Chapter 3 – Determination of Law

Objectives:

1. *Reduce non-linear laws to linear forms.*
2. *Determine unknown constants in the laws from the graph.*
3. *Estimate values of variables by interpolation within the range of validity of the law.*

3.1 The Straight Line Graph [Revision]

In this chapter we will first introduce the straight-line equation and then discuss the components that make up this equation. We will conclude by applying this straight-line idea to experimental data to verify that these data do follow a certain law.

The equation of a straight line takes the form:

$$\boxed{y = mx + c}$$

where **y** is the dependent variable on the vertical axis,
m is the constant that gives the slope of the line,
x is the independent variable on the horizontal axis, and
c is the constant where the line cuts the vertical axis

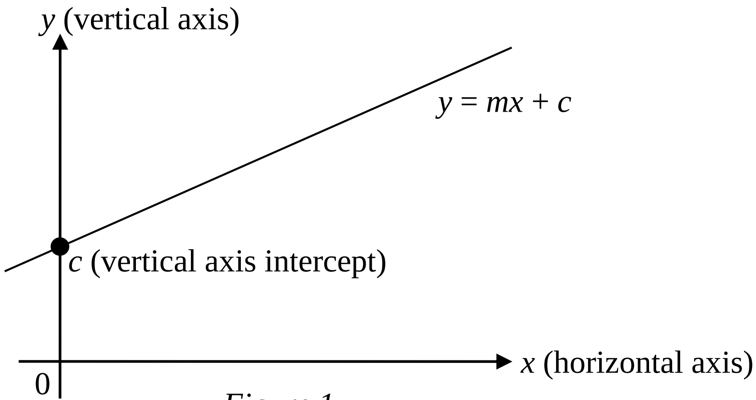


Figure 1

Being able to recognize a straight-line form is essential especially when we have to verify the laws governing some physical quantities such as temperature, resistance, the stress or strain on a material and so forth.

To compute the value of the gradient, we pick any two points, $\mathrm{A}\left(x_1,\ y_1\right)$ and $\mathrm{B}\left(x_2, y_2\right)$, on the line and use the formula:

$$\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}}$$

Example 1: Find the equation of the straight line passing through the points (3, 5) and (–4, 2).

(Answer: $y = \frac{3}{7}x + \frac{26}{7}$)

Compressed Scale

Sometimes if our data are numerically too big, positively or negatively, it would be impossible to start the scales for one or both axes at 0. In such cases, we would have to "compress" the scales as shown in *Figure 2* and *Figure 3*.

If the horizontal scale is compressed, like they are in these two figures, then the vertical-axis intercept cannot be read from the graph. This is because the portion of the line near the vertical axis is missing. In both graphs, the portion of line from 0 to 110 is missing. As such, reading the vertical-axis intercept directly from the graph would be incorrect. The vertical-axis intercept has to be calculated and this will be illustrated in *Examples 2 & 3* later in this chapter.

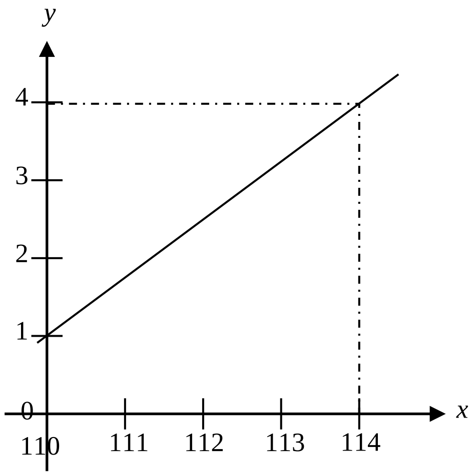


Figure 2

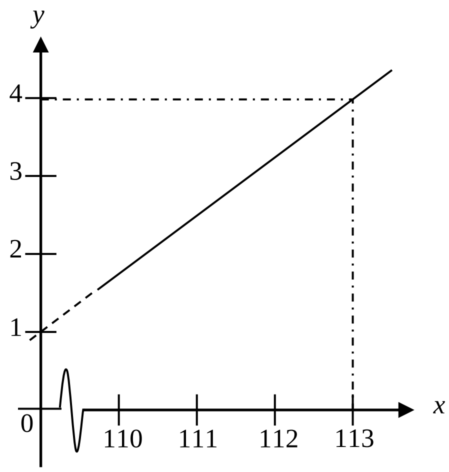


Figure 3

Please note that we will calculate the vertical-axis intercept only if the **<u>horizontal scale is compressed</u>**. If the vertical scale is compressed and the horizontal scale is not, we can still read the vertical-axis intercept directly from our graph.

