Chapter 2 - Complex Numbers

2.1 Introduction

Consider the quadratic equation $x^2+2x+2=0$.

To solve, recall that the formula for obtaining the roots of a quadratic equation

$$ax^2+bx+c=0 \text{ is: } x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Applying this formula to the above equation, we have

$$x=\frac{-2\pm\sqrt{2^2-4(1)(2)}}{2(1)}=\frac{-2\pm\sqrt{-4}}{2}$$

$$=\frac{-2\pm\sqrt{4}\sqrt{-1}}{2}=\frac{-2\pm2\sqrt{-1}}{2}=-1\pm\sqrt{-1}$$

The value of $\sqrt{-1}$ cannot be evaluated as a real number. We denote $\sqrt{-1}$ by *j*. Hence the solution to the above equation is $x=-1\pm j$, which are examples of complex numbers in Cartesian or rectangular form.

2.2 Complex Numbers in Cartesian (Rectangular) Form

1 Definition of the Number *j*

$$\boxed{j=\sqrt{-1} \quad \text{and} \quad j^2=-1}$$

Using j, the square root of a negative real number is defined.

That is, for $x>0$, $\sqrt{-x}=\sqrt{-1}\sqrt{x}=j\sqrt{x}$.

Example 1 : Simplify

(a) $\sqrt{-16}$ (b) $\sqrt{-\frac{49}{36}}$ (c) j^3 (d) j^6

2 Definition of a Complex Number

A complex number *z*, in Cartesian or rectangular form is defined as

$$z = x + jy$$

where

x = Re(z) = real part of z
y = Im(z) = the imaginary part of z
$j = \sqrt{-1}$.

Note: both x and y are real numbers.

If $x = 0$ and $y \neq 0$, then $z = jy$, which is called a *pure imaginary number*.
If $x \neq 0$ and $y = 0$, then $z = x$, which is a *real number*.

Example 2 : Find the real and imaginary parts of the following complex numbers.
(a) $z = 2 - j4$ (b) $z = j6$

Example 3 : Given $z = a + 3 + j(1 - b)$.
(a) Find the value of a if z is an imaginary number.
(b) Find the value of b if z is a real number.

2.2.3 The Conjugate of a Complex Number

The conjugate of $z = x + jy$ is the complex number $\bar{z} = x - jy$, with the same real part as z but with imaginary part opposite in sign from that of z.

Example 4 : Write down the complex conjugates of :
(a) $-3 - j4$ (b) $j8$ (c) $-7 + j9$ (d) 6

2.2.4 Equality of Complex Numbers

The complex numbers $x_1 + jy_1$ and $x_2 + jy_2$ are equal if and only if both their real and imaginary parts are equal, i.e.

$$\boxed{x_1 + jy_1 = x_2 + jy_2 \text{ if and only if } x_1 = x_2 \text{ and } y_1 = y_2}$$

Example 5 : Find real values of x and y given
(a) $x + j2 = 5 - jy$ (b) $x + jx + jy = 1 + j3$

2.2.5 Operations of Complex Numbers in Cartesian Form

Let us take a look at the four operations on the following complex numbers:

$$z_1 = x_1 + jy_1 \quad \text{and} \quad z_2 = x_2 + jy_2$$

where x_1, y_1, x_2 and y_2 are real numbers.

(a) Addition

$$\begin{aligned} z_1 + z_2 &= (x_1 + jy_1) + (x_2 + jy_2) \\ &= (x_1 + x_2) + j(y_1 + y_2) \end{aligned}$$

Example 6 : Simplify (a) $j3 + j7$ (b) $(2 - j5) + (-3 + j4)$

(b) Subtraction

$$\begin{aligned} z_1 - z_2 &= (x_1 + jy_1) - (x_2 + jy_2) \\ &= (x_1 - x_2) + j(y_1 - y_2) \end{aligned}$$

Example 7 : Simplify $(6 - j2) - (4 + j5)$

(c) Multiplication

$$\begin{aligned} z_1 z_2 &= (x_1 + jy_1)(x_2 + jy_2) \\ &= x_1x_2 + jx_1y_2 + jy_1x_2 + j^2 y_1y_2 \qquad \left(\text{Note : } j^2 = -1\right) \\ &= x_1x_2 + jx_1y_2 + jy_1x_2 - y_1y_2 \\ &= (x_1x_2 - y_1y_2) + j(x_1y_2 + y_1x_2) \end{aligned}$$

Example 8 : Find (a) $(3 + j2)(5 - j3)$ (b) $(2 + j3)^2$

Note: The product of a complex number $z = x + jy$ and its conjugate $\overline{z} = x - jy$ is always a real number.

$$\boxed{\begin{aligned} z\overline{z} &= (x + jy)(x - jy) = x^2 - j^2y^2 \\ &= x^2 + y^2 \end{aligned}}$$

Example 9 : Simplify $(3 + j4)(3 - j4)$.

(d) Division

To obtain the quotient of two complex numbers, the numerator and denominator are multiplied by the conjugate of the denominator.

$$\begin{aligned} \frac{z_1}{z_2} = \frac{x_1 + jy_1}{x_2 + jy_2} &= \frac{x_1 + jy_1}{x_2 + jy_2} \cdot \frac{x_2 - jy_2}{x_2 - jy_2} \\ &= \frac{x_1x_2 - jx_1y_2 + jy_1x_2 - j^2y_1y_2}{(x_2)^2 + (y_2)^2} \qquad \left(\text{Note : } j^2 = -1\right) \\ &= \frac{x_1x_2 + y_1y_2}{(x_2)^2 + (y_2)^2} + j\frac{y_1x_2 - x_1y_2}{(x_2)^2 + (y_2)^2} \end{aligned}$$

Example 10 : Simplify $\dfrac{2 + j3}{-1 - j}$.

