Chapter 2 : Integrating Functions of Linear Function & Using Trigo. Identities

Objectives :

1. Integrate functions of a linear function

2. Integrate trigonometric functions using trigonometric identities.

2.1 Revision on Integration

2.1.1 Integration is the process of finding anti-derivatives.

If $\frac{d}{dx}(F(x)) = f(x)$,

then $\int f(x)\,dx = F(x) + C$, where C is an arbitrary constant.

2.1.2 **Standard Integrals**: (Fill in the blanks)

- Algebraic Function

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad , \quad n \neq -1$$

- Reciprocal Function

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

- Exponential Function

$$\int e^x dx = e^x + C$$

- Trigonometric Function

$$\int \sin(x)dx = -\cos(x) + C$$

$$\int \cos(x)dx = \sin(x) + C$$

$$\int \sec^2(x)dx = \tan(x) + C$$

$$\int \csc^2(x)dx = -\cot(x) + C$$

$$\int \sec(x)\tan(x)dx = \sec(x) + C$$

$$\int \csc(x)\cot(x)dx = -\csc x + C$$

$$\int \tan(x)dx = -\ln|\cos x| + C$$

$$\int \cot(x)dx = \ln|\sin x| + C$$

$$\int \sec(x)dx = \ln|\sec x + \tan x| + C$$

$$\int \csc(x)dx = -\ln|\csc x + \cot x| + C$$

2.1.3 **Properties of Indefinite integral**

- $\int k \cdot f(x)dx = k\int f(x)dx$, k is a constant.
- $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

***Example 1*:**

(a) $\int \left(x - \frac{1}{x^2} - 3 \right) dx =$

(b) $\int \left(\frac{5}{x} + \sqrt{x} - 5e^{2x} \right) dx =$

(c) $\int \left(\sin x - \cos x + 3\sec^2 x \right) dx =$

(d) $\int \tan x \left(1 + \sec x \right) dx =$

2.1.4 Definite Integrals

If $\int f(x)dx = F(x) + C,$ then $\int_a^b f(x)\ dx = F(b) - F(a)$

In definite integral, whenever trigonometric functions, such as $\sin x$ or $\cos x$, is involved, the limits for x are measured in **radians**. Therefore, when evaluating a definite integral involving trigonometric function, your calculator has to be in radian mode.

***Example 2*:** **Evaluate:**

(a) $\int_0^2 \left(x + 2 \right) dx$

(b) $\int_0^{0.5} \left(3e^{-2t} - \frac{1}{2}\cos \pi t \right) dt$

Note

This is the wrong move

$$\int \cos(\pi t)\ dt = \frac{\sin(\cancel{\pi} t)}{\cancel{\pi}} + C$$

$$\neq \sin(t) + C$$

2.2 Integration of Functions of a Linear Function

2.2.1 Linear Function

A function $f(x) = ax + b$, where a and b are constants and $a \neq 0$, is known as a <u>linear function</u> of x. E.g. $2x + 1$, $-5x + 1$, $\frac{x}{\sqrt{2}} - 3$ are linear functions of x.

2.2.2 Functions of a Linear Function

Functions of a linear function are functions that are in terms of $(ax + b)$,
e.g. $(5x+2)^3$ is a cubic function of the linear function $5x + 2$,

$\cos\left(\frac{x}{2} - \frac{\pi}{4}\right)$ is a cosine function of the linear function $\frac{x}{2} - \frac{\pi}{4}$, also written as $\frac{1}{2}x - \frac{\pi}{4}$.

2.2.3 Integration of Algebraic Functions of a Linear Function

Consider the differentiation of the function $(ax+b)^{n+1}$,

$$\begin{aligned}\frac{d}{dx}(ax+b)^{n+1} &= (n+1)\cdot(ax+b)^n \frac{d}{dx}(ax+b)\\ &= (n+1)\cdot(ax+b)^n \cdot a\\ &= a(n+1)\cdot(ax+b)^n\end{aligned}$$

