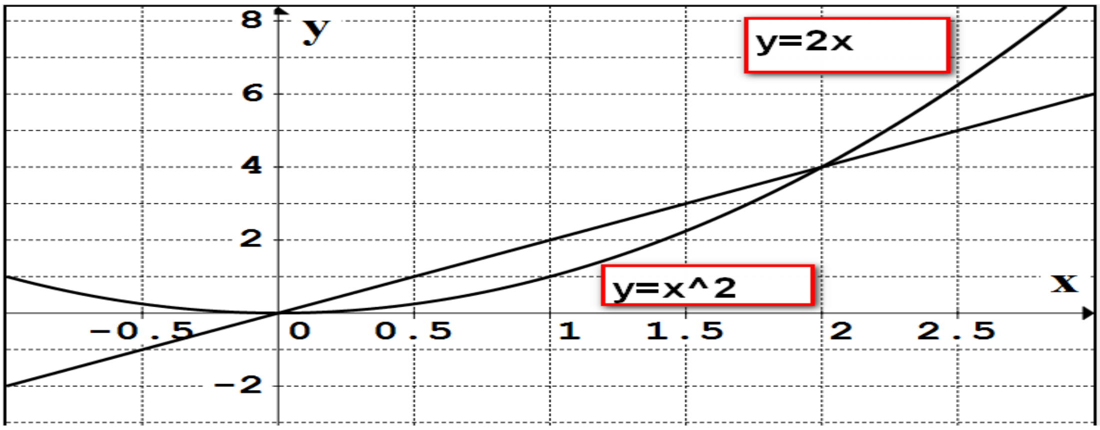
6. INEQUALITIES AND ABSOLUTE VALUES

6.1 SOLVING INEQUALITIES



Let's look at the graph above. Suppose that we want to determine the set of points such that the quadratic graph is **underneath** the straight line graph. Mathematically, this is expressed as "Find all the values of x such that $x^2 \le 2x$ expressing your answer in interval notation."

Solving this problem above is called ***solving an inequality.***

EXAMPLE 1 Determine the set of real numbers such that $x^2 \le 2x$.

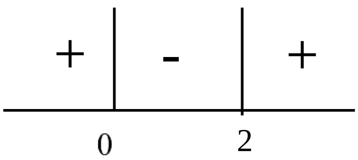
Solution:
Rewrite the inequality as $x^2 - 2x \le 0$. Then we factorize the LHS obtaining

$$x(x-2) \le 0.$$

Now if $x < 0$, then $(x-2) < 0$.
Hence, in the LHS above where the two expressions " x " and " $x-2$ " are multiplied together, we are actually multiplying two negative numbers.

Hence, $x(x-2)$ is going to be a positive numbers. In other words, values of $x < 0$ **are not** solutions of the inequality above.

When $0 < x < 2$, $x(x-2)$ is negative and when $x > 2$, $x(x-2)$ is positive.
Summarize the results in a table:



<u>Conclusion and answer to the problem:</u>

The solution to the inequality $x^2 \le 2x$ is [0,2] in interval notation.

EXAMPLE 2 Solve the following inequalities.

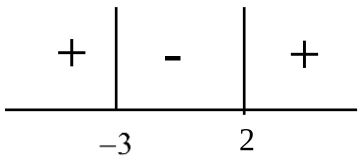
a) $x^2 - 3x > 0$ $\qquad$ $Ans: x < 0$ or $x > 3$

b) $x^2 - 5x \leq -4$ $\qquad$ $Ans: 1 \leq x \leq 4$

c) $x^2 \geq 6 - x$ $\qquad$ $Ans: x \leq -3$ or $x \geq 2$

c)

$$x^2 \geq 6 - x$$
$$x^2 + x - 6 \geq 0$$
$$(x+3)(x-2) \geq 0$$



From the diagram, the range of values which satisfy the inequality are $x \leq -3$ or $x \geq 2$.

