Chapter 1: Partial Derivatives

Objectives:

1. Describe functions of several variables.
2. Define partial derivatives.
3. Use partial derivatives to approximate rate of change.
4. Apply partial derivatives to approximate errors and percentage errors.

1.1 Revision on Differentiation

Differentiation allows us to find the gradient or rate of change of a function.

1.1.1 Let's start with three basic differentiation formulae, together with chain rule.

$$\frac{d}{dx}x^n = nx^{n-1}$$
$$\frac{d}{dx}\ln x = \frac{1}{x}$$
$$\frac{d}{dx}e^x = e^x$$

Let $u = u(x)$ and $y = y(u)$

Chain Rule: $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx}$

Example 1:

(a) $\dfrac{d}{dx}\left(5x^2 + 3x^{-\frac{1}{2}}\right) =$

(b) $\dfrac{d}{dx}\left[\left(x^3 + 7\right)^{10}\right] =$

(c) $\dfrac{d}{dx}\left(2\ln x\right) =$

(d) $\dfrac{d}{dx}\left(\dfrac{1}{e^x}\right) =$

(e) $\dfrac{d}{dx}\left(\dfrac{4}{x} - 6 + 5e^{3-4x}\right) =$

1.1.2 Differentiation of trigonometric functions can also be carried out using formulae such as those below.

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$
$$\frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\sec x = \sec x \tan x$$
$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

***Example 2*:**

(a) $\frac{d}{dx}\left[7\sin(5x)\right] =$

(b) $\frac{d}{dx}\left[12\sec(3x)\right] =$

(c) $\frac{d}{dx}\left[5\ln(\cos x)\right] =$

1.1.3 Product rule and quotient rule of differentiation are used to differentiate products and quotients of functions respectively.

Product Rule of Differentiation:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Quotient Rule of Differentiation:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

***Example 3*:**

(a) $\frac{d}{dx}(x\sin x) =$

(b) $\frac{d}{dx}\left(\frac{e^x}{1+x}\right) =$

1.2 Partial Derivatives

So far, in your previous study of calculus, you have been introduced to the notion of a derivative which measure the rate of change of $f(x)$ with respect to one independent variable x. In this chapter, we will extend it to finding the derivatives of function of **two or more independent variables.**

The following are examples of functions with two or more independent variables.

Volume of a cylinder, $V = \pi r^2 h$

Ideal gas law, $PV = \mathrm{n}RT$

Ohms law, $V = IR$

Cobb-Douglas production function, $Q = AL^{\alpha}K^{\beta}$

Let $f(x, y)$ be a function of the two variables x and y. To find the rate of change of $f(x, y)$ w.r.t. both x and y, the technique called **partial differentiation** will be involved. Since $f(x, y)$ is dependent on two variables, we have to, first of all, determine how $f(x, y)$ changes with x keeping y constant and how $f(x, y)$ changes with y keeping x constant. Summing up the effects, the rate of change of $f(x, y)$ w.r.t. both x and y can be evaluated.

Notation for partial derivatives:

The *partial derivative of* $f(x, y)$ *with respect to* x is written as $\frac{\partial f}{\partial x}$.

$\frac{\partial f}{\partial x}$ is the derivative of $f(x, y)$, where y is treated as the constant and $f(x, y)$ is treated as a function of x alone.

The *partial derivative of* $f(x, y)$ *with respect to* y is written as $\frac{\partial f}{\partial y}$.

$\frac{\partial f}{\partial y}$ is the derivative of $f(x, y)$, where x is treated as the constant and $f(x, y)$ is treated as a function of y alone.

