Chapter 6 – Applications of The Derivatives

Objectives:

1. *Solve application problems on equation of a tangent line, rates of change and related rates of change.*
2. *Explain the stationary points on a curve.*
3. *Distinguish between relative maximum and minimum points.*
4. *Solve optimisation problems.*

6.1 Introduction

In this Chapter, we will look at some common applications of differentiation. We will solve problems on equation of tangent line, rates of change and related rates of change. We will use the derivatives to locate the extreme values of a function, and apply the concept to solve optimisation problems.

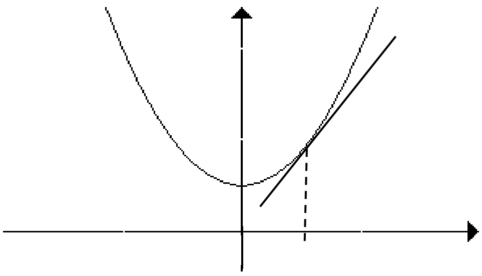
6.2 Equation of a tangent line

Recall that the derivative $\frac{dy}{dx}$ is the gradient of the tangent to a curve at any point.

Example 1 : Given the function $f(x) = 3x^2 + 1$.

(a) Calculate the slope of the tangent line at x = 2 .

(b) Find the equation of the tangent line at the given point.



6.3 Rates of Change

Consider a function $y = f(x)$.

The instantaneous rate of change of y with respect to (w.r.t.) x at some point **P** is defined as $\frac{dy}{dx}$, i.e. the derivative of y w.r.t. x.

If $\frac{dy}{dx}$ has a positive value, then we will have a rate of increase of y w.r.t x (Figure 1).

If $\frac{dy}{dx}$ has a negative value, then we will have a rate of decrease of y w.r.t. x (Figure 2).

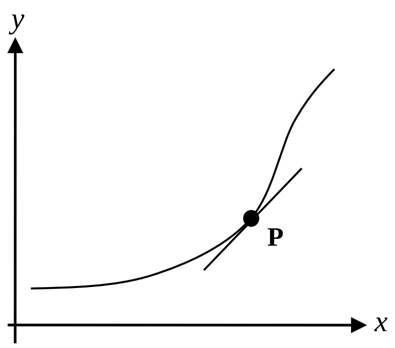


Figure 1

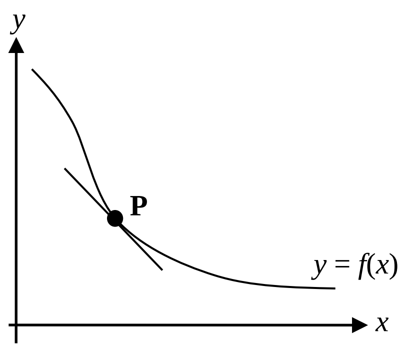


Figure 2

There are many practical applications involving rates of change in science and engineering. Listed below are just some of the applications:

- Velocity v is the rate of change of displacement s w.r.t. time t i.e. $v = \dfrac{ds}{dt}$;
- Acceleration a is the rate of change of velocity v i.e. $a = \dfrac{dv}{dt}$;
- Power P is the rate at which work W is done i.e. $P = \dfrac{dW}{dt}$;
- Current i is the amount of charge q passing through a point per unit time i.e. $i = \dfrac{dq}{dt}$;
- Current i through a capacitance C when the applied voltage v varies with time t i.e. $i = C\dfrac{dv}{dt}$;
- Voltage v across an inductor L, through which a current i passes, is given by $v = L\dfrac{di}{dt}$.

The following are some handy tips that might come in useful when solving "rates of change" problems:

1. Translate the phrase "rate of change of ☆" mathematically into "$\dfrac{d}{d\_}$(☆)" ;
2. The phrase "with respect to ☺" shows that ☺ is the independent variable and thus should appear beside the d in the denominator ;
3. Generally, if the phrase "with respect to ☺" is missing, then assume that the rate of change is with respect to time t. This is in keeping with the English definition of the word "rate".

Example 2 : (**Electrical**) Find the rate of change of the current after 2 milliseconds if the current is given by $i = 14\cos 800t$.

Example 3 : (**Electrical**) Find the current i passing through an 18μF-capacitor if the voltage across capacitor is $v = 35\sin\left(8000t + \frac{5\pi}{36}\right)$ volts.

Example 4 : (**Civil**) When a constant pressure p is applied internally to a cylinder that has an internal radius r and an external radius R, the stress s is given by

$$s = \left(\frac{R^2 + r^2}{R^2 - r^2}\right)p \ .$$

Assuming that R is constant, find the rate of change of the stress with respect to the internal radius.

