PHY 320 - Assignment 7

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1 Problem 1d

Find the center of mass of the volume bounded by a paraboloid of revolution $z = \frac{(x^2+y^2)}{b}$ and the plane $z = b$.

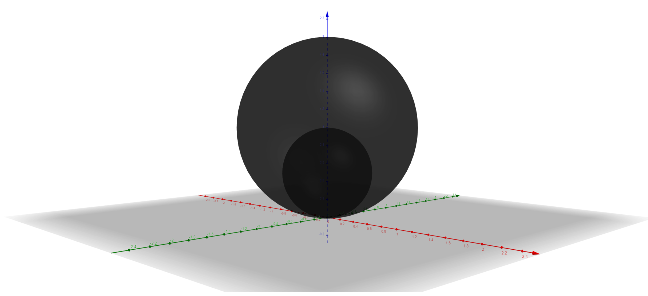
Solution From symmetry, the center of mass lies on the z axis. With b being a parameter that determines the width of the paraboloid, and volume is constrained by $z = b$. b has to be part of the final expression.

$$\begin{aligned} z_{cm} &= \frac{\int \rho z dV}{\int \rho dV} \qquad\qquad dV = \pi r^2 dz = \pi(x^2+y^2)dz = \pi b z\, dz \\ z_{cm} &= \frac{\int_0^b z^2 \pi b\, dz}{\int_0^b \pi b z\, dz} = \frac{\int_0^b z^2\, dz}{\int_0^b z\, dz} = \frac{2b}{3} \end{aligned} \tag{1}$$

2 Problem 3

A solid uniform sphere of radius a has a spherical cavity of radius $a/2$ centered at a point $a/2$ from the center of the sphere. Find the center of mass.

Solution Volume inside $x^2+y^2+(z-a)^2 = a^2$, outside $x^2+y^2+(z-\frac{a}{2})^2 = \left(\frac{a}{2}\right)^2$. From symmetry, the center of mass lies on the z axis.



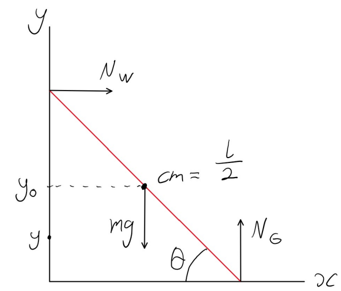
$$\begin{aligned} x^2+y^2+z^2-2az+a^2 &= a^2 & x^2+y^2+z^2-az+\left(\frac{a}{2}\right)^2 &= \left(\frac{a}{2}\right)^2 \\ x^2+y^2+z^2 &= 2az & x^2+y^2+z^2 &= az \\ r^2 &= 2arcos\theta & r^2 &= arcos\theta \\ r_a &= 2acos\theta & r_{\frac{a}{2}} &= acos\theta \end{aligned} \tag{2}$$

$$\begin{aligned} z_{cm} = \frac{\int \rho z dV}{\int \rho dV} &= \frac{\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_{a\cos\theta}^{2a\cos\theta} r^3\cos\theta\sin\theta\, dr\, d\theta\, d\phi}{\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_{a\cos\theta}^{2a\cos\theta} r^2\sin\theta\, dr\, d\theta\, d\phi} = \frac{\int_0^{\frac{\pi}{2}}\left[\frac{r^4}{4}\cos\theta\sin\theta\right]_{r=a\cos\theta}^{r=2a\cos\theta} d\theta}{\int_0^{\frac{\pi}{2}}\left[\frac{r^3}{3}\sin\theta\right]_{r=a\cos\theta}^{r=2a\cos\theta} d\theta} \\ &= \frac{\int_0^{\frac{\pi}{2}}\left[\frac{(2a\cos\theta)^4}{4}\cos\theta\sin\theta - \frac{(a\cos\theta)^4}{4}\cos\theta\sin\theta\right] d\theta}{\int_0^{\frac{\pi}{2}}\left[\frac{(2a\cos\theta)^3}{3}\sin\theta - \frac{(a\cos\theta)^3}{3}\sin\theta\right] d\theta} \\ &= \frac{\frac{15a^4}{4}\int_0^{\frac{\pi}{2}}\cos^5\theta\sin\theta\, d\theta}{\frac{7a^3}{3}\int_0^{\frac{\pi}{2}}\cos^3\theta\sin\theta\, d\theta} = \frac{\frac{15a^4}{4}\left[\frac{\cos^6\theta}{6}\right]_0^{\theta=\frac{\pi}{2}}}{\frac{7a^3}{3}\left[\frac{\cos^4\theta}{4}\right]_0^{\theta=\frac{\pi}{2}}} = \frac{\frac{15a^4}{4}(-\frac{1}{6})}{\frac{7a^3}{3}(-\frac{1}{4})} = \frac{15a}{14} \approx 1.071a \end{aligned} \tag{3}$$

