1. SEQUENCES AND SERIES

1.1 SEQUENCES

A sequence is a set of numbers which are written in some particular order. For example, take the numbers

$$1,\ 3,\ 5,\ 7,\ \ldots$$

Here, we seem to have a rule. We start with the number 1 and then each successive number is obtained by adding 2 to give the next number.

$$1,\ 4,\ 9,\ 16,\ 25,\ \ldots$$

This is the sequence of square numbers.
Each number of the sequence is called a term.

A sequence is a set of numbers written in a particular order and each number can be obtained from the previous number according to some rule.

1.2 SERIES

A series is something we obtain from a sequence by adding all the terms together.

For example, let us consider the sequence of numbers

$$2,\ 4,\ 6,\ 8,\ 10,\ \ldots$$

By adding up all the terms in the above sequence we obtain the series

$$2 + 4 + 6 + 8 + 10 + \ldots$$

The sum of the terms of a sequence is called a series.

1.3 ARITHMETIC PROGRESSIONS (AP)

1.3.1 GENERAL TERM OF AN AP

Consider the sequence

$$3, \quad 5, \quad 7, \quad 9, \quad 11, \ldots$$

The first term of the sequence is 3 and each subsequent term is obtained by adding a constant 2, to the previous term. We say the terms progress arithmetically and a sequence with this property is called Arithmetic Progression (A.P) or Arithmetic Sequence.

In the general case, if the first term is denoted by a and the constant (called the common difference) is d, the general arithmetic progression would be:

$$a, \quad (a+d), \quad (a+2d), \quad (a+3d), \quad [a+(n-1)d], \ldots$$

Where $a+(n-1)d$ is the expression for the nth term of the sequence. Often we denote nth term of the sequence by T_n and so in this case we would have

$$T_n = a+(n-1)d$$

> An arithmetic progression, or AP, is a sequence where each new term after the first is obtained by adding a constant *d*, called the common difference, to the preceding term. If the first term of the sequence is *a* then the arithmetic progression is
>
> $$a, \quad (a+d), \quad (a+2d), \quad (a+3d), \ldots$$
>
> where the nth term is given by $T_n = a+(n-1)d$.

EXAMPLE 1 Find the fifth term (T_5) of the following arithmetic sequences.

(a) $2, 5, 8, \ldots$

(b) first term is 37 and the common difference is -4.

(Ans: (a) 14, (b) 21)

EXAMPLE 2 The fifth term of an arithmetic progression is 10 while the 15th term is 40. Write down the first 5 terms of the AP. (Ans: –2, 1, 4, 7, 10)

1.3.2 SUM OF AN ARITHMETIC PROGRESSION

Given the AP: $a, \quad (a+d), \quad (a+2d) \ , \ldots, \ \left[a+(n-2)d\right], \ \left[a+(n-1)d\right]$

Let S_n denote the sum of the first *n* terms of an AP:

$$S_n = a + (a+d) + \ldots + \left[a+(n-2)d\right] + \left[a+(n-1)d\right] \cdots(1)$$

Writing the AP in reverse order, we have

$$S_n = \left[a+(n-1)d\right] + \left[a+(n-2)d\right] + \cdots + (a+d) + a \cdots(2)$$

$$(1)+(2): \quad 2S_n = \left[2a+(n-1)d\right] + \left[2a+(n-1)d\right] + \cdots + \left[2a+(n-1)d\right] + \left[2a+(n-1)d\right]$$

$$2S_n = n\left[2a+(n-1)d\right]$$

$$S_n = \frac{n}{2}\left[2a+(n-1)d\right]$$

We can also find an expression for the sum in terms of the *a*, *n* and the *nth* term, T_n:

$$S_n = \frac{n}{2}\left[a+a+(n-1)d\right] = \frac{n}{2}\left[a+T_n\right]$$

The sum of the terms of an arithmetic progression gives an arithmetic series. If the starting value is *a* and the common difference is *d* then the sum of the first *n* terms is

$$S_n = \frac{n}{2}\left[2a+(n-1)d\right] \quad \text{or} \quad \frac{n}{2}\left[a+T_n\right]$$

EXAMPLE 3

(a) Find the sum of the first 50 terms of the sequence $1, 3, 5, 7, 9, \ldots$.

(b) Find S_8 of the AP in which the first term is 37 and the common difference is -4.

(Ans: (a) 2500, (b) 184)

EXAMPLE 4

Find the sum of the series

$$1 + 3.5 + 6 + 8.5 + \ldots + 101$$

(Ans: 2091)

EXAMPLE 5
The sum of the first 8 terms of an arithmetic progression is 56 and the sum of the first 20 terms is 260. Find the first term and the common difference of the AP. (Ans: $a = 3.5$, $d = 1$)

1.4 GEOMETRIC PROGRESSIONS (GP)

1.4.1 GENERAL TERM OF A GP

Consider the sequence

$$1, \quad 2, \quad 4, \quad 8, \quad 16, \ldots$$

Each term of the above sequence can be obtained by multiplying the previous term by 2.

