Greedy Algorithms

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Textbooks: Algorithms by Jeff Erickson /Algorithms by Sanjoy Dasgupta, Christos Papadimitriou, Umesh Vazirani.

Selecting Breakpoints

Consider the following scheduling problem.

- Road trip from Pohang to Seoul along fixed route.
- Fuel capacity $= C$. Refueling stations at certain points along the way.
- Goal: make as few refueling stops as possible.

Greedy algorithm (optimal): Go as far as you can before refueling.

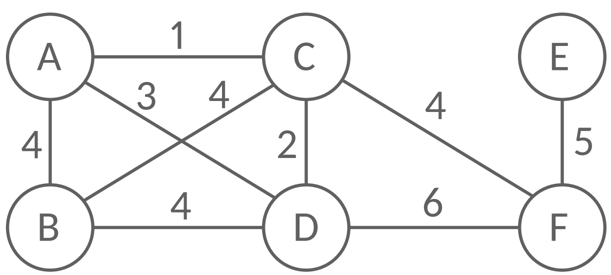
```
function SelectBreakpoints(B)
Sort breakpoints so that 0 = b_0 ≤ b_1 ≤ ··· ≤ b_n = L
S = {0}
x = 0
while x ≠ b_n do
    let p be the largest integer such that b_p ≤ x + C
    if b_p = x then
        return "no solution"
    x = b_p
    S = S ∪ {p}
return S
```

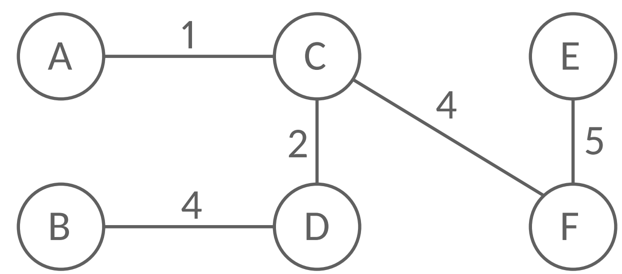
Greedy Algorithms

thinking ahead vs. choosing immediate advantage

Greedy algorithms build up a solution piece by piece, always choosing the next piece that offers the most obivious and immediate benefit.

What is the cheapest possible (connected) network of the following graph?

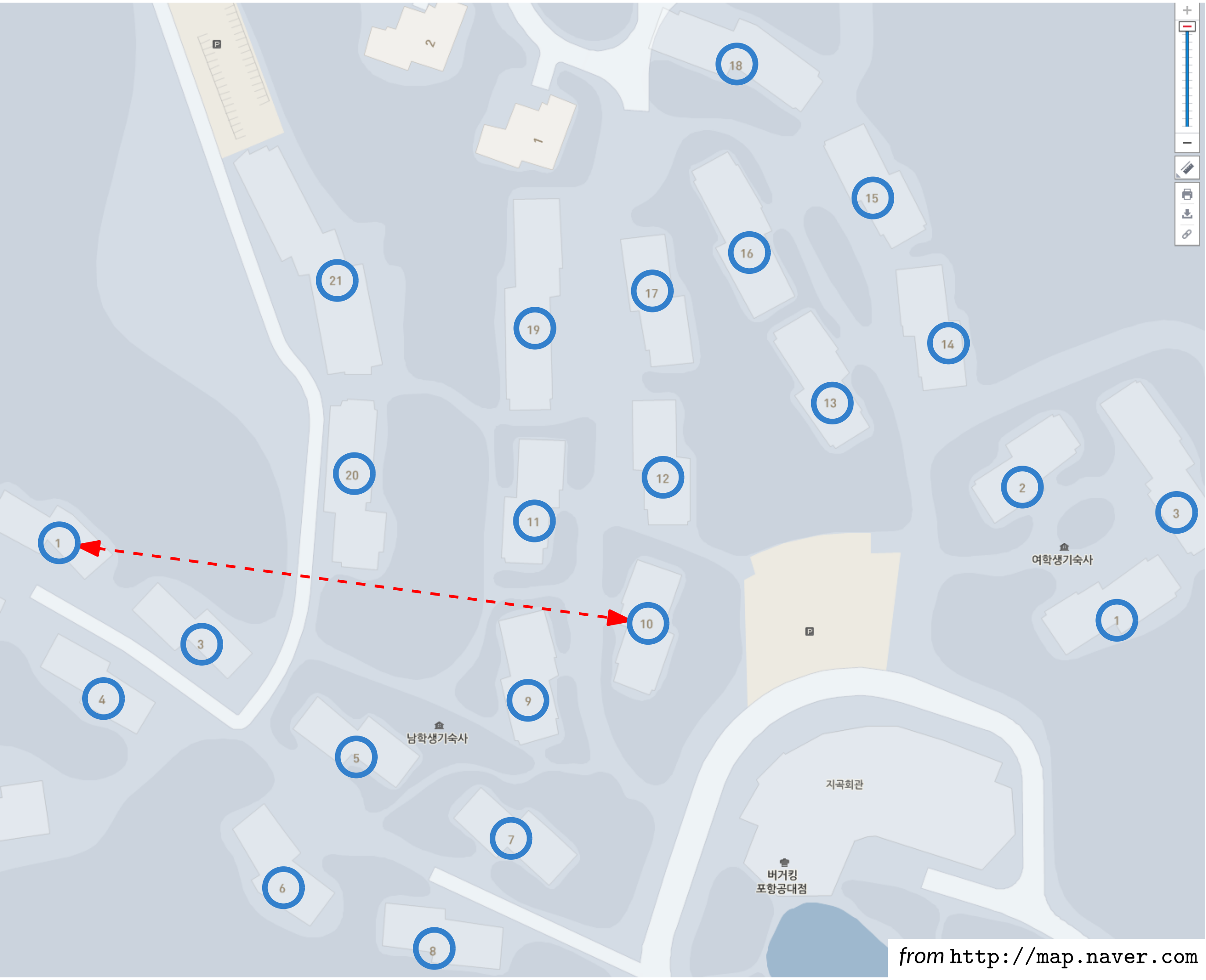




Property 1 Removing a cycle edge does not disconnect a graph.

Connected and acyclic graphs $\Rightarrow$ *Trees* (undriected graphs).

Minimum Spanning Trees



All pairwise distances are given.

Minimum Spanning Trees

from `http://map.naver.com`

When Euclidean distances are used.

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When geodesic distances are used.

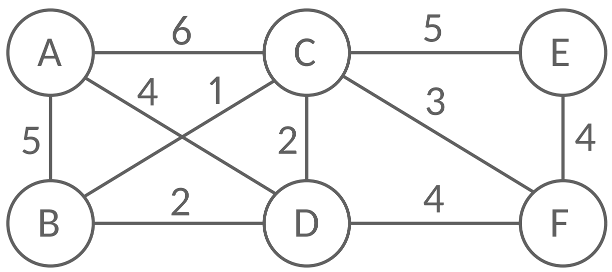
Input: An undirected graph $G = (V, E)$; edge weights w_e.
Output: A tree $T = (V, E')$, with $E' \subseteq E$, that minimizes

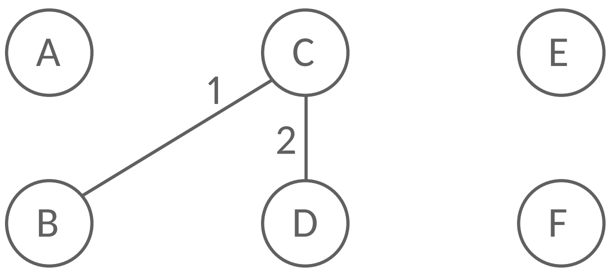
$$\text{weight}(T) = \sum_{e \in E'} w_e.$$

A greedy approach (Kruskal's algorithm): starts with the empty graph and selects edges from E according to the following rule.

Repeatedly add the next lightest edge that doesn't produce a cycle.

That is, take the most obvious immediate advantage in every decision!





