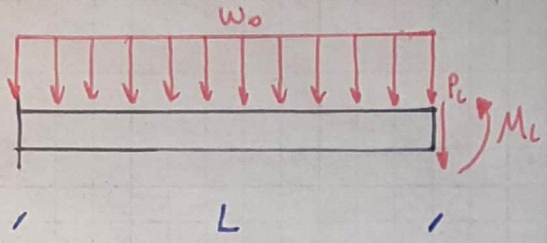
CESG 506 | Homework 4 | Kristinn Hlíðar ~~Grétarsson~~

4-1



1. The potential energy stored in the deformed beam is

$$\Pi(v) = \frac{1}{2}\int_L EI\,\Phi^2\,dx - \int_L w_o\,v\,dx + \bar{M}_L\,\theta(L) - \bar{P}_L\,v(L)$$

w/ $\theta(x) = v'(x)$ & $\Phi(x) = -\theta'(x) = -v''(x)$

Find the Euler equations i.e. strong form represented by $\Pi(v) \to \min$
Use $M(x) = EI\,\Phi(x)$ & $V(x) = M'(x)$ to simplify your expressions

$$\delta\Pi(v,\delta v) = \int_L \underbrace{EI\,\Phi}_{M(x)}\,\delta\Phi - \int_L w_o\,\delta v\,dx + \bar{M}_L\,\delta\theta(L) - \bar{P}_L\,\delta v(L) = 0$$

$$= -M\delta\theta\Big|_0^L - \int_L -\frac{dM}{dx}\,\delta\theta\,dx - \int_L w_o\,\delta v\,dx + \bar{M}_L\,\delta\theta(L) - \bar{P}_L\,\delta v(L) = 0$$

$$= -M\delta\theta\Big|_0^L + V\delta v\Big|_0^L - \int_L \frac{dV}{dx}\,\delta v\,dx - \int_L w_o\,\delta v\,dx + \bar{M}_L\,\delta\theta(L) - \bar{P}_L\,\delta v(L) = 0$$

$$= -\int_L \left(\frac{dV}{dx} + w_o\right)\delta v\,dx - \big(M(L)\delta\theta(L) - M(0)\delta\theta(0)\big) + \big(V(L)\delta v(L) - V(0)\delta v(0)\big)\ldots$$

$$\ldots + \bar{M}_L\,\delta\theta(L) - \bar{P}_L\,\delta v(L) = 0$$

$$= -\int_L \underbrace{(V' + w_o)}_{=0 \text{ in } \Omega,\ 0\le x\le L}\delta v\,dx + \underbrace{(\bar{M}_L - M(L))}_{=0}\delta\theta(L) + \underbrace{M(0)\,\delta\theta(0)}_{=0} + \underbrace{(V(L) - \bar{P}_L)}_{=0}\delta v(L) - V(0)\underbrace{\delta v(0)}_{=0} = 0$$

$$\Rightarrow \quad V'(x) + w_o = M''(x) + w_o = EI\,\Phi''(x) + w_o = -EI\,\theta'''(x) + w_o$$

$$-EI\,v''''(x) + w_o = 0$$

$$\underline{\underline{EI\,v^{iv}(x) = w_o}} \qquad v(0) = 0 \qquad v'(0) = 0$$

$$M(L) = \bar{M}_L \qquad V(L) = \bar{P}_L$$

4-1 continued

$$EI\, v''''(x) = w_0 \qquad \text{w/} \quad v(0)=0 \qquad M(L) = -EI\, v''(L) = \bar{M}_L$$

$$v'(0)=0 \qquad V(L) = -EI\, v'''(L) = \bar{P}_L$$

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2.

$$\tilde{v}(x) = a + bx + cx^2 + dx^3$$

$$\tilde{v}'(x) = b + 2cx + 3dx^2$$

we are interested in a function spanning $0 \leq x \leq L$ which scales propotionatelly to L. Change $\tilde{v}$ to:

$$\tilde{v}(\xi) = a + b\xi + c\xi^2 + d\xi^3 \qquad \text{with} \quad \xi = \frac{x}{L} \qquad d\xi = \frac{dx}{L} \Rightarrow dx = L\,d\xi$$

$$\frac{d}{dx}\tilde{v}(\xi) = L \cdot \tilde{v}'(\xi) = b + 2c\xi + 3d\xi^2 \qquad 0 \leq \xi \leq 1$$

