Chapter 8: Vectors

Objectives

At the end of this topic, you will be able to:

- Define vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$
- List properties of vector addition and scalar multiplication
- Define dot product (scalar product) of two vectors and state its properties
- Define cross product (vector product) of two vectors
- Find equations of lines and planes in space

8.1 Vectors

(A) Vectors and Scalars

A *vector* is a quantity that has both magnitude and direction.

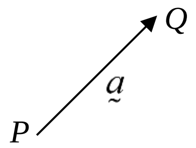
Some examples are velocity (speed in a certain direction), displacement (a movement in a certain direction), force, acceleration, … etc.

A *scalar* is a quantity that has magnitude but no direction.

Some examples are mass, length, temperature, electric charge, work, … etc.

(B) Representation of Vectors and Notation

A vector is usually represented by a directed line segment.



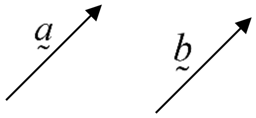
The directed line segment shown in the above figure is the vector from points P to Q and is denoted by $\overrightarrow{PQ}$, or simply by a single letter $\underset{\sim}{a}$.

The magnitude of the vector $\overrightarrow{PQ}$ is specified by the length of the line segment PQ and is denoted by $\left\|\overrightarrow{PQ}\right\|$ or $\|\underset{\sim}{a}\|$. The direction of a vector is the angle it makes with the positive x-axis, measured anticlockwise.

(C) Equality of Vectors

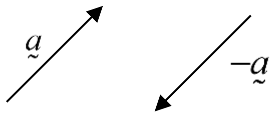
Two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are equal *if and only if* they have the same magnitude and direction.

$\underset{\sim}{a} = \underset{\sim}{b} \iff \underset{\sim}{a}$ and $\underset{\sim}{b}$ have the same direction and $\|\underset{\sim}{a}\| = \|\underset{\sim}{b}\|$



(D) Negative Vectors

The negative vector $-\underset{\sim}{a}$ is a vector having the same magnitude as $\underset{\sim}{a}$ but a direction opposite to that of $\underset{\sim}{a}$.



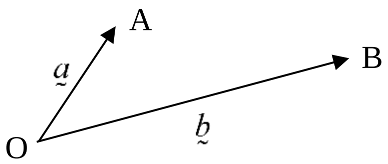
$\|-\underset{\sim}{a}\| = \|\underset{\sim}{a}\|$

(E) Zero Vector

The *zero* or *null* vector denoted by $\underset{\sim}{0}$ is the vector with zero magnitude and no particular direction.

(F) Position Vector

A position vector is a vector that starts from the origin. We say that the position vector of the point A = $\overrightarrow{OA} = \underset{\sim}{a}$, and the position vector of the point B $\overrightarrow{OB} = \underset{\sim}{b}$



(G) Unit Vectors

A *unit vector* is a vector whose magnitude is 1.

If $\underset{\sim}{a}$ is any non-zero vector, the unit vector with the same direction as $\underset{\sim}{a}$ is denoted by $\underset{\sim}{\hat{a}}$.

Since $\underset{\sim}{a} = \|\underset{\sim}{a}\|\,\underset{\sim}{\hat{a}}$, it follows that the *unit vector with the same direction as* $\underset{\sim}{a}$ is given by

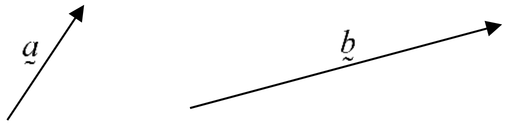
$$\boxed{\underset{\sim}{\hat{a}} = \frac{\underset{\sim}{a}}{\|\underset{\sim}{a}\|}}$$

Hence, a vector of magnitude *m* in the same direction as $\underset{\sim}{a}$ is equal to $m\underset{\sim}{\hat{a}}$.

(H) Arithmetic Operations on Vectors

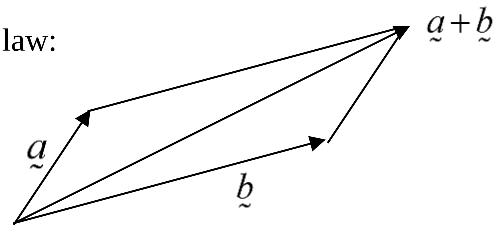
(i) Addition of Vectors

Let $\underset{\sim}{a}$ and $\underset{\sim}{b}$ be two vectors.

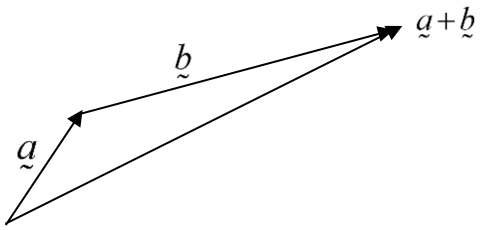


The *resultant* or the *vector sum* of the two vectors is $\underset{\sim}{a}+\underset{\sim}{b}$ and can be obtained using the parallelogram law or the triangle law.

The parallelogram law:



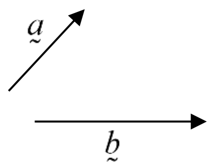
The triangle law:

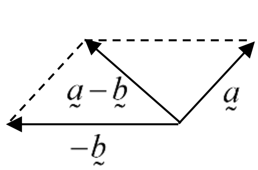


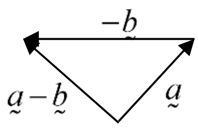
Can you show that $\underset{\sim}{a}+\underset{\sim}{b}$ and $\underset{\sim}{b}+\underset{\sim}{a}$ are equal?

(ii) Subtraction of Vectors

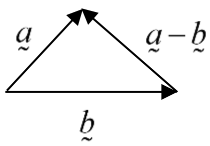
The difference of two vectors, $\underset{\sim}{a}-\underset{\sim}{b}$ is defined as the vector sum $\underset{\sim}{a}+(-\underset{\sim}{b})$







This is the same as

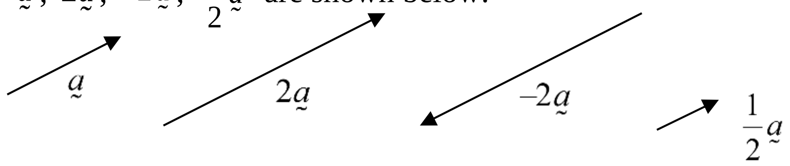


(iii) Scalar Multiplication

Let λ be a non-zero real number (i.e, a scalar) and $\underset{\sim}{a}$ a non-zero vector.
The vector $\lambda\underset{\sim}{a}$ is a vector parallel to $\underset{\sim}{a}$.

