7. INTEGRATION

7.1 REVISION ON INTEGRATION

7.1.1 INDEFINITE INTEGRAL

Integration is the **reverse process** of differentiation.

In general terms, the operation of integration can be summarised as follows:

If $\dfrac{d}{dx}F(x) = f(x)$,

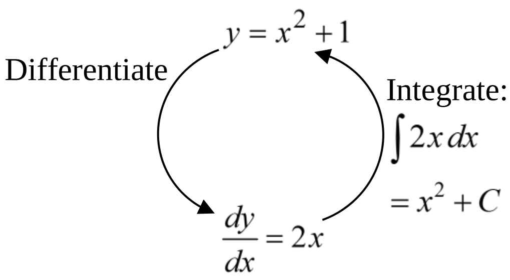
then $\int f(x)\,dx = F(x) + C$

where

- $\int f(x)\,dx$ is called the **indefinite integral** of $f(x)$ with respect to (w.r.t.) x,
- $f(x)$ is called the **integrand**,
- $F(x)$ is called an **anti-derivative** of the function $f(x)$, and
- C is called the **constant of integration**.

For example, $\dfrac{d}{dx}\left(x^2+1\right) = 2x+0$, then

$\int 2x\,dx = x^2 + C$



<u>Some Standard Integral</u>

1. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C, \quad n \neq -1$
2. $\int k\,dx = kx + C$ (not given in formulae card)
3. $\int \dfrac{1}{x}\,dx = \ln|x| + C$
4. $\int e^x dx = e^x + C$
5. $\int \sin x\,dx = -\cos x + C$
6. $\int \cos x\,dx = \sin x + C$

A more complete standard integral can be found in the formulae card.

EXAMPLE 1 Find

(a) $\displaystyle\int\left(\sqrt{x}+\frac{3}{x}\right)dx$ (b) $\displaystyle\int\left(2+3x^4-\frac{3}{\sqrt{x}}\right)dx$ (c) $\displaystyle\int\left(\pi+\sin 2x\right)dx$

(d) $\displaystyle\int\left(5\sec^2 3\theta+\frac{2}{\theta^2}\right)d\theta$ (e) $\displaystyle\int\left(2e^{x/2}-e^{-x}\right)dx$ (f) $\displaystyle\int\sin\left(3x+1\right)dx$

EXAMPLE 2

A curve passes through the point (1, 7) and the gradient of the curve is given by $(5-2x)$.
Fine the equation of the curve. (Ans: $y=-x^2+5x+3$)

7.1.2 DEFINITE INTEGRAL

If $f(x)$ is continuous on the interval $[a,b]$, then

$$\boxed{\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b)-F(a)}$$

where

$f(x)$ is the integrand;
x is the variable of integration;
a is the lower limit of integration;
b is the upper limit of integration;
$a \le x \le b$ the interval of integration;
and

$\int_a^b f(x)\,dx$ is called the Definite Integral of f w.r.t. x from $x=a$ to $x=b$.

To find the value of the **definite integral** $\int_a^b f(x)\,dx$, where $f(x)$ is continuous over the interval [a, b], we first find an anti-derivative of $f(x)$. Call it $F(x)$. Substitute the upper and lower limit b and a into $F(x)$ to obtain the values $F(b)$ and $F(a)$. Then do the subtraction $F(b) - F(a)$.

$$\boxed{\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b)-F(a)}$$

EXAMPLE 3 Evaluate:

(a) $\int_1^3 x^3 dx$ (b) $\int_{\pi/3}^{\pi} \cos 2x\, dx$ (c) $\int_0^1 e^{2x} dx$

Solution (a) $\int_1^3 x^3 dx = \left[\frac{x^4}{4}\right]_1^3 = \frac{1}{4}\left[3^4 - 1^4\right] = \frac{1}{4}(81-1) = 20$

$$\int_{\pi/3}^{\pi} \cos 2x\, dx = \left[\frac{\sin 2x}{2}\right]_{\frac{\pi}{3}}^{\pi}$$

(b)
$$= \left[\frac{\sin(2\pi)}{2}\right] - \left[\frac{\sin\left(\frac{2\pi}{3}\right)}{2}\right]$$

$$= 0 - \frac{\sqrt{3}}{4} = -\frac{\sqrt{3}}{4}$$

(c) $\int_0^1 e^{2x} dx = \left[\frac{e^{2x}}{2}\right]_0^1 = \frac{1}{2}\left(e^2 - e^0\right) = 3.195$

EXAMPLE 4 Evaluate

(a) $\int_0^2 (x+2)\, dx$ (b) $\int_{-1}^1 (4u-3)^2\, du$

(c) $\int_0^{0.5} \left(3e^{-2t} - \frac{1}{2}\cos \pi t\right) dt$ (d) $\int_2^4 \left(5\sin 3x + \frac{2}{x}\right) dx$

7.2 INTEGRATION BY SUBSTITUTION

Consider at the following integrals:

- $\int 3x^2(x^3+1)^8\,dx$
- $\int \dfrac{3x^2-1}{x^3-x}\,dx$
- $\int x\,e^{x^2}\,dx$

These integrals might look complicated, but they can each reduced to standard integral by a simple substitution. This technique is known as *'Integration by Substitution'* which is one of the very useful techniques of integrations.

Differential of a Function

The differential of $y = f(x)$ is defined as

$$\boxed{dy = \frac{dy}{dx}\cdot dx} \quad \text{or} \quad \boxed{dy = f'(x)\cdot dx}$$

EXAMPLE 5 Find the differential of the following functions:

(a) $y = 4x^2 + 3x - 7$ (b) $u = 3\sin 4t$

Solution (a) $\dfrac{dy}{dx} = 8x + 3$

The differential of y is $dy = (8x + 3)\,dx$

(b) $\dfrac{du}{dt}$ = (3cos 4t) 4 = 12 cos 4t

The differential of u is $du = (12\cos 4t)\,dt$

7.2.1 INTEGRATION BY SUBSTITUTION OF THE FORM

$$\int [f(x)]^n \cdot f'(x)\, dx$$

We notice that one function of the product is the differential coefficient of the other function. We can solve the problem by a substitution which leads the integral to one of the standard integrals.

Let $u = f(x)$, then $\dfrac{du}{dx} = f'(x)$. Expressing in differential form: $du = f'(x)\, dx$

$\int \left[f(x) \right]^n \cdot f'(x)\, dx = \int u^n\, du = \dfrac{u^{n+1}}{n+1} + C$, where $n \neq -1$ (Standard Integral).

Hence
$$\boxed{\int \left[f(x) \right]^n \cdot f'(x)\, dx = \frac{\left[f(x) \right]^{n+1}}{n+1} + C \quad \text{where } n \neq -1}$$

or
$$\boxed{\int u^n\, du = \frac{u^{n+1}}{n+1} + C \quad \text{where } n \neq -1}$$

EXAMPLE 6 **(a)** $\int (x^2+3)^5 2x\, dx$ $\qquad$ $Ans: \dfrac{1}{6}\left(x^2+3\right)^6 + C$

(b) $\int 3x \sqrt{1-2x^2}\, dx$ $\qquad$ $Ans: -\dfrac{1}{2}\left(1-2x^2\right)^{3/2} + C$

EXAMPLE 7 $\int (e^x+1)^3 e^x\,dx$

Solution

Let $u = e^x + 1$

$du = e^x dx$

$$\int (e^x+1)^3 e^x\,dx = \int (u)^3 du \qquad \text{(change } x \text{ to } u\text{)}$$

$$= \frac{u^4}{4} + C \qquad \text{(apply standard integral)}$$

$$= \frac{\left(e^x+1\right)^4}{4} + C \qquad \text{(leave answer in terms of } x\text{)}$$

Integration by substitution of the form $\int [f(x)]^n \cdot f'(x)\,dx$ can be summarised by the following example.