2.2.6 Applications

Example 11 : Solve the following equations for complex number X.

(a) $3X^2 + X - 1 = 0$

(b) $3X^2 + X + 1 = 0$

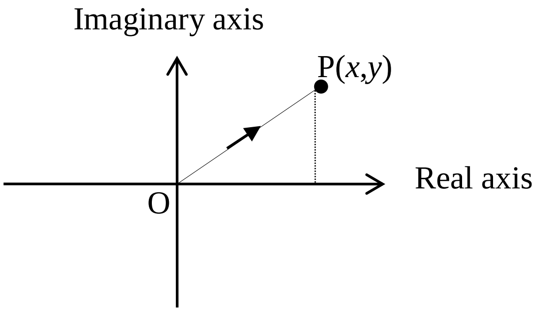
(c) $X^2 + X + j2X + j = 0$

(d) $\dfrac{X+3}{-1-j} = X - 1$

(e) $X\overline{X} + \overline{X} = 2X - j$

2.2.7 The Argand Diagram

A complex number can be plotted using a rectangular coordinate system in which the horizontal axis is the real axis and the vertical axis the imaginary axis. Such a coordinate system defines what is called the **complex plane** (or **Argand diagram**). To plot a complex number $x + jy$ in the complex plane, simply locate a point with a horizontal coordinate of x and a vertical coordinate of y.

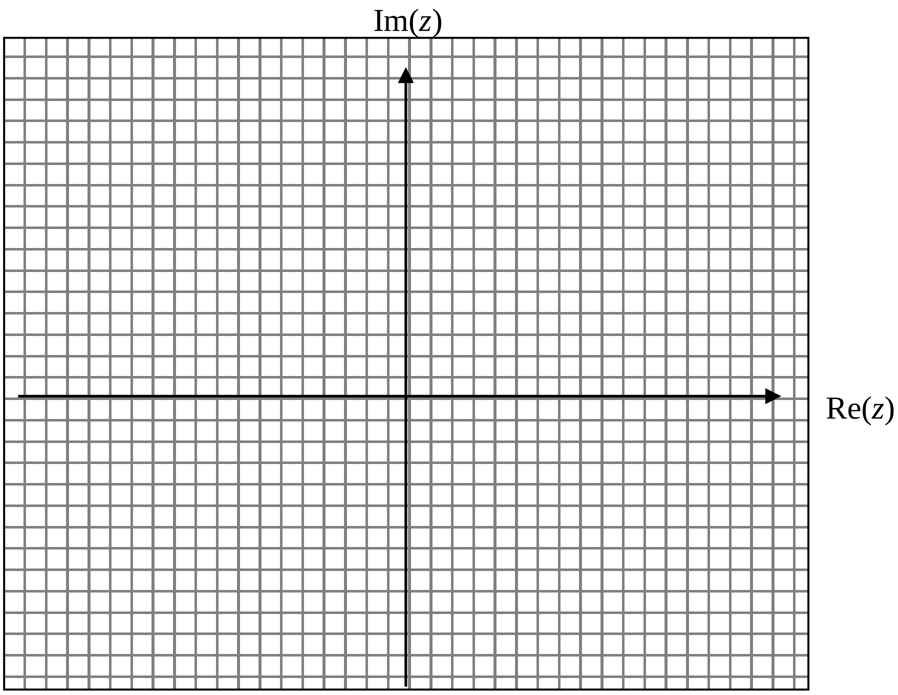


The complex number $\boldsymbol{z}$ is represented by the ***point P*** or the ***vector OP***.

Example 12 : Given $Z = -3 + j\,7$.
Plot each of the following complex numbers on the Argand diagram:
(a) Z (b) jZ (c) $-jZ$ (d) $\overline{Z}$

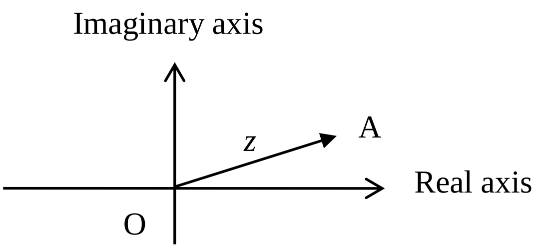
Explain the effect of j multiply to the complex number Z.
How do we plot $\overline{Z}$ from Z ?

Solution

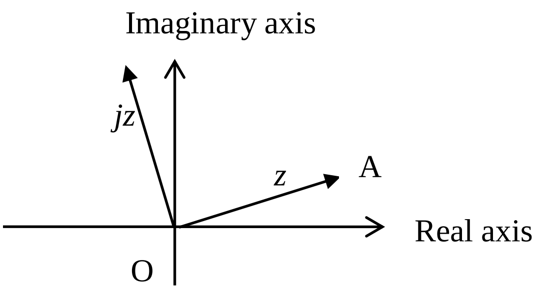


2.2.8 The j - operator

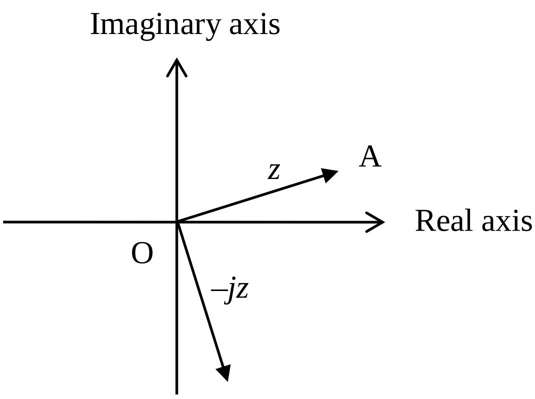
Let's consider a complex number z represented by the vector OA .



