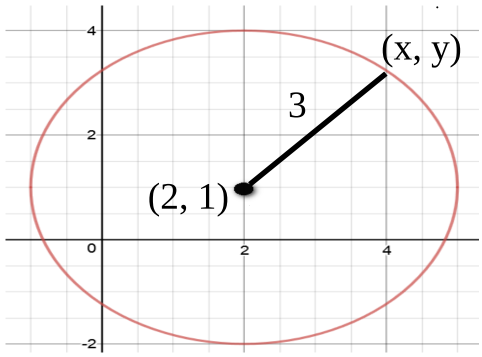
2 FUNCTIONS

2.0 Equation of a Circle



Let's consider the circle shown at the right. This circle has its center at point (2, 1) and it has a radius of 3.

All the points (x, y) on the circle are a distance of 3 units away from the center of the circle. We can express this information as an equation with the help of the Pythagorean Theorem. The right triangle shown in the figure has legs of length $x - 2$ and $y - 1$ and hypotenuse of length 3. We write:

$$(x-2)^2 + (y-1)^2 = 3^2$$

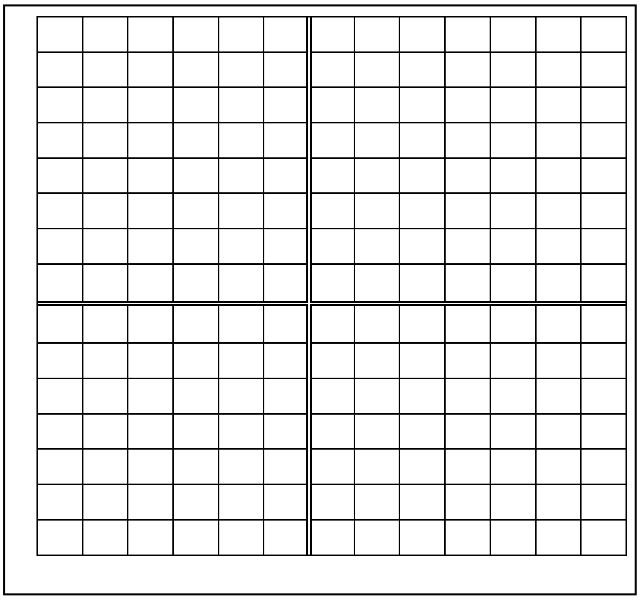
We can generalize this equation for a circle with center at point (h, k) and radius r.

$$(x-h)^2 + (y-k)^2 = r^2$$

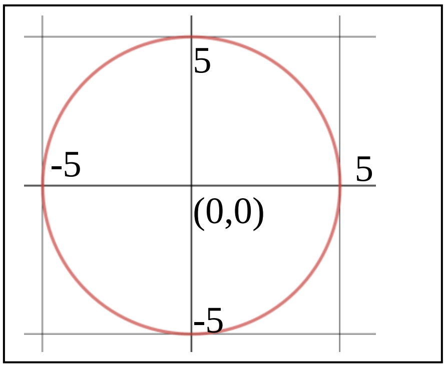
EXAMPLE

(a) Graph the circle:

$$(x-1)^2 + (y+2)^2 = 16$$



(b) Find the equation of the following circle



2.1 DEFINITION OF A FUNCTION

2.1.1 Definition. If a variable y depends on a variable x in such a way that each value of x determines exactly one value of y, then we say that ***y is a function of x***.

2.1.1 INDEPENDENT AND DEPENDENT VARIABLES

For a given input x, the output of a function f is called the ***value*** of f at x or the ***image*** of x under f and write $y = f(x)$. This equation expresses y as a function of x; the variable x is called the ***independent variable*** of f, and the variable y is called the ***dependent variable*** of f.

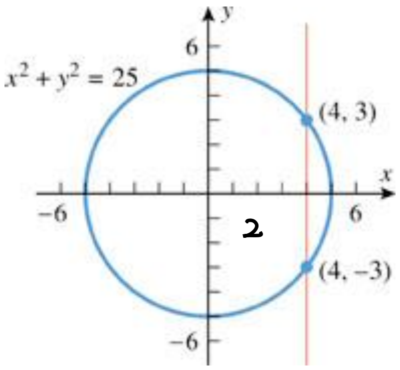
EXAMPLE 1 The equation $y = 2x^2 - x + 2$ states that y is a function of x since each value of x gives exactly one value of y. The independent variable is x and the dependent variable is y.

