3. DIFFERENTIATION AND ITS APPLICATION

3.1 REVISION ON RULES OF DIFFERENTIATION

In this Chapter, we will look at some common applications of differentiation. We will use the derivatives to find the gradients of tangent/normal, locate the extreme values of a function, and apply the concept to solve some optimisation problems.

Before that, let revise some of the rules on differentiation.

3.1.1 Derivative of the power function

$$\boxed{\frac{d}{dx}\left(x^n\right) = n\,x^{n-1}}$$

EXAMPLE 1

Find the following derivatives.

(a) $y = x^3 - \pi^3$ (b) $y = x^3 - \dfrac{1}{x^4}$ (c) $y = 10 - 2\sqrt{x}$

Remember that the derivative can be denoted by $\dfrac{dy}{dx}$, $f\,'(x)$, $\dfrac{d}{dx}\left[f(x)\right]$ and $y\,'$.

The derivative $\dfrac{d}{dx}(\;\;)$ can be considered as an operator. The $\boldsymbol{\dfrac{d}{dx}}$ in front of an expression indicates that the expression is to be differentiated.

3.1.2 Derivative of trigonometric functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Note : x is measured in radians

EXAMPLE 2

Find the following derivatives:

(a) $u = \sin 3x$

(b) $y = \sin\left(t^2 + 4t\right)$

3.1.3 Differentiating $y = a^x$

$$\boxed{\frac{d}{dx}\left(a^u\right) = a^u \frac{du}{dx} \ln a}$$

EXAMPLE 3

Differentiate the following with respect to x :

(a) $y = 2^x$

(b) $y = 5^{-2x}$

3.1.4 Differentiating $y = e^x$

In general, $$\boxed{\frac{d}{dx}(e^u) = e^u \frac{du}{dx}}$$

EXAMPLE 4

Differentiate the following expressions with respect to x:

(a) $t = 2e^x$

(b) $y = e^{3x}$

3.1.5 Differentiating $y = \ln x$

$$\boxed{\frac{d}{dx}\left[\ln f(x)\right] = \frac{1}{f(x)} \bullet \frac{df}{dx}}$$

EXAMPLE 5

Differentiate the following with respect to x:

(a) $y = 2 + \ln(x)$

(b) $y = 2\ln(2x+3)$

(c) $u = \ln(x^3 - 3x^2 + 6)$

3.1.6 Differentiating $y = \log_a x$

In this section, we will discuss how to differentiate the general logarithmic function. Recall the change of base formula below ($a > 0$, $a \neq 1$) :

$$\log_a x = \frac{\ln x}{\ln a}$$

We will use it to first change the '$\log_a$' to the natural log 'ln', then perform the differentiation, since we know how to differentiate the natural logarithmic function.

EXAMPLE 6

Differentiate the following with respect to x :

(a) $y = \log_2(2x+1)$ (b) $y = 4\log(x+2)$

When differentiating logarithmic functions, we should break down, whenever possible, the given logarithmic expression into simpler ones by using the following laws of logarithms:

- $\log_a mn = \log_a m + \log_a n$
- $\log_a \left(\dfrac{m}{n}\right) = \log_a m - \log_a n$
- $\log_a m^p = p \log_a m$

EXAMPLE 7

Differentiate the following with respect to x :

(a) $u = 7\ln \dfrac{6e^{5x^4}}{\left(x^3+1\right)^2}$ (b) $s = 6\ln \sqrt[3]{5x^2+1}$

3.1.7 Rules of Differentiation

Let $u \equiv u(x)$ and $v \equiv v(x)$, and k is a constant

1. **Constant Multiple Rule**
$$\frac{d}{dx}(k\,u) = k\frac{du}{dx}$$

2. **Sum and Difference Rule**
$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

3. **Product Rule**
$$\frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}$$

4. **Quotient Rule**
$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

5. **Chain Rule**
If $y \equiv y(u)$ and $u \equiv u(x)$ then $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

EXAMPLE 8

Find the derivatives with respect to their respective variables.

(a) $y = 4\cos x - 3x + \dfrac{7}{x^2}$

(b) $f(x) = (4x-2)^5$

(c) $q = (5\theta + 3)\sin\theta$

(d) $p = \dfrac{3r+1}{\tan r}$

3.2 THE DERIVATIVE AND THE GRADIENT OF A GRAPH

In figure 3.1, a secant line intersects the curve $y = f(x)$ at $P\ (x_1, y_1)$ and $Q\ (x_2, y_2)$.