3 Problem 17

A uniform ladder leans against a smooth vertical wall. If the floor is smooth, and the initial angle between the floor and the ladder is θ_0, show that the ladder, in sliding down, will lose contact with the wall when the angle between the floor and the ladder is $\sin^{-1}(\frac{2}{3}\sin\theta_0)$.

Solution The ladder loses contact with the wall when the normal force from the wall equals 0, $N_W = 0$.



$$m\ddot{x} = N_W \quad m\ddot{y} = N_G - mg \quad mgy_0 = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I\omega^2 + mgy \tag{4}$$

$$\begin{aligned} x_{cm} &= \frac{l}{2}\cos\theta & y_{cm} &= \frac{l}{2}\sin\theta \\ \dot{x}_{cm} &= -\frac{l}{2}\dot{\theta}\sin\theta & \dot{y}_{cm} &= \frac{l}{2}\dot{\theta}\cos\theta \end{aligned} \tag{5}$$

$$\begin{aligned} \ddot{x}_{cm} &= -\frac{l}{2}\left[\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right] \\ I_{rod} &= \frac{ml^2}{12} \end{aligned} \tag{6}$$

$$\begin{aligned} mgy_0 &= \frac{1}{2}m(\dot{x}_{cm}^2 + \dot{y}_{cm}^2) + \frac{1}{2}I\omega^2 + mgy \\ \frac{mgl}{2}\sin\theta_0 &= \frac{1}{2}m\left((-\frac{l}{2}\dot{\theta}\sin\theta)^2 + (\frac{l}{2}\dot{\theta}\cos\theta)^2\right) + \frac{1}{2}\left(\frac{ml^2}{12}\right)\dot{\theta}^2 + mg\frac{l}{2}\sin\theta \\ \frac{mgl}{2}(\sin\theta_0 - \sin\theta) &= \frac{ml^2\dot{\theta}^2}{8} + \frac{ml^2\dot{\theta}^2}{24} = \frac{ml^2\dot{\theta}^2}{6} \\ \dot{\theta} &= \left(\frac{3g}{l}(\sin\theta_0 - \sin\theta)\right)^{\frac{1}{2}} \\ \ddot{\theta} &= \left(\frac{1}{2}\right)\left(\frac{3g}{l}(\sin\theta_0 - \sin\theta)\right)^{-\frac{1}{2}}\left(-\frac{3g\dot{\theta}}{l}\cos\theta\right) = -\frac{3g}{2l}\cos\theta \end{aligned} \tag{7}$$

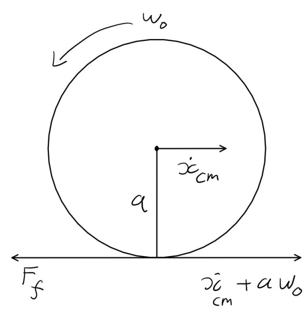
$$\begin{aligned} m\ddot{x} &= -\frac{ml}{2}\left[\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right] = -\frac{ml}{2}\left[\left(-\frac{3g}{2l}\cos\theta\right)\sin\theta + \left(\frac{3g}{l}(\sin\theta_0 - \sin\theta)\right)\cos\theta\right] \\ 0 &= \frac{3mg}{4}\cos\theta\sin\theta - \frac{3mg}{2}(\sin\theta_0 - \sin\theta)\cos\theta = \frac{3mg}{2}\cos\theta\left(\frac{1}{2}\sin\theta - \sin\theta_0 + \sin\theta\right) \end{aligned} \tag{8}$$

$$\begin{aligned} \frac{1}{2}\sin\theta - \sin\theta_0 + \sin\theta &= 0 \\ \frac{3}{2}\sin\theta &= \sin\theta_0 \\ \theta &= \sin^{-1}\left(\frac{2}{3}\sin\theta_0\right) \end{aligned} \tag{9}$$

4 Problem 20

A billiard ball of radius a is initially spinning about a horizontal axis with angular speed ω_0 and with zero forward speed. If the coefficient of sliding friction between the ball and the billiard table is μ_k, find the distance the ball travels before slipping ceases to occur.

