Chapter 7 : Fourier Series

Objectives:

1. Define periodic functions.
2. Obtain the Fourier Series of a periodic function.
3. Able to recognise odd and even functions.
4. Obtain the Fourier series of an odd or even periodic function by using the properties of odd and even functions.

7.1 Periodic Function

A *periodic function* is a function which repeats itself at regular intervals.

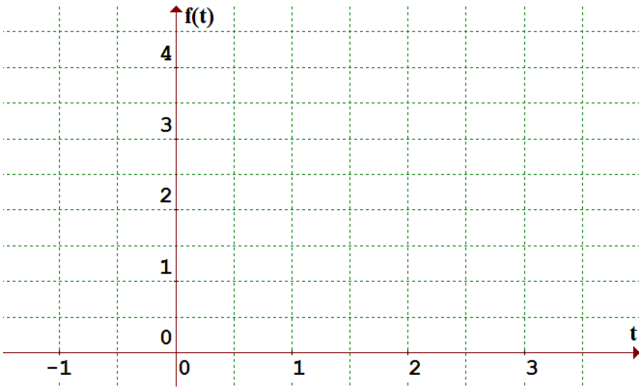
A function $f(t)$ is said to be periodic with period T if $f(t+T) = f(t)$.

Example 1: Sketch 2 cycles of the following periodic functions.

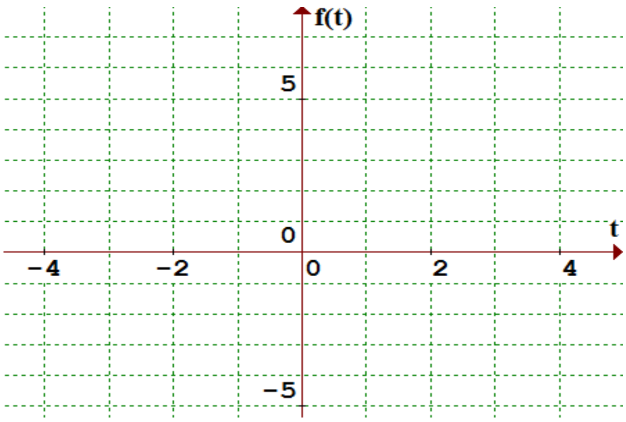
(a) $f(t) = \begin{cases} 2 & , \quad -1 < t < 0 \\ 4 & , \quad 0 < t < 1 \end{cases}$ $\qquad f(t+2) = f(t)$

(b) $f(t) = \begin{cases} t+3 & , \quad 0 < t < 2 \\ t-7 & , \quad 2 < t < 4 \end{cases}$ $\qquad f(t+4) = f(t)$

Solution
(a)



(b)



7.2 Fourier Series

Any periodic function of period T can be expressed by an infinite series of the form:

$$\begin{aligned} f(t) &= a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \\ &= a_0 + a_1 \cos \omega_0 t + a_2 \cos 2\omega_0 t + a_3 \cos 3\omega_0 t + \cdots \\ &\quad + b_1 \sin \omega_0 t + b_2 \sin 2\omega_0 t + b_3 \sin 3\omega_0 t + \cdots \end{aligned}$$

where a_0, a_n, and b_n are constants for $n = 1, 2, 3, \ldots$ and $\omega_0 = \frac{2\pi}{T}$, provided that

1. it is a single valued function,
2. if it is discontinuous there are a finite number of discontinuous in the period of T,
3. it has a finite number of positive and negative maxima and minima in any one period,
4. it has a finite average value for the period T.

When these conditions (called the DIRICHLET conditions) are satisfied, the above series exists. It is called the TRIGONOMETRIC FORM OF A FOURIER SERIES. It converges to the value $f(a)$ if $f(t)$ is continuous at $t = a$ and it converges to $\frac{1}{2}\big(f(a+) + f(a-)\big)$ if $f(t)$ is not continuous at $t = a$. Note the use of $a+$ to denote approaching a from the right and $a-$ to denote approaching a from the left.

a_0 is referred to as the *d-c component*.

a_1 is referred to as the *amplitude of the fundamental cosine component*.

b_1 is referred to as the *amplitude of the fundamental sine component*.

a_n is referred to as the *amplitude of the n^{th} cosine component*.

b_n is referred to as the *amplitude of the n^{th} sine component*.

