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Convergence of Kac–Moody Eisenstein series over a function field

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Abstract

We establish everywhere convergence in a natural domain for Eisenstein series on a symmetrizable Kac–Moody group over a function field. Our method is different from that of the affine case which does not directly generalize. In comparison with the analogous result over the real numbers, everywhere convergence is achieved without any additional condition on the root system.

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1 Introduction

Eisenstein series are amongst the most fundamental examples of automorphic forms. They are of significant consequence in number theory, and have applications in other areas of mathematics and mathematical physics. The theory of Eisenstein series on reductive groups was developed by Langlands in [23, 24] and led ultimately to the Principle of Functoriality. Furthermore, as surveyed in [19, Section 8], Eisenstein

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series form the basis of an established technique for proving cases of functoriality, namely, the Langlands–Shahidi method.

Motivated by potential arithmetic applications such as generalizing the Langlands–Shahidi method, we study Eisenstein series over Kac–Moody groups. In a series of pioneering works, affine Kac–Moody Eisenstein series over $\mathbb{R}$ were extensively studied by Garland [8–17]. Some of the results in the affine case were generalized to number fields by Liu [26], rank 2 hyperbolic Eisenstein series over $\mathbb{R}$ by Carbone–Lee–Liu [4], and general symmetrizable Kac–Moody Eisenstein series over $\mathbb{R}$ by Carbone–Garland–Lee–Liu–Miller [3]. Entirety of certain cuspidal Kac–Moody Eisenstein series over $\mathbb{R}$ is established for the affine case in [18], for the rank 2 hyperbolic case in [4], and for more general cases in [5]. Applications of Kac–Moody Eisenstein series to string theory appeared in [6, 7].

Classical Eisenstein series over function fields were studied by Harder [20]. Important results for affine Kac–Moody Eisenstein series over function fields were announced by Braverman–Kazhdan [1], utilising on a geometric formalism introduced by Kapranov [21] and Patnaik [27]. A more algebraic framework was established by Lee–Lombardo in the affine case [25]. In this paper we are interested in general Kac–Moody Eisenstein series over function fields.

As for general Eisenstein series over symmetrizable Kac–Moody groups, it has been a challenging problem to establish everywhere convergence of the series on the natural domain, while almost everywhere convergence of the series follows from the convergence of their constant terms. Over $\mathbb{R}$, the main result of [3] establishes everywhere convergence under some restrictions. Over function fields, everywhere convergence of the affine Kac–Moody Eisenstein series follows from that of its constant term through an explicit description of the unipotent subgroup as was shown in [25]. However, such an explicit description is not available for general Kac–Moody groups, and the method in [25] does not directly generalize.

In this paper, we circumvent the obstacle and establish everywhere convergence of the Kac–Moody Eisenstein series over a function field on a natural domain, i.e., for the Godement range of the spectral parameter and for the Tits cone range of the group element. Our approach involves constructing a subset of positive measure inside the unipotent group through various representation-theoretic properties. It requires convergence of the constant terms as a prerequisite which we obtain through the similar argument in [3].

In order to state our main result more precisely, we need to introduce some notation. Let G be a symmetrizable Kac–Moody group over a function field F with the Iwasawa decomposition $G_{\mathbb{A}} = U_{\mathbb{A}} H_{\mathbb{A}} \mathbb{K}$, where $\mathbb{A}$ is the adèle ring of F, U is a pro-unipotent subgroup, H is a torus, and $\mathbb{K}$ is an analogue of a maximal compact subgroup. Let $\mathfrak{g}$ be the Kac–Moody algebra of G, and $\mathfrak{h}$ be its Cartan subalgebra. Denote by $\rho \in \mathfrak{h}^*_{\mathbb{C}}$ the Weyl vector and by $\mathcal{C}^* \subset \mathfrak{h}^*_{\mathbb{R}}$ the set of strictly dominant weights. Finally, let $H_{\mathfrak{C}} \subset H_{\mathbb{A}}$ be the subgroup corresponding the Tits cone $\mathfrak{C} \subset \mathfrak{h}_{\mathbb{R}}$. The main result of this paper can be stated as follows:

Theorem 1.1 *If $\lambda \in \mathfrak{h}^*_{\mathbb{C}}$ satisfies $Re(\lambda - \rho) \in \mathcal{C}^*$, then the Kac–Moody Eisenstein series $E_\lambda(g)$ converges absolutely for $g \in U_{\mathbb{A}} H_{\mathfrak{C}} \mathbb{K}$.*

In comparison with the real case considered in [3], we do not impose conditions on the root system of G.

We conclude the introduction with an outline of what follows. In Sect. 2, we define an adelic Kac–Moody group $G_{\mathbb{A}}$ over a function field and recall its Iwasawa decomposition (Proposition 2.5). In Sect. 3, we define Eisenstein series on $G_{\mathbb{A}}$ and establish convergence of their constant terms (Proposition 3.8). In Sect. 4, we prove the main result Theorem 1.1 after establishing a crucial lemma (Lemma 4.1).

2 Preliminaries

In this section, we fix notation and recall the Iwasawa decomposition.

