Chapter 11 : Inverse Laplace Transform

Objectives :

1. Find inverse Laplace transforms using standard results.

2. Use techniques like linearity property, completing the square and partial fractions to evaluate inverse Laplace transforms.

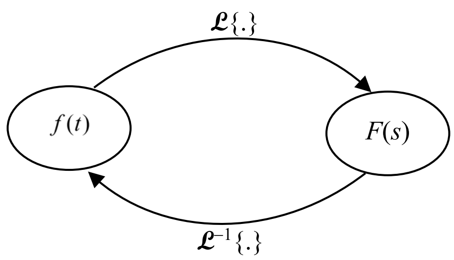
11.1 Definition of the Inverse Laplace Transform

If the Laplace transform of a function $f(t)$ is $F(s)$, then $f(t)$ is called an **inverse Laplace transform** of $F(s)$. That is,

$$\text{if } \mathcal{L}\{f(t)\} = F(s)\text{, then } f(t) = \mathcal{L}^{-1}\{F(s)\}$$

where $\mathcal{L}^{-1}$ is called the **inverse Laplace transformation operator**.

The relationship between $f(t)$ and $F(s)$ is depicted graphically here:



When finding the inverse Laplace transform of $F(s)$, the following strategies are normally used, sometimes in combinations, before referring to the Laplace transforms **formulae table**:

- Linearity rule
- Completing the square
- Partial fractions

These methods will be gradually introduced and demonstrated in the latter sections of this chapter.

11.2 Evaluation of Inverse Laplace Transform

Since $\mathcal{L}\left\{e^{3t}\right\} = \dfrac{1}{s-3}$, then inversely, $\mathcal{L}^{-1}\left\{\dfrac{1}{s-3}\right\} = e^{3t}$

Similarly, since $\mathcal{L}\left\{t^2\right\} = \dfrac{2}{s^3}$, then inversely, $\mathcal{L}^{-1}\left\{\dfrac{2}{s^3}\right\} = t^2$

The linearity property works for the inverse Laplace transform operator as well.

Theorem: Linearity Property

If $f_1(t)$ and $f_2(t)$ are functions of t, a and b are constants, $F_1(s)$ and $F_2(s)$ are Laplace transforms of $f_1(t)$ and $f_2(t)$ respectively, then

$$\mathcal{L}^{-1}\{a\,F_1(s)+b\,F_2(s)\}=a\,f_1(t)+b\,f_2(t)$$

Formula $\boxed{1}$

1	$\frac{1}{s}$

Example 1a: $\mathcal{L}^{-1}\left\{\frac{3}{s}\right\}$

Example 1b: $\mathcal{L}^{-1}\left\{\frac{\pi}{s}\right\}$

Formula $\boxed{2}$

t^n (n is a positive integer)	$\frac{n!}{s^{n+1}}$

Example 2a: $\mathcal{L}^{-1}\left\{\frac{24}{s^5}\right\}$

Example 2b: $\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\}$

Example 2c: $\mathcal{L}^{-1}\left\{\frac{3}{s^4}\right\}$

Formula $\boxed{3}$

e^{at}	$\frac{1}{s-a}$

Example 3a: $\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$

Example 3b: $\mathcal{L}^{-1}\left\{\frac{4}{s-2}\right\}$

Example 3c: $\mathcal{L}^{-1}\left\{\frac{1}{3s+2}\right\}$

Formula $\boxed{4}$	$\sin at$	$\dfrac{a}{s^2+a^2}$
Formula $\boxed{5}$	$\cos at$	$\dfrac{s}{s^2+a^2}$

***Example 4*:** $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+4}\right\}$

***Example 5*:** $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+4}\right\}$

Formula $\boxed{6}$	$t\sin at$	$\dfrac{2as}{\left(s^2+a^2\right)^2}$
Formula $\boxed{7}$	$t\cos at$	$\dfrac{s^2-a^2}{\left(s^2+a^2\right)^2}$

