# Chapter 5 : Integration by Parts & Simpson's Rule

**Objectives :**
*1. Derive the formula for integration by parts.*
*2. State the guideline (LIATE) and apply the integration by parts formula.*
*3. Use Simpson's rule to obtain approximate values of definite integrals.*

## 5.1 Integration by Parts

If $u$ and $v$ are both functions of $x$,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\int \frac{d}{dx}(uv)\,dx = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

i.e. $uv = \int u\,dv + \int v\,du$

Thus $$\boxed{\int u\,dv = u\cdot v - \int v\,du} \quad \cdots\cdots\cdots\cdots(1)$$

The method of "***integration by parts***" is used to integrate:

- products of two different types of functions,
  such as $x^2 \sin x$, $x e^x$, $x \ln x$, $e^x \sin x$, $x \tan^{-1} x, \ldots$ etc.
- single functions that cannot be directly integrated with standard formulae,
  such as $\ln x$, $\sin^{-1} x$, $\tan^{-1} x$, $\cos^{-1} x$, $\ldots$ etc.

In this method, we make use of formula (1) to change the original integral $\int u\,dv$ to a different integral $\int v\,du$, which can be integrated easily.

One important step in carrying out the above formula is to select appropriate $u$ and $dv$.
To select $u$, we use the acronym **L I A T E** to guide us.

This acronym spells out the priority of $u$ according to the following order:

**L**ogarithm expression

**I**nverse Trigonometric expression

**A**lgebraic expression

**T**rigonometric expression

**E**xponential expression

### 5.1.1 Integrals of the Type $\int x^n e^{ax}\,dx$, $\int x^n \sin ax\,dx$, $\int x^n \cos ax\,dx$ where $n$ is a positive integer and $a$, a constant

Let $u = x^n$, $dv = e^{ax}dx$ or $dv = \sin ax\,dx$ or $dv = \cos ax\,dx$

***Example 1:*** Find $\int x e^{2x}\,dx$ *Ans:* $\dfrac{x e^{2x}}{2} - \dfrac{e^{2x}}{4} + C$

*Solution 1* $\int x e^{2x}\,dx$

$=$

$=$

$u =$ $dv =$

$du =$ $v =$

$-\int$

*Solution 2* $\int x e^{2x}\,dx =$

$\underline{u}$ $\underline{dv}$

***Example 2:*** Find (a) $\int x \sin 2x\,dx$ *Ans*: $-\dfrac{1}{2}x\cos 2x + \dfrac{1}{4}\sin 2x + C$

(b) $\int x^2 e^x dx$ *Ans*: $x^2e^x - 2xe^x + 2e^x + C$

### 5.1.2 Integrals of the Type $\int x^n \ln(ax+b)\,dx$ or $\int x^n \tan^{-1}(ax+b)\,dx$ where $n$ is a non-negative integer and $a$, $b$ are constants

Let $u = \ln(ax+b)$ or $u = \tan^{-1}(ax+b)$, $dv = x^n dx$

***Example 3*:** Find $\int \ln(2x+1)\,dx$ *Ans:* $x\left[\ln(2x+1)\right] - x + \frac{1}{2}\ln\left|2x+1\right| + C$

| *Solution* | $\int \ln(2x+1)\,dx$ | $\underline{u}$ | $\underline{dv}$ |
|---|---|---|---|
| | $=$ | | |

**Integration by parts formula for definite integrals**

$$\boxed{\int_a^b u\ dv = \left[u \cdot v\right]_a^b - \int_a^b v\ du} \quad \cdots\cdots\cdots\cdots(2)$$

***Example 4:*** Find $\int_1^e x \ln x\,dx$ *Ans*: 2.097

***Example 5:*** Find $\int \tan^{-1} x\ dx$ *Ans*: $x \tan^{-1} x - \frac{1}{2}\ln\left|1+x^2\right| + C$

### 5.1.3 Integrals of the Type $\int e^{ax} \sin bx \, dx$ or $\int e^{ax} \cos bx \, dx$ where $a$ and $b$ are constants

