Chapter 1 – Matrices and Determinants

1 Objectives:

1. *Define a matrix, its order and elements.*
2. *Define some special types of matrices.*
3. *Explain transpose of a matrix.*
4. *Define and explain the equality rule of matrices.*
5. *Perform matrix operations such as addition, subtraction, scalar multiplication and matrix multiplication.*
6. *Define and evaluate 2nd order determinants.*
7. *Define and evaluate higher order determinants.*
8. *Use Cramer's rule to solve simultaneous linear equations with two or three unknowns.*
9. *Define the inverse of a square matrix.*
10. *Use the inverse matrix method to solve system of linear equations with two unknowns.*

1.1 Introduction

In our daily life, information can often be conveniently presented as an array of rows and columns. Bus timetable and football league results often use this form of presentation. Such an arrangement of information is called a matrix. Below are some examples of matrices.

Eg1 The weekly expenditures in dollars of a department for January and February are given below.

January Week	Payroll	Marketing	Miscellaneous
1	10000	5000	500
2	10000	3000	400
3	10000	4000	300
4	10000	6000	600

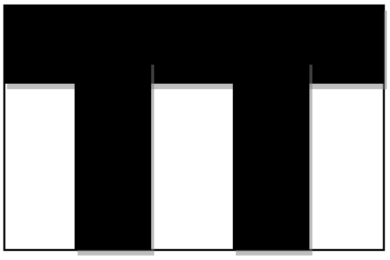
February Week	Payroll	Marketing	Miscellaneous
1	12000	6000	700
2	12000	3000	300
3	12000	5000	500
4	12000	4000	200

Matrix addition can be used to find the total expenditures for the three categories Payroll, Marketing and Miscellaneous during the corresponding weeks of January and February.

Eg2 A building contractor accepts summer orders for 135 houses, 3 condominiums and 1 high rise building. The construction materials (in appropriate units) that go into each of these buildings are listed in the table below. Using matrices, we can find how much of each raw material will be needed for all the summer contracts.

Material	House	Condominium	High rise
Lumber	10	400	500
Glass	5	150	1000
Steel	0	50	2000
Concrete	0	100	1000
Labour	20	1000	5000

Eg3 Matrices can be used in digital image processing. A digital image is an image by an array of numbers. Consider a very simple example, the mathematical representation of the symbol π. We can picture the symbol in the figure below:



We could represent the image of π by a 3×5 matrix as follows:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Note that the image is divided into a number of areas, or fields, in the above 15 of them. A number 0 is assigned as the field to be unshaded and the number 1, shaded.

Eg4 A scientist, trying to grow lobsters in a controlled environment, mixes two grains *A* and *B* in varying amounts to make the ideal lobster food. The scientist makes three mixes *I*, *II* and *III* according to the following mixtures:

$$\boldsymbol{M} = \begin{array}{c} \begin{array}{ccc} I & II & III \end{array} \\ \begin{bmatrix} 45 & 30 & 15 \\ 15 & 30 & 45 \end{bmatrix} \end{array} \begin{array}{l} \\ Grain\ A \\ Grain\ B \end{array}$$

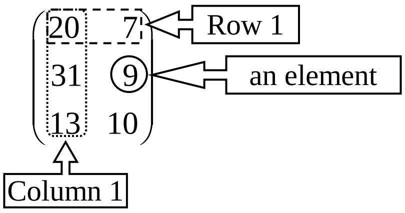
The nutritional value of each of the two grains is given by the values in the 3×2 matrix.

$$\boldsymbol{N} = \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} 5 & 10 \\ 50 & 30 \\ 10 & 5 \end{bmatrix} \end{array} \begin{array}{c} \\ \text{protein} \\ \text{carbohydrates} \\ \text{fat} \end{array}$$

Using matrices, we can find the amount of carbohydrates, protein or fat in the above mixes.

1.2 Definition of a Matrix

A **matrix** is a rectangular array of numbers enclosed by a pair of large brackets. A matrix which has 3 rows and 2 columns is shown below:



The above matrix is known as a 3×2 matrix, or a matrix of **order** 3×2.
In general, a matrix with $\boldsymbol{m}$ rows and $\boldsymbol{n}$ columns is called an $\boldsymbol{m}\times\boldsymbol{n}$ matrix or a matrix of **order** $\boldsymbol{m}\times\boldsymbol{n}$. The entries in a matrix are called the **elements** of the matrix.

Note:

(i) Matrices are denoted by capital letters.
(ii) The elements are enclosed in large square brackets or large round brackets.
(iii) $\boldsymbol{a_{ij}}$ represents an element in the $\boldsymbol{i}^{\text{th}}$ row and $\boldsymbol{j}^{\text{th}}$ column of a matrix.
(iv) The order of a matrix is also called the **size** or **dimension** of the matrix.

Example 1 : The matrix A is given as $\boldsymbol{A}=\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix}$.

(a) State the order of $\boldsymbol{A}$.
(b) Find the elements $\boldsymbol{a}_{12}$ and $\boldsymbol{a}_{21}$.

1.3 Special Types of Matrices

1.3.1 Row Matrix

A matrix with only one row is called a **row matrix**.

For example, $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$ is a 1×4 row matrix.

1.3.2 Column Matrix

A matrix with only one column is called a **column matrix**.

For example, the matrix $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ is a 4×1 column matrix.

1.3.3 Square Matrix

A **square matrix** is a matrix where the number of rows is equal to the number of columns.

For example, the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is a 3×3 square matrix.

1.3.4 Symmetric Matrix

A square matrix, such that $\boldsymbol{a_{ij}} = \boldsymbol{a_{ji}}$ for all values of i and j, is called a **symmetric matrix**.

For example, the following matrix is a symmetric matrix:

a_{ij}

a_{ji} $\begin{bmatrix} 1 & 2 & 3 \\ \boxed{2} & 5 & 6 \\ \boxed{3} & \boxed{6} & 9 \end{bmatrix}$

the principal diagonal acts as a "mirror"

e.g. $a_{12} = a_{21} = 2$

Example 2 : Find the values of a, b and c such that the matrix $\begin{bmatrix} 1 & 0 & 1 \\ c & b & 2b \\ a & a+c & 2 \end{bmatrix}$ is a symmetric matrix.

1.3.5 Diagonal Matrix

A **diagonal matrix** is a square matrix in which all the elements not on the principal diagonal are zeros.

For example, the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a 3×3 diagonal matrix.

1.3.6 Identity Matrix

An **identity matrix** is a diagonal matrix where every diagonal element is 1.

For example, the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a 3×3 identity matrix.

Note: An $\boldsymbol{n} \times \boldsymbol{n}$ identity matrix is denoted by $\mathbf{I_n}$ or simply by **I** if the order is obvious from the context.

1.3.7 Zero Matrix

The **zero matrix** is a matrix where every element is 0.

For example, the matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a 2×3 zero matrix.