Gradient of secant line, $m_s = \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1}$

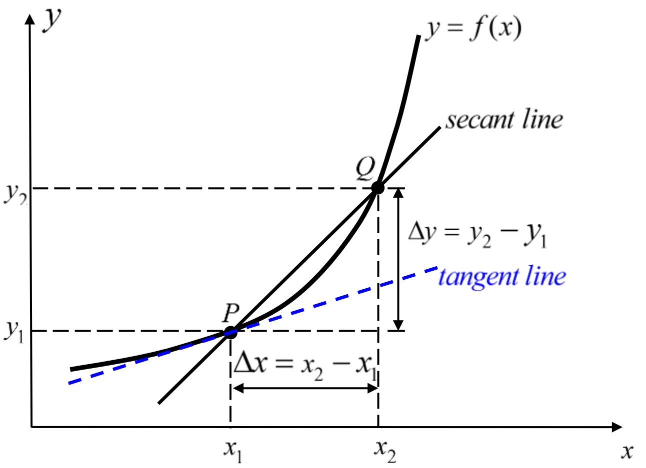


Figure 3.1

As Q approaches P (i.e. as x_2 approaches x_1, written as $x_2 \to x_1$), the gradient of the secant line approaches the gradient of the tangent line.

The gradient of the tangent line at P is also the gradient of the graph at P.

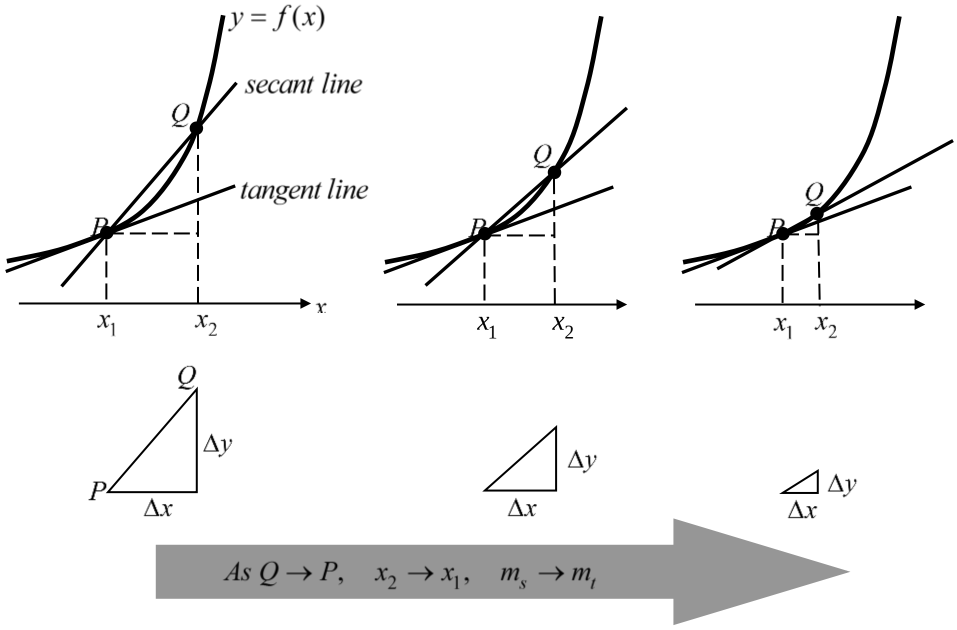


Figure3.2 Secant Line Approaches Tangent

If $y = x^2$, the gradient of the tangent line at $x = 2$ can be estimated by $\dfrac{y_2 - y_1}{x_2 - x_1}$.

Taking $x_1 = 2$, $x_2 = 2.5, 2.1, 2.01, 2.001$, $\dfrac{y_2 - y_1}{x_2 - x_1}$ approaches the limiting value 4.

x_1	x_2	y_1	y_2	$\dfrac{y_2 - y_1}{x_2 - x_1}$
2.0	2.5	4.0	6.25	4.5
2.0	2.1	4.0	4.41	4.1
2.0	2.01	4.0	4.0401	4.01
2.0	2.001	4.0	4.004001	4.001

Similarly, when $x_1 = 2$, $x_2 = 1.5, 1.9, 1.99, 1.999$, $\dfrac{y_2 - y_1}{x_2 - x_1}$ approaches the limiting value 4.

x_1	x_2	y_1	y_2	$\dfrac{y_2 - y_1}{x_2 - x_1}$
2.0	1.5	4.0	3.25	3.5
2.0	1.9	4.0	3.61	3.9
2.0	1.99	4.0	3.9601	3.99
2.0	1.999	4.0	3.996001	3.999

In general, the gradient of tangent line at point P (x_1, y_1) is the limiting value of $\dfrac{y_2 - y_1}{x_2 - x_1}$ as $x_2 \to x_1$.

