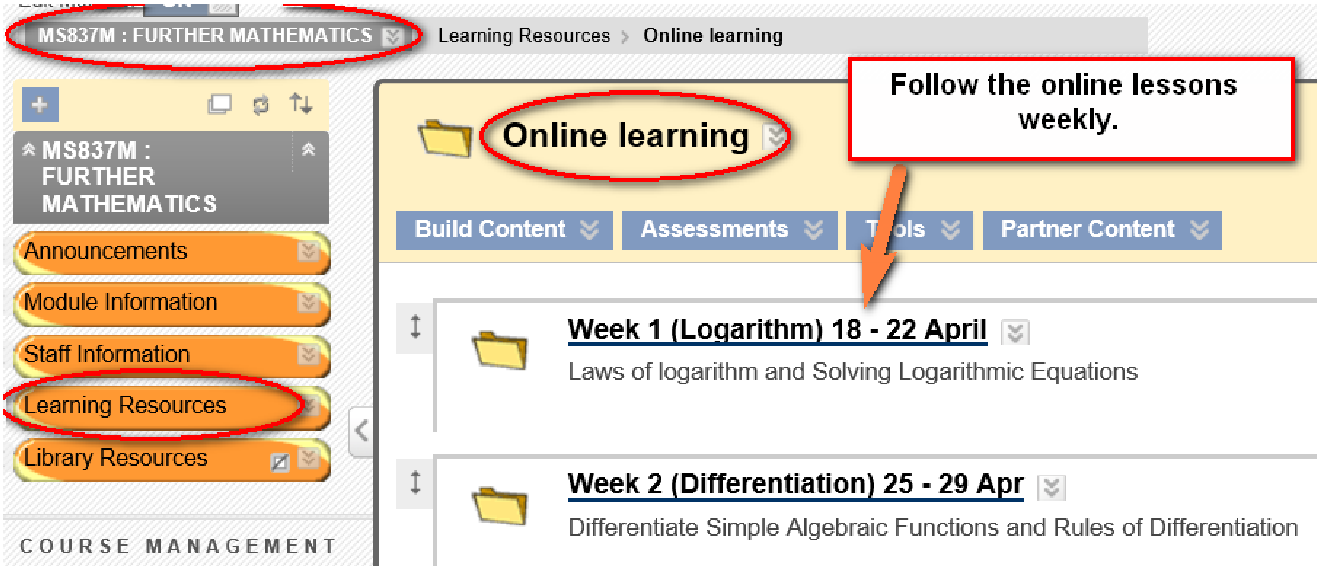
MS837M Online Learning

Refer to the study plan in
Blackboard-> MS837M: Further Mathematics -> Learning Resources-> Online learning



Chapter 1: Logarithmic Functions

Objectives :

1. *Define logarithm of any number to any base.*
2. *Define common logarithm and natural logarithm.*
3. *Convert logarithmic form to exponential form.*
4. *State the laws of logarithms.*
5. *Use the laws to simplify expressions.*
6. *Solve exponential equations.*
7. *Use the laws to solve equations.*

1.1 Logarithm of a number

DEFINITION

If $\quad y = a^x$

then $\quad \log_a y = x$

i.e. x is the logarithm of number y to the base a.

$y = a^x$ is said to be the **exponential form**, and $\log_a y = x$ is said to be in the **logarithmic form**.

For $\log_a y = x$ to be defined:

(i) $y > 0$

(ii) $a > 0$ but $a \neq 1$

Example 1: (a) $2^3 = 8$

=> $3 = \log_2 8$

i.e. 3 is the logarithm of 8 to base 2.

(b) $316.22 = 10^{2.5}$

=> $\log_{10} 316.22 = 2.5$

Example 2: Write each of the following in exponential form :

(a) $\log_6 216 = 3$ (b) $\log_3 \frac{1}{27} = -3$

1.2 The Common Logarithm and Natural Logarithm

Logarithms may be expressed in terms of any "base".

- Logarithms to **base 10** are called "**common logarithms**". When the base is 10 , this number is generally omitted.

  log N denotes the logarithm of N to the **base 10**.

- Logarithms to **base *e*** are called "**natural logarithms**" . *e* has approximately the value **2.718**.

  log*e* N is written as **ln N** .

Example 3: Write each of the following in exponential form :

(a) $\ln x = 5$ (b) $\log 100 = 2$

Example 4: Solve the following logarithmic equations : (Answer: (c) 8 (d) 10.54)

(a) $\log_2 x = 4$ (b) $\ln x = 2$

(c) $\log(x+2) = 1$ (d) $\ln(2x-1) = 3$

1.2.1 Properties of Logarithm

In general, for $a > 0$ but $a \neq 1$,

$\boxed{\log_a a = 1}$ since $a^1 = a$;

$\boxed{\log_a 1 = 0}$ since $a^0 = 1$.

Example 5: Evaluate each of the following :

(a) $\log_3 3$ (b) $\log_2 1$

(c) log 10 (d) log 1

(e) ln 1 (f) $\ln e$

Example 6: Find each of the following using a calculator :

(a) log 2 (b) log (–2)

(c) $\ln 11.18$ (d) $\ln 0.0037$

1.3 Laws of Logarithms

(A) Product Law

$$\log_a(XY) = \log_a X + \log_a Y$$

(B) Quotient Law

$$\log_a\left(\frac{X}{Y}\right) = \log_a X - \log_a Y$$

<Caution>

$$\log_a(x-y) \neq \log_a x - \log_a y$$
$$\log_a(x+y) \neq \log_a x + \log_a y$$

Example 7: Express $\log_a 3x + \log_a 7x^3$ as a single logarithm.

Example 8: Express $\ln xy$ as a sum of logarithms.

Example 9: Express $\log_2\left(\frac{b}{a}\right)$ as a difference of logarithms.

Example 10: Express $\log_a 32 - \log_a 16$ as a single logarithm.

(C) Power Law

$$\log_a X^n = n\log_a X$$

Example 11: Evaluate each of the following without using calculator:

(a) $\log_2 8$ (b) $\log 100$

Example 12: Express $\log_b\left(\dfrac{x^3y}{z^5}\right)$ in terms of sum and difference of logarithms.

(Answer: $3\log_b x+\log_b y-5\log_b z$)

Example 13: Express $\log_2\left(\dfrac{\sqrt{y}}{4x^3}\right)$ in terms of sum and difference of logarithms.

(Answer: $0.5\log_2 y-2-3\log_2 x$)

Example 14: Simplify $\dfrac{1}{2}\ln x+3\ln(x-1)$ to a single logarithm. (Answer: $\ln\left[\sqrt{x}(x-1)^3\right]$)

Example 15: Simplify $\log\left(x^2-xy\right)-\log(x-y)$ to a single logarithm. (Answer: $\log x$)

Example 16: Find the value of $\log 2 + \log 5$ without using calculator. (Answer: 1)

Example 17: Find the value of $\log_3 135 - \log_3 5$ without using calculator. (Answer: 3)

1.4 Change of Base

A general formula for changing from base a to base b can be derived as follows:

Let $\log_a N = x$

Change to exponential form $N = a^x$

Take logarithm to the base *b*: $\log_b N = \log_b a^x$

Apply power law of logarithm: $\log_b N = x \log_b a$

$$\frac{\log_b N}{\log_b a} = x$$

Hence,

$$\boxed{\log_a N = \frac{\log_b N}{\log_b a}}$$

Example 18: Find $\log_3 140$.

1.5 Solving Exponential Equations

An equation which contains a variable in an exponent (power) is called an **exponential equation (**e.g $3^x = 27$). Some exponential equations can be solved by expressing both sides of the equation as a power of the same base. However, if both sides cannot be rewritten to the same base, we can "take log of both sides".

