Chapter 9 : Solve First Order Differential Equations using Integrating Factor

Objective :

1. Solve first order differential equations by using the method of "integration factor"
2. Solve application problems: R-C and R-L series circuit problems.

9.1 Standard Linear Differential Equation

A **standard linear** first order differential equation (DE) is in the form:

$$\frac{dy}{dx}+P(x)y=Q(x) \qquad \cdots\cdots(1)$$

where $P(x)$ and $Q(x)$ are functions of x.

If $P(x)=0$, then DE takes the form $\frac{dy}{dx}=Q(x)$ which can be solved by direct integration.
However if $P(x)\neq 0$, then it cannot be solved by direct integration. The solution in this case requires a certain strategy.

An ordinary DE may not always be given in standard linear form (1). We then need to rearrange the terms to put it into the desired linear form.

For example: $x\frac{dy}{dx}=-y+3x^2$ is not in standard linear form.

Rearranging, it becomes:

which is in the standard linear form (for $x > 0$)

Comparing with (1): $P(x)=$ and $Q(x)=$

In this example, since $P(x)\neq 0$, thus the DE cannot be solved by direct integration.

But, multiplying "x" to the DE, we get:

Since $\frac{d}{dx}(xy)=x\frac{dy}{dx}+y$, DE becomes:

which can now be directly integrated. Thus $xy=$

So the key for solving this particular DE is the "x" we used to multiply to both sides of the equation. This "x" which enabled the left-hand side to be rewritten as $x\frac{dy}{dx}+y=\frac{d}{dx}(xy)$ is known as the **integrating factor** for the DE. So how do we find the integrating factor for any linear DE?

9.2 Integrating Factor

Let $\mu = \mu(x)$ be the integrating factor of the linear DE: $\dfrac{dy}{dx} + P(x)y = Q(x) \quad \cdots(1)$

such that when $\mu(x)$ is multiplied to (1), the LHS can become $\dfrac{d}{dx}(\mu y)$.

This gives:

This is a DE which can be solved by separating the variables, hence:

$$\int \frac{1}{\mu}\, d\mu = \int P(x)\, dx$$

$$\Rightarrow \ \ln \mu = \int P(x)\, dx$$

$$\Rightarrow \ \mu = e^{\int P(x)\, dx}$$

Thus, the **integrating factor** (**I.F.**) is: $\mu(x) = e^{\int P(x)\, dx}$

Note that the I.F. for the DE depends on $P(x)$.

Let us use this to find the I.F. for the DE example we considered earlier: $\dfrac{dy}{dx} + \dfrac{1}{x} y = 3x$

Since $P(x) = \dfrac{1}{x}$, the I.F. for the DE is then given by: $\mu(x) = e^{\int P(x)\, dx} = e^{\int \frac{1}{x}\, dx} = e^{\ln x} = x$

In summary, we can solve a linear DE with the help of an integrating factor as follow:

Step 1 Rewrite the first order linear differential equation in the standard form

$$\frac{dy}{dx} + P(x)y = Q(x).$$

Step 2 Find the integrating factor $\mu(x) = e^{\int P(x)\, dx}$.

Step 3 Multiply both sides of the differential equation by the I.F.

$$\mu\frac{dy}{dx} + \mu P(x)y = \mu Q(x).$$

Step 4 Rewrite the above equation as

$$\frac{d}{dx}(\mu y) = \mu Q(x).$$

Step 5 Integrate both sides of the equation to obtain the solution

$$y(x) = \frac{1}{\mu}\int \mu Q(x)\ dx.$$

Remarks:

1. It is important to simplify the expression for the integrating factor $\mu(x)$ at Step 2 as much as possible as this will simplify the manipulations in the subsequent steps. In particular, note that $e^{\ln f(x)} = f(x)$.
2. It is good practice to check using product rule that $\frac{d}{dx}(\mu y) = \mu \frac{dy}{dx} + \mu P(x) y$ at Step 4. If this cannot be shown, it suggests that the $\mu(x)$ obtained at Step 2 is in error.

Example 1: Find the general solution of the differential equation $2\frac{dy}{dx} + 4y = 2e^x$

Solution Step 1 Rewrite the first order linear differential equation in the standard form:

where $P(x) =$ and $Q(x) =$.

Step 2 Find the integrating factor $\mu(x) = e^{\int P(x)\,dx}$.

The I.F. is $\mu(x) = e^{\int P(x)\,dx}$

Step 3 Multiply both sides of the differential equation by the I.F. found above

Thus

Note: A quick check using product rule:

This suggests that I.F. obtained earlier is correct.

