Chapter 10 : Laplace Transform

Objectives :

1. Understand the definition of Laplace transform.
2. Know the Laplace transforms of common functions.
3. Use standard results and apply basic theorems.

10.1 Introduction

The Laplace transform has a key role to play in the modern approach to the analysis and design of engineering systems. It is an example of a class called integral transforms, in which Fourier transform is another widely used example.

In Laplace transform, a function $f(t)$ of one variable t (time) is changed into a function $F(s)$ of another variable s. In doing so, it transforms a *differential* equation in the t (time) domain into an *algebraic* equation in the s domain, which makes solving easier. After obtaining the solution of the algebraic equation in s, the solution of the original differential equation in t can be obtained by using the inverse Laplace transform.

Another advantage of using the Laplace transform to solve differential equations is that initial conditions play an important role in the transformation process, so they are automatically incorporated into the solution. Hence, it is an ideal tool for solving initial-value problems such as those occurring in the investigation of electrical circuits and mechanical vibrations.

A further important advantage is that the method of Laplace transform enables us to deal with situations where the function is discontinuous or periodic, or even impulsive.

10.2 Definition and Notation of the Laplace Transform

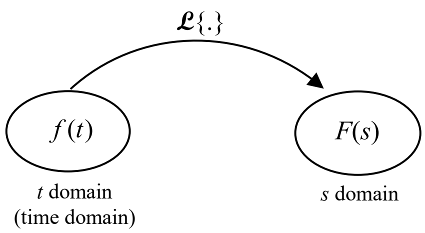
Let $f(t)$ be a function defined for $t > 0$. The Laplace transform of $f(t)$, denoted by $\mathcal{L}\{f(t)\}$, is defined by

$$\mathcal{L}\left\{f(t)\right\} = \int_0^\infty e^{-st} f(t)\, dt \text{, where } s > 0$$

The Laplace transform of $f(t)$ is said to exist if the above integral converges for some value of s; otherwise it does not exist.

It is usual to represent the Laplace transform of a function by the corresponding capital letter. That is, $\mathcal{L}\{f(t)\} = F(s)$. We may similarly write $\mathcal{L}\{q(t)\} = Q(s)$, $\mathcal{L}\{i(t)\} = I(s)$, $\mathcal{L}\{v(t)\} = V(s)$, etc.

The symbol $\mathcal{L}$ denotes the Laplace transform operator. The relationship between $f(t)$ and $F(s)$ is depicted graphically here:



10.3 Laplace Transforms of Simple Functions

Let us use the **definition** stated in 8.2 to prove the following common functions.

***Example 1*:** $\mathcal{L}\{1\} = \frac{1}{s}$

Proof:
$$\begin{aligned}\mathcal{L}\{1\} &= \int_0^\infty e^{-st}(1)\,dt && [\text{ taking } f(t) \text{ as the value 1 }] \\ &= -\frac{1}{s}\left[e^{-st}\right]_0^\infty && \left[\text{ Let } [\;]_0^\infty \text{ denote } \lim_{b\to\infty}[\;]_0^b \right] \\ &= -\frac{1}{s}\lim_{b\to\infty}\left[e^{-st}\right]_0^b = -\frac{1}{s}\lim_{b\to\infty}\left(e^{-sb}-1\right) \\ &= -\frac{1}{s}(0-1) = \frac{1}{s} && \text{since } e^{-sb} \to 0 \text{ as } b \to \infty \text{ if } s > 0.\end{aligned}$$

***Example 2*:** $\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}, s > a$

Proof:
$$\mathcal{L}\left\{e^{at}\right\} = \int_0^\infty e^{-st}e^{at}\,dt$$
$$= \int_0^\infty e^{-(s-a)t}\,dt = -\frac{1}{s-a}\left[e^{-(s-a)t}\right]_0^\infty$$

Since $e^{-(s-a)b} \to 0$ as $b \to \infty$ if $(s-a) > 0$ or $s > a$,

$$\mathcal{L}\left\{e^{at}\right\} = -\frac{1}{s-a}(0-1) = \frac{1}{s-a}.$$

Example 3: $\mathcal{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}}$ for positive integer *n*.