Hence, the gradient of the tangent at P can be written as $\lim\limits_{\Delta x \to 0}\left(\dfrac{y_2 - y_1}{x_2 - x_1}\right)$, where $\Delta x = x_2 - x_1$

The limiting value of $\dfrac{y_2 - y_1}{x_2 - x_1}$ as $\Delta x \to 0$ is known as the **derivative** of the function.

$$\boxed{\frac{dy}{dx} = \lim_{\Delta x \to 0}\left(\frac{y_2 - y_1}{x_2 - x_1}\right), \text{ where } \Delta x = x_2 - x_1}$$

The notation $\dfrac{dy}{dx}$ is read as 'the derivative of y with respect to x'. If $y = f(x)$, then the derivative may also be denoted by $f'(x)$, $\dfrac{d}{dx}\big(f(x)\big)$ and y'. The process of finding $\dfrac{dy}{dx}$ is called differentiation.

EXAMPLE Find the gradient of the curve $y = x^2 + 1$ at the point $x = 1$.

EXAMPLE 9 (To be taught in class)

Prove the following by differentiation from first principles. ($f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$)

(a) If $f(x) = 2x$, prove that $f'(x) = 2$

(b) If $f(x) = x^2 + 1$, prove that $f'(x) = 2x$

3.3 EQUATIONS OF THE TANGENT AND NORMAL

3.3.1 GRADIENTS OF TANGENT AND NORMAL

Recall that the derivative is the gradient of the tangent to a curve at any point *P*. It is also the gradient of the graph at point *P*.

However, *P* may represent any point, which means that the value of the derivative changes from one point on a graph to another point.

