Work, energy and power

PRE-CLASS (1 TO 14)

IN-CLASS (16 ONWARDS)

Learning outcomes of pre-class slides

After reading through the slides, students should be able to

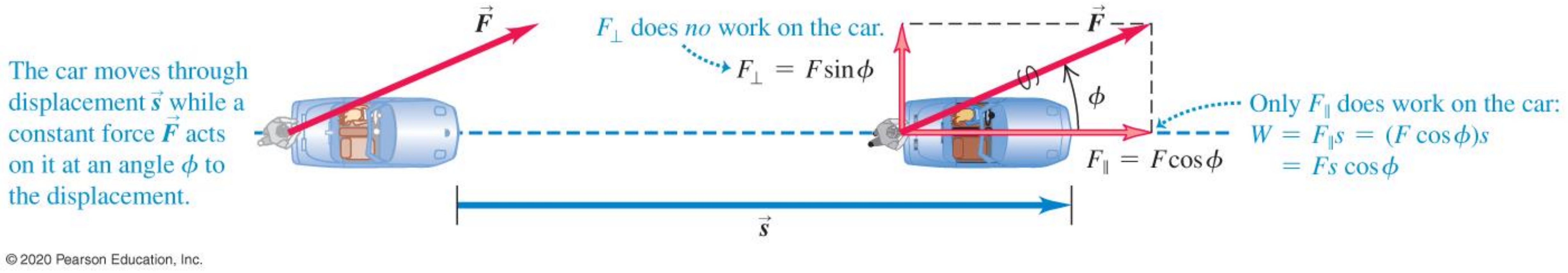
- define work done by a constant force
- apply work-energy theorem in situations with constant force
- state the principle of conservation of energy

An alternative approach to mechanics

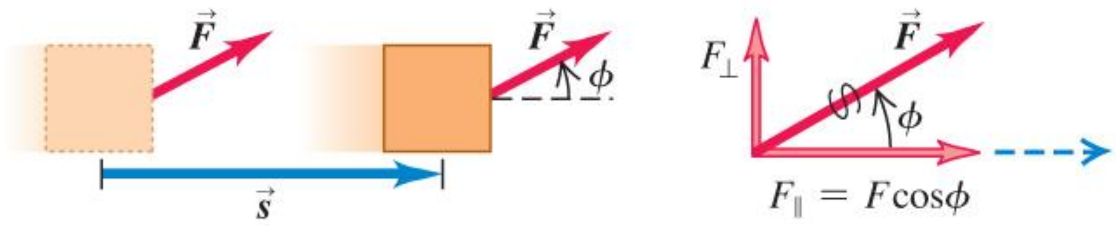
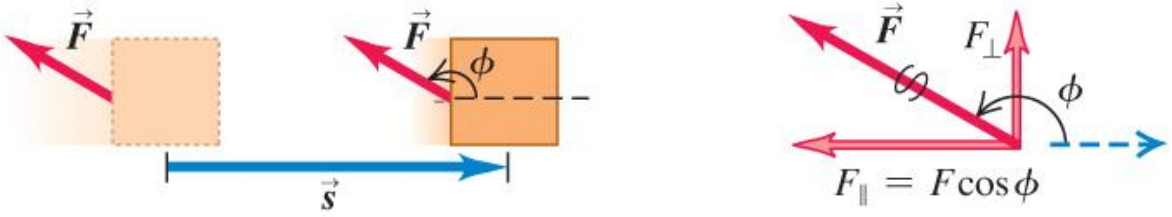
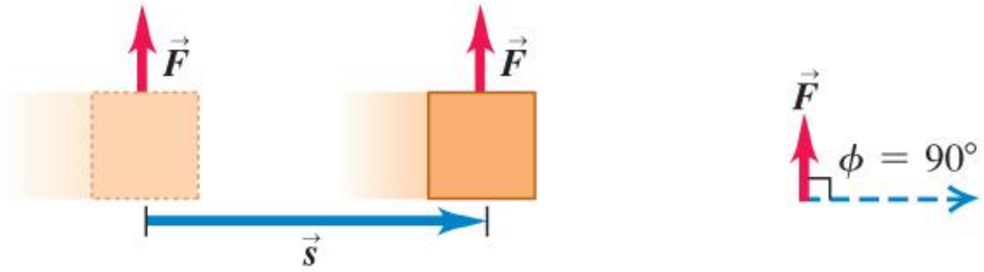
- Many problems in mechanics involve forces which are not constant.
- Solving this type of problems using Newton's laws can be very difficult.
- An alternate method based on the concepts of work, energy and power can greatly simplify calculations.

Work done by a constant force

- The work done by a constant force is defined as $W = \vec{F} \cdot \vec{s} = Fs\cos\phi$ where ϕ is the angle between the force and the displacement.

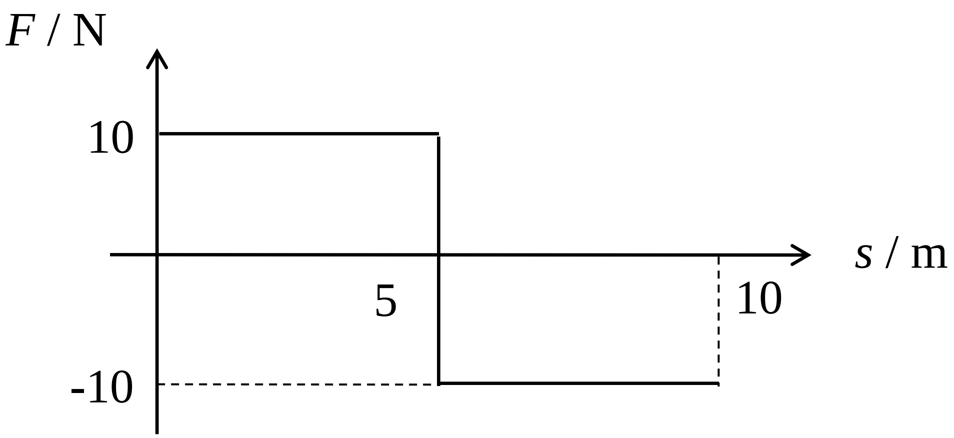


Work can be positive, negative or zero

Direction of Force (or Force Component)	Situation	Force Diagram
(a) **Force $\vec{F}$ has a component in direction of displacement:** $W = F_{\parallel}s = (F\cos\phi)s$ Work is *positive*.		
(b) **Force $\vec{F}$ has a component opposite to direction of displacement:** $W = F_{\parallel}s = (F\cos\phi)s$ Work is *negative* (because $F\cos\phi$ is negative for $90° < \phi < 180°$).		
(c) **Force $\vec{F}$ (or force component $F_{\perp}$) is perpendicular to direction of displacement:** The force (or force component) does *no* work on the object.		

