SOME BRANCHING FORMULAS FOR
KAC–MOODY LIE ALGEBRAS

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Abstract. In this paper we give some branching rules for the fundamental representations of Kac–Moody Lie algebras associated to T-shaped graphs. These formulas are useful to describe generators of the generic rings for free resolutions of length three described in [7]. We also make some conjectures about the generic rings.

1. Introduction

Let $T_{p,q,r}$ be a T-shaped graph defined as follows.
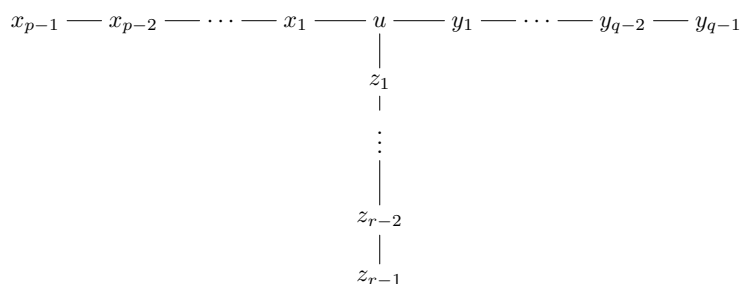
There are two other notations for vertices that we will be using. Sometimes the indices will be indexed by the set $\{0, 1, \ldots, p-1, 1', \ldots, (q-1)', 1'', \ldots, (r-1)''\}$ corresponding to the vertices $u, x_1, \ldots, x_{p-1}, y_1, \ldots, y_{q-1}, z_1, \ldots, z_{r-1}$, respectively. Sometimes we denote vertices by natural numbers from $[1, p+q+r-2]$, in the order listed above.

The main result of [7] associates to every graph $T_{p,q,r}$ the format of free resolutions of length three over commutative rings, and constructs a particular generic ring $\hat{R}_{gen}$ for the resolutions of that format which deforms to a ring $\hat{R}_{spec}$ on which the Kac–Moody Lie algebra $\mathfrak{g}(T_{p,q,r})$ associated to $T_{p,q,r}$ acts.

The correspondence is as follows. For the free resolutions of format $(r_1, r_1 + r_2, r_2 + r_3, r_3)$, i.e., those with the ranks of differentials given by (r_1, r_2, r_3), we have

$$(p, q, r) = (r_1 + 1, r_2 - 1, r_3 + 1).$$

In this context it is important to consider the grading on $\mathfrak{g}(T_{p,q,r})$ associated to the simple root corresponding to the vertex z_1; more precisely

$$\mathfrak{g}(T_{p,q,r}) = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i(T_{p,q,r}),$$

where $\mathfrak{g}_i(T_{p,q,r})$ is the span of roots where the simple root corresponding to vertex z_1 occurs with multiplicity i.

The generators of the ring $\hat{R}_{gen}$ involve the fundamental representations of the algebra $\mathfrak{g}(T_{p,q,r})$. Especially important are those corresponding to extremal vertices x_{p-1}, y_{q-1}, z_{r-1}. Thus it is important to find the restrictions of these representations to the Lie subalgebra of $\mathfrak{g}(T_{p,q,r})$ corresponding to the graph we obtain from $T_{p,q,r}$ by omitting the vertex z_1.

In this note we calculate these restrictions. Note that the graph $T_{p,q,r}$ with the node z_1 removed is a disjoint union of two Dynkin graphs of types A_{p+q-1} and of type A_{r-2}. We denote these Lie algebras by $\mathfrak{sl}(F_1)$ and $\mathfrak{sl}(F_3)$ where F_1 is a vector space over $\mathbb{C}$ of dimension $p + q$ and F_3 is a vector space over $\mathbb{C}$ of dimension $r - 1$. We also denote by $\mathfrak{gl}(F_1)$ and $\mathfrak{gl}(F_3)$ the corresponding general linear Lie algebras.

This paper is organized as follows. In Section 2 we recall the results of [7], and list the formulas for the restrictions of representations we need for the tables in the following sections, and state some conjectures. We employ the Bourbaki notation, but always present the graph $T_{p,q,r}$ as above. In Section 3 we deal with the graphs of type D_n. In Section 4 we deal with E_6, in Section 5 with E_7 and in Section 6 with E_8. In the last section we illustrate the usefulness of the tables by observing some general patterns from the tables and discussing some plausible conjectures.

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2. The free resolutions of length 3 and Kac–Moody algebras related to T-shaped graphs

The problem motivating calculations presented in this note has to do with generic rings for finite free resolutions. For a partition λ, we denote by S_λ the Schur functor associated with λ. In particular, S_n denotes the symmetric power. The exterior power will be denoted by $\bigwedge^n$.

We consider the free acyclic complexes $\mathbb{F}_\bullet$ (i.e., complexes whose only (possible) nonzero homology group is $H_0(\mathbb{F}_\bullet)$) of the form

$$\mathbb{F}_\bullet: \ 0 \to F_3 \to F_2 \to F_1 \to F_0$$

over commutative Noetherian rings R, with $\operatorname{rank}F_i = f_i$ ($0 \leq i \leq 3$), $\operatorname{rank}(d_i) = r_i$ ($1 \leq i \leq 3$). The quadruple (f_0, f_1, f_2, f_3) is *the format* of the complex $\mathbb{F}_\bullet$. We always have $f_i = r_i + r_{i+1}$ ($0 \leq i \leq 3$).

For the resolutions of such format (f_0, f_1, f_2, f_3) we say that a pair $(R_{gen}, \mathbb{F}_\bullet^{gen})$ where R_{gen} is a commutative ring and $\mathbb{F}_\bullet^{gen}$ is a free acyclic complex over R_{gen} is a *generic resolution* of this format if two conditions are satisfied:

(1) The complex $\mathbb{F}_\bullet^{gen}$ is acyclic over R_{gen};
(2) For every acyclic free complex $\mathbb{G}_\bullet$ over a Noetherian ring S there exists a ring homomorphism $\phi: R_{gen} \to S$ such that

$$\mathbb{G}_\bullet = \mathbb{F}_\bullet^{gen} \otimes_{R_{gen}} S.$$

Of particular interest is whether the ring R_{gen} is Noetherian, because it can be shown quite easily that a non-Noetherian (non-unique) generic pair always exists. For complexes of length 2 the existence of the pairs $(R_{gen}, \mathbb{F}_\bullet^{gen})$ was established by Hochster ([2]). He also proved that this generic ring is Noetherian.

In [7] the particular generic rings $\hat{R}_{gen}$ were constructed for all formats (f_0, f_1, f_2, f_3). The main result of [7] is (Theorem 9.1 there):

Theorem 2.1. *For every format (f_0, f_1, f_2, f_3) there exists a generic pair*

$$(\hat{R}_{gen}, \mathbb{F}_\bullet^{gen}) := ((j_{gen})_* \mathcal{O}_{X_{gen} \setminus D_3}, \mathbb{F}_\bullet^a \otimes_{R_a} (j_{gen})_* \mathcal{O}_{X_{gen} \setminus D_3}).$$

The generic ring $\hat{R}_{gen}$ deforms to a ring $\hat{R}_{spec}$ which carries a multiplicity free action of $\mathfrak{g}(T_{p,q,r}) \times \mathfrak{gl}(F_2) \times \mathfrak{gl}(F_0)$, where $f_3 = r - 1$, $f_2 = q + r$, $f_1 = p + q$, $r_1 = p - 1$. If the defect Lie algebra $\mathbb{L}(r_1 + 1, F_3, F_1)$ is finite dimensional, then the generic ring $\hat{R}_{gen}$ is Noetherian.

It is natural to call the format (f_0, f_1, f_2, f_3) Dynkin if the diagram $T_{p,q,r}$ is a Dynkin graph. The rings $\hat{R}_{gen}$ have an explicit decomposition to representations of $\prod_{i=0}^3 GL(F_i)$ which we now describe.

For a sextuple $\mu = (a, b, c, \alpha, \beta, \gamma)$ with $a \geq 0$, where α is a partition with $\leq r_3 - 1$ parts, β is a partition with $\leq r_2$ parts and γ is a partition with $\leq r_1 - 1$ parts, we define

$$\sigma(\mu) := (a - b + c + \alpha_1, \ldots, a - b + c + \alpha_{r_3-1}, a - b + c),$$

$$\tau(\mu) := (c + \gamma_1, \ldots, c + \gamma_{r_1-1}, c, c - b, c - b - \beta_{r_2-1}, \ldots, c - b - \beta_1),$$

$$\theta(\mu) := (b - c + \beta_1, \ldots, b - c + \beta_{r_2-1}, b - c, -a + b - c, -a + b - c - \alpha_{r_3-1},$$
$$\ldots, -a + b - c - \alpha_1),$$

$$\phi(\mu) := (0^{f_0-r_1}, -c, -c - \gamma_{r_1-1}, \ldots, -c - \gamma_1).$$

The Buchsbaum–Eisenbud multipliers ring (see [7]) has a decomposition

$$R_a = \oplus_\mu S_{\phi(\mu)} F_0 \otimes S_{\tau(\mu)} F_1 \otimes S_{\theta(\mu)} F_2 \otimes S_{\sigma(\mu)} F_3.$$

For given α, β, t we define the weight $\lambda(\sigma, \tau, t)$ of $\mathfrak{g}(T_{p,q,r})$ as follows. We label the vertices of $T_{p,q,r}$ on the third arm by the coefficients of fundamental weights in σ, i.e.,

$$\lambda_{p+q+i} = \sigma_{r-1-i} - \sigma_{r-i}$$

for $i = 1, \ldots, r-2$. We also label the vertices at the center and the first two arms by coefficients of fundamental weights in τ, i.e.,

$$\lambda_0 = \tau_p - \tau_{p+1},$$

$$\lambda_i = \tau_{p-i} - \tau_{p-i+1}$$

for $1 \le i \le p-1$, and

$$\lambda_i = \tau_i - \tau_{i+1}$$

for $i = p+1, \ldots, p+q-1$. Finally, we put

$$\lambda_{p+q} = a.$$

Finally we set $t := a + 1$.

Then we have ([7], Proposition 9.3):

Proposition 2.2. *We have a* $\mathfrak{gl}(F_0) \times \mathfrak{gl}(F_2) \times \mathfrak{g}(T_{p,q,r})$ *decomposition of a deformation* $\hat{R}_{spec}$ *of* $\hat{R}_{gen}$

$$\hat{R}_{spec} = \oplus_\mu S_{\phi(\mu)} F_0 \otimes S_{\theta(\mu)} F_2 \otimes V_{\lambda(\sigma(\mu), \tau(\mu), a)},$$

where V_λ *is the irreducible lowest weight module of weight* λ *for* $\mathfrak{g}(T_{p,q,r})$. *The module* V_λ *is the highest weight representation for the opposite Borel subalgebra. It is also irreducible. The ring* $\hat{R}_{gen}$ *has a decomposition as a representation of* $\prod_{i=0}^3 GL(F_i)$

$$\hat{R}_{gen} = \oplus_\mu S_{\phi(\mu)} F_0 \otimes S_{\theta(\mu)} F_2 \otimes \mathrm{res}(V_{\lambda(\sigma(\mu), \tau(\mu), a)}),$$

where res *denotes the restriction from* $\mathfrak{g}(T_{p,q,r})$ *to* $\mathfrak{gl}(F_1) \times \mathfrak{gl}(F_3)$.

Let us denote by $W(\mu)$ the isotypic component

$$S_{\phi(\mu)} F_0 \otimes S_{\theta(\mu)} F_2 \otimes \mathrm{res}(V_{\lambda(\sigma(\mu), \tau(\mu), a)}).$$

The representation $W(\mu)$ is equipped with the grading

$$W(\mu) = \oplus_{i \ge 0} W(\mu)(i)$$

induced by the grading

$$V_{\lambda(\sigma(\mu), \tau(\mu), a)} = \oplus_{i \ge 0} V_{\lambda(\sigma(\mu), \tau(\mu), a)}(i).$$

Notice that the graded summand $W(\mu)(0)$ gives the corresponding summand in R_a.

One also has the description of the generators of $\hat{R}_{gen}$ ([7], Proposition 10.1).

