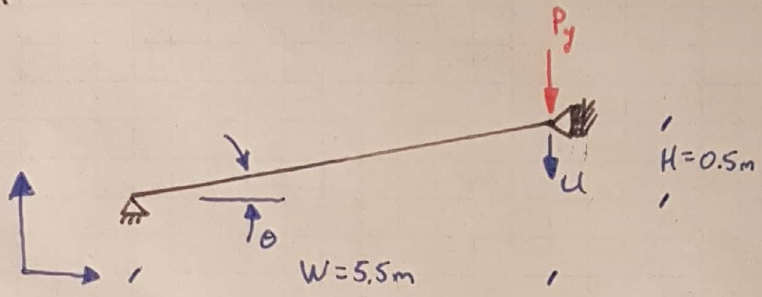
2020-04-15

Assignment 1

CESG 506

KRISTINN HLÍÐAR GRÉTARSSON

1-1



$EA = 2100\,kN$

1a
$$\epsilon = \frac{l-L}{L}$$

1b
$$E = \frac{1}{2}\frac{l^2-L^2}{L^2}$$

1c
$$\varepsilon = \frac{1}{2}\frac{l^2-L^2}{l^2}$$

1d
$$\tilde{\varepsilon} = \frac{1}{2}\ln\left(\frac{l^2}{L^2}\right)$$

1. Derive force-displacement relation for all possible combinations (8 options) of kinematic relations and equilibrium on the undeformed and the deformed configuration and compare them in a single plot for $0 \le u \le 2.5H$. Include the linear solution in the same plot

$$\underline{L} = \begin{Bmatrix} 5.5 \\ 0.5 \end{Bmatrix} \qquad \underline{l} = \begin{Bmatrix} 5.5 \\ 0.5-u \end{Bmatrix}$$

$$L^2 = \underline{L}\cdot\underline{L} = 30.5 \qquad l^2 = \underline{l}\cdot\underline{l} = 30.5 + u^2 - u$$

$$f = EA\cdot\epsilon \qquad P_y = f\cdot\sin(\theta) \qquad \sin(\theta) = \frac{H-u}{l}$$