Example 5 : (**Life Sciences**) Assume that the total number (in millions) of bacteria present in a culture at a certain time t (in hours) is given by $B(t) = 2t(t-10)^2 + 50$. Find the rate at which the population of bacteria is changing when t = 8 hours. Explain your answers in terms of the population of bacteria.

6.4 Related Rate of Change

In this section we will extend what we have learnt previously about rates of change. This time however, the rate of change of our event is affected by the rate of change of a related event.

To solve this type of problems, we often have to apply the **Chain Rule**:

If $y = f(x)$ and $x = g(t)$,

Then $\boxed{\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt}}$

As in the previous section, the following tips might come in useful when solving "related rates of change" problems:

1. Highlight the equation given, the letters in use and any other given numbers together with their units. Sometimes you may be required to set up the equation by yourself.
2. Identify which letters denote variables and which denote constants. Associate any given numbers with the letters.
3. Generally, values for constants can be substituted into the equation first, whilst those for variables can only be substituted **after** differentiation is done.
4. Pay attention to the units attached to the numbers as they usually give a hint as to what quantity or rates they represent.
5. Read the question carefully and establish what rate you have to find.
6. Set up the Chain Rule to relate the rates you have been given and those which you need to find.

Example 6: (**Electrical**) When resistors R_1 and R_2 are connected in parallel, the total resistance of the combination is $R = \dfrac{R_1 R_2}{R_1 + R_2}$.

(a) If R_1 remains constant at $10\ \Omega$ and R_2 increases at $0.2\ \Omega/s$, find the rate of change of R at $R_2 = 4$ ohms.

(b) If R_2 remains constant at $4\ \Omega$ and R_1 decreases at the rate of $0.5\ \Omega/s$, find the rate of change of R at $R_1 = 10$ ohms.

Example 7 : (**Chemical**) In an experiment, a spherical balloon is leaking air at a rate of 500 cm^3/s. How fast is the radius of this balloon shrinking when the radius is 10 cm? (Volume of a sphere : $v = \dfrac{4}{3}\pi r^3$)

Example 8 : (**Civil**) A square sheet of metal of side 8 cm is heated in a furnace. If each side expands at a rate of 0.2 mm/s, how fast is the area expanding after 5 seconds in the furnace?

6.5 Maximum, Minimum and Inflection Points

6.5.1 Increasing and Decreasing Functions

A function is said to be increasing when its graph rises as it goes from left to right. A function is decreasing when its graph falls as it goes from left to right. The increasing/decreasing concept can be associated with the slope of the tangent line. In Figure 3, we see that on the interval $(-\infty, a)$ the graph of f is rising, $f(x)$ is increasing, and the slope of the graph is positive, i.e. $f'(x) > 0$. On the interval (a, ∞), the graph of f is falling, $f(x)$ is decreasing, and the slope of the graph is negative, i.e. $f'(x) < 0$.

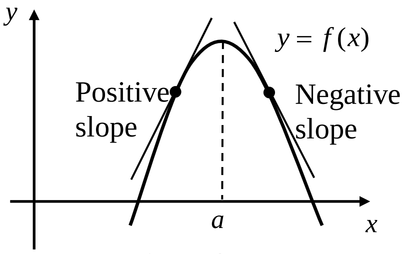


Figure 3

Example 9 : Find the intervals on which the function $f(x) = 4x - x^2$ is increasing and the intervals on which it is decreasing.

6.5.2 Maximum and Minimum Points

When a graph of a continuous function changes from rising to falling, a relative **maximum** occurs, and when the graph changes from falling to rising, a relative **minimum** occurs.

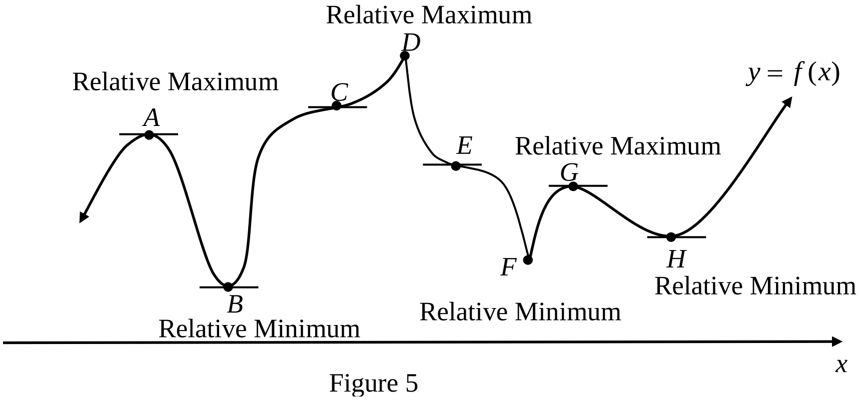


Figure 5

The graph will be "rounded" at the relative maxima or minima if $f'(x) = 0$. They are points *A*, *B*, *G* and *H* in Figure 5. The graph will be "pointed" at the relative maxima or minima when $f'(x)$ is not defined. They are points *D* and *F*. The slope of the tangent line is not defined at such points as the tangent line is either vertical or does not exist.