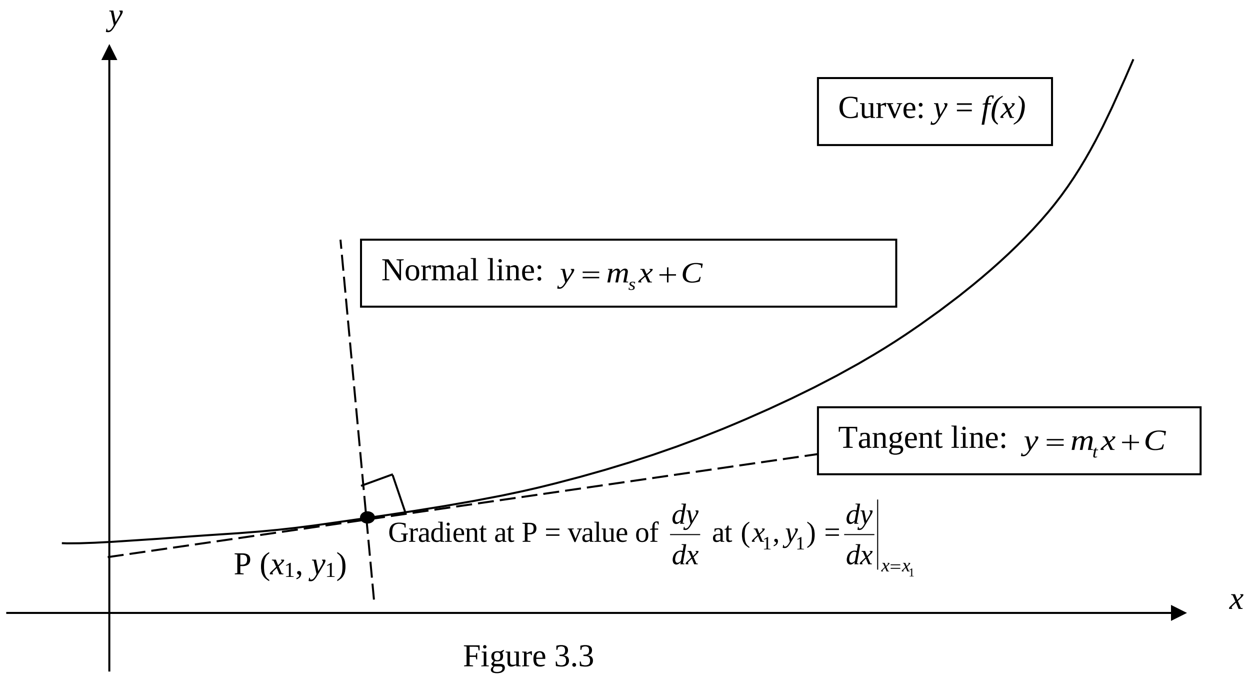


Figure 3.3

The tangent line and the normal line are perpendicular to each other. From coordinate geometry, when two lines are perpendicular, the product of their gradient is -1.

Gradient of a tangent line at (x_1, y_1) = value of $\frac{dy}{dx}$ at (x_1, y_1) Gradient of tangent line $\times$ Gradient of normal line $= -1$

3.3.2 TO FIND EQUATION OF TANGENT TO A CURVE

EXAMPLE 10

Find the equation of the tangent to the curve $y = x^3 - 2x^2 + 5$ at the point (2,5).

Step 1 : First find the derivative $\frac{dy}{dx}$, that is the gradient function at any point.

Step 2 : Substitute in the point given i.e. (2,5). That is the gradient at that point and that is the gradient of the tangent line.

Step 3 : Since the equation of the tangent line is $y = mx + c$, where m is the gradient, substitute in the gradient found in step 2 and the point given (2,5) and solve for c. The end result is the tangent line equation.

EXAMPLE 11

Find the equation of the tangent of the curve $y = 3 - 2x + 4x^3$ at the point (-1,1).

(Ans: $y = 10x + 11$)

3.3.3 TO FIND EQUATION OF NORMAL TO A CURVE

EXAMPLE 12

Find the equation of the normal to the curve $y = x^3 - 2x^2 + 5$ at the point (2,5).

Step 1 : First find the derivative $\frac{dy}{dx}$, that is the gradient function at any point.

Step 2 : Substitute in the point given i.e. (2,5). That is the gradient at that point and that is the gradient of the tangent line. Since at any point, gradient of tangent $\times$ gradient of normal $= -1$, we can find the gradient of the normal as $-\frac{1}{\text{gradient of tangent}}$.

Step 3 : Since the equation of the normal line is $y = mx + c$, where *m* is the gradient, substitute in the gradient found in step 2 and the point given (2,5) and solve for *c*. The end result is the normal line equation.

EXAMPLE 13

Find the equation of the normal of the curve $y = (3x+1)^2$ at the point (0,1).

(Ans: $y = -\frac{1}{6}x + 1$)

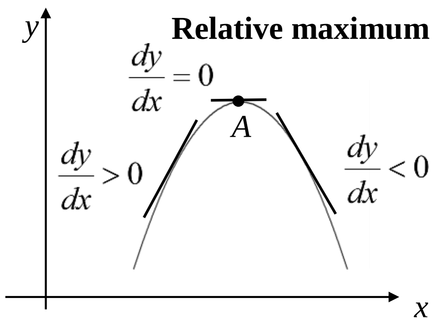
3.4 OPTIMISATION

One very important application of differential calculus is the solving of maximum-minimum problems. We will use the first and second derivative tests to identify stationary points of curves and determine their nature. Finally, we will look at some practical applications involving maxima and minima problems.

3.4.1 First-Derivative Test

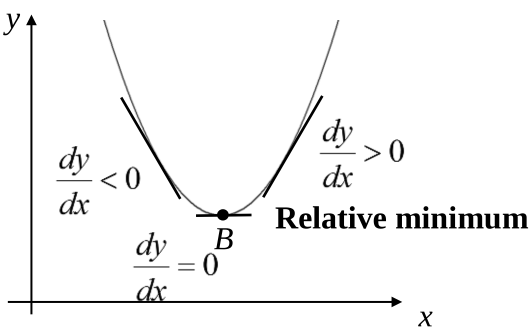
From the diagrams below, we see that the derivative changes sign from positive to negative when passing through a relative maximum point and from negative to positive when passing through a relative minimum point.

A relative maximum



In passing through the point A, $\frac{dy}{dx}$ decreases from positive to negative.