- When z is multiplied by j, the vector OA is rotated 90° in the ***anticlockwise*** direction.

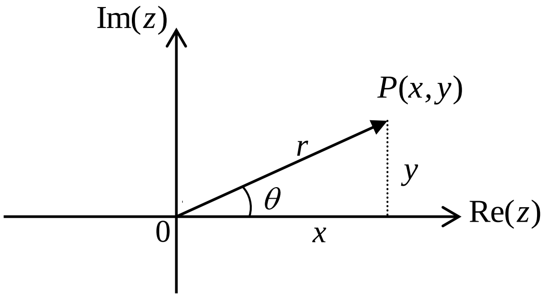


- When z is multiplied by $-j$, the vector OA is rotated 90° in the **clockwise** direction.



2.3 The Polar Form of a Complex Number

Let a complex number $z = x + jy$ be represented by the point $P(x, y)$ in the Argand diagram.



Connect the point P to the origin with an arrow line of length r making an angle of θ with the positive real axis.

Then r, also denoted by $|z|$, is called the **magnitude** or **modulus** of z.

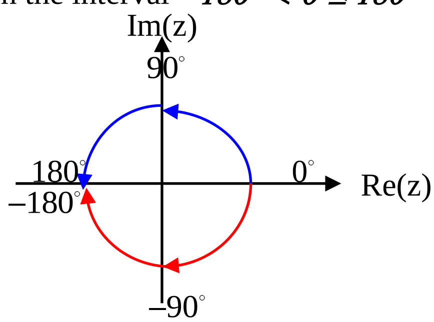
Using the Pythagoras Theorem, $\boxed{|z| = r = |x + jy| = \sqrt{x^2 + y^2}}$

The angle θ is called the **argument** of the complex number z and is written as $\arg(z)$.

$$\boxed{\arg(x + jy) = \arg(\text{z}) = \theta \quad , \quad \text{where } \tan\theta = \frac{y}{x}}$$

The principal values of θ lie in the interval $-180° < \theta \le 180°$ (or $-\pi < \theta \le \pi$)

In finding arg(z):

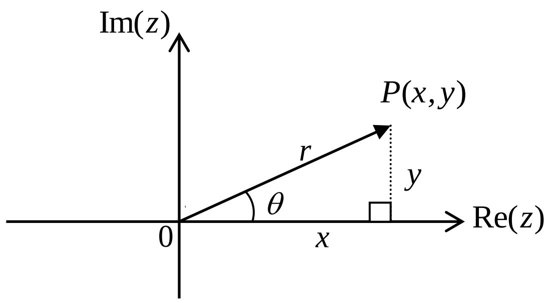


When an angle is measured in the <u>anticlockwise</u> direction, the angle will be <u>positive</u>. When an angle is measured in the **clockwise** direction, the angle will be **negative**.

Try these:

Find the principal values of the following angles:

(a) $135°$ (b) $280°$ (c) $200°$



Referring to the diagram above again, $x = r\cos\theta$ and $y = r\sin\theta$

Therefore, $z = x + jy$ (rectangular form of the complex number)

$= r\cos\theta + j\,r\sin\theta$

$= r\left(\cos\theta + j\sin\theta\right)$ (trigonometric form of the complex number)

$= r\angle\theta$ or $|z|\angle\theta$ (polar form of the complex number)

2.3.1 Conversions (Rectangular Form to Polar Form and Vice Versa)

(a) Polar Form to Rectangular Form

Example 13 : (a) Write $z = 4\angle -150°$ in rectangular form.

(b) Given that $|z| = 4$ and $\arg(z) = \pi/3$, write z in rectangular form.

(b) Rectangular Form to Polar Form

Assuming a complex number, $z = x + jy$

Step (i) : Find the <u>Magnitude</u> of the complex number, $|z|$.

Using the Pythagoras Theorem, $|z| = r = |x + jy| = \sqrt{x^2 + y^2}$

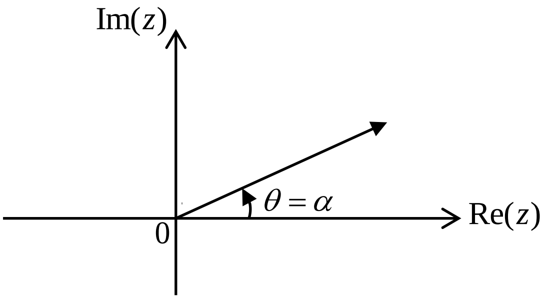
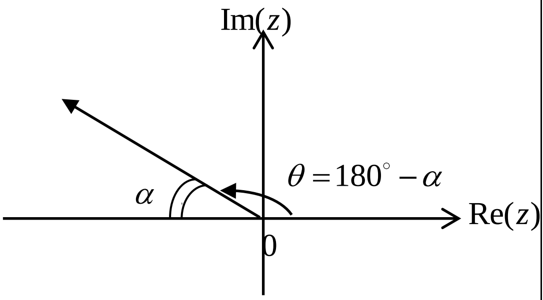
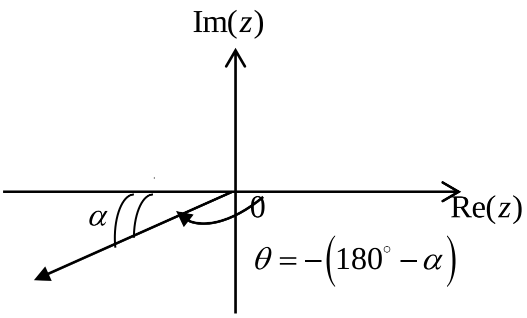
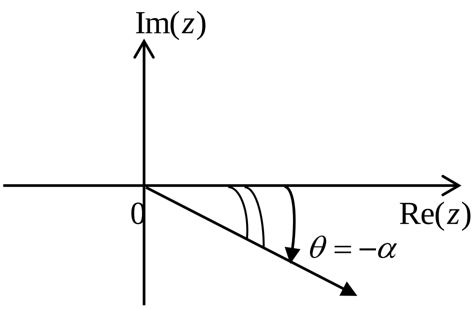
Step (ii) : Find the <u>Argument</u> of the complex number, θ.

