Chapter 8 : Solve First Order Differential Equations by Separating Variables

Objectives :

1. Define differential equation and state the order of a differentiatl equation.
2. Verify that a given function is a solution of a differential equation.
3. Describe the general solution and arbitrary constants.
4. Solve first order differential equations by "separating" the variables.
5. Solve application problems: Newton's law of cooling.

8.1 Definitions

8.1.1 Differential Equations

A differential equation is an equation that contains derivative(s). It is used to model various activities/happenings around us. It is used to examine the motion of a particle in Physics, to calculate compound interest in Economics, to model electrical circuit in engineering and so on.

8.1.2 Ordinary Differential Equation

An ***ordinary differential equation*** is a differential equation involving only **one** independent variable. Examples of ordinary differential equations are :

$$2\frac{dy}{dx}+6y=x+1 \quad , \qquad 3\frac{d^2y}{dx^2}+5\frac{dy}{dx}-8y=\sin x$$

8.1.3 Order of a Differential Equation

The ***order*** of a differential equation is the order of the highest derivative in the differential equation.

***Example 1*:** Identify the order of each of the following differential equations:

(a) $\frac{dy}{dx}+3y=x$ (b) $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3=0$ (c) $\left(\frac{d^3y}{dx^3}\right)^2+6\frac{dy}{dx}=0$

Solution

8.2 Solutions of a Differential Equation

Any relation between the dependent and independent variables of a differential equation, not containing any derivatives, which *satisfies* the differential equation is called a **solution** of the equation.

Example 2: Show that $y = 5e^{2x}$ is a solution of the differential equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} = 2y$.

Solution

8.2.1 General Solution

The *general solution* of a differential equation of order n is the solution containing n arbitrary constants.

8.2.2 Particular Solution

A *particular solution* of a differential equation is one in which the arbitrary constants in the general solution have assigned values.

8.3 Solving First Order Differential Equations by Direct Integration

A first order differential equation of the form $\frac{dy}{dx} = f(x)$ can be solved by direct integration. Thus if $\frac{dy}{dx} = f(x)$, then $y = \int f(x)\, dx$.

Example 3: Find the general solution of the differential equation $x^2 \frac{dy}{dx} = 2x^4 + \frac{1}{2}$.

Find also the particular solution for which $y = 2$ when $x = 1$.

Solution The equation can be written as

$$\therefore\ y = \int \left(2x^2 + \frac{1}{2x^2} \right) dx =$$

The general solution is $y =$

When $x = 1$, $y = 2$,

$\rightarrow\ C =$

The particular solution is $y(x) =$

8.4 Separation of Variables

Variable separable technique is commonly used to solve first order differential equations.

Equations of the form:

$$\frac{dy}{dx} = f(x)g(y) \qquad \text{or} \qquad \frac{dy}{dx} = \frac{f(x)}{g(y)}$$

can be re-written as:

$$\frac{1}{g(y)}\,dy = f(x)\,dx \qquad \text{or} \qquad g(y)\,dy = f(x)\,dx$$

All the terms in x can be collected on one side of the equation and all the terms in y on the other side.

The general solution is then obtained by integrating each side of the equation separately.

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx \qquad \text{or} \qquad \int g(y)\,dy = \int f(x)\,dx$$

Example 4: Solve the differential equation $\dfrac{dy}{dx} = 2y$ for y in terms of x.

Solution First rewrite the differential equation in differential form

$$dy =$$

Then separating the variables, we have

$$\frac{dy}{y} =$$

Integrate each side of the equation,

$$\int \frac{dy}{y} =$$

***Example 5*:** Solve $y\, e^{x+3y} \dfrac{dy}{dx} = 1$

Ans: $y \cdot \dfrac{e^{3y}}{3} - \dfrac{1}{9} e^{3y} = -e^{-x} + C$

Solution

Example 6: Solve $2xy\dfrac{dy}{dx} + 3(y^2 + 1) = 0$ given that $y(1) = 2$.

Ans: $\ln\left|y^2 + 1\right| = -3\ln|x| + \ln 5$ or $x^3(y^2 + 1) = 5$

Solution

8.5 Application: Newton's Law of Cooling

If we leave a glass of hot water on a table, it cools to the surrounding room temperature. **Newton's Law of Cooling** states that the rate of cooling (how fast is the decrease in temperature) is proportional to the temperature difference between the hot water and the room. This is in general true.