Proposition 2.3. *The generators of the semigroup of weights in $\hat{R}_{gen}$ are as follows*:

(1) $\alpha = (1^i), \beta = \gamma = a = b = c = 0$ *for* $1 \leq i \leq r_3 - 1$,
(2) $a = 1$, $\alpha = \beta = \gamma = b = c = 0$,
(3) $\beta = (1^j), \alpha = \gamma = a = b = c = 0$ *for* $1 \leq j \leq r_2 - 1$,
(4) $b = 1$, $\alpha = \beta = \gamma = a = c = 0$,
(5) $\gamma = (1^k), \alpha = \beta = a = b = c = 0$ *for* $1 \leq k \leq r_1 - 1$,
(6) $c = 1$, $\alpha = \beta = \gamma = a = b = 0$.

The rings $\hat{R}_{spec}$ and $\hat{R}_{gen}$ are generated by the corresponding representations $W(\mu)$.

In fact we expect that the ring $\hat{R}_{gen}$ is generated by much smaller set of representations (see [7] for motivation).

Conjecture 2.4. The ring $\hat{R}_{gen}$ is generated by the six representations $W(\mu)$ corresponding to the following weights.

(1) $\alpha = (1)$, $\beta = \gamma = a = b = c = 0$, corresponding to $i = 1$ in Proposition 2.3(1),
(2) $a = 1$, $\alpha = \beta = \gamma = b = c = 0$,
(3) $\beta = (1)$, $\alpha = \gamma = a = b = c = 0$, corresponding to $j = 1$ in Proposition 2.3(3),
(4) $b = 1$, $\alpha = \beta = \gamma = a = c = 0$,
(5) $\gamma = (1)$, $\alpha = \beta = a = b = c = 0$, corresponding to $k = 1$ in Proposition 2.3(5),
(6) $c = 1$, $\alpha = \beta = \gamma = a = b = 0$.

For the formats with $r_1 = 1$, i.e., resolutions of cyclic modules, we actually have even stronger expectation.

Conjecture 2.5. Let us assume that $r_1 = 1$. The ring $\hat{R}_{gen}$ is generated by the five representations $W(\mu)$ corresponding to the following weights.

(1) $\alpha = (1)$, $\beta = \gamma = a = b = c = 0$, corresponding to $i = 1$ in Proposition 2.3(1),
(2) $a = 1$, $\alpha = \beta = \gamma = b = c = 0$,
(3) $\beta = (1)$, $\alpha = \gamma = a = b = c = 0$, corresponding to $j = 1$ in Proposition 2.3(3),
(4) $b = 1$, $\alpha = \beta = \gamma = a = c = 0$,
(5) $c = 1$, $\alpha = \beta = \gamma = a = b = 0$.

Moreover, the first representation is redundant if $r_3 = 1$, and the second one is redundant if $r_3 > 1$. The last representation is just a variable a_1, so it is completely understood.

Particularly important are three critical representations. The first one is the one corresponding to $\alpha = (1)$, $\beta = \gamma = a = b = c = 0$ (or to $a = 1$, $\alpha = \beta = \gamma = b = c = 0$ if $r_3 = 1$). We denote it by $W(d_3)$. Similarly, the representation

$W(d_2)$ is the one corresponding to $\beta = (1)$, $\alpha = \gamma = a = b = c = 0$. Finally, $W(d_1)$ is the representation corresponding to $\gamma = (1)$, $\alpha = \beta = a = b = c = 0$ (in the case $r_1 = 1$ we replace it by $W(a_2)$, i.e., $b = 1$, $\alpha = \beta = \gamma = a = c = 0$).

All representations $W(\mu)$ have a grading

$$W(\mu) = \oplus_{i \geq 0} W_i(\mu).$$

So in order to understand the generators of $\hat{R}_{gen}$ in the case $r_1 = 1$ we need to understand the restrictions to $\mathfrak{gl}(F_3) \times \mathfrak{gl}(F_1)$ of three fundamental representations of $\mathfrak{g}(T_{p,q,r})$ corresponding to extremal nodes of the diagram $T_{p,q,r}$. This is carried out case by case in the following sections.

We end with some remarks about the tables. For each situation we list the representations in graded components of representations $V(\omega_{x_p})$, $V(\omega_{y_q})$, $V(\omega_{z_r})$. The bold-faced first column of a table shows degrees in the graded decomposition, and the middle and last column with numeric values have decomposition multiplicities. Note that each table actually serves two formats, as p and q can be swapped. But this amounts to changing F_1 to F_1^*. Some of the representations are very big, so in some cases we list just half of the representation. The other half then can be read off by duality.

3. The type D_n

The first format is $(1, n, n, 1)$. The graph $T_{p,q,r}$ is:

We number the vertices as follows:

The Lie algebra $\mathfrak{g}(T_{p,q,r})$ is $\mathfrak{g}(D_n) = \mathfrak{so}(2n) = \mathfrak{so}(F_1 \oplus F_1^*)$ $(\dim F_1 = n)$, with the grading

$$\mathfrak{g}(D_n) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

where $\mathfrak{g}_0 = \mathfrak{sl}(F_3) \times \mathfrak{sl}(F_1) \times \mathbb{C} = \mathfrak{sl}(F_1) \times \mathbb{C}$ and $\mathfrak{g}_1 = F_3^* \otimes \bigwedge^2 F_1$. The orthogonal space is $U := F_1 \oplus F_1^*$.
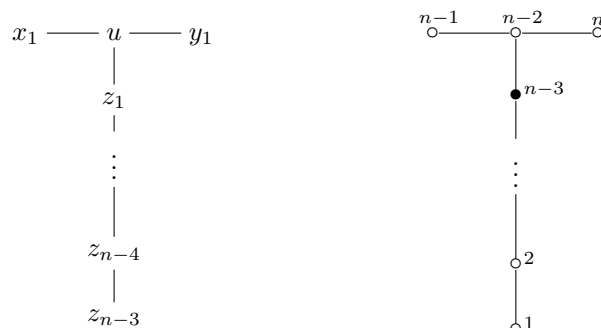
It is not difficult to see that

$$V(\omega_1) = F_1^* \oplus F_3^* \otimes F_1,$$

$$V(\omega_{n-1}) = \oplus_{k=0}^{\frac{n}{2}} S_{1-k} F_3^* \otimes \bigwedge^{2k} F_1,$$

$$V(\omega_n) = \oplus_{k=0}^{\frac{n}{2}} S_{-k} F_3 \otimes \bigwedge^{2k+1} F_1.$$

The next format is $(1, 4, n, n - 3)$. The graph $T_{p,q,r}$ with the distinguished root z_1 and the labeling of the vertices are as follows:

The corresponding Lie algebra is

$$\mathfrak{g}(T_{p,q,r}) = \mathfrak{so}(2n).$$

The orthogonal space in question is
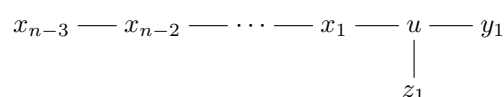
$$U := F_3 \oplus \bigwedge^2 F_1 \oplus F_3^*,$$

with $\dim F_3 = n - 3$ and $\dim F_1 = 4$. We have

$$V(\omega_1) = F_3 \oplus \bigwedge^2 F_1 \oplus F_3^* \otimes \bigwedge^4 F_1,$$

$$V(\omega_{n-1}) = \bigwedge^{even} (F_3) \otimes F_1 \oplus \bigwedge^{odd} (F_3) \otimes F_1^*,$$

$$V(\omega_n) = \bigwedge^{even} (F_3^*) \otimes F_1 \oplus \bigwedge^{odd} (F_3^*) \otimes F_1^*.$$

Finally we have the format $(n - 3, n, 4, 1)$. The graph $T_{p,q,r}$ is:
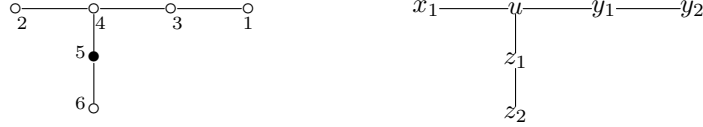

We number the vertices as follows:
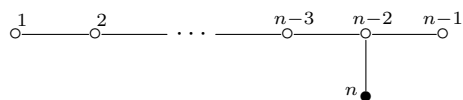
The orthogonal space is $U = F_1 \oplus F_1^*$. This case can be obtained from the first format $(1, n, n, 1)$ with F_1 and F_1^* swapped.

4. The type E_6

4.1. E_6 graded by α_5

This format is $(1,5,6,2)$.

4.1.1. $V(\omega_6)$. This representation is of dimension 27 and has 4 graded components.

0	F_1^*	$(0,0)$	$(0,0,0,0,-1)$	1
1	$F_3^* \otimes F_1$	$(1,0)$	$(1,0,0,0,0)$	1
2	$\bigwedge^2 F_3^* \otimes \bigwedge^3 F_1$	$(1,1)$	$(1,1,1,0,0)$	1
3	$S_{2,1} F_3^* \otimes \bigwedge^5 F_1$	$(2,1)$	$(1,1,1,1,1)$	1

4.1.2. $V(\omega_1)$. This representation is of dimension 27 and has 4 graded components.

0	F_3	$(0,-1)$	$(0,0,0,0,0)$	1
1	$\bigwedge^2 F_1$	$(0,0)$	$(1,1,0,0,0)$	1
2	$F_3^* \otimes \bigwedge^4 F_1$	$(1,0)$	$(1,1,1,1,0)$	1
3	$\bigwedge^2 F_3^* \otimes S_{2,1,1,1,1} F_1$	$(1,1)$	$(2,1,1,1,1)$	1

4.1.3. $V(\omega_2)$. This representation is of dimension 78 and has 5 graded components.

0	F_1	$(0,0)$	$(1,0,0,0,0)$	1			
1	$F_3^* \otimes \bigwedge^3 F_1$	$(1,0)$	$(1,1,1,0,0)$	1			
2	$S_2 F_3^* \otimes \bigwedge^5 F_1$	$(2,0)$	$(1,1,1,1,1)$	1	$\bigwedge^2 F_3^* \otimes \bigwedge^5 F_1$	$(1,1)$ $(1,1,1,1,1)$	1
	$\bigwedge^2 F_3^* \otimes S_{2,1^3} F_1$	$(1,1)$	$(2,1,1,1,0)$	1			
3	$S_{2,1} F_3^* \otimes S_{2^2,1^3} F_1$	$(2,1)$	$(2,2,1,1,1)$	1			
4	$S_{2,2} F_3^* \otimes S_{2^4,1} F_1$	$(2,2)$	$(2,2,2,2,1)$	1			

4.2. E_6 graded by α_2

This format is $(2,6,5,1)$.

4.2.1. $V(\omega_6)$. There are 3 graded components.

0	F_1	$(1,0,0,0,0,0)$	1
1	$\bigwedge^4 F_1$	$(1,1,1,1,0,0)$	1
2	$S_{2,1^5} F_1$	$(2,1,1,1,1,1)$	1

4.2.2. $V(\omega_1)$. There are 3 graded components.

0	F_1^*	$(0,0,0,0,0,-1)$	1
1	$\bigwedge^2 F_1$	$(1,1,0,0,0,0)$	1
2	$\bigwedge^5 F_1$	$(1,1,1,1,1,0)$	1

4.2.3. $V(\omega_2)$. There are 5 graded components.

0	$\mathbb{C}$	$(0,0,0,0,0,0)$	1			
1	$\bigwedge^3 F_1$	$(1,1,1,0,0,0)$	1			
2	$S_{2,1^4} F_1$	$(2,1,1,1,1,0)$	1	$\bigwedge^6 F_1$	$(1,1,1,1,1,1)$	1
3	$S_{2^3,1^3} F_1$	$(2,2,2,1,1,1)$	1			
4	$S_{2^6} F_1$	$(2,2,2,2,2,2)$	1			

5. The type E_7

5.1. E_7 graded by α_5

This format is $(1,5,7,3)$.
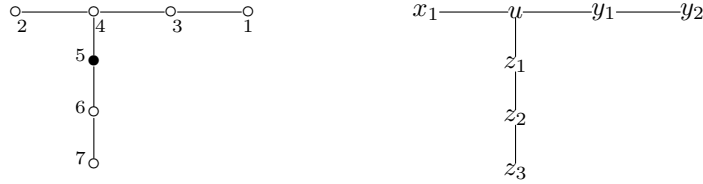
5.1.1. $V(\omega_7)$. This representation is of dimension 56 and has 6 graded components.

