Vectors

PRE-CLASS (1 TO 16)

IN-CLASS (18 ONWARDS)

Scalars

- A physical quantity that is completely described by its magnitude is called a scalar.
- Examples of scalars are :
  - Mass
  - Volume
  - Work and energy
  - Time
  - Density
  - Temperature
- Obviously mass, volume, temperature, time, etc. cannot have directions.

Scalars

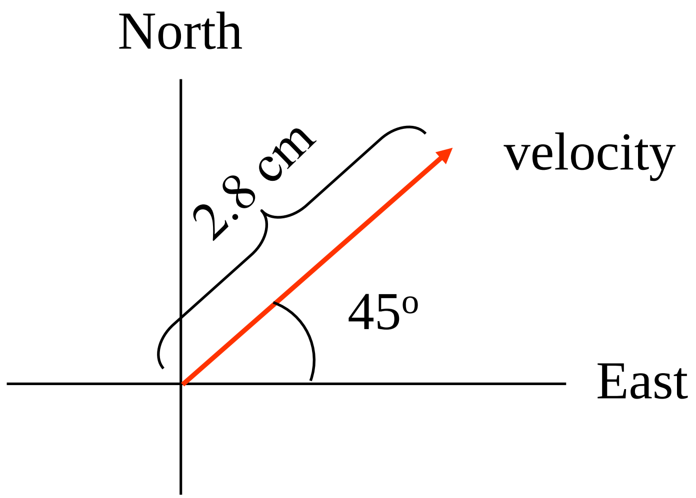
- Scalars can be added/subtracted using algebraic addition/subtraction.
- For example, 2.0 kg + 3.0 kg = 5.0 kg.
- Some physical quantities like length and area are treated as vectors by giving them a direction.

Vectors

- A physical quantity that is completely described only if both its magnitude and direction are specified is known as a vector.
- If we ask someone to apply a force of 10 N, he cannot act until he knows the direction to apply the force. Hence force is a vector.
- Other examples of vectors are :
  - velocity
  - acceleration
  - displacement
  - momentum

Graphical representation of vectors

- A vector can be represented as an arrow drawn to scale.
- The length of the arrow gives the magnitude of the vector and the arrow head gives the direction of the vector.
- The figure below shows how a velocity vector is represented as an arrow.



Scale: 1 cm = 10 m/s

In words : Velocity = 28 m/s in the direction of North East or 45° with respect to East.

Specifying the direction of a vector

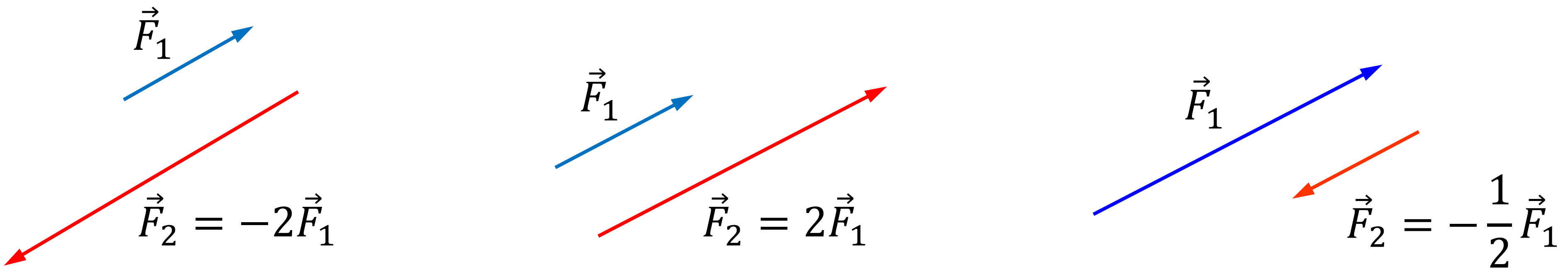
- The direction of a vector may be stated in words or numerically.
- In words,
  - weight of an object is 20 N vertically down.
  - force is 10 N to the right.
  - acceleration is 2.0 m/s^2 north-east.
- Numerically,
  - velocity is 2.0 m/s, 30° anti-clockwise from the +x-axis.
  - momentum is 10 $\hat{\imath}$ kg m/s, where $\hat{\imath}$ is a vector of size 1 pointing in the +x-axis.

Vector notations

- A vector can be written as a letter with an arrow on top such as $\vec{A}$.
- It can also be written in boldface, such as $\boldsymbol{A}$.
- The magnitude or size of a vector is always positive and can be written as $|\vec{A}|$, $|\vec{\boldsymbol{A}}|$, or simply as A.
- In our slides, we shall use a letter with an arrow on top to represent a vector and the same letter without the arrow to indicate its magnitude.
- E.g. the size of a force $\vec{F}$ is 50 N or simply $F = 50$ N.

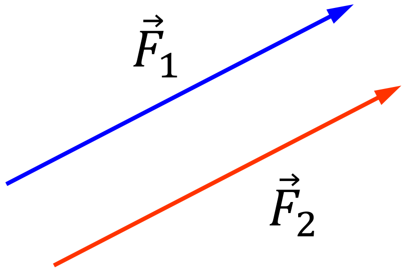
Scalar multiplication

- When a vector $\vec{A}$ is multiplied with a scalar m, the resultant vector $\vec{B} = m\vec{A}$
  - has the same or opposite direction as $\vec{A}$ depending on the sign of the scalar m.
  - is larger or smaller than $\vec{A}$ depending whether the scalar is greater or less than one.
- For example, in the formula $\vec{F} = m\vec{a}$, the vector $\vec{F}$ is 10 times the vector $\vec{a}$ if m is 10 kg.

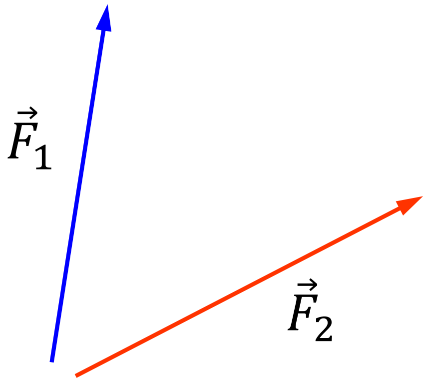


Equality of vectors

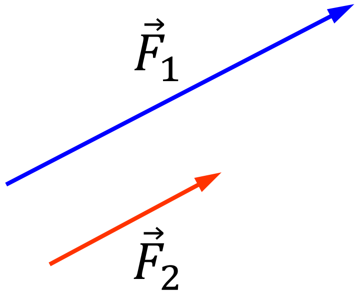
- Two vectors are equal if they have the same magnitude, are parallel and in the same direction.
- In the figure below, only the first pair of vectors are equal.



$\vec{F}_1 = \vec{F}_2$ if $F_1 = F_2$



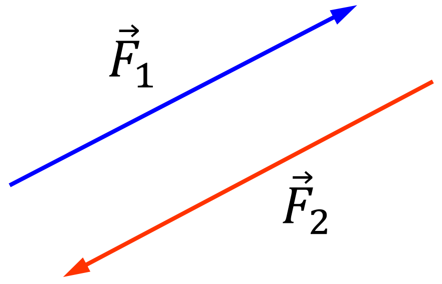
$\vec{F}_1 \neq \vec{F}_2$ even if $F_1 = F_2$



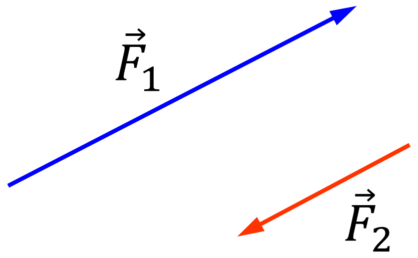
$\vec{F}_1 \neq \vec{F}_2$ since $F_1 \neq F_2$

Negative vectors

- Two vectors are negative of each other if they have the same magnitude, are parallel but in opposite directions.
- In the figure below, only the vectors in the first pair of vectors are negative of each other.



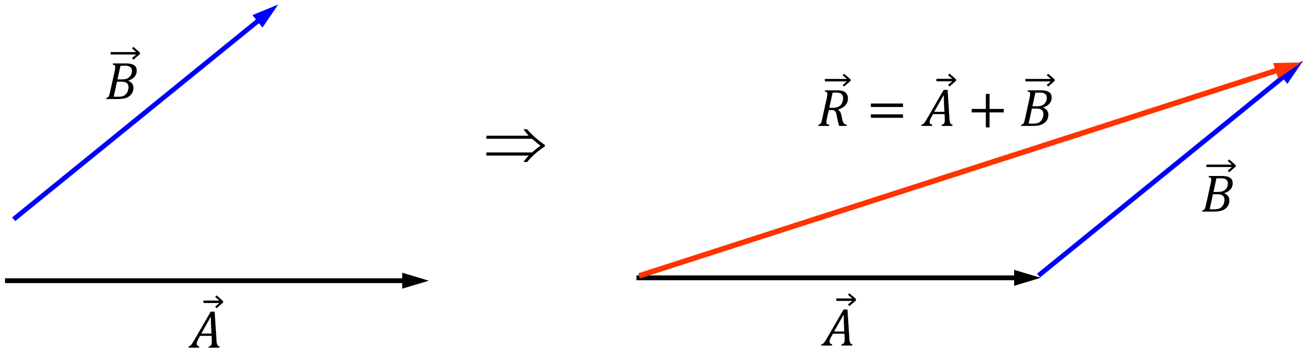
$\vec{F}_1 = -\vec{F}_2$ if $F_1 = F_2$



$\vec{F}_1 \neq -\vec{F}_2$

Triangle rule for adding two vectors

- The resultant of adding two vectors is obtained by
  - placing the tail of the second vector at the head of the first vector
  - drawing a line joining the tail of the first vector to the head of the second vector.
- It does not matter which vector is first and which is second.