***Example 6*:** $\mathcal{L}^{-1}\left\{\dfrac{3s}{\left(s^2+16\right)^2}\right\}$

***Example 7*:** $\mathcal{L}^{-1}\left\{\dfrac{s^2-16}{\left(s^2+16\right)^2}\right\}$

***Example 8*:** Find $\mathcal{L}^{-1}\left\{\dfrac{3}{s}-\dfrac{7}{s-5}+\dfrac{1}{s^4}\right\}$

***Example 9*:** Find $\mathcal{L}^{-1}\left\{\dfrac{s+2}{s^4}\right\}$

***Example 10*:** Find $\mathcal{L}^{-1}\left\{\dfrac{-\frac{1}{5}s+\frac{4}{5}}{s^2+4}\right\}$

11.3 Inversion using First Shift Theorem

11.3.1 First Shift Theorem

This is *first shift theorem* expressed in the inverse form:

First Shift Theorem (Formula $\boxed{8}$)

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)$$

We can also write: $\mathcal{L}^{-1}\{F(s-a)\} = \mathcal{L}^{-1}\left\{F(s)\big|_{s\to s-a}\right\} = e^{at} f(t)$

For example,

since we know that $\mathcal{L}^{-1}\left\{\dfrac{6}{s^4}\right\} = t^3$,

and that $\dfrac{6}{(s-2)^4}$ is just $\dfrac{6}{s^4}$ with 's' replaced by $(s-2)$,

hence $\mathcal{L}^{-1}\left\{\dfrac{6}{(s-2)^4}\right\} = e^{2t}\mathcal{L}^{-1}\left\{\dfrac{6}{s^4}\right\} =$

Alternatively, we can write: $\mathcal{L}^{-1}\left\{\dfrac{6}{(s-2)^4}\right\} = \mathcal{L}^{-1}\left\{\dfrac{6}{s^4}\bigg|_{s\to s-2}\right\} =$

More Examples on First Shift Theorem

***Example 11*:** Find $\mathcal{L}^{-1}\left\{\dfrac{2}{(s-3)^5}\right\}$

***Example 12*:** Find $\mathcal{L}^{-1}\left\{\dfrac{3}{(s+2)^2}\right\}$

***Example 13*:** Find $\mathcal{L}^{-1}\left\{\dfrac{3}{(s-5)^2+4}\right\}$

Example 14: Find $\mathcal{L}^{-1}\left\{\dfrac{s+1}{(s+1)^2+25}\right\}$

Example 15: Find $\mathcal{L}^{-1}\left\{\dfrac{s}{(s+1)^2+25}\right\}$

Example 16: Find $\mathcal{L}^{-1}\left\{\dfrac{2s+1}{(s+2)^3}\right\}$

11.3.2 Complete the Squares Method

In rational expressions (i.e. fractions) where the denominator is a quadratic function of the form as^2+bs+c which cannot be factorised, we will perform "completing the square" method for the denominator.

Briefly, to complete the square: $$s^2+ks=\left(s+\frac{k}{2}\right)^2-\left(\frac{k}{2}\right)^2$$

For example,

to find $\mathcal{L}^{-1}\left\{\dfrac{2(s+1)}{s^2+2s+10}\right\}$, we would want to "complete the square" for the quadratic denominator $s^2+2s+10$ which cannot be factorised.

Let $k =$: $s^2+2s+10=\left(s+\frac{\quad}{2}\right)^2-\left(\frac{\quad}{2}\right)^2+10=$

Hence, $\mathcal{L}^{-1}\left\{\dfrac{2(s+1)}{s^2+2s+10}\right\}=\mathcal{L}^{-1}\left\{\dfrac{2(s+1)}{\qquad}\right\}$

$$=\mathcal{L}^{-1}\left\{\left.\frac{\quad}{\quad}\right|_{s\to s+1}\right\}$$

More Examples on Completing the Squares Method

***Example 17*:** Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+4s+8}\right\}$

***Example 18*:** Find $\mathcal{L}^{-1}\left\{\dfrac{\frac{1}{5}s-\frac{2}{5}}{s^2+2s+2}\right\}$

11.4 Inversion by Resolving into Partial Fractions

A proper rational function of the form $\dfrac{p(s)}{q(s)}$ can be written as the sum of partial fractions having the forms $\dfrac{A}{as+b}$, $\dfrac{A}{(as+b)^2}$ or $\dfrac{As+B}{as^2+bs+c}$. By finding the inverse Laplace transform of each of the partial fractions, we can then evaluate $\mathcal{L}^{-1}\left\{\dfrac{p(s)}{q(s)}\right\}$.