3.2 Determination of Law

In this section we will now look at the usefulness of learning the straight-line form. Usually when we are given an equation connecting two variables, for example, $F = \frac{9}{5}C + 32$, it is relatively easy to plot the graph.

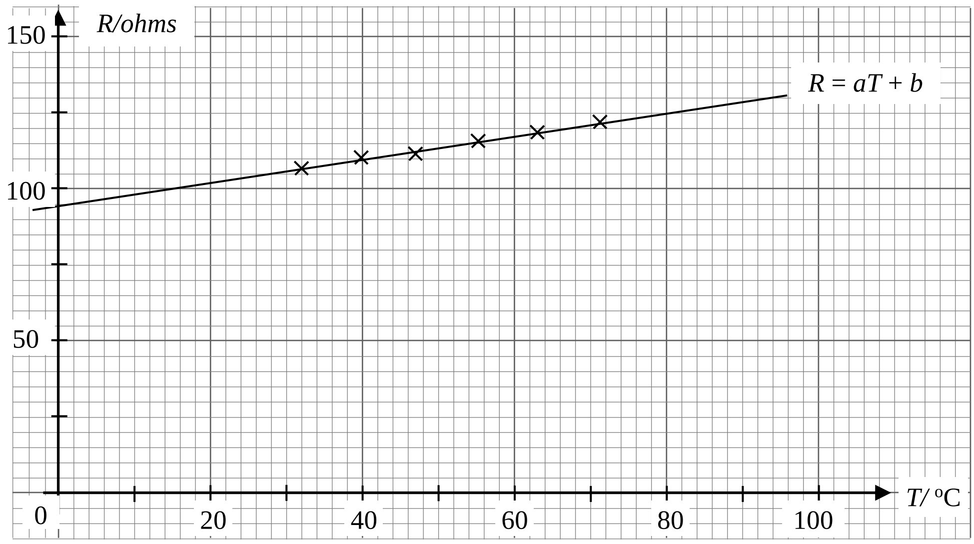
However, in reality we are usually faced with the reverse problem. We would have the data and the graph for a phenomenon we are studying. Our task would then be to find an equation that will fit the data.

Example 2: In an experiment, the resistance R of a conductor was measured at various temperatures T and the data tabulated as shown:

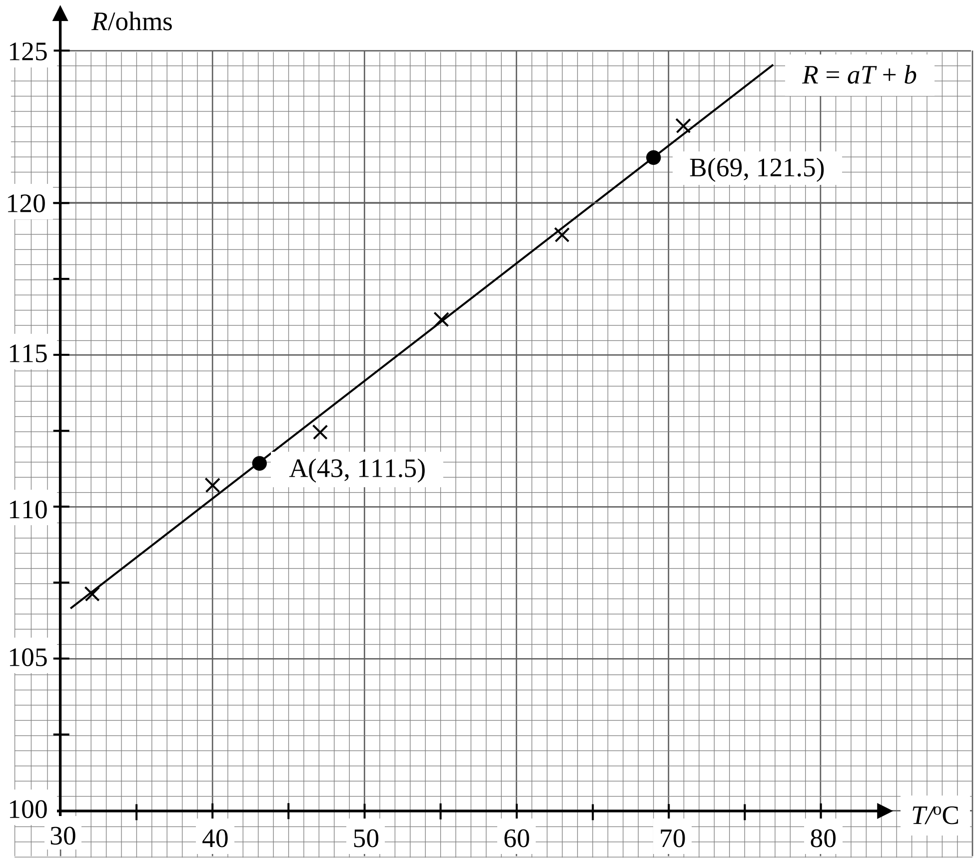
R (ohms)	107.2	110.7	112.5	116.3	119.0	122.5
***T* (°C)**	32	40	47	55	63	71

