PHY 313 — Satellites & Space Science

Name: Khalifa Salem Almatrooshi
Student Number: @00090847

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Assignment: HW 1

Problem 1:

Knowing that the solar energy flux at Earth is $\boldsymbol{1.37\ kW\ m^{-2}}$,

(a) determine the average temperature on the Earth's surface;

Solution We can use Stefan-Boltzmann's Law to find the effective temperature of the Earth's surface, in other words, the black-body temperature where the emissivity $\boldsymbol{e}$ in the equation is equal to $\mathbf{1}$.

$$P = \sigma e A T^4 = \sigma A T_{\text{eff}}^4$$

Many factors contribute to the Earth's surface temperature, with the most important ones being the irradiance received from the Sun $\boldsymbol{S(r_E)}$, and the irradiance radiated by the Earth itself $\boldsymbol{Q}$.

$$P = \sigma A_{\text{surface}} T_{\text{eff}}^4 = A_{\text{incident}} S(r_E) + A_{\text{surface}} Q$$
$$\sigma 4\pi R_E^2 T_{\text{eff}}^4 = \pi R_E^2 S(r_E) + 4\pi R_E^2 Q$$
$$T_{\text{eff}} = \left[\frac{1}{4\sigma}\left(S(r_E) + 4Q\right)\right]^{1/4}$$

The value for $\boldsymbol{Q}$ is known to be around $\boldsymbol{0.06\ kWm^{-2}}$. While $\boldsymbol{S(r_E)}$ is easily calculated by the following formula.

$$S(r) = S_E\left(\frac{r_E}{r}\right)^2 (1 - A)$$

$$\begin{aligned} S(r_E) &= (1.37 \times 10^3\ Wm^{-2})\left(\frac{1\ AU}{1\ AU}\right)^2 (1 - 0.30) \\ &= (1.37 \times 10^3\ Wm^{-2})(0.70) \\ S(r_E) &= 9.59 \times 10^2\ Wm^{-2} \end{aligned}$$

$$T_{\text{eff}} = \left[\frac{1}{4(5.6703 \times 10^{-8}\ Wm^{-2}K^{-4})}\left((9.59 \times 10^2\ Wm^{-2}) + 4(0.06 \times 10^3\ Wm^{-2})\right)\right]^{1/4}$$
$$T_{\text{eff}} = \left[\frac{1.20 \times 10^3}{2.2681 \times 10^{-7}}\right]^{1/4} K \approx 270\ K$$

(b) do the same for Mars, assuming it behaves the same as Earth in every regard.

Solution The only difference for Mars would be the distance from the Sun $\boldsymbol{r_M}$. The radius of Mars $\boldsymbol{R_M}$ is not needed in this case as it cancels out when finding the $\boldsymbol{T_{\text{eff}}}$, as shown above with $\boldsymbol{R_E}$. The same $\boldsymbol{Q}$ and albedo $\boldsymbol{A}$ is used.

$$T_{\text{eff}} = \left[\frac{1}{4\sigma}\left(S(r_M) + 4Q\right)\right]^{1/4}$$

$$\begin{aligned} S(r_M) &= (1.37 \times 10^3\ Wm^{-2})\left(\frac{1\ AU}{1.52\ AU}\right)^2 (1 - 0.30) \\ S(r_M) &= 4.15 \times 10^2\ Wm^{-2} \end{aligned}$$

$$T_{\text{eff}} = \left[\frac{1}{4(5.6703 \times 10^{-8}\ Wm^{-2}K^{-4})}\left((4.15 \times 10^2\ Wm^{-2}) + 4(0.06 \times 10^3\ Wm^{-2})\right)\right]^{1/4}$$
$$T_{\text{eff}} = \left[\frac{6.55 \times 10^2}{2.2681 \times 10^{-7}}\right]^{1/4} K \approx 232\ K$$

Problem 2:

Derive the dependence of the acceleration of gravity on height/distance from Earth, and compare the "apparent weight" (in kg) of a **500** $\boldsymbol{kg}$ satellite and a **75** $\boldsymbol{kg}$ astronaut in space at altitudes of:

(a) **400** $\boldsymbol{km}$

Solution First to derive the dependence of the acceleration of gravity on height/distance from Earth using Newton's Law of Universal Gravitation.

$$F_g = G\frac{m_1 m_2}{r^2}$$

For an object on the surface of the Earth, $\boldsymbol{m_1 = M_E}$, $\boldsymbol{m_2 = m}$, and $\boldsymbol{r = R_E}$.

$$ma = G\frac{M_E m}{R_E^2}$$

$$a = \frac{GM_E}{R_E^2} = \frac{(6.6743 \times 10^{-11}\ Nm^2kg^{-2})(5.972 \times 10^{24}\ kg)}{(6371 \times 10^3\ m)^2} = 9.82\ ms^{-2} = g$$

For an object a few kilometers off the surface of the Earth like a satellite or an astronaut, its as simple as adding the object's altitude in the denominator.