7.3 Some Special Integrals

The following integrals are used throughout the theory of Fourier series.

(i) $\displaystyle\int_K^{K+T} \cos n\omega_0 t \, dt = 0$

(ii) $\displaystyle\int_K^{K+T} \sin n\omega_0 t \, dt = 0$

(iii) $\displaystyle\int_K^{K+T} \cos m\omega_0 t \cdot cos\ n\omega_0 t \, dt = \begin{cases} 0 & , \text{ if } m \neq n \\ T/2 & , \text{ if } m = n \neq 0 \\ T & , \text{ if } m = n = 0 \end{cases}$

(iv) $\displaystyle\int_K^{K+T} \sin m\omega_0 t \cdot \sin n\omega_0 t \, dt = \begin{cases} 0 & , \text{ if } m \neq n \\ T/2 & , \text{ if } m = n \end{cases}$

(v) $\displaystyle\int_K^{K+T} \cos m\omega_0 t \cdot \sin n\omega_0 t \, dt = 0$

where m, n are positive integers, K is a constant, and $\omega_0 = \dfrac{2\pi}{T}$.

Proof (iii) when $m \neq n$

$$\int_K^{K+T} \cos\, m\,\omega_0 t \cdot cos\ n\,\omega_0 t\, dt$$

$$= \frac{1}{2}\int_K^{K+T} [\cos(m+n)\omega_0 t + \cos(m-n)\omega_0 t]\, dt$$

$$= \frac{1}{2}\left[\frac{\sin(m+n)\omega_0 t}{(m+n)\omega_0} + \frac{\sin(m-n)\omega_0 t}{(m-n)\omega_0}\right]_K^{K+T} \qquad \text{(where } T = \frac{2\pi}{\omega_0}\text{)}$$

$$= \frac{1}{2}\left[\frac{\sin(m+n)(\omega_0 K + 2\pi)}{(m+n)\omega_0} + \frac{\sin(m-n)(\omega_0 K + 2\pi)}{(m-n)\omega_0}\right]$$

$$-\frac{1}{2}\left[\frac{\sin(m+n)\omega_0 K}{(m+n)\omega_0} + \frac{\sin(m-n)\omega_0 K}{(m-n)\omega_0}\right]$$

$= 0$ if $m \neq n$

Note: $\sin[(2\pi(m+n) + \omega_0 K(m+n)] = \sin(m+n)\omega_0 K$

7.4 Determination of the Fourier Coefficients

$$f(t) = a_0 + a_1 \cos\omega_0 t + a_2 \cos 2\omega_0 t + a_3 \cos 3\omega_0 t + \cdots$$
$$+ \ b_1 \sin\omega_0 t + \ b_2 \sin 2\omega_0 t + \ b_3 \sin 3\omega_0 t + \cdots$$

To find a_0, integrate both sides of the above equation over one period :

$$\int_K^{K+T} f(t)\, dt = \int_K^{K+T} a_0\, dt + a_1 \int_K^{K+T} \cos\ \omega_0 t\, dt + a_2 \int_K^{K+T} \cos\ 2\omega_0 t\, dt + \cdots$$

$$+ \ b_1 \int_K^{K+T} \sin\omega_0 t\, dt + b_2 \int_K^{K+T} \sin 2\omega_0 t\, dt + \cdots$$

$$= a_0 [\ t\ ]_K^{K+T} = a_0 T$$

$$\boxed{a_0 = \frac{1}{T}\int_K^{K+T} f(t)\ dt}$$

To find a_n, multiply both sides by $\cos\ n\,\omega_0 t$ and then integrate

$$\int_K^{K+T} f(t)\cos\ n\,\omega_0 t\, dt = a_0 \int_K^{K+T} \cos n\,\omega_0 t\, dt$$
$$+ a_1 \int_K^{K+T} \cos\omega_0 t \cos n\,\omega_0 t\, dt + \cdots + a_n \int_K^{K+T} \cos^2 n\,\omega_0 t\ dt$$