Show that the variables R and T follow the equation $R = aT + b$. From the graph drawn, determine the values of a and b.

Solution: In this example, if both scales start at 0, the graph will look as follows:



Notice the large gap between the horizontal axis and the plotted line, whilst the plotted points are clustered in the upper portion of our graph paper. Accuracy is compromised in this case. So for greater accuracy and to better distribute these points we will start our R-axis scale at 100 and our T-axis scale at 30.



R/ohms
125
120
115
110
105
100
30
40
50
60
70
80
T/°C
R = aT + b
B(69, 121.5)
A(43, 111.5)

Since the graph drawn is a straight line, R and T do follow the equation $R = aT + b$.

We are not able to read the R-axis intercept from the graph because the horizontal axis has been compressed. As such we will calculate the gradient first and use it to help us find the R-axis intercept.

To find the gradient, pick two points on the line, say, A (43, 111.5) and B (69,121.5).

Then,

gradient =

$\Rightarrow \quad a =$

To find b, we will use the calculated value for a and one of the points, say A(43, 111.5), in the equation $R = aT + b$:

$\therefore$ the equation of the line is ____

3.3 Reduction of Non-Linear Equations to Linear Form

Frequently, the relationship between the variables is not a linear one, for example $L = 0.03T^2 + 3$ or $i = 1750e^{-0.25t}$. In such cases, the non-linear equation can be modified into a linear form $Y = mX + C$, so that the law can be verified by plotting Y against X and the constants m and C can be found from the graph.

We will now see how we can modify some non-linear equations into straight-line forms. Recall, the straight-line equation is

$$Y = mX + C$$

which is really of the form:

(*dependent variable*) = (constant 1) (*independent variable*) + (constant 2)

Note that '*dependent variable*' and '*independent variable*' should not contain any of the unknown constants given in the original equation. Otherwise we will not be able to plot our graph, as '*dependent variable*' is the vertical axis while '*independent variable*' is the horizontal axis.

Similarly, 'constant 1' and 'constant 2' can be a combination of the unknown constants given in the original equation. They should not contain any variables as this would contradict the fact that they are constants and thus do not change their values. With this in mind, let us work on the following examples.

Example 3: [Converting polynomial function to linear function]

Reduce the following non-linear equations to linear form:

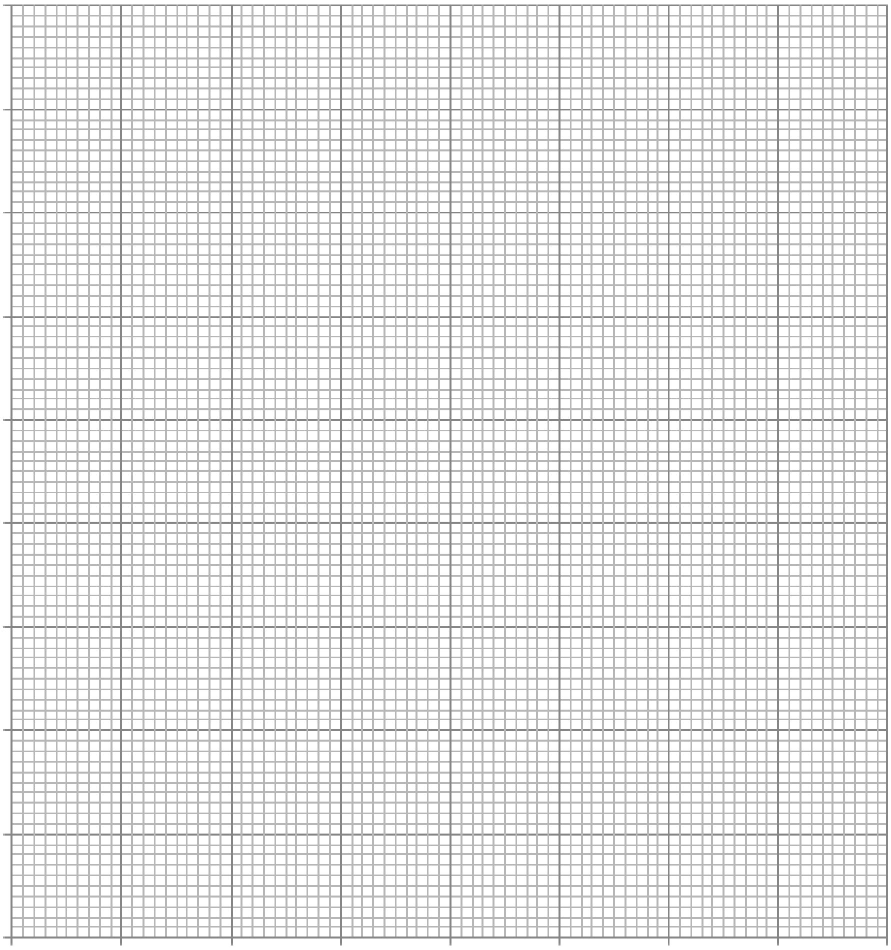
(a) $y = ax^2 + b$ (b) $y = \dfrac{a}{x} + b$

where a and b are constants.

Example 4: The variables x and y are believed to be related by a law of the form $y = ax^2 + b$, where a and b are constants.
Show by plotting a suitable straight line graph that the law is obeyed. Hence find the approximate values of a and b.

x	2	3	4	5	6
y	22	45	76	115	162

Solution:



Since a straight line can be fitted through the points, therefore the variables x and y are related by the law $y = ax^2 + b$.

From the graph, the vertical axis-intercept is __________ .

Hence b = __________ .

Gradient =

Hence a = __________ .

Example 5: [Converting polynomial function to linear function]

Reduce the following non-linear equations to linear forms:

(a) $y = ax^2 + bx$ (b) $xy = ax + by$

where a and b are constants.

Example 6: [More examples on converting non-linear functions to linear function]

Reduce the following non-linear equations to linear forms:

(a) $y = ax^2 + \dfrac{b}{x^3}$ (b) $y = \dfrac{ax}{b + x^2}$

(c) $y = 1 + ax + bx^2$

where a and b are constants.

Example 7: [Converting power and exponential functions to linear function]
Reduce the following non-linear equations to linear forms:

(a) $y = ax^b$ (b) $y = ab^x$

(c) $y = ae^{bx}$

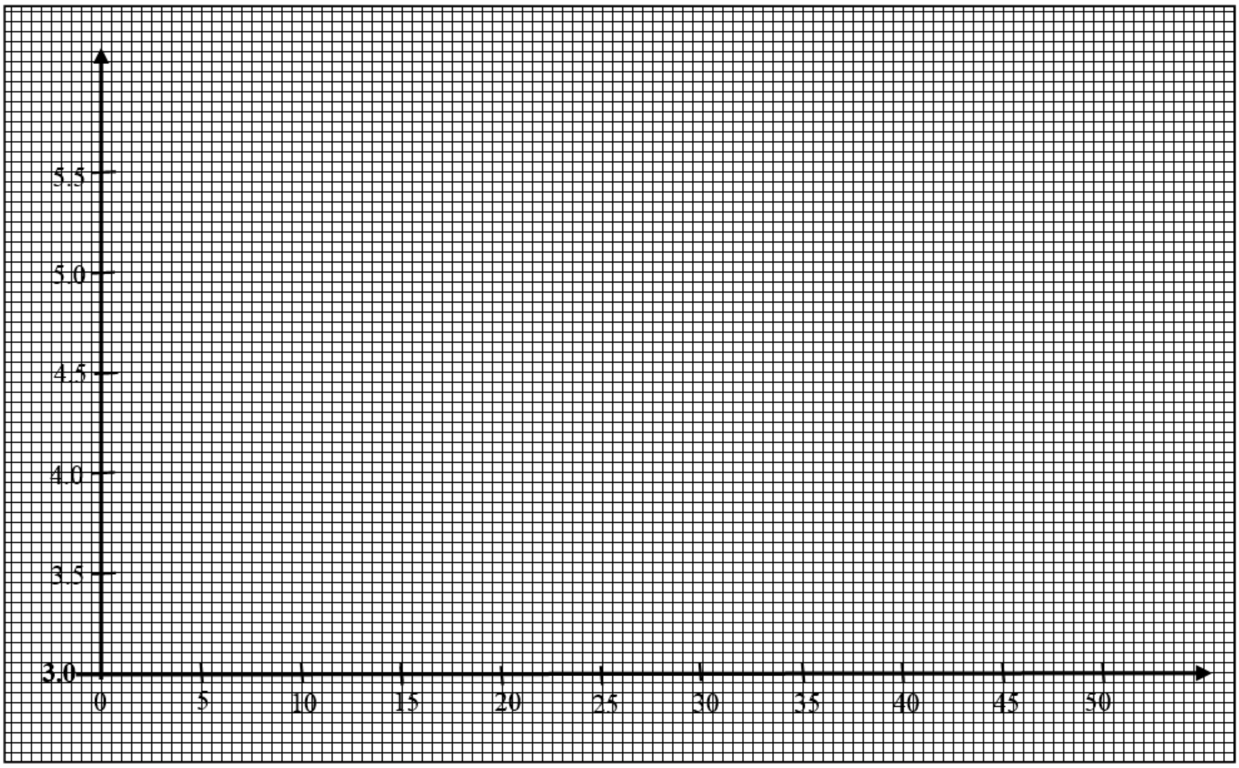
Where a and b are constants.

