PHY 350 — Quantum Mechanics

Name: Khalifa Salem Almatrooshi **Due Date:** 23 April 2024
Student Number: @00090847 **Assignment:** 3

Problem 1: Exercise 5.3 & 5.4

If $\hat{L}_\pm$ and $\hat{E}_\pm$ are defined by $\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$ and $\hat{R}_\pm = \hat{X}_x \pm i\hat{Y}_y$, prove the following commutators:

(a) $[\hat{L}_\pm, \hat{R}_\pm] = \pm 2\hbar\hat{Z}$,

Solution The book seems to be wrong for the first two parts.

$$\begin{aligned}
[\hat{L}_\pm, \hat{R}_\pm] &= [\hat{L}_x \pm i\hat{L}_y, \hat{X}_x \pm i\hat{Y}_y] && (1)\\
&= [\hat{L}_x, \hat{X}_x] \pm [\hat{L}_x, i\hat{Y}_y] \pm [i\hat{L}_y, \hat{X}_x] \pm [i\hat{L}_y, i\hat{Y}_y] && (2)\\
&= [\hat{L}_x, \hat{X}_x] \pm i[\hat{L}_x, \hat{Y}_y] \pm i[\hat{L}_y, \hat{X}_x] \mp [\hat{L}_y, \hat{Y}_y] && (3)\\
&= 0 \pm i(i\hbar\hat{Z}_z) \pm i(-i\hbar\hat{Z}_z) \mp 0 && (4)\\
&= \pm i^2\hbar\hat{Z}_z \mp i^2\hbar\hat{Z}_z && (5)\\
&= \boxed{0} && (6)
\end{aligned}$$

(b) $[\hat{L}_\pm, \hat{R}_\mp] = 0$,

Solution

$$\begin{aligned}
[\hat{L}_\pm, \hat{R}_\mp] &= [\hat{L}_x \pm i\hat{L}_y, \hat{X}_x \mp i\hat{Y}_y] && (7)\\
&= [\hat{L}_x, \hat{X}_x] \mp [\hat{L}_x, i\hat{Y}_y] \pm [i\hat{L}_y, \hat{X}_x] \mp [i\hat{L}_y, i\hat{Y}_y] && (8)\\
&= [\hat{L}_x, \hat{X}_x] \mp i[\hat{L}_x, \hat{Y}_y] \pm i[\hat{L}_y, \hat{X}_x] \pm [\hat{L}_y, \hat{Y}_y] && (9)\\
&= 0 \mp i(i\hbar\hat{Z}_z) \pm i(-i\hbar\hat{Z}_z) \pm 0 && (10)\\
&= \mp i^2\hbar\hat{Z}_z \mp i^2\hbar\hat{Z}_z && (11)\\
&= \boxed{\pm 2\hbar\hat{Z}_z} && (12)
\end{aligned}$$

(c) $[\hat{L}_z, \hat{R}_\pm] = \pm\hbar\hat{R}_\pm$,

Solution

$$\begin{aligned}
[\hat{L}_z, \hat{R}_\pm] &= [\hat{L}_z, \hat{X}_x \pm i\hat{Y}_y] && (13)\\
&= [\hat{L}_z, \hat{X}_x] \pm [\hat{L}_z, i\hat{Y}_y] && (14)\\
&= (i\hbar\hat{Y}_y) \pm i(-i\hbar\hat{X}_x) && (15)\\
&= \pm\hbar\hat{X}_x + i\hbar\hat{Y}_y && (16)
\end{aligned}$$

We can factor put the $\pm\hbar$, factoring out the $\pm$ from the second term causes it turn to $\pm$.

$$\begin{aligned}
&= \pm\hbar(\hat{X}_x \pm i\hat{Y}_y) && (17)\\
&= \boxed{\pm\hbar\hat{R}_\pm} && (18)
\end{aligned}$$

(d) $[\hat{L}_z, \hat{Z}] = 0$,

Solution sol

$$\begin{aligned}
[\hat{L}_z, \hat{Z}] &= [\hat{X}\hat{P}_y - \hat{Y}\hat{P}_x, \hat{Z}] && (19)\\
&= [\hat{X}\hat{P}_y, \hat{Z}] - [\hat{Y}\hat{P}_x, \hat{Z}] && (20)\\
&= \hat{X}[\hat{P}_y, \hat{Z}] - \hat{Y}[\hat{P}_x, \hat{Z}] && (21)\\
&= \boxed{0} && (22)
\end{aligned}$$

Problem 2: Exercise 5.11

Consider the wave function

$$\Psi(\theta, \phi) = 3\sin\theta\cos\theta e^{i\phi} - 2\left(1-\cos^2\theta\right)e^{2i\phi}. \tag{23}$$

(a) Write $\Psi(\theta, \phi)$ in terms of the spherical harmonics.

Solution We can find the spherical harmonics in the book, by looking at our wave function, I single out the relevant equations.

$$Y_{2,\pm1}(\theta,\phi) = \mp\sqrt{\frac{15}{8\pi}}e^{\pm i\phi}\sin\theta\cos\theta \tag{24}$$

$$Y_{2,\pm2}(\theta,\phi) = \sqrt{\frac{15}{32\pi}}e^{\pm 2i\phi}\sin^2\theta \tag{25}$$

$$\begin{aligned}\Psi(\theta, \phi) &= 3\sin\theta\cos\theta e^{i\phi} - 2\sin^2\theta e^{2i\phi} \\ &= \boxed{-3\sqrt{\frac{8\pi}{15}}Y_{2,1}(\theta,\phi) - 2\sqrt{\frac{32\pi}{15}}Y_{2,2}(\theta,\phi)}\end{aligned} \tag{26}$$

(b) Is $\Psi(\theta, \phi)$ an eigenstate of $\hat{L}^2$ or $\hat{L}_z$?

