Theorem (★★)

Let $u = g(x)$. Then we have

$$\int f(g(x))g'(x)dx = \int f(u)du,$$

and

$$du = g'(x)dx.$$

Proof.

$$\begin{aligned} \frac{d}{dx}\int f(u)du &= \left(\frac{d}{du}\int f(u)du\right)\cdot\frac{du}{dx} \\ &= f(u)\cdot\frac{du}{dx} \\ &= f(g(x))g'(x) \end{aligned}$$

Theorem

Let $u = g(x)$. Then we have

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

- Once you make the change of variables from x to u, completely forget about x.
- Examples

- What is the value of the following limit?

$$\lim_{x\to 3} \frac{1}{x-3} \int_3^x \frac{\sin t}{t} dt$$

A. 1

B. 3

C. $\sin 3$

D. $(\sin 3)/3$

E. Does Not Exist

6.1 Velocity and Net Change

- $s(t)$: position, $v(t)$: velocity, $a(t)$: acceleration.
- We have

$$s(t) \underset{\int}{\overset{\prime}{\rightleftharpoons}} v(t) \underset{\int}{\overset{\prime}{\rightleftharpoons}} a(t).$$

- The displacement is the net change in the position, and can be calculated by

$$s(b) - s(a) = \int_a^b v(t)dt.$$

- The distance traveled is given by

$$\int_a^b |v(t)|dt.$$

- We have

$$s(t) = s(0) + \int_0^t v(x)dx,$$

and

$$v(t) = v(0) + \int_0^t a(x)dx.$$

(Fundamental Theorem of Calculus–Practical Meaning)

- *The integral of a rate of change is the net change, i.e.*

$$\int_a^b Q'(t)dt = Q(b) - Q(a).$$

- *In particular, we have*

$$Q(t) = Q(0) + \int_0^t Q'(x)dx.$$