Singapore Polytechnic, School of Mathematics and Science
Academic Year 2020/2021 Semester 2 **Further Mathematics**
Mid-Semester Test **Duration: 1.5 hour**

Instructions

1. All SP examination rules are to be complied with.
2. This paper consists of 4 pages.
3. Answer ALL the questions. Unless otherwise stated, leave your answers in 2 decimal places.
4. Except for graphs and diagrams, no solutions are to be written in pencil.

Additional Formulae

Log of a power: $\log_b a^x = x \log_b a$

Completing the square: $x^2 + kx = \left(x + \frac{k}{2}\right)^2 - \left(\frac{k}{2}\right)^2$

1. (a) A sum \$$P$ is deposited in a bank at a compound interest rate of $q\%$ per annum and the interest is paid yearly. The investor leaves the principal and interest in the account untouched.

(i) If the amount in the bank double after 15 years, find the interest rate offered by the bank. (5 marks)

(ii) \$10,000 is deposited in the bank with 3% interest, what is the minimum number of years for the amount in the bank to reach more than \$30,000. (5 marks)

(b) The equation of a circle, C, is $x^2 + 10x + y^2 - 10y + 25 = 0$

(i) By completing the square or otherwise, find the coordinates of the centre of C and find the radius of C. (4 marks)

The equation of another circle, D, is $(x-5)^2 + (y-5)^2 = 25$.

(ii) Sketch the circles C and D on the same set of axes. Label the centres of the two circles and all intercepts clearly. (6 marks)

(iii) From part (ii) or otherwise, find the distance between the centres of the two circles. (2 marks)

2. The parametric equations of a curve are

$$x = 2t + \ln t, \qquad y = t + 4, \quad \text{where } t \text{ takes all positive values.}$$

(a) Show that $\dfrac{dy}{dx} = \dfrac{t}{2t+1}$. (5 marks)

(b) Find the equation of the tangent to the curve at the point for which $t = 1$. (6 marks)

(c) Given $\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$, obtain and simplify an expression for $\dfrac{d^2y}{dx^2}$. (6 marks)

(d) Find the value of t at the point of intersection of the curve with the line $x - 2y = 8$. Leave your answer in term of e. (4 marks)

3. (a) Two functions are defined as follows:

$$f(x) = \frac{1}{2 - \sqrt{x-3}}$$

$$g(x) = 4 + \sin x$$

(i) Determine the domain of $f(x)$. (3 marks)

(ii) Determine the domain and range of $g(x)$. (6 marks)

(iii) Find the value of $(g \circ f)(3)$. (4 marks)

(b) The function f is defined as $f(x) = ax + b$, $a \neq -1$ for the domain $[0, 6]$.

(i) Given that $y = f(x)$ passes through the point (1, 2) and the graph of $y = f(x)$ and $y = f^{-1}(x)$ intersect at the point whose x-coordinate is 4, find the value of a and of b. (6 marks)

(ii) With the values of a and b found in (i), write down the corresponding range of $f(x)$. (2 marks)

4. (a) Find the slope of the curve $y = \sin^3(2x)$ at $x = 1$. (4 marks)

(b) The curve $y = \dfrac{6-2x}{x+1}$ in figure 1 (not drawn to scale) cuts the x-axis at P and the y-axis at Q. The **normal** to the curve at P crosses the y-axis at R. Find the length QR. (11 marks)