$$+b_1\int_K^{K+T}\sin\omega_0 t\cos n\omega_0 t\,dt+\cdots+b_n\int_K^{K+T}\sin n\,\omega_0 t\cos n\,\omega_0 t\;dt$$

$$=a_n\left[\frac{T}{2}\right]$$

$$\boxed{a_n=\frac{2}{T}\int_K^{K+T}f(t)\cos n\,\omega_0 t\,dt}$$

Similarly $\boxed{b_n=\frac{2}{T}\int_K^{K+T}f(t)\sin n\,\omega_0 t\,dt}$

Example 2: A function $f(t)$ is defined by:

$$f(t)=\begin{cases}1 & , \quad 0<t<\pi \\ 0 & , \quad \pi<t<2\pi\end{cases} \qquad \text{and} \qquad f(t+2\pi)=f(t)\,.$$

Obtain the Fourier series of $f(t)$ up to and including the third harmonic.

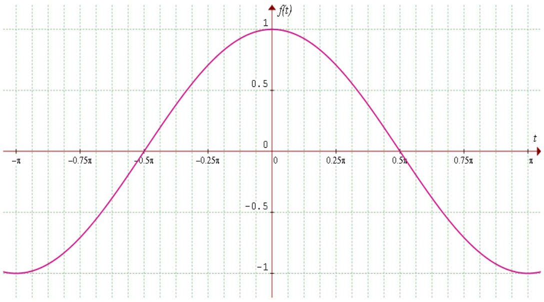
$$Ans\text{: } f(t)=\frac{1}{2}+\frac{2}{\pi}\sin t+\frac{2}{3\pi}\sin 3t+\ldots$$

Solution

7.5 Odd and Even Functions

7.5.1 Even Function

A function $f(t)$, defined in the interval $-\frac{T}{2} < t < \frac{T}{2}$, is said to be *even* if $f(-t) = f(t)$ for every value of t in the interval, e.g. $f(t) = \cos t$.



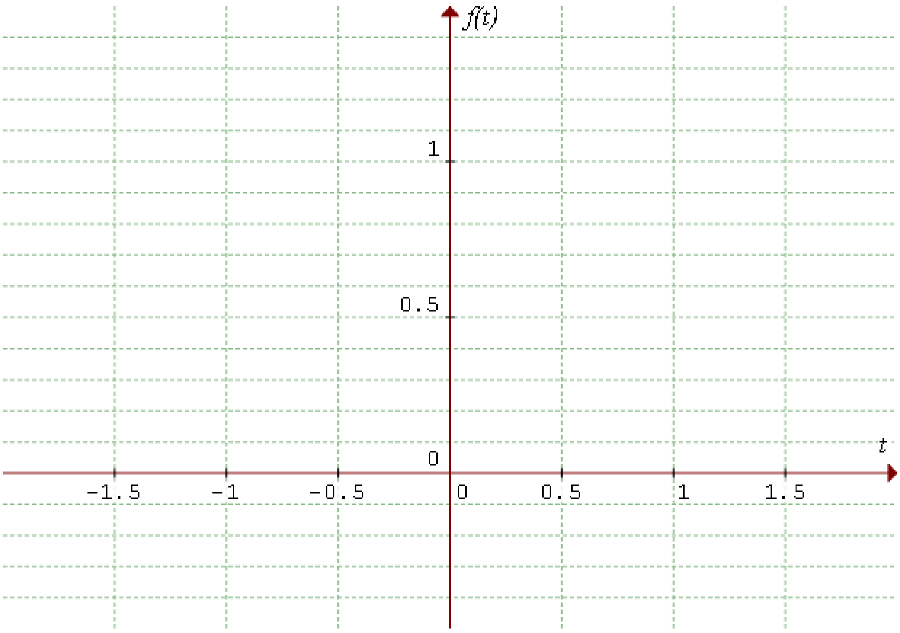
The graph of an even periodic function is *symmetrical about the vertical axis*.