The argument of a complex number depends on the location of the complex number in the complex plane.

To find θ, we first have to find the basic or reference angle α, where

$$\alpha = \tan^{-1}\frac{|y|}{|x|}.$$

Then, draw the complex number in rectangular form in an argand diagram to help visualisation. The complex number may be in quadrant I, II, III or IV. In the different quadrants, the argument θ is then found differently, as seen in the diagram below.

If it is in Quadrant I,	If it is in Quadrant II,
	
If it is in Quadrant III,	If it is in Quadrant IV,
	

Step (iii) : Write z in polar form , $z = r\angle\theta$ or $|z|\angle\theta$

Example 14 : Convert the following complex numbers to polar forms.

(a) $z = 1 - j$ (b) $z = -1 + j$ (c) $z = -1 - j$

Example 15 : Given $z = 1 - j2$. Evaluate the following.

(a) $| z + j |$ (b) $\arg(z - 3)$

Example 16 : Given $z = 2 + jy$ and $| z + 2 | = 5$, find the value(s) of the real number y.

.

Example 17 : Given $z = x + j2$. Find the value(s) of the real number x if

(a) $\arg(z - j) = \dfrac{\pi}{3}$ (in Quadrant I) (b) $\arg(z - j3) = -\dfrac{2\pi}{3}$

2.3.2 Operations of Complex numbers in Polar Form

(a) Addition and Subtraction

Addition and subtraction of complex numbers can only be done in **rectangular form**.

(b) Multiplication

Let $z_1 = r_1\angle\theta_1$ and $z_2 = r_2\angle\theta_2$

$$
\begin{aligned}
z_1 z_2 &= \left(r_1\angle\theta_1\right)\left(r_2\angle\theta_2\right) = r_1\left(\cos\theta_1 + j\sin\theta_1\right) \times r_2\left(\cos\theta_2 + j\sin\theta_2\right) \\
&= r_1 r_2\left(\cos\theta_1\cos\theta_2 + j\sin\theta_1\cos\theta_2 + j\cos\theta_1\sin\theta_2 + j^2\sin\theta_1\sin\theta_2\right) \\
&= r_1 r_2\left[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + j\left(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2\right)\right] \\
&= r_1 r_2\left[\cos\left(\theta_1+\theta_2\right) + j\sin\left(\theta_1+\theta_2\right)\right] \\
&= r_1 r_2\angle\left(\theta_1+\theta_2\right)
\end{aligned}
$$

Hence $\boxed{\left(r_1\angle\theta_1\right)\left(r_2\angle\theta_2\right) = r_1 r_2\angle\left(\theta_1+\theta_2\right)}$

To multiply two complex numbers in polar form, multiply their moduli and add their arguments.

Example 18 : Simplify the following and express the answers polar forms.

(a) $4(\cos 125^\circ + j\sin 125^\circ) \times 2(\cos 75^\circ + j\sin 75^\circ)$

(b) $\left(2\angle\dfrac{\pi}{6}\right)\left(12\angle\dfrac{\pi}{12}\right)$

(c) Division

Let $z_1 = r_1\angle\theta_1$ and $z_2 = r_2\angle\theta_2$

$$\frac{z_1}{z_2} = \frac{r_1\angle\theta_1}{r_2\angle\theta_2} = \frac{r_1\left(\cos\theta_1 + j\sin\theta_1\right)}{r_2\left(\cos\theta_2 + j\sin\theta_2\right)} = \frac{r_1\left(\cos\theta_1 + j\sin\theta_1\right)}{r_2\left(\cos\theta_2 + j\sin\theta_2\right)} \times \frac{\left(\cos\theta_2 - j\sin\theta_2\right)}{\left(\cos\theta_2 - j\sin\theta_2\right)}$$

$$= \frac{r_1}{r_2} \cdot \frac{\cos\theta_1\cos\theta_2 + j\sin\theta_1\cos\theta_2 - j\cos\theta_1\sin\theta_2 - j^2\sin\theta_1\sin\theta_2}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$= \frac{r_1}{r_2} \cdot \frac{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + j\left(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2\right)}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + j\sin(\theta_1 - \theta_2)]$$

$$= \frac{r_1}{r_2}\angle(\theta_1 - \theta_2)$$

Hence $$\boxed{\frac{z_1}{z_2} = \frac{r_1\angle\theta_1}{r_2\angle\theta_2} = \frac{r_1}{r_2}\angle(\theta_1 - \theta_2), \qquad z_2 \neq 0}$$

To divide two complex numbers in polar form, divide their moduli and subtract their arguments.

Example 19 : Simplify (a) $\dfrac{9(\cos 82^\circ + j\sin 82^\circ)}{3(\cos 12^\circ + j\sin 12^\circ)}$ (b) $\dfrac{2+j}{3-j5}$

Example 20 : Simplify the following expressions and leave your answers in rectangular forms.