- If $\lambda > 0$, $\lambda\underset{\sim}{a}$ is in the same direction as $\underset{\sim}{a}$.
- If $\lambda < 0$, $\lambda\underset{\sim}{a}$ is in the opposite direction from $\underset{\sim}{a}$.
- $\|\lambda\underset{\sim}{a}\| = |\lambda|\,\|\underset{\sim}{a}\|$

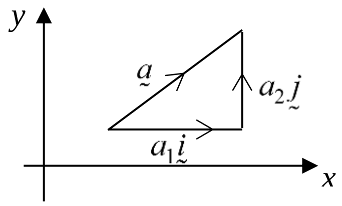
Vectors $\underset{\sim}{a}$, $2\underset{\sim}{a}$, $-2\underset{\sim}{a}$, $\frac{1}{2}\underset{\sim}{a}$ are shown below:



Thus, if $\underset{\sim}{a} \neq \underset{\sim}{0}$ and $\underset{\sim}{b} \neq \underset{\sim}{0}$, $\boxed{\underset{\sim}{a} \text{ and } \underset{\sim}{b} \text{ are parallel} \Leftrightarrow \underset{\sim}{a} = \lambda\underset{\sim}{b}}$ where λ is a scalar and $\lambda \neq 0$.

(I) Vectors in Two-Dimensional Space $\mathbb{R}^2$

Let us consider a two-dimensional coordinate frame.



A unit vector in the positive direction of the x-axis is denoted by $\underset{\sim}{i}$.

A unit vector in the positive direction of the y-axis is denoted by $\underset{\sim}{j}$.

For every vector $\underset{\sim}{a}$ in the xy-plane, if a_1 and a_2 are the components of $\underset{\sim}{a}$ in the x and y directions respectively, then $\underset{\sim}{a}$ can be expressed in the following forms**:**

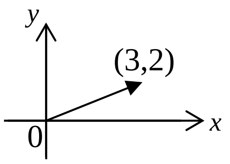
$\underset{\sim}{i}\ \underset{\sim}{j}$ **- notation**	**Vector notation**	**Matrix notation**
$\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j}$	$\underset{\sim}{a} = \langle a_1, a_2 \rangle$	$\underset{\sim}{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

By Pythagoras' theorem, the magnitude of $\underset{\sim}{a}$ is $\boxed{\|\underset{\sim}{a}\| = \sqrt{a_1^{\,2} + a_2^{\,2}}}$.

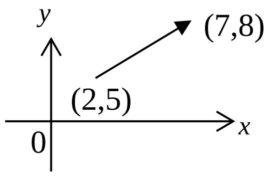
Example 1

Write the following vectors in $x\underset{\sim}{i} + y\underset{\sim}{j}$ form.

(a)



(b)



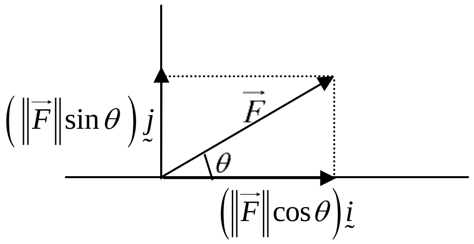
Example 2

If $\underset{\sim}{a} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$, find

(a) $\|\underset{\sim}{a}\|$ (b) $\hat{\underset{\sim}{a}}$ (c) a vector of magnitude 20 in the direction of $\underset{\sim}{a}$

(J) Resolution of a Vector into Components in Two Perpendicular Directions

In applications, we often resolve a vector into its components in two perpendicular directions.

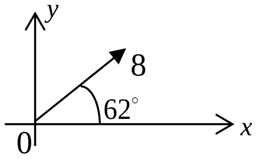


$$\boxed{\vec{F} = \left(\|\vec{F}\|\cos\theta\right)\underset{\sim}{i} + \left(\|\vec{F}\|\sin\theta\right)\underset{\sim}{j}}$$

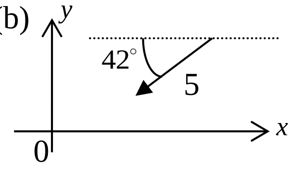
Example 3 (to be taught in class)

Write the following vectors in $x\underset{\sim}{i} + y\underset{\sim}{j}$ form.

(a)

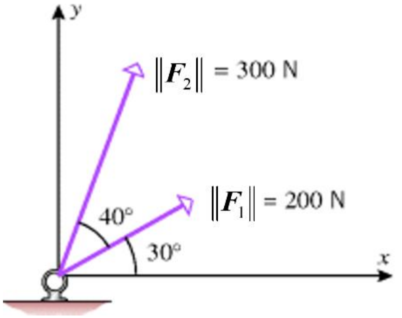


(b)



Example 4 (to be taught in class)

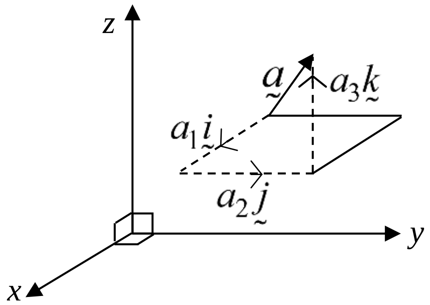
Suppose that two forces are applied to an eye bracket, as shown below. Find the magnitude of the resultant and the angle θ that it makes with the positive x-axis.



(K) Vectors in Three-Dimensional Space $\mathbb{R}^3$

Let us consider a three-dimensional coordinate frame.

$\|\underset{\sim}{i}\| = \|\underset{\sim}{j}\| = \|\underset{\sim}{k}\| = 1$



The axes are placed in such a way that they follow the Right-hand Rule.

The vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are unit vectors in the positive directions of the *x*-, *y*- and *z*-axis, respectively.

For every vector $\underset{\sim}{a}$ in $\mathbb{R}^3$, if a_1, a_2 and a_3 are the components of $\underset{\sim}{a}$ in the *x*, *y* and *z* directions respectively, then $\underset{\sim}{a}$ can be expressed in the following form:

$\underset{\sim}{i}\,\underset{\sim}{j}\,\underset{\sim}{k}$ - notation	Vector notation	Matrix notation
$\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$	$\underset{\sim}{a} = \langle a_1, a_2, a_3 \rangle$	$\underset{\sim}{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$

By Pythagoras' theorem, the magnitude of $\underset{\sim}{a}$ is $\boxed{\|\underset{\sim}{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}}$.

<u>Example 5</u>

If $\underset{\sim}{a} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$, find

(a) $\|\underset{\sim}{a}\|$

(b) a vector of magnitude 5 in the direction of $\underset{\sim}{a}$.

