Chapter 5 – Implicit Differentiation & Higher-Order Derivatives

Objectives:

1. *Define higher-order derivatives.*
2. *Evaluate higher-order derivatives of algebraic, exponential, logarithmic and trigonometric functions.*
3. *Perform differentiation involving implicit functions.*

5.1 Introduction

In this chapter we will look at differentiating a given expression more than once and also a method of differentiation known as implicit differentiation.

The three main rules of differentiation, namely the Chain Rule, the Product Rule and Quotient Rule are still applicable when we do higher order derivatives and implicit differentiation.

5.2 Higher-Order Differentiation

5.2.1 Define Higher-Order Derivatives

Higher order derivatives are essential in the study of series – Taylor's series, Maclaurin's series and Fourier series, just to name a few. These series are important in the fields of science and engineering as many processes can be mathematically approximated by them.

In chapter 5, we differentiated an expression only once. However it is possible to carry out the process of differentiation as many times as we want. Consider the following:

$$y = x^3$$

$$\frac{dy}{dx} = \frac{d}{dx}(x^3) = 3x^2$$ looking at $3x^2$, we know we can differentiate it to get $6x$.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(3x^2) = 6x$$ similarly, looking at $6x$, we can differentiate it to get 6

$$\frac{d^3y}{dx^3} = \frac{d}{dx}(6x) = 6$$ finally, the constant 6 can be differentiated to get 0

$$\frac{d^4y}{dx^4} = \frac{d}{dx}(6) = 0$$

Summary:

If $y = f(x)$

$\boldsymbol{\dfrac{dy}{dx} = f'(x)}$	is the notation for the first order derivative of y with respect to x. Another way of writing it is $\boldsymbol{y'}$ (read as "y prime").
$\boldsymbol{\dfrac{d^2y}{dx^2} = f''(x)}$	is the notation for the second order derivative of y with respect to x. We read it as "dee two y by dee x squared". Another way of writing it is $\boldsymbol{y''}$. The superscript '2' on the 'd' and 'x' gives the order of the derivative and tells us that we have differentiated y twice. What we really did was to differentiate the expression for $\dfrac{dy}{dx}$.
$\boldsymbol{\dfrac{d^3y}{dx^3} = f'''(x)}$	is the notation of the third order derivative of y with respect to x, which is read as "dee three y by dee x cubed". It tells us that we have differentiated thrice.

Thus if we were to carry out the process of differentiation n times, the n^{th} order derivative of y with respect to x would be written as $\boldsymbol{\dfrac{d^ny}{dx^n}}$ **or** $\boldsymbol{f^{(n)}(x)}$ (which reads as "dee 'n' y by dee x 'n' ") or as $\boldsymbol{y^{(n)}}$.

Example 1 : Given that $y = x^4 + 2x^3 - 3x^2 + x - 5$, obtain the first, second and third order derivatives of y with respect to x.

5.3 Differentiation of Implicit Functions

In this section we will first explain what implicit functions are. We will compare them with explicit functions so as to give our readers a fair idea of the differences between both functions. Our discussion will then move on to how the Chain Rule is used to differentiate implicit functions.

5.3.1 Implicit Functions

So far, we have dealt mostly with explicit functions in the topics we have covered.

Explicit functions are those functions whereby one variable (usually y) can be isolated and expressed clearly in terms of the other variables (usually x).

In other words, we can make one variable, explicitly, the subject. For example, $y = 3x^2 + 4x - 5$, $v = e^t \sin t$ and $r = t \ln t$ are examples of explicit functions.

However, for some equations, it is impossible or difficult to express one variable in terms of the other. Consider the following equations,

$$y^2 - 4xy + 3x = 8$$
$$\ln(x + y) - 2y = xy$$
$$2\sin x \cos y = x + y$$

Such functions are called **implicit functions**.

5.3.2 Implicit Differentiation

Sometimes it may be necessary for us to differentiate implicit functions, for example, to find the gradient of a curve at a point. In such cases, we will use the Chain Rule to differentiate.

