Chapter 4 : Integration of Rational Functions

Objectives :

1. Find the integrals by resolving proper rational functions into partial fractions.

2. Find the integrals by completing the square for quadratic denominators.

4.1 Proper and Improper Fractions

A function of the form $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ is a polynomial in x of degree n.

Given the polynomial $f(x)$ of degree *n* and another polynomial $g(x)$ of degree *m*, then the **rational expression** $\dfrac{f(x)}{g(x)}$ is a **proper** fraction if $n < m$, and an **improper** fraction if $n \geq m$.

4.2 Partial Fractions

A **proper fraction** $\dfrac{f(x)}{g(x)}$ can be expressed as a sum of simpler fractions if $g(x)$ can be factorised. These simpler fractions are called **Partial Fractions**. Each partial fraction corresponds to a factor of $g(x)$.

4.2.1 Rules of Partial Fraction

The rules of partial fractions are as follows:

Rule 1 The fraction $\dfrac{f(x)}{g(x)}$ must be a proper fraction. (If it is not, then first divide out by long division.)

Rule 2 Factorise the denominator $g(x)$ into its prime factors. This is important since the factors obtained determine the form of the partial fractions.

Rule 3 Corresponding to a **linear factor** $\boxed{ax + b}$ in the denominator, there is a partial fraction of the form $\boxed{\dfrac{A}{ax+b}}$.

e.g. $\dfrac{3x-2}{(x+1)(2x-1)} = \dfrac{A}{x+1} + \dfrac{B}{2x-1}$

Rule 4 Corresponding to a **repeated linear factor** $\boxed{(ax+b)^n}$ in the denominator, there will be *n* partial fractions $\boxed{\dfrac{A_1}{(ax+b)} + \dfrac{A_2}{(ax+b)^2} + \cdots + \dfrac{A_n}{(ax+b)^n}}$.

e.g. $\dfrac{3x-2}{(x+3)^2} = \dfrac{A}{x+3} + \dfrac{B}{(x+3)^2}$

Rule 5 Corresponding to an irreducible **quadratic factor** $\boxed{ax^2 + bx + c}$ in the denominator, there will be a partial fraction of the form $\boxed{\dfrac{Ax+B}{ax^2+bx+c}}$.

e.g. $\dfrac{3x-2}{(x+1)(x^2+x+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+x+1}$

Note: All the constants *A's* and *B's* can to be determined by
(i) "Cover-up method" and/or (ii) "Equating coefficients of like terms"

Example 1: Write down the following partial fractions, without evaluating the constants.

Original Fraction	**Forms of the Partial Fractions**
$\dfrac{x}{(x+1)(x+3)}$	
$\dfrac{x}{(x+1)(x+3)^2}$	
$\dfrac{x}{(x+1)(x^2+3)}$	

Example 2: Resolve $\dfrac{3}{(x+1)(x-2)}$ into partial fractions. *Ans :* $\dfrac{-1}{x+1} + \dfrac{1}{x-2}$

4.3 Integration of Rational Functions

A rational function has the form of a fraction $\frac{f(x)}{g(x)}$ where both $f(x)$ and $g(x)$ are polynomials in x.

4.3.1 Integrals of the Form $\int \frac{f(x)}{g(x)}\, dx$, where $g(x)$ can be factorised

If the polynomial $g(x)$ can be factorised, we use the method of partial fractions.

Example 3: Find $\int \frac{5x+3}{x^3-2x^2-3x}\, dx$.

$Ans: -\ln|x| + \frac{3}{2}\ln|x-3| - \frac{1}{2}\ln|x+1| + C$

Solution

$$\frac{5x+3}{x^3-2x^2-3x} = \frac{5x+3}{x(x^2-2x-3)}$$

$$= \frac{5x+3}{x(x-3)(x+1)} =$$

By "cover-up" rule,

$A =$ $B =$ $C =$

Example 4: Find $\int \frac{x+1}{(x+2)(x-1)^2}\, dx$

$Ans: -\frac{1}{9}\ln|x+2| + \frac{1}{9}\ln|x-1| - \frac{2}{3(x-1)} + C$

Example 5: Find $\displaystyle\int \frac{2x-1}{(x+2)(x^2+1)}\,dx$ *Ans* : $-\ln\left|x+2\right|+\dfrac{1}{2}\ln\left|x^2+1\right|+C$

Solution

4.3.2 Integrals of the Form $\int \frac{f(x)}{g(x)}\,dx$, where $g(x)$ cannot be factorised

Here we will consider the cases where $g(x) = ax^2 + bx + c$. If $g(x)$ cannot be factorised, we will "complete the square" for the denominator.

