Revision Tutorial

I. Partial Differentiation

Basic

1. Find the partial derivatives $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for each of the following functions.

   (a) $z = x^5 + yx + \ln(x + 2y)$

   (b) $z = e^{2x}\sin(y)$

   (c) $z = x^2 \sin^2 y$

   (d) $z = x^3 + 5x^2 y + 2y^3 + 6$

   (e) $z = x^2 y + 2xy^2 - 2x$

2. Find $f_x(x, y)$ and $f_y(x, y)$ for each of the following.

   (a) $f(x, y) = xy + e^{9y} - \cos(3x)$

   (b) $f(x, y) = 3x\sqrt{x^2 + 5y^2}$

   (c) $f(x, y) = y^4 + 3xy + \ln(y)$

3. Let $f(x, y) = 2x^2 + xy + \sin(y)$. Evaluate $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at $(0, \dfrac{1}{2})$.

4. Find the indicated partial derivatives.

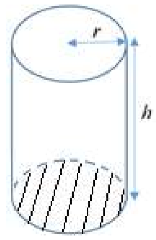
   (a) $f(x, y) = \sqrt{x^2 + y^2}$, $f_x(3, 4)$ (b) $f(x, y) = \dfrac{x}{y+1}$, $f_y(3, 2)$

Intermediate to challenging

5. If $z = x^2 y - y^2$ where $x = t^2$ and $y = 2t$, calculate $\dfrac{dz}{dt}$ by using partial differentiation with chain rule. Leave your answer in terms of t.

6. The total surface area S of a cone of base radius r and perpendicular height h is given by $S = \pi r^2 + \pi r\sqrt{r^2 + h^2}$. If r and h are each increasing at the rate of 0.25 cm/sec, find the rate at which S is increasing at the instant when $r = 3$ cm and $h = 4$cm.

7. Figure on the right shows a cylindrical-shaped tank with height, h and radius, r. When the height of the cylindrical tank is increasing at a rate of 0.03 m/s and the radius is increasing at a rate of 0.02 m/s, what is the rate of change of the volume of the tank at the instant where $r = 0.2$ m and $h = 1.5$ m?



8. The magnitude of the resultant force R of two forces P and Q acting on an object and inclined at an angle θ is given by

$$R=\sqrt{P^2+2PQ\cos\theta+Q^2}$$

(a) Show that $\dfrac{\partial R}{\partial P}=\dfrac{P+Q\cos\theta}{R}$.

(b) Find $\dfrac{\partial R}{\partial \theta}$.

(c) Suppose the force Q remains constant at 15 N. Find the rate at which the resultant force R is changing if the force P is increasing at a rate of 0.2 N/s and θ is decreasing at a rate of 0.2 rad/s at the instant when P = 25 N and $\theta=\dfrac{\pi}{3}$.

II. Integrate functions of linear functions and using trigo identities

Basic

1. Integrate the following functions of linear function:

(a) $\int (1-2x)^2\, dx$ (b) $\int \sqrt{4-3x}\, dx$

(c) $\int \dfrac{1}{(2x-3)^5}\, dx$ (d) $\int \dfrac{1}{8x+3}\, dx$

(e) $\int \dfrac{4}{25-4x}\, dx$ (f) $\int \cos\left(3x-\dfrac{\pi}{6}\right) dx$

(g) $\int \sin(2x+1)\, dx$ (h) $\int e^{\frac{x}{2}+5}\, dx$

2. Find the values of the following integrals.

(a) $\displaystyle\int_{4.5}^{10.5} \frac{2}{\sqrt{2x-5}}\, dx$ (b) $\displaystyle\int_{-2/3}^{0} \frac{1}{e^{3x+2}}\, dx$

Intermediate to challenging

3. Find the following integrals:

(a) $\int 2\sin x\cos x\, dx$ (b) $\int \dfrac{1}{\cos^2(2x)}\, dx$

(c) $\int 2\tan^2 2x\, dx$ (d) $\int 2\sin 3x\cos 5x\, dx$

(e) $\int 3\sin\dfrac{3t}{2}\sin\dfrac{5t}{2}\, dt$ (f) $\int \sin^2\theta\cos 3\theta\, d\theta$

4. Find the root-mean-square (rms) value of

(a) $f(t)=1+3e^{-t}$ from $t=0$ to $t=2$

(b) $y=2(\sin x+\cos x)$ from $x=0$ to $x=\pi$

5. The current in an electronic circuit is given by $i = \sin 2t + \cos 3t$. By means of integration, find the RMS value of i for $0 \le t \le \frac{\pi}{4}$.