Example 3 Sketch one cycle of the function

$$f(t) = \begin{cases} 0 & , \quad -2 < t < -1 \\ 1 & , \quad -1 < t < 1 \\ 0 & , \quad 1 < t < 2 \end{cases} \quad \text{and} \quad f(t+4) = f(t).$$

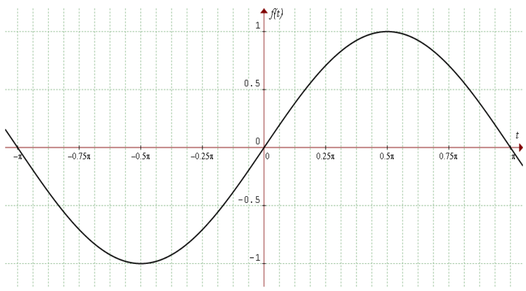
Is $f(t)$ an even function?

Solution



7.5.2 Odd Function

A function $f(t)$, defined in the interval $-\dfrac{T}{2} < t < \dfrac{T}{2}$, is said to be *odd* if $f(-t) = -f(t)$ for every value of t in the interval, e.g. $f(t) = \sin t$. For this definition to be consistent, $f(0) = 0$.



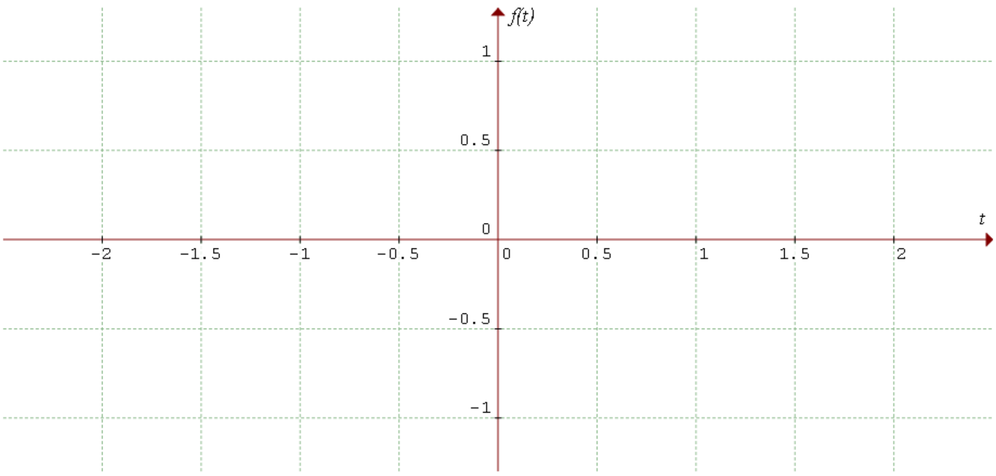
The graph of an odd periodic function is *symmetrical about the origin*.

Example 4 Sketch one cycle of the function

$$f(t) = \begin{cases} -t-1 & , \quad -1 < t < 0 \\ 0 & , \quad t = 0 \\ -t+1 & , \quad 0 < t < 1 \end{cases} \qquad \text{and} \qquad f(t+2) = f(t).$$

Is $f(t)$ an odd function?

Solution



7.6 Fourier Series for Odd and Even Functions

Let $f(t)$ be a periodic function of period T. The Fourier series of $f(t)$ is given by

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right)$$

where
$$\omega_0 = \frac{2\pi}{T}$$
$$a_0 = \frac{1}{T}\int_K^{K+T} f(t)\,dt$$
$$a_n = \frac{2}{T}\int_K^{K+T} f(t)\cdot\cos n\omega_0 t\,dt$$
$$b_n = \frac{2}{T}\int_K^{K+T} f(t)\cdot\sin n\omega_0 t\,dt$$

(a) If $f(t)$ is an **even** periodic function, then
$$a_0 = \frac{2}{T}\int_0^{T/2} f(t)\,dt$$
$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cdot\cos n\omega_0 t\,dt$$
$$b_n = 0$$

(b) If $f(t)$ is an **odd** periodic function, then
$$a_0 = 0$$
$$a_n = 0$$
$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\cdot\sin n\omega_0 t\,dt$$

Note: the Fourier coefficients formulae for odd and even periodic functions will **not** be provided during examination. Students are required to derive these formulae from the given ones in the formulae card.

