Kinematics 2 – Motion in 2D/3D

PRE-CLASS (2 TO 18)

IN-CLASS (20 ONWARDS)

Learning outcomes (pre-class)

At the end of the pre-class slides, students are to be able to

- define position, displacement, average velocity, instantaneous velocity, average acceleration, and instantaneous acceleration in 2D or 3D.
- apply the above definitions to solve problems involving motions in 2D or 3D.

Motion in two and three dimensions

- Examples include a merry-go-round undergoes a circular motion, a base ball when hit may move in path that is not a straight line.

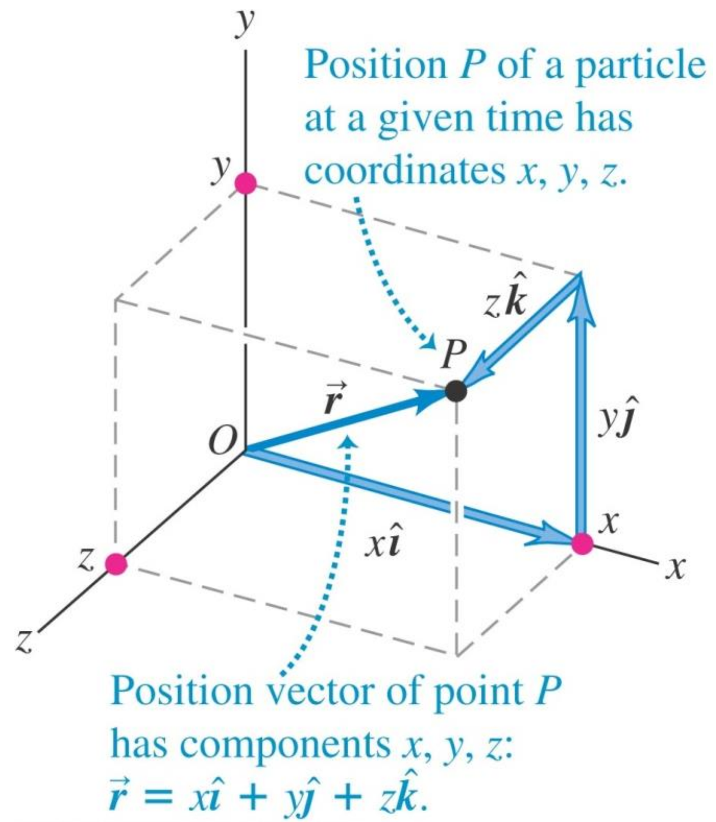
- The equations that we set up for studying motion in one dimension will be extended to two and three dimensions.

Position vector

- The position vector of a particle is a vector drawn from the origin of a co-ordinate system to where the particle is located at *P*.

- The vector is given by

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$



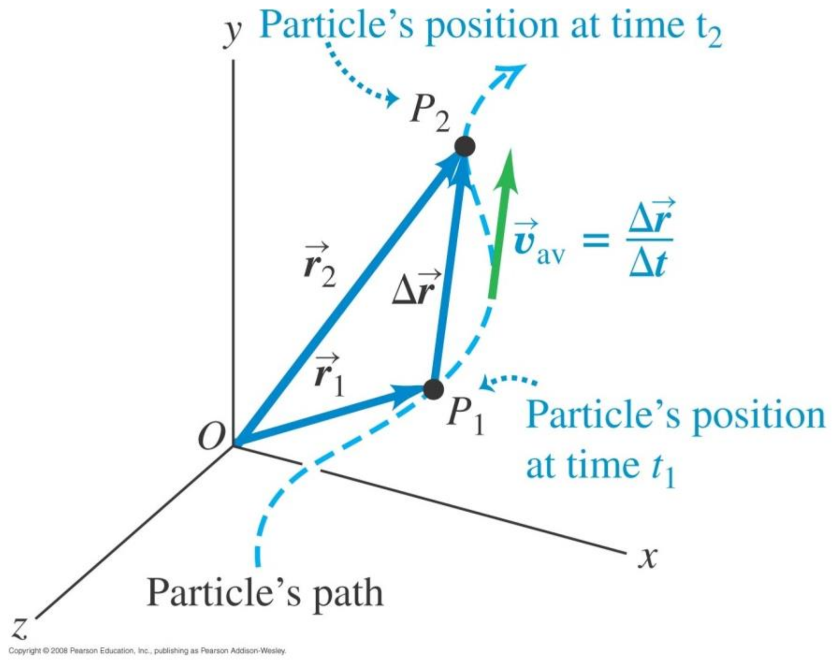
Displacement vector

- The displacement of a particle that moves from P_1 to P_2 is defined as

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$$

- For a curved path, the magnitude of the displacement vector is less than the distance travelled.

- Displacement vector is independent of the path, i.e. depends only on the start and end points.

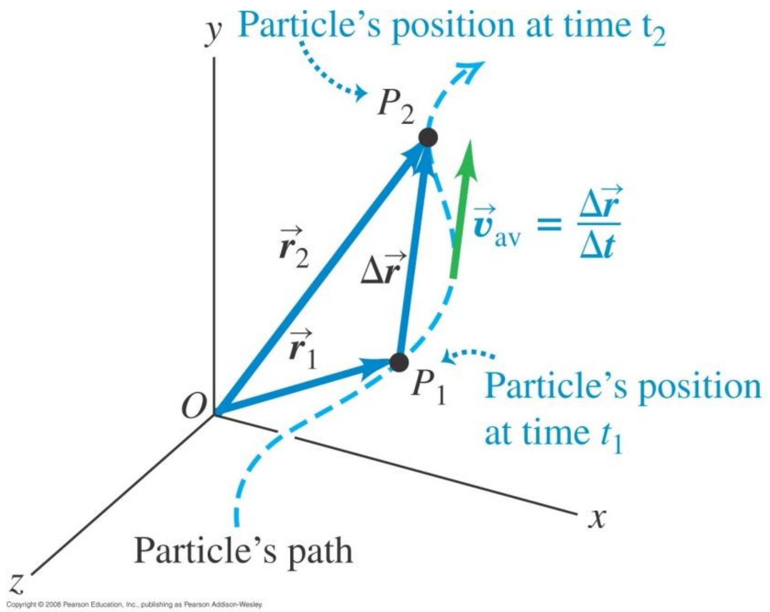


Average velocity

- The average velocity is defined as

$$\vec{v}_{av} = \frac{\Delta\vec{r}}{\Delta t}$$

- It has the same direction as the displacement vector.

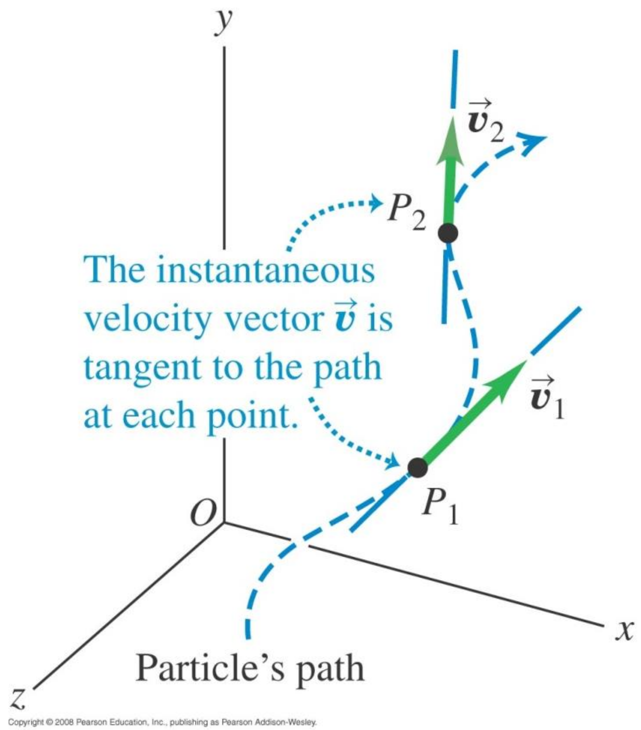


Instantaneous velocity

- The instantaneous velocity is defined as

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

- Its direction is along the tangent to the path at each point of the path.



Instantaneous velocity

- The components of the instantaneous velocity are

$$v_x = \frac{dx}{dt}, \qquad v_y = \frac{dy}{dt}, \qquad v_z = \frac{dz}{dt}$$

- In terms of unit vectors:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}$$

- The **magnitude** of the instantaneous velocity is the **speed** of the object.

