Chapter 3 : Integration by Substitution

Objectives :
1. Find integrals by means of an appropriate substitution

3.1 Differential of a Function

Look at the following integrals:

- $\int 3x^2(x^3+1)^8\,dx$
- $\int \dfrac{3x^2-1}{x^3-x}\,dx$
- $\int x\,e^{x^2}\,dx$

These integrals might look complicated, but they can be integrated using the technique of ***"Integration by Substitution"***.

Integration by substitution enables us to reduce a given integral to one with which we are familiar. The technique is very powerful and covers a great range of problems. Unfortunately, it is not possible to give a general rule for choosing the required substitution, but this will come with experience gained through practice.

The differential of $y = f(x)$ is defined as

$$\boxed{dy = \frac{dy}{dx}\cdot dx} \quad \text{or} \quad \boxed{dy = f'(x)\cdot dx}$$

Example 1: Find the differential of the following functions:

(a) $y = 4x^2 + 3x - 7$ (b) $u = 3\sin 4t$

Solution:

(a) $\dfrac{dy}{dx} =$

The differential of y is

(b) $\dfrac{du}{dt} =$

The differential of u is

3.2 Integration by substitution of the form $\int [f(x)]^n \cdot f'(x)\,dx$

We notice that one function of the product is the differential coefficient of the other function. We can solve the problem by a substitution which leads the integral to one of the standard integrals.

Let $u = f(x)$, then $\dfrac{du}{dx} = f'(x)$. Expressing in differential form: $du = f'(x)dx$

$\int [f(x)]^n \cdot f'(x)\,dx = \int u^n\,du = \dfrac{u^{n+1}}{n+1} + C$, where $n \neq -1$ (Standard Integral)

Hence $\boxed{\int [f(x)]^n \cdot f'(x)\,dx = \dfrac{[f(x)]^{n+1}}{n+1} + C \quad \text{where } n \neq -1}$

or $\boxed{\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C \quad \text{where } n \neq -1}$

Example 2: $\int (x^2+3)^5\, 2x\,dx$ $\qquad Ans: \dfrac{1}{6}(x^2+3)^6 + C$

Choose substitution: Let $u =$

Find differential: $du =$

Substitute x by u completely: $\int (x^2+3)^5\, 2x\,dx =$

(Integrate, then substitute x back)

Example 3: $\int 3x\sqrt{1-2x^2}\,dx$ $\qquad Ans: -\dfrac{1}{2}(1-2x^2)^{3/2} + C$

***Example 4*:** $\int (e^x+1)^3 \, e^x \, dx$ $\qquad$ $Ans: \frac{1}{4}\left(e^x+1\right)^4 + C$

Solution

Integration by substitution of the form $\int [f(x)]^n \cdot f'(x) \, dx$ can be summarised by the following example.

To find $\int 6x^2\left(2x^3-3\right)^7 dx$:

	Recommended Procedure	In this example:
Step 1	Choose *u* as some expression that appears in the integrand. (This may require some trial and error to find the correct expression for *u*)	Let $u = 2x^3 - 3$
Step 2	Find $\frac{du}{dx}$ and obtain the differential of u.	$\frac{du}{dx} = 6x^2$ *or* $du = 6x^2 dx$
Step 3	Substitute the values of *u* and *du* into the original integral.	$\int 6x^2 \cdot \left(2x^3-3\right)^7 dx$ $= \int \left(2x^3-3\right)^7 \cdot 6x^2 dx$ $= \int u^7 du$
Step 4	Integrate w.r.t. u (Using standard formulae)	$= \frac{u^8}{8} + C$
Step 5	Write the answer in terms of x	$= \frac{\left(2x^3-3\right)^8}{8} + C$

3.3 Integration by substitution of the form $\displaystyle\int \frac{f'(x)}{f(x)}\,dx$

Let's look at an integral in which the numerator is the differential of the denominator.

Let $u = f(x)$

then $du = f'(x)\,dx$

$$\int \frac{f'(x)}{f(x)}\,dx = \int \frac{1}{u}\,du = \ln|u| + C \quad \text{(Standard Integral)}$$

i.e. $$\boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C}$$

Example 5**:** Find $\displaystyle\int \frac{3x^2-1}{x^3-x}\,dx$ *Ans:* $\ln\left|x^3-x\right| + C$

Solution:

Example 6**:** $\displaystyle\int \frac{2x^2}{x^3-4}\,dx$ $Ans:\ \dfrac{2}{3}\ln|x^3-4|+C$

Example 7: $\displaystyle\int \frac{e^{2x}}{e^{2x}+1}\,dx$ $Ans:\ \dfrac{1}{2}\ln|e^{2x}+1|+C$

3.4 Integration by substitution of the form $\int e^{f(x)} \cdot f'(x)\,dx$

Let $u = f(x)$, then $du = f'(x)\,dx$

$$\int e^{f(x)} f'(x)\,dx = \int e^u \,du = e^u + C \quad \text{(Standard Integral)}$$

i.e. $$\boxed{\int e^{f(x)} f'(x)\,dx = e^{f(x)} + C}$$

Example 8: $\int 3x^2 e^{x^3}\,dx$ $\qquad$ $Ans:\ e^{x^3} + C$

3.5 Other Substitutions by Inspection

Some integrals do not fit in any of the types previously studied. Other substitutions are then needed.

