5 MATHEMATICAL INDUCTION

5.1 INTRODUCTION TO MATHEMATICAL INDUCTION

Mathematical Induction (MI) is a mathematical proof technique which is used to prove the truth of a mathematical statement without having to verify for every case.

Mathematical Induction is a technique of proving mathematical statements like:

(a) $1+2+3+...+n=\frac{1}{2}n(n+1)$ where $n \in Z^+$.

(b) $\frac{1}{2}+\frac{1}{6}+...+\frac{1}{n(n-1)}=1-\frac{1}{n}$ where $n \geq 2$ and $n \in Z^+$.

(c) $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$, where $n \in Z^+$.

5.2 PROOF BY MATHEMATICAL INDUCTION

Suppose we want to prove that a statement is true for all positive integers.

Let $Z^+ = \{1, 2, 3, ...\}$ be the set of all positive integers.

Let P_n, where $n \in Z^+$, be the proposition or statement to be proved.

The proof procedure is outlined below:

STEP 1 (Base step):	Prove that P_1 is true ($n=1$ must be proven first).
STEP 2 :	Assume that P_n is true for some $n \in Z^+$.
STEP 3 (Induction step):	Prove that P_{n+1} is true (based on the assumption that P_n is true).

Then we can conclude that, by the principle of mathematical induction, P_n is true for all $n \in Z^+$.

Example 1

Prove by mathematical induction that $1^2+2^2+3^2+...+n^2=\dfrac{n(n+1)(2n+1)}{6}$ for all positive integers *n*.

Solution:

STEP 1: Prove that P_1 is true.

$$\text{LHS} = 1^2 = 1$$
$$\text{RHS} = \frac{1(1+1)(2+1)}{6} = \frac{1(2)(3)}{6} = 1$$
$$\therefore \text{LHS} = \text{RHS}$$
$$\therefore P_1 \text{ is true.}$$

STEP 2: Assume P_n is true for some $n \in Z^+$.

$$i.e.\quad P_n:\quad 1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$$

STEP 3: Prove that P_{n+1} is true.

$$\text{i.e.}\qquad P_{n+1}:\ 1^2+2^2+3^2+...+n^2+(n+1)^2=\frac{(n+1)(n+2)(2n+3)}{6}$$

$$\begin{aligned}\text{LHS} &= 1^2+2^2+3^2+\cdots+n^2+(n+1)^2=\frac{n(n+1)(2n+1)}{6}+(n+1)^2\\
&= \frac{n(n+1)(2n+1)+6(n+1)^2}{6}\\
&= \frac{(n+1)\left[n(2n+1)+6(n+1)\right]}{6}\\
&= \frac{(n+1)\left[2n^2+n+6n+6\right]}{6}\\
&= \frac{(n+1)\left[2n^2+7n+6\right]}{6}\\
&= \frac{(n+1)(n+2)(2n+3)}{6}\\
&= \text{RHS}\end{aligned}$$

$$\therefore P_n \text{ is true} \Rightarrow P_{n+1} \text{ is true.}$$

Since P_1 is true, it follows by the principle of mathematical induction that P_n is true for all positive integers *n*.

Example 2

Prove that the sum to *n* terms of the series $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$.

Solution:

Let P_n be $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$

STEP 1 (Base step): Prove that P_1 is true.

STEP 2: Assume P_n is true i.e. $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$

STEP 3 (Induction step): Prove that P_{n+1} is true.

$$\sum_{k=1}^{n+1} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \frac{1}{k(k+1)} + \underline{\hspace{6cm}}$$

Example 3

Prove that $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2\left(n+1\right)^2$, where $n \in Z^+$, is true by mathematical induction.

Example 4

The terms of a sequence u_1, u_2, u_3, ... u_n,... are given by the following rules:

$u_1 = 1$, $u_2 = 4$, $u_3 = 9$;
$u_n = 3u_{n-1} - 3u_{n-2} + u_{n-3}$ for $n \geq 4$

Prove that $u_n = n^2$ for all n = 1, 2, 3,...

Solution:

Use the principle of mathematical induction.

<u>STEP 1</u>:

$u_1 = 1^2 = 1$

$u_2 = 2^2 = 4$

$u_3 = 3^2 = 9$

Hence $u_n = n^2$ for n = 1, 2, 3.

