1.3 Inverse, Exponential, and Logarithmic Functions

Definition

An ***exponential function*** *is a function of the form*

$$f(x) = a^x$$

where a is a positive constant.

- Some graphs of exponential functions
- 1. If $a > 1$ then $f(x) = a^x$ is increasing.
  2. If $0 < a < 1$ then $f(x) = a^x$ is decreasing.
- The functions $f(x) = a^x$ and $g(x) = (1/a)^x$ are symmetric about the y-axis.

Assume that n, m are positive integers and $x, y \in \mathbb{R}$.

- $a^n = a \cdot a \cdot \cdots \cdot a$ (n times)
- $a^{-n} = \dfrac{1}{a^n}$
- $a^{n/m} = \sqrt[m]{a^n} = (\sqrt[m]{a})^n$
- $a^{x+y} = a^x\, a^y$
- $a^{x-y} = \dfrac{a^x}{a^y}$
- $(a^x)^y = a^{xy}$
- $(ab)^x = a^x b^x$

Question: What is the best (or most natural) choice of a for $f(x) = a^x$?

A. $a = 1$ B. $a = 2$ C. $a = 3$ D. $a = 10$ E. None of A-D

Answer: $a = e$

Definition

$$\begin{aligned} e &= \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots \\ &= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots \end{aligned}$$

- Approximately, $e = 2.718281828459....$ It is an irrational number.
- See the graphs.

Recall the definition of a one-to-one function.

Definition

Let f be a one-to-one function. Then its inverse function f^{-1} is defined by

$$f^{-1}(b) = a \Leftrightarrow f(a) = b.$$

- See the diagram.
- 1. domain of f^{-1} = range of f
  2. range of f^{-1} = domain of f
- $f^{-1}(f(x)) = x, \qquad f(f^{-1}(x)) = x$
- Example

- How to find the inverse function
  1. Write $y = f(x)$.
  2. Solve for x in terms of y so that $x = f^{-1}(y)$.
  3. Interchange x and y and obtain $y = f^{-1}(x)$.
- Example
- The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.
- See the graph.

Definition

Assume that $a > 0$ and $a \neq 1$. The logarithmic function with base a, denoted by $y = \log_a x$, is defined to be the inverse function of $y = a^x$.

- See the graphs.
- 1. If $a > 1$ then $f(x) = \log_a x$ is increasing.
  2. If $0 < a < 1$ then $f(x) = \log_a x$ is decreasing.
- The functions $f(x) = \log_a x$ and $g(x) = \log_{1/a} x$ are symmetric about the x-axis.

- We write $\log_e x = \ln x$.
- $\log_a(a^x) = x, \qquad a^{\log_a x} = x$
- $a^0 = 1, \qquad \log_a 1 = 0$
- 1. $\log_a(xy) = \log_a x + \log_a y$
  2. $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
  3. $\log_a(x^r) = r\log_a x$
- $a^1 = a, \qquad \log_a a = 1, \qquad \ln e = \log_e e = 1$
- $\log_a b = \dfrac{\ln b}{\ln a}, \qquad \log_a b = \dfrac{\log_c b}{\log_c a}$
- Examples

- Table of the values
- Standard triangles
- $\sin^2\theta + \cos^2\theta = 1, \quad 1 + \tan^2\theta = \sec\theta, \quad 1 + \cot^2\theta = \csc^2\theta$
- $\sin 2\theta = 2\sin\theta\cos\theta$
- $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
- $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}, \qquad \sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

- Solve the trigonometric equations for $0 \leq x < 2\pi$.
  1. $2\sin x - 1 = 0$
  2. $\cos 2x = \sin 2x$
  3. $2\sin^2 x = \cos x + 1$

- We define $y = \sin^{-1} x$, $y = \cos^{-1} x$ and $y = \tan^{-1} x$ to be the inverses of $y = \sin x$, $y = \cos x$ and $y = \tan x$, respectively, after restricting the domains appropriately.
- 1. domain of $\sin^{-1} = [-1, 1]$, range of $\sin^{-1} = [-\pi/2, \pi/2]$
  2. domain of $\cos^{-1} = [-1, 1]$, range of $\cos^{-1} = [0, \pi]$
  3. domain of $\tan^{-1} = (-\infty, \infty)$, range of $\tan^{-1} = [-\pi/2, \pi/2]$
- See the graphs.

- Evaluate or simplify the expressions.
  1. $\sin^{-1}(\sqrt{3}/2)$
  2. $\cos^{-1}(-1/\sqrt{2})$
  3. $\tan^{-1}(-\sqrt{3})$
  4. $\cos(\tan^{-1} x)$
  5. $\cot(\cos^{-1}(x/4))$