Example 8: A liquid which is cooling is believed to follow a law of the form $\theta = \theta_0 e^{kt}$ where θ_0 and k are constants and θ is the temperature of the body at time *t*. An experiment conducted gives the following data:

θ (^{o}C)	83	58	41.5	32	26
t (min)	16.7	25	32.5	37	43.5

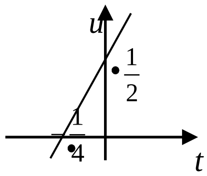
Show graphically the values obtained satisfy the law and determine the approximate values of θ_0 and k from the linear graph.

Solution:



Tutorial : Multiple Choice Questions

1. The equation of the straight line in the diagram on the right is:



(a) $u = 4t + 1$ (b) $u = 4t - 1$

(c) $2u = 4t + 1$ (d) $2u = 4t - 1$

2. Given the equation $y = ax^2 + bx + 2$ where a and b are constants, in order to obtain a straight line to determine a and b, we plot

(a) y against x^2 (b) $\dfrac{y}{x}$ against x

(c) $(y - 2)$ against x^2 (d) $\dfrac{y-2}{x}$ against x

3. Given that $y = a\,e^x + b$, in order to obtain a straight-line graph to determine a and b, we plot

(a) y against e^x (b) $\ln y$ against $\ln(ae^x)$

(c) $\ln(y - b)$ against x (d) $\ln y$ against x

4. Given the equation $y = e^{h-kt} + 3$ where h and k are constants, in order to obtain a straight line to determine h and k, we plot

(a) $\ln y$ against t (b) y against e^t
(c) $\ln(y-3)$ against t (d) $(y-3)$ against e^t

Tutorial 3

1. Express each of the following non-linear equations in linear form, stating the terms to be plotted on the vertical and horizontal axis. (a and b are constants).

(a) $y = \dfrac{a}{x^2} + b$ (b) $y = ax^3 + bx^2$

(c) $\dfrac{1}{p} = \dfrac{a}{q} + \dfrac{1}{b}$ (d) $v = at^2 + bt + 2$

(e) $y = ax^{-b}$ (f) $y = a^{x+b}$

(g) $y = 10^{a(x+b)}$ (h) $y = ab^{-x}$

2. In a wastewater treatment plant, the approximate number of a type of bacteria, B is checked regularly and recorded in the table below.

Bacteria, B ($\times 10^3$)	5	28.5	41.0	113.0	253.6	450.0
Time, t (hours)	1.0	2.5	3.0	5.0	7.5	10.0

It is thought that the growth is related according to the law $B = mt^2 + c$, where m and c are constants. By plotting a suitable graph, show this to be true and evaluate the value of m and c.

3. The table below shows the variation in the coefficient of viscosity of a particular fluid, z, at various temperatures, t.

t (°C)	0	6	12	18	24
z	40.0	23.3	13.6	7.9	4.6

i) By plotting a suitable straight line graph, show that z and t are related by the equation $z = ae^{-bt}$ and hence find the values of the constants a and b.

ii) Estimate the coefficient of viscosity when the temperature is 20 °C.