Example 1

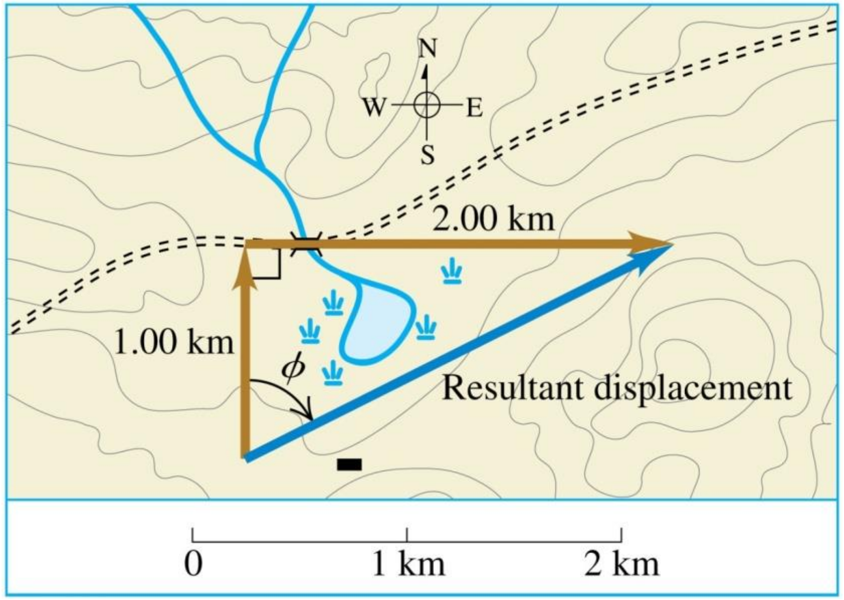
A cross-country skier skis 1.00 km north and then 2.00 km east on a horizontal snow field. How far and in what direction is she from the starting point, i.e. what is her final displacement?

The first displacement is 1.00 km north.

The second displacement is 2.00 km east.

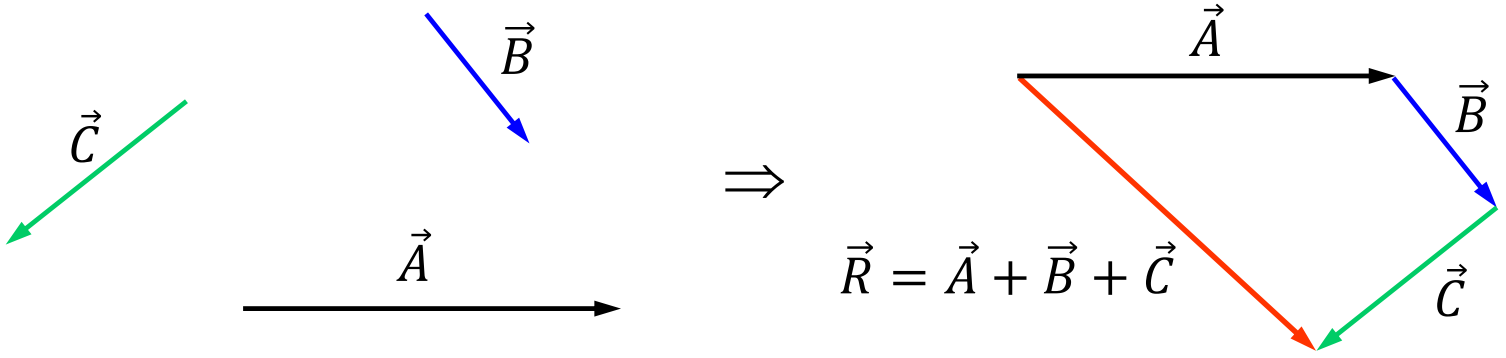
The resultant or final displacement is
$\sqrt{1.00^2 + 2.00^2} = 2.24$ km

The direction of the final displacement is
$\phi = \tan^{-1}(2.00) = 63.4°$ east of north or
$90° - 63.4° = 26.6°$ north of east.



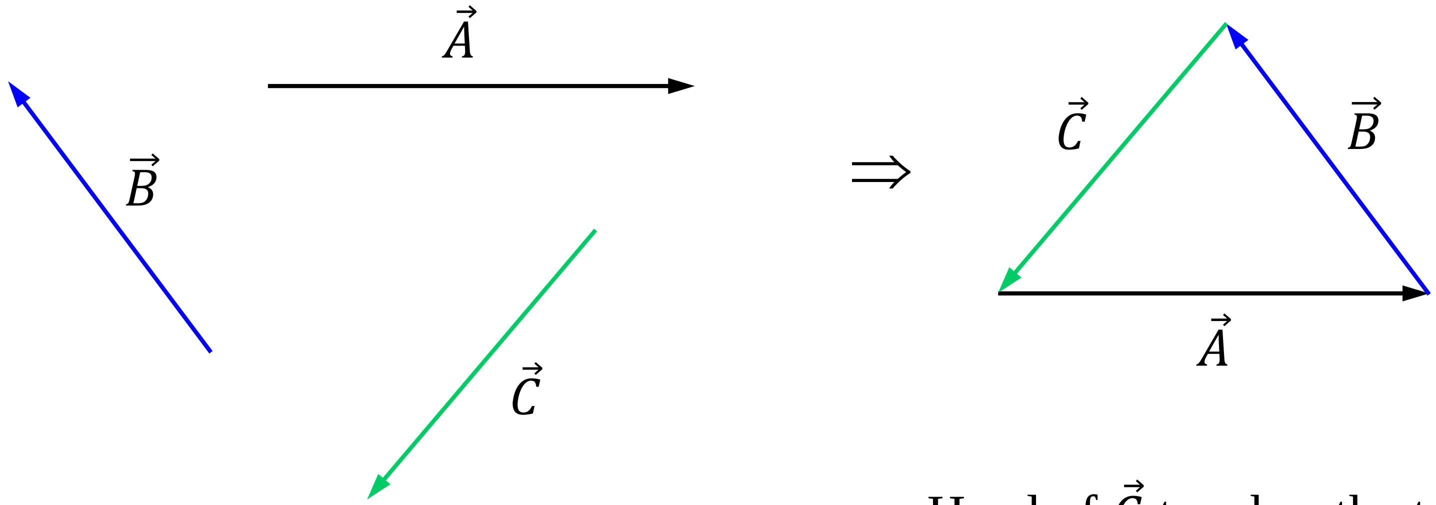
Polygon rule for adding 3 or more vectors

- This is an extension of the triangle rule for adding more than two vectors by drawing.
- The resultant is the line joining the tail of the first vector to the head of the last vector after putting all the vectors touching head to tail.



Example 2

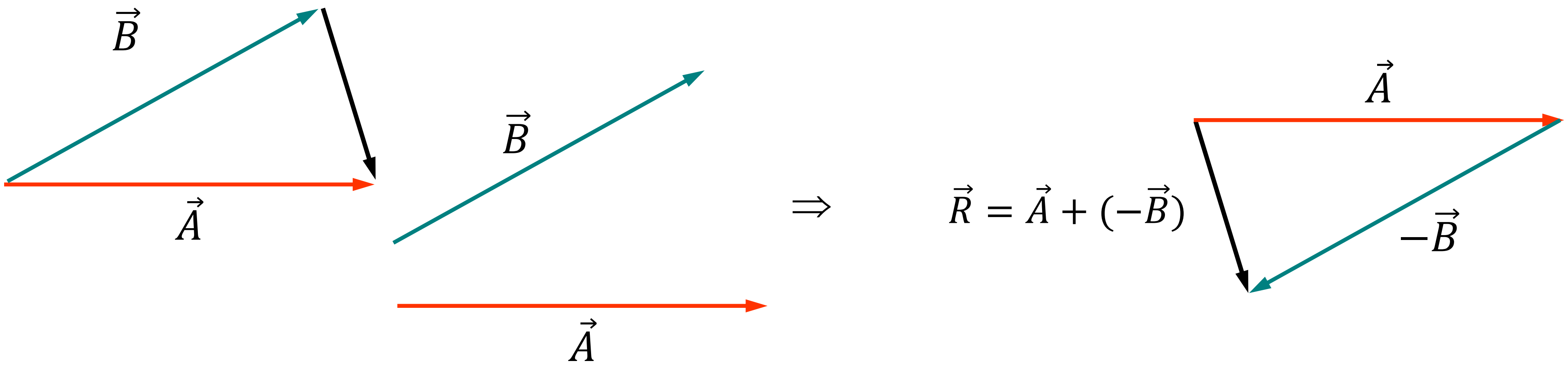
- Add the force vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$. Comment on your answer.