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

Motion in *x*-*y* plane

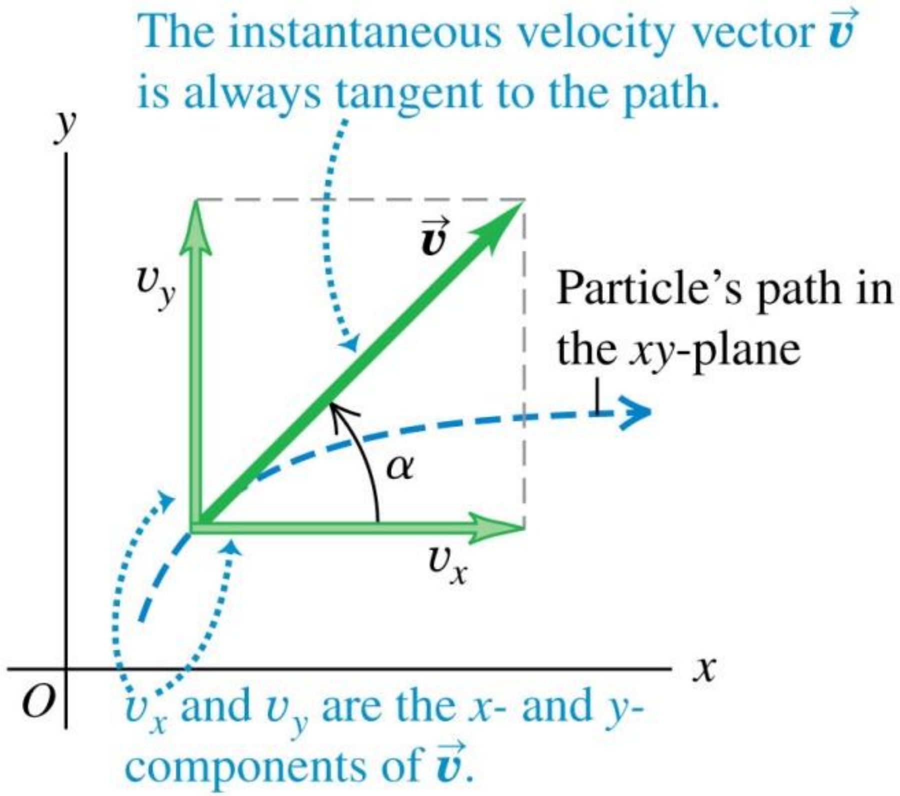
- When a particle moves in the *x*-*y* plane (see right figure) then speed is

$$v = \sqrt{v_x^2 + v_y^2}$$

- The direction of the velocity vector is angle α from v_x where α is

$$\alpha = \tan^{-1}\frac{v_y}{v_x}$$

- The velocity is always **tangent** to the particle's path.

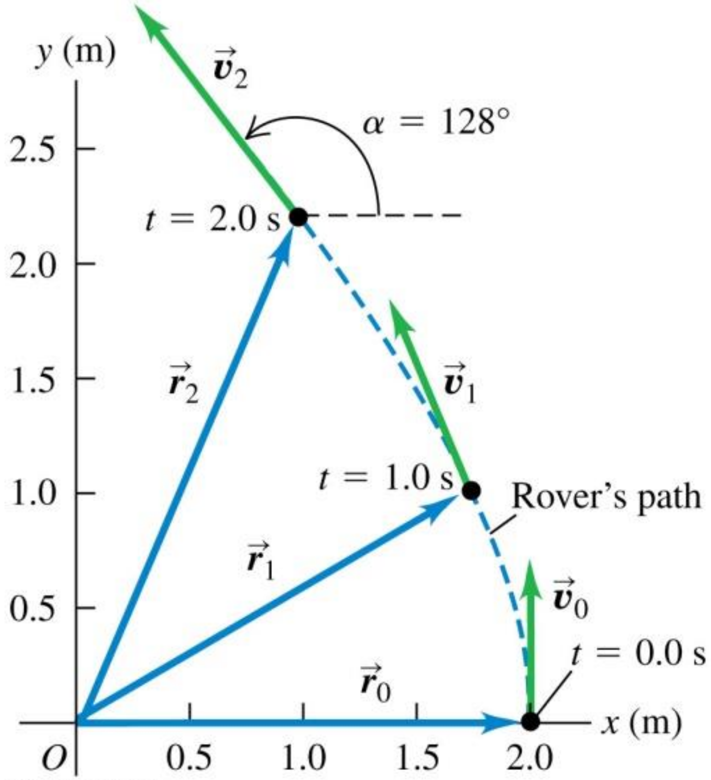


Example 1

A robotic vehicle, or rover, is exploring the surface of Mars. The landing craft is the origin of the coordinates, and the surrounding Martian surface lies in the *x*-*y* plane. The rover, which we represent as a point, has *x*- and *y*-coordinates that vary with time:

$$x = 2.0 - 0.25t^2 \qquad y = 1.0t + 0.025t^3$$

a) Find the rover's coordinates and its distance from the lander at $t = 2.0$ s.
b) Find the rover's displacement and average velocity vectors during the time interval from $t = 0.0$ s to $t = 2.0$ s.
c) Derive a general expression for the rover's instantaneous velocity vector.
d) Express the instantaneous velocity at $t = 2.0$ s in component form and also in terms of magnitude and direction.



Example 1 (solution)

A robotic vehicle, or rover, is exploring the surface of Mars. The landing craft is the origin of the coordinates, and the surrounding Martian surface lies in the *x*-*y* plane. The rover, which we represent as a point, has *x*- and *y*-coordinates that vary with time:

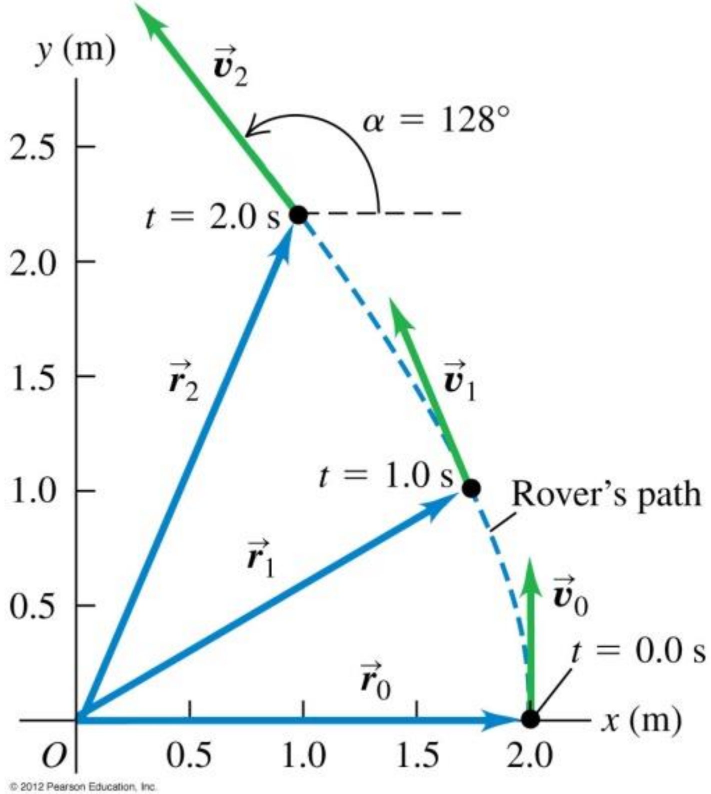
$$x = 2.0 - 0.25t^2 \qquad y = 1.0t + 0.025t^3$$

a) Find the rover's coordinates and its distance from the lander at $t = 2.0$ s.

$$x(2) = 2.0 - 0.25(2^2) = 1.0\ \text{m}$$
$$y(2) = 1.0(2) + 0.025(2^3) = 2.2\ \text{m}$$

$$\text{Coordinate of rover at } t = 2.0\ \text{s} = (1.0\ \text{m}, 2.2\ \text{m})$$

$$\text{Distance} = \sqrt{x^2 + y^2} = \sqrt{1^2 + 2.2^2} = 2.4\ \text{m}$$



Example 1 (solution)

A robotic vehicle, or rover, is exploring the surface of Mars. The landing craft is the origin of the coordinates, and the surrounding Martian surface lies in the *x*-*y* plane. The rover, which we represent as a point, has *x*- and *y*-coordinates that vary with time:

$$x = 2.0 - 0.25t^2 \qquad y = 1.0t + 0.025t^3$$

b) Find the rover's displacement and average velocity vectors during the time interval from *t* = 0.0 s to *t* = 2.0 s.