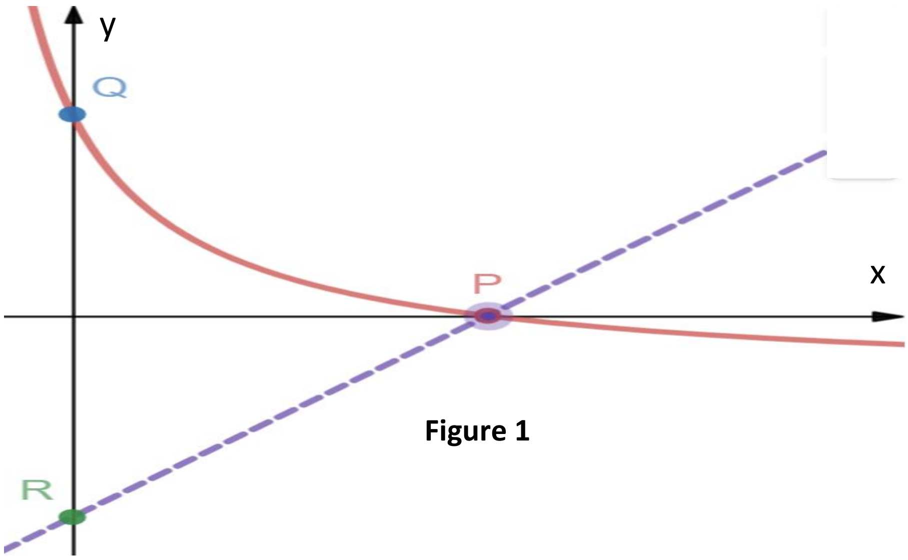


Figure 1

5. (a) The equation of a curve is $y = 20x(x+1)^4$.

(i) Find the coordinates of the stationary points of the curve and determine the nature of the stationary points. (6 marks)

(ii) Sketch the graph of $y = 20x(x+1)^4$ for $-1.5 \le x \le 0$. Label the coordinates of the stationary points clearly. (6 marks)

(b) The derivative of $f(x) = 3x^2 + x + 1$ from first principles can be derived from

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

(i) Determine $f(x+\Delta x) - f(x)$ in terms of Δx. (6 marks)

(ii) Hence, find the derivative of $f(x) = 3x^2 + x + 1$ from first principles. (3 marks)

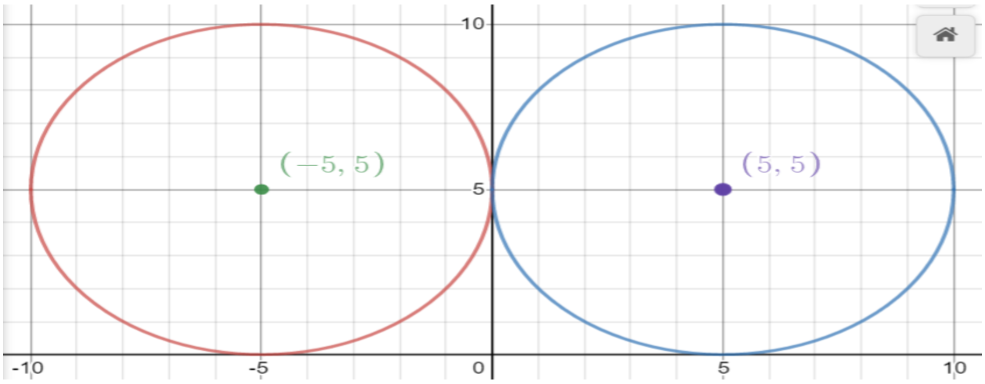
~ End of paper ~

Answers (MST 20/21 S2)

1. (a) (i) 4.73% (ii) 38 years

(b) (i) Centre $(-5,5)$, radius = 5

(ii)



1 (b) (iii) Distance = 10 units

2. (b) $3y = x+13$ (c) $\dfrac{t}{(2t+1)^3}$ (d) $t = e^{16}$

3. (a) (i)
$$D_f = \{x : x \geq 3 \text{ and } x \neq 7\} = [3,7)\cup(7,+\infty)$$

$$D_g = (-\infty, \infty)$$

(ii)
$$R_g = \{g(x) : 3 \leq g(x) \leq 5\} = [3,5]$$

(iii)
$$(g\circ f)(x) = g\left(\frac{1}{2-\sqrt{x-3}}\right) = 4+\sin\left(\frac{1}{2-\sqrt{x-3}}\right)$$

$$(g\circ f)(3) = 4+\sin\left(\frac{1}{2}\right) = 4.48$$

3. (b)(i) $a = \dfrac{2}{3}$ and $b = \dfrac{4}{3}$ (ii) $R_f = \left[\dfrac{4}{3}, 5\dfrac{1}{3}\right]$