Head of $\vec{C}$ touches the tail of $\vec{A}$.
The resultant force is zero.

Triangle rule – Subtracting two vectors

- It is easier to treat vector subtraction as vector addition by reversing the vector to be subtracted.
- For example, to find $\vec{A} - \vec{B}$, we write it as $\vec{A} + (-\vec{B})$.

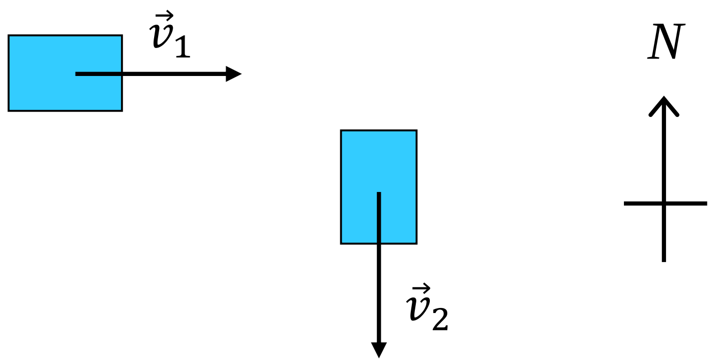


Example 3

An object is moving east at 11.0 m/s. It then changes direction to move at 11.0 m/s south. What is the change in its velocity?

Change in velocity is defined as

$$\Delta\vec{v} = \vec{v}_2 - \vec{v}_1 = \vec{v}_2 + (-\vec{v}_1)$$



The magnitude of $\Delta\vec{v}$ is

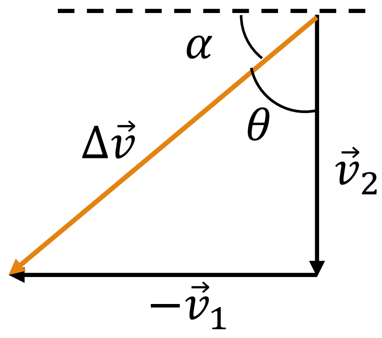
$$\Delta\vec{v} = \sqrt{11.0^2 + 11.0^2} = 15.6 \text{ m/s}$$

The angle θ is

$$\tan^{-1}\left(\frac{11.0}{11.0}\right) = 45°$$

The direction is angle $\alpha = 90° - \theta$

$$\alpha = 90° - 45° = 45° \text{ south of west}$$



End of pre-class slides

Vector components – Resolution of vectors

- Since vectors can be added to form a resultant, the reverse is also possible.
- Any vector can be resolved (break up) into its components.
- We usually resolve a vector into mutually perpendicular components.

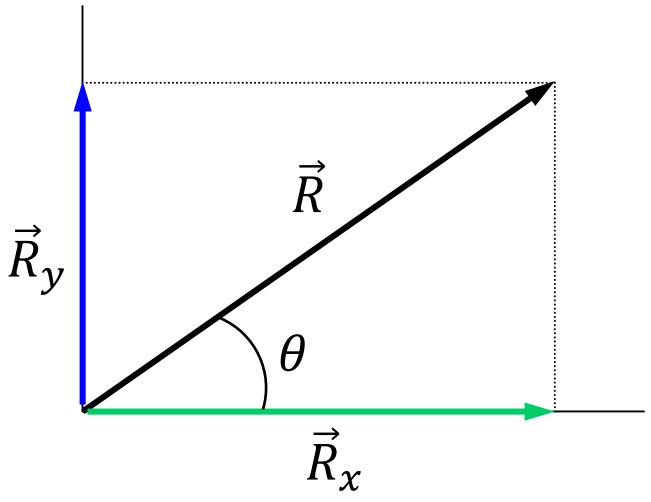
Resolving vectors in 2D

- A vector $\vec{R}$ in 2D can be resolved into two mutually perpendicular vectors, $\vec{R}_x$ and $\vec{R}_y$ such that $\vec{R} = \vec{R}_x + \vec{R}_y$.

where $R_x = R\cos\theta$ and $R_y = R\sin\theta$

The magnitude of $\vec{R}$ is $R = \sqrt{R_x^2 + R_y^2}$.

The direction of $\vec{R}$ is $\theta = \tan^{-1}\frac{R_y}{R_x}$.

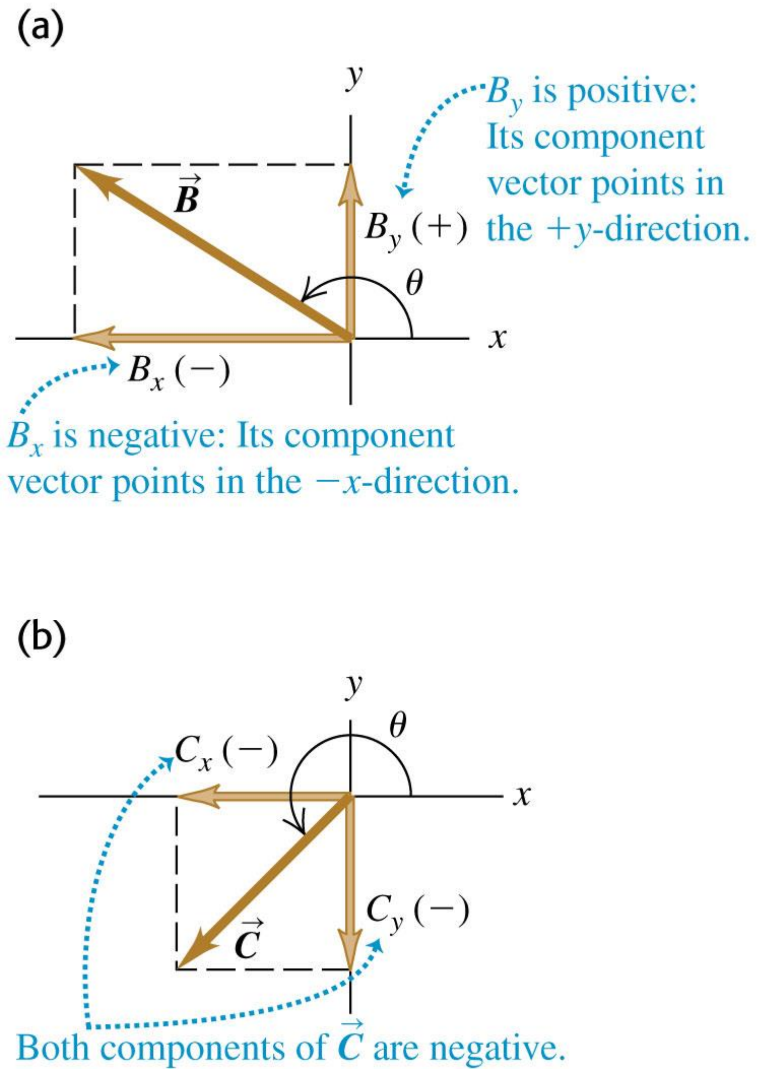


Positive and negative components

- The components of a vector can be positive or negative, depending on which quadrant of the circle it is in.

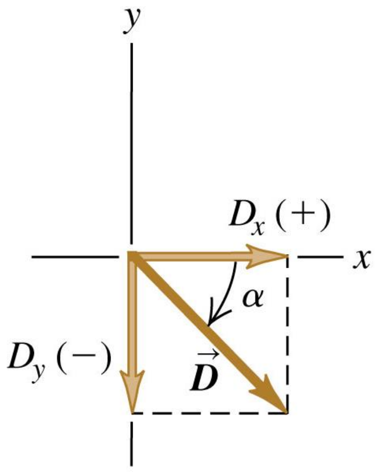
- In (a), $\vec{B}$ has a positive y component and a negative x component.

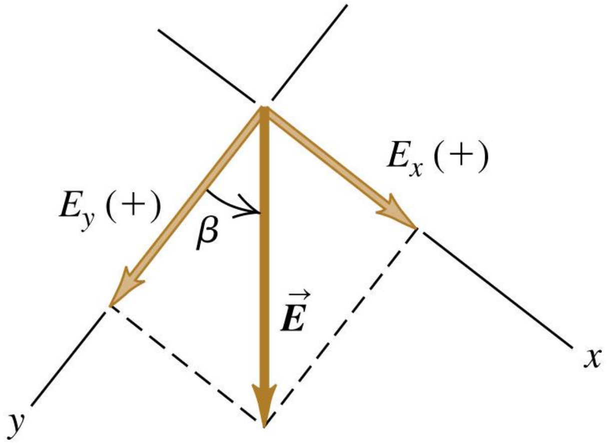
- In (b), $\vec{C}$ has a negative y component and a negative x component.