The **reverse process** (i.e. integration) gives:

$$\int a(n+1)\cdot(ax+b)^n \, dx = (ax+b)^{n+1} + C_1$$

$$a(n+1)\int (ax+b)^n \, dx = (ax+b)^{n+1} + C_1$$

$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

Hence $$\boxed{\int (ax+b)^n \, dx = \frac{1}{a}\cdot\frac{(ax+b)^{n+1}}{(n+1)} + C \quad , \qquad n \neq -1}$$

Example 3: Find: (a) $\int (3x+1)^9 \, dx$ (b) $\int \frac{1}{(3x+4)^2} \, dx$ (c) $\int \frac{2}{\sqrt{1-3u}} \, du$

Solution: (a) $\int (3x+1)^9 \, dx =$

(b) $\int \frac{1}{(3x+4)^2}\,dx =$

(c) $\int \frac{2}{\sqrt{1-3u}}\,du =$

2.2.4 Integration of the Reciprocal of a Linear Function

Consider the differentiation of the function $\ln(ax+b)$,

$$\frac{d}{dx}\ln(ax+b) = \frac{1}{(ax+b)}\cdot\frac{d}{dx}(ax+b)$$

$$= \frac{a}{ax+b}$$

The reverse process gives $\int \frac{a}{ax+b}\,dx = \ln(ax+b) + C_1$

Hence $\boxed{\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C}$

Example 4: Find : (a) $\int_0^1 \frac{1}{3x+4}\,dx$ (b) $\int \frac{2}{1-4x}\,dx$ (c) $\int \frac{x+2}{x+1}\,dx$

Solution:

(a) $\int_0^1 \frac{1}{3x+4}\,dx =$

(b) $\int \frac{2}{1-4x}\,dx =$

(c) $\int \frac{x+2}{x+1}\,dx =$

2.2.5 Integration of Exponential Functions of a Linear Function

Consider the differentiation of the function e^{ax+b} ,

$$\frac{d}{dx}\left(e^{ax+b}\right) = e^{ax+b} \cdot \frac{d}{dx}(ax+b)$$
$$= e^{ax+b} \cdot a$$

The reverse process gives $\int a \cdot e^{ax+b} dx = e^{ax+b} + C_1$

Hence $\boxed{\int e^{ax+b} dx = \frac{1}{a} \cdot e^{ax+b} + C}$

Example 5: Find : (a) $\int e^{7x+2} dx$ (b) $\int \frac{dx}{e^{3+x}}$ (c) $\int \sqrt{e^{x+3}}\, dx$

Solution:

(a) $\int e^{7x+2} dx =$

(b) $\int \frac{dx}{e^{3+x}} =$

(c) $\int \sqrt{e^{x+3}}\, dx =$

2.2.6 Summary (Integration of Functions of a Linear Function)

In general,

if	$\int f(x)\,dx = F(x) + C \quad \leftarrow \{\text{Basic Formula}\}$
then	$\int f(ax+b)\, dx = \frac{1}{a} \cdot F(ax+b) + C$

That is, the steps to integrate function of a linear function are outlined as follow:

1. apply the basic formula (standard integral), then
2. include the extra factor " $\frac{1}{a}$ " in the result.

Example 6: Find $\int 3^{2x+7} dx$

Solution: $\int 3^{2x+7} dx =$

<u>Standard Integral</u> (given in formulae card)
$\int k^x \; dx = \frac{k^x}{\ln k} + C$, where k is a constant

2.2.7 Integration of Trigonometric Functions of a Linear Function

Example 7: Using the method of section 2.2.6 , find the following integrals.

(a) $\int \cos(2x+\pi)\ dx =$

(b) $\int_0^1 \sin(3x+1)\,dx =$

(c) $\int 2\sec^2(\pi t-1)\ dt =$

Standard Integral
(given in formulae card)

$$\int \cos(x)\ dx = \sin(x) + C$$

$$\int \sin(x)\ dx = -\cos(x) + C$$

$$\int \sec^2(x)\ dx = \tan(x) + C$$

In general, there is always a factor of " $\frac{1}{a}$ " in the result of integration of trigonometric function of a linear function.