In the general case, if the first term is denoted by a and the ratio of any term to its preceding term is r (called the common ratio), the sequence can then be written as

$$a, \quad ar, \quad ar^2, \quad ar^3, \quad ar^4, \ldots$$

It follows that *nth* term T_n is

$$T_n = ar^{n-1}$$

Sequence such as the one above is known as Geometric Progression (GP) or Geometric Sequence.

> A geometric progression, or GP, is a sequence where each new term after the first is obtained by multiplying the preceding term by a constant *r*, called the common ratio. If the first term of the sequence is *a* then the geometric progression is
>
> $$a, \quad ar, \quad ar^2, \quad ar^3, \quad ar^3, \ldots$$
>
> where the nth term is given by $T_n = ar^{n-1}$.

EXAMPLE 6 Find the fifth term (T_5) of the following geometric sequences

(a) $2,\ -6,\ 18,\ldots$

(b) the first term is 27 and the common ratio is $\frac{2}{3}$.

(Ans: (a) 162, (b) $5\frac{1}{3}$)

EXAMPLE 7

How many terms are there in the geometric progression $2,\ 4,\ 8,\ \ldots,128$?

(Ans: $n = 7$)

1.4.2 SUM OF A GEOMETRIC PROGRESSION

Writing the sum of the first *n* terms of a GP as S_n, it follows that:

$$S_n = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1} \qquad \cdots(1)$$

$(1)\times r:$ $$rS_n = ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1} + ar^n \qquad \cdots(2)$$

$(1)-(2):$

$$S_n - rS_n = a - ar^n$$
$$S_n(1-r) = a(1-r^n)$$
$$S_n = \frac{a(1-r^n)}{(1-r)} \quad \text{or} \quad \frac{a(r^n-1)}{(r-1)} \qquad r \neq 1$$

The sum of the terms of a geometric progression gives a geometric series. If the starting value is *a* and the common ratio is *r* then the sum of the first *n* terms is

$$S_n = \frac{a(1-r^n)}{(1-r)} \text{ or } \frac{a(r^n-1)}{(r-1)} \qquad \text{provided } r \neq 1.$$

EXAMPLE 8

(a) Find the sum of the first 6 terms of the geometric series 2 + 6 + 18 + 54 +....

(b) Find S_5 of the geometric progression in which the first term is 8 and the common ratio is $-\frac{1}{2}$. (Ans: (a) 728 (b) 5.5)

EXAMPLE 9

In a GP, $T_3 = 32$ and $T_6 = 4$, find *a* and *r* and the sum of the first eight terms of the GP.

(Ans: $a = 128, r = \frac{1}{2}, S_8 = 255$)

1.4.3 SUM TO INFINITY OF A GP

Consider the infinite geometric series:

$$18 + 1.8 + 0.18 + 0.018 + \ldots$$

The first term is 18 and common ratio is $\frac{1}{10}$.

For this GP

$$S_2 = 19.8$$
$$S_4 = 19.998$$
$$S_6 = 19.999\ 98$$
$$S_8 = 19.999\ 999\ 8$$

Clearly the sum S_n approaches the value 20. By taking a sufficiently large value of *n*, we can make S_n as near to 20 as we wish. We say that S_n tends towards a limiting value of 20 as *n* approaches infinity and this is written as

$$S_n \to 20, \quad \text{as } n \to \infty.$$

The above sum (known as sum to infinity) can be obtained by a formula. We know that

$$S_n = \frac{a\left(1-r^n\right)}{\left(1-r\right)}.$$

Suppose that $-1 < r < 1$ and n is very large ($n \to \infty$) then S_n can be written as

$$S_\infty = \frac{a}{\left(1-r\right)} \quad \text{since } r^n \to 0 \text{ as } n \to \infty .$$

The sum to infinity of a geometric series with starting value *a* and the common ratio *r* is given by

$$S_\infty = \frac{a}{(1-r)} \qquad \text{where } -1 < r < 1.$$

EXAMPLE 10

Find the sum to infinity of the geometric series $16 + 12 + 9 + \ldots$ (Ans: 64)

EXAMPLE 11

The sum to infinity of a GP is twice the sum of the first two terms. Find possible values of the common ratio. (Ans: $\pm\frac{1}{\sqrt{2}}$)

EXAMPLE 12

If the population of a country is 55 million and is decreasing at 2.4% per annum, what will be the population at the end of 5 years? (Ans: 48.71 million)

1.5 SIGMA NOTATION (SUMMATION), $\sum$

The sum

$$1^2+2^2+3^2+4^2+5^2$$

can be written in a compact form using the **sigma notation** $\sum$. That is,

$$1^2+2^2+3^2+4^2+5^2=\sum_{i=1}^{5} i^2$$

The above is read as "The summation of i^2" where i runs from 1 to 5.