To find $\int 6x^2\left(2x^3 - 3\right)^7 dx$:

	Recommended Procedure	In this example:
Step 1	Choose u as some expression that appears in the integrand. (This may require some trial and error to find the correct expression for u)	Let $u = 2x^3 - 3$
Step 2	Find $\frac{du}{dx}$ and obtain the differential of u.	$\frac{du}{dx} = 6x^2$ *or* $du = 6x^2 dx$
Step 3	Substitute the values of u and du into the original integral.	$\int 6x^2 \cdot \left(2x^3-3\right)^7 dx$ $= \int \left(2x^3-3\right)^7 \cdot 6x^2\,dx$ $= \int u^7 du$
Step 4	Integrate w.r.t. u (Using standard formulae)	$= \frac{u^8}{8} + C$
Step 5	Write the answer in terms of x	$= \frac{\left(2x^3-3\right)^8}{8} + C$

7.2.2 INTEGRATION BY SUBSTITUTION OF THE FORM

$$\int \frac{f'(x)}{f(x)}\,dx$$

Let's look at an integral in which the numerator is the differential of the denominator.

Let $\quad u = f(x)$ then $du = f'(x)\,dx$

$$\int \frac{f'(x)}{f(x)}\,dx = \int \frac{1}{u}\,du = \ln|u| + C \quad \text{(Standard Integral)}$$

i.e.

$$\boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C}$$

EXAMPLE 8 Find $\displaystyle\int \frac{3x^2-1}{x^3-x}\,dx$

Solution Let $f(x) = x^3 - x$, then $f'(x) = 3x^2 - 1$;

$$\int \frac{3x^2-1}{x^3-x}\,dx \quad \text{corresponds to} \quad \int \frac{f'(x)}{f(x)}\,dx;$$

Hence, $\displaystyle\int \frac{3x^2-1}{x^3-x}\,dx = \ln|x^3-x| + C$.

EXAMPLE 9 **(a)** $\displaystyle\int \frac{2x^2}{x^3-4}\,dx$ $\qquad$ $Ans: \dfrac{2}{3}\ln|x^3-4|+C$

(b) $\displaystyle\int \frac{e^{2x}}{e^{2x}+1}\,dx$ $\qquad$ $Ans: \dfrac{1}{2}\ln|e^{2x}+1|+C$

7.2.3 INTEGRATION BY SUBSTITUTION OF THE FORM

$$\int e^{f(x)} \cdot f'(x)\, dx$$

EXAMPLE 10 $\int 3x^2 e^{x^3}\, dx$ $Ans:\ e^{x^3} + C$

Some integrals do not fit in any of the types previously studied. Other substitutions are then needed.

EXAMPLE 11 $\int \frac{t}{\sqrt{t-3}}\, dt$ $Ans:\ \frac{2}{3}(t-3)^{3/2} + 6(t-3)^{1/2} + C$

(To be taught in class)

7.2.4 INTEGRATION BY SUBSTITUTION & THE DEFINITE INTEGRAL

When evaluating a definite integral involving substitution (i.e. change of variable from *x* to *u*), it is more convenient to change the limits for *x* to the corresponding values of *u*.

EXAMPLE 12 Evaluate $\int_0^2 x e^{x^2}\, dx$

Solution

$$\int_0^2 x e^{x^2}\, dx$$
$$= \frac{1}{2}\int_0^2 2x e^{x^2}\, dx$$
$$= \frac{1}{2}\int_0^4 e^u\, du$$
$$= \frac{1}{2}\left[e^u\right]_0^4$$
$$= \frac{1}{2}\left(e^4 - 1\right)$$
$$= 26.80$$

Let $u = x^2 \Rightarrow du = 2x\,dx$

Since $u = x^2$:

When $x = 0$, $u = 0^2 = 0$

When $x = 2$, $u = 2^2 = 4$

7.3 INTEGRATION BY PARTS

If u and v are both functions of x,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\int \frac{d}{dx}(uv)\,dx = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

i.e. $uv = \int u\,dv + \int v\,du$

Thus $$\boxed{\int u\,dv = u\cdot v - \int v\,du}$$ ……………(1)

The method by "*integration by parts*" is used to integrate products of two different types of functions such as $x^2\sin x,\ xe^x,\ x\ln x,\ e^x\sin x,\ x\tan^{-1}x$, ... etc or single functions such as $\ln x,\ \sin^{-1}x,\ \tan^{-1}x,\ \cos^{-1}x,$ ...... etc.

In this method, u and v in formula (1) must be selected appropriately so that

(i) u can be differentiated to give du;

(ii) dv can be readily integrated to give v; and

(iii) the integral $\int v\,du$ is no more difficult than the original integral $\int u\,dv$.

A helpful aid to make an appropriate selection is to select u using the acronym **LIATE**.

This acronym spells out the priority of u according to the following order:

L ogarithm expression,

I nverse Trigonometric expression,

A lgebraic expression,

T rigonometric expression,

E xponential expression.

EXAMPLE 13 Find $\int x e^{2x}\, dx$

Solution 1 $\int x e^{2x}\, dx$

$$= x\left(\frac{e^{2x}}{2}\right) - \int \frac{e^{2x}}{2}\, dx$$

$$= \frac{x e^{2x}}{2} - \frac{e^{2x}}{4} + C$$

$u = x$ $\qquad dv = e^{2x}\, dx$

$du = dx \xrightarrow[-\int]{} v = \dfrac{e^{2x}}{2}$

Solution 2 $\int x e^{2x}\, dx = \dfrac{x e^{2x}}{2} - \dfrac{e^{2x}}{4} + C$

$\underline{u}$		$\underline{dv}$
x		e^{2x}
	$+$	
1		$\dfrac{e^{2x}}{2}$
	$-$	
0		$\dfrac{e^{2x}}{4}$

EXAMPLE 14

Find (a) $\int x \sin 2x\, dx$ Ans: $-\dfrac{1}{2} x \cos 2x + \dfrac{1}{4} \sin 2x + C$

(b) $\int x^2 e^x dx$ Ans: $x^2 e^x - 2xe^x + 2e^x + C$

EXAMPLE 15 Find $\int \ln(2x+1)\,dx$

Solution

$$\int \ln(2x+1)\,dx$$

$$= x\left[\ln(2x+1)\right] - \int \frac{2x}{2x+1}\,dx$$

$$= x\left[\ln(2x+1)\right] - \int \frac{2x+1-1}{2x+1}\,dx$$

$$= x\left[\ln(2x+1)\right] - \int \left(1 - \frac{1}{2x+1}\right)dx$$

$$= x\left[\ln(2x+1)\right] - x + \frac{1}{2}\ln\left|2x+1\right| + C$$

<u>*u*</u>		<u>*dv*</u>
$\ln(2x+1)$		1
	$+$	
$\dfrac{2}{2x+1}$	$\longrightarrow$	x
	$-\int$	

EXAMPLE 16 **(a)** Find $\int_1^e x\ln x\,dx$ Ans: 2.097

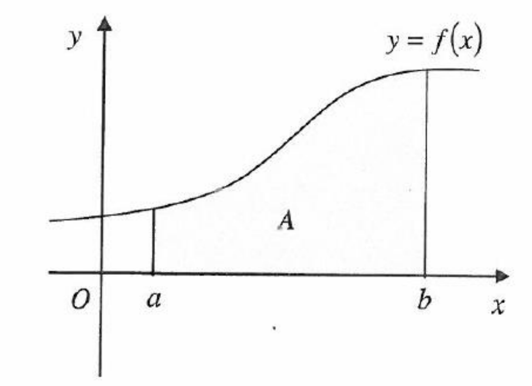
(b) Find $\int \tan^{-1} x\,dx$ Ans: $x\tan^{-1}x - \frac{1}{2}\ln\left|1+x^2\right| + C$

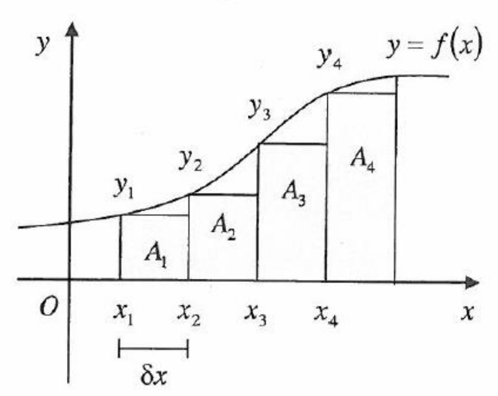
7. INTEGRATION

7.4 DEFINITE INTEGRAL AS THE AREA UNDER A CURVE

7.4.1 AREA BETWEEN A CURVE AND THE X-AXIS

Consider the area of the region bounded by a curve $y = f(x)$, two vertical lines, and the x-axis as illustrated in the diagrams.





