PHY 313 — Satellites & Space Science

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Assignment: HW 3

Problem 1:

A rocket with structure 6.0 tons in mass needs to take a payload of 500 kg to an altitude of 36 000 km above Earth and place it in orbit.

(a) Calculate the minimum amount of fuel needed to do this in a single-stage process if the propulsion has an I_{sp} of 450 s.

Solution The altitude of the orbit suggests that it is a geosynchronous orbit. We need to first find the ΔV needed to reach that orbit.

$$\Delta V = C \ln\left(\frac{m_{\text{i}}}{m_{\text{f}}}\right) \Rightarrow m_{\text{i}} = m_{\text{f}} \exp\left(\frac{\Delta V}{C}\right) \qquad I_{\text{sp}} = \frac{C}{g}$$

$$\Delta\vec{V}_{\text{total}} = \Delta\vec{V}_{\text{orbit}} + \Delta\vec{V}_{\text{gravity}} + \Delta\vec{V}_{\text{tangential}}$$

ΔV_{orbit} :

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

$$v = \sqrt{\frac{GM}{r}}$$

$$\Delta V_{orbit} = \sqrt{\frac{(6.67 \times 10^{-11}\,\text{N}\,\text{kg}^{-2}\,\text{m}^2)(5.97 \times 10^{24}\,\text{kg})}{(36\,000 \times 10^3\,\text{m}) + (6.37 \times 10^6\,\text{m})}}$$

$$\Delta V_{orbit} = 3.07 \times 10^3\,\text{m}\,\text{s}^{-1}$$

$\Delta V_{\text{gravity}}$:

$$K_{\text{i}} + U_{\text{i}} = K_{\text{f}} + U_{\text{f}}$$

$$\frac{1}{2}mv^2 - \frac{GMm}{R_{\text{E}}} = 0 - \frac{GMm}{R_{\text{E}} + h}$$

$$v = \sqrt{2GM\left(\frac{1}{R_{\text{E}}} - \frac{1}{R_{\text{E}} + h}\right)}$$

$$\Delta V_{\text{gravity}} = \sqrt{2(6.67 \times 10^{-11}\,\text{N}\,\text{kg}^{-2}\,\text{m}^2)(5.97 \times 10^{24}\,\text{kg})\left(\frac{1}{6.37 \times 10^6\,\text{m}} - \frac{1}{42.37 \times 10^6\,\text{m}}\right)}$$

$$\Delta V_{\text{gravity}} = 1.03 \times 10^4\,\text{m}\,\text{s}^{-1}$$

$\Delta V_{\text{tangential}}$:

$$v = \omega r \qquad r = R_{\text{E}} \cos(L)$$

$$\Delta V_{\text{tangential}} = \omega_{\text{E}} R_{\text{E}} \cos(L)$$

For geosynchronous orbits, it is optimal to launch closer to the equator.
Therefore we can assume that we launch from the equator for the minimum amount of fuel.
If launched from any other latitude, more fuel will be needed for compensation midflight
to reach the fundamental plane.

$$\Delta V_{\text{tangential}} = \left(\frac{2\pi}{86\,400\,\text{s}}\right)(6.37 \times 10^6\,\text{m})\cos(0^\circ) = 4.63 \times 10^2\,\text{m}\,\text{s}^{-1}$$

$$\Delta V_{\text{total}} = 3.07 \times 10^3\,\text{m}\,\text{s}^{-1} + 1.03 \times 10^4\,\text{m}\,\text{s}^{-1} - 4.63 \times 10^2\,\text{m}\,\text{s}^{-1} = 1.29 \times 10^4\,\text{m}\,\text{s}^{-1}$$

The directions are aligned for simplicity.

$$m_{\text{i}} = (6500\,\text{kg}) \exp\left(\frac{1.29 \times 10^4\,\text{m}\,\text{s}^{-1}}{(450\,\text{s})(9.81\,\text{m}\,\text{s}^{-2})}\right) = 120\,782\,\text{kg}$$

$$m_{\text{fuel}} = 120\,782\,\text{kg} - 6500\,\text{kg} = 114\,282\,\text{kg}$$

(b) Now recalling that multi-staging saves fuel, calculate how much fuel will be needed if we divide the process into 2 stages and 3 stages, each time discarding the empty tank/compartment/stage. Assume the same I_{sp} applies to all stages.

i. 2 stages;

Solution We still need the same ΔV to reach the target orbit. We can assume equal proportion in fuel and structure for each stage.