0	F_3	$(0,0,-1)$	$(0,0,0,0,0)$	1
1	$\bigwedge^2 F_1$	$(0,0,0)$	$(1,1,0,0,0)$	1
2	$F_3^* \otimes \bigwedge^4 F_1$	$(1,0,0)$	$(1,1,1,1,0)$	1
3	$\bigwedge^2 F_3^* \otimes S_{2,1^4} F_1$	$(1,1,0)$	$(2,1,1,1,1)$	1
4	$\bigwedge^3 F_3^* \otimes S_{2^3,1^2} F_1$	$(1,1,1)$	$(2,2,2,1,1)$	1
5	$S_{2,1,1} F_3^* \otimes S_{2^5} F_1$	$(2,1,1)$	$(2,2,2,2,2)$	1

5.1.2. $V(\omega_1)$. This representation is of dimension 133 and has 7 graded components.

0	F_1^*	$(0,0,0)$	$(0,0,0,0,-1)$	1			
1	$F_3^* \otimes F_1$	$(1,0,0)$	$(1,0,0,0,0)$	1			
2	$\bigwedge^2 F_3^* \otimes \bigwedge^3 F_1$	$(1,1,0)$	$(1,1,1,0,0)$	1			
3	$\bigwedge^3 F_3^* \otimes S_{2,1^3} F_1$	$(1,1,1)$	$(2,1,1,1,0)$	1	$\bigwedge^3 F_3^* \otimes \bigwedge^5 F_1$	$(1,1,1)$ $(1,1,1,1,1)$	1
	$S_{2,1} F_3^* \otimes \bigwedge^5 F_1$	$(2,1,0)$	$(1,1,1,1,1)$	1			
4	$S_{2,1,1} F_3^* \otimes S_{2^2,1^3} F_1$	$(2,1,1)$	$(2,2,1,1,1)$	1			
5	$S_{2,2,1} F_3^* \otimes S_{2^4,1} F_1$	$(2,2,1)$	$(2,2,2,2,1)$	1			
6	$S_{2,2,2} F_3^* \otimes S_{3,2^4} F_1$	$(2,2,2)$	$(3,2,2,2,2)$	1			

5.1.3. $V(\omega_2)$. This representation is of dimension 912 and has 10 graded components.
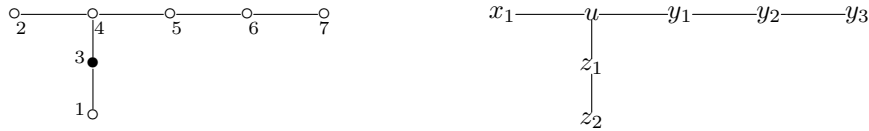
0	F_1	$(0,0,0)$	$(1,0,0,0,0)$	1				
1	$F_3^* \otimes \bigwedge^3 F_1$	$(1,0,0)$	$(1,1,1,0,0)$	1				
2	$S_2 F_3^* \otimes \bigwedge^5 F_1$	$(2,0,0)$	$(1,1,1,1,1)$	1	$\bigwedge^2 F_3^* \otimes \bigwedge^5 F_1$	$(1,1,0)$	$(1,1,1,1,1)$	1
	$\bigwedge^2 F_3^* \otimes S_{2,1^3} F_1$	$(1,1,0)$	$(2,1,1,1,0)$	1				
3	$S_{2,1} F_3^* \otimes S_{2^2,1^3} F_1$	$(2,1,0)$	$(2,2,1,1,1)$	1	$\bigwedge^3 F_3^* \otimes S_{3,1^4} F_1$	$(1,1,1)$	$(3,1,1,1,1)$	1
	$\bigwedge^3 F_3^* \otimes S_{2^2,1^3} F_1$	$(1,1,1)$	$(2,2,1,1,1)$	1	$\bigwedge^3 F_3^* \otimes S_{2^3,1} F_1$	$(1,1,1)$	$(2,2,2,1,0)$	1
4	$S_{2,2} F_3^* \otimes S_{2^4,1} F_1$	$(2,2,0)$	$(2,2,2,2,1)$	1	$S_{2,1,1} F_3^* \otimes S_{2^4,1} F_1$	$(2,1,1)$	$(2,2,2,2,1)$	2
	$S_{2,1,1} F_3^* \otimes S_{3,2,2,1,1} F_1$	$(2,1,1)$	$(3,2,2,1,1)$	1				
5	$S_{2,2,1} F_3^* \otimes S_{3,2^4} F_1$	$(2,2,1)$	$(3,2,2,2,2)$	2	$S_{2,2,1} F_3^* \otimes S_{3,3,2,2,1} F_1$	$(2,2,1)$	$(3,3,2,2,1)$	1
	$S_{3,1,1} F_3^* \otimes S_{3,2^4} F_1$	$(3,1,1)$	$(3,2,2,2,2)$	1				
6	$S_{2,2,2} F_3^* \otimes S_{4,3,2^3} F_1$	$(2,2,2)$	$(4,3,2,2,2)$	1	$S_{2,2,2} F_3^* \otimes S_{3^4,1} F_1$	$(2,2,2)$	$(3,3,3,3,1)$	1
	$S_{2,2,2} F_3^* \otimes S_{3^3,2^2} F_1$	$(2,2,2)$	$(3,3,3,2,2)$	1	$S_{3,2,1} F_3^* \otimes S_{3^3,2^2} F_1$	$(3,2,1)$	$(3,3,3,2,2)$	1
7	$S_{3,2,2} F_3^* \otimes S_{4,3^3,2} F_1$	$(3,2,2)$	$(4,3,3,3,2)$	1	$S_{3,2,2} F_3^* \otimes S_{3^5} F_1$	$(3,2,2)$	$(3,3,3,3,3)$	1
	$S_{3,3,1} F_3^* \otimes S_{3^5} F_1$	$(3,3,1)$	$(3,3,3,3,3)$	1				
8	$S_{3,3,2} F_3^* \otimes S_{4^2,3^3} F_1$	$(3,3,2)$	$(4,4,3,3,3)$	1				
9	$S_{3,3,3} F_3^* \otimes S_{4^4,3} F_1$	$(3,3,3)$	$(4,4,4,4,3)$	1				

5.2. E_7 graded by α_3

This format is $(1,6,7,2)$.

5.2.1. $V(\omega_7)$. There are 5 graded components.

0	F_1^*	$(0,0)$	$(0,0,0,0,0,-1)$	1
1	$F_3^* \otimes F_1$	$(1,0)$	$(1,0,0,0,0,0)$	1
2	$\bigwedge^2 F_3^* \otimes \bigwedge^3 F_1$	$(1,1)$	$(1,1,1,0,0,0)$	1
3	$S_{2,1} F_3^* \otimes \bigwedge^5 F_1$	$(2,1)$	$(1,1,1,1,1,0)$	1
4	$S_{2,2} F_3^* \otimes S_{2,1^5} F_1$	$(2,2)$	$(2,1,1,1,1,1)$	1

5.2.2. $V(\omega_1)$. There are 7 graded components.

0	F_3	$(0,-1)$	$(0,0,0,0,0,0)$	1			
1	$\bigwedge^2 F_1$	$(0,0)$	$(1,1,0,0,0,0)$	1			
2	$F_3^* \otimes \bigwedge^4 F_1$	$(1,0)$	$(1,1,1,1,0,0)$	1			
3	$\bigwedge^2 F_3^* \otimes S_{2,1^4} F_1$	$(1,1)$	$(2,1,1,1,1,0)$	1	$\bigwedge^2 F_3^* \otimes \bigwedge^6 F_1$	$(1,1)$	$(1,1,1,1,1,1)$	1
	$S_2 F_3^* \otimes \bigwedge^6 F_1$	$(2,0)$	$(1,1,1,1,1,1)$	1			
4	$S_{2,1} F_3^* \otimes S_{2,2,1^4} F_1$	$(2,1)$	$(2,2,1,1,1,1)$	1			
5	$S_{2,2} F_3^* \otimes S_{2^4,1,1} F_1$	$(2,2)$	$(2,2,2,2,1,1)$	1			
6	$S_{3,2} F_3^* \otimes S_{2^6} F_1$	$(3,2)$	$(2,2,2,2,2,2)$	1			

5.2.3. $V(\omega_2)$. We have 9 graded components.

0	F_1	(0,0)	(1,0,0,0,0,0)	1				
1	$F_3^* \otimes \bigwedge^3 F_1$	(1,0)	(1,1,1,0,0,0)	1				
2	$\bigwedge^2 F_3^* \otimes S_{2,1,1,1} F_1$	(1,1)	(2,1,1,1,0,0)	1	$\bigwedge^2 F_3^* \otimes \bigwedge^5 F_1$	(1,1)	(1,1,1,1,1,0)	1
	$S_2 F_3^* \otimes \bigwedge^5 F_1$	(2,0)	(1,1,1,1,1,0)	1				
3	$S_{2,1} F_3^* \otimes S_{2,2,1,1,1} F_1$	(2,1)	(2,2,1,1,1,0)	1	$S_{2,1} F_3^* \otimes S_{2,1^5} F_1$	(2,1)	(2,1,1,1,1,1)	2
4	$S_{2,2} F_3^* \otimes S_{2^3,1^3} F_1$	(2,2)	(2,2,2,1,1,1)	1	$S_{2,2} F_3^* \otimes S_{3,2,1^4} F_1$	(2,2)	(3,2,1,1,1,1)	1
	$S_{2,2} F_3^* \otimes S_{2,2,2,2,1} F_1$	(2,2)	(2,2,2,2,1,0)	1	$S_{3,1} F_3^* \otimes S_{2^3,1^3} F_1$	(3,1)	(2,2,2,1,1,1)	1
5	$S_{3,2} F_3^* \otimes S_{3,2^3,1^2} F_1$	(3,2)	(3,2,2,2,1,1)	1	$S_{3,2} F_3^* \otimes S_{2^5,1} F_1$	(3,2)	(2,2,2,2,2,1)	2
6	$S_{3,3} F_3^* \otimes S_{3^2,2^3,1} F_1$	(3,3)	(3,3,3,2,2,1)	1	$S_{3,3} F_3^* \otimes S_{3,2^5} F_1$	(3,3)	(3,2,2,2,2,2)	1
	$S_{4,2} F_3^* \otimes S_{3,2^5} F_1$	(4,2)	(3,2,2,2,2,2)					
7	$S_{4,3} F_3^* \otimes S_{3^3,2^3} F_1$	(4,3)	(3,3,3,3,2,2)	1				
8	$S_{4,4} F_3^* \otimes S_{3^5,2} F_1$	(4,4)	(3,3,3,3,3,2)	1				

5.3. E_7 graded by α_2

This format is $(2, 7, 6, 1)$.

Dynkin diagram: nodes $1 - 3 - 4 - 5 - 6 - 7 \quad x_2 - x_1 - u - y_1 - y_2 - y_3$, with node 2 attached below node 4, and z_1 attached below u.
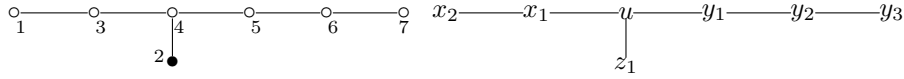
5.3.1. $V(\omega_7)$. There are 4 graded components.

0	F_1^*	(0,0,0,0,0,0,−1)	1
1	$\bigwedge^2 F_1$	(1,1,0,0,0,0,0)	1
2	$\bigwedge^5 F_1$	(1,1,1,1,1,0,0)	1
3	$S_{2,1^6} F_1$	(2,1,1,1,1,1,1)	1

5.3.2. $V(\omega_1)$. There are 5 graded components.

0	F_1	(1,0,0,0,0,0,0)	1			
1	$\bigwedge^4 F_1$	(1,1,1,1,0,0,0)	1			
2	$S_{2,1^5} F_1$	(2,1,1,1,1,1,0)	1	$\bigwedge^7 F_1$	(1,1,1,1,1,1,1)	1
3	$S_{2^3,1^4} F_1$	(2,2,2,1,1,1,1)	1			
4	$S_{2^6,1} F_1$	(2,2,2,2,2,2,1)	1			

5.3.3. $V(\omega_2)$. There are 8 graded components.