Example 1

The diagram shows a force-displacement graph of an object. The force acts in the direction of the displacement for the first 5 m and opposite to the displacement in the next 5 m. What is the work done for the whole journey?



Solution:

For the first 5 m, $\phi = 0°$

$$W = \vec{F} \cdot \vec{s} = Fs\cos 0° = 10 \times 5 = 50\ \text{J}$$

For the next 5 m, $\phi = 180°$

$$W = \vec{F} \cdot \vec{s} = Fs\cos 180° = -10 \times 5 = -50\ \text{J}$$

Total work done is zero.

Example 2

a) Steve exerts a steady force of magnitude 210 N on a stalled car so that it moves a distance of 18 m. The car has a flat tire. So to make the car track straight Steve must push at an angle of 30° to the direction of motion. How much work does Steve do?

b) Steve pushes another stalled car with a steady force $\vec{F} = (160\hat{\imath} - 40\hat{\jmath})$ N. The displacement of the car is $\vec{s} = (14\hat{\imath} + 11\hat{\jmath})$ m. How much work does Steve do?

Solution:

(a) $W = \vec{F} \cdot \vec{s} = Fs\cos 30° = 210 \times 18 \cos 30° = 3.3\ \mathrm{kJ}$

(b) $W = \vec{F} \cdot \vec{s} = (160\hat{\imath} - 40\hat{\jmath}) \cdot (14\hat{\imath} + 11\hat{\jmath}) = 160 \times 14 - 40 \times 11 = 1.8\ \mathrm{kJ}$

Work done due to many forces

If there are many forces acting on **an object**, we can use one of the following two approaches to calculate the total work :

i) Compute the work done by each separate force and add them together.

ii) Compute the net force (vector sum) and find the work done by the net force (applicable to a single object only).

Energy

- Energy is defined as the ability to do work.
- Hence, energy and work are equivalent.
- The SI unit of energy is the joule (J).
- The different types of energy are light, heat, sound, electrical, chemical and mechanical.
- Mechanical energy comprises kinetic energy and potential energy.

Principle of conservation of energy

- The principle of conservation of energy states that energy
  - can neither be created nor destroyed in any process.
  - can be converted from one type to another (including work).
  - can be transferred from one body to another with the total amount remaining constant.
- The total energy of an isolated system (one that is not interacting with other systems) is constant.

Total Work

- Suppose a constant **net force** F_{net} acting on a particle of mass *m* along the +*x*-axis results in a displacement *s* along the +*x*-axis such that it has an initial velocity v_1 and final velocity v_2 then from kinematics.

$$v_2^2 = v_1^2 + 2a_x s \quad \text{or} \quad a_x = \frac{v_2^2 - v_1^2}{2s}$$

- Multiplying the above equation with *m*: $ma_x = m\dfrac{v_2^2 - v_1^2}{2s}$
- Since $F_{net} = ma_x$, so $F_{net} = m\dfrac{v_2^2 - v_1^2}{2s}$
- Multiply both sides with *s* and using $W_{total} = F_{net}s$, we get

$$F_{net}s = W_{total} = \frac{1}{2}m(v_2^2 - v_1^2)$$

Kinetic energy and Work-energy theorem

- The quantity $\frac{1}{2}mv^2$ is known as kinetic energy (KE or K).
- The SI unit of kinetic energy is joule (J).
- The work-energy theorem states that the work done by the net force on an object equals the change in the kinetic energy of the object ($W_{net} = \Delta K$).
- Kinetic energy cannot be negative because of the factor v^2.
- If the object at rest (i.e., $v = 0$), then its kinetic energy is 0.
- Note that the direction in which the object moved is not considered in the calculation of its kinetic energy.

Work-energy theorem

- From $W_{total} = K_2 - K_1 = \Delta K$,

  if $W_{total} > 0$, then $K_2 > K_1$ (the object is speeding up)

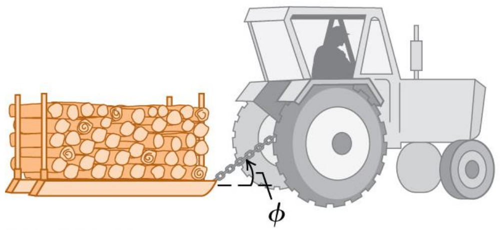
  if $W_{total} < 0$, then $K_2 < K_1$ (the object is slowing down)

  if $W_{total} = 0$, then $K_2 = K_1$. (the object is moving at constant speed)

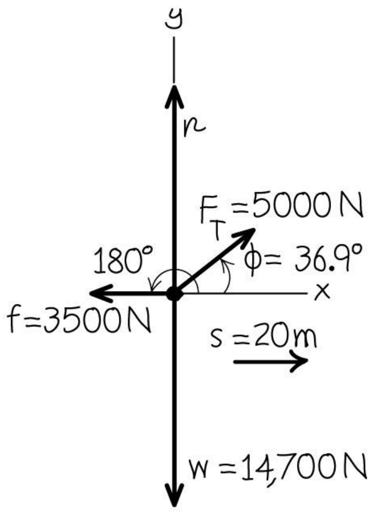
- The work-energy theorem applies even when the forces are not constant and the particle's trajectory is curved.

Example 3

A farmer hitches his tractor to a sled loaded with firewood and pulls it a distance of 20 m along level ground (see figure). The total weight of the sled and load is 14700 N. The tractor exerts a constant force of 5000 N at an angle of 36.9° above the horizontal. A friction force of 3500 N opposes the sled's motion. (a) Find the work done by each force acting on the sled and (b) the total work done by all the forces. (c) Suppose the sled's initial speed is 2.0 m/s. What is the speed of the sled after it moves 20.0 m?