6. If the current in an electric circuit is given by $i = I_p \sin \omega t$ where I_p is the maximum current. Show that the root mean square (RMS) value of the current from $t = 0$ to $t = \frac{2\pi}{\omega}$ is $\frac{I_p}{\sqrt{2}}$.

III. Integration by substitution

Basic

1. Integrate the following by suitable substitution:

(a) $\int x\left(x^2-3\right)^4 dx$ (b) $\int \frac{x}{(4-x^2)^2} dx$

(c) $\int \sin^2\theta \cos\theta \, d\theta$ (d) $\int 3x^2\left(x^3-10\right)^8 dx$

(e) $\int \frac{x}{1-2x^2} dx$ (f) $\int \frac{dx}{x \ln x}$

(g) $\int t\, e^{3-2t^2} dt$ (h) $\int y\, e^{\frac{y^2}{3}} dy$

(i) $\int \frac{5e^{2x}}{\sqrt{1-e^{2x}}} dx$ (j) $\int t^3 \sin t^4 \, dt$

2. Find the values of the following integrals.

(a) $\int_0^{1/2} y\sqrt{\frac{1}{4}-y^2}\, dy$ (b) $\int_1^2 \frac{e^{1/t}}{t^2} dt$

(c) $\int_0^4 \frac{4x}{\sqrt{2x+1}} dx$ (d) $\int_0^{\frac{\pi}{4}} \frac{\cos 2x}{1+\sin^2 2x} dx$

Intermediate to challenging

3. Integrate the following:

(a) $\int \frac{2e^x}{e^x-5} dx$ (b) $\int \frac{2x}{\sqrt{1-2x^2}} dx$ (c) $\int \frac{2t+3}{\left(4t-5\right)^5} dt$

(d) $\int \sin^3 x \, dx$ (Hint: use $\sin^2 x = 1-\cos^2 x$ and let $u = \cos x$)

(e) $\int x\sqrt{4-x}\; dx$ (Hint: let $u = 4-x$ and represent x in term of u)

(f) $\int e^{2x}\sqrt{1+4e^x}\; dx$

IV. Integration by partial fraction

Basic

1. Find the following integrals:

(a) $\displaystyle\int \frac{-x+7}{(x+3)(3x-1)}\,dx$ (b) $\displaystyle\int \frac{3}{(x+1)(x-2)}\,dx$

(c) $\displaystyle\int \frac{5x+3}{x(x-3)(x+1)}\,dx$

Intermediate to challenging

2. Find the following integrals:

(a) $\displaystyle\int \frac{x^2-6x+2}{(x+1)(2x-1)^2}\,dx$ (b) $\displaystyle\int \frac{3x^2-x+8}{x\left(x^2+4\right)}\,dx$

3. (a) Express $\dfrac{7x^2+x-4}{\left(x^2+2\right)(1-x)}$ as a sum of partial fractions.

(b) Hence find $\displaystyle\int \frac{7x^2+x-4}{\left(x^2+2\right)(1-x)}\,dx$

4. (a) Given that $\dfrac{5x+4}{(x-1)(x+2)^2}=\dfrac{A}{x-1}+\dfrac{B}{x+2}+\dfrac{C}{(x+2)^2}$, find the value of constants A, B and C.

(b) Hence, determine $\displaystyle\int \frac{5x+4}{(x-1)(x+2)^2}\,dx$.

5. Integrate the following:

(a) $\displaystyle\int \frac{x^2-3x+6}{x^3+3x}\,dx$ (b) $\displaystyle\int_4^5 \frac{3x-4}{x^3-4x^2+4x}\,dx$

V. Integration by completing the square

Basic

1. By “completing the square”, find the integrals:

(a) $\displaystyle\int \frac{3}{x^2+6x+12}\,dx$ (b) $\displaystyle\int \frac{x-5}{x^2-10x+50}\,dx$

2. (a) If $x^2+6x+13=(x+a)^2+b$, where a and b are constants. Find the values of a and b.

   (b) Hence, evaluate $\int_0^1 \frac{3}{x^2+6x+13}\,dx$.

3. By completing the square for x^2-x+1, find $\int \frac{1-x}{x^2-x+1}\,dx$.

Intermediate to challenging

4. By completing the square for $x^2-4x+68$, find $\int \frac{1}{x^2-4x+68}\,dx$. Hence, determine $\int \frac{1}{2x^2-8x+136}\,dx$.

5. (a) By completing the square for $x^2-6x+12$, find $\int \frac{1}{x^2-6x+12}\,dx$.

   (b) Hence, determine $\int \frac{x^2-6x+13}{x^2-6x+12}\,dx$.