EXAMPLE 3 Find the range of values of x that satisfy the inequality $\dfrac{2x-5}{x-2} \leq 1$.

(Ans: $2 < x \leq 3$)

6.2 ABSOLUTE VALUES

6.2.1 DEFINITION OF ABSOLUTE VALUES

The **absolute value** of a real number *x*, denoted by $|x|$, is defined as

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

EXAMPLE 4 (a) (i) $|2| = 2$ (ii) $|-2| = 2$ (iii) $|0| = 0$

(b) If $f(x) = |x^2 - 1|$, find the values of $f(0)$ and $f(2)$.

Sketch $f(x) = |x^2 - 1|$ for $-2 \leq x \leq 2$.

(c) Simplify $\left|\sqrt{3} - \sqrt{5}\right| + \left|\sqrt{5} + \sqrt{3}\right|$, leaving your answer in surd form.

EXAMPLE 5 **(a)** Solve $|x - 2| = 3$

SOLUTION

When $(x-2) \geq 0$, $|x-2| = x - 2 = 3 \Rightarrow x = 5$

When $(x-2) < 0$, $|x-2| = -(x-2) = 3 \Rightarrow -x + 2 = 3 \Rightarrow x = -1$

The solutions are $x = 5$ or $x = -1$.

EXAMPLE 6 **(b)** Solve $|x-1|=2$.

Sketch the graphs of $y=|x-1|$ and $y=2$ on the same diagram.

EXAMPLE 7 Solve $|2x+3|=|3x-8|$

SOLUTION Since $|a|=|b| \Rightarrow a=b \text{ or } a=-b$, we have

$2x+3=3x-8 \quad or \quad 2x+3=-(3x-8)$

$\Rightarrow x=11 \qquad or \quad x=1$

The solutions are x = 11 or x = 1.

6.2.2 RELATIONSHIP BETWEEN SQUARE ROOTS AND ABSOLUTE VALUES

THEOREM 6.6 For any real number a, $|a|=\sqrt{a^2}$

EXAMPLE 8 Solve $\sqrt{(x+2)^2}=3$ (Ans: $x=1 \text{ or } x=-5$)

6.2.3 PROPERTIES OF ABSOLUTE VALUES

THEOREM 6.7 For any real numbers a and b,

(i) $|-a|=|a|$ (ii) $|ab|=|a||b|$

(iii) $\frac{|a|}{|b|}=\left|\frac{a}{b}\right|$, if $b \neq 0$

EXAMPLE 9 (i) $|-2|=|2|=2$ (ii) $|a^{-1}|=\left|\frac{1}{a}\right|=\frac{|1|}{|a|}=|a|^{-1}$

6.2.4 SOLVING INEQUALITIES INVOLVING ABSOLUTE VALUES

THEOREM 6.8

For $k > 0$

(i) $|x-a| < k$ is equivalent to $-k < x-a < k$

(ii) $|x-a| > k$ is equivalent to $x-a > k$ or $x-a < -k$

PROOF (i) If $(x-a) \geq 0$, then $|x-a| = x-a$. Thus

$$|x-a| < k \Rightarrow x-a < k$$

If $(x-a) < 0$, then $|x-a| = -(x-a)$. Thus

$$|x-a| < k \Rightarrow -(x-a) < k$$

$$\Rightarrow x-a > -k \text{ or } -k < x-a$$

since multiplication by a negative number reverses the inequality sign.
In all cases, we have $-k < x-a < k$.
(ii) The proof is similar.