$$\tilde{v}(0) = a = v_i \qquad \Rightarrow \quad a = v_i$$

$$\tilde{v}'(0) = b/L = \theta_i \qquad \Rightarrow \quad b = L\theta_i$$

$$\tilde{v}(1) = v_i + L\theta_i + c + d = v_j \qquad \Rightarrow c = v_j - v_i - L\theta_i - d \quad (1)$$

$$\tilde{v}'(1) = L\theta_i + 2c + 3d = L\theta_j \quad (2)$$

insert (1) into (2)

$$L\theta_i + 2(v_j - v_i - L\theta_i - d) + 3d = L\theta_j$$

$$-L\theta_i + 2v_j - 2v_i + d = L\theta_j \qquad \Rightarrow d = L(\theta_j + \theta_i) + 2(v_i - v_j) \quad (3)$$

insert (3) into (1)

$$c = v_j - v_i - L\theta_i - (L(\theta_j + \theta_i) + 2(v_i - v_j))$$

$$= 3(v_j - v_i) - 2L\theta_i - L\theta_j$$

Rewrite $\tilde{v}$

$$\tilde{v}(\xi) = v_i + L\theta_i\xi + (3(v_j - v_i) - 2L\theta_i - L\theta_j)\xi^2 + (L(\theta_j + \theta_i) + 2(v_i - v_j))\xi^3$$

Re organize in terms of $v_i, v_j, \theta_i, \theta_j$

$$\tilde{v}(\xi) = v_i\underbrace{(1 - 3\xi^2 + 2\xi^3)}_{N_1} + \theta_i\underbrace{L(\xi - 2\xi^2 + \xi^3)}_{N_2} + v_j\underbrace{(3\xi^2 - 2\xi^3)}_{N_3} + \theta_j\underbrace{L(\xi^3 - \xi^2)}_{N_4}$$

Shape functions match those of beams

4-1

2. continued

$\tilde{v}(\xi) = v_i N_1(\xi) + \theta_i N_2(\xi) + v_j N_3(\xi) + \theta_j N_4(\xi)$

$\delta\tilde{v}(\xi) = \delta v_i N_1(\xi) + \delta\theta_i N_2(\xi) + \delta v_j N_3(\xi) + \delta\theta_j N_4(\xi)$

$\tilde{v}'(\xi) = v_i N_1'(\xi) + \theta_i N_2'(\xi) + v_j N_3'(\xi) + \theta_j N_4'(\xi)$

$\delta\tilde{v}'(\xi) = \delta v_i N_1'(\xi) + \delta\theta_i N_2'(\xi) + \delta v_j N_3'(\xi) + \delta\theta_j N_4'(\xi)$

$\tilde{v}''(\xi) = v_i N_1''(\xi) + \theta_i N_2''(\xi) + v_j N_3''(\xi) + \theta_j N_4''(\xi)$

$\delta\tilde{v}''(\xi) = \delta v_i N_1''(\xi) + \delta\theta_i N_2''(\xi) + \delta v_j N_3''(\xi) + \delta\theta_j N_4''(\xi)$

$\tilde{\pi}(v_i, \theta_i, v_j, \theta_j) = \pi(\tilde{v}) = \frac{1}{2}\int_L EI\tilde{\Phi}^2\, dx - \int_L w_0 \tilde{v}\, dx + \bar{M}_L \theta_j - \bar{P}_L v_j$

with:

$\tilde{\Phi} = -\tilde{v}''$

and B.C

$v_i = 0$

$\theta_i = 0$

$\Rightarrow \tilde{\pi}(0, 0, v_j, \theta_j) = -\frac{1}{2}\int_L EI\, \tilde{v}''^2 dx - \int_L w_0\, \tilde{v}\, dx + \bar{M}_L \theta_j - \bar{P}_L v_j$

$= -\frac{1}{2}\int_L EI\left(v_j N_3''(\xi) + \theta_j N_4''(\xi)\right)^2 dx - \int_L w_0\left(v_j N_3(\xi) + \theta_j N_4(\xi)\right) dx \ldots$

$\ldots + \bar{M}_L \theta_j - \bar{P}_L v_j$

4-1

3.