$$mg = G\frac{M_E m}{(R_E + h)^2}$$

$$g(h) = G\frac{M_E}{(R_E + h)^2} = \frac{GM_E}{R_E^2}\frac{1}{(1 + \frac{h}{R_E})^2} = \frac{g}{(1 + \frac{h}{R_E})^2}$$

Now to use the derived dependence to find the acceleration of gravity at each altitude and the apparent weight for a 500 kg satellite and a 75 kg astronaut. This assumes that they are both stationary and not in orbit around the Earth, so the only force acting on them is gravity.

$$g(400\ km) = \frac{9.82\ ms^{-2}}{(1 + \frac{400 \times 10^3\ m}{6371 \times 10^3\ m})^2} = 8.69\ ms^{-2}$$

$$\text{Satellite: } F = (500\ kg)(8.69\ ms^{-2}) = 4350\ N$$

$$\text{Astronaut: } F = (75\ kg)(8.69\ ms^{-2}) = 652\ N$$

(b) **2000** $\boldsymbol{km}$

Solution

$$g(2000\ km) = \frac{9.82\ ms^{-2}}{(1 + \frac{2000 \times 10^3\ m}{6371 \times 10^3\ m})^2} = 5.69\ ms^{-2}$$

$$\text{Satellite: } F = (500\ kg)(5.69\ ms^{-2}) = 2850\ N$$

$$\text{Astronaut: } F = (75\ kg)(5.69\ ms^{-2}) = 427\ N$$

(c) **36000** $\boldsymbol{km}$

Solution

$$g(36000\ km) = \frac{9.82\ ms^{-2}}{(1 + \frac{36000 \times 10^3\ m}{6371 \times 10^3\ m})^2} = 0.22\ ms^{-2}$$

$$\text{Satellite: } F = (500\ kg)(0.22\ ms^{-2}) = 110\ N$$

$$\text{Astronaut: } F = (75\ kg)(0.22\ ms^{-2}) = 16.5\ N$$

Problem 3:

Determine the escape velocities from:

(a) the surface of the Moon;

Solution In class we derived the escape velocity $\boldsymbol{v_{esc}}$ using the work-kinetic energy theorem. One step back from this is to start from Newton's Law of Universal Gravitation $\boldsymbol{F_g}$, and calculating the potential energy function $\boldsymbol{U}$ integrated from the surface of a celestial body to infinity.

$$F_g = \frac{GMm}{r^2} \qquad U = -\int F\,ds$$

$$U_g = -\int_R^\infty \frac{GMm}{r^2}\,dr = -GMm\left[-\frac{1}{r}\right]_R^\infty = GMm\left[\frac{1}{\infty} - \frac{1}{R}\right]$$
$$U_g = \frac{GMm}{R}$$

For an object to escape the potential energy well of a celestial body, it must do work equal to said potential energy to an infinite distance away. Work here is the required kinetic energy.

$$W = U_g$$
$$\frac{1}{2}mv^2 = \frac{GMm}{R}$$
$$v = \sqrt{\frac{2GM}{R}}$$

$$v_{esc}(Moon) = \sqrt{\frac{2(6.6743 \times 10^{-11}\ Nm^2kg^{-2})(7.342 \times 10^{22}\ kg)}{(1737.4 \times 10^3\ m)}} = 2.375 \times 10^3\ ms^{-1}$$

(b) the surface of Mars

Solution

$$v_{esc}(Mars) = \sqrt{\frac{2(6.6743 \times 10^{-11}\ Nm^2kg^{-2})(6.4171 \times 10^{23}\ kg)}{(3389.5 \times 10^3\ m)}} = 5.0271 \times 10^3\ ms^{-1}$$

Problem 4:

Consider a probe/satellite in a circular low Mars orbit, **200 *km*** above the planet's surface.

(a) What is the orbital velocity of the satellite?

Solution It is fairly easy to derive the formula for the orbital velocity of a satellite.

$$F_g = a_c$$
$$\frac{GMm}{(R+h)^2} = \frac{mv^2}{(R+h)}$$
$$v = \sqrt{\frac{GM}{(R+h)}}$$

$$v(200\ km) = \sqrt{\frac{(6.6743 \times 10^{-11}\ Nm^2kg^{-2})(6.4171 \times 10^{23}\ kg)}{((3389.5 \times 10^3\ m) + (200 \times 10^3\ m))}} = 3.45 \times 10^3\ ms^{-1}$$

(b) What is the probe's period in this low Mars orbit?

Solution Due to the circular orbit, we can assume that $\boldsymbol{v = \frac{2\pi r}{T}}$ can be applied in this case.

$$v(200\ km) = \frac{2\pi(R+h)}{T}$$

$$T = \frac{2\pi(3589.5 \times 10^3\ m)}{(3.45 \times 10^3\ ms^{-1})} = 6.54 \times 10^3\ s = 109\ mins$$

(c) What is the altitude (measured from the surface) of a satellite around Mars if it is to be in a synchronous orbit?

