Chapter 4 – Differentiation of Exponential ,Logarithmic and Inverse Trigonometric Functions

Objectives:

1. *State the derivatives of logarithmic , exponential and inverse trigonometric functions.*
2. *Apply rules of differentiation to find derivatives of functions involving algebraic, trigonometric, logarithmic , exponential and inverse trigonometric functions.*

4.1 Introduction

We have learnt how to differentiate algebraic and trigonometric expressions and also some rules governing differentiation. We will now look at differentiating logarithmic, exponential and inverse trigonometric functions, as these functions are widely used in the fields of engineering and science.

4.2 Recap

4.2.1 Derivative of the power function

$$\boxed{\frac{d}{dx}\left(x^n\right) = n\,x^{n-1}}$$

Example 1 : Find the following derivatives.

(a) $y = x^3 - \pi^3$ (b) $y = x^3 - \dfrac{1}{x^4}$ (c) $y = 10 - 2\sqrt{x}$

4.2.2 Derivative of trigonometric functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Note : x is measured in radians

Example 2 : Find the following derivatives:

(a) $u = \sin 3x$ (b) $y = \sin\left(t^2 + 4t\right)$

4.2.3 Rules of Differentiation

Let $u \equiv u(x)$ and $v \equiv v(x)$, and k is a constant

1. **Constant Multiple Rule**

$$\frac{d}{dx}(k\,u) = k\frac{du}{dx}$$

2. **Sum and Difference Rule**

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

3. **Product Rule**

$$\frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}$$

4. **Quotient Rule**

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

5. **Chain Rule**

If $y \equiv y(u)$ and $u \equiv u(x)$ then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Example 3 : Find the derivatives with respect to their respective variables.

(a) $y = 4\cos x - 3x + \dfrac{7}{x^2}$ (b) $f(x) = (4x-2)^5$

(c) $q = (5\theta + 3)\sin\theta$ (d) $p = \dfrac{3r+1}{\tan r}$

4.3 Differentiation of Logarithmic Functions

In this section we will state the formula for differentiating a logarithmic function. Our focus is on the natural logarithmic function, $\ln x$, as the formula for this function is more simplified than the general logarithmic function, $\log_a x$ ($a > 0$, $a \neq 1$). Next, we will show you how to differentiate $\log_a x$ using a change of base.

4.3.1 Differentiating $y = \ln x$

If $y = \ln x$, then $\dfrac{dy}{dx} = \dfrac{1}{x}$.

In general, if $y = \ln\left(f(x)\right)$, where $f(x)$ is a function of *x*, to differentiate *y*, we will have to use Chain Rule as follows:

$$\boxed{\frac{d}{dx}\left[\ln f(x)\right] = \frac{1}{f(x)} \bullet \frac{df}{dx}}$$

Example 4 : Find the derivatives of the following functions with respect to their respective variables.

(a) $y = 2\ln(2t+3)$

(b) $u = \ln(x^3 - 3x^2 + 6)$

(c) $u = 4\sin x \ln\left(x^2 - 3\right)$

(d) $x = \dfrac{\ln t}{3 + \tan t}$

(e) $f(x) = \left(\ln 2x\right)^4$

4.3.2 Differentiating $y = \log_a x$

In this section, we will discuss how to differentiate the general logarithmic function. Recall the change of base formula below $(a > 0\,,\ a \neq 1\ )$:

$$\log_a x = \frac{\ln x}{\ln a}$$

We will use it to first change the '$\log_a$' to the natural log 'ln', then perform the differentiation, since we know how to differentiate the natural logarithmic function.

Example 5 : Find $\dfrac{dy}{dx}$ given that,

(a) $y = \log_2(2x+1)$ (b) $y = 4\log(x+2)$

When differentiating logarithmic functions, we should break down, whenever possible, the given logarithmic expression into simpler ones by using the following laws of logarithms:

- $\log_a mn = \log_a m + \log_a n$
- $\log_a\left(\dfrac{m}{n}\right) = \log_a m - \log_a n$
- $\log_a m^p = p\log_a m$

This will make the expression easier to differentiate as the next set of examples will show.

Example 6 : Differentiate the following with respect to *x*.

(a) $u = 7\ln\dfrac{6e^{5x^4}}{\left(x^3+1\right)^2}$ (b) $s = 6\ln\sqrt[3]{5x^2+1}$

4.4 Differentiation of Exponential Functions

In this section, we will first look at the derivative of the basic exponential form $y = a^x$ ($a > 0$, $a \neq 0$). After which we will see how we can use the Chain Rule to get the general form. We will then look at the derivative of $y = e^x$ and its general form.

4.4.1 Differentiating $y = a^x$

Let us make used of the derivative of the logarithmic function together with the Chain Rule to obtain the derivative of $y = a^x$.

