Singapore Polytechnic, School of Mathematics and Science
Academic Year 2019/2020 Semester 2 **Further Mathematics**
Mid-Semester Test **Duration: 1.5 hour**

Instructions

1. All SP examination rules are to be complied with.
2. This paper consists of 3 pages.
3. Answer ALL the questions. Unless otherwise stated, leave your answers in 2 decimal places.
4. Except for graphs and diagrams, no solutions are to be written in pencil.

Additional Formulae

Change of base: $\log_b a = \dfrac{\log_c a}{\log_c b}$

Log of a power: $\log_b a^x = x \log_b a$

Area of circle = πr^2

Circumference of circle = $2\pi r$

1. Andrew makes one donation per year to a certain charity. He starts by donating \$500 in the first year. In each subsequent year, the value of his donation is 1.05 times the value of his previous year's donation.

   (a) Find the value of Andrew's donation in the 15th year. Round your answer to the nearest dollar.

   (b) Over the years, Andrew has donated a total of \$104,674. Find the number of years Andrew has donated. (10 marks)

2. (a) The parametric equations of a curve are

$$x = 2e^t, \qquad y = \ln(t+1).$$

   (i) Find $\dfrac{dy}{dx}$. (5 marks)

   (ii) Find the equation of the tangent to the curve at the point for which $t = 0$. (5 marks)

   (iii) Find the Cartesian equation of the curve. (4 marks)

   (b) A curve C has parametric equations

$$x = \sin t + 1, \qquad y = \cos t - 2.$$

   (i) Find the Cartesian equation of the curve. (5 marks)

   (ii) Hence, sketch the curve C. (6 marks)

3. (a) Two functions are defined as follows:

$$f(x) = \frac{3}{x^2 + 5x + 6}$$

$$g(x) = 3 + \sqrt{x-2}$$

   (i) Determine the domain of $f(x)$. (5 marks)

   (ii) Determine the domain and range of $g(x)$. (6 marks)

   (iii) Find an expression for $(f \circ g)(x)$. You do not need to simplify the expression obtained. (4 marks)

   (iv) Find $g^{-1}(x)$ and state its domain and range. (4 marks)

3. (b) The function f is defined as follow:

$$f(x)=\begin{cases}2x+1, & 0\le x<1\\ -2x+5, & 1\le x\le 2\end{cases}$$

(i) Sketch $y=f(x)$. (6 marks)

(ii) State the range of f. (3 marks)

(iii) Explain why f is a function. (2 marks)

4 (a) Find the slope of each of the following curves.

(i) $y=3\sin(2x)+\dfrac{7}{x^3}$ (4 marks)

(ii) $y=\log_5(3x+1)$ (5 marks)

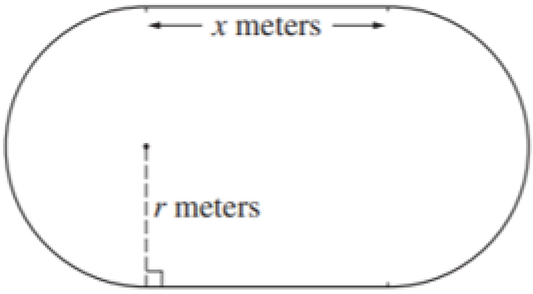
(b) A curve has equation $y=\dfrac{2}{(2x+1)^3}$. Find the equation of the **normal** to the curve at the point where the line $x=0$ intersects the curve. (11 marks)

(c) The diagram shows the first lane of a competitive running track which consists of two straight sections each of length x meters, and two semicircular sections each of radius r meters. The straight sections are perpendicular to the diameters of the semicircular sections. The perimeter of this first lane is 400 meters. Ignore the thickness of the lane.

(i) Show that the area, A m^2, of the region enclosed by the lane is given by

$$A=400r-\pi r^2.$$ (6 marks)

(ii) Given that x and r can vary, show that, when A has a stationary value, there are no straight sections in the track. Determine the stationary value is a maximum or minimum. (9 marks)



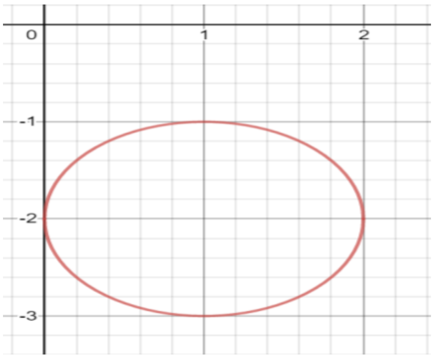
~ End of paper ~

Answers (MST 19/20 S2)

1. (a) \$990 (b) 50 years

2. (a)(i) $\dfrac{dy}{dx}=\dfrac{1}{2e^t(t+1)}$ (ii) $y=\dfrac{1}{2}x-1$ (iii) $y=\ln\left(\ln\left(\dfrac{x}{2}\right)+1\right)$

(b) $(x-1)^2+(y+2)^2=1$; Curve C is a circle with centre(1,-2) and radius of 1 unit.



3. (a) (i) $D_f=\{x : x\neq -3 \text{ or } x\neq -2\}$

$=(-\infty,-3)\cup(-3,-2)\cup(-2,\infty)$ (can leave answer in EITHER notation)

(ii)

$D_g=\{x : x\geq 2\}$

$=[2,\infty)$

$R_g=\{g(x) : g(x)\geq 3\}$

$=[3,\infty)$

(iii) $(f\circ g)(x)=\dfrac{3}{\left(3+\sqrt{x-2}\right)^2+5\left(3+\sqrt{x-2}\right)+6}$

(iv) $g^{-1}(x)=(x-3)^2+2$

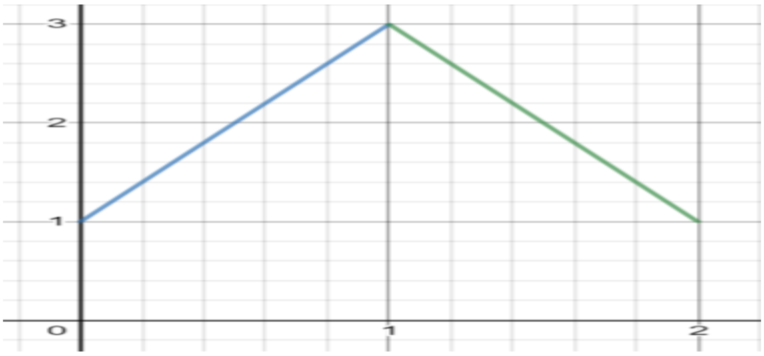
$D_{g^{-1}}=R_g$

$=[3,\infty)$

$R_{g^{-1}}=D_g$

$=[2,\infty)$

3. (b) (i)



(ii) $R_f = [1,3]$

(iii) f passes the vertical line test.

4. (a) (i) $\dfrac{dy}{dx} = 6\cos(2x) - \dfrac{21}{x^4}$ (ii) $\dfrac{dy}{dx} = \dfrac{3}{(3x+1)\ln 5}$

(b) $12y = x + 24$

(c) (ii) stationary point: $r = \dfrac{400}{2\pi}$

$$\frac{d^2A}{dr^2} = -2\pi < 0 \text{ (max)}$$

$$\begin{aligned} x &= \frac{400 - 2\pi r}{2} \\ &= 200 - \pi\left(\frac{400}{2\pi}\right) \\ &= 0 \end{aligned}$$

Since x =0 m when A is max, there are no straight sections.