Step 4 Rewrite the above equation as $\frac{d}{dx}(\mu y) = \mu Q(x)$.

Hence above equation is

Step 5 Integrate both sides of the equation to obtain the solution:

Hence the general solution is $y(x) =$.

Example 2: Solve $x\frac{dy}{dx}-4y=x^6e^x$ for y in terms of x given that $y(1)=4$.

Ans: $y(x)=x^4(xe^x-e^x+4)$

Solution

Example 3: Solve $\frac{dy}{dx}+2xy=x$

Ans: $y(x)=\frac{1}{2}+C\,e^{-x^2}$

Solution

9.3 Application: Electrical Circuits

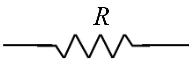
Below are some electrical formulae we will be using to form DE in electrical circuits.

9.3.1 Current and Charge in a Circuit

The current i (in amperes) is the rate of change of the charge q (in coulombs) flowing through a circuit at any time t (in seconds), that is $\boxed{i = \dfrac{dq}{dt}}$

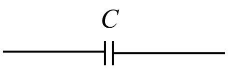
The charge q can be expressed as $\boxed{q = \int i\,dt}$

9.3.2 Voltage Across a Resistor



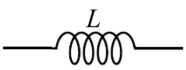
The potential differential v_R across a resistor of resistance R (in ohms) is given by $\boxed{v_R = R\,i}$

9.3.3 Voltage Across a Capacitor



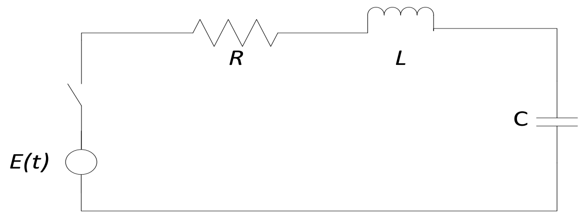
The voltage v_C (in volts) across the plates of a capacitor with capacitance C (in farads), is given by $\boxed{v_C = \dfrac{q}{C}}$

9.3.4 Voltage Across an Inductor



A voltage v_L (in volts) across the inductor with inductance L (in henrys) is given by $\boxed{v_L = L\dfrac{di}{dt}}$

9.3.5 *RLC* Series Circuit



For a RLC series circuit, when the switch is closed at time $t = 0$, by Kirchoff 's voltage law,

$$v_R + v_L + v_C = E(t)$$

that is
$$L\frac{di}{dt} + Ri + \frac{q}{C} = E(t)$$

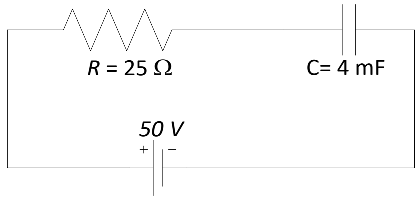
***Example 4*:** A circuit has in series a voltage source of 30 V, a resistor of $60\,\Omega$ and an inductor of 3 H. If the initial current is zero, find the current at time $t > 0$. Hence, find the steady-state current.

Ans: $i(t) = \frac{1}{2} - \frac{1}{2}e^{-20t}$, $\frac{1}{2}$ A

Example 5: Assuming that the charge on the capacitor is zero at $t=0$, find:

(a) the charge and current at any time t ;

(b) the steady-state charge.



Solution

(a) $v_R + v_C = V$

Tutorial 9

1. Solve the following linear differential equations:

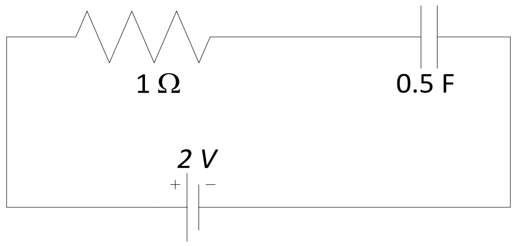
   (a) $\dfrac{dy}{dx}+\dfrac{3y}{x}=6x^2$

   (b) $\dfrac{dy}{dx}+3y=3x^2e^{-3x}$

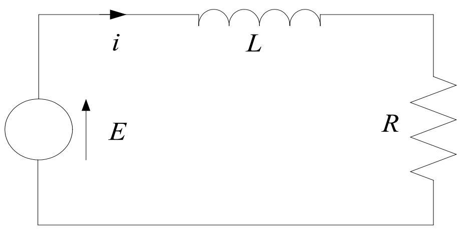
   (c) $y'+\dfrac{y}{x}-\sin^2 x=0$

   (d) $\dfrac{dy}{dx}+5x=x-xy$ given that $y=1$ when $x=0$.