Note that $\partial \neq \mathrm{d}$

***Example 4*:** Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of the following.

(a) $f(x, y) = (5x^3)y^2$

[Ans: $15x^2y^2$, $10x^3y$]

(b) $f(x,y) = (4x+3y-5)^8$

[Ans: $32(4x+3y-5)^7$, $24(4x+3y-5)^7$]

(c) $f(x,y) = \sin\left(\dfrac{x}{1+y}\right)$

$$\left[\text{Ans:} \frac{1}{1+y}\cos\left(\frac{x}{1+y}\right), -\frac{x}{(1+y)^2}\cos\left(\frac{x}{1+y}\right)\right]$$

The value of $\dfrac{\partial f}{\partial x}$ at a point $(x, y) = (a, b)$ is denoted by $\left.\dfrac{\partial f}{\partial x}\right|_{\substack{x=a\\y=b}}$ or $f_x(a,b)$.

The value of $\dfrac{\partial f}{\partial y}$ at point $(x, y) = (a, b)$ is denoted by $\left.\dfrac{\partial f}{\partial y}\right|_{\substack{x=a\\y=b}}$ or $f_y(a,b)$.

***Example 5*:** Given that $f(a,b) = \dfrac{a-b}{a+b}$. Evaluate $\dfrac{\partial f}{\partial a}$ and $\dfrac{\partial f}{\partial b}$ when $a = 1$ and $b = 1$.

$$\left[\text{Ans: } \frac{1}{2}, -\frac{1}{2}\right]$$

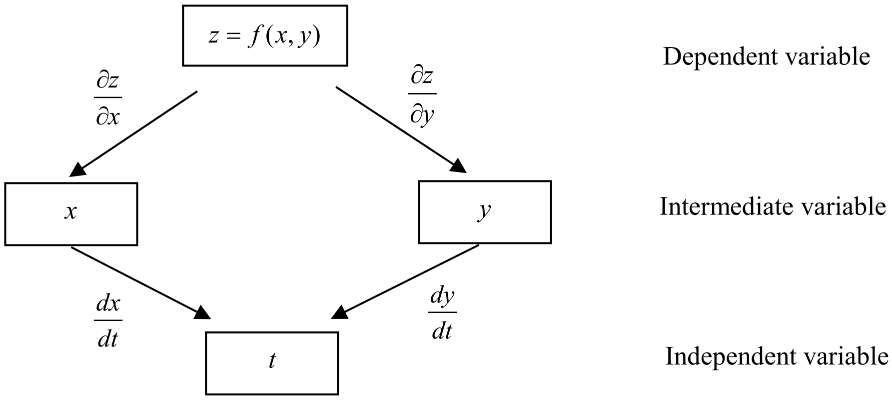
1.3 The Chain Rule for Functions of Two Variables

Sometimes, to differentiate $z = f(x, y)$ with respect to t, where $x = g(t)$ and $y = h(t)$, can generate functions whose formulas are too complicated for convenient substitution or for which formulas are not readily available.

To find derivative under circumstances like this, we will use Chain Rule. The Chain Rule for functions of two variables is as follows:

$$\boxed{\frac{dz}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}}$$

A convenient way to remember this rule is to use a diagram:

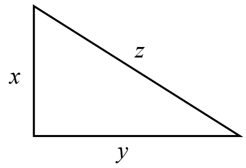


Example 6: If $z = x^2 y + 3xy^4$, where $x = \sin 2t$ and $y = \cos 2t$, find $\dfrac{dz}{dt}$ when $t = 0$.

$$\left[\text{Ans: } \left.\frac{dz}{dt}\right|_{t=0} = 6\right]$$

Example 7: In the right-angle triangle shown below, x is increasing at a rate of 3 cm/s while y is decreasing at a rate of 4 cm/s. Calculate the rate at which z is changing when $x = 3$ cm and $y = 5$ cm.

[Ans: −1.88cm/s]



Example 8: The pressure P (in kilopascals, volume V (in litres) and temperature T (in kelvins) of a mole of ideal gas are related by the equation $PV = 8.31T$. Find the rate at which the pressure is changing when the temperature is 300 K and increasing at a rate of 0.1 K/s. Given that the volume is 100 L and increasing at a rate of 0.2 L/s.