(L) Addition, Subtraction and Scalar Multiplication in Component Form

THEOREM

Given that $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$ and $\underset{\sim}{b} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k}$, then

	$\underset{\sim}{i}\,\underset{\sim}{j}\,\underset{\sim}{k}$ - notation	Vector notation	Matrix notation
Addition $\underset{\sim}{a} + \underset{\sim}{b}$	$(a_1+b_1)\underset{\sim}{i} + (a_2+b_2)\underset{\sim}{j} + (a_3+b_3)\underset{\sim}{k}$	$\langle a_1, a_2, a_3\rangle + \langle b_1, b_2, b_3\rangle = \langle a_1+b_1, a_2+b_2, a_3+b_3\rangle$	$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \end{bmatrix}$
Subtraction $\underset{\sim}{a} - \underset{\sim}{b}$	$(a_1-b_1)\underset{\sim}{i} + (a_2-b_2)\underset{\sim}{j} + (a_3-b_3)\underset{\sim}{k}$	$\langle a_1, a_2, a_3\rangle - \langle b_1, b_2, b_3\rangle = \langle a_1-b_1, a_2-b_2, a_3-b_3\rangle$	$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1-b_1 \\ a_2-b_2 \\ a_3-b_3 \end{bmatrix}$
Scalar Multiplication $\lambda\underset{\sim}{a}$	$\lambda a_1\underset{\sim}{i} + \lambda a_2\underset{\sim}{j} + \lambda a_3\underset{\sim}{k}$	$\lambda\langle a_1, a_2, a_3\rangle = \langle \lambda a_1, \lambda a_2, \lambda a_3\rangle$	$\lambda\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \lambda a_1 \\ \lambda a_2 \\ \lambda a_3 \end{bmatrix}$

Example 6

If $\underset{\sim}{a} = \underset{\sim}{i} + 5\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} - 6\underset{\sim}{j} + \underset{\sim}{k}$, find

(a) $\underset{\sim}{a} + \underset{\sim}{b}$ (b) $\underset{\sim}{a} - \underset{\sim}{b}$ (c) $7\underset{\sim}{a}$

(M) Position Vectors

The *position vector* of a point P in space is the vector from the origin to the point P which is denoted by the vector $\overrightarrow{OP}$. If P has coordinates (x, y, z), then the position vector of P is

$$\boxed{\overrightarrow{OP} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k} \text{ or } \langle x, y, z\rangle \text{ or } \begin{pmatrix} x \\ y \\ z \end{pmatrix}} \text{ and } \boxed{\left\|\overrightarrow{OP}\right\| = \sqrt{x^2 + y^2 + z^2}} .$$

Example 7

The coordinates of points P and Q are points (−3, 7, 8) and (2, 5, 5) respectively. Find

(a) the position vector of P (b) the position vector of Q (c) $\overrightarrow{PQ}$ (d) $\left\|\overrightarrow{PQ}\right\|$

8.2 Dot Product (Scalar Product)

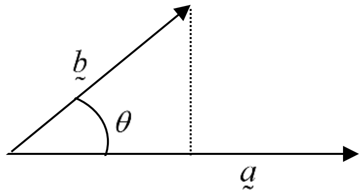
Dot Product is also known as *Scalar Product* or *Inner Product*.

(A) Definition of the Dot Product

DEFINITION

The *scalar product* or *dot product* of two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ denoted by $\underset{\sim}{a} \cdot \underset{\sim}{b}$ is defined as

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = \| \underset{\sim}{a} \| \| \underset{\sim}{b} \| \cos\theta$$



where θ is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.
Note that $\underset{\sim}{a} \cdot \underset{\sim}{b}$ is a scalar, not a vector.

Example 8

If vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are inclined at 60° to each other and $\|\underset{\sim}{a}\| = 3$, $\|\underset{\sim}{b}\| = 8$, find $\underset{\sim}{a} \bullet \underset{\sim}{b}$.

(B) Properties of the Dot Product

THEOREM

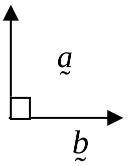
(1) $\underset{\sim}{a}\cdot\underset{\sim}{b}=\underset{\sim}{b}\cdot\underset{\sim}{a}$ (scalar product is commutative)

(2) $\underset{\sim}{a}\cdot(\underset{\sim}{b}+\underset{\sim}{c})=\underset{\sim}{a}\cdot\underset{\sim}{b}+\underset{\sim}{a}\cdot\underset{\sim}{c}$ (scalar product is distributive over addition)

(3) $\lambda(\underset{\sim}{a}\cdot\underset{\sim}{b})=(\lambda\underset{\sim}{a})\cdot\underset{\sim}{b}=\underset{\sim}{a}\cdot(\lambda\underset{\sim}{b})$ where $\lambda\in\mathbb{R}$

(4) If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are non-zero vectors,

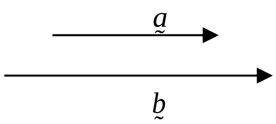
- $\boxed{\underset{\sim}{a}\cdot\underset{\sim}{b}=0 \iff \underset{\sim}{a} \text{ and } \underset{\sim}{b} \text{ are perpendicular (orthogonal) to each other.}}$



$$\underset{\sim}{a}\cdot\underset{\sim}{b}=|\underset{\sim}{a}||\underset{\sim}{b}|\cos 90^0=0$$

Hence $\underset{\sim}{i}\cdot\underset{\sim}{j}=\underset{\sim}{j}\cdot\underset{\sim}{k}=\underset{\sim}{k}\cdot\underset{\sim}{i}=0$

- $\boxed{\underset{\sim}{a}\cdot\underset{\sim}{b}=|\underset{\sim}{a}||\underset{\sim}{b}| \iff \underset{\sim}{a} \text{ and } \underset{\sim}{b} \text{ are in the same direction, i.e. } \theta=0.}$

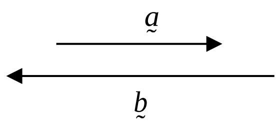


$$\underset{\sim}{a}\cdot\underset{\sim}{b}=|\underset{\sim}{a}||\underset{\sim}{b}|\cos 0^0=|\underset{\sim}{a}||\underset{\sim}{b}|$$

Hence $\underset{\sim}{a}\cdot\underset{\sim}{a}=|\underset{\sim}{a}|^2$,

$\underset{\sim}{i}\cdot\underset{\sim}{i}=|\underset{\sim}{i}||\underset{\sim}{i}|=1,$ $\underset{\sim}{j}\cdot\underset{\sim}{j}=|\underset{\sim}{j}||\underset{\sim}{j}|=1$ and $\underset{\sim}{k}\cdot\underset{\sim}{k}=|\underset{\sim}{k}||\underset{\sim}{k}|=1$

- $\boxed{\underset{\sim}{a}\cdot\underset{\sim}{b}=-|\underset{\sim}{a}||\underset{\sim}{b}| \iff \underset{\sim}{a} \text{ and } \underset{\sim}{b} \text{ are in opposite directions, i.e. } \theta=180^\circ}$



$$\underset{\sim}{a}\cdot\underset{\sim}{b}=|\underset{\sim}{a}||\underset{\sim}{b}|\cos 180^0=-|\underset{\sim}{a}||\underset{\sim}{b}|$$

(C) Dot Product in Component Form

Let $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$ and $\underset{\sim}{b} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k}$, then

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = \left(a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}\right) \cdot \left(b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k}\right).$$

Since $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are orthogonal unit vectors, the dot products of the basis vectors are all zero except for

$$\underset{\sim}{i} \cdot \underset{\sim}{i} = \underset{\sim}{j} \cdot \underset{\sim}{j} = \underset{\sim}{k} \cdot \underset{\sim}{k} = 1.$$

Hence the dot product $\underset{\sim}{a} \cdot \underset{\sim}{b}$ can be written in the following notations:

$\underset{\sim}{i}\,\underset{\sim}{j}\,\underset{\sim}{k}$ **- notation**	**Vector notation**	**Matrix notation**
$\underset{\sim}{a} \cdot \underset{\sim}{b}$ $= a_1b_1 + a_2b_2 + a_3b_3$	$\underset{\sim}{a} \cdot \underset{\sim}{b}$ $= \langle a_1, a_2, a_3 \rangle \bullet \langle b_1, b_2, b_3 \rangle$ $= a_1b_1 + a_2b_2 + a_3b_3$	$\underset{\sim}{a} \cdot \underset{\sim}{b}$ $= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ $= \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ $= a_1b_1 + a_2b_2 + a_3b_3$

Note that all notations arrive at the same result.

