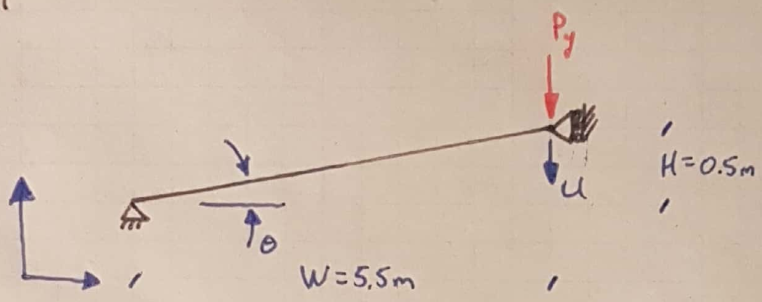
1-1




$EA = 2100\,kN$

1a
$$\epsilon = \frac{l-L}{L}$$

1b
$$E = \frac{1}{2}\,\frac{l^2-L^2}{L^2}$$

1c
$$e = \frac{1}{2}\,\frac{l^2-L^2}{l^2}$$

1d
$$\tilde{\varepsilon} = \frac{1}{2}\ln\left(\frac{l^2}{L^2}\right)$$

1. Derive force-displacement relation for all possible combinations (8 options) of kinematic relations and equilibrium on the undeformed and the deformed configuration and compare them in a single plot for $0 \le u \le 2.5H$. Include the linear solution in the same plot

$$\underline{L} = \begin{Bmatrix} 5.5 \\ 0.5 \end{Bmatrix} \qquad \underline{l} = \begin{Bmatrix} 5.5 \\ 0.5-u \end{Bmatrix}$$

$$L^2 = \underline{L}\cdot\underline{L} = 30.5 \qquad l^2 = \underline{l}\cdot\underline{l} = 30.5 + u^2 - u$$

$$f = EA\cdot\epsilon \qquad P_y = f\cdot\sin(\theta) \qquad \sin(\theta) = \frac{H-u}{L}$$

Undeformed

1a $P_y = EA\cdot\frac{l-L}{L}\cdot\frac{H}{L}$

1b $P_y = EA\cdot\frac{l^2-L^2}{2L^2}\,\frac{H}{L}$

1c $P_y = EA\cdot\frac{l^2-L^2}{2l^2}\,\frac{H}{L}$

1d $P_y = \frac{EA}{2}\ln\left(\frac{l^2}{L^2}\right)\frac{H}{L}$

Deformed

$P_y = EA\cdot\frac{l-L}{L}\cdot\frac{H-u}{l}$

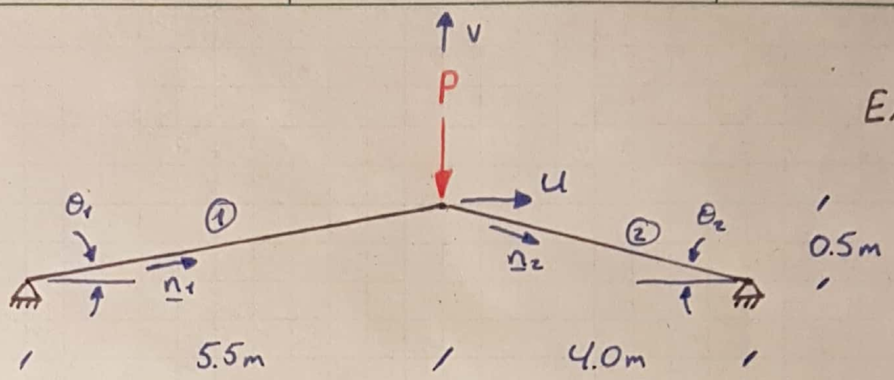
$P_y = EA\cdot\frac{l^2-L^2}{2L^2}\,\frac{H-u}{l}$

$P_y = EA\,\frac{l^2-L^2}{2l^2}\,\frac{H-u}{l}$

$P_y = \frac{EA}{2}\ln\left(\frac{l^2}{L^2}\right)\frac{H-u}{l}$

1-2

$EA = 2100\,kN$




1. Using Henkey strain, a linear relation $\sigma = E \cdot \varepsilon$ $A = \text{const.}$ and equilibrium on the deformed system, derive the relationship betweend displacement and forces on the free node i.e.

$$\begin{Bmatrix} P_x \\ P_y \end{Bmatrix} = \begin{Bmatrix} F_x(u,v) \\ F_y(u,v) \end{Bmatrix}$$

$$\epsilon = \frac{1}{2}\ln(\lambda^2) = \frac{1}{2}\ln\left(\frac{\ell^2}{L^2}\right) \quad \text{(Henkey strain)}$$