The result of completing the square may be written as a formula, generally

$$\boxed{x^2 + kx = \left(x + \frac{k}{2}\right)^2 - \left(\frac{k}{2}\right)^2}$$

Example 6: Find $\int \frac{1}{x^2 + 4x + 13}\,dx$

Solution $x^2 + 4x + 13 = \left(\quad + \frac{}{2}\right)^2 - \left(\frac{}{2}\right)^2 +$

$$\therefore \int \frac{1}{x^2 + 4x + 13}\,dx = \int \frac{1}{(\quad + \quad)^2 +}\,dx$$

$$=$$

$$=$$

Example 7: Find $\int \frac{x}{x^2 - 2x + 5}\,dx$ *Ans:* $\frac{1}{2}\ln\left[(x-1)^2 + 4\right] + \frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right) + C$

Tutorial 4

Express the following in partial fractions:

1. $\dfrac{6s^2+7s-49}{(s-4)(s+1)(2s-3)}$

2. $\dfrac{x}{(3-x)(9+x^2)}$

By "partial fraction method", find the following integrals:

3. $\displaystyle\int \frac{-x+7}{(x+3)(3x-1)}\,dx$

4. $\displaystyle\int \frac{x^2-6x+2}{(x+1)(2x-1)^2}\,dx$

5. $\displaystyle\int \frac{3s^2-s+8}{s(s^2+4)}\,ds$

6. $\displaystyle\int \frac{x-3}{(x-4)^2}\,dx$

7. $\displaystyle\int \frac{2x}{(x+2)(x-1)^2}\,dx$

8. $\displaystyle\int_4^5 \frac{3x-4}{x^3-4x^2+4x}\,dx$

9. By "completing the square", find the integrals:

(a) $\displaystyle\int \frac{3}{x^2+6x+12}\,dx$ (b) $\displaystyle\int \frac{x}{x^2-10x+50}\,dx$

Miscellaneous Exercises

*10. $\displaystyle\int_2^3 \frac{1}{(x^2+x)(x-1)^2}\,dx$

*11. $\displaystyle\int \frac{x^2}{1-x^4}\,dx$

*12. $\displaystyle\int_1^2 \frac{1}{x^4+x^2}\,dx$

*13. $\displaystyle\int_0^1 \frac{5x}{(x^2+1)(x+2)}\,dx$

*14. $\displaystyle\int \frac{1}{x^{1/2}-x^{1/4}}\,dx$

*15. $\displaystyle\int \frac{x^4+1}{x^2+3x+2}\,dx$

*16. $\displaystyle\int \frac{x^2-3x+6}{x^3+3x}\,dx$

*17. $\displaystyle\int \frac{3x^2-x+8}{x(x^2+4)}\,dx$

*18. $\displaystyle\int \frac{1-x}{x^2-x+1}\,dx$

*19. Let $I=\displaystyle\int \frac{P(x)}{x^3+1}\,dx$, where $P(x)$ is a polynomial in x.

(i) Find I when $P(x)=x^2$.

(ii) By writing $x^3+1=(x+1)(x^2+Ax+B)$, where A and B are constants, find I when

(a) $P(x)=x^2-x+1$

(b) $P(x)=x+1$

(iii) Using the results of parts (i) and (ii), or otherwise, find I when $P(x)=1$.