(a) $\dfrac{1}{2\angle 35^\circ} + \dfrac{2}{3\angle(-75^\circ)}$ (b) $\dfrac{100\angle 125^o}{25\angle 45^o} - 15\angle 36^o$

2.4 The Exponential Form of a Complex Number

We have already expressed a complex number z in

$$z = x + jy \qquad \text{Cartesian (rectangular) form}$$
$$= r(\cos\theta + j\sin\theta) \qquad \text{trigonometric form}$$
$$= r\angle\theta \qquad \text{polar form}$$

Using Euler's Formula: $\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$

$$z = r(\cos\theta + j\sin\theta) = r\,e^{j\theta}$$

which is known as the exponential form of the complex number.

As before, $r = |z|$ is the modulus of z, and
θ is the argument expressed in **radians**, $-\pi < \theta \le \pi$

Since $e^{-j\theta} = \cos(-\theta) + j\sin(-\theta) = \cos\theta - j\sin\theta$,

therefore, if $z = r(\cos\theta + j\sin\theta) = re^{j\theta}$
then $\bar{z} = r(\cos\theta - j\sin\theta) = re^{-j\theta}$

Recap: Radians vs. Degrees

Recall that π radians $= 180^\circ$. Hence 1 radian $= \dfrac{180^\circ}{\pi}$ and $1^\circ = \dfrac{\pi}{180}$.

Try these:

Convert the following angles from radians to degrees.
(a) 2π
(b) -3

Convert the following angles from degrees to radians.
(a) 135°
(b) -45°

Example 21 : Express the following expressions in exponential forms.
(a) $-1 + j$ (b) $12\angle(-14^\circ)$ (c) $3(\cos 50^\circ - j\sin 50^\circ)$

Example 22 : Express the following expressions in rectangular forms.

(a) $4e^{-j\frac{\pi}{3}}$ (b) $e^{1+j0.2}$ (c) $e^{\frac{1}{2}-j3}$

2.4.1 Operations of Complex Numbers in Exponential Form

Products, quotients and powers of complex numbers in exponential form are obtainable by using the laws of exponents.

(a) Multiplication

$$\boxed{r_1 e^{j\theta_1} \cdot r_2 e^{j\theta_2} = r_1 r_2 e^{j(\theta_1+\theta_2)}}$$

Example 23 : Simplify $7e^{j3} \times 3e^{j2}$

(b) Division

$$\boxed{\frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1-\theta_2)}}$$

Example 24 : Simplify $\dfrac{12e^{j2.5}}{3e^{j1.7}}$

Example 25 **:** Express $-je^{j2\theta}(1-e^{-j4\theta})$ in terms of sine only.

2.5 De Moivre's Theorem

De Moivre's theorem can be used to find powers of complex numbers. The theorem involves raising $\cos\theta + j\sin\theta$ to the power *n*.

De Moivre's Theorem

$(\cos\theta + j\sin\theta)^n = \cos n\theta + j\sin n\theta$ where *n* is an integer

(De Moivre's theorem for fractional powers will not be discussed here.)

Since any complex number *z* can be expressed as $z = r(\cos\theta + j\sin\theta) = r\angle\theta = re^{j\theta}$, by using De Moivre's theorem,

$$
\begin{aligned}
z^n &= \left[r(\cos\theta + j\sin\theta)\right]^n = (r\angle\theta)^n = \left(re^{j\theta}\right)^n \\
&= r^n(\cos n\theta + j\sin n\theta) = r^n\angle n\theta = r^n e^{jn\theta}
\end{aligned}
$$

Example 26 : Simplify the following expressions and leave your answers in rectangular forms.

(a) $(5\angle 70°)^3$ (b) $\left[3(\cos 10° + j\sin 10°)\right]^4$ (c) $(2 + j3)^4$

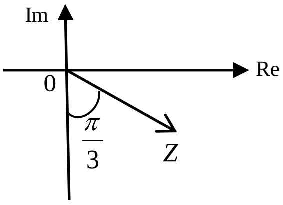
(d) $\left(5e^{-j2}\right)^3$

Example 27 : Given $Z_1 = 1 + j$, $Z_2 = 2\angle 1$ and $Z_3 = j$.

Evaluate $\dfrac{Z_1^4}{Z_2 - Z_3}$ and leave your answer in polar form.

<u>Tutorial</u> : Multiple Choice Questions

1. The complex number Z is shown in the Argand diagram below.

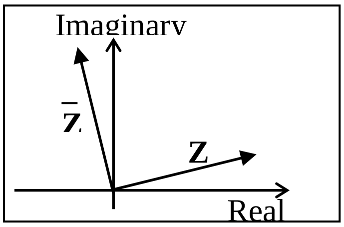


The principal argument of $\overline{Z}$ is:

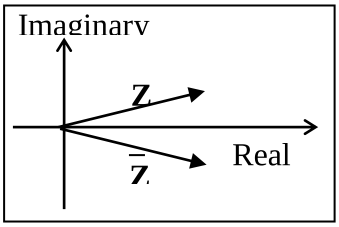
(a) $\frac{\pi}{6}$ (b) $-\frac{\pi}{6}$ (c) $\frac{\pi}{3}$ (d) $-\frac{\pi}{3}$

2. Which of the following diagrams is **CORRECT** ?

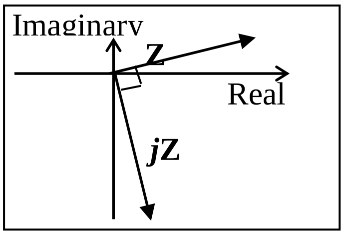
(a)