2.3 Integration using Trigonometric Identities

2.3.1 Integrals of Product of Sine and Cosine Functions

Apply the following **Product to Sum** identities:

$$\sin x\ \cos y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$$

$$\cos x\ \sin y = \frac{1}{2}\left[\sin(x+y) - \sin(x-y)\right]$$

$$\cos x\ \cos y = \frac{1}{2}\left[\cos(x+y) + \cos(x-y)\right]$$

$$\sin x\ \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$$

Remember also that

$$\sin(-A) = -\sin A \quad \text{and} \quad \cos(-A) = \cos A$$

Example 8: Find $\int \sin 2x \cos 3x\ dx$ *Ans*: $-\frac{\cos 5x}{10} + \frac{\cos x}{2} + C$

2.3.2 Integrals of Even Powers of Sine and Cosine Functions

If the integral is of the form $\int sin^m x\, cos^n x\, dx$, and both m and n are even, we use the formulae for **Reducing Power**:

$$\boxed{\begin{aligned}\sin^2 x &= \frac{1-\cos 2x}{2} \\ \cos^2 x &= \frac{1+\cos 2x}{2}\end{aligned}}$$

One of the exponents m or n may be zero.

Example 9: Find $\int 5\sin^2 3x\, dx$ *Ans*: $\frac{5}{2}\left(x-\frac{\sin 6x}{6}\right)+C$

Example 10: Find $\int \sin^2 x\cos^2 x\, dx$ *Ans*: $\frac{1}{8}x-\frac{1}{32}\sin 4x+C$

2.4 Application: Root-Mean-Square (RMS) Value

The **root-mean-square (rms) value** of a function $y = f(x)$ over an interval $x = a$ to $x = b$ is defined as

$$\boxed{y_{rms} = \sqrt{\frac{1}{b-a}\int_a^b [f(x)]^2\,dx}}$$

Note that y_{rms} is non-negative.

Example 11: Find the rms value of the voltage $y = 2x + 1$ over the interval $x = 1$ to $x = 4$.

Example 12: Find the rms value of the voltage $v = 3\sin 2t$. *Ans*: $\dfrac{3}{\sqrt{2}}$