[Ans: Pressure is decreasing at a rate of 0.042 kPa/s]

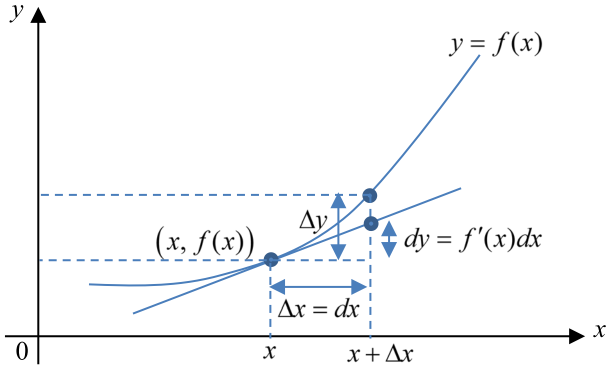
1.4 An Application of Partial Derivatives – Error analysis

1.4.1 Small change/error

If y is a function of one variable x, that is, $y = f(x)$, then we have $\frac{dy}{dx} = f'(x) \approx \frac{\Delta y}{\Delta x}$, where Δx is a small change in x and Δy is the corresponding small change in y.

When Δx is relatively small, we use the differential dy as an approximation to the actual increment Δy.
Making Δy the subject, we get $\Delta y \approx f'(x)\Delta x$.

The relation between Δy and dy is shown below:



Similarly, if u is a function of more than one variable, say x and y, or $u = f(x, y)$ and there are small changes/errors Δx and Δy in x and y respectively, then the corresponding small change/error in u is given by the expression:

$$\boxed{\Delta u \approx \frac{\partial u}{\partial x} \cdot \Delta x + \frac{\partial u}{\partial y} \cdot \Delta y}$$

***Example 9*:** The radius, r, and height, h, of a right circular cylinder were measured to be 3.5 cm and 7.8 cm respectively, with possible error of 0.1 cm in each measurement. Use partial differentiation to estimate the possible error, correct to 2 decimal places, in the computation of the surface area S. [Note: Surface area, S, of the cylinder is given by $S = 2\pi r^2 + 2\pi rh$].

Solution: [Ans: 11.50 cm^2]

1.4.2 Relative change/error

To find relative change/error of u , we divide by u throughout and change to percentage form.

That is:

$$\Delta u \approx \frac{\partial u}{\partial x}\cdot\Delta x+\frac{\partial u}{\partial y}\cdot\Delta y$$

$$\Rightarrow\ \frac{\Delta u}{u}\approx\frac{\partial u}{\partial x}\cdot\frac{\Delta x}{u}+\frac{\partial u}{\partial y}\cdot\frac{\Delta y}{u}$$

$$\Rightarrow\ \frac{\Delta u}{u}(\times 100\%)\approx\frac{\partial u}{\partial x}\cdot\frac{\Delta x}{u}(\times 100\%)+\frac{\partial u}{\partial y}\cdot\frac{\Delta y}{u}(\times 100\%)$$

Example 10: Two variables x and y are related by the equation $z = x\,y^{1.4}$. Use partial differentiation to approximate the percentage change in z when x is increased by 2.3% and y is decreased by 0.8%.

Solution: [Ans: 1.18 %]