Proof:

In addition to the above functions, the Laplace transforms of other functions can be determined in much the same way. The final results have been collated and presented into a **formulae table** which can be found at the end of this book as well as in the Maths Formulae Card. We shall make references to these **standard results** from this point onwards.

Formula $\boxed{2}$

t^n (n is a positive integer)	$\dfrac{n!}{s^{n+1}}$

Example 4: $\mathcal{L}\left\{t^2\right\}$

Example 5: $\mathcal{L}\left\{t^5\right\}$

Formula $\boxed{3}$

e^{at}	$\dfrac{1}{s-a}$ $(s > a)$

Example 6: $\mathcal{L}\left\{e^{3t}\right\}$

Example 7: $\mathcal{L}\left\{e^{-2t}\right\}$

Example 8: $\mathcal{L}\left\{e^{\frac{t}{2}}\right\}$

Formula $\boxed{4}$	$\sin at$	$\dfrac{a}{s^2+a^2}$
Formula $\boxed{5}$	$\cos at$	$\dfrac{s}{s^2+a^2}$

Example 9: $\mathcal{L}\left\{\sin 3t\right\}$

Example 10: $\mathcal{L}\left\{\cos t\right\}$

Formula $\boxed{6}$	$t\sin at$	$\dfrac{2as}{\left(s^2+a^2\right)^2}$
Formula $\boxed{7}$	$t\cos at$	$\dfrac{s^2-a^2}{\left(s^2+a^2\right)^2}$

Example 11: $\mathcal{L}\left\{t\sin 3t\right\}$

Example 12: $\mathcal{L}\left\{t\cos 4t\right\}$

10.4 Some Basic Theorems

Theorem: Linearity Property

If $f_1(t)$ and $f_2(t)$ are functions of t and, a and b are constants, then

$$\mathcal{L}\{a\,f_1(t)+b\,f_2(t)\}=a\,F_1(s)+b\,F_2(s)$$

where $F_1(s)$ and $F_2(s)$ are Laplace transforms of $f_1(t)$ and $f_2(t)$ respectively.

***Example 13*:** $\mathcal{L}\{5\sin 3t\}=$

***Example 14*:** $\mathcal{L}\{e^{3t}+\cos t\}=$

More Examples

***Example 15*:** Find $\mathcal{L}\{\pi-2t^3+e^{4t}\}$

***Example 16*:** Find $\mathcal{L}\{6e^{-t}-3+7\,t\sin\pi t\}$

***Example 17*:** Find $\mathcal{L}\{3t(t-2)\}$

Example 18: Find $\mathcal{L}\left\{e^{t+2}\right\}$

Example 19: Find $\mathcal{L}\left\{\sin\left(3t+\frac{\pi}{4}\right)\right\}$

Example 20: Find $\mathcal{L}\left\{\sin^2 3t\right\}$

Example 21: Find $\mathcal{L}\left\{t\sin 2t\cos 2t\right\}$

First Shift Theorem (Formula $\boxed{8}$)

If $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\left\{e^{at} f(t)\right\} = \int_0^{\infty} e^{-st} e^{at} f(t)\, dt$$
$$= \int_0^{\infty} e^{-(s-a)t} f(t)\, dt = F(s-a), \quad s > a$$

The transform $\mathcal{L}\left\{e^{at} f(t)\right\}$ is the same as $\mathcal{L}\{f(t)\}$ with 's' everywhere in the result replaced by $(s - a)$.

Hence, we can also write: $\mathcal{L}\left\{e^{at} f(t)\right\} = F(s)\big|_{s\to s-a} = F(s-a)$

For example,

since $\mathcal{L}\left\{t^3\right\} = \dfrac{3!}{s^{3+1}} = \dfrac{6}{s^4}$, then to obtain $\mathcal{L}\left\{e^{2t}\, t^3\right\}$,

we replace 's' with $(s - 2)$ and get $\mathcal{L}\left\{e^{2t} t^3\right\} =$

Alternatively, we can write: $\mathcal{L}\left\{e^{2t} t^3\right\} = \left.\dfrac{6}{s^4}\right|_{s\to s-2} =$