$$\delta\pi(\tilde{v},\delta\tilde{v}) = \int_L EI\tilde{\Phi}\delta\tilde{\Phi}\,dx - \int_L w_o\delta\tilde{v}\,dx + \bar{M}_L\delta\theta_j - \bar{P}_L\delta v_j = 0$$

$$= \int_L EI\left(v_j N_3'' + \theta_j N_4''\right)\left(\delta v_j N_3'' + \delta\theta_j N_4''\right)dx - \int_L w_o\left(\delta v_j N_3 + \delta\theta_j N_4\right)dx \ldots$$

$$\ldots + \bar{M}_L\delta\theta_j - \bar{P}_L\delta v_j = 0$$

separate in terms of δv_j & $\delta\theta_j$

$$= \delta v_j\left(\int_L EI\left(N_3''N_3''v_j + N_4''N_3''\theta_j\right)dx - \int_L w_o N_3\,dx - \bar{P}_L\right)\ldots$$

$$\ldots\ \delta\theta_j\left(\int_L EI\left(N_3''N_4''v_j + N_4''N_4''\theta_j\right)dx - \int_L w_o N_4\,dx + \bar{M}_L\right) = 0$$

or written as

$$\begin{Bmatrix}\delta v_j\\ \delta\theta_j\end{Bmatrix}\begin{Bmatrix} v_j\int_L EI\,N_3''N_3''\,dx + \theta_j\int_L EI\,N_4''N_3''\,dx - \int_L w_o N_3\,dx - \bar{P}_L \\ v_j\int_L EI\,N_3''N_4''\,dx + \theta_j\int_L EI\,N_4''N_4''\,dx - \int_L w_o N_4\,dx + \bar{M}_L\end{Bmatrix} = 0$$

δv_j & $\delta\theta_j$ are both arbitrary, therfor

$$v_j\int_L EI\,N_3''N_3''\,dx + \theta_j\int_L EI\,N_4''N_3''\,dx - \int_L w_o N_3\,dx - \bar{P}_L = 0$$

&

$$v_j\int_L EI\,N_3''N_4''\,dx + \theta_j\int_L EI\,N_4''N_4''\,dx - \int_L w_o N_4\,dx + \bar{M}_L = 0$$

Using Mathematica to calculate integrals we get

$$v_j\frac{12EI}{L^3} - \theta_j\frac{6EI}{L^2} - \frac{w_o\cdot L}{2} - \bar{P}_L = 0 \qquad (1)$$

&

$$-v_j\frac{6EI}{L^2} + \theta_j\frac{4EI}{L} + \frac{w_o\cdot L^2}{12} + \bar{M}_L = 0 \qquad (2)$$

4-1

3. continued

Solving v_j & θ_j

to ① we do $\times \frac{L}{2}$ to get:

$$v_j \frac{6EI}{L^2} - \theta_j \frac{3EI}{L} - \frac{w_0 L^2}{4} - \frac{\bar{P}_L \cdot L}{2} = 0 \quad ③$$

② + ③:

$$v_j \underbrace{\left(\frac{6EI}{L^2} - \frac{6EI}{L^2}\right)}_{=0} + \theta_j \underbrace{\left(\frac{4EI}{L} - \frac{3EI}{L}\right)}_{\frac{EI}{L}} + \frac{w_0 L^2}{12} - \frac{w_0 L^2}{4} - \frac{\bar{P}_L \cdot L}{2} + M_L = 0$$

$$\Rightarrow \theta_j = \left(\frac{w_0 L^2}{6} + \frac{\bar{P}_L \cdot L}{2} - \bar{M}_L\right)\frac{L}{EI} = \frac{w_0 L^3}{6EI} + \frac{\bar{P}_L \cdot L^2}{2EI} - \frac{\bar{M}_L \cdot L}{EI}$$

$$\boxed{\theta_j = \frac{w_0 L^3}{6EI} + \frac{\bar{P}_L \cdot L^2}{2EI} - \frac{\bar{M}_L L}{EI}} \quad ④$$

insert ④ into ①

$$v_j\left(\frac{12EI}{L^3}\right) - \left(\frac{w_0 L^3}{6EI} + \frac{\bar{P}_L L^2}{2EI} - \frac{\bar{M}_L L}{EI}\right)\frac{6EI}{L^2} - \frac{w_0 L}{2} - \bar{P}_L = 0$$

$$v_j\left(\frac{12EI}{L^3}\right) = w_0 L - \frac{w_0 L}{2} + 3\bar{P}_L - \frac{6\bar{M}_L}{L} + \bar{P}_L = \frac{3w_0 L}{2} + 4\bar{P}_L - \frac{6\bar{M}_L}{L}$$

$$\boxed{v_j = \frac{w_0 L^4}{8EI} + \frac{\bar{P}_L L^3}{3EI} - \frac{\bar{M}_L L^2}{2EI}}$$

