2.3 Techniques for Computing Limits

Proposition (Limit Laws)

Suppose that the limits $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ *exist. Then*

1. $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
2. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
3. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ *if* $\lim_{x\to a} g(x) \neq 0$
4. $\lim_{x\to a}[f(x)]^{m/n} = \left[\lim_{x\to a} f(x)\right]^{m/n}$

- Examples

★ $\frac{0}{0}$-type : factorization and cancellation

★ $\frac{0}{0}$-type with $\sqrt{\ }$: multiplication by the companion (or conjugate)

Theorem (Squeeze Thm)

If $f(x) \leq g(x) \leq h(x)$ *and* $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$

then $\lim_{x \to a} g(x) = L.$

- Example

1. $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$ What is the value of $\lim_{x \to 0} f(x)$?

   A. 0 B. 1 C. Does not exist

2. $f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$ What is the value of $\lim_{x \to 0} f(x)$?

   A. 0 B. 1 C. Does not exist

3. For $x > 0$, $f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{m}{n},\ (m, n) = 1, \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$

   $\lim_{x \to 1} f(x) = ?$ $\lim_{x \to e} f(x) = ?$

2.4 Infinite Limits

Definition

1. *We write*

$$\lim_{x \to a} f(x) = \infty$$

if the values of $f(x)$ can be made arbitrarily large as $x \to a$.

2. *We write*

$$\lim_{x \to a} f(x) = -\infty$$

if the values of $f(x)$ can be made arbitrarily large negative as $x \to a$.

3. *One-sided infinite limits are defined similarly.*

- Examples

Definition

The line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if

$$\lim_{x \to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \pm\infty \quad \text{or} \quad \text{both.}$$

- Examples
- Graphs
- Find vertical asymptotes.
  1. $y = \tan x$
  2. $y = \ln x$
  3. $y = \dfrac{1}{x^2 - 1}$