Chapter 2

Signals and Spectra

(Part 2 of 5)

2.3 Fourier Series

- A signal $x(t)$ can be described by

Time Domain representation, x(t)	Frequency Domain representation, $X(f)$
Function of time	Function of frequency
Signal waveform	Signal spectrum
- Signal shape - Instantaneous magnitude	- Amplitude and phase of various frequency components
Observed by Oscilloscope	Observed by Spectrum Analyzer

2.3 Fourier Series

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Fourier series and Fourier transform

Mathematical tools used to determine signal spectrum of periodic and non-periodic signal, respectively:

Fourier series

For periodic signals

A periodic signal is written as a sum of trigonometric or exponential functions with specific frequencies.

Fourier transform

For all signals

A signal is written as a continuous integral of trigonometric or exponential functions with a continuum of possible frequency

2.3 Fourier Series

$x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(2\pi n f_0 t + \varphi_n)$

Fourier series

- A periodic signal x(t) of period T_0 can be expended into a trigonometric Fourier series:

$$\mathbf{x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)}$$

or

$$\mathbf{x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(2\pi n f_0 t + \varphi_n)}$$

$\omega_0 = 2\pi f_0$ (rad/sec)
$f_0 = 1/T_0$ Hz, fundamental frequency of x(t)
nf_0 is n^{th} harmonic frequency

where

$$a_0 = \frac{1}{T}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)dt$$

$$a_n = \frac{2}{T}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\cos(n\omega_0 t)\,dt$$

$$b_n = \frac{2}{T}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\sin(n\omega_0 t)\,dt$$

and

$A_0 = a_0$,

$$A_n = \sqrt{{a_n}^2 + {b_n}^2}$$

$$\varphi_n = -\tan^{-1}\left(\frac{b_n}{a_n}\right)$$

Not tested

2.3 Fourier Series

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Fourier series

$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \varphi_n)$$

$\omega_0 = 2\pi f_0$

$$= A_0 + \underbrace{A_1\cos(\omega_0 t + \varphi_1) + A_2\cos(2\omega_0 t + \varphi_2) + A_3\cos(n\omega_0 t + \varphi_3) + \ldots}$$

DC component at f=0 : A_0

AC components: $A_n \cos(n\omega_0 t + \varphi_n)$
A_1: fundamental frequency components at f_0
A_n: harmonic frequency components at nf_0 (n=2,3,4…).
φ_n: phase of the n*th* harmonic frequency component

2.3 Fourier Series

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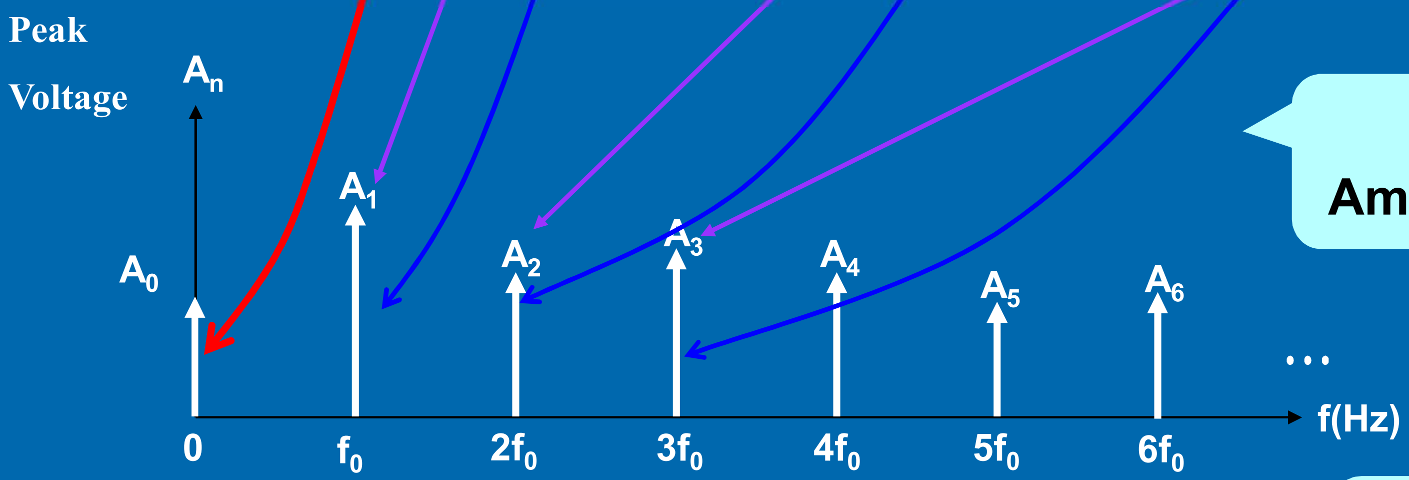
Single-sided Amplitude spectrum

- Graphical presentation of A_n vs frequency

$$v(t) = A_0 + A_1\cos(\omega_0 t + \varphi_1) + A_2\cos(2\omega_0 t + \varphi_2) + A_3\cos(n\omega_0 t + \varphi_3) + \ldots$$