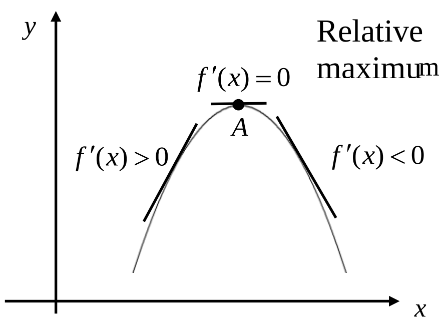
Critical points are those points $(x, f(x))$ on $y = f(x)$ where $f'(x) = 0$ or at which $f(x)$ is not differentiable. The critical points where $f'(x) = 0$ are called **stationary points** of $f(x)$.

Note: It does not imply that every critical point produces a relative maximum or minimum. In Figure 5, points *C* and *E* are critical points (the slope is zero), but the function does not have a relative maximum or minimum at either of these values. Points *C* and *E* are known as **Points of Inflection**.

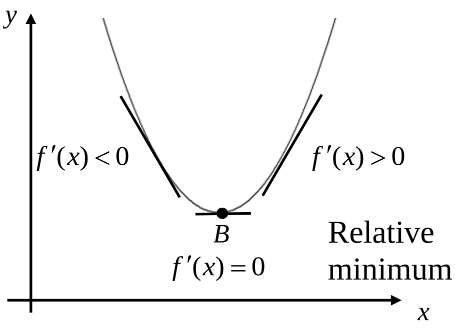
6.5.3 First-Derivative Test

From the discussion of increasing and decreasing values for *f*, we see that the derivative changes sign from positive to negative when passing through a relative maximum point and from negative to positive when passing through a relative minimum point

A relative maximum



A relative minimum



In passing through the point *A*, $f'(x)$ decreases from positive to negative.

In passing through the point *B*, $f'(x)$ increases from negative to positive.

Type of points	Sign of $f'(x)$ immediately before and after		
	Before stationary point	Stationary point	After stationary point
Maximum	**(+)** /	**(0)** —	\ **(–)**
Minimum	**(–)** \	**(0)** —	/ **(+)**

You may perform the following steps to determine the maximum or minimum points of a function $y = f(x)$:

(a) Differentiate the function $f(x)$.

(b) Let $f'(x)=0$ and solve for the values(s) of x (stationary points).

(c) Look at the slope of the tangent on either side of the stationary point(s). We must choose the test points close to the stationary point(s). If the test points are too far away, the curve might have already changed direction.

(d) If the slope of tangent is positive to the left of a stationary point and negative to the right, then the stationary point is a maximum. The reverse is true for a minimum point.

Example 10 : Find all the stationary points of the following functions and determine if each of the stationary points is relative maximum or minimum ; or point of inflection.

(a) $f(x) = 3x^4 - 1$ (b) $f(x) = 3x^4 + 4x^3$

6.5.4 Second-Derivative Test

It is apparent that a curve is concave down at a maximum point and concave up in a minimum point. Therefore, at $x = c$, where c is a critical point of f, we have

1. A **relative minimum** point at $(c,\ f(c))$ if $f''(c) > 0$
2. A **relative maximum** point at $(c,\ f(c))$ if $f''(c) < 0$
3. There is no conclusion from this test if $f''(c) = 0$ or if $f''(c)$ is undefined. (Instead, we must use the first-derivative test)

Example 11 : Find all the stationary points of the function $f(x) = x^3 - 3x^2 - 9x + 2$ and determine if each of the stationary points is relative maximum or minimum.

6.6 Optimisation

One very important application of differential calculus is the solving of maximum-minimum problems. In this section, we shall look at how the methods can be used to solve some applied optimisation problems.

Suggested Steps for Solving Applied Optimisation Problems:

1. Read the problem carefully. Where relevant and possible, sketch a diagram.
2. Label the diagram with appropriate variables and constants to show the relationships between them.
3. Identify the quantity which is to be optimised and call it Q.
4. Translate the problem to a mathematical equation involving Q.
5. Express Q as a function of one variable only, say, x. For that to happen, sometimes you may need to make substitutions.
6. Find the derivative $\dfrac{dQ}{dx}$.
7. Set the derivative to zero and solve for x.
8. Determine the desired maximum or minimum value using the techniques introduced in the preceding sections.