Solution A wave function is an eigenstate of an operator if it satisfies $\hat{A}\Psi = \lambda\Psi$, where λ is a constant. $\hat{L}^2\left|l,m\right\rangle = \hbar^2 l(l+1)\left|l,m\right\rangle$, $\hat{L}_z\left|l,m\right\rangle = m\hbar\left|l,m\right\rangle$

$$\begin{aligned}\hat{L}^2\Psi(\theta,\phi) &= -3\sqrt{\frac{8\pi}{15}}\left[2(2+1)\hbar^2\right]Y_{2,1} - 2\sqrt{\frac{32\pi}{15}}\left[2(2+1)\hbar^2\right]Y_{2,2} \\ &= 6\hbar^2\Psi(\theta,\phi) \quad \boxed{\text{Eigenstate}}\end{aligned} \tag{27}$$

$$\begin{aligned}\hat{L}_z\Psi(\theta,\phi) &= -3\sqrt{\frac{8\pi}{15}}\left[1\times\hbar\right]Y_{2,1} - 2\sqrt{\frac{32\pi}{15}}\left[2\times\hbar\right]Y_{2,2} \\ &\neq \lambda\Psi(\theta,\phi) \quad \boxed{\text{Not an eigenstate}}\end{aligned} \tag{28}$$

(b) Find the probability of measuring $2\hbar$ for the z-component of the orbital angular momentum.

Solution The only state with $2\hbar$ is $Y_{2,2}$.

$$\begin{aligned}P_2 &= \left|\left\langle 2,2|\Psi\right\rangle\right|^2 \\ &= \left|-3\sqrt{\frac{8\pi}{15}}\left\langle 2,2|2,1\right\rangle - 2\sqrt{\frac{32\pi}{15}}\left\langle 2,2|2,2\right\rangle\right|^2 \\ &= \left|-2\sqrt{\frac{32\pi}{15}}\right|^2 \\ &= \boxed{\frac{128\pi}{15}}\end{aligned} \tag{29}$$

Problem 3: Exercise 5.16

Consider a system which is described by the state

$$\Psi(\theta,\phi) = \sqrt{\frac{3}{8}}Y_{1,1}(\theta,\phi) + \sqrt{\frac{1}{8}}Y_{1,0}(\theta,\phi) + AY_{1,-1}(\theta,\phi), \tag{30}$$

where A is a real constant.

(a) Calculate A so that $|\Psi\rangle$ is normalized.

Solution

$$\begin{aligned} \langle\Psi|\Psi\rangle = \frac{3}{8} + \frac{1}{8} + A^2 = 1 \\ A = \sqrt{\frac{1}{2}} \end{aligned} \tag{31}$$

(b) Find $\hat{L}_+\Psi(\theta,\phi)$.

Solution $\hat{L}_\pm |l,m\rangle = \hbar\sqrt{l(l+1) - m(m\pm 1)}\,|l,m\pm 1\rangle$.

$$\begin{aligned} \hat{L}_+\Psi(\theta,\phi) &= \sqrt{\frac{3}{8}}\hat{L}_+|1,1\rangle + \sqrt{\frac{1}{8}}\hat{L}_+|1,0\rangle + \sqrt{\frac{1}{2}}\hat{L}_+|1,-1\rangle \\ &= \sqrt{\frac{3}{8}}\hbar\sqrt{1(1+1) - 1(1+1)}\,|1,2\rangle \\ &\quad + \sqrt{\frac{1}{8}}\hbar\sqrt{1(1+1) - 0(0+1)}\,|1,1\rangle \\ &\quad + \sqrt{\frac{1}{2}}\hbar\sqrt{1(1+1) + 1(-1+1)}\,|1,0\rangle \\ &= 0 + \frac{\hbar}{2}|1,1\rangle + \hbar|1,0\rangle \\ &= \boxed{\hbar\left(\frac{1}{2}|1,1\rangle + |1,0\rangle\right)} \end{aligned} \tag{32}$$

(c) Calculate the expectation values of $\hat{L}_x$ and $\hat{L}^2$ in the state $|\Psi\rangle$.

Solution

$$\hat{L}_\pm = \hat{L}_x \pm i\hat{l}_y \quad \Rightarrow \quad \hat{L}_x = \frac{1}{2}\left[\hat{L}_+ + \hat{L}_-\right] \tag{33}$$

$$\begin{aligned} \left\langle \hat{L}_+ \right\rangle &= \left[\sqrt{\frac{3}{8}}|1,1\rangle + \sqrt{\frac{1}{8}}|1,0\rangle + \sqrt{\frac{1}{2}}|1,-1\rangle\right]\left[\sqrt{\frac{3}{8}}(0)\,|1,2\rangle + \sqrt{\frac{1}{8}}\sqrt{2}\hbar\,|1,1\rangle + \sqrt{\frac{1}{2}}\sqrt{2}\hbar\,|1,0\rangle\right] \\ &= \hbar\left[\sqrt{\frac{3}{8}\times\frac{1}{8}\times 2} + \sqrt{\frac{1}{8}\times\frac{1}{2}\times 2}\right] \\ &= \frac{\sqrt{6}+2\sqrt{2}}{8}\hbar \end{aligned} \tag{34}$$

$$\begin{aligned} \left\langle \hat{L}_- \right\rangle &= \left[\sqrt{\frac{3}{8}}|1,1\rangle + \sqrt{\frac{1}{8}}|1,0\rangle + \sqrt{\frac{1}{2}}|1,-1\rangle\right]\left[\sqrt{\frac{3}{8}}\sqrt{2}\hbar\,|1,0\rangle + \sqrt{\frac{1}{8}}\sqrt{2}\hbar\,|1,-1\rangle + \sqrt{\frac{1}{2}}(0)\,|1,-2\rangle\right] \\ &= \hbar\left[\sqrt{\frac{1}{8}\times\frac{3}{8}\times 2} + \sqrt{\frac{1}{2}\times\frac{1}{8}\times 2}\right] \\ &= \frac{\sqrt{6}+2\sqrt{2}}{8}\hbar \end{aligned} \tag{35}$$