Single-sided Amplitude spectrum

Discrete (line) spectrum

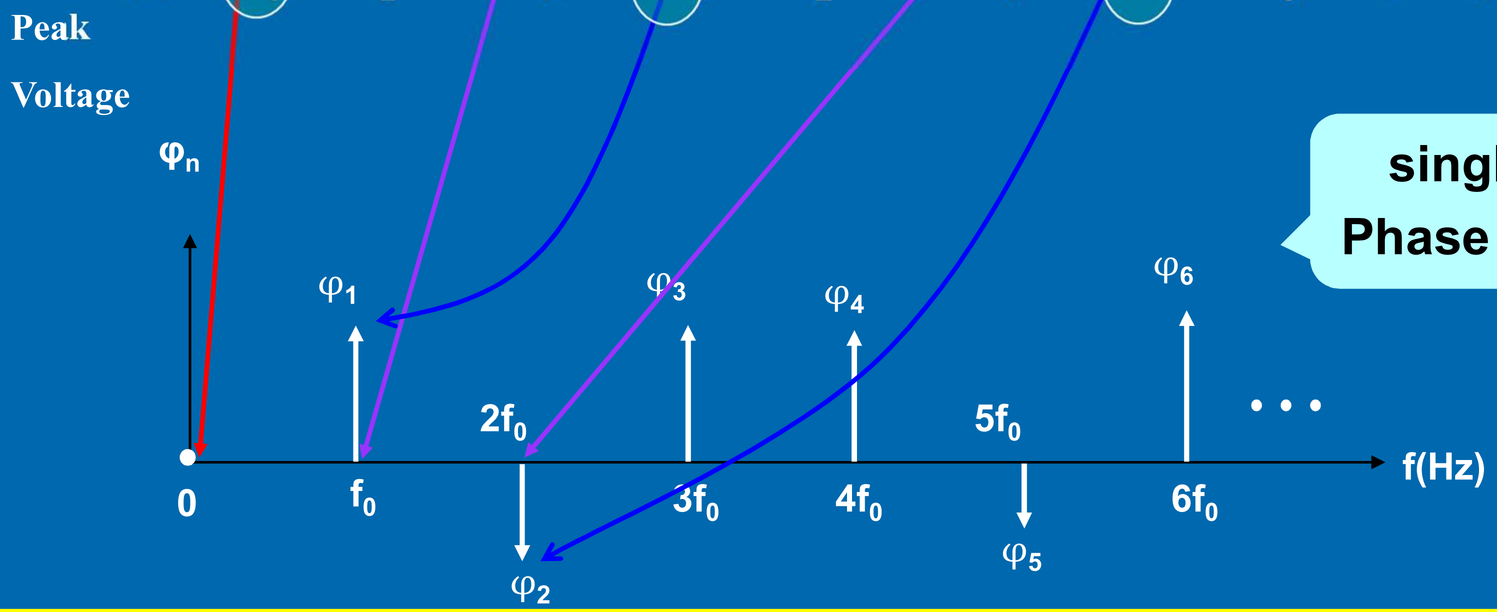
2.3 Fourier Series

Not tested

Single-sided phase spectrum

- Graphical presentation of φ_n vs frequency

$$x(t) = A_0 + A_1\cos(\omega_0 t + \varphi_1) + A_2\cos(2\omega_0 t + \varphi_2) + A_3\cos(n\omega_0 t + \varphi_3) + \ldots$$



single-sided Phase spectrum

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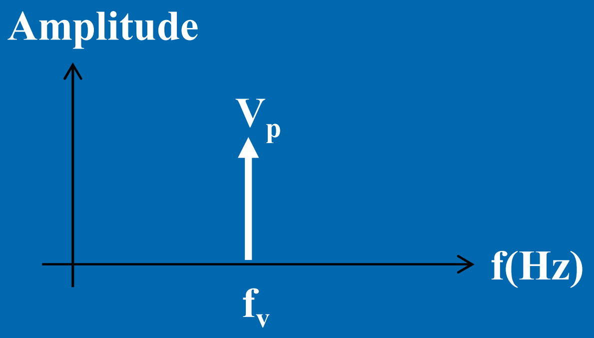
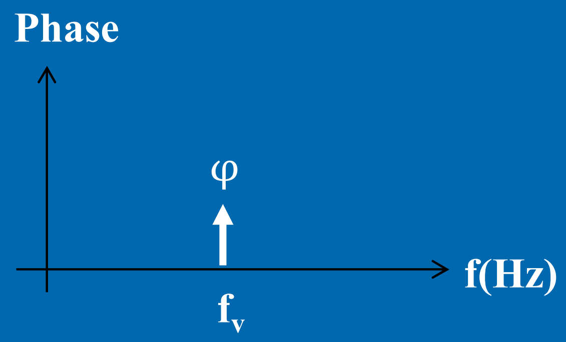
Single-sided amplitude and phase spectrum of sinusoidal signal

$$v(t) = V_P \cos(2\pi f_v t + \varphi)$$

- Sinusoidal signal contains only one frequency component.
- A sinusoidal signal is thus known as single-tone signal.