4-1

4.

$$\tilde{v}(\xi) = \left(\frac{W_0 L^4}{8EI} + \frac{\bar{P}_L L^3}{3EI} - \frac{\bar{M}_L L^2}{2EI}\right)(3\xi^2 - 2\xi^3) + \left(\frac{W_0 L^3}{6EI} + \frac{\bar{P}_L L^2}{2EI} - \bar{M}_L\right) L\,(\xi^3 - \xi^2)$$

with $\xi = \frac{x}{L}$ it can be written as

$$\tilde{v}(x) = \left(\frac{W_0 L^4}{8EI} + \frac{\bar{P}_L L^3}{3EI} - \frac{\bar{M}_L L^2}{2EI}\right)\left(3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3\right) + \left(\frac{W_0 L^3}{6EI} + \frac{\bar{P}_L L^2}{2EI} - \bar{M}_L\right) L\left(\left(\frac{x}{L}\right)^3 - \left(\frac{x}{L}\right)^2\right)$$

4-2

1. $$G(v,\delta v) = -\int_L M(x)\,\delta\phi(x) + \int_L w_0\,\delta v(x)\,dx - \bar{M}_L\,\delta\theta(L) + \bar{P}_L\,\delta v(L) = 0$$

where

$$M(x) = EI\,\phi(x), \quad \theta(x) = v'(x), \quad \phi(x) = -\theta'(x) = -v''(x)$$

$$V(x) = M'(x) \quad \delta\theta(x) = \delta v'(x), \quad \delta\phi(x) = -\delta\theta'(x) = -\delta v''(x)$$

Find strong form (Euler equations) represented by $G(v,\delta v) = 0 \;\; \forall\, \delta v(x)$

$$G(v,\delta v) = -\int_L M(x)(-\delta v''(x))\,dx + \int_L w_0\,\delta v(x)\,dx - \bar{M}_L\,\delta v'(L) + \bar{P}_L\,\delta v(L) = 0$$

IBP

$$= M(x)\,\delta v'(x)\Big|_0^L - \int \frac{dM}{dx}\,\delta v'(x)\,dx + \int_L w_0\,\delta v(x)\,dx - \bar{M}_L\,\delta v'(L) + \bar{P}_L\,\delta v(L) = 0$$

IBP

$$= M(x)\,\delta v'(x)\Big|_0^L - V(x)\,\delta v(x)\Big|_0^L + \int_L \frac{dV}{dx}\,\delta v(x)\,dx + \int_L w_0\,\delta v(x)\,dx - \bar{M}_L\,\delta v'(L) + \bar{P}_L\,\delta v(L) = 0$$

collect by δv & $\delta v' = \delta\theta$

$$= \int_L (V' + w_0)\,\delta v(x)\,dx + (M(L) - \bar{M}_L)\,\delta\theta(L) - M(0)\,\delta\theta(0) + (\bar{P}_L - V(L))\,\delta v(L) + V(0)\,\delta v(0) = 0$$

$EI\,v''''(x) = w_0$ on Ω $0 \le x \le L$

$v(0) = v'(0) = 0$

$V(L) = \bar{P}_L$

$M(L) = \bar{M}_L$

4-2

2. Use the adjusted cubic polynomial from P4-1 (2), build a similar function for the virtual displacement, $\delta v(x)$, and solve for all remaining degrees of freedom.

Same process as in 4-1 (3) but now we start from

$$G(v,\delta v) = -\int_L M(x)\,\delta v(x)\,dx + \int_L w_0\,\delta v(x)\,dx - \bar{M}_c\,\delta\theta(L) + \bar{P}_c\,\delta v(L) = 0$$

instead of potential energy and linear relation of $M = EI\,\phi(x)$

to find min potential energy

otherwise the same process.