A relative minimum



In passing through the point B, $\frac{dy}{dx}$ increases from negative to positive.

3 Type of points	Sign of $\frac{dy}{dx}$ immediately before and after		
	Before stationary point	Stationary point	After stationary point
Maximum	**(+)**		**(–)**
Minimum	**(–)**		**(+)**

You may perform the following steps to determine the maximum or minimum points of a function $y = f(x)$:

(a) Differentiate the function $y = f(x)$.

(b) Let $\frac{dy}{dx} = 0$ and solve for the values(s) of x (known as stationary points).

(c) Look at the slope of the tangent on either side of the stationary point(s). We must choose the test points close to the stationary point(s). If the test points are too far away, the curve might have already changed direction.

(d) If the slope of tangent is positive to the left of a stationary point and negative to the right, then the stationary point is a maximum. The reverse is true for a minimum point.

EXAMPLE 15

Find the coordinates of the stationary points of the curve $f(x) = 3x^4 - 1$.

Determine the nature of the point by applying the first derivative test.

(Ans: min(0, -1))

3.4.2 Second-Derivative Test

It is apparent that a curve is concave down at a maximum point and concave up in a minimum point. Therefore, at $x = c$, where c is a critical point of f, we have

1. A **relative minimum** point at $(c,\ f(c))$ if $f''(c) > 0$
2. A **relative maximum** point at $(c,\ f(c))$ if $f''(c) < 0$
3. There is no conclusion from this test if $f''(c) = 0$ or if $f''(c)$ is undefined. (Instead, we must use the first-derivative test)

EXAMPLE 16

Find all the stationary points of the function $f(x) = x^3 - 3x^2 - 9x + 2$ and determine if each of the stationary points is relative maximum or minimum.

(Ans: max(-1, 7), min(3, -25))

3.4.3 Maximum and Minimum Problems

In the previous section, we saw how the first and second derivatives can be used to find the maximum and minimum point of a given curve. Let us now apply what we have learned to help us to find the largest and least value of a variable in a practical problem.

Suggested Steps for Solving Applied Optimisation Problems:

1. Identify the quantity which is to be optimised and call it P.
2. Express P in terms of the variables and use the information given in the problem and the principle of substitution to rewrite P in terms of a single variable, say x.
3. Find the derivative $\dfrac{dP}{dx}$.
4. Set the derivative to zero and solve for x.
5. Determine the desired maximum or minimum value using the techniques introduced in the preceding sections (first or second derivative test).

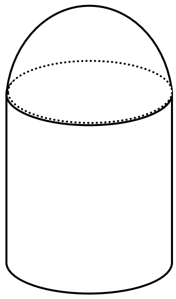
EXAMPLE 17 (To be taught in class)

The total cost of producing k smartphones at a factory is $\$\left(k^2 + 500k + 10{,}000\right)$ on a daily basis and the retail price of **each** smartphone is set at $\$\left(750 - \dfrac{1}{4}k\right)$.

What should be the daily output production of the factory in order to achieve maximum profit?
(Ans: Max profit when $k = 100$)

EXAMPLE 18

The diagram shows a solid consisting of a hemisphere and a circular cylinder with common radius *r cm*. Given that the solid has a fixed volume of 450π cm^3. Prove that its surface area, *A* is given by



$$A = \frac{900\pi}{r} + \frac{5}{3}\pi r^2$$

Given that *r* may vary, calculate the stationary value of *A* and the corresponding height of the solid. Determine whether this value of *A* is a maximum or a minimum. (Ans: $r = 6.463$, height=12.9 *cm*, min $A = 656$.)

TUTORIAL 3
Applications of Differentiation

Revision on Differentiation

1. Differentiate with respect to x and simplify your answers wherever possible.

(a) $5x^4 + \dfrac{1}{2x^2}$ (b) $\sqrt{x} + 8$ (c) 7sin5x (d) 12 sec3x

(e) $\ln(7x^3 + 5)$ (f) $5\ln\cos x$ (g) e^{3x} (h) $2e^{3-4x}$

2a. Differentiate with respect to x and simplify your answers wherever possible.