Example 5: Find the Fourier series of the function $f(t) = t^2$ for $-\pi < t < \pi$ and $f(t + 2\pi) = f(t)$.

Solution

Tutorial 7

Section A

1. Sketch the waveforms of the following periodic functions.

(a) $f(x)=\begin{cases}0 & , \quad 0<x<2 \\ 1 & , \quad 2<x<6 \\ 2 & , \quad 6<x<10\end{cases}$ and $f(x+10)=f(x)$

(b) $f(t)=\begin{cases}t & , \quad 0<t<2 \\ 0 & , \quad 2<t<4\end{cases}$ and $f(t+4)=f(t)$

2. A periodic function $f(t)$ of period 4 is defined as

$$f(t)=\begin{cases}t-1 & , \quad -1<t<1 \\ 2 & , \quad 1<t<3\end{cases} \quad \text{and} \quad f(t+4)=f(t).$$

Find:

(a) the d.c. component (i.e. a_0)

(b) the second sine harmonic (i.e. $b_2\sin\left(2\omega t\right)$), and

(c) the third cosine harmonic (i.e. $a_3\cos\left(3\omega t\right)$) of the Fourier series of $f(t)$.

3. A periodic function $f(t)$ is defined by

$$f(t)=\begin{cases}3 & , \quad 0<t<\pi \\ -1 & , \quad \pi<t<2\pi\end{cases} \quad \text{and} \quad f(t+2\pi)=f(t).$$

Obtain the Fourier series of $f(t)$ up to and including the third harmonic.

4. $f(t)$ is a periodic function defined over one period as follows:

$$f(t)=\begin{cases}2 & , \quad 0<t<\pi \\ -1 & , \quad \pi<t<3\pi/2 \\ 0 & , \quad 3\pi/2<t<2\pi\end{cases} \quad \text{and} \quad f(t+2\pi)=f(t).$$

Find the Fourier series of $f(t)$ as far as the third harmonic.

5. A function $f(t)$ is defined by:

$$f(t)=\begin{cases}t & , \quad 0<t<1 \\ 0 & , \quad 1<t<2\end{cases} \quad \text{and} \quad f(t+2)=f(t).$$

Obtain the Fourier series of $f(t)$ up to and including the third harmonics.

Section B

1. Sketch the following periodic functions and state whether the functions are even, odd or "neither even nor odd".

   (a) $f(t)=\begin{cases} t+1 & , \quad -1<t<0 \\ -t+1 & , \quad 0<t<1 \end{cases}$ and $f(t+2)=f(t)$

   (b) $f(t)=t \quad , \quad -\pi<t<\pi$ and $f(t+2\pi)=f(t)$

   (c) $f(t)=t^2 \quad , \quad 0<t<2$ and $f(t+2)=f(t)$

2. A function $f(t)$ is defined by:

$$f(t)=\begin{cases} -0.5 & , \quad -2<t<-1 \\ 0.5 & , \quad -1<t<1 \\ -0.5 & , \quad 1<t<2 \end{cases} \qquad \text{and} \qquad f(t+4)=f(t).$$

   (a) Sketch the graph and indicate whether $f(t)$ even or odd.

   (b) Find the Fourier Series of $f(t)$ as far as the third harmonic.

3. A periodic function $f(t)$ is defined as

$$f(t)=\begin{cases} -1 & , \quad -2<t<0 \\ 0 & , \quad t=0 \\ 1 & , \quad 0<t<2 \end{cases} \qquad \text{and} \qquad f(t+4)=f(t).$$

   Find the Fourier series of $f(t)$ as far as the third harmonic.
   (First find out whether the function is odd or even and then make full use of the advantages of this classification).