Example 9

If $\underset{\sim}{a} = \underset{\sim}{i} + 8\underset{\sim}{j} + 7\underset{\sim}{k}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} - 2\underset{\sim}{j} + 5\underset{\sim}{k}$, find $\underset{\sim}{a} \cdot \underset{\sim}{b}$.

Example 10

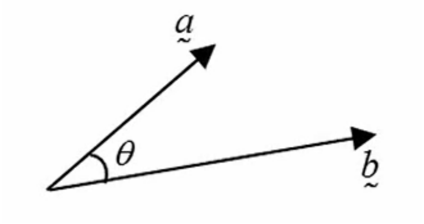
Find the value of p if $\underset{\sim}{u} = 2\underset{\sim}{i} + p\underset{\sim}{j} + \underset{\sim}{k}$ is perpendicular to $\underset{\sim}{v} = 4\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$

(D) Angle Between Vectors

From the definition of dot product, we can drive that the angle $\theta \in [0, \pi]$ between two non-zero vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is given by:

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = \|\underset{\sim}{a}\| \|\underset{\sim}{b}\| \cos\theta$$

$$\cos\theta = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\|\underset{\sim}{a}\| \|\underset{\sim}{b}\|}$$



Example 11

Find the angle between the vectors $\underset{\sim}{u} = 4\underset{\sim}{i} - 3\underset{\sim}{j}$ and $\underset{\sim}{v} = 2\underset{\sim}{i} + 5\underset{\sim}{j}$. Ans: 105^0

(E) Work Done

The work done by a force $\vec{F}$ in moving an object is given by

$$\boxed{W = \vec{F} \cdot \vec{S}}$$ where $\vec{S}$ is the displacement vector.

Example 12

If $\vec{F} = 2\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}$ N and $\vec{S} = 2\underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k}$ m , find the work done and the angle between $\vec{F}$ and $\vec{S}$.

Solution $W = \vec{F} \cdot \vec{S}$

$$= \left(2\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}\right) \cdot \left(2\underset{\sim}{i} + 2\underset{\sim}{j} - 4\underset{\sim}{k}\right)$$

$$= 2(2) + 1(2) + (-3)(-4)$$

$$= 4 + 2 + 12$$

$$= 18 \quad \text{J}$$

To find the angle between $\vec{F}$ and $\vec{S}$ we use

$$\vec{F} \cdot \vec{S} = \left|\vec{F}\right|\left|\vec{S}\right| \cos\theta$$

$$\cos\theta = \frac{\vec{F} \cdot \vec{S}}{\left|\vec{F}\right|\left|\vec{S}\right|} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{\vec{F} \cdot \vec{S}}{\left|\vec{F}\right|\left|\vec{S}\right|}\right)$$

$$|\vec{F}| = \sqrt{2^2 + 1^2 + 3^2} = \sqrt{4 + 1 + 9} = \sqrt{14}$$

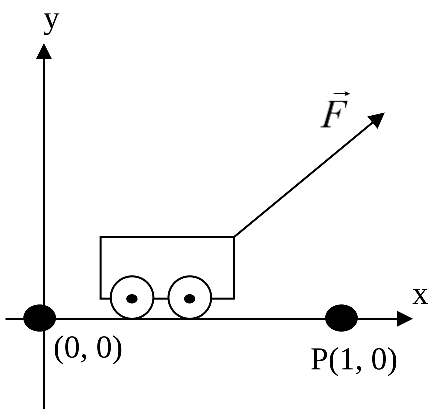
$$|\vec{S}| = \sqrt{2^2 + 2^2 + 4^2} = \sqrt{4 + 4 + 16} = \sqrt{24}$$

$$\therefore \theta = \cos^{-1}\left(\frac{18}{\sqrt{14}\sqrt{24}}\right) = \cos^{-1} 0.9819 = 10.89^\circ$$

The angle between $\vec{F}$ and $\vec{S}$ is 10.89°.

Example 13

Find the work done by a force $\vec{F}$ of magnitude 5 N acting in the direction of $\underset{\sim}{i} + \underset{\sim}{j}$ in moving an object from the origin to the point $P(1,0)$, distance being measured in meters. (Ans: 3.536 J)



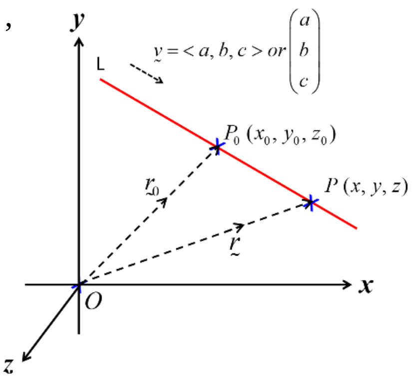
8.3 Lines and Line Segments in $\mathbb{R}^3$

(A) Vector and Parametric Equations of Lines

Consider a line L in $\mathbb{R}^3$ parallel to a nonzero vector $\underset{\sim}{v} = \langle a, b, c\rangle$ and passing through a fixed point $P_0(x_0, y_0, z_0)$, with position vector $\overrightarrow{OP_0} = \underset{\sim}{r}_0$.

Let $P(x, y, z)$ be an arbitrary point on the line L with position vector, that is $\overrightarrow{OP} = \underset{\sim}{r}$.