Note: An $\boldsymbol{m} \times \boldsymbol{n}$ zero matrix is denoted by $\mathbf{0_{mn}}$ or simply by **0** if the order is obvious from the context. The zero matrix is also called the **null matrix**.

1.4 Matrix Operations

1.4.1 Transpose of a Matrix

The **transpose** A^{T} of an $\boldsymbol{m} \times \boldsymbol{n}$ matrix A is the $\boldsymbol{n} \times \boldsymbol{m}$ matrix whose rows are the corresponding columns of A.

Theorem 1.1

1. $(\boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}} = \boldsymbol{A}$
2. If $\boldsymbol{A}$ is a symmetric matrix, then $\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A}$.

Example 3 : Find the transpose of the following matrices .

(a) $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ (b) $\boldsymbol{B} = \begin{bmatrix} 3 & 0 \\ -1 & 6 \\ 4 & 1 \end{bmatrix}$

1.4.2 Equality of Matrices

Two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are equal, i.e. $\boldsymbol{A} = \boldsymbol{B}$, if

(i) order of $\boldsymbol{A}$ = order of $\boldsymbol{B}$

(ii) $a_{ij} = b_{ij}$ for all i and j

Example 4 : Find the values of w, x, y and z, if $\begin{bmatrix} w+1 & x \\ \frac{y}{3} & 2z \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$.

1.4.3 Matrix Addition and Subtraction

Given two matrices $\boldsymbol{A} = \left(a_{ij}\right)_{m\times n}$ and $\boldsymbol{B} = \left(b_{ij}\right)_{m\times n}$

(i) $\boldsymbol{A} + \boldsymbol{B} = \left(a_{ij} + b_{ij}\right)_{m\times n}$

(ii) $\boldsymbol{A} - \boldsymbol{B} = \left(a_{ij} - b_{ij}\right)_{m\times n}$

Important: The order of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ must be the same.

Theorem 1.2

If $\boldsymbol{O}$ is the zero matrix with the same order as the matrix $\boldsymbol{A}$, then

1. $\boldsymbol{O} + \boldsymbol{A} = \boldsymbol{A} + \boldsymbol{O} = \boldsymbol{A}$
2. $\boldsymbol{A} - \boldsymbol{A} = \boldsymbol{O}$

Example 5 : Given $\boldsymbol{A} = \begin{bmatrix} 4 & 6 \\ 8 & 10 \end{bmatrix}$, and $\boldsymbol{B} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, find

(a) $\boldsymbol{A} + \boldsymbol{B}$

(b) $\boldsymbol{A} - \boldsymbol{B}$

Example 6 : The weekly expenditures in dollars of a department for January and February are given below.

January Week	Payroll	Marketing	Miscellaneous
1	10000	5000	500
2	10000	3000	400
3	10000	4000	300
4	10000	6000	600

February Week	Payroll	Marketing	Miscellaneous
1	12000	6000	700
2	12000	3000	700
3	12000	5000	500
4	12000	4000	200

Use matrix addition to find the total expenditures for the three categories Payroll, Marketing and Miscellaneous during the corresponding weeks of January and February.

1.4.4 Scalar Multiplication

When a matrix $\boldsymbol{A} = \left(a_{ij}\right)_{m\times n}$ is multiplied by a scalar or a number k, each element a_{ij} in the matrix is multiplied by the scalar k, i.e. $k\boldsymbol{A} = k\left(a_{ij}\right)_{m\times n} = \left(ka_{ij}\right)_{m\times n}$.

Example 7 : Evaluate $5\begin{bmatrix} 0 & -3 & 1 \\ 2 & 4 & 13 \end{bmatrix}$.

Example 8 : Find the values of x and y such that $3\begin{bmatrix} x & 0 \\ 2 & 2y \end{bmatrix} - 4\begin{bmatrix} 3 & -1 \\ 1 & y \end{bmatrix} = 2\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$.

1.4.5 Matrix Multiplication

1.4.5.1 Matrix Conformability

Not any two matrices can be used for matrix multiplication. Two matrices can be multiplied only when they are conformable for matrix multiplication.

Two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are said to be **conformable for multiplication**, if the number of columns in $\boldsymbol{A}$ is equal to the number of rows in $\boldsymbol{B}$, i.e.

$$\boldsymbol{A}_{m\times n} \times \boldsymbol{B}_{n\times p} = \boldsymbol{C}_{m\times p}$$

equal

Example 9 : Are the two matrices $\boldsymbol{A}_{3\times 5}$ and $\boldsymbol{B}_{5\times 4}$ conformable for matrix multiplication $\boldsymbol{AB}$? If yes, what is the order of $\boldsymbol{AB}$?

Example 10 : Are the two matrices $\boldsymbol{A}_{3\times 5}$ and $\boldsymbol{B}_{3\times 2}$ conformable for matrix multiplication $\boldsymbol{AB}$? If yes, what is the order of $\boldsymbol{AB}$?

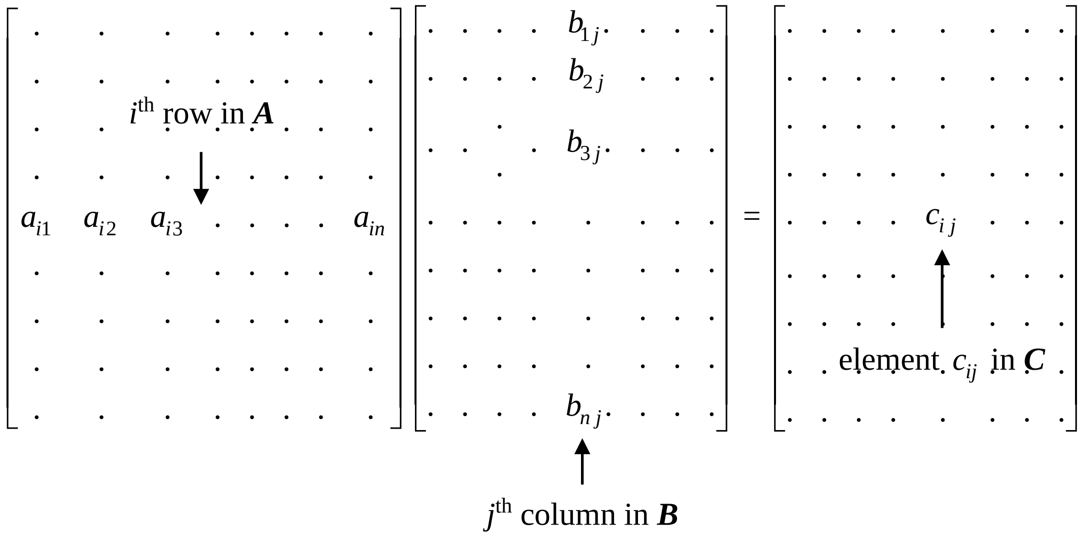
1.4.5.2 Matrix Multiplication

If $\boldsymbol{A}$ is a $1\times n$ row matrix, and $\boldsymbol{B}$ is an $n\times 1$ column matrix, then $\boldsymbol{AB}$ is a 1×1 matrix. To find the product, multiply each element in $\boldsymbol{A}$ (from left to right) by the corresponding element in $\boldsymbol{B}$ (from top to bottom) and then add the results.