Minimum Spanning Trees - Kruskal

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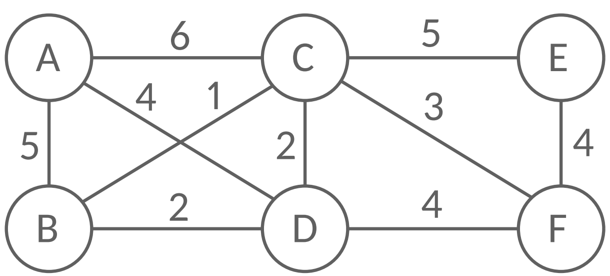

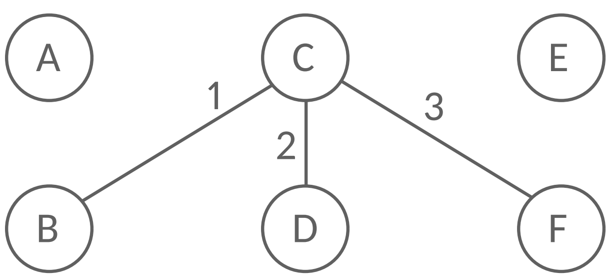

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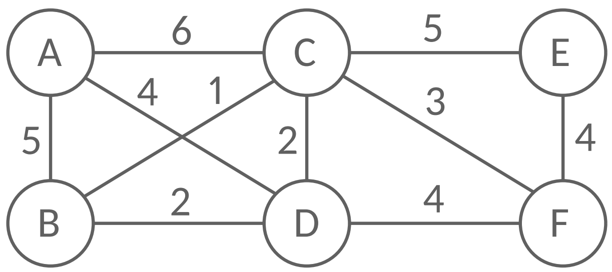

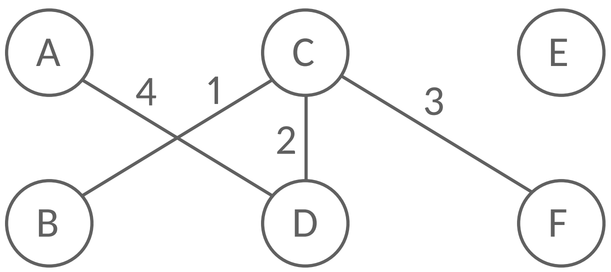

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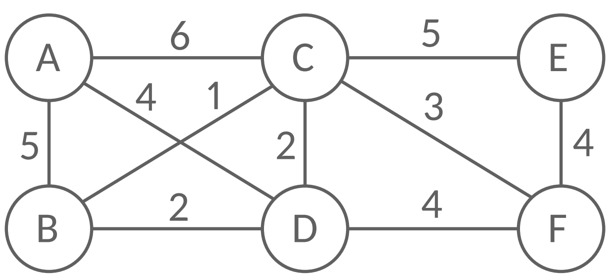
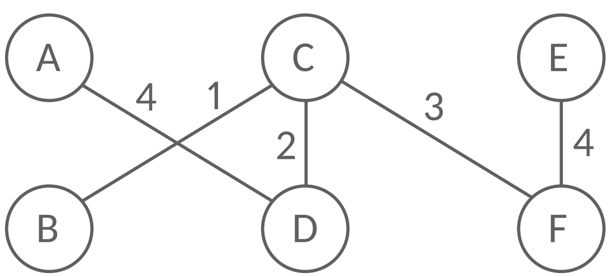




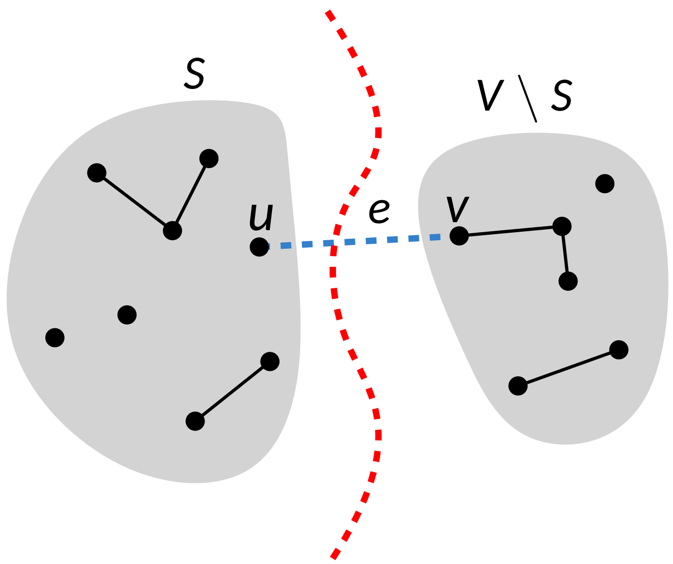
Minimum Spanning Trees - Kruskal

A greedy approach (Kruskal's algorithm): starts with the empty graph and selects edges from E according to the following rule.

Repeatedly add the next lightest edge that doesn't produce a cycle.

The correctness of Kruskal's method follows from a certain cut property.

Cut property Suppose edges in $X \subset E$ are part of a minimum spanning tree T of $G = (V, E)$. Pick any subset of nodes S for which no edge in X crosses between S and $V \setminus S$, and let e be the lightest edge across this partition. Then $X \cup \{e\}$ is part of some MST.



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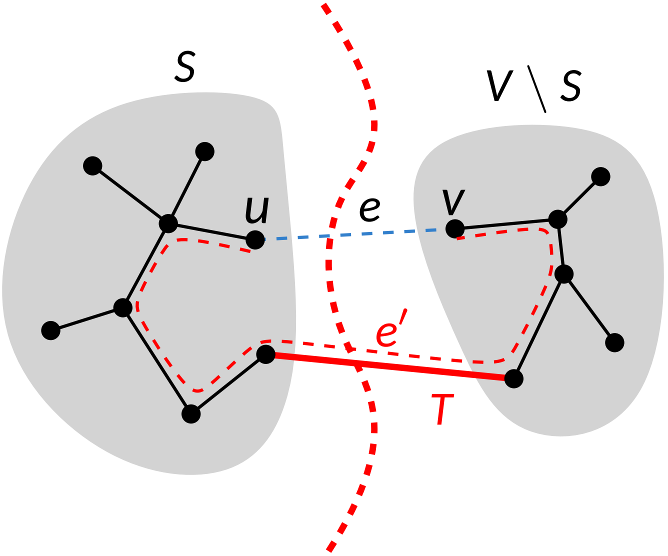
Proof. Assume $e = (u, v) \notin T$. We can construct a different MST T' containing $X \cup \{e\}$ by altering T slightly.

- u and v are connected by a path in T which contains an edge e' crossing S and $V \setminus S$.



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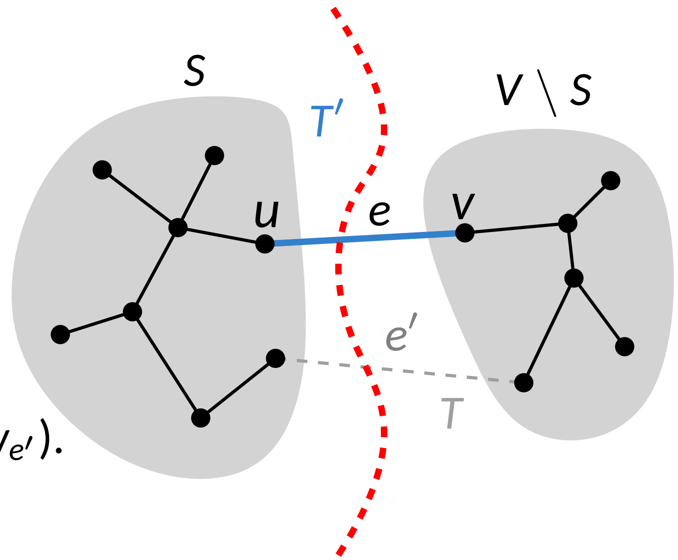
- u and v are connected by a path in T which contains an edge e' crossing S and $V \setminus S$.
- Construct a tree T' from T - remove e' and add e.
- T' is a spanning tree with $\text{cost}(T') \leqslant \text{cost}(T)$ ($\because w_e \leqslant w_{e'}$). So T' is an MST of G.



Minimum Spanning Trees - Kruskal

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function **Kruskal**(G, w)

for all $u \in V$ **do**
    makeset(u)
$X = \{\}$
Sort the edges in E by weight
for all edges $(u, v) \in E$, in increasing order of weight **do**
    if find$(u) \neq$ find(v) **then**
        add edge (u, v) to X
        union(u, v)

At any given moment,

- the set X forms a partial solution, a collection of trees.
- the next edge e to be added is the lightest one among edges connecting two of these trees.

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$|V|$ `makeset`, $2|E|$ `find`, $|V| - 1$ `union` operations.

