Singapore Polytechnic, School of Mathematics and Science
Academic Year 2021/2022 Semester 2 **Further Mathematics**
Mid-Semester Test **Duration: 1 hour 40 minutes**

Instructions
1. All SP examination rules are to be complied with.
2. This paper consists of 4 pages.
3. Answer ALL the questions. Unless otherwise stated, leave your answers in 2 decimal places.
4. Except for graphs and diagrams, no solutions are to be written in pencil.

Additional Formulae

Log of a power: $\log_b a^x = x\log_b a$

Completing the square: $x^2 + kx = \left(x+\frac{k}{2}\right)^2 - \left(\frac{k}{2}\right)^2$

Differentiation from first principles: $f'(x) = \lim\limits_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

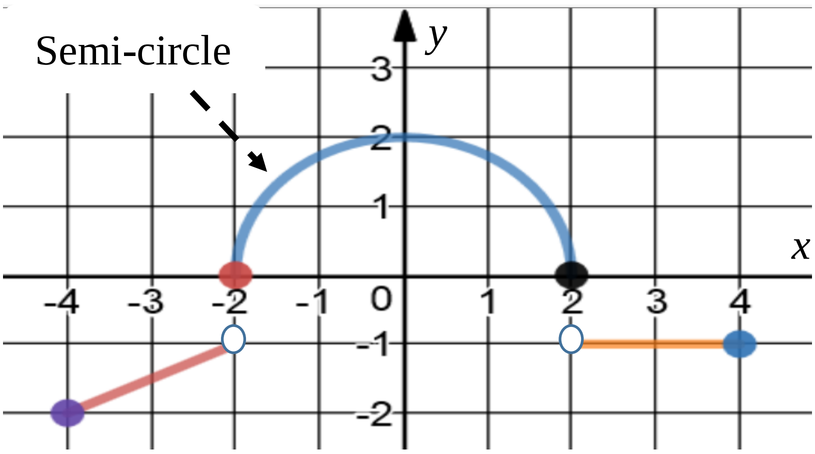
1 (a) A woman deposits \$200 into her son's savings account on his first birthday. On his second birthday she deposits \$225, \$250 on his third birthday, and so on.

(i) How much money would she deposit into her son's account on his 17th birthday? (3 marks)

(ii) How much in total would she have deposited after her son's 17th birthday? (2 marks)

(iii) At which birthday the amount in the saving account first exceed \$20,000? (6 marks)

(b) Write down the piecewise function f that is graphed below. (9 marks)



2. The parametric equations of a curve are

$$x = \frac{2}{t} + 1, \quad y = t^2 + 1$$

(a) Find an expression for $\frac{dy}{dx}$ in terms of t. (4 marks)

(b) Find the coordinates of the point at which the normal to the curve at $x = 2$ meets the y-axis. (6 marks)

(c) Find the values of t at each of the points of intersection of the curve with the line $2y - x(y-1) = 2$. (4 marks)

(d) Find the Cartesian equation of the curve, (4 marks)

(e) From the Cartesian equation, determine the domain and range of the curve. (5 marks)

3. (a) The width, w, of the rectangular LCD monitor to the height, h, is 16 : 9. w and h are measured in inches.

   (i) Express the diagonal length d of the monitor in terms of w only. (3 marks)

   (ii) If the domain of d is $\{w:\ 16 \le w \le 24\}$, find its range. (5 marks)

   (iii) Determine the minimum space to mount the biggest monitor on the wall. (3 marks)

   (b) A function is even if $f(x) = f(-x)$ for all x.

   A function is odd if $f(-x) = -f(x)$ for all x.

   Determine whether the following functions is an odd or even function. Show all workings clearly.