*4. A periodic function $f(t)$ of period 2 is defined over **half** a period as follows:

$$f(t)=t^2 \quad , \quad 0<t<1$$

   If $f(t)$ is an even function,

   (i) sketch the graph of $f(t)$ for $-1<t<1$,

   (ii) find the Fourier Series of $f(t)$ as far as the third harmonic.

*5. A periodic function $f(t)$ of period 2π is defined over **half** a period as follows:

$$f(t)=2t \quad , \quad 0<t<\pi.$$

   If $f(t)$ is an odd function,

   (i) sketch the graph of $f(t)$ for $-\pi<t<\pi$,

   (iii) find the Fourier Series of $f(t)$ as far as the third harmonic.

Miscellaneous Exercises

*1. Show that the Fourier series of $f(t)=t \quad -\pi<t<\pi$ and $f(t+2\pi)=f(t)$

is $f(t)=2\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{\sin(nt)}{n}$.

Hence, deduce that $1-\dfrac{1}{3}+\dfrac{1}{5}-\dfrac{1}{7}+\ldots=\dfrac{\pi}{4}$.

*2. Show that the Fourier series of $f(t)=\begin{cases}0 & , \quad -1<t<0 \\ t+1 & , \quad 0<t<1\end{cases}$ and $f(t+2)=f(t)$

is $f(t)=\dfrac{3}{4}-\dfrac{2}{\pi^2}\sum_{n=1}^{\infty}\dfrac{\cos(2n-1)\pi t}{(2n-1)^2}+\dfrac{3}{\pi}\sum_{n=1}^{\infty}\dfrac{\sin(2n-1)\pi t}{2n-1}-\dfrac{1}{\pi}\sum_{n=1}^{\infty}\dfrac{\sin 2n\pi t}{2n}$.

*3. A periodic function $f(t)$ of period 4 is defined as

$$f(t)=4-t^2 \quad , \quad -2\le t\le 2 \quad \text{and} \quad f(t+4)=f(t).$$

(a) Sketch the graph of $f(t)$ for the interval $-2\le t\le 2$.

(b) Show that the Fourier series of $f(t)$ is

$$f(t)=\frac{8}{3}+\frac{16}{\pi^2}\left(\cos\frac{\pi t}{2}-\frac{1}{4}\cos\pi t+\frac{1}{9}\cos\frac{3\pi t}{2}+\cdots\right).$$

(c) Choose a suitable value for t in the Fourier series of $f(t)$ above to show that

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}=\frac{\pi^2}{12}.$$

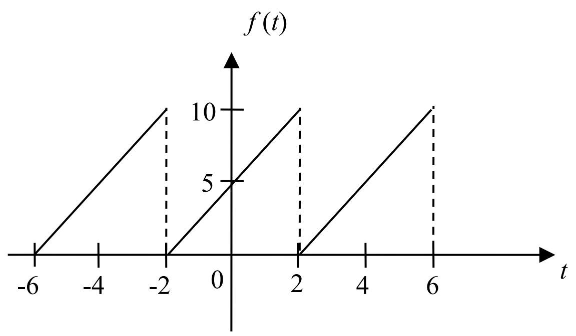
*4. Sketch the graphs of the following periodic functions for two periods and classify them as even, odd or "neither even nor odd". Find their Fourier series.

(a) $f(t)=|t| \quad , \quad -1<t<1$ and $f(t+2)=f(t)$

(b) $f(t)=t-t^3 \quad , \quad -1<t<1$ and $f(t+2)=f(t)$

Multiple Choice Questions

1.



In the figure above, $f(t)$ is a periodic function. The period of $f(t)$ is

(a) 2 (b) 4

(c) 6 (d) 10

2. The d.c. component a_0 of the Fourier series of $f(t)$ (as shown in the figure in MCQ 1) is

(a) 0	(b) 2
(c) 5	(d) 10

3. The trigonometric Fourier series representation of the periodic function $f(t)$ of period 2π is given by $f(t) = \dfrac{4}{\pi^2}\left(\cos t + \dfrac{1}{9}\cos 3t + \dfrac{1}{25}\cos 5t + \cdots\right) + \dfrac{1}{\pi}\left(\sin t - 2\sin 3t + 3\sin 5t + \cdots\right) + \cdots$.
Then $f(t)$ is