0	$\mathbb{C}$	(0,0,0,0,0,0,0)	1			
1	$\bigwedge^3, F_1$	(1,1,1,0,0,0,0)	1			
2	$S_{2,1^4} F_1$	(2,1,1,1,1,0,0)	1	$\bigwedge^6 F_1$	(1,1,1,1,1,1,0)	1
3	$S_{3,1^6} F_1$	(3,1,1,1,1,1,1)	1	$S_{2^3,1^3,} F_1$	(2,2,2,1,1,1,0)	1
	$S_{2^2,1^5} F_1$	(2,2,1,1,1,1,1)	1			
4	$S_{3,2^3,1^3} F_1$	(3,2,2,2,1,1,1)	1	$S_{2^6} F_1$	(2,2,2,2,2,2,0)	1
	$S_{2^5,1^2} F_1$	(2,2,2,2,2,1,1)	1			
5	$S_{3^2,2^4,1} F_1$	(3,3,2,2,2,2,1)	1	$S_{3,2^6} F_1$	(3,2,2,2,2,2,2)	1
6	$S_{3^4,2^3,} F_1$	(3,3,3,3,2,2,2)	1			
7	$S_{3^7} F_1$	(3,3,3,3,3,3,3)	1			

6. The type E_8

6.1. E_8 graded by α_5

This format is $(1, 5, 8, 4)$.
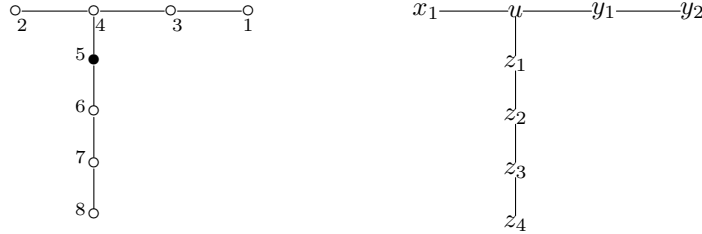
6.1.1. $V(\omega_8)$. This representation is of dimension 248 and has 11 graded components.

0	F_3	$(0,0,0,-1)$	$(0,0,0,0,0)$	1				
1	$\bigwedge^2 F_1$	$(0,0,0,0)$	$(1,1,0,0,0)$	1				
2	$F_3^* \otimes \bigwedge^4 F_1$	$(1,0,0,0)$	$(1,1,1,1,0)$	1				
3	$\bigwedge^2 F_3^* \otimes S_{2,1^4} F_1$	$(1,1,0,0)$	$(2,1,1,1,1)$	1				
4	$\bigwedge^3 F_3^* \otimes S_{2^3,1^2} F_1$	$(1,1,1,0)$	$(2,2,2,1,1)$	1				
5	$\bigwedge^4 F_3^* \otimes S_{3,2^3,1} F_1$	$(1,1,1,1)$	$(3,2,2,2,1)$	1	$\bigwedge^4 F_3^* \otimes S_{2^5} F_1$	$(1,1,1,1)$	$(2,2,2,2,2)$	1
	$S_{2,1,1} F_3^* \otimes S_{2^5} F_1$	$(2,1,1,0)$	$(2,2,2,2,2)$	1				
6	$S_{2,1^3} F_3^* \otimes S_{3^2,2^3} F_1$	$(2,1,1,1)$	$(3,3,2,2,2)$	1				
7	$S_{2^2,1^2} F_3^* \otimes S_{3^4,2} F_1$	$(2,2,1,1)$	$(3,3,3,3,2)$	1				
8	$S_{2^3,1} F_3^* \otimes S_{4,3^4} F_1$	$(2,2,2,1)$	$(4,3,3,3,3)$	1				
9	$S_{2^4} F_3^* \otimes S_{4^3,3^2} F_1$	$(2,2,2,2)$	$(4,4,4,3,3)$	1				
10	$S_{3,2^3} F_3^* \otimes S_{4^5} F_1$	$(3,2,2,2)$	$(4,4,4,4,4)$	1				

6.1.2. $V(\omega_1)$. This representation is of dimension 3875 and has 17 graded components.

0	F_1^*	$(0,0,0,0)$	$(0,0,0,0,-1)$	1				
1	$F_3^* \otimes F_1$	$(1,0,0,0)$	$(1,0,0,0,0)$	1				
2	$\bigwedge^2 F_3^* \otimes \bigwedge^3 F_1$	$(1,1,0,0)$	$(1,1,1,0,0)$	1				
3	$\bigwedge^3 F_3^* \otimes S_{2,1^3} F_1$	$(1,1,1,0)$	$(2,1,1,1,0)$	1	$\bigwedge^3 F_3^* \otimes \bigwedge^5 F_1$	$(1,1,1,0)$	$(1,1,1,1,1)$	1
	$S_{2,1} F_3^* \otimes \bigwedge^5 F_1$	$(2,1,0,0)$	$(1,1,1,1,1)$	1				
4	$\bigwedge^4 F_3^* \otimes S_{2^2,1^3} F_1$	$(1,1,1,1)$	$(2,2,1,1,1)$	1	$\bigwedge^4 F_3^* \otimes S_{2^3,1} F_1$	$(1,1,1,1)$	$(2,2,2,1,0)$	1
	$\bigwedge^4 F_3^* \otimes S_{3,1^4} F_1$	$(1,1,1,1)$	$(3,1,1,1,1)$	1	$S_{2,1,1} F_3^* \otimes S_{2^2,1^3} F_1$	$(2,1,1,0)$	$(2,2,1,1,1)$	1
5	$S_{2,1,1,1} F_3^* \otimes S_{2^4,1} F_1$	$(2,1,1,1)$	$(2,2,2,2,1)$	2	$S_{2,1,1,1} F_3^* \otimes S_{3,2^2,1^2} F_1$	$(2,1,1,1)$	$(3,2,2,1,1)$	1
	$S_{2,2,1} F_3^* \otimes S_{2^4,1} F_1$	$(2,2,1,0)$	$(2,2,2,2,1)$	1				
6	$S_{2^2,1^2} F_3^* \otimes S_{3,2^4} F_1$	$(2,2,1,1)$	$(3,2,2,2,2)$	2	$S_{2^2,1^2} F_3^* \otimes S_{3^2,2^2,1} F_1$	$(2,2,1,1)$	$(3,3,2,2,1)$	1
	$S_{3,1^3} F_3^* \otimes S_{3,2^4} F_1$	$(3,1,1,1)$	$(3,2,2,2,2)$	1	$S_{2,2,2} F_3^* \otimes S_{3,2^4} F_1$	$(2,2,2,0)$	$(3,2,2,2,2)$	1
7	$S_{2^3,1} F_3^* \otimes S_{3^3,2^2} F_1$	$(2,2,2,1)$	$(3,3,3,2,2)$	2	$S_{2^3,1} F_3^* \otimes S_{4,3,2^3} F_1$	$(2,2,2,1)$	$(4,3,2,2,2)$	1
	$S_{2^3,1} F_3^* \otimes S_{3^4,1} F_1$	$(2,2,2,1)$	$(3,3,3,3,1)$	1	$S_{3,2,1,1} F_3^* \otimes S_{3^3,2^2} F_1$	$(3,2,1,1)$	$(3,3,3,2,2)$	1
8	$S_{2^4} F_3^* \otimes S_{4,3^3,2} F_1$	$(2,2,2,2)$	$(4,3,3,3,2)$	2	$S_{2^4} F_3^* \otimes S_{3^5} F_1$	$(2,2,2,2)$	$(3,3,3,3,3)$	2
	$S_{2^4} F_3^* \otimes S_{4^2,3,2^2} F_1$	$(2,2,2,2)$	$(4,4,3,2,2)$	1	$S_{3,2,2,1} F_3^* \otimes S_{3^5} F_1$	$(3,2,2,1)$	$(3,3,3,3,3)$	2
	$S_{3,2,2,1} F_3^* \otimes S_{4,3^3,2} F_1$	$(3,2,2,1)$	$(4,3,3,3,2)$	1	$S_{3^2,1^2} F_3^* \otimes S_{3^5} F_1$	$(3,3,1,1)$	$(3,3,3,3,3)$	1

(The table continues on the next page.)

9	$S_{3,2^3}F_3^* \otimes S_{4^2,3^3}F_1$	(3,2,2,2)	(4,4,3,3,3)	2	$S_{3,2^3}F_3^* \otimes S_{4^3,3,2}F_1$	(3,2,2,2)	(4,4,4,3,2)	1
	$S_{3,2^3}F_3^* \otimes S_{5,3^4}F_1$	(3,2,2,2)	(5,3,3,3,3)	1	$S_{3^2,2,1}F_3^* \otimes S_{4^2,3^3}F_1$	(3,3,2,1)	(4,4,3,3,3)	1
10	$S_{3^2,2^2}F_3^* \otimes S_{4^4,3}F_1$	(3,3,2,2)	(4,4,4,4,3)	2	$S_{3^2,2^2}F_3^* \otimes S_{5,4^2,3^2}F_1$	(3,3,2,2)	(5,4,4,3,3)	1
	$S_{4,2^3}F_3^* \otimes S_{4^4,3}F_1$	(4,2,2,2)	(4,4,4,4,3)	1	$S_{3^3,1} \otimes S_{4^4,3}F_1$	(3,3,3,1)	(4,4,4,4,3)	1
11	$S_{3^3,2}F_3^* \otimes S_{5,4^4}F_1$	(3,3,3,2)	(5,4,4,4,4)	2	$S_{3^3,2}F_3^* \otimes S_{5^2,4^2,3}F_1$	(3,3,3,2)	(5,5,4,4,3)	1
	$S_{4,3,2^2}F_3^* \otimes S_{5,4^4}F_1$	(4,3,2,2)	(5,4,4,4,4)	1				
12	$S_{3^4}F_3^* \otimes S_{5^3,4^2}F_1$	(3,3,3,3)	(5,5,5,4,4)	1	$S_{3^4}F_3^* \otimes S_{6,5,4^3}F_1$	(3,3,3,3)	(6,5,4,4,4)	1
	$S_{3^4}F_3^* \otimes S_{5^4,3}F_1$	(3,3,3,3)	(5,5,5,5,3)	1	$S_{4,3^2,2}F_3^* \otimes S_{5^3,4^2}F_1$	(4,3,3,2)	(5,5,5,4,4)	1
13	$S_{4,3^3}F_3^* \otimes S_{5^5}F_1$	(4,3,3,3)	(5,5,5,5,5)	1	$S_{4,3^3}F_3^* \otimes S_{6,5^3,4}F_1$	(4,3,3,3)	(6,5,5,5,4)	1
	$S_{4^2,3,2}F_3^* \otimes S_{5^5}F_1$	(4,4,3,2)	(5,5,5,5,5)	1				
14	$S_{4^2,3^2}F_3^* \otimes S_{6^2,5^3}F_1$	(4,4,3,3)	(6,6,5,5,5)	1				
15	$S_{4^3,3}F_3^* \otimes S_{6^4,5}F_1$	(4,4,4,3)	(6,6,6,6,5)	1				
16	$S_{4^4}F_3^* \otimes S_{7,6^4}F_1$	(4,4,4,4)	(7,6,6,6,6)	1				