4. (a)
$$\frac{dy}{dx} = 3\sin^2(2x)\cos(2x)2$$

$$x = 1, \frac{dy}{dx} = 6\sin^2(2)\cos(2) = -2.06$$

(b) $Q(0,6)$, $R(0,-6)$, $QR = 12$ units

5. (a) (i) $\frac{dy}{dx}=(x+1)^3(100x+20)$, $x=-1, y=0, \max(-1,0)$,

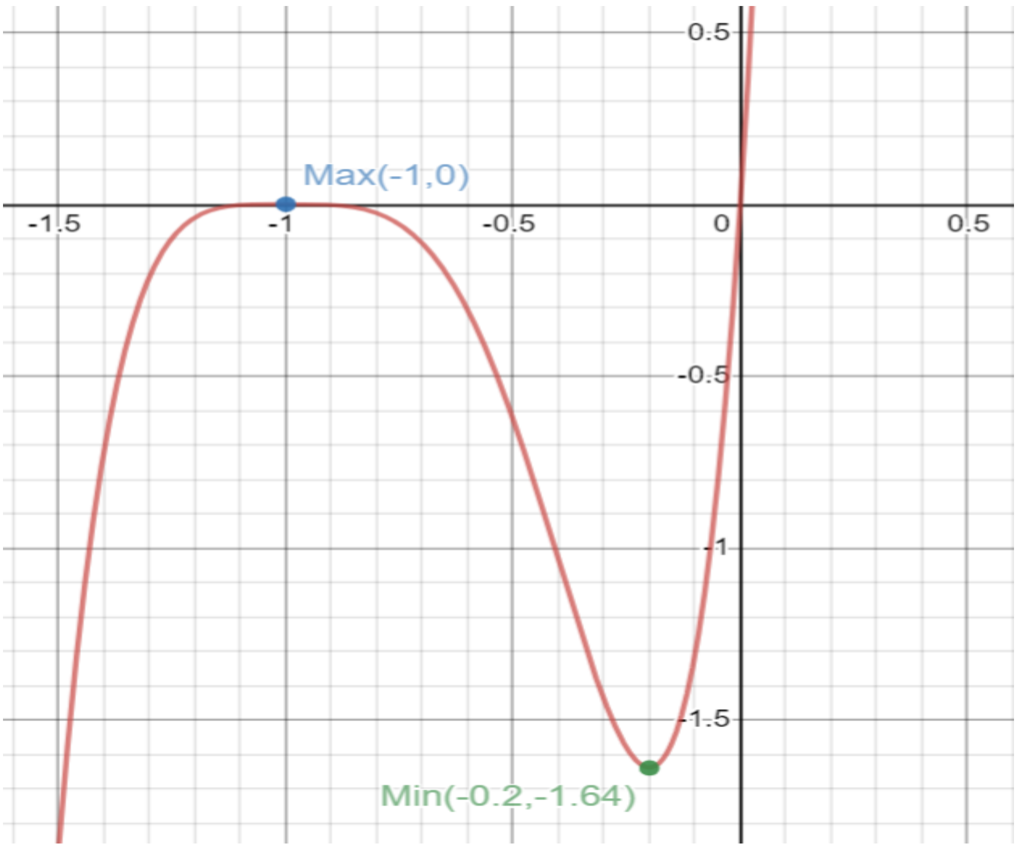
$x=-0.2, y=-1.64, \min(-0.2,-1.64)$

Note, cant use 2nd derivative test for x=-1

$x=-1, \frac{d^2y}{dx^2}=0$ (not conclusive hence use 1st derivative test) :

	x^-	x	x^+
Gradient ($\frac{dy}{dx}$) (x = -1)	+	0	-
Gradient ($\frac{dy}{dx}$) (x = -0.2)	-	0	+

5a(ii)



5 (b) (i) $f(x)=6x\Delta x+3\Delta x^2+\Delta x$ (ii) $\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=6x+1$