Let $A = (a_{ij})_{i,j\in I}$ be a non-singular[1] symmetrizable generalized Cartan matrix, indexed by $I = \{1, \dots, n\}$, let $\left(\mathfrak{h}_{\mathbb{C}}, \Delta, \Delta^{\vee}\right)$ be a realisation of A with $\Delta = \{\alpha_1, ..., \alpha_n\} \subset \mathfrak{h}_{\mathbb{C}}^*$ (resp. $\Delta^{\vee} = \{\alpha_1^{\vee}, ..., \alpha_n^{\vee}\} \subset \mathfrak{h}_{\mathbb{C}}$), and let $\mathfrak{g}_{\mathbb{C}}$ be the associated complex Kac–Moody algebra. Since A is non-singular, we have $\mathfrak{h}_{\mathbb{C}} = \mathrm{Span}_{\mathbb{C}}\{\alpha_1^{\vee}, ..., \alpha_n^{\vee}\}$ (resp. $\mathfrak{h}_{\mathbb{C}}^* = \mathrm{Span}_{\mathbb{C}}\{\alpha_1, ..., \alpha_n\}$). We denote by $Q \subseteq \mathfrak{h}_{\mathbb{C}}^*$ (resp. $Q^{\vee} \subseteq \mathfrak{h}_{\mathbb{C}}$) the integral linear span of Δ (resp. $\Delta^{\vee}$).

Let Φ be the set of roots of $\mathfrak{g}_{\mathbb{C}}$, let Φ_+ (resp. Φ_-) be the set of positive (resp. negative) roots with respect to Δ, and let W be the Weyl group of $\mathfrak{g}_{\mathbb{C}}$ generated by the simple reflections w_i ($i \in I$) corresponding to the simple roots α_i. A root $\alpha \in \Phi$ is called a *real* root if there exists $w \in W$ such that $w\alpha$ is a simple root. A root α which is not real is referred to as *imaginary*. For each real root α, written as $w\alpha_i$ for some $w \in W$ and $i \in I$, its associated coroot is well-defined by the formula $\alpha^{\vee} = w\alpha_i^{\vee}$. For $w \in W$, we define

$$\Phi_w = \Phi_+ \cap w^{-1}\Phi_-. \tag{2.1}$$

Denote by e_i and f_i, $i \in I$, the Chevalley generators of $\mathfrak{g}_{\mathbb{C}}$. Let $\mathcal{U}_{\mathbb{C}}$ be the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$, and let $\mathcal{U}_{\mathbb{Z}} \subseteq \mathcal{U}_{\mathbb{C}}$ be the $\mathbb{Z}$–subalgebra generated by the set:

$$\left\{\frac{e_i^m}{m!}, \frac{f_i^m}{m!}, \binom{h}{m} : i \in I, h \in Q^{\vee}, m \in \mathbb{Z}_{\geq 0}\right\},$$

where $\binom{h}{m} := \frac{h(h-1)\cdots(h-m+1)}{m!}$. Fix a nonzero dominant integral weight $\Lambda \in \mathfrak{h}_{\mathbb{C}}^*$, let (V, π) be the corresponding irreducible highest weight representation of $\mathfrak{g}_{\mathbb{C}}$, and let $v \in V$ be a non-zero highest weight vector. We set $V_{\mathbb{Z}} = \mathcal{U}_{\mathbb{Z}} \cdot v$ and, for a field F, we set $V_F = F \otimes_{\mathbb{Z}} V_{\mathbb{Z}}$.

For all $i \in I$, the Chevalley generators e_i, f_i are locally nilpotent on V_F. Given $i \in I$ and $r, s \in F$, we introduce:

$$u_{\alpha_i}(r) = \exp(\pi(re_i)), \quad u_{-\alpha_i}(s) = \exp(\pi(sf_i)) \in \mathrm{Aut}\,(V_F)\,. \tag{2.2}$$

[1] This condition excludes the affine case which is well understood.

We define G_F^0 to be the subgroup of Aut (V_F) generated by $u_{\alpha_i}(r)$ and $u_{-\alpha_i}(s)$, with r, s varying in F and i varying in I.

Choose a coherently ordered basis $\mathfrak{X} = \{v_1, v_2, \dots\}$ of $V_{\mathbb{Z}}$, as defined in [2, Section 5]. For $t \in \mathbb{Z}_{>0}$, let $A_t \subset G_F^0$ be the following subset:

$$A_t = \left\{ g \in G_F^0 : g v_i = v_i,\ i = 1, 2, \dots, t \right\}.$$

The sets A_t, indexed by $t \in \mathbb{Z}_{>0}$, form a base of open neighborhoods of the identity to define a topology τ on G_F^0.

Definition 2.1 The *Kac–Moody group* G_F is the completion of G_F^0 with respect to τ.

The group G_F in Definition 2.1 is usually referred to as the representation theoretic construction of a Kac–Moody group. One may define a Kac–Moody group through other means, for example, see [22, 28].

Let B_F^0 be the subgroup of G_F^0 consisting of the elements represented by upper triangular matrices with respect to $\mathfrak{X}$. For $i \in I$ and $s \in F^\times$, we set:

$$w_i(s) = u_{\alpha_i}(s) u_{-\alpha_i}(-s^{-1}) u_{\alpha_i}(s) \in G_F^0, \tag{2.3}$$

where u_{α_i} is as in Eq. (2.2). For a real root α, choose once for all an expression $\alpha = w_{i_1} \cdots w_{i_\ell} \alpha_i$ for some $i \in I$, and define the corresponding one-parameter subgroup

$$u_\alpha(s) = w u_{\alpha_i}(s) w^{-1} \in \mathrm{Aut}(V_F) \qquad (s \in F),$$

where $w = w_{i_1}(1) \cdots w_{i_\ell}(1)$.

