PHY 313 — Satellites & Space Science

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Problem 1:

A satellite is in an orbit with a semi-major axis of $6750\,\mathrm{km}$, an inclination of $35°$, and an eccentricity of 0.05. Calculate the J2 perturbations to the longitude of the ascending node and to the argument of perigee.

Solution   The formulas are taken from the slides. https://www.desmos.com/calculator/azhgfmdir8

$$\dot{\Omega}_{\mathrm{J}_2} = -1.5\sqrt{\frac{GM}{a^3}}\,J_2\left(\frac{R_{\mathrm{E}}}{a}\right)^2\cos(i)\left(1-e^2\right)^{-2} = -1.3591\times10^{-6}\,\mathrm{degrees/sec} = -0.117\,43\,\mathrm{degrees/day}$$

$$\approx -2.064\,74\times10^{14}\,a^{-\frac{7}{2}}\cos(i)(1-e^2)^{-2} \approx \boxed{-6.727\,\mathrm{degrees/day}}$$

The negative sign implies that the ascending node moves opposite to the direction of flight.

$$\dot{\omega}_{\mathrm{J}_2} = 0.75\sqrt{\frac{GM}{a^3}}\,J_2\left(\frac{R_{\mathrm{E}}}{a}\right)^2\left(4-5\sin^2(i)\right)\left(1-e^2\right)^{-2} = -1.8234\times10^{-6}\,\mathrm{degrees/sec} = -0.157\,55\,\mathrm{degrees/day}$$

$$\approx 1.032\,37\times10^{14}\,a^{-\frac{7}{2}}(4-5\sin^2(i))(1-e^2)^{-2} \approx \boxed{9.670\,\mathrm{degrees/day}}$$

The initial equation for both parameters fails to give the same value as the approximate one. Could be a missing parameter.

$$[\dot{\Omega}_{\mathrm{J}_2}] = \frac{(\mathrm{m^{3/2}\,kg^{-1/2}\,s^{-2/2}})(\mathrm{kg^{1/2}})}{(\mathrm{m^{3/2}})} = \mathrm{s^{-1}} \Rightarrow \mathrm{degrees/sec}$$

Problem 2:

Estimate the fraction of free electrons to neutral atoms/molecules on Earth at room temperature and in the magnetosphere, assuming both gases/plasmas are in equilibrium (ignore radiation).

Solution   Manipulating the Saha equation similar to how it was discussed in lecture.

$$\frac{n_{\mathrm{e}}^2}{n-n_{\mathrm{e}}} = \left(\frac{2\pi m_{\mathrm{e}}k_{\mathrm{B}}T}{h^2}\right)^{\frac{3}{2}}\frac{2g_i}{g_0}\exp\left(-\frac{E_i}{k_{\mathrm{B}}T}\right)$$

Let the ratio $\dfrac{n_e}{n} = x_i$.

$$\frac{n^2\left(\frac{n_{\mathrm{e}}}{n}\right)^2}{n\left(1-\frac{n_{\mathrm{e}}}{n}\right)} = \left(\frac{2\pi m_{\mathrm{e}}k_{\mathrm{B}}T}{h^2}\right)^{\frac{3}{2}}\frac{2g_i}{g_0}\exp\left(-\frac{E_i}{k_{\mathrm{B}}T}\right)$$

$$\frac{x_i^2}{1-x_i} = \frac{1}{n}\left[\left(\frac{2\pi m_{\mathrm{e}}k_{\mathrm{B}}T}{h^2}\right)^{\frac{3}{2}}\frac{2g_i}{g_0}\exp\left(-\frac{E_i}{k_{\mathrm{B}}T}\right)\right]$$

Multiply the right side of the equation by $\dfrac{c^3}{c^3}$.

$$\frac{x_i^2}{1-x_i} = \frac{1}{n}\left[\left(\frac{2\pi (m_{\mathrm{e}}c^2)k_{\mathrm{B}}T}{(hc)^2}\right)^{\frac{3}{2}}\frac{2g_i}{g_0}\exp\left(-\frac{E_i}{k_{\mathrm{B}}T}\right)\right]$$

The transformations simplify the expression and allows the use of eV in computation.

$$k = 8.62\times10^{-5}\,\mathrm{eV\,K^{-1}}$$
$$m_{\mathrm{e}}c^2 = 0.511\,\mathrm{MeV}$$
$$hc = 1.24\times10^{-6}\,\mathrm{eV\,m}$$

Starting with the ratio on Earth at room temperature, $T \approx 20\,°\mathrm{C} = 293.15\,\mathrm{K}$. At a glance, this scenario will not produce a plasma because of the low temperature. To show that I will focus on the exponential term. E_i is the average ionization energy of the given gas. For dry air, there is a well-known value for the average energy needed to form an electron-ion pair, $E = 33.97\,\mathrm{eV/ion}$ pair. With this we can quickly show if the first guess was correct.

$$\exp\left(-\frac{33.97\,\mathrm{eV}}{(86.2\,\mu\mathrm{eV\,K^{-1}})(293.15\,\mathrm{K})}\right) = 0$$

The exponential term will dominate and cause the whole expression to equal 0. This means that there are no free electrons present in this scenario.