The area A can be approximated as follows:

(1) Divide the area A into a number of n of thin vertical strips, each with equal width of $\Delta x = \dfrac{(b-a)}{n}$.

(2) For each vertical strip, its area, ΔA, can be approximated by a rectangle with width Δx and height $y_i = f(x_i)$:

$\Delta A \approx y_i \bullet \Delta x = f(x_i) \bullet \Delta x$

for $i = 1, 2, 3, \ldots n$

(3) Thus:

$$A = A_1 + A_2 + A_3 + \ldots + A_n$$

$$\approx y_1 \Delta x + y_2 \Delta x + y_3 \Delta x + \ldots + y_n \Delta x$$
$$= (y_1 + y_2 + y_3 + \ldots y_n)\Delta x$$
$$= \sum_{i=1}^{n} y_i \Delta x \text{ or } \sum_{i=1}^{n} f(x_i)\Delta x$$

For example:

$n = 4$ and $\delta x = \dfrac{(b-a)}{4}$

$$A_1 \approx y_1 \delta x$$
$$A_2 \approx y_2 \delta x$$
$$A_3 \approx y_3 \delta x$$
$$A_4 \approx y_4 \delta x$$

$$A = A_1 + A_2 + A_3 + A_4$$
$$= (y_1 + y_2 + y_3 + \ldots y_n)\delta x$$
$$\approx (y_1 + y_2 + y_3 + y_4)\delta x$$
$$= \sum_{i=1}^{4} y_i \delta x \text{ or } \sum_{i=1}^{4} f(x_i)\delta x$$

The approximation $A = \sum_{i=1}^{n} y_i \Delta x$ or $\sum_{i=1}^{n} f(x_i)\Delta x$ (known as **Riemann sum**) gets better as the number of n of thin vertical strips increases: $n \to \infty$

However, as $n \to \infty$. $\Delta x \to 0$

Hence, $A = \lim_{n\to\infty}\sum_{i=1}^{n} y_i \Delta x$ or $\lim_{\Delta x\to 0}\sum_{x=a}^{b} f(x)\Delta x$

The limiting process of finding the area A under the curve $y = f(x)$ is called integration and the Limit of Riemann Sum $A = \lim_{\Delta x\to 0}\sum_{x=a}^{x=b} f(x)\Delta x$ is later denoted by the elongated 'S' as $\int_a^b f(x)\,dx$:

$$\boxed{A = \lim_{\Delta x\to 0}\sum_{x=a}^{x=b} f(x)\cdot\Delta x = \int_a^b f(x)\,dx}$$

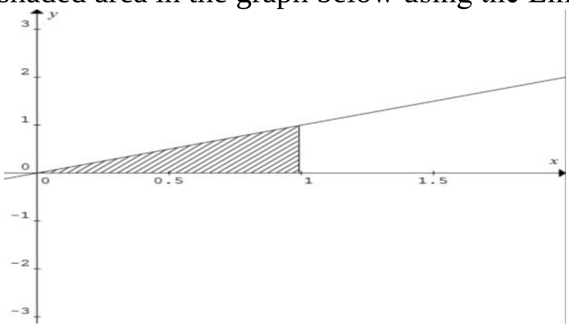
Using the above Limit of the Riemann Sum to find the area under the curve is quite tedious if $f(x)$ is complicated. Instead we can use a shortcut method using the antiderivative. The fundamental theorem of calculus is a theorem that links the concept of the derivative of a function with the concept of the integral:

The Fundamental Theorem of Calculus states that

$$\int_a^b f(x)\,dx = F(b) - F(a) \text{ where } F(x) \text{ is an antiderivative of } f(x)$$

The fundamental theorem of calculus relates differentiation and integration, showing that these two operations are essentially inverses of one another. Before the discovery of this theorem, it was not recognized that these two operations are related. The proof of the Fundamental Theorem of Calculus can be found at the Appendix 2 at the end of this chapter.

Example 18 Find the shaded area in the graph below using the Limit of Riemann's Sum.



Solution:

Divide the interval [0, 1] in n subintervals of equal width $\Delta x = \dfrac{1-0}{n} = \dfrac{1}{n}$.

The heights of each rectangle (refer to the above diagram) are given by
Let $f(x) = x$.

$$f(0)=0, \quad f(\frac{1}{n})=\frac{1}{n}, \quad f(\frac{2}{n})=\frac{2}{n}, \quad f(\frac{3}{n})=\frac{3}{n}, \quad f(\frac{n-1}{n})=\frac{n-1}{n}.$$

It follows that

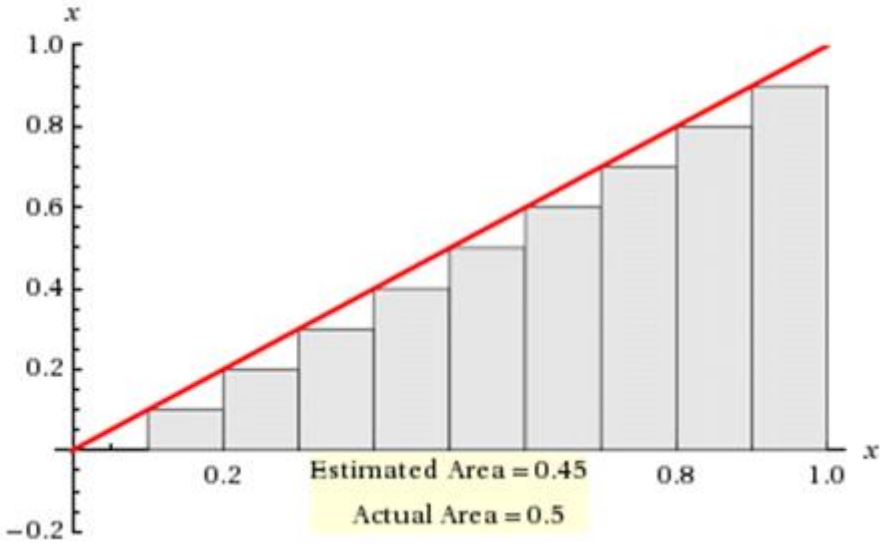
$$\text{Area} = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(x_k\right) \Delta x$$

$$= \lim_{n\to\infty}\left[0.\frac{1}{n}+\frac{1}{n}.\frac{1}{n}+\frac{2}{n}.\frac{1}{n}+\ldots+\frac{n-1}{n}.\frac{1}{n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n^2}(1+2+3+\ldots+(n-1))\right]$$

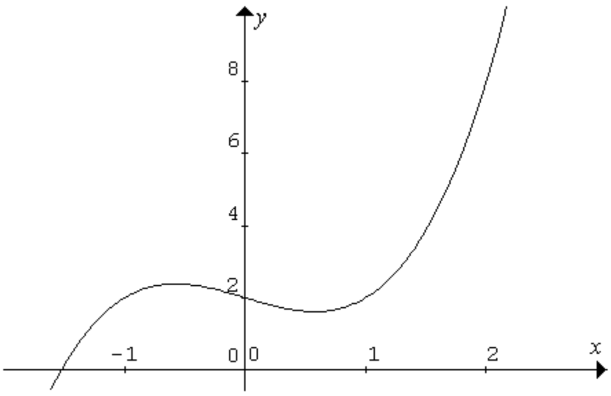
$$= \lim_{n\to\infty}\left[\frac{1}{n^2}\bullet\frac{(n-1)(n)}{2}\right] \qquad \text{since } (1+2+3+\ldots+(n-1))=\frac{(n-1)n}{2}$$

$$= \lim_{n\to\infty}\left[\frac{n^2-n}{2n^2}\right] = \lim_{n\to\infty}\left[\frac{1}{2}-\frac{1}{2n}\right]=\frac{1}{2}$$



Riemann Sum
http://mathworld.wolfram.com/RiemannSum.html

EXAMPLE 19 Find the area under the curve $y=x^3-x+2$ from $x=-1$ to $x=2$.