   (i) $f(x) = x^4 - 2x^2 + 1$ (3 marks)

   (ii) $f(x) = \sin(2x)$ (3 marks)

   (ii) $f(x) = \dfrac{e^x - 1}{e^x + 1}$ (5 marks)

   (c) Given $f(x) = \dfrac{e^x - 1}{e^x + 1}$. Find $f^{-1}(x)$ (5 marks)

4. (a) A curve C has equation $y = \dfrac{2x+3}{4x^2+7}$

(i) Find $\dfrac{dy}{dx}$ (5 marks)

(ii) Hence show that y is increasing when $4x^2 + 12x - 7 < 0$.

Hint: For an increasing function, $\dfrac{dy}{dx} > 0$. (3 marks)

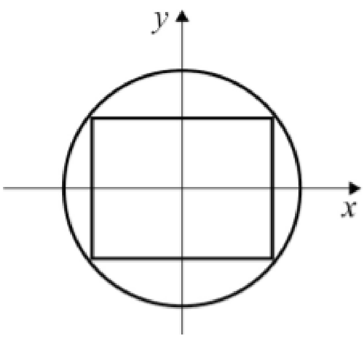
(iii) The equation of tangent to the curve C at $x = 0$ intersects the line $y = x + k$ at x-coordinate, $x = 1$. Find the value of k. (7 marks)

(b) Find the derivative of $f(x) = 2x^2 - x + 5$ from first principles. (5 marks)

(c) The design of the logo of a company is a rectangle inscribed in a circle as shown in the diagram below. The radius of the circle is 5 cm. The company has requested the rectangle must be as large as possible.

Use differentiation to find the maximum area of the rectangle.

You can assume the area you have calculated is a maximum value. That is, you are not required to perform the first or second derivative test.



(10 marks)

- End of Paper -

Answers (MST 21/22 S2)

1. (a) (i) $T_{17} = \$600$ (ii) $S_{17} = \$6,800$ (iii) $n = 33.19$ hence 34th birthday

(b) $f(x) = \begin{cases} \frac{1}{2}x, & -4 \le x < -2 \\ \sqrt{4-x^2}, & -2 \le x \le 2 \\ -1, & 2 < x \le 4 \end{cases}$

2. (a) $\dfrac{dy}{dt} = -t^3$ (b) (0, 19/4) (c) $t = 0$ or $t = 2$

(d) $y = \left(\dfrac{2}{x-1}\right)^2 + 1, \quad D = (-\infty,1) \cup (1,\infty) \quad R = (1,\infty)$

3. (a) (i) $d = \dfrac{\sqrt{337}}{16}w$ (ii) $R_d = \left[\sqrt{337}, \dfrac{3\sqrt{337}}{2}\right]$

(a) (iii) Min. wall space needed is $24 \times 13.5 = 324 \text{ in}^2$

(b) (i) even since
$$\begin{aligned} f(-x) &= (-x)^4 - 2(-x)^2 + 1 \\ &= x^4 - 2x^2 + 1 \\ &= f(x) \end{aligned}$$

(ii) odd since
$$\begin{aligned} f(-x) &= \sin(-2x) \\ &= -\sin(2x) \\ &= -f(x) \end{aligned}$$

(iii) odd since
$$\begin{aligned} f(-x) &= -\left(\frac{e^x - 1}{e^x + 1}\right) \\ &= -f(x) \end{aligned}$$

(c) $f^{-1}(x) = \ln\left(\dfrac{x+1}{1-x}\right)$

4. (a) (i) $\dfrac{dy}{dx} = \dfrac{14 - 24x - 8x^2}{\left(4x^2+7\right)^2}$ (ii) $\dfrac{dy}{dx} = -\dfrac{2\left(4x^2+12x-7\right)}{\left(4x^2+7\right)^2}$

(ii) Since $4x^2 + 12x - 7 < 0$, then $\dfrac{dy}{dx}$ is always > 0 due to the negative sign and the perfect square at the denominator.

(iii) $k = -0.29$

(b) $f'(x) = \lim_{\Delta x \to 0} \left(4x + 2\Delta x - 1\right) = 4x - 1$

(c) Max Area $= 50 \text{ cm}^2$