Suppose we wish to solve the equation $e^x = 0.25$.
We shall use the *natural* logarithms here because of the e^x in the equation.

$$\ln(e^x) = \ln(0.25)$$
$$\Rightarrow\ x \cdot \ln e = \ln(0.25)$$
$$\Rightarrow\ x = \ln(0.25)$$
$$\Rightarrow\ x = -1.386$$

Let's now solve the equation $10^{3x} = 4.2$. Common logarithm is a good choice here because its base is 10.

$$\log(10^{3x}) = \log(4.2)$$
$$3x \cdot \log 10 = \log 4.2$$
$$3x = \log 4.2$$
$$x = \frac{\log 4.2}{3} = \frac{0.6232}{3} = 0.208$$

We wonder what will happen if we use natural logarithm.

$$\ln(10^{3x}) = \ln(4.2)$$
$$3x \cdot \ln 10 = \ln 4.2$$
$$3x = \frac{\ln 4.2}{\ln 10}$$
$$x = \frac{\ln 4.2}{3\ln 10} = \frac{1.4351}{3(2.3026)} = 0.208$$

Hence, we are free to choose any system of logarithm we want to use.

When using the operation "*take log of both sides"*, be careful that you do not commit this very common mistake:

If $\square + \blacktriangle + \bigcirc = ▱$

$\log_a \square + \log_a \blacktriangle + \log_a \bigcirc = \log_a ▱$ (WRONG!)

Always remember that the operation is to take log of both <u>sides</u>:

$\log_a(\square + \blacktriangle + \bigcirc) = \log_a ▱$ (CORRECT)

For equations of the type $\boldsymbol{e^x = a}$ and $\boldsymbol{a^x = b}$, we can take log of both sides.

Example 19: Solve the following for x :

(a) $5^x = 125$

(b) $e^{-0.3x} = \dfrac{1}{2}$

(c) $10^{2x+3} = 200$

(d) $7e^{1.5x} = 2e^{2.4x}$

(Answer: (a) 3; (b) 2.31; (c) −0.349; (d) 1.392)

1.6 Solving Logarithmic Equations

Equations containing variables in logarithmic expressions, such as $\log_2 x = 3$ and $\log x + \log(x+2) = 1$, are called **logarithmic equations**.

To solve logarithmic equations, we first try to obtain a single logarithmic expression on one side. We can then solve the equation by rewriting it in its equivalent exponential form or by "dropping log from both sides".

(1) Conversion to exponential form

If $\log_a \square = \blacktriangle$, then $\square = a^{\blacktriangle}$. (From definition of logarithms)
For example, if $\log_2 x = 0.6$, than $x = 2^{0.6}$.

Two important special cases:

$$\log \square = \blacktriangle \quad \Rightarrow \quad \square = 10^{\blacktriangle};$$
$$\ln \square = \blacktriangle \quad \Rightarrow \quad \square = e^{\blacktriangle}.$$

For those readers who find this operation confusing or hard-to-remember, we offer the following 'mechanical' way of doing this step: To remove the log from the equation $\log_4 x = 0.7$, move the base 4 to the other side and *push* the number 0.7 *up* like this:

$$\cancel{\log}_4 \, x \;=\; 0.7 \;\uparrow$$

$x = 4^{0.7}$ You see, we get what we wanted!

For equations of the type $\boldsymbol{\ln x = a}$ and $\boldsymbol{\log_a x = b}$, we remove log from the equation by rewriting it in its equivalent exponential form.

(2) Drop log from both sides.
If $\log_a \square = \log_a \blacktriangle$, then $\square = \blacktriangle$.
For equations of the type $\boldsymbol{\log_a x = \log_a y}$, we drop log from both sides.

Example 20: Solve the following equations for real x:

(a) $\log_4(x+5) - \log_4 x = 2$

(b) $\ln x + \ln(x+3) = 1$

(c) $\ln x + \ln(x-4) = \ln(x+6)$

(d) $2\log x - \log(x-1) = \log(x+8)$

Important: Logarithmic expressions are defined only for logarithms of positive real numbers. Hence, always check your solutions in the original equation, and exclude any proposed solution that produces the logarithm of a negative number.

Tutorial 1

1. Write each of the following exponential equations in **logarithmic form**:

   (a) $10^b = Y$ (b) $e^x = 7.15$ (c) $3^x = M + 2$

2. Write each of the following logarithmic equations in **exponential form**:

   (a) $\log_2 X = 3.28$ (b) $\log Y = 0.28$ (c) $\ln I = 1 - 6t$

3. Express each of the following expressions in terms of sum and/or differences of logarithms.

   (a) $\log_a\left(6xy^2\right)$ (b) $\ln\left(\frac{\sqrt{x}}{b^2}\right)$ (c) $\ln\left(\frac{5a}{2b^3}\right)$

   (d) $\log\left(\sqrt{xy}\right)$ (e) $\ln\left[\frac{\sqrt[3]{x-6}}{2x\left(x^2-4\right)}\right]$ (f) $\log\left[\frac{10^3(4-x)^2}{x^5}\right]$

4. Express each of the following as a single logarithm and if possible, simplify **without using calculator**:

   (a) $\log 20 + \log 50$ (b) $\log_3 54 - \log_3 6$

   (c) $\ln(x^2-9) - \ln(x+3)$ (d) $2\ln(x+1) + \ln(x-1) - \ln(x^2-1)$

5. Evaluate the following **without using a calculator** (Hint: Use Power Law).

   (a) $\log_4 16$ (b) $\log_a a^3$ (c) $\log_x \frac{1}{x^2}$

6. Do a change of base and then use your calculator to compute the following:

   (a) $\log_2 3$ (b) $\log_4 100$ (c) $\log_5 200$

7. Solve the following for x :

   (a) $e^{-x} = 0.4$ (b) $3^x = 5$ (c) $2^{x+1} = 7$

   (d) $6(2)^x = 1$ (e) $5e^{2x+1} - 8 = 0$ (f) $10^{-x} = 5^{x-1}$

8. Solve the following logarithmic equations for real x:

   (a) $\log_3 x = -2$ (b) $\log(x-3) = 1$ (c) $\log(10+3x) = 2$

   (d) $\ln x = -4$ (e) $\ln(x+5) = 3$ (f) $\ln(8-7x) = 1$

9. Solve the following logarithmic equations for real x:

(a) $\ln x + \ln(x + 2) = 1$ (b) $\log x + \log(x - 9) = 1$

(c) $\log_2(x+3) + \log_2(x-3) = 4$ (d) $\log_3(x+1) - \log_3 x = 2$

(e) $\log(x^2 - 1) - 2 = \log(x + 1)$ (f) $\log(10 - 2t)^3 - 3\log(3 - t) = 6$

10. Solve the following logarithmic equations for real x:

(a) $\log(x + 5) - \log(x - 1) = \log 2$

(b) $\ln(x^2 - 2) = \ln(3x - 4)$

(c) $\ln x + \ln(x + 4) = \ln 12$

(d) $\ln(x + 5) - \ln(x - 3) = \ln 2$

(e) $\log(x + 8) + \log(x - 1) = 2\log x$

Tutorial 1

1. (a) $b = \log Y$ (b) $x = \ln 7.15$ (c) $x = \log_3(M + 2)$

2. (a) $X = 2^{3.28}$ (b) $Y = 10^{0.28}$ (c) $I = e^{1-6t}$

3. (a) $\log_a 6 + \log_a x + 2\log_a y$ (b) $\frac{1}{2}\ln x - 2\ln b$ (c) $\ln 5 + \ln a - \ln 2 - 3\ln b$

(d) $\frac{1}{2}(\log x + \log y)$ (e) $\frac{1}{3}\ln(x-6) - \ln 2 - \ln x - \ln(x+2) - \ln(x-2)$

(f) $3 + 2\log(4-x) - 5\log x$

4. (a) 3 (b) 2 (c) $\ln(x-3)$ (d) $\ln(x+1)$

5. (a) 2 (b) 3 (c) –2

6. (a) 1.58 (b) 3.32 (c) 3.29

7. (a) 0.92 (b) 1.46 (c) 1.81
(d) –2.58 (e) –0.264 (f) 0.411

8. (a) 1/9 (b) 13 (c) 30
(d) 0.0183 (e) 15.1 (f) 0.755

9. (a) 0.928 (b) $x = 10$ or $x = -1$ (NA) (c) $x = 5$ or $x = -5$ (NA)
(d) 1/8 (e) 101 (f) 2.959

