3.5 Derivatives as Rates of Change

- $f'(x) =$ (instantaneous) rate of change of f

Example

Assume that $s = f(t)$ is the position function. Then

- $v = \dfrac{ds}{dt} = f'(t)$ *is the velocity, and* $|v| = |f'(t)|$ *is the speed.*
- $a = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2} = f''(t)$ *is the acceleration.*

$$\text{position} \xrightarrow{\ '\ } \text{velocity} \xrightarrow{\ '\ } \text{acceleraton}$$

Example

Let $C(x)$ be the cost function.

- *The average cost is $\bar{C}(x) = C(x)/x$.*
- *The marginal cost is defined to be $C'(x)$. It is the approximate cost to produce one additional item after producing x items.*

3.6 The Chain Rule

- Consider the composite of functions

$$x \xrightarrow{g} u \xrightarrow{f} y.$$

- Then we have

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \frac{\Delta u}{\Delta x} \\ &= \left(\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \right) \left(\lim_{\Delta u \to 0} \frac{\Delta u}{\Delta x} \right) \\ &= \frac{dy}{du} \frac{du}{dx}.\end{aligned}$$

- Equivalently, we have $u = g(x)$ and $y = f(g(x))$ and

$$[f(g(x))]' = f'(g(x))g'(x).$$

Proposition (★★)

Assume that $u = g(x)$ *and* $y = f(g(x))$. *Then we have*

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}, \qquad \text{or}$$

$$[f(g(x))]' = f'(g(x))g'(x).$$

- Examples

- What is the derivative of $f(x) = e^{\sec 3\theta}$?

a. $e^{\sec 3\theta}$

b. $e^{\sec 3\theta} \sec 3\theta$

c. $e^{\sec 3\theta} \sec 3\theta \tan 3\theta$

d. $3e^{\sec 3\theta} \sec 3\theta \tan 3\theta$

e. $9e^{\sec 3\theta} \sec 3\theta \tan 3\theta$

Example

- $[f(ax+b)]' = a\, f'(ax+b)$
- $[(g(x))^n]' = n(g(x))^{n-1}\, g'(x)$
- $[a^x]' = (\ln a)\, a^x$

(pf.) $[a^x]' = [e^{(\ln a)x}]' = (\ln a)e^{(\ln a)x} = (\ln a)\, a^x$