Let $u = e^{ax}$, $dv = \sin bx \, dx$ or $dv = \cos bx \, dx$

alternatively

Let $u = \sin bx$ *or* $u = \cos bx$, $dv = e^{ax} dx$

***Example 6:*** Find $\int e^{2x} \sin x \, dx$ $\quad$ *Ans*: $-\frac{1}{5}e^{2x} \cos x + \frac{2}{5}e^{2x} \sin x + C$

## 5.2 Numerical Integration

We have learnt how to deal with various types of integrals. But there are still some integrals, that seem simple enough, yet it cannot be integrated by any of the standard method we have studied.

For example,

1. $\int_0^1 e^{-x^2}\, dx$ , 2. $\int_1^2 \ln\sqrt{1+x^3}\, dx$ .

When a definite integral $\int_a^b f(x)\,dx$ does not fit into the 'normal patterns' of the standard integral, a numerical approximation of the integral is often sought after. One of such approximate methods is called **Simpson's rule**.

Numerical methods of integration are used to approximate definite integrals $\int_a^b f(x)\,dx$ when:

**(i) the integrand $f(x)$ cannot be integrated by any standard integrals or integration methods that we have encountered so far.**

E.g. Evaluate $\int_2^4 \sqrt{1+x^3}\, dx$ $\left[\text{Here } f(x)=\sqrt{1+x^3}\right]$

**(ii) the integrand $f(x)$ is not known explicitly but only as a set of points.**

E.g. Given the corresponding data in $x$ and $y$ in the table below:

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $y=f(x)$ | 0.25 | 0.78 | 1.28 | 1.43 | 0.96 | 0.32 | 0.18 |

Since $y=f(x)$ is not defined, $\int_1^7 y\,dx$ cannot be integrated. Therefore the value of the integral $\int_1^7 y\,dx$ can only be approximated by numerical method.

## 5.3 Simpson's Rule

The two most popular numerical methods of integration are Trapezoid rule and Simpson's rule. However, we will learn only Simpson's rule in our module.

The definite integral $\int_a^b f(x)\,dx$ represents the area between the curve $y=f(x)$ and the $x$-axis from $x=a$ to $x=b$. So even if $\int f(x)\,dx$ cannot be found in terms of elementary functions, an approximate value for $\int_a^b f(x)\,dx$ can be obtained by evaluating the appropriate area.

In Simpson's rule, this area is considered to be made up of areas of $n$ vertical strips of equal width $h$, where $n$ is an even integer.

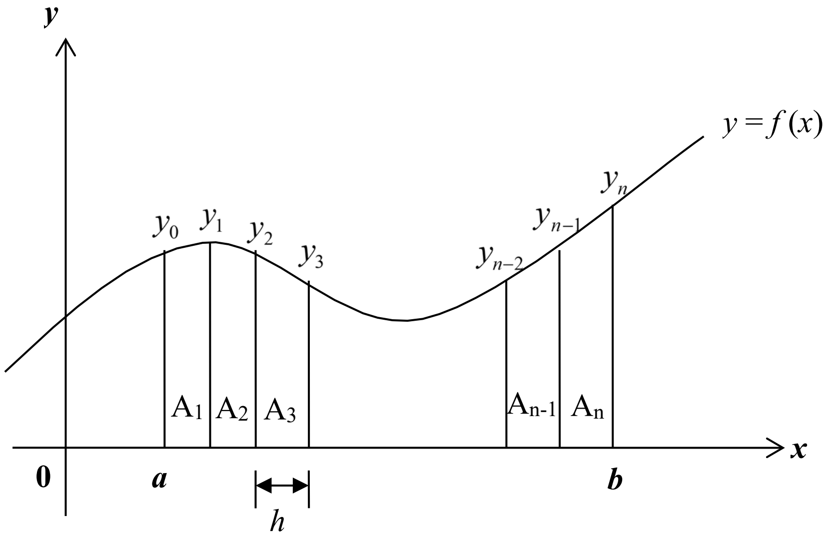
To find the area under the curve $y = f(x)$ between $x = a$ and $x = b$:

(a) Divide the area into an *even* number ($n$) of strips each of width $h = \dfrac{b-a}{n}$.

