2.6 Continuity

Definition

A function f is continuous at a if

$$\lim_{x \to a} f(x) = f(a).$$

Remark

1. *Practically (and naively), the function f is continuous at a if the graph of f is continuous in the usual sense.*
2. *The definition requires three things:*
   1. *$f(a)$ is defined*
   2. *$\lim_{x \to a} f(x)$ exits*
   3. *Two values are equal*

- Basic functions are continuous at every point in their domains.
  1. A polynomial function is continuous everywhere.
  2. A rational function $y = \dfrac{f(x)}{g(x)}$ is continuous at x for which $g(x) \neq 0$.
  3. $y = \sqrt{x}$: continuous at $x > 0$
  4. $y = \sqrt[3]{x}$: continuous everywhere
  5. $y = \sin x$, $y = \cos x$: continuous everywhere
  6. $y = \tan x$: continuous at $x \neq \frac{\pi}{2} + n\pi$
  7. $y = e^x$: continuous everywhere
  8. $y = \ln x$: continuous at $x > 0$
  9. $y = \tan^{-1} x$: continuous everywhere

- Other examples

Proposition

1. *If f and g are continuous at a, then the following functions are also continuous at a:*

$$f+g, \quad f-g, \quad fg \quad \text{and} \quad \frac{f}{g} \quad (g(a) \neq 0).$$

2. *If f is continuous at $b = \lim_{x \to a} g(x)$, then*

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(b).$$

3. *If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ is continuous at a.*

Definition

1. *A function f is continuous from the right at a if*

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is continuous from the left at a if

$$\lim_{x \to a^-} f(x) = f(a).$$

2. *A function f is continuous on an interval if it is continuous at every point in the interval.*

- Examples

Theorem (Intermediate Value Thm)

Suppose that f is continuous on $[a, b]$, and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists $c \in (a, b)$ such that $f(c) = N$.

- It is an important property of a continuous function.
- Application: the existence of a root in an interval

- The gravitational force exerted by the earth at a distance r from the center is

$$F(r) = \begin{cases} \frac{GMr}{R^3} & \text{if } r < R \\ \frac{GM}{r^2} & \text{if } r \geq R \end{cases}$$

where M is the mass of the earth, R is its radius, and G it the gravitational constant. Is F a continuous function of r?

A. Yes B. No