In general, let $\boldsymbol{A}$ be a $m\times n$ matrix and $\boldsymbol{B}$ be a $n\times p$ matrix. Then $\boldsymbol{C}=\boldsymbol{AB}$ is an $m\times p$ matrix**,** such that the element c_{ij} in the i^{th} row and j^{th} column is the sum of the products of the corresponding elements of the i^{th} row (from left to right) of $\boldsymbol{A}$ and the j^{th} column (from top to bottom) of $\boldsymbol{B}$, i.e.

$$c_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+\ldots+a_{in}b_{nj}$$

where $a_{i1}, a_{i2}, \ldots, a_{in}$ are the elements in the i^{th} row of $\boldsymbol{A}$ and $b_{j1}, b_{j2}, \ldots, b_{nj}$ are the elements in the j^{th} column of $\boldsymbol{B}$. This is illustrated below.



where $c_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+\ldots+a_{in}b_{nj}$.

> **Theorem 1.3**
> Let $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{O}$ (zero matrix) and $\boldsymbol{I}$ (identity matrix) be conformable matrices and k be any scalar.
> 1. $(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC})$
> 2. $\boldsymbol{A}(\boldsymbol{B}+\boldsymbol{C}) = \boldsymbol{AB}+\boldsymbol{AC}$
> 3. $k(\boldsymbol{AB}) = (k\boldsymbol{A})\boldsymbol{B} = \boldsymbol{A}(k\boldsymbol{B})$
> 4. $\boldsymbol{AO} = \boldsymbol{O}$
> 5. $\boldsymbol{AI} = \boldsymbol{A}$
> 6. $(\boldsymbol{AB})^{\mathrm{T}} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}$

(i) In general, matrix multiplication is not commutative: $\boldsymbol{AB} \neq \boldsymbol{BA}$.

(ii) In general, the cancellation law is not valid: $\boldsymbol{AB}=\boldsymbol{AC}$ does not imply $\boldsymbol{B}=\boldsymbol{C}$.

Example 11 : Evaluate

(a) $\begin{bmatrix} 2 & 4 & 6 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$ (b) $\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}\begin{bmatrix} 2 & 4 & 6 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 4 \\ 3 & 1 \\ 5 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Example 12 : A fast food chain has three outlets A, B and C. The average daily sales and profits of hamburgers, potato chips and drinks in each outlet are given in the following tables.

	Units sold		
	Outlet A	Outlet B	Outlet C
Hamburgers	800	500	600
Potato chips	900	700	800
Drinks	600	800	900

	Unit profit ($)		
	Outlet A	Outlet B	Outlet C
Hamburgers	0.20	0.40	0.30
Potato chips	0.40	0.50	0.60
Drinks	0.50	0.30	0.40

(a) Write the above information on units sold and unit profit of each product into 2 separate 3×3 matrices.

(b) Use your matrices in part (a) to find the total profit of
  (i) each product
  (ii) each outlet
  You may need to transpose the matrices.

1.5 Determinant of a Square Matrix

1.5.1 Introduction

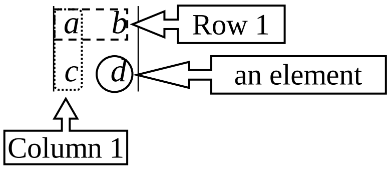
The **determinant** of a square matrix **A** is denoted by **det (A)** or $|\mathbf{A}|$. It is an algebraic operation that transforms a square matrix into a scalar. This operation is very useful in the analysis and solution of systems of linear equations.

The symbol Δ (read as 'delta'), which is the Greek capital "D", is usually used to denote a determinant.

1.5.2 Second Order Determinant

A 2nd order determinant is written as $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, where a, b, c and d are numbers or variables.

Some useful terms for a determinant:



1.5.3 Evaluating Second Order Determinant

To evaluate a 2nd order determinant, we use the following:

$$\boxed{\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc}$$

Example 13 : Evaluate

(a) $\begin{vmatrix} 6 & -4 \\ 5 & 1 \end{vmatrix}$ (b) $\begin{vmatrix} 7 & 10 \\ -2 & 3 \end{vmatrix}$

1.5.4 Evaluating Third and Higher Order Determinants

1.5.4.1 Minors and Cofactors

Consider the third order determinant $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Minor of an Element

The minor of an element a_{ij} is a smaller determinant formed by deleting the row i and column j from the original determinant.

To find the minor of element a_{12}, first remove row 1 and column 2 as follows:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Then form a determinant with the remaining elements, i.e.

$$\text{Minor of } a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{23}a_{31}$$

Cofactor of an Element

The cofactor of an element a_{ij}, denoted as A_{ij}, is defined as follows:

$$\boxed{A_{ij} = \text{cofactor of } a_{ij} = (-1)^{i+j}(\text{minor of } a_{ij})}$$

$$\text{Cofactor of } a_{12} = (-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$= -1(a_{21}a_{33} - a_{23}a_{31})$$

Example 14 : Find the minors and cofactors of the circled elements in the following determinants.

(a) $\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & -7 & 4 \\ 2 & -3 & 5 \\ -1 & 6 & 8 \end{vmatrix}$

1.5.4.2 Evaluate Third and Fourth Order Determinants

Given a 3rd order determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, expand along row 1:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\left(+\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}\right) + a_{12}\left(-\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}\right) + a_{13}\left(+\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}\right)$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Note:
1. In the formula above, the elements in row 1 were chosen for expansion.
   In fact, any one row or column can be used to do the expansion.
2. Regardless of whichever row or column is chosen, the result should be the same.
3. As a rule of thumb, choose the row or column with the most 0's or 1's.

Note : You may use Sarrus' Rule to evaluate a <u>third order</u> determinant.

$$\boxed{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - gec - hfa - idb}$$

Example 15 : Evaluate $\begin{vmatrix} 4 & 6 & -8 \\ 2 & 5 & -1 \\ 1 & -3 & 2 \end{vmatrix}$

Example 16 : Find x if $\begin{vmatrix} 2 & x & 3 \\ 1 & 3 & -1 \\ 2 & -2 & 5 \end{vmatrix} = 9$.

1.6 Inverse Matrices

1.6.1 Definition of an inverse matrix

Let $\boldsymbol{A}$ be a square matrix of order $n \times n$. If there exists a matrix $\boldsymbol{B}$ such that $\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}$, where $\boldsymbol{I}$ is the $n \times n$ identity matrix, then $\boldsymbol{B}$ is called the inverse matrix of $\boldsymbol{A}$, and is written as $\boldsymbol{A}^{-1}$.

