Linear Programming

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Textbooks: Algorithms by Jeff Erickson /Algorithms by Sanjoy Dasgupta, Christos Papadimitriou, Umesh Vazirani.

Linear Programming

In many optimization tasks, we seek a solution that

- satisfies certain <u>constraints</u>, and
- is the best possible, with respect to <u>some criterion</u> (objective function).

> *Linear programming* describes
> a broad class of optimization tasks in which both the constraints and the optimization criterion are linear functions.

Given a set of variables, we want to assign real values to them so as to

- satisfy a set of linear equations and/or linear inequalities involving the variables, and
- maximize or minimize a given linear objective function.

The task is, from the context, to

(A) identify variables, linear equations, linear inequalities and a linear objective function involving the variables, and Formulate problem in linear programming!

(B) find real values to the variables that maximize or minimize the linear objective function. Let simplex algorithm do this.

Profit Maximization

Two fountain pen products,
- POSPEN : 1K Won profit per piece, $\leqslant$200 pieces demand per day
- POSPEN Gold : 6K Won profit per piece, $\leqslant$300 pieces demand per day
- Workforce can produce $\leqslant$400 pieces per day.

What is the optimal level of production?

Variables:
- x_1: #. POSPEN to produce per day.
- x_2: #. POSPEN Gold to produce per day.

Constraints:
- Default ranges of variables: $x_1, x_2 \geqslant 0$.
- Demands for POSPEN ($\leqslant$ 200) and POSPEN Gold ($\leqslant$ 300) per day.
- Capacity of the workforce per day: $\leqslant$ 400.

Objective function: optimal production (maximum profit) you can make per day.

Objective function	$\max\ x_1 + 6x_2$
Constraints	$x_1 \leq 200$
	$x_2 \leq 300$
	$x_1 + x_2 \leq 400$
	$x_1, x_2 \geq 0$

Profit Maximization

Objective function	$\max\ x_1 + 6x_2$
Constraints	$x_1 \leq 200$
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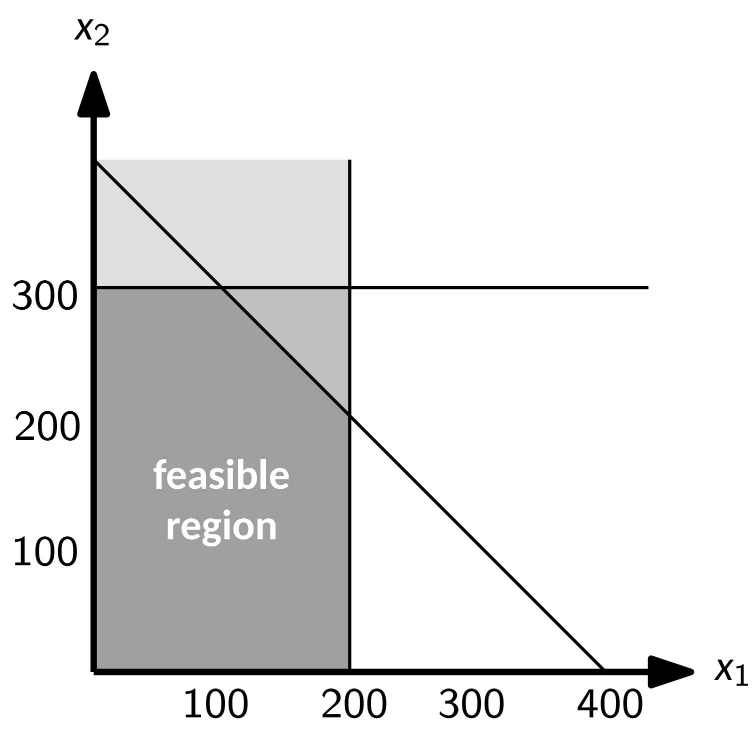
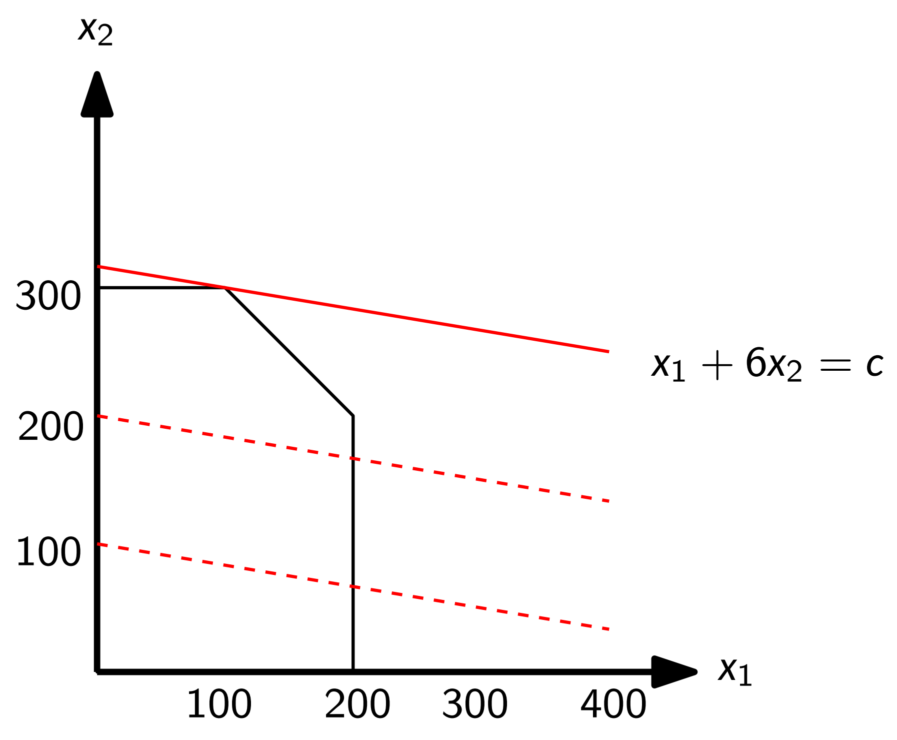
	$x_1 + x_2 \leq 400$
	$x_1, x_2 \geq 0$