For $t \in \mathbb{Z}_{>0}$ we let V_t be the span of the $v_m \in \mathfrak{X}$ for $m \le t$, so that $B_F^0 V_t \subseteq V_t$ for each t. Let $B_{F,t}$ be the image of B_F^0 in $\mathrm{Aut}(V_t)$. For $t' \ge t$, we have surjective homomorphisms $\pi_{tt'} : B_{F,t'} \longrightarrow B_{F,t}$. We define $B_F \le G_F$ to be the projective limit of the projective family $\{B_{F,t}, \pi_{tt'}\}$.

Let N_F be the subgroup of G_F generated by $w_i(s), i \in I, s \in F^\times$, and let $U_F \subset B_F$ be the subgroup of elements acting as unipotent upper triangular matrices with respect to $\mathfrak{X}$. We define, for $i \in I$ and $s \in F^\times$,

$$h_i(s) = w_i(s) w_i(1)^{-1}. \tag{2.4}$$

We denote by $H_F \subset G_F$ the group generated by $h_i(s)$ as i varies over I and s varies over $F^\times$.

An important fact about the structure of G_F is established by the following proposition.

Proposition 2.2 [2, Theorem 6.1] *The pair* (B_F, N_F) *is a* BN*-pair for the group* G_F.

We identify the Weyl group W with the group $N_F/(B_F \cap N_F)$ in such a way that w_i is represented by $w_i(1)$. From a standard property of a BN-pair we obtain

Corollary 2.3 *The group G_F admits a Bruhat decomposition, that is:*

$$G_F = \bigcup_{w \in W} B_F w B_F, \tag{2.5}$$

in which the union is disjoint.

From now on, let F be the function field of a smooth projective curve X defined over a finite field $\mathbb{F}_q$. For a place v of F, we denote the corresponding completion by F_v. Denoting by $\mathcal{O}_v$ the ring of integers in F_v, we define $G^0_{\mathcal{O}_v}$ to be the subgroup of $\mathrm{Aut}\left(V_{F_v}\right)$ generated by $u_{\alpha_i}(r)$ and $u_{-\alpha_i}(s)$, with r, s varying in $\mathcal{O}_v$ and i varying in I, and define $G_{\mathcal{O}_v}$ to be its completion. For convenience, we will write:

$$G_v = G_{F_v}, \quad U_v = U_{F_v}, \quad H_v = H_{F_v}, \quad K_v = G_{\mathcal{O}_v}.$$

A proof of the following proposition is essentially the same as the affine case, and we do not reproduce it here.

Proposition 2.4 [9] *For each place v of the field F, the group G_v admits an Iwasawa decomposition. That is:*

$$G_v = U_v H_v K_v. \tag{2.6}$$

In particular, we may write each $g_v \in G_v$ as a product $g_v = u_v h_v k_v$ with $u_v \in U_v$, $h_v \in H_v$, and $k_v \in K_v$. We note that such an expression for g_v is not unique. For a choice of expression $g_v = u_v h_v k_v$, we will use the notation $\mathrm{Iw}_{H_v}(g)$ to denote h_v.

Let $\mathcal{V}$ be the set of places of F. The adèle ring $\mathbb{A}$ of F is the restricted direct product over $\mathcal{V}$ of F_v with respect to the subrings $\mathcal{O}_v$. We introduce the following subgroups:

$$U_{\mathbb{A}} = \prod_{v \in \mathcal{V}}{}' U_v, \quad H_{\mathbb{A}} = \prod_{v \in \mathcal{V}}{}' H_v, \quad \mathbb{K} = \prod_{v \in \mathcal{V}} K_v, \tag{2.7}$$

in which the first product is restricted with respect to $U_v \cap K_v$, the second product is restricted with respect to $H_v \cap K_v$, and the third product is unrestricted. For later use, we also define $B_{\mathbb{A}} \subset G_{\mathbb{A}}$ to be the restricted direct product $\prod'_{v \in \mathcal{V}} B_v$ with respect to $B_v \bigcap K_v$.

Taking a product of Eq. (2.6) over $v \in \mathcal{V}$, we deduce the following proposition.

Proposition 2.5 *The group $G_{\mathbb{A}}$ admits an Iwasawa decomposition, that is:*

$$G_{\mathbb{A}} = U_{\mathbb{A}} H_{\mathbb{A}} \mathbb{K}. \tag{2.8}$$

As in the local case, an expression $g = uhk$ as per Eq. (2.8) is not uniquely determined. For $g \in G_{\mathbb{A}}$ and a choice of expression $g = uhk$, we will use the notation

$$\mathrm{Iw}_H(g) = h. \tag{2.9}$$

3 Kac–Moody Eisenstein series

In this section the Kac–Moody Eisenstein series will be defined, and convergence of its constant term will be established. As a result, we will obtain almost everywhere convergence of the Eisenstein series.

We maintain the notations from Sect. 2. Furthermore we let $\langle\ ,\ \rangle$ denote the natural pairing between $\mathfrak{h}_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}^*$, so that $\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij}$ is the (i, j)-entry of the generalized Cartan matrix A. Recall that the fundamental weights form the basis of $\mathfrak{h}_{\mathbb{C}}^*$ dual to $\Delta^\vee \subset \mathfrak{h}_{\mathbb{C}}$. We denote the integral span of the fundamental weights by $P \subset \mathfrak{h}_{\mathbb{C}}^*$. Since A is non-singular, there exists a unique $\rho \in \mathfrak{h}_{\mathbb{C}}^*$ such that, for all $i \in I$, we have $\langle \rho, \alpha_i^\vee \rangle = 1$. Set $\mathfrak{h}_{\mathbb{R}} = \mathrm{Span}_{\mathbb{R}}\{\alpha_1^\vee, \ldots, \alpha_r^\vee\}$ (resp. $\mathfrak{h}_{\mathbb{R}}^* = \mathrm{Span}_{\mathbb{R}}\{\alpha_1, \ldots, \alpha_r\}$).