Undeformed

1a $P_y = EA\cdot\frac{l-L}{L}\cdot\frac{H}{L}$

1b $P_y = EA\cdot\frac{l^2-L^2}{2L^2}\,\frac{H}{L}$

1c $P_y = EA\cdot\frac{l^2-L^2}{2l^2}\,\frac{H}{L}$

1d $P_y = \frac{EA}{2}\ln\left(\frac{l^2}{L^2}\right)\frac{H}{L}$

Deformed

$P_y = EA\cdot\frac{l-L}{L}\cdot\frac{H-u}{l}$

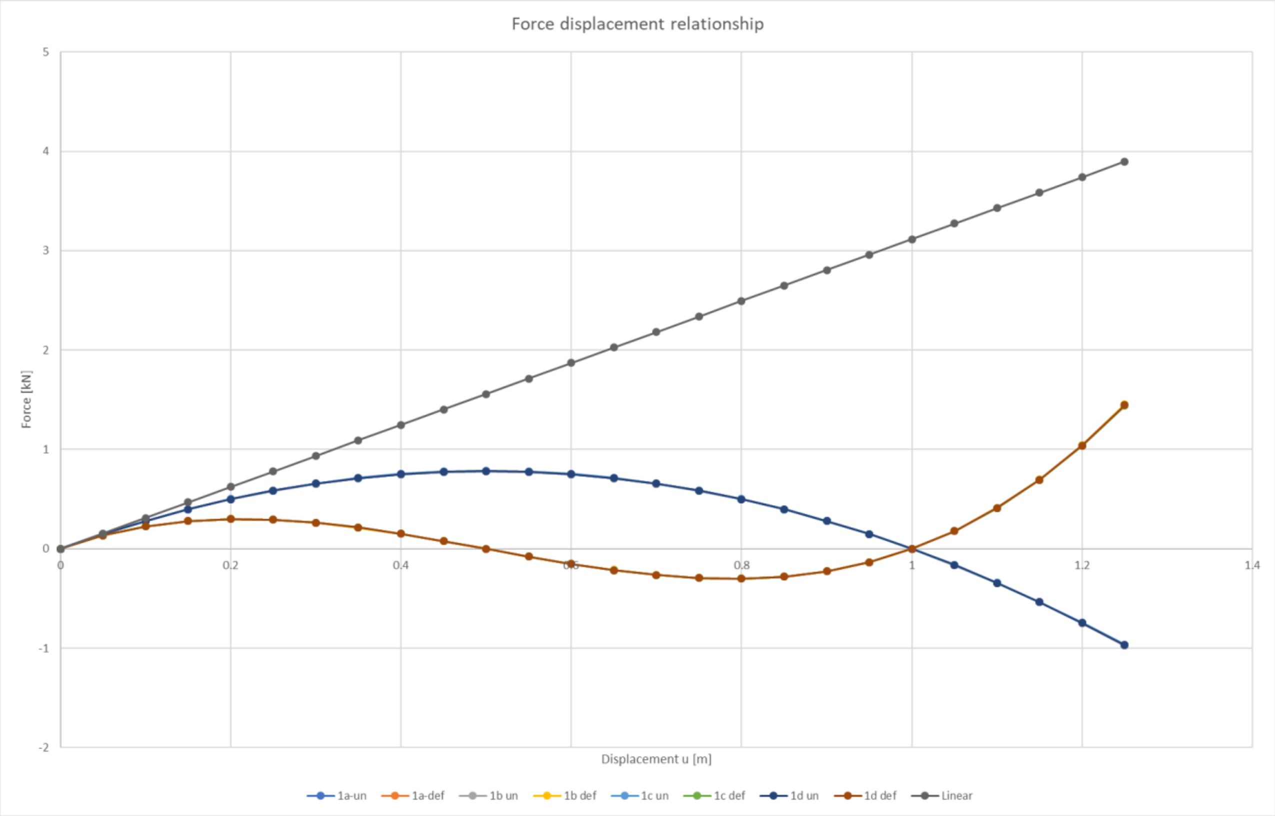
$P_y = EA\cdot\frac{l^2-L^2}{2L^2}\,\frac{H-u}{l}$

$P_y = EA\,\frac{l^2-L^2}{2l^2}\,\frac{H-u}{l}$

$P_y = \frac{EA}{2}\ln\left(\frac{l^2}{L^2}\right)\frac{H-u}{l}$

Problem 1-1

1.


Force displacement relationship
Force [kN]
Displacement u [m]
5
4
3
2
1
0
-1
-2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1a-un
1a-def
1b un
1b def
1c un
1c def
1d un
1d def
Linear

Problem 1-1

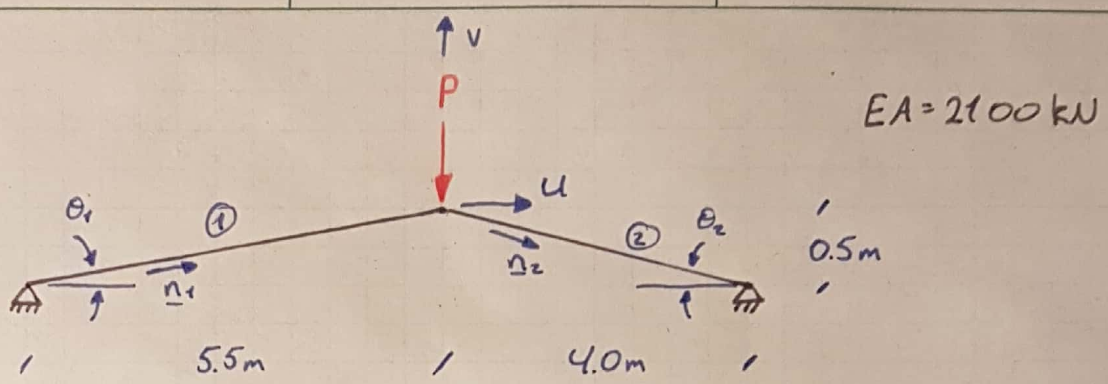
2.