(b) Free-body diagram for sled



(a) $W_{\text{weight}} = W_{\text{normal}} = 0$ (angle between force and s is 90°)

$W_{\text{friction}} = 3500 \times 20 \times \cos 180° = -70 \text{ kJ}$

$W_{\text{tractor}} = 5000 \times 20 \times \cos 36.9° = 80 \text{ kJ}$

(b) $W_{\text{total}} = 0 + 0 + 80 \text{ kJ} - 70 \text{ kJ} = 10 \text{ kJ}$

(c) Mass of sled, $m = \frac{W}{g} = \frac{14700}{9.80} = 1500 \text{ kg}$

By work-energy theorem, $W_{\text{total}} = \Delta K$

$$10000 = \frac{1}{2}(1500)(v^2 - 2^2)$$

Hence $v = 4.2$ m/s

End of pre-class slides

Learning Outcomes

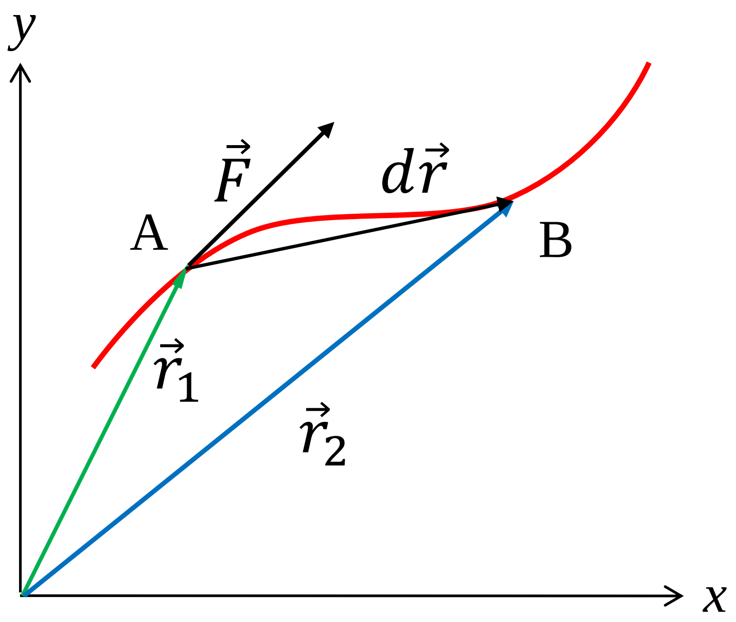
At the end of the lesson, students should be able to

- solve problems involving work done by varying force
- apply Hooke’s law and work done by spring to solve relevant problems
- calculate power and efficiency in energy related problems

Work done by non-constant force

- If the force is not constant, the infinitesimal work done is $dW = \vec{F} \cdot d\vec{r}$
- Suppose an object moves from A to B under a variable net force.
- Since $\vec{F} = F_x\,\hat{\imath} + F_y\,\hat{\jmath} + F_z\,\hat{k}$ and $d\vec{r} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}$, then the total work done is

$$W = \int F_x\,dx + \int F_y\,dy + \int F_z\,dz$$



Example 4

A force $\vec{F} = 4x\,\hat{\imath}$ N acts on a particle moving along the curve $y = 4x^2$ from $x = 1$ to $x = 4$. Calculate the work done by the force.

Solution:

The position vector of the particle is $\vec{r} = x\,\hat{\imath} + 4x^2\,\hat{\jmath}$

Infinitesimal displacement is $d\vec{r} = dx\,\hat{\imath} + 8x\,dx\,\hat{\jmath}$

Work done by the force:

$$W = \int \vec{F} \cdot d\vec{r} = \int (4x\,\hat{\imath}) \cdot (dx\,\hat{\imath} + 8x\,dx\,\hat{\jmath}) = \int_1^4 4x\,dx$$

$$W = \int_1^4 4x\,dx = [2x^2]_1^4 = 2(16 - 1) = 30 \text{ J}$$

Work done on spring

- The force by spring is not constant. The force needed to stretch it increases as the spring becomes longer.

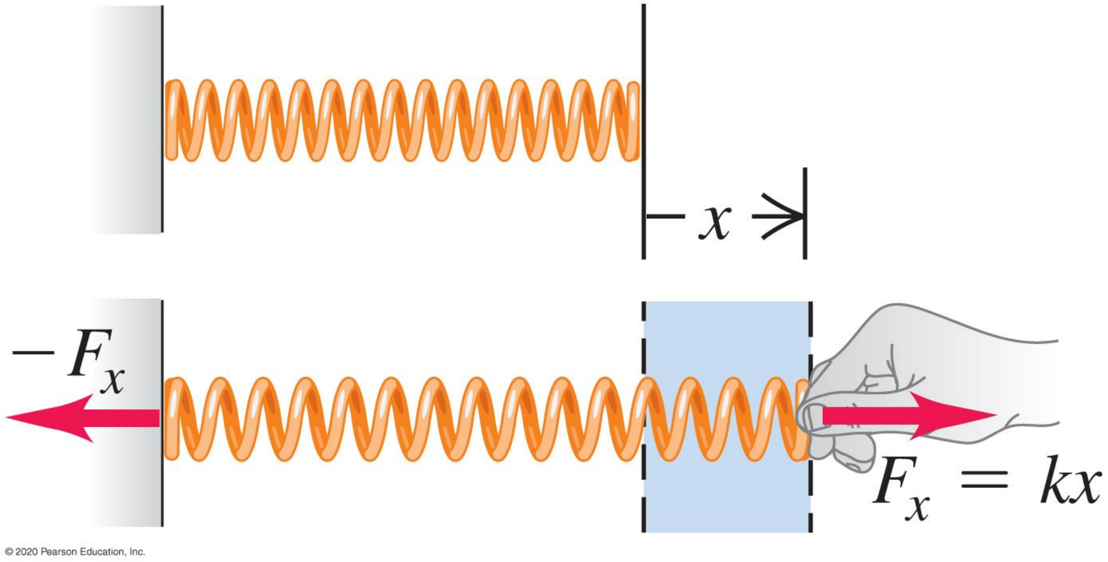
- Hooke's law states that the tension in a spring is directly proportional to the elongation x of the spring, i.e., $\vec{F}_{sp} = -k\vec{x}$, where k is the spring constant.