Now for the magnetosphere, I will use the values given in the slides. $n \approx 0.5\,\mathrm{atoms/cm^3}$, $T \approx 10^4\,\mathrm{K}$. In this case, I will assume that the magnetosphere is mostly composed of hydrogen atoms. $E_i = 13.6\,\mathrm{eV}$. Quick test with the exponential term.

$$\exp\left(-\frac{13.6\,\mathrm{eV}}{(86.2\,\mu\mathrm{eV\,K^{-1}})(10^4\,\mathrm{K})}\right) = 1.4061 \times 10^{-7}$$

$$\frac{x_i^2}{1-x_i} = \frac{1}{0.5 \times 10^6\,\mathrm{atoms/m^3}}\left[\left(\frac{2\pi(0.511\,\mathrm{MeV})(8.62 \times 10^{-5}\,\mathrm{eV\,K^{-1}})(10^4\,\mathrm{K})}{(1.24 \times 10^{-6}\,\mathrm{eV\,m})^2}\right)^{\frac{3}{2}} \frac{2(1)}{(2)}(1.4061 \times 10^{-7})\right] = 6.7911 \times 10^{14}$$

Since the ratio is large, the 1 minus will only flip the sign. The sign does not matter in ratios.

$$\frac{x_i^2}{x_i} = x_i \approx 6.7911 \times 10^{14}$$

Can be found quadratically.

$$x_i^2 + 6.7911 \times 10^{14} x_i - 6.7911 \times 10^{14} = 0$$
$$x_i = \frac{-(6.7911 \times 10^{14}) \pm \sqrt{(6.7911 \times 10^{14})^2 + (2.716\,44 \times 10^{15})}}{2}$$
$$x_i^+ = 0 \qquad x_i^- = -6.7911 \times 10^{14}$$

Problem 3:

(a) Determine the Debye length in the solar corona with typical physical parameters.

Solution   Looking at the Debye length variables, we only need the temperature and the number density at the solar corona. Comparing values between slides and online gives, $T \approx 10^6$ K and $n \approx 10^{14}$ atoms/m^3

$$\lambda_D = \left(\frac{\epsilon_0 k_{\mathrm{B}} T}{n e^2} \right)^{\frac{1}{2}} = \left(\frac{(55.263\,\mathrm{e}^2/\mathrm{eV\,\mu m})(8.62 \times 10^{-5}\,\mathrm{eV\,K^{-1}})(10^6\,\mathrm{K})}{(10^{14}\,\mathrm{atoms/m^3})e^2} \right)^{\frac{1}{2}} \approx 6.90 \times 10^{-3}\,\mathrm{m}$$

(b) Determine the electrostatic potential from a point charge of 5 nC at 0.01 m and 0.1 m in the ionosphere.

Solution   Simple to do.

$$\phi_{\mathrm{C}}(r) = \frac{Q}{4\pi\epsilon_0 r}$$

$$\phi_{\mathrm{C}}(0.01\,\mathrm{m}) = \frac{(5\,\mathrm{nC})}{4\pi(8.85 \times 10^{-12}\,\mathrm{F\,m^{-1}})(0.01\,\mathrm{m})} = 4496\,\mathrm{V}$$

$$\phi_{\mathrm{C}}(0.1\,\mathrm{m}) = \frac{(5\,\mathrm{nC})}{4\pi(8.85 \times 10^{-12}\,\mathrm{F\,m^{-1}})(0.1\,\mathrm{m})} = 449.6\,\mathrm{V}$$