The two examples below illustrate the difference between explicit differentiation and implicit differentiation.

<u>Explicit Differentiation</u> vs <u>Implicit Differentiation</u>

Find $\dfrac{d}{dx}\left(x^3\right)$

$$\frac{d}{dx}\left(x^3\right) = 3x^2 \frac{dx}{dx} = 3x^2$$

same

simplify to 1

Find $\dfrac{d}{dx}\left(u^3\right)$

use Chain rule here

$$\frac{d}{dx}\left(u^3\right) = \frac{d}{du}\left(u^3\right) \times \frac{du}{dx} = 3u^2 \frac{du}{dx}$$

different

change dx to du

don't forget!

Essentially there are only two steps involved:

(i) differentiate each term with respect to the required variable

(ii) make the differential coefficient the subject.

We will use the following example to explain with details how to differentiate an implicit function.

Example 2 : (a) $\dfrac{d}{dx}\left(y^4\right)$ (b) $\dfrac{d}{dx}\left(3\sin(4y)\right)$

(c) $\dfrac{d}{dt}\left(x^2+4t^2\right)$ (d) $\dfrac{d}{d\theta}\left(\theta^2 e^{3t}\right)$

Example 3 : Find $\dfrac{dy}{dx}$ for the following implicit functions.

(a) $y^2-x^2-3y=0$ (b) $y^2+e^{3y}=4x-5$

(c) $5\ln y = x^2\sin 2y + e^{4x+3}$

Tutorial 5

1. Obtain the first and second order derivatives of the following with respect to their respective variables. Simplify your answers whenever possible.

   (a) $y = x^2 + \dfrac{4}{x^2} - 5$ (b) $z = \dfrac{1}{t} + 3\sin(4t+1)$

2. If $p = 3s^6 + 4s^3 - 5$, find $\dfrac{dp}{ds}$, $\dfrac{d^2p}{ds^2}$ and $\dfrac{d^3p}{ds^3}$.

3. If $h(t) = at^3 - 2t^2 + 1$ and $h'''(t) = 12$, find the value of $h(-1)$.

4. If $f(x) = 6\sin 4x$, find $f'(0.3)$ and $f''(0.3)$.

5. Use implicit differentiation to find $\dfrac{dy}{dx}$ for the following. Simplify your answers whenever possible.

   (a) $2x + 3y^2 = 4$ (b) $5y^4 - 2y = e^x + \pi$

   (c) $6y + \ln y - \dfrac{4}{x^2} = 10$ (d) $(x-5)^2 + (y-3)^2 = 9$

   (e) $\ln y^3 - \tan^{-1} y = x + 1$ (f) $x^3 + 2xy^2 + y^3 = 3$

   (g) $ye^x + 2(x+y) = \ln 3$ (h) $x + 2y + 4\ln(xy) = 60$

   (i) $y^2 + (5x-y)^4 = e^{2x}$ (j) $e^{xy} + \sin(4y) = 5 - x^2$

   (k) $y\cos^2 x = x\sin y$ (l) $x\ln(5y+2) - 3y^2 = 2 + e$.

   (m) $2\tan^{-1}(xy) + x = 0$

6. Find the slope of the tangent line to the curve $\ln(1+xy) = x + y + 3$ at $x = 0$.

7. Given that $x^2 \sin\theta - 3x = \sec\theta$, determine the value of $\dfrac{dx}{d\theta}$ when $\theta = \pi$.

8. Given the curve $xy + y^2 + 2 = 0$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at the point $y = 1$.

9. (**Electrical**) The impressed voltage in a circuit, at time t seconds, is given by $E(t) = 6\ln\left(1 + 0.25t^2\right)$ volts. Find the time t when $E'(t) = 0$. Hence find the value of $E''(t)$ at this point of time.

10. (**Civil**) When using an external focussing instrument, the horizontal stadia multiplier H is given by $H = 100S\cos^2\theta + 1$, where S is the stadia intercept and θ is the vertical angle. If S = 0.8 m, find $\frac{dH}{d\theta}$ and $\frac{d^2H}{d\theta^2}$. Simplify your answers.