Example 9: $\int \left(t\sqrt{t-1}\right) dt$ $\qquad$ $Ans:\ \frac{2}{5}(t-1)^{5/2} + \frac{2}{3}(t-1)^{3/2} + C$

3.6 Integration By Substitution & The Definite Integral

When evaluating a definite integral involving substitution (i.e. change of variable from x to u), it is necessary to change the limits for x to the corresponding values of u.

Example 10: Evaluate $\int_0^2 x e^{x^2}\, dx$ *Ans:* 26.80

Solution:

Tutorial 3

1. Find the following integrals, by using the suitable substitutions.

(a) $\int 2x\left(x^2+1\right)^5 dx$ (b) $\int x\left(x^2-3\right)^4 dx$ (c) $\int t\,e^{3-2t^2} dt$

(d) $\int \frac{x}{1-2x^2} dx$ (e) $\int \frac{x}{(4-x^2)^2} dx$ (f) $\int \sin^2\theta\cos\theta\, d\theta$

(g) $\int \frac{dx}{x\ln x}$ (h) $\int \frac{5e^{2x}}{\sqrt{1-e^{2x}}} dx$ *(i) $\int \frac{x+1}{\sqrt{x+2}} dx$

*(j) $\int \frac{1}{\sqrt{x}+x} dx$ (k) $\int t^3 \sin t^4\, dt$

2. Evaluate the following definite integrals.

(a) $\int_0^{1/2} y\sqrt{\frac{1}{4}-y^2}\, dy$ (b) $\int_1^2 \frac{e^{1/t}}{t^2} dt$ (c) $\int_0^4 \frac{4x}{\sqrt{2x+1}} dx$

3. A 1.25F capacitor, that has an initial voltage of 25.0V, is charged with a current that varies with time according to the equation $i = t\sqrt{t^2+6.83}$. The formula for the voltage across a capacitor is $V_c = \frac{1}{C}\int i\, dt$ volts .

(a) Show that the general equation of the voltage across the capacitor is given by

$V_c = 0.267\left(t^2+6.83\right)^{\frac{3}{2}} + k,$ where k is a constant

(b) Find the value of k .

(c) Hence, find the voltage across the capacitor at 1.00s.

4. If a circular disk of radius r carries a uniform electrical charge, then the electric potential on the axis of the disk at a point a from its centre is given by the equation

$$V = k\int_0^r \frac{x}{\sqrt{x^2+a^2}} dx$$

where k is a constant depending on the charge density. Integrate to find V as a function of r and a.

5. Find the root-mean-square (rms) value of $i = t^{\frac{1}{2}}e^{-t^2}$ A from $t = 1$s to $t = 2$s.

Miscellaneous Exercises

1. Find the results of the integrals.

*(a) $\int \sin^3 x\, dx$ (Hint: use $\sin^2 x = 1-\cos^2 x$ and let $u = \cos x$)

*(b) $\int \left(27e^{9x}+e^{12x}\right)^{1/3} dx$ *(c) $\int \frac{3}{x\ln x} dx$

*(d) $\int x\sqrt{4-x}\, dx$ (Hint: let $u = 4-x$ and represent x in term of u)

*(e) $\int e^{2x}\sqrt{1+4e^x}\, dx$ *(f) $\int \tan^3 x\, dx$

*(g) $\int \sec^6 t\, dt$

*(h) $\int \tan^3 x \sec x\, dx$

*(i) $\int \sin^2 x \cos^4 x\, dx$

*(j) $\int \cos^4 2x \sin^3 2x\, dx$

*(k) $\int \sin^3 \theta \cos^3 \theta\, d\theta$

2. Integrate the following:

*(a) $\int \frac{x^3}{\sqrt{1-x^2}}\, dx$

*(b) $\int x(2x-5)^3\, dx$

*(c) $\int t^3\sqrt{1-t^2}\, dt$

*(d) $\int \frac{dx}{3+\sqrt{x+2}}$

*(e) $\int \frac{2x+1}{(x-3)^6}\, dx$

*(f) $\int \frac{\sqrt{x^3-4}}{x}\, dx$

*3. Evaluate $\int_0^{\frac{\pi}{4}} \frac{\cos 2x}{1+\sin^2 2x}\, dx$.