To conclude, for an invertible matrix $\boldsymbol{A}$,

$$\boxed{\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}}$$

Theorem 1.4

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be invertible matrices. Let c be a nonzero scalar and m be a positive integer.

1. $\left(\boldsymbol{A}^{-1}\right)^{-1} = \boldsymbol{A}$
2. $\left(c\boldsymbol{A}\right)^{-1} = \dfrac{1}{c}\boldsymbol{A}^{-1}$
3. $\left(\boldsymbol{AB}\right)^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$
4. $\left(\boldsymbol{A}^{\mathrm{T}}\right)^{-1} = \left(\boldsymbol{A}^{-1}\right)^{\mathrm{T}}$
5. $\left(\boldsymbol{A}^{-1}\right)^{m} = \left(\boldsymbol{A}^{m}\right)^{-1}$

1.6.2 Inverse of a 2×2 matrix

Let $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 invertible matrix.

The inverse of matrix $\boldsymbol{A}$, $\boldsymbol{A}^{-1}$ can be found using the formula below:

$$\boxed{\boldsymbol{A}^{-1} = \frac{1}{|\boldsymbol{A}|}\operatorname{adj}(\boldsymbol{A})}$$

Note : (i) $|\boldsymbol{A}|$ is the determinant of matrix $\boldsymbol{A}$, $|\boldsymbol{A}| = ad - bc$

(ii) $\operatorname{adj}(\boldsymbol{A})$ is the adjoint of matrix $\boldsymbol{A}$.

$$\begin{aligned}
\operatorname{adj}(\boldsymbol{A}) &= \begin{bmatrix} \text{Cofactor of } a & \text{Cofactor of } b \\ \text{Cofactor of } c & \text{Cofactor of } d \end{bmatrix}^{T} \\
&= \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^{T} \\
&= \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
\end{aligned}$$

If the determinant of a square matrix is **zero**, then the inverse of the matrix does not exist and is called a non-invertible or **singular matrix**.

Hence, for a 2×2 invertible matrix $\boldsymbol{A}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$\boxed{\boldsymbol{A}^{-1}=\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

Example 17 : Find the inverse matrix of $\boldsymbol{A}=\begin{bmatrix} 5 & 1 \\ -3 & 2 \end{bmatrix}$.

Example 18 : Given that $\boldsymbol{A}=\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $\boldsymbol{B}=\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, find matrix $\boldsymbol{C}$ such that $\boldsymbol{ACA}^{-1}=\boldsymbol{B}$.

Example 19 : Determine whether the matrix $\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is singular.

Example 20 : Find the value of k such that the matrix $\begin{bmatrix} k & 1 & 0 \\ 4k & 3 & 2 \\ 8 & 5 & 0 \end{bmatrix}$ is singular.

1.7 Solution of system of equations

1.7.1 Inverse Matrix Method

Inverse matrices may be used to solve systems of simultaneous linear equations.

The system of linear equations with 2 unknowns x and y

$$\begin{aligned} a_1x + b_1y &= k_1 \\ a_2x + b_2y &= k_2 \end{aligned}$$

can be written in the matrix form as

$$\boldsymbol{AX} = \boldsymbol{B}$$

where $\boldsymbol{A} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, $\boldsymbol{X} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$.

Matrix $\boldsymbol{A}$ is called the **coefficient** matrix.

If the inverse $\boldsymbol{A}^{-1}$ exists, then

$$\left(\boldsymbol{A}^{-1}\boldsymbol{A}\right)\boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{B} \quad \Rightarrow \quad \boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{B}$$

This is the solution in matrix form.

If $\boldsymbol{A}^{-1}$ does not exist, then the above method cannot be applied and the system either has no solution or the solution is not unique.

In a similar manner, the above method can be extended to solve linear system of equations with three unknowns.

Example 21 : Solve the following system of linear equations using the inverse matrix method.

$$\begin{aligned} 2x + y &= 9 \\ x - 3y &= 8 \end{aligned}$$

1.7.2 Cramer's Rule

Consider the following 3×3 system of equations i.e. a system of three linear equations with three unknowns x, y and z:

$$\begin{aligned} a_1x + b_1y + c_1z &= k_1 \\ a_2x + b_2y + c_2z &= k_2 \\ a_3x + b_3y + c_3z &= k_3 \end{aligned}$$

First, define a "system" determinant, Δ, using the coefficients of x, y and z as follows,

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Note that the original arrangement of the coefficients is retained.

Next, define 3 other determinants – Δ_x , Δ_y and Δ_z – as follows,

$$\Delta_x = \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}, \qquad \Delta_y = \begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}, \qquad \Delta_z = \begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}$$

where k_1, k_2 and k_3 are the right-hand-side constants.

Finally find the unknowns x, y and z using these four determinants as follows,

$$x = \frac{\Delta_x}{\Delta}, \qquad y = \frac{\Delta_y}{\Delta}, \qquad z = \frac{\Delta_z}{\Delta}$$

provided $\Delta \neq 0$.

The above method for solving simultaneous linear equations is called Cramer's Rule, after the Swiss mathematician Gabriel Cramer, who gave the general rule for solving an n×n system in his paper *Introduction to the Analysis of Algebraic Curves* (1750).

Do note that Cramer's Rule can be used to solve system of n equations involving n unknowns. That means it can also be used to solve 2 equations involving 2 unknowns.

Example 22 : Use Cramer's Rule to solve the following system of three linear equations with three unknowns

$$x + 2y - z = 4$$
$$2x - 4y - 3z = 6$$
$$3x - 6y - z = 2$$

Example 23 : The curve of the equation $y = ax^3 + bx^2 + cx + d$ passes throught the point $(1, -8)$, intercepts the y-axis at $y = -6$ and has roots at $x = -1$ and 2.

(i) Form a system of equations.

(ii) Use Cramer's Rule to solve for the constants a, b, c and d.

<u>Tutorial</u> : True/False questions

State whether the following statements are true or false.

1. The terms "determinant" and "matrix" have the same meaning.

2. When you multiply matrix $\boldsymbol{A}$ by the identity matrix $\boldsymbol{I}$, you will obtain A^{-1}.

3. One can always find the determinant of a matrix.

4. The product of the two matrices $\boldsymbol{AB}$ is always equal to the product of the two matrices $\boldsymbol{BA}$.

5. The matrix product $\begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}\begin{bmatrix} 3 & 1 & 6 \end{bmatrix}$ will yield a square matrix.

6. The matrix product $\begin{bmatrix} 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \\ 9 \end{bmatrix}$ is undefined.

7. Matrix multiplication is only possible if the number of columns in the first matrix equals the number of rows in the second matrix.

8. The inverse of a matrix is a unique matrix of the same dimensions which, when multiplied by the original matrix produces the transpose of that matrix.