$$\hat{L}_x = \frac{1}{2}\left[\frac{\sqrt{6}+2\sqrt{2}}{8}\hbar + \frac{\sqrt{6}+2\sqrt{2}}{8}\hbar\right]$$
$$= \boxed{\frac{\sqrt{6}+2\sqrt{2}}{8}\hbar} \tag{36}$$

$$\left\langle \hat{L}^2 \right\rangle = \left[\sqrt{\frac{3}{8}}\left|1,1\right\rangle + \sqrt{\frac{1}{8}}\left|1,0\right\rangle + \sqrt{\frac{1}{2}}\left|1,-1\right\rangle\right]\left[\sqrt{\frac{3}{8}}\hat{L}^2\left|1,1\right\rangle + \sqrt{\frac{1}{8}}\hat{L}^2\left|1,0\right\rangle + \sqrt{\frac{1}{2}}\hat{L}^2\left|1,-1\right\rangle\right]$$
$$= \left[\sqrt{\frac{3}{8}}\left|1,1\right\rangle + \sqrt{\frac{1}{8}}\left|1,0\right\rangle + \sqrt{\frac{1}{2}}\left|1,-1\right\rangle\right]\left[\sqrt{\frac{3}{8}}2\hbar^2\left|1,1\right\rangle + \sqrt{\frac{1}{8}}2\hbar^2\left|1,0\right\rangle + \sqrt{\frac{1}{2}}2\hbar^2\left|1,-1\right\rangle\right]$$
$$= 2\hbar^2\left[\frac{3}{8} + \frac{1}{8} + \frac{1}{2}\right]$$
$$= \boxed{2\hbar^2} \tag{37}$$

(d) Find the probability associated with a measurement that gives zero for the z-component of the angular momentum.

Solution

$$\hat{L}_z\Psi = \sqrt{\frac{3}{8}}(1\times\hbar)\left|1,1\right\rangle + \sqrt{\frac{1}{8}}(0\times\hbar)\left|1,0\right\rangle + \sqrt{\frac{1}{2}}(-1\times\hbar)\left|1,-1\right\rangle \tag{38}$$

The only state with 0 is $Y_{1,0}$.

$$P_0 = \left|\left\langle 1,0|\Psi\right\rangle\right|^2$$
$$= \left|\sqrt{\frac{3}{8}}\left\langle 1,0|1,1\right\rangle + \sqrt{\frac{1}{8}}\left\langle 1,0|1,0\right\rangle + \sqrt{\frac{1}{2}}\left\langle 1,0|1,-1\right\rangle\right|^2$$
$$= \left|\sqrt{\frac{1}{8}}\right|^2$$
$$= \boxed{\frac{1}{8}} \tag{39}$$

(e) Calculate $\left\langle\Phi\right|\hat{L}_z\left|\Psi\right\rangle$ and $\left\langle\Phi\right|\hat{L}_-\left|\Psi\right\rangle$ where

$$\Phi(\theta,\phi) = \sqrt{\frac{8}{15}}Y_{1,1}(\theta,\phi) + \sqrt{\frac{4}{15}}Y_{1,0}(\theta,\phi) + \sqrt{\frac{3}{15}}Y_{2,-1}(\theta,\phi). \tag{40}$$

Solution

$$\left\langle\Phi\right|\hat{L}_z\left|\Psi\right\rangle = \left[\sqrt{\frac{8}{15}}\left|1,1\right\rangle + \sqrt{\frac{4}{15}}\left|1,0\right\rangle + \sqrt{\frac{3}{15}}\left|2,-1\right\rangle\right]\left[\sqrt{\frac{3}{8}}\hbar\left|1,1\right\rangle - \sqrt{\frac{1}{2}}\hbar\left|1,-1\right\rangle\right]$$
$$= \hbar\left[\sqrt{\frac{8}{15}}\times\sqrt{\frac{3}{8}}\right]$$
$$= \boxed{\frac{\sqrt{5}}{5}\hbar} \tag{41}$$

$$\left\langle\Phi\right|\hat{L}_-\left|\Psi\right\rangle = \left[\sqrt{\frac{8}{15}}\left|1,1\right\rangle + \sqrt{\frac{4}{15}}\left|1,0\right\rangle + \sqrt{\frac{3}{15}}\left|2,-1\right\rangle\right]\left[\sqrt{\frac{3}{8}}\sqrt{2}\hbar\left|1,0\right\rangle + \sqrt{\frac{1}{8}}\sqrt{2}\hbar\left|1,-1\right\rangle\right]$$
$$= \hbar\left[\sqrt{\frac{4}{15}}\times\sqrt{\frac{3}{8}}\times\sqrt{2}\right]$$
$$= \boxed{\frac{\sqrt{5}}{5}\hbar} \tag{42}$$

Problem 4: Exercise 5.28

Consider a system which is given in the following angular momentum eigenstates $|l, m\rangle$:

$$|\Psi\rangle = \sqrt{\frac{1}{7}}\,|1,-1\rangle + A\,|1,0\rangle + \sqrt{\frac{2}{7}}\,|1,1\rangle\,, \tag{43}$$

where A is a real constant.

(a) Calculate A so that $|\Psi\rangle$ is normalized.

Solution

$$\begin{aligned}\langle\Psi|\Psi\rangle = \frac{1}{7} + A^2 + \frac{2}{7} &= 1\\ A &= \sqrt{\frac{4}{7}}\end{aligned} \tag{44}$$

(b) Calculate the expectation values of $\hat{L}_x$, $\hat{L}_y$, $\hat{L}_z$, and $\hat{L}^2$ in the state $|\Psi\rangle$.