4-2

3. $\hat{v}(x) = C_0 + C_1 \cdot \sin\left(\frac{\pi x}{2L}\right) + C_2 \cdot \cos\left(\frac{\pi x}{2L}\right)$

Use trig. series and find all remaining degrees of freedom using PVD

$$\hat{v}'(x) = 0 + \frac{C_1 \pi}{2L}\cos\left(\frac{\pi x}{2L}\right) - \frac{C_2 \pi}{2L}\sin\left(\frac{\pi x}{2L}\right)$$

Fulfill B.Cs.

$$\hat{v}(0) = C_0 + C_2 = v_i = 0 \quad \Rightarrow \quad C_0 = -C_2$$

$$\hat{v}'(0) = \frac{C_1 \pi}{2L} = \theta_i = 0 \quad \Rightarrow \quad C_1 = 0$$

$$\hat{v}(x) = C_0 + (-C_0)\cos\left(\frac{\pi x}{2L}\right) = C_0\left(1 - \cos\left(\frac{\pi x}{2L}\right)\right)$$

$$\delta\hat{v}(x) = \delta C_0\left(1 - \cos\left(\frac{\pi x}{2L}\right)\right)$$

$$\hat{v}'(x) = \frac{C_0 \pi}{2L}\sin\left(\frac{\pi x}{2L}\right)$$

$$\delta\hat{v}'(x) = \frac{\delta C_0 \pi}{2L}\sin\left(\frac{\pi x}{2L}\right)$$

$$\hat{v}''(x) = \frac{C_0 \pi^2}{4L^2}\cos\left(\frac{\pi x}{2L}\right)$$

$$\delta\hat{v}''(x) = \frac{\delta C_0 \pi^2}{4L^2}\cos\left(\frac{\pi x}{2L}\right)$$

$$-\int_L EI(-\hat{v}'')(-\delta\hat{v}'')\,dx + \int_L w_0\,\delta\hat{v}\,dx - \bar{M}_L\,\delta\hat{v}'(L) + \bar{P}_L\,\delta\hat{v}(L) = 0$$

$$-\int_L EI\,C_0\frac{\pi^2}{4L^2}\cos\left(\frac{\pi x}{2L}\right)\delta C_0\frac{\pi^2}{4L^2}\cos\left(\frac{\pi x}{2L}\right)dx + \int_L w_0\,\delta C_0\left(1 - \cos\left(\frac{\pi x}{2L}\right)\right)dx \;\ldots$$

$$\ldots\; - \bar{M}_L\,\delta C_0\frac{\pi}{2L} + \bar{P}_L\,\delta C_0 = 0$$

Collect by δC_0

$$\delta C_0\left(-\frac{EI\pi^4}{32L^3}C_0 + \frac{(\pi-2)L}{\pi}w_0 - \frac{\bar{M}_L\pi}{2L} + \bar{P}_L\right) = 0 \qquad \forall\,\delta C_0$$

we can now solve for C_0

$$C_0 = \left(\frac{\bar{M}_L\pi}{2L} - \bar{P}_L - \frac{(\pi-2)L\,w_0}{\pi}\right)\frac{-32L^3}{EI\pi^4} = \frac{32L^3\bar{P}_L}{EI\pi^4} + \frac{32\left(1-\frac{2}{\pi}\right)L^4 w_0}{EI\pi^4} - \frac{16\bar{M}_L L^2}{EI\pi^3}$$

$$C_0 = \frac{32\left(1-\frac{2}{\pi}\right)L^4 w_0}{EI\pi^4} + \frac{32L^3\bar{P}_L}{EI\pi^4} - \frac{16\bar{M}_L L^2}{EI\pi^3}$$

4-2

3. continued

$$\hat{v}_j = \hat{v}(L) = \frac{32\left(1-\frac{2}{\pi}\right)L^4 w_o}{EI\,\pi^4} + \frac{32L^3\bar{P}_L}{EI\,\pi^4} - \frac{16\bar{M}_L L^2}{EI\,\pi^3}$$

$$\hat{\theta}_j = \hat{v}'(L) = C_o \cdot \frac{\pi}{2L} = \frac{32\left(1-\frac{2}{\pi}\right)L^3 w_o}{2EI\,\pi^3} + \frac{16L^2\bar{P}_L}{EI\,\pi^3} - \frac{8\bar{M}_L L}{EI\,\pi^2}$$