(a) an even function	(b) an odd function
(c) an odd function plus constant	(d) a function with no symmetry

4. The d.c. component a_0 of the Fourier series of $f(t)$ (as given in MCQ 3) is

(a) 0	(b) 1
(c) 2	(d) $\dfrac{4}{\pi^2}$

*5. If the Fourier series of a periodic function $f(t)$ of period π is given by

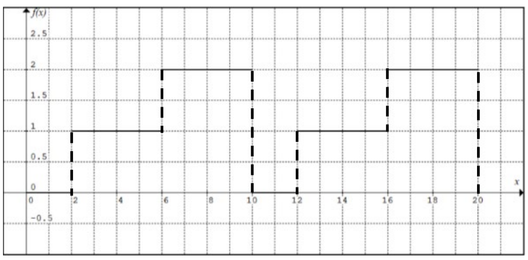
$f(t) = \dfrac{2}{\pi} + \dfrac{4}{3\pi}\cos 2t - \dfrac{4}{15\pi}\cos 4t + \dfrac{4}{35\pi}\cos 6t + \cdots$, then the value of $\int_0^{\pi} f(t)\cos 2t\,dt$ is given by

(a) $\dfrac{4}{3\pi}$	(b) $\dfrac{4}{3}$
(c) $\dfrac{2}{3}$	(d) none of the above

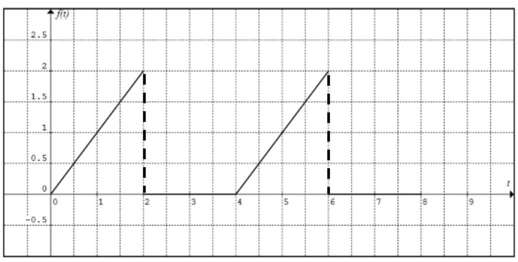
Answers

Section A

1. (a)



(b)



2. (a) $\dfrac{1}{2}$ (b) $\dfrac{1}{\pi}\sin(\pi t)$ (c) $\dfrac{2}{\pi}\cos\left(\dfrac{3\pi}{2}t\right)$

3. $a_0 = 1$, $a_n = 0$, $b_n = \dfrac{4}{n\pi}\left[1-\cos(n\pi)\right]$

$$f(t) = 1 + \frac{8}{\pi}\sin t + \frac{8}{3\pi}\sin 3t + \cdots$$

4. $a_0 = \dfrac{3}{4}$, $a_n = -\dfrac{1}{n\pi}\sin\left(\dfrac{3n\pi}{2}\right)$, $b_n = \dfrac{1}{\pi}\left[-\dfrac{3}{n}\cos(n\pi) + \dfrac{2}{n} + \dfrac{1}{n}\cos\left(\dfrac{3n\pi}{2}\right)\right]$

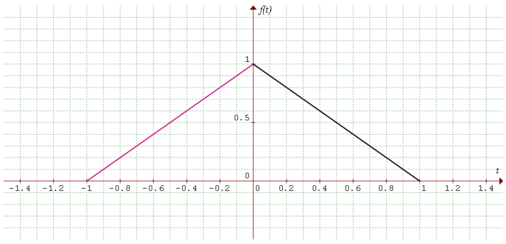
$$f(t) = \frac{3}{4} + \frac{1}{\pi}\cos t - \frac{1}{3\pi}\cos 3t + \cdots + \frac{5}{\pi}\sin t - \frac{1}{\pi}\sin 2t + \frac{5}{3\pi}\sin 3t + \cdots$$

5. $a_0 = \dfrac{1}{4}$, $a_n = \dfrac{1}{(n\pi)^2}\left[\cos(n\pi)-1\right]$, $b_n = -\dfrac{1}{n\pi}\cos(n\pi)$

