0.1 Central bead is the closest bead to the center of mass

Let our chain be defined as a sequence of N uncorrelated and independent random variables in the following way
$p_1 = n_1$
$p_2 = p_1 + n_2 = n_1 + n_2$
$p_3 = p_2 + n_3 = n1 + n_2 + n_3$
.
.
$p_N = \sum_{i=1}^{i=N} n_i$
where $n_i \sim N(0,1) \forall i = 1..N$ and N is an odd integer.

We define the chain center of mass as:

$$p_{cm} = \frac{1}{N} \sum_{i=1}^{i=N} p_i = \frac{1}{N}(Nn_1 + (N-1)n_2 + (N-3)n_3 + ...n_N) = \sum_{i=1}^{N} (\frac{N-i+1}{N})n_i$$

The question we address here is for which index $k \in [1..N]$ does the point p_k is closest to p_{cm}?

Each p_i is distributed normally with mean $\mu_i = 0$ (since the sum of means of the preceding points is zero) and $\sigma_i = \sqrt{\sigma_{i-1}^2 + 1 + 2\rho\sigma_{i-1}}$, where ρ is the correlation coefficient. Since each two subsequent points are uncorrelated, by definition $\rho = 0$. More specifically, the ρ in the expression for the standard deviation of point i is given by $E[(n_i - \mu_i)(p_{i-1} - \mu_{i-1})]/\sigma_{n_i}\sigma_{i-1} = E[n_i p_{i-1}]/\sigma_j$, where E is the expectation. Since n_i is independent of p_{i-1}, we get

$$E[n_i p_{i-1}]) = E[n_i]E[p_{i-1}] = 0$$

Therefore we get

$$\sigma_i = \sqrt{i}$$

The center of mass p_{cm} has mean $\mu_{cm} = 0$ and standard deviation that can be written as

$$\sigma_{cm} = \sum_{i=1}^{i=N} \left(\frac{N-i+1}{N}) \right)^2 = \frac{N(N+1)(2N+1)}{6N^2}$$

We now look for the point p_j for which the distribution of the random variable

$$Y(j) = p_{cm} - p_j$$

is the most "concentrated" around zero. [this part should be better characterized]. In this sense, the standard deviation of $Y(j)$ should be the smallest among all $i = 1..N$.

the random variable $Y(j)$ can be written as

$$Y(j) = \sum_{k=1}^{N} \left(1 + \frac{1-k}{N}\right) n_k - \sum_{k=1}^{j} n_k = \sum_{k=1}^{j} \frac{1-k}{N} n_k + \sum_{k=j+1}^{N} (1 + \frac{1-k}{N}) n_k$$

The standard deviation of Y_j is

$$\sigma_{Y(j)} = \sqrt{\sum_{k=1}^{j} \left(\frac{1-k}{N}\right)^2 + \sum_{k=j+1}^{N} \left(\frac{N+1-k}{N}\right)^2} = \frac{1}{N} \sqrt{\sum_{k=1}^{j-1} k^2 + \sum_{k=1}^{N-j} k^2}$$

To find the j that minimizes this expression, we can disregard the square root, and denote $Q(j) = \sum_{k=1}^{j-1} k^2$, so $Q(N-j+1) = \sum_{k=1}^{N-j} k^2$. Differentiate to find $Q'(j) - Q'(N-j+1) = 0$ we see that when $j = \frac{N+1}{2}$

$$Q'(\frac{N+1}{2}) - Q'(N - \frac{N+1}{2} + 1) = Q'(\frac{N+1}{2})$$