Recall that F is the function field of a smooth projective curve X defined over a finite field $\mathbb{F}_q$ and that, for a place v of F, we denote the corresponding completion by F_v. For clarity, we use the notation $h_{i,v}$ for h_i in Eq. (2.4) when it is applied to an element of $F_v^\times$. By construction we may write each $h \in H_{\mathbb{A}}$ as $h = (h_v)_{v \in \mathcal{V}}$, where $h_v = \prod_{i \in I} h_{i,v}(t_{i,v}) \in H_v$ and $t_{i,v} \in F_v^\times$. For $v \in \mathcal{V}$, we denote by $|\cdot|_v$ the normalized absolute value on F_v. For $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$, we write

$$h^\lambda = \prod_{i \in I} \prod_{v \in \mathcal{V}} |t_{i,v}|_v^{\langle \lambda, \alpha_i^\vee \rangle}. \tag{3.1}$$

By definition, for all but finitely many $v \in \mathcal{V}$, we have $h_v \in H_v \cap K_v$.

Lemma 3.1 *For $v \in \mathcal{V}$, if $h_v = \prod_{i \in I} h_{i,v}(t_{i,v}) \in H_v \cap K_v$ then $t_{i,v} \in \mathcal{O}_v{}^\times$ for all $i \in I$.*

Proof For $i \in I$, denote the corresponding fundamental weight by Λ_i. Recall that we fixed a regular dominant integral weight $\Lambda = \sum_{i \in I} m_i \Lambda_i$ ($m_i \in \mathbb{Z}_{>0}$) to construct G_v is in Sect. 2. Choose a highest weight vector $v_0 \in V_{\mathbb{Z}}$ (with weight Λ) and consider $1 \otimes v_0 \in V_{F_v} = F_v \otimes V_{\mathbb{Z}}$. Then we have:

$$h_v(1 \otimes v_0) = \prod_{i \in I} t_{i,v}^{m_i} \otimes v_0.$$

Since K_v preserves $V_{\mathcal{O}_v}$, we deduce that $\sum_{i \in I} m_i \,\mathrm{ord}_v(t_{i,v}) \geq 0$, where ord_v is the v-adic valuation on F_v. On the other hand, since $H_v \cap K_v$ is a group, we have $h_v^{-1} = \prod_{i \in I} h_{\alpha_i}(t_{i,v}^{-1}) \in H_v \cap K_v$. By the same argument, we deduce that $\sum_{i \in I} m_i \,\mathrm{ord}_v(t_{i,v}) \leq 0$. Consequently, we have

$$\sum_{i \in I} m_i \,\mathrm{ord}_v(t_{i,v}) = 0. \tag{3.2}$$

Since Λ is regular, i.e., $m_j > 0$ for each j, we can choose a weigh vector $v_j \in V_{\mathbb{Z}}$ with weight $\Lambda - \alpha_j$ for each $j \in I$. By applying h_v and h_v^{-1} to v_j, we obtain

$$\sum_{i \in I} (m_i - a_{ij}) \,\mathrm{ord}_v(t_{i,v}) = 0, \tag{3.3}$$

where $A = (a_{ij})$ is the generalized Cartan matrix. Combining (3.2) and (3.3), we have

$$\sum_{i \in I} a_{ij} \operatorname{ord}_v(t_{i,v}) = 0, \qquad \text{for each } j.$$

Since A is nonsingular, we have $\operatorname{ord}_v(t_{i,v}) = 0$ and $t_{i,v} \in \mathcal{O}^\times$ for all $i \in I$, as desired. □

By Lemma 3.1, the infinite product in Eq. (3.1) converges. Denoting the adèle $(t_{i,v})_{v \in \mathcal{V}} \in \mathbb{A}$ by t_i and the adelic norm by $|\cdot|$, we have

$$h^\lambda = \prod_i |t_i|^{\langle \lambda, \alpha_i^\vee \rangle}, \qquad \lambda \in \mathfrak{h}_\mathbb{C}^*. \tag{3.4}$$

Lemma 3.2 *The function $g \mapsto \mathrm{Iw}_H(g)^\lambda$ is well-defined on $G_\mathbb{A}$ for $\lambda \in \mathfrak{h}_\mathbb{C}^*$, where the notation $\mathrm{Iw}_H(g)$ is introduced in (2.9).*

Proof Assume that $g = u_1h_1k_1 = u_2h_2k_2$ with respect to the Iwasawa decomposition. Rearranging $u_1h_1k_1 = u_2h_2k_2$, we get

$$k_1k_2^{-1} = (u_1h_1)^{-1}u_2h_2 \in B_\mathbb{A} \cap \mathbb{K}.$$

Since $B_\mathbb{A} \cap \mathbb{K} = (U_\mathbb{A} \cap \mathbb{K}) \rtimes (H_\mathbb{A} \cap \mathbb{K})$, we may write $k_1k_2^{-1} = u_3h_3$ for some $u_3 \in U_\mathbb{A} \cap \mathbb{K}$ and $h_3 \in H_\mathbb{A} \cap \mathbb{K}$. Therefore, since $H_\mathbb{A}$ normalizes $U_\mathbb{A}$, we may write

$$u_2h_2k_2 = u_1h_1k_1 = u_1h_1u_3h_3k_2 = u_1u_4h_1h_3k_2$$

for some $u_4 \in U_\mathbb{A}$, and obtain $h_2 = h_1h_3$ from $H_\mathbb{A} \cap U_\mathbb{A}$ being trivial. To conclude, note that

$$\mathrm{Iw}_H(u_2h_2k_2)^\lambda = h_2^\lambda = (h_1h_3)^\lambda = h_1^\lambda h_3^\lambda = h_1^\lambda = \mathrm{Iw}_H(u_1h_1k_1)^\lambda.$$

