2.7 Precise Definitions of Limits (Revisited)

Definition

We write

$$\lim_{x \to a} f(x) = L$$

if for every $\varepsilon > 0$ *there is a number* $\delta > 0$ *such that*

$$|f(x) - L| < \varepsilon \qquad \textit{whenever} \quad 0 < |x - a| < \delta.$$

- $|f(x) - L| < \varepsilon$: small neighborhood of L
- $0 < |x - a| < \delta$: small (punctured) neighborhood of a.
- Given ε (challenge), we find δ (defense) from the given ε.

Definition

We write

$$\lim_{x \to a} f(x) = \infty$$

if for every $M > 0$ *there is a number* $\delta > 0$ *such that*

$$f(x) > M \qquad \textit{whenever} \quad 0 < |x - a| < \delta.$$

- M represents a big number
- Given M (challenge), we find δ (defense) from the given M.

3.1 Introducing the Derivative

Definition

1. *The average rate of change of f on $[a, b]$ is defined by*

$$m_{ave} = \frac{f(b) - f(a)}{b - a}.$$

2. *The instantaneous rate of change of f at $x = a$ is defined by*

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Remark

Geometrically, m_{ave} = *the slope of the secant line, and*

m = *the slope of the tangent line at* $x = a$.

- We write

$$f'(a) = m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

and we call $f'(a)$ the derivative of f at $x = a$.

- Thus we have

$$\begin{aligned} f'(a) &= (\text{derivative of } f \text{ at } x = a) \\ &= (\text{slope of the tangent line to } y = f(x) \text{ at } x = a) \\ &= (\text{instantaneous rate of change of } f \text{ at } x = a) \end{aligned}$$

- Setting $x = a + h$, we see that $h \to 0$ as $x \to a$ and obtain

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Definition

1. *If $f'(a)$ exists, we say that f is differentiable at $x = a$.*
2. *If f is differentiable at every point of an interval I, we say that f is differentiable on I.*

- Examples

- We can consider the correspondence $a \mapsto f'(a)$ as a function.

Definition

The ***derivative*** *of f is the function*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided that f is differentiable.

- If we know $f'(x)$ then we obtain $f'(1)$, $f'(2)$, $f'(3)$, $\ldots$ by substituting $x = 1, 2, 3, \ldots$.
- Example

- We set

$$\Delta y = f(x+h) - f(x) \quad \text{and} \quad \Delta x = h,$$

and write

$$f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

- We also write

$$f'(a) = \left. \frac{dy}{dx} \right|_{x=a}.$$