<u>More Examples on First Shift Theorem</u>

***Example 22*:** Find $\mathcal{L}\left\{e^{t} \sin 3t\right\}$

***Example 23*:** Find $\mathcal{L}\left\{t\, e^{t} \sin 3t\right\}$

***Example 24*:** Find $\mathcal{L}\left\{2t\, e^{5t}\right\}$

***Example 25*:** Find $\mathcal{L}\left\{5e^{-2t} \cos 3t\right\}$

***Example 26*:** Find $\mathcal{L}\left\{\left(e^{2t} - e^{-2t}\right)\cos 2t\right\}$

10.5 Laplace Transforms of Derivatives

If we are to use Laplace transform methods to solve differential equations, we need to find the Laplace transforms of the derivatives such as $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$. Note that $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$ can also be written as $y'(t)$ and $y''(t)$ respectively. Now,

$$\mathcal{L}\{y'(t)\} = \int_0^\infty e^{-st}y'(t)\,dt$$

$$= \left[e^{-st}y(t)\right]_0^\infty + s\int_0^\infty e^{-st}y(t)\,dt, \text{ using integration by parts.}$$

As $b \to \infty$, $e^{-sb}y(b) = \frac{y(b)}{e^{sb}} \to 0$ since e^{sb} for $s > 0$ grow at a greater rate than $|y(b)|$ in most cases, so

$$\mathcal{L}\{y'(t)\} = 0 - y(0) + s\int_0^\infty e^{-st}y(t)\,dt = s\mathcal{L}\{y(t)\} - y(0).$$

We can use this result to find the Laplace transform of $y''(t)$.

$$\mathcal{L}\{y''(t)\} = \mathcal{L}\left\{\left(y'(t)\right)'\right\} = s\mathcal{L}\{y'(t)\} - y'(0)$$

$$= s\left(s\mathcal{L}\{y(t)\} - y(0)\right) - y'(0) = s^2\mathcal{L}\{y(t)\} - sy(0) - y'(0)$$

These results are summarized in the **formulae table**:

Theorems: Derivatives (Formulae $\boxed{9}$ & $\boxed{10}$)

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = s\mathcal{L}\{y\} - y(0) \text{ and } \mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0)$$

where $y(0)$ and $y'(0)$ are values of y and y' respectively when $t = 0$.

Example 27: Given $y' = 6t, y(0) = 5$, find $\mathcal{L}\{y\}$.

Example 28: Given $y(0) = 3,$ and $y'(0) = 7$, find $\mathcal{L}\left\{\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 13y\right\}$.

Tutorial 10

Find the following:

1. (a) $\mathcal{L}\left\{4-9e^{-4t}\right\}$ (b) $\mathcal{L}\{7+\pi\}$

   (c) $\mathcal{L}\{\cos \pi t\}$ (d) $\mathcal{L}\left\{5t^3+3\sin 2t\right\}$

   (e) $\mathcal{L}\{2\sin 4t-9\cos 6t\}$ (f) $\mathcal{L}\{A\sin \omega t\}$, where A and ω are constants.

2. (a) $\mathcal{L}\{(t+1)(t+2)\}$ (b) $\mathcal{L}\left\{e^{2t+3}\right\}$

   (c) $\mathcal{L}\left\{\sin\left(t+\dfrac{\pi}{6}\right)\right\}$ (d) $\mathcal{L}\left\{\cos 2\left(t-\dfrac{\pi}{4}\right)\right\}$

   (e) $\mathcal{L}\left\{2\sin^2 t\right\}$ (f) $\mathcal{L}\left\{(\sin t-\cos t)^2\right\}$

3. (a) $\mathcal{L}\{t\sin 2t\}$ (b) $\mathcal{L}\left\{t\cos^2 3t\right\}$

   (c) $\mathcal{L}\{t\sin 2t\sin 5t\}$

4. (a) $\mathcal{L}\left\{e^{3t}\sin t\right\}$ (b) $\mathcal{L}\left\{e^{-2t}\cos\sqrt{3}\,t-t^2e^{-2t}\right\}$

   (c) $\mathcal{L}\left\{t\,e^{2t}\cos 5t\right\}$ (d) $\mathcal{L}\left\{t\,e^{3t}\sin 2t\right\}$

   (e) $\mathcal{L}\left\{e^{4t}\sin 3t\cos 2t\right\}$

5. (a) Given $y'=t^2, y(0)=1$, find $\mathcal{L}\{y\}$.