$$x(0) = 2.0\text{ m}, \quad y(0) = 0\text{ m}$$

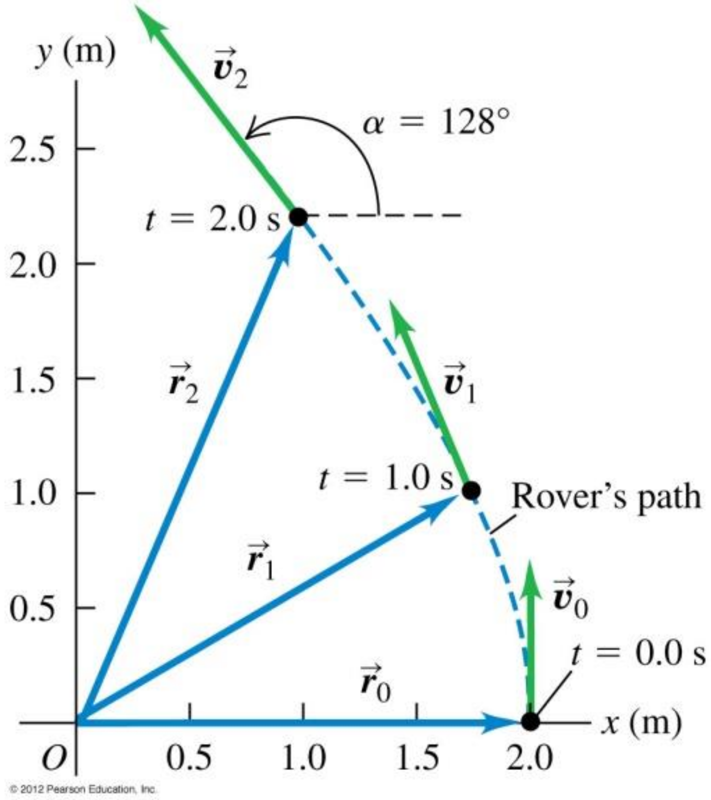
$$\vec{r}(2) = (1.0\,\hat{\imath} + 2.2\,\hat{\jmath})\text{ m} \quad \text{and} \quad \vec{r}(0) = 2.0\,\hat{\imath}$$

Displacement vector:

$$\Delta\vec{r} = \vec{r}(2) - \vec{r}(0) = (1.0\,\hat{\imath} + 2.2\,\hat{\jmath}) - 2.0\,\hat{\imath} = (-1.0\,\hat{\imath} + 2.2\,\hat{\jmath})\text{ m}$$

Average velocity vector:

$$\vec{v}_{av} = \frac{\Delta\vec{r}}{\Delta t} = \frac{1}{2}(-1.0\,\hat{\imath} + 2.2\,\hat{\jmath}) = (-0.5\,\hat{\imath} + 1.1\,\hat{\jmath})\text{ m/s}$$



Example 1 (solution)

A robotic vehicle, or rover, is exploring the surface of Mars. The landing craft is the origin of the coordinates, and the surrounding Martian surface lies in the *x*-*y* plane. The rover, which we represent as a point, has *x*- and *y*-coordinates that vary with time:

$$x = 2.0 - 0.25t^2 \qquad y = 1.0t + 0.025t^3$$

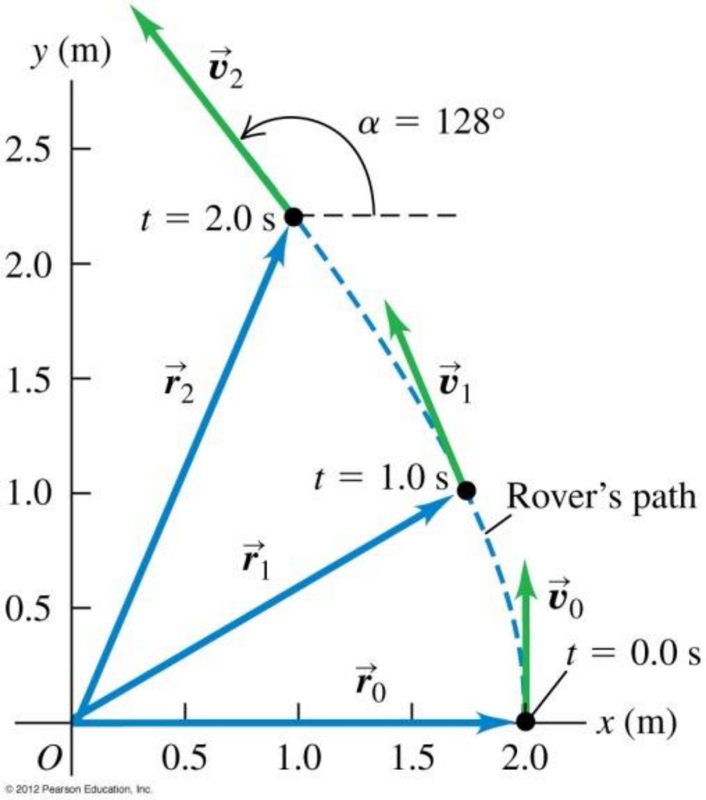
c) Derive a general expression for the rover's instantaneous velocity vector.

$$v_x = \frac{dx}{dt} = \frac{d}{dt}(2.0 - 0.25t^2) = -0.5t$$

$$v_y = \frac{dy}{dt} = \frac{d}{dt}(1.0t + 0.025t^3) = 1.0 + 0.075t^2$$

Instantaneous velocity vector:

$$\vec{v} = v_x\,\hat{\imath} + v_y\,\hat{\jmath} = -0.5t\,\hat{\imath} + (1.0 + 0.075t^2)\,\hat{\jmath}$$



Example 1 (solution)

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$$x = 2.0 - 0.25t^2 \qquad y = 1.0t + 0.025t^3$$

d) Express the instantaneous velocity at *t* = 2.0 s in component form and also in terms of magnitude and direction.

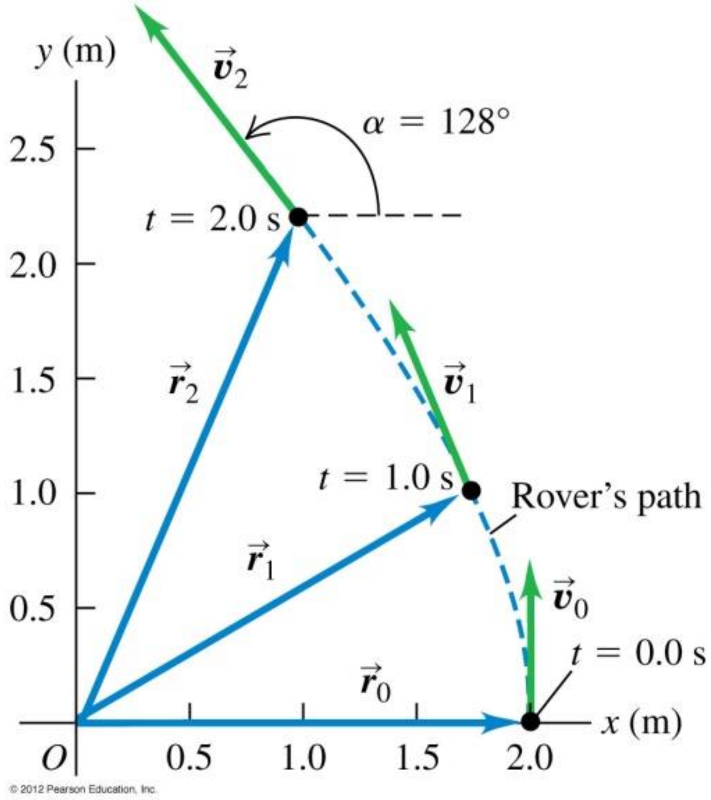
$$v_x = -0.5t \quad v_y = 1.0 + 0.075t^2$$

At $t = 2.0$ s,

$v_x = -0.5(2) = -1.0$ m/s $\qquad v_y = 1.0 + 0.075(2^2) = 1.3$ m/s

Magnitude of velocity, $v = \sqrt{v_x^2 + v_y^2} = \sqrt{1^2 + 1.3^2} = 1.6$ m/s

Direction, $\alpha = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{1.3}{-1.0}\right) = 128°$ (see diagram)

