text{ or } x > 0$ $x^2 + 4x = 6x + 3 \Rightarrow x^2 - 2x - 3 = 0$ $\Rightarrow (x+1)(x-3) = 0 \Rightarrow x = -1 \text{ or } 3$ +ve -ve +ve when $x = 0, (x+1)(x-3) < 0$

No.	SOLUTION

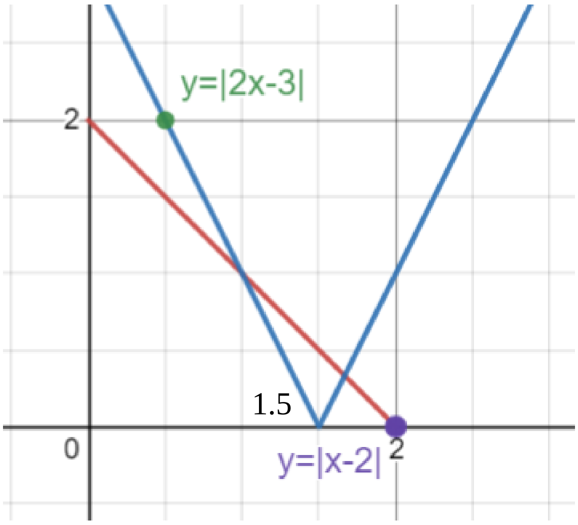


$\therefore -1 \le x \le 3$

Hence, for $0 < x^2 + 4x$ and $x^2 + 4x \le 6x + 3$

$0 < x \le 3$

(b)



Find coordinate of intersection point:

$$2x-3=-(x-2)$$
$$3x=5$$
$$x=\frac{5}{3}$$

$$-(2x-3)=-(x-2)$$
$$-2x+3=-x+2$$
$$x=1$$

From the graph, for the y values of the red line to be >= than the blue line (i.e. $|x-2| \ge |2x-3|$) then

$$1 \le x \le \frac{5}{3}$$

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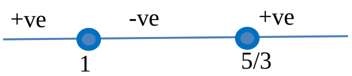
	OR (solve the inequality) $\lvert x-2\rvert \geq \lvert 2x-3\rvert \Rightarrow (x-2)^2 \geq (2x-3)^2$ $x^2-4x+4-4x^2+12x-9 \geq 0 \Rightarrow 3x^2-8x+5 \leq 0$ $\Rightarrow (x-1)(3x-5) \leq 0$  when $x=0, (x-1)(3x-5) > 0$ $\therefore 1 \leq x \leq \frac{5}{3}$