Amplitude spectrum



Phase spectrum

Example 2.4

Plot the single-sided amplitude and phase spectrum of the following sinusoidal signal.

$$v(t) = 3.5\cos(2000\pi t + \frac{\pi}{3})$$

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Solution

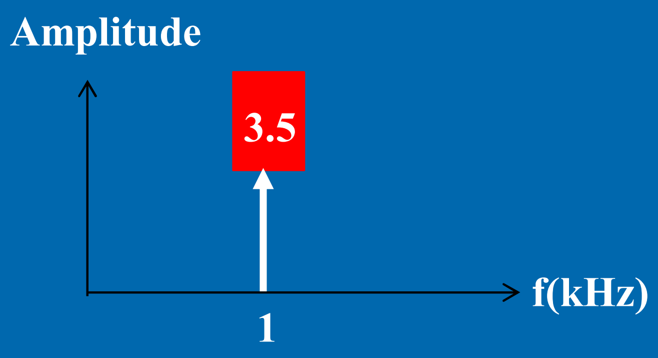
Amplitude

Phase

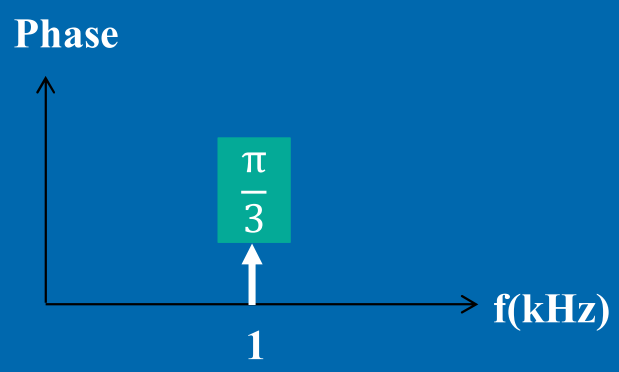
$$v(t) = \boxed{3.5}\cos(2000\pi t + \boxed{\frac{\pi}{3}})$$

$2\pi f t = 2\pi(1000)t$

$f = 1000$ Hz



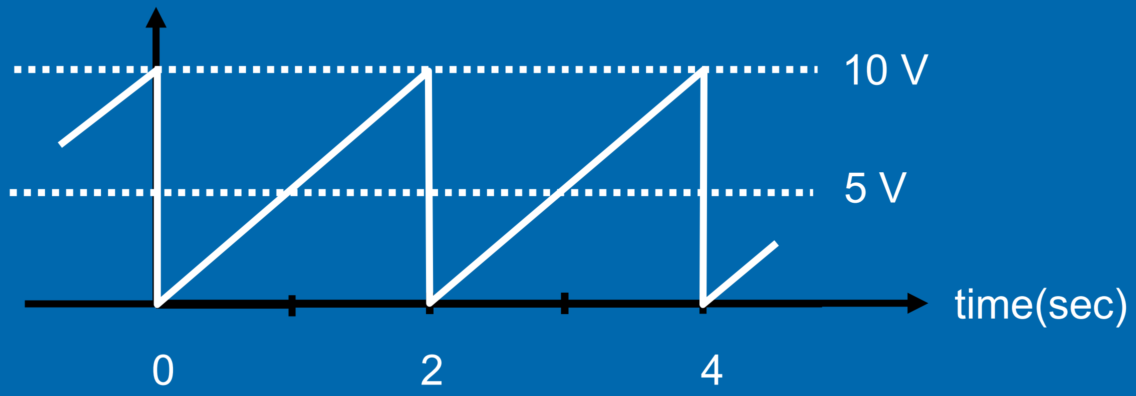
Amplitude spectrum



Phase spectrum

Example 2.5

Plot the single-sided amplitude and phase spectrum of a sawtooth signal.



The Fourier Series of the above waveform is given below:

$$f(t) = 5 + \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\cos\left(n\omega_0 t + \frac{\pi}{2}\right)$$

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Solution

The period of $f(t)$ is 2 s. Therefore, $f_0 = \frac{1}{T} = \frac{1}{2}$ = 0.5 Hz

Expanding the Fourier series of f(t):

$$f(t) = 5 + \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\cos\left(n\omega_0 t + \frac{\pi}{2}\right)$$

$$= 5 + \underbrace{\boxed{\frac{10}{\pi}}\cos\left(2\pi f_0 t + \boxed{\frac{\pi}{2}}\right)}_{n=1} + \frac{10}{\pi}\cdot\frac{1}{2}\cos\left(4\pi f_0 t + \frac{\pi}{2}\right) + \ldots.$$

where $\omega_0 = 2\pi f_0$

Amplitude

Phase

Solution

The period of *f*(*t*) is 2 s. Therefore, $f_0 = \frac{1}{T} = \frac{1}{2}$ Hz

Expanding the Fourier series of f(t):