Tutorial 2

Section A: Basic Integration

1. Find the following integrals:

(a) $\int \left(x - \frac{1}{x^2} \right) dx$

(b) $\int \left(e^{5x} + \frac{3}{e^{3x}} \right) dx$

(c) $\int \frac{x}{3}(2x + \sqrt{x})\, dx$

(d) $\int \left(x^2+2\right)\left(4x-3\right) dx$

(e) $\int e^x \left(2e^x + \frac{1}{e^{3x}} \right) dx$

(f) $\int 2 \tan 3x \, dx$

(g) $\int \cot 6x \, dx$

(h) $\int \frac{2}{9+x^2}\, dx$

2. Evaluate the following definite integrals:

(a) $\int_1^3 x^3 dx$

(b) $\int_2^5 \, dx$

(c) $\int_1^{10} \frac{1}{2x} dx$

(d) $\int_0^1 e^{2x} dx$

(e) $\int_{\pi/3}^{\pi} \cos 2x \, dx$

(f) $\int_1^4 (x^2+3x)\, dx$

(g) $\int_1^2 \left(x^2 + \frac{1}{x} - 3 \right) dx$

(h) $\int_{-2}^{-1} \left(4e^{-2x} + \frac{3}{x} \right) dx$

(i) $\int_0^1 (5x - \sin 3x) dx$

(j) $\int_{\pi/6}^{\pi/3} \left(\sin 3x - \cos 4x \right) dx$

(k) $\int_2^4 \left(5\sin 3x + \frac{2}{x} \right) dx$

Section B: Integration of functions of linear function

1. Find the following integrals.

(a) $\int (3x+2)^4 dx$

(b) $\int \left(1-2x\right)^2 dx$

(c) $\int \sqrt{4-3x}\ \ dx$

(d) $\int \frac{1}{(2x-3)^5}\, dx$

(e) $\int \sin\left(2x+1\right) dx$

(f) $\int \cos\left(3x - \frac{\pi}{6} \right) dx$

(g) $\int e^{\frac{x}{2}+5}\, dx$

(h) $\int 5e^{3x-2}\, dx$

(i) $\int \frac{1}{8x+3}\, dx$

(j) $\int \frac{3}{2x-25}\, dx$

(k) $\int \frac{1}{2-x} dx$

(l) $\int \frac{4}{25-4x} dx$

2. Evaluate the following definite integrals.

(a) $\int_{-1}^1 (4x-3)^2 dx$

(b) $\int_{4.5}^{10.5} \frac{2}{\sqrt{2x-5}}\, dx$

(c) $\int_{-2/3}^{0} \frac{1}{e^{3x+2}}\, dx$

Section C: Integration using trigo. identities

1. Find the following integrals:

(a) $\int 2\sin x \cos x \, dx$

(b) $\int \frac{1}{\cos^2(2x)} \, dx$

(c) $\int 2\tan^2 2x \, dx$

(d) $\int 2\sin 3x \cos 5x \, dx$

(e) $\int 3\sin\frac{3t}{2}\sin\frac{5t}{2} \, dt$

(f) $\int \sin^2\theta \cos 3\theta \, d\theta$

*(g) $\int \cos^4 x \, dx$

*(h) $\int_0^{\pi/2} \sin^4 x \, dx$

2. Find the root-mean-square (rms) value of

(a) $f(t) = 1 + 3e^{-t}$ from $t = 0$ to $t = 2$

(b) $y = 2(\sin x + \cos x)$ from $x = 0$ to $x = \pi$

*3. If a and b are integers, find the following integrals for each of the following 3 cases

(i) $a \neq b$, (ii) $a = b \neq 0$, (iii) $a = b = 0$:

(a) $\int \cos ax \cos bx \, dx$

(b) $\int \sin ax \sin bx \, dx$

(c) $\int \sin ax \cos bx \, dx$

*4. If m and n are integers, use the results of question 3 to show that

(a) $$\int_0^{2\pi} \cos mx \cos nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \neq 0 \\ 2\pi & \text{if } m = n = 0 \end{cases},$$

(b) $$\int_0^{2\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \neq 0 \\ 0 & \text{if } m = n = 0 \end{cases},$$

(c) $$\int_0^{2\pi} \cos mx \sin nx \, dx = 0.$$

Miscellaneous Exercises

*1. Find the results of the integrals.

(a) $\int \frac{(x-2)^2}{x}\,dx$ (b) $\int \left(2e^{-x} + e^{x}\right)^2 dx$

(c) $\int \frac{e^{2x}+2e^{x}}{e^{2x}}\,dx$ (d) $\int \left(1-e^{-x}\right)^2 dx$

*(e) $\int \frac{x}{x-1}\,dx$ *(f) $\int \sin^2 x \cos^4 x\,dx$

*(g) $\int \cos^4 2x \sin^3 2x\,dx$ *(h) $\int \sin^3\theta \cos^3\theta\,d\theta$

*(i) $\int \cot^2 \pi x\,dx$

*2. If the current in an electric circuit is given by $i = I_p \sin \omega t$, where I_p is the maximum current, show that the rms value of the current from $t = 0$ to $t = \frac{2\pi}{\omega}$ is $\frac{I_p}{\sqrt{2}}$.

Multiple Choice Questions

1. Given that $\frac{d}{dx}\left(\sin^3 x \tan x\right) = \sin x \tan^2 x + 3\sin^3 x$, which of the following is equivalent to $\int 3\sin^3 x\,dx$?

(a) $\int \sin^3 x \tan x\,dx$ (b) $\sin^3 x \tan x - \int \sin x \tan^2 x\,dx$

(c) $\int \left(\sin^3 x \tan x - \sin x \tan^2 x\right)dx$ (d) $\sin^3 x \tan x - \sin x \tan^2 x + C$

2. Given the expression $\int f(x)dx = [f(x)]^2 + C$, where C is an arbitrary constant, which of the following could be $f(x)$?