Tutorial 1

Section A

1. Differentiate with respect to x and simplify your answers wherever possible.

(a) $5x^4 + \dfrac{1}{2x^2}$ (b) $\sqrt{x} + 8$

(c) $7\sin 5x$ (d) $2\cos 3x$

(e) $\ln(7x^3 + 5)$ (f) $2\ln(e^x + 1)$

(g) e^{3x} (h) $2e^{\cos 2x}$

(i) $(x^3 + 7)^{10}$ (j) $\sqrt{5x - 7}$

2. Differentiate with respect to x and simplify your answers wherever possible.

(a) $x^4 \sin 2x$ (b) $e^x \ln x$

(c) $x^3 \ln x$ (d) $e^x \sin 2x$

(e) $\dfrac{2x+1}{x-3}$ (f) $\dfrac{2x}{x+3}$

(g) $\dfrac{e^x}{x}$ (h) $\dfrac{3x}{(1+\sin x)}$

Section B

1. The volume V of a right circular cylinder is given by the formula $V = \pi r^2 h$, where r is the radius and h is the height.
   (a) Find the formula for the instantaneous rate of change of V with respect to r if r changes and h remains constant.
   (b) Find the formula for the instantaneous rate of change of V with respect to h if h changes and r remains constant.
   (c) Suppose that h has a constant value of 4 cm. but r varies. Find the rate of change of V with respect to r at the point where $r = 6$ cm.
   (d) Suppose that r has a constant value of 8 cm but h varies. Find the instantaneous rate of change of V with respect to h at the point where $h = 10$cm.

2. Find the first partial derivatives of the function.

   (a) $f(x, y) = 5x^2 - 3y^2 + 10$ (b) $f(x, y) = 4x^3 + y^3 - 5x^2 y$

   (c) $z(x, y) = \sin x + \cos y$ (d) $z(x, y) = x^2\sqrt{1 - y^2}$

   (e) $z = \ln(xy)$ (f) $u = \ln\sqrt{x^2 - y^2}$

   (g) $z(x, y) = x^2 \sin y$ (h) $z = \left(1 + x^2 y\right)e^{3y}$

   (i) $f(x, y) = x^2 \sin(xy) - 3y^3$ (j) $f(r, s) = r \cdot \ln\left(r^2 + s^2\right)$

3. Find the indicated partial derivatives.

   (a) $f(x, y) = \sqrt{x^2 + y^2}$, $f_x(3, 4)$ (b) $f(x, y) = \dfrac{x}{y+1}$, $f_y(3, 2)$

4. Determine the slope of the tangent line in the $x-$ and $y-$directions to the surface $z = z(x, y)$ at the stated point P.

   (a) $z(x, y) = x^3 + 2xy + 2y^2$, $P(1, 2)$ (b) $z(x, y) = x\sin(xy) + 3$, $P(1, \frac{\pi}{2})$

5. The voltage V in a circuit that satisfy the law $V = IR$ is slowly dropping as battery wears out. At the same time, the resistance R is increasing as the resistor heats up. Find how the current is changing at the instant when $R = 600$ ohm, $I = 0.04$ amp, $\frac{dR}{dt} = 0.5$ ohm/sec and $\frac{dV}{dt} = -0.01$ volt/sec.

6. The radius r of a closed cylindrical can decreased at a rate of 0.02cm/s when $r = 5$cm and the height $h = 15$cm. Find the rate of change of h such that the surface area of the can remains unchanged.

7. The total surface area S of a cone of base radius r and perpendicular height h is given by $S = \pi r^2 + \pi r\sqrt{r^2 + h^2}$. If r and h are each increasing at the rate of 0.25 cm/sec, find the rate at which S is increasing at the instant when $r = 3$ cm and $h = 4$cm.

8. The power P dissipated in a resistor is given by $P = \frac{E^2}{R}$ where E is voltage and R is resistance. If $E = 200$ volts and $R = 8$ ohms, use partial differentiation to approximate the change in power resulting from a drop of 5 volts in E and an increase of 0.2 ohm in R.

9. The diameter and height of a right circular cylinder are found by measurement to be 8 cm and 12.5 cm respectively, with possible error of +0.05 cm in each measurement. Use partial differentiation to find the possible approximate error in the computed volume.

10. The radius r and height h of a right circular cylinder are measured with possible errors of 1% and 2% respectively. Use partial differentiation, approximate the possible percentage error in measuring the volume. [Volume V, of a cylinder is given by $V = \pi r^2 h$]

11. The radius r and height h of a right circular cylinder are measured with possible errors of 4% and 2% respectively. Use partial differentiation to approximate the possible percentage error in measuring the volume.

12. The 2 shorter sides of a right triangle are measured and found to be 6 cm and 8 cm with a possible error of +0.1 cm in each measurement. Find the approximate percentage error in the computed value of the hypotenuse, using partial differentiation.