6.1.3. $V(\omega_2)$. This representation is of dimension 147250 and has 25 graded components. We exhibit the first 13, as the others can be determined by duality. (The representation is graded self-dual.)
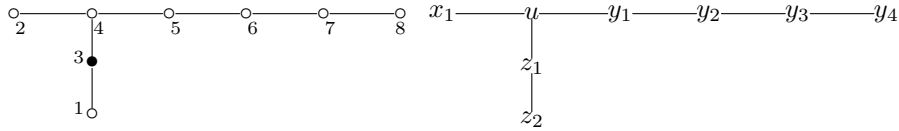
0	F_1	(0,0,0,0)	(1,0,0,0,0)	1				
1	$F_3^* \otimes \bigwedge^3 F_1$	(1,0,0,0)	(1,1,1,0,0)	1				
2	$S_2 F_3^* \otimes \bigwedge^5 F_1$	(2,0,0,0)	(1,1,1,1,1)	1	$\bigwedge^2 F_3^* \otimes \bigwedge^5 F_1$	(1,1,0,0)	(1,1,1,1,1)	1
	$\bigwedge^2 F_3^* \otimes S_{2,1^3}F_1$	(1,1,0,0)	(2,1,1,1,0)	1				
3	$S_{2,1}F_3^* \otimes S_{2^2,1^3}F_1$	(2,1,0,0)	(2,2,1,1,1)	1	$\bigwedge^3 F_3^* \otimes S_{3,1^4}F_1$	(1,1,1,0)	(3,1,1,1,1)	1
	$\bigwedge^3 F_3^* \otimes S_{2^3,1}F_1$	(1,1,1,0)	(2,2,2,1,0)	1	$\bigwedge^3 F_3^* \otimes S_{2^2,1^3}F_1$	(1,1,1,0)	(2,2,1,1,1)	1
4	$S_{2^2}F_3^* \otimes S_{2^4,1}F_1$	(2,2,0,0)	(2,2,2,2,1)	1	$S_{2,1,1}F_3^* \otimes S_{2^4,1}F_1$	(2,1,1,0)	(2,2,2,2,1)	2
	$S_{2,1,1}F_3^* \otimes S_{3,2,2,1,1}F_1$	(2,1,1,0)	(3,2,2,1,1)	1	$\bigwedge^4 F_3^* \otimes S_{3,2^2,1^2}F_1$	(1,1,1,1)	(3,2,2,1,1)	2
	$\bigwedge^4 F_3^* \otimes S_{2^4,1}F_1$	(1,1,1,1)	(2,2,2,2,1)	2	$\bigwedge^4 F_3^* \otimes S_{3,2^3}F_1$	(1,1,1,1)	(3,2,2,2,0)	1
5	$S_{3,1,1}F_3^* \otimes S_{3,2^4}F_1$	(3,1,1,0)	(3,2,2,2,2)	1	$S_{2,2,1}F_3^* \otimes S_{3,2^4}F_1$	(2,2,1,0)	(3,2,2,2,2)	2
	$S_{2,2,1}F_3^* \otimes S_{3,3,2,2,1}F_1$	(2,2,1,0)	(3,3,2,2,1)	1	$S_{2,1^3}F_3^* \otimes S_{3,2^4}F_1$	(2,1,1,1)	(3,2,2,2,2)	4
	$S_{2,1^3}F_3^* \otimes S_{3^2,2^2,1}F_1$	(2,1,1,1)	(3,3,2,2,1)	2	$S_{2,1^3}F_3^* \otimes S_{4,2^3,1}F_1$	(2,1,1,1)	(4,2,2,2,1)	1
	$S_{2,1^3}F_3^* \otimes S_{3^3,1^2}F_1$	(2,1,1,1)	(3,3,3,1,1)	1				
6	$S_{3,2,1}F_3^* \otimes S_{3^3,2^2}F_1$	(3,2,1,0)	(3,3,3,2,2)	1	$S_{3,1^3}F_3^* \otimes S_{3^3,2^2}F_1]$	(3,1,1,1)	(3,3,3,2,2)	2
	$S_{3,1^3}F_3^* \otimes S_{4,3,2^3}F_1$	(3,1,1,1)	(4,3,2,2,2)	1	$S_{2,2,2}F_3^* \otimes S_{3^4,1}F_1$	(2,2,2,0)	(3,3,3,3,1)	1
	$S_{2,2,2}F_3^* \otimes S_{4,3,2^3}F_1$	(2,2,2,0)	(4,3,2,2,2)	1	$S_{2,2,2}F_3^* \otimes S_{3^3,2^2}F_1$	(2,2,2,0)	(3,3,3,2,2)	1
	$S_{2^2,1^2}F_3^* \otimes S_{3^3,2^2}F_1$	(2,2,1,1)	(3,3,3,2,2)	5	$S_{2^2,1^2}F_3^* \otimes S_{3^4,1}F_1$	(2,2,1,1)	(3,3,3,3,1)	2
	$S_{2^2,1^2}F_3^* \otimes S_{4,3,2^3}F_1$	(2,2,1,1)	(4,3,2,2,2)	2	$S_{2^2,1^2}F_3^* \otimes S_{4,3^2,2,1}F_1$	(2,2,1,1)	(4,3,3,2,1)	1
7	$S_{4,1,1,1}F_3^* \otimes S_{3^5}F_1$	(4,1,1,1)	(3,3,3,3,3)	1	$S_{3,3,1}F_3^* \otimes S_{3^5}F_1$	(3,3,1,0)	(3,3,3,3,3)	1
	$S_{3,2,2}F_3^* \otimes S_{3^5}F_1$	(3,2,2,0)	(3,3,3,3,3)	1	$S_{3,2,2}F_3^* \otimes S_{4,3^3,2}F_1$	(3,2,2,0)	(4,3,3,3,2)	1
	$S_{3,2,1^2}F_3^* \otimes S_{3^5}F_1$	(3,2,1,1)	(3,3,3,3,3)	4	$S_{3,2,1^2}F_3^* \otimes S_{4,3^3,2}F_1$	(3,2,1,1)	(4,3,3,3,2)	3
	$S_{3,2,1^2}F_3^* \otimes S_{4^2,3,2^2}F_1$	(3,2,1,1)	(4,4,3,2,2)	1	$S_{2^3,1}F_3^* \otimes S_{4,3^3,2}F_1$	(2,2,2,1)	(4,3,3,3,2)	6
	$S_{2^3,1}F_3^* \otimes S_{3^5}F_1$	(2,2,2,1)	(3,3,3,3,3)	4	$S_{2^3,1}F_3^* \otimes S_{4^2,3,2^2}F_1$	(2,2,2,1)	(4,4,3,2,2)	2
	$S_{2^3,1}F_3^* \otimes S_{5,3^2,2^2}F_1$	(2,2,2,1)	(5,3,3,2,2)	1	$S_{2^3,1}F_3^* \otimes S_{4^2,3^2,1}F_1$	(2,2,2,1)	(4,4,3,3,1)	1
8	$S_{4,2,1^2}F_3^* \otimes S_{4^2,3^3}F_1$	(4,2,1,1)	(4,4,3,3,3)	1	$S_{3,3,2}F_3^* \otimes S_{4^2,3^3}F_1$	(3,3,2,0)	(4,4,3,3,3)	1
	$S_{3^2,1^2}F_3^* \otimes S_{4^2,3^3}F_1$	(3,3,1,1)	(4,4,3,3,3)	2	$S_{3^2,1^2}F_3^* \otimes S_{4^3,3,2}F_1$	(3,3,1,1)	(4,4,4,3,2)	1
	$S_{3,2^2,1}F_3^* \otimes S_{4^2,3^3}F_1$	(3,2,2,1)	(4,4,3,3,3)	6	$S_{3,2^2,1}F_3^* \otimes S_{5,3^4}F_1$	(3,2,2,1)	(5,3,3,3,3)	3
	$S_{3,2^2,1}F_3^* \otimes S_{4^3,3,2}F_1$	(3,2,2,1)	(4,4,4,3,2)	3	$S_{3,2^2,1}F_3^* \otimes S_{5,4,3^2,2}F_1$	(3,2,2,1)	(5,4,3,3,2)	1
	$S_{2^4}F_3^* \otimes S_{4^2,3^3}F_1$	(2,2,2,2)	(4,4,3,3,3)	5	$S_{2^4}F_3^* \otimes S_{4^3,3,2}F_1$	(2,2,2,2)	(4,4,4,3,2)	4
	$S_{2^4}F_3^* \otimes S_{5,3^4}F_1$	(2,2,2,2)	(5,3,3,3,3)	3	$S_{2^4}F_3^* \otimes S_{5,4,3^2,2}F_1$	(2,2,2,2)	(5,4,3,3,2)	2
	$S_{2^4}F_3^* \otimes S_{4^4,1}F_1$	(2,2,2,2)	(4,4,4,4,1)	1	$S_{2^4}F_3^* \otimes S_{5,4^2,2^2}F_1$	(2,2,2,2)	(5,4,4,2,2)	1
9	$S_{4,3,1,1}F_3^* \otimes S_{4^4,3}F_1$	(4,3,1,1)	(4,4,4,4,3)	1	$S_{4,2^2,1}F_3^* \otimes S_{4^4,3}F_1$	(4,2,2,1)	(4,4,4,4,3)	2
	$S_{4,2^2,1}F_3^* \otimes S_{5,4^2,3^2}F_1$	(4,2,2,1)	(5,4,4,3,3)	1	$S_{3,3,3}F_3^* \otimes S_{4^4,3}F_1$	(3,3,3,0)	(4,4,4,4,3)	1
	$S_{3^2,2,1}F_3^* \otimes S_{4^4,3}F_1$	(3,3,2,1)	(4,4,4,4,3)	6	$S_{3^2,2,1}F_3^* \otimes S_{5,4^2,3^2}F_1$	(3,3,2,1)	(5,4,4,3,3)	3
	$S_{3^2,2,1}F_3^* \otimes S_{5^2,3^3}F_1$	(3,3,2,1)	(5,5,3,3,3)	1	$S_{3^2,2,1}F_3^* \otimes S_{5,4^3,2}F_1$	(3,3,2,1)	(5,4,4,4,2)	1
	$S_{3,2^3}F_3^* \otimes S_{4^4,3}F_1$	(3,2,2,2)	(4,4,4,4,3)	8	$S_{3,2^3}F_3^* \otimes S_{5,4^2,3^2}F_1$	(3,2,2,2)	(5,4,4,3,3)	7
	$S_{3,2^3}F_3^* \otimes S_{5,4^3,2}F_1$	(3,2,2,2)	(5,4,4,4,2)	2	$S_{3,2^3}F_3^* \otimes S_{6,4,3^3}F_1$	(3,2,2,2)	(6,4,3,3,3)	1
	$S_{3,2^3}F_3^* \otimes S_{5^2,3^3}F_1$	(3,2,2,2)	(5,5,3,3,3)	1	$S_{3,2^3}F_3^* \otimes S_{5^2,4,3,2}F_1$	(3,2,2,2)	(5,5,4,3,2)	1

(The table continues on the next page.)