(a) $f(x) = x+1$ (b) $f(x) = 2x+1$

(c) $f(x) = \frac{1}{2}x+1$ (d) $f(x) = x+\frac{1}{2}$

Answers

Section A

1. (a) $\frac{x^2}{2}+\frac{1}{x}+C$ (b) $\frac{1}{5}e^{5x}-\frac{1}{e^{3x}}+C$ (c) $\frac{2}{9}x^3+\frac{2}{15}x^{\frac{5}{2}}+C$

   (d) $x^4-x^3+4x^2-6x+\mathrm{C}$ (e) $\mathrm{e}^{2x}-\frac{1}{2}\mathrm{e}^{-2x}+\mathrm{C}$ (f) $-\frac{2}{3}\ln|\cos 3x|+\mathrm{C}$

   (g) $\frac{1}{6}\ln|\sin 6x|+C$ (h) $\frac{2}{3}\tan^{-1}\frac{x}{3}+C$

2. (a) 20 (b) 3 (c) 1.15

   (d) 3.195 (e) $-\frac{\sqrt{3}}{4}$ (f) 43.5

   (g) 0.026 (h) 92.34 (i) 1.84

   (j) 0.766 (k) 1.581

Section B

1. (a) $\frac{1}{15}(3x+2)^5+C$ (b) $-\frac{1}{6}(1-2x)^3+C$ (c) $-\frac{2}{9}(4-3x)^{3/2}+C$

   (d) $-\frac{1}{8(2x-3)^4}+C$ (e) $-\frac{1}{2}\cos(2x+1)+C$ (f) $\frac{1}{3}\sin\left(3x-\frac{\pi}{6}\right)+C$

   (g) $2e^{\frac{x}{2}+5}+C$ (h) $\frac{5}{3}e^{3x-2}+C$ (i) $\frac{1}{8}\ln|8x+3|+C$

   (j) $\frac{3}{2}\ln|2x-25|+C$ (k) $-\ln|2-x|+C$ (l) $-\ln|25-4x|+C$

2. (a) 86/3 (b) 4 (c) 0.2882

Section C

1. (a) $-\frac{1}{2}\cos 2x+C$ (b) $\frac{1}{2}\tan 2x+C$

   (c) $\tan 2x-2x+C$ (d) $\frac{\cos 2x}{2}-\frac{\cos 8x}{8}+C$

   (e) $\frac{3}{2}(\sin t-\frac{1}{4}\sin 4t)+C$ (f) $\frac{\sin 3\theta}{6}-\frac{\sin 5\theta}{20}-\frac{1}{4}\sin\theta+C$

   (g) $\frac{1}{8}(3x+2\sin 2x+\frac{1}{4}\sin 4x)+C$ (h) $\frac{3\pi}{16}$

2. (a) 2.41 (b) 2

3. (a) (i) $\frac{1}{2}\left[\frac{\sin[(a-b)x]}{a-b}+\frac{\sin[(a+b)x]}{a+b}\right]+C$; (ii) $\frac{1}{2}\left[x+\frac{\sin 2ax}{2a}\right]+C$; (iii) $x+C$

(b) (i) $\frac{1}{2}\left[\frac{\sin[(a-b)x]}{a-b}-\frac{\sin[(a+b)x]}{a+b}\right]+C$; (ii) $\frac{1}{2}\left[x-\frac{\sin 2ax}{2a}\right]+C$; (iii) 0

(c) (i) $-\frac{1}{2}\left[\frac{\cos[(a-b)x]}{a-b}+\frac{\cos[(a+b)x]}{a+b}\right]+C$; (ii) $-\frac{\cos 2ax}{4a}+C$; (iii) 0

Miscellaneous Exercises

1. (a) $\frac{x^2}{2}-4x+4\ln|x|+C$ (b) $-2e^{-2x}+4x+\frac{e^{2x}}{2}+C$

(c) $x-2e^{-x}+C$ (d) $x+2e^{-x}-\frac{1}{2}e^{-2x}+C$

(e) $x+\ln|x-1|+C$ (f) $\frac{1}{16}\left(x-\frac{1}{12}\sin 6x-\frac{1}{4}\sin 4x+\frac{1}{4}\sin 2x\right)+C$

(g) $\frac{1}{32}\left(-\frac{3}{4}\cos 2x-\frac{1}{4}\cos 6x+\frac{1}{20}\cos 10x+\frac{1}{28}\cos 14x\right)+C$

(h) $\frac{1}{32}\left(-\frac{3}{2}\cos 2\theta+\frac{1}{6}\cos 6\theta\right)+C$ (i) $-\frac{1}{\pi}\cot \pi x-x+C$

MCQ

1. (b) 2. (c)