(a) $(6 - x^2)^7$ (b) $\dfrac{2}{(3x+1)}$ (c) $x^4 \sin 2x$ (d) $e^x \ln x$

(e) $\dfrac{3x}{(1+\sin x)}$ (f) $\dfrac{2x+1}{x-3}$ (g) $x^3 \ln x$ (h) $e^x \sin 2x$

(i) $\dfrac{e^x}{x}$ (j) $\dfrac{2x}{x+3}$ (k) $\log_3 x^2$ (l) $\log \sin x$

(m) $\log_4\left[e^{2x}\cos(3x)\right]$

2b. The process of finding the derivative using the definition

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}, \ \Delta x \neq 0$$

is known as differentiation from first principles.
Differentiate the following functions from first principles.

(a) $f(x) = x + 2$ (b) $f(x) = 3x^2 + x$

(c) $f(x) = x^n$ (n is a constant, hint: use of binomial theorem is required)

Equations of tangent and normal

3 The equation of a curve is $y = x^3 - 3x^2 - 9x + 27$.
Find the equation of the tangent to the curve at the point where $x = 4$.

4 (a) A curve is given by $y = (3x-1)\sqrt{x+1}$. Calculate its gradient when $x = 3$.

(b) The curve $y = \ln(3x + 2) + x$ passes through the point P where $x = 1$. Find the equation of the normal to the curve at the point P.

5 (a) Find the equation of the tangent to the curve $y = \sqrt{x^2 - 6x + 25}$ at the point (0, 5).

(b) Find the equation of the normal to the curve $y = \dfrac{2x+4}{x-1}$ at the point where the curve meets the x-axis.

6. Given that $y = (x+2)\sqrt{2x-3}$,

(a) show that $\frac{dy}{dx}$ can be written in the form $\frac{ax+b}{\sqrt{2x-3}}$, and state the values of a and b.

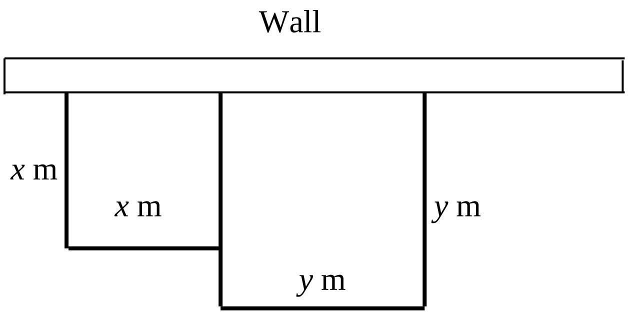
(b) find the coordinates of the point at which the normal to the curve at $x = 2$ cuts the x-axis.

Maximum and Minimum Problems

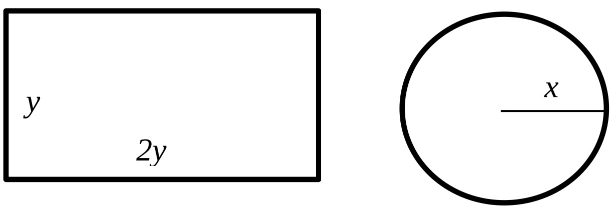
7 A man has 26 m of fencing to make two square enclosures using existing wall as one side of each enclosure. The fences are highlighted in bold and the dimensions of each enclosure are x m and y m $(x < y)$ as shown below.

Show that the total area of the two enclosures is $A = x^2 + \frac{(26-2x)^2}{9}$.

Hence, calculate the stationary value of this area and determine whether this is a maximum or a minimum value.



8 A wire 3 m long, is cut into two pieces to form a rectangle and a circle. The sides of the rectangle are y (m) and $2y$ (m). The radius of the circle is x (m).



(a) Express y in terms of x. Show that the total area A (m^2) of the rectangle and the circle is given by

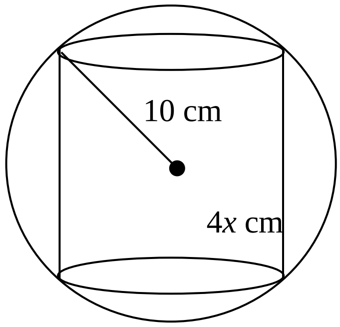
$$A = \frac{1}{18}(3-2\pi x)^2 + \pi x^2$$

(b) Find the radius of the circle such that the total area is a minimum.