Tutorial : Multiple Choice Questions

1. Which one of the following statements is false?
   (a) A matrix with m rows and n columns is called an $m \times n$ matrix.
   (b) The element in the i^{th} row and j^{th} column of a matrix is denoted by a_{ij}.
   (c) The dimension of a matrix refers to the number of elements in the matrix.
   (d) The order of a matrix is sometimes called the size of the matrix.

2. Which of the following statements are true about matrices $\boldsymbol{A}$ and $\boldsymbol{B}$?
   (i) $(\boldsymbol{A}+\boldsymbol{B})^{\mathrm{T}} = \boldsymbol{A}^{\mathrm{T}} + \boldsymbol{B}^{\mathrm{T}}$
   (ii) $(\boldsymbol{AB})^{\mathrm{T}} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}$
   (iii) $(\boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}} = \boldsymbol{A}$
   (iv) $\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A}$ if $\boldsymbol{A}$ is a symmetric matrix

   (a) (i) and (ii) (b) (ii) and (iii)
   (c) (i), (iii) and (iv) (d) All of the above

3. When you multiply a matrix by the identity matrix, you will get
   (a) an inverse matrix (b) a transpose matrix
   (c) an adjoint matrix (d) an original matrix

4. If we interchange the rows and columns of a matrix, we have its
   (a) transpose (b) adjoint
   (c) cofactor (d) inverse

5. If $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$ is a 2×2 matrix, then the order of matrix $\boldsymbol{A}$ could be
   (a) 2×3 (b) 3×2
   (c) 1×3 (d) 3×1

6. In order for two matrices to be added, the matrices
   (a) Must be of the same size (b) Must be a square
   (c) Must both be invertible (d) Must be identical

7. Which of the following conditions are necessary for a matrix to have an inverse?
   (i) Square matrix
   (ii) Non singular matrix
   (iii) Singular matrix
   (iv) Conformable matrix

   (a) (i) and (ii) (b) (i) and (iii)
   (c) (i) and (iv) (d) (ii) and (iv)

8. The inverse of a matrix is found by which of the following:
   (a) Multiplying the matrix by the identity matrix
   (b) Dividing the adjoint of the matrix by the determinant
   (c) Dividing the cofactor matrix by its determinant
   (d) Dividing the matrix by its transpose

9. In order for a square matrix to have an inverse, it must have a/an ________

(a) cofactor (b) adjoint
(c) transpose matrix (d) non zero determinant

10. For a non-square matrix, which one of the following can be found?

(a) adjoint (b) cofactor
(c) dimension (d) inverse

11. Given that $\boldsymbol{A}$ is a matrix. If $\boldsymbol{A} = 3\boldsymbol{A}^{-1}$, then

(a) $\boldsymbol{A}^{-1} = \boldsymbol{I}$ (b) $\boldsymbol{A}^{2} = \boldsymbol{I}$
(c) $\boldsymbol{A}^{-1} = \frac{1}{3}\boldsymbol{I}$ (d) $\boldsymbol{A}^{2} = 3\boldsymbol{I}$

12. $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are three matrices such that $\boldsymbol{AB} = \boldsymbol{C}$. Then $\boldsymbol{B}$ =

(a) $\boldsymbol{C}^{-1}\boldsymbol{A}$ (b) $\boldsymbol{CA}$
(c) $\boldsymbol{A}^{-1}\boldsymbol{C}$ (d) $\boldsymbol{AC}$

13. For any two non singular square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, $(\boldsymbol{AB})^{-1}$ =

(a) $\boldsymbol{AB}$ (b) $\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$
(c) $\boldsymbol{A}^{-1}\boldsymbol{B}^{-1}$ (d) $\boldsymbol{A}^{-1}\boldsymbol{B}$

14. If $\boldsymbol{A}$ is a matrix of order $m \times n$ and k is a real number, then the order of $k\boldsymbol{A}$ is of which of the following?

(a) $km \times n$ (b) $m \times kn$
(c) $km \times kn$ (d) $m \times n$

15. Which of the following statement(s) is/are TRUE for determinants?

(I) The number of rows must be equal to the number of columns.
(II) The number of rows may be different from the number of columns.
(III) Every element in a determinant has a minor.
(IV) Addition, subtraction, multiplication and division of two determinants will result in a new determinant.

(a) (I) and (II) (b) (I) , (II) , (III) , (IV)
(c) (III) and (IV) (d) (I) and (III)

Tutorial 1a (Properties of Matrices)

1. State the order of the following matrices.

(a) $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$

2. Find the elements a_{23}, a_{22} and a_{32} of the matrix $\begin{bmatrix} 1 & 0 & -1 \\ 3 & 5 & -2 \\ 4 & 7 & -3 \end{bmatrix}$.

Tutorial 1b (Special matrices and Matrix Operations)

1. Find the transpose of each of the following matrices.

(a) $\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

2. Find the value of k such that the matrix $\begin{bmatrix} 1 & 0 & k+1 \\ 0 & k & 2 \\ 2k-1 & 2 & 3 \end{bmatrix}$ is a symmetric matrix.

3. Find the values of a and b such that $\begin{bmatrix} 1 & a & b \\ b & 2 & a+b \\ a & 4 & 3 \end{bmatrix}$ is a symmetric matrix.

4. Given $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$ and $\boldsymbol{C} = \begin{bmatrix} 1 & 4 \\ -2 & 0 \\ 5 & -1 \end{bmatrix}$.

Evaluate : (a) $\boldsymbol{A} + \boldsymbol{B}$ (b) $\boldsymbol{B}^T + \boldsymbol{C}$ (c) $2\boldsymbol{A} + 3\boldsymbol{B}$

5. Find matrix $\boldsymbol{A}$ such that $\boldsymbol{A} + \begin{bmatrix} 2 & 1 & -3 \\ 1 & 5 & 0 \end{bmatrix} = 2\begin{bmatrix} -1 & 4 & 3 \\ -2 & 0 & 4 \end{bmatrix}$.

6. Find the values of a and b such that $3\begin{bmatrix} a & 2 \\ -1 & 2b \end{bmatrix} + 2\begin{bmatrix} a+b & 2 \\ 3 & a-b \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 3 & 6 \end{bmatrix}$.

7. If $\boldsymbol{A}$ is a 2×3 matrix and the matrix product $\boldsymbol{AB}$ is a 2×4 matrix, find the order of matrix $\boldsymbol{B}$.