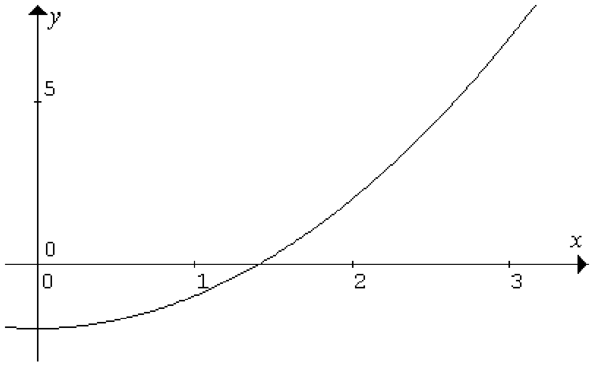
$$\text{Area} = \int_{-1}^{2} \left(x^3 - x + 2\right) dx \; = \left[\frac{x^4}{4} - \frac{x^2}{2} + 2x\right]_{-1}^{2}$$

$$= \left[\frac{2^4}{4} - \frac{2^2}{2} + 2(2)\right] - \left[\frac{(-1)^4}{4} - \frac{(-1)^2}{2} + 2(-1)\right]$$

$$= [4 - 2 + 4] - \left[\frac{1}{4} - \frac{1}{2} - 2\right] = 8\frac{1}{4}$$

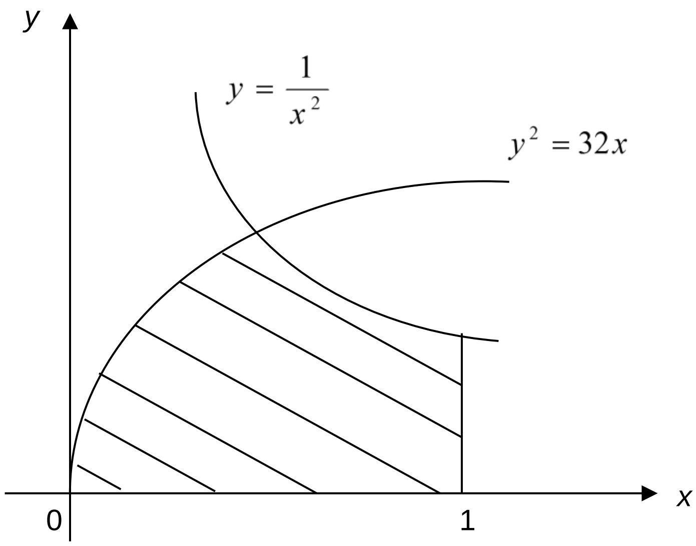
EXAMPLE 20

Find the area bounded by the curve $y = x^2 - 2$ and the x-axis, from $x = 0$ to $x = 3$.



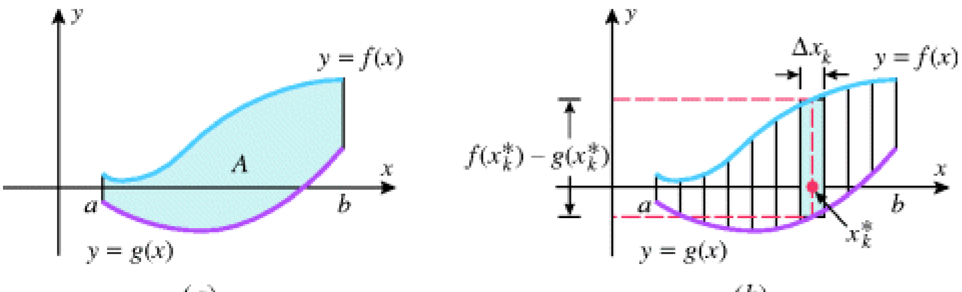
EXAMPLE 21
(To be taught in class)
The diagram shows part of the graph of the curves $y^2 = 32x$ and $y = \dfrac{1}{x^2}$.
Find the area of the shaded region bounded by the curve and the line $x = 1$.

(Ans: $2\frac{1}{3}$)



7.4.2 AREA BETWEEN TWO CURVES

Suppose the curves $y = f(x)$ and $y = g(x)$ intersect at the points $x = a$ and $x = b$ as shown below.



(a) (b)

We may form a Riemann sum to estimate the area between these two curves as

$$\sum_{k=1}^{n} (f(x_k^*) - g(x_k^*))\Delta x_k$$

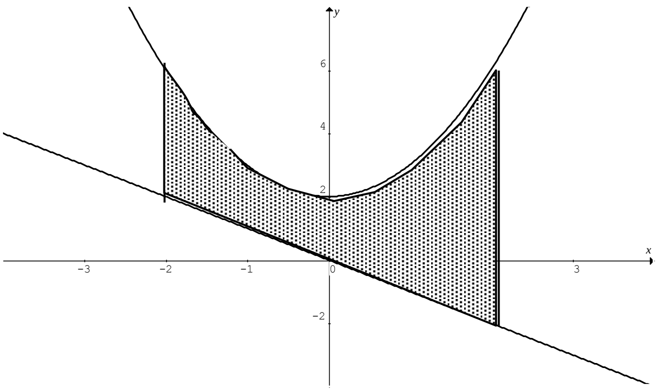
Thus, taking the limit as $n \to \infty$ (i.e. we take narrower and narrower strips) we obtain that the area between two curves is given by

$$\boxed{A = \int_a^b [\, f(x) - g(x)]\, dx \quad \text{or} \quad \int_a^b [y_1 - y_2]\, dx}$$

Note that as long as $f(x) \geq g(x)$ for $a \leq x \leq b$, the area we get is the unsigned area under the graph.

EXAMPLE 22

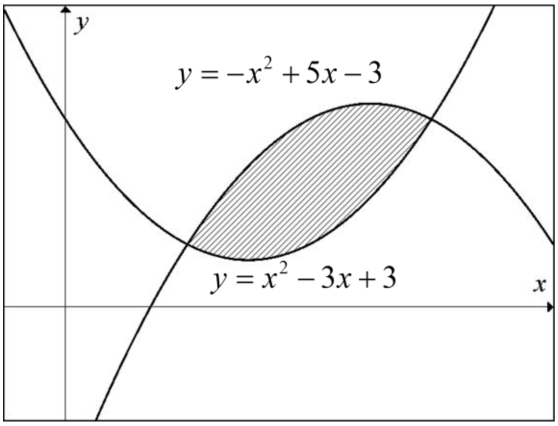
Find the area bounded by the curves $y = x^2 + 2$, $y = -x$, $x = -2$ and $x = 2$.



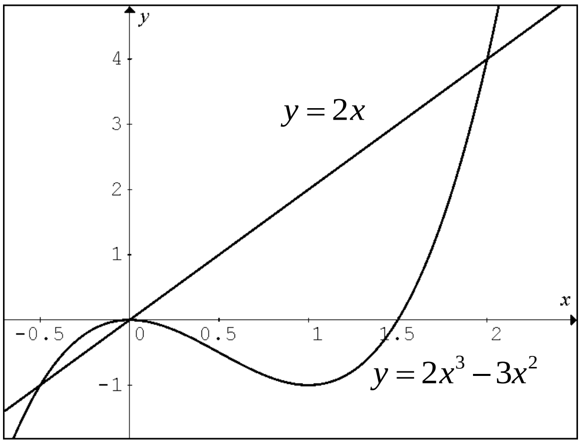
Solution $$A = \int_a^b (y_1 - y_2)\, dx = \int_{-2}^{2} [(x^2+2) - (-x)]\, dx = \int_{-2}^{2} (x^2 + x + 2)\, dx$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]_{-2}^{2} = \left(\frac{8}{3} + 2 + 4\right) - \left(-\frac{8}{3} + 2 - 4\right) = \frac{16}{3} + 8 = \frac{40}{3}$$

EXAMPLE 23 Find the shaded area.
(Ans: 2.67)