What formulation is better would depend on what range I expect to be working in and how accurate the calculations need to be. If I know that I will be working with low loads that will cause small deflections and be in the range where the system is close to linear. I would choose to use linear formulation with the undeformed equilibrium. That makes the calculations much simpler and saves computational power/time.

On the other hand, if I expect the deflections to be large enough for the system to start behaving in considerably non-linear manner. I would choose to use a formulation based on the deformed equilibrium. When it comes to choosing a strain, I wouldn't worry much about which one I would choose as all of them give really similar results. I would go for strain formulation that would allow for quicker computation.

For larger deformations I would go for a strain that does not blow up to infinity anywhere close to my working range.

1-2



1. Using Henkey strain, a linear relation $\sigma = E \cdot \varepsilon$ $A = \text{const.}$ and equilibrium on the deformed system, derive the relationship betweend displacement and forces on the free node i.e.

$$\begin{Bmatrix} P_x \\ P_y \end{Bmatrix} = \begin{Bmatrix} F_x(u, v) \\ F_y(u, v) \end{Bmatrix}$$

$$\epsilon = \frac{1}{2} \ln(\lambda^2) = \frac{1}{2} \ln\left(\frac{\ell^2}{L^2}\right) \quad \text{(Henkey strain)}$$

$L_1^2 = 5.5^2 + 0.5^2 = 30.5\,m^2$

$\ell_1^2 = (5.5+u)^2 + (0.5+v)^2$

$= 30.5 + u^2 + v^2 + 11u + v$

$L_2^2 = 4.0^2 + 0.5^2 = 16.25\,m^2$

$\ell_2^2 = (4.0-u)^2 + (0.5+v)^2$

$= 16.25 + u^2 + v^2 - 8u + v$

$$P = -(f_{1.j} + f_{2.i}) = -(f_1 \cdot \underline{n}_1 + (-f_2\, \underline{n}_2)) = -(f_1 \cdot \underline{n}_1 - f_2\, \underline{n}_2) = f_2\, \underline{n}_2 - f_1\, \underline{n}_1$$

Same for both elements

$$P = EA\left(\frac{\underline{\ell}_2}{\|\underline{\ell}_2\|}\epsilon_2 - \frac{\underline{\ell}_1}{\|\underline{\ell}_1\|}\epsilon_1\right)$$

$$\underline{n}_k = \frac{\underline{V}_k}{\|\underline{V}_k\|}$$

$$\underline{\ell}_1 = \begin{Bmatrix} 5.5 \\ 0.5 \end{Bmatrix} + \begin{Bmatrix} u \\ v \end{Bmatrix} \qquad \underline{\ell}_2 = \begin{Bmatrix} 4.0 \\ -0.5 \end{Bmatrix} + \begin{Bmatrix} -u \\ -v \end{Bmatrix} = \begin{Bmatrix} 4.0 \\ -0.5 \end{Bmatrix} - \begin{Bmatrix} u \\ v \end{Bmatrix}$$

$\|\underline{\ell}_1\| = \ell_1 \qquad \|\underline{\ell}_2\| = \ell_2$

$$P = EA\left(\frac{\epsilon_2}{\ell_2}\begin{Bmatrix} 4.0-u \\ -0.5-v \end{Bmatrix} - \frac{\epsilon_1}{\ell_1}\begin{Bmatrix} 5.5+u \\ 0.5+v \end{Bmatrix}\right) = \begin{Bmatrix} P_x \\ P_y \end{Bmatrix}$$

Also written as

$$f_2\, \underline{n}_2 - f_1\, \underline{n}_1 = P$$

1-2

2. Find the tangent stiffness matrix of the system. Keep it simple and don't substitute long expressions into your answer

Tangent stiffness for element is written as

$$[K^e] = \begin{bmatrix} k^e & -k^e \\ -k^e & k^e \end{bmatrix}$$

where

$$k^e = \frac{EA}{l} \underline{n} \otimes \underline{n} + \frac{\tilde{f}}{l}\left(\underline{1} - \underline{n} \otimes \underline{n}\right)$$ where $\tilde{f}$ is force in element