Since $\overrightarrow{P_0P}$ is parallel to the line and hence parallel to the vector $\underset{\sim}{v}$, then



$$\overrightarrow{P_0P} = \lambda \underset{\sim}{v}, \qquad \lambda \in \mathbb{R}$$

$$\overrightarrow{OP} - \overrightarrow{OP_0} = \lambda \underset{\sim}{v}$$

$$\overrightarrow{OP} = \overrightarrow{OP_0} + \lambda \underset{\sim}{v}$$

$$\underset{\sim}{r} = \underset{\sim}{r}_0 + \lambda \underset{\sim}{v}, \qquad \lambda \in \mathbb{R}$$

We know $\quad \underset{\sim}{r} = \underset{\sim}{r}_0 + \lambda \underset{\sim}{v}, \qquad \lambda \in \mathbb{R}$

Where $\quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + \lambda \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \qquad \lambda \in \mathbb{R}$

Therefore,

$$\begin{aligned} x &= x_0 + \lambda a, \\ y &= y_0 + \lambda b, \\ z &= z_0 + \lambda c, \qquad \lambda \in \mathbb{R} \end{aligned}$$

VECTOR EQUATION OF A LINE

$$\underset{\sim}{r} = \underset{\sim}{r}_0 + \lambda \underset{\sim}{v} \quad , \ \lambda \in \mathbb{R}$$

Where
$\underset{\sim}{r}$ is the position vector of any point on the line,
$\underset{\sim}{r}_0$ is the position vector of a known point on the line,
$\underset{\sim}{v}$ is a non-zero vector parallel to the line.

Note:
(1) The vector equation of line L is not unique as there are many choices of $\underset{\sim}{r}_0$.
(2) $\underset{\sim}{v}$ is called a **direction vector** of line L.

PARAMETRIC EQUATIONS OF A LINE

$$x = x_0 + \lambda a, \quad y = y_0 + \lambda b, \quad z = z_0 + \lambda c \qquad \text{where } \lambda \in \mathbb{R}$$

Example 14

Find vector and parametric equations of the line
(a) passing through (−2, 0) and parallel to $\underset{\sim}{v} = \langle 2, 4 \rangle$.
(b) passing through (−3, 2, −3) and parallel to $\underset{\sim}{v} = \underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{k}$.

Example 15

Find parametric and vector equations of the line L passing through the points P (−3,2,−3) and Q(5,0,7). Where does the line intersect the xy-plane?

Example 16

Consider the two lines in $\mathbb{R}^3$:

$$\begin{aligned} &L_1 : x = -3 + 4\lambda, \quad y = 9 - 4\lambda, \quad z = -6 + 5\lambda \\ &L_2 : x = 10 + 8\mu, \quad y = 1 - 3\mu, \quad z = 6 + \mu \end{aligned} \text{. Where } \lambda, \mu \in \mathbb{R}$$

Determine whether the lines are parallel and whether they intersect.

(B) Vector and Parametric Equations of a Line Segment

A line segment is a section of a line. Instead of extending infinitely where $\lambda \in \mathbb{R}$, a line segment begins and ends with two points, and is only valid for certain values of λ.

Let $A_1(x_1, y_1, z_1)$ and $A_2(x_2, y_2, z_2)$ be two known points on a line. To find the vector/parametric equation of a line segment,

(i) Find the vector/parametric equation of the entire line (see section [A] above)

(ii) Substitute the point A_1 into the equation to find λ_1 .

(iii) Substitute the point A_2 into the equation to find λ_2.

(iv) Thus, the line segment between A_1 and A_2 is the part of the line for $\lambda_1 \le \lambda \le \lambda_2$.

VECTOR EQUATION OF A LINE SEGMENT

$$\underset{\sim}{r} = \underset{\sim}{r}_0 + \lambda \underset{\sim}{v} \quad , \ \lambda \in [\lambda_1, \lambda_2]$$

PARAMETRIC EQUATIONS OF A LINE SEGMENT

$$x = x_0 + \lambda a, \quad y = y_0 + \lambda b, \quad z = z_0 + \lambda c \quad \text{where } \lambda \in [\lambda_1, \lambda_2]$$

Example 17

Parametrize the line segment joining the points $P\,(-3,2,\text{-}3)$ and $Q\,(5,0,7)$.

8.4 The Vector Product or Cross Product

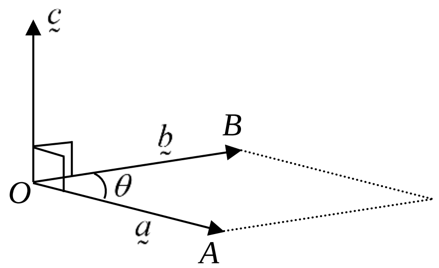
(A) Definition of the Cross Product

Given two vectors $\underset{\sim}{a}, \underset{\sim}{b} \in \mathbb{R}^3$, let $\underset{\sim}{c} \in \mathbb{R}^3$ be a third vector with the following properties.

(i) $\underset{\sim}{c}$ is orthogonal to both $\underset{\sim}{a}$ and $\underset{\sim}{b}$;

(ii) $\underset{\sim}{c}$ points in the direction of an advancing right-handed screw when it is turned from $\underset{\sim}{a}$ to $\underset{\sim}{b}$;

(iii) $\underset{\sim}{c}$ has a magnitude $\|\underset{\sim}{a}\|\|\underset{\sim}{b}\|\sin\theta$ which is the area of the parallelogram formed by $\underset{\sim}{a}$ and $\underset{\sim}{b}$.



The vector $\underset{\sim}{c}$ is called the **vector product** or **cross product** of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ and is denoted by $\underset{\sim}{a} \times \underset{\sim}{b}$.

Unlike scalar product, the vector product is not commutative but anti-commutative, that is, $\underset{\sim}{a} \times \underset{\sim}{b} = -\underset{\sim}{b} \times \underset{\sim}{a}$.

However, the vector product does obey the distributive law: $\underset{\sim}{a} \times (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \times \underset{\sim}{b} + \underset{\sim}{a} \times \underset{\sim}{c}$.

The vector product can also be expressed in component form.

Let $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$ and $\underset{\sim}{b} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k}$, then

$$\begin{aligned} \underset{\sim}{a} \times \underset{\sim}{b} &= \left(a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}\right) \times \left(b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k}\right) = \ldots \\ &= (a_2b_3 - a_3b_2)\underset{\sim}{i} + (a_3b_1 - a_1b_3)\underset{\sim}{j} + (a_1b_2 - a_2b_1)\underset{\sim}{k} \end{aligned}$$

DEFINITION

If $\underset{\sim}{a}$ and $\underset{\sim}{b} \in \mathbb{R}^3$, given by $\underset{\sim}{a} = \langle a_1, a_2, a_3 \rangle$ and $\underset{\sim}{b} = \langle b_1, b_2, b_3 \rangle$, then the *cross product* $\underset{\sim}{a} \times \underset{\sim}{b}$ is the vector defined by

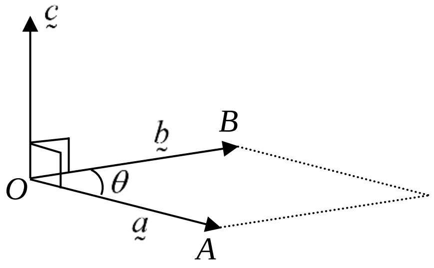
$$\underset{\sim}{a} \times \underset{\sim}{b} = \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \rangle$$

$\underset{\sim}{a} \times \underset{\sim}{b}$ can also be written as a determinant: $\underset{\sim}{a} \times \underset{\sim}{b} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Example 18a

$\underset{\sim}{c}$ is orthogonal to both $\underset{\sim}{a}$ and $\underset{\sim}{b}$ hence $\underset{\sim}{c} = \underset{\sim}{a} \times \underset{\sim}{b}$.