EXAMPLE 2 The circle $x^2 + y^2 = 25$ does not give y as a function of x since each value of x gives 2 values of y, namely, $y = \sqrt{25 - x^2}$ and $y = -\sqrt{25 - x^2}$.

2.1.2 THE VERTICAL LINE TEST

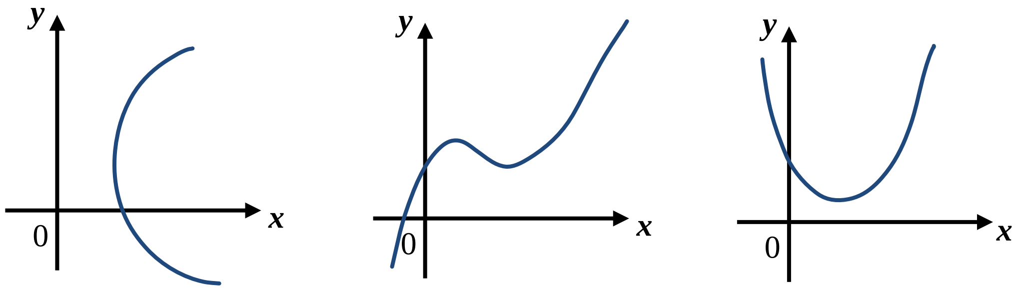
2.1.2 The Vertical Line Test.
A curve in the xy-plane is the graph of some function f if and only if no vertical line intersects the curve more than once.

EXAMPLE 3A The graph of $x^2 + y^2 = 25$ shows that it is not a function.



EXAMPLE 3B

Which of the following are graphs of functions?



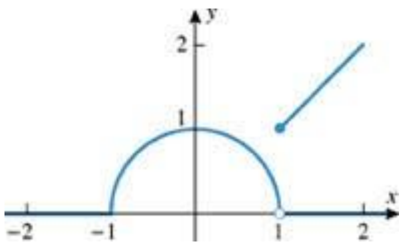
2.1.3 FUNCTIONS DEFINED PIECEWISE

A function is defined ***piecewise*** if the formula for f changes, depending on the value of x.

EXAMPLE 4A The graph of the function defined piecewise by the formula

$$f(x)=\begin{cases} 0, & x \le -1 \\ \sqrt{1-x^2}, & -1 < x < 1 \\ x, & x \ge 1 \end{cases}$$

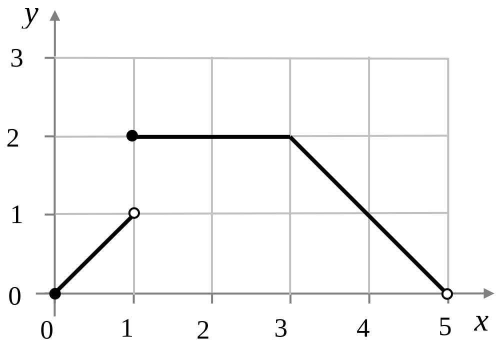
is sketched below.



Find the values of

(a) $f(-2)$ (b) $f(-1)$ (c) $f(0)$ (d) $f(2)$

EXAMPLE 4B Write down the function f that is graphed below.



2.1.4 DOMAIN AND RANGE

<u>Notations</u>

<u>Set Builder Notation</u>
$\{x\text{: conditions}\}$ is read as "*the set of all x such that <u>conditions</u>*"

<u>Example 1</u>
$\{x : x \in \mathbb{R}, 2 < x < 3\}$ is read as

____.

<u>Interval Notation</u>

$[a,b]$ denotes <u>closed interval</u> from *a* to *b*, i.e. $\{x : a \le x \le b\}$. Square brackets indicate that the end points are included in the interval.

