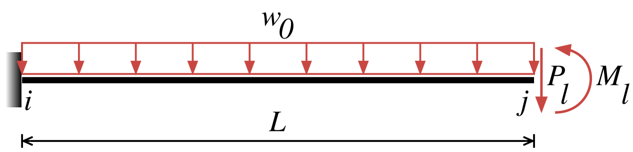
This assignment will review variational principles and approximation techniques. The main focus will be on linear beam theory such that solutions can be obtained easily. Only the last problem will illustrate their use on nonlinear problems.

Problem 4-1: Linear beam problem – Minimum of Potential Energy

Given: Throughout this assignment, we will consider the straight beam with boundary conditions and loading as shown.



Find:

1. The potential energy stored in a deformed beam is

$$\Pi(v) = \frac{1}{2}\int_L EI\,\phi^2\,dx - \int_L w_0\,v\,dx + \bar{M}_\ell\,\theta(L) - \bar{P}_\ell\,v(L)$$

   with rotation $\theta(x) = v'(x)$ and curvature $\phi(x) = -\theta'(x)$.

   Find the Euler equations, i.e., the strong form (differential equations and boundary conditions) represented by $\Pi(v) \to \min$. Use $M(x) = EI\,\phi(x)$ and $V(x) = M'(x)$ to simplify your expressions.

2. Identify a cubic approximation function, $\tilde{v}(x)$, using $v_i = v(0)$, $\theta_i = v(0)$, $v_j = v(L)$, $\theta_j = v(L)$, as nodal degrees of freedom. Adjust $\tilde{v}(x)$ to satisfy the essential boundary conditions (those on $\partial\Omega_u$). Express the potential energy for this approximation function as

$$\tilde{\Pi}(v_i, \theta_i, v_j, \theta_j) = \Pi(\tilde{v})$$

3. Identify all remaining degrees of freedom using $\tilde{\Pi}(v_i, \theta_i, v_j, \theta_j) \to \min$.

4. Write out the final function for $\tilde{v}(x)$. This is worth doing once, just to see the entire procedure of finding the approximate solution. Later on, we will leave this task to the computer.

Problem 4-2: Linear beam problem – Principle of Virtual Displacements

Given: The same beam as shown in Problem 4-1.

Find:

1. The principle of virtual displacement for the beam can be expressed as

$$G(v;\delta v) = -\int_L M(x)\,\delta\phi(x)\,dx + \int_L w_0\,\delta v(x)\,dx - \bar{M}_\ell\,\delta\theta(L) + \bar{P}_\ell\,\delta v(L) = 0\ ,$$

   where $M(x) = EI\,\phi(x)$ is the (real) moment, $\theta(x) = v'(x)$ the (real) rotation, and $\phi(x) = -\theta'(x)$ the (real) curvature. The virtual functions are $\delta\theta(x) = \delta v'(x)$ as the virtual rotation, and $\delta\phi(x) = -\delta\theta'(x)$ as the virtual curvature.

   Find the Euler equations, i.e., the strong form (differential equations and boundary conditions) represented by $G(v;\delta v) = 0$ for all $\delta v(x)$. Show that both the MPE and PVD yield identical equations, though the PVD does not require the linear elastic moment-curvature relation while the MPE does.

2. Use the adjusted cubic polynomial from Problem 4-1(2), build a similar function for the virtual displacement, $\delta v(x)$, and solve for all remaining degrees of freedom.

3. Use the trigonometric series

$$v^*(x) = c_0 + c_1 \sin\left(\frac{\pi x}{2L}\right) + c_2 \cos\left(\frac{\pi x}{2L}\right)$$

   instead of the cubic polynomial to represent both $\tilde{v}(x)$ and $\delta v(x)$ (with necessary adjustments!) and find all remaining degrees of freedom using the PVD.

Problem 4-3: Linear beam model – Comparison

Given: Your solution for Problems 4-1 and 4-2.

Find: Using the results from Problems 4-1 and 4-2,

1. Compare results from all three approximations against the exact solution at $x = L$ and $x = L/2$. The exact solution is obtained using the governing equation:

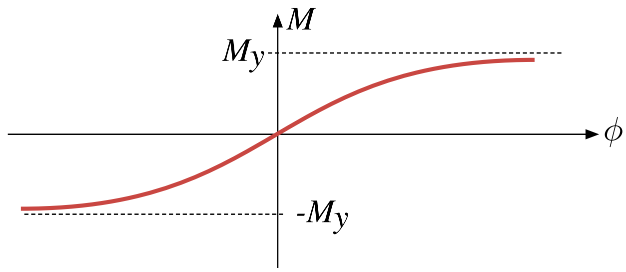
$$\begin{aligned}
&EI\,v''''(x) - w_0 = 0 && \text{on } \Omega : 0 < x < L \\
&v(0) = 0 \text{ and } \theta(0) = 0 && \partial\Omega_u : x = 0 \\
&V(L) = \bar{P}_\ell \text{ and } M(L) = \bar{M}_\ell && \partial\Omega_f : x = L
\end{aligned}$$

   The best way to compare these results are to compare coefficients of

$$v(x) = [\dots]\,\frac{\bar{P}_\ell L^3}{EI} + [\dots]\,\frac{\bar{M}_\ell L^2}{EI} + [\dots]\,\frac{w_0 L^4}{EI}$$

Problem 4-4: Nonlinear elastic, geometrically linear beam model

Given: Now let us consider a non-linear moment-curvature relation as shown.



Find:

1. Such a moment-curvature relationship can be expressed by a function

$$M(\phi) = A \arctan(B\phi) \ .$$

   Find the coefficients A and B such that

   (a) $\dfrac{dM}{d\phi}\Big|_{\phi=0} = EI$

   (b) $\lim\limits_{\phi\to\infty} M(\phi) = M_y$.

2. Formulate the weak form equilibrium (PVD) for the nonlinear material model and a general approximation function

$$\tilde{v}(x) = \sum_{i=1}^{N} N_i(x)\, q_i$$

   with $N_i(x)$ as the i-th (given) shape function and q_i as the respective unknown degree of freedom. Assume that the given function already satisfies all essential boundary conditions.

   Note: do NOT expand $M(\phi)$ or the expression will become uselessly complicated.

3. Show that

$$G(v; \delta v) =: -\{q_1 \;\; q_2 \;\; \cdots \;\; q_N\} \cdot \{\boldsymbol{R}\} = 0$$

   uniquely defines the residual force vector, $\boldsymbol{R}$, and applies the equilibrium condition $\boldsymbol{R} = \boldsymbol{0}$.

   Note: the zero in the PVD is a scalar zero while the equilibrium condition $\boldsymbol{R} = \boldsymbol{0}$ contains an N-dimensional zero vector.