10	$S_{4,3,2,1}F_3^* \otimes S_{5,4^4}F_1$	(4,3,2,1)	(5,4,4,4,4)	3	$S_{4,3,2,1}F_3^* \otimes S_{5^2,4^2,3}F_1$	(4,3,2,1)	(5,5,4,4,3)	1
	$S_{4,2^3}F_3^* \otimes S_{5,4^4}F_1$	(4,2,2,2)	(5,4,4,4,4)	5	$S_{4,2^3}F_3^* \otimes S_{5^2,4^2,3}F_1$	(4,2,2,2)	(5,5,4,4,3)	2
	$S_{4,2^3}F_3^* \otimes S_{6,4^3,3}F_1$	(4,2,2,2)	(6,4,4,4,3)	1	$S_{4,2^3}F_3^* \otimes S_{5^3,3^2}F_1$	(4,2,2,2)	(5,5,5,3,3)	1
	$S_{3^3,1}F_3^* \otimes S_{5,4^4}F_1$	(3,3,3,1)	(5,4,4,4,4)	4	$S_{3^3,1}F_3^* \otimes S_{5^2,4^2,3}F_1$	(3,3,3,1)	(5,5,4,4,3)	3
	$S_{3^3,1}F_3^* \otimes S_{6,4^3,3}F_1$	(3,3,3,1)	(6,4,4,4,3)	1	$S_{3^2,2^2}F_3^* \otimes S_{5,4^4}F_1$	(3,3,2,2)	(5,4,4,4,4)	9
	$S_{3^2,2^2}F_3^* \otimes S_{5^2,4^2,3}F_1$	(3,3,2,2)	(5,5,4,4,3)	7	$S_{3^2,2^2}F_3^* \otimes S_{6,4^3,3}F_1$	(3,3,2,2)	(6,4,4,4,3)	2
	$S_{3^2,2^2}F_3^* \otimes S_{5^3,3^2}F_1$	(3,3,2,2)	(5,5,5,3,3)	2	$S_{3^2,2^2}F_3^* \otimes S_{5^3,4,2}F_1$	(3,3,2,2)	(5,5,5,4,2)	1
	$S_{3^2,2^2}F_3^* \otimes S_{6,5,4,3,3^2}F_1$	(3,3,2,2)	(6,5,4,3,3)	1				
11	$S_{5,2^3}F_3^* \otimes S_{5^3,4^2}F_1$	(5,2,2,2)	(5,5,5,4,4)	1	$S_{4^2,2,1}F_3^* \otimes S_{5^3,4^2}F_1$	(4,4,2,1)	(5,5,5,4,4)	1
	$S_{4,3^2,1}F_3^* \otimes S_{5^3,4^2}F_1$	(4,3,3,1)	(5,5,5,4,4)	2	$S_{4,3^2,1}F_3^* \otimes S_{5^4,3}F_1$	(4,3,3,1)	(5,5,5,5,3)	1
	$S_{4,3^2,1}F_3^* \otimes S_{6,5,4^3}F_1$	(4,3,3,1)	(6,5,4,4,4)	1	$S_{4,3,2^2}F_3^* \otimes S_{5^3,4^2}F_1$	(4,3,2,2)	(5,5,5,4,4)	7
	$S_{4,3,2^2}F_3^* \otimes S_{6,5,4^3}F_1$	(4,3,2,2)	(6,5,4,4,4)	3	$S_{4,3,2^2}F_3^* \otimes S_{5^4,3}F_1$	(4,3,2,2)	(5,5,5,5,3)	2
	$S_{4,3,2^2}F_3^* \otimes S_{6,5^2,4,3}F_1$	(4,3,2,2)	(6,5,5,4,3)	1	$S_{3^3,2}F_3^* \otimes S_{5^3,4^2}F_1$	(3,3,3,2)	(5,5,5,4,4)	9
	$S_{3^3,2}F_3^* \otimes S_{6,5,4^3}F_1$	(3,3,3,2)	(6,5,4,4,4)	6	$S_{3^3,2}F_3^* \otimes S_{5^4,3}F_1$	(3,3,3,2)	(5,5,5,5,3)	5
	$S_{3^3,2}F_3^* \otimes S_{6,5^2,4,3}F_1$	(3,3,3,2)	(6,5,5,4,3)	3	$S_{3^3,2}F_3^* \otimes S_{6^2,4^2,3}F_1$	(3,3,3,2)	(6,6,4,4,3)	1
	$S_{3^3,2}F_3^* \otimes S_{7,4^4}F_1$	(3,3,3,2)	(7,4,4,4,4)	1				
12	$S_{5,3,2^2}F_3^* \otimes S_{5^5}F_1$	(5,3,2,2)	(5,5,5,5,5)	2	$S_{5,3,2^2}F_3^* \otimes S_{6,5^3,4}F_1$	(5,3,2,2)	(6,5,5,5,4)	1
	$S_{4^2,3,1}F_3^* \otimes S_{5^5}F_1$	(4,4,3,1)	(5,5,5,5,5)	2	$S_{4^2,3,1}F_3^* \otimes S_{6,5^3,4}F_1$	(4,4,3,1)	(6,5,5,5,4)	1
	$S_{4^2,2^2}F_3^* \otimes S_{5^5}F_1$	(4,4,2,2)	(5,5,5,5,5)	4	$S_{4^2,2^2}F_3^* \otimes S_{6,5^3,4}F_1$	(4,4,2,2)	(6,5,5,5,4)	2
	$S_{4^2,2^2}F_3^* \otimes S_{6^2,5,4^2}F_1$	(4,4,2,2)	(6,6,5,4,4)	1	$S_{4,3^2,2}F_3^* \otimes S_{6,5^3,4}F_1$	(4,3,3,2)	(6,5,5,5,4)	9
	$S_{4,3^2,2}F_3^* \otimes S_{5^5}F_1$	(4,3,3,2)	(5,5,5,5,5)	7	$S_{4,3^2,2}F_3^* \otimes S_{6^2,5,4^2}F_1$	(4,3,3,2)	(6,6,5,4,4)	3
	$S_{4,3^2,2}F_3^* \otimes S_{7,5^2,4^2}F_1$	(4,3,3,2)	(7,5,5,4,4)	1	$S_{4,3^2,2}F_3^* \otimes S_{6^2,5^2,3}F_1$	(4,3,3,2)	(6,6,5,5,3)	1
	$S_{3^4}F_3^* \otimes S_{6,5^3,4}F_1$	(3,3,3,3)	(6,5,5,5,4)	8	$S_{3^4}F_3^* \otimes S_{6^2,5,4^2}F_1$	(3,3,3,3)	(6,6,5,4,4)	4
	$S_{3^4}F_3^* \otimes S_{5^5}F_1$	(3,3,3,3)	(5,5,5,5,5)	4	$S_{3^4}F_3^* \otimes S_{7,5^2,4^2}F_1$	(3,3,3,3)	(7,5,5,4,4)	2
	$S_{3^4}F_3^* \otimes S_{6^2,5^2,3}F_1$	(3,3,3,3)	(6,6,5,5,3)	2	$S_{3^4}F_3^* \otimes S_{7,6,4^3}F_1$	(3,3,3,3)	(7,6,4,4,4)	1
	$S_{3^4}F_3^* \otimes S_{6^3,4,3}F_1$	(3,3,3,3)	(6,6,6,4,3)	1				

6.2. E_8 graded by α_3

This format is $(1,7,8,2)$.

6.2.1. $V(\omega_8)$. There are 9 graded components.

0	F_1^*	(0,0)	(0,0,0,0,0,0,−1)	1				
1	$F_3^* \otimes F_1$	(1,0)	(1,0,0,0,0,0,0)	1				
2	$\bigwedge^2 F_3^* \otimes \bigwedge^3 F_1$	(1,1)	(1,1,1,0,0,0,0)	1				
3	$S_{2,1}F_3^* \otimes \bigwedge^5 F_1$	(2,1)	(1,1,1,1,1,0,0)	1				
4	$S_{3,1}F_3^* \otimes \bigwedge^7 F_1$	(3,1)	(1,1,1,1,1,1,1)	1	$S_{2,2}F_3^* \otimes S_{2,1^5}F_1$	(2,2)	(2,1,1,1,1,1,0)	1
	$S_{2,2}F_3^* \otimes \bigwedge^7 F_1$	(2,2)	(1,1,1,1,1,1,1)	1				
5	$S_{3,2}F_3^* \otimes S_{2^2,1^5}F_1$	(3,2)	(2,2,1,1,1,1,1)	1				
6	$S_{3,3}F_3^* \otimes S_{2^4,1^3}F_1$	(3,3)	(2,2,2,2,1,1,1)	1				
7	$S_{4,3}F_3^* \otimes S_{2^6,1}F_1$	(4,3)	(2,2,2,2,2,2,1)	1				
8	$S_{4,4}F_3^* \otimes S_{3,2^6}F_1$	(4,4)	(3,2,2,2,2,2,2)	1				

6.2.2. $V(\omega_1)$. There are 15 graded components.

0	F_3	(0,−1)	(0,0,0,0,0,0,0)	1				
1	$\bigwedge^2 F_1$	(0,0)	(1,1,0,0,0,0,0)	1				
2	$F_3^* \otimes \bigwedge^4 F_1$	(1,0)	(1,1,1,1,0,0,0)	1				
3	$S_2 F_3^* \otimes \bigwedge^6 F_1$	(2,0)	(1,1,1,1,1,1,0)	1	$\bigwedge^2 F_3^* \otimes S_{2,1^4} F_1$	(1,1)	(2,1,1,1,1,0,0)	1
	$\bigwedge^2 F_3^* \otimes \bigwedge^6 F_1$	(1,1)	(1,1,1,1,1,1,0)	1				
4	$S_{2,1} F_3^* \otimes S_{2,1^6} F_1$	(2,1)	(2,1,1,1,1,1,1)	2	$S_{2,1} F_3^* \otimes S_{2,2,1,1,1,1,1} F_1$	(2,1)	(2,2,1,1,1,1,0)	1
5	$S_{3,1} F_3^* \otimes S_{2^3,1^4} F_1$	(3,1)	(2,2,2,1,1,1,1)	1	$S_{2,2} F_3^* \otimes S_{2^3,1^4} F_1$	(2,2)	(2,2,2,1,1,1,1)	1
	$S_{2,2} F_3^* \otimes S_{3,2,1^5} F_1$	(2,2)	(3,2,1,1,1,1,1)	1	$S_{2,2} F_3^* \otimes S_{2,2,2,2,1,1} F_1$	(2,2)	(2,2,2,2,1,1,0)	1
6	$S_{3,2} F_3^* \otimes S_{2^5,1^2} F_1$	(3,2)	(2,2,2,2,2,1,1)	2	$S_{3,2} F_3^* \otimes S_{3,2^3,1^3} F_1$	(3,2)	(3,2,2,2,1,1,1)	1
	$S_{3,2} F_3^* \otimes S_{2,2,2,2,2,2} F_1$	(3,2)	(2,2,2,2,2,2,0)	1				
7	$S_{4,2} F_3^* \otimes S_{2^7} F_1$	(4,2)	(2,2,2,2,2,2,2)	1	$S_{4,2} F_3^* \otimes S_{3,2^5,1} F_1$	(4,2)	(3,2,2,2,2,2,1)	1
	$S_{3,3} F_3^* \otimes S_{2^7} F_1$	(3,3)	(2,2,2,2,2,2,2)	2	$S_{3,3} F_3^* \otimes S_{3,2^5,1} F_1$	(3,3)	(3,2,2,2,2,2,1)	2
	$S_{3,3} F_3^* \otimes S_{3^2,2^3,1^2} F_1$	(3,3)	(3,3,2,2,2,1,1)	1				
8	$S_{4,3} F_3^* \otimes S_{3^2,2^5} F_1$	(4,3)	(3,3,2,2,2,2,2)	2	$S_{4,3} F_3^* \otimes S_{4,2^6} F_1$	(4,3)	(4,2,2,2,2,2,2)	1
	$S_{4,3} F_3^* \otimes S_{3^3,2^3,1} F_1$	(4,3)	(3,3,3,2,2,2,1)	1				
9	$S_{5,3} F_3^* \otimes S_{3^4,2^3} F_1$	(5,3)	(3,3,3,3,2,2,2)	1	$S_{4,4} F_3^* \otimes S_{3^4,2^3} F_1$	(4,4)	(3,3,3,3,2,2,2)	1
	$S_{4,4} F_3^* \otimes S_{4,3^2,2^4} F_1$	(4,4)	(4,3,3,2,2,2,2)	1	$S_{4,4} F_3^* \otimes S_{3^5,2,1} F_1$	(4,4)	(3,3,3,3,3,2,1)	1
10	$S_{5,4} F_3^* \otimes S_{3^6,2} F_1$	(5,4)	(3,3,3,3,3,3,2)	2	$S_{5,4} F_3^* \otimes S_{4,3^4,2^2} F_1$	(5,4)	(4,3,3,3,3,2,2)	1
11	$S_{6,4} F_3^* \otimes S_{4,3^6} F_1$	(6,4)	(4,3,3,3,3,3,3)	1	$S_{5,5} F_3^* \otimes S_{4^2,3^4,2} F_1$	(5,5)	(4,4,3,3,3,3,2)	1
	$S_{5,5} F_3^* \otimes S_{4,3^6} F_1$	(5,5)	(4,3,3,3,3,3,3)	1				
12	$S_{6,5} F_3^* \otimes S_{4^3,3^4} F_1$	(6,5)	(4,4,4,3,3,3,3)	1				
13	$S_{6,6} F_3^* \otimes S_{4^5,3^2} F_1$	(6,6)	(4,4,4,4,4,3,3)	1				
14	$S_{7,6} F_3^* \otimes S_{4^7} F_1$	(7,6)	(4,4,4,4,4,4,4)	1				