9 A solid right cylinder is removed from a solid sphere of radius 10 cm as shown. If the height of the cylinder is $4x$ cm, show that the volume V of the cylinder is

$$V = 400\pi x - 16\pi x^3.$$

Find the maximum volume of the cylinder as x varies. Give your answer to the nearest whole number.
[Volume of cylinder = $\pi r^2 h$]



10 A producer of computer graphics software finds that selling price p of its software is related to the number of x copies of its software sold annually by the demand equation, $x = 10000 - 200p$, while its total cost in producing and marketing these x copies is given by the function $C(x) = 50000 + 5x$. Find the price p for which profits will be a maximum. Find the maximum profit earned by selling at this price.

11 Dalto Pizza currently sells 1000 pizzas per week at $18 per pizza. It is planning to reduce the unit price of each pizza. It estimates that for every $1 discount in price, it can sell 100 more pizzas each week.
(a) Form the weekly revenue function of Dalto Pizza in terms of p, the new unit price of the pizza.

(b) What should the new unit price be in order to maximize weekly revenue? What is the maximum weekly revenue?

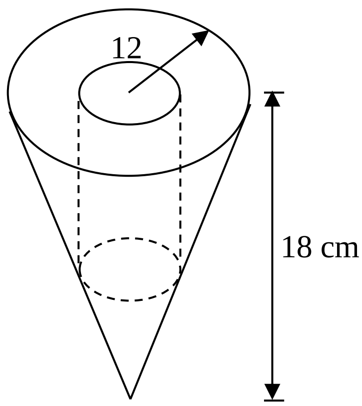
12 *Near East Inc.*, a corporate gift specialist, just received an order of 500 hampers. To meet the order, the general manager is considering engaging inexperienced packers who each charges a flat fee of $100. Each packer can pack 2 hampers in 1 hour. A supervisor who charges $10 per hour will also be engaged. A flat commission fee of $50 is applicable. What is the minimum manpower cost if this option is chosen?

Miscellaneous Exercises

1. Find the gradient of the normal to the curve $y = e^{\sin(\ln x)}$ at the point where the x-coordinate is 2.

2. The equation of a curve is $y = x + \dfrac{p}{x^2}$. The normal to the curve at the point $x = 2$ passes through the point (0, 2). Find the values of p. (MA1301 1314)

3. A cylinder is placed inside a circular cone of radius 12 cm and height 18 cm so that its base is level with the base of the cone as shown in the diagram (not drawn to scale) below. When the radius of the cylinder is x cm, its height is y cm.

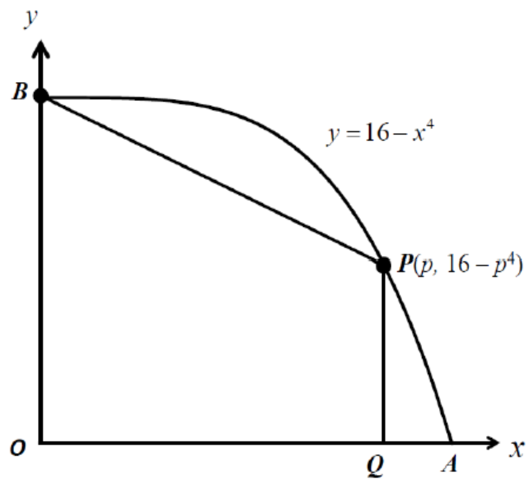


(i) Show that the volume of the cylinder, V, is given by

$$V = 18\pi x^2 - \frac{3}{2}\pi x^3.$$

(ii) Find the value of x that gives the maximum possible volume of the cylinder and find this volume.

4. In the diagram, the curve $y = 16 - x^4$ meets the x-axis and y-axis at the points A and B respectively. P is a variable point on the arc AB and Q is the foot of perpendicular from P to the x-axis. O is the origin.



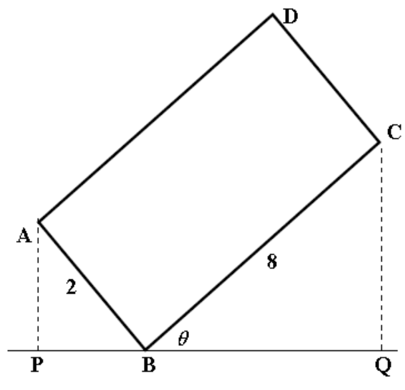
(i) Find the coordinates of A and B.

(ii) Find the equation of the tangent to the curve at the point A.

(iii) Show that the area of the trapezium OBPQ is $\frac{1}{2}\left(32p - p^5\right)$.