$L_1^2 = 5.5^2 + 0.5^2 = 30.5\,m^2$

$\ell_1^2 = (5.5+u)^2 + (0.5+v)^2$

$= 30.5 + u^2 + v^2 + 11u + v$

$L_2^2 = 4.0^2 + 0.5^2 = 16.25\,m^2$

$\ell_2^2 = (4.0-u)^2 + (0.5+v)^2$

$= 16.25 + u^2 + v^2 - 8u + v$

$$P = -(f_{1.j} + f_{2.i}) = -(f_1 \cdot \underline{n}_1 + (-f_2\, \underline{n}_2)) = -(f_1 \cdot \underline{n}_1 - f_2\, \underline{n}_2) = f_2\, \underline{n}_2 - f_1\, \underline{n}_1$$

same for both elements

$$P = EA\left(\frac{\underline{\ell}_2}{\|\underline{\ell}_2\|}\epsilon_2 - \frac{\underline{\ell}_1}{\|\underline{\ell}_1\|}\epsilon_1\right)$$

$$\underline{n}_k = \frac{\underline{V}_k}{\|\underline{V}_k\|}$$

$$\underline{\ell}_1 = \begin{Bmatrix} 5.5 \\ 0.5 \end{Bmatrix} + \begin{Bmatrix} u \\ v \end{Bmatrix} \qquad \underline{\ell}_2 = \begin{Bmatrix} 4.0 \\ -0.5 \end{Bmatrix} + \begin{Bmatrix} -u \\ -v \end{Bmatrix} = \begin{Bmatrix} 4.0 \\ -0.5 \end{Bmatrix} - \begin{Bmatrix} u \\ v \end{Bmatrix}$$

$\|\underline{\ell}_1\| = \ell_1 \qquad \|\underline{\ell}_2\| = \ell_2$

$$P = EA\left(\frac{\epsilon_2}{\ell_2}\begin{Bmatrix} 4.0-u \\ -0.5-v \end{Bmatrix} - \frac{\epsilon_1}{\ell_1}\begin{Bmatrix} 5.5+u \\ 0.5+v \end{Bmatrix}\right) = \begin{Bmatrix} P_x \\ P_y \end{Bmatrix}$$

Also written as

$$f_2\, \underline{n}_2 - f_1\, \underline{n}_1 = P$$

1-2

2. Find the tangent stiffness matrix of the system. Keep it simple and don't substitute long expressions into your answer

Tangent stiffness for element is written as

$$[K^e] = \begin{bmatrix} k^e & -k^e \\ -k^e & k^e \end{bmatrix}$$

where

$$k^e = \frac{EA}{l}\, \underline{n} \otimes \underline{n} + \frac{\tilde{f}}{l}\left(\underline{\underline{1}} - \underline{n} \otimes \underline{n}\right)$$

where $\tilde{f}$ is force in element

Our global stiffness matrix is then

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} k^1 & -k^1 & 0 \\ -k^1 & k^1 + k^2 & -k^2 \\ 0 & -k^2 & k^2 \end{bmatrix} \end{array}$$

with nodes 1 & 3 being supported

node 2 unsupported

$u_1$ & $u_3 = \underline{0}$, we are therefor only interested in $K_{ff} = k^1 + k^2$

$$k^1 = \frac{EA}{l_1}\, \underline{n}_1 \otimes \underline{n}_1 + \frac{\tilde{f}_1}{l_1}\left(\underline{\underline{1}} - \underline{n} \otimes \underline{n}\right)$$

$$\underline{n}_1 = \begin{Bmatrix} \cos(\theta_1) \\ \sin(\theta_1) \end{Bmatrix} = \begin{Bmatrix} C_1 \\ S_1 \end{Bmatrix} \qquad \tilde{f}_1 = EA\epsilon_1$$

$$= \frac{EA}{l_1}\begin{bmatrix} C_1^2 & C_1 S_1 \\ C_1 S_1 & S_1^2 \end{bmatrix} + \frac{EA\epsilon_1}{l_1}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} C_1^2 & C_1 S_1 \\ C_1 S_1 & S_1^2 \end{bmatrix}\right)$$

$$= \frac{EA(1-\epsilon_1)}{l_1}\begin{bmatrix} C_1^2 & C_1 S_1 \\ C_1 S_1 & S_1^2 \end{bmatrix} + \frac{EA\epsilon_1}{l_1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