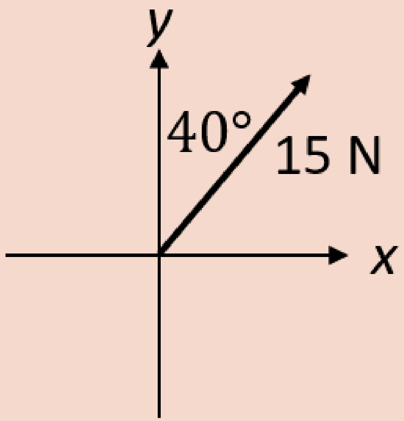
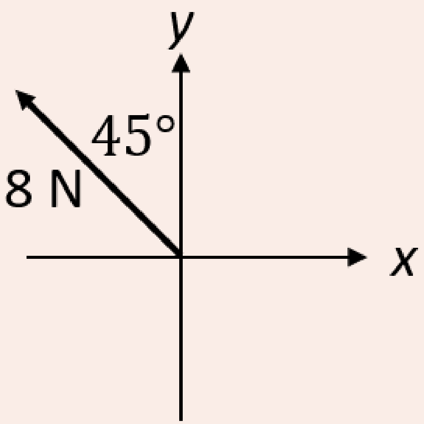
Example 4

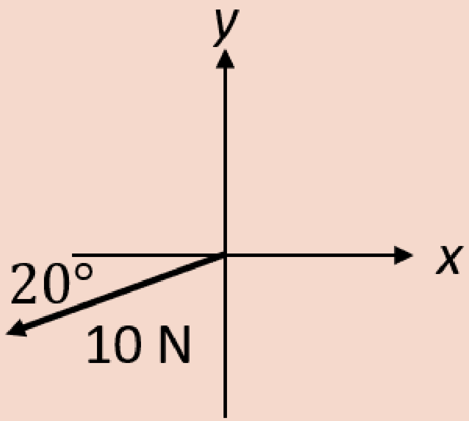
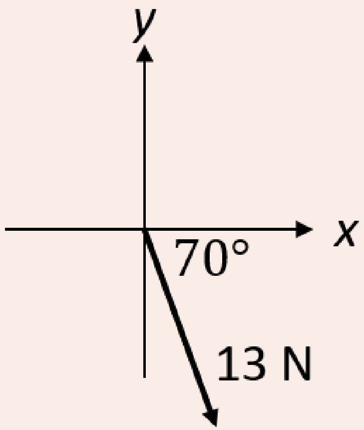
What are the x- and y- components of vectors $\vec{D}$ and $\vec{E}$ in the below figures? The magnitude of $\vec{D}$ is 3.00 m and the angle α is 45°. The magnitude of $\vec{E}$ is 4.50 m and the angle β is 37.0°.





Question

Vector	Components
	*x*-component: *y*-component:
	*x*-component: *y*-component:

Vector	Components
	*x*-component: *y*-component:
	*x*-component: *y*-component:

Resolving vectors in 3D

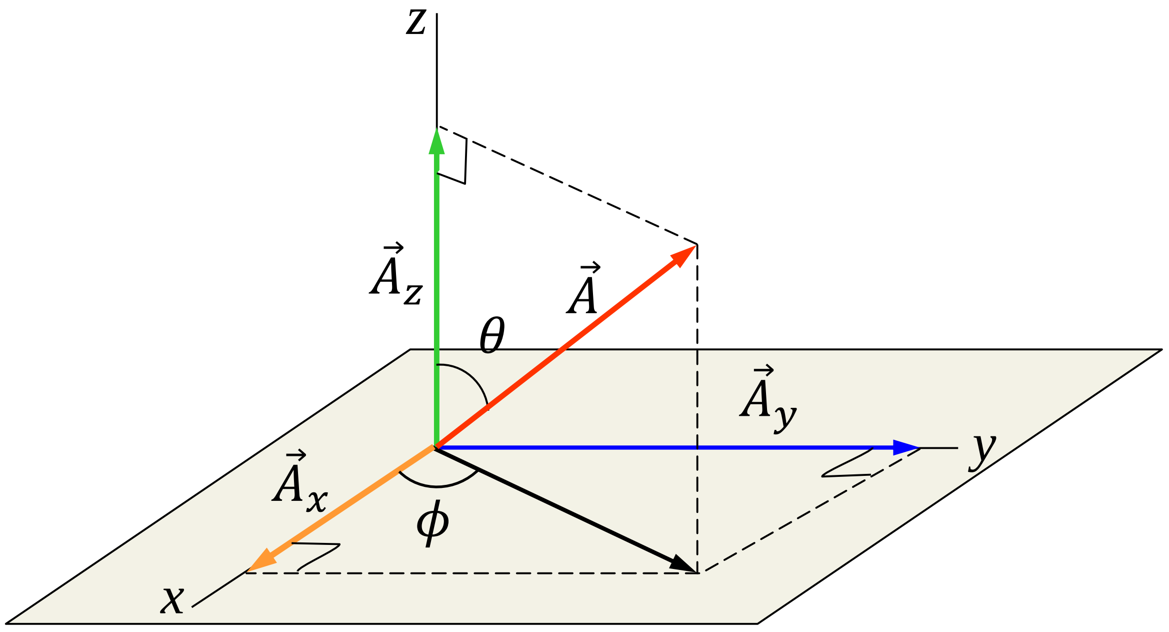
- A vector $\vec{A}$ in 3D can be resolved into three mutually perpendicular components as follow :

$$\vec{A} = \vec{A}_x + \vec{A}_y + \vec{A}_z$$

where $A_x = A \sin\theta \cos\phi$

$A_y = A \sin\theta \sin\phi$

$A_z = A \cos\theta$



The magnitude of $\vec{A}$ is $\sqrt{A_x^2 + A_y^2 + A_z^2}$.

The direction of $\vec{A}$ is given by

$$\phi = \tan^{-1}\frac{A_y}{A_x} \quad \text{and} \quad \theta = \tan^{-1}\frac{\sqrt{A_x^2+A_y^2}}{A_z}$$

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Vectors as column matrices

- A vector can also be represented as a column matrix where each row consists of its x, y, z components.

$$\vec{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

- The above notation is useful in solving problems.

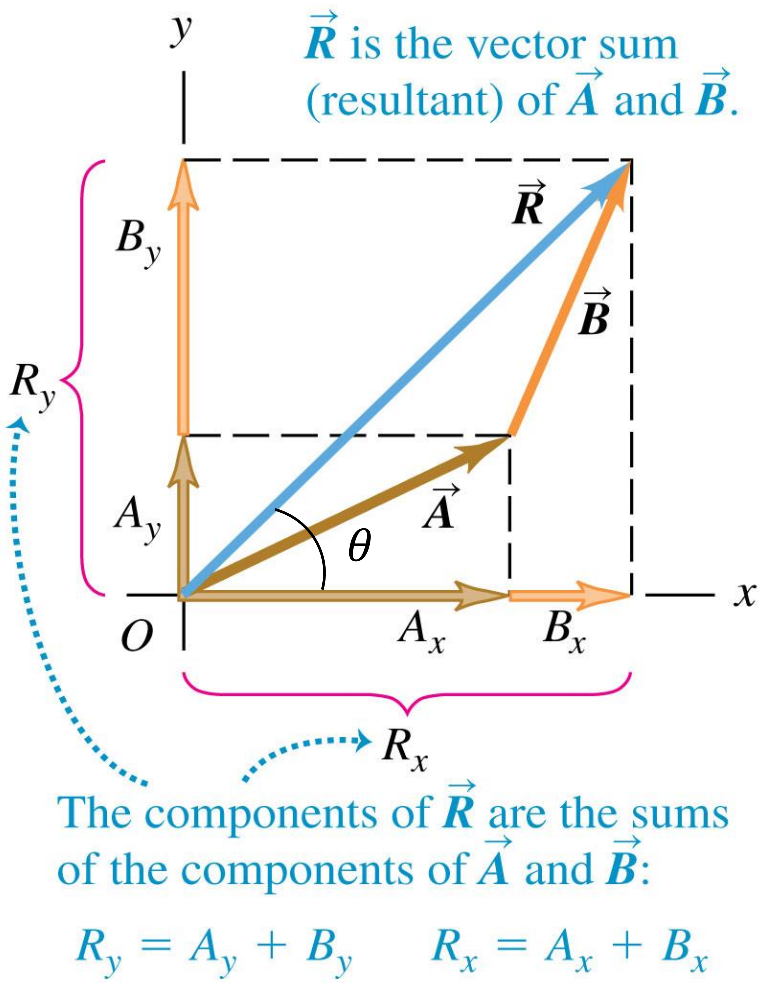
Vectors as column matrices

- We can use column matrices to find the sum of two vectors $\vec{A}$ and $\vec{B}$ by first adding their x and y components and then adding the resultants of the components using vector addition.

$$\begin{pmatrix} R_x \\ R_y \end{pmatrix} = \begin{pmatrix} A_x \\ A_y \end{pmatrix} + \begin{pmatrix} B_x \\ B_y \end{pmatrix}$$