(b) Number and evaluate each ordinate: $y_0,\ y_1,\ y_2,\ \cdots,\ y_n$.

(c) The area is then given by:

$$\boxed{\begin{array}{l}\displaystyle\int_a^b f(x)\,dx \approx \frac{1}{3}\,h\left[y_0 + y_n + 4\left(y_1 + y_3 + y_5 + \ldots + y_{n-1}\right) + 2\left(y_2 + y_4 + \ldots + y_{n-2}\right)\right] \\ \text{where } h = \dfrac{b-a}{n} \text{ and } n \text{ is an even integer.}\end{array}}$$




***Example 7:*** By using Simpson's rule with 6 strips, determine the approximate value of the definite integral $\displaystyle\int_0^{\pi} \frac{1}{1+\sin x}\,dx$ correct to 2 decimal places. *Ans:* 2.00

*Solution* Let $f(x) = \dfrac{1}{1+\sin x}$ and $h = \dfrac{b-a}{n} =$ [*Note* : $x$ must be in radians]

| $x$ | 0 | $1\cdot\left(\frac{\quad}{\quad}\right)$ | $2\cdot\left(\frac{\quad}{\quad}\right)$ | $3\cdot\left(\frac{\pi}{6}\right)$ | $4\cdot\left(\frac{\pi}{6}\right)$ | $5\cdot\left(\frac{\pi}{6}\right)$ | $\pi$ |
|---|---|---|---|---|---|---|---|
| $\dfrac{1}{1+\sin x}$ | 1 | 0.6667 | | | 0.5359 | | |

By Simpson's rule,

$$\int_0^{\pi} \frac{1}{1+\sin x}\,dx \approx \frac{h}{3}\left[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$= \frac{1}{3}\cdot\left(\frac{\pi}{6}\right)\left[2 + 4(\qquad + \qquad + \qquad) + 2(\qquad + \qquad)\right]$$

$$= \frac{\pi}{18}(\qquad\qquad) =$$

***Example8:*** Evaluate $\int_1^2 \frac{\sin x}{x}\,dx$ using Simpson's rule with 4 strips correct to 3 decimal places. *Ans*: 0.659

*Solution*

| | | | | | |
|---|---|---|---|---|---|
| | | | | | |
| | | | | | |

***Example 9*:** The voltage of a supply in mV at regular intervals of 0.01s over a half cycle is found to be

0, 19.5, 35, 45, 40.5, 25, 20.5, 29, 27, 12.5 and 0

By Simpson's rule, find the r.m.s. value of the voltage over the half cycle, correct to 2 decimal places. *Ans*: 28.37 (mV)

*Solution* The r.m.s. value of the voltage is $V_{rms} = \sqrt{\frac{1}{b-a}\int_a^b V^2 dt}$

| | | | | | | |
|---|---|---|---|---|---|---|
| *t* | | | | | | |
| *V* | 0 | 19.5 | 35 | 45 | 40.5 | 25 |
| | | | | | | |

| | | | | | |
|---|---|---|---|---|---|
| *t* | | | | | |
| *V* | 20.5 | 29 | 27 | 12.5 | 0 |
| | | | | | |

# Tutorial 5

## <u>Section A</u>: Integration by Parts

Find the following integrals:

1. $\int x\cos x\,dx$
2. $\int (x^2+x)e^{2x}dx$
3. $\int x^2 \sin 3x\,dx$
4. $\int_0^1 x\,e^{-5x}\,dx$
5. $\int_1^e x^2 \ln x\,dx$
6. $\int e^{5x}\cos 2x\,dx$
7. $\int \theta \sin^2\theta\,d\theta$
8. $\int \ln(1-4x)\,dx$