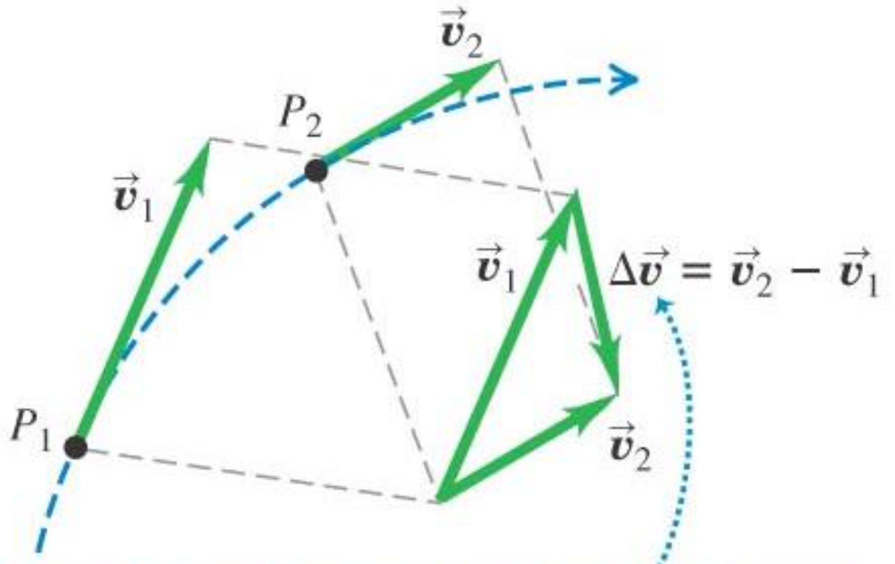


Average acceleration

- The average acceleration is the change in instantaneous velocity $\Delta\vec{v}$ divided by the time interval Δt.

$$\vec{a}_{av} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1} = \frac{\Delta\vec{v}}{\Delta t}$$

- Its direction is in the direction of $\Delta\vec{v}$.



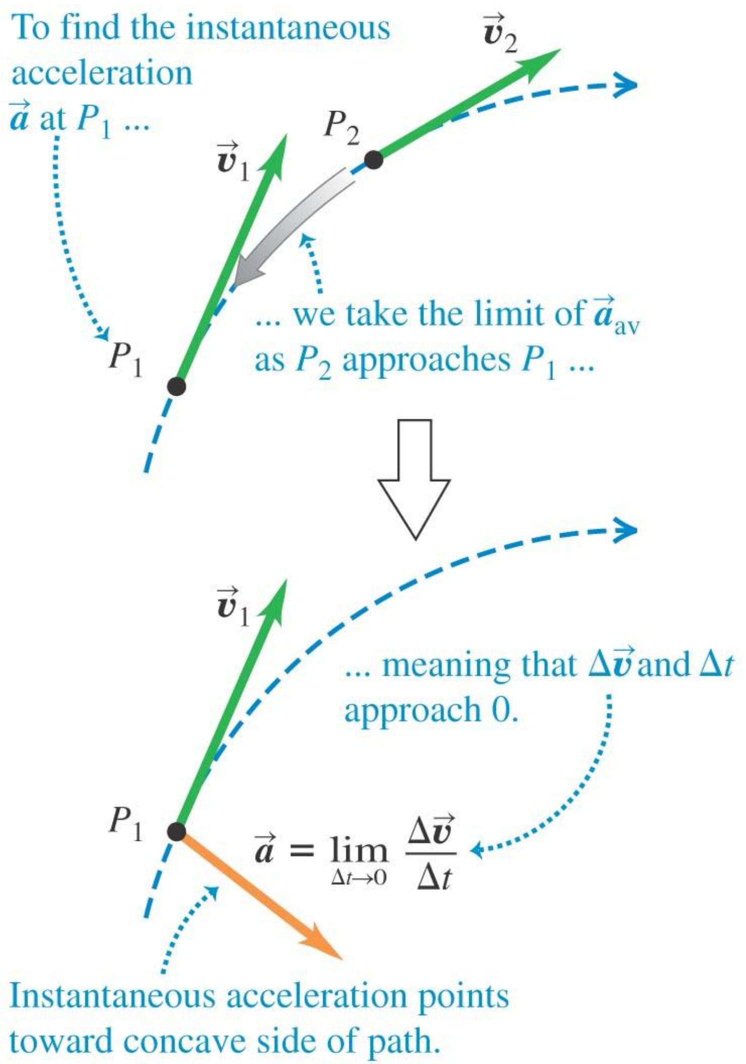
To find the car's average acceleration between P_1 and P_2, we first find the change in velocity $\Delta\vec{v}$ by subtracting $\vec{v}_1$ from $\vec{v}_2$. (Notice that $\vec{v}_1 + \Delta\vec{v} = \vec{v}_2$.)

Instantaneous acceleration

- The instantaneous acceleration is defined as :

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

- Its direction is towards the **concave** side of the path.



Instantaneous acceleration

- The components of the instantaneous acceleration are

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}, \qquad a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}, \qquad a_z = \frac{dv_z}{dt} = \frac{d^2z}{dt^2}$$

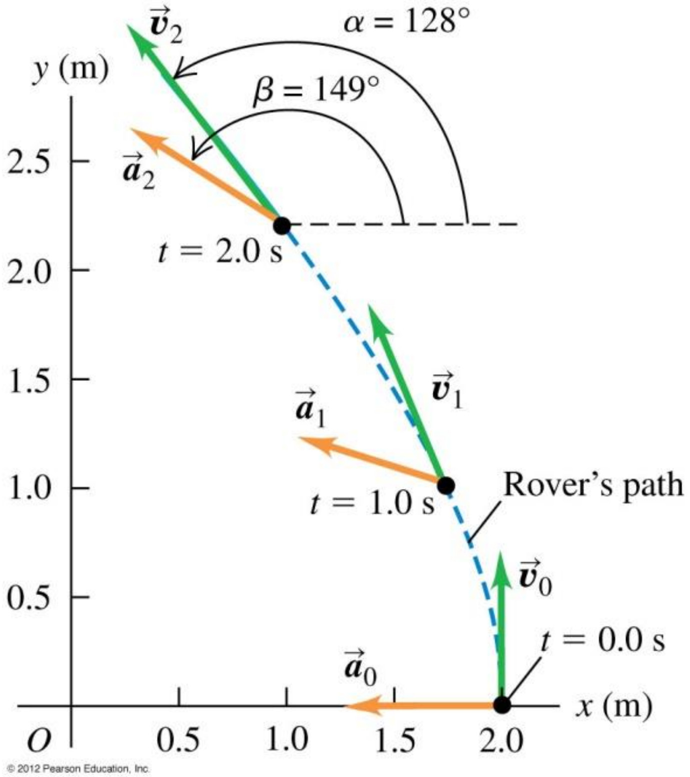
- We can also write the acceleration vector in terms of unit vectors as

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\hat{\imath} + \frac{dv_y}{dt}\hat{\jmath} + \frac{dv_z}{dt}\hat{k}$$

Example 2 (will go through in class)

A robotic vehicle, or rover, is exploring the surface of Mars. The landing craft is the origin of the coordinates, and the surrounding Martian surface lies in the *x*-*y* plane. The rover, which we represent as a point, has *x*- and *y*-coordinates that vary with time :

$$x = 2.0 - 0.25t^2 \qquad y = 1.0t + 0.025t^3$$



a) Find the components of the average acceleration during the time interval from t = 0.0 s to t = 2.0 s.

[Ans: $a_{av,x} = -0.50$ m/s^2, $a_{av,y} = 0.15$ m/s^2]

b) Find the instantaneous acceleration at t = 2.0 s.

[Ans: $\vec{a} = (-0.50\ \hat{\imath} + 0.30\ \hat{\jmath})$ m/s^2]

End of pre-class slides

Recap of last week material

- What are the directions of acceleration of the car in these situations?

Car	Initial velocity	Changes in motion	Acceleration
A	Moving to the right	Constant velocity	
B	Moving to the right	Speeding up	
C	Moving to the left	Slowing down	
D	Moving to the right	Slowing down	
E	Moving to the left	Speeding up	

- What are the kinematics equations for constant acceleration motion?