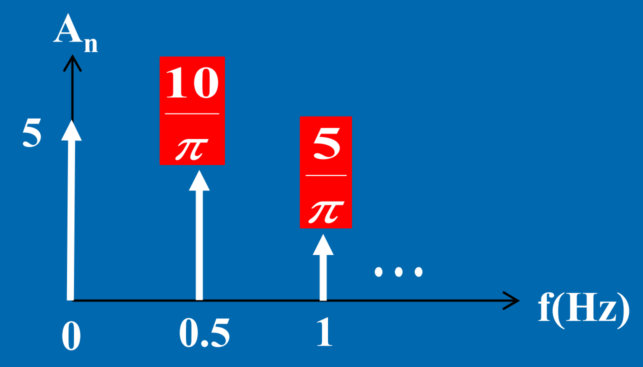
$$f(t) = 5 + \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\cos\left(n\omega_0 t + \frac{\pi}{2}\right)$$

$$= 5 + \underbrace{\frac{10}{\pi}\cos\left(2\pi f_0 t + \frac{\pi}{2}\right)}_{n=1} + \underbrace{\boxed{\frac{10}{\pi}\cdot\frac{1}{2}}\cos\left(4\pi f_0 t + \boxed{\frac{\pi}{2}}\right)}_{n=2} + \ldots.$$

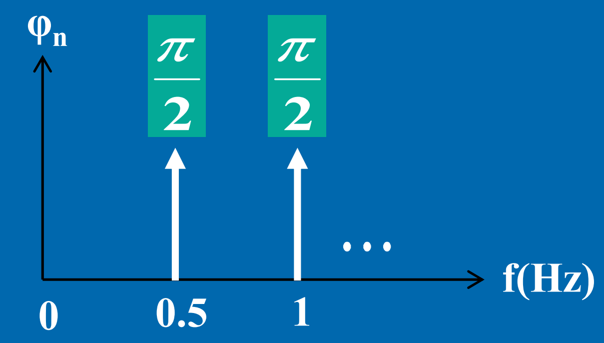
freq = 2π (2f_0)t

Amplitude

Phase



(a) Amplitude spectrum



(b) Phase spectrum

2.3 Fourier Series

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Exponential form of Fourier Series

- **A periodic signal x(t) expressed as a complex exponential Fourier series:**

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\ {}_0 t}$$

Consist of both positive and negative frequency components: $\pm n\omega_0$.

where

$$C_n = \frac{1}{T}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)e^{-jn\omega_0 t}dt = |C_n|\, e^{j\varphi_n}$$

Not tested

$\omega_0 = 2\pi f_0$ (rad/sec)

2.3 Fourier Series

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Exponential form of Fourier Series

$$C_n = \boxed{|C_n|}\, e^{j\boxed{\varphi_n}}$$

$|C_n| \geq 0$

Phase of *nth* harmonic component

Amplitude of *nth* harmonic component

plot of φ_n vs frequency is the phase spectrum

plot of $|C_n|$ vs frequency is the amplitude spectrum

Double-sided spectrum

- The coefficients a_n and b_n are related to the coefficient C_n:

Not tested

$$C_n = \begin{cases} \frac{1}{2}a_n - \frac{1}{2}jb_n & n \geq 1 \\ a_0 & n = 0 \\ \frac{1}{2}a_{|n|} + \frac{1}{2}jb_{|n|} & n \leq -1 \end{cases} \quad \Rightarrow \quad \begin{cases} |C_0| = a_0 = A_0 \\ |C_n| = \dfrac{\sqrt{a_n^2 + b_n^2}}{2} = \dfrac{A_n}{2} \quad \text{for } n \neq 0 \end{cases}$$

2.3 Fourier Series

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Exponential form of Fourier Series

- **Conversion between single-sided and double-sided amplitude spectrum:**

$$|C_0| = a_0 = A_0$$

$$|C_n| = \frac{\sqrt{a_n^2 + b_n^2}}{2} = \frac{A_n}{2} \quad \text{for } n \neq 0$$