1-2 2. continued

$$k^2 = \frac{EA}{l_2}\, \underline{n}_2 \otimes \underline{n}_2 + \frac{\tilde{f}_2}{l_2}\left(\underline{\underline{1}} - \underline{n}_2 \otimes \underline{n}_2\right)$$

$$\underline{n}_2 = \begin{Bmatrix} \cos(\theta_2) \\ -\sin(\theta_2) \end{Bmatrix} = \begin{Bmatrix} C_2 \\ -S_2 \end{Bmatrix} \qquad \tilde{f}_2 = EA \cdot \epsilon_2$$

$$= \frac{EA}{l_2}\begin{bmatrix} C_2^2 & -C_2S_2 \\ -C_2S_2 & S_2^2 \end{bmatrix} + \frac{EA\epsilon_2}{l_2}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} C_2^2 & -C_2S_2 \\ -C_2S_2 & S_2^2 \end{bmatrix}\right)$$

$$= \frac{EA(1-\epsilon_2)}{l_2}\begin{bmatrix} C_2^2 & -C_2S_2 \\ -C_2S_2 & S_2^2 \end{bmatrix} + \frac{EA\epsilon_2}{l_2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$K_{ff}$ then becomes

$$K_{ff} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

where

$$K_{11} = \frac{EA(1-\epsilon_1)}{l_2} C_1^2 + \frac{EA\epsilon_1}{l_2} + \frac{EA(1-\epsilon_2)}{l_2} C_2^2 + \frac{EA\epsilon_2}{l_2}$$

$$= EA\left(\frac{C_1^2(1-\epsilon_1) + \epsilon_1}{l_1} + \frac{C_2^2(1-\epsilon_2) + \epsilon_2}{l_2}\right)$$

$$K_{21} = K_{12} = \frac{EA(1-\epsilon_1)C_1S_1}{l_1} + \frac{EA(1-\epsilon_2)(-C_2S_2)}{l_2}$$

$$= EA\left(\frac{C_1S_1(1-\epsilon_1)}{l_1} - \frac{C_2S_2(1-\epsilon_2)}{l_2}\right)$$

$$K_{22} = \frac{EA(1-\epsilon_1)}{l_1} S_1^2 + \frac{EA\epsilon_1}{l_1} + \frac{EA(1-\epsilon_2)}{l_2} S_2^2 + \frac{EA\epsilon_2}{l_2}$$

$$= EA\left(\frac{S_1^2(1-\epsilon_1) + \epsilon_1}{l_1} + \frac{S_2^2(1-\epsilon_2) + \epsilon_2}{l_2}\right)$$

Terms $\epsilon, l$ have been written out on previous pages.

1-2

3. Formulate the condition $P_x = 0$ to the system of equations. Will this allow you to eliminate the horizontal displacement from the system of equations? Why? Why not?

From 1-2 1. we have

$$\frac{EA\epsilon_2}{l_2}\begin{Bmatrix}4.0-u\\-0.5-v\end{Bmatrix} - \frac{EA\epsilon_1}{l_1}\begin{Bmatrix}5.5+u\\0.5+v\end{Bmatrix} = \begin{Bmatrix}P_x\\P_y\end{Bmatrix}$$

Now setting $P_x = 0$ gives

$$\frac{EA\epsilon_2}{l_2}\begin{Bmatrix}4.0-u\\-0.5-v\end{Bmatrix} - \frac{EA\epsilon_1}{l_1}\begin{Bmatrix}5.5+u\\0.5+v\end{Bmatrix} = \begin{Bmatrix}0\\P_y\end{Bmatrix}$$

Looking only at force in x direction

$$EA\underbrace{\left(\frac{\epsilon_2}{l_2}(4.0-u) - \frac{\epsilon_1}{l_1}(5.5+u)\right)}_{=0 \text{ to fulfill condition}} = 0$$

Solving for $u$ gives

$$u = \frac{5.5\cdot\frac{\epsilon_1}{l_1} - 4.0\frac{\epsilon_2}{l_2}}{-\left(\frac{\epsilon_1}{l_1} + \frac{\epsilon_2}{l_2}\right)}$$

But $l_1, l_2, \epsilon_1, \epsilon_2$ are all functions of $u$, we can therefor not eliminate $u$.

Side note, if system was symmetric and $l_1 = \begin{Bmatrix}a\\b\end{Bmatrix}$ $l_2 = \begin{Bmatrix}a\\-b\end{Bmatrix}$

we would get

$$\frac{\epsilon_2}{l_2}(a-u) - \frac{\epsilon_1}{l_1}(a+u) \qquad \text{w/ } \epsilon_2 = \epsilon_1 = \epsilon, \; l_2 = l_1 = l$$

$$\Rightarrow 2u\frac{\epsilon}{l} = P_x = 0$$

Which would only be fulfilled at $u = 0$ and we could eliminate $u$