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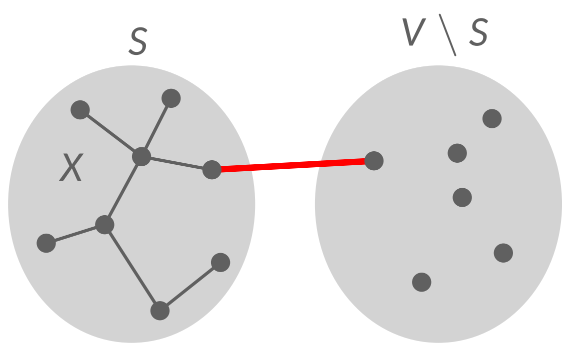
Minimum Spanning Trees - Prim

Prim's algorithm: choose the next edge s.t. the intermediate set X of edges forms a subtree, that is, the subtree X grows by the lightest edge between a vertex $v \in S$ and a vertex $z \notin S$.

function **Prim**(G, w)

for all $u \in V$ **do**
  cost$(u) = \infty$; prev$(u) = \texttt{nil}$;
Pick any initial node u_0 and set cost$(u_0) = 0$
$H =$ makequeue(V)
while H is not empty **do**
  $v =$ deletemin(H)
  for each edge $(v, z) \in E$ **do**
    if cost$(z) > w(v, z)$ **then**
      cost$(z) = w(v, z)$; prev$(z) = v$;
      decreasekey(H, z)



$O(|E| \log |V|)$

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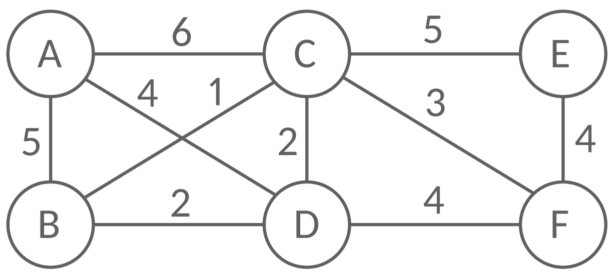
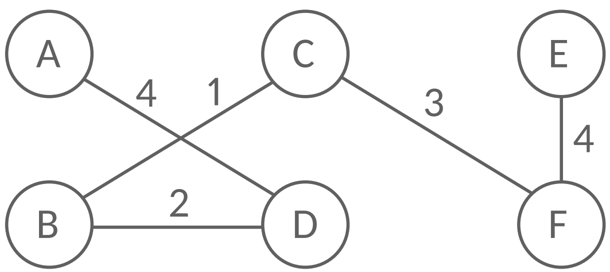




Set S	A	B	C	D	E	F
{}	$0/\texttt{nil}$	$\infty/\texttt{nil}$	$\infty/\texttt{nil}$	$\infty/\texttt{nil}$	$\infty/\texttt{nil}$	$\infty/\texttt{nil}$
A		$5/A$	$6/A$	$4/A$	$\infty/\texttt{nil}$	$\infty/\texttt{nil}$
A, D		$2/D$	$2/D$		$\infty/\texttt{nil}$	$4/D$
A, D, B			$1/B$		$\infty/\texttt{nil}$	$4/D$
A, D, B, C					$5/C$	$3/C$
A, D, B, C, F					$4/F$	

cost$(z)/$prev(z)

Datastructure for Disjoint Sets

Maintaining a collection of disjoint sets under the operation of **union**:

- makeset(x): create a new set containing x.
- find(x): return the *root* node of the set containing x.
- union(x, y): form a union of two sets containing x and y.

For a node x,

- $\pi(x)$ denotes a pointer to its parent, and
- rank(x) denotes the height of the subtree hanging from x.

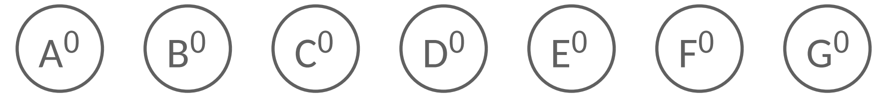
```
function makeset(x)
  π(x) ← x
  rank(x) ← 0
```

```
function find(x)
  while x ≠ π(x) do
      x ← π(x)
  return x
```

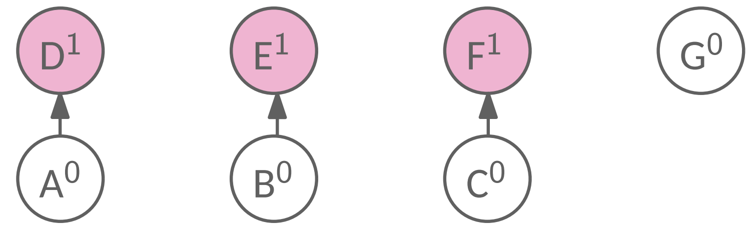
```
function union(x, y)
  r_x ← find(x); r_y ← find(y)
  if r_x = r_y then
      return
  if rank(r_x) > rank(r_y) then
      π(r_y) ← r_x
  else
      π(r_x) ← r_y
      if rank(r_x) = rank(r_y) then
          rank(r_y) ← rank(r_y) + 1
```

Datastructure for Disjoint Sets

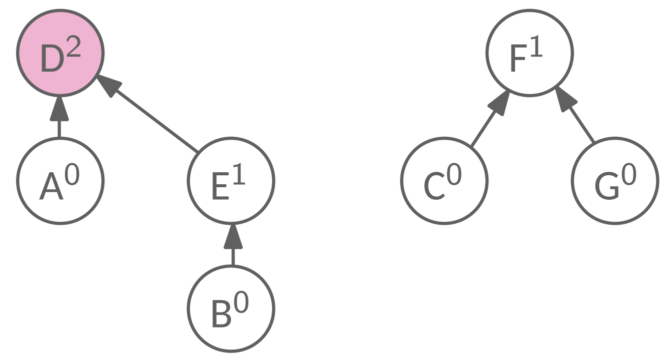
After $\texttt{makeset}(A), \texttt{makeset}(B), \ldots, \texttt{makeset}(G)$:



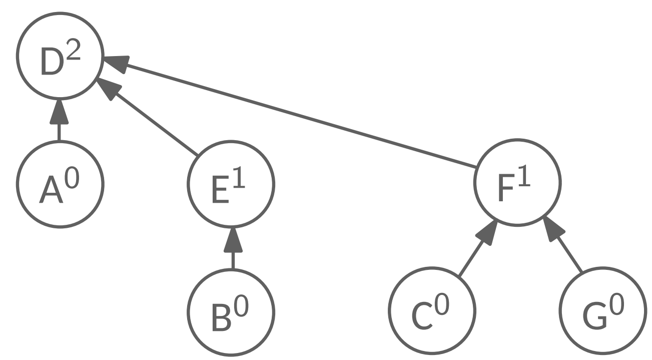
After $\texttt{union}(A, D), \texttt{union}(B, E), \texttt{union}(C, F)$:



After $\texttt{union}(C, G), \texttt{union}(E, A)$:



After $\texttt{union}(B, G)$:



Properties of union and find DS

A few properties of union-find data structures.

1. For any x, $\text{rank}(x) \leqslant \text{rank}(\pi(x))$.
2. If x is not a root, $\text{rank}(x)$ will never change again.

   Rank changes only for roots. A nonroot node never becomes a root.
3. Any root node of rank k has at least 2^k nodes in its tree.
4. If there are n elements overall, there can be at most $n/2^k$ nodes of rank k.

   $\rightarrow$ the max. rank is $\log n$, so the tree height $\leqslant \log n$.

Property 3. Proof by induction on k.

- It holds for $k = 0$.
- Assume that the claim holds for $k - 1$.
- A node x of rank k is created only by merging two roots of rank $k - 1$, each subtree consisting of $\geqslant 2^{k-1}$ nodes. Thus, x has $\geqslant 2^k$ nodes in its subtree.

Kruskal's algorithm takes

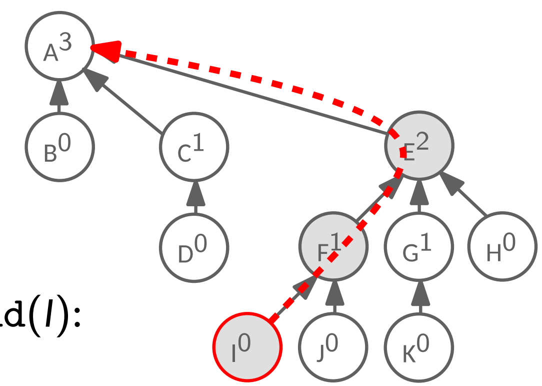
- $O(|E| \log |V|)$ for sorting, plus
- $O(|E| \log |V|)$ for union and find operations.