The magnitude of $\vec{R}$ is $$R = \sqrt{R_x^2 + R_y^2}$$

The direction of $\vec{R}$ is $$\theta = \tan^{-1}\frac{R_y}{R_x}$$



Example 5

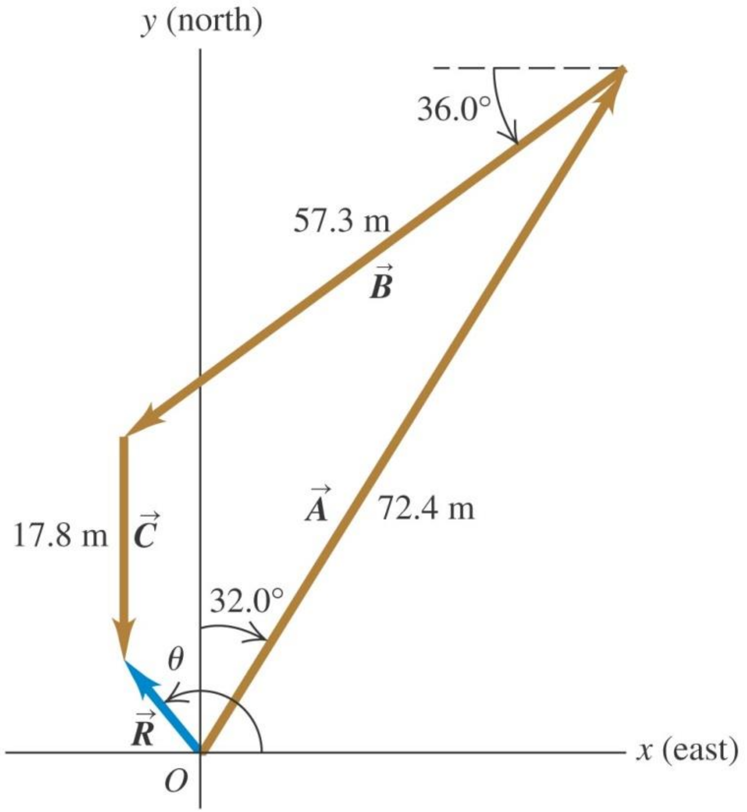
An object starts from a point O goes to point A which is described by vector $\vec{A}$: 72.4 m, 32.0° east of north. It then moves to point B which is described by vector $\vec{B}$: 57.3 m, 36.0° south of west. Finally it moves to point C which is described by $\vec{C}$: 17.8 m due south. Find the resultant displacement vector.

$$\vec{A} = \begin{pmatrix} 72.4 \sin 32° \\ 72.4 \cos 32° \end{pmatrix} \qquad \vec{B} = \begin{pmatrix} -57.3 \cos 36° \\ -57.3 \sin 36° \end{pmatrix} \qquad \vec{C} = \begin{pmatrix} 0 \\ -17.8 \end{pmatrix}$$

$$\vec{R} = \vec{A} + \vec{B} + \vec{C} = \begin{pmatrix} 38.37 \\ 61.40 \end{pmatrix} + \begin{pmatrix} -46.36 \\ -33.68 \end{pmatrix} + \begin{pmatrix} 0 \\ -17.8 \end{pmatrix} = \begin{pmatrix} -7.99 \\ 9.92 \end{pmatrix}$$

$$R = \sqrt{(-7.99)^2 + (9.92)^2} = 12.7 \text{ m}$$

$$\tan\theta = \frac{9.92}{-7.99} \quad \Rightarrow \quad \theta = 129°$$

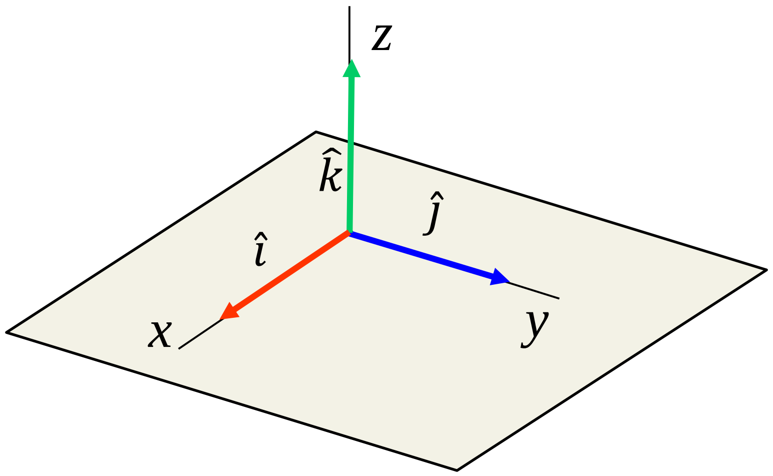


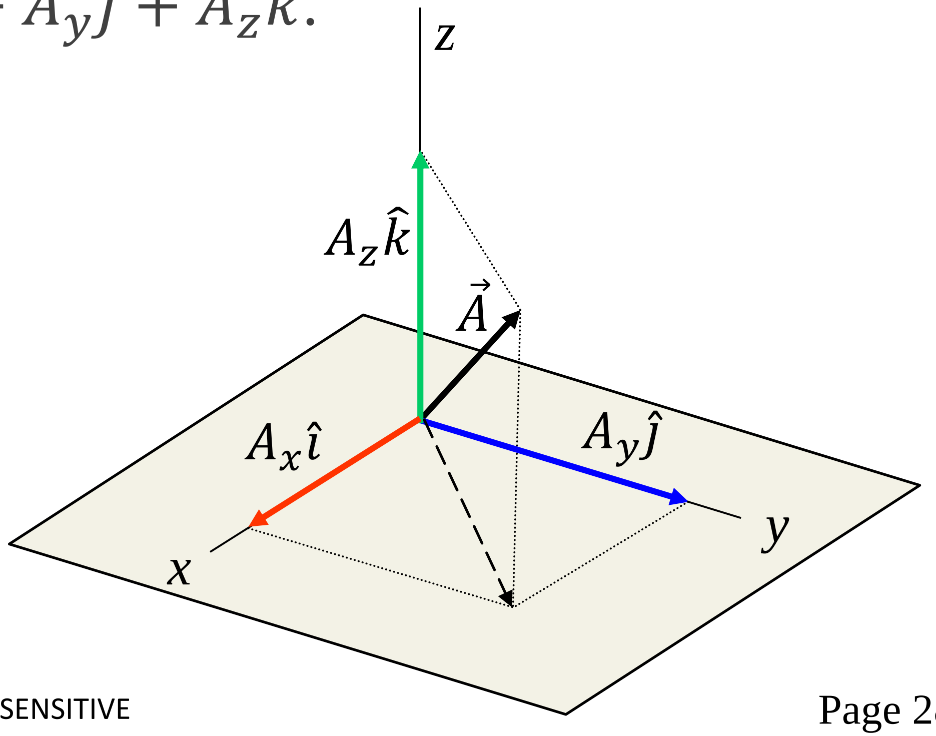
Unit vector

- A unit vector has magnitude 1.
- It is used to indicate the direction of a vector.
- Unit vectors are denoted with small letters and a hat on top.
- E.g. the unit vector pointing in the same direction as the vector $\vec{A}$ is $\hat{a}$.
- It follows that $\vec{A} = A\hat{a}$.

Unit vectors in 3D

- In 3D, it is usual to use the unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$.
- These vectors have size 1 and point in the direction of positive x, y and z respectively.
- Any vector $\vec{A}$ can be written as $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$.





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Vectors in column matrices and unit vectors

- Vectors in 2D:

$$\vec{A} = A_x\,\hat{\imath} + A_y\,\hat{\jmath} = \begin{pmatrix} A_x \\ A_y \end{pmatrix} = A_x \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\hat{\imath}} + A_y \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\hat{\jmath}}$$

- Vectors in 3D:

$$\vec{A} = A_x\,\hat{\imath} + A_y\,\hat{\jmath} + A_z\,\hat{k} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = A_x \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\hat{\imath}} + A_y \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{\hat{\jmath}} + A_z \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\hat{k}}$$

Vector addition and subtraction using unit vectors

- Addition/Subtraction: $\vec{C} = \vec{A} \pm \vec{B} = (A_x \pm B_x)\hat{\imath} + \left(A_y \pm B_y\right)\hat{\jmath} + (A_z \pm B_z)\hat{k}$

Example 6:

Given the position vectors $\vec{A} = 3\hat{\imath} - 4\hat{\jmath} + 4\hat{k}$ and $\vec{B} = 2\hat{\imath} + 3\hat{\jmath} - 7\hat{k}$, find

a) $\vec{A} + \vec{B}$

b) $\vec{A} - \vec{B}$

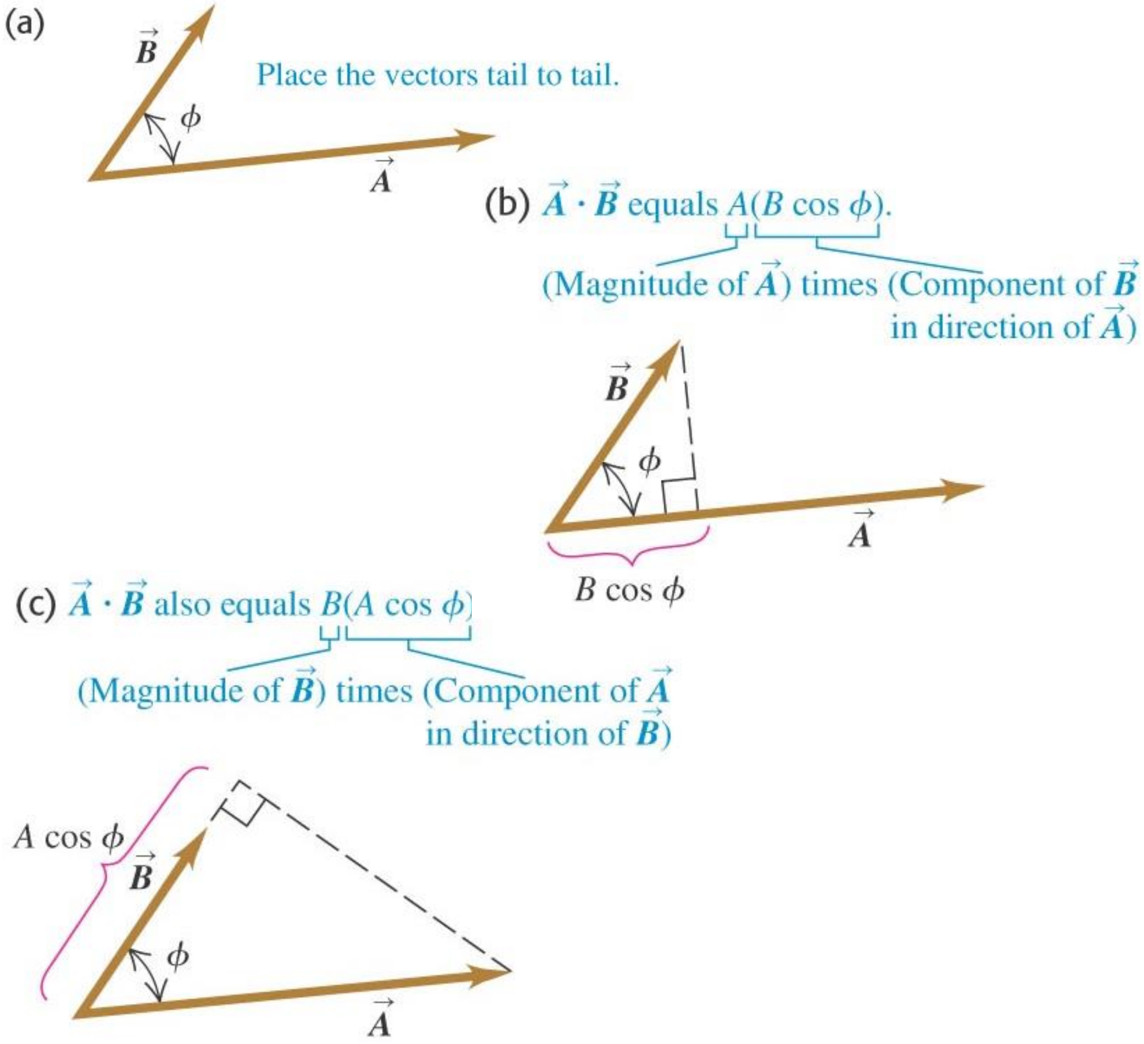
in terms of x, y and z components.

Product of vectors

- Some physical quantities can be expressed into products of vectors.
- However, vectors have both magnitudes and directions. We cannot perform ordinary multiplication with vectors.
- 2 types of products: **dot (scalar) product** and **cross (vector) product**.
- The <u>dot product</u> yields a <u>scalar</u> quantity while the <u>cross product</u> gives a <u>vector</u> quantity.

Dot (scalar) product of two vectors

- The dot product of two vectors $\vec{A}$ and $\vec{B}$ is defined as $C = \vec{A} \cdot \vec{B} = AB\cos\phi$ where ϕ is the angle between $\vec{A}$ and $\vec{B}$.
- The result C is a scalar.
- Note that A and B are positive quantities.
- Since $AB\cos\phi = BA\cos\phi$, so $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
- The dot product can be positive, negative or zero, depending on the angle ϕ.

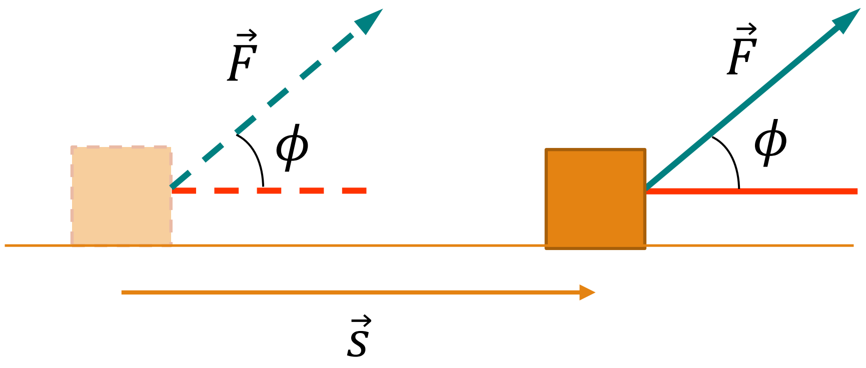


Application of dot product

- If a constant force $\vec{F}$ causes an object to undergo a displacement $\vec{s}$, the work done, which is a scalar, is given by $W = \vec{F} \cdot \vec{s} = Fs\cos\phi$.

- Work is maximum when the force is parallel to the displacement and work is zero when the force is perpendicular to the displacement.

- Work can be zero, positive or negative depending on ϕ .



Example 7

Determine the magnetic flux Φ_B, where Φ_B is defined as $\Phi_B = \vec{B} \cdot \vec{A}$. Assume that $\vec{B}$ and $\vec{A}$ are at 60° to each other and B is 1.0 tesla (T) and A is 2.0 m^2.

$$\Phi_B = \vec{B} \cdot \vec{A} = BA \cos 60°$$
$$= 1.0 \times 2.0 \times 0.5$$
$$= 1.0 \text{ T m}^2$$
$$= 1.0 \text{ weber (Wb)}$$

Dot product in unit vector notation

- Dot product with unit vectors:
  - $\hat{\imath} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{\jmath} = \hat{k} \cdot \hat{k} = 1 \times 1 \cos 0° = 1$
  - $\hat{\imath} \cdot \hat{\jmath} = \hat{\jmath} \cdot \hat{k} = \hat{k} \cdot \hat{\imath} = 1 \times 1 \cos 90° = 0$
- Dot product between $\vec{A} = A_x\,\hat{\imath} + A_y\,\hat{\jmath} + A_z\,\hat{k}$ and $\vec{B} = B_x\,\hat{\imath} + B_y\,\hat{\jmath} + B_z\,\hat{k}$

$$
\begin{aligned}
\vec{A} \cdot \vec{B} &= \left(A_x\,\hat{\imath} + A_y\,\hat{\jmath} + A_z\,\hat{k}\right) \cdot \left(B_x\,\hat{\imath} + B_y\,\hat{\jmath} + B_z\,\hat{k}\right) \\
&= A_xB_x\,\hat{\imath} \cdot \hat{\imath} + \cancel{A_xB_y\,\hat{\imath} \cdot \hat{\jmath}} + \cancel{A_xB_z\,\hat{\imath} \cdot \hat{k}} \\
&\quad + \cancel{A_yB_x\,\hat{\jmath} \cdot \hat{\imath}} + A_yB_y\,\hat{\jmath} \cdot \hat{\jmath} + \cancel{A_yB_z\,\hat{\jmath} \cdot \hat{k}} \\
&\quad + \cancel{A_zB_x\,\hat{k} \cdot \hat{\imath}} + \cancel{A_zB_y\,\hat{k} \cdot \hat{\jmath}} + A_zB_z\,\hat{k} \cdot \hat{k} \\
&= A_xB_x + A_yB_y + A_zB_z
\end{aligned}
$$

Example 8

A force $\vec{F} = 4\hat{\imath} + 3\hat{\jmath} - 5\hat{k}$ N acts on an object and produces a displacement $\vec{s} = 6\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ m. What is the work done?

By definition,

$$W = \vec{F} \cdot \vec{s}$$

$$W = \left(4\hat{\imath} + 3\hat{\jmath} - 5\hat{k}\right) \cdot \left(6\hat{\imath} + 2\hat{\jmath} + 3\hat{k}\right)$$

$$W = 4(6) + 3(2) + (-5)(3) = 15 \text{ J}$$

Application of dot product

- With the unit vector notation, the dot product definition becomes

$$\vec{A} \cdot \vec{B} = AB\cos\phi = A_x B_x + A_y B_y + A_z B_z$$

- We can determine the angle between two 3D vectors using dot product.