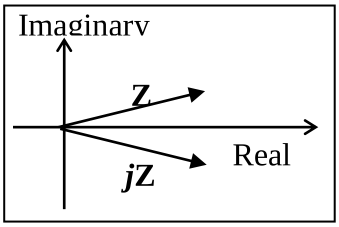
(b)



(c)



(d)



3. Given $Z = a + jb$, where a and b are not equal to zero, and $j = \sqrt{-1}$.
Which of the followings is TRUE ?

(a) $\arg(-jZ) = \arg(Z)$ (b) $\arg(jZ) = \arg(Z)$
(c) Z^2 is a real number (d) $|\,jZ\,| = |\,Z\,|$

4. Given the complex number $Z = r\angle\theta$, where $r > 0$ and $0 < \theta < 90^{\circ}$.
If $j = \sqrt{-1}$ then

(a) $\arg(jZ) = -\theta$ (b) $\arg(jZ) = \theta$
(c) $\arg(jZ) = \theta - 90^{o}$ (d) $\arg(jZ) = \theta + 90^{o}$

5. If $\overline{Z}$ is the conjugate of $Z = a + jb$, where a and b are not equal to zero.
Which of the followings is TRUE ?

(a) $|\,\overline{Z}\,| = |\,Z\,|$ (b) $\arg(\overline{Z}) = \arg(Z)$
(c) $\dfrac{\overline{Z}}{Z}$ is a real number (d) $\overline{Z}\,Z = 0$

Tutorial 2a (Complex Numbers in Rectangular Form)

1. Simplify the following and leave your answers in $x + jy$ form :

(a) $(2 - j) + (5 + j4) - (3 - j2)$ (b) $(4 + j3) - (2 - j5) + j3$ (c) $4j^3 - j^5 + 2j^2 - j^8$

(d) $j^2\, 3(j^5 - j4 + 8)$ (e) $(6 + j5)(-3 + j2)$ (f) $(3 + j2)(5 - j3)$

(g) $\dfrac{3 - j4}{5 + j2}$ (h) $\dfrac{2 - j7}{1 + j}$ (i) $\sqrt{-4} + 2\sqrt{9} - \sqrt{-25}$

2. Given that $z_1 = 3 + j2$ and $z_2 = 2 - j$, evaluate the following :

(a) $\overline{z_1 + z_2}$ (b) $\overline{z_1 z_2}$ (c) $\text{Re}(z_1)\text{Im}(z_2)$ (d) $\dfrac{\text{Im}(\overline{z_1})}{\text{Re}(\overline{z_2})}$

3. If $z = -1 + j3$, plot and label the given points on the same Argand diagram.

(a) z (b) jz (c) $-jz$ (d) $j^2 z$

4. Given $z_1 = 3 + j2$ and $z_2 = 4 - j$, plot the given points on the same Argand diagram.

(a) z_1 and $\overline{z_1}$ (b) z_2 and $\overline{z_2}$

5. Solve the following equations for real values of x and y :

(a) $2x - jy = \dfrac{1}{4 - j}$ (b) $x + jy = (2 + j3)(3 - j4)$

(c) $x - jy = \dfrac{3 + j2}{j}$ (d) $3x + 2 = j(y + 2)$

(e) $2 + x + jy = (x - jy)(5 + j6)$ (f) $\dfrac{4}{x + jy} + \dfrac{2}{x - jy} = j3$

6. Solve the following for the complex number z :

(a) $z^2 + 4z + 8 = 0$ (b) $z^2 + 36 = 0$

7. Solve the following for complex number z :

(a) $\dfrac{z + j}{z + j2} = \dfrac{1 + j}{3 + j2}$ (b) $z\bar{z} + j2z = 12 + j6$

8. Given the matrix $A = \begin{pmatrix} x - 1 & j \\ j2y - jx & j \end{pmatrix}$, where $j = \sqrt{-1}$.

Find the real values of x and y such that matrix A is a singular matrix.

*9. The impedance Z of a parallel circuit is given by $\dfrac{1}{Z} = \dfrac{1}{R + j\omega L} + j\omega C$,

where $R = 1$ ohm , $L = 2$ henries , $C = 0.5$ farad and $\omega > 0$.

What is the value of ω for resonance (i.e. when Z is real) ?

*10. The characteristic impedance Z of a transmission line is given by $Z^2 = \dfrac{R + j\omega L}{G + j\omega C}$, where R , w , L , G and C are real constants. If Z^2 is real, find L in terms of R , C and G.

*11. In the relationship $(R_1 + j\omega L)\left(R_2 - \dfrac{j}{\omega C}\right) = \dfrac{R_3}{R_4}$, all quantities are real except j . Show that $\omega = \sqrt{\dfrac{R_1}{R_2 LC}}$ and express R_1 in terms of C, L, R_2, R_3 and R_4 .

Tutorial 2b (Complex Numbers in Polar Form)

1. Convert the following complex numbers to polar form:

   (a) $2 + j2$ (b) $-4 + j4$ (c) $3 - j3$ (d) $-3 - j3\sqrt{3}$

2. Convert the following complex numbers to rectangular form:

   (a) $2\angle 5°$ (b) $1\angle(-150°)$ (c) $10\angle\left(\dfrac{\pi}{2}\right)$ (d) $5\angle(-1.5)$

3. Simplify the following and express the answer in polar form:

   (a) $(3\angle 2.5)(2\angle -0.27)(0.5\angle 1.97)$ (b) $\dfrac{4\angle(-3)}{(6\angle 2)(5\angle(-1.9))}$

   (c) $(1.2 - j6.6)(-2.5 + j3.9)(j5)$ (d) $5\angle 35° + 1\angle 65° - 2\angle -30°$

4. When two elements of an electrical circuit are connected in parallel, the total impedance Z of the circuit is given by

$$Z = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

where Z_1 and Z_2 are the impedance of the two elements. Given that $Z_1 = 2 - j3$ and $Z_2 = 3 + j4$, find the total impedance Z in polar form.