Given that $\underset{\sim}{a} = \langle 5,1,4 \rangle$ and $\underset{\sim}{b} = \langle -1,0,2 \rangle$, find $\underset{\sim}{c}$. (ans: $\underset{\sim}{c} = \langle 2,-14,1 \rangle$)



Video link

https://qrgo.page.link/KLgS1

Example 18b (try on your own)

Given that $\underset{\sim}{a} = \langle 1,2,3 \rangle$ and $\underset{\sim}{b} = \langle -2,0,1 \rangle$, find $\underset{\sim}{a} \times \underset{\sim}{b}$. (ans: $\langle 2,-7,4 \rangle$)

Video link

https://qrgo.page.link/KZV7G

8.5 Planes in $\mathbb{R}^3$

A plane is a two-dimensional flat surface which extends infinitely large with no thickness.

(A) Vector and Point – Normal Equations of Planes

Suppose plane *M* passes through a point $P_0(x_0, y_0, z_0)$ with position vector $\overrightarrow{OP_0} = \underset{\sim}{r}_0$, and is normal (perpendicular) to the non-zero vector $\underset{\sim}{n} = \langle a, b, c \rangle$.

Let $P(x, y, z)$ be any point on the plane *M* with position vector, that is $\overrightarrow{OP} = \underset{\sim}{r}$.

Since $\overrightarrow{P_0P}$ is perpendicular to $\underset{\sim}{n}$,

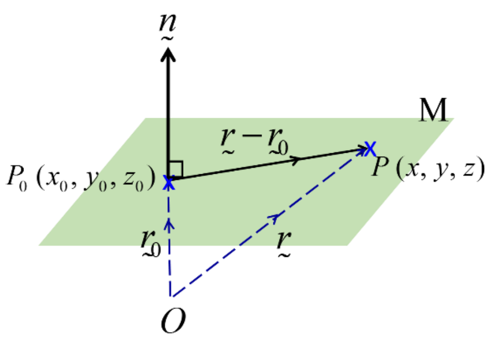
$$\Rightarrow \boxed{\underset{\sim}{n} \bullet \overrightarrow{P_0P} = 0} \leftarrow \text{Vector Equation}$$

$$\Rightarrow (\overrightarrow{OP} - \overrightarrow{OP_0}) \bullet \underset{\sim}{n} = 0$$

$$\Rightarrow (\underset{\sim}{r} - \underset{\sim}{r}_0) \bullet \underset{\sim}{n} = 0 \quad \text{or} \quad \Rightarrow \boxed{\underset{\sim}{r} \bullet \underset{\sim}{n} = \underset{\sim}{r}_0 \bullet \underset{\sim}{n}}$$

$$\Rightarrow \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \right) \bullet \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} \bullet \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$



$$\therefore \boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0} \leftarrow \text{Point-normal Equation}$$

or

$$ax + by + cz + d = 0 \quad \text{where} \quad d = -(ax_0 + by_0 + cz_0) = -\underset{\sim}{r}_0 \bullet \underset{\sim}{n}$$

THEOREM

The plane in $\mathbb{R}^3$ that passes through the point $P_0(x_0, y_0, z_0)$ and is normal to the non-zero vector $\underset{\sim}{n} = \langle a, b, c \rangle = a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$ has equations:

In vector form: $\underset{\sim}{n} \bullet \overrightarrow{P_0P} = 0$ or $\underset{\sim}{r} \bullet \underset{\sim}{n} = \underset{\sim}{r}_0 \bullet \underset{\sim}{n}$

In point-normal form: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

REMARKS

If *a*, *b*, *c* and *d* are constants and *a*, *b* and *c* are not all zeros, then $\langle a, b, c \rangle$ is normal to the plane $ax + by + cz + d = 0$.

Example 19

Find an equation of the plane passing through P_0 $(-3, 0, 7)$ and perpendicular to $\underset{\sim}{n} = \langle 5, 2, -1 \rangle$.

Example 20

Find an equation of the plane through the points $P_1(1,2,-1)$, $P_2(2,0,1)$, and $P_3(0,3,2)$.

Example 21

Determine whether the planes $3x - 4y + 5z = 0$ and $-6x + 8y - 10z = 7$ are parallel.

Example 22

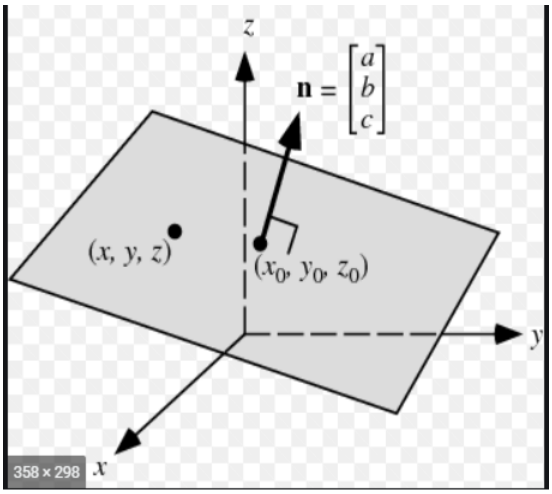
Determine whether the line $x=1+2\lambda, \quad y=-2+3\lambda, \quad z=-3-\lambda$ is parallel to the plane $x-2y+4z=12$.

Example 23

Find the point where the line $x=\frac{8}{3}+2\mu, \quad y=-2\mu, \quad z=1+\mu$ intersects the plane $3x+2y+6z=6$.

Example 24 (to be taught in class)

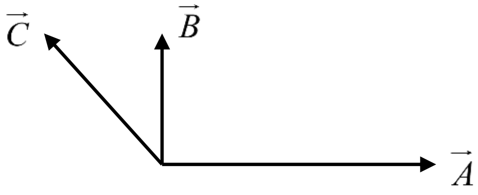
Find the equation of a 3D plane that passes through $(1,-4,3)$ and it has a normal vector of $<2,3,-1>$. If Q(1, 2, z) lies on the given plane, find z. (Answer: $2x+3y-z=-13$, z = 21)



Tutorial 8.1:

1) Based on vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ shown below, sketch the vectors:

(a) $\vec{A}+\vec{B}$ (b) $2\vec{A}-3\vec{C}$ (c) $\vec{A}-2\vec{B}+\vec{C}$



2) Sketch the vectors with their initial points at the origin.

(a) $2\underset{\sim}{i}+3\underset{\sim}{j}$ (b) $2\underset{\sim}{i}-5\underset{\sim}{j}+\underset{\sim}{k}$ (c) $\langle -1,0,4\rangle$

3) Find the terminal point of $\underset{\sim}{v}=-2\underset{\sim}{i}+5\underset{\sim}{j}+\underset{\sim}{k}$ if the initial point is $(0,3,-4)$.

4) If $\underset{\sim}{u}=2\underset{\sim}{i}-\underset{\sim}{j}$, $\underset{\sim}{v}=4\underset{\sim}{i}+5\underset{\sim}{j}$, calculate

(a) $\underset{\sim}{u}+\underset{\sim}{v}$ (b) $5\underset{\sim}{u}-3\underset{\sim}{v}$ (c) $\|-2\underset{\sim}{u}+\underset{\sim}{v}\|$ (d) $\|\underset{\sim}{u}\|+\|2\underset{\sim}{v}\|$

5) Let $\underset{\sim}{v}=-3\underset{\sim}{i}+\underset{\sim}{j}$.

(a) Find $\|\underset{\sim}{v}\|$ and the direction of $\underset{\sim}{v}$. (b) The unit vector in the direction of $\underset{\sim}{v}$.

6) The force $\vec{F}$ with a magnitude of 4 N is acting in the direction of $3\underset{\sim}{i}-4\underset{\sim}{j}+12\underset{\sim}{k}$. Find $\vec{F}$.

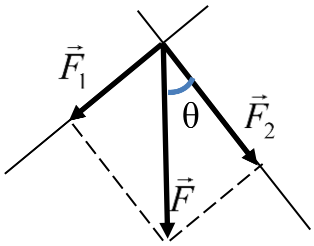
7) Given the points P(-1,1) and Q(2,5). Find the position vectors of P and Q and hence find the vector $\overrightarrow{PQ}$.

8) The process of breaking a vector into its components is called **resolving into components.**

For example, if we express a vector $\vec{F}$ into $\vec{F}_1 + \vec{F}_2$, we are resolving $\vec{F}$ into its $\vec{F}_1$ and $\vec{F}_2$ components.

$\vec{F}$ is the weight of the object. Resolve $\vec{F}$ into components which are at right angles to one another: One along $\vec{F}_1$ and the other along $\vec{F}_2$. (See the diagram below.)

Assume that $\|\vec{F}\|$ = 5 N, $\theta = 30^\circ$. Calculate the magnitudes of its components.

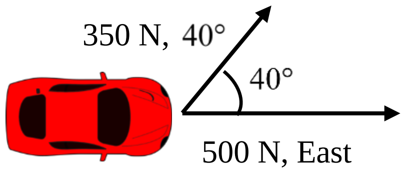


9) Two persons pull horizontally on ropes attached to a car stuck in mud. One person pulls with a force of 500 N directly east of the car, and the other person pulls with a force of 350 N at $40°$ from the first force, as shown in the figure.

(a) Express the two force vectors in $x\underset{\sim}{i} + y\underset{\sim}{j}$ form.

(b) Find the resultant force on the car.

(c) Find the magnitude and the direction of the resultant force.



Answers

3) $(-2, 8, -3)$

4) (a) $6\underset{\sim}{i} + 4\underset{\sim}{j}$ (b) $-2\underset{\sim}{i} - 20\underset{\sim}{j}$ (c) 7 (d) $\sqrt{5} + \sqrt{164}$

5) (a) $\sqrt{10}$, $\theta = 161.57^\circ$ (b) $\frac{1}{\sqrt{10}}(-3\underset{\sim}{i} + \underset{\sim}{j})$

6) $\vec{F} = 0.923\underset{\sim}{i} - 1.231\underset{\sim}{j} + 3.692\underset{\sim}{k}$

7) $\overrightarrow{OP} = -\underset{\sim}{i} + \underset{\sim}{j}$; $\overrightarrow{OQ} = 2\underset{\sim}{i} + 5\underset{\sim}{j}$; $\overrightarrow{PQ} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$

8) $|\vec{F}_1| = 2.5$ N; $|\vec{F}_2| = 4.33$ N

9) (a) $\vec{F}_1 = 500\underset{\sim}{i}$, $\vec{F}_2 = 268.12\underset{\sim}{i} + 224.98\underset{\sim}{j}$ (b) $\vec{F} = 768.12\underset{\sim}{i} + 224.98\underset{\sim}{j}$

(c) $|\vec{F}| = 800.39N$,

The resultant force acts an angle of $16.33°$ from the first force.

Tutorial 8.2:

1) Find the following dot products.

(a) $2\underset{\sim}{i}\bullet\underset{\sim}{i}$ (b) $\left(\underset{\sim}{i}+\underset{\sim}{j}\right)\bullet\underset{\sim}{k}$ (c) $3\underset{\sim}{i}\bullet 4\underset{\sim}{j}$ (d) $\left(5\underset{\sim}{i}-3\underset{\sim}{j}\right)\bullet\left(-2\underset{\sim}{i}+\underset{\sim}{j}\right)$

2) Find the angle between the following vectors.

(a) $\underset{\sim}{u}=\underset{\sim}{i}-3\underset{\sim}{j}+5\underset{\sim}{k}$ and $\underset{\sim}{v}=2\underset{\sim}{i}+5\underset{\sim}{j}-12\underset{\sim}{k}$ (b) $\underset{\sim}{a}=6\underset{\sim}{i}+\underset{\sim}{j}+3\underset{\sim}{k}$ *and* $\underset{\sim}{b}=5\underset{\sim}{i}-7\underset{\sim}{k}$

3) Find the work done by the force $\vec{F}$.

(a) $\vec{F}=2\underset{\sim}{i}-5\underset{\sim}{j}+3\underset{\sim}{k}$, moving an object from the origin to the point $P(8,1,0)$.

(b) $\vec{F}=6\underset{\sim}{i}+3\underset{\sim}{j}-\underset{\sim}{k}$, moving an object from the point $P(-1,2,2)$ to $Q(3,-5,-4)$.

4) Given two points $A(3,-2,1)$ and $B(1,1,-4)$ and a force $\vec{F}$ of 8 N acting in the direction of $2\underset{\sim}{i}+3\underset{\sim}{j}-\sqrt{3}\underset{\sim}{k}$.

(a) Find the displacement vector $\overrightarrow{AB}$.

(b) Find the force $\vec{F}$.

(c) Find the angle between $\vec{F}$ and $\overrightarrow{AB}$.

(d) Find the work done by $\vec{F}$ in displacing an object from *A* to *B*.

5) A boat travels 100 m due north while the wind exerts a force of 400 N toward the northeast. How much work does the wind do?

6) A force $\vec{F}=4\underset{\sim}{i}-6\underset{\sim}{j}+\underset{\sim}{k}$ newtons is applied to a point that moves a distance of 15 meters in the direction of $\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{k}$. How much work is done?

Answers

1) (a) 2 (b) 0 (c) 0 (d) -13

2) (a) 159.74^{o} (b) 81.13^{o}

3) (a) 11 J (b) 9 J

4) (a) $-2\underset{\sim}{i}+3\underset{\sim}{j}-5\underset{\sim}{k}$ (b) $4\underset{\sim}{i}+6\underset{\sim}{j}-2\sqrt{3}\underset{\sim}{k}$ (c) 56.36° (d) 27.32 joules

5) $20000\sqrt{2}$ J

6) $-5\sqrt{3}$ J

Tutorial 8.3:

1) Find parametric equations for the line through P_1 and P_2 and also for the line segment joining those points.

   (a) $P_1(3,-2)$, $P_2(5,1)$ (b) $P_1(5,-2,1)$, $P_2(2,4,2)$

2) Find parametric equations for the line whose vector equation is given.

   (a) $\langle x,y\rangle = \langle 2,-3\rangle + \lambda\langle 1,-4\rangle$ (b) $x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k} = \underset{\sim}{k} + \mu\left(\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}\right)$

3) Find a point P on the line and a vector $\underset{\sim}{v}$ parallel to the line by inspection.

   (a) $x\underset{\sim}{i} + y\underset{\sim}{j} = \left(2\underset{\sim}{i} - \underset{\sim}{j}\right) + \lambda\left(4\underset{\sim}{i} - \underset{\sim}{j}\right)$ (b) $\langle x,y,z\rangle = \langle -1,2,4\rangle + \mu\langle 5,7,-8\rangle$

4) Express the given parametric equations of a line using bracket notation and also using $\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{k}$ notation.

   (a) $x = -3+\lambda$, $y = 4+5\lambda$ (b) $x = 2-\mu$, $y = -3+5\mu$, $z = \mu$

5) Find the parametric equation of the line through $(-2,0,5)$ that is parallel to the line given by $x = 1+2\lambda$, $y = 4-\lambda$, $z = 6+2\lambda$.

6) Find the intersections of the line $x = -2$, $y = 4+2\lambda$, $z = -3+\lambda$ with the *xy*-plane, the *xz*-plane, and the *yz*-plane.

7) Where does the line $x = 1+\lambda$, $y = 3-\lambda$, $z = 2\lambda$ intersect the cylinder $x^2 + y^2 = 16$?

8) Show that the lines L_1 and L_2 intersect, and find their point of intersection.

   L_1: $x = 2+\lambda, \quad y = 2+3\lambda, \quad z = 3+\lambda$

   L_2: $x = 2+\mu, \quad y = 3+4\mu, \quad z = 4+2\mu$

Answers

1) (a) $x = 3+2\lambda, \quad y = -2+3\lambda$; line segment: $0 \le \lambda \le 1$

   (b) $x = 5-3\lambda, \quad y = -2+6\lambda, \quad z = 1+\lambda$; line segment: $0 \le \lambda \le 1$

2) (a) $x = 2+\lambda, \quad y = -3-4\lambda$ (b) $x = \mu, \quad y = -\mu, \quad z = 1+\mu$

3) (a) $P(2,-1)$, $\underset{\sim}{v} = 4\underset{\sim}{i} - \underset{\sim}{j}$ (b) $P(-1,2,4)$, $\underset{\sim}{v} = 5\underset{\sim}{i} + 7\underset{\sim}{j} - 8\underset{\sim}{k}$

4) (a) $\langle -3,4\rangle + \lambda\langle 1,5\rangle$; $-3\underset{\sim}{i} + 4\underset{\sim}{j} + \lambda\left(\underset{\sim}{i} + 5\underset{\sim}{j}\right)$

   (b) $\langle 2,-3,0\rangle + \mu\langle -1,5,1\rangle$; $2\underset{\sim}{i} - 3\underset{\sim}{j} + \mu\left(-\underset{\sim}{i} + 5\underset{\sim}{j} + \underset{\sim}{k}\right)$

5) $x = -2+2\lambda, \quad y = -\lambda, \quad z = 5+2\lambda$

6) (-2,10,0); (-2,0,-5); the line does not intersect the *yz*-plane

7) (0,4,-2), (4,0,6) 8) (1,-1,2)

Tutorial 8.4:

1) Find an equation of the plane that passes through the point $P(2,6,1)$ and has the vector $\underset{\sim}{n} = \langle 1,4,2 \rangle$ as a normal in (i) point-normal form (ii) vector form.

2) Find an equation of the plane that passes through the points $(-2,1,1)$, $(0,2,3)$ and $(1,0,-1)$ in vector form. Convert the equation into point-normal form.

3) Determine if the planes are parallel, perpendicular or neither.

(a) $\begin{array}{l} 2x-8y-6z-2=0 \\ -x+4y+3z-5=0 \end{array}$ (b) $\begin{array}{l} 3x-2y+z=1 \\ 4x+5y-2z=4 \end{array}$ (c) $\begin{array}{l} x-y+3z-2=0 \\ 2x+z=1 \end{array}$

4) Determine if the line and the plane are parallel, perpendicular or neither.

(a) $\begin{array}{l} x=4+2\lambda,\ y=-\lambda,\ z=-1-4\lambda \\ 3x+2y+z-7=0 \end{array}$ (b) $\begin{array}{l} x=\lambda,\ y=2\lambda,\ z=3\lambda \\ x-y+2z=5 \end{array}$

(c) $\begin{array}{l} x=-1+2\lambda,\ y=4+\lambda,\ z=1-\lambda \\ 4x+2y-2z=7 \end{array}$

5) Determine whether the line and plane intersect. If so, find the coordinates of the intersection.

(a) $\begin{array}{l} x=\lambda,\ y=\lambda,\ z=\lambda \\ 3x-2y+z-5=0 \end{array}$ (b) $\begin{array}{l} x=2-\lambda,\ y=3+\lambda,\ z=\lambda \\ 2x+y+z=1 \end{array}$

6) Find the equation of the plane through the origin that is parallel to the plane $4x-2y+7z+12=0$.

7) Let L_1 and L_2 be the lines whose parametric equations are

$L_1:\quad x=1+2t,\quad y=2-t,\quad z=4-2t$

$L_2:\quad x=9+\mu,\quad y=5+3\mu,\quad z=-4-\mu$

(a) Show that L_1 and L_2 intersect at the point (7,-1,-2).

(b) Find, to the nearest degree, the acute angle between L_1 and L_2 at their intersection.

(c) Find parametric equations for the line that is perpendicular to L_1 and L_2 and passes through their point of intersection.

Answers

1) $x+4y+2z=28,\quad \underset{\sim}{r} \bullet \langle 1,4,2 \rangle = 28$

2) $\underset{\sim}{r} \bullet \langle 0,2,-1 \rangle = 1,\quad 2y-z=1$

3) (a) Parallel (b) Perpendicular (c) Neither

4) (a) Parallel (b) Neither (c) Perpendicular

5) (a) $\left(\frac{5}{2},\frac{5}{2},\frac{5}{2}\right)$ (b) No intersection

6) $4x-2y+7z=0$ or $\underset{\sim}{r} \bullet \langle 4,-2,7 \rangle = 0$

7) (b) 84° (c) $x=7+7\lambda,\quad y=-1,\quad z=-2+7\lambda$ or $x=7+\lambda,\quad y=-1,\quad z=-2+\lambda$