- The direction of the force by spring is opposite to the direction of the force that pulls/pushes it.

- For example, when you pull the spring, the spring pulls you back. Similarly, when you pushes on the spring, the spring pushes back on you.

Work done on spring

- When the spring is being pulled to the right, the hand exerts a force to the right of magnitude kx.
- The wall also pulls on the spring to the left with magnitude kx.
- However, only the hand is doing work as the leftward force does not move.
- Since the force by hand is not constant, we need to integrate to find the work done by hand.

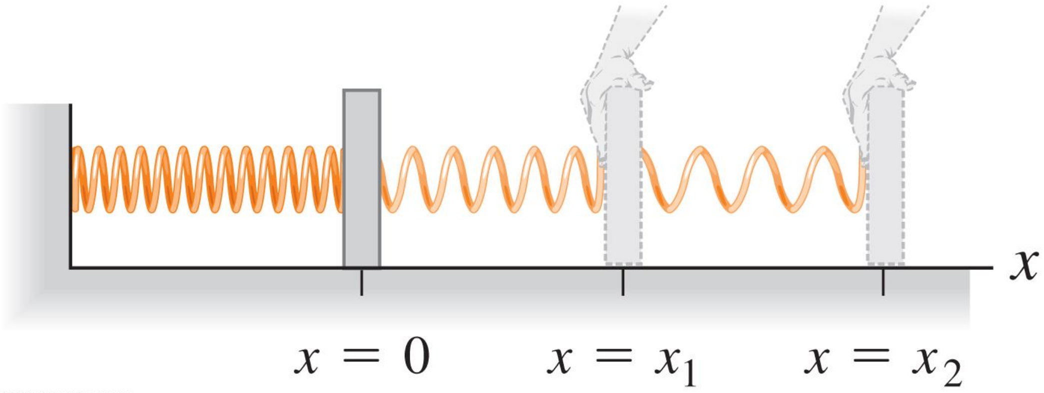


Work done on spring

- The work done on the spring by hand to stretch the spring x_1 to x_2 is

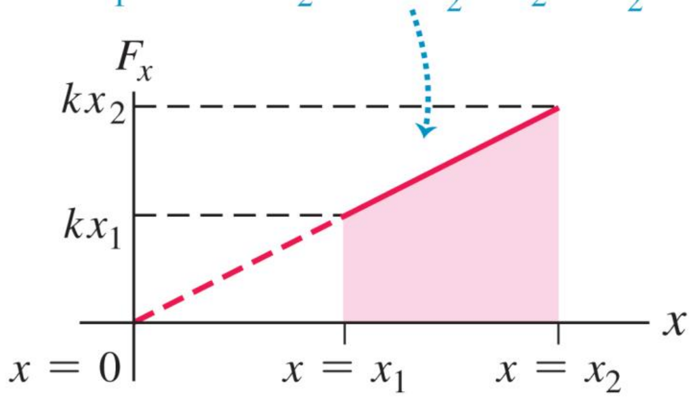
$$W_{hand} = \int \vec{F} \cdot d\vec{r} = \int_{x_1}^{x_2} kx\, dx = \frac{1}{2}k[x^2]_{x_1}^{x_2} = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2$$

(a) Stretching a spring from elongation x_1 to elongation x_2



(b) Force-versus-distance graph

The trapezoidal area under the graph represents the work done on the spring to stretch it from $x = x_1$ to $x = x_2$: $W = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2$.



Work done on spring

- The work done **on** the spring by hand to stretch the spring x_1 to x_2 is

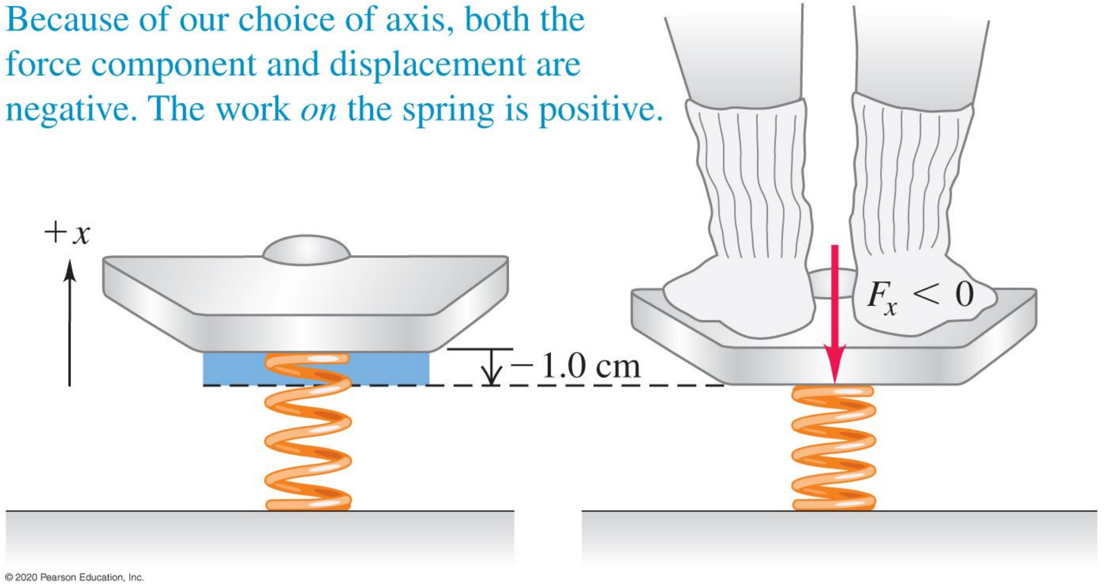
$$W_{hand} = \int \vec{F} \cdot d\vec{r} = \int_{x_1}^{x_2} kx\, dx = \frac{1}{2}k[x^2]_{x_1}^{x_2} = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2$$

- Since the spring force is opposite to the force by the hand, the work done **by** the spring on the hand to stretch the spring x_1 to x_2 is

$$W_{sp} = -W_{hand} = -\left(\frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2\right)$$

Example 5

A woman weighing 600 N steps on a bathroom scale that contains a stiff spring. In equilibrium, the spring is compressed 1.00 cm under her weight. Find the force constant of the spring and the total work done on it during compression.