□

For a place v of F, the natural inclusion $F \to F_v$ induces a map $\iota_v : G_F \to G_v$. Therefore, we get a map $\iota = \prod \iota_v : G_F \to \prod G_v$. Define $\Gamma = \{\gamma \in G_F : \iota(\gamma) \in G_\mathbb{A}\}$. Whenever appropriate, elements of Γ will be identified with their images in $G_\mathbb{A}$ via ι.

Remark 3.3 As the definition of Γ suggests, it is not always true that the image of ι is contained in $G_\mathbb{A}$. In the affine case, it is discussed, for example, in [25, Example 3.13].

Now we define the Kac–Moody Eisenstein series.

Definition 3.4 Given $\lambda \in \mathfrak{h}_\mathbb{C}^*$, the *Eisenstein series* E_λ is defined to be:

$$E_\lambda(g) \quad = \sum_{\gamma \in \Gamma \cap B_\mathbb{A} \backslash \Gamma} \mathrm{Iw}_H(\gamma g)^{\lambda+\rho}, \qquad g \in G_\mathbb{A}. \tag{3.5}$$

Note that we may regard E_λ as a function on $(\Gamma \cap U_{\mathbb{A}} \backslash U_{\mathbb{A}}) \times H_{\mathbb{A}}$. In the rest of this section we will define the constant term of E_λ and establish convergence of the constant term. The result will be crucial in proving the main theorem.

As in [11, 25], the space $\Gamma \cap U_{\mathbb{A}} \backslash U_{\mathbb{A}}$ admits a projective limit measure du, which is a $U_{\mathbb{A}}$-invariant probability measure.

Definition 3.5 Given $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$, the *constant term* of E_λ is defined to be:

$$E_\lambda^\sharp(g) = \int_{\Gamma \cap U_{\mathbb{A}} \backslash U_{\mathbb{A}}} E_\lambda(ug)du, \qquad g \in G_{\mathbb{A}}. \tag{3.6}$$

Remark 3.6 In the proof of Proposition 3.8, we only need $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$. In this case, we interpret the infinite sum E_λ as a function taking values in $\mathbb{R}_+ \cup \{\infty\}$, and $E_\lambda^\sharp$ is well-defined. After convergence is established for $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$, the above definition of $E_\lambda^\sharp$ will be valid for $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ by dominance.

Applying the Gindikin–Karpelevich formula, a formal calculation as in [11, 23] yields

$$E_\lambda^\sharp(g) = \sum_{w \in W} \mathrm{Iw}_H(g)^{w\lambda+\rho} c(\lambda, w), \quad c(\lambda, w) = \prod_{\alpha \in \Phi_w} \frac{\zeta_X\left(\langle \lambda, \alpha^\vee \rangle\right)}{\zeta_X\left(1 + \langle \lambda, \alpha^\vee \rangle\right)}, \tag{3.7}$$

where Φ_w is defined in Eq. (2.1) and $\zeta_X(s)$ is the zeta function of the smooth projective curve X defined over the finite field $\mathbb{F}_q$.

We let $\mathcal{C} = \{x \in \mathfrak{h}_{\mathbb{R}} : \langle \alpha_i, x \rangle > 0, \ i \in I\}$ (resp. $\mathcal{C}^* = \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* : \langle \lambda, \alpha_i^\vee \rangle > 0, \ i \in I\}$), and let $\mathfrak{C} = \mathrm{int}\left(\bigcup_{w \in W} w\bar{\mathcal{C}}\right)$ (resp. $\mathfrak{C}^* = \mathrm{int}\left(\bigcup_{w \in W} w\bar{\mathcal{C}}^*\right)$) where $\bar{\mathcal{C}}$ (resp. $\bar{\mathcal{C}}^*$) denotes the closure. For $h = (h_v)_{v \in \mathcal{V}} \in H_{\mathbb{A}}$ with $h_v = \prod_{i \in I} h_{i,v}(t_{i,v})$, we set $t_i = (t_{i,v})_{v \in \mathcal{V}} \in \mathbb{A}^\times$ for $i \in I$ as before and write $h = \prod_{i \in I} h_i(t_i)$ by abusing notation slightly. Using the map $(t_1, \ldots, t_n) \mapsto \prod_{i \in I} h_{\alpha_i}(t_i)$, we may identify $(\mathbb{A}^\times)^n$ with $H_{\mathbb{A}}$. In particular, $H_{\mathbb{A}}$ has the Haar measure.