   (b) $\mathcal{L}\left\{\dfrac{dv}{dt}+3v-13\sin 2t\right\}, \quad v(0)=6$

   (c) $\mathcal{L}\left\{\dfrac{di}{dt}+5i+6e^{-2t}\right\}, \quad i(0)=0$

   (d) $\mathcal{L}\left\{\dfrac{d^2y}{dt^2}+2\dfrac{dy}{dt}+5y-e^{-2t}\cos 3t\right\}, \quad y(0)=1, \quad y'(0)=-2$

*6. Use $\mathcal{L}\{t\cos 3t\}$ to evaluate $\int_0^\infty te^{-2t}\cos 3t\,dt$.

Multiple Choice Questions

1. $\mathcal{L}\left\{e^{-3t-5}\right\}$ is equal to

(a) $\dfrac{1}{e^3(s+5)}$ (b) $\dfrac{e^3}{s+5}$

(c) $\dfrac{1}{e^5(s+3)}$ (d) $\dfrac{e^5}{s+3}$

2. $\mathcal{L}\left\{\left(1-e^{-t}\right)\cos 2t\right\}$ is equal to

(a) $\left(\dfrac{1}{s}-\dfrac{1}{s+1}\right)\left(\dfrac{s}{s^2+4}\right)$ (b) $\dfrac{s}{s^2+4}-\dfrac{s}{(s+1)(s^2+4)}$

(c) $\dfrac{s}{s^2+4}-\dfrac{s}{(s+1)^2+4}$ (d) $\dfrac{s}{s^2+4}-\dfrac{s+1}{(s+1)^2+4}$

Answers

1. (a) $\dfrac{4}{s}-\dfrac{9}{s+4}$ (b) $\dfrac{7+\pi}{s}$ (c) $\dfrac{s}{s^2+\pi^2}$

(d) $\dfrac{30}{s^4}+\dfrac{6}{s^2+4}$ (e) $\dfrac{8}{s^2+16}-\dfrac{9s}{s^2+36}$ (f) $\dfrac{A\omega}{s^2+\omega^2}$

2. (a) $\dfrac{2}{s^3}+\dfrac{3}{s^2}+\dfrac{2}{s}$ (b) $\dfrac{e^3}{s-2}$ (c) $\dfrac{\sqrt{3}+s}{2\left(s^2+1\right)}$ or $\dfrac{0.866+0.5s}{s^2+1}$

(d) $\dfrac{2}{s^2+4}$ (e) $\dfrac{1}{s}-\dfrac{s}{s^2+4}$ (f) $\dfrac{1}{s}-\dfrac{2}{s^2+4}$

3. (a) $\dfrac{4s}{(s^2+4)^2}$ (b) $\dfrac{1}{2}\left[\dfrac{1}{s^2}+\dfrac{s^2-36}{(s^2+36)^2}\right]$ (c) $\dfrac{1}{2}\left[\dfrac{s^2-9}{(s^2+9)^2}-\dfrac{s^2-49}{(s^2+49)^2}\right]$

4. (a) $\dfrac{1}{(s-3)^2+1}$ (b) $\dfrac{s+2}{(s+2)^2+3}-\dfrac{2}{(s+2)^3}$ (c) $\dfrac{(s-2)^2-25}{\left[(s-2)^2+25\right]^2}$

(d) $\dfrac{4(s-3)}{\left[(s-3)^2+4\right]^2}$ (e) $\dfrac{1}{2}\left[\dfrac{5}{(s-4)^2+25}+\dfrac{1}{(s-4)^2+1}\right]$

5. (a) $\mathcal{L}\{y\}=\dfrac{2}{s^4}+\dfrac{1}{s}$ (b) $(s+3)\mathcal{L}\{v\}-6-\dfrac{26}{s^2+4}$ (c) $(s+5)\mathcal{L}\{i\}+\dfrac{6}{s+2}$

(d) $(s^2+2s+5)\mathcal{L}\{y\}-s-\dfrac{s+2}{(s+2)^2+9}$

6. $-\dfrac{5}{169}$

MCQ

1. c 2. d