VI. Integration by parts

Basic

1. Find the following integrals:

   (a) $\int x^2 \sin 3x\,dx$

   (b) $\int_0^1 x e^{-5x}\,dx$

   (c) $\int_1^e x^2 \ln x\,dx$

   (d) $\int e^{5x}\cos 2x\,dx$

   (e) $\int \ln(1-4x)\,dx$

Intermediate to challenging

2. Integrate the following:

   (a) $\int \frac{\ln(x)}{(2x+1)^3}\,dx$

   (b) $\int \frac{x\sin^{-1}(2x)}{\sqrt{1-4x^2}}\,dx$

3. Evaluate $\int_1^3 \sqrt{x}\tan^{-1}\sqrt{x}\,dx$

VII. Simpson's Rules

1. (a) Use Simpson's Rule with 4 strips to find an approximate value for the integral $\int_1^2 \sqrt{1+\frac{1}{x}}\,dx$. Give your answer correct to 4 decimal places.

   (b) Explain briefly whether by increasing the number of strips to 7 can increase the accuracy of the final answer using Simpson's Rule.

2. By using Simpson's rule with 6 equal intervals, find the approximate value of $\int_0^1 \left(\sqrt{x}+x\right)^{\frac{1}{3}} dx$, accurate to 3 decimal places.

3. By using Simpson's rule with 6 equal intervals, find the approximate value of $\int_0^1 \ln\left(1+e^x\right) dx$, accurate to 3 decimal places. (Show your workings clearly.)

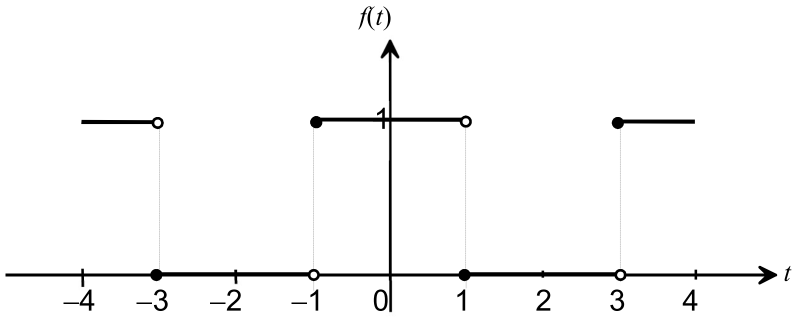
4. By using Simpson's rule with 6 equal intervals, find the approximate value of $\int_1^2 \sin\left(\ln x\right) dx$, accurate to 3 decimal places. (Show your workings clearly.)

VIII. Fourier Series

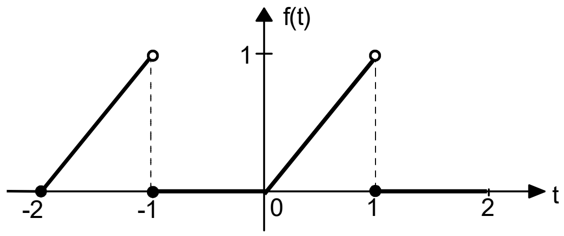
Basic

1. Sketch the graphs of the following periodic functions for two periods, and identify whether the function is even or odd or neither.
   (a) $f(t)=|t|,\ -1<t<1;\ f(t+2)=f(t)$
   (b) $f(t)=t-t^3,\ -1<t<1;\ f(t+2)=f(t)$
   (c) $f(t)=\begin{cases} 2 & -1\le t<0 \\ 4 & 0\le t<1 \end{cases};\ f(t+2)=f(t)$

2. Find the period of each of these functions and then give an analytic definition of the function. Indicate clearly whether the function is even, odd or neither.
   (a)



(b)



Intermediate and challenging

3. Find the Fourier series as far as the third harmonic for each of the following.

   (a) $f(t) = |t|,\ -1 < t < 1;\ f(t+2) = f(t)$

   (b) $f(t) = t - t^3,\ -1 < t < 1;\ f(t+2) = f(t)$

4. A periodic function $f(t)$ of period 4 is defined as $f(t) = 4 - t^2 \quad -2 \le t \le 2$.

   (i) Sketch the waveform of $f(t)$ for the interval $-2 \le t \le 2$.

   (ii) Show that the trigonometric Fourier series of $f(t)$ is given by

$$f(t) = \frac{8}{3} + \frac{16}{\pi^2}\left(\cos\frac{\pi t}{2} - \frac{1}{4}\cos \pi t + \frac{1}{9}\cos\frac{3\pi t}{2} + \cdots\right)$$

5. The current flowing over an inductor with inductance 0.1 H has the periodic waveform shown below. Find the Fourier series for the voltage $V_L(t)$ across the inductor as far as the second harmonic. [Hint: Find the Fourier series for $i(t)$ and use $V_L(t) = L\dfrac{d}{dt}i(t)$]