<u>STEP 2</u>: Assume $u_k = k^2$ is true for a positive integer $k \geq 4$

<u>STEP 3</u>: To prove that $u_{k+1} = (k+1)^2$ is true.

From $u_n = 3u_{n-1} - 3u_{n-2} + u_{n-3}$, we have

$$
\begin{aligned}
u_{k+1} &= 3u_k - 3u_{k-1} + u_{k-2} \\
&= 3k^2 - 3(k-1)^2 + (k-2)^2 \\
&= 3k^2 - 3(k^2 - 2k + 1) + k^2 - 4k + 4 \\
&= 3k^2 - 3k^2 + 6k - 3 + k^2 - 4k + 4 \\
&= k^2 + 2k + 1 \\
&= (k+1)^2
\end{aligned}
$$

Hence $u_{k+1} = (k+1)^2$ is true if $u_k = k^2$ is true for a positive integer $k \geq 4$

Since $u_n = n^2$ is true for n = 1, 2, and 3, it follows by mathematical induction that $u_n = n^2$ is true for all positive integers n .

TUTORIAL 5

MATHEMATICAL INDUCTION

Prove by mathematical induction the following results:

1. $\displaystyle\sum_{k=1}^{n}\frac{1}{(2k)^2-1}=\frac{n}{2n+1}$ for all $n\in Z^+$.

2. $\displaystyle\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+...+\frac{1}{n(n+1)}=\frac{n}{n+1}$ for all $n\in Z^+$.

3. $\displaystyle\sum_{r=1}^{n}r(2^{r-1})=1+(n-1)2^n$ for all $n\in Z^+$.

4. $\displaystyle\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)...\left(1-\frac{1}{n+1}\right)=\frac{1}{n+1}$ for all $n\in Z^+$.

5. $\displaystyle\sum_{k=1}^{n}(3k+2)=\frac{1}{2}(3n^2+7n)$ for all $n\in Z^+$.

6. $\displaystyle\sum_{r=1}^{n}r(r+2)=\frac{1}{6}n(n+1)(2n+7)$ for all $n\in Z^+$.

7. $\displaystyle\sum_{k=1}^{n}\frac{1}{(3k-2)(3k+1)}=\frac{n}{3n+1}$ for all $n\in Z^+$.

8. $\displaystyle\sum_{r=1}^{n}\frac{1}{r(r+2)}=\frac{n(3n+5)}{4(n+1)(n+2)}$ for all $n\in\mathbb{Z}^+$.

9. $\displaystyle\sum_{r=1}^{n}r^2(r-1)=\frac{n}{12}(n^2-1)(3n+2)$ for all $n\in\mathbb{Z}^+$.

10. $n!>2^n$ for all positive integers $n\geq 4$.

Miscellaneous Exercises

11. Given $y = 2e^{3x}$, find an expression of $\dfrac{d^n y}{dx^n}$ for every positive integer n and hence use mathematical induction to prove your result for all $n \in Z^+$.

12. Show by mathematical induction that the statement below is true for all positive integers n.

$$\sum_{r=1}^{n}(2r+1)(3r+1)=\frac{n}{2}(4n^2+11n+9)$$

13. Use mathematical induction to prove that for all positive integers n,

$$\sum_{r=1}^{n}(2r-1)\cdot 2^{-r}=3-\frac{2n+3}{2^n}.$$

Hence, show that $\displaystyle\sum_{r=1}^{n} r\cdot 2^{-r}=2-\frac{n+2}{2^n}$. (MA1301 0910)

14. Prove by induction that for any positive integer n,

$$\sum_{r=1}^{n}\left(r^2+r+1\right)=\frac{1}{3}n\left(n^2+3n+5\right).$$

Use the above result to evaluate the sum

$1\times 2+2\times 3+3\times 4+.....+2009\times 2010.$ (MA1301 1011)

15. Prove by induction that for any positive integer n, $\displaystyle\sum_{r=1}^{n}\frac{1}{4r^2-1}=\frac{n}{2n+1}$.

Deduce that for any positive integer n, $\displaystyle\sum_{r=1}^{n}\frac{1}{r^2}<2.$ (MA1301 1112)