Step 1: Let $y = a^x$, where $a > 0$ and $a \neq 1$.

Step 2: Take the natural log of both sides and differentiate with respect to *x*,

$$\ln y = \ln a^x$$

$$\ln y = x \ln a$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln a)$$

Step 3: Apply the Chain Rule to the left hand side of this equation.

$$\frac{d}{dy}(\ln y)\frac{dy}{dx} = \ln a \frac{d}{dx}(x) \qquad \text{note that 'ln } a\text{' is a constant}$$

$$\frac{1}{y}\frac{dy}{dx} = \ln a$$

$$\frac{dy}{dx} = y \ln a = a^x \ln a \quad \text{since } y = a^x$$

Thus the standard derivative is : $\frac{d}{dx}\left(a^x\right) = a^x \ln a$

In general, to differentiate $y = a^u$, where u is a function of x, we will have to use the Chain Rule as follows

$$\boxed{\frac{d}{dx}\left(a^u\right) = a^u \frac{du}{dx} \ln a}$$

Example 7 : Differentiate the following with respect to x :

(a) $y = 2^x$ (b) $y = 5^{-2x}$

4.4.2 Differentiating $y = e^x$

If we replace the base a with Euler's constant e in $y = a^x$ and apply the rule we just derived in 4.4.1, we will get, $\frac{d}{dx}\left(e^x\right) = e^x \ln e$. Since $\ln e = 1$, this simplifies to

$$\frac{d}{dx}(e^x) = e^x$$

In general, $\boxed{\frac{d}{dx}(e^u) = e^u \frac{du}{dx}}$

Example 8 : Differentiate the following expressions with respect to x:

(a) $t = 2e^x$ (b) $y = e^{3x}$

(c) $R = 5x^2 e^{-3x^2}$ (d) $y = \frac{5e^{2x}}{1-6x}$

4.5 Differentiation of Inverse Trigonometric Functions

In this section we will state the formula for differentiating inverse trigonometric functions. Our focus is on $y = \sin^{-1} x$, $y = \cos^{-1} x$ and $y = \tan^{-1} x$. Next, we will show you how to differentiate $y = \csc^{-1} x$, $y = \sec^{-1} x$ and $y = \cot^{-1} x$.

4.5.1 Introduction to $y = \sin^{-1} x$, $y = \cos^{-1} x$ and $y = \tan^{-1} x$

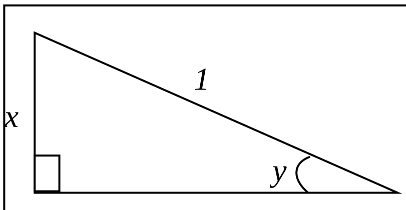


Figure 1

From *figure 1* , by trigonometric ratio,

$$\sin y = \frac{x}{1}$$

Then , $y = \sin^{-1} x$ or arcsin x

For example, $\sin 30^o = \dfrac{1}{2}$

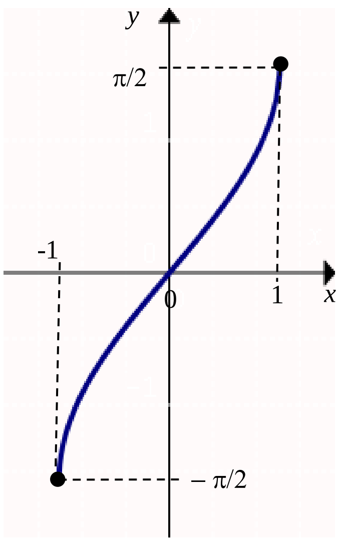
then $\sin^{-1} \dfrac{1}{2} = 30^o$

Similarly, if $\cos y = x$, then $y = \cos^{-1} x$ or y = arccos x.

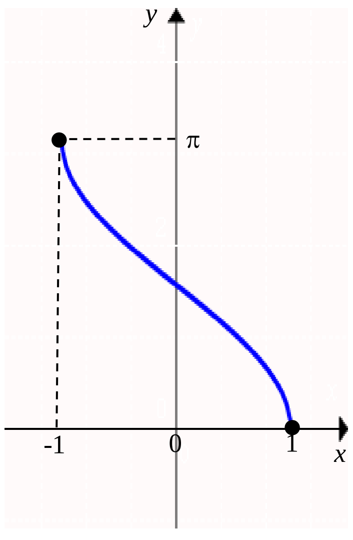
If $\tan y = x$, then $y = \tan^{-1} x$ or y = arctan x

Note : $\sin^{-1} x \neq \dfrac{1}{\sin x}$

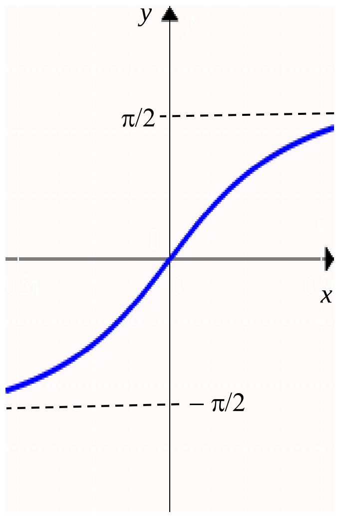
The following graphs of inverse trigonometric functions are for your reference only.