6.2.3. $V(\omega_2)$. There are 21 graded components. We exhibit the first 11. The others can be found by duality.
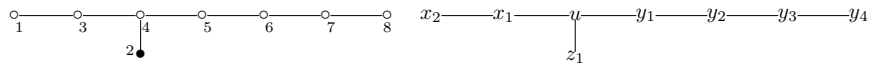
0	F_1	(0,0)	(1,0,0,0,0,0,0)	1				
1	$F_3^* \otimes \bigwedge^3 F_1$	(1,0)	(1,1,1,0,0,0,0)	1				
2	$\bigwedge^2 F_3^* \otimes \bigwedge^5 F_1$	(2,0)	(1,1,1,1,1,0,0)	1	$\bigwedge^2 F_3^* \otimes S_{2,1,1,1} F_1$	(1,1)	(2,1,1,1,0,0,0)	1
	$S_2 F_3^* \otimes \bigwedge^5 F_1$	(1,1)	(1,1,1,1,1,0,0)	1				
3	$S_3 F_3^* \otimes \bigwedge^7 F_1$	(3,0)	(1,1,1,1,1,1,1)	1	$S_{2,1} F_3^* \otimes S_{2,1^5} F_1$	(2,1)	(2,1,1,1,1,1,0)	2
	$S_{2,1} F_3^* \otimes \bigwedge^7 F_1$	(2,1)	(1,1,1,1,1,1,1)	2	$S_{2,1} F_3^* \otimes S_{2,2,1,1,1,1} F_1$	(2,1)	(2,2,1,1,1,0,0)	1
4	$S_{3,1} F_3^* \otimes S_{2^2,1^5} F_1$	(3,1)	(2,2,1,1,1,1,1)	2	$S_{3,1} F_3^* \otimes S_{2^3,1^3} F_1$	(3,1)	(2,2,2,1,1,1,0)	1
	$S_{2,2} F_3^* \otimes S_{2^2,1^5} F_1$	(2,2)	(2,2,1,1,1,1,1)	3	$S_{2,2} F_3^* \otimes S_{3,2,1^4} F_1$	(2,2)	(3,2,1,1,1,1,0)	1
	$S_{2,2} F_3^* \otimes S_{3,1^6} F_1$	(2,2)	(3,1,1,1,1,1,1)	1	$S_{2,2} F_3^* \otimes S_{2,2,2,2,1} F_1$	(2,2)	(2,2,2,2,1,0,0)	1
	$S_{2,2} F_3^* \otimes S_{2^3,1^3} F_1$	(2,2)	(2,2,2,1,1,1,0)	1				
5	$S_{4,1} F_3^* \otimes S_{2^4,1^3} F_1$	(4,1)	(2,2,2,2,1,1,1)	1	$S_{3,2} F_3^* \otimes S_{2^4,1^3} F_1$	(3,2)	(2,2,2,2,1,1,1)	4
	$S_{3,2} F_3^* \otimes S_{3,2^2,1^4} F_1$	(3,2)	(3,2,2,1,1,1,1)	2	$S_{3,2} F_3^* \otimes S_{2^5,1} F_1$	(3,2)	(2,2,2,2,2,1,0)	2
	$S_{3,2} F_3^* \otimes S_{3^2,1^5} F_1$	(3,2)	(3,3,1,1,1,1,1)	1	$S_{3,2} F_3^* \otimes S_{3,2^3,1^2} F_1$	(3,2)	(3,2,2,2,1,1,0)	1
6	$S_{4,2} F_3^* \otimes S_{2^6,1} F_1$	(4,2)	(2,2,2,2,2,2,1)	4	$S_{4,2} F_3^* \otimes S_{3,2^4,1^2} F_1$	(4,2)	(3,2,2,2,2,1,1)	2
	$S_{4,2} F_3^* \otimes S_{3^2,2^2,1^3} F_1$	(4,2)	(3,3,2,2,1,1,1)	1	$S_{4,2} F_3^* \otimes S_{3,2^5} F_1$	(4,2)	(3,2,2,2,2,2,0)	1
	$S_{3,3} F_3^* \otimes S_{3,2^4,1^2} F_1$	(3,3)	(3,2,2,2,2,1,1)	4	$S_{3,3} F_3^* \otimes S_{2^6,1} F_1$	(3,3)	(2,2,2,2,2,2,1)	4
	$S_{3,3} F_3^* \otimes S_{3^2,2^2,1^3} F_1$	(3,3)	(3,3,2,2,1,1,1)	2	$S_{3,3} F_3^* \otimes S_{4,2^3,1^3} F_1$	(3,3)	(4,2,2,2,1,1,1)	1
	$S_{3,3} F_3^* \otimes S_{3^2,2^3,1} F_1$	(3,3)	(3,3,2,2,2,1,0)	1	$S_{3,3} F_3^* \otimes S_{3,2^5} F_1$	(3,3)	(3,2,2,2,2,2,0)	1
7	$S_{5,2} F_3^* \otimes S_{3^2,2^4,1} F_1$	(5,2)	(3,3,2,2,2,2,2)	1	$S_{5,2} F_3^* \otimes S_{3,2^6} F_1$	(5,2)	(3,2,2,2,2,2,2)	1
	$S_{4,3} F_3^* \otimes S_{3^2,2^4,1} F_1$	(4,3)	(3,3,2,2,2,2,1)	6	$S_{4,3} F_3^* \otimes S_{3,2^6} F_1$	(4,3)	(3,2,2,2,2,2,2)	6
	$S_{4,3} F_3^* \otimes S_{4,2^5,1} F_1$	(4,3)	(4,2,2,2,2,2,1)	2	$S_{4,3} F_3^* \otimes S_{3^3,2^2,1^2} F_1$	(4,3)	(3,3,3,2,2,1,1)	2
	$S_{4,3} F_3^* \otimes S_{4,3,2^3,1^2} F_1$	(4,3)	(4,3,2,2,2,1,1)	1	$S_{4,3} F_3^* \otimes S_{3^4,1^3} F_1$	(4,3)	(3,3,3,3,1,1,1)	1
	$S_{4,3} F_3^* \otimes S_{3^3,2^3} F_1$	(4,3)	(3,3,3,2,2,2,0)	1				
8	$S_{5,3} F_3^* \otimes S_{3^3,2^4} F_1$	(5,3)	(3,3,3,2,2,2,2)	4	$S_{5,3} F_3^* \otimes S_{3^4,2^2,1} F_1$	(5,3)	(3,3,3,3,2,2,1)	3
	$S_{5,3} F_3^* \otimes S_{4,3,2^5} F_1$	(5,3)	(4,3,2,2,2,2,2)	2	$S_{5,3} F_3^* \otimes S_{4,3^2,2^3,1} F_1$	(5,3)	(4,3,3,2,2,2,1)	1
	$S_{4,4} F_3^* \otimes S_{3^3,2^4} F_1$	(4,4)	(3,3,3,2,2,2,2)	5	$S_{4,4} F_3^* \otimes S_{4,3,2^5} F_1$	(4,4)	(4,3,2,2,2,2,2)	4
	$S_{4,4} F_3^* \otimes S_{3^4,2^2,1} F_1$	(4,4)	(3,3,3,3,2,2,1)	4	$S_{4,4} F_3^* \otimes S_{4,3^2,2^3,1} F_1$	(4,4)	(4,3,3,2,2,2,1)	2
	$S_{4,4} F_3^* \otimes S_{5,2^6} F_1$	(4,4)	(5,2,2,2,2,2,2)	1	$S_{4,4} F_3^* \otimes S_{4^2,2^4,1} F_1$	(4,4)	(4,4,2,2,2,2,1)	1
	$S_{4,4} F_3^* \otimes S_{4,3^3,2,1^2} F_1$	(4,4)	(4,3,3,3,2,1,1)	1	$S_{4,4} F_3^* \otimes S_{3^5,2} F_1$	(4,4)	(3,3,3,3,3,2,0)	1
	$S_{4,4} F_3^* \otimes S_{3^5,1^2} F_1$	(4,4)	(3,3,3,3,3,1,1)	1				

9	$S_{6,3}F_3^* \otimes S_{4,3^3,2^3}F_1$	(6,3)	(4,3,3,3,2,2,2)	1	$S_{6,3}F_3^* \otimes S_{3^6,1}F_1$	(6,3)	(3,3,3,3,3,3,1)	1
	$S_{6,3}F_3^* \otimes S_{3^5,2^2}F_1$	(6,3)	(3,3,3,3,3,2,2)	1	$S_{5,4}F_3^* \otimes S_{3^5,2^2}F_1$	(5,4)	(3,3,3,3,3,2,2)	7
	$S_{5,4}F_3^* \otimes S_{4,3^3,2^3}F_1$	(5,4)	(4,3,3,3,2,2,2)	6	$S_{5,4}F_3^* \otimes S_{3^6,1}F_1$	(5,4)	(3,3,3,3,3,3,1)	4
	$S_{5,4}F_3^* \otimes S_{4,3^4,2,1}F_1$	(5,4)	(4,3,3,3,3,2,1)	3	$S_{5,4}F_3^* \otimes S_{4^2,3,2^4}F_1$	(5,4)	(4,4,3,2,2,2,2)	2
	$S_{5,4}F_3^* \otimes S_{5,3^2,2^4}F_1$	(5,4)	(5,3,3,2,2,2,2)	1	$S_{5,4}F_3^* \otimes S_{4^2,3^2,2^2,1}F_1$	(5,4)	(4,4,3,3,2,2,1)	1
10	$S_{6,4}F_3^* \otimes S_{4,3^5,2}F_1$	(6,4)	(4,3,3,3,3,3,2)	6	$S_{6,4}F_3^* \otimes S_{3^7}F_1$	(6,4)	(3,3,3,3,3,3,3)	3
	$S_{6,4}F_3^* \otimes S_{4^2,3^3,2^2}F_1$	(6,4)	(4,4,3,3,3,2,2)	2	$S_{6,4}F_3^* \otimes S_{5,3^4,2^2}F_1$	(6,4)	(5,3,3,3,3,2,2)	1
	$S_{6,4}F_3^* \otimes S_{4^3,3,2^3}F_1$	(6,4)	(4,4,4,3,2,2,2)	1	$S_{6,4}F_3^* \otimes S_{4^2,3^4,1}F_1$	(6,4)	(4,4,3,3,3,3,1)	1
	$S_{5,5}F_3^* \otimes S_{4,3^5,2}F_1$	(5,5)	(4,3,3,3,3,3,2)	7	$S_{5,5}F_3^* \otimes S_{4^2,3^3,2^2}F_1$	(5,5)	(4,4,3,3,3,2,2)	5
	$S_{5,5}F_3^* \otimes S_{3^7}F_1$	(5,5)	(3,3,3,3,3,3,3)	3	$S_{5,5}F_3^* \otimes S_{5,3^4,2^2}F_1$	(5,5)	(5,3,3,3,3,2,2)	2
	$S_{5,5}F_3^* \otimes S_{4^2,3^4,1}F_1$	(5,5)	(4,4,3,3,3,3,1)	2	$S_{5,5}F_3^* \otimes S_{5,4,3^2,2^3}F_1$	(5,5)	(5,4,3,3,2,2,2)	1
	$S_{5,5}F_3^* \otimes S_{4^3,3^2,2,1}F_1$	(5,5)	(4,4,4,3,3,2,1)	1	$S_{5,5}F_3^* \otimes S_{4^3,3,2^3}F_1$	(5,5)	(4,4,4,3,2,2,2)	1

6.3. E_8 graded by α_2

This format is $(2,8,7,1)$.

$$
\underset{1}{\circ}-\underset{3}{\circ}-\underset{4}{\circ}-\underset{5}{\circ}-\underset{6}{\circ}-\underset{7}{\circ}-\underset{8}{\circ} \qquad x_2-\!-x_1-\!-u-\!-y_1-\!-y_2-\!-y_3-\!-y_4
$$
$$
\underset{2}{\bullet} \qquad\qquad\qquad\qquad z_1
$$

6.3.1. $V(\omega_8)$. There are 7 graded components.

0	F_1^*	(0,0,0,0,0,0,0,-1)	1			
1	$\bigwedge^2 F_1$	(1,1,0,0,0,0,0,0)	1			
2	$\bigwedge^5 F_1$	(1,1,1,1,1,0,0,0)	1			
3	$S_{2,1^6}F_1$	(2,1,1,1,1,1,1,0)	1	$\bigwedge^8 F_1$	(1,1,1,1,1,1,1,1)	1
4	$S_{2^3,1^5}F_1$	(2,2,2,1,1,1,1,1)	1			
5	$S_{2^6,1^2}F_1$	(2,2,2,2,2,2,1,1)	1			
6	$S_{3,2^7}F_1$	(3,2,2,2,2,2,2,2)	1			

6.3.2. $V(\omega_1)$. There are 11 graded components.