Multiple Choice Questions

1. The maximum number of partial fractions that $\dfrac{x^4-16}{(2x+1)^3(x^2-1)}$ can be expressed to is ____________________

(a) 2 (b) 3
(c) 4 (d) 5

2. The expression $\dfrac{x}{(x-2)(x+1)}$ (in partial fractions) is equivalent to ____________________

(a) $\dfrac{1}{3}\left[\dfrac{2}{x-2}-\dfrac{1}{x+1}\right]$ (b) $\dfrac{1}{3}\left[\dfrac{2}{x-2}+\dfrac{1}{x+1}\right]$
(c) $\dfrac{1}{3}\left[\dfrac{1}{x+1}-\dfrac{2}{x-2}\right]$ (d) $-\dfrac{1}{3}\left[\dfrac{2}{x-2}+\dfrac{1}{x+1}\right]$

3. $\dfrac{x+3}{(2x-1)(x^2+9)}$ can be expressed in the form ____________________

(a) $\dfrac{A}{2x-1}+\dfrac{B}{x+3}$ (b) $\dfrac{A}{2x-1}+\dfrac{B}{x+3}+\dfrac{C}{(x+3)^2}$
(c) $\dfrac{A}{2x-1}+\dfrac{Bx}{x^2+9}$ (d) $\dfrac{A}{2x-1}+\dfrac{Bx+C}{x^2+9}$

4. $\dfrac{x(3x-1)}{(x+1)(x^2+4)}$ can be expressed in the partial fractions as ____________________

(a) $\dfrac{A}{x+1}+\dfrac{B}{x^2+4}$ (b) $\dfrac{A}{x+1}+\dfrac{Bx}{x^2+4}$
(c) $\dfrac{A}{x+1}+\dfrac{Bx+C}{x^2+4}$ (d) $\dfrac{A}{x+1}+\dfrac{B}{x+2}+\dfrac{C}{(x+2)^2}$

Answers

1. $\dfrac{3}{s-4}-\dfrac{2}{s+1}+\dfrac{4}{2s-3}$

2. $\dfrac{1}{6}\left(\dfrac{1}{3-x}+\dfrac{x-3}{9+x^2}\right)$

3. $-\ln|x+3|+\dfrac{2}{3}\ln|3x-1|+C$

4. $\ln|x+1|-\dfrac{3}{4}\ln|2x-1|+\dfrac{1}{4(2x-1)}+C$

5. $2\ln|s|+\dfrac{1}{2}\ln|s^2+4|-\dfrac{1}{2}\tan^{-1}\dfrac{s}{2}+C$

6. $\ln|x-4|-\dfrac{1}{(x-4)}+C$

7. $\dfrac{4}{9}\ln|x-1|-\dfrac{2}{3(x-1)}-\dfrac{4}{9}\ln|x+2|+C$

8. 0.349

9. (a) $\sqrt{3}\tan^{-1}\left(\dfrac{x+3}{\sqrt{3}}\right)+C$ (b) $\dfrac{1}{2}\ln\left([x-5]^2+25\right)+\tan^{-1}\left(\dfrac{x-5}{5}\right)+C$

Miscellaneous Exercises

10. 0.0637

11. $\dfrac{1}{4}\ln\left|\dfrac{x+1}{x-1}\right|-\dfrac{1}{2}\tan^{-1}(x)+C$

12. 0.1782

13. 0.667

14. $2x^{1/2}+4x^{1/4}+4\ln\left|x^{1/4}-1\right|+C$

15. $\dfrac{x^3}{3}-\dfrac{3x^2}{2}+7x-17\ln|x+2|+2\ln|x+1|+C_1$

16. $2\ln|x|-\dfrac{1}{2}\ln(x^2+3)-\sqrt{3}\tan^{-1}\dfrac{x}{\sqrt{3}}+C$

17. $2\ln|x|+\dfrac{1}{2}\ln|x^2+4|-\dfrac{1}{2}\tan^{-1}\dfrac{x}{2}+C$

18. $-\dfrac{1}{2}\ln|x^2-x+1|+\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2x-1}{\sqrt{3}}+C$

19. (i) $\dfrac{1}{3}\ln|x^3+1|+C$ (ii) (a) $\ln|x+1|+C$ (b) $\dfrac{2}{\sqrt{3}}\tan^{-1}\dfrac{2x-1}{\sqrt{3}}+C$

(iii) $\dfrac{1}{2}\ln|x+1|-\dfrac{1}{6}\ln|x^3+1|+\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2x-1}{\sqrt{3}}+C$

MCQ

1. (d) 2. (b) 3. (d) 4. (c)