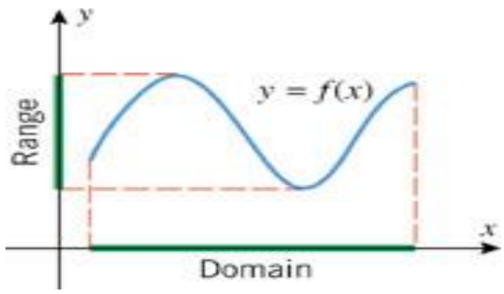
(a,b) denotes <u>open interval</u> from *a* to *b*, i.e. $\{x : a < x < b\}$. Parentheses indicate that the end points are not included in the interval.

$[a,b)$ or $(a,b]$ denotes <u>half-closed (or half-open) interval</u> from *a* to *b*. Only one end point is included.

If x and y are related by the equation $y = f(x)$, then the set of all allowable inputs (x-values) is called the ***domain*** of f, and the set of outputs (y-values) that result when x varies over the domain is called the ***range*** of f.

EXAMPLE 5 If $f(0) = 1$, $f(1) = 3$, $f(2) = -2$, and $f(3) = 5$, then the domain is the set $\{0, 1, 2, 3\}$ and the range is the set $\{1, 3, -2, 5\}$.

The domain and range of a function f can be pictured by projecting the graph of $y = f(x)$ onto the coordinate axes as shown below.



2.1.4 Definition. If a real-valued function of a real variable is defined by a formula, and if no domain is stated explicitly, then it is to be understood that the domain consists of all real numbers for which the formula yields a real value. This is called the ***natural domain*** of the function.

EXAMPLE 6 Find the natural domain of the following functions.

(i) $f(x) = \sqrt{x-1}$, $D_f =$

(ii) $g(x) = \dfrac{1}{(x-1)(x-3)}$ $D_g =$

(iii) $h(x) = \dfrac{1}{x^2+1}$ $D_h =$

(iv) $f(x)=\dfrac{x}{(x-2)^2}$ $D_f =$

(v) $g(x)=\sqrt{x^2-4}$ $D_g =$

EXAMPLE 7 Find the natural domain of

(a) $f(x)=x^3$

(b) $f(x)=\dfrac{1}{(x-1)(x-2)}$

(c) $f(x)=\sqrt{x^2-4x+3}$

SOLUTION (A). The function f has real values for all real x, so its natural domain is the interval $(-\infty,+\infty)$.

SOLUTION (B). The function f has real values for all real x, except x = 1 and x = 2, where divisions by zero occur. Thus, the natural domain is

$$\{x : x \neq 1 and\, x \neq 2\} = (-\infty,1)\cup(1,2)\cup(2,+\infty)$$

SOLUTION (C). The function f has real values, except when the expression inside the radical is negative. Thus the natural domain consists of all real numbers x such that

$$x^2-4x+3=(x-1)(x-3)\geq 0$$

This inequality is satisfied if $x \leq 1$ or $x \geq 3$ (verify), so the natural domain of f is

$$\{x : x \leq 1 \; or \; x \geq 3\} = (-\infty,1]\cup[3,+\infty)$$

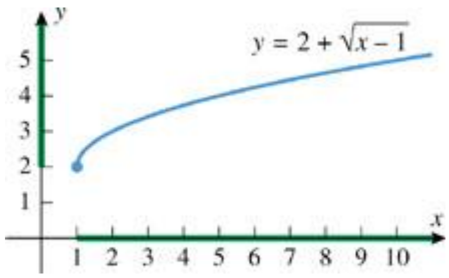
EXAMPLE 8 Find the domain and range of

(a) $y = 2 + \sqrt{x-1}$

(b) $f(x) = 1 + \dfrac{3}{x-1}$ (Hint: For range, graph $f(x)$)

$$\text{Ans: } \begin{aligned} D_f &= \{x : x \neq 1\} = (-\infty, 1) \cup (1, +\infty) \\ R_f &= \{f(x) : f(x) \neq 1\} = (-\infty, 1) \cup (1, +\infty) \end{aligned}$$

SOLUTION (a). Since no domain is stated explicitly, the domain of y is the natural domain $[1, +\infty)$. As x varies over the interval $[1, +\infty)$, the value of $\sqrt{x-1}$ varies over the interval $[0, +\infty)$, so the value of $y = 2 + \sqrt{x-1}$ varies over the interval $[2, +\infty)$, which is the range of f. The domain and range are highlighted on the x- and y-axes in the figure below.



SOLUTION (b).