Example 12 : A rectangular area is to be enclosed using an existing wall as one side and 100m of fencing for the other three sides. What would be the necessary dimensions of the enclosure so that the area is the maximum possible?

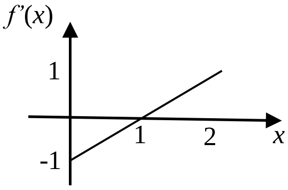
<u>Tutorial</u> : <u>MCQ</u>

1. Given $y = f(x)$ and $f''(0) = 4$ At $x = 0$, the function $f(x)$ ______________

(a) is equal to 0 (b) is equal to 0

(c) concaves downward (d) concaves upward

2. The diagram of $f'(x)$ over $0 \le x \le 2$ is given below. At $x = 1$, $f(x)$ has ____________



(a) a maximum value (b) a minimum value

(c) a point of inflexion (d) none of the above

3. The curve $y = f(x)$ has a maximum point at $x = a$, which of the following is *NOT* true?

(a) $\frac{dy}{dx}$ is an increasing function (b) $\frac{dy}{dx}$ is a decreasing function

(c) $\frac{d^2y}{dx^2} < 0$ at $x = a$ (d) $\frac{dy}{dx} > 0$ for $x < a$ & $\frac{dy}{dx} < 0$ for $x > a$

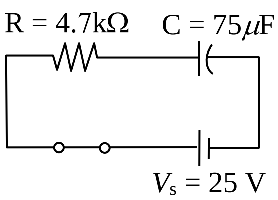
Tutorial 6.1

1. Find $\frac{dy}{dx}$ and the slope of a tangent to the curve of $y = \frac{\tan^{-1} x}{x^2 + 1}$ at $x = 1$.

2. Given the function $y = e^{\sin x + \cos x}$, find the gradient and the equation of the tangent line at $x = \frac{\pi}{2}$.

3. (**Electrical**) The charge q on a certain capacitor as a function of time t is given by $q = 4000e^{-62.5t}$ μC. Find the current i. [Note : $i = \frac{dq}{dt}$]

4. (**Electrical**) For the series RC-circuit shown, find the current through the capacitor at $t = 3$ seconds if the voltage across it is $v = V_s\left(1 - e^{-\frac{t}{RC}}\right)$. [Note : $i = C\frac{dv}{dt}$]



5. (**Electrical**) If the current through a 40 μH inductor in a parallel RL circuit is $i = 2.5\cos\left(1.5 \times 10^6 t + \frac{\pi}{4}\right)$ amperes, what is the voltage (v) at time t seconds? [Note : $v = L\frac{di}{dt}$]

6. (**Electrical**) The work done in a certain circuit is given by the equation $W = \frac{1}{4}\sin\left(400t + \frac{\pi}{4}\right) + 50\sqrt{2}\,t$ joules. Obtain an expression for the power P.

7. (**Mechanical**) The displacement y of the end of a robot arm for welding, at time t seconds, is given by $y = 12\sin 0.5t$ metres. What is the rate of change of the displacement?

8. (**Mechanical**) The displacement (y) of a mass oscillating at the end of a spring is given by $y = 0.60\cos 8t + 0.125\sin 8t$, where t is the time in seconds.
   (a) Find the initial displacement.
   (b) Find the velocity function in terms of t.
   (c) Show that the initial velocity is 1m/s.
   (d) Find the velocity at $t = 2s$.

9. (**Mechanical**) A belt makes an angle of θ radians as it passes around a pulley. If the tension T on one side of the belt is $T = 48\,e^{0.32\theta}$ newtons, find the rate of change of the tension with respect to the angle of contact.

10. (**Life Sciences**) In studying the inhibitory action of glucose on the rate at which maltase catalyses the conversion of p-niotrophenyl-α-D-glucoside, the speed of conversion in the controlled experiment was found to be $v = \frac{0.22s}{0.25 + s}$, where s is the concentration of the substrate. What is the rate of change of this conversion speed with respect to the concentration of the substrate, when $s = 0.05$?

11. (**Life Sciences**) The gross photosynthetic rate p of a leaf was experimentally found to be $p = \left(3 + \frac{1}{2I}\right)^{-1}$, where I is the intensity of the light. Find the rate of change of p with respect to I and simplify your answer.