Single-sided → Double-sided

$$|C_0| = A_0$$

$$|C_n| = \frac{A_n}{2} \quad \text{for } n \neq 0$$

Double-sided → Single-sided

$$A_0 = |C_0|$$

$A_n = 2|C_n|$ for $n \neq 0$

2.3 Fourier Series

Spectrum of rectangular waveform

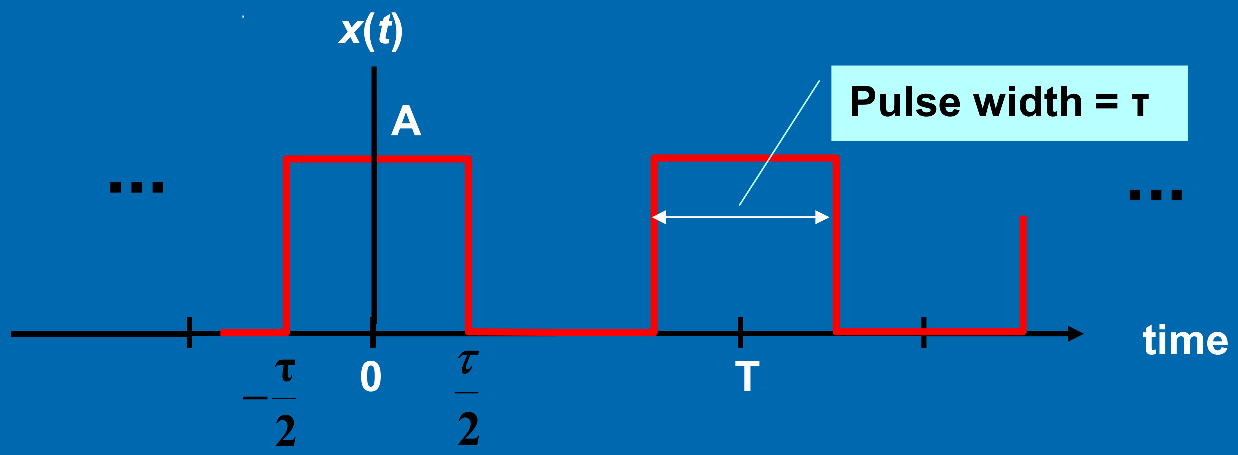
1. Data
2. Sampling pulse train

Example 2.6

Obtain the frequency spectrum of a rectangular waveform, $x(t)$, shown below using the complex exponential Fourier Series. Consider two cases:

a) $\tau = \dfrac{T}{2}$ (50% duty cycle) — Square wave

b) $\tau = \dfrac{T}{5}$ (20% duty cycle) — rectangular wave



Duty cycle = $\dfrac{\tau}{T} \times 100\%$

Spectrum of rectangular waveform

Solution

$$C_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jn\omega_0 t}dt$$

Let us now consider the period, $-\frac{\tau}{2} \le t \le T - \frac{\tau}{2}$

$$C_n = \frac{1}{T}\int_{-\frac{\tau}{2}}^{T-\frac{\tau}{2}} x(t)e^{-jn\omega_0 t}dt$$

$$C_n = \frac{1}{T}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} x(t)e^{-jn\omega_0 t}dt$$

$$= -\frac{A}{jn\omega_0 T}\left[e^{-jn\omega_0 t}\right]_{-\frac{\tau}{2}}^{\frac{\tau}{2}} = -\frac{2A}{n\omega_0 T}\left[\frac{e^{jn\frac{\omega_0}{2}\tau} - e^{-jn\frac{\omega_0}{2}\tau}}{j2}\right]$$

Recall

Euler's identity for sine function

sin z = $\frac{e^{jz} - e^{-jz}}{2j}$

Spectrum of rectangular waveform

Solution (cont'd)

$$= \frac{2A}{n\omega_0 T} \sin\frac{n\omega_0\tau}{2} \qquad = \frac{A\tau}{T}\frac{\sin\frac{n\omega_0\tau}{2}}{\frac{n\omega_0\tau}{2}}$$

$$\left.\begin{aligned}\omega_0 &= 2\pi f_0 = \frac{2\pi}{T}\\ \frac{n\omega_0\tau}{2} &= \pi\frac{n\tau}{T}\end{aligned}\right\}$$

$$= \frac{A\tau}{T}\frac{\sin\pi\frac{n\tau}{T}}{\pi\frac{n\tau}{T}}$$

x

Let $x = \frac{n\tau}{T}$

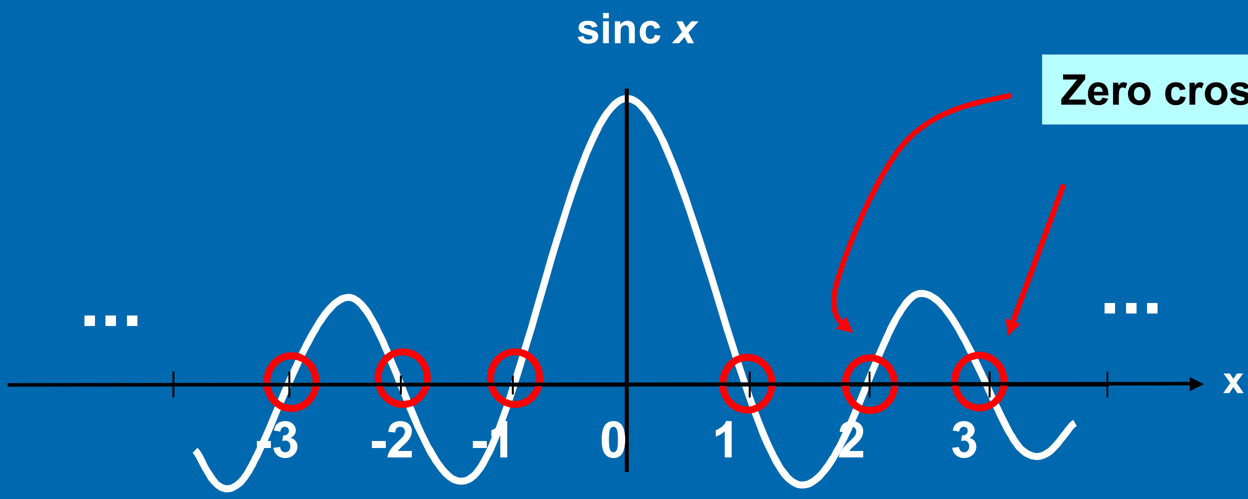
$$C_n = \frac{A\tau}{T}\,\text{sinc}\,\frac{n\tau}{T}$$

x

$$\text{sinc } x = \frac{\sin \pi x}{\pi x}$$

Spectrum of rectangular waveform

<u>Solution</u> (cont'd)