2.2 OPERATIONS ON FUNCTIONS

2.2.1 ARITHMETIC OPERATIONS ON FUNCTIONS

2.2.1 Definition Given functions *f* and *g*, we define

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)g(x)$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

For the functions $f+g$, $f-g$ and fg we define the domain to be the intersection of the domains of *f* and *g*, and for the function $\frac{f}{g}$ we define the domain to be the intersection of the domains of *f* and *g* but with the points where $g(x)=0$ excluded (to avoid division by zero).

EXAMPLE 9 Let $f(x) = 1+\sqrt{x+2}$ *and* $g(x) = x-2$

Find the domains and formulas for the functions $f+g$, $f-g$, fg, $\frac{f}{g}$ and $5f$.

SOLUTION. First, we will find the formulas and then the domains. The formulas are

$$(f+g)(x) = f(x)+g(x) = \left(1+\sqrt{x+2}\right)+(x-2) = x-1+\sqrt{x+2}$$

$$(f-g)(x) = f(x)-g(x) = \left(1+\sqrt{x+2}\right)-(x-2) = 3-x+\sqrt{x+2}$$

$$(f\ g)(x) = f(x)g(x) = \left(1+\sqrt{x+2}\right)(x-2)$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{1+\sqrt{x+2}}{x-2}$$

$$5f(x) = 5\left(1+\sqrt{x+2}\right) = 5+5\sqrt{x+2}$$

The domains of *f* and *g* are $[-2,+\infty)$ and $(-\infty,+\infty)$ respectively (their natural domains). Thus, it follows that the domains of $f+g$, $f-g$ and $f\ g$ are the intersection of these two domains, namely, $[-2,+\infty)\cap(-\infty,+\infty) = [-2,+\infty)$.

Moreover, since $g(x)=0$ if $x=2$, the domain of $\frac{f}{g}$ with $x=2$ removed is $[-2,2)\cup(2,+\infty)$. Finally, the domain of $5f$ is the same as the domain of f.

2.2.2 COMPOSITION OF FUNCTIONS

2.2.2 Definition Given functions f and g, the ***composition*** of f with g, denoted by $f \circ g$, is the function defined by

$$(f \circ g)(x) = f(g(x))$$

EXAMPLE 10 Let $f(x) = x^2 + 1$ and $g(x) = \sqrt{x}$. Find

(a) $(f \circ g)(x)$

(b) $(g \circ f)(x)$

SOLUTION

(a) The formula for $(f \circ g)(x)$ is

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f\left(\sqrt{x}\right) \\ &= \left[\sqrt{x}\right]^2 + 1 \\ &= x + 1\end{aligned}$$

(b) The formula for $(g \circ f)(x)$ is

Compositions can also be defined for three or more functions; for example, $(f \circ g \circ h)(x)$ is computed as

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

2.3 INVERSE FUNCTIONS

2.3.1 Definition If the functions *f* and *g* satisfy the two conditions

$$g\left(f(x)\right) = x \text{ for every x in the domain of } f.$$

$$f\left(g(y)\right) = y \text{ for every y in the domain of } g.$$

then we say that ***f is an inverse of g*** and ***g is an inverse of f*** or that ***f and g are inverse functions***. Moreover, if a function *f* has an inverse, then that inverse is unique. The inverse of *f* is denoted by the symbol f^{-1}. Note that

$$f^{-1}(f(x)) = x \qquad \text{for every } x \text{ in the domain of } f$$

$$f(f^{-1}(x)) = x \qquad \text{for every } x \text{ in the domain of } f^{-1}$$

2.3.2 DOMAIN AND RANGE OF INVERSE FUNCTIONS

We have the following relationships between the domains and ranges of *f* and f^{-1}:

$$\text{domain of } f^{-1} = \text{range of } f$$

$$\text{range of } f^{-1} = \text{domain of } f$$

2.3.3 A METHOD FOR FINDING INVERSE FUNCTIONS

2.3.2 Theorem If an equation $y = f(x)$ can be solved for *x* as a function of y, then *f* has an inverse and that inverse is $x = f^{-1}(y)$.

Methods for Finding Inverse Functions

Theorem. If an equation $y = f(x)$ can be solved for x as a function of y, then *f* has an inverse and that inverse is $x = f^{-1}(y)$.