- (Not tested) Another application of dot product is to find the length of the projection of vector $\vec{A}$ in the direction of vector $\vec{B}$ or vice versa.
  - Step 1: Find the angle θ between vectors $\vec{A}$ and $\vec{B}$
  - Step 2: Length of the projection of vector $\vec{A}$ onto vector $\vec{B} = A\cos\theta$

Example 9

Find the angle between the two vectors $\vec{A} = 2\hat{\imath} + 3\hat{\jmath} + \hat{k}$ and $\vec{B} = -4\hat{\imath} + 2\hat{\jmath} - \hat{k}$.

$$\vec{A} \cdot \vec{B} = \left(2\hat{\imath} + 3\hat{\jmath} + \hat{k}\right) \cdot \left(-4\hat{\imath} + 2\hat{\jmath} - \hat{k}\right)$$

$$\vec{A} \cdot \vec{B} = (2)(-4) + (3)(2) + (1)(-1)$$

$$= -8 + 6 - 1 = -3$$

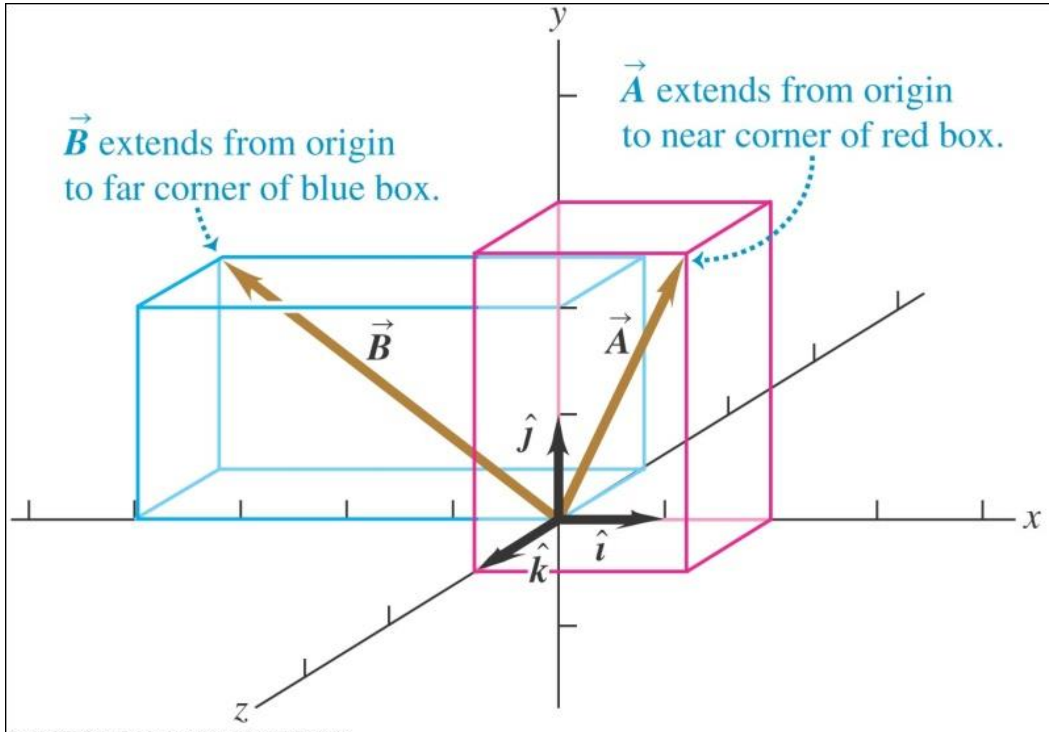
$$A = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$$

$$B = \sqrt{4^2 + 2^2 + 1^2} = \sqrt{21}$$

$$\vec{A} \cdot \vec{B} = AB\cos\phi$$

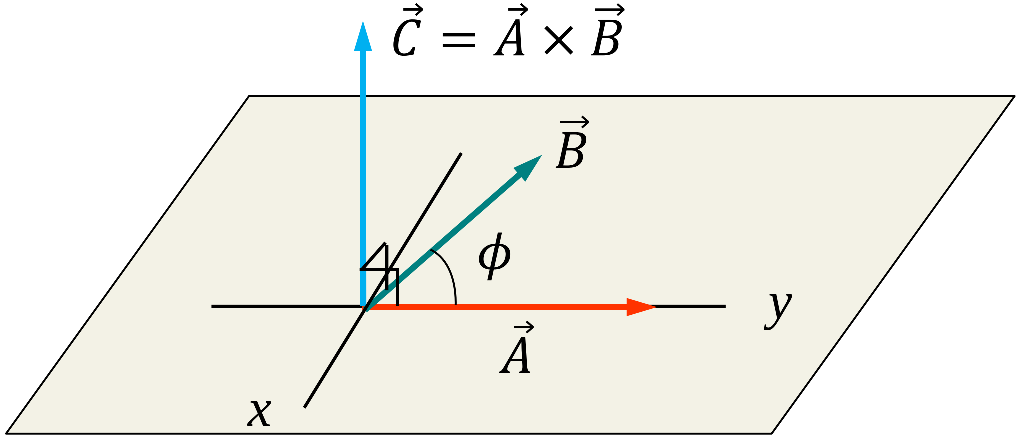
$$\cos\phi = \frac{\vec{A} \cdot \vec{B}}{AB} = \frac{-3}{\sqrt{14}\sqrt{21}} = -0.175$$

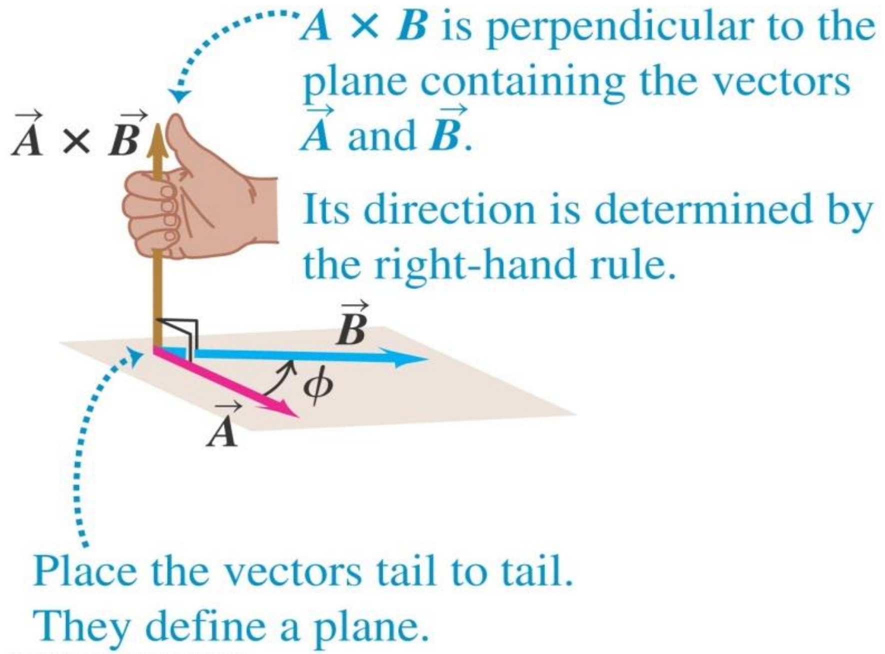
$$\phi = \cos^{-1} -0.175 = 100°$$



Cross (vector) product

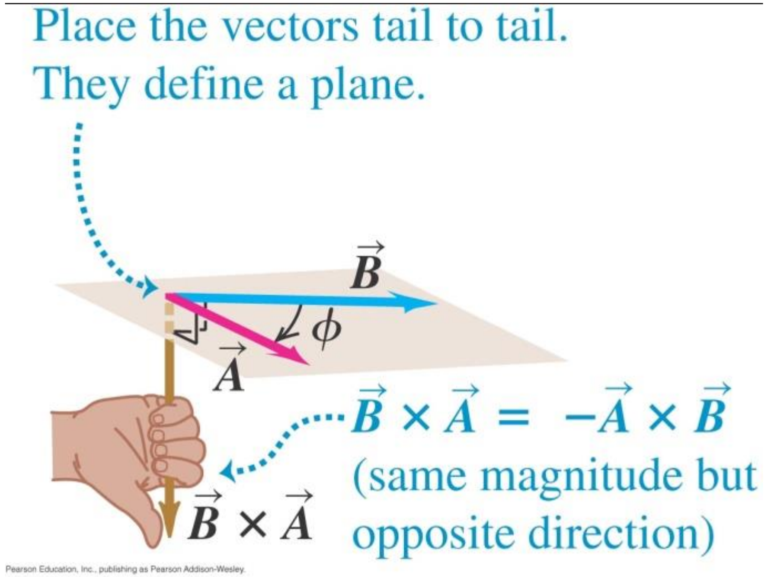
- The cross product of two vectors $\vec{A}$ and $\vec{B}$ is defined by $\vec{C} = \vec{A} \times \vec{B}$.
- The magnitude of $\vec{C}$ is given by $C = AB \sin \phi$, where $0° \leq \phi \leq 180°$.
- The direction of $\vec{C}$ is perpendicular to both $\vec{A}$ and $\vec{B}$ and can be obtained using the right hand grip rule.