The sigma notation is illustrated in more detail below:

Value of *i* for the last term
(known as upper limit)

Symbol for summation (known as sigma) $\longrightarrow \displaystyle\sum_{i=1}^{n} u_i \longleftarrow$ General term

Value of *i* for the first term
(known as lower limit)

i is known as the index of the summation.

An infinite series can also be represented in the same form $u_1+u_2+u_3+\ldots=\sum_{i=1}^{\infty} u_i$.

Example 13: Express each of the following as addition sums.

(a) $\sum_{i=1}^{3} 4i$

Solution $\sum_{i=1}^{3} 4i = 4\times 1 + 4\times 2 + 4\times 3$

Sub *i* =   Sub *i* =   Sub *i* =

(b) $\sum_{i=3}^{6} (-1)^i i^2 =$

(c) $\sum_{i=1}^{\infty} (2i+1) =$

1.5.1 ADDITION AND MULTIPLE RULES OF SIGMA NOTATION

Addition rule: $$\sum_{i=1}^{n}(u_i + v_i) = \sum_{i=1}^{n}u_i + \sum_{i=1}^{n}v_i$$

Multiple rule: $$\sum_{i=1}^{n}ku_i = k\sum_{i=1}^{n}u_i \quad \text{where k is a constant.}$$

Example 14:

If $\sum_{i=1}^{20} i^2 = 2870$ and $\sum_{i=1}^{20} i^3 = 44100$, use the addition and multiple rules to calculate

(a) $\sum_{i=1}^{20} i^2(i-2)$

(b) $\sum_{i=1}^{20}\left(\frac{1}{100}i^3 + i^2 + 1\right)$

TUTORIAL 1

SERIES AND SEQUENCES

Arithmetic Progressions

1. In each of the following AP, find

(a) the common difference (b) the *n*th term (c) the 10th term

(i) $1,\ 3,\ 5,\ldots$ (ii) $-25,\ -20,\ -15,\ldots$ (iii) $-\frac{1}{8},\ -\frac{1}{4},\ -\frac{3}{8},\ldots$

2. The sixth term of an A.P. is 32 while the tenth term is 48. Find the common difference and the 21st term.

3. Which term of the A.P. 6, 13, 20, 27, … is 111?

4. (a) Find the sum of the first 12 terms of $2,\ 4,\ 6,\ldots$
(b) Find the sum of the first 20 terms of $7,\ 3,\ -1,\ldots$

5. The fourth term of an A.P. is 1 and the sum of the first 8 terms is 24. Find the sum of the first 3 terms of the progression.

6. In an arithmetic progression whose first term is -27, the tenth term is equal to the sum of the first 9 terms. Calculate the common difference.

7. The first term of an A.P. is 3 and its *n*th term is 23. If the sum of the first *n* terms of the AP is 351, find *n*.

Geometric Progressions

8. (a) Find the 10th and 20th terms of the GP with first term 3 and common ratio 2.
(b) Find the 7th term of the GP $2,\ -6,\ 18,\ldots$

9 (a) Find the sum of the first five terms of the GP with first term 3 and common ratio 2.
(b) The sum of the first 3 terms of a geometric series is $\frac{37}{8}$. The sum of the first six terms is $\frac{3367}{512}$. Find the first term and common ratio.

10 Find x if the numbers $x + 3$, $5x - 3$ and $7x + 3$ are three consecutive terms of a G.P. of positive terms. With this value of x and given that $x + 3$, $5x - 3$ and $7x + 3$ are the third, fourth, and fifth terms of the G.P., find the sum of the first 8 terms of the progression.

11. The third and sixth terms of a geometric progression are 9 and $2\frac{2}{3}$ respectively.

Calculate the common ratio, the first term and the sum to infinity of the progression.

12. The sum of an infinite sequence is 12 and its first term is 3. Find the first 4 terms of the G.P.

13. The sum of the first 3 terms of a G.P. is 27 and the sum of the fourth, fifth and sixth terms is -1. Find the common ratio and the sum to infinity of the G.P.

14. Given that $x+18$, $x+4$, and $x-8$ are the first three terms of a geometric progression, find the value of x. Hence, find the
    (a) common ratio,
    (b) fifth term,
    (c) sum to infinity.

