5.2 Definite Integrals

- A Riemann sum is actually approximating the net area.

Definition

1. *The net area is the sum of the areas that lie above the x-axis minus the sum of the areas that lie below the x-axis.*
2. *The definite integral of f from a to b is the net area of the region bounded by the graph of f and the x-axis between $x = a$ and $x = b$.*
3. *We write*

$$\int_a^b f(x)dx$$

to denote the definite integral of f from a to b.

- How can we calculate the definite integral $\int_a^b f(x)dx$?
- A net area can be calculated as the limit of a Riemann sum.
- Thus we have

$$\begin{aligned}\int_a^b f(x)dx &= \text{net area} \\ &= \text{the limit of a Riemann sum} \\ &= \lim_{n\to\infty}\sum_{k=1}^{n} f(\bar{x}_k)\Delta x,\end{aligned}$$

  if the limit exists.
- If $\int_a^b f(x)dx$ exists, the function f is said to be integrable.
- Sometimes, we can use familiar area formulas.

Definition

Suppose that f *is integrable on* $[a,b]$. *We define*

$$\int_b^a f(x)dx = -\int_a^b f(x)dx \quad \textit{and} \quad \int_a^a f(x)dx = 0.$$

Proposition

1. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
2. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$

Proposition

$$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$$

- A sprinter practices by running various distances back and forth in a straight line in a gym. Her velocity at t seconds is given by the function $v(t)$. What does $\int_0^{60} |v(t)| dt$ represent?

  A. The total distance the sprinter ran in one minute

  B. The sprinter's average velocity in one minute

  C. The sprinter's distance from the starting point after one minute

  D. None of the above

5.3 Fundamental Theorem of Calculus

- In general, it is impossible to calculate $\int_a^b f(x)dx$ as the limit of Riemann sums.
- If $f(t)$ is a velocity function, then $\int_a^b f(t)dt$ represents the net change in position from $t = a$ to $t = b$.
- Since an antiderivative $F(t)$ of $f(t)$ is a position function, we have

$$\begin{aligned} \int_a^b f(t)dt &= \text{distance} \\ &= \text{(ending position)} - \text{(starting position)} \\ &= F(b) - F(a). \end{aligned}$$

Theorem (Fundamental Theorem of Calculus (★★))

1. *If F is an antiderivative of f, then*

$$\int_a^b f(x)dx = F(b) - F(a).$$

2. $$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$

- The part (2) is clear, since

$$\frac{d}{dx}\int_a^x f(t)dt = \frac{d}{dx}[F(x) - F(a)] = f(x).$$

- Examples