Let $T(t)$ = temperature of glass at time t, T_s = temperature of the surrounding, then,

$$\frac{dT}{dt} = -k(T - T_s)$$

where k is a positive constant determined by other factors, e.g., shape of the glass, the material it is made from, whether it is open or closed and so on. The negative sign in front of k indicates that the temperature T decreases with time t.

$$\int \frac{dT}{T - T_s} = -\int k\,dt$$

$$\ln\left|T - T_s\right| = -kt + C$$

$$T - T_s = e^{-kt+C} = e^{-kt}e^{C} = Ae^{-kt}, \text{ where } A = e^{C}$$

$\therefore$ General solution is $T(t) = T_s + Ae^{-kt}$

Note: Since k is positive, as $t \to \infty$, $e^{-kt} \to 0$ and hence $T \to T_s$. The limiting value of T is called the **ultimate temperature** of the body. The body will cool down to its surrounding temperature T_s after a very long time. **Ultimate** temperature is also known as **Final** temperature, **Limiting** temperature or T**erminal** temperature. Condition for ultimate temperature is $\frac{dT}{dt} = 0$.

***Example 7*:** A cup of boiling water is allowed to cool in a room where the temperature is maintained constant at $30°C$. The cooling process follows Newton's law of cooling whereby the rate of change of temperature of the water is proportional to the difference between the water and its surrounding temperature. If after 3 minutes, the water temperature is dropped to $78°C$.

(a) Set up the differential equation that depicts the cooling process of the water.

(b) Find the particular solution of the differential equation in part (a).

(c) Find the water temperature after 10 minutes.

Ans: $50°C$

Tutorial 8

1. What are the orders of the following differential equations?

(a) $y\dfrac{d^2y}{dx^2}+\left[\dfrac{dy}{dx}\right]^2=0$ (b) $\left[\dfrac{dy}{dx}\right]^3-y=x$

2. Verify that the function $y=x^2+x$ is a solution of the differential equation

$$\frac{d^2y}{dx^2}+x\frac{dy}{dx}-2=2y-x$$

(Refer to Example 2 on Pg 8-2 if you have problem in doing this question)

3. Solve the following differential equations by separating variables:

(a) $\dfrac{dy}{dx}=2xy$ (b) $\dfrac{dy}{dt}=e^{t-y+2}$

(c) $\left(1+x^2\right)\dfrac{dy}{dx}=xy$ (d) $\dfrac{dy}{dx}-x^2+1=0$

4. Find the general solution for each of the following differential equations.

(a) $\dfrac{dy}{dx}-\dfrac{2x}{3y^2+1}=0$ (b) $ye^{y^2-x}\dfrac{dy}{dx}=x$

5. Find the particular solution for each of the following differential equations.

(a) $2x^2y\dfrac{dy}{dx}=-(y+1)$; $y=0$ *when* $x=1$

(b) $\cos y+(1+e^{-x})\sin y\dfrac{dy}{dx}=0$; for $y=\dfrac{\pi}{4}$ when $x=0$.

(c) $x^2(y+1)+y^2(x-1)\dfrac{dy}{dx}=0$; for $y=0$ when $x=0$.

6. The variation of resistance R ohms of a copper conducted with temperature $\theta°C$ is given by

$$\frac{dR}{d\theta}=4\times10^{-3}R$$

If $R=60$ ohms at $\theta=0°C$,

(a) solve the differential equation for R,

(b) find the resistance of the copper conductor at $20°C$.

7. A cup of boiling coffee is allowed to cool in a room where the temperature is maintained constant at $25°C$. The cooling process follows Newton's law of cooling. If after 2 minutes, the coffee temperature is dropped to $80°C$.

(a) Set up the differential equation that depicts the cooling process of the coffee;

(b) Find the particular solution of the differential equation in part (a);

(c) Find the coffee temperature after 8 minutes.

8. If a body cools from $100°C$ to $80°C$ in 10 minutes in air, which is maintained at $20°C$. The cooling process follows Newton's law of cooling.
   (a) Set up the differential equation that depicts the cooling process of the body.
   (b) Solve the equation in part (a) using given conditions.
   (c) How long will it takes the body to cool down from $80°C$ to $60°C$?

Miscellaneous Exercises

1. Solve the following differential equation $\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2$, $y(0)=2,\ y'(0)=1$

   [*Hint*: let $z = \frac{dy}{dx}$]

2. Solve the differential equation $dy = (9x+4y+1)^2\,dx$. [*Hint*: Let $z = 9x+4y+1$]

3. By using the substitution $y = \frac{1}{v}$, show that the differential equation $-xy^2 = y^2 + y^3\frac{dy}{dx}$ can be reduced to the form of $\frac{1}{v^3}\frac{dv}{dx} = 1+x$. Find the particular solution that passes through the origin.