$y = \sin^{-1} x$



$y = \cos^{-1} x$



$y = \tan^{-1} x$

4.5.2 Differentiating $y = \sin^{-1} x$, $y = \cos^{-1} x$ and $y = \tan^{-1} x$

Let $y = \sin^{-1} x$ where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

Then $\sin y = x$

Differentiating both sides w.r.t. **y** ,

$$\frac{d}{dy}[\sin y] = \frac{d}{dy}[x]$$

$$\cos y = \frac{dx}{dy}$$

$$\frac{dy}{dx} = \frac{1}{\cos y} \quad \ldots (1)$$

From the trigonometric identities $\cos^2 y = 1 - \sin^2 y$

$$\cos y = \pm\sqrt{1 - \sin^2 y}$$

$$= \pm\sqrt{1 - x^2}$$

Since $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $\cos y$ must be positive, so

$$\cos y = \sqrt{1 - x^2}$$

Substituting this into equation (1) gives:-

$$\boxed{\frac{d}{dx}\left[\sin^{-1} x\right] = \frac{1}{\sqrt{1 - x^2}}}$$

In general, to differentiate $y = \sin^{-1} u$, where u is a function of x, we will have to use the Chain Rule as follows

$$\boxed{\frac{d}{dx}\left[\sin^{-1} u\right] = \frac{1}{\sqrt{1 - u^2}}\frac{du}{dx}}$$

Following the same argument, where u is a function of x ,we can show that in general

$$\frac{d}{dx}\left[\cos^{-1}u\right] = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}\left[\tan^{-1}u\right] = \frac{1}{1+u^2}\frac{du}{dx}$$

Example 9 : Differentiate the following expressions with respect to x :

(a) $y=\sin^{-1}(3x)$ (b) $y=2\tan^{-1}(x^2)$

(c) $y=e^{2x}\sin^{-1}(4x)$

Tutorial 4

1. Differentiate the following functions with respect to x. Simplify your answers whenever possible.

(a) $y=\ln\left(x^2+5\right)$ (b) $y=4\ln x+3\ln\left(1-x\right)$ (c) $y=5+4\ln\left(3x^2-1\right)$

(d) $y=\log(3x)$ (e) $y=5\log_3(x+2)$ (f) $y=(2\ln x)^3$

(g) $y=x\ln(3x)$ (h) $y=\dfrac{\ln\left(2x+5\right)}{x^2}$ (i) $y=\sqrt{\ln(2x+1)}$

2. Differentiate the following functions with respect to x. Simplify your answers whenever possible.

(a) $s=\ln\sqrt{5+x^3}$ (b) $s=\dfrac{1}{2}\ln\left(x^2+7\right)^6$

(c) $y=\ln\left[\left(5x-4\right)\sqrt{x^2+3}\right]$ (d) $s=2\ln\left(\dfrac{2x+1}{2x-1}\right)$

3. Differentiate the following functions with respect to the given variables. Simplify your answers, where possible.

(a) $y=4^{2x}$ (b) $y=2^{4x+1}$ (c) $w=2e^{2x}+4e^{-2x}$

(d) $w=\sqrt{e^x}+\dfrac{1}{e^{2x}}-e^{\frac{1}{2}}$ (e) $w=3e^{2x+7}+4e^{x^2}+e^{\frac{1}{x}}$ (f) $y=\left(e^{2x}-4\right)^3$

4. Differentiate the following functions with respect to the given variables. Simplify your answers, where possible.

(a) $\cos^{-1}\left(2x\right)$ (b) $\tan^{-1}x^2$

(c) $2\sin^{-1}3x+\cos^{-1}3x+\tan^{-1}e$ (d) $\ln\left[\sin^{-1}x\right]$ (e) $e^{\sin^{-1}x}$

5. Use Product or Quotient Rule to differentiate the following with respect to the given variables and simplify your answers whenever possible.

(a) $y=xe^{2x}$ (b) $y=3e^{2x}\ln\left(x+2\right)$ (c) $y=\dfrac{\sin 2x}{e^x}$

(d) $y=5x\sin^{-1}x$ (e) $y=\sqrt{1-x^2}\sin^{-1}x$ (f) $y=\dfrac{1+4x^2}{\tan^{-1}2x}$

6. (**Electrical**) The impressed voltage in a circuit is given by $E(t)=6\ln\left(1+0.25t^2\right)$ volts. Find $E'(t)$ as a function of time t.