Solution

$$\begin{aligned}\left\langle \hat{L}^2 \right\rangle &= \left[\sqrt{\frac{1}{7}}\,|1,-1\rangle + \sqrt{\frac{4}{7}}\,|1,0\rangle + \sqrt{\frac{5}{7}}\,|1,1\rangle\right]\left[2\hbar^2\sqrt{\frac{1}{7}}\,|1,-1\rangle + 2\hbar^2\sqrt{\frac{4}{7}}\,|1,0\rangle + 2\hbar^2\sqrt{\frac{2}{7}}\,|1,1\rangle\right]\\ &= 2\hbar^2\left[\frac{1}{7}+\frac{4}{7}+\frac{2}{7}\right]\\ &= \boxed{2\hbar^2}\end{aligned} \tag{45}$$

$$\begin{aligned}\left\langle \hat{L}_z \right\rangle &= \left[\sqrt{\frac{1}{7}}\,|1,-1\rangle + \sqrt{\frac{4}{7}}\,|1,0\rangle + \sqrt{\frac{5}{7}}\,|1,1\rangle\right]\left[\sqrt{\frac{1}{7}}(-1\times\hbar)\,|1,-1\rangle + \sqrt{\frac{4}{7}}(0\times\hbar)\,|1,0\rangle + \sqrt{\frac{2}{7}}(1\times\hbar)\,|1,1\rangle\right]\\ &= \hbar\left[\frac{1}{7}+\frac{2}{7}\right]\\ &= \boxed{\frac{3}{7}\hbar}\end{aligned} \tag{46}$$

$$\begin{aligned}\left\langle \hat{L}_+ \right\rangle &= \left[\sqrt{\frac{1}{7}}\,|1,-1\rangle + \sqrt{\frac{4}{7}}\,|1,0\rangle + \sqrt{\frac{5}{7}}\,|1,1\rangle\right]\left[\sqrt{\frac{1}{7}}\sqrt{2}\hbar\,|1,0\rangle + \sqrt{\frac{4}{7}}\sqrt{2}\hbar\,|1,1\rangle + \sqrt{\frac{5}{7}}(0)\,|1,2\rangle\right]\\ &= \hbar\left[\sqrt{\frac{4}{7}\times\frac{1}{7}\times 2} + \sqrt{\frac{5}{7}\times\frac{4}{7}\times 2}\right]\\ &= \boxed{\frac{2\sqrt{10}+2\sqrt{2}}{7}\hbar}\end{aligned} \tag{47}$$

$$\begin{aligned}\left\langle \hat{L}_- \right\rangle &= \left[\sqrt{\frac{1}{7}}\,|1,-1\rangle + \sqrt{\frac{4}{7}}\,|1,0\rangle + \sqrt{\frac{5}{7}}\,|1,1\rangle\right]\left[\sqrt{\frac{1}{7}}(0)\,|1,-2\rangle + \sqrt{\frac{4}{7}}\sqrt{2}\hbar\,|1,-1\rangle + \sqrt{\frac{5}{7}}\sqrt{2}\hbar\,|1,0\rangle\right]\\ &= \hbar\left[\sqrt{\frac{1}{7}\times\frac{4}{7}\times 2} + \sqrt{\frac{4}{7}\times\frac{5}{7}\times 2}\right]\\ &= \boxed{\frac{2\sqrt{10}+2\sqrt{2}}{7}\hbar}\end{aligned} \tag{48}$$

$$\begin{aligned}\left\langle \hat{L}_x \right\rangle &= \frac{1}{2}\left[\frac{2\sqrt{10}+2\sqrt{2}}{7}\hbar + \frac{2\sqrt{10}+2\sqrt{2}}{7}\hbar\right]\\ &= \boxed{\frac{2\sqrt{10}+2\sqrt{2}}{7}\hbar}\end{aligned} \tag{49}$$

$$\begin{aligned}\left\langle \hat{L}_y \right\rangle &= \frac{1}{2}\left[\frac{2\sqrt{10}+2\sqrt{2}}{7}\hbar - \frac{2\sqrt{10}+2\sqrt{2}}{7}\hbar\right]\\ &= \boxed{0}\end{aligned} \tag{50}$$

(c) Find the probability associated with a measurement that gives $1\hbar$ for the z-component of the angular momentum.

Solution The only state with $1\hbar$ is $Y_{1,1}$.

$$\begin{aligned}
P_1 &= |\langle 1,1|\Psi\rangle|^2 \\
&= \left|\sqrt{\frac{1}{7}}\langle 1,1|1,-1\rangle + \sqrt{\frac{4}{7}}\langle 1,1|1,0\rangle + \sqrt{\frac{5}{7}}\langle 1,1|1,1\rangle\right|^2 \\
&= \left|\sqrt{\frac{5}{7}}\right|^2 \\
&= \boxed{\frac{5}{7}}
\end{aligned} \tag{51}$$

(d) Calculate $\langle 1,m|\hat{L}_+^2|\Psi\rangle$ and $\langle 1,m|\hat{L}_-^2|\Psi\rangle$.

Solution We need to apply the operators twice, then multiply by the bra.

$$\hat{L}_+|\Psi\rangle = \sqrt{\frac{1}{7}}\sqrt{2}\hbar|1,0\rangle + \sqrt{\frac{4}{7}}\sqrt{2}\hbar|1,1\rangle \tag{52}$$

$$\begin{aligned}
\hat{L}_+^2|\Psi\rangle &= \sqrt{\frac{1}{7}}\sqrt{2}\hbar\sqrt{2}\hbar|1,1\rangle + \sqrt{\frac{4}{7}}\sqrt{2}\hbar(0)|1,2\rangle \\
&= \sqrt{\frac{1}{7}}2\hbar^2|1,1\rangle
\end{aligned} \tag{53}$$

$$\begin{aligned}
\langle 1,m|\hat{L}_+^2|\Psi\rangle &= \sqrt{\frac{4}{7}}\hbar^2\langle 1,m|1,1\rangle \\
&= \boxed{\sqrt{\frac{4}{7}}\hbar^2\delta_{m,1}}
\end{aligned} \tag{54}$$

