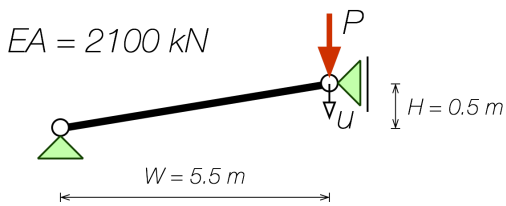
With the two previous assignments, you explored three simple nonlinear systems. You solved Problem 1-1 using displacement control (in its most primitive form), and Problem 1-2 using incremental load control (prescribing the load level) and as Problem 2-1 using the general form of displacement control.

This week, you'll be exploring path following techniques for tracking equilibrium paths beyond a snap-through[1] or a snap-back[2] point.

Problem 3-1: Introducing an arc-length method for a single degree of freedom problem.



Consider Problem 1-1 from Assignment #1. Let's restrict this study to the use of Henky strain, $\varepsilon = \ln(\ell/L)$, and assume the cross section area, A, does not change during the deformation process.

Use your solutions for $F(u)$ from this past assignment and

1. find the tangent stiffness
$$K_T = \frac{d\,F(u)}{du} \ . \tag{1a}$$
For this single d.o.f.-problem, the stiffness matrix K_T degenerates to a scalar and traditional differentiation may be used.

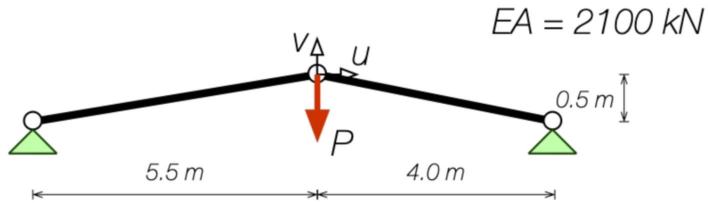
2. Arc-length method: Develop a Newton method for solving the system
$$R(\gamma, u) = \gamma(s)\bar{P} - F(u(s)) = 0 \tag{1b}$$
with reference load, $\bar{P} = 0.30$ kN, for load intensity γ and displacement u using the arc-length constraint
$$\Delta s^2 = \Delta u \cdot \Delta u + \alpha(\Delta\gamma)^2 \tag{1c}$$
with
$$\Delta u = u_{n+1} - u_n \quad \text{and} \quad \Delta\gamma = \gamma_{n+1} - \gamma_n \ . \tag{1d}$$
Initialize the arc-length Δs to the value found from a single load-controlled step for $\gamma_1 = 0.25$. Test your algorithm with $\alpha = 0$, $\alpha = 0.5$, and $\alpha = 1$.

3. Show the obtained solution for $0 \leq \gamma \leq 2.0$. Overlay the computed discrete solutions (plot just dots for the discrete solution) on the solution from Assignment #1 to verify that your algorithm is producing correct results.

4. Provide a plot of $|R|$ versus iteration counter[3], and a plot of $|R|$ versus arc-length, s. Use a semi-log-y plot in both cases.

[1] This is where load control will fail.
[2] This is where displacement control will fail.
[3] Keep counting as you go from increment to increment. This plot should look like saw teeth.

Problem 3-2: Two degree of freedom (2 DOF) problem – Arc-length method.



This problem expands Problems 1-2 and 2-1 from Assignments #1 and #2, respectively, such that you shall track the entire equilibrium path from $(u = 0, v = 0)$ and $\gamma = 0$ at least through $\gamma \geq 2.0$ using an arc-length method.

1. Using Henky strain, a linear relation $\sigma = E\varepsilon$, $A = const.$, and equilibrium on the deformed system, derive the relationship between displacements and forces on the free node. Adjust it for path following (using the arc-length method) as

$$\mathbf{R}(\gamma, \mathbf{u}) = \gamma(s)\bar{\mathbf{P}} - \mathbf{F}(\mathbf{u}(s)) = \mathbf{0} \tag{2a}$$

   with reference load, $\bar{\mathbf{P}} = -(0.99 \text{ kN})\,\mathbf{j}$, load intensity γ and displacement $\mathbf{u}$.