Note:
sinc x = 1 for x = 0 i.e. sinc 0 = 1
and sinc x has zero crossings at $x = \pm 1, \pm 2, \pm 3, \ldots$
i.e. sinc 1 = 0, sinc -1 = 0, sinc 2 = 0, sinc -2 = 0 etc

Spectrum of rectangular waveform

Solution (cont'd)

Therefore,

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} \frac{A\tau}{T} \operatorname{sinc} \frac{n\tau}{T} e^{jn\omega_0 t}$$

substituting $C_n = \frac{A\tau}{T} \operatorname{sinc} \frac{n\tau}{T}$

$$= \underbrace{\boxed{\frac{A\tau}{T} \sum_{n=-\infty}^{\infty} \operatorname{sinc} \frac{n\tau}{T}}}_{\text{amplitude}} e^{jn\omega_0 t}$$

frequency

E.g. n =0 , dc

n =1 , freq = f_0

n =2 , freq = $2f_0$

2.3 Fourier Series

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Spectrum of rectangular waveform

- **Consider now the first case where**

i.e. $\tau = \dfrac{T}{2}$ $\qquad \dfrac{\tau}{T} = \dfrac{1}{2}$ $\qquad$ **Square wave**

Substitutie $\dfrac{\tau}{T} = \dfrac{1}{2}$ into the C_n equation, $\boxed{C_n = \dfrac{A\tau}{T}\,\mathrm{sinc}\,\dfrac{n\tau}{T}}$

$$C_n = \frac{A}{2}\,\mathrm{sinc}\,\frac{n}{2}$$

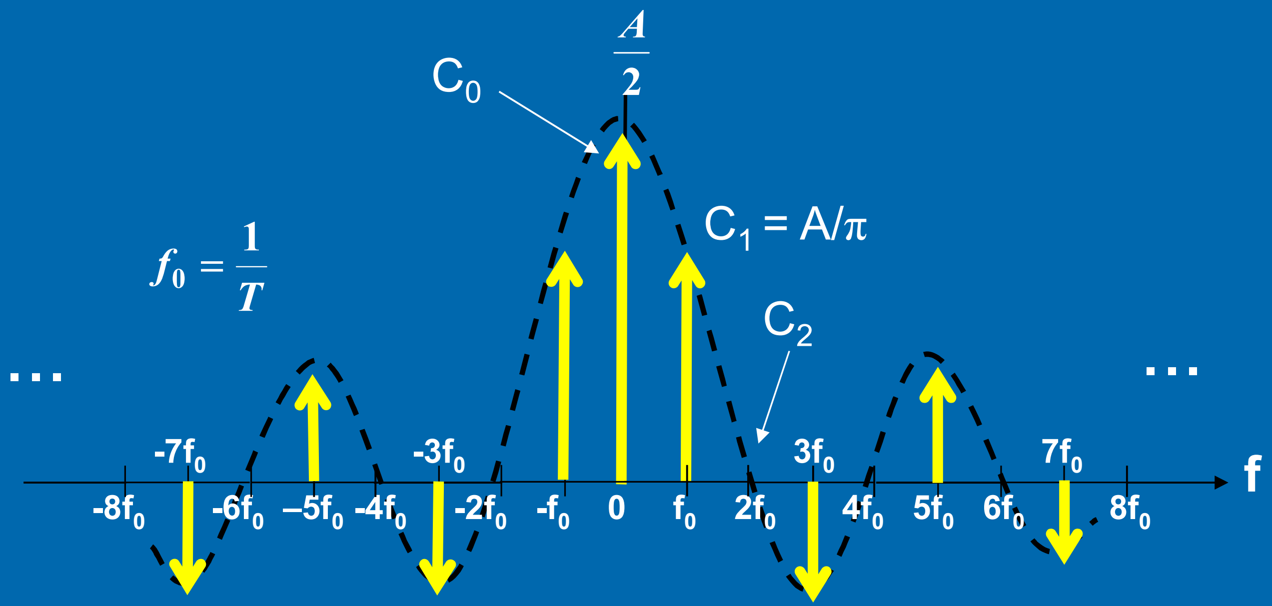
Substitute C_n in x(t) equation, $\boxed{x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}}$

$$\therefore \quad x(t) = \frac{A}{2}\sum_{n=-\infty}^{\infty} \mathrm{sinc}\,\frac{n}{2}\, e^{jn\omega_0 t}$$

2.3 Fourier Series

Spectrum of rectangular waveform

Spectrum of **Square wave**



$$n = 0,\ C_0 = \frac{A}{2}\operatorname{sinc}\frac{0}{2} = \frac{A}{2}$$

$$n = 1,\ C_1 = \frac{A}{2}\operatorname{sinc}\frac{1}{2} = \frac{A}{\pi}$$

$$n = 2,\ C_2 = \frac{A}{2}\operatorname{sinc}\frac{2}{2} = 0$$

$$\operatorname{sinc}\frac{1}{2} = \frac{\sin \pi \frac{1}{2}}{\pi \frac{1}{2}} = \frac{2}{\pi}$$