12. (**Life Sciences**) It is found that the heartbeat rate b of an average healthy person of height h , in centimetres ($75 \le h \le 190$) can be approximated by the equation $b = \frac{940}{\sqrt{h}}$ beats per minute. Obtain an expression for the rate of change of b with respect to the height h .

13. (**Environmental Science**) A tuning fork vibrating at 225 Hz produces a sound wave given by the equation $y = 0.05\sin 450\pi t$ millimetres, where t is time in seconds. What is the rate of change of y?

14. (**Business**) A company in the computer software business makes a monthly profit of $P = 150x - \frac{x^2}{100} - 60,000$ for $0 \le x \le 5,000$, where x is the number of units sold per month. Find the rate of change of the profit with respect to the number of units sold.

15. (**Business**) The mathematical model for continuous compound interest is $A = Pe^{rt}$, where A is the amount after t years, P is the principal sum invested and r is the annual compounded interest rate. If a person invests \$5,000.00 when the rate is 6%, compute the rate of growth for that investment after 5 years.

16. (**Business**) The price-demand for the jerseys of a certain football club is given by $p = 15 - \ln x^4$, where x is the number of jerseys sold per month ($1 \le x \le 40$). What is the rate of change of price with respect to the number of jerseys sold?

17. (**Civil**) The deflection y of a 5 metres-long cantilever beam at some distance x from the built-in end is given by the equation $y = \dfrac{wx^2}{24EI}\left(150 - 20x + x^2\right)$, where w is the load on the beam, E is the modulus of elasticity and I is the moment of inertia. Find the rate of change of the deflection with respect to the distance x.

18. (**Chemical**) In a certain chemical reaction, the concentration of a reactant X after t minutes is given by the equation $C = 0.05e^{-kt}$ mol/L, where constant $k = 0.08$ mol per litre per minute. Find the rate of change of the concentration 5 minutes after the reaction was started.

Tutorial 6.2

1. (**Electrical**) A parallel RL circuit has impedance $Z = \dfrac{RX}{R+X}$ ohms. If R is fixed at 4 ohms, while X is allowed to decrease at a rate of 2 ohms per second, find the rate at which Z is changing.

2. (**Electrical**) The impedance Z ohms of a circuit in series with reactance X Ω is given by
$$Z = \sqrt{16 + X^2}.$$
If X decreases at 2 ohms/s, find the rate of change of Z when $X = 3$ ohms.

3. (**Electrical**) The resistance R of a certain conducting rod varies with the temperature T according to the equation $R = 0.24T^2 + 1.08T + 18.4$ ohms. If resistance R is increasing at 3Ω/s, how fast is the temperature changing?

4. (**Chemical**) The heat capacity C of carbon monoxide varies with absolute temperature T according to $C = 26.53 + 7.70\times10^{-3}T - 1.17\times10^{-6}T^2$. If the temperature is decreasing at 1 kelvin per second, obtain an expression for the rate of change of C.

5. (**Chemical**) At a certain height h metres above ground level, the atmospheric pressure $p = p_o e^{-1650h}$ pascals, where $p_o = 1.013\times10^5$ pascals. If h is increasing at a rate of 2 m/min, how fast is the pressure changing?

6. (**Civil**) The length of a certain rod at some temperature T°C is given by $x = 100 + 0.002T + 0.000\,02T^2$ centimetres. If the temperature of the rod is increasing at a rate of 0.1°C/s, how fast is the length increasing when the temperature is about 50°C?

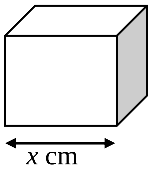
*7. (**Electrical**) The power P dissipated in a variable resistor R by a battery of e.m.f. E and internal resistance r is given by $P = \dfrac{E^2R}{(R+r)^2}$. If E and r are fixed, while R is allowed to decrease at a rate of 0.5 Ω/s, obtain an expression for the rate of change of P in terms of R, E and r.

8. **(Mechanical)** A gas in a cylinder has its state changed according to the law $PV^{1.2} = C$, where P is the pressure, V is the volume, and C is a constant.
Initially, $P = 16\, kN/m^2$ and $V = 0.78\text{ m}^3$.
   (a) In a compression process, the volume is reduced to $V = 0.52\text{ m}^3$, what is the pressure?
   (b) From the given law $PV^{1.2} = C$, find an expression for $\dfrac{dP}{dV}$.
   (c) If the gas volume is decreasing at a rate of 0.05 m^3/s, find the rate of change of pressure when the volume is 0.52 m^3.

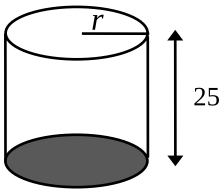
9. (**Environmental Science**) A circular oil slick is expanding at a rate of 2 m^2/h. Find the rate at which its diameter is expanding when its radius is 1.5 m.