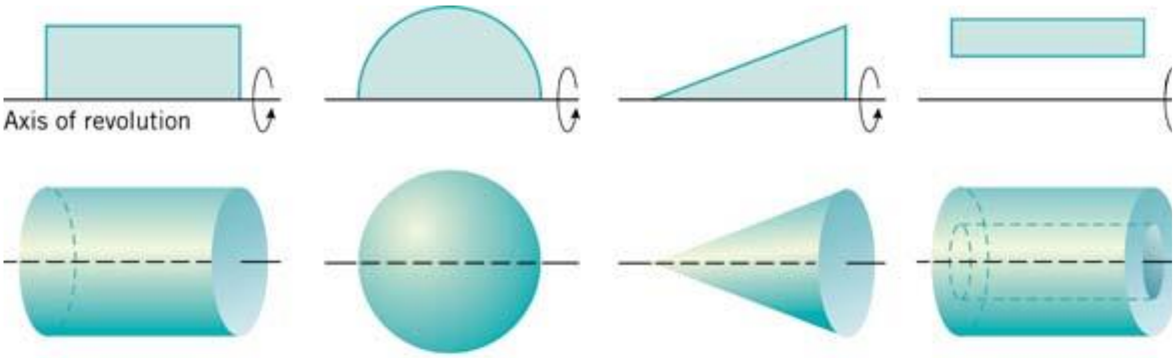


EXAMPLE 24
Find the area enclosed by the line $y = 2x$ and the curve $y = 2x^3 - 3x^2$. (Ans: $\frac{131}{32}$)

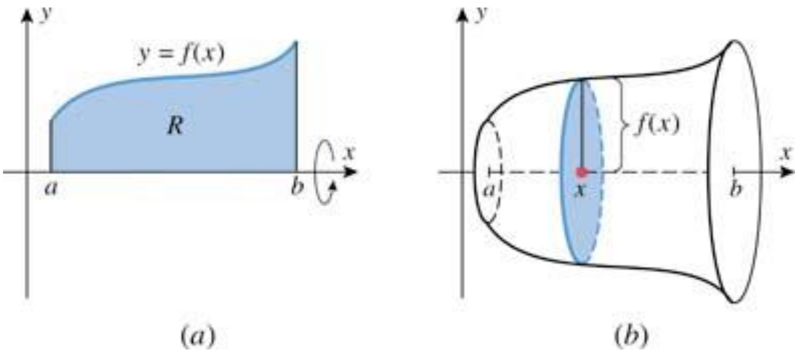


7.5 VOLUME OF SOLID OF REVOLUTION

A 3 dimensional solid object can be obtained from a 2 dimensional area by rotating the said area around a fixed axis (either the x or y axis). The resulting object is known as **solid of revolution.** Many familiar solids are of this type



Some familiar solids of revolution



How do we compute its volume?

Once again, the idea is to divide the region R (see above) into thin fundamental strips. When we perform the rotation, the resulting solid of revolution is approximated by thin **fundamental cylinders** (or slabs) with thickness Δx_k .

The Riemann sum

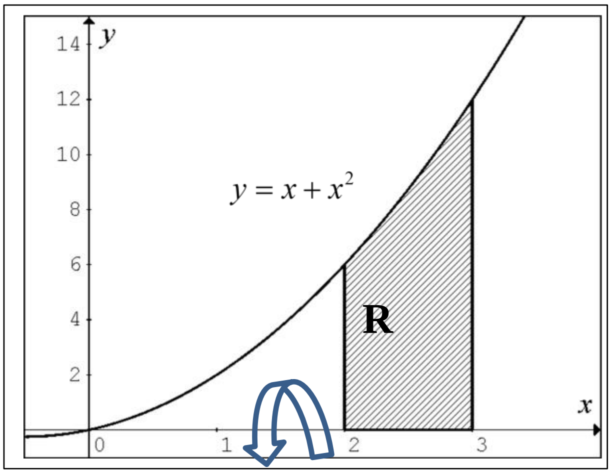
$$\sum_{k=1}^{n} \pi (f(x_k^*))^2 \Delta x_k$$

represents the approximate volume of this solid of revolution by thin slabs. So if we allow $n \to \infty$, the Riemann sum will approach the true volume of the solid of revolution, i.e.

$$\boxed{V = \int_a^b \pi f(x)^2 \, dx \quad \text{or} \quad \int_a^b \pi y^2 \, dx}$$

EXAMPLE 25

Find the volume of the solid of revolution formed by rotating the area enclosed by the curve $y = x + x^2$, the x-axis and the ordinates $x = 2$ and $x = 3$ through one revolution about the x-axis. (Ans: $81\frac{1}{30}\pi$)



EXAMPLE 26 (To be taught in class)

By means of integration, prove that the volume of a sphere is given by $V = \frac{4}{3}\pi r^3$, where r is the radius of the sphere.

TUTORIAL 7

Applications of Integration

Revision on Integration

1. Find the following integrals:

(a) $\int\left(x^2+\frac{1}{x}-3\right)dx$ (b) $\int\left(x-\frac{1}{x^2}\right)dx$

(c) $\int(\sin 3x-\cos 4x)\,dx$ (d) $\int\left(e^{5x}+\frac{3}{e^{3x}}\right)dx$

(e) $\int 2\sin x\cos x\,dx$ (f) $\int\frac{x}{3}(2x+\sqrt{x})\,dx$

(g) $\int\left(x^2+2\right)(4x-3)\,dx$ (h) $\int\frac{(x-2)^2}{x}\,dx$

(i) $\int(3x+2)^2dx$ (j) $\int \mathrm{e}^x\left(2\mathrm{e}^x+\frac{1}{\mathrm{e}^{3x}}\right)dx$

(k) $\int\left(2\mathrm{e}^{-x}+\mathrm{e}^x\right)^2dx$ (l) $\int\frac{\mathrm{e}^{2x}+2\mathrm{e}^x}{\mathrm{e}^{2x}}\,dx$

(m) $\int 2\tan 3x\,dx$ (n) $\int\frac{1}{\cos^2(2\theta)}\,d\theta$

(o) $\int\cot 6x\,dx$ (p) $\int\frac{2}{9+x^2}\,dx$

Integration by Substitutions

2. Find the results of the integrals.

(a) $\int x\left(x^2+1\right)^4dx$ (b) $\int t\,e^{3-2t^2}\,dt$ (c) $\int\frac{x}{1-2x^2}\,dx$

(d) $\int\frac{x}{(4-x^2)^2}\,dx$ (e) $\int\sin^2\theta\cos\theta\,d\theta$ (f) $\int\frac{dx}{x\ln x}$

(g) $\int\frac{5e^{2x}}{\sqrt{1-e^{2x}}}\,dx$ (h) $\int\frac{x+1}{\sqrt{x+2}}\,dx$ (i) $\int\frac{1}{\sqrt{x}+x}\,dx$

3. Find the values of the following integrals.

(a) $\int_0^{\frac{1}{2}} y\sqrt{\frac{1}{4}-y^2}\,dy$ (b) $\int_1^2\frac{e^{\frac{1}{t}}}{t^2}\,dt$ (c) $\int_0^4\frac{4x}{\sqrt{2x+1}}\,dx$

Integration by parts

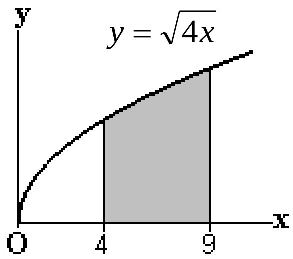
4. $\int x \cos x dx$ 5. $\int (x^2 + x)e^{2x} \, dx$ 6. $\int x^2 \sin 3x \, dx$

7. $\int_0^1 x e^{-5x} \, dx$ 8. $\int_1^e x^2 \ln x \, dx$

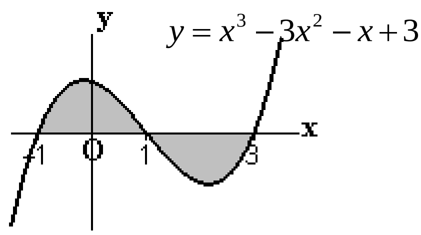
Area under the curve

9. Find the shaded areas.

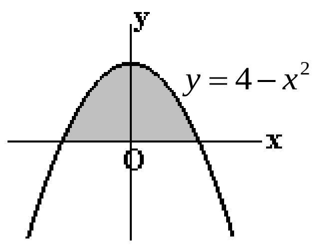
(a)



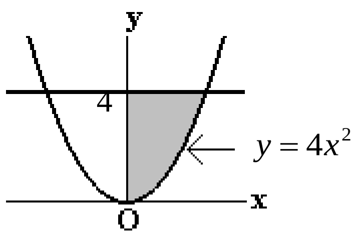
(b)



