2.1 The Idea of Limits

Example

Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

$$m = \lim_{Q \to P} m_{PQ} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

Thus we have

$$y - 1 = 2(x - 1) \quad \text{or} \quad y = 2x - 1.$$

Example

Suppose that the position of an object is given by $s = 4.9t^2$. Find the (instantaneous) velocity of the object at $t = 5$.

$$v = \lim_{t \to 5} v_{\mathrm{ave}} = \lim_{t \to 5} \frac{4.9t^2 - 4.9(5)^2}{t - 5} = 49$$

Actually, the above two problems are of the same kind, since the velocity at $t = 5$ is the same as the slope of the tangent line to $s = 4.9t^2$ at $t = 5$.

2.2 Definitions of Limits

Definition

We write

$$\lim_{x \to a} f(x) = L$$

if the values of $f(x)$ are getting closer (or always equal) to L as x approaches a from the both sides.

- Examples
- We do not consider $x = a$. Thus it is possible to have $f(a) \neq \lim_{x \to a} f(x)$.
- A limit does not always exist.

Definition

1. *We write*

$$\lim_{x \to a^+} f(x) = L$$

if the values of $f(x)$ are getting closer (or always equal) to L as x approaches a from the right.

2. *We write*

$$\lim_{x \to a^-} f(x) = L$$

if the values of $f(x)$ are getting closer (or always equal) to L as x approaches a from the left.

- Example

- Define $[x]$ to be the largest integer that is less than or equal to x.

  For example, $[1.2] = 1$, $[2] = 2$ and $[\pi] = 3$.

- What is the value of $[-1.12]$?

  A. -1 B. -2

- What is the value of $\lim_{x \to 1^-} [x]$?

  A. 0 B. 1 C. 2

Proposition

$$\lim_{x \to a} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to a^-} f(x) = L \quad \textit{and} \quad \lim_{x \to a^+} f(x) = L$$

- The limit $\lim_{x \to 1} [x]$ does not exist.
- The limit $\lim_{x \to 0} \sin(1/x)$ does not exist.