0	F_1	(1,0,0,0,0,0,0,0)	1			
1	$\bigwedge^4 F_1$	(1,1,1,1,0,0,0,0)	1			
2	$S_{2,1^5}F_1$	(2,1,1,1,1,1,0,0)	1	$\bigwedge^7 F_1$	(1,1,1,1,1,1,1,0)	1
3	$S_{3,1^7}F_1$	(3,1,1,1,1,1,1,1)	1	$S_{2^3,1^4}F_1$	(2,2,2,1,1,1,1,0)	1
	$S_{2^2,1^6}F_1$	(2,2,1,1,1,1,1,1)	1			
4	$S_{3,2^3,1^4}F_1$	(3,2,2,2,1,1,1,1)	1	$S_{2^6,1}F_1$	(2,2,2,2,2,2,1,0)	1
	$S_{2^5,1^3}F_1$	(2,2,2,2,2,1,1,1)	1			
5	$S_{3^2,2^4,1^2}F_1$	(3,3,2,2,2,2,1,1)	1	$S_{3,2^6,1}F_1$	(3,2,2,2,2,2,2,1)	2
	$S_{2^8}F_1$	(2,2,2,2,2,2,2,2)	1			
6	$S_{4,3,2^6}F_1$	(4,3,2,2,2,2,2,2)	1	$S_{3^4,2^3,1}F_1$	(3,3,3,3,2,2,2,1)	1
	$S_{3^3,2^5}F_1$	(3,3,3,2,2,2,2,2)	1			
7	$S_{4,3^4,2^3}F_1$	(4,3,3,3,3,2,2,2)	1	$S_{3^7,1}F_1$	(3,3,3,3,3,3,3,1)	1
	$S_{3^6,2^2}F_1$	(3,3,3,3,3,3,2,2)	1			
8	$S_{4^2,3^5,2}F_1$	(4,4,3,3,3,3,3,2)	1	$S_{4,3^7}F_1$	(4,3,3,3,3,3,3,3)	1
9	$S_{4^4,3^4}F_1$	(4,4,4,4,3,3,3,3)	1			
10	$S_{4^7,3}F_1$	(4,4,4,4,4,4,4,3)	1			

6.3.3. $V(\omega_2)$. There are 17 graded components. We present the first 9. The others can be filled by duality.

0	$\mathbb{C}$	(0,0,0,0,0,0,0,0)	1			
1	$\bigwedge^3 F_1$	(1,1,1,0,0,0,0,0)	1			
2	$S_{2,1^4}F_1$	(2,1,1,1,1,0,0,0)	1	$\bigwedge^6 F_1$	(1,1,1,1,1,1,0,0)	1
3	$S_{3,1^6}F_1$	(3,1,1,1,1,1,1,0)	1	$S_{2^3,1^3}F_1$	(2,2,2,1,1,1,0,0)	1
	$S_{2^2,1^5}F_1$	(2,2,1,1,1,1,1,0)	1	$S_{2,1^7}F_1$	(2,1,1,1,1,1,1,1)	2
4	$S_{3,2^3,1^3}F_1$	(3,2,2,2,1,1,1,0)	1	$S_{3,2^2,1^5}F_1$	(3,2,2,1,1,1,1,1)	2
	$S_{2^6}F_1$	(2,2,2,2,2,2,0,0)	1	$S_{2^5,1^2}F_1$	(2,2,2,2,2,1,1,0)	1
	$S_{2^4,1^4}F_1$	(2,2,2,2,1,1,1,1)	2			
5	$S_{4,2^4,1^3}F_1$	(4,2,2,2,2,1,1,1)	1	$S_{3^3,2,1^4}F_1$	(3,3,3,2,1,1,1,1)	1
	$S_{3^2,2^4,1}F_1$	(3,3,2,2,2,2,1,0)	1	$S_{3^2,2^3,1^3}F_1$	(3,3,2,2,2,1,1,1)	1
	$S_{3,2^6}F_1$	(3,2,2,2,2,2,2,0)	1	$S_{3,2^5,1^2}F_1$	(3,2,2,2,2,2,1,1)	4
	$S_{2^7,1}F_1$	(2,2,2,2,2,2,2,1)	2			
6	$S_{4,3^2,2^3,1^2}F_1$	(4,3,3,2,2,2,1,1)	1	$S_{4,3,2^5,1}F_1$	(4,3,2,2,2,2,2,1)	2
	$S_{4,2^7}F_1$	(4,2,2,2,2,2,2,2)	3	$S_{3^4,2^3}F_1$	(3,3,3,3,2,2,2,0)	1
	$S_{3^4,2^2,1^2}F_1$	(3,3,3,3,2,2,1,1)	2	$S_{3^3,2^4,1}F_1$	(3,3,3,2,2,2,2,1)	3
	$S_{3^2,2^6}F_1$	(3,3,2,2,2,2,2,2)	3			
7	$S_{5,3^2,2^5}F_1$	(5,3,3,2,2,2,2,2)	1	$S_{4^2,3^2,2^3,1}F_1$	(4,4,3,3,2,2,2,1)	1
	$S_{4^2,3,2^5}F_1$	(4,4,3,2,2,2,2,2)	1	$S_{4,3^5,1^2}F_1$	(4,3,3,3,3,3,1,1)	1
	$S_{4,3^4,2^2,1}F_1$	(4,3,3,3,3,2,2,1)	2	$S_{4,3^3,2^4}F_1$	(4,3,3,3,2,2,2,2)	4
	$S_{3^7}F_1$	(3,3,3,3,3,3,3,0)	1	$S_{3^6,2,1}F_1$	(3,3,3,3,3,3,2,1)	3
	$S_{3^5,2^3}F_1$	(3,3,3,3,3,2,2,2)	3			
8	$S_{5,4,3^3,2^3}F_1$	(5,4,3,3,3,2,2,2)	1	$S_{5,3^5,2^2}F_1$	(5,3,3,3,3,3,2,2)	2
	$S_{4^4,2^4}F_1$	(4,4,4,4,2,2,2,2)	1	$S_{4^3,3^3,2,1}F_1$	(4,4,4,3,3,3,2,1)	1
	$S_{4^3,3^2,2^3}F_1$	(4,4,4,3,3,2,2,2)	1	$S_{4^2,3^5,1}F_1$	(4,4,3,3,3,3,3,1)	2
	$S_{4^2,3^4,2^2}F_1$	(4,4,3,3,3,3,2,2)	4	$S_{4,3^6,2}F_1$	(4,3,3,3,3,3,3,2)	5
	$S_{3^8}F_1$	(3,3,3,3,3,3,3,3)	1			

7. Some patterns and observations

In this section we observe some general patterns that can be deduced from the tables. They will form a basis for a conjecture.

Proposition 7.1. *Let us restrict to the formats of resolutions of cyclic modules, i.e., $p = 2$. Then the tensors giving the first graded components of three critical representations are*:

(1) $W_1(d_3) = F_2^* \otimes \bigwedge^2 F_1$ *gives the tensor of multiplicative structure $F_1 \otimes F_1 \to F_2$ on $\mathbb{F}_\bullet$,*

(2) $W_1(d_2) = F_2 \otimes F_3^* \otimes F_1$ *gives the tensor of multiplicative structure $F_1 \otimes F_2 \to F_3$ on $\mathbb{F}_\bullet$,*

(3) $W_1(a_2) = F_3^* \otimes \bigwedge^3 F_1$ *gives the tensor of multiplicative structure $\bigwedge^3 F_1 \to F_3$ on $\mathbb{F}_\bullet$.*

Proof. The formulas follow immediately from the parabolic BGG complex. See [5], Section 9.2. $\square$

For other formats we have a similar interpretation.

Consider the format with the ranks (r_1, r_2, r_3). We have a comparison map from the Buchsbaum–Rim complex to the complex $\mathbb{F}_\bullet$.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & F_3 & \overset{d_3}{\longrightarrow} & F_2 & \overset{d_2}{\longrightarrow} & F_1 & \overset{d_1}{\longrightarrow} & F_0 \\
& & \uparrow{\scriptstyle v_3} & & \uparrow{\scriptstyle v_1} & & \uparrow{\scriptstyle =} & & \uparrow{\scriptstyle =} \\
\cdots \longrightarrow & \bigwedge^{r_1+2} F_1 \otimes \otimes F_0^* \otimes \bigwedge^{r_1} F_0^* & \longrightarrow & \bigwedge^{r_1+1} F_1 \otimes \bigwedge^{r_1} F_0^* & \longrightarrow & F_1 & \overset{d_1}{\longrightarrow} & F_0
\end{array}
$$

Similarly we have a lifting

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & F_3 & \overset{d_3}{\longrightarrow} & F_2 & \overset{d_2}{\longrightarrow} & F_1 & \overset{d_1}{\longrightarrow} & F_0 \\
& & & {\scriptstyle v_2}\swarrow & \uparrow{\scriptstyle q_1} & & & & \\
& & & \bigwedge^{r_1} F_1 \otimes F_2 & & & & &
\end{array}
$$

of the cycle $q_1 = v_1 d_1 - (\bigwedge^{r_1} d_1 \otimes d_2)$, which we denote v_2.

Proposition 7.2. *Let us deal with a general format. Then the tensors giving the first graded components of three critical representations are*:

(1) $W_1(d_3) = F_2^* \otimes \bigwedge^{r_1+1} F_1$ *gives the tensor v_1 in the diagram above,*

(2) $W_1(d_2) = F_2 \otimes F_3^* \otimes \bigwedge^{r_1} F_1$ *gives the tensor v_2 in the diagram above,*

(3) $W_1(a_2) = F_0^* \otimes F_3^* \otimes \bigwedge^{r_1+2} F_1$ *gives the tensor v_3 in the diagram above.*

Proof. The formulas follow immediately from the parabolic BGG complex. See [5], Section 9.2. $\qquad\square$

For the Dynkin formats with $p = 2$, notice that, because three critical representations are finite dimensional, they have top graded components. Moreover, since critical representations are either self dual or (in the case of D_n or E_6) possibly dual to each other, we have the following.

Proposition 7.3. *Let us consider Dynkin format, with the exception of $(1, n - 1, 1)$ with n odd. Then the top components $W_{top}(d_3)$, $W_{top}(d_2)$, $W_{top}(a_2)$ give tensors which can be interpreted as the differentials in the complex*

$$
\mathbb{F}_\bullet^{top} : F_3^* \overset{\partial_3}{\to} F_2 \overset{\partial_2}{\to} F_1^* \overset{\partial_1}{\to} F_0,
$$

except for the format $(1, 4, 2)$ when we get a complex

$$
\mathbb{F}_\bullet^{top} : F_3^* \to F_2^* \to F_1^* \to F_0.
$$

Proof. This follows from a careful examination of the tables. The fact that we get such complexes follows from the decomposition of $\hat{R}_{gen}$ given in Proposition 2.2 into irreducibles. One can see we do not have in $\hat{R}_{gen}$ representations which could be tensors giving compositions $\partial_2 \partial_3$ and $\partial_1 \partial_2$. $\qquad\square$

The case of almost complete intersections (proved in [1]) is the basis for the following conjecture.

Conjecture 7.4 ([1]). Assume we deal with Dynkin format. The open set U_{CM} of points in $Spec(\hat{R}_{gen})$ at which $\mathbb{F}_{\bullet}^{gen}$ is a resolution of a Cohen-Macaulay module consists of points where the complex $\mathbb{F}_{\bullet}^{top}$ is split exact.

8. Appendix

In this appendix, we explain how one can generate the tables in the previous sections using SageMath [6]. Our method uses *crystals* introduced by M. Kashiwara [4]. A systematic introduction to the theory of crystals can be found in [3].

As all the cases are similar, we only present a SageMath code for the table in Section 5.1.1.

```
sage: La=RootSystem(['E',7]).weight_space().fundamental_weights()
sage: B=crystals.LSPaths(La[7])
sage: for x in B:
....:     if x.is_highest_weight([1,2,3,4,6,7]):
....:         a=x.to_highest_weight()[1]
....:         l=len(a); d=0
....:         for j in range (0,l):
....:             if a[j]== 5:
....:                 d=d+1
....:         print d, ",", x.weight()
....:
0 , Lambda[7]
1 , Lambda[4] - Lambda[5]
2 , Lambda[1] - Lambda[5] + Lambda[6]
3 , Lambda[2] - Lambda[5] + Lambda[7]
4 , Lambda[3] - Lambda[5]
5 , -Lambda[5] + Lambda[6]
```

Each line of the outcome shows the degree and highest weight of an irreducible component. One can convert a highest weight into a pair of partitions. For example, the highest weight

```
Lambda[1] - Lambda[5] + Lambda[6]
```

of degree 2 corresponds to $(1,0,0)$ $(1,1,1,1,0)$ as `Lambda[6]` becomes the first fundamental weight of $\mathfrak{gl}(3)$ and `Lambda[1]` the last fundamental weight of $\mathfrak{gl}(5)$. (See the Dynkin diagram in Section 5.1.)

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