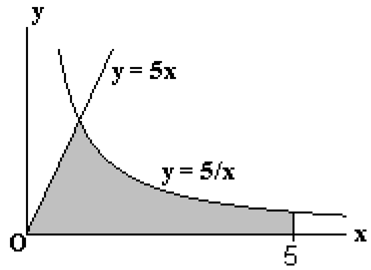
(c)



(d)

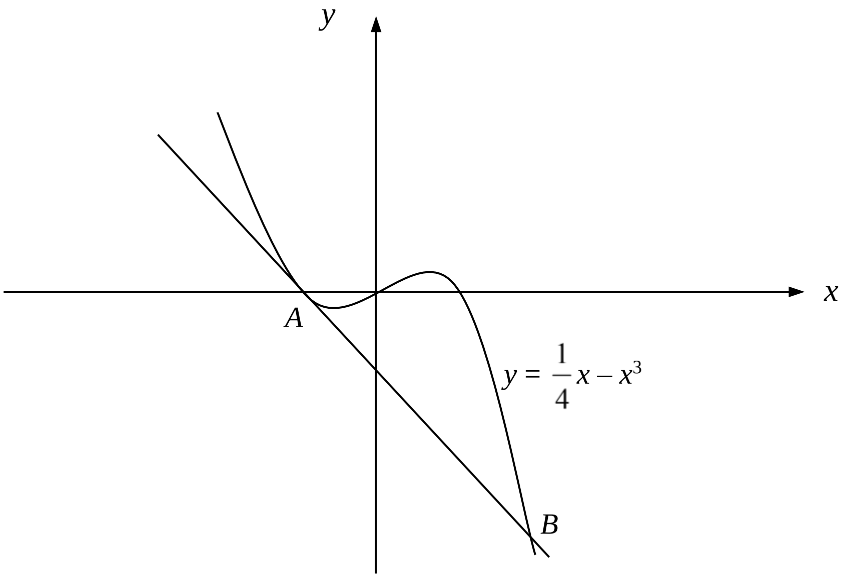


(e)



10. Find the area bounded by the curve $y = \dfrac{1}{x}$, the lines $x = 1$ and $x = 2$.

11. The diagram shows a sketch of the graph of the curve $y = \frac{1}{4}x - x^3$ together with the tangent to the curve at the point $A(k, 0)$.



Find

(i) the value of k.

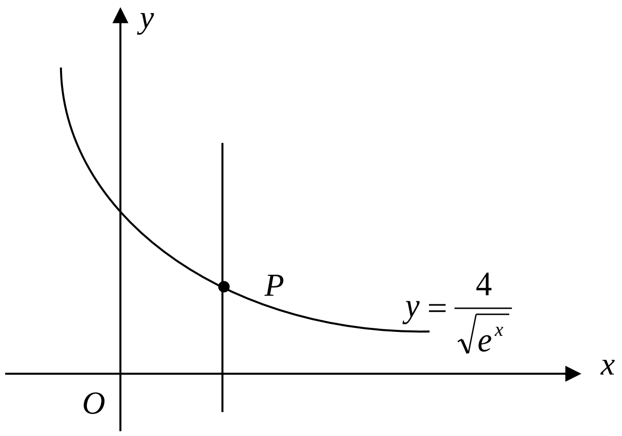
(ii) the equation of the tangent to the curve at A, and verify that the point B where the tangent cuts the curve again has coordinates $\left(1, -\frac{3}{4}\right)$

(iii) the area of the region bounded by the curve and the tangent, giving your answer as a fraction in its lowest terms.

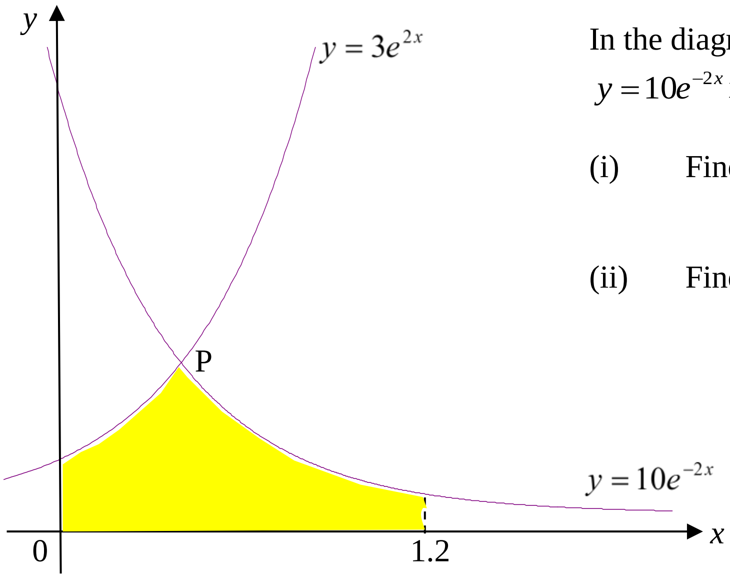
12. In the diagram, the gradient of the tangent to the curve $y = \frac{4}{\sqrt{e^x}}$ at point P is $-\frac{1}{2}$.

i) Show that the x- coordinate of P is $\ln 16$

ii) Find the area of region bounded by the curve $y = \frac{4}{\sqrt{e^x}}$, the line $x = \ln 16$ and the coordinate axes.



13.



In the diagram, the 2 curves $y = 3e^{2x}$ and $y = 10e^{-2x}$ intersect each other at P.

(i) Find the coordinates of P.

(ii) Find the area of the shaded region.

<u>Riemann Sum</u>

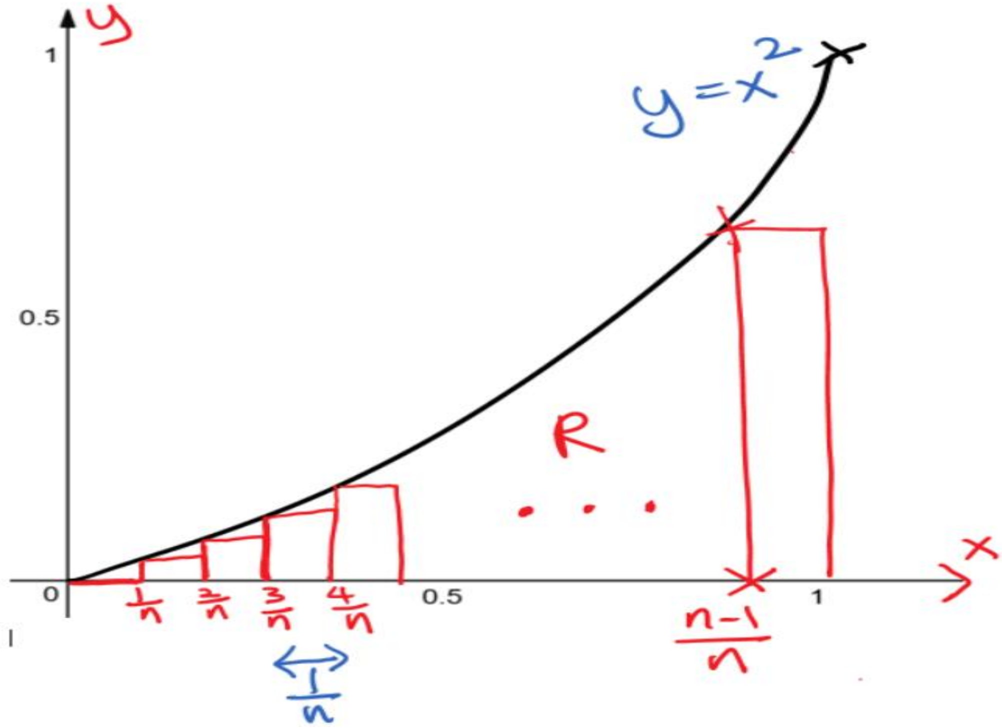
14. The diagram below shows a region R bounded by the curve $y = x^2$, x-axis, y-axis and x = 1. The region is divided into n strips, each of equal width $\frac{1}{n}$.

(a) Find the exact area of R using standard integration.

(b) By using the Limit of Riemann's Sum, find the area of the region R.