4. The gradient of a curve at the point (x, y) is given by $\frac{y}{\ln y}$. Given that the curve passes through the point $(0, e)$, find the equation of the curve.

5. Given that $\frac{d^2y}{dx^2} = \frac{1}{x}$, find y in terms of x.

6. Obtain the general solution of each of the given differential equations.
   (a) $\tan x = -y\cos^2 x\frac{dy}{dx}$
   (b) $\frac{dy}{dx}\left(1+\sin^2 x\right) = \sin y\cos x$
   (c) $\frac{dy}{dx} = \frac{y^2\sqrt{1-y^2}}{\sqrt{25-4x^2}}$ [Hint: let $y = \sin\theta$]
   (d) $y\frac{dy}{dx} = \frac{\sqrt{y^4+1}}{2+\sin x}$ [Hint: let $y^2 = \tan\theta$ and $u = \tan\frac{x}{2}$]

Multiple Choice Questions

1. Which of the following differential equations cannot be solved by separating the variables?
   (a) $\frac{dy}{dx} = \frac{y}{x}$
   (b) $\frac{dy}{dx} = \frac{x}{y}$
   (c) $\frac{dy}{dx} = xy$
   (d) $\frac{dy}{dx} = x+y$

2. Which of the following differential equations can be solved by separating the variables?

(a) $\dfrac{dy}{dx}=\dfrac{xe^x \sin y}{\cos y}$ (b) $\dfrac{dy}{dx}=\dfrac{x^2+x-1}{xe^y-\sin y}$

(c) $\dfrac{dy}{dx}=\dfrac{e^{x^2}}{\tan y}$ (d) $\dfrac{dy}{dx}=\dfrac{x+y}{x-y}$

3. Which of the following is not a solution to the differential equation $\dfrac{dy}{dx}=ky$, where k is a constant?

(a) $\ln y = kx + c$ (b) $y = ce^x + k$

(c) $\ln(cy) = kx$ (d) $y = ce^{kx}$

4. The expression $e^{\frac{1}{2}\ln|1+x|}$ can be simplified as

(a) $\dfrac{1}{2}(1+x)$ (b) $\sqrt{1+x}$

(c) $e^{\frac{1}{2}}(1+x)$ (d) $\dfrac{1}{\sqrt{1+x}}$

Answers

1. (a) 2 (b) 1

3. (a) $\ell n|y| = x^2 + C$ or $y(x) = Ae^{x^2}$

   (b) $e^y = e^{t+2} + C$ or $y(t) = \ell n\left(e^{t+2} + C\right)$

   (c) $\ell n|y| = \frac{1}{2}\ell n\left|1 + x^2\right| + C$ or $y(x) = A\sqrt{1+x^2}$

   (d) $y(x) = \frac{x^3}{3} - x + C$

4. (a) $y^3 + y = x^2 + C$ (b) $e^{y^2} = 2e^x(x-1) + C$

5. (a) $2(y - \ell n|y+1|) = \frac{1}{x} - 1$

   (b) $(1+e^x)\sec y = 2\sqrt{2}$ or $(1+e^x) = 2\sqrt{2}\cos y$

   (c) $x^2 + y^2 + 2x - 2y + 2\,\ell n|x-1| + 2\,\ell n|y+1| = 0$

6. (a) $R(\theta) = 60e^{0.004\theta}$ ohms (b) $R = 65$ ohms

7. (b) $T(t) = 25 + 75e^{-0.155t}$ $(°C)$ (c) $46.7°C$

8. (b) $T(t) = 20 + 80e^{-0.029t}$ $(°C)$ (c) 14.1 min

Miscellaneous Exercises

1. $y(x) = 2 + \frac{1}{\sqrt{2}}\ln\left|\frac{x+\sqrt{2}}{x-\sqrt{2}}\right|$

2. $3\tan\left(6x + c\right) = 2\left(9x + 4y + 1\right)$

3. $x^2 + 2x + y^2 = 0$

4. $\left(\ln y\right)^2 = 2x + 1$

5. $y(x) = x\ln x + Ax + B$

6. (a) $y^2 + \sec^2 x = C$

   (b) $\tan^{-1}\left(\sin x\right) + \ln\left|\csc y + \cot y\right| = C$

   (c) $\frac{1}{2}\sin^{-1}\frac{2x}{5} + \frac{\sqrt{1-y^2}}{y} = C$

   (d) $\frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{2}{\sqrt{3}}\left(\tan\frac{x}{2} + \frac{1}{2}\right)\right] = \frac{1}{2}\ln\left(\sqrt{y^4+1} + y^2\right) + C$

MCQ

1. d 2. a 3. b 4. b