Chapter 12 : Solving Second Order Differential Equations

Objectives :

1. Define homogeneous and non-homogeneous second order ordinary differential equation with constant coefficients.
2. Solve differential equations using auxiliary equation.
3. Solve differential equations using Laplace transform method.

12.1 Introduction

A **2nd order linear differential equation with constant coefficients** is an equation of the form

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)\ \text{, where } a(\neq 0)\ ,\ b \text{ and } c \text{ are constants.}$$

If $f(x)=0$, the equation is said to be **homogeneous**.
If $f(x)\neq 0$, the equation is said to be **nonhomogeneous or inhomogeneous.**

12.2 Solving 2nd Oder Linear Homogeneous Differential Equations Using Auxiliary Equation

The general form of the equation is

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0 \quad .... \quad (1)$$

where a, b and c are constants. Note that $f(x)=0$; the equation is a **homogeneous** equation.

To solve this equation, we observe that the function $y=e^{\lambda x}$ satisfies equation (1), i.e. it is a solution of the differential equation. Substitute $y=e^{\lambda x}$ into equation (1) , we have

$$e^{\lambda x}(a\lambda^2+b\lambda+c)=0 \quad .... \quad (2)$$

Since $e^{\lambda x}$ cannot be zero, equation (2) is satisfied only if λ satisfies the quadratic equation

$$a\lambda^2+b\lambda+c=0 \quad .... \quad (3)$$

Equation (3) is called the <u>**auxiliary equation**</u> or <u>**characteristic equation**</u>**.**

$$\therefore \quad \lambda \quad = \quad \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

If λ_1 and λ_2 are the roots of auxiliary equation (3), then we get two solutions $y_1=e^{\lambda_1 x}$ and $y_2=e^{\lambda_2 x}$.

It can be shown that if y_1 and y_2 are the solutions to the differential equation (1), then $y=Ay_1+By_2$ (where A and B are constant) is also a solution.

The different forms of general solution of differential equation (1) depending on the nature of the roots of equation (3) are summarized below:

	Nature of Root(s)	**General Solution**
Case 1	Two distinct real roots λ_1 and λ_2 ($b^2-4ac>0$)	$y=Ae^{\lambda_1 x}+Be^{\lambda_2 x}$
Case 2	One repeated real root λ ($b^2-4ac=0$)	$y=e^{\lambda x}(Ax+B)$
Case 3	Complex roots $\lambda=\alpha \pm \beta j$ ($b^2-4ac<0$)	$y=e^{\alpha x}\left[A\cos(\beta x)+B\sin(\beta x)\right]$

Example 1: Find the general solution to the differential equation

$$9\frac{d^2y}{dx^2}-6\frac{dy}{dx}+y=0$$

Ans: $y(x)=e^{\frac{1}{3}x}\left(Ax+B\right)$

Example 2 : Find the general solution to the differential equation

$$2\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=0$$

Ans: $y(x)=e^{-1.25x}\left[A\cos(1.20x)+B\sin(1.20x)\right]$

Example 3 : Find the particular solution to the differential equation

$4y'' - 9y' = 0$ given that $y(0) = -1$ and $y'(0) = 1$. Ans: $y(x) = \frac{4}{9}e^{\frac{9}{4}x} - \frac{13}{9}$

12.3 Solving 2nd Order Linear Non-homogeneous Differential Equations With Laplace Transform Method

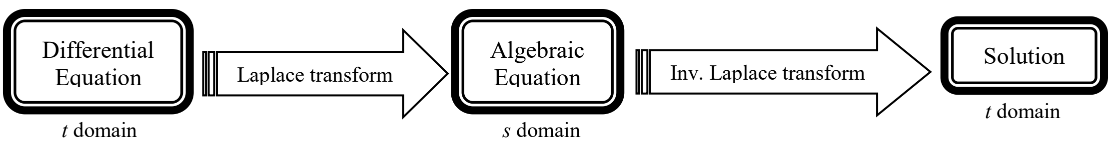
The Laplace transform is useful in solving ordinary linear differential equations with constant coefficients. The general second-order linear differential equation is as shown, subjected to initial conditions:

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t), \quad t \ge 0$$

Note that $f(t) \neq 0$; this equation is a **nonhomogeneous** equation. To solve the differential equation by the method of Laplace transforms, four distinct steps are required:

1. Take Laplace transform of each term in the differential equation.
2. Insert the given initial conditions.
3. Re-arrange the algebraic equation to obtain the transform of the solution, i.e. $Y(s)$.
4. Determine the inverse Laplace transform to obtain the solution, i.e. $y(t)$.

The concept is summarized here:



The actual detailed process will be illustrated in the *Example 4*.

Example 4: Solve the differential equation $\dfrac{d^2y}{dt^2}+5\dfrac{dy}{dt}+6y=2e^{-t}$, given that $y(0)=1$ and $y'(0)=0$.

Solution:

Step 1: Take Laplace transform of each term in the differential equation.

Use linearity property	$\mathcal{L}\left\{\dfrac{d^2y}{dt^2}\right\}+5\mathcal{L}\left\{\dfrac{dy}{dt}\right\}+6\mathcal{L}\{y\}=2\mathcal{L}\{e^{-t}\}$
Recall the Laplace transforms of differentials mentioned in 3.2	

Step 2: Insert the given initial conditions.

Given $y(0)=1$ and $y'(0)=0$, and recall the convention $\mathcal{L}\{y\}=Y(s)$	$s^2Y(s)-s+5\left[sY(s)-1\right]+6Y(s)=\dfrac{2}{s+1}$

Step 3: Re-arrange the algebraic equation to obtain the transform of the solution, i.e. $Y(s)$.