5. (a) Let $z_1 = 3 + jy$. Draw z_1 on Argand diagram and rewrite z_1 in polar form if :
   (i) $y > 0$ (ii) $y < 0$

   (b) Let $z_2 = -3 + jy$. Draw z_2 on Argand diagram and rewrite z_2 in polar form if :
   (i) $y > 0$ (ii) $y < 0$

   Leave the answers in terms of x or y.

6. If $z_1 = -1 + j5$ and $z_2 = 2 - j3$, find $|z_1 - z_2|$ and $\arg(z_1 + z_2)$.

7. Given $z = 3 + jy$ and $\arg(z + j2) = 45°$, find the value(s) of the real number y.

8. Solve the following question without using calculator.

If $z = 1 + j\sqrt{3}$, $w = 1 + j$

(a) Calculate $\frac{z}{w}$, express your answers in both rectangular and trigonometric forms and keeping them in the surd form too.

(b) Hence, use your answers in part (a) to deduce the exact value of $\cos 15°$.

*9. Given that $|z - 3| = 4$ and the argument of z is $\frac{\pi}{4}$, find z in Rectangular form.

Tutorial 2c (Complex Numbers in Exponential Form)

1. Convert the following complex numbers to exponential form :

(a) $5 - j3$ (b) $5.2 \angle\left(\frac{\pi}{2}\right)$ (c) $9(\cos 2.5 + j\sin 2.5)$ (d) $\sqrt{2} + j$

2. Convert the following complex numbers to rectangular form :

(a) $3.8e^{-j1.3}$ (b) $6e^{j\pi}$ (c) $e^{2.5 - j2.2}$

3. Simplify the following and express the answer in exponential form :

(a) $3e^{-j\frac{\pi}{6}}(\sqrt{2} + j7)$ (b) $2e^{-j} - 2\angle -\frac{\pi}{3}$ (c) $\frac{1 - j4 - 2\angle 1}{e^{-j}}$

4. Solve the equation $\frac{Z+1}{Z+2-j} = j$ for complex number Z. Express Z in exponential form.

5. In finding the Thevenin equivalent for a complex circuit, the open-circuit voltage V_0 must be found, where

$$V_0 = V_2 + \frac{Z_2}{Z_1 + Z_2}(V_1 - V_2)$$

Given that $V_1 = 2\angle 0$, $V_2 = e^{j\frac{\pi}{4}}$, $Z_1 = 1 + j2$ and $Z_2 = 2 + j3$, find V_0 in exponential form.

6. Given that $2e^{-j\frac{\pi}{2}} - 3e^{j\pi} = a + jb$, solve for a and b which are both real.

7. Express $3\left(\frac{e^{j4\omega\theta} + e^{-2j\omega\theta}}{e^{j\omega\theta}}\right)$ in terms of cosine only.

Tutorial 2d (De Moivre's Theorem)

1. Evaluate and express your answers in rectangular form.

   (a) $\left[2\left(\cos 15° + j\sin 15°\right)\right]^3$ (b) $\left(3\angle 50°\right)^6$ (c) $\left(2e^{j\frac{\pi}{24}}\right)^4$

2. Simplify the following and express the answer in polar form:

   (a) $\left(3\cos\frac{\pi}{4} - j3\sin\frac{\pi}{4}\right)^5$ (b) $\dfrac{(\sqrt{3}\ -j)^3\ -j}{2\left(\cos\frac{\pi}{6} - j\sin\frac{\pi}{6}\right)}$

3. Given that $Z_1 = (\sqrt{3} - j)^6$ and $Z_2 = (-1 - j\sqrt{3})^6$, express $Z_1 - Z_2$ in the form $r\angle\theta$ and determine the values of r and θ.

*4. Given that $Z = 2^{1/3}\left(\cos\frac{\pi}{4} - j\sin\frac{\pi}{4}\right)$, find the real numbers a and b if $e^{a+jb} = Z^3 + \dfrac{1}{Z^3}$

*5. Simplify the following expression and leave your answer in rectangular form:

$$\frac{(1-j4)^3}{3e^{-j1.74} + 2\angle 2}$$

ANSWERS

Eg 1 : (a) $j4$ (b) $j7/6$ (c) $-j$ (d) -1
Eg 2 : (a) $Re(z) = 2$, $Im(z) = -4$ (b) $Re(j6) = 0$, $Im(j6) = 6$
Eg 3 : (a) $a = -3$ (b) $b = 1$
Eg 4 : (a) $-3 + j4$ (b) $-j8$ (c) $-7 - j9$ (d) 6
Eg 5 : (a) $x = 5$, $y = -2$ (b) $x = 1$, $y = 2$
Eg 6 : (a) $j10$ (b) $-1 - j$
Eg 7 : $2 - j7$
Eg 8 : (a) $21 + j$ (b) $-5 + j12$
Eg 9 : 25
Eg 10 : $\frac{1}{2}(-5 - j)$
Eg 11 : (a) $X = 0.434$ or -0.768 (b) $X = \frac{1}{6}\left(-1 \pm j\sqrt{11}\right)$

