Execises Week 10 (MATH 310):

Ex1:

$$\hat{a} = Q_{j-1}Q_{j-1}^t\vec{a} = Q_{j-1}\begin{bmatrix} \vec{q}_1^t\vec{a} \\ \vec{q}_2^t\vec{a} \\ \vdots \\ \vec{q}_{j-1}^t\vec{a} \end{bmatrix} = \underbrace{\begin{bmatrix} \vec{q}_1 & \vec{q}_2 & \cdots & \vec{q}_{j-1} \end{bmatrix}}_{(Q_{j-1})} \begin{bmatrix} \vec{q}_1^t\vec{a} \\ \vec{q}_2^t\vec{a} \\ \vdots \\ \vec{q}_{j-1}^t\vec{a} \end{bmatrix}$$

$$= \left(\sum_{k=1}^{j-1}\vec{q}_k\vec{q}_k^t\right)\vec{a}$$

The two matrices $Q_{j-1}Q_{j-1}^t$ and $\sum_{k=1}^{j-1}\vec{q}_k\vec{q}_k^t$ are equal because $\boxed{Q_{j-1}Q_{j-1}^t\vec{e}_k = \left(\sum_{k=1}^{j-1}\vec{q}_k\vec{q}_k^t\right)\vec{e}_k \quad k=1,\dots,j-1}$ shows that all their colums are identical ($\vec{e}_k$ is the k-th standard basis vector.)

Ex2:

$P_j = I_m - Q_{j-1}Q_{j-1}^t$ is a projection because

1) $P_j * P_j = I_m - Q_{j-1}Q_{j-1}^t \cancel{- Q_{j-1}Q_{j-1}^t + Q_{j-1}\underbrace{Q_{j-1}^t Q_{j-1}}_{I}Q_{j-1}^t} = I_m - Q_{j-1}Q_{j-1}^t = P_j$

2 $P_j^t = I_m^t - (Q_{j-1}^t)^t Q_{j-1}^t = I_m - Q_{j-1}Q_{j-1}^t = P_j$

Ex 3:

To verify that P_j is the projection onto $Col(Q_{j-1})^{\perp}$ $= Col([\vec{q}_1\ \vec{q}_2 \cdots \vec{q}_{j-1}])^{\perp}$ we must show that

i) $\vec{q}^{\,t}(P_j\vec{a}) = 0$ for any $\vec{q} \in Col(Q_{j-1})$

$\boxed{\text{and}}$

ii) $\vec{p} = P_j\vec{p}$ for any $\vec{p} \in Col(Q_{j-1})^{\perp}$

i): $q^t \overbrace{(I - Q_{j-1}Q_{j-1}^t)}^{P_j}\vec{a} = (\vec{q}^{\,t} - \vec{q}^{\,t}Q_{j-1}Q_{j-1}^t)\vec{a}$

& because $\boxed{\vec{q}^{\,t} = (Q_{j-1}Q_{j-1}^t\vec{q})^t = q^tQ_{j-1}Q_{j-1}^t}$

$= \vec{0}^{\,t}\vec{a} = 0.$

ii)

$P_j\vec{p} = (I - Q_{j-1}Q_{j-1}^t)p = \vec{p} - \underbrace{Q_{j-1}Q_{j-1}^t p}_{=0} = \vec{p}.$

Ex. 4:

$$P_{\perp q_k} \overset{def}{=} I - \vec{q}_k \vec{q}_k^{\,t}$$

Hence $\vec{q}_k^{\,t}(I - \vec{q}_k \vec{q}_k^{\,t})A = (\vec{q}_k^{\,t} - \underbrace{\vec{q}_k^{\,t}\vec{q}_k q_k^t}_{q_k^t})A = (q_k^t - q_k^t)A = \vec{0}$

$\boxed{\text{OK for } P_2 = I - \vec{q}_1\vec{q}_1^{\,t} = I - Q_1 Q_1^t \;\blacksquare}$

$\boxed{\text{By induction}}$ $P_{\perp \vec{q}_{j-1}} P_{\perp \vec{q}_{j-2}} \cdots P_{\perp \vec{q}_2} P_{\perp \vec{q}_1} \overset{I.H.}{=} (I - \vec{q}_{j-1} \vec{q}_{j-1}^{\,t}) P_{j-1}$

$$= (I - \vec{q}_{j-1}\vec{q}_{j-1}^{\,t})(I - \sum_{k=1}^{j-2} \vec{q}_k \vec{q}_k^{\,t}) = I - \sum_{k=1}^{j-2} \vec{q}_k \vec{q}_k^{\,t} - \vec{q}_{j-1}\vec{q}_{j-1}^{\,t} + \vec{q}_{j-1}\vec{q}_{j-1}^{\,t} \sum_{k=1}^{j-2} \vec{q}_k \vec{q}_k^{\,t}$$

$$= I - \sum_{k=1}^{j-1} \vec{q}_k \vec{q}_k^{\,t} + \underbrace{\vec{q}_{j-1} \sum_{k=1}^{j-2} \underbrace{\vec{q}_{j-1}^{\,t} \vec{q}_k}_{0} \vec{q}_k^{\,t}}_{0\text{-matrix}} = \underbrace{I - \sum_{k=1}^{j-1} \vec{q}_k \vec{q}_k^{\,t}}_{P_j} = I - Q_{j-1} Q_{j-1}^t \;\blacksquare$$

$P_{j-1} = I - Q_{j-2} Q_{j-2}^t = I - \sum_{k=1}^{j-2} q_k q_k^t$

Ex 5:

See "MGS.jl".

Ex 6:

See "Execise6solution.jl"

Ex 7:

See "Exercise7solution.jl".

Ex 8:

The rank k (truncated) pseudo-inverse of X ($= U_r S_r V_r^t$) is

$$X_k^+ = V_k S_k^{-1} U_k^t$$

The bi-diagonal factorization

$$\tilde{X}_k \stackrel{def}{=} T_k B_k W_k^t$$

has the pseudo-inverse

$$\tilde{X}_k^+ = W_k B_k^{-1} T_k^t$$

& from $\boxed{B_k = P_k^t W_k}$

$$\vec{\beta}_k = W_k B_k^{-1} T_k^t \vec{y}_0 \boxed{= \tilde{X}_k^+ \vec{y}_0}$$

, we're done.

Ex 9:

See "RR_PCR_Execrcises_worked.jl".