EXAMPLE 10 (i) $|x-2| < 3$ (ii) $|x-2| > 4$ (iii) $\dfrac{2}{|2x-3|} \geq 3$

(Ans (i) $-1 < x < 5$ (ii) $x > 6$ *or* $x < -2$)

(iii) We have

$$\frac{2}{|2x-3|} \ge 3 \Rightarrow x \ne \frac{3}{2} (division\, by\, zero\ is\ not\, allowed)$$

$$\Leftrightarrow 2 \ge 3|2x-3|$$

$$\Leftrightarrow \frac{2}{3} \ge |2x-3|$$

$$\Leftrightarrow |2x-3| \le \frac{2}{3}$$

$$\Leftrightarrow -\frac{2}{3} \le 2x-3 \le \frac{2}{3}$$

$$\Leftrightarrow -\frac{2}{3}+3 \le 2x \le \frac{2}{3}+3$$

$$\Leftrightarrow \frac{7}{3} \le 2x \le \frac{11}{3}$$

$$\Leftrightarrow \frac{7}{6} \le x \le \frac{11}{6}$$

But we must exclude the point $x = \frac{3}{2}$.

Hence $\frac{7}{6} \le x < \frac{3}{2}$ or $\frac{3}{2} < x \le \frac{11}{6}$.

Or $\left[\frac{7}{6}, \frac{3}{2}\right) \cup \left(\frac{3}{2}, \frac{11}{6}\right]$ in interval notation

EXAMPLE 11: Summary

1. $x^2 + x - 6 = 0$

2. $x^2 + x - 6 > 0$

3. $|x-3| = 5$

4. $|x-3| > 5$

(Ans: 1. $x = 2$ or -3, 2. $x > 2$ or $x < -3$, 3. $x = 8$ or -2, 4. $x > 8$ or $x < -2$)

TUTORIAL 6

INEQUALITIES & ABSOLUTE VALUES

Solving Inequalities

1. Solve the following inequalities.

a. $x^2 - 9 < 0$

b. $(x+3)(x-2) < 0$

c. $x^3 - x \geq 0$

d. $(x^2 - 1)(x+5) < 0$

e. $\dfrac{1}{x-1} < 1$

f. $\dfrac{x-1}{x+2} > 2$

2. Find the set of values of x for which $(x - 12)^2 > 2x$.

Absolute Values

3. If $f(x) = \left|x^2 - x - 2\right|$, find the values of $f(0)$ and $f(2)$.

Sketch $f(x) = \left|x^2 - x - 2\right|$ for $-2 \leq x \leq 2$.

4. Sketch the graphs $y = |2x+4|$ and $y = x+3$ for $-3 \leq x \leq 0$ on the same diagram.

Hence from the graph, state the number of solutions of the equation $|2x+4| = x+3$.

5. Solve for x:

(a) $|x-3| = 5$

(b) $2|3x-1| = 4$

(c) $\sqrt{(x+5)^2} = 4$

(d) $|6x-7| = |3+2x|$

(e) $\dfrac{|x+5|}{|2-x|} = 6$

6. Solve for x :

(a) $|2x-3| \leq 6$

(b) $|5-2x| \geq 4$

(c) $\dfrac{3}{|2x-1|} \geq 4$

(d) $3 \leq |x-2| \leq 7$

(e) $4x^2 - 3x - 1 > 0$

(f) $|x+2| > |x-1|$ (Hint: square both sides to remove absolute signs)

(g) $|x-1| < |2x+1|$

7. Express the function $f(x) = |x| + x$ in piecewise form with no absolute values. Sketch the graph of f.

8. Sketch the graph of $y = -|2x+3|$ for $-3 \leq x \leq 1$.

(a) State the corresponding range of values of y.

(b) Find the range of values of x for which $y > -1$.

Miscellaneous Exercises

1. (a) Sketch, on the same diagram, the graphs of $y = 3x^2 + 1$ and $y = |x - 3|$.

   Hence, solve the inequality $3x^2 + 1 > |x - 3|$.

   (b) Use the Factor Theorem to find the linear factors of $2x^3 - 9x^2 - 2x + 24$.

