EP0604 Further Mathematics

Singapore Polytechnic, School of Mathematics and Science
Academic Year 2021/2022 Semester 1 **Further Mathematics**
Mid-Semester Test **Duration: 1.5 hour**

Instructions

1. All SP examination rules are to be complied with.
2. This paper consists of 4 pages.
3. Answer ALL the questions. Unless otherwise stated, leave your answers in 2 decimal places.
4. Except for graphs and diagrams, no solutions are to be written in pencil.

Additional Formulae

Log of a power: $\log_b a^x = x \log_b a$

Differentiation from first principles: $f'(x) = \lim_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

1. (a) A tailor is cutting off pieces of string from a long roll of string. The first piece he cuts off is 256 cm long and each successive piece is $\frac{3}{4}$ as long as the preceding piece.

(i) Show that the total length of string cut off can never be greater than 1024 cm.
(4 marks)

(ii) Find the maximum number of pieces must be cut off before the total length cut off is greater than 1000 cm? You must show sufficient working to justify your answer.
(6 marks)

(b) In figure 1, the line segment from $A(-4,3)$ to $B(2,1)$ is the diameter of the circle. Find the equation of the circle.

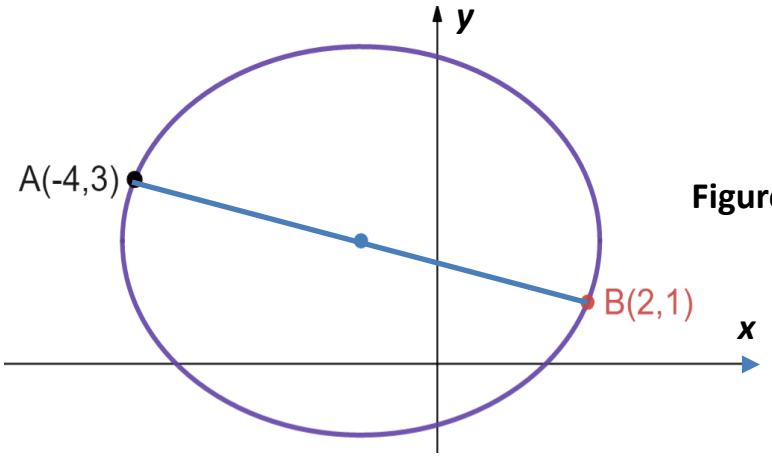


Figure 1

(8 marks)

2. (a) The position of an object at time t seconds, $t \geq 0$, is given by the parametric equations

$$x = 2t^2 + 1, \qquad y = 3 - t, \quad \text{where } t \geq 0.$$

(i) What is the position of the object at $t = 5$ seconds? (2 marks)

(ii) Sketch the position of the object for $0 \leq t \leq 3$. (4 marks)

(iii) Does the object pass through the point $(9,1)$? If so, when? (4 marks)

(iv) Find the equation of the normal to the curve at the point for which $t = 1$. (7 marks)

(b) A curve C is defined by the parametric equations

$$x = asin(t), \qquad y = \frac{1}{2}acos^2(2t), \qquad \text{where } a \text{ is a positive constant.}$$

Find a Cartesian equation of C.
(Hint: Apply the double angle formula of trigonometry) (7 marks)

3. (a) $f(x)$ and $g(x)$ are real-valued functions with the rules

$$f(x) = \ln(x - 3)$$
$$g(x) = 2 + \sqrt{x - 2}$$

(i) What is the domain and range of $f(x)$? (4 marks)

(ii) What is the domain and range of $g(x)$? (4 marks)

(b) The function f is such that $f(x) = \frac{1}{x-1}$ for $x \neq 1$.

(i) Prove that $(f \circ f^{-1})(x) = x$. (4 marks)

(ii) Prove that $(f \circ f)(x) = \frac{x-1}{2-x}$. (4 marks)

(iii) Find $(f \circ f \circ f)(x)$. (4 marks)

4.

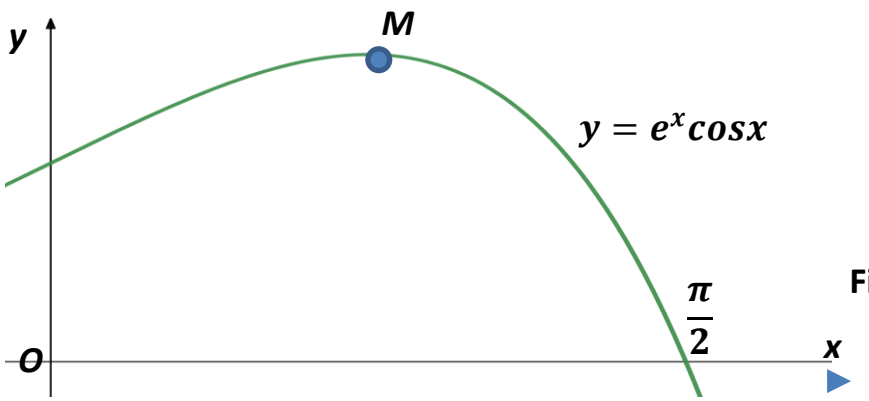


Figure 2

Figure 2 shows the curve $y = e^x cosx$ for $0 \leq x \leq \frac{\pi}{2}$, and its maximum point M.