- **The zero crossings are found at *n* = ±2; ±4; ±6; ... (i.e. when *n*/2 is an integer). Even harmonics are suppressed.**

2.3 Fourier Series

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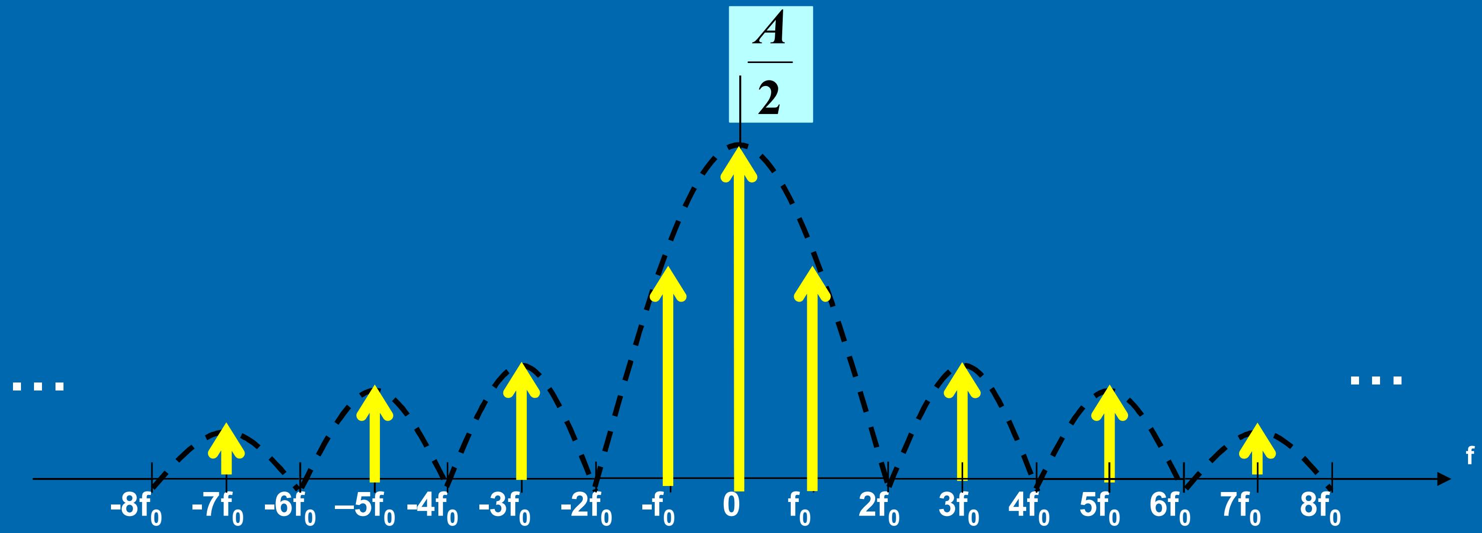
Spectrum of rectangular waveform

- If only the **amplitude spectrum** is considered, the negative components should be inverted.

Amplitude spectrum of **Square wave**



2.3 Fourier Series

Spectrum of rectangular waveform

- Consider now the second case where

$$\tau = \frac{T}{5} \quad \text{i.e.} \quad \frac{\tau}{T} = \frac{1}{5}$$

Rectangular wave

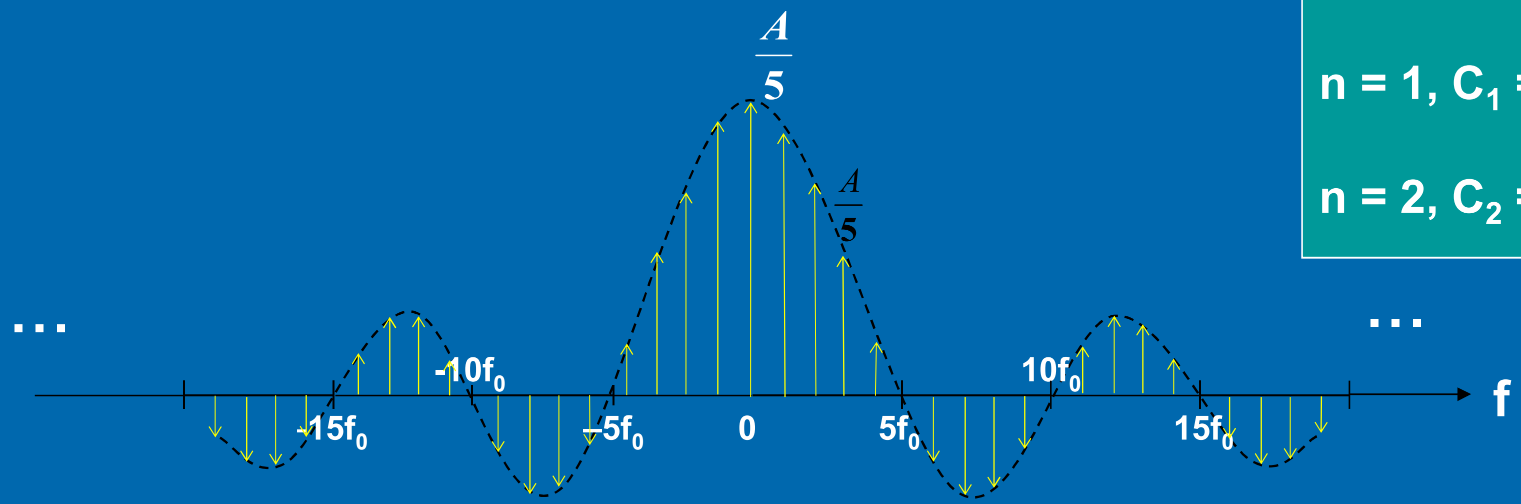
Substituting $\frac{\tau}{T} = \frac{1}{5}$ into the C_n equation, $C_n = \frac{A\tau}{T} \operatorname{sinc} \frac{n\tau}{T}$ we get