    (a) Find the smallest positive angle θ at which $\frac{dH}{d\theta}$ = 0.

    (b) What is the value of $\frac{d^2H}{d\theta^2}$ at this value of θ?

11. **(Life Sciences)** Biologists have proposed that the rate of production x of photosynthesis is related to the light intensity y by the formula $x(b + y^2) = ay$, where a and b are positive constants. Use implicit differentiation to find $\frac{dy}{dx}$ and simplify your answer.

*12. Show that the function $y = A\sin(\omega t + \alpha) + B\cos(\omega t + \alpha)$ satisfies the equation $\frac{d^2y}{dt^2} + \omega^2 y = 0$. Note that A, B, ω, and α are constants.

ANSWERS

Eg 2 : (a) $4y^3\dfrac{dy}{dx}$ (b) $12\cos(4y)\dfrac{dy}{dx}$ (c) $2x\dfrac{dx}{dt}+8t$ (d) $\theta e^{3t}\left(2+3\theta\dfrac{dt}{d\theta}\right)$

Eg 3 : (a) $\dfrac{dy}{dx}=\dfrac{2x}{2y-3}$ (b) $\dfrac{dy}{dx}=\dfrac{4}{2y+3e^{3y}}$

(c) $\dfrac{dy}{dx}=\dfrac{2y(x\sin(2y)+2e^{4x+3})}{5-2x^2y\cos(2y)}$

Tutorial 5

1. (a) $2\left(x-\dfrac{4}{x^3}\right)$; $2\left(1+\dfrac{12}{x^4}\right)$ (b) $12\cos(4t+1)-\dfrac{1}{t^2}$; $\dfrac{2}{t^3}-48\sin(4t+1)$

2. $18s^5+12s^2$; $90s^4+24s$; $360s^3+24$

3. $h=-3$

4. 8.697; − 89.476

5. (a) $\dfrac{dy}{dx}=-\dfrac{1}{3y}$ (b) $\dfrac{dy}{dx}=\dfrac{e^x}{20y^3-2}$ (c) $\dfrac{dy}{dx}=\dfrac{-8y}{6x^3y+x^3}$

(d) $\dfrac{dy}{dx}=\dfrac{5-x}{y-3}$ (e) $\dfrac{dy}{dx}=\dfrac{y(1+y^2)}{3+3y^2-y}$ (f) $\dfrac{dy}{dx}=\dfrac{-3x^2-2y^2}{4xy+3y^2}$

(g) $\dfrac{dy}{dx}=\dfrac{-ye^x-2}{e^x+2}$ (h) $\dfrac{dy}{dx}=\dfrac{-xy-4y}{2xy+4x}$ (i) $\dfrac{dy}{dx}=\dfrac{e^{2x}-10(5x-y)^3}{y-2(5x-y)^3}$

(j) $\dfrac{dy}{dx}=\dfrac{-2x-ye^{xy}}{xe^{xy}+4\cos(4y)}$ (k) $\dfrac{dy}{dx}=\dfrac{\sin y+2y\sin x\cos x}{\cos^2 x-x\cos y}$

(l) $\dfrac{dy}{dx}=\dfrac{-(5y+2)\ln(5y+2)}{5x-6y(5y+2)}$ (m) $\dfrac{dy}{dx}=\dfrac{-1-x^2y^2-2y}{2x}$

6. $\dfrac{1+xy-y}{x-1-xy}$; -4

7. $-\dfrac{1}{27}$

8. $\left.\dfrac{dy}{dx}\right|_{y=1}=1$, $\left.\dfrac{d^2y}{dx^2}\right|_{y=1}=4$

9. 0; 3

10. $-80\sin 2\theta$; $-160\cos 2\theta$; $\dfrac{\pi}{2}$; 160

11. $\dfrac{y^2+b}{a-2xy}$