Solution A synchronous orbit is one where a satellite orbits in the same period and direction as the rotation of the celestial body. For a satellite to achieve a synchronous orbit around Mars, it's time period must equal to one rotational period of Mars, in other words, a Martian day. Find that and then use Kepler's Law of Periods.

$$T_{Mars} = 1477\ mins = 8.862 \times 10^4\ s$$

$$T^2 = \frac{4\pi^2}{GM} a^3$$

$$a = \left[\frac{GM}{4\pi^2} T^2\right]^{1/3}$$

$$a = \left[\frac{(6.6743 \times 10^{-11}\ Nm^2kg^{-2})(6.4171 \times 10^{23}\ kg)}{4\pi^2} (8.862 \times 10^4\ s)^2\right]^{1/3} = 2.04 \times 10^7\ m$$

Problem 5:

Starting from gravitation/pressure balance in the atmosphere, derive the dependence of pressure and density on height and determine the scale height:

(a) assuming the temperature is constant;

Solution One model for Earth's lower atmosphere starts from hydrostatic equilibirum between the bunched up particles near the surface and the particles attracted by gravity ontop. This create a pressure differential that pushes outwards, which translates to a net force.

$$dm = \rho(h)\ dV$$

$$F_{net} \Rightarrow -dm\ g = \rho(h)\ A\ dh\ g$$

$$-dF = -dP\ A = \rho(h)\ A\ dh\ g$$

$$dP = -\rho(h)\ g\ dh$$

Because we assume that the temperature is constant, the following formula allows us to easily find the pressure.

$$PV = nkT = \left(\frac{M}{\mu m_u}\right)\ kT$$

$$P = \left(\frac{\rho(h)}{\mu m_u}\right)\ kT$$

Dividing this equation into the hydrostatic equilibrim equation.

$$\frac{dP}{P} = -\left(\frac{g\mu m_u}{kT}\right)\ dh$$

Integrating.

$$\int_{P_0}^{P} \frac{dP}{P} = \int_0^h -\left(\frac{g\mu m_u}{kT}\right)\ dh$$

$$\ln(P) - \ln(P_0) = -\left(\frac{g\mu m_u}{kT}\right)\ h$$

$$\ln\left(\frac{P}{P_0}\right) = -\left(\frac{g\mu m_u}{kT}\right) h$$

We can now define the scale height of Earth' atmosphere as $H_p = \left(\frac{kT}{g\mu m_u}\right)$

$$P(h) = P_0\ \exp\left(-\frac{g\mu m_u}{kT} h\right) = P_0\ \exp\left(-\frac{h}{H_p}\right)$$

Pressure and density are related by Boyle's Law $P \propto \rho \Rightarrow P = \rho RT$, the following equation is valid.

$$\rho(h) = \rho_0\ \exp\left(-\frac{h}{H_p}\right)$$

(b) assuming the system is adiabatic.

Solution For an adiabatic process, the pressure and temperature are related in the following way: $\boldsymbol{T \propto P^{1-\frac{1}{\gamma}}}$.

$$\frac{dT}{T} = \frac{\gamma - 1}{\gamma}\frac{dP}{P}$$

$$\frac{\gamma}{\gamma - 1}\frac{dT}{T} = -\left(\frac{g\mu m_u}{kT}\right)\,dh$$

$$\int_{T_0}^{T} dT = \int_0^h -\left(\frac{g\mu m_u}{k}\right)\left(\frac{\gamma - 1}{\gamma}\right)\,dh$$

$$T - T_0 = -\left(\frac{g\mu m_u}{k}\right)\left(\frac{\gamma - 1}{\gamma}\right)\,h$$

$$T = T_0 - \left(\frac{g\mu m_u}{k}\frac{\gamma - 1}{\gamma}\right)h$$

$$T = T_0\left[1 - \left(\frac{g\mu m_u}{kT_0}\frac{\gamma - 1}{\gamma}\right)h\right]$$

Let $H_T = \dfrac{kT_0}{g\mu m_u}$

$$T = T_0\left[1 - \left(\frac{h}{H_T}\frac{\gamma - 1}{\gamma}\right)\right]$$

Since $P \propto T^{\frac{\gamma}{\gamma - 1}}$

$$P(h) = P_0\left[1 - \left(\frac{\gamma - 1}{\gamma}\frac{h}{H_T}\right)\right]^{\frac{\gamma}{\gamma - 1}}$$

$$\rho(h) = \rho_0\left[1 - \left(\frac{\gamma - 1}{\gamma}\frac{h}{H_T}\right)\right]^{\frac{\gamma}{\gamma - 1}}$$

The value for γ is the ratio of specific heats of the atmosphere, which is typically taken as 1.4. If $\gamma = 1$, we get the equation for if the temperature is constant, meaning an isothermal atmosphere. An approximation can be taken where $\gamma \to 1$, which reduces the equation to the same format as the previous part's answer.