7. (**Mechanical**) The time variation of a certain mechanical strain is given by the equation $E = 80 + 60\sin 15t - 45\cos 30t$. Find $\dfrac{dE}{dt}$.

8. (**Life Sciences**) The model for a certain population over time t years is given as $P(t) = \dfrac{578}{1+4e^{-t/3}}$. Find $P'(6)$ when $t = 6$ years.

9. (**Chemical**) The mass m (in grams) of a certain substance is believed to dissolve in one litre of water at temperature T °C according to the law $m(T) = 2(10^{0.02T})$. Differentiate m with respect to T.

10. (**Environmental Science**) The acoustical intensity of a sound wave is given by $I(t) = A\cos(2\pi ft - \alpha)$, where f is the frequency of the sound. Find $I'(t)$ for a wave with the following values: $A = 0.027$ W/cm^2, $f = 240$ Hz and $\alpha = 0.80$.

11. (**Environmental Science**) The amount of radioactive radium after t years is given by $N = N_o e^{-4.279\times10^{-4}t}$, where N_o is the original amount of radioactive radium. Find $\dfrac{dN}{dt}$ if $N_o = 10$g.

12. If $f(t) = 2e^{-t}\cos 2t$, find $f'(0)$.

13. In the design of a cone-type clutch, an equation that relates the cone angle θ and the applied force F is $\theta = \sin^{-1}\left(\dfrac{Ff}{R}\right)$, where R is the frictional resistance and f is the coefficient of friction. Assume that R and f are constants, find $\dfrac{d\theta}{dF}$.

ANSWERS

Tutorial 4

1. (a) $\dfrac{2x}{x^2+5}$ (b) $\dfrac{4}{x}-\dfrac{3}{1-x}$ (c) $\dfrac{24x}{3x^2-1}$

   (d) $\dfrac{1}{x\ln 10}$ (e) $\dfrac{5}{(x+2)\ln 3}$ (f) $\dfrac{24(\ln x)^2}{x}$

   (g) $1+\ln(3x)$ (h) $\dfrac{2}{x^3}\left[\dfrac{x}{2x+5}-\ln(2x+5)\right]$ (i) $\dfrac{1}{(2x+1)\sqrt{\ln(2x+1)}}$

2. (a) $\dfrac{3x^2}{2(5+x^3)}$ (b) $\dfrac{6x}{x^2+7}$ (c) $\dfrac{5}{5x-4}+\dfrac{x}{x^2+3}$

   (d) $\dfrac{4}{2x+1}-\dfrac{4}{2x-1}$

3. (a) $4^{2x}\,2\ln 4$ (b) $2^{4x+1}\,4\ln 2$ (c) $4\left(e^{2x}-2e^{-2x}\right)$

   (d) $2\left(\dfrac{1}{4}e^{\frac{x}{2}}-e^{-2x}\right)$ (e) $6e^{2x+7}+8xe^{x^2}-\dfrac{e^{\frac{1}{x}}}{x^2}$ (f) $6e^{2x}\left(e^{2x}-4\right)^2$

4. (a) $-\dfrac{2}{\sqrt{1-4x^2}}$ (b) $\dfrac{2x}{1+x^4}$

   (c) $\dfrac{3}{\sqrt{1-9x^2}}$ (d) $\dfrac{1}{\sin^{-1}x\sqrt{1-x^2}}$ (e) $\dfrac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$

5. (a) $e^{2x}(2x+1)$ (b) $3e^{2x}\left[\dfrac{1}{x+2}+2\ln(x+2)\right]$ (c) $\dfrac{2\cos(2x)-\sin(2x)}{e^x}$

   (d) $5\sin^{-1}x+\dfrac{5x}{\sqrt{1-x^2}}$ (e) $1-\dfrac{x\sin^{-1}x}{\sqrt{1-x^2}}$ (f) $\dfrac{8x\tan^{-1}(2x)-2}{\left(\tan^{-1}2x\right)^2}$

6. $E'(t)=\dfrac{dE}{dt}=\dfrac{3t}{1+0.25t^2}$

7. $450(2\cos 15t+3\sin 30t)$

8. 43.9

9. (0.04 ln 10) $10^{0.02T}$ g/°C

10. $-12.96\pi\, sin(480\pi t-0.8)$ W/cm^{2}s

11. $-4.279\times10^{-3}e^{-4.279\times10^{-4}t}$

12. $f'(0)=-2$

13. $\dfrac{d\theta}{dF}=\dfrac{f}{\sqrt{R^2-F^2f^2}}$