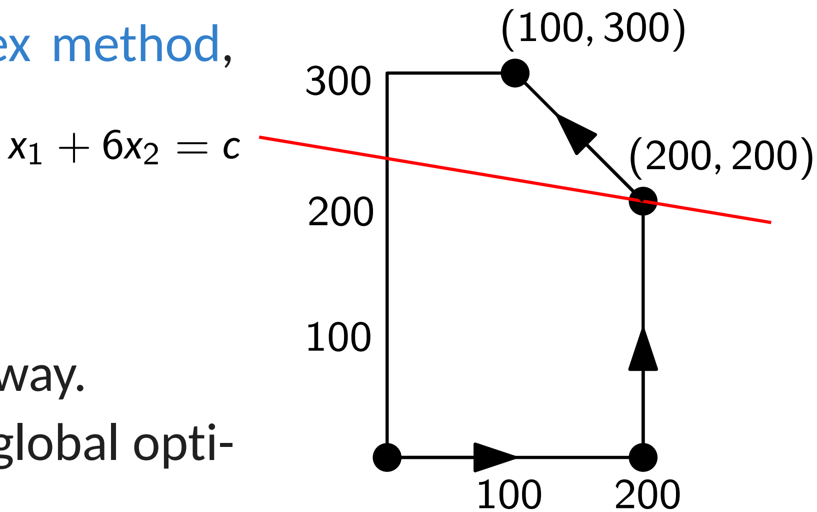
Profit Maximization - Simplex Method

The optimum is achieved at a vertex of the feasible region, with some exceptions

- *infeasible* : the constraints are so tight that it is impossible to satisfy all of them.
- *unbounded* : the constraints are so loose that it is possible to achieve arbitrary high values.

Linear programs (LPs) can be solved by the simplex method, which

- starts at a vertex,
- repeatedly looks for an adjacent vertex of better objective value,
- *hill-climbing* : steadily increases profit along the way.
- A local optimal vertex (no better neighbors) is a global optimum vertex.



Profit Maximization

Three fountain pen products,
- POSPEN : 1K Won profit per piece, $\leqslant$200 pieces demand per day
- POSPEN Gold : 6K Won profit per piece, $\leqslant$300 pieces demand per day
- POSPEN Limited : 13K profit per piece
- Workforce can produce $\leqslant$400 pieces per day.
- Gold and Limited use the same packaging machinery($\leqslant$600 packagings per day), Limited uses it three times as much

What is the optimal level of production?

Identify variables, linear constraints, and a linear objective function.