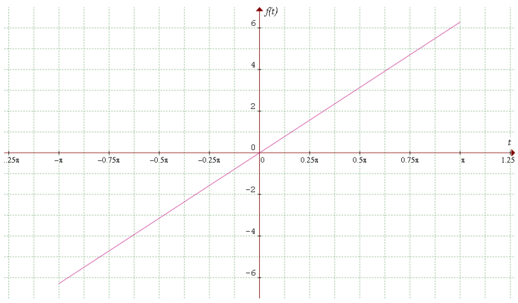
$$f(t) = \frac{1}{4} - \frac{2}{\pi^2}\cos\pi t - \frac{2}{9\pi^2}\cos 3\pi t + \ldots + \frac{1}{\pi}\sin\pi t - \frac{1}{2\pi}\sin 2\pi t + \frac{1}{3\pi}\sin 3\pi t + \ldots$$

Section B

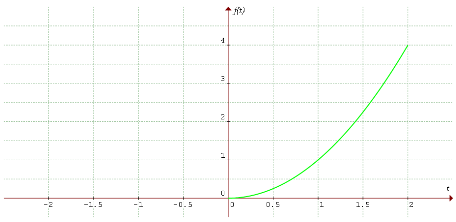
1. (a) even



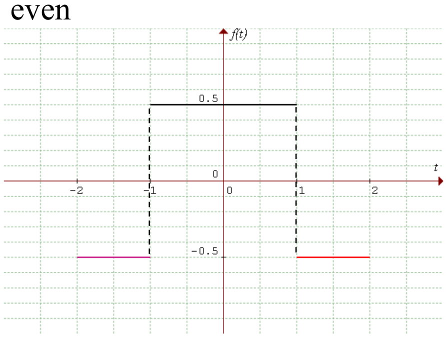
(b) odd



(c) neither even nor odd



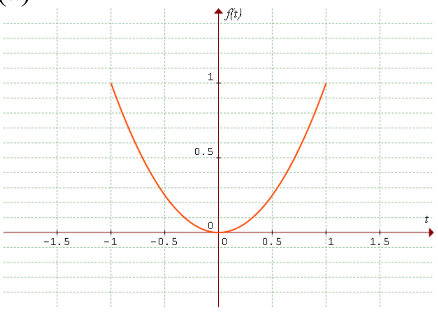
2. (a) even



(b) $a_0 = 0$, $a_n = \dfrac{2}{n\pi}\sin\left(\dfrac{n\pi}{2}\right)$, $f(t) = \dfrac{2}{\pi}\cos\dfrac{\pi t}{2} - \dfrac{2}{3\pi}\cos\dfrac{3\pi t}{2} + \ldots$

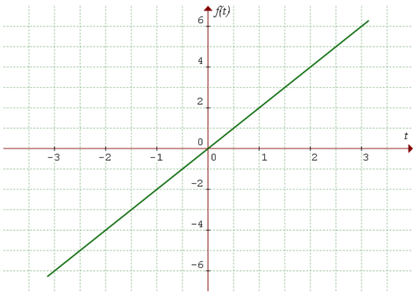
3. $b_n = \dfrac{-2}{n\pi}\left[\cos(n\pi)-1\right]$, $f(t) = \dfrac{4}{\pi}\sin\dfrac{\pi t}{2} + \dfrac{4}{3\pi}\sin\dfrac{3\pi t}{2} + \cdots$

4. (a)



(b) $a_0 = \frac{1}{3}$, $a_n = \frac{4}{(n\pi)^2}\cos(n\pi)$, $b_n = 0$, $f(t) = \frac{1}{3} - \frac{4}{\pi^2}\cos\pi t + \frac{1}{\pi^2}\cos 2\pi t - \frac{4}{9\pi^2}\cos 3\pi t + \ldots$

5. (a)



(b) $a_0 = 0$, $a_n = 0$, $b_n = -\frac{4}{n}\cos(n\pi)$, $f(t) = 4\sin t - 2\sin 2t + \frac{4}{3}\sin 3t + \cdots$

Miscellaneous Exercises

1. (iii) $t = 0$

2. (a) even, $f(t) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi t}{2n-1}$

(b) odd, $f(t) = \frac{12}{\pi^3}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^3}\sin(n\pi t)$

MCQ

1. b 2. c 3. d 4. a 5. C