10. **(Mechanical)** A ladder 3 metres long is leaning against the side of a wall. If the foot of the ladder slides away from the wall at a speed of 30 cm/s, how fast is the top end sliding down the wall when the foot of the ladder is 1 metre away from the wall?

11. **(Physics)** The surface area of a cube with side x cm, is increasing at a rate of 16 cm^2/s. When the total surface area of the cube reaches 216 cm^2,
   (a) show that the volume of the cube is 216 cm^3;
   (b) find the rate at which its volume is increasing.



*12. **(Physics)** A cylindrical container opened at one end is made from thin material with radius r cm and height 25 cm as shown below. The volume of the container is V (cm^3) and the total external surface area is A (cm^2).
   (a) Express A and V in terms of r. Hence show that $A = 10\sqrt{\pi V} + \dfrac{V}{25}$
   (b) If the volume is increasing at the rate of 0.3 cm^3/min, determine the rate at which the external surface area A is changing at the moment when the volume of the container is 900 cm^3.



*13. It is known that $z = 3x^2y^3$. If x and y are increasing at the rate of 4 unit/s and 2 unit/s respectively, find the rate of change of z with respect to time t at the point $x = 1$ and $y = 1$.
(Hint : Consider differentiating with respect to the variable t)

Problem-solving Assignment
The goal of this series of problem-solving assignments is to develop problem-solving skills, not just to test your ability to get the answer. It's ok to try hard and not succeed at first (only your effort is evaluated), but you must try.

Question
When resistors R_1 and R_2 are connected in parallel, the total resistance of the combination is

$$R = \frac{R_1 R_2}{R_1 + R_2}.$$

If R_2 increases at $0.2\Omega/s$ and R_1 decreases at the rate of $0.5\Omega/s$, find the rate of change of R at $R_2 = 4$ ohms and $R_1 = 10$ ohms.

1. Understand the problem • Identify the variables. • State the given rates. • Identify the unknown that you are asked to find.	
2. Devise a plan • Identify the relevant concept/technique that can be applied.	
3. Implement the plan • Carry out the plan, showing each step clearly.	
4. Look back • Ask yourself - "Does it answer the question that was asked?" - "Does the answer make sense?" • Determine if there is any other easier way of finding the solution.	

Tutorial 6.3

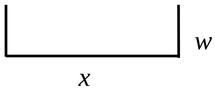
1. Find all the relative maximum and minimum points of the following functions.

(a) $f(x)=x^3+9x^2+15x-25$ (b) $f(x)=x^3-3x^2+1$

(c) $f(x)=(x+1)^3$ (d) $h(x)=\dfrac{x^2+5x+3}{x-1}$

(e) $g(x)=\dfrac{x^2-2x+4}{x-2}$

2. A piece of wire 160mm is bent to form the shape of a rectangle without the top as shown in the diagram on the right. Find the width w and length x which gives the maximum cross-sectional area.



3. **(Mechanical)** The torgue Γ on the crankshaft of an engine is given by $\Gamma=8+3\sin 2\theta-4\cos 2\theta$. Find the value of θ between $0°$ and $180°$ for which Γ is a maximum or minimum and find the corresponding values of Γ.

4. **(Mechanical)** A rectangular box is made of a very thin sheet of metal and is open at the top end. It is given that the volume of the box is 125 cm^3, its width is x cm and its length is $5x$ cm. Show that the total internal surface area S of the open box is $S=5\left(x^2+\dfrac{60}{x}\right)$ cm^2. Find the value of x for which the total internal surface area of the open box is a minimum.

5. **(Environmental Science)** A certain underwater cable has a core of copper wires covered by insulation. The speed of transmission of a signal along the cable is $S=x^2\ln\dfrac{1}{x}$, where x is the ratio of the radius of the core to the thickness of the insulation. What value of x gives the greatest signal speed?

*6. **(Environmental Science)** The total area of sheet metal used in the construction of a closed cylindrical can is 24π mm^2. If the base radius of a can is r mm, show that the volume is given by $V=(12-r^2)\pi r$ mm^3. Determine the maximum volume of the can.

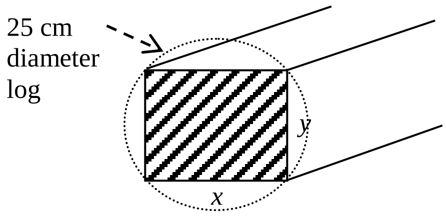
*7. **(Environmental Science)** A window frame is made in the shape of a rectangle with a semi-circle of radius r (m) on top of the breadth of the rectangle. The breadth of the rectangle is the same as the diameter of the semi-circle. If the total area is to be 8 m^2, show that the perimeter P meters of the frame is $P=\dfrac{8}{r}+r\left(\dfrac{\pi}{2}+2\right)$. Find the minimum cost of producing the frame if 1 m costs \$5.