We define $H_{\mathcal{C}}$ (resp. $H_{\mathfrak{C}}$) to be

$$\left\{ \prod_{i \in I} h_i(t_i) \in H_{\mathbb{A}} : \sum_{i \in I} \log |t_i| \, \alpha_i^\vee \in \mathcal{C} \text{ (resp.} \mathfrak{C}) \right\}. \tag{3.8}$$

We note from (3.1) that

$$H_{\mathcal{C}} = \left\{ h \in H_{\mathbb{A}} : h^{\alpha_i} > 1, i \in I \right\}. \tag{3.9}$$

More generally, for any $\mathcal{K} \subset \mathfrak{C}$, we define

$$H_{\mathcal{K}} = \left\{ \prod_{i \in I} h_i(t_i) \in H_{\mathbb{A}} : \sum_{i \in I} \log |t_i| \, \alpha_i^\vee \in \mathcal{K} \right\}.$$

The following is a function-field analogue of [3, Theorem 3.5]. Since the proof is similar, we omit it.

Lemma 3.7 *If $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ satisfies $\mathrm{Re}(\lambda) \in \mathcal{C}^*$, then, for any $M > 0$ and any compact $\mathcal{K} \subset \mathfrak{C}$, the sum $\sum_{w \in W} M^{\ell(w)} h^{w\lambda}$ converges absolutely and uniformly for h in $H_{\mathcal{K}}$.*

Now we state the main result of this section.

Proposition 3.8 *If $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ satisfies $\mathrm{Re}(\lambda - \rho) \in \mathcal{C}^*$, then $E_{\lambda}^{\sharp}(g)$ converges absolutely for $g \in U_{\mathbb{A}} H_{\mathfrak{C}} \mathbb{K}$. Moreover, for any compact $\mathcal{K} \subset \mathfrak{C}$, the convergence is uniform for $\mathrm{Iw}_H(g)$ in $H_{\mathcal{K}}$.*

Proof For a smooth projective curve X defined over $\mathbb{F}_q$, one has $\lim_{\mathrm{Re}(s) \to \infty} \frac{\zeta_X(s)}{\zeta_X(s+1)} = 1$. This follows immediately from the well-known fact that $\zeta_X(s)$ can be written as a product $\prod_{n=1}^{\infty} \left(1 - \frac{1}{q^{ns}}\right)^{-a_n}$, where a_n is the number of closed points of degree n. Alternatively, this follows directly from the well-known Weil conjecture

$$\zeta_X(s) = \frac{\prod_{i=1}^{2g}(1 - \beta_i q^{-s})}{(1 - q^{-s})(1 - q^{1-s})},$$

where g is the genus of X, and β_i, $i = 1, \ldots, 2g$, are algebraic integers with absolute value $\sqrt{q}$.

We may assume that $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$. Fix a constant $M > 1$, and let $S \geq 1$ be a constant such that $|\zeta_X(s)/\zeta_X(s+1)| \leq M$ for all $\mathrm{Re}(s) > S$. Since, for all $i \in I$, we have $\langle \lambda, \alpha_i^{\vee} \rangle > 1$, we deduce that $\langle \lambda, \alpha^{\vee} \rangle > S$ for all but finitely many positive roots α. Therefore, for all $w \in W$, the function $c(\lambda, w)$ defined in Eq. (3.7) is bounded by $M^{\ell(w)}$ times a constant independent of w, since the cardinality of Φ_w is equal to $\ell(w)$. Now the assertion of the proposition follows from Lemma 3.7. □

As a corollary, we obtain almost everywhere convergence of $E_{\lambda}(g)$ by applying Tonelli's theorem.

Corollary 3.9 *If $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ satisfies $Re(\lambda - \rho) \in \mathcal{C}^*$, then, for any compact $\mathcal{K} \subset \mathfrak{C}$, there exists a measure zero subset S_0 of $(\Gamma \cap U_{\mathbb{A}}) \backslash U_{\mathbb{A}} H_{\mathcal{K}}$ such that $E_{\lambda}(g)$ converges absolutely for $g \in U_{\mathbb{A}} H_{\mathcal{K}} \mathbb{K}$ off the set $S_0 \mathbb{K}$.*

4 Everywhere convergence

In this section, we establish everywhere convergence of E_{λ}.

We start with a lemma which is crucial for the proof of the main theorem.

Lemma 4.1 *If $h \in H_{\bar{\mathfrak{C}}}$ and $U_{\mathbb{K}} := \mathbb{U}_{\mathbb{A}} \cap \mathbb{K}$, then the image of $hU_{\mathbb{K}}h^{-1}$ in $\Gamma \cap U_{\mathbb{A}} \backslash U_{\mathbb{A}}$ has positive measure.*

Proof Since a fundamental domain of $\mathbb{A}/F$ can be chosen inside $\prod_{v \in \mathcal{V}} \mathcal{O}_v$, we see that $U_{\mathbb{K}}$ has positive measure in $\Gamma \cap U_{\mathbb{A}} \backslash U_{\mathbb{A}}$. By assumption, we have that

$$h = w^{-1} h_1 w$$

for some $w \in W$ and $h_1 \in H_{\bar{\mathcal{C}}}$. Write $U^w = U_{\mathbb{A}} \cap w^{-1} U_{\mathbb{A}} w$ and $U_w = U_{\mathbb{A}} \cap w^{-1} U_{\mathbb{A}}^- w$, where $U_{\mathbb{A}}^-$ is the opposite unipotent group. Then we may decompose the space $U_{\mathbb{A}}$ as a product:

$$U_{\mathbb{A}} = U^w U_w. \tag{4.1}$$

Note that $U_w \cong \mathbb{A}^{\ell(w)}$. Likewise,

$$U_{\mathbb{K}} = U_{\mathbb{K}}^w U_{w,\mathbb{K}},$$

where $U_{\mathbb{K}}^w := U^w \cap \mathbb{K}$ and $U_{w,\mathbb{K}} := U_w \cap \mathbb{K}$.