Our global stiffness matrix is then

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & k^1 & -k^1 & 0 \\ 2 & -k^1 & k^1 + k^2 & -k^2 \\ 3 & 0 & -k^2 & k^2 \end{array}$$

with nodes 1 & 3 being supported

node 2 unsupported

u_1 & $u_3 = \underline{0}$, we are therefor only interested in $K_{ff} = k^1 + k^2$

$$k^1 = \frac{EA}{l_1} \underline{n}_1 \otimes \underline{n}_1 + \frac{\tilde{f}_1}{l_1}\left(\underline{1} - \underline{n} \otimes \underline{n}\right)$$

$$\underline{n}_1 = \begin{Bmatrix} \cos(\theta_1) \\ \sin(\theta_1) \end{Bmatrix} = \begin{Bmatrix} C_1 \\ S_1 \end{Bmatrix} \qquad \tilde{f}_1 = EA\epsilon_1$$

$$= \frac{EA}{l_1} \begin{bmatrix} C_1^2 & C_1 S_1 \\ C_1 S_1 & S_1^2 \end{bmatrix} + \frac{EA\epsilon_1}{l_1}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} C_1^2 & C_1 S_1 \\ C_1 S_1 & S_1^2 \end{bmatrix} \right)$$

$$= \frac{EA(1-\epsilon_1)}{l_1} \begin{bmatrix} C_1^2 & C_1 S_1 \\ C_1 S_1 & S_1^2 \end{bmatrix} + \frac{EA\epsilon_1}{l_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

1-2 2. continued

$$\underline{k}^2 = \frac{EA}{l_2}\, \underline{n}_2 \otimes \underline{n}_2 + \frac{\tilde{f}_2}{l_2}\left(\underline{\underline{1}} - \underline{n}_2 \otimes \underline{n}_2\right)$$

$$\underline{n}_2 = \begin{Bmatrix} \cos(\theta_2) \\ -\sin(\theta_2) \end{Bmatrix} = \begin{Bmatrix} c_2 \\ -s_2 \end{Bmatrix} \qquad \tilde{f}_2 = EA \cdot \epsilon_2$$

$$= \frac{EA}{l_2}\begin{bmatrix} c_2^2 & -c_2 s_2 \\ -c_2 s_2 & s_2^2 \end{bmatrix} + \frac{EA\epsilon_2}{l_2}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} c_2^2 & -c_2 s_2 \\ -c_2 s_2 & s_2^2 \end{bmatrix}\right)$$

$$= \frac{EA(1-\epsilon_2)}{l_2}\begin{bmatrix} c_2^2 & -c_2 s_2 \\ -c_2 s_2 & s_2^2 \end{bmatrix} + \frac{EA\epsilon_2}{l_2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

K_{ff} then becomes

$$K_{ff} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

where

$$K_{11} = \frac{EA(1-\epsilon_1)}{l_2} c_1^2 + \frac{EA\epsilon_1}{l_2} + \frac{EA(1-\epsilon_2)}{l_2} c_2^2 + \frac{EA\epsilon_2}{l_2}$$

$$= EA\left(\frac{c_1^2(1-\epsilon_1) + \epsilon_1}{l_1} + \frac{c_2^2(1-\epsilon_2) + \epsilon_2}{l_2}\right)$$

$$K_{21} = K_{12} = \frac{EA(1-\epsilon_1)c_1 s_1}{l_1} + \frac{EA(1-\epsilon_2)(-c_2 s_2)}{l_2}$$

$$= EA\left(\frac{c_1 s_1 (1-\epsilon_1)}{l_1} - \frac{c_2 s_2 (1-\epsilon_2)}{l_2}\right)$$

$$K_{22} = \frac{EA(1-\epsilon_1)}{l_1} s_1^2 + \frac{EA\epsilon_1}{l_1} + \frac{EA(1-\epsilon_2)}{l_2} s_2^2 + \frac{EA\epsilon_2}{l_2}$$

$$= EA\left(\frac{s_1^2(1-\epsilon_1) + \epsilon_1}{l_1} + \frac{s_2^2(1-\epsilon_2) + \epsilon_2}{l_2}\right)$$

Terms ϵ, l have been written out on previous pages.