13. Electrical power P is given by $P = \frac{E^2}{R}$, where E is voltage and R is resistance. Approximate the percent error in calculating power if the percentage errors in measuring E and R are 2% and 3%, respectively.

Miscellaneous Exercises

1. Find the first partial derivatives of the function.
   (a) $f(x, y, z) = 2x^3y + z^2$ (b) $f(x, y, z) = xyz + xy + z$

2. Two straight roads intersect at right angles. Car A, moving on one of the roads, approaches the intersection at 55 km/h and car B, moving on the other road, approaches intersection at 70km/h. At what rate is the distance between the cars decreasing when A is 3 km from the intersection and B is 4 km from the intersection?

3. The total volume of a cone is given by $V = \frac{\pi}{3}h^3\tan^2(\theta)$, where V (m^3) is the volume of a cone, θ (rad) is the semi-vertical angle and h (m) is height. If $\frac{dh}{dt} = 0.001$ m/s and $\frac{d\theta}{dt} = 0.002$ rad/s, find the rate of change of V when $\theta = \frac{\pi}{4}$ and $h = 0.20$m.

4. The time rate Q of flow of fluid through a cylindrical tube (such as a windpipe) with radius r and length l is given by $Q = \frac{\pi p r^4}{8l\eta}$ where η is the viscosity of the fluid and p is the difference in the pressure at the two ends of the tube. Suppose the length of the tube remains constant, while the radius increases at the rate of $\frac{1}{10}$ m/s and the pressure decreases at the rate of $\frac{1}{5}$ N/m^3/s. Find the rate of change of Q with respect to time t.

5. A container with molten metal has a mass of m (kg). A force F (N) acting on an angle θ (rad) measured from the vertical is required to tilt the container. Given that $F = \frac{2.5m}{1 + 0.8\tan\theta}$, find $\frac{\partial F}{\partial m}$ and $\frac{\partial F}{\partial \theta}$. If θ is increasing at $\frac{\pi}{90}$ rad/s and m is decreasing at 1.2 kg/s, find the rate of change of F when $\theta = \frac{\pi}{4}$ rad and $m = 100$kg.

6. For a real gas, van der Waal's equation states that
$$\left(P + \frac{n^2 a}{V^2}\right)(V - nb) = nRT,$$
where P is the pressure of the gas, V is the volume of the gas, T is the temperature of the gas (in degree Kelvin), n is the number of moles of gas, R is the universal gas constant and a and b are constants. Compute $\frac{\partial P}{\partial V}$ and $\frac{\partial T}{\partial P}$.

7. The inductance L (microhenrys) of a certain wire in free space is

$$L = 0.00021\left(\ln\frac{2h}{r} - 0.75\right)$$

where h is the length (mm) of the wire and r (mm) is the radius of a circular cross section. Use partial differentiation to approximate L when $r = 2 \pm \frac{1}{16}$ mm and $h = 100 \pm \frac{1}{100}$ mm.

8. The deflection y at the centre of a circular plate is given by $y = \frac{kwd^4}{t^3}$ where w = total load, d = diameter of plate, t = thickness and k is a constant. Use partial differentiation to calculate the approximate percentage change in y if w is increased by 3%, d is decreased by 2.5% and t is increased by 4%.

Multiple Choice Questions

1. Given that $f(x, y) = 0$, then $\frac{dy}{dx}$ equals to ___________

(a) $\frac{f_x}{f_y}$ (b) $\frac{f_y}{f_x}$

(c) $-\frac{f_x}{f_y}$ (d) $-\frac{f_y}{f_x}$

2. If partial differentiation is performed on a function, then this function must have
   (a) only one independent variable.
   (b) more than one dependent variable.
   (c) two or more independent variables.
   (d) equal number of dependent and independent variables.