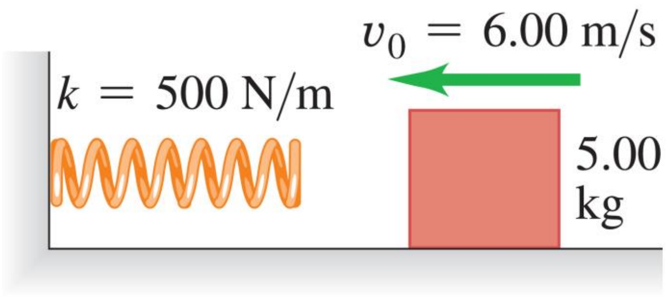


Example 6

A 5.00-kg block is moving at $v_0 = 6.00$ m/s along a frictionless, horizontal surface towards a spring with force constant $k = 500$ N/m that is attached to a wall. The spring has negligible mass.

(a) Find the maximum distance the spring will be compressed.

(b) If the spring is to compress by no more than 0.550 m, what should be the maximum value of v_0?



Power

- We often need to know how quickly the work was done.
- Therefore we define a quantity power which is the rate at which work is done.
- When ΔW amount of work is done during a time interval Δt, the average work done per unit time or the average power is

$$P_{avg} = \frac{\Delta W}{\Delta t}$$

- The SI unit of power is joule per second or watt (W).
- The kilowatt (kW) and megawatt (MW) are commonly used in electricity.

Power in terms of force and velocity

- Suppose a force $\vec{F}$ acts on a body while it undergoes a displacement $d\vec{s}$.
- The infinitesimal work done is $dW = \vec{F} \cdot d\vec{s}$.
- The instantaneous rate of work done is

$$\frac{dW}{dt} = \vec{F} \cdot \frac{d\vec{s}}{dt} = \vec{F} \cdot \vec{v}$$

- Hence instantaneous power $P = \vec{F} \cdot \vec{v}$.

Example 7

A 50.0-kg marathon runner runs up the stairs to the top of Chicago's Willis tower whose height is 443 m. To lift herself to the top in 15.0 minutes, what must be her average power output?

Example 8

A pump is required to lift 790 kg of water per minute from a well 14.1 m deep and eject it with a speed of 17.5 m/s.

(a) How much work is done per minute in lifting the water?

(b) How much work is done in giving the water the kinetic energy it has when ejected?

(c) What must be the power output of the pump?

Domestic unit of power

- The domestic unit of electricity is the kilowatt-hour (kWh).
- One kWh is the energy used by a device at a rate of 1000 watts in one hour.
- $1\ \text{kWh} = 1000\ \text{W} \times 3600\ \text{s} = 3.6 \times 10^6\ \text{J} = 3.6\ \text{MJ}$.
- The cost of electricity is based on the number of kilowatt-hour (kWh) of electrical energy used.

Efficiency of machines

- By the principle of conversation of energy, energy input equals useful energy output plus wasted energy due to friction and other work.

$$\text{Efficiency} = \frac{\text{useful work done}}{\text{work input}} \times 100\%$$

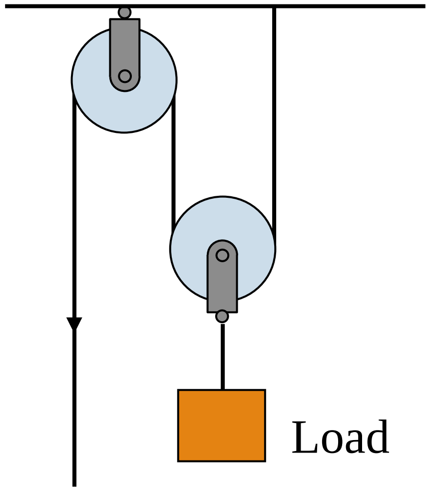
- Other versions of the efficiency formula include:

$$\text{Efficiency} = \frac{\text{useful energy output}}{\text{energy input}} \times 100\%$$

$$\text{Efficiency} = \frac{\text{useful power output}}{\text{power input}} \times 100\%$$

Useful and useless work

- The idea of useful work depends on the context.
- In the double pulley system shown below, the useful work is the work to raise the load.
- However, the lower pulley and rope have weight and work must be done to raise them in order to lift the load.
- The work done to raise the lower pulley and rope are useless work.



Example 9

The Grand Coulee Dam is 1270 m long and 170 m high. The electrical power output from generators at its base is approximately 2000 MW. How many cubic meters of water must flow from the top of the dam per second to produce this amount of power if 92% of the work done on the water by gravity is converted to electrical energy? (Each cubic meter of water has a mass of 1000 kg.)

Potential energy and conservation of energy

Learning outcomes

At the end of the lesson, students should be able to

- use conservation of energy to solve problems involving gravitational potential energy, elastic potential energy, and/or work done by friction
- recognize that gravitational force and spring force are conservative forces, and friction is an example of non-conservative forces

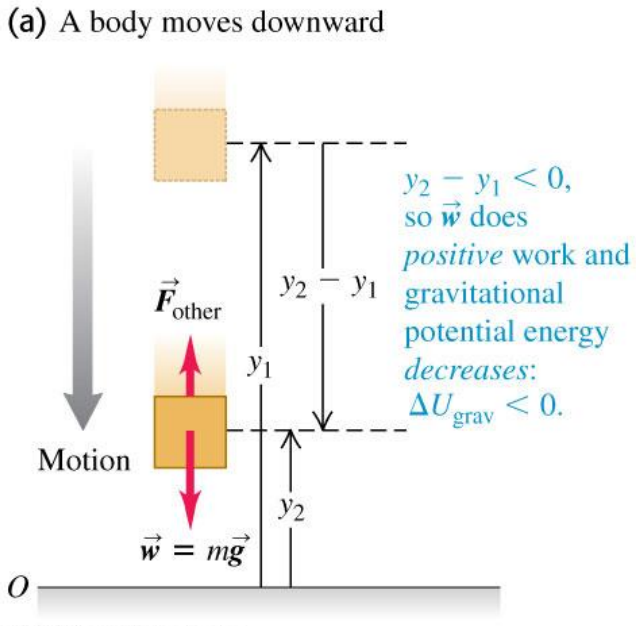
Work and Gravitational Potential Energy

- A body of mass m moves vertically down from a height y_1 above the origin to a lower height y_2.