Procedure for Finding the Inverse of a Function *f*

Step 1 Write down the equation $y = f(x)$.
Step 2 Solve this equation for *x* as a function of *y*.
Step 3 Replace *x* by $f^{-1}(\text{x})$ and *y* by *x*.

Example 11

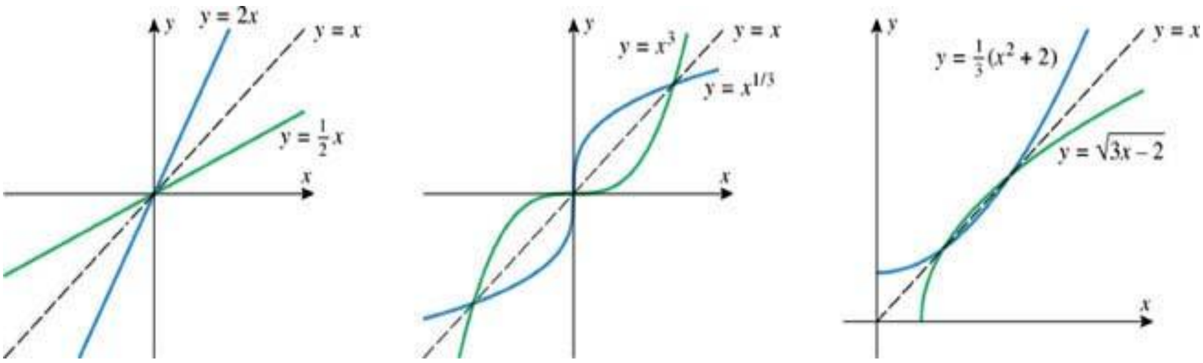
Find a formula for the inverse of $f(x)=\sqrt{3x+2}$ with *x* as the independent variable, and state the domain and range of f^{-1}.

[Ans: $f^{-1}(x)=\frac{1}{3}\left(x^2-2\right), \quad x\geq 0, \ D_{f^{-1}}=[0,+\infty), \ R_{f^{-1}}=\left[-\frac{2}{3},+\infty\right)$ **]**

2.3.4 GRAPHS OF INVERSE FUNCTIONS

2.3.3 Theorem If f has an inverse, then the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of one another about the line $y = x$; that is, each graph is the mirror image of the other with respect to that line.

Graphs of some inverse functions are given below.



Example 12

The graph of $y = h^{-1}(x)$ is a smooth curve passing through the points $(0,0)$ $(1,1)$ and $(3,9)$. Sketch $y = h^{-1}(x)$ and $y = h(x)$ for $0 \le x \le 3$.

Example 13 (To be taught in class)

Let $f(x)=1+\sqrt{x-3}$ and $g(x)=\sqrt{x^2-5x+6}$.

i. Find the domain and range of *f*.

ii. Find the domain and range of *g*.

iii. Find f^{-1}, the inverse function of f .

iv. Find the domain of $f+g$.

v. Find a formula for $(f\circ g)(x)$.

Answers:

(i) $D_f=[3,+\infty)$, $R_f=[1,+\infty)$

(ii) $D_g=(-\infty,2]\cup[3,+\infty)$, $R_g=[0,\ +\infty)$

(iii) $f^{-1}(x)=3+(x-1)^2$

(iv) $D_{f+g}=[3,+\infty)$

(v) $fg(x)=1+\sqrt{\sqrt{x^2-5x+6}-3}$

TUTORIAL 2
FUNCTIONS

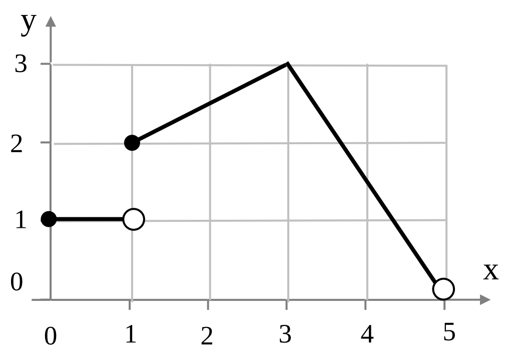
1. (a) Find the center and radius of the following circles. Sketch the graph of the circle.