Answers

Section A

1. (a) $20x^3 - \frac{1}{x^3}$ (b) $\frac{1}{2\sqrt{x}}$ (c) $35\cos 5x$ (d) $-6\sin 3x$

   (e) $\frac{21x^2}{7x^3+5}$ (f) $\frac{2e^x}{e^x+1}$ (g) $3e^{3x}$ (h) $-4e^{\cos 2x}\sin 2x$

   (i) $30x^2(x^3+7)^9$ (j) $\frac{5}{2\sqrt{5x-7}}$

2. (a) $2x^3(x\cos 2x + 2\sin 2x)$ (b) $e^x\left(\frac{1}{x} + \ln x\right)$ (c) $x^2(1 + 3\ln x)$

   (d) $e^x(\sin 2x + 2\cos 2x)$ (e) $-\frac{7}{(x-3)^2}$ (f) $\frac{6}{(x+3)^2}$

(g) $\dfrac{e^x(x-1)}{x^2}$ (h) $\dfrac{3(1+\sin x - x\cos x)}{(1+\sin x)^2}$

Section B

1. (a) $2\pi rh$ (b) πr^2 (c) 48π (d) 64π
2. (a) $f_x(x,y)=10x$; $f_y(x,y)=-6y$
   (b) $f_x(x,y)=12x^2-10xy$; $f_y(x,y)=3y^2-5x^2$
   (c) $z_x(x,y)=\cos x$; $z_y(x,y)=-\sin y$
   (d) $z_x(x,y)=2x\sqrt{1-y^2}$; $z_y(x,y)=\dfrac{-x^2y}{\sqrt{1-y^2}}$
   (e) $\dfrac{\partial z}{\partial x}=\dfrac{1}{x}$; $\dfrac{\partial z}{\partial y}=\dfrac{1}{y}$
   (f) $\dfrac{\partial u}{\partial x}=\dfrac{x}{x^2-y^2}$; $\dfrac{\partial u}{\partial y}=\dfrac{-y}{x^2-y^2}$ *or* $\dfrac{y}{y^2-x^2}$
   (g) $z_x(x,y)=2x\sin y$; $z_y(x,y)=x^2\cos y$
   (h) $\dfrac{\partial z}{\partial x}=2xye^{3y}$; $\dfrac{\partial z}{\partial y}=x^2e^{3y}+3e^{3y}\left(1+x^2y\right)$
   (i) $f_x(x,y)=2x\sin(xy)+x^2y\cos(xy)$; $f_y(x,y)=x^3\cos(xy)-9y^2$
   (j) $f_r(r,s)=\ln\left(r^2+s^2\right)+\dfrac{2r^2}{r^2+s^2}$; $f_s(r,s)=\dfrac{2rs}{r^2+s^2}$
3. (a) $\dfrac{3}{5}$ (b) $-\dfrac{1}{3}$
4. (a) $z_x(1,2)=7$; $z_y(1,2)=10$ (b) $z_x(1,\frac{\pi}{2})=1$; $z_y(1,\frac{\pi}{2})=0$
5. -0.00005 amp/sec 6. 0.1 cm/sec 7. 11.93cm²/sec
8. Power decreases by 375 watts 9. 3.3π cm^3 10. 4%
11. 10% 12. 1.4% 13. 1%

Miscellaneous

1. (a) $f_x(x,y,z)=6x^2y$; $f_y(x,y,z)=2x^3$; $f_z(x,y,z)=2z$
   (b) $f_x(x,y,z)=yz+y$; $f_y(x,y,z)=xz+x$; $f_z(x,y,z)=xy+1$
2. 89 km/h 3. 0.00019m³ / sec 4. $\dfrac{\pi r^3}{40l\eta}(2p-r)$ m^3/s
5. $\dfrac{2.5}{1+0.8\tan\theta}$; $-\dfrac{2m\sec^2\theta}{(1+0.8\tan\theta)^2}$; -5.98N/sec
6. $-\dfrac{nRT}{(V-nb)^2}+\dfrac{2n^2a}{V^3}$ N/m^5 ; $\dfrac{1}{nR}(V-nb)$ Kelvin m^2/N
7. $L\approx 8.096\times10^{-4}\pm6.6\times10^{-6}$ microhenrys 8. –19%

MCQ

1. (c) 2. (c)