10. (a) 7 (b) $x = 2$ or $x = 1$ (NA) (c)) $x = 2$ or $x = -6$ (NA)
(d) 11 (e) 8/7

Chapter 2: Differentiation

Objectives :

1. *State the derivative of the function $y = x^n$ for all real n.*
2. *State the rules*

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$$

$$\frac{d}{dx}\left[k\,f(x)\right] = k\frac{d}{dx}f(x)$$ *, where k is a constant*

3. *State product rule.*
4. *State quotient rule.*
5. *State chain rule.*
6. *State the derivatives of trigonometric functions.*
7. *Apply the rules to find derivatives of algebraic & trigonometric functions.*
8. *Apply the rules to find derivatives of logarithm & exponential functions.*

Derivative of the power function

The derivative of the power function, $y = x^n$, where n is a real constant is:

$$\boxed{\frac{d}{dx}\left[x^n\right] = n\,x^{n-1}}$$

In words, to differentiate a power of the variable, we "bring down the power and decrease the power by 1."

The derivative can be denoted by $\frac{dy}{dx}$, $f'(x)$, $\frac{d}{dx}\left[f(x)\right]$ and y'.

The derivative $\frac{d}{dx}(\ )$ can be considered as an operator. The $\boldsymbol{\frac{d}{dx}}$ in front of an expression indicates that the expression is to be differentiated.

Constant Rule, Constant Multiple Rule, and Linearity Rule

Constant Rule

$\frac{d}{dx}(k) = 0$, where k is a constant.

Constant Multiple Rule

$\frac{d}{dx}\left[k\,f(x)\right] = k\frac{d}{dx}\left[f(x)\right]$ where k is a constant.

Linearity Rule

If $u = f(x)$ and $v = g(x)$, then

$$\frac{d}{dx}(au + bv) = a\frac{du}{dx} + b\frac{dv}{dx}.$$

Product Rule

When y, a function of x, is a **product** of two functions u and v, both of which are also functions of x, then the derivative of y with respect to x can be evaluated using the Product Rule as follows:

Product Rule

If $y = uv$, then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

Quotient Rule

On the other hand, if y is such that it is a **quotient** of two functions, u and v, then the derivative of y w.r.t x can be evaluated using the Quotient Rule:

Quotient Rule

If $y = \dfrac{u}{v}$, then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

Chain Rule

If $y = f(u)$ and $u = g(x)$ then $y = f[\,g(x)\,]$.

We say that y is a **composite function** of x.

Example 8:

(a) If $y = \sqrt{u}$ and $u = 3x^2 - 2$,

then $y = \sqrt{3x^2 - 2}$ is a composite function.

(b) If $y = u^2$ and $u = \sin x$,

then $y = (\sin x)^2 = \sin^2 x$ is a composite function.

For $y = f(u)$, we can find $\dfrac{dy}{du}$.

For $u = g(x)$, we can find $\dfrac{du}{dx}$.

So, to find $\dfrac{dy}{dx}$, we apply the Chain Rule

Chain Rule

If $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx}$$

Differentiation of Trigonometric Functions

In the previous sections we learnt how to differentiate algebraic expressions and also the rules of differentiation. We will now look at differentiating trigonometric functions.

$$\frac{d}{dx}(\sin x) = \cos x\,; \qquad \frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\tan x) = \sec^2 x\,; \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x\,; \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Notice that if the trigonometric function starts with 'c', for example 'cosine', its derivative will have a negative sign.

We remind readers that any numerical computation involving the derivatives of the trigonometric functions must be done in **radians**. This is because their derivative formulae were essentially obtained based on the use of the unit radians.

Differentiation of Logarithmic Functions

In this section we will state the formula for differentiating a logarithmic function. Our focus is on the natural logarithmic function, $\ln x$, as the formula for this function is more simplified than the general logarithmic function, $\log_a x$. Next, we will show you how to differentiate $\log_a x$ using a change of base.

Differentiating $y = \ln x$

If $y = \ln x$, then $\dfrac{dy}{dx} = \dfrac{1}{x}$.

In general, if $y = \ln\big(f(x)\big)$, where $f(x)$ is a function of *x*, to differentiate *y*, we will have to use Chain Rule as follows:

$$\frac{d}{dx}\big(\ln f(x)\big) = \frac{1}{f(x)}\frac{df}{dx}$$

Example 1 If $y = \ln(1-x)$, find $\dfrac{dy}{dx}$.

Solution $$\frac{dy}{dx} = \frac{1}{1-x}\frac{d}{dx}(1-x) = \frac{1}{1-x}(-1) = \frac{1}{x-1}$$

Example 2 Differentiate the following functions with respect to their respective variables.

(a) $u = 4\sin x \ln\left(x^2 - 3\right)$

(b) $x = \dfrac{\ln t}{3 + \tan t}$

Solution (a)

$$\begin{aligned}\frac{du}{dx} &= \ln\left(x^2-3\right)\frac{d}{dx}(4\sin x) + 4\sin x\frac{d}{dx}\ln\left(x^2-3\right)\\ &= 4\cos x\ln\left(x^2-3\right) + 4\sin x\frac{1}{x^2-3}\frac{d}{dx}(x^2-3)\\ &= 4\cos x\ln\left(x^2-3\right) + 4\sin x\frac{1}{x^2-3}(2x)\\ &= 4\cos x\ln\left(x^2-3\right) + \frac{8x\sin x}{x^2-3}\end{aligned}$$

(b)

$$\frac{dx}{dt} = \frac{(3+\tan t)\dfrac{d}{dt}\left(\ln t\right) - \ln t\dfrac{d}{dt}(3+\tan t)}{(3+\tan t)^2}$$

$$= \frac{\dfrac{1}{t}(3+\tan t) - \ln t(\sec^2 t)}{(3+\tan t)^2} = \frac{3+\tan t - t\ln t\sec^2 t}{t(3+\tan t)^2}$$

Differentiating $y = e^x$

If we replace the base a with Euler's constant e in $y = a^x$ and apply the rule we just derived in 6.3.1, we will get, $\dfrac{d}{dx}\left(e^x\right) = e^x \ln e$. Since $\ln e = 1$, this simplifies to

$$\frac{d}{dx}(e^x) = e^x$$

In general, $\dfrac{d}{dx}(e^u) = e^u\dfrac{du}{dx}$

Example 7 Differentiate with respect to x:

(a) $t = 2e^x$

(b) $y = e^{3x}$

Solution (a) $\dfrac{dt}{dx} = 2\dfrac{d}{dx}(e^x) = 2e^x$

(b) $\dfrac{dy}{dx} = \dfrac{d}{dx}(e^{3x}) = e^{3x}\dfrac{d}{dx}(3x) = 3e^{3x}$

Tutorial 2

1. Find the derivative of each of the following functions:

(a) $f(x) = x^4$ (b) $g(x) = 1/x^5$

(c) $h(x) = \sqrt[3]{x}$ (d) $k(x) = 1/\sqrt{x}$

2. Find each of the following derivatives.

(a) $\dfrac{d}{dx}\left(x^9\right)$ (b) $\dfrac{d}{dx}\left(\sqrt{x^3}\right)$

(c) $\dfrac{d}{dx}\left(\dfrac{1}{x^{-4}}\right)$ (d) $\dfrac{d}{dx}\left(\dfrac{1}{\sqrt[3]{x^2}}\right)$

3. (a) Find $\dfrac{dy}{dx}$, if $y = 4x^2 + \dfrac{3}{x^3} - 6\sqrt{x}$.

(b) Find $\dfrac{ds}{dr}$, if $s = 2\sqrt{r} + \dfrac{5}{r^2} - \pi$.

(c) Find $\dfrac{d}{dr}(ar^3 + br)$ where $a, b \in \mathbb{R}$ are constants.