(c) Plot the electric potential due to this point charge, with and without Debye shielding.
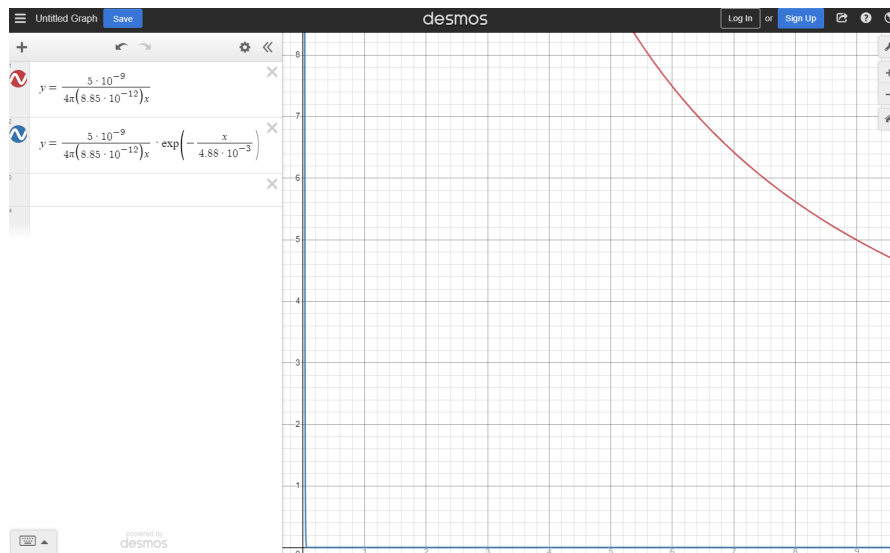
Solution   First we need to find the Debye length in the ionosphere with typical physical paramaters. From the slides: $T \approx 500$ K and $n \approx 10^{11}$ atoms/m^3.

$$\lambda_D = \left(\frac{\epsilon_0 k_{\mathrm{B}} T}{n e^2} \right)^{\frac{1}{2}} = \left(\frac{(55.263\,\mathrm{e}^2/\mathrm{eV\,\mu m})(8.62 \times 10^{-5}\,\mathrm{eV\,K^{-1}})(500\,\mathrm{K})}{(10^{11}\,\mathrm{atoms/m^3})e^2} \right)^{\frac{1}{2}} \approx 4.88 \times 10^{-3}\,\mathrm{m}$$

The modified electric potential due to a point charge:

$$\phi_{\mathrm{C}}(r) = \frac{Q}{4\pi\epsilon_0 r} \exp\left(-\frac{e}{\lambda_D} \right)$$

I used desmos to plot the graph. https://www.desmos.com/calculator/nslvejzedv

Problem 4:

Calculate the (typical) beta number for a plasma in the photosphere, in a solar coronal loop, in the solar wind, and in the Earth's magnetosphere and ionosphere (look up the values that you need).

Solution Only need to plug in the typical physical parameters for each scenario.

$$\beta = \frac{P_{\text{pl}}}{P_{\text{B}}} = \frac{nk_{\text{B}}T}{B^2/2\mu_0}$$

Photosphere: $T \approx 5800\,\text{K}$, $n \approx 10^{23}\,\text{atoms/m}^3$, $B \approx 10^{-3}\,\text{T}$. Complicated behavior across the surface of the sun, with differential rotation that causes magnetic field lines to intersect.

$$\beta \approx 20\,137$$

Solar coronal loop: $T \approx 5 \times 10^6\,\text{K}$, $n \approx 10^{18}\,\text{atoms/m}^3$, $B \approx 0.1\,\text{T}$. Charged particles flow through the loop.

$$\beta \approx 1.7359 \times 10^{-2}$$

Solar wind: $T \approx 5 \times 10^5\,\text{K}$, $n \approx 5 \times 10^6\,\text{atoms/m}^3$, $B \approx 6\,\text{nT}$. Complicated behavior over large distances.

$$\beta \approx 2.411$$

Magnetosphere: $T \approx 10^4\,\text{K}$, $n \approx 0.5 \times 10^6\,\text{atoms/m}^3$, $B \approx 0.04\,\text{T}$. Charged particles follow the magnetic field lines.

$$\beta \approx 1.08 \times 10^{-16}$$

Ionosphere: $T \approx 500\,\text{K}$, $n \approx 10^{11}\,\text{atoms/m}^3$, $B \approx 10^{-8}\,\text{T}$. Complicated behavior across this section of the atmosphere. Many phenomena occur like the sporadic e-like layers.

$$\beta \approx 17.4$$

Problem 5:

(a) What is the gyro-frequency and the Larmor radius of an electron in a solar flare? Why does the plasma follow the magnetic field lines?.