8. **(Mechanical)** The cross-section of a rectangular beam of width x(cm) and depth y(cm) is shown below. The strength of the beam is given as $Q=8y^2x$.

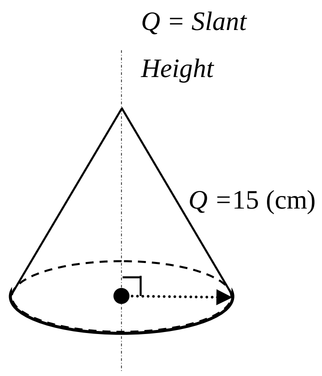
(a) If the beam is cut out from a log of diameter 25 (cm) as shown, prove that $Q=5000x-8x^3$.

(b) Hence, find the value of x which gives the strongest beam.

(c) Find the corresponding value of y which gives the strongest beam.



*9. **(Mechanical)** The slant height Q of a cone with a circular base is held constant at 15 (cm) as shown. Find the maximum volume. (Volume of cone is $V=\frac{1}{3}\pi r^2 h$)



10. **(Electrical)** The power developed in watts in an alternating-current circuit is given by $P=\dfrac{100R}{R^2+25}$. Compute the value of R which makes P a maximum.

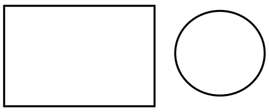
*11. **(Electrical)** The power output of a circuit is given by $P=\dfrac{E^2R}{(R+r)^2}$, where R is the variable external resistance and r, the constant internal resistance and E is the e.m.f. of the battery. If E is a constant, show that the power output is maximum when $R = r$.

*12. **(Electrical)** At time t seconds, the total current i flowing to a parallel combination of a resistance R and a capacitance C is given by $i=\dfrac{V}{R}+C\dfrac{dV}{dt}$, where V is the supply voltage. If $R = 20\ \Omega$, $C = 0.02$ F and $V=4e^{-2t}+2e^{-5t}$ volts,

(i) show that $i=0.04e^{-2t}-0.1e^{-5t}$ amperes.

(ii) Find the maximum value of i and the time when this occurs.

***13. (Mechanical)** A wire 3 m long, is cut into 2 pieces to form a rectangle and a circle. The rectangle has a length that is double of its breadth. Find the radius of the circle such that the total area of both the rectangle and the circle is minimum.



*14. **(Electrical)** The voltage across an inductor is given by

$$v=L\frac{di}{dt} \quad \text{volts}$$

where L= 2 henrys and $i=2-2e^{-t}-2te^{-t}$.

Find the time when the minimum/maximum voltage is reached.

15. (**Environmental Science)** Under certain conditions the work done per unit volume of steam is given by $W = P_1(1 + \ln r) - P_2\, r$, where P_1 and P_2 are given constants. Find r when W is maximum.

16. **(Chemical)** In an autocatalytic chemical reaction a substance is formed that causes an increase in its rate of formation. The reaction rate R (in g/s) is related to the amount x (in g) of the substance present at any time according to the equation $R = 6.2\,x\,(25.2 - x)$. For what value of x will the reaction rate R be a maximum?

*17. **(Life Sciences)** When a person coughs, the trachea or windpipe contracts. Using the principles of fluid flow given in the equations

$$r_0 - r = ap \quad \text{and} \quad v = bp\pi r^2$$

where $r_0 =$ normal radius of the trachea, r = radius of trachea during cough, p = increase in air pressure in the trachea during cough, v = velocity of air through the trachea during cough and $a, b, r_0 =$ positive constants,

(a) make v the subject in terms a, b, r_0 and r.

(b) determine how much the trachea should contract in order to create the greatest air velocity.