   Find the tangents $\partial\mathbf{R}/\partial\gamma$ and $\partial\mathbf{R}/\partial\mathbf{u}$ as you will need them in what follows.

   You should have all the necessary parts for this question from Assignment #1.

2. Develop a, or adjust your existing Newton method to incrementally find $\gamma(s)$ and $\mathbf{u}(s)$ using the arc length constraint

$$\Delta s^2 = \Delta\mathbf{u} \cdot \Delta\mathbf{u} + \alpha(\Delta\gamma)^2 \tag{2b}$$

   with

$$\Delta\mathbf{u} = \mathbf{u}_{n+1}^{(k)} - \mathbf{u}_n \quad \text{and} \quad \Delta\gamma = \gamma_{n+1}^{(k)} - \gamma_n \tag{2c}$$

   and $0 \leq \alpha \leq 1$. $\mathbf{u}_{n+1}^{(k)}$ and $\gamma_{n+1}^{(k)}$ are the k-the approximation for displacement and load factor, respectively. $\mathbf{u}_n$ and γ_n are the converged solution from the previous step and, as such, treated as known constants in this algorithm.

3. Use the load stepping algorithm from Assignment #1 and solve for $\gamma_1 = 0.200$. Use that load factor and the obtained displacement $\mathbf{u}_1$ to define the target arc length as

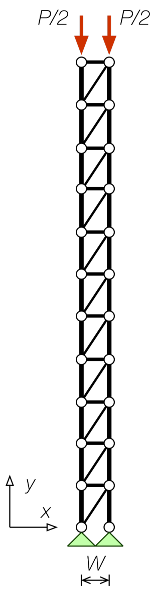
$$\Delta s^2 = \mathbf{u}_1 \cdot \mathbf{u}_1 + \alpha(\gamma_1)^2 \ . \tag{2d}$$

   Use this target for all subsequent steps and follow the path until $\gamma \geq 2.0$. You may need to experiment with the parameter α in order to get past the snap-through point.

4. Plot load level $\gamma(s)$ versus vertical displacement $v(s)$, as well as $\gamma(s)$ versus horizontal displacement $u(s)$ for the converged points on the equilibrium path. Plotting this solution on top of the solution obtained using displacement control (from Assignment #2) must show identical curves, though the points will be at different locations along that curve.

Problem 3-3: Tracking the path of a truss-column.

With your arc-length method in place and working nicely, use your truss element to create a more realistic (though ideally elastic) system as shown:



Parameters

Height: $H = 5.0$ m
Width: $W = H/20$

Vertical members: $EA = 2000$ kN

Horizontal ties: $EA = 5000$ kN

Diagonals: $EA = 5000$ kN

1. System setup:
   - Complete the figure with node numbers and element labels. You may download a larger version of the image from the Assignment3 section under files on canvas.
   - Create a table (in your software, not needed on paper) that lists the global numbers for nodes i and j for each element. Add element properties as you see useful for your code, e.g., EA. This is your mesh definition. This may be represented as a matrix in MATLAB or a list of dictionaries in Python.
   - Create an assembly function that takes the above list and creates a system force vector $\mathbf{F}$ and a system tangent stiffness matrix $\mathbf{K}_T$. (you should have 44 d.o.f.s)
2. Estimate the buckling load of the column using a beam model. You can easily get the bending stiffness EI from treating the truss as a beam with holes in it and concentrated areas (the vertical members) on top and bottom flange. Use this buckling load as reference load.
3. Use the arc-length method to trace the deformation of the beam to a point when the first loaded node hits ground level. This will create very large displacements and (unrealistic) large elastic strain in several members. Pick the target arc length, Δs, such that you have at least 5 increments before reaching the buckling load ($\gamma = 1.0$) or a maximum of the load factor, γ – whichever occurs first.
4. Provide force–displacement plots for the top nodes (may fit into one or two plots).
5. Show a trace plot of the top nodes, i.e., plot their paths $\{x(s), y(s)\}$. Make sure to use equal scale factors for both axes.
6. If you feel like some more programming/plotting, use your mesh definition table and create a deformed plot of the system. This may be a smart thing to do in collaboration to share the effort.