(d) Find $\dfrac{d}{dt}(\pi t^3 + \sqrt{2}\, t)$.

4. For $f(x) = 2x^5 - 4x^3 + 3x^2 + 6x - 9$, find $f'(x)$ and hence the value $f'(1)$.

5. If $f(x) = 2.75x^2 - 5.02x$, find $f'(3.36)$.

6. For $f(x) = 2\sqrt{x} + \tan x$, find $f'(x)$.

7. If $g(t) = 2\sin t + 3\cos t$, find $g'(t)$.

8. For each of the following functions, find $f'(x)$ using the product rule:

(a) $f(x) = (2x^3 - 3x)(x^2 + 1)$ (b) $f(x) = (x+2)\left(1 - \dfrac{1}{x^2}\right)$

(c) $f(x) = (x^2 + 1)(\sqrt{x} + 2)$ (d) $f(x) = 5x\left(\sqrt{x} - \dfrac{1}{x}\right)$

(e) $f(x) = x\sin x$ (f) $f(x) = (x+5)\cos x$

(g) $f(x) = x^2 \tan x$ (h) $f(x) = \cos x \sin x$

9. By using the quotient rule, find the derivative of the following functions:

(a) $y = \dfrac{1}{1+x^2}$ (b) $y = \dfrac{x+3}{x-3}$

(c) $y = \dfrac{x^2}{2+3x^5}$ (d) $f(x) = \dfrac{x^2}{\sin x}$

(e) $f(x) = \dfrac{1+\cos x}{1-\cos x}$

10. Find $\dfrac{dy}{dx}$ by using chain rule:

(a) $y = (2x+1)^8$ (b) $y = (x^2+1)^{15}$

(c) $y = \sqrt{1-x}$ (d) $y = \sqrt{x^2+x+1}$

(e) $y = \csc 5x$ (f) $y = \sec 3x$

(g) $y = 6\tan \dfrac{1}{2}x$ (h) $y = 2\cot 0.1x$

(i) $y = 2\sin\left(3x - \dfrac{\pi}{4}\right)$ (j) $y = \pi \cos(6x + 0.71)$

(k) $y = \sin 3x^2$ (l) $y = \cos^3 x$

*(m) $y = \sin^2(3x)$ *(n) $y = \cos^4\left(\dfrac{x}{2}\right)$

11. Differentiate the following functions with respect to its variables. Simplify your answers whenever possible.

(a) $s = 0.5\sin 4x + 2x$ (b) $h = \dfrac{2}{3}\cos 6t - 5t$

(c) $y = 4\cos 7x - 2\sec 5x$ (d) $w = 300\sin\left(\dfrac{t}{150}\right) + \tan(2t)$

12. Differentiate the following with respect to its variables and simplify your answers whenever possible.

(a) $u = (2r+1)\sqrt{r^2-3}$ (b) $s = \sin t \cos 3t$

(c) $y = \left(x^3+1\right)\tan 2x$ (d) $x = \dfrac{1-u}{(1+u)^2}$

(e) $y = \dfrac{x}{\sin 2x}$ (f) $w = \dfrac{\sin u}{\cos 3u}$

13. Differentiate the following functions with respect to x. Simplify your answers whenever possible.

(a) $y = \ln 3x$ (b) $y = 3\ln\left(x^2+5\right)$

(c) $y = 4\ln x + x\ln 4$ (d) $y = 5 - 4\ln\left(3x^2-1\right)$

(e) $y = (2\ln x)^3$ (f) $y = \sin x \ln(x+1)$

(g) $y = \dfrac{\ln(2x+5)}{4x^2}$

14. Differentiate the following functions with respect to the given variables. Simplify your answers, where possible.

(a) $w = 2e^{2x} + 4e^{-2x}$ (b) $w = \sqrt{e^x} + \dfrac{1}{e^{2x}} - e^{\frac{1}{2}}$

(c) $w = 3e^{2x+7} + 4e^{x^2} + e^{\frac{1}{x}}$ (d) $w = 5\left(e^{2x} - 4\right)^3$

(e) $w = 2\sqrt{e^{3x-1}}$

Answers

1. (a) $4x^3$ (b) $-\dfrac{5}{x^6}$ (c) $\dfrac{1}{3x^{2/3}}$ (d) $-\dfrac{1}{2x^{3/2}}$

2. (a) $9x^8$ (b) $\dfrac{3}{2}\sqrt{x}$ (c) $4x^3$ (d) $-\dfrac{2}{3x^{5/3}}$

3. (a) $8x - \dfrac{9}{x^4} - \dfrac{3}{\sqrt{x}}$ (b) $\dfrac{1}{\sqrt{r}} - \dfrac{10}{r^3}$ (c) $3ar^2 + b$ (d) $3\pi t^2 + \sqrt{2}$

4. $f'(x) = 10x^4 - 12x^2 + 6x + 6$; $f'(1) = 10$

5. 13.46

6. $\dfrac{1}{\sqrt{x}} + \sec^2 x$ 7. $2\cos t - 3\sin t$

8. (a) $10x^4 - 3x^2 - 3$ (b) $\dfrac{1}{x^2} + \dfrac{4}{x^3} + 1$ (c) $\dfrac{5}{2}x^{3/2} + 4x + \dfrac{1}{2}x^{-1/2}$

(d) $5x \cdot \left(\dfrac{1}{2\sqrt{x}} + \dfrac{1}{x^2}\right) + \left(\sqrt{x} - \dfrac{1}{x}\right) \cdot 5 = \dfrac{15}{2\sqrt{x}}$ (e) $\sin x + x\cos x$

(f) $-(x+5)\sin x + \cos x$ (g) $x^2\sec^2 x + 2x\tan x$ (h) $\cos^2 x - \sin^2 x = \cos 2x$

9. (a) $\dfrac{-2x}{(1+x^2)^2}$ (b) $\dfrac{-6}{(x-3)^2}$ (c) $\dfrac{(2+3x^5)(2x) - x^2(15x^4)}{(2+3x^5)^2} = \dfrac{4x - 9x^6}{(2+3x^5)^2}$

(d) $\dfrac{2x\sin x - x^2\cos x}{\sin^2 x}$ (e) $\dfrac{-2\sin x}{(1-\cos x)^2}$

10. (a) $16(2x+1)^7$ (b) $30x(x^2+1)^{14}$ (c) $-\dfrac{1}{2\sqrt{1-x}}$

(d) $\dfrac{2x+1}{2\sqrt{x^2+x+1}}$ (e) $-5 \csc 5x \cot 5x$ (f) $3 \sec 3x \tan 3x$

(g) $3\sec^2\left(\dfrac{x}{2}\right)$ (h) $-0.2\csc^2 0.1x$ (i) $6\cos\left(3x - \dfrac{\pi}{4}\right)$

(j) $-6\pi \sin(6x + 0.71)$ (k) $6x\cos\left(3x^2\right)$ (l) $-3\cos^2 x\sin x$

(m) $6\sin(3x)\cos(3x) = 3\sin(6x)$ (n) $-2\cos^3\left(\dfrac{t}{2}\right)\sin\left(\dfrac{t}{2}\right)$

11. (a) $\frac{ds}{dx} = 2(\cos 4x + 1)$ (b) $\frac{dh}{dt} = -4\sin 6t - 5$ (c) $\frac{dy}{dx} = -28\sin 7x - 10\sec 5x \tan 5x$

(d) $\frac{dw}{dt} = 2\cos\frac{t}{150} + 2\sec^2 2t$

12. (a) $\frac{du}{dr} = 2\sqrt{r^2-3} + \frac{r(2r+1)}{\sqrt{r^2-3}}$ (b) $\frac{ds}{dt} = -3\sin t(\sin 3t) + \cos 3t(\cos t)$

(c) $\frac{dy}{dx} = 2(x^3+1)\sec^2 2x + 3x^2 \tan 2x$ (d) $\frac{dx}{du} = \frac{u^2-2u-3}{(1+u)^4} = \frac{u-3}{(1+u)^3}$