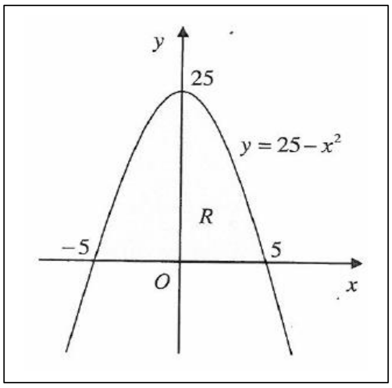
(you may use this result: $1^2 + 2^2 + \ldots + (n-1)^2 = \frac{n(n+1)(2n+1)}{6} - n^2$)

You should notice that the answers for (a) and (b) are the same.



Volume of solid of revolution

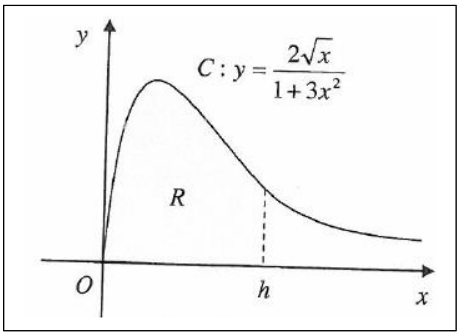
15. A finite region R is bounded by the curve $y = 25 - x^2$ and the x-axis. Find the volume of the solid obtained when R is rotated through one revolution about the x-axis.



16. The curve C has equation $y = \dfrac{2\sqrt{x}}{1+3x^2}$, where $x \ge 0$.

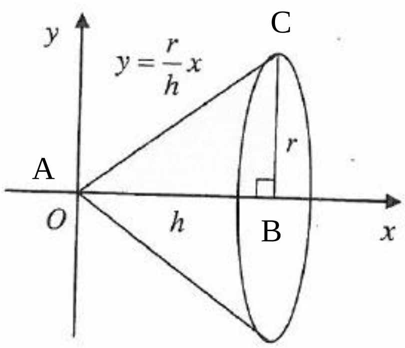
The region R is bounded by the curve C, the x-axis, and the vertical line $x = h$. The region R is rotated through 2π radians about the x-axis.

Show that the volume V of the solid formed is $V = \dfrac{2\pi h^2}{1+3h^2}$.

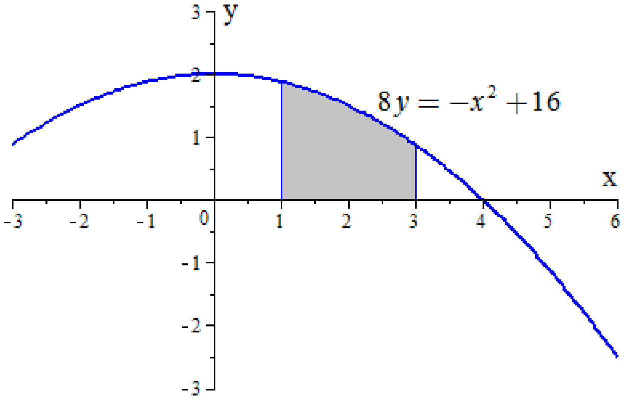


17. The diagram shows a right circular cone with base radius r and the height h. The volume of the cone can be generated by revolving the right-angled triangle ABC about the height of the cone through 2π radians. By using integration, establish the formula:

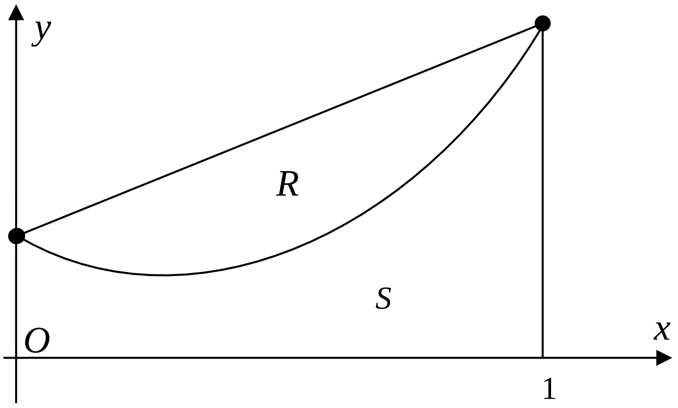
$V = \dfrac{1}{3}\pi r^2 h$, where V is the volume of the cone.



18. Find the volume of the solid generated when the area enclosed by the curve $8y = -x^2 + 16$, $x = 1$, $x = 3$ and the x-axis (see diagram below) is revolved about the x-axis.



19.



As shown in the above diagram, the region R is enclosed by the curve $y = e^{2x} - 3x$, $0 \le x \le 1$, and the line segment joining the points on the curve whose x-coordinate are 0 and 1. The region S is bounded by the curve, the axes and the line $x = 1$.
Find

I. the x-coordinate of the stationary point on the curve
II. the exact area of the region R
III. the exact volume of revolution formed when the region S is rotated completely about the x-axis. (MA1301 2014)

Miscellaneous Exercises

Find the results of the integrals.

1. $\int \sin^3 x dx$ (Hint: use $\sin^2 x = 1 - \cos^2 x$ and let $u = \cos x$)

2. $\int \left(27e^{9x} + e^{12x}\right)^{1/3} dx$ 3. $\int \frac{3}{x \ln x} dx$

4. $\int x\sqrt{4-x}\, dx$ (Hint: let $u = 4 - x$ and represent x in term of u)

5. $\int e^{2x} \sqrt{1+4e^x}\, dx$

6 (a) $\int \frac{x^3}{\sqrt{1-x^2}} dx$ (b) $\int x(2x-5)^3 dx$

(c) $\int t^3 \sqrt{1-t^2}\, dt$ (d) $\int \frac{dx}{3+\sqrt{x+2}}$

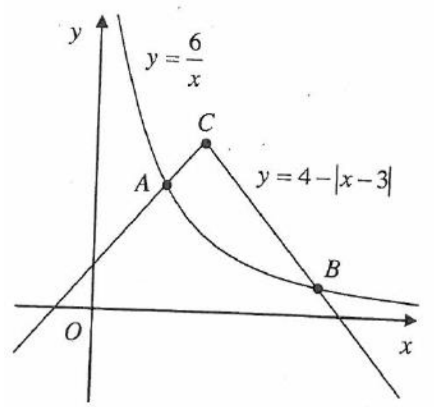
(e) $\int \frac{2x+1}{(x-3)^6} \mathrm{dx}$ (f) $\int \frac{\sqrt{x^3-4}}{x} \mathrm{dx}$

7 The diagram shows the graph of $y = 4 - |x-3|$ and part of the graph $y = \frac{6}{x}$.

A and B are points of intersection of the graphs, and C is the vertex of the graph of $y = 4 - |x-3|$.

(i) State the coordinates of A, B and C.

(ii) The bounded region ABC is rotated completely about the x-axis. Find the volume of the solid formed.



ANSWERS

Revision on Integration

1 (a) $\frac{x^3}{3} + \ln|x| - 3x + C$ (b) $\frac{x^2}{2} + \frac{1}{x} + C$ (c) $-\frac{\cos 3x}{3} - \frac{\sin 4x}{4} + C$

(d) $\frac{1}{5}e^{5x} - \frac{1}{e^{3x}} + C$ (e) $-\frac{1}{2}\cos 2x + C$ (f) $\frac{2}{9}x^3 + \frac{2}{15}x^{\frac{5}{2}} + C$

(g) $x^4 - x^3 + 4x^2 - 6x + \mathrm{C}$ (h) $\frac{x^2}{2} - 4x + 4\ln|x| + \mathrm{C}$ (i) $3x^3 + 6x^2 + 4x + \mathrm{C}$

(j) $\mathrm{e}^{2x} - \frac{1}{2}\mathrm{e}^{-2x} + \mathrm{C}$ (k) $-2e^{-2x} + 4x + \frac{e^{2x}}{2} + C$ (l) $x - 2e^{-x} + \mathrm{C}$

(m) $-\frac{2}{3}\ell n|\cos 3x| + \mathrm{C}$ (n) $\frac{1}{2}\tan 2\theta + C$ (o) $\frac{1}{6}\ell n|\sin 6x| + C$