8. Evaluate the following matrix products wherever possible.

(a) $\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 6 & 7 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 5 & 4 & 8 \\ 6 & 7 & 9 \end{bmatrix}$

(c) $\begin{bmatrix} 5 & 6 \\ 4 & 7 \\ 8 & 9 \end{bmatrix}^T\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 6 & 4 & 8 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \\ 1 \\ 7 \end{bmatrix}$

(e) $\begin{bmatrix} 3 \\ 5 \\ 1 \\ 7 \end{bmatrix}\begin{bmatrix} 2 & 6 & 4 & 8 \end{bmatrix}$

(f) $\begin{bmatrix} 5 & 8 & -4 \\ 6 & 9 & -5 \\ 4 & 7 & -3 \end{bmatrix}\begin{bmatrix} 3 & 2 & 5 \\ 4 & -1 & 3 \\ 9 & 6 & 5 \end{bmatrix}$

(g) $\begin{bmatrix} -3 & 4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & -3 \\ 6 & 0 \\ 8 & 9 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \end{bmatrix}$

(h) $\begin{bmatrix} 5 & 2 & 0 & -3 \\ 4 & -2 & 7 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & -3 \\ 6 & 0 \\ 8 & 9 \end{bmatrix}\begin{bmatrix} 6 & 8 \end{bmatrix}$

9. Given $\boldsymbol{A} = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 0 & -2 \\ 1 & 4 \end{bmatrix}$, show that $\boldsymbol{AB} \neq \boldsymbol{BA}$.

Hence explain why $\boldsymbol{A}^2 - \boldsymbol{B}^2 \neq (\boldsymbol{A} + \boldsymbol{B})(\boldsymbol{A} - \boldsymbol{B})$.

10. Find the values of $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{r}$ such that $\begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{q} & -7 \\ \boldsymbol{p} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 6 & -3\boldsymbol{r} \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$.

11. Given $\boldsymbol{A} = \begin{bmatrix} k & k \\ -2 & k \\ 1 & 1 \end{bmatrix}$, find the value(s) of k such that $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$ is a diagonal matrix.

*12. A chemical company manufactures three types of chemicals C1, C2 and C3 in three factories F1, F2 and F3. The factory outputs in tons per hour are shown in the table below.

	C1	C2	C3
F1	1	2	3
F2	2	3	2
F3	3	1	2

Use matrix multiplication to find the total quantities of C1, C2 and C3 that will be manufactured if F1 operates for 30 hours, F2 for 40 hours and F3 for 50 hours.

*13. A computer company sells two models A and B of a certain type of personal microcomputer. On average, the company sells 1000 units of model A microcomputers and 900 units of model B microcomputers per month. A model A microcomputer uses 8 Type I chips, 4 Type II chips and 3 Type III chips. A model B microcomputer uses 12 Type I chips, 4 Type II chips and 5 Type III chips. A Type I chip costs $40, a Type II chip costs $50 and a Type III chip costs $60 respectively.

Use matrix multiplication to find
(a) the total number of each type of chip used in the microcomputers sold per month, and
(b) the total cost of the chips used in the microcomputers sold per month.

Tutorial 1c (Determinants)

1. Identify the matrices and determinants.

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (b) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ (c) $\begin{Bmatrix} 1 & 2 \\ 3 & 4 \end{Bmatrix}$ (d) $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

2. Evaluate the following determinants.

(a) $\begin{vmatrix} 5 & 9 \\ 1 & 4 \end{vmatrix}$ (b) $\begin{vmatrix} 4 & -1 & 1 \\ 8 & -3 & -2 \\ -5 & 2 & -3 \end{vmatrix}$ (c) $\begin{vmatrix} 3 & -2 & 1 \\ -4 & 8 & -9 \\ 2 & -7 & 4 \end{vmatrix}$

3. Expand the following determinants and find the value(s) for x.

(a) $\begin{vmatrix} 2 & x \\ x-3 & x-2 \end{vmatrix} = 0$ (b) $\begin{vmatrix} x & x \\ 1 & x-1 \end{vmatrix} = \begin{vmatrix} 3 & 3 \\ 1 & 6 \end{vmatrix}$

(c) $\begin{vmatrix} 4 & x+1 & 2 \\ x & 5 & 3 \\ -1 & 0 & 1 \end{vmatrix} = 6$ (d) $\begin{vmatrix} x-1 & 0 & 0 \\ 4 & x+2 & 0 \\ 4 & 4 & x-3 \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix}$

(e) $\begin{vmatrix} 2x & 2 \\ -x & -1 \end{vmatrix} + \begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} = \begin{vmatrix} x & 1 \\ 0 & -2 \end{vmatrix}$ (f) $\begin{vmatrix} 1 & 2 \\ 2x & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}\begin{vmatrix} 3 & -2 \\ 1 & 2 \end{vmatrix}$

4. Let $\boldsymbol{A} = \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix}$.

Evaluate : (i) $\boldsymbol{A}^2$ (ii) $|\boldsymbol{A}|$ (iii) $|\boldsymbol{A}^2|$ (iv) $|\boldsymbol{A}|^2$ (v) $|\boldsymbol{A}^4|$

*5. Evaluate $\begin{vmatrix} 4 & 3 & 1 & 0 \\ -1 & 2 & -3 & 5 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & -3 & 5 \end{vmatrix}$.

Tutorial 1d (Inverse Matrices)

1. State which of the following matrices are singular.

(a) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

2. Find the value(s) of $\boldsymbol{k}$ such that the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 1 & \boldsymbol{k} & 0 \\ 1 & 2 & 3 \end{bmatrix}$ is a non-singular.

3. Find the inverse of each of the following matrices.

(a) $\begin{bmatrix} 1 & -3 \\ 5 & 8 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 6 \\ -1 & -3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} -5 & 2 \\ -1 & 1 \end{bmatrix}$

4. Given $\boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, find matrix $\boldsymbol{C}$ such that $\boldsymbol{AC} = \boldsymbol{BA}$.

*5. Let $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$. Find the matrix product $\boldsymbol{AB}$. Hence find the inverse $\boldsymbol{A}^{-1}$ of the matrix $\boldsymbol{A}$.

*6. Given that $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & -1 \\ -2 & -1 & -2 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 11 & -5 & -3 \\ -8 & 4 & 2 \\ -7 & 3 & 1 \end{bmatrix}$, find the matrix product $\boldsymbol{AB}$.

Hence determine the matrix inverse of $\boldsymbol{A}$.

*7. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible square matrices such that $\boldsymbol{AB} = 2\boldsymbol{I}$ where $\boldsymbol{I}$ is the identity matrix, find the matrix inverse of $\boldsymbol{A}$ in terms of the matrix $\boldsymbol{B}$.

*8. If $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are invertible square matrices such that $\boldsymbol{A} = \boldsymbol{BCB}^{-1}$, show that $\boldsymbol{C} = \boldsymbol{B}^{-1}\boldsymbol{AB}$.