Make $Y(s)$ the subject of the formula *Some useful tips:* • *Factorise the quadratic expressions?* • *Combine common denominator?*	$\left(s^2+5s+6\right)Y(s)=\dfrac{2}{s+1}+s+5$

Step 4: Determine the inverse Laplace transform to obtain the solution, i.e. $y(t)$.

Inverse Laplace transform $Y(s)$ to get $y(t)$	$y(t)=\mathcal{L}^{-1}\{Y(s)\}$ $=\mathcal{L}^{-1}\left\{\dfrac{s^2+6s+7}{(s+1)(s+2)(s+3)}\right\}$
Before performing inversion, we need to first resolve into partial fractions.	Let $\dfrac{s^2+6s+7}{(s+1)(s+2)(s+3)}=\dfrac{A}{s+1}+\dfrac{B}{s+2}+\dfrac{C}{s+3}$ Use cover-up rule to find A, B and C: (*do your own working*) $A=1,\ B=1,\ C=-1$
Perform inversion to get solution	$\therefore y(t)=\mathcal{L}^{-1}\left\{\dfrac{1}{s+1}+\dfrac{1}{s+2}-\dfrac{1}{s+3}\right\}$ $=e^{-t}+e^{-2t}-e^{-3t}$

Example 5: Given the differential equation $\frac{d^2i}{dt^2}+6\frac{di}{dt}+9i=6t^2e^{-3t}$ with initial conditions $i(0) = 0$ and $i'(0) = 0$, find $i(t)$. *Ans*: $i(t) = \frac{1}{2}e^{-3t}t^4$

Tutorial 12

Section A

Find the general solution to each differential equation:

1. $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$
2. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$
3. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$

Solve each differential equation.
[Use the given boundary conditions to find the constants of integration.]

4. $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$, when $y(0) = 0$ and $y'(0) = 3$.
5. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$, when $y(0) = 5$ and $y'(0) = -9$.
6. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$, when $y(1) = 1 + e^2$ and $y'(1) = 2e^2$.
7. $\frac{d^2y}{dx^2} - 4y = 0$, when $y(0) = 1$ and $y'(0) = -1$.
8. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$, when $y(0) = 0$ and $y'(0) = 1$.
9. $2\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$ when $y(0) = 1$ and $y'(0) = 1$.

Section B

1. Solve the initial-value problem $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} = 4, \quad y(0) = -1, \; y'(0) = 2$

2. Solve the following differential equation using Laplace transform method:
   $q'' + 9q = 0$, where $q(0) = 0$ and $q'(0) = 2$

3. (a) Resolve $\frac{8}{(s+2)^2(s^2+4)}$ into partial fractions of the form $\frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{Cs+D}{s^2+4}$.

   (b) Hence, use the result from part (a) to solve the differential equation for $v(t)$:
   $v'' + 4v' + 4v = 4\sin 2t$, where $v(0) = 1$ and $v'(0) = 0$

4. (a) Express $\frac{2}{(s^2+1)(s^2+2s+5)}$ as a sum of partial fractions.

   (b) Show that $\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+5}\right\} = e^{-t}\left(\cos 2t - \frac{1}{2}\sin 2t\right)$.

(c) Given the differential equation $\frac{d^2y}{dt^2}+y=e^{-t}\sin 2t$ where $y(0)=0$ and $y'(0)=1$,

(i) use the result from part (a) to show that

$$\mathcal{L}\{y(t)\}=Y(s)=\frac{-s+7}{5(s^2+1)}+\frac{s}{5(s^2+2s+5)}$$

(ii) Hence, use the result from (b) to solve for $y(t)$.

*5. Use Laplace transform method to solve the following differential equation for $y(t)$:

$$\frac{d^2y}{dt^2}+9y=\cos 2t\,,\ \text{where}\ \ y(0)=1\ \ \text{and}\ \ y\left(\frac{\pi}{2}\right)=-1$$

Multiple Choice Questions

1. If the differential equation $4\frac{d^2y}{dx^2}+8\frac{dy}{dx}+ky=0$ has a general solution of the form $y(x)=e^{\alpha x}\left[A\cos(\beta x)+B\sin(\beta x)\right]$, where α, β, A and B are constants, then the value of the constant k is ________.

(a) <4 (b) ≤ 4
(c) >4 (d) ≥ 4

Answers

Section A

1. $y(x)=Ae^{3x}+Be^{-2x}$
2. $y(x)=(A+Bx)e^{2x}$
3. $y(x)=e^{-2x}\left(A\cos 3x+B\sin 3x\right)$
4. $y(x)=3xe^{-3x}$
5. $y(x)=(5-14x)e^{x}$
6. $y(x)=1+e^{2x}$
7. $y(x)=\frac{1}{4}\left(3e^{-2x}+e^{2x}\right)$
8. $y(x)=e^{-x}\sin x$
9. $y(x)=e^{-\frac{1}{2}x}\left(\cos\frac{3}{2}x+\sin\frac{3}{2}x\right)$

Section B

1. $y(t)=-3-2t+2e^{2t}$
2. $q(t)=\frac{2}{3}\sin 3t$
3. (a) $\frac{1}{2(s+2)}+\frac{1}{(s+2)^2}-\frac{s}{2(s^2+4)}$ (b) $v(t)=\frac{3}{2}e^{-2t}+3te^{-2t}-\frac{1}{2}\cos 2t$
4. (a) $\frac{-s+2}{5(s^2+1)}+\frac{s}{5(s^2+2s+5)}$ (c) $y(t)=\frac{7}{5}\sin t-\frac{1}{5}\cos t+\frac{1}{5}e^{-t}\left(\cos 2t-\frac{1}{2}\sin 2t\right)$
5. $y(t)=\frac{4}{5}\cos 3t+\frac{4}{5}\sin 3t+\frac{1}{5}\cos 2t$

MCQ

1. (c)