$$\hat{L}_-|\Psi\rangle = \sqrt{\frac{4}{7}}\sqrt{2}\hbar|1,-1\rangle + \sqrt{\frac{5}{7}}\sqrt{2}\hbar|1,0\rangle \tag{55}$$

$$\begin{aligned}
\hat{L}_-^2|\Psi\rangle &= \sqrt{\frac{4}{7}}\sqrt{2}\hbar(0)|1,-2\rangle + \sqrt{\frac{5}{7}}\sqrt{2}\hbar\sqrt{2}\hbar|1,-1\rangle \\
&= \sqrt{\frac{5}{7}}2\hbar^2|1,-1\rangle
\end{aligned} \tag{56}$$

$$\begin{aligned}
\langle 1,m|\hat{L}_-^2|\Psi\rangle &= \sqrt{\frac{20}{7}}\hbar^2\langle 1,m|1,-1\rangle \\
&= \sqrt{\frac{20}{7}}\hbar^2\delta_{m,-1}
\end{aligned} \tag{57}$$

Problem 5: Exercise 5.31

Consider a spin $\frac{3}{2}$ particle whose Hamiltonian is given by

$$\hat{H} = \frac{\varepsilon_0}{\hbar^2}(\hat{S}_x^2 - \hat{S}_y^2) - \frac{\varepsilon_0}{\hbar^2}(\hat{S}_z^2) \tag{58}$$

where ϵ_0 is a constant having the dimensions of energy.

(a) Find the matrix of the Hamiltonian and diagonalize it to find the energy levels.

Solution Need to find the matrix for $\hat{S}_x, \hat{S}_y$, and $\hat{S}_z$. For long matrix calculations I used https://matrixcalc.org/.

$$\hat{S}_z = \begin{bmatrix} \left\langle \frac{3}{2}, \frac{3}{2} \right| \frac{3}{2}\hbar \left| \frac{3}{2}, \frac{3}{2} \right\rangle & \left\langle \frac{3}{2}, \frac{3}{2} \right| \frac{1}{2}\hbar \left| \frac{3}{2}, \frac{1}{2} \right\rangle & \left\langle \frac{3}{2}, \frac{3}{2} \right| \frac{-1}{2}\hbar \left| \frac{3}{2}, \frac{-1}{2} \right\rangle & \left\langle \frac{3}{2}, \frac{3}{2} \right| \frac{-3}{2}\hbar \left| \frac{3}{2}, \frac{-3}{2} \right\rangle \\ \left\langle \frac{3}{2}, \frac{1}{2} \right| \frac{3}{2}\hbar \left| \frac{3}{2}, \frac{3}{2} \right\rangle & \left\langle \frac{3}{2}, \frac{1}{2} \right| \frac{1}{2}\hbar \left| \frac{3}{2}, \frac{1}{2} \right\rangle & \left\langle \frac{3}{2}, \frac{1}{2} \right| \frac{-1}{2}\hbar \left| \frac{3}{2}, \frac{-1}{2} \right\rangle & \left\langle \frac{3}{2}, \frac{1}{2} \right| \frac{-3}{2}\hbar \left| \frac{3}{2}, \frac{-3}{2} \right\rangle \\ \left\langle \frac{3}{2}, \frac{-1}{2} \right| \frac{3}{2}\hbar \left| \frac{3}{2}, \frac{3}{2} \right\rangle & \left\langle \frac{3}{2}, \frac{-1}{2} \right| \frac{1}{2}\hbar \left| \frac{3}{2}, \frac{1}{2} \right\rangle & \left\langle \frac{3}{2}, \frac{-1}{2} \right| \frac{-1}{2}\hbar \left| \frac{3}{2}, \frac{-1}{2} \right\rangle & \left\langle \frac{3}{2}, \frac{-1}{2} \right| \frac{-3}{2}\hbar \left| \frac{3}{2}, \frac{-3}{2} \right\rangle \\ \left\langle \frac{3}{2}, \frac{-3}{2} \right| \frac{3}{2}\hbar \left| \frac{3}{2}, \frac{3}{2} \right\rangle & \left\langle \frac{3}{2}, \frac{-3}{2} \right| \frac{1}{2}\hbar \left| \frac{3}{2}, \frac{1}{2} \right\rangle & \left\langle \frac{3}{2}, \frac{-3}{2} \right| \frac{-1}{2}\hbar \left| \frac{3}{2}, \frac{-1}{2} \right\rangle & \left\langle \frac{3}{2}, \frac{-3}{2} \right| \frac{-3}{2}\hbar \left| \frac{3}{2}, \frac{-3}{2} \right\rangle \end{bmatrix} \tag{59}$$

Applying the orthogonality condition, only the diagonal terms will survive.

$$\hat{S}_z = \frac{\hbar}{2}\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} \tag{60}$$

$$\hat{S}_z = \frac{\hbar^2}{4}\begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix} \tag{61}$$

For $\hat{S}_x$ and $\hat{S}_y$, the ladder operators $\hat{S}_\pm$ need to be found first. I will skip adding the s quantum number in the matrix for simplicity. Also the a_{mn} denotes the constant from $\hat{S}_\pm$.