1-2

3. Formulate the condition $P_x = 0$ to the system of equations. Will this allow you to eliminate the horizontal displacement from the system of equations? Why? Why not?

From 1-2 1. we have

$$\frac{EA\,\epsilon_2}{l_2}\begin{Bmatrix}4.0-u\\-0.5-v\end{Bmatrix} - \frac{EA\,\epsilon_1}{l_1}\begin{Bmatrix}5.5+u\\0.5+v\end{Bmatrix} = \begin{Bmatrix}P_x\\P_y\end{Bmatrix}$$

Now setting $P_x = 0$ gives

$$\frac{EA\,\epsilon_2}{l_2}\begin{Bmatrix}4.0-u\\-0.5-v\end{Bmatrix} - \frac{EA\,\epsilon_1}{l_1}\begin{Bmatrix}5.5+u\\0.5+v\end{Bmatrix} = \begin{Bmatrix}0\\P_y\end{Bmatrix}$$

Looking only at force in x direction

$$EA\underbrace{\left(\frac{\epsilon_2}{l_2}(4.0-u) - \frac{\epsilon_1}{l_1}(5.5+u)\right)}_{=0 \text{ to fulfill condition}} = 0$$

Solving for u gives

$$u = \frac{5.5\cdot\frac{\epsilon_1}{l_1} - 4.0\frac{\epsilon_2}{l_2}}{-\left(\frac{\epsilon_1}{l_1} + \frac{\epsilon_2}{l_2}\right)}$$

But $l_1, l_2, \epsilon_1, \epsilon_2$ are all functions of u, we can therefor not eliminate u.

Side note, if system was symmetric and $l_1 = \begin{Bmatrix}a\\b\end{Bmatrix}$ $l_2 = \begin{Bmatrix}a\\-b\end{Bmatrix}$

we would get

$$\frac{\epsilon_2}{l_2}(a-u) - \frac{\epsilon_1}{l_1}(a+u) \qquad \text{w/ } \epsilon_2 = \epsilon_1 = \epsilon,\ l_2 = l_1 = l$$

$$\Rightarrow 2u\frac{\epsilon}{l} = P_x = 0$$

Which would only be fulfilled at $u=0$ and we could eliminate u

Problem 1-2

4.

P_{cr} was found by controlling vertical displacement and finding the first local maximum as node 2 is moved down wards. Each step of the way horizontal displacement was iterated on to get total horizontal force equal to zero.

P_{cr} = -0.9817134398668483 kN

Results from each iteration step during the process

Load step = 0
#0 u=0.0000e+00 v=0.0000e+00 Rx=0.000000e+00 Ry=0.000000e+00

Load step = 0.25
#0 u=0.0000e+00 v=0.0000e+00 Rx=0.000000e+00 Ry=2.454284e-01
#1 u=-8.1190e-04 v=-2.4224e-02 Rx=-1.716990e-02 Ry=1.787191e-02
#2 u=-8.5612e-04 v=-2.6212e-02 Rx=-1.135897e-04 Ry=1.188102e-04
#3 u=-8.5642e-04 v=-2.6226e-02 Rx=-5.160655e-09 Ry=5.379573e-09
#4 u=-8.5642e-04 v=-2.6226e-02 Rx=6.616929e-14 Ry=-3.969047e-14

Load step = 0.5
#0 u=-8.5642e-04 v=-2.6226e-02 Rx=6.616929e-14 Ry=2.454284e-01
#1 u=-1.7640e-03 v=-5.4771e-02 Rx=-2.397555e-02 Ry=2.350293e-02
#2 u=-1.8342e-03 v=-5.8018e-02 Rx=-3.059614e-04 Ry=2.969055e-04
#3 u=-1.8352e-03 v=-5.8060e-02 Rx=-5.138651e-08 Ry=4.960239e-08
#4 u=-1.8352e-03 v=-5.8060e-02 Rx=-7.549517e-14 Ry=2.203793e-14

