3.1 Introducing the Derivative (Continued)

- Recall

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \\ &= (\text{slope of the tangent line to } y = f(x) \text{ at } x = a) \\ &= (\text{instantaneous rate of change of } f \text{ at } x = a) \end{aligned}$$

- If $f'(a)$ exists, we say that f is differentiable at $x = a$.

- Is the function $f(x) = |x|$ differentiable at $x = 0$?

  A. Yes B. No

- If $f(x)$ is differentiable, it is smooth (to the first degree). In particular, it does not have any kink, discontinuity or vertical tangent.

Theorem

If f is differentiable at a, then f is continuous at a.

- Continuity does not imply differentiability.
- Examples: $y = |x|, \quad y = \sqrt[3]{x}$
- We can (roughly) sketch the graph of f' from the graph of f.

3.2 Rules of Differentiation

Proposition

1. $[cf(x)]' = cf'(x)$
2. $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$

- Proofs
- Let $f(x) = c$, a constant function. What is $f'(x)$?

A. $f'(x) = c$ B. $f'(x) = cx$ C. $f'(x) = 0$

Proposition (★★)

1. $[c]' = 0$
2. $[x^n]' = nx^{n-1}$ *(n: any real number)*
3. $[e^x]' = e^x$

- We need:
  1. Binomial Theorem (n: positive integer)

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2 + \cdots + y^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

  2. $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

Definition

1. *The second derivative of f is the derivative of f', and denoted by*

$$f''(x) = f^{(2)}(x) = \frac{d^2 f}{dx^2}.$$

2. *The third derivative of f is the derivative of f'', and denoted by*

$$f'''(x) = f^{(3)}(x) = \frac{d^3 f}{dx^3}.$$

3. *Similarly, we define the n-th derivative of f:*

$$f^{(n)}(x) = \frac{d^n f}{dx^n} = [f^{(n-1)}(x)]'.$$

- Examples

3.3 The Product and Quotient Rules

- Is it true that $[f(x)g(x)]' = f'(x)g'(x)$?

  (Hint: Consider the case $f(x) = x$ and $g(x) = x$.)

  A. Yes B. No

Proposition (★★)

1. $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$
2. $$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

- Proofs
- Examples