(p) $\frac{2}{3}\tan^{-1}\frac{x}{3} + C$

Integration by Simple Substitution

2. (a) $\frac{1}{10}(x^2+1)^5 + \mathrm{C}$ (b) $-\frac{1}{4}\mathrm{e}^{3-2t^2} + \mathrm{C}$ (c) $-\frac{1}{4}\ln\left|1-2x^2\right| + \mathrm{C}$

(d) $\frac{1}{2(4-x^2)} + C$ (e) $\frac{1}{3}\sin^3\theta + C$ (f) $\ln(\ln x) + C$

(g) $-5\sqrt{1-e^{2x}} + C$ (h) $\frac{2}{3}(x+2)^{3/2} - 2(x+2)^{1/2} + \mathrm{C}$ (i) $2\ln\left(1+\sqrt{x}\right) + \mathrm{C}$

3. (a) 1/24 (b) 1.07 (c) 13.33

Integration by parts

4. $x\sin x + \cos x + C$ 5. $\frac{1}{2}x^2e^{2x} + C$

6. $-\frac{x^2}{3}\cos 3x + \frac{2}{9}x\sin 3x + \frac{2}{27}\cos 3x + C$ 7. 0.0384 8. 4.575

<u>Area under the curve</u>

9. (a) $25\frac{1}{3}$ (b) 8 (c) $\frac{32}{3}$ (d) $2\frac{2}{3}$ (e) 10.55

10. $\ln 2$

11. (i) $-\frac{1}{2}$ (ii) $y=-\frac{1}{2}x-\frac{1}{4}$ (iii) $\frac{27}{64}$ square units

12. 6

13. (i) (0.301, 5.48) (ii) 3.52

14. 1/3

<u>Volume of solid of revolution</u>

15. $\frac{10000\pi}{3}$

18. 13.90

19. (i) $x=\frac{1}{2}\ln\left(\frac{3}{2}\right)$ (ii) 1 (iii) $\pi\left(\frac{e^4}{4}-\frac{3}{2}e^2+\frac{5}{4}\right)$

Miscellaneous Exercises

1. $-\cos x+\frac{\cos^3 x}{3}+C$

2. $\frac{1}{4}\left(27+e^{3x}\right)^{\frac{4}{3}}+C$

3. $3\ln\left|\ln x\right|+C$

4. $\frac{2}{5}(4-x)^{5/2}-\frac{8}{3}(4-x)^{3/2}+C$

5. $\frac{1}{40}\left(1+4e^x\right)^{5/2}-\frac{1}{24}\left(1+4e^x\right)^{3/2}+C$

6. (a) $\frac{1}{3}\left(1-x^2\right)^{\frac{3}{2}}-\left(1-x^2\right)^{\frac{1}{2}}+C$

(b) $\frac{1}{80}(2x-5)^4(8x+5)+C$

(c) $\frac{1}{5}\left(1-t^2\right)^{\frac{5}{2}}-\frac{1}{3}\left(1-t^2\right)^{\frac{3}{2}}+C$

(d) $2\left(3+\sqrt{x+2}\right)-6\ln\left(3+\sqrt{x+2}\right)+C$

or $2\sqrt{x+2}-6\ln\left(3+\sqrt{x+2}\right)+C_1$

(e) $\frac{1-5x}{10(x-3)^5}+C$

(f) $\frac{2}{3}\left(x^3-4\right)^{\frac{1}{2}}-\frac{4}{3}\tan^{-1}\frac{\sqrt{x^3-4}}{2}+C$

7 (i) A(2,3), B(6,1), C(3,4) (ii) 67.02

<u>Appendix 1:</u> Integrals of the Type $\int e^{ax} \sin bx\, dx$ or $\int e^{ax} \cos bx\, dx$ where *a* and *b* are constants

Let $u = e^{ax}$; $dv = \sin bx$ or $dv = \cos bx\ dx$

or let $u = \sin bx$ or $u = \cos bx$; $dv = e^{ax}\, dx$

EXAMPLE Find $\int e^{2x} \sin x\, dx$ Ans: $\frac{2}{5}\sin x\, e^{2x} - \frac{1}{5}\cos x\, e^{2x} + C$

Solution: Apply integration by parts:

<u>u</u>	<u>dv</u>
$\sin x$	e^{2x}
$\cos x$	$\frac{e^{2x}}{2}$
$-\int$	

$$\int e^{2x} \sin x\, dx = \sin x\left(\frac{e^{2x}}{2}\right) - \int (\cos x)\left(\frac{e^{2x}}{2}\right) dx$$
$$= \sin x\left(\frac{e^{2x}}{2}\right) - \frac{1}{2}\int e^{2x} \cos x dx$$

Apply integration by parts again to $\int e^{2x} \cos x dx$:

<u>u</u>	<u>dv</u>
$\cos x$	e^{2x}
$-\sin x$	$\frac{e^{2x}}{2}$
$-\int$	

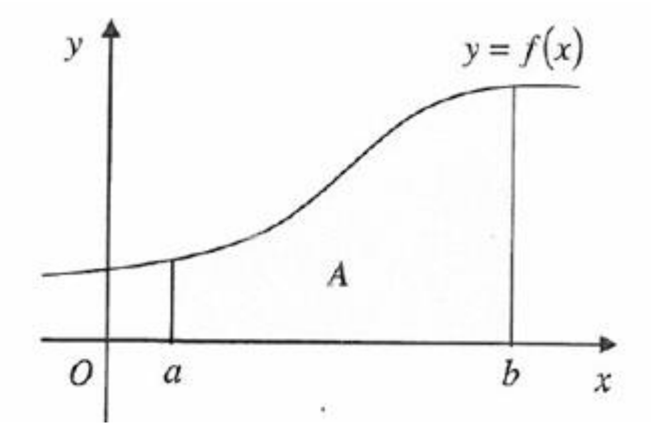
$$\int e^{2x} \sin x\, dx = \sin x\left(\frac{e^{2x}}{2}\right) - \frac{1}{2}\left[\cos x\left(\frac{e^{2x}}{2}\right) - \int -\sin x\left(\frac{e^{2x}}{2}\right) dx\right]$$
$$= \sin x\left(\frac{e^{2x}}{2}\right) - \frac{1}{4}\cos x\, e^{2x} - \frac{1}{4}\int \sin x\, e^{2x} dx$$

Let $I = \int \sin x\, e^{2x} dx$ and make I the subject of formula:

$$I = \sin x\left(\frac{e^{2x}}{2}\right) - \frac{1}{4}\cos x\, e^{2x} - \frac{1}{4} I$$
$$I + \frac{1}{4} I = \sin x\left(\frac{e^{2x}}{2}\right) - \frac{1}{4}\cos x\, e^{2x}$$
$$\frac{5}{4} I = \sin x\left(\frac{e^{2x}}{2}\right) - \frac{1}{4}\cos x\, e^{2x}$$
$$I = \frac{4}{5}\left[\sin x\left(\frac{e^{2x}}{2}\right) - \frac{1}{4}\cos x\, e^{2x}\right] + C$$
$$I = \frac{2}{5}\sin x e^{2x} - \frac{1}{5}\cos x\, e^{2x} + C \quad \text{(ans)}$$

Exercise: $\int e^{5x} \cos 2x\, dx$ (Ans: $\frac{1}{29} e^{5x}(2\sin 2x + 5\cos 2x) + C$)

Appendix 2: Proof of Fundamental Theorem of Calculus



As x changes so will the area A, and the rate the area is changing with x is given by

$$\begin{aligned}\frac{dA}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{f(\bar{x})\Delta x}{\Delta x} \\ &= \lim_{\Delta x \to 0} f(\bar{x}) \qquad \text{where } x \le \bar{x} \le x + \Delta x\end{aligned}$$

As $\Delta x \to 0, \bar{x} \to x$ and hence $\frac{dA}{dx} = f(x)$ which means that $A(x)$ is an antiderivative of $f(x)$, i.e., $A(x) = F(x) + c$

Since $A(a) = F(a) + c = 0 \Rightarrow c = -F(a)$.
Hence $A(x) = F(x) - F(a)$ and $A(b) = F(b) - F(a)$ which we defined as the definite integral $\int_a^b f(x)dx$.

We use the notation $\int_a^b f(x)dx = \left[F(x)\right]_a^b = F(b) - F(a)$.