Projectile Motion

Learning outcomes

At the end of the session, students are to be able to

- describe the characteristics of projectile motion under no air resistance
- explain how different factors affect the projectile motion under no air resistance
- apply kinematics equations to solve problems related to projectile motion under no air resistance

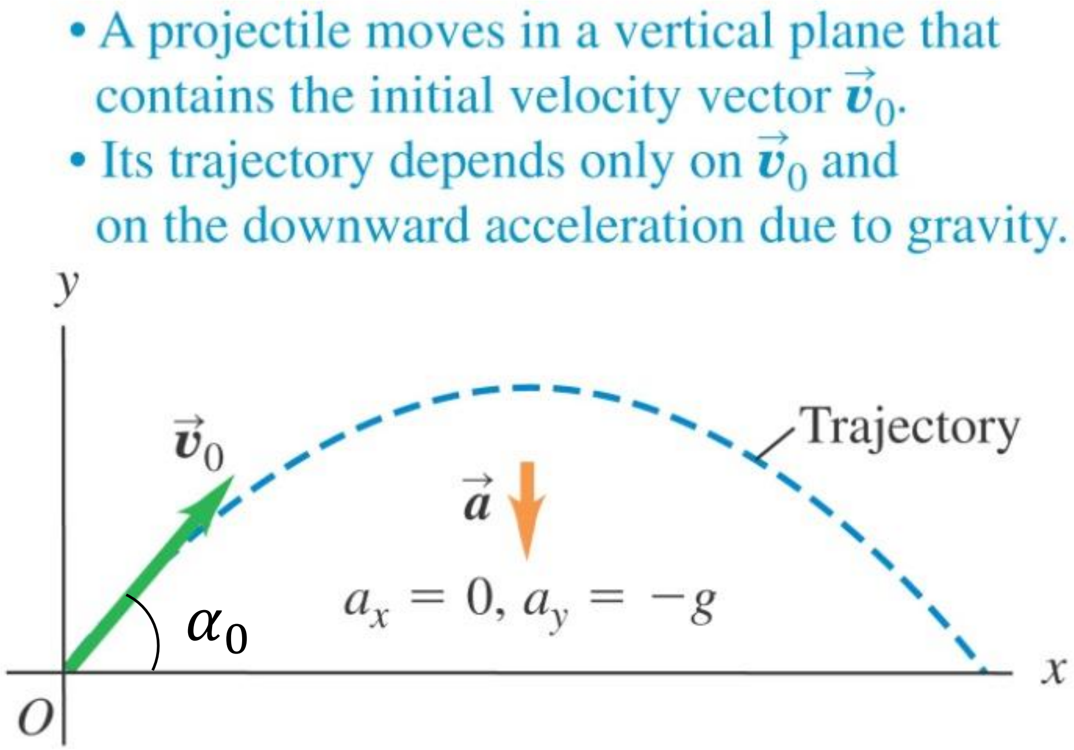
Projectile motion

- A projectile is any body that is given an initial velocity and then follows a path determined entirely by the effects of gravity and air resistance.

- We will set up equations for a projectile motion assuming no air resistance.

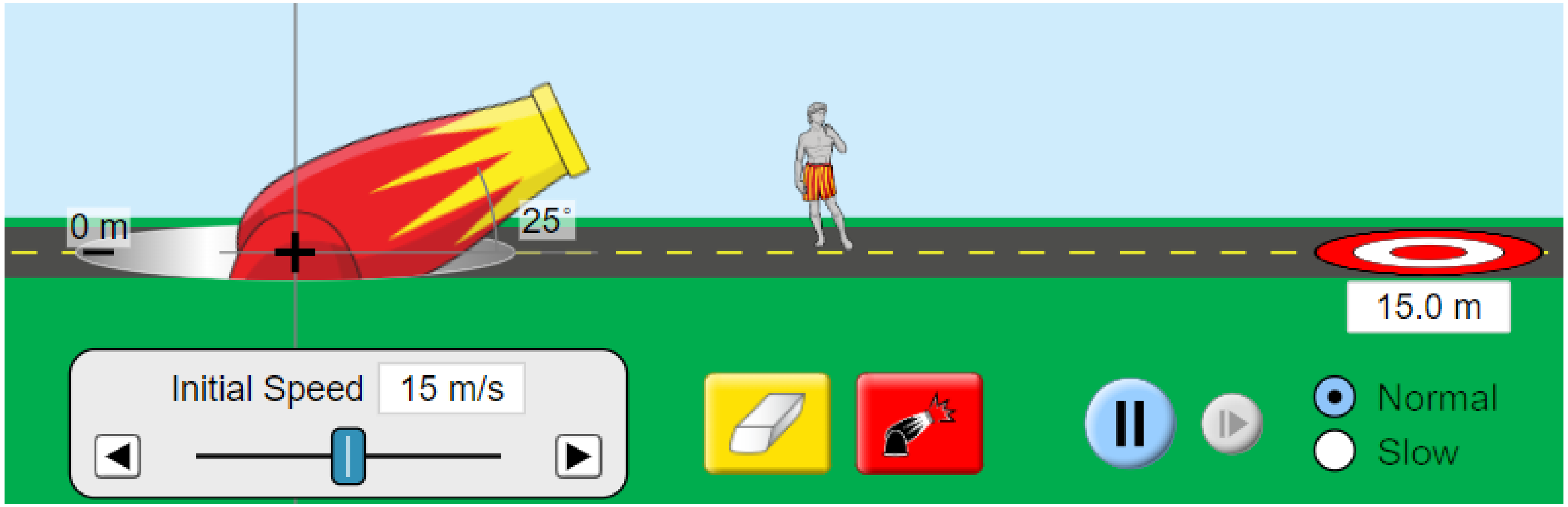
- The initial velocity v_0 can be resolved into the x and y components in terms of the angle α_0, i.e.

$$v_{0x} = v_0 \cos \alpha_0$$
$$v_{0y} = v_0 \sin \alpha_0$$



Projectile motion (In-class Activity)

- PhET simulation: [Projectile Motion (colorado.edu)]
- Instructions:
  - On the first page, select "Intro".
  - Adjust the height of the cannon so that it is 0 m. You should see the below picture on your screen.



Projectile motion (In-class Activity)

- **Task 1:** Investigate the changes in velocity and acceleration vectors during projectile motion under no air resistance. (guided by me)

- **Task 2:** Investigate the factors affecting the horizontal range, maximum height reached, and the time of flight in projectile motion under no air resistance.

- **Task 3:** Investigate how the launch height of a horizontal cannon affects the horizontal range and the time of flight in the projectile motion.

Projectile motion (In-class Activity Task 1, 10 mins)

Describe the changes in the following quantities (both **magnitude** and **direction**) when the projectile is in the air.

Possible words for magnitude: *increasing*, *decreasing*, *constant*, *zero*
Possible words for direction: *up*, *down*, *left*, *right*

	When the projectile is moving up	When the projectile is at the highest point	When the projectile is moving down
Horizontal component of velocity, v_x			
Vertical component of velocity, v_y			
Acceleration			

Projectile motion (In-class Activity Task 2, 10 mins)

Task 2: Factors affecting the horizontal range, maximum height reached, and the time of flight in projectile motion

Instructions: Fix the launch speed at 15 m/s, height = 0 m. Adjust the launch angle between 30° and 60° in steps of 5°.

a) How does the horizontal range change with launch angle? At what launch angle is the range maximum?

b) How does the time of flight of the cannon ball change with launch angle?

c) How does the maximum height reached by the cannon ball change with launch angle?

d) Any other observations would you like to highlight?

Projectile motion (In-class Activity Task 3, 10 mins)

Task 3: Investigate how the launch height of a horizontal cannon affects the horizontal range and the time of flight in the projectile motion.

Instructions: Fix launch speed at 15 m/s, launch angle at 0°, adjust the height of the cannon.

(a) How does the time of flight of the cannon ball change with the height of the cannon?

(b) How does the horizontal range of the cannon ball change with the height of the cannon?

Projectile motion

- If there is no air resistance, the horizontal velocity $v_0 \cos \alpha_0$ is constant.
- The horizontal distance travelled in time t is $x = (v_0 \cos \alpha_0)t$.
- Because of gravity, the vertical velocity varies as $v_y = (v_0 \sin \alpha_0) - gt$
- The vertical displacement in time t is $\Delta y = (v_0 \sin \alpha_0)t - \frac{1}{2} g t^2$
- Eliminating t from the equation for x and y, we have,

$$y = (\tan \alpha_0)x - \frac{g}{2v_0^2 \cos^2 \alpha_0} x^2$$

- This shows that the object moves in a parabolic path.