$$\hat{S}_+ = \begin{bmatrix} \left\langle \frac{3}{2} \right| \hat{S}_+ \left| \frac{3}{2} \right\rangle & \left\langle \frac{3}{2} \right| \hat{S}_+ \left| \frac{1}{2} \right\rangle & \left\langle \frac{3}{2} \right| \hat{S}_+ \left| \frac{-1}{2} \right\rangle & \left\langle \frac{3}{2} \right| \hat{S}_+ \left| \frac{-3}{2} \right\rangle \\ \left\langle \frac{1}{2} \right| \hat{S}_+ \left| \frac{3}{2} \right\rangle & \left\langle \frac{1}{2} \right| \hat{S}_+ \left| \frac{1}{2} \right\rangle & \left\langle \frac{1}{2} \right| \hat{S}_+ \left| \frac{-1}{2} \right\rangle & \left\langle \frac{1}{2} \right| \hat{S}_+ \left| \frac{-3}{2} \right\rangle \\ \left\langle \frac{-1}{2} \right| \hat{S}_+ \left| \frac{3}{2} \right\rangle & \left\langle \frac{-1}{2} \right| \hat{S}_+ \left| \frac{1}{2} \right\rangle & \left\langle \frac{-1}{2} \right| \hat{S}_+ \left| \frac{-1}{2} \right\rangle & \left\langle \frac{-1}{2} \right| \hat{S}_+ \left| \frac{-3}{2} \right\rangle \\ \left\langle \frac{-3}{2} \right| \hat{S}_+ \left| \frac{3}{2} \right\rangle & \left\langle \frac{-3}{2} \right| \hat{S}_+ \left| \frac{1}{2} \right\rangle & \left\langle \frac{-3}{2} \right| \hat{S}_+ \left| \frac{-1}{2} \right\rangle & \left\langle \frac{-3}{2} \right| \hat{S}_+ \left| \frac{-3}{2} \right\rangle \end{bmatrix} \tag{62}$$

$$= \begin{bmatrix} \left\langle \frac{3}{2} \right| a_{11} \left| \frac{5}{2} \right\rangle & \left\langle \frac{3}{2} \right| a_{12} \left| \frac{3}{2} \right\rangle & \left\langle \frac{3}{2} \right| a_{13} \left| \frac{1}{2} \right\rangle & \left\langle \frac{3}{2} \right| a_{14} \left| \frac{-1}{2} \right\rangle \\ \left\langle \frac{1}{2} \right| a_{21} \left| \frac{5}{2} \right\rangle & \left\langle \frac{1}{2} \right| a_{22} \left| \frac{3}{2} \right\rangle & \left\langle \frac{1}{2} \right| a_{23} \left| \frac{1}{2} \right\rangle & \left\langle \frac{1}{2} \right| a_{24} \left| \frac{-1}{2} \right\rangle \\ \left\langle \frac{-1}{2} \right| a_{31} \left| \frac{5}{2} \right\rangle & \left\langle \frac{-1}{2} \right| a_{32} \left| \frac{3}{2} \right\rangle & \left\langle \frac{-1}{2} \right| a_{33} \left| \frac{1}{2} \right\rangle & \left\langle \frac{-1}{2} \right| a_{34} \left| \frac{-1}{2} \right\rangle \\ \left\langle \frac{-3}{2} \right| a_{41} \left| \frac{5}{2} \right\rangle & \left\langle \frac{-3}{2} \right| a_{42} \left| \frac{3}{2} \right\rangle & \left\langle \frac{-3}{2} \right| a_{43} \left| \frac{1}{2} \right\rangle & \left\langle \frac{-3}{2} \right| a_{44} \left| \frac{-1}{2} \right\rangle \end{bmatrix} \tag{63}$$

$$= \begin{bmatrix} 0 & \left\langle \frac{3}{2} \right| a_{12} \left| \frac{3}{2} \right\rangle & 0 & 0 \\ 0 & 0 & \left\langle \frac{1}{2} \right| a_{23} \left| \frac{1}{2} \right\rangle & 0 \\ 0 & 0 & 0 & \left\langle \frac{-1}{2} \right| a_{34} \left| \frac{-1}{2} \right\rangle \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{64}$$

$$\hat{S}_+ = \frac{\hbar}{2}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{65}$$

$$\hat{S}_- = \begin{bmatrix} \langle \frac{3}{2} | \hat{S}_- | \frac{3}{2} \rangle & \langle \frac{3}{2} | \hat{S}_- | \frac{1}{2} \rangle & \langle \frac{3}{2} | \hat{S}_- | \frac{-1}{2} \rangle & \langle \frac{3}{2} | \hat{S}_- | \frac{-3}{2} \rangle \\ \langle \frac{1}{2} | \hat{S}_- | \frac{3}{2} \rangle & \langle \frac{1}{2} | \hat{S}_- | \frac{1}{2} \rangle & \langle \frac{1}{2} | \hat{S}_- | \frac{-1}{2} \rangle & \langle \frac{1}{2} | \hat{S}_- | \frac{-3}{2} \rangle \\ \langle \frac{-1}{2} | \hat{S}_- | \frac{3}{2} \rangle & \langle \frac{-1}{2} | \hat{S}_- | \frac{1}{2} \rangle & \langle \frac{-1}{2} | \hat{S}_- | \frac{-1}{2} \rangle & \langle \frac{-1}{2} | \hat{S}_- | \frac{-3}{2} \rangle \\ \langle \frac{-3}{2} | \hat{S}_- | \frac{3}{2} \rangle & \langle \frac{-3}{2} | \hat{S}_- | \frac{1}{2} \rangle & \langle \frac{-3}{2} | \hat{S}_- | \frac{-1}{2} \rangle & \langle \frac{-3}{2} | \hat{S}_- | \frac{-3}{2} \rangle \end{bmatrix} \tag{66}$$

