2.5 Limits at Infinity

Definition

1. *We write*

$$\lim_{x\to\infty} f(x) = L$$

if the values of $f(x)$ approaches (or always equals) L as x becomes arbitrarily large.

2. *We write*

$$\lim_{x\to-\infty} f(x) = L$$

if the values of $f(x)$ approaches (or always equals) L as x becomes arbitrarily large negative.

Definition

Then line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \qquad \textit{or} \qquad \lim_{x\to-\infty} f(x) = L.$$

- Examples
  1. $y = \dfrac{1}{x^r}, \quad r > 0$
  2. $y = \tan^{-1} x$

Definition

We define

$$\lim_{x\to\infty} f(x) = \infty, \lim_{x\to\infty} f(x) = -\infty, \lim_{x\to-\infty} f(x) = \infty, \lim_{x\to-\infty} f(x) = -\infty$$

in an obvious way.

★ $\frac{\infty}{\infty}$-type

1. numerator $\gg$ denominator $\Rightarrow$ $\lim = \infty$ or $\lim = -\infty$
2. numerator $\ll$ denominator $\Rightarrow$ $\lim = 0$
3. numerator $\sim$ denominator $\Rightarrow$ $\lim =$ the ratio of the coefficients

- $x^m \gg x^n$ if $m > n$
- Be careful about $\sqrt{\ }$ if $x \to -\infty$.

★ $(\infty - \infty)$-type with $\sqrt{\ }$: make a fraction using the companion

- $\lim_{x\to\infty} e^x = \infty, \qquad \lim_{x\to-\infty} e^x = 0$
- $\lim_{x\to\infty} e^{-x} = 0, \qquad \lim_{x\to-\infty} e^{-x} = \infty$
- $\lim_{x\to\infty} \ln x = \infty, \qquad \lim_{x\to 0^+} \ln x = -\infty$

1. What is the value of $\lim_{x \to \infty} \frac{x}{e^x}$?

   A. 0 B. 1 C. $1/e$ D. ∞ E. None of A-D

2. What is the value of $\lim_{x \to \infty} \frac{x}{\ln x}$?

   A. 0 B. 1 C. $1/e$ D. ∞ E. None of A-D

2.7 Precise Definitions of Limits

Problem

For $x > 0$, *we define* $f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{m}{n},\ (m,n) = 1, \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$

Prove that $\lim_{x \to e} f(x) = 0$.

- We cannot appeal to our eyes any more.
- In order to develop a rigorous argument, we need first to have precise definitions.

Definition

We write

$$\lim_{x \to a} f(x) = L$$

if for every $\varepsilon > 0$ *there is a number* $\delta > 0$ *such that*

$$|f(x) - L| < \varepsilon \qquad \textit{whenever} \quad 0 < |x - a| < \delta.$$

- $|f(x) - L| < \varepsilon$: small neighborhood of L
- $0 < |x - a| < \delta$: small (punctured) neighborhood of a.
- Given ε (challenge), we find δ (defense) from the given ε.
- Examples

Definition

We write

$$\lim_{x \to a} f(x) = \infty$$

if for every $M > 0$ *there is a number* $\delta > 0$ *such that*

$$f(x) > M \qquad \textit{whenever} \quad 0 < |x - a| < \delta.$$

- M represents a big number
- Given M (challenge), we find δ (defense) from the given M.