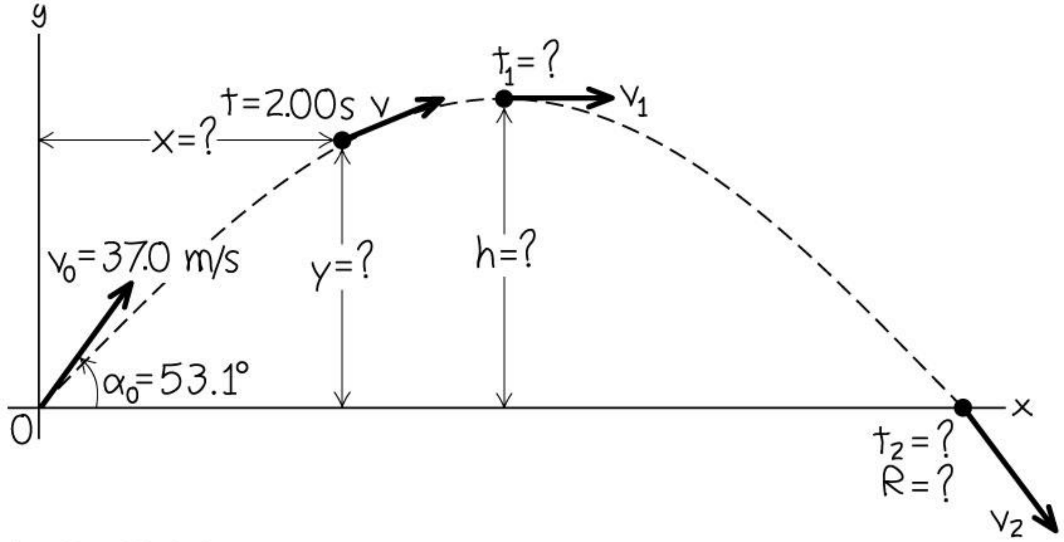
Example 3

A batter hits a baseball so that it leaves the bat at speed v_0 = 37.0 m/s at an angle α_0 = 53.1°.

a) Find the position of the ball and its velocity at t = 2.00 s.

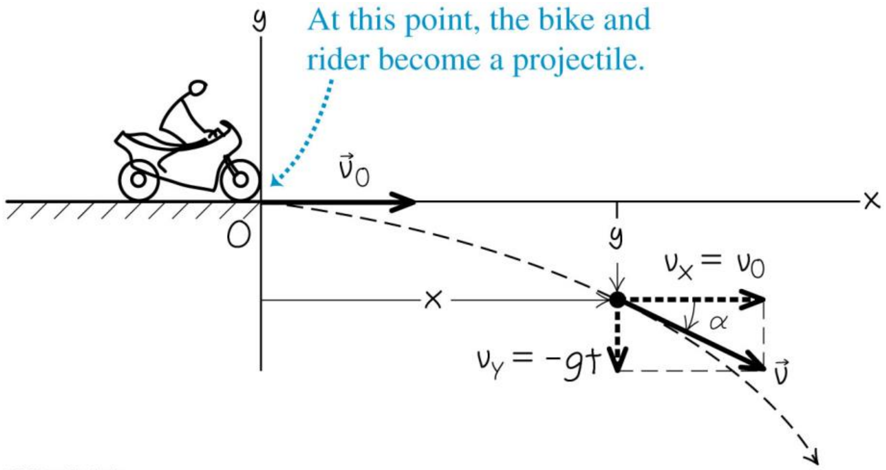
b) Find the time when the ball reaches the highest point of its flight and its height h, at this time.

c) Find the horizontal range R which is the maximum horizontal distance travelled by the ball.



Example 4

A motorcycle stunt rider rides off the edge of a cliff. Just at the edge his velocity is horizontal, with magnitude 9.0 m/s. Find the motorcycle's (a) position (b) distance from the edge of the cliff and (c) velocity 0.5 s after it leaves the edge of the cliff.

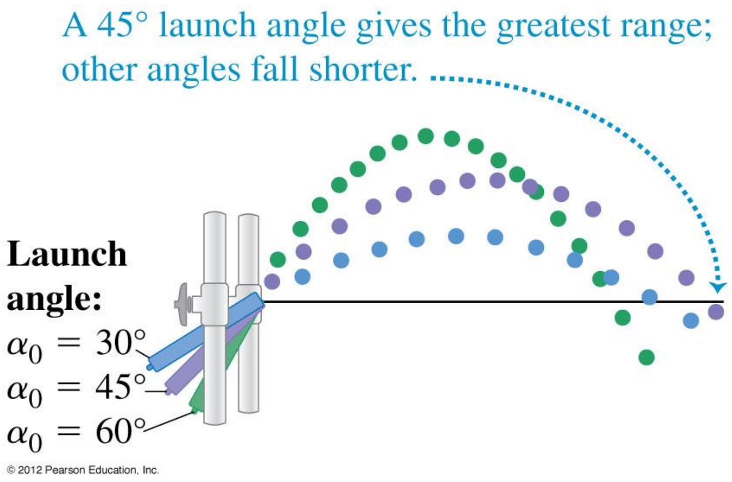


Example 5

Find the maximum height h and horizontal range R of a projectile launched with speed v_0 at an initial angle α_0 between 0º and 90º .

a) For a given v_0, what value of α_0 gives maximum height?

b) What value of α_0 gives maximum horizontal range?



Problem solving strategy for projectile motion problems

1. Draw a sketch of the situation. Define your coordinate system.

2. List down the unknown and known quantities, and decide which unknowns are your target variables.

3. Look at the relevant equations for horizontal and vertical motion to see which one is suitable for finding the unknowns.

   Horizontal motion: $x = (v_0 \cos \alpha_0)t$

   Equations for vertical motion:
   $$v_y = (v_0 \sin \alpha_0) - gt$$
   $$y = (v_0 \sin \alpha_0)t - \frac{1}{2}gt^2$$
   $$v_y^2 = (v_0 \sin \alpha_0)^2 - 2gy$$

Relative Motion

Learning outcomes

At the end of the session, students are to be able to

- recognize that experimental observations depends on reference frame
- define the relative velocity of one object in a particular reference frame in a vector equation
- solve relative motion problems using the vector equation and relevant mathematical tools

Relative motion

- The velocity of a moving body seen by an observer is called the relative velocity.
- The observer may be stationary or moving.
- To study relative velocity we need a frame of reference.
- A frame of reference is a coordinate system plus a time scale.
- Note: the observer is at rest in his/her own reference frame.

Relative motion in 1D

Consider the following scenarios for two students A and B who are 10 m apart initially:

- S1: B is stationary and A is moving at 2 m/s towards B.
- S2: Both A and B move to the right at 2 m/s.
- S3: A and B move to the right with $v_A = 1$ m/s and $v_B = 2$ m/s.
- S4: A and B move towards each other with equal speeds of 2 m/s.

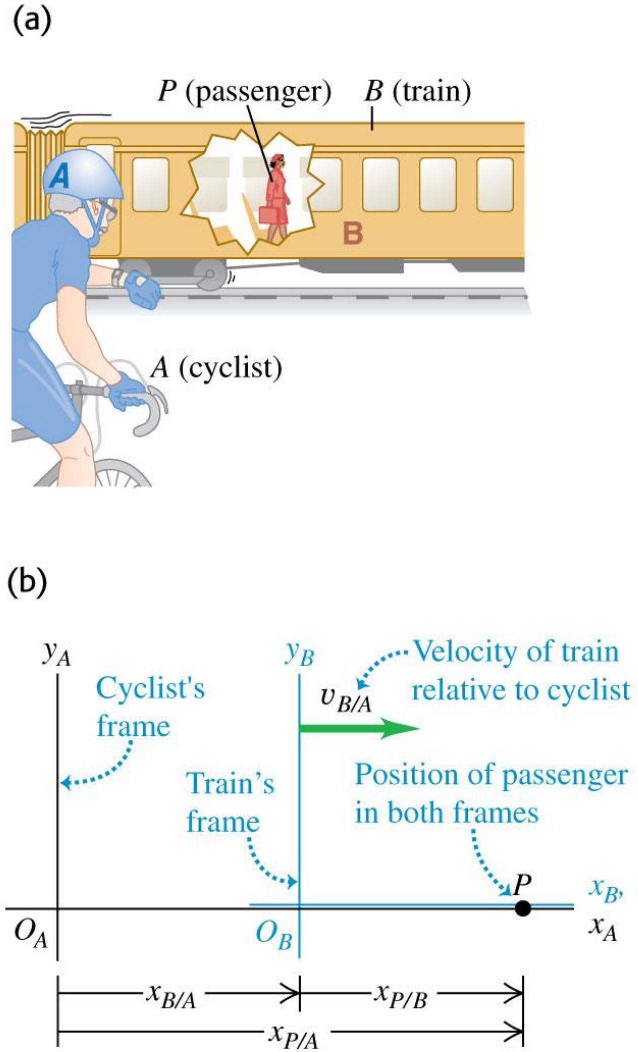
Question: For each scenario, what is the velocity of B from A's perspective? How do you arrive at your answers? (Discuss with a friend)

Relative motion

- One way to look at relative motion is to
  - simulate the motion of different objects in a common reference frame,
  - make measurements from different observers' perspectives, and
  - use the definitions of position, displacement and velocity to calculate them.
- Always remember that the observer is at rest in his/her own reference frame.