2. For the circuit shown in the figure below, find the charge $q(t)$ on the capacitor, given that $q(0)=0$ coulomb.



3. Consider the circuit in the figure below with inductance 1 H, resistance 5000 Ω and a voltage source of 12 V. Assume that no current flows in the circuit when the switch is closed at $t=0$.
   (a) Find the current in the circuit at any time t.
   (b) Find the steady-state current (i.e. when t tends to infinity.).



4. The current i (amperes) flowing through an RL circuit at time t (seconds) satisfies the differential equation

$$L\frac{di}{dt}+Ri=V$$

where R, L and V are constants. Given that $i=0$ at $t=0$, find
   (a) the current i, at any time t, in terms of R, L and V;
   (b) an expression for the steady-state current.

*5. A RL series circuit has a 35 volt supply connected to an inductor with inductance L (henries) and a resistor with resistance R (ohms). The current i (amperes) flowing through the circuit at time t (seconds) satisfies the differential equation

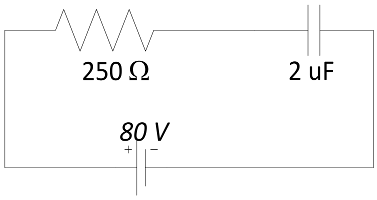
$$L\frac{di}{dt}+Ri=V$$

where V is the voltage supply.

(a) Find the current i, at any time t, in terms of R and L given that $i=0$ when $t=0$.

(b) If the circuit is to have an instantaneous current of 5 mA at a time of 22 ms from switch-on and the resistance $R=4.7$ kΩ, determine the required value of the inductance L.

*6. For the circuit on the right, the capacitor is initially discharged. How long after the switch is closed will the capacitor voltage be 76 volts? Determine the current in the resistor at that time.



Miscellaneous Exercises

1. Solve the differential equation $2xy\frac{dy}{dx}=4x^2+3y^2$. [*Hint*: Let $u=y^2$]

2. Show that the following equations are linear. Hence solve them.

(a) $xy'=e^x\left(1-xy\right)-y$

(b) $\left(1+y^2\right)^2=2y\left[1+x\left(1+y^2\right)\right]\frac{dy}{dx}$

(c) $\cos^2 y+\left[\frac{2x}{\tan y}-\frac{\cos^2 y}{\tan y-y}\right]\frac{dy}{dx}=0;\quad y\left(0\right)=\frac{\pi}{4}$

Multiple Choice Questions

1. Reduce $x\frac{dy}{dx}-\frac{y}{x^2}=\ln x$ to linear form and identify $P(x)$ and $Q(x)$.

(a) $P\left(x\right)=-\frac{1}{x^2}$ and $Q\left(x\right)=\ln x$

(b) $P\left(x\right)=-\frac{1}{x^3}$ and $Q\left(x\right)=\ln x$

(c) $P\left(x\right)=-\frac{1}{x^2}$ and $Q\left(x\right)=\frac{\ln x}{x}$

(d) $P\left(x\right)=-\frac{1}{x^3}$ and $Q\left(x\right)=\frac{\ln x}{x}$

Answers

1. (a) $y(x) = x^3 + Cx^{-3}$

   (b) $y(x) = (x^3 + C)e^{-3x}$

   (c) $y\,x = \frac{1}{4}\left(x^2 - x\sin 2x - \frac{1}{2}\cos 2x\right) + C$

   (d) $y(x) = 5e^{-\frac{x^2}{2}} - 4$

2. $q(t) = 1 - e^{-2t}$ coulombs

3. (a) $i(t) = \frac{3}{1250}\left(1 - e^{-5000t}\right)$ amperes (b) $\frac{3}{1250}$ amperes

4. (a) $i(t) = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right)$ amperes (b) $\frac{V}{R}$ amperes

5. (a) $i(t) = \frac{35}{R}\left(1 - e^{-\frac{R}{L}t}\right)$ amperes (b) 92.91 henries

6. 1.50 ms, 16 mA

Miscellaneous Exercises

1. $y^2 = -4x^2 + Cx^3$

2. (a) $xy = 1 + Ce^{-e^x}$ (b) $x(y) = -\frac{1}{2\left(1+y^2\right)} + C\left(1+y^2\right)$ (c) $x\tan^2 y = \ln\left|\frac{\tan y - y}{1 - \frac{\pi}{4}}\right|$

MCQ

1. d