Solution The ball spinning about a horizontal axis implies a force along said axis. Slipping occurs when translational motion exceeds rotational motion, so slipping ceases to occur $\dot{x}_{cm} - a\omega = 0$ at the point of contact between the ball and the ground.



$$m\ddot{x}_{cm} = F_f = \mu_k mg \quad I_{ball} = \frac{2ma^2}{5} \tag{10}$$

$$\begin{aligned} &\ddot{x}_{cm} = \mu_k g \quad \dot{x}_{cm} = \mu_k gt \quad x_{cm} = \frac{1}{2}\mu_k gt^2 \\ &\dot{\omega} = \frac{\mu_k g}{a} \qquad \omega = \frac{\mu_k gt}{a} \end{aligned} \tag{11}$$

In terms of rotational motion, the frictional force opposes angular velocity at the point of contact between the ball and the ground.

$$\begin{aligned} I_{ball}\dot{\omega} &= -a\mu_k mg \\ \dot{\omega} &= -\frac{5\mu_k g}{2a} \\ \omega - \omega_0 &= -\frac{5\mu_k g}{2a}t \\ \omega &= \omega_0 - \frac{5\mu_k g}{2a}t \end{aligned} \tag{12}$$

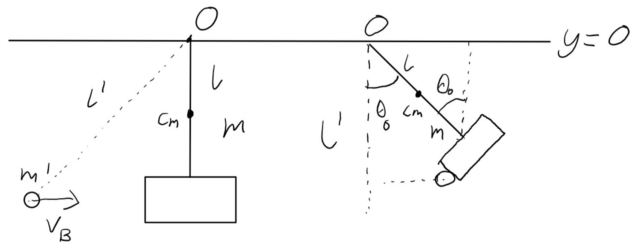
$$\begin{aligned} \dot{x}_{cm} - a\omega &= 0 \\ \mu_k gt - a\omega_0 - \frac{5\mu_k g}{2}t &= 0 \\ \frac{7\mu_k g}{2}t &= a\omega_0 \\ t &= \frac{2a\omega_0}{7\mu_k g} \end{aligned} \tag{13}$$

$$x_{cm} = \frac{1}{2}\mu_k gt^2 = \frac{1}{2}\mu_k g\frac{4a^2\omega_0^2}{49\mu_k^2 g^2} = \frac{2a^2\omega_0^2}{49\mu_k g} \tag{14}$$

5 Problem 24

A ballistic pendulum is made of a long plank of length l and mass m. It is free to swing about one end O and is initially at rest in a vertical position. A bullet of mass m' is fired horizontally into the pendulum at a distance l' below O, the bullet coming to rest in the plank. If the resulting amplitude of oscillation of the pendulum is θ_0, find the speed of the bullet.

Solution Conservation of angular momentum and energy.



$$\begin{aligned}
L: &\quad I\dot{\theta} = m' v_B l' \\
E: &\quad \frac{1}{2} I \dot{\theta}^2 - mg\frac{l}{2} - m'gl' = -mg\frac{l}{2}\cos\theta_0 - m'gl'\cos\theta_0 \\
I: &\quad I = I_{plank} + I_{bullet} = \frac{1}{3}ml^2 + m'l'^2
\end{aligned} \tag{15}$$

$$\begin{aligned}
\frac{1}{2}I\dot{\theta}^2 &= g\left[m\frac{l}{2} + m'l' - \cos\theta_0\left(m\frac{l}{2} + m'l'\right)\right] = g\left(m\frac{l}{2} + m'l'\right)(1-\cos\theta_0) \\
\frac{1}{2}I\frac{m'^2 v_B^2 l'^2}{I^2} &= g\left(m\frac{l}{2} + m'l'\right)(1-\cos\theta_0) \\
v_B^2 &= \frac{2g}{m'^2 l'^2} I \left(m\frac{l}{2} + m'l'\right)(1-\cos\theta_0) \\
v_B^2 &= \frac{2g}{m'^2 l'^2}\left(\frac{1}{3}ml^2 + m'l'^2\right)\left(m\frac{l}{2} + m'l'\right)(1-\cos\theta_0) \\
v_B &= \frac{1}{m'l'}\left[2g\left(\frac{1}{3}ml^2 + m'l'^2\right)\left(m\frac{l}{2} + m'l'\right)(1-\cos\theta_0)\right]^{\frac{1}{2}}
\end{aligned} \tag{16}$$