   (i) $(x-4)^2+(y-1)^2=25$ (ii) $(x+1)^2+(y-2)^2=4$

   (b) Find $f(0)$, $f(2)$, $f(-2)$, $f(3)$, $f(\sqrt{2})$ and $f(3t)$ for the functions

   (i) $f(x)=3x^2-2$ (ii) $f(x)=\begin{cases}\dfrac{1}{x}, & x>3\\ 2x, & x\le 3\end{cases}$

   (c) Write down the piecewise function f that is graphed below.



   (d) Find the domain for each of the functions.

   (i) $f(x)=\sqrt{x-1}$ (ii) $g(x)=\sqrt{x^2+5x+6}$

   (iii) $h(x)=\dfrac{1}{x^2+5x+6}$ (iv) $k(x)=x^2$

2. Find the domain and range for each of the functions

   (a) $f(x)=\dfrac{1}{x-3}$ (b) $g(x)=\sqrt{x^2-3}$

   (c) $h(x)=\sqrt{x^2-2x+5}$ (d) $k(x)=5x+7$

3. Find $f+g, f-g, fg$, and $\dfrac{f}{g}$ for the functions $f(x)=2\sqrt{x-1}$ and $g(x)=\sqrt{x-1}$.

4. Find $f\circ g$ and $g\circ f$ for the functions $f(x)=x^2$ and $g(x)=\sqrt{1-x}$.

5. Find the inverse function $f^{-1}(x)$ for each of the following functions.

(a) $f(x)=7x-6$ (b) $f(x)=\sqrt[3]{2x-1}$

(c) $f(x)=\dfrac{3}{x^2},\ x<0$ (d) $f(x)=\dfrac{x+1}{x-1}$

6. Let $f(x)=\dfrac{1}{x-3}$ and $g(x)=\sin^2(x)$.

(i) Find the domain and range of f.

(ii) Find the domain and range of g.

(iii) Find a formula for $(f\circ g)(x)$. Hence find $(f\circ g)(1)$.

7. Given $f(x)=3-\sqrt{x+1}$, $h(x)=5x^2$.

(a) Find a formula for $f^{-1}(x)$ and state the domain and range of the function f^{-1}.

(b) Find $(f\circ h)(x)$.

8 (a) Let $f(x)=\dfrac{1}{1-x}$ and $g(x)=\sqrt{2x+1}$.

i. Find the domain and range of f.

ii. Find the domain and range of g.

iii. Find $g(2t+1)$.

iv. Show that $f^2(x)=f^{-1}(x)$. [Note: $f^2(x)=(f\circ f)(x)$]

(b) The function h is defined as

$$h(x)=\begin{cases}2x^2-1, & x<1\\ x+4, & x\geq 1\end{cases}$$

i. Evaluate $h(3)-h(0)$.

ii. Find the x-coordinates where the function h cuts the x-axis.

9 Let $f(x)=3\sqrt{x-1}$ and $g(x)=x^2$.

i. Find the domain and range of f.

ii. Find the domain and range of g.

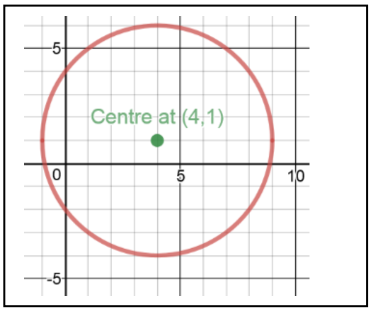
iii. Solve $(g\circ f)(x)=18$.

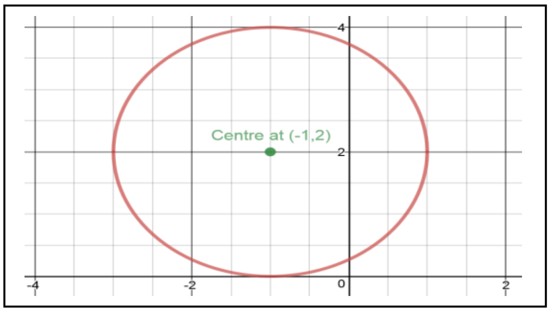
iv. Find a formula for $f^{-1}(x)$ and state the domain and range of $f^{-1}(x)$.