4. In the following table, N and R are connected by the law $R = aN + bN^2$ where a and b are constants.

N	50	75	100	125	150
R	18	36.5	61	92	129

i) Draw a suitable straight line graph to show that the values obey the above law, and use the graph to determine the values of a and b.

ii) Hence, find the value of R when $N = 90$.

5. The following experimentally determined values of x and y are believed to follow a law of the form $y = \dfrac{a}{1 - bx^2}$. Show graphically that this is so and find the values for a and b.

x	6	8	10	11	12
y	5.50	6.76	9.10	11.60	16.67

*6. In a RC circuit, where R stands for resistance in ohms and C stands for capacitance in farads, the voltage across the resistor , V_R, is believed to be governed by the law $V_R = Ve^{-t/RC}$ when the capacitor is charging. R, C and V are all constants. An experiment was conducted and the measurements tabulated as follows:

t(s)	10	20	30	40	50	60
V_R (V)	7.28	4.41	2.68	1.62	0.99	0.60

i) Show the law is obeyed by plotting a suitable straight-line graph.

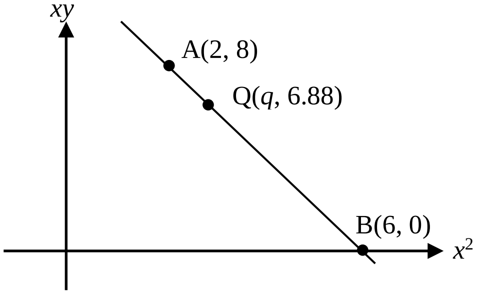
ii) Decay time constant is defined to be RC. From the graph drawn, determine the values of V and the decay time constant.

iii) Express in terms of the decay time constant, RC, the time that it will take for V_R to fall to $\frac{1}{e}$ of its initial value.

iv) Hence, for another RC circuit where the resistor is 300kΩ and the capacitor is 200μF,would it take a shorter or longer time for the voltage across the resistor to fall to $\frac{1}{e}$ of its initial value? Please explain why it would take a shorter or longer time.

v) Estimate how long it will take for V_R to fall from $\frac{1}{e^2}$ of the initial value to $\frac{1}{e^3}$ of the initial value.

*7.



Variables x and y and related in such a way that, when xy is plotted against x^2, a straight line is produced which passes through the points A(2, 8), B(6,0) and Q(q, 6.88), as shown in the diagram. Find

(i) y in term of x,

(ii) the value of q,

(iii) the value of x and y at the point of Q.

Answers

Multiple Choice Questions

1. (c)
2. (d)
3. (a)
4. (c)

Tutorial 3

1. (a) Vert. axis $= y$; Hor. axis $= \frac{1}{x^2}$; Gradient $= a$; Vert. intercept $= b$

   (b) Vert. axis $= \frac{y}{x^2}$; Hor. axis $= x$; Gradient $= a$; Vert. intercept $= b$

   (c) Vert. axis $= \frac{1}{p}$; Hor. axis $= \frac{1}{q}$; Gradient $= a$; Vert. intercept $= \frac{1}{b}$

   (d) Vert. axis $= \frac{v-2}{t}$; Hor. axis $= t$; Gradient $= a$; Vert. intercept $= b$

   (e) Vert. axis $= \log y$; Hor. axis $= \log x$; Gradient $= -$ b; Vert. intercept $= \log a$

   (f) Vert. axis $= \log y$; Hor. axis $= x$; Gradient $= \log a$; Vert. intercept $= b \log a$

   (g) Vert. axis $= \log y$; Hor. axis $= x$; Gradient $= a$; Vert. intercept $= ab$

   (h) Vert. axis $= \log y$; Hor. axis $= x$; Gradient $= -\log b$; Vert. intercept $= \log a$

2. $m = 4.50$; $c = 0$

3. (i) $a = 40$; $b = 0.09$ (ii) 6.6

4. (i) $a = 0.12$; $b = 0.0049$ (ii) 50.8

5. $a = 4.3$; $b = 0.00504$

6. (ii) $V = 12$; $RC = 20$ (iii) $t = RC$

   (iv) Longer, because RC for the other circuit is 60.

   (v) 20

7. (i) $y = -2x + \frac{12}{x}$ (ii) $q = 2.56$

   (iii) $x = \pm 1.6,\ y = \pm 4.3$