Path Compression

If the time for sorting doesn't count,

- we keep the tree short by *path compression*, that is,
- for each find(x), we make the nodes on the path to point the root.

find(I):



function **find**(x)

if $x \neq \pi(x)$ **then**
    $\pi(x) \leftarrow$ find($\pi(x)$)
return $\pi(x)$

~~function **find**(x)~~

~~**while** $x \neq \pi(x)$ **do**~~
~~ $x \leftarrow \pi(x)$~~
~~return x~~

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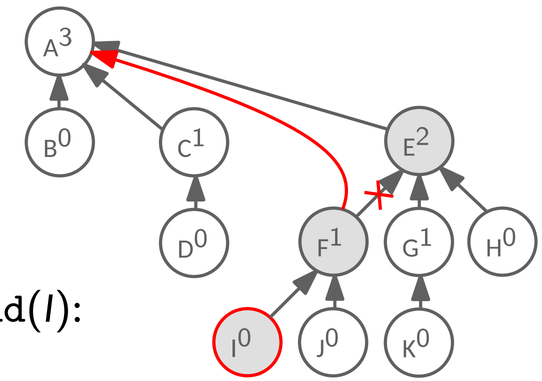

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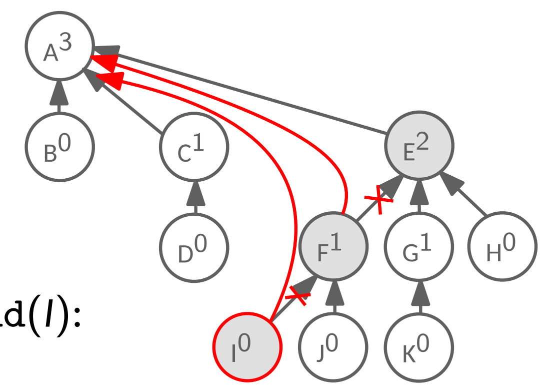

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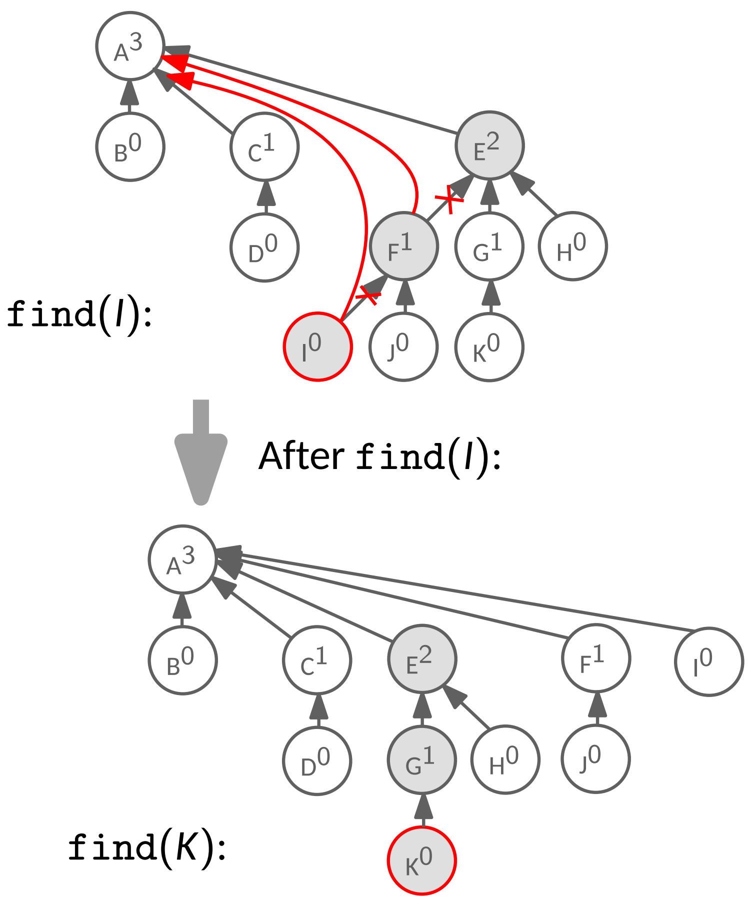

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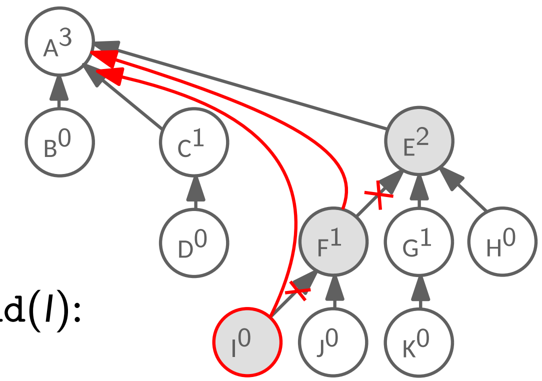

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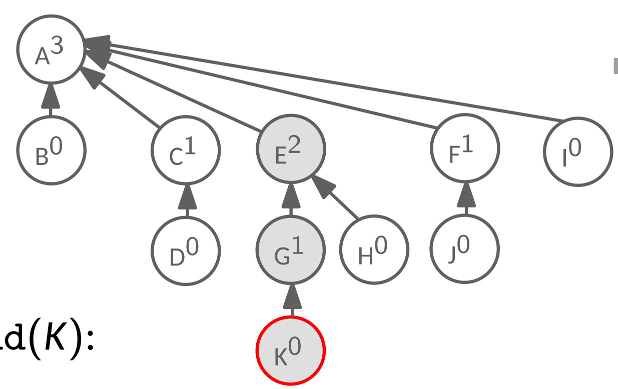
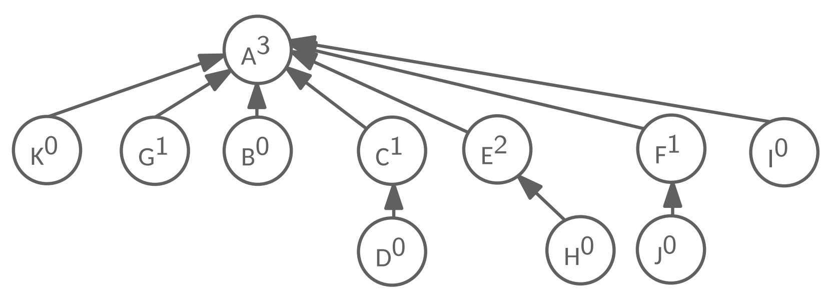
After find(I):



find(K):

After find(K):



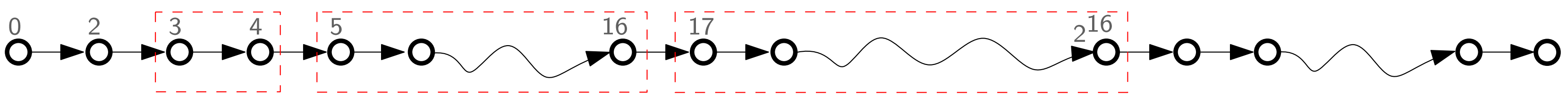
Path Compression

We now look at an intermixed sequence of find and union operations, starting from an empty data structure.

- # union operations is at most $n - 1$ for n nodes.
- We analyze the average time per operation $\rightarrow$ amortized cost

Ranks, ranging $0 \leqslant \text{rank}(x) \leqslant \log n$, form groups of interval $\{k+1, k+2, \ldots, 2^k\}$ for k, a power of 2, from 0.

$$\underset{2^0}{\{1\}}, \underset{2^1}{\{2\}}, \underset{2^2}{\{3,4\}}, \underset{2^4}{\{5,6,\ldots,16\}}, \underset{2^{16}=65536}{\{17,18,\ldots,2^{16}\}}, \underset{2^{2^{16}}}{\{65537,65538,\ldots,2^{65536}\}}, \ldots$$



groups $= \log^* n$

$\log^* n$: the number of successive log operations that need to be applied to n to bring it down to 1 (or below). In other words,

$$\log^* n = \begin{cases} 0 & \text{for } n \leqslant 1, \\ 1 + \log^*(\log_2 n) & \text{for } n > 1. \end{cases}$$

$$\log^* 2^{2^2} = 1 + \log^* 2^2 = 1 + 1 + \log^* 2 = 1 + 1 + 1 + \log^* 1 = 3.$$

Path Compression

Each find operation is given $\log^* n$ dollars in its pocket.