Tutorial 1e (Solution of System of Equations)

1. Solve the following systems for their unknowns using inverse matrix method:

(a) $2y = 5x - 4$
$y = 2x - 1$

(b) $3a + 2b = -2$
$5a - 3b = 3$

(c) $\frac{p}{2} + \frac{q}{3} - 3 = 0$
$\frac{p}{5} + \frac{q}{2} - \frac{23}{10} = 0$

(d) $2x^2 - 3y^2 = 5$
$4x^2 + y^2 = 17$

2. Solve the following systems for their unknowns using Cramer's Rule:

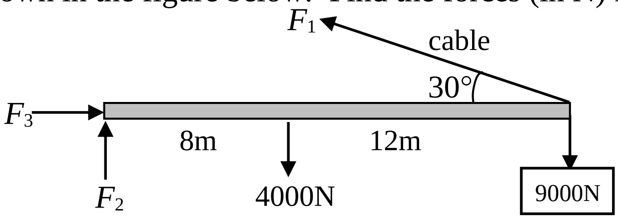
(a) $6u + 4v + 2w = 5$
$4u + 6v - w = -2$
$3u - 5v + 4w = 8$

(b) $x - 3y - 3z = 19$
$3x - 5y - 2z = 14$
$5x - 8y - z = 7.5$

(c) $\frac{1}{x} - \frac{1}{y} + \frac{2}{z} = 6$
$\frac{1}{x} + \frac{2}{y} - \frac{1}{z} = -3$
$\frac{3}{x} + \frac{1}{y} + \frac{1}{z} = 4$

3. (**Electrical**) A two-mesh electrical circuit is analysed, producing the following equations:
$$11I_1 - 10I_2 = 30$$
$$-20I_1 + 21I_2 = -40$$
Use inverse matrix method to find the values of I_1 and I_2.

4. (**Mechanical**) A 20-m crane arm with a supporting cable and a 9000-N box suspended from its end has forces acting on it as shown in the figure below. Find the forces (in N) for the following equations:
$F_1 + 2.0F_2 = 26\,000$
$0.87F_1 - F_3 = 0$
$3.0F_1 - 4.0F_2 = 54\,000$



*5. (**Business**) The demand and supply functions for two related products (P_1:pens, P_2:paper) are given by the equations:
$Q_{d1} = 30 - P_1 + 4P_2$, $Q_{s1} = 3P_1 - 6$
$Q_{d2} = 36 + 3P_1 - 2P_2$, $Q_{s2} = 12P_2 - 3$
Find the equilibrium quantities P_1 and P_2 (demand, Q_d =supply , Q_s).

6. (**Business**) The equilibrium condition for each product is that $Q_d = Q_s$. If the equilibrium condition in : (i) the goods market is given by the equation $Y = C + I$, where $C = 237.8 + 0.2Y$ and $I = 10 - 0.4r$.
(ii) the money market is given by the equation $M_d = M_s$, where $M_d = 100 + 0.1Y - 0.3r$ and $M_s = 129.225$.

(a) Write the equilibrium equations for each market in the form $aY + br = c$, where a, b and c are constants.

(b) Solve for the equilibrium levels of income (Y) and interest rate (r), for which the product and money markets are simultaneously in equilibrium.

7. (**Chemical**) To get 1 litre of a 20% saline solution, x ml of an 18% saline solution and y ml of a 25% saline solution are mixed together. This gives rise to the following system of equations:

$$x + y = 1000$$
$$0.18x + 0.25y = 0.2(1000)$$

Solve for the volumes, x and y.

8. (**Civil**) To test the elastic deformation of a beam, weights w_1, w_2 and w_3 are applied to it and the total deflection is measured. The following equations are obtained. Solve for w_1, w_2 and w_3.

$$0.01w_1 + 0.02w_2 + 0.04w_3 = 2.0$$
$$0.02w_1 + 0.01w_2 + 0.02w_3 = 2.5$$
$$0.04w_1 + 0.02w_2 + 0.01w_3 = 3.0$$

9. (**Life Sciences**) Equations connecting the lens system in a position transducer are:

$$\frac{4}{u_1} + \frac{6}{v_1} + \frac{9}{v_2} = 6$$
$$\frac{15}{u_1} + \frac{11}{v_1} + \frac{2}{v_2} = 8\frac{1}{12}$$

If $v_1 = v_2$, find the values of u_1, v_1 and v_2.

*10. (**Chemical**) 100 kg of a new alloy is to be made by combining x kg of alloy X, y kg of alloy Y and z kg of alloy Z. The compositions of the three alloys are as follows: alloy X is 60% copper, 30% lead and 10% manganese; alloy Y is 50% copper and 50% lead; alloy Z is 50% copper, 30% lead and 20% manganese. The new alloy will be 54.4% copper, 37.2% lead and 8.4% manganese.

(a) We obtain the equation $0.6x + 0.5y + 0.5z = 54.4$ for copper.
Obtain two other equations in terms of x, y and z.

(b) Calculate how much of alloy Z is required.

*11. Given that $A = \begin{bmatrix} 5 & -14 & 2 \\ -10 & -5 & -10 \\ 10 & 2 & -11 \end{bmatrix}$, find the matrix product AA^{T}.

Hence deduce the matrix inverse of A and use the result to solve the system of linear equations. (Hint : re-arrange equation first)

$$5x - 14y + 2z = 1$$
$$10x + 5y + 10z = 1$$
$$10x + 2y - 11z = 0$$

Problem-solving Assignment

The goal of this series of problem-solving assignments is to develop problem-solving skills, not just to test your ability to get the answer. It's ok to try hard and not succeed at first (only your effort is evaluated), but you must try.

Question

At the beginning of 2017, a customer invested a total of \$100,000 in four investment plans : A, B, C and D. At the end of 2017, the amount of profit he made from plans A, B and C are 2%, 3% and 6% respectively, giving a total profit of \$4000. He made a loss of 6% in plan D , which is \$900. The amount of money invested in plan B is twice the amount of money invested in plan A.
Find the amount invested in each plan at the beginning of the year.

1. Understand the problem • Identify the unknown that you are asked to find. • State the given conditions and quantities.	
2. Devise a plan • Break down the problem into smaller parts. • The following are some strategies that may be useful: o Write an equation that describes the relationship between the unknown and given quantities for each given piece of information. • Formulate a system of equations with three unknowns. • Identify which are the relevant techniques (Cramer's Rule, substitution or elimination) that can be applied.	
3. Implement the plan • Carry out the plan, showing each step clearly.	
4. Look back Substitute your answer back into the problem and check if it satisfies the given conditions.	

ANSWERS

Eg 1 : (a) 2 x 3 order (b) $a_{12} = 2$, $a_{21} = 4$

Eg 2 : *a = 1* , *b = ½* , *c = 0*

Eg 4 : *w = 0* , *x = 2* , *y = –9* , *z = 2*

Eg 5 : (a) $\begin{bmatrix} 5 & 8 \\ 11 & 14 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$

Eg 6 :

Week	Payroll	Marketing	Miscellaneous
1	22000	11000	1200
2	22000	6000	1100
3	22000	9000	800
4	22000	10000	800

Eg 8 : *x = 6* , *y = 4* *Eg 9* : Yes; 3 x 4 *Eg 10* : Not Conformable.