$$\Delta V = \Delta V_1 + \Delta V_2$$

$$\Delta V_1 = (4414.5\,\mathrm{m\,s^{-1}}) \ln\left(\frac{6000 + 500 + 114282}{6000 + 500 + 57140}\right) = 2.83 \times 10^3\,\mathrm{m\,s^{-1}}$$

$$\Delta V_2 = 1.29 \times 10^4\,\mathrm{m\,s^{-1}} - 2.83 \times 10^3\,\mathrm{m\,s^{-1}} = 1.007 \times 10^4\,\mathrm{m\,s^{-1}}$$

$$m_\mathrm{f} = (3000 + 500 + 57140) \exp\left(-\frac{1.007 \times 10^4\,\mathrm{m\,s^{-1}}}{4414.5\,\mathrm{m\,s^{-1}}}\right) = 6196\,\mathrm{kg}$$

$$\text{Mass of fuel left} = 6196 - 3500 = 2696\,\mathrm{kg}$$

$$\text{Required fuel compared to single stage} = 114282 - 2696 = 111\,586\,\mathrm{kg}$$

ii. 3 stages,

Solution Similar reasoning to previous part.

$$\Delta V = \Delta V_1 + \Delta V_2 + \Delta V_3$$

$$\Delta V_1 = (4414.5\,\mathrm{m\,s^{-1}}) \ln\left(\frac{6000 + 500 + 114282}{6000 + 500 + 76187}\right) = 1.67 \times 10^3\,\mathrm{m\,s^{-1}}$$

$$\Delta V_2 = (4414.5\,\mathrm{m\,s^{-1}}) \ln\left(\frac{4000 + 500 + 76187}{4000 + 500 + 38093}\right) = 2.82 \times 10^3\,\mathrm{m\,s^{-1}}$$

$$\Delta V_3 = 1.29 \times 10^4\,\mathrm{m\,s^{-1}} - (1.67 \times 10^3\,\mathrm{m\,s^{-1}} + 2.82 \times 10^3\,\mathrm{m\,s^{-1}}) = 8.41 \times 10^3\,\mathrm{m\,s^{-1}}$$

$$m_\mathrm{f} = (2000 + 500 + 38093) \exp\left(-\frac{8.41 \times 10^3\,\mathrm{m\,s^{-1}}}{4414.5\,\mathrm{m\,s^{-1}}}\right) = 6041\,\mathrm{kg}$$

$$\text{Mass of fuel left} = 6041 - 2500 = 3541\,\mathrm{kg}$$

$$\text{Required fuel compared to single stage} = 114282 - 3541 = 110\,741\,\mathrm{kg}$$

(c) Calculate how much energy (in Joules and in equivalent gasoline liters) will be needed in each case (1 stage, 2 stages, 3 stages).

Solution Conservation of Energy will work with all cases. We can find the energy needed to reach the target altitude, and the energy to place the satellite into orbit. Starting with one stage.

$$E_{\text{total}} = K_{\text{altitude}} + K_{\text{orbit}}$$

$$\text{Energy density of gasoline } = 34.2 \times 10^6\,\mathrm{J\,L^{-1}}$$

$$\begin{aligned}
K_{\text{altitude}} - \frac{GMm}{R_{\mathrm{E}}} &= 0 - \frac{GMm}{R_{\mathrm{E}} + h} \\
K_{\text{altitude}} &= GMm\left(\frac{1}{R_{\mathrm{E}}} - \frac{1}{R_{\mathrm{E}} + h}\right) \\
&= (GM)(120\,780\,\mathrm{kg})\left(\frac{1}{6.37 \times 10^6\,\mathrm{m}} - \frac{1}{6.37 \times 10^6\,\mathrm{m} + 36\,000 \times 10^3\,\mathrm{m}}\right) \\
&= 6.42 \times 10^{12}\,\mathrm{J}
\end{aligned}$$

This assumes that the mass is constant, but this is not true as the propellant is used up to the target altitude. Comparing with a rocket that only has the structure and payload.

$$\begin{aligned}
K_{\text{altitude}} &= (GM)(6500\,\mathrm{kg})\left(\frac{1}{6.37 \times 10^6\,\mathrm{m}} - \frac{1}{6.37 \times 10^6\,\mathrm{m} + 36\,000 \times 10^3\,\mathrm{m}}\right) \\
&= 3.45 \times 10^{11}\,\mathrm{J}
\end{aligned}$$

There is an order of magnitude of difference, which is practically significant. Now when the satellite reaches the target altitude, it needs energy to enter the orbit. It would still need some of the fuel, but the proportion of that is dependent on the mission itself. Lets assume we still have all the fuel.

$$\begin{aligned}
v &= \sqrt{\frac{GM}{R_{\mathrm{E}} + h}} \\
K_{\text{orbit}} = \frac{1}{2}mv^2 &= \frac{GMm}{2(R_{\mathrm{E}} + h)} = \frac{GM(120\,780\,\mathrm{kg})}{2(6.37 \times 10^6\,\mathrm{m} + 36\,000 \times 10^3\,\mathrm{m})} \\
&= 5.68 \times 10^{11}\,\mathrm{J}
\end{aligned}$$

Comparing with rocket that only has the structure and payload.

$$\begin{aligned}
K_{\text{orbit}} &= \frac{GM(6500\,\mathrm{kg})}{2(6.