4-3

1. Compare solutions from 4-1 and 4-2 to exact solution at $x=L$ and $x=L/2$

For exact solution solve

$$EI\,v''''(x) = W_0 \quad \text{or} \quad v''''(x) = \frac{W_0}{EI}$$

$$V(0) = v'(0) = 0$$

$$V(L) = -EI\,v'''(x) = \bar{P}_L$$

$$M(L) = -EI\,v''(x) = \bar{M}_L$$

$$v'''(x) = \frac{W_0}{EI}x + C_1 \qquad v''(x) = \frac{W_0}{2EI}x^2 + C_1 x + C_2$$

$$v'(x) = \frac{W_0}{6EI}x^3 + 2C_1 x^2 + C_2 x + C_3 \qquad V(x) = \frac{W_0}{24EI}x^4 + \frac{1}{6}C_1 x^3 + \frac{1}{2}C_2 x^2 + C_3 x + C_4$$

$$V(0) = C_4 = 0 \Rightarrow C_4 = 0$$

$$V'(0) = C_3 = 0 \Rightarrow C_3 = 0$$

$$V(L) = -EI\left(\frac{W_0}{EI}L + C_1\right) = \bar{P}_L$$

$$\Rightarrow C_1 = \frac{-\bar{P}_L}{EI} - \frac{W_0 L}{EI}$$

$$M(L) = -EI\left(\frac{W_0}{2EI}L^2 + L\left(\frac{-\bar{P}_L}{EI} - \frac{W_0 L}{EI}\right) + C_2\right) = \bar{M}_L$$

$$\Rightarrow C_2 = -\frac{\bar{M}_L}{EI} - \frac{W_0 L^2}{2EI} + \frac{\bar{P}_L L}{EI} + \frac{W_0 L^2}{EI} = \frac{W_0 L^2}{2EI} + \frac{\bar{P}_L L}{EI} - \frac{\bar{M}_L}{EI}$$

$$V(x) = \frac{W_0}{24EI}x^4 + \frac{1}{6}\left(\frac{-\bar{P}_L - W_0 L}{EI}\right)x^3 + \frac{1}{2}\left(\frac{W_0 L^2}{2EI} + \frac{\bar{P}_L L}{EI} - \frac{\bar{M}_L}{EI}\right)x^2$$

$$V(x) = \frac{W_0(x^4 - 4Lx^3 + 6L^2x^2)}{24EI} + \frac{\bar{P}_L(3Lx^2 - x^3)}{6EI} - \frac{\bar{M}_L x^2}{2EI}$$

$$V'(x) = \frac{W_0(x^3 - 3Lx^2 + 3L^2x)}{6EI} + \frac{\bar{P}_L(2Lx - x^2)}{2EI} - \frac{\bar{M}_L x}{2EI}$$

4-3

1. continued 1

Comparing results

at $x = L$

Exact: $$v(L) = \frac{w_0 L^4}{8EI} + \frac{\bar{P}_L L^3}{3EI} - \frac{\bar{M}_L L^2}{2EI}$$

$$\theta(L) = \frac{w_0 L^3}{6EI} + \frac{\bar{P}_L L^2}{2EI} - \frac{\bar{M}_L L}{EI}$$

Cubic: $$v(L) = \frac{w_0 L^4}{8EI} + \frac{\bar{P}_L L^3}{3EI} - \frac{\bar{M}_L L^2}{2EI}$$

$$\theta(L) = \frac{w_0 L^3}{6EI} + \frac{\bar{P}_L L^2}{2EI} - \frac{\bar{M}_L L}{EI}$$

Trig: $$v(L) = \frac{32\left(1-\frac{2}{\pi}\right) w_0 L^4}{EI \pi^4} + \frac{32 L^3 \bar{P}_L}{EI \pi^4} - \frac{16 \bar{M}_L L^2}{EI \pi^3}$$

$$= 0.11938 \frac{w_0 L^4}{EI} + 0.32851 \frac{\bar{P}_L L^3}{EI} - 0.51603 \frac{\bar{M}_L L^2}{EI}$$

$$\theta(L) = \frac{16\left(1-\frac{2}{\pi}\right) w_0 L^3}{EI \pi^3} + \frac{16 L^2 \bar{P}_L}{EI \pi^3} - \frac{8 \bar{M}_L L}{EI \pi^2}$$

$$= 0.18751 \frac{w_0 L^3}{EI} + 0.51603 \frac{\bar{P}_L L^2}{EI} - 0.81057 \frac{\bar{M}_L}{EI}$$

4-3

1. continued 2

at $x = L/2$

Exact: $v(L/2) = \frac{17 w_o L^4}{384 EI} + \frac{5 \bar{P}_L L^3}{48 EI} - \frac{\bar{M}_L L^2}{8 EI}$

$\theta(L/2) = \frac{7 w_o L^3}{48 EI} + \frac{3 \bar{P}_L L^2}{8 EI} - \frac{\bar{M}_L L}{2 EI}$