$$C_n = \frac{A}{5} \operatorname{sinc} \frac{n}{5}$$

Substitute C_n in x(t) equation, $x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$

$$\therefore \quad x(t) = \frac{A}{5} \sum_{n=-\infty}^{\infty} \operatorname{sinc} \frac{n}{5} e^{jn\omega_0 t}$$

Spectrum of rectangular waveform



$$n = 0,\ C_0 = \frac{A}{5}$$

$$n = 1,\ C_1 = \frac{A}{5}\operatorname{sinc}\frac{1}{5}$$

$$n = 2,\ C_2 = \frac{A}{5}\operatorname{sinc}\frac{2}{5}$$

Note :The zero crossings are now at $\pm 5f_o$; $\pm 10f_o$; $\pm 15f_o$; ...

Change of pulse width changes the zero crossings

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Example 2.7

Plot the double-sided amplitude spectrum of the signal given in example 2.4.

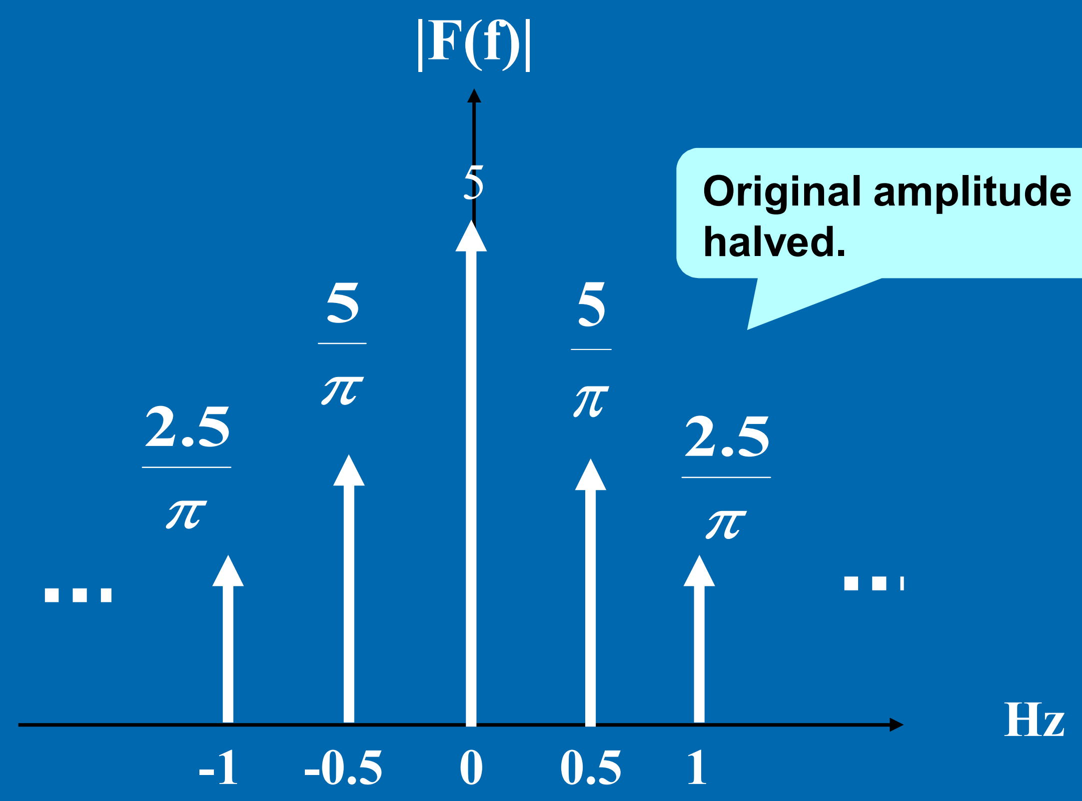
Solution

$$A_0 = 5, \quad A_1 = \frac{10}{\pi}, \quad A_2 = \frac{5}{\pi}, \ldots, A_n = \frac{10}{n\pi}$$

$$|C_0| = A_0$$

$$|C_n| = \frac{A_n}{2} \quad \text{for } n \neq 0$$

$$C_0 = 5, \quad C_1 = \frac{10}{2\pi}, \quad C_2 = \frac{5}{2\pi}, \ldots, C_n = \frac{10}{2n\pi}$$



Original amplitude is halved.

End

CHAPTER 2

(Part 2 of 5)