Objective function	$\max\ x_1 + 6x_2 + 13x_3$
Constraints	$x_1 \leq 200$
	$x_2 \leq 300$
	$x_1 + x_2 + x_3 \leq 400$
	$x_2 + 3x_3 \leq 600$
	$x_1, x_2, x_3 \geq 0$

Diet Problem

There are n different types of food, $F_1, \dots, F_n$, and each food has some of each of m nutrients, $N_1, \dots, N_m$. Suppose

- a_{ij} = amount of nutrient N_j in a unit of food F_i, for $i = 1, \dots n, j = 1, \dots, m$.
- r_j = daily requirement of nutrient N_j, for $j = 1, \dots, m$.
- x_i = daily consumption of food F_i, for $i = 1, \dots, n$, in units.
- p_i = price per unit of food F_i, for $i = 1, \dots, n$.

A daily diet is represented by a choice of an n-vector $x \geqslant 0$. We want to find the least expensive diet that is nutritionally adequate. Formulate this problem in standard form.

Minimize

$$p_1 x_1 + p_2 x_2 + \cdots + p_n x_n.$$

Subject to

$$a_{1j} x_1 + a_{2j} x_2 + \cdots + a_{nj} x_n \geqslant r_j \quad \text{for } j = 1, \dots m.$$

$$x_1 \geqslant 0, x_2 \geqslant 0, \dots x_n \geqslant 0.$$

Assignment Problem

There are n persons available for m jobs. The value of person i working 1 day at job j is a_{ij}, for $i = 1, \dots, n$, and $j = 1, \dots, m$. A person can switch to different jobs, but only one person is allowed on a job at a time Choose an assignment of persons to jobs to maximize the total value.

Let x_{ij} denote the proportion of the amount of time that person i spends on job j, for $i = 1, \dots, n$ and $j = 1, \dots, m$. Then,

$$\begin{aligned}
\sum_{j=1}^{m} x_{ij} &\leqslant 1 \quad \text{for } i = 1, \dots, n, \\
\sum_{i=1}^{n} x_{ij} &\leqslant 1 \quad \text{for } j = 1, \dots, m, \text{ and} \\
x_{ij} &\geqslant 0 \quad \text{for } i = 1, \dots, n \text{ and } j = 1, \dots, m.
\end{aligned}$$

We want to maximize the total value,

$$\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} x_{ij}.$$

Production Planning

A company produces handcraft ceramic jars. The monthly demand estimates are $440 \leqslant d_i \leqslant 920$ for $i = 1, \ldots, 12$. We have currently 30 employees, each makes 20 jars per month and gets salary of 2M Won.

- *overtime* : 80% extra pay. $\leqslant$30% overtime.
- *hiring and firing* : 320K Won and 400K Won, respectively.
- *storing* : 8K Won per jar per month. No stored jars in the beginning and no extra jars at the end of the year.

Formulate these conditions using some variables!

- w_i = number of workers during ith month; w_0 = 30.
- x_i = number of jars made during ith month.
- o_i = number of jars made by overtime in month i.
- h_i, f_i = number of workers hired and fired, respectively, at beginning of month i.
- s_i = number of jars stored at end of month i; s_0 = 0.

What are the constraints?
What is the objective function, minimizing the total cost?

Production Planning

All variables must be nonnegative: $w_i, x_i, o_i, h_i, f_i, s_i \geqslant 0, \quad i = 1, \ldots, 12$

#. jars made per month: $x_i = 20w_i + o_i$

#. workers can change at the beginning of each month: $w_i = w_{i-1} + h_i - f_i$