- The work done on the mass by gravitational force is

$$W_{grav} = Fs = w(y_1 - y_2) = mgy_1 - mgy_2$$

- We define the quantity mgy as the gravitational potential energy (referred to GPE or U_{grav}), i.e., $U_{grav} = mgy$



Work and Gravitational Potential Energy

- The work done by force of gravity is expressed as

$$W_{grav} = mgy_1 - mgy_2 = U_{grav,1} - U_{grav,2}$$

$$W_{grav} = -\left(U_{grav,2} - U_{grav,1}\right) = -\Delta U_{grav}$$

- **Note that the change in GPE is relevant rather than GPE at the two points.**
- **Therefore, we can choose the reference level according to our convenience**.

Conservation of mechanical energy

- From the work-energy theorem, $W_{grav} = K_2 - K_1$.

- Since $W_{grav} = -\Delta U_{grav}$, so $K_2 - K_1 = -\Delta U$, then

$$\Delta K + \Delta U = \Delta(K + U) = 0$$

- Hence $K + U =$ constant.

- The quantity $K + U$ is the total mechanical energy of the system.

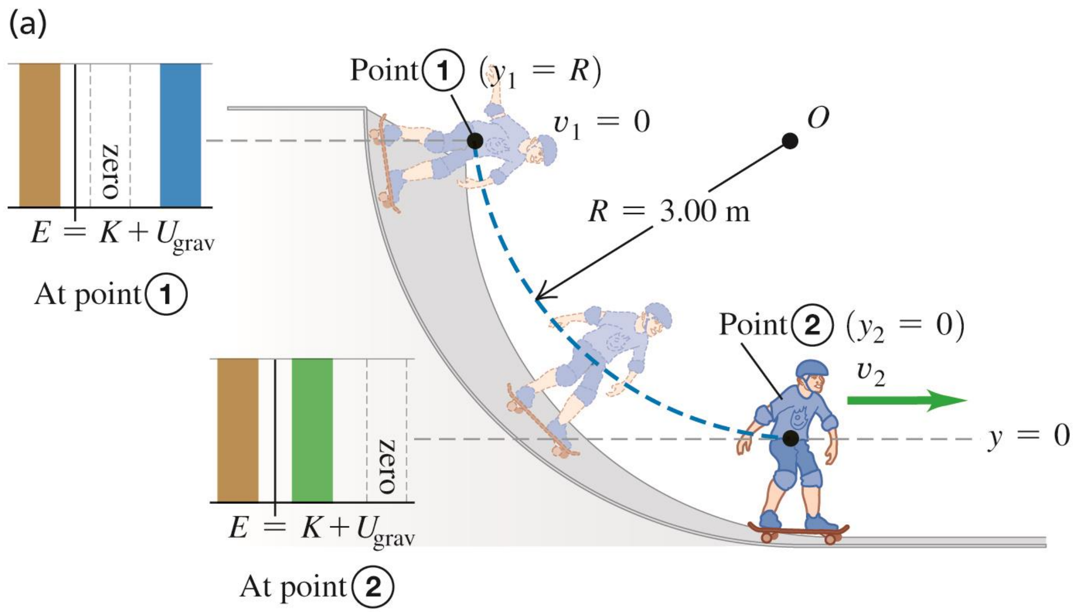
- At any height, $K + U$ is constant.

Example 10

A man skateboards from rest down a curved, frictionless ramp. If we treat the man and his skateboard as a particle, he moves through a quarter circle with radius $R = 3.00$ m (see figure). The mass of the person and the skate board is 25.0 kg.

(a) Find his speed at the bottom of the ramp and

(b) Find the normal force that acts on him at the bottom of the curve.



Elastic potential energy

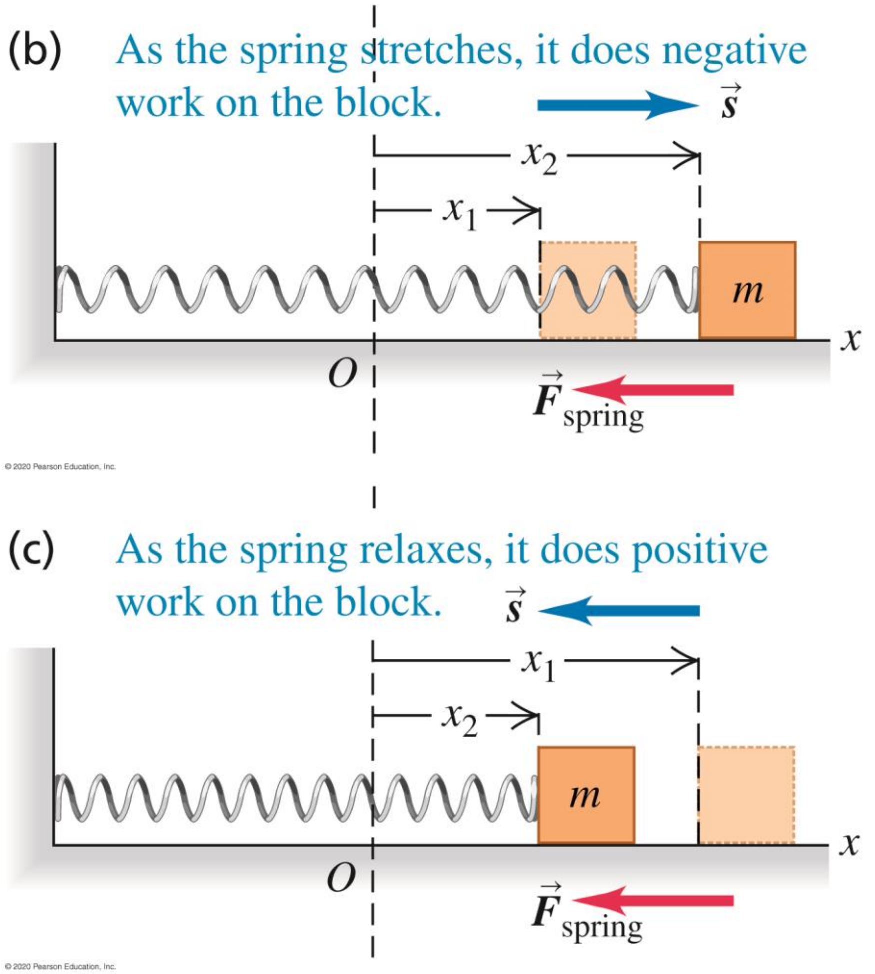
- In slide 21, the work done by spring is

$$W_{sp} = -\left(\frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2\right)$$