$$= \begin{bmatrix} \langle \frac{3}{2} | a_{11} | \frac{1}{2} \rangle & \langle \frac{3}{2} | a_{12} | \frac{-1}{2} \rangle & \langle \frac{3}{2} | a_{13} | \frac{-3}{2} \rangle & \langle \frac{3}{2} | a_{14} | \frac{-5}{2} \rangle \\ \langle \frac{1}{2} | a_{21} | \frac{1}{2} \rangle & \langle \frac{1}{2} | a_{22} | \frac{-1}{2} \rangle & \langle \frac{1}{2} | a_{23} | \frac{-3}{2} \rangle & \langle \frac{1}{2} | a_{24} | \frac{-5}{2} \rangle \\ \langle \frac{-1}{2} | a_{31} | \frac{1}{2} \rangle & \langle \frac{-1}{2} | a_{32} | \frac{-1}{2} \rangle & \langle \frac{-1}{2} | a_{33} | \frac{-3}{2} \rangle & \langle \frac{-1}{2} | a_{34} | \frac{-5}{2} \rangle \\ \langle \frac{-3}{2} | a_{41} | \frac{1}{2} \rangle & \langle \frac{-3}{2} | a_{42} | \frac{-1}{2} \rangle & \langle \frac{-3}{2} | a_{43} | \frac{-3}{2} \rangle & \langle \frac{-3}{2} | a_{44} | \frac{-5}{2} \rangle \end{bmatrix} \tag{67}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ \langle \frac{1}{2} | a_{21} | \frac{1}{2} \rangle & 0 & 0 & 0 \\ 0 & \langle \frac{-1}{2} | a_{32} | \frac{-1}{2} \rangle & 0 & 0 \\ 0 & 0 & \langle \frac{-3}{2} | a_{43} | \frac{-3}{2} \rangle & 0 \end{bmatrix} \tag{68}$$

$$\hat{S}_- = \frac{\hbar}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix} \tag{69}$$

$$\hat{S}_x = \frac{\hbar}{2} \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix} \tag{70}$$

$$\hat{S}_y = \frac{i\hbar}{2} \begin{bmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix} \tag{71}$$

$$\hat{S}_x^2 = \frac{\hbar^2}{4} \begin{bmatrix} 3 & 0 & 2\sqrt{3} & 0 \\ 0 & 7 & 0 & 2\sqrt{3} \\ 2\sqrt{3} & 0 & 7 & 0 \\ 0 & 2\sqrt{3} & 0 & 3 \end{bmatrix} \tag{72}$$

$$\hat{S}_y^2 = \frac{-\hbar^2}{4} \begin{bmatrix} -3 & 0 & 2\sqrt{3} & 0 \\ 0 & -7 & 0 & 2\sqrt{3} \\ 2\sqrt{3} & 0 & -7 & 0 \\ 0 & 2\sqrt{3} & 0 & -3 \end{bmatrix} \tag{73}$$

$$\hat{H} = \frac{\varepsilon_0}{4} \begin{bmatrix} 0 & 0 & 4\sqrt{3} & 0 \\ 0 & 0 & 0 & 4\sqrt{3} \\ 4\sqrt{3} & 0 & 0 & 0 \\ 0 & 4\sqrt{3} & 0 & 0 \end{bmatrix} - \frac{\varepsilon_0}{4} \begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix} \tag{74}$$

$$\hat{H} = \varepsilon_0 \begin{bmatrix} -9 & 0 & 4\sqrt{3} & 0 \\ 0 & -1 & 0 & 4\sqrt{3} \\ 4\sqrt{3} & 0 & -1 & 0 \\ 0 & 4\sqrt{3} & 0 & -9 \end{bmatrix} \tag{75}$$

Now to diagonalize the Hamiltonian, the eigenvalues and the eigenvectors need to be found according to $\hat{H} = PDP^{-1}$, where P is the matrix fo the eigenvectors, and D is a diagonal matrix of the eigenvalues. To find the eigenvalues we find $det(\hat{H} - \lambda\hat{I}) = 0$.

$$det(\hat{H} - \lambda\hat{I}) = \begin{vmatrix} -9-\lambda & 0 & 4\sqrt{3} & 0 \\ 0 & -1-\lambda & 0 & 4\sqrt{3} \\ 4\sqrt{3} & 0 & -1-\lambda & 0 \\ 0 & 4\sqrt{3} & 0 & -9-\lambda \end{vmatrix} = 0 \tag{76}$$

Expanding along the first row.

$$0 = (-9-\lambda)\begin{vmatrix} -1-\lambda & 0 & 4\sqrt{3} \\ 0 & -1-\lambda & 0 \\ 4\sqrt{3} & 0 & -9-\lambda \end{vmatrix} + (4\sqrt{3})\begin{vmatrix} 0 & -1-\lambda & 4\sqrt{3} \\ 4\sqrt{3} & 0 & 0 \\ 0 & 4\sqrt{3} & -9-\lambda \end{vmatrix} \tag{77}$$

For the first determinant, expand along the second row. The second determinant, along the second row.

$$\begin{aligned} 0 &= (-9-\lambda)\left[(-1-\lambda)\left[(-1-\lambda)(-9-\lambda) - (4\sqrt{3})(4\sqrt{3})\right]\right] \\ &\quad - (4\sqrt{3})\left[(4\sqrt{3})\left[(-1-\lambda)(-9-\lambda) - (4\sqrt{3})(4\sqrt{3})\right]\right] && (78) \\ 0 &= (-9-\lambda)\left[(-1-\lambda)\left[\lambda^2 + 10\lambda + 9 - 48\right]\right] \\ &\quad - (4\sqrt{3})\left[(4\sqrt{3})\left[\lambda^2 + 10\lambda + 9 - 48\right]\right] && (79) \\ 0 &= (\lambda^2 + 10\lambda - 39)^2 && (80) \\ 0 &= (\lambda-3)^2(\lambda+13)^2 && (81) \\ \lambda &= 3, 3, -13, -13 && (82) \end{aligned}$$

With the eigenvalues, we input them back into the original equation $\hat{H} - \lambda\hat{I} = 0$ to find the eigenvectors.

$$|\lambda_{1,2}\rangle = 3 \quad : \quad \begin{pmatrix} -12 & 0 & 4\sqrt{3} & 0 \\ 0 & -4 & 0 & 4\sqrt{3} \\ 4\sqrt{3} & 0 & -4 & 0 \\ 0 & 4\sqrt{3} & 0 & -12 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{83}$$

$$\begin{aligned} -12a \quad + 4\sqrt{3}c \quad &= 0 \\ -\quad 4b \quad + 4\sqrt{3}d &= 0 \\ 4\sqrt{3}a \quad - \quad 4c \quad &= 0 \\ 4\sqrt{3}b \quad - \quad 12d &= 0 \end{aligned} \tag{84}$$