#. jars stored at the end of each month: $s_i = s_{i-1} + x_i - d_i$

#. jars made by overtime in each month: $o_i = 6w_i$

$\frac{2000}{20} \times 1.8$

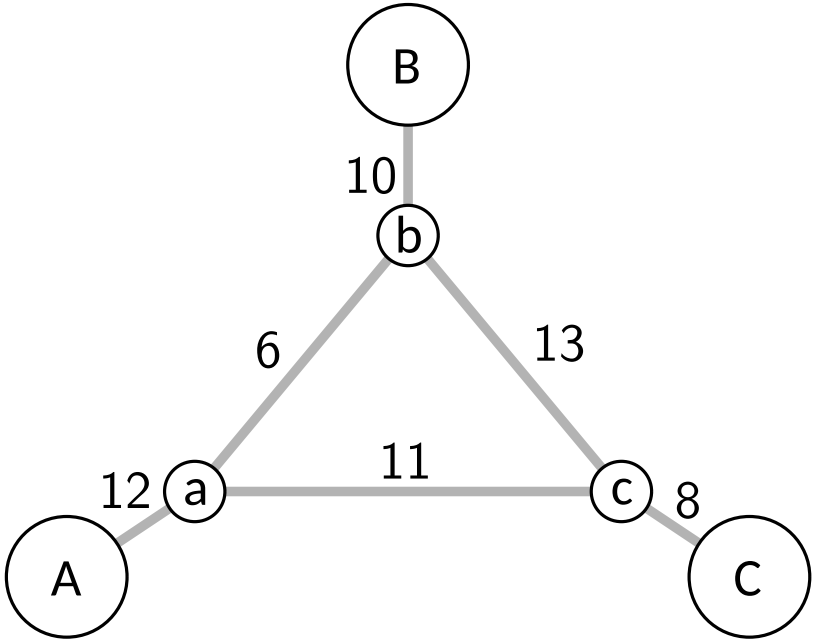
The objective function that minimizes the total cost:

$$\min \quad 2000 \sum_i w_i + 320 \sum_i h_i + 400 \sum_i f_i + 8 \sum_i s_i + \underbrace{180}_{\frac{2000}{20} \times 1.8} \sum_i o_i.$$

A network service provider faces the following problem.

- each connection requires at least two units of bandwidth.
- Connection A-B pays 3K Won, B-C pays 2K Won, and A-C pays 4K Won per unit.
- *two-way routing* : a long path, a short path, or by combination.

How do we route these connections to maximize the network's revenue?



Bandwidth Allocation

- x_{AB} : short-path bandwidth allocated to the connection between A and B.
- x'_{AB} : long-path bandwidth allocated to the connection between A and B.

Formulate these conditions using the variables!

$$\max\ 3x_{AB} + 3x'_{AB} + 2x_{BC} + 2x'_{BC} + 4x_{AC} + 4x'_{AC}$$

$$\begin{aligned}
x_{AB} + x'_{AB} + x_{BC} + x'_{BC} &\leq 10 \quad [\text{edge } (b, B)] \\
x_{AB} + x'_{AB} + x_{AC} + x'_{AC} &\leq 12 \quad [\text{edge } (a, A)] \\
x_{BC} + x'_{BC} + x_{AC} + x'_{AC} &\leq 8 \quad [\text{edge } (c, C)] \\
x_{AB} + x'_{BC} + x'_{AC} &\leq 6 \quad [\text{edge } (a, b)] \\
x'_{AB} + x_{BC} + x'_{AC} &\leq 13 \quad [\text{edge } (b, c)] \\
x'_{AB} + x'_{BC} + x_{AC} &\leq 11 \quad [\text{edge } (a, c)] \\
x_{AB} + x'_{AB} &\geq 2 \\
x_{BC} + x'_{BC} &\geq 2 \\
x_{AC} + x'_{AC} &\geq 2 \\
x_{AB}, x'_{AB}, x_{BC}, x'_{BC}, x_{AC}, x'_{AC} &\geq 0
\end{aligned}$$

Any LP $\longrightarrow$ Standard Form

problems	maximization or minimization
constraints	inequalities and equalities
variables	nonnegative and unrestricted variables

$\downarrow$

a standard form

minimization

equations

nonnegative

$\sum_{i=1}^{n} a_i x_i \leqslant b$

$\downarrow$

create a variable s

$\sum_{i=1}^{n} a_i x_i + s = b$

$s \geqslant 0$

unrestricted x

$\downarrow$

create x^+ and x^-.

$x = x^+ - x^-$

$x^+, x^- \geqslant 0.$

Any LP $\longrightarrow$ Standard Form

$$
\begin{aligned}
\max\ & x_1 + 6x_2 \\
x_1 &\leqslant 200 \\
x_2 &\leqslant 300 \\
x_1 + x_2 &\leqslant 400 \\
x_1, x_2 &\geqslant 0
\end{aligned}
\quad\longrightarrow\quad
\begin{aligned}
\min\ & -x_1 - 6x_2 \\
x_1 + s_1 &= 200 \\
x_2 + s_2 &= 300 \\
x_1 + x_2 + s_3 &= 400 \\
x_1, x_2, s_1, s_2, s_3 &\geqslant 0
\end{aligned}
$$