## <u>Section B</u>: Simpson's Rule (Give all your final answers correct to 2 decimal places.)

1. Estimate the following integrals by Simpson's rule, using the number of intervals indicated:

   (a) $\int_0^1 \sqrt{1+x^3}\,dx$ (n = 8)  (b) $\int_0^{\pi/2} \sqrt{\sin\theta}\,d\theta$ (n = 6)

   (c) $\int_{\pi/2}^{3\pi/2} \frac{\sin x}{x}\,dx$ (n = 6)  (d) $\int_0^2 e^{x^2}dx$ (n=4)

2. Given that $V = 2\pi\int_a^b rh\,dr$ where the values of $r$ and $h$ are given in the following table:

| $r$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $h$ | 0.599 | 1.072 | 1.415 | 1.588 | 1.579 | 1.428 | 1.003 |

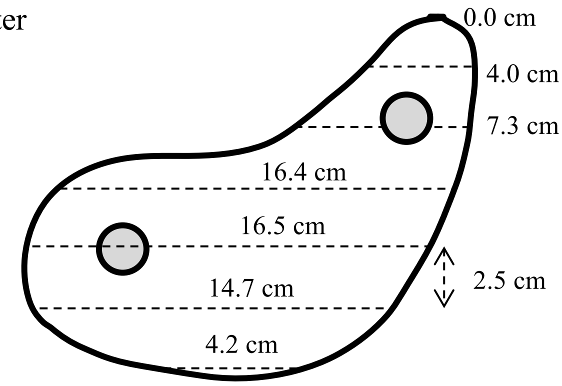
   Use Simpson's rule to find the approximate value of $V$.

3. When a battery is applied to the sending end of a long telegraph line, the growth of received current at 5 ms intervals is given by

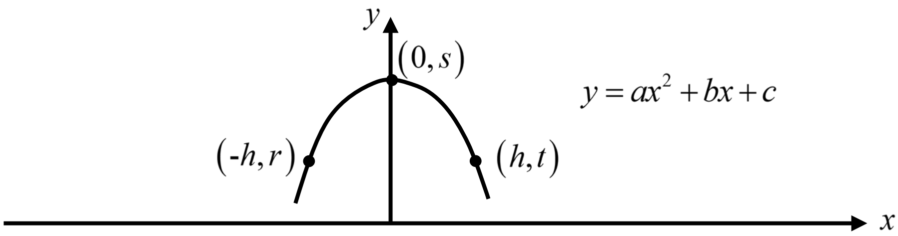
| Time(ms) | 0 | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 |
|---|---|---|---|---|---|---|---|---|---|
| Current (mA) | 0 | 1.5 | 7 | 13 | 16 | 18 | 19 | 19.5 | 20 |

   Use Simpson's rule to evaluate the r.m.s. value of the current in the complete 40 ms interval.

4. The widths of a bell crank are measured at 2.5 cm intervals as shown in the diagram. Find the approximate area, accurate to two decimal places, of the bell crank if the two connector holes are each 1.5 cm in diameter

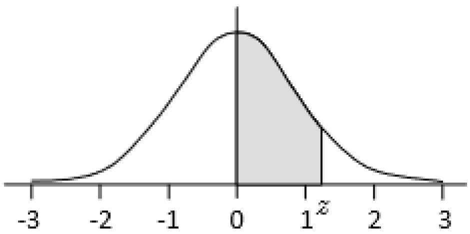


* 5.




Show that the area under a quadratic curve $y = ax^2 + bx + c$ in the interval $[-h, h]$ is $\frac{h}{3}(r + 4s + t)$, where $r$, $s$ and $t$ are the $y$-coordinates of the three points shown in the graph above. Hence or otherwise, prove the Simpson's rule formula.

*6. A very large number of random variables observed in nature follow a frequency distribution that is approximately bell-shaped. Statistician called this a normal probability distribution.