   Hence, solve the inequality $\frac{1}{2x^3 - 9x^2 - 2x + 24} \le 0$. (MS1301 1314)

2. Express $6x^3 - 25x^2 + 18x + 9$ as a product of three linear factors.

   Hence, solve the inequality $\frac{6x^2 + 12x + 9}{6x^3 - 25x^2 + 18x + 9} \le 0$ (MS1301 0506)

3. Solve the inequality $\frac{3x - 1}{|x - 1|} \ge 4$. (MS1301 0910)

4. Solve the inequality $2x + \frac{5}{x} \ge 7$. (MS1301 1011)

5. On the same diagram, sketch the graphs $y = |x + 2|$ and $y = 2|x|$.

   Hence, solve the inequality $2|x| - |x + 2| = 0$. (MS1301 0607)

6. (a) Prove that $2x^2 + 2 > 3x$ for all real values of x.

   (b) For what values of x is $|2x - 3| > 5$? (MS1301 0708)

7. Solve the inequality $x + |2x - 1| < 4$. (MS1301 0809)

ANSWERS

1 a) $-3 < x < 3$ b) $-3 < x < 2$ c) $1 \le x$ or $-1 \le x \le 0$

d) $x < -5$ or $-1 < x < 1$ e) $x < 1$ or $x > 2$ f) $-5 < x < -2$

2 $x < 8$ or $x > 18$

3 2, 0 4 2

5 (a) $x = -2$ or 8 (b) $x = 1$ or $-\frac{1}{3}$ (c) $x = -9$ or -1

(d) $x = \frac{1}{2}$ or $\frac{5}{2}$ (e) $x = 1$ or $\frac{17}{5}$

6 (a) $-\frac{3}{2} \le x \le \frac{9}{2}$, $[-\frac{3}{2},\frac{9}{2}]$ (b) $x \le \frac{1}{2} or\, x \ge \frac{9}{2}$, $(-\infty,\frac{1}{2}]\cup[\frac{9}{2},\infty)$

(c) $\frac{1}{8} \le x < \frac{1}{2} or \frac{1}{2} < x \le \frac{7}{8}$, $[\frac{1}{8},\frac{1}{2})\cup(\frac{1}{2},\frac{7}{8}]$

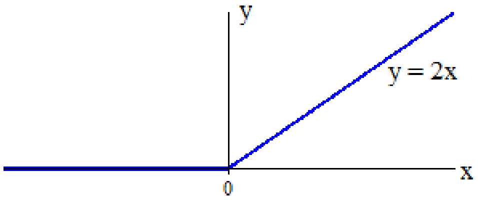
(d) $-5 \le x \le -1 or\, 5 \le x \le 9$, $[-5,-1]\cup[5,9]$ (e) $x < -\frac{1}{4} or\, x > 1$ or $\left(-\infty,-\frac{1}{4}\right)\cup(1,\infty)$

(f) $x > -\frac{1}{2}$ (g) $x < -2 or\, x > 0$

7

$$f(x) = |x| + x = \begin{cases} x + x, \text{ if } x \ge 0 \\ -x + x, \text{ if } x < 0 \end{cases}$$

$$= \begin{cases} 2x, \text{ if } x \ge 0 \\ 0, \text{ if } x < 0 \end{cases}$$



8 (a) From the graph, $-5 \le y \le 0$ (b) By solving the inequality, $-2 < x < -1$

Miscellaneous Exercises

1 (a) $x < -1 \,or\, x > \frac{2}{3}$ (b) $(x-2)(x-4)(2x+3)$, $x < -\frac{3}{2} \,or\, 2 < x < 4$

2 $(x-3)(2x-3)(3x+1)$, $x < -\frac{1}{3}$ or $\frac{3}{2} < x < 3$

3 $\frac{5}{7} \le x < 1$ *or* $1 < x \le 3$ 4 $0 < x \le 1$ *or* $x \ge \frac{5}{2}$

5 $x = -\frac{2}{3}$ or $x = 2$ 6 (b) $x > 4$ *or* $x < -1$

7 $-3 < x < \frac{5}{3}$