Hence, determine the maximum area of trapezium OBPQ as p varies. Give your answer in three significant figures. (MA1301 1314)

5 In the given diagram, the side BC of a rectangle $ABCD$ is inclined at an angle θ to the horizontal line passing through the points P, B and Q. P is the foot of perpendicular from A to the horizontal and Q is the foot of perpendicular from C to the horizontal. It is given that AB = 2cm and BC = 8cm.



(i) Express the length of PQ in terms of θ and hence, find the greatest possible length of PQ as θ varies.

(ii) Show that the area, y cm^2, of the trapezium $APQC$ is given by

$$y = p + q\sin 2\theta$$

where p and q are constants.

(iii) Hence, write down the value of θ for which y has the maximum value.

(MA1301 0910)

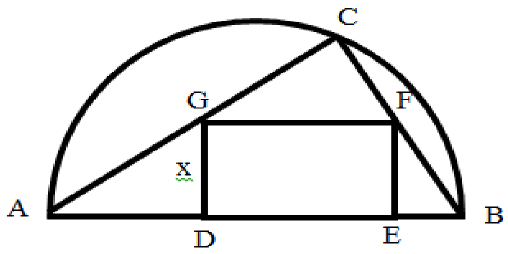
6 The diagram shows a semi-circle with diameter *AB*. C is a point on the semi-circle. A rectangle *DEFG* is drawn within the triangle *ABC* where *DE* on the side *AB*, point *F* and point *G* are on the sides *BC* and *AC* respectively.

Given that *AC*=8cm, *BC*=6cm, find,

(i) the perpendicular distance from *C* to *AB*, leaving your answer in the exact value form.

(ii) Let *DG* be x, show that the area of rectangle *DEFG*, A m^2, is given by

$$A = 10x - \frac{25}{12}x^2$$



(iii) Calculate the value of x for which *A* has a stationary value. Determine whether this stationary value of *A* is a maximum or a minimum.

ANSWERS

1. (a) $20x^3 - \frac{1}{x^3}$ (b) $\frac{1}{2\sqrt{x}}$ (c) 35cos5x

(d) 36sec3xtan3x (e) $\frac{21x^2}{7x^3+5}$ (f) $-5\tan x$

(g) $3e^{3x}$ (h) $-8e^{3-4x}$

2a (a) $-14x(6-x^2)^6$ (b) $-\frac{6}{(3x+1)^2}$ (c) $2x^3(x\cos 2x + 2\sin 2x)$

(d) $e^x\left(\frac{1}{x} + \ln x\right)$ (e) $\frac{3(1+\sin x - x\cos x)}{(1+\sin x)^2}$ (f) $-\frac{7}{(x-3)^2}$

(g) $x^2(1+3\ln x)$ (h) $e^x(\sin 2x + 2\cos 2x)$ (i) $\frac{e^x(x-1)}{x^2}$

(j) $\frac{6}{(x+3)^2}$ (k) $\frac{2}{\ln 3}\frac{1}{x}$ (l) $\frac{1}{\ln 10}\cot x$ (m) $\frac{1}{\ln 4\cos 3x}[2\cos 3x - 3\sin 3x]$

3 $y = 15x - 53.$ 4(a) 8 4(b) $y = -\frac{5}{8}x + \ln 5 + \frac{13}{8}$

5 (a) $5y + 3x = 25$ 5(b) $2y = 3x + 6$ 6 (a) a = 3, b = -1 (b) (22, 0)

7 A= 52(minimum) 8(b) $\frac{3}{2\pi+9}$ m or 0.196 m

9. 2418

10 Price = \$27.50, Profit = \$51, 250

11 (a) $R = 2800p - 100p^2$ (b) $p = 14$, Maximum Revenue =\$19600 12. \$1,050

Miscellaneous Exercises

1 - 1.37 2 p = - 4 or 8 3 (ii) x = 8, Max Vol. $\approx$ 1206 cm^3

4 (i) A(2, 0), B(0, 16) (ii) $y = -32x + 64$ (iii) 20.4

5 (i) $PQ = 2\sin\theta + 8\cos\theta$, largest value of $PQ = \sqrt{68}$

(ii) $y = 8 + 17\sin 2\theta$ (iii) $\theta = \frac{\pi}{4}$

6 (i) $4\frac{4}{5}cm$ (ii) A is maximum at x= $2\frac{2}{5}$ cm