Recall that we fixed a regular dominant integral weight $\Lambda \in \mathfrak{h}_{\mathbb{C}}^*$ and that (V, π) is the irreducible highest weight representation of $\mathfrak{g}_{\mathbb{C}}$ with highest weight Λ. The group G_v is defined as a subgroup of $\mathrm{Aut}(V_{F_v})$ for each $v \in \mathcal{V}$, and the adelic group $G_{\mathbb{A}}$ is a restricted product of G_v. For each $v \in \mathcal{V}$, let $\mathfrak{X}_v$ denote the coherently ordered basis for V_{F_v} induced from $\mathfrak{X}$ for $V_{\mathbb{Z}}$, and consider $x_{v,\mu} \in \mathfrak{X}_v$ with weight μ. Let U_v^w be the local component of U^w at the place v. For $u_v \in U_v^w$, we have that $w u_v w^{-1} \in U_v$, thus we can write

$$w u_v x_{v,\mu} = (w u_v w^{-1}) w x_{v,\mu} = \sum_{\nu \succeq w\mu} \tilde{x}_{v,\nu}, \qquad \text{or} \qquad u_v x_{v,\mu} = \sum_{\nu \succeq w\mu} w^{-1} \tilde{x}_{v,\nu},$$

where $\tilde{x}_{v,\nu}$ are of weight ν and $\succeq$ is the usual partial order, i.e., $\nu \succeq w\mu$ if and only if $\nu - w\mu$ is a non-negative integral sum of simple roots α_i $(i \in I)$. For an integral weight λ of $\mathfrak{g}_{\mathbb{C}}$ and $h_v = \prod_{i \in I} h_i(t_{i,v}) \in H_v$, $t_{i,v} \in F_v^\times$, define

$$h_v^\lambda = \prod_{i \in I} t_{i,v}^{\langle \lambda, \alpha_i^\vee \rangle} \in F_v^\times.$$

Write $h = (h_v)_{v \in \mathcal{V}} = (w^{-1} h_{1,v} w)_{v \in \mathcal{V}} \in H_{\overline{\mathfrak{C}}}$. Then we have

$$\begin{aligned} h_v^{-1} u_v h_v x_{v,\mu} &= h_v^\mu \cdot h_v^{-1} u_v x_{v,\mu} = h_v^\mu \cdot w^{-1} h_{1,v}^{-1} w u_v x_{v,\mu} \\ &= h_v^\mu \cdot w^{-1} h_{1,v}^{-1} \sum_{\nu \succeq w\mu} \tilde{x}_{v,\nu} = \sum_{\nu \succeq w\mu} h_v^{\mu - w^{-1}\nu} \cdot w^{-1} \tilde{x}_{v,\nu} \\ &= \sum_{\nu \succeq w\mu} h_{1,v}^{w\mu - \nu} \cdot w^{-1} \tilde{x}_{v,\nu} \end{aligned}$$

at each place $v \in \mathcal{V}$. Considering u_v as an infinite matrix, we see that the each entry in the block corresponding to (μ, ν) has been multiplied by $h_{1,v}^{w\mu-\nu}$ after conjugation by h_v^{-1}. Globally, it follows from (3.9) that $h_1^{w\mu-\nu} \leq 1$. In particular, taking $u = (u_v)_{v \in \mathcal{V}} \in U_{\mathbb{K}}^w$, we see that

$$h^{-1} U_{\mathbb{K}}^w h \subset U_{\mathbb{K}}^w.$$

It follows that $U_{\mathbb{K}}^{w} \subset hU_{\mathbb{K}}^{w}h^{-1}$ hence

$$U_{\mathbb{K}}^{w} \cdot hU_{w,\mathbb{K}}h^{-1} \subset hU_{\mathbb{K}}h^{-1}.$$

Since $U_w \cong \mathbb{A}^{\ell(w)}$ is locally compact, its open compact subgroups $hU_{w,\mathbb{K}}h^{-1}$ and $U_{w,\mathbb{K}}$ are commensurable. As $U_{\mathbb{K}} = U_{\mathbb{K}}^{w}U_{w,\mathbb{K}}$ has positive measure in $\Gamma \cap U_{\mathbb{A}}$, so do $U_{\mathbb{K}}^{w} \cdot hU_{w,\mathbb{K}}h^{-1}$ and $hU_{\mathbb{K}}h^{-1}$. □

Now we prove the main theorem.

Theorem 4.2 (Theorem 1.1) *If $\lambda \in \mathfrak{h}_{\mathbb{C}}^{*}$ satisfies $Re(\lambda - \rho) \in \mathcal{C}^{*}$, then the series $E_{\lambda}(g)$ converges absolutely for $g \in U_{\mathbb{A}}H_{\mathfrak{C}}\mathbb{K}$.*