Each node x is given 2^k dollars if $\text{rank}(x)$ lies in the interval $\{k+1, \dots, 2^k\}$.

- # nodes with rank $> k$ is $\frac{n}{2^{k+1}} + \frac{n}{2^{k+2}} + \cdots \leqslant \frac{n}{2^k}$,
- the total money given to nodes in this interval is $\leqslant n$ dollars, and
- **the total money given to all nodes is $\leqslant n \log^* n$ dollars.**

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Consider the chain of nodes by find(u), and let x be a node in the chain. There are two categories of nodes x on the chain, depending on the parents $\pi(x)$:

A. x and $\pi(x)$ belong to different intervals.

find(u) pays 1 dollar using its pocket money.
At most $\log^* n$ such nodes.

B. x and $\pi(x)$ belong to the same interval.

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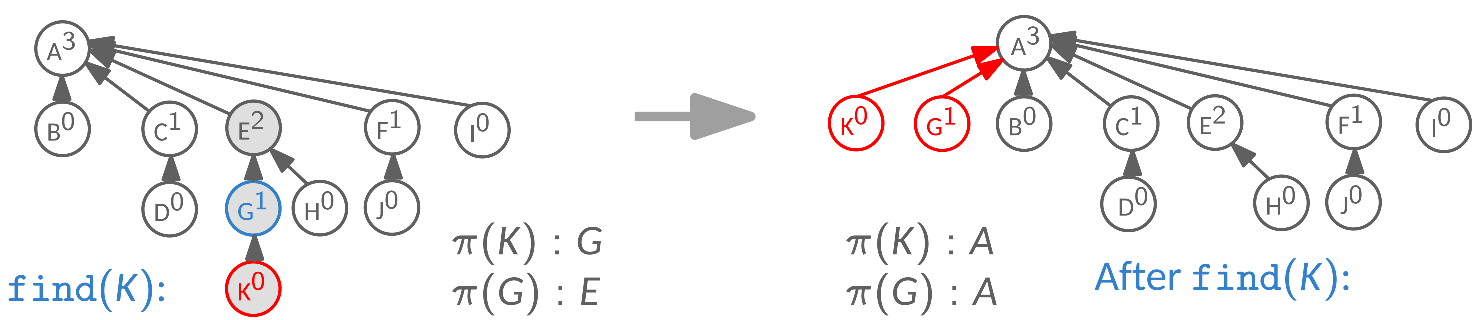
Each time x pays 1 dollar, $\pi(x)$ changes to the root node of higher rank.
($\text{rank}(\pi(x))$ increases at least by 1.)



Path Compression

Each find operation takes $O(\log^* n)$ steps plus some additional amount of time.

Consider the chain of nodes by find(u), and let x be a node in the chain. There are two categories of nodes x on the chain, depending on the parents $\pi(x)$:

A. x and $\pi(x)$ belong to different intervals.

find(u) pays 1 dollar using its pocket money. At most $\log^* n$ such nodes.

B. x and $\pi(x)$ belong to the same interval.

x pays 1 dollar using its pocket money.

(1) Each time x pays 1 dollar, $\pi(x)$ changes to a node of higher rank.

(2) If rank(x) lies in the interval $\{k+1, k+2, \ldots, 2^k\}$, it has to pay at most 2^k dollars before rank$(\pi(x))$ is in a higher interval;

(3) Once rank$(\pi(x))$ is in a higher group than rank(x), it remains so. Thus, x and $\pi(x)$ belong to A and x never has to pay again! (Instead, find pays.)

(4) Once a root node becomes nonroot, it stays as nonroot and its rank remains the same.

An intermixed sequence of m find and $n-1$ union operations can be done in $O(m \log^* n + n \log^* n) = O(m \log^* n)$ time $(m \geqslant n)$.

Path Compression

- (1972) Fischer derived an upper bound of $(m \log \log n)$.
- (1973) Hopcroft and Ullman improved the bound to $O(m \log^* n)$.
- (1975) Tarjan obtained the actual worst-case bound, $\Theta(m\alpha(m, n))$, where $\alpha(m, n)$ is a functional inverse of Ackerman's function which grows very slow. For example, $\alpha(m, n) \leqslant 3$ for $n < 2^{16} = 65,536$.

Remarks. Path compression requires two passes over the find path, one to find the tree root and another to perform the compression. Tarjan and Leeuwen studied a number of one-pass variants, some of which run in $O(m\alpha(m, n))$ time. For example, the following program implements *path halving*.

```
function find(x)
while π(π(x)) ≠ π(x) do
    x ← π(x) ← π(π(x))
return π(x)
```

Is the bound, $O(m\alpha(m, n))$ tight? Yes, there are sequences of set operations that actually take $\Omega(m\alpha(m, n))$ time (Tarjan 1975).

Interval Scheduling

Consider the following scheduling problem.

- Job j starts at s_j and finishes at f_j.
- Two jobs compatible if they don't overlap.
- Goal: find maximum subset of mutually compatible jobs.

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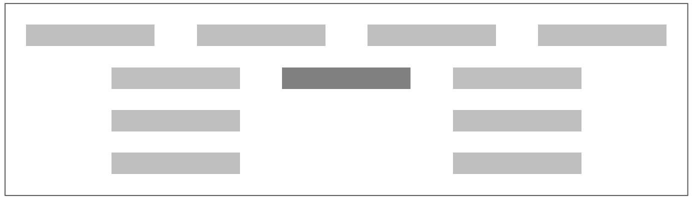
Greedy template. Consider jobs in some order. Take each job provided it is compatible with the ones already taken.

- Earliest start time: in ascending order of start time s_j.
- Earliest finish time: in ascending order of finish time f_j.
- Shortest interval: in ascending order of interval length $f_j - s_j$.
- Fewest conflicts: in ascending order of conflicts c_j.



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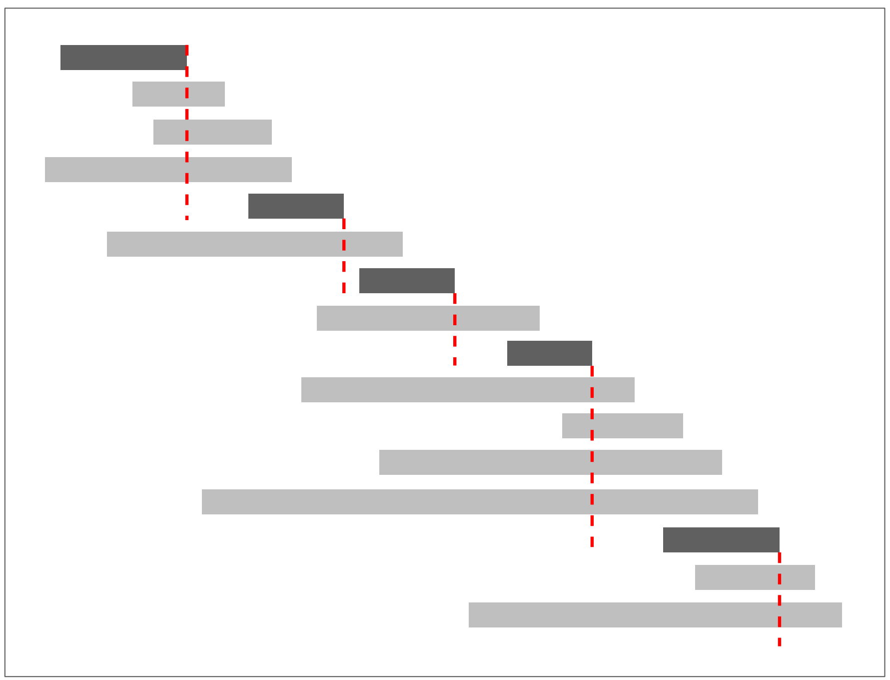

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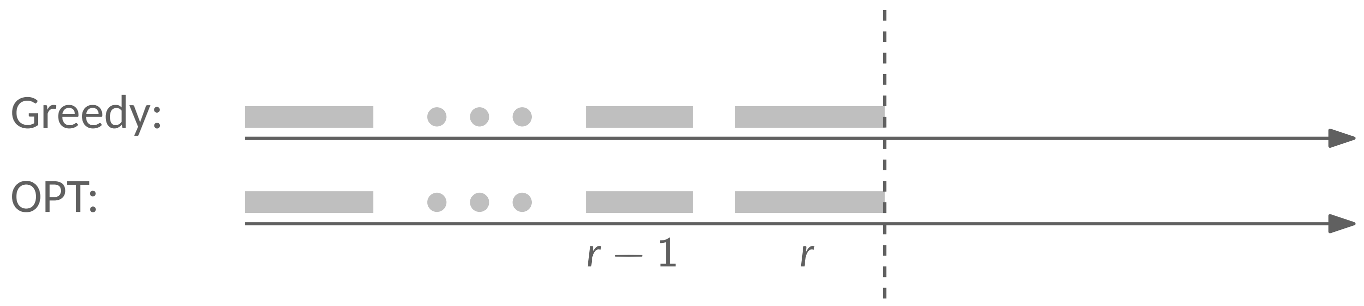
Theorem The greedy algorithm above is optimal.