- Define elastic potential energy $U_{el} = \frac{1}{2}kx^2$, so

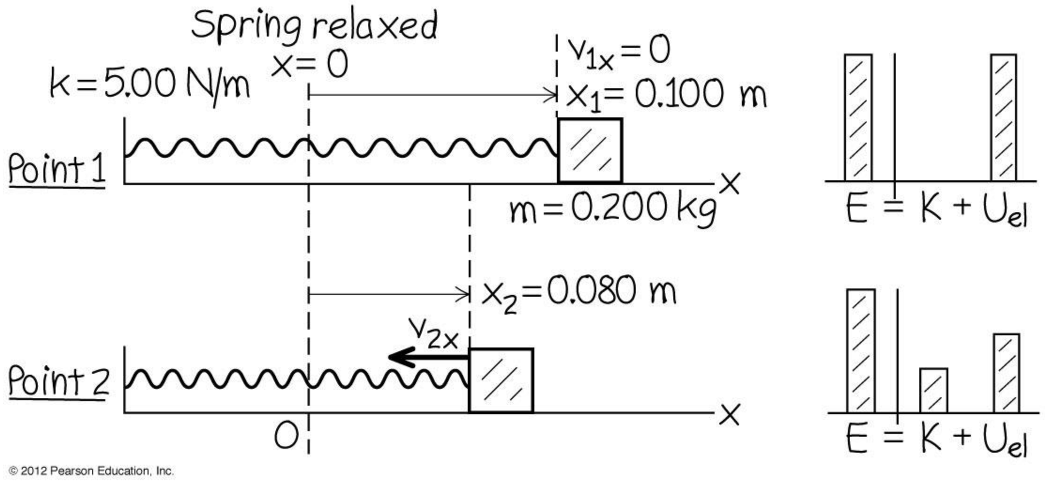
$$W_{sp} = -\left(U_{el,2} - U_{el,1}\right) = -\Delta U_{el}$$

- Note that the above equation has similar form as $W_{grav} = -\Delta U_{grav}$.



Example 11

A mass of 0.200 kg sits on a frictionless horizontal surface connected to a spring with force constant 5.00 N/m. The mass is pulled so that it stretches the spring 0.100 m and then released from rest. The mass moves back toward it equilibrium position. What is its *x*-velocity when $x = 0.080$ m?

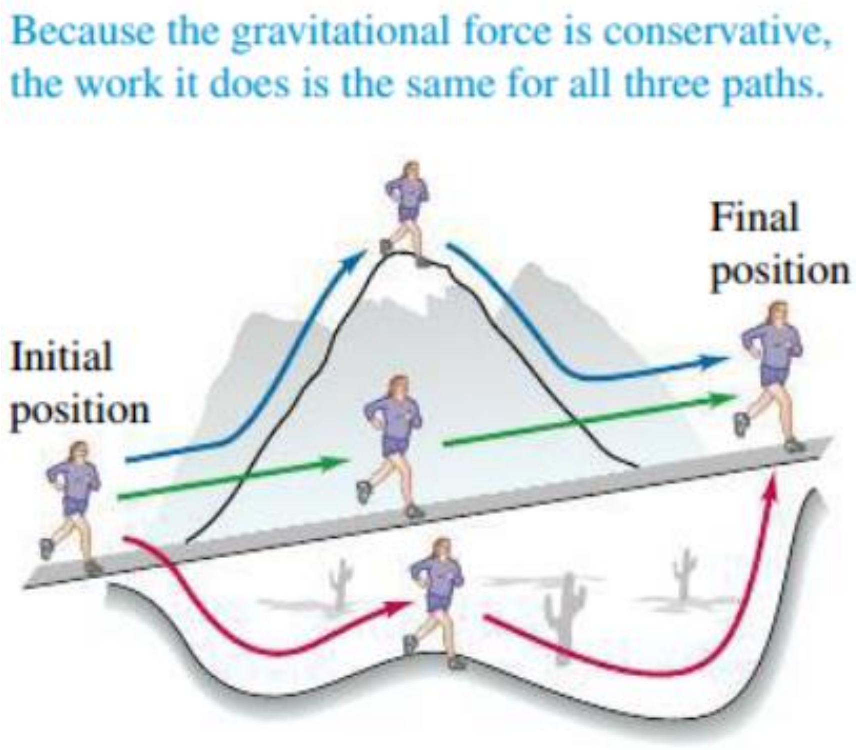


Conservative systems and forces

- Systems where the total mechanical energy is conserved are known as conservative systems.
- The forces which act in a conservative system are known as conservative forces.
- Examples of conservative forces are gravity and tension in a spring.

Work done by a conservative force

- The work done by a conservative force can be expressed as the difference between the initial and final values of a potential energy function.
- It is independent of the path taken by the object.
- When the start and the end points are the same, the total work done by the conservative force is zero.



Conservative force and potential energy

- The work done by a conservative force is related to the change in potential energy of the system by $W = -\Delta U$.

- If Δx is the displacement of a particle, and if F_x is constant, we can write

$$W = F_x \Delta x = -\Delta U \quad \Rightarrow \quad F_x = -\frac{\Delta U}{\Delta x}$$

- In the limit $\Delta x \to 0$,

$$F(x) = -\lim_{\Delta x \to 0} \frac{\Delta U}{\Delta x} = -\frac{dU}{dx}$$

- Hence we can calculate the conservative force if we know the potential energy function.