(e) $\frac{dy}{dx} = \frac{\sin 2x - 2x\cos 2x}{\sin^2 2x}$ (f) $\frac{dw}{du} = \frac{\cos 3u \cos u + 3\sin u(\sin 3u)}{\cos^2 3u}$

13. (a) $\frac{1}{x}$ (b) $\frac{6x}{x^2+5}$ (c) $\frac{4}{x} + \ln 4$ (d) $\frac{24x}{1-3x^2}$

(e) $\frac{24(\ln x)^2}{x}$ (f) $\frac{\sin x}{x+1} + \cos x \ln(x+1)$ (g) $\frac{1}{2x^3}\left[\frac{x}{2x+5} - \ln(2x+5)\right]$

14. (a) $4\left(e^{2x} - 2e^{-2x}\right)$ (b) $2\left(\frac{1}{4}e^{\frac{x}{2}} - e^{-2x}\right)$ (c) $6e^{2x+7} + 8xe^{x^2} - \frac{e^{\frac{1}{x}}}{x^2}$

(d) $30e^{2x}\left(e^{2x}-4\right)^2$ (e) $3\sqrt{e^{3x-1}}$

Chapter 3 – Integration

Objectives:

1. *Define integration is the reverse of differentiation.*
2. *State the standard integrals.*
3. *Integrate functions involving algebraic, trigonometric, exponential and integration leading to inverse trigonometric functions.*
4. *Obtain y from* $\frac{dy}{dx} = f(x)$ or $\frac{dx}{dy} = f(y)$, *and relate it to practical problems in engineering.*

3.1 Introduction

So far, when given a function $y = f(x)$, we can differentiate it to obtain the derivative $\frac{dy}{dx}$. This procedure is called **differentiation**. In this chapter, we will look at the reverse process, that is, obtaining the function $y = f(x)$ given the derivative $\frac{dy}{dx}$. This reverse process is called **integration**. Take note that the process of integration is not as straightforward as differentiation. And in this chapter, we will only deal with some standard integrals.

3.2 Integration as the reverse of differentiation

Consider the derivative $\frac{dy}{dx} = 2x$. We will now try to find a function $y \equiv y(x)$ whose derivative is $2x$. One answer is $y = x^2$, since $\frac{dy}{dx} = \frac{d}{dx}\left(x^2\right) = 2x$.

However there are other answers too, such as $y = x^2 + 1$, $y = x^2 - 10$, and so on since

$$\frac{d}{dx}\left(x^2 + 1\right) = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(1) = 2x + 0 = 2x,$$
$$\frac{d}{dx}(x^2 - 10) = \frac{d}{dx}(x^2) - \frac{d}{dx}(10) = 2x + 0 = 2x$$

In fact, there are infinitely many answers of the form $y = x^2 + C$ where C is a constant, as we know that $\frac{d}{dx}(C) = 0$. We say that the function $2x$ has been integrated to obtain another function $x^2 + C$ where C is a constant. And we express this as $\int 2x\,dx = x^2 + C$.
Since there is an arbitrary constant C in the expression $x^2 + C$, we say that this expression is an **indefinite integral**.

In general terms the operation of integration can be summarised as follows:

If $\dfrac{d}{dx}F(x) = f(x)$,

then $\int f(x)\,dx = F(x) + C$

where

- $\int f(x)\,dx$ is called the **indefinite integral** of $f(x)$ with respect to (w.r.t.) x,
- $f(x)$ is called the **integrand**,
- $F(x)$ is called an **anti-derivative** of the function $f(x)$, and
- C is called the **constant of integration**.

Since integration and differentiation are inverse processes of each other, we have

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x) \qquad \text{and} \qquad \int\left[\frac{d}{dx}f(x)\right]dx = f(x)$$

3.3 Integration of Power Functions

We will now establish a rule for $\int x^n dx$ where n is a constant and $n \neq -1$.

Using differentiation, we have $\dfrac{d}{dx}\left(\dfrac{x^{n+1}}{n+1}\right) = \dfrac{1}{n+1}\dfrac{d}{dx}\left(x^{n+1}\right) = \dfrac{1}{n+1}(n+1)x^n = x^n$

This means that $\boxed{\int x^n dx = \dfrac{x^{n+1}}{n+1} + C}$

Important:

The above integration formula $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C$ is invalid when $n = -1$. This means that the formula cannot be used to find $\int \dfrac{1}{x}\,dx$.

However we know that $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$ for $x > 0$. Hence, applying the definition of integration, we can write $\int \dfrac{1}{x}\,dx = \ln x + C$ for $x > 0$. To take care of the general case where x can be positive or negative, we have the following general result

$\boxed{\int \dfrac{1}{x}\,dx = \ln|x| + C}$ where $|x|$ is the absolute value of x.

3.4 Integration of Constants

We wish to establish a formula for $\int k\,dx$ where k is a constant.

From $k = k(1) = kx^0$, we have

$$\int k\,dx = \int kx^0\,dx = k\int x^0\,dx = k\left(\frac{x^{0+1}}{0+1}\right) + C = kx + C$$

Therefore we have $\boxed{\int k\,dx = kx + C}$

Example 1 : Find the following integrals.

(a) $\int 3x^2 dx$ (b) $\int \frac{1}{x^2}\,dx$ (c) $\int \frac{1}{u}\,du$

3.5 Integration of Trigonometric Functions

3.5.1 Standard Integration Formula for Trigonometric Functions

From $\frac{d}{dx}(\sin x) = \cos x$, we have $\boxed{\int \cos x\,dx = \sin x + C}$

Similarly, the following differentiation formulas give rise to the following corresponding integration formulas (where x is in radians) :

$$\frac{d}{dx}(\cos x) = -\sin x \quad \rightarrow \quad \boxed{\int \sin x\,dx = -\cos x + C}$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \quad \rightarrow \quad \boxed{\int \sec^2 x\,dx = \tan x + C}$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x \quad \rightarrow \quad \boxed{\int \csc^2 x\,dx = -\cot x + C}$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \quad \rightarrow \quad \boxed{\int \sec x \tan x\,dx = \sec x + C}$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x \quad \rightarrow \quad \boxed{\int \csc x \cot x\,dx = -\csc x + C}$$

3.5.2 Integration for Trigonometric Functions of Multiple Angles

What about the indefinite integral $\int \cos 2x\,dx$? What is the rule for integration when there is a coefficient (in this case 2) attach to the variable *x*?

Our knowledge of differentiation tells us that

$$\frac{d}{dx}(\sin 2x) = 2\cos 2x \text{ or } \frac{d}{dx}\left(\frac{\sin 2x}{2}\right) = \cos 2x.$$

This latter result shows us that.

$$\int \cos 2x\,dx = \frac{1}{2}\sin 2x + C$$

In general, if m is a nonzero constant, then $\frac{d}{dx}\left(\frac{\sin mx}{m}\right) = \cos mx$ and therefore

$$\boxed{\int \cos mx\,dx = \frac{\sin mx}{m} + C}$$

Similarly, we can obtain the following integration formulae (m is a nonzero constant.):

$$\boxed{\int \sin mx\,dx = -\frac{\cos mx}{m} + C} \qquad \boxed{\int \sec^2 mx\,dx = \frac{\tan mx}{m} + C}$$

$$\boxed{\int \csc^2 mx\,dx = -\frac{\cot mx}{m} + C} \qquad \boxed{\int \sec mx \tan mx\,dx = \frac{\sec mx}{m} + C}$$

$$\boxed{\int \csc mx \cot mx\,dx = -\frac{\csc mx}{m} + C}$$

Example 2 : Find:

(a) $\int \cos(\pi x)\,dx$

(b) $\int \sin\left(\frac{x}{2}\right)dx$

(c) $\int \sec(\pi x)\tan(\pi x)\,dx$

3.6 Integration of Exponential Functions

Let a be a positive constant and $a \neq 1$. From the differentiation formula:

$$\frac{d}{dx}a^x = a^x \ln a \qquad \text{or} \qquad \frac{d}{dx}\left(\frac{a^x}{\ln a}\right) = a^x$$

we obtain the integration formula $\boxed{\int a^x dx = \frac{a^x}{\ln a} + C}$

For most applications, the positive constant a is equal to the base of the natural logarithms e. Using the above result, we have

$$\boxed{\int e^x dx = \frac{e^x}{\ln e} + C = e^x + C} \qquad \text{where } \ln e = 1$$