Proof. Assume greedy is not optimal, and let's see what happens.
Let $i_1, i_2, \ldots, i_k$ denote the set of jobs selected by greedy. Let $j_1, j_2, \ldots, j_m$ denote the set of jobs in the optimal solution with $i_1 = j_1, i_2 = j_2, \ldots, i_r = j_r$ for the largest possible value of r.



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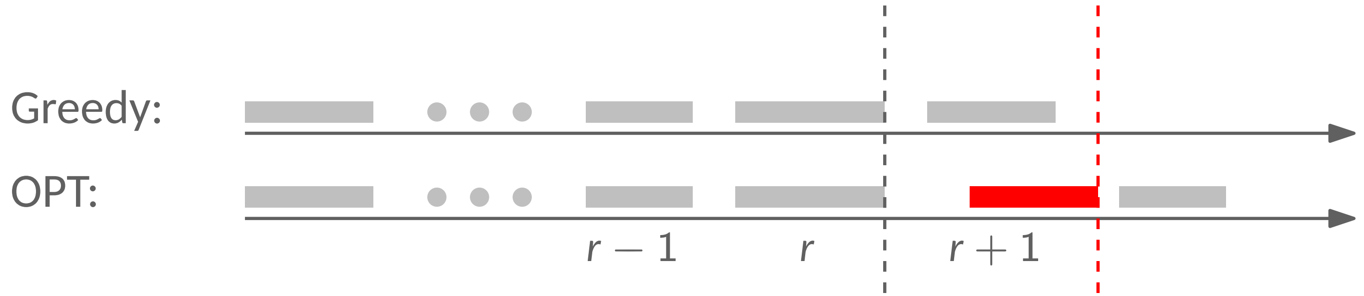
By the greedy choice, job i_{r+1} finishes before j_{r+1}. So in the optimal solution, we can replace job j_{r+1} with job i_{r+1}. The resulting solution is still feasible and optimal, but contradicts the maximality of r.



Interval Partitioning

There are n lectures. Lecture j starts at s_j and finishes at f_j.
Goal: find minimum number of classrooms to schedule all lectures so that no two lectures occur at the same time in the same classroom.

The depth of a set of open intervals is the maximum number of intervals that contain any given time. Then the number of classrooms needed $\geqslant$ depth.

Does there always exist a schedule equal to depth of intervals?

function **IntervalPartition**(x, y)

Sort lectures by starting time
$d = 0$
for $j = 1$ to n **do**
  if lecture j is compatible with a classroom k opened so far **then**
    schedule lecture j in classroom k
  else
    allocate a new classroom $d + 1$
    schedule lecture j in classroom $d + 1$
    $d = d + 1$

Implementation: $O(n \log n)$. For each classroom k, maintain the finish time of the last job added. Keep the classrooms in a priority queue.

Interval Partitioning

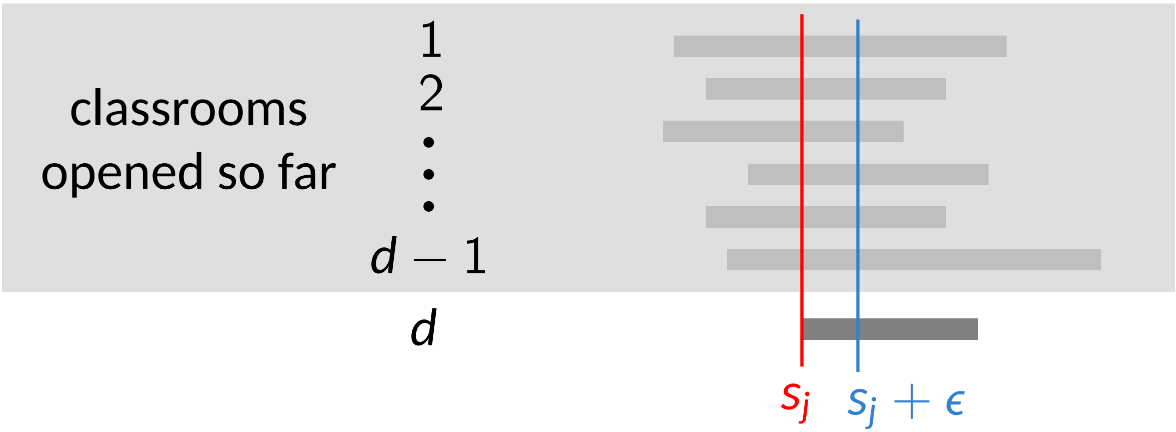
Theorem. Greedy algorithm is optimal.

Proof. Let d be the number of classrooms that the greedy algorithm allocates.

Classroom d is opened because we needed to schedule a job, say j, that is incompatible with all $d-1$ other classrooms.

Since we sorted lectures by their start time, all these incompatibilities are caused by lectures that start earlier than or at s_j.

Thus, we have d lectures overlapping at time $s_j + \varepsilon$, which implies that any valid schedule uses at least d classrooms.



Huffman Encoding

Consider a string consisting of 130 million characters and the alphabet $\{A, B, C, D\}$.
`CDADBBADDDAABCCACDBBABABBCDCDCCCDDACCBDADCAB...`
Most economic way to write this long string *in binary*?

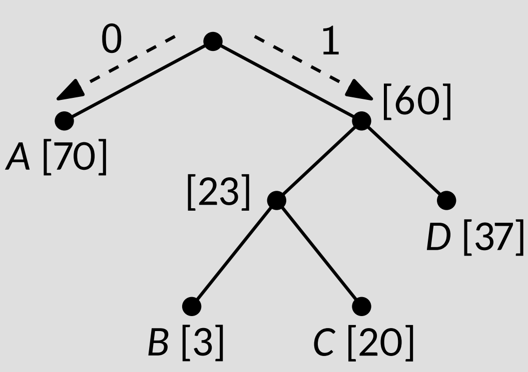
- **Two bits per symbol.** $\{A = 00, B = 01, C = 10, D = 11\} \Rightarrow$ 260 Mbits.
  `10110011010100111111000001...`
- Any **better encoding** than this?

What about using **variable-length encoding** with respect to the frequency.

But we need to devise a way to guarantee that decoding is **unique** $\Rightarrow$ **prefix-free**!

(No code word in the system is a prefix of any other code word in the system.)

Symbol	Codeword
A	0
B	100
C	101
D	11



A prefix-free encoding and its coding tree (full binary).

Huffman Encoding

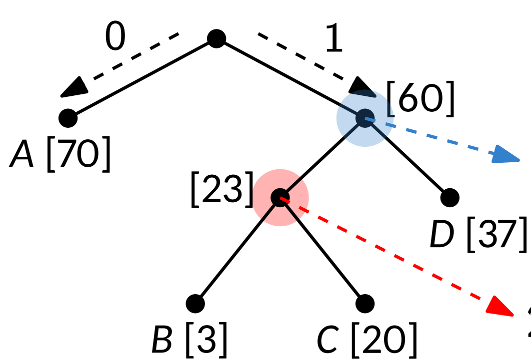
Given the frequencies $f_1, \dots, f_n$ of n symbols, find the optimal coding tree?
In other words, we want to find a (*full* binary) coding tree minimizing

$$\text{cost of tree} = \sum_{i=1}^{n} f_i \cdot (\text{depth(\# bits) of } i\text{th symbol in tree})$$

Another interpretation of the cost function is to define the *frequency* $f(v)$ of an internal node v as $f(v) = \sum_{\text{leaf } i \text{ in the subtree of } v} f_i$.

Then, $f(v)$ = number of times v is visited during encoding and decoding. Thus,

$$\text{cost of tree} = \sum_{\text{nonroot internal node } v} f(v) + \sum_{i=1}^{n} f_i$$



$60 = 3 + 20 + 37$

$23 = 3 + 20$

$\text{cost of tree} = 70*1+3*3+20*3+37*2 = 213$

$\text{cost of tree} = 23+60+70+3+20+37 = 213$

Huffman: Merge the two least frequent letters and recurse.