Relative velocity in 1D

- In Fig. (a), a cyclist (A) is observing the motion of the passenger P in a moving train (B).
- Define the following vectors:
  - $\vec{x}_{P/A}$ = position of P relative to A
  - $\vec{x}_{B/A}$ = position of B relative to A
  - $\vec{x}_{P/B}$ = position of P relative to B
- At $t = 0$ s, O_A, O_B and P coincides.
- After some time (see Fig. (b)), the x position of P relative to A is $\vec{x}_{P/A} = \vec{x}_{P/B} + \vec{x}_{B/A}$



Relative velocity in 1D

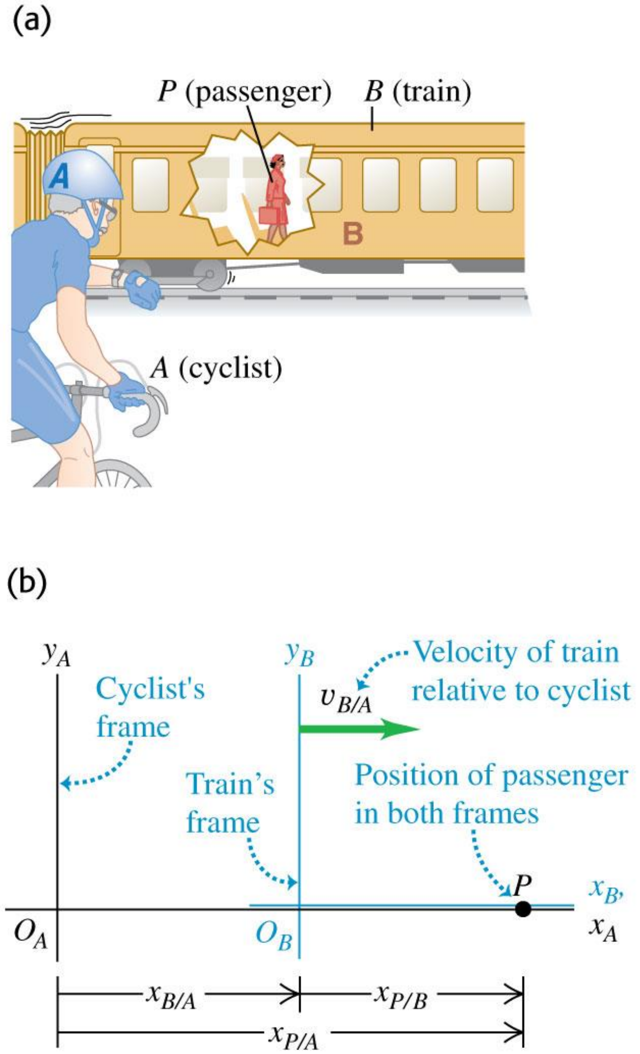
- The velocity of P relative to A is obtained by differentiation.

$$\frac{d\vec{x}_{P/A}}{dt} = \frac{d\vec{x}_{P/B}}{dt} + \frac{d\vec{x}_{B/A}}{dt}$$

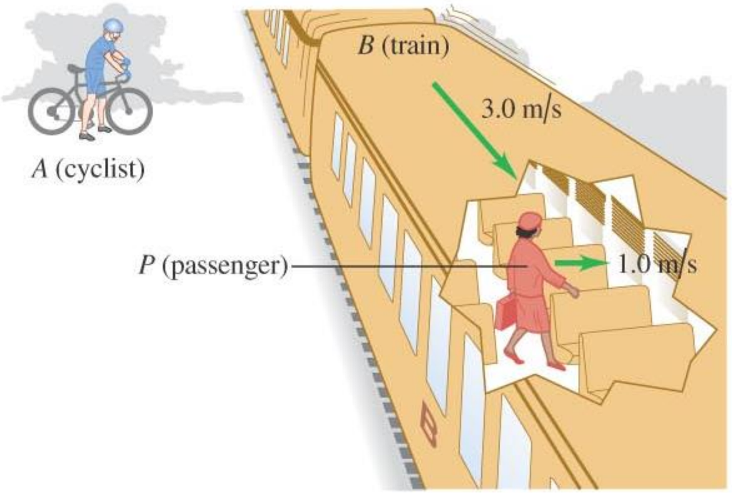
$$\vec{v}_{P/A} = \vec{v}_{P/B} + \vec{v}_{B/A}$$

- One way to remember the ordering is to imagine the subscripts as a fraction and split it as $P/A = P/B \times B/A$.

- Note: $\vec{v}_{A/P} = -\vec{v}_{P/A}$



Relative velocity in 2D and 3D



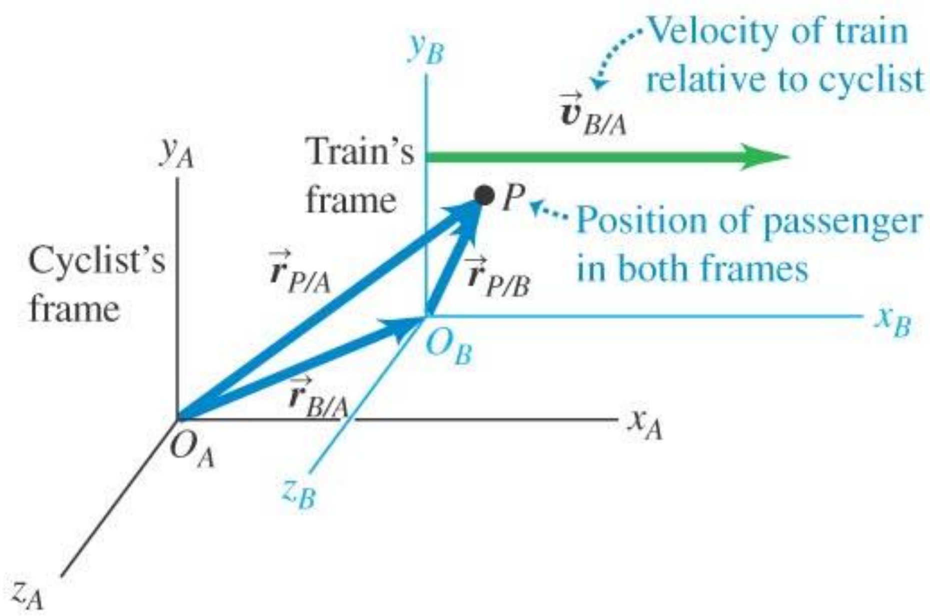
- In 2D and 3D, the velocity of P relative to A is obtained by differentiating the position vector

$$\vec{r}_{P/A} = \vec{r}_{P/B} + \vec{r}_{B/A}$$

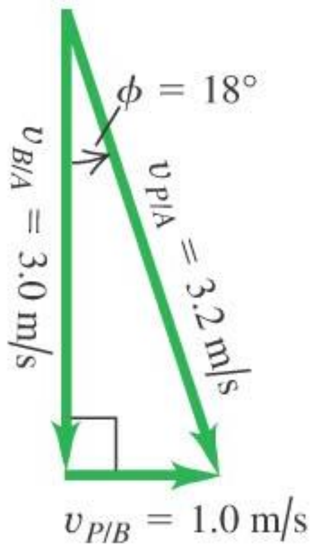
$$\frac{d\vec{r}_{P/A}}{dt} = \frac{d\vec{r}_{P/B}}{dt} + \frac{d\vec{r}_{B/A}}{dt}$$

$$\vec{v}_{P/A} = \vec{v}_{P/B} + \vec{v}_{B/A}$$

(b)