We have $c = \sqrt{3}a$ and $d = \frac{\sqrt{3}}{3}b$, a and b are arbitrary, making a solution set. Set $a = b = 1$.

$$|\lambda_{1,2}\rangle = A\begin{pmatrix} a \\ b \\ \sqrt{3}a \\ \frac{\sqrt{3}}{3}b \end{pmatrix} = A_1\begin{pmatrix} 1 \\ 0 \\ \sqrt{3} \\ 0 \end{pmatrix} + A_2\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{\sqrt{3}}{3} \end{pmatrix} \tag{85}$$

$$\begin{aligned} |\lambda_1\rangle = A_1\begin{pmatrix} 1 \\ 0 \\ \sqrt{3} \\ 0 \end{pmatrix} \qquad & \qquad \langle\lambda_1|\lambda_1\rangle = A_1^2\begin{pmatrix} 1 & 0 & \sqrt{3} & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ \sqrt{3} \\ 0 \end{pmatrix} = 1 \\ \Rightarrow \qquad & \qquad A_1^2(1+0+3+0) = 1 \\ |\lambda_1\rangle = \frac{1}{2}\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \qquad & \qquad A_1 = \frac{1}{2} \end{aligned} \tag{86}$$

$$\begin{aligned} |\lambda_2\rangle = A_2\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{\sqrt{3}}{3} \end{pmatrix} \qquad & \qquad \langle\lambda_2|\lambda_2\rangle = A_2^2\begin{pmatrix} 0 & 1 & 0 & 0\frac{\sqrt{3}}{3} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{\sqrt{3}}{3} \end{pmatrix} = 1 \\ \Rightarrow \qquad & \qquad A_2^2\left(0+1+0+\frac{1}{3}\right) = 1 \\ |\lambda_2\rangle = \frac{\sqrt{3}}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{\sqrt{3}}{3} \end{pmatrix} \qquad & \qquad A_2 = \frac{\sqrt{3}}{2} \end{aligned} \tag{87}$$

$$|\lambda_{1,2}\rangle = \frac{1}{2}\begin{pmatrix}1\\0\\\sqrt{3}\\0\end{pmatrix} + \frac{\sqrt{3}}{2}\begin{pmatrix}0\\1\\0\\\frac{\sqrt{3}}{3}\end{pmatrix} \tag{88}$$

$$|\lambda_{3,4}\rangle = -13 \quad : \quad \begin{pmatrix}4 & 0 & 4\sqrt{3} & 0\\0 & 12 & 0 & 4\sqrt{3}\\4\sqrt{3} & 0 & 12 & 0\\0 & 4\sqrt{3} & 0 & -4\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \tag{89}$$

$$\begin{aligned}4a \quad + 4\sqrt{3}c \quad &= 0\\12b \quad + 4\sqrt{3}d &= 0\\4\sqrt{3}a \quad + \quad 12c \quad &= 0\\4\sqrt{3}b \quad + \quad 4d &= 0\end{aligned} \tag{90}$$

We have $c = -\frac{\sqrt{3}}{3}a$ and $d = -\sqrt{3}b$, a and b are arbitrary, making a solution set. Set $a = b = 1$.

$$|\lambda_{3,4}\rangle = A\begin{pmatrix}a\\b\\-\frac{\sqrt{3}}{3}a\\-\sqrt{3}b\end{pmatrix} = A_1\begin{pmatrix}1\\0\\-\frac{\sqrt{3}}{3}\\0\end{pmatrix} + A_2\begin{pmatrix}0\\1\\0\\-\sqrt{3}\end{pmatrix} \tag{91}$$

$$\begin{aligned}&|\lambda_3\rangle = A_1\begin{pmatrix}1\\0\\-\frac{\sqrt{3}}{3}\\0\end{pmatrix} & & \langle\lambda_3|\lambda_3\rangle = A_1^2\begin{pmatrix}1 & 0 & -\frac{\sqrt{3}}{3} & 0\end{pmatrix}\begin{pmatrix}1\\0\\-\frac{\sqrt{3}}{3}\\0\end{pmatrix} = 1\\ & &\Rightarrow \quad & A_1^2\left(1+0+\frac{1}{3}+0\right) = 1\\&|\lambda_3\rangle = \frac{\sqrt{3}}{2}\begin{pmatrix}1\\0\\-\frac{\sqrt{3}}{3}\\0\end{pmatrix} & & A_1 = \frac{\sqrt{3}}{2}\end{aligned} \tag{92}$$

$$\begin{aligned}&|\lambda_4\rangle = A_2\begin{pmatrix}0\\1\\0\\-\sqrt{3}\end{pmatrix} & & \langle\lambda_4|\lambda_4\rangle = A_2^2\begin{pmatrix}0 & 1 & 0 & 0-\sqrt{3}\end{pmatrix}\begin{pmatrix}0\\1\\0\\-\sqrt{3}\end{pmatrix} = 1\\ & &\Rightarrow \quad & A_2^2\,(0+1+0+3) = 1\\&|\lambda_4\rangle = \frac{1}{2}\begin{pmatrix}0\\1\\0\\-\sqrt{3}\end{pmatrix} & & A_2 = \frac{1}{2}\end{aligned} \tag{93}$$

$$|\lambda_{3,4}\rangle = \frac{\sqrt{3}}{2}\begin{pmatrix}1\\0\\-\frac{\sqrt{3}}{3}\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\1\\0\\-\sqrt{3}\end{pmatrix} \tag{94}$$

(b) Find the eigenvectors and verify that the energy levels are doubly degenerate.

Solution From part (a) the eigenvectors were found, and the fact that each eigenvalue produces two eigenvectors points to the energy levels being doubly degenerate. Explicitly shown in $\hat{H}\Psi = E\Psi$, where E corresponds to the λ's that I found.