15. The fourth term of a geometric progression is 162, and the seventh term is 6. Find the first term, the sixth term, and the sum of the first 7 terms.

16. The fourth and seventh terms of a geometric progression are respectively 6 and $-\frac{3}{4}$.

    (a) Find the first and the sixth terms of the progression.
    (b) Find the sum of the first 8 terms.

Practical problems of arithmetic and geometric series

17 A company is offering a job with a salary of $30,000 for the first year and a 5% raise each year after that. If that 5% raise continues every year, find the amount of money you would earn in a 20-year career.

18 An auditorium has 20 seats on the first row, 24 seats on the second row, 28 seats on the third row, and so on and has 30 rows of seats. How many seats are in the theatre?

19 A man who weighs 125 kg is told he must reduce his weigh to 80 kg. He goes on a diet and finds he is losing 1.5 kg weight every week. Find his weigh on the 11th weeks, and find how long it will take to get his weigh down to 80 kg.

Sigma Notation (Summation)

20 Write the series in the form $u_1 + u_2 + u_3 + u_4 + \ldots\ldots$

(a) $\sum_{i=1}^{\infty}\left(\frac{3}{2}\right)^{i}$ (b) $\sum_{i=0}^{\infty}3\,(-1)^{i}$ (c) $\sum_{i=1}^{\infty}\frac{3^{i-2}}{4^{i}}$

(d) $\sum_{i=4}^{\infty}(-\frac{3}{4})^{i}$ (e) $\sum_{i=0}^{\infty}(-1)^{i+1}\,\frac{5}{2^{i}}$

Miscellaneous Exercises

1. The sum of the first $2n$ terms of a geometric progression is K times the sum of the first n terms, where $K > 2$. Prove that the common ratio, r of the geometric progression is given by

$$1+r^n = K$$ (MA1301 0506)

2. Find the sum of all multiples of 11 between 100 and 1000. (MA1301 0506)

3. The *nth* term of an arithmetic progression is $\frac{3n-1}{6}$. Prove that the sum of the first n terms of the progression is $\frac{n(3n+1)}{12}$. (MA1301 0708)

4. The sum to infinity of a convergent geometric progression is S and the sum to infinity of the square of the terms is $2S$. Given further that the sum of the first two terms is $\frac{25}{8}$, find the value of S. (MA1301 1112)

5. It is given that 3^x, 2^{x+1}, 4^{x-1} are the first three terms of a geometric progression.
   (i) Find, to 3 significant figures, the value of x.
   (ii) Find, to the nearest integer, the sum to infinity of the progression.
   (MA1301 1314)

6. A geometric series has first term 0.27 and common ratio $\frac{4}{3}$. Find the least number of terms the series can have if its sum exceeds 550.

ANSWERS

1	(i) (a) $d=2$	(b) $T_n = 2n-1$	(c) $T_{10} = 19$
	(ii) (a) $d=5$	(b) $T_n = -30+5n$	(c) $T_{10} = 20$
	(iii) (a) $d=-\frac{1}{8}$	(b) $T_n = -\frac{1}{8}n$	(c) $T_{10} = -\frac{5}{4}$
2	$d=4$, $T_{21}=92$		
3	The 16th term.		
4	(a) 156	(b) -620	
5	-21	6 d =8	7 n =27
8	(a) 1536, 1,572,864	(b) 1458	

9 (a) 93 (b) 2, $\frac{3}{4}$

10 $x = 3;\ S_8 = \frac{765}{2}$

11 $r = \frac{2}{3}; a = \frac{81}{4} S_\infty = \frac{243}{4}$

12 $3, \frac{9}{4}, \frac{27}{16}, \frac{81}{64}$

13 $r = -\frac{1}{3}; S_\infty = \frac{729}{28}$

14 (a) $r = \frac{6}{7}$ (b) $T_5 = \frac{2592}{49}$ (c) $S_\infty = 686$

15 $T_1 = 4374; T_6 = 18; S_7 = 6558$

16 (a) $T_1 = -48; T_6 = \frac{3}{2}$ (b) $S_8 = -\frac{255}{8}$

17 \$991,978.62

18 2340

19 110 kg, 31 weeks

20 (a) $\frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \frac{81}{16} + \cdots$ (b) 3 – 3 + 3 - 3 + ….

(c) $\frac{1}{12} + \frac{1}{16} + \frac{3}{64} + \frac{9}{256} + \cdots$

(d) $\frac{81}{256} - \frac{243}{1024} + \frac{729}{4096} - \frac{2187}{16384} + \cdots$

(e) $5\left(-1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \cdots\right)$

Miscellaneous Exercises

2 44550

4 $S = \frac{10}{3}$

5 (i) 2.52 (ii) 57

6 n = 23