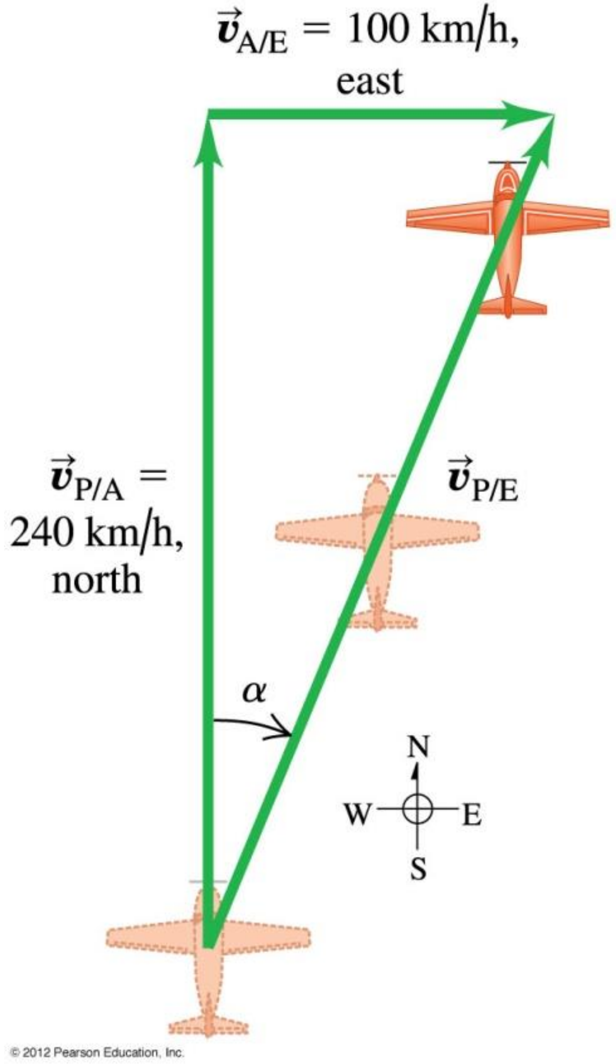


(c) Relative velocities (seen from above)



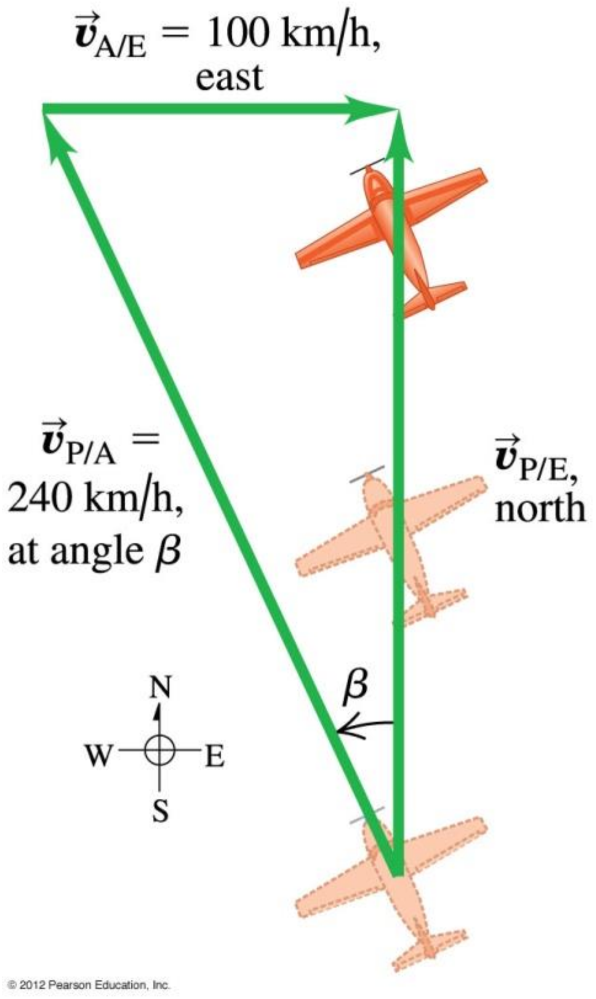
Example 6a

a) An air-plane's compass indicates that it is headed due north and its airspeed indicator shows that it is moving through the air at 240 km/h. If there is a 100-km/h wind from west to east, what is the velocity of the airplane relative to the earth?



Example 6b

b) With the same wind and airspeed, in what direction should the airplane head to travel due north? What will be the velocity of the airplane relative to the earth?



Optional slides (not tested)

Motion with a non-uniform acceleration

- The following equations are useful in the case of non-uniform acceleration.

$$\vec{a} = \frac{d\vec{v}}{dt}$$

- Integrating the above equation, we get $\vec{v} = \vec{v}_0 + \int_0^t \vec{a}\, dt$

$$\vec{v} = \frac{d\vec{r}}{dt}$$

- Integrating, we get $\vec{r} = \vec{r}_0 + \int_0^t \vec{v}\, dt$

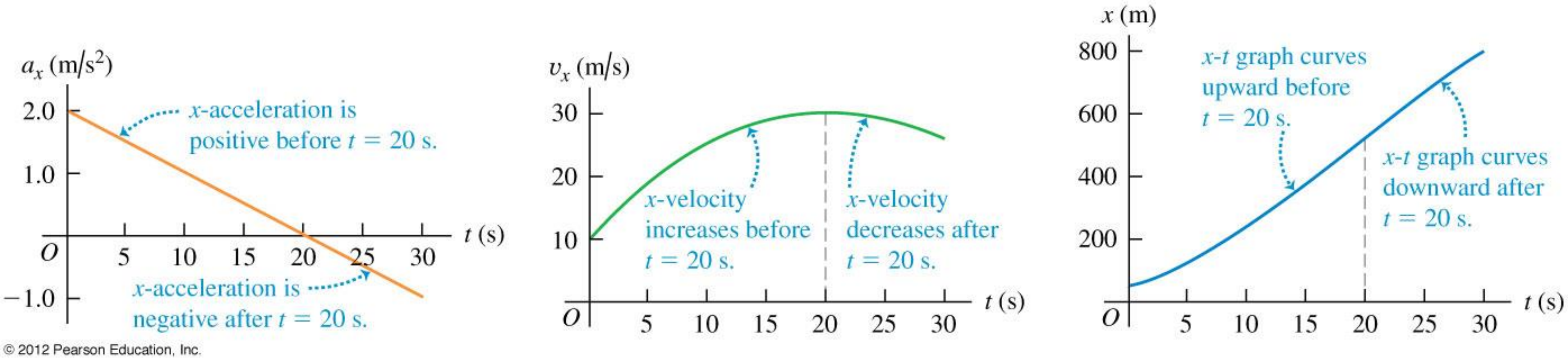
Example 7

Sally is driving along a straight highway. At t = 0, when she is moving at 10 m/s in the positive x-direction, she passes a sign-post at x = 50 m. Her x-acceleration as a function of time is
$a_x = 2.0 - 0.10t$ m/s^2.

a) Find her x-velocity v_x and position x as functions of time.

b) When is her x-velocity maximum?

c) What is that maximum x-velocity?



Example 7 (solution)

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Solution for (a):

$$v_x = v_0 + \int a\, dt = 10 + \int_0^t (2.0 - 0.10t)\, dt = 10 + [2t - 0.05t^2]_0^t = 10 + 2t - 0.05t^2$$

$$x = x_0 + \int v_x\, dt = 50 + \int_0^t (10 + 2t - 0.05t^2)\, dt = 50 + 10t + t^2 - \frac{0.05}{3} t^3$$

Example 7 (solution)

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a) Find her x-velocity v_x and position x as functions of time.

b) When is her x-velocity maximum?

c) What is that maximum x-velocity?

Solution for (b):

v_x is maximum when $a_x = 0$.

$$a_x = 0 \quad \Rightarrow \quad 2.0 - 0.10t = 0$$

$$t = \frac{2.0}{0.10} = 20 \text{ s}$$

Example 7 (solution)

Sally is driving along a straight highway. At t = 0, when she is moving at 10 m/s in the positive x-direction, she passes a sign-post at x = 50 m. Her x-acceleration as a function of time is $a_x = 2.0 - 0.10t$ m/s^2.

a) Find her x-velocity v_x and position x as functions of time.

b) When is her x-velocity maximum?

c) What is that maximum x-velocity?

Solution for (c):

v_x is maximum when $t = 20$ s.

$$v_x = 10 + 2t - 0.05t^2 = 10 + 2(20) - 0.05(20)^2 = 30 \text{ m/s}$$

End of chapter