If we substitute x by mx where m is a nonzero constant, the integration formulas become:

$$\boxed{\int a^{mx} dx = \int \left(a^m\right)^x dx = \frac{\left(a^m\right)^x}{\ln\left(a^m\right)} + C = \frac{a^{mx}}{m \ln a} + C}$$

and

$$\boxed{\int e^{mx} dx = \frac{e^{mx}}{m} + C}$$

Example 3 : Find: (a) $\int 2^{3x} dx$ (b) $\int e^{2x} dx$

3.7 Integration Leading to Inverse Trigonometric Functions

Recall that $\dfrac{d}{dx}\left[\sin^{-1} x\right] = \dfrac{1}{\sqrt{1-x^2}}$

Hence, $\dfrac{d}{dx}\left[\sin^{-1}\dfrac{x}{a}\right] = \dfrac{1}{\sqrt{1-\left[\dfrac{x}{a}\right]^2}}\left[\dfrac{1}{a}\right]$, where a is a constant.

$$= \sqrt{\frac{a^2}{a^2-x^2}}\left[\frac{1}{a}\right] \qquad = \frac{1}{\sqrt{a^2-x^2}}$$

Since integration is the reverse process of differentiation, we can use the above result to get

$$\boxed{\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left[\frac{x}{a}\right] + C}$$ where a is any constant.

Similarly, $\dfrac{d}{dx}\left[\tan^{-1}\dfrac{x}{a}\right] = \dfrac{1}{1+\left[\dfrac{x}{a}\right]^2}\left[\dfrac{1}{a}\right]$

$$= \frac{a^2}{a^2+x^2}\left[\frac{1}{a}\right] \qquad = \frac{a}{a^2+x^2}$$

$$\frac{d}{dx}\left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right] = \frac{1}{a^2+x^2}$$

Thus, $$\boxed{\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left[\frac{x}{a}\right] + C}$$ where a is any constant.

Example 4 : Find : (a) $\displaystyle\int \frac{dx}{\sqrt{4-x^2}}$ (b) $\displaystyle\int \frac{dt}{7+t^2}$

3.8 Rules of Integration

Indefinite integrals have the following properties:

(i) $\int k\, f(x)\,dx = k\int f(x)\,dx + C$, where k is a constant.

(ii) $\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

(iii) $\int [f(x)-g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$

In other words, the process of integration satisfies the linearity property

$$\int [a\, f(x)+b\, g(x)]\,dx = a\int f(x)\,dx + b\int g(x)\,dx$$

where a and b are constants.

Example 5 : Find :

(a) $\int \left(\sqrt{x}+\frac{3}{x}\right)dx$

(b) $\int \left(2+3x^4-\frac{3}{\sqrt{x}}\right)dx$

(c) $\int (\pi+\sin 2x)\,dx$

(d) $\int \left(5\sec^2 3\theta+\frac{2}{\theta^2}\right)d\theta$

(e) $\int \left(2e^{x/2}-e^{-x}\right)dx$

(f) $\int \left(3r+10^{5r}-\frac{1}{e^{2r}}\right)dr$

(g) $\int x(3x-2)\,dx$

(h) $\int (2x-3)^2\,dx$

Tutorial 3

1. Find the following indefinite integrals:

(a) $\int 10\,dx$ (b) $\int t^5 dt$ (c) $\int \frac{1}{2x}dx$

(d) $\int \frac{x+1}{x}dx$ (e) $\int \left(x+\frac{1}{x}\right)^2 dx$ (f) $\int \frac{x^4-3x+2}{x^2}dx$

(g) $\int \cos \pi x\,dx$ (h) $\int 2\sin\left(\frac{u}{2}\right)du$ (i) $\int (\sec^2 2x+\csc^2 3x)\,dx$

(j) $\int \frac{4}{9+x^2}dx$ (k) $\int \frac{2}{\sqrt{3-x^2}}dx$

2. Find the following indefinite integrals:

(a) $\int 3^x\,dx$ (b) $\int e^{3y}\,dy$ (c) $\int e^x\left(e^x+\frac{2}{e^{2x}}\right)dx$

(d) $\int \frac{1-2e^{-x}+e^x}{e^x}dx$ (e) $\int (e^x-e^{-x})^2 dx$ (f) $\int (x^e+e^x+e)\,dx$

*(g) $\int e^x\left(10^x+e\right)\,dx$ (h) $\int \frac{1}{\sqrt{(4-w)(4+w)}}\,dw$

(i) $\int \left(\frac{1}{\sqrt{25-3t^2}} - \frac{1}{3t^2+25}\right)dt$

<u>Tutorial</u> : MCQ

1. Given the function $y=f(x)$ and its derivative $\frac{dy}{dx}=1$, which of the following best describes the function $y=f(x)$?

(a) $y=f(x)$ is a trigonometric function (b) $y=f(x)$ is a linear function
(c) $y=f(x)$ is a logarithmic function (d) $y=f(x)$ is an exponential function

2. If $\frac{d}{dx}F(x)=f(x)$, then ________.

(a) $\int F'(x)\,dx = f(x)+C$ (b) $\int f'(x)\,dx = F(x)+C$
(c) $\int F(x)\,dx = f(x)+C$ (d) $\int f(x)\,dx = F(x)+C$

Tutorial 3 (Answers)

1. (a) $10x + C$ (b) $\frac{t^6}{6} + C$ (c) $\frac{1}{2}\ln|x| + c$

(d) $x + \ln|x| + C$ (e) $\frac{x^3}{3} + 2x - \frac{1}{x} + C$ (f) $\frac{x^3}{3} - 3\ln|x| - \frac{2}{x} + C$

(g) $\frac{\sin \pi x}{\pi} + C$ (h) $-4\cos\frac{u}{2} + C$ (i) $\frac{\tan 2x}{2} - \frac{\cot 3x}{3} + C$

(j) $\frac{4}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$ (k) $2\sin^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$

2. (a) $\frac{3^x}{\ln 3} + C$ (b) $\frac{e^{3y}}{3} + C$ (c) $\frac{e^{2x}}{2} - \frac{2}{e^x} + C$

(d) $-e^{-x} + e^{-2x} + x + C$ (e) $\frac{e^{2x}}{2} - 2x - \frac{e^{-2x}}{2} + C$ (f) $\frac{x^{e+1}}{e+1} + e^x + ex + C$

(g) $\frac{e^x 10^x}{1 + \ln 10} + e^{x+1} + C$ (h) $\sin^{-1}\left(\frac{w}{4}\right) + C$

(i) $\frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{\sqrt{3}\,t}{5}\right) - \frac{1}{5\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}\,t}{5}\right) + C$

MCQ

1. (b) 2. (d)

Chapter 4 – Definite Integrals and Area Under a Curve

Objectives:

1. *Define and explain definite integral* $\int_a^b f(x)dx = F(b) - F(a)$.
2. *Evaluate definite integrals.*
3. *Explain that* $\int_a^b y\,dx$ *denotes the net-signed area bounded by the curve* $y = f(x)$ *between the ordinates x = a and x = b.*
4. *Find the area under a curve.*
5. *Explain the case where negative area is involved.*

4.1 Introduction

Calculus is traditionally divided into two branches: the differential calculus and the integral calculus. In differential calculus we study the concept of the derivative; in integral calculus the concept of the definite integral. In this lesson we shall learn what definite integral is and see how such a mathematical idea, first used by the great Greek mathematicians more than 2000 years ago, developed into a mathematical tool of great beauty and immense power.