(c) $X = -0.5 - j1.866$ or $-0.5 - j0.134$ (d) $X = \frac{1}{5}(-3 + j4)$

(e) $X = 0.127 + j\frac{1}{3}$ or $0.873 + j\frac{1}{3}$

Eg 13 : (a) $-3.46 - j2$ (b) $2 + j2\sqrt{3}$
Eg 14 : (a) $\sqrt{2}\angle -45^o$ (b) $\sqrt{2}\angle 135^o$ (c) $\sqrt{2}\angle -135^o$
Eg 15 : (a) $\sqrt{2}$ (b) $-3\pi/4$
Eg 16 : ± 3
Eg 17 : (a) $\frac{1}{\sqrt{3}}$ (b) $-\frac{1}{\sqrt{3}}$

Eg 18 : (a) $8\angle(-160°)$ (b) $24\angle\frac{\pi}{4}$

Eg 19 : (a) $3\angle 70°$ (b) $\sqrt{\frac{5}{34}}\angle 85.6°$

Eg 20 : (a) $0.5821 + j0.3572$ (b) $-11.44 - j4.88$

Eg 21 : (a) $\sqrt{2}\, e^{j3\pi/4}$ (b) $12\, e^{-j0.2443}$ (c) $3e^{-j5\pi/18}$

Eg 22 : (a) $2 - j3.464$ (b) $2.66 + j0.540$ (c) $-1.63 - j0.233$

Eg 23 : $21e^{-j1.2832}$

Eg 24 : $4e^{j0.8}$

Eg 25 : $2\sin 2\theta$

Eg 26 : (a) $-108.25 - j62.5$ (b) $62.05 + j\,52.1$ (c) $-119 - j120$
(d) $120.02 + j\,34.9$

Eg 27 : $3.129\angle 2.578$

Multiple Choice Questions

1. a 2. b 3. d 4. d 5. A

Tutorial 2a

1.(a) $4 + j5$ (b) $2 + j11$ (c) $-3 - j5$ (d) $-24 + j9$
(e) $-28 - j3$ (f) $21 + j$ (g) $\frac{7}{29} - j\frac{26}{29}$ (h) $-\frac{1}{2}(5 + j9)$
(i) $6 - j\,3$

2.(a) $5 - j$ (b) $8 - j$ (c) -3 (d) -1

5.(a) $x = \frac{2}{17},\ y = -\frac{1}{17}$ (b) $x = 18,\ \ y = 1$ (c) 2 , 3 (d) –2/3 , –2
(e) 1/5 , 1/5 (f) 0 , – 2/3

6.(a) $z = -2 + j2,\ z = -2 - j2$ (b) $\pm j6$

7.(a) $-\frac{1}{5}(1 + j2)$ (b) $3(1 + j)\,,\ 3 - j$ 8. $x = 1\ ,\ y = ½$

9. $\frac{\sqrt{3}}{2}$ 10. $L = RC/G$ 11. $R_1 = \frac{R_3C - LR_4}{R_2R_4C}$

Tutorial 2b

1.(a) $2\sqrt{2}\angle 45°$ (b) $4\sqrt{2}\angle 135°$ (c) $3\sqrt{2}\angle(-45°)$ (d) $6\angle\left(-\frac{2\pi}{3}\right)$

2.(a) $1.99 + j0.174$ (b) $-\frac{\sqrt{3}}{2} - j\frac{1}{2}$ (c) $j10$ (d) $0.354 - j\,4.987$

3.(a) $3\angle(-2.083)$ (b) $0.133\angle(-3.1)$ (c) $155.357\angle 132.96°$ (d) $5.528\angle 59.72°$

4. $3.54\angle(-0.253)$ 5(a) (i) $\sqrt{9 + y^2}\angle\left(\tan^{-1}\left|\frac{y}{3}\right|\right)$ (ii) $\sqrt{9 + y^2}\angle\left(-\tan^{-1}\left|\frac{y}{3}\right|\right)$

5(b) (i) $\sqrt{9+y^2}\angle\left(180^o-\tan^{-1}\left|\frac{y}{3}\right|\right)$ (ii) $\sqrt{9+y^2}\angle\left(-\left(180^o-\tan^{-1}\left|\frac{y}{3}\right|\right)\right)$

6. (a) $\sqrt{73}$ 6(b) 63.4^o or 1.107 7. $y = 1$

8.(a) $\frac{1+\sqrt{3}}{2}+j\frac{\sqrt{3}-1}{2}$; $\sqrt{2}\cos 15°+j\sqrt{2}\sin 15°$ 8(b) $\frac{1+\sqrt{3}}{2\sqrt{2}}$ 9. $3.9(1+j)$

Tutorial 2c

1.(a) $5.83e^{-j0.54}$ (b) $5.2e^{j\frac{\pi}{2}}$ (c) $9e^{j2.5}$ (d) $1.73e^{j0.615}$
2.(a) $1.016-j3.662$ (b) -6 (c) $-7.169-j9.850$

3.(a) $21.424e^{j0.848}$ (b) $0.0944e^{j0.547}$ (c) $5.684e^{-j0.585}$

4. $\sqrt{2}\,e^{j\frac{3\pi}{4}}$ 5. $1.50e^{j0.155}$ 6. $a=3,\ b=-2$ 7. $6\cos 3w\theta$

Tutorial 2d

1. (a) $4\sqrt{2}+j4\sqrt{2}$ (b) $\frac{729}{2}-j\frac{729\sqrt{3}}{2}$ (c) $8\sqrt{3}+j8$
2. (a) $243\angle\left(\frac{3\pi}{4}\right)$ (b) $4.5\angle\left(-\frac{\pi}{3}\right)$
3. $128\angle\pi$
4. $a = 0.723$, $b = -2.601$ 5. $1.18-j39.89$