Eg 11 : (a) [84] (b) $\begin{bmatrix} 6 & 12 & 18 \\ 12 & 24 & 36 \\ 18 & 36 & 54 \end{bmatrix}$ (c) $\begin{bmatrix} 29 & 7 \\ 5 & 8 \end{bmatrix}$ (d) Not Conformable.

Eg 12 : (a) $\begin{bmatrix} 800 & 500 & 600 \\ 900 & 700 & 800 \\ 600 & 800 & 900 \end{bmatrix}$ and $\begin{bmatrix} 0.20 & 0.40 & 0.30 \\ 0.40 & 0.50 & 0.60 \\ 0.50 & 0.30 & 0.40 \end{bmatrix}$

(b)(i)

Product	Total
Hamburgers	\$540
Potato Chips	\$1190
Drinks	\$900

(b)(ii)

Outlet	Total
A	\$820
B	\$790
C	\$1020

Eg 13 : (a) 26 (b) 41

Eg 14 : (a) Minor = 4
Cofactor = 4
(b) Minor of – 1 = – 23
Cofactor – 1 = – 23
(c) Minor of 5 = – 1
Cofactor of 5= 1

Eg 15 : 86

Eg 16 : $x = -1$

Eg 17 : $A^{-1} = \dfrac{1}{13}\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}$ *Eg 18* : $C = \begin{bmatrix} -1 & -2 \\ 4 & 5 \end{bmatrix}$ *Eg 19* : Non-singular

Eg 20 : *k* = 8/5 *Eg 21* : *x = 5* , *y = – 1* *Eg 22* : *x=1, y=½, z=* –2

Eg 23 : *a=1, b=2, c=* –5 , *d = –* 6

True/False Questions

1. False 2. False 3. False 4. False
5. True 6. False 7. True 8. False

Multiple Choice Questions

1. c	2.d	3. d	4.a	5.b	6.a	7.a	
8.b	9.d	10.c	11.d	12.c	13.b	14. d	15. d

Tutorial 1a

1.(a) 1×3 order 1.(b) 2×3 order 1.(c) 3×2 order 1.(d) 4×1 order

2. $a_{23}=-2$, $a_{22}=5, a_{32}=7$

Tutorial 1b

1(a) $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ 1(b) $\begin{bmatrix}1&4\\2&5\\3&6\end{bmatrix}$ 1(c) $\begin{bmatrix}1&4&7\\2&5&8\\3&6&9\end{bmatrix}$ 2. $k = 2$ 3. $a = 2, b = 2$

4(a) $\begin{bmatrix}0&3&5\\3&3&2\end{bmatrix}$ 4(b) $\begin{bmatrix}0&5\\-1&2\\7&-2\end{bmatrix}$ 4(c) $\begin{bmatrix}-1&7&12\\7&8&3\end{bmatrix}$ 5. $A=\begin{bmatrix}-4&7&9\\-5&-5&8\end{bmatrix}$

6. $a = 1, b = 1$ 7. 3×4 order

8(a) $\begin{bmatrix}5&4\\27&26\end{bmatrix}$ 8(b) $\begin{bmatrix}5&4&8\\27&26&42\end{bmatrix}$ 8(c) Not conformable

8(d) $[96]$ 8(e) $\begin{bmatrix}6&18&12&24\\10&30&20&40\\2&6&4&8\\14&42&28&56\end{bmatrix}$ 8(f) $\begin{bmatrix}11&-22&29\\9&-27&32\\13&-17&26\end{bmatrix}$

8(g) [54] 8(h) Not conformable

9. $AB=\begin{bmatrix}-1&-10\\1&0\end{bmatrix}$, $BA=\begin{bmatrix}-4&-2\\11&3\end{bmatrix}$ 10. $p = 1, q = -3, r = -5$

11. $k = 1$ 12. C1=260, C2=230, C3=270 tonnes

13(a) 18800 Type I chips, 7600 Type II chips, 7500 Type III chips, 13(b) $1582000

Tutorial 1c

1. (a), (b) are matrices ; (d) is determinant

2(a) 11 2(b) 19 2(c) –77

3(a) $x = 1$ or 4 3(b) $x = -3$ or 5 3(c) $x = -7$ or 3

3(d) $x = 1$ or -2 or 3 3(e) $x = -1$ 3(f) $x = 9/4$

4(i) $\begin{pmatrix}1&-2\\6&-3\end{pmatrix}$ 4(ii) 3 4(iii) 9 4(iv) 9 4(v) 81

5. 2

Tutorial 1d

1. (a), (c)

2. $k \neq 0$

3(a) $\frac{1}{23}\begin{bmatrix} 8 & 3 \\ -5 & 1 \end{bmatrix}$

3(b) no inverse

3(c) $\begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix}$

3(d) $-\frac{1}{3}\begin{bmatrix} 1 & -2 \\ 1 & -5 \end{bmatrix}$

4. $\begin{bmatrix} -2 & -2 \\ 6 & 5 \end{bmatrix}$

5. $AB = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}, A^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$

6. $AB = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, A^{-1} = \frac{1}{2}\begin{bmatrix} 11 & -5 & -3 \\ -8 & 4 & 2 \\ -7 & 3 & 1 \end{bmatrix}$

7. $A^{-1} = \frac{1}{2}B$

Tutorial 1e

1.(a) $x = 2,\ y = 3$

1.(b) $a = 0,\ b = -1$

1.(c) $p = 4,\ q = 3$

1.(d) $x = \pm 2,\ y = \pm 1$

2.(a) $u = -\frac{1}{2},\ v = \frac{1}{2},\ w = 3$

2.(b) $x = 2.5,\ y = 1.5,\ z = -7$

2.(c) $x = 1,\ y = -1,\ z = \frac{1}{2}$

3. $I_1 = 7.419,\ I_2 = 5.161$

4. $F_1 = 21\,200,\ F_2 = 2\,400,\ F_3 = 18\,444$

5. $P_1 = 15,\ P_2 = 6;\quad Q_{d1} = 39,\ Q_{d2} = 69$

6.(a) $0.8Y + 0.4r = 247.8,\ 0.1Y - 0.3r = 29.225$

6.(b) $Y = 307.25,\ r = 5$

7. $x = 714.29,\ y = 285.71$

8. $w_1 = 100,\ w_2 = -83\frac{1}{3},\ w_3 = 66\frac{2}{3}$

9. $u_1 = 4,\ v_1 = v_2 = 3$

10.(a) $0.3x + 0.5y + 0.3z = 37.2,\ 0.1x + 0.2z = 8.4$

10.(b) $z = 20$ kg

11. $225I,\quad A^{-1} = \frac{1}{225}\begin{bmatrix} 5 & -10 & 10 \\ -14 & -5 & 2 \\ 2 & -10 & -11 \end{bmatrix},\ x = \frac{1}{15}, y = -\frac{1}{25}, z = \frac{4}{75}$

Problem-solving Assignment: A= \$11000 , B = \$ 22000 , C= \$ 52000 , D = \$ 15000