For example,
to find $\mathcal{L}^{-1}\left\{\dfrac{9s+14}{(s-2)(s^2+4)}\right\}$, we must first resolve $\dfrac{9s+14}{(s-2)(s^2+4)}$ into its partial fractions.

Let $\dfrac{9s+14}{(s-2)(s^2+4)}=\dfrac{A}{s-2}+\dfrac{Bs+C}{s^2+4}$ .... (1)

Use cover-up rule to find A: $A=\left.\dfrac{9s+14}{s^2+4}\right|_{s=2}=$

Compare coefficients to find B and C:

Multiply (1) by $(s-2)(s^2+4)$: $9s+14=A(s^2+4)+(Bs+C)(s-2)$

Expand:

Compare s^2 terms: $0 = A + B$

Compare constants: $14 = 4A - 2C$

Solve:

Hence, $\mathcal{L}^{-1}\left\{\dfrac{9s+14}{(s-2)(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\dfrac{\qquad}{s-2} + \dfrac{\qquad}{s^2+4}\right\}$

More Examples on Partial Fractions Method

***Example 19*:** Find $\mathcal{L}^{-1}\left\{\dfrac{7s-6}{(s+2)(s-3)}\right\}$

***Example 20*:** Find $\mathcal{L}^{-1}\left\{\dfrac{2}{(s-1)(s+1)^2}\right\}$

One last question to ponder...

What method should we use to find $\mathcal{L}^{-1}\left\{\dfrac{6s-4}{s^2-8s+15}\right\}$?

Tutorial 11

1. Find the following:

(a) $\mathcal{L}^{-1}\left\{\frac{2}{s}-\frac{8}{s^3}+\frac{16}{s^5}\right\}$

(b) $\mathcal{L}^{-1}\left\{\frac{1}{s+6}-\frac{3s}{s^2+25}+\frac{1}{s^2+49}\right\}$

(c) $\mathcal{L}^{-1}\left\{\frac{s^2-100}{\left(s^2+100\right)^2}-\frac{4s}{\left(s^2+81\right)^2}\right\}$

(d) $\mathcal{L}^{-1}\left\{\frac{1}{2s-3}\right\}$

(e) $\mathcal{L}^{-1}\left\{\frac{3(1+s)}{s^5}\right\}$

(f) $\mathcal{L}^{-1}\left\{\frac{3s+2}{s^2+36}\right\}$

2. Use *first shift theorem* to find the following:

(a) $\mathcal{L}^{-1}\left\{\frac{6}{(s-1)^3}\right\}$

(b) $\mathcal{L}^{-1}\left\{\frac{3}{(s-2)^2+9}\right\}$

(c) $\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+25}\right\}$

(d) $\mathcal{L}^{-1}\left\{\frac{2(s-5)}{(s-5)^2+49}\right\}$

3. Use the methods of completing the square or partial fractions to find the following:

(a) $\mathcal{L}^{-1}\left\{\frac{2}{s^2+6s+13}\right\}$

(b) $\mathcal{L}^{-1}\left\{\frac{s-1}{s^2-4s+20}\right\}$

(c) $\mathcal{L}^{-1}\left\{\frac{s+2}{s^2+6s+9}\right\}$

(d) $\mathcal{L}^{-1}\left\{\frac{2s+3}{s^2-2s+5}\right\}$

(e) $\mathcal{L}^{-1}\left\{\frac{s-\frac{3}{2}}{2s^2-6s+\frac{13}{2}}\right\}$

(f) $\mathcal{L}^{-1}\left\{\frac{s^2-2s+3}{s(s-1)(s-2)}\right\}$

(g) $\mathcal{L}^{-1}\left\{\frac{s^2+1}{(s-1)(s^2+2)}\right\}$

(h) $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2(s^2+9)}\right\}$

*4. Find the following:

(a) $\mathcal{L}^{-1}\left\{\frac{4e^{-3}}{2s-1}\right\}$

(b) $\mathcal{L}^{-1}\left\{\frac{s-1}{4s^2+60}\right\}$

*5. Find the following:

(a) $\mathcal{L}^{-1}\left\{\frac{s-3}{(s-2)^2+2(s-2)+1}\right\}$

(b) $\mathcal{L}^{-1}\left\{\frac{1}{s^2+s+1}\right\}$

*6. Find $\mathcal{L}\left\{e^{2t}\cos 3t\right\}$. Hence, given that $\mathcal{L}^{-1}\left\{F(s+3)\right\}=e^{3(1-t)}\cos 3t$, find $F(s-2)$.

Multiple Choice Questions

1. If $\mathcal{L}^{-1}\{F(s)\} = \sin 2t$, then $\mathcal{L}^{-1}\{F(s+\pi)\}$ is equal to

 (a) $\sin 2t$ (b) $-\sin 2t$
 (c) $e^{-\pi t}\sin 2t$ (d) $e^{\pi t}\sin 2t$

2. If $\mathcal{L}^{-1}\{F(s+2)\} = e^{2(1-t)}t^3$, then $\mathcal{L}^{-1}\{F(s)\}$ is equal to

 (a) t^3 (b) e^2t^3
 (c) $e^{2(1+t)}t^3$ (d) $e^{2(1-2t)}t^3$

3. When performing the following transformations, which one does NOT involve *First Shift Theorem*?

 (a) $\mathcal{L}^{-1}\left\{\dfrac{se^{-s}}{(s^2+9)^2}\right\}$ (b) $\mathcal{L}\left\{\left(e^{-t}-e^{3t}\right)\sin 2t\right\}$

 (c) $\mathcal{L}^{-1}\left\{\dfrac{s-1}{(s-3)^3}\right\}$ (d) $\mathcal{L}\left\{\int_0^t t^2\left(e^t+t-3\right)dt\right\}$

Answers

1. (a) $2-4t^2+\dfrac{2}{3}t^4$ (b) $e^{-6t}-3\cos 5t+\dfrac{1}{7}\sin 7t$ (c) $t\cos 10t-\dfrac{2}{9}t\sin 9t$

 (d) $\dfrac{1}{2}e^{\frac{3}{2}t}$ (e) $\dfrac{1}{8}t^4+\dfrac{1}{2}t^3$ (f) $3\cos 6t+\dfrac{1}{3}\sin 6t$

2. (a) $3t^2e^t$ (b) $e^{2t}\sin 3t$ (c) $e^{-2t}\cos 5t$ (d) $2e^{5t}\cos 7t$

3. (a) $e^{-3t}\sin 2t$ (b) $e^{2t}\left(\cos 4t+\dfrac{1}{4}\sin 4t\right)$ (c) $e^{-3t}(1-t)$

 (d) $e^t\left(2\cos 2t+\dfrac{5}{2}\sin 2t\right)$ (e) $\dfrac{1}{2}e^{\frac{3}{2}t}\cos t$ (f) $\dfrac{3}{2}-2e^t+\dfrac{3}{2}e^{2t}$

 (g) $\dfrac{2}{3}e^t+\dfrac{1}{3}\left(\cos\sqrt{2}\,t+\dfrac{1}{\sqrt{2}}\sin\sqrt{2}\,t\right)$ (h) $\dfrac{1}{9}+\dfrac{1}{9}t-\dfrac{1}{9}\cos 3t-\dfrac{1}{27}\sin 3t$

4. (a) $2e^{\frac{t}{2}-3}$ (b) $\dfrac{1}{4}\left(\cos\sqrt{15}\,t-\dfrac{1}{\sqrt{15}}\sin\sqrt{15}\,t\right)$

5. (a) $e^t(1-2t)$ (b) $\dfrac{2}{\sqrt{3}}e^{-\frac{t}{2}}\sin\dfrac{\sqrt{3}}{2}t$

6. $e^3\left(\dfrac{s-2}{(s-2)^2+9}\right)$

MCQ

1. c 2. b 3. a