4.2 Define Definite Integrals

If $f(x)$ is continuous on the interval $[a, b]$, then

$$\boxed{\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)}$$

where

$f(x)$ is the integrand;
x is the variable of integration;
a is the lower limit of integration;
b is the upper limit of integration;
$a \le x \le b$ the interval of integration;
and
$\int_a^b f(x)\,dx$ is the integral of the function $f(x)$ over the interval $[a, b]$.

4.3 Evaluate Definite Integrals

To find the value of the **definite integral** $\int_a^b f(x)\,dx$, where $f(x)$ is continuous over the interval $[a, b]$, we first find an anti-derivative of $f(x)$. Call it $F(x)$. Substitute the upper and lower limit b and a into $F(x)$ to obtain the values $F(b)$ and $F(a)$. Then do the subtraction $F(b) - F(a)$.

$$\boxed{\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)}$$

4.4 Properties of the Definite Integral

1. $\int_a^b k\, f(x)\,dx = k\int_a^b f(x)\,dx$, where k is constant

   Example: $\int_0^1 6\cos x\,dx = 6\int_0^1 \cos x\,dx$.

2. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

   Example: $\int_0^1 (x + \sqrt{3x} - e^x)\,dx = \int_0^1 x\,dx + \int_0^1 \sqrt{3x}\,dx - \int_0^1 e^x dx$.

3. Let c be a value inside the interval $[a, b]$

   $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

   It is possible to split the integral $\int_a^b f(x)\,dx$ into several parts. For example, we could write, if there is a need for it,

   $$\int_0^6 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^5 f(x)\,dx + \int_5^6 f(x)\,dx.$$

4. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

5. In $\int_a^b f(x)\,dx$, x is a *dummy* variable. It means that x can be substituted by any symbol without affecting the value of the definite integral. Thus

   $$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

Example 1 : Evaluate:

(a) $\int_{1}^{3} x^3 dx$ (b) $\int_{\pi/3}^{\pi} \cos 2x \, dx$ (c) $\int_{0}^{1} e^{2x} dx$

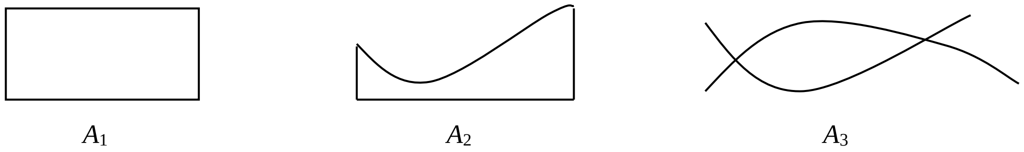
Example 2 : Evaluate:

(a) $\int_{0}^{2} (x+2) \, dx$ (b) $\int_{-1}^{1} (4u-3)^2 du$

(c) $\int_{0}^{0.5} \left(3e^{-2t} - \frac{1}{2}\cos \pi t\right) dt$ (d) $\int_{2}^{4} \left(5\sin 3x + \frac{2}{x}\right) dx$

4.5 Applications of the Definite Integrals – Area

Finding areas is an important part of mathematics. In schools, we learned very early how to find the area of squares, of rectangles, of triangles, and many other shapes. We all know that some areas are easy to find and some areas are difficult to find. Consider the following three figures and the areas they enclose.



Area A_1 is easy to find; but areas A_2 and A_3 are difficult to find. The definite integral is a powerful and versatile tool for tackling the problem of areas.

Note: Definite Integral vs Area

Before we begin to apply definite integrals to the problem of areas, it is important to note that although the *definite integral* $\int_a^b f(x)dx$ is often illustrated by the *area* under a curve, these two are entirely different concepts. $\int_a^b f(x)dx$ must be understood as the 'limit of a sum' whereas the area is a quantity associated with a plane figure like rectangle or a circle. Furthermore note also that $\int_a^b f(x)dx$ is a number which can be positive, negative or zero; the area, however, is non-negative (always positive, can be zero, but never negative) by definition.

4.5.1 The Classic Case: Area Under a Curve

Consider the area of the region bounded by a curve $y = f(x)$, two vertical lines, and the x-axis as illustrated in Figure 2. Note that in this "classic case", $f(x) \geq 0$ throughout the interval $[a, b]$. In other words, the curve does not cross to the negative side of the y-axis in the interval $[a, b]$. This type of area is generally known as the 'area under a curve'.

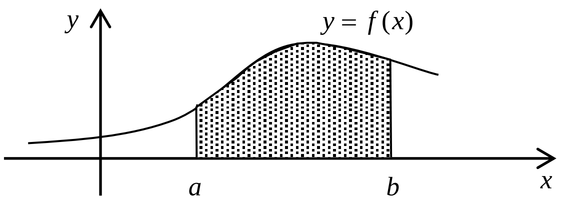


Figure 2

1st Step: Identify the quantity we are interested in. It is the area A.

2nd Step: Find an expression for ΔA
Slice the area vertically into many small areas as shown in Figure 3.
ΔA, the small area, can be approximated by a rectangle of height y_i and width Δx.

$$\Delta A \approx y_i \cdot \Delta x = f(x_i) \cdot \Delta x$$

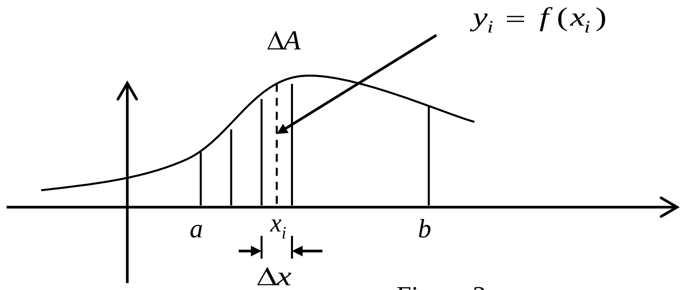


Figure 3

3rd Step: Sum them up.
We now have the approximate area.

$$A = \sum \Delta A \approx \sum f(x_i) \cdot \Delta x$$

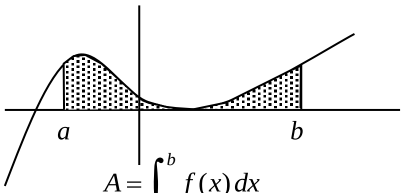
4th Step: Carry out the limiting operation.
By carrying out the limiting operation $n \to \infty$, the approximation becomes an equality (an equation) and we have

$$A = \lim_{n\to\infty} \sum_{x=a}^{x=b} f(x_i) \cdot \Delta x = \int_a^b f(x)\,dx\,.$$

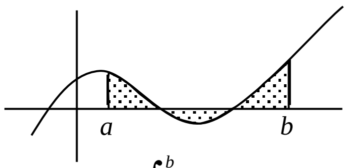
$$A = \int_a^b f(x)\,dx \qquad f(x) \geq 0 \quad \text{in} \quad [\text{a, b}]$$

Important

This formula is applicable only to the standard 'area under the curve' type of areas. And it is valid only under the assumption that the function $f(x)$ is non-negative over the interval [*a*, *b*]. It is very easy to check this condition visually; just make sure that the curve is above the *x*-axis (touching the *x*-axis is all right) and that it does not cross the *x*-axis to the other side in the interval [*a*, *b*].

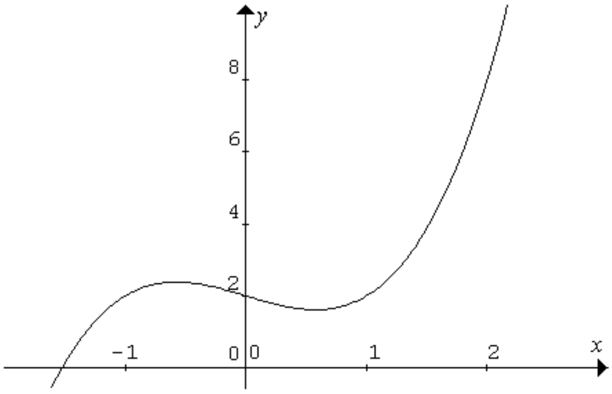


$$A = \int_a^b f(x)\,dx$$

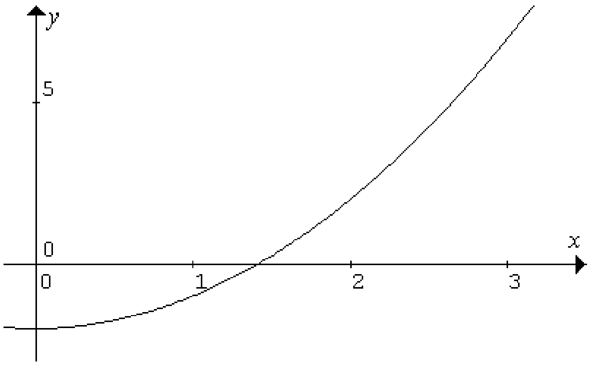


$$A \neq \int_a^b f(x)\,dx$$

Example 3 : Find the area under the curve $y = x^3 - x + 2$ from $x = -1$ to $x = 2$.