The function that describes the standardized normal curve is $f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$. The probability for $z$ taking on values from $a$ to $b$, denoted by $P(a < z < b)$ is given by the area under the curve between $z = a$ and $z = b$. Hence

$$P(a < z < b) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz$$

Unfortunately, $f(z)$ does not have an antiderivative in terms of elementary functions. Hence we have to use numerical integration method like the Simpson's Rule to evaluate this integral. Use Simpson's Rule to evaluate $P(0 < z < 1.2) = \int_0^{1.2} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \, dz$ using $n = 6$.

**Miscellaneous Exercises**

*1. Integrate the following:

(a) $\int \frac{\ln(x)}{(2x+1)^3} \, dx$ (b) $\int \frac{x \sin^{-1}(2x)}{\sqrt{1-4x^2}} \, dx$

*2. Evaluate $\int_1^3 \sqrt{x} \tan^{-1} \sqrt{x} \, dx$

*3. Show that $\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x dx$. Hence integrate $\int x^3 e^x dx$.

*4. By writing $\sin^n x = \sin^{n-1} x \sin x$, derive the reduction formula for

$$\int \sin^n x \, dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x \, dx$$

Hence evaluate $\int \sin^5 x \, dx$.

## Multiple Choice Questions

1. To find $\int x \sec^2(5x)\, dx$ using 'integration by parts', we choose

(a) $u = x$ and $dv = \sec^2(5x)\, dx$ (b) $u = \sec^2(5x)$ and $dv = x\, dx$

(c) $u = x dx$ and $dv = \sec^2(5x)$ (d) $u = \sec^2(5x)\, dx$ and $dv = x$

2. The number of panels or strips to be considered in Simpson's rule must be ________.

(a) Odd (b) Even

3. The exact solution of a definite integral can be obtained using the Simpson's rule.

(a) True (b) False

4. A definite integral $\int_0^3 \sqrt{1-x^2}\, dx$ is evaluated using the Simpson's rule with 8 strips. Which of the following could be used to increase the accuracy of the final answer?

(a) Evaluate the definite integral by integrating the function $\sqrt{1-x^2}$ and substituting the limits of integration.

(b) Use the trapezoid method instead of Simpson's rule using the same number of strips.

(c) Reduce the number of strips from 8 to 4.

(d) Increase the number of strips from 8 to 16.

# Answers

## Section A

1. $x\sin x+\cos x+C$
2. $\frac{1}{2}x^2e^{2x}+C$
3. $-\frac{x^2}{3}\cos 3x+\frac{2}{9}x\sin 3x+\frac{2}{27}\cos 3x+C$
4. 0.0384
5. 4.575
6. $\frac{1}{29}e^{5x}(2\sin 2x+5\cos 2x)+C$
7. $\frac{\theta^2}{4}-\frac{\theta}{4}\sin 2\theta-\frac{1}{8}\cos 2\theta+C$
8. $x\,\ell n(1-4x)-x-\frac{1}{4}\ell n\left(1-4x\right)+C$

## Section B

1. (a) 1.11; (b) 1.19; (c) 0.24 (d) 17.35

2. 159.62 3. 14.77 mA 4. 156.63 $\text{cm}^2$ 6. 0.3849 or 38.49%

## Miscellaneous Exercises

1. (a) $-\frac{\ln|x|}{4\left(2x+1\right)^2}+\frac{1}{4}\left[\ln|x|-\ln|2x+1|+\frac{1}{2x+1}\right]+C$

   (b) $\frac{1}{4}\left[-\sin^{-1}\left(2x\right)\sqrt{1-4x^2}+2x\right]+C$

2. $\frac{1}{3}\left(2\sqrt{3}\ \pi-\frac{\pi}{2}-2+\ln 2\right)$ or 2.668

3. $x^3e^x-3x^2e^x+6xe^x-6e^x+C$

4. $-\frac{1}{5}\sin^4 x\cos x-\frac{4}{15}\sin^2 x\cos x-\frac{8}{15}\cos x+C$

## MCQ

1. a 2. b 3. b 4. d