(a) Prove that the coordinates of M is $\left(\frac{\pi}{4}, 1.55\right)$. (8 marks)

(b) Find the coordinates of the point at which the normal to the curve at (0, 1) cuts the x-axis. (6 marks)

5. (a) A submarine cable linking 2 regions will help boost digital connectivity between the regions. In figure 3, x is a variable, it costs \$$a$ per km to lay the cable across the ocean and \$$b$ per km to lay the cable on land, where a and b are positive constants.

(i) Find the formula of the cost, y, to lay the cable from point A to point B in terms of x, a and b.

(ii) Find the value of x in terms of a and b for which y has a stationary value, given that $a > b$. You are NOT required to show this value of x makes y a minimum.

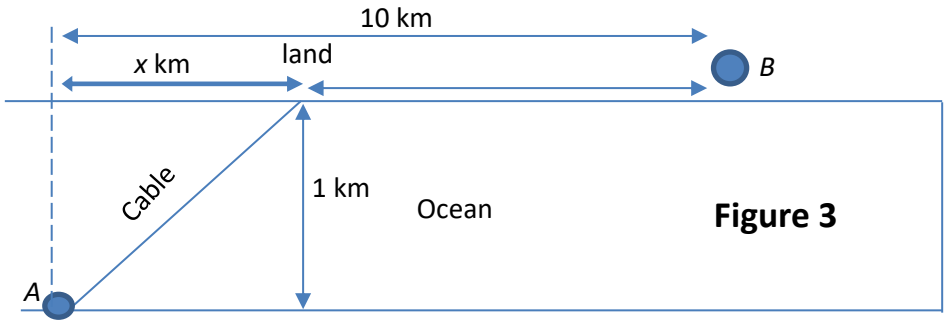


Figure 3

(14 marks)

(b) Find the derivative of $f(x) = x^n$ using first principles, given n is an arbitrary constant. (10 marks)

~ End of paper ~

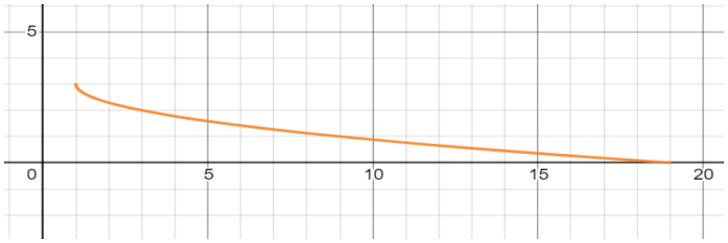
EP0604 Further Mathematics

Answers (MST 21/22 S1)

1. (a) (i) $S_\infty = \frac{256}{1-\frac{3}{4}} = 1024$ (ii) 13 pieces

(b) Equ of circle: $(x+1)^2 + (y-2)^2 = 1$

2. (a)(i) $(51, -2)$ (ii)



(iii) with y = 9, find t (t =+/- 2) check when t =2 whether it gives x = 1. (iv) $y = 4x - 10$

(b) $y = \frac{1}{2}a\left[1 - 2(\frac{x}{a})^2\right]^2$

3. (a) (i) $D_f = (3, \infty), R_f = (-\infty, \infty)$ (ii) $D_g = [2, \infty), R_f = [2, \infty)$

(b) (iii) $(f \circ f \circ f)(x) = \frac{2-x}{2x-3}$

4. (a) $y = e^x cosx$

$\frac{dy}{dx} = e^x(-sinx) + cosxe^x$

$e^x(-sinx + cosx) = 0$

$e^x = 0(N.A.)\ or - sinx + cosx = 0$

sinx = cosx

divide by cosx: $\frac{sinx}{cosx} = 1$ (this is a standard way to solve sinx = cosx)

$tanx = 1$

$x = \frac{\pi}{4}$ $y = e^{\frac{\pi}{4}}\cos(\frac{\pi}{4})$=1.55

(b) Equ of normal: $y = -x + 1$

Cut x-axis, y=0, x =1

5. (a) (i) $y = a\sqrt{1+x^2} + b(10-x)$ (ii) $x = +\frac{b}{\sqrt{a^2-b^2}}$ or $-\frac{b}{\sqrt{a^2-b^2}}$ $(N.A.)$

(b) $f'(x) = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$

By applying binomial theorem, $f'(x) = nx^{n-1}$. Full solutions refer to blackboard.