ANSWERS

Eg 1 : (a) slope/gradient $= \left.\dfrac{dy}{dx}\right|_{x=2} = 12$ (b) Equation of tangent line , $y = 12x - 11$

Eg 2 : $-$ 11195.2 A/s

Eg 3 : $i = 5.04\cos\left(8000t + \dfrac{5\pi}{36}\right)$

Eg 4 : $\dfrac{ds}{dr} = \dfrac{4prR^2}{\left(R^2 - r^2\right)^2}$

Eg 5 : $-$56 million bacteria per hour

Eg 6 : (a) 0.102 Ω/s (b) $-$ 0.0408 Ω/s

Eg 7 : $-$ 0.398 cm/s

Eg 8 : 32.4 mm^2/s

Eg 9 : $f(x)$ is increasing wrt x for $x < 2$, $f(x)$ is decreasing wrt x for $x > 2$

Eg 10 : (a) (0,–1) min (b) $(-1, -1)$ min ; (0,0) point of inflection

Eg 11 : $(-1, 7)$ max ; $(3, -25)$ min

Eg 12 : 50m by 25m

MCQ

1. d 2.b 3.a

Tutorial 6.1

1. $\dfrac{dy}{dx}=\dfrac{1-2x\tan^{-1}x}{\left(x^2+1\right)^2}$; $\left.\dfrac{dy}{dx}\right|_{x=1}=-0.143$

2. Gradient = – e

   Equation of the tangent line is $y=e\left(1+\dfrac{\pi}{2}\right)-ex$ or $y=6.988-ex$

3. $i=\dfrac{dq}{dt}=-0.25\,e^{-62.5t}$ A

4. i = 1.07 μA

5. $v=-150\sin\left(1.5\times10^6t+\dfrac{\pi}{4}\right)$

6. $P=50\left[2\cos\left(400t+\dfrac{\pi}{4}\right)+\sqrt{2}\right]$

7. $\dfrac{dy}{dt}$ = 6 cos 0.5t m/s

8. (a) 0.6 m (b) $v=-4.8\sin(8t)+\cos(8t)$ (d) v = 0.424 m/s

9. $\dfrac{dT}{d\theta}=15.36\,e^{0.32\theta}$ N/rad

10. $\dfrac{dv}{ds}$ = 0.611 or $\dfrac{11}{18}$

11. $\dfrac{dp}{dI}=\dfrac{2}{(6I+1)^2}$

12. $\dfrac{db}{dh}=-\dfrac{470}{\sqrt{h^3}}$

13. $\dfrac{dy}{dt}$ = 22.5π cos 450πt

14. $\dfrac{dP}{dx}=150-\dfrac{x}{50}$

15. $\dfrac{dA}{dt}$ = \$404.96 per year

16. $\dfrac{dp}{dx}=-\dfrac{4}{x}$

17. $\dfrac{dy}{dx}=\dfrac{wx}{6EI}\left(75-15x+x^2\right)$

18. $\dfrac{dC}{dt}=-2.681\times10^{-3}$ mol L^{-1} min^{-1}

Tutorial 6.2

1. $\dfrac{dZ}{dt}=-\dfrac{32}{(4+X)^2}$

2. – 6/5 Ω/s

3. $\dfrac{dT}{dt}=\dfrac{25}{4T+9}$

4. $10^{-3}\left(2.34\times10^{-3}T-7.70\right)$

5. $-334.29\times10^6e^{-1650h}$ Pa/min

6. 0.0004 cm/s

7. $\dfrac{dP}{dt}=\dfrac{E^2(R-r)}{2(R+r)^3}$ W/s

8. (a) 26.03 (kN/m^2) (b) $-\dfrac{14.25}{V^{2.2}}$ (c) 3.00 kN/m^2s

9. $\dfrac{4}{3\pi}$ m/h

10. −10.607 cm/s

11. 24cm^3/s

12. (b) 0.1cm^2/min

13. 42 units/s

Problem-solving Assignment : $0.061\Omega/s$

Tutorial 6.3

1. (a) $(-5, 0)$ maximum $(-1, -32)$ minimum
   (b) $(0, 1)$ maximum, $(2, -3)$ minimum
   (c) $(-1, 0)$ point of inflection
   (d) $(-2, 1)$ maximum, $(4, 13)$ minimum
   (e) $(0, -2)$ maximum, $(4, 6)$ minimum

2. 40 mm, 80 mm
3. $\Gamma(\text{min}) = 3,\ 161.57°$, $\Gamma(\text{max.}) = 13,\ 71.57°$
4. $x = 3.11$cm
5. $e^{-\frac{1}{2}}$
6. 16π mm^3

7. $53.44
8.(b) x= 14.43 cm (c) y=20.42cm
9. 1360.35cm^3

10. $R = 5\Omega$
12. (ii) 7.07 (mA)
13. $\dfrac{3}{2\pi + 9}$ m or 0.196 m
14. $t = 1$ second
15. $r = \dfrac{P_1}{P_2}$
16. 12.6 g
17. (a) $v = \dfrac{b}{a}\pi r_0 r^2 - \dfrac{b}{a}\pi r^3$ (b) $r = \dfrac{2}{3}r_0$