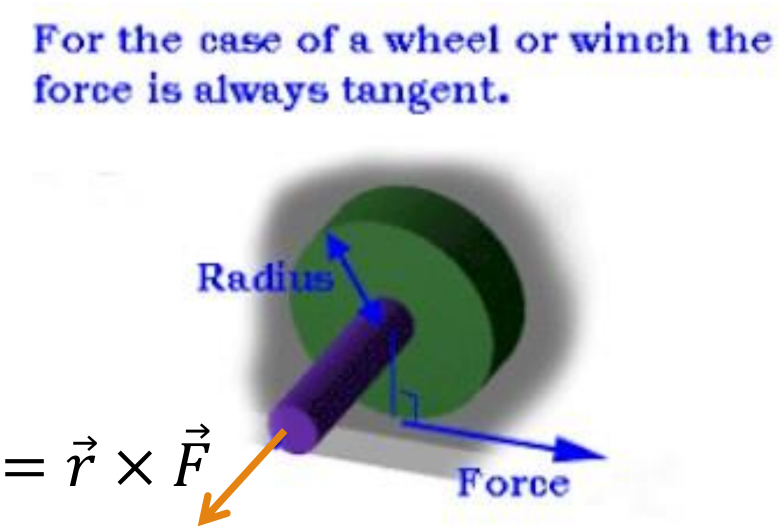
Right hand grip rule

- If $\vec{B}$ is rotated about a perpendicular line until it is aligned with $\vec{A}$, the thumb points in the direction of $\vec{B} \times \vec{A}$.
- In other words, $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$.



Application of cross product

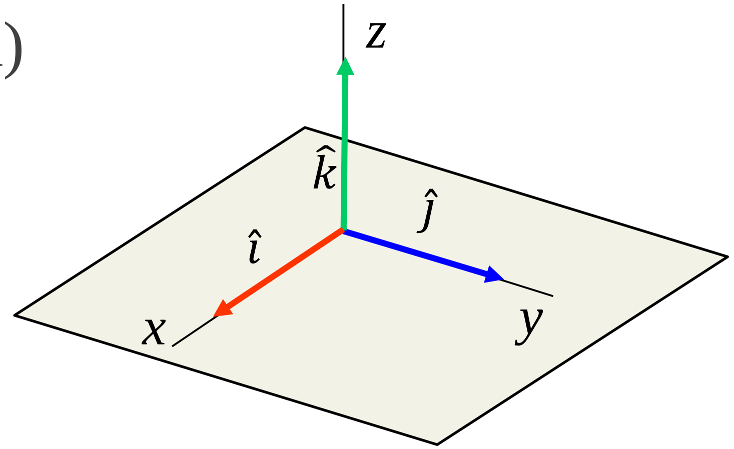
- The vector $\vec{\tau} = \vec{r} \times \vec{F}$ is known as torque where $\vec{r}$ is the radius vector and $\vec{F}$ is the force vector.
- The magnitude of $\vec{\tau}$ is known as the moment (turning effect) of the force.
- The direction of $\vec{\tau}$ is given by the right hand grip rule.



$$\vec{\tau} = \vec{r} \times \vec{F}$$

Cross product in unit vector notations

- Cross products (follow right-handed coordinate system)
  - $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 1 \times 1 \sin 0° = 0$
  - $\hat{\imath} \times \hat{\jmath} = 1 \times 1 \sin 90°\, \hat{k} = \hat{k} = -\hat{\jmath} \times \hat{\imath}$
  - $\hat{\jmath} \times \hat{k} = 1 \times 1 \sin 90°\, \hat{\imath} = \hat{\imath} = -\hat{k} \times \hat{\jmath}$
  - $\hat{k} \times \hat{\imath} = 1 \times 1 \sin 90°\, \hat{\jmath} = \hat{\jmath} = -\hat{\imath} \times \hat{k}$



- Cross product between $\vec{A} = A_x\, \hat{\imath} + A_y\, \hat{\jmath} + A_z\, \hat{k}$ and $\vec{B} = B_x\, \hat{\imath} + B_y\, \hat{\jmath} + B_z\, \hat{k}$

$$\vec{A} \times \vec{B} = \left(A_x\, \hat{\imath} + A_y\, \hat{\jmath} + A_z\, \hat{k}\right) \times \left(B_x\, \hat{\imath} + B_y\, \hat{\jmath} + B_z\, \hat{k}\right)$$

$$= \cancel{A_x B_x\, \hat{\imath} \times \hat{\imath}} + A_x B_y\, \hat{\imath} \times \hat{\jmath} + A_x B_z\, \hat{\imath} \times \hat{k}$$

$$+ A_y B_x\, \hat{\jmath} \times \hat{\imath} + \cancel{A_y B_y\, \hat{\jmath} \times \hat{\jmath}} + A_y B_z\, \hat{\jmath} \times \hat{k}$$

$$+ A_z B_x\, \hat{k} \times \hat{\imath} + A_z B_y\, \hat{k} \times \hat{\jmath} + \cancel{A_z B_z\, \hat{k} \times \hat{k}}$$

Cross product in unit vector notations

- Cross product between $\vec{A}$ and $\vec{B}$:

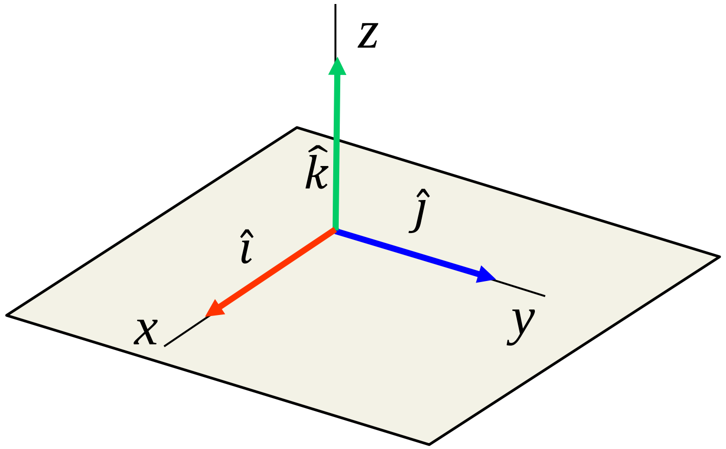
$$\vec{A} \times \vec{B} = \left(A_x\,\hat{\imath} + A_y\,\hat{\jmath} + A_z\,\hat{k}\right) \times \left(B_x\,\hat{\imath} + B_y\,\hat{\jmath} + B_z\,\hat{k}\right)$$

$$= \quad A_xB_y\,\hat{\imath} \times \hat{\jmath} + A_xB_z\,\hat{\imath} \times \hat{k}$$

$$+ \quad A_yB_x\,\hat{\jmath} \times \hat{\imath} + A_yB_z\,\hat{\jmath} \times \hat{k}$$

$$+ \quad A_zB_x\,\hat{k} \times \hat{\imath} + A_zB_y\,\hat{k} \times \hat{\jmath}$$

$$= \left(A_yB_z - A_zB_y\right)\hat{\imath} - \left(A_xB_z - A_zB_x\right)\hat{\jmath} + \left(A_xB_y - A_yB_x\right)\hat{k}$$



Reference:

$\hat{\imath} \times \hat{\jmath} = \hat{k} = -\hat{\jmath} \times \hat{\imath}$

$\hat{\jmath} \times \hat{k} = \hat{\imath} = -\hat{k} \times \hat{\jmath}$

$\hat{k} \times \hat{\imath} = \hat{\jmath} = -\hat{\imath} \times \hat{k}$

Cross product in unit vector notations

- Cross product between $\vec{A}$ and $\vec{B}$ can be re-expressed as a determinant

$$\vec{A} \times \vec{B} = \left(A_x\,\hat{\imath} + A_y\,\hat{\jmath} + A_z\,\hat{k}\right) \times \left(B_x\,\hat{\imath} + B_y\,\hat{\jmath} + B_z\,\hat{k}\right)$$

$$= \left(A_yB_z - A_zB_y\right)\hat{\imath} - \left(A_xB_z - A_zB_x\right)\hat{\jmath} + \left(A_xB_y - A_yB_x\right)\hat{k}$$

$$= \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix}\hat{\imath} - \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix}\hat{\jmath} + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}\hat{k}$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Example 10

A force $\vec{F} = 4\hat{\imath} + 3\hat{\jmath} - 5\hat{k}$ N acts on an object with position vector $\vec{r} = 6\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ m. What is the torque about the origin?

By definition,

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\vec{\tau} = \left(6\hat{\imath} + 2\hat{\jmath} + 3\hat{k}\right) \times \left(4\hat{\imath} + 3\hat{\jmath} - 5\hat{k}\right)$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 6 & 2 & 3 \\ 4 & 3 & -5 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 3 & -5 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 6 & 3 \\ 4 & -5 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 6 & 2 \\ 4 & 3 \end{vmatrix} \hat{k}$$

$$= -19\hat{\imath} + 42\hat{\jmath} + 10\hat{k} \text{ Nm}$$

Vector operations using unit vectors (Summary)

- Addition/Subtraction

$$\vec{C} = \vec{A} \pm \vec{B} = (A_x \pm B_x)\hat{\imath} + \left(A_y \pm B_y\right)\hat{\jmath} + (A_z \pm B_z)\hat{k}$$

- Dot product

$$C = \vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

- Cross product

$$\vec{C} = \vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} \hat{\imath} - \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} \hat{\jmath} + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} \hat{k}$$

$$= \left(A_y B_z - A_z B_y\right)\hat{\imath} - (A_x B_z - A_z B_x)\hat{\jmath} + \left(A_x B_y - A_y B_x\right)\hat{k}$$

End of chapter