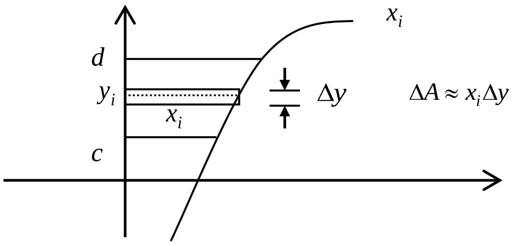


Example 4 : Find the area bounded by the curve $y = x^2 - 2$ and the *x*-axis, from *x* = 0 to $x = 3$.



4.5.2 Reversing the Role of *x* and *y*

For certain type of areas, it may be easier to slice the area *horizontally* instead of vertically.



$\Delta A \approx x_i \Delta y = g(y_i)\Delta y$

$$\boxed{A = \int_c^d g(y)\,dy}$$

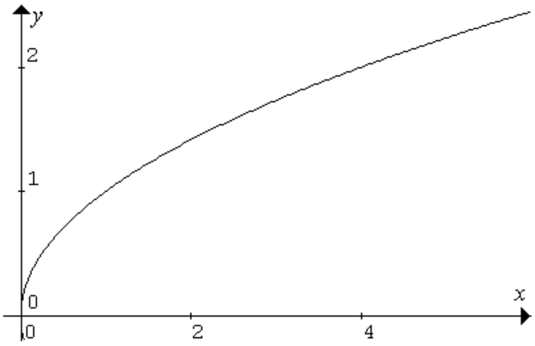
Important

This formula is applicable only to the standard 'area under the curve' type of areas. And it is valid only under the assumption that the function $g(y)$ is non-negative over the interval $[c, d]$.

Example 5 **:** Find the area of the region bounded by the curve $x = y^2$,

(i) the *y*-axis, $y = 0$ and $y = 2$.

(ii) the *x*-axis, $x = 2$ and $x = 4$.



Tutorial 4

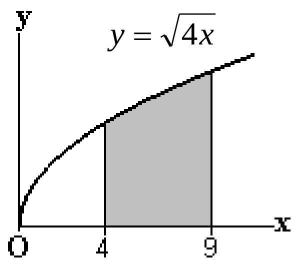
1. Evaluate the following integrals.

(a) $\int_{2}^{3} x\,dx$ (b) $\int_{2}^{5} dx$ (c) $\int_{1}^{4} (x^2+3x)\,dx$

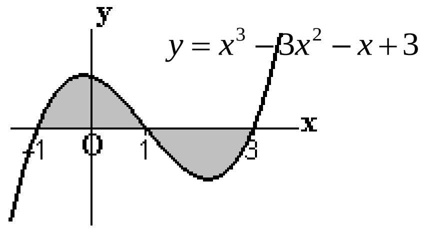
(d) $\int_{1}^{10} \frac{1}{2x}\,dx$ (e) $\int_{-2}^{-1}\left(4e^{-2x}+\frac{3}{x}\right)dx$ (f) $\int_{0}^{1}(5x-\sin 3x)dx$

2. Find the shaded areas.

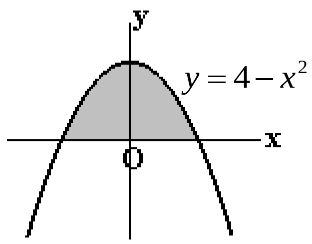
(a)



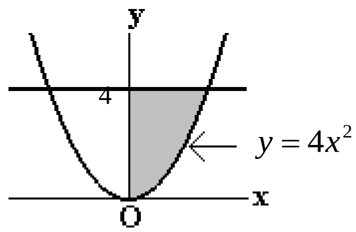
(b)



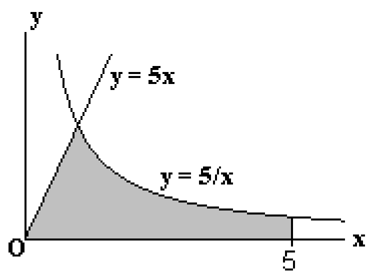
(c)



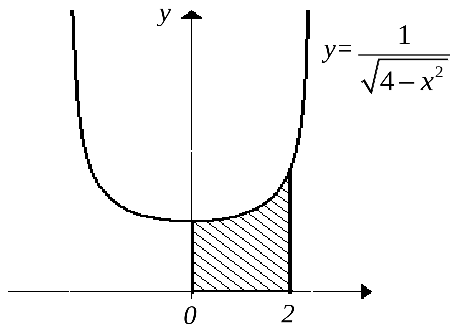
(d)



(e)



(f)



3. Find the area bounded by the given curves:
Note: $x = 0$ is the y-axis and $y = 0$ is the x-axis. Sketch the graphs for each question.

(a) $y=2x+3$; $y-axis$, from $y=0$ to $y=5$

(b) $y=4-x^2$; $x-axis$, from $x=0$ to $x=4$

(c) $y=5\sin 3x$; $x-axis$, from $x=\frac{\pi}{4}$ to $x=\frac{2\pi}{3}$

4. Find the area bounded by the curve and the x-axis:

(a) $y=x^2-2x-3$ (b) $y=x(x-1)(x-3)$

Tutorial : MCQ

1. Given $\int_{-1}^{k} dx = 1$, then $k =$ ______.

   (a) 0   (b) -1 (c) 1   (d) 2

2. Given $w = \int_{-1}^{k} x\,dx$, where k is a positive constant. By carrying out integration and simplification, $w =$ ______.

   (a) $0.5(k^2 - 1)$ (b) $0.5(k^2 + 1)$ (c) $k + 1$ (d) $k - 1$

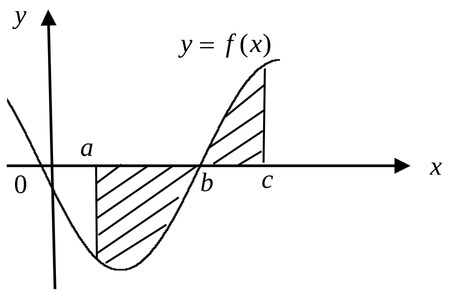
3. Which of the following integrals gives the area of the shaded region?

   (a) $\int_{a}^{c} f(x)\,dx$

   (b) $\int_{a}^{b} f(x)\,dx + \int_{b}^{c} f(x)\,dx$

   (c) $\left|\int_{a}^{b} f(x)\,dx\right| + \int_{b}^{c} f(x)\,dx$

   (d) $\int_{a}^{b} f(x)\,dx + \left|\int_{b}^{c} f(x)\,dx\right|$



ANSWERS

Tutorial 4

1.	(a) 2.5	(b) 3	(c) 43.5	(d) 1.15	(e) 92.34	(f) 1.84
2.	(a) $25\frac{1}{3}$	(b) 8	(c) $\frac{32}{3}$	(d) $2\frac{2}{3}$	(e) 10.55	(f) $\frac{\pi}{2}$
3.	(a) 13/4	(b) 16	(c) 3.82			
4.	(a) $10\frac{2}{3}$	(b) $3\frac{1}{12}$				

MCQ

1. (a) 2. (a) 3. (c)