Cubic: $v(L/2) = \frac{w_o L^4}{24 EI} + \frac{5 \bar{P}_L L^3}{48 EI} - \frac{\bar{M}_L L^2}{8 EI}$

$\theta(L/2) = \frac{7 w_o L^3}{48 EI} + \frac{3 \bar{P}_L L^2}{8 EI} - \frac{\bar{M}_L L}{2 EI}$

Trig: $v(L/2) = 0.034964 \frac{w_o L^4}{EI} + 0.0962188 \frac{\bar{P}_L \cdot L^3}{EI} - 0.15114 \frac{\bar{M}_L L^2}{EI}$

$\theta(L/2) = 0.132592 \frac{w_o L^3}{EI} + 0.364884 \frac{\bar{P}_L L^2}{EI} - 0.573159 \frac{\bar{M}_L L}{EI}$

4-4

1. with $M(\phi) = A \cdot \arctan(B\phi)$

Find the coefficients A & B such that

a) $\left.\frac{dM}{d\phi}\right|_{\phi=0} = EI$

b) $\lim_{\phi \to \infty} M(\phi) = M_y$

$$\frac{dM}{d\phi} = \frac{A \cdot B}{B^2 \cdot \phi^2 + 1}$$

$$\left.\frac{dM}{d\phi}\right|_{\phi=0} = A \cdot B = EI \quad ①$$

$$\lim_{\phi \to \infty} M(\phi) = A \cdot \frac{\pi}{2} = M_y$$

$$\Rightarrow A = \frac{2M_y}{\pi} \quad ②$$

Insert ② into ①

$$\frac{2M_y}{\pi} \cdot B = EI$$

$$\Rightarrow B = \frac{EI \cdot \pi}{2M_y}$$

$$\underline{\underline{M(\phi) = \frac{2M_y}{\pi} \tan^{-1}\left(\frac{EI\pi}{2M_y}\phi\right)}}$$

4-4

2. Formulate the weak form equilibrium (PVD) for the nonlinear material model and a general approximation function

$$\tilde{v}(x) = \sum_{i=1}^{N} N_i(x) q_i$$

with $N_i(x)$ as the i-th (given) shape function and q_i as the respective unknown degree of freedom. Assume that the given function already satisfies all essential boundary conditions.

We have

$$\tilde{v}(x) = \sum_{i=1}^{N} N_i(x)\, q_i \qquad \delta\tilde{v}(x) = \sum_{i=1}^{N} N_i(x)\,\delta q_i$$

$$\tilde{v}'(x) = \sum_{i=1}^{N} N_i'(x)\, q_i \qquad \delta\tilde{v}'(x) = \sum_{i=1}^{N} N_i'(x)\,\delta q_i$$

$$\tilde{v}''(x) = \sum_{i=1}^{N} N_i''(x)\, q_i \qquad \delta\tilde{v}''(x) = \sum_{i=1}^{N} N_i''(x)\,\delta q_i$$

Moment is now described with

$$M(x) = \tilde{M}(\tilde{v}'')$$

with $\tilde{v}''(x, q_1, q_2 \dots q_N)$

Create new description of moment as a function of q's

$\bar{M}(q_1, q_2, q_3 \dots q_N)$ or short hand $\bar{M}(\underline{q})$

$$G(\tilde{v}, \delta\tilde{v}) = \int_L \bar{M}(\underline{q})\,\delta\tilde{v}''\,dx + \int_L w_0\,\delta\tilde{v}\,dx = 0$$

$$\Rightarrow \int_L \bar{M}(\underline{q}) \sum_{i=1}^{N} N_i''(x)\,\delta q_i\,dx + \int_L w_0 \sum_{i=1}^{N} N_i(x)\,\delta q_i\,dx = 0$$

$$\Rightarrow \sum_{i=1}^{N} \delta q_i \left(\int_L \bar{M}(\underline{q}) N_i''(x)\,dx + \int_L w_0 N_i(x)\,dx \right) = 0$$

4-4.3

δq_i is arbitrary and therefor

$$R_i(\underline{q}) = \int_L \bar{M}(\underline{q}) N_i''(x)\,dx + \int_L w_0 N_i(x)\,dx = 0$$

which gives

$$G(v, \delta v) =: -\{\delta q_1 \dots \delta q_N\} \cdot \{\underline{R}\}$$