*4. By using the substitution $x = \tan\theta$, or otherwise, find $\int \frac{1}{(1+x^2)^2}\, dx$.

*5. By using the substitution $t-1 = \sin\theta$, or otherwise, find $\int \sqrt{1-(t-1)^2}\, dt$.

Multiple Choice Questions

1. To find the integral $\int \frac{x-2}{\sqrt{x^2-4x+1}}\, dx$ by substitution method, we should let

(a) $u = x-2$

(b) $u = x^2-4x+1$

(c) $u = 2x-4$

(d) $u = x$

2. Which of the following integrals **cannot** be found using the substitution method?

(a) $\int \frac{1}{1+x^2}\, dx$

(b) $\int \frac{x}{1+x^2}\, dx$

(c) $\int x^2 e^{x^3}\, dx$

(d) $\int 4\cos^2 x \sin x\, dx$

3. To find $\int x\sqrt{x^2+1}\, dx$,

(a) let $u = x$

(b) let $u = \sqrt{x}$

(c) let $u = x+1$

(d) let $u = x^2+1$

Answers

1. (a) $\frac{(x^2+1)^6}{6}+C$ (b) $\frac{1}{10}\left(x^2-3\right)^5+C$ (c) $-\frac{1}{4}e^{3-2t^2}+C$

   (d) $-\frac{1}{4}\ln\left|1-2x^2\right|+C$ (e) $\frac{1}{2(4-x^2)}+C$ (f) $\frac{1}{3}\sin^3\theta+C$

   (g) $ln\left|ln\,x\right|+C$ (h) $-5\sqrt{1-e^{2x}}+C$

   (i) $\frac{2}{3}(x+2)^{3/2}-2(x+2)^{1/2}+C$ (j) $2\ln\left(1+\sqrt{x}\right)+C$

   (k) $-\frac{1}{4}\cos t^4+C$
2. (a) 1/24 (b) 1.07 (c) 13.33
3. (b) 20.2 V (c) 26.0V
4. $V=k\left(\sqrt{r^2+a^2}-a\right)$
5. 0.18 A

Miscellaneous Exercises

1. (a) $-\cos x+\frac{\cos^3 x}{3}+C$ (b) $\frac{1}{4}\left(27+e^{3x}\right)^{\frac{4}{3}}+C$ (c) $3\ln\left|\ln x\right|+C$

   (d) $\frac{2}{5}(4-x)^{5/2}-\frac{8}{3}(4-x)^{3/2}+C$ (e) $\frac{1}{40}(1+4e^x)^{5/2}-\frac{1}{24}(1+4e^x)^{3/2}+C$

   (f) $\frac{1}{2}\tan^2 x+\ln\left|\cos x\right|+C$ (g) $\frac{1}{5}\tan^5 t+\frac{2}{3}\tan^3 t+\tan t+C$

   (h) $\frac{1}{3}\sec^3 x-\sec x+C$ (i) $\frac{x}{16}-\frac{\sin 4x}{64}+\frac{\sin^3 2x}{48}+C$

   (j) $-\frac{1}{10}\cos^5 2x+\frac{1}{14}\cos^7 2x+C$ (k) $\frac{1}{6}\cos^6\theta-\frac{1}{4}\cos^4\theta+C$

   or $\frac{1}{4}\sin^4\theta-\frac{1}{6}\sin^6\theta+C$

2. (a) $\frac{1}{3}\left(1-x^2\right)^{\frac{3}{2}}-\left(1-x^2\right)^{\frac{1}{2}}+C$

   (b) $\frac{1}{20}(2x-5)^5+\frac{5}{16}(2x-5)^4+C$ or $\frac{1}{80}(2x-5)^4(8x+5)+C_1$

   (c) $\frac{1}{5}\left(1-t^2\right)^{\frac{5}{2}}-\frac{1}{3}\left(1-t^2\right)^{\frac{3}{2}}+C$

   (d) $2\left(3+\sqrt{x+2}\right)-6\ln\left(3+\sqrt{x+2}\right)+C$ or $2\sqrt{x+2}-6\ln\left(3+\sqrt{x+2}\right)+C_1$

   (e) $\frac{-1}{2}(x-3)^{-4}-\frac{7}{5}(x-3)^{-5}+C$ or $\frac{1-5x}{10(x-3)^5}+C_1$

   (f) $\frac{2}{3}\left(x^3-4\right)^{\frac{1}{2}}-\frac{4}{3}\tan^{-1}\frac{\sqrt{x^3-4}}{2}+C$
3. $\frac{\pi}{8}$
4. $\frac{1}{2}\left[\tan^{-1}x+\frac{x}{1+x^2}\right]+C$

5. $\frac{1}{2}\left[\sin^{-1}(t-1)+(t-1)\sqrt{1-(t-1)^2}\right]+C$

MCQ

1. (b) 2. (a) 3. (d)