Proof We may consider E_{λ} as a function on $(\Gamma \cap U_{\mathbb{A}} \backslash U_{\mathbb{A}}) \times H_{\mathbb{A}}$. Assume that $E_{\lambda}(uh) = \infty$ for some $u \in U_{\mathbb{A}}$ and $h \in H_{\mathfrak{C}}$. By Lemma 4.1, $U' := hU_{\mathbb{K}}h^{-1}$ has positive measure in $\Gamma \cap U_{\mathbb{A}} \backslash U_{\mathbb{A}}$. Then we have

$$E_{\lambda}(uu'h) = E_{\lambda}(uh(h^{-1}u'h)) = E_{\lambda}(uh) = \infty$$

for any $u' \in U'$. Since finite dimensional unipotent groups are unimodular and $\Gamma \cap U_{\mathbb{A}} \backslash U_{\mathbb{A}}$ admits the projective limit measure, uU' and U' have the same measure in $\Gamma \cap U_{\mathbb{A}} \backslash U_{\mathbb{A}}$, which is positive. This is a contradiction to Corollary 3.9 and completes the proof. □

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References

1. Braverman, A., Kazhdan, D.: Representations of affine Kac–Moody groups over local and global fields: a suvery of some recent results. European Congress of Mathematics. European Mathematical Society, Zürich, pp. 91–117 (2013)
2. Carbone, L., Garland, H.: Existence of lattices in Kac–Moody groups over finite fields. Commun. Contemp. Math. **5**(5), 1–55 (2003)
3. Carbone, L., Garland, H., Lee, K.-H., Liu, D., Miller, S.: On the convergence of Kac–Moody Eisenstein series. Am. J. Math. **146**(5), 1253–1274 (2024)
4. Carbone, L., Lee, K.-H., Liu, D.: Eisenstein series on rank 2 hyperbolic Kac–Moody groups. Math. Ann. **367**, 1173–1197 (2017)
5. Carbone, L., Lee, K.-H., Liu, D.: Entirety of cuspidal Eisenstein series on Kac–Moody groups. Algebra Number Theory **16**(5), 1099–1119 (2022)
6. Fleig, P., Kleinschmidt, A.: Eisenstein series for infinite-dimensional U-duality groups. J. High Energy Phys. **2012**, 54 (2012)
7. Fleig, P., Kleinschmidt, A., Persson, D.: Fourier expansions of Kac–Moody Eisenstein series and degenerate Whittaker vectors. Commun. Number Theor. Phys. **08**, 41–100 (2014)
8. Garland, H.: The arithmetic theory of loop algebras. J. Algebra **53**(2), 480–551 (1978)
9. Garland, H.: The arithmetic theory of loop groups. Inst. Hautes Études Sci. Publ. Math. **52**, 5–136 (1980)
10. Garland, H.: Eisenstein series on arithmetic quotients of loop groups. Math. Res. Lett. **6**(5–6), 723–733 (1999)
11. Garland, H.: Certain Eisenstein series on loop groups: convergence and the constant term. Algebraic Groups and Arithmetic, pp. 275–319. Tata Institute of Fundamental Research, Mumbai (2004)

12. Garland, H.: Absolute convergence of Eisenstein series on loop groups. Duke Math. J. **135**(2), 203–260 (2006)
13. Garland, H.: Eisenstein series on loop groups: Maass-Selberg relations. I. Algebraic groups and homogeneous spaces. Tata Institute of Fundamental Research Studies in Mathematics, pp. 275–300. Tata Institute of Fundamental Research, Mumbai (2007)
14. Garland, H.: Eisenstein series on loop groups: Maass-Selberg relations. II. Am. J. Math. **129**(3), 723–784 (2007)
15. Garland, H.: Eisenstein series on loop groups: Maass-Selberg relations. III. Am. J. Math. **129**(5), 1277–1353 (2007)
16. Garland, H.: Eisenstein series on loop groups: Maass-Selberg relations. IV. Lie algebras, vertex operator algebras and their applications, Contemporary Mathematics, vol. 442, pp. 115–158. American Mathematical Society, Providence, RI (2007)
17. Garland, H.: On extending the Langlands-Shahidi method to arithmetic quotients of loop groups. Representation theory and mathematical physics, Contemporary Mathematics, vol. 557, pp. 151–167. American Mathematical Society, Providence, RI (2011)
18. Garland, H., Miller, S.D., Patnaik, M.M.: Entirety of cuspidal Eisenstein series on loop groups. Am. J. Math. **139**, 461–512 (2017)
19. Gelbart, S.S., Miller, S.D.: Riemann's zeta function and beyond,. Bull. Am. Math. Soc. (N.S.) **41**(1), 59–112 (2004)
20. Harder, G.: Chevalley groups over function fields and automorphic forms. Ann. Math. **2**(100), 249–306 (1974)
21. Kapranov, M.: The elliptic curve in the S-duality theory and Eisenstein series for Kac–Moody groups, math.AG/0001005
22. Kumar, S.: Kac–Moody Groups, Their Flag Varieties and Representation Theory, Progress in Mathematics 204. Birkhäuser Boston Inc, Boston, MA (2002)
23. Langlands, R.P.: Euler Products, Yale Mathematical Monographs 1. Yale University Press, New Haven (1971)
24. Langlands, R.P.: On the Functional Equations Satisfied by Eisenstein Series. Lecture Notes in Mathematics, vol. 544. Springer, Berlin (1976)
25. Lee, K.-H., Lombardo, P.: Eisenstein series on affine Kac–Moody groups over function fields. Trans. Am. Math. Soc. **366**(4), 2121–2165 (2014)
26. Liu, D.: Eisenstein series on loop groups. Tran. Am. Math. Soc. **367**(3), 2079–2135 (2015)
27. Patnaik, M.M.: Geometry of loop Eisenstein series. Ph.D. thesis, Yale University (2008)
28. Tits, J.: Uniqueness and presentation of Kac–Moody groups over fields. J. Algebra **105**, 542–573 (1987)