Solution Using the equations and understanding the conditions predicted by the β value. For a solar flare: $T \approx 10^6 \sim 10^7$ K, $n \approx 10^{16} \sim 10^{20}$ atoms/m^3, $B \approx 0.1$ T.

$$\omega = \frac{q\|B\|}{m} = \frac{(1.6 \times 10^{-19}\,\text{C})(0.1\,\text{T})}{(9.11 \times 10^{-31}\,\text{kg})} = \boxed{1.7563 \times 10^{10}\,\text{s}^{-1}}$$

$$R_\text{L} = \frac{v_\perp}{\omega}$$

To find $v_\perp$ we use the equipartition principle.

$$K = K_\| + K_\perp = 3\left(\frac{1}{2}k_\text{B}T\right)$$

$$= \frac{1}{2}mv_\|^2 + \frac{1}{2}mv_\perp^2$$

$$= 1\left(\frac{1}{2}k_\text{B}T\right) + 2\left(\frac{1}{2}k_\text{B}T\right)$$

$$\frac{1}{2}mv_\perp^2 = 2\left(\frac{1}{2}k_\text{B}T\right)$$

$$v_\perp = \sqrt{\frac{2k_\text{B}T}{m}} = \sqrt{\frac{2(1.38 \times 10^{-23}\,\text{J K}^{-1})(5 \times 10^6\,\text{K})}{(9.11 \times 10^{-31}\,\text{kg})}} = 1.2308 \times 10^7\,\text{m s}^{-1}$$

$$R_\text{L} = \frac{1.2308 \times 10^7\,\text{m s}^{-1}}{1.7563 \times 10^{10}\,\text{s}^{-1}} = \boxed{7.0079 \times 10^{-4}\,\text{m}}$$

In the previous problem, we found that $\beta \approx 1.7359 \times 10^{-2}$ for a solar loop/flare. For $\beta << 1$, the model predicts that the plasma magnetic pressure P_B dominates the plasma thermal pressure P_pl because the magnetic field strength is strong. This forces the plasma to follow the magnetic field lines.

(b) Taking the Earth's magnetic field as a dipole (inverse cube dependence on radial distance), estimate the Larmor radii at a distance of $5R_\text{E}$ in the equatorial plane, for protons and electrons with $1\,\text{keV}$ energy, knowing the magnitude of $\boldsymbol{B}_\text{E}$ on the equator at Earth's surface $= 3 \times 10^{-5}$ T.

Solution Similar steps to previous part. First to find the magnetic field strength at $5R_\text{E}$. For temperature $T \approx 10^5 \sim 10^6$ K for average particle conditions around Earth.

$$\text{We have } B_\text{E}(a) \propto \frac{1}{r_a^3} \quad \Rightarrow \quad \frac{B_\text{E}(a)}{B_\text{E}(b)} = \left(\frac{r_b}{r_a}\right)^3$$

$$B(5R_\text{E}) = B(R_\text{E})\left(\frac{R_\text{E}}{5R_\text{E}}\right)^3 = \frac{3 \times 10^{-5}\,\text{T}}{125} = 2.4 \times 10^{-7}\,\text{T}$$

Converting the energy to temperature.

$$K = \frac{3}{2}k_\text{B}T$$

$$T = \frac{2K}{3k_\text{B}} = \frac{2(10^3\,\text{eV})}{3(8.62 \times 10^{-5}\,\text{eV K}^{-1})} = 7.7340 \times 10^6\,\text{K}$$

$$v_\perp(e) = \sqrt{\frac{2(1.38 \times 10^{-23}\,\text{J K}^{-1})(7.7340 \times 10^6\,\text{K})}{(9.11 \times 10^{-31}\,\text{kg})}} = 1.5307 \times 10^7\,\text{m s}^{-1}$$

$$R_\text{L}(e) = \frac{mv_\perp(e)}{q\|B\|} = \frac{(9.11 \times 10^{-31}\,\text{kg})(1.5307 \times 10^7\,\text{m s}^{-1})}{(1.60 \times 10^{-19}\,\text{C})(2.4 \times 10^{-7}\,\text{T})} = \boxed{3.6314 \times 10^2\,\text{m}}$$

$$v_\perp(p) = \sqrt{\frac{2(1.38 \times 10^{-23}\,\text{J K}^{-1})(7.7340 \times 10^6\,\text{K})}{(1.67 \times 10^{-27}\,\text{kg})}} = 3.5752 \times 10^5\,\text{m s}^{-1}$$

$$R_\text{L}(p) = \frac{mv_\perp(p)}{q\|B\|} = \frac{(1.67 \times 10^{-27}\,\text{kg})(3.5752 \times 10^5\,\text{m s}^{-1})}{(1.60 \times 10^{-19}\,\text{C})(2.4 \times 10^{-7}\,\text{T})} = \boxed{1.5548 \times 10^4\,\text{m}}$$

Both particles have large larmor radii, indicating that they loosely spiral around the field lines, somewhere in the van allen radiation belt. This is largely because of the weak magnetic field strength at $5R_\text{E}$.