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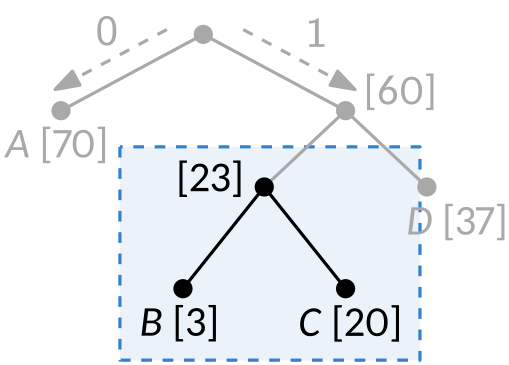
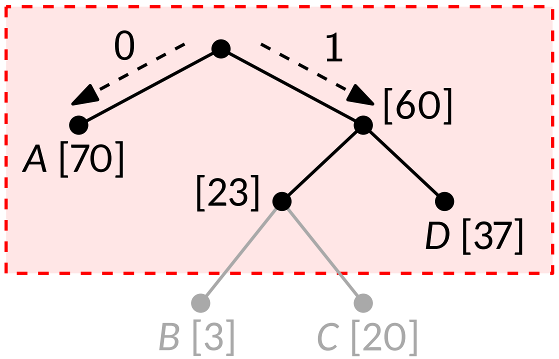




Lemma. Let x and y be the two least frequent letters. There is an optimal code tree in which x and y are siblings.

Proof. Let T be an optimal tree, with depth d. Because T is a full binary tree, it has two leaves at depth d that are siblings. Suppose they are not x and y, but some other letters a and b.

Let T' be the code tree obtained by swapping x and a, and let $\Delta = d - \text{depth}_T(x)$. Then $\text{cost}(T') = \text{cost}(T) + \Delta \cdot (f[x] - f[a])$.

Since $f[x] \leq f[a]$ and $\Delta \geq 0$, $\text{cost}(T') \leq \text{cost}(T)$. Since T is an optimal code tree, T' is also an optimal code tree.

Similarly, by swapping y and b, we can get another optimal code tree T''. Then x and y are siblings in T''.

Huffman Encoding

Lemma. Every Huffman code is an optimal prefix-free binary code.

Proof. $f[1:n]$: Input frequencies such that $f[1]$ and $f[2]$ are the two smallest frequencies. By the previous lemma, **1 and 2 are deepest siblings** in some optimal code tree for $f[1:n]$.

Let T' be the Huffman tree for $f[3:n+1]$, an optimal code tree for $f[3:n+1]$, where $f[n+1] = f[1] + f[2]$. Let T be the coding tree obtained from T' **by replacing the leaf $n+1$ with an internal node with two child nodes 1 and 2.**

We show that T is optimal for $f[1:n]$ by expressing $\text{cost}(T)$ in terms of $\text{cost}(T')$.

$$\begin{aligned}
\text{cost}(T) &= \sum_{i=1}^{n} f[i] \cdot \text{depth}(i) \qquad \text{depth}(i) = \text{depth of the leaf labeled } i \text{ in either } T \text{ or } T'. \\
&= \sum_{i=3}^{n+1} f[i] \cdot \text{depth}(i) + f[1] \cdot \text{depth}(1) + f[2] \cdot \text{depth}(2) - f[n+1] \cdot \text{depth}(n+1) \\
&= \text{cost}(T') + (f[1] + f[2]) \cdot \text{depth}(T) - f[n+1] \cdot (\text{depth}(T) - 1) \\
&= \text{cost}(T') + f[1] + f[2] + (f[1] + f[2] - f[n+1]) \cdot (\text{depth}(T) - 1) \\
&= \text{cost}(T') + f[1] + f[2]
\end{aligned}$$

Minimizing $\text{cost}(T)$ is equivalent to minimizing $\text{cost}(T')$. Attaching leaves labeled 1 and 2 to the leaf in T' labeled $n+1$ gives an optimal code tree for $f[1:n]$.

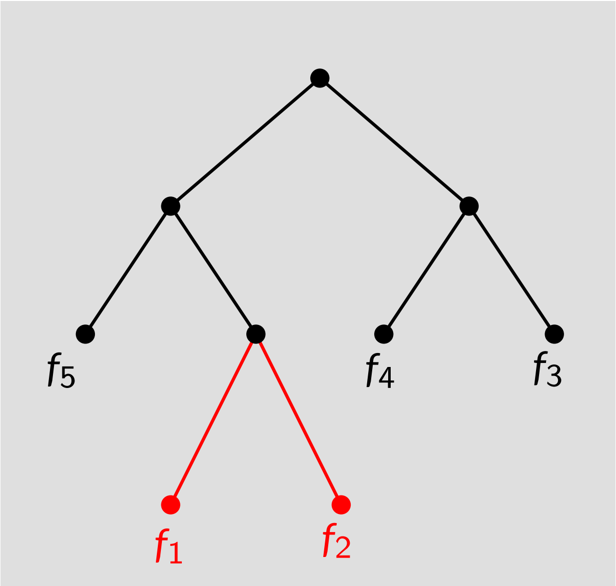
Huffman Encoding

$$\text{cost of tree} = \sum_{i=1}^{n} f_i \cdot (\text{depth(\# bits) of } i\text{th symbol in tree}) \tag{1}$$

$$= \sum_{\text{nonroot internal node } v} f(v) + \sum_{i=1}^{n} f_i \tag{2}$$

Huffman: Merge the two least frequent letters and recurse.

(1): two symbols with the smallest frequencies must be at the **bottom** of the optimal tree.
(Otherwise we can always find a better coding tree.)



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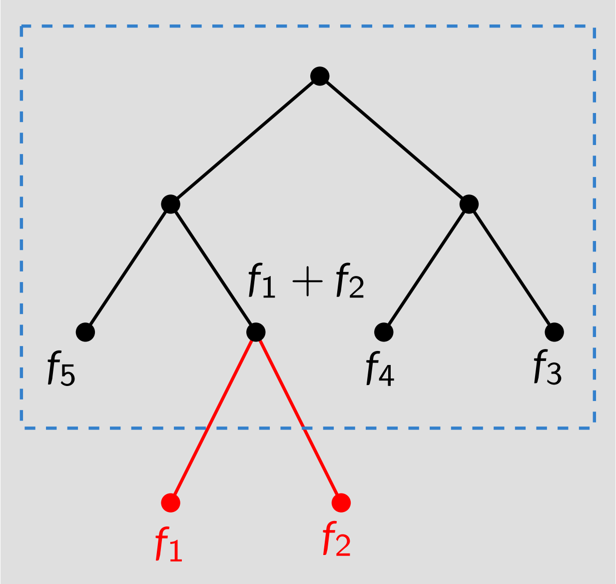
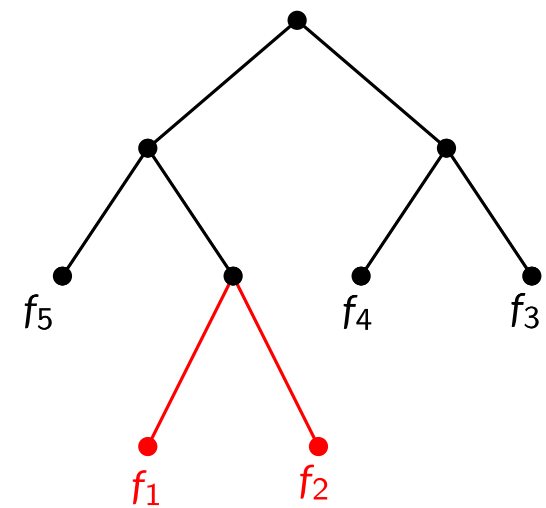


function **Huffman**(f)

makequeue(H)
for $i = 1$ to n **do**
  insert($H, i, f[i]$) (* number i with key $f[i]$ *)
for $k = n + 1$ to $2n - 1$ **do**
  $\boxed{i = \text{deletemin}(H), j = \text{deletemin}(H)}$ (1)
  Create a node numbered k with child nodes i, j.
  $f[k] = f[i] + f[j]$
  insert($H, k, f(k)$)