Load step = 0.75
#0 u=-1.8352e-03 v=-5.8060e-02 Rx=-7.549517e-14 Ry=2.454284e-01
#1 u=-2.9006e-03 v=-9.3938e-02 Rx=-3.812335e-02 Ry=3.452191e-02
#2 u=-3.0409e-03 v=-1.0063e-01 Rx=-1.316545e-03 Ry=1.150251e-03
#3 u=-3.0458e-03 v=-1.0087e-01 Rx=-1.674827e-06 Ry=1.446128e-06
#4 u=-3.0458e-03 v=-1.0087e-01 Rx=-3.090861e-12 Ry=2.350675e-12
#5 u=-3.0458e-03 v=-1.0087e-01 Rx=8.482104e-13 Ry=-3.963496e-14

Load step = 0.99
#0 u=-3.0458e-03 v=-1.0087e-01 Rx=8.482104e-13 Ry=2.356112e-01
#1 u=-4.4038e-03 v=-1.5143e-01 Rx=-7.633330e-02 Ry=6.121922e-02
#2 u=-4.9093e-03 v=-1.7656e-01 Rx=-1.886744e-02 Ry=1.366535e-02
#3 u=-5.1060e-03 v=-1.8654e-01 Rx=-2.985580e-03 Ry=2.025239e-03
#4 u=-5.1469e-03 v=-1.8863e-01 Rx=-1.312852e-04 Ry=8.686342e-05
#5 u=-5.1488e-03 v=-1.8873e-01 Rx=-2.884631e-07 Ry=1.899347e-07
#6 u=-5.1488e-03 v=-1.8873e-01 Rx=-1.173284e-12 Ry=8.997247e-13
#7 u=-5.1488e-03 v=-1.8873e-01 Rx=-4.494183e-13 Ry=1.421085e-14

Load step = 0.999

#0 u=-5.1488e-03 v=-1.8873e-01 Rx=-4.494183e-13 Ry=8.835421e-03

#1 u=-5.3659e-03 v=-1.9907e-01 Rx=-3.220339e-03 Ry=2.080566e-03

#2 u=-5.4500e-03 v=-2.0338e-01 Rx=-5.605904e-04 Ry=3.537781e-04

#3 u=-5.4716e-03 v=-2.0449e-01 Rx=-3.705937e-05 Ry=2.311145e-05

#4 u=-5.4732e-03 v=-2.0457e-01 Rx=-2.090722e-07 Ry=1.300174e-07

#5 u=-5.4732e-03 v=-2.0457e-01 Rx=-6.854961e-12 Ry=4.194978e-12

#6 u=-5.4732e-03 v=-2.0457e-01 Rx=8.526513e-13 Ry=1.676437e-14

1-2

5.

I could compute to 1×10^{-13} kN error. As soon as I went to 1×10^{-14} kN my code did not converge which was a cut of point for all the load steps.

Load step = 0 0, 0.000000e+00 Load step = 0.25 0, 2.454284e-01 1, 2.478327e-02 2, 1.643730e-04 3, 7.454674e-09 4, 7.716028e-14 Load step = 0.5 0, 2.454284e-01 1, 3.357401e-02 2, 4.263394e-04 3, 7.142108e-08 4, 7.864598e-14	Load step = 0.75 0, 2.454284e-01 1, 5.143104e-02 2, 1.748247e-03 3, 2.212765e-06 4, 3.883181e-12 5, 8.491359e-13 Load step = 0.99 0, 2.356112e-01 1, 9.784971e-02 2, 2.329640e-02 3, 3.607670e-03 4, 1.574200e-04 5, 3.453783e-07 6, 1.478546e-12 7, 4.496429e-13	Load step = 0.999 0, 8.835421e-03 1, 3.833972e-03 2, 6.628880e-04 3, 4.367535e-05 4, 2.462026e-07 5, 8.036686e-12 6, 8.528161e-13