10. If $f(x)=\dfrac{e^x-1}{e^x+1}$ for all real values of x, prove that $f(-x)=-f(x)$. (MA1301 0809)

ANSWERS

1(a) (i) centre at (4,1), radius = 5 (ii) centre at (– 1,2), radius = 2





1(b)(i) $f(0)=-2,\ f(2)=10,\ f(-2)=10,\ f(3)=25,\ f(\sqrt{2})=4,\ f(3t)=27t^2-2$

1(b)(ii) $f(0)=0,\ f(2)=4,\ f(-2)=-4,\ f(3)=6,\ f(\sqrt{2})=2\sqrt{2},\ f(3t)=\begin{cases}\dfrac{1}{3t}, & t>1\\ 6t, & t\le 1\end{cases}$

1(c) $f(x)=\begin{cases}1, & 0\le x<1\\ \dfrac{1}{2}x+\dfrac{3}{2}, & 1\le x<3\\ -\dfrac{3}{2}x+7\dfrac{1}{2}, & 3\le x<5\end{cases}$

1(d)(i) $D_g=[1,\infty)$ or $\{x:x\ge 1\}$

(ii) $D_g=(-\infty,-3]\cup[-2,\infty)$ or $\{x:x\le -3 \text{ or } x\ge -2\}$

(iii) $D_g=(-\infty,-3)\cup(-3,-2)\cup(-2,\infty)$ or $\{x:x\ne -3 \text{ and } x\ne -2\}$

(iv) $D_g=(-\infty,\infty)$ or $\{x:x\in\mathbb{R}\}$

2 (a) $x\ne 3$ or $(-\infty,3)\cup(3,\infty)$; $y\ne 0$ or $(-\infty,0)\cup(0,\infty)$

(b) $x\le -\sqrt{3}$ or $x\ge\sqrt{3}$ or $(-\infty,-\sqrt{3}]\cup[\sqrt{3},\infty)$; $y\ge 0$ or $[0,\infty)$

(c) all x or $(-\infty,\infty)$; $y\ge 2$ or $[2,\infty)$

(d) all x or $(-\infty,\infty)$; all y or $(-\infty,\infty)$

3 $(f+g)(x)=3\sqrt{x-1};\ (f-g)(x)=\sqrt{x-1};$

$(fg)(x)=2x-2;\ \left(\dfrac{f}{g}\right)(x)=2$

4 $(f\circ g)(x)=1-x;\ (g\circ f)(x)=\sqrt{1-x^2}$

5 (a) $f^{-1}(x)=\dfrac{1}{7}(x+6)$ (b) $f^{-1}(x)=\dfrac{1}{2}(x^3+1)$

(c) $f^{-1}(x)=-\sqrt{\dfrac{3}{x}}$ (d) $f^{-1}(x)=\dfrac{x+1}{x-1}$

6 (i) $x\ne 3;\ y\ne 0$ (ii) all x; $0\le y\le 1$

(iii) $\dfrac{1}{\sin^2 x-3}$;–0.4363

7 (a) $f^{-1}(x)=(3-x)^2-1$, $x\le 3$, $y\ge -1$

(b) $(f\circ h)(x)=3-\sqrt{5x^2+1}$

8 (a) (i) $D_f=(-\infty,1)\cup(1,+\infty)$, $R_f=(-\infty,0)\cup(0,+\infty)$

(ii) $D_g=\left[-\frac{1}{2},+\infty\right)$, $R_g=[0,\ +\infty)$

(iii) $g(2t+1)=\sqrt{4t+3}$

(iv) $f^2(x)=f^{-1}(x)=1-\frac{1}{x}$

(b) (i) 8 (ii) $x=\pm\sqrt{\frac{1}{2}}$

9 (i) $D_f=[1,+\infty)$ or $\{x:x\ge 1\}$, $R_f=[0,+\infty)$ or $\{y:y\ge 0\}$

(ii) $D_g=(-\infty,+\infty)$ or $\{x:x\in\Re\}$, $R_g=[0,+\infty)$ or $\{y:y\ge 0\}$

(iii) $x=3$

(iv) $f^{-1}(x)=\left(\frac{x}{3}\right)^2+1$ $D_{f^{-1}}=R_f=[0,+\infty)$, $R_{f^{-1}}=D_f=[1,+\infty)$