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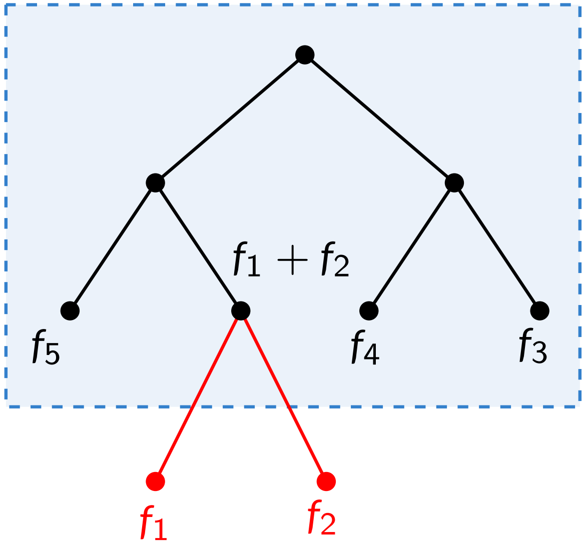

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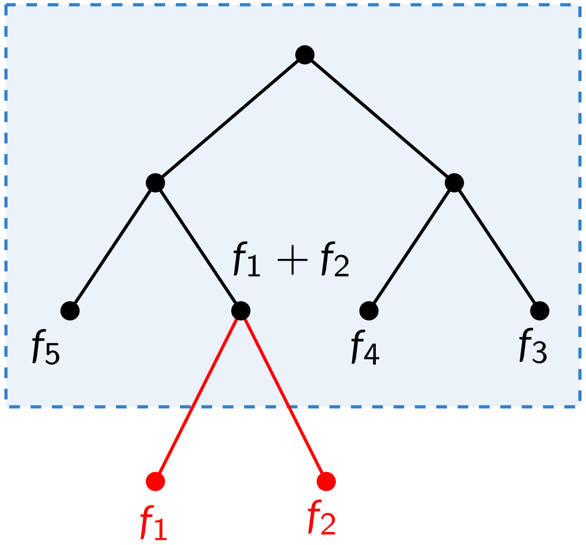

$O(n \log n)$ time if a binary heap is used.

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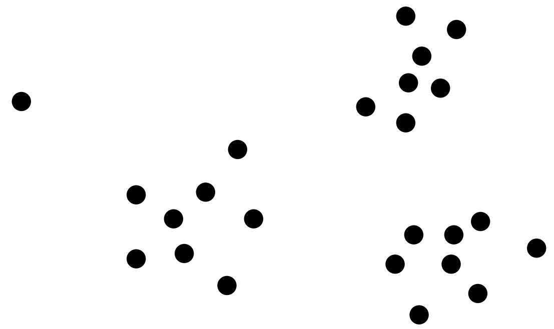
Clustering of Maximum Spacing

Clustering Given a set of n objects, classify them into *coherent* groups.



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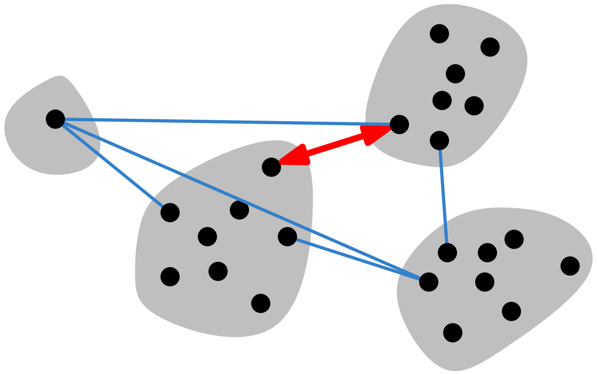


k**-clustering** divides objects into k nonempty groups.

Distance functions must reflect closeness of two objects.
Spacing is the min. distance between any pair of points in different clusters.

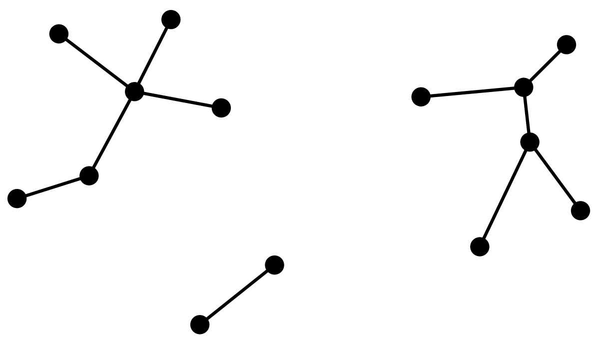
Clustering of max. spacing. Given k, find a k-clustering of maximum spacing.

Clustering of Maximum Spacing

Clustering of max. spacing. Given k, find a k-clustering of maximum spacing.

Single-link k-clustering algorithm.
- form a graph on the vertex set U, corresponding to n clusters.
- find the closest pair of objects s.t. each object is in a different cluster, and add an edge between them.
- repeat $n - k$ times until there are exactly k clusters.

Kruskal's algorithm, except we stop when there are k connected components.



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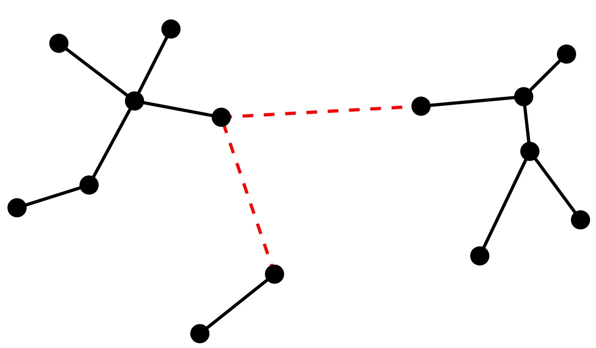
Equivalent to finding an MST and then deleting the $k - 1$ most expensive edges.

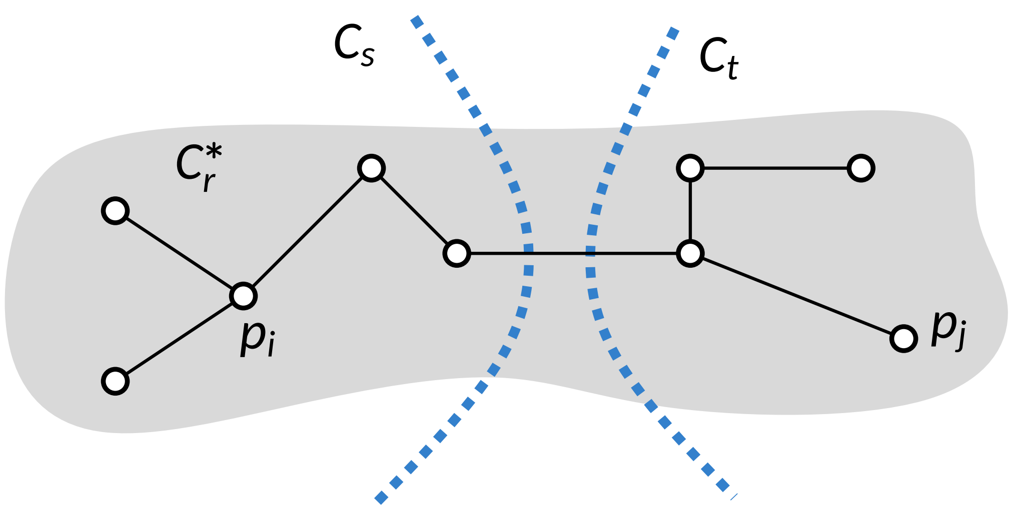


Clustering of Maximum Spacing

Theorem Let $\mathcal{C}^*$ denote the clustering $C_1^*, \ldots, C_k^*$ formed by deleting the k most expensive edges of a MST. $\mathcal{C}^*$ is a k-clustering of max. spacing.

Proof. The spacing of $\mathcal{C}^*$ is the length d^* of the $(k-1)$st most expensive edge. Let $\mathcal{C}$ denote some other clustering $C_1, \ldots, C_k$, and d denote its spacing.

Let p_i, p_j be two points in the same cluster in $\mathcal{C}^*$, say C_r^*, but in different clusters in $\mathcal{C}$, say $p_i \in C_s$ and $p_j \in C_t$.



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Consider the path from p_i to p_j in C_r^*. Some edge (p, q) on the path spans two different clusters, C_s and C_t in $\mathcal{C}$.

Since all edges on the path have length $\leqslant d^*$, we have $d \leqslant d^*$ ($\because$ p and q are in different clusters and therefore $d \leqslant |pq|$).