Conservative force and potential energy

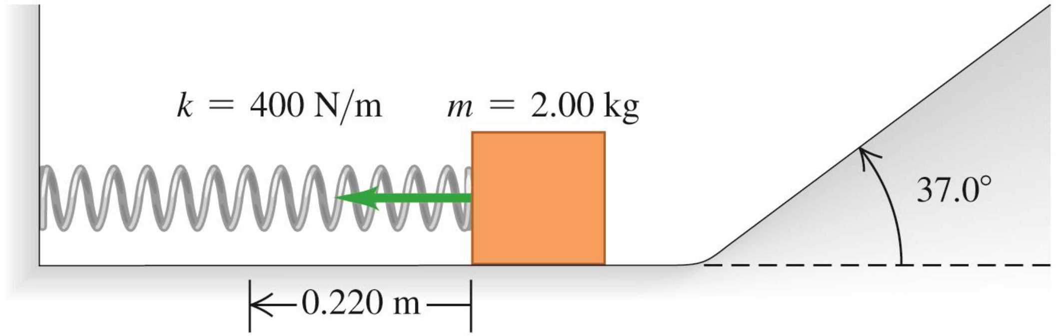
- The conservative force always acts to reduce the system's potential energy.
- If the potential energy function is known we will be able to find the conservative force.
- For example the potential energy stored in a spring is $U = \frac{1}{2}kx^2$.
- The conservative force in the spring is $F = -\frac{dU}{dx} = -kx$, which is the same as that given by Hooke's law.

Example 12

A 2.00-kg block is pushed against a spring with negligible mass and force constant $k = 400$ N/m, compressing it 0.220 m. When the block is released, it moves along a frictionless, horizontal surface and then up a frictionless incline with slope 37.0°.

(a) What is the speed of the block as it slides along the horizontal surface after having left the spring?

(b) How far does the block travel up the incline before starting to slide back down?



Example 13

A 2.90-kg fish is attached to the lower end of a vertical spring that has negligible mass and force constant 865 N/m. The spring is neither stretched nor compressed initially. The fish is released from rest.

(a) What is its speed after it has descended 0.0490 m from its initial position?

(b) What is the maximum speed of the fish as it descends?

Non-conservative force

- Examples of non-conservative forces are friction and the force unleashed in exploding fire crackers.

- The work done by a non-conservative force cannot be represented in terms of a potential energy function.

- If we toss an object onto a rough horizontal surface, it will come to rest because of work done by friction, which is converted to thermal energy of the object and the surface. i.e., $\Delta E_{int} = -W_{non-conservative\ forces}$

- $W_{non-conservative\ forces}$ is negative because friction acts opposite to the displacement of the object. Hence ΔE_{int} is positive.

Non-conservative force

- If we include the loss of the initial total mechanical energy due to non-conservative forces, we can write **the law of conservation of energy** as

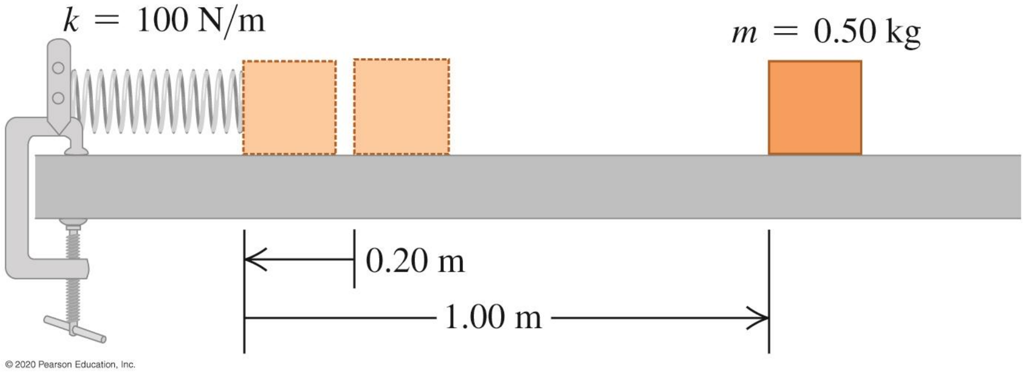
$$K_1 + U_1 - \Delta E_{int} = K_2 + U_2$$

- Or, we can write $\Delta K + \Delta U + \Delta E_{int} = 0$

- This is the general form of law of conservation of energy.

Example 14

A block with mass 0.50 kg is forced against a horizontal spring of negligible mass, compressing the spring a distance of 0.20 m. When released, the block moves on a horizontal tabletop for 1.00 m before coming to rest. The force constant is $k = 100$ N/m. What is the coefficient of kinetic friction μ_k between the block and the tabletop?

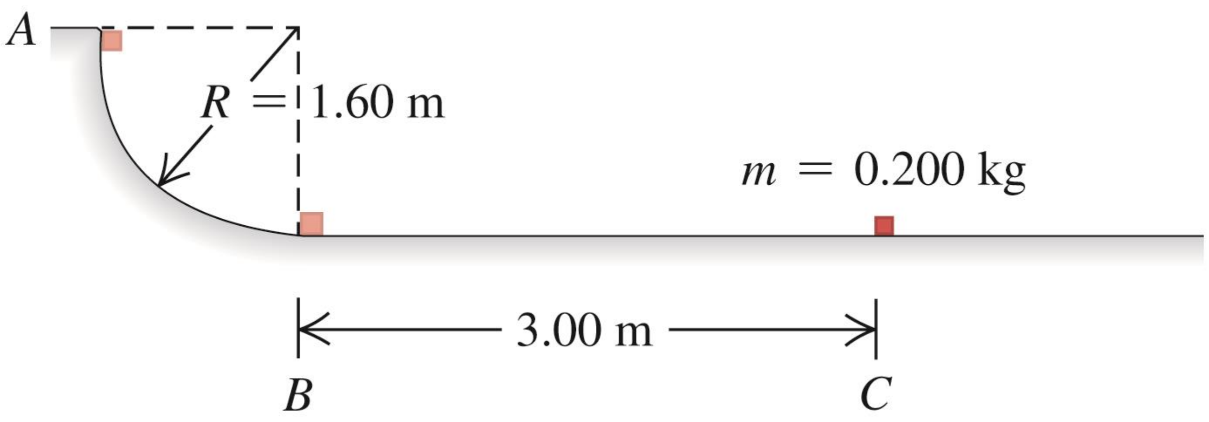


Example 15

A small 0.200-kg package is released from rest at point A on a track that is one-quarter of a circle with radius 1.60 m. It slides down the track and reaches point B with a speed of 4.90 m/s. Then, it slides on a level surface for 3.00 m and stops at point C.

(a) What is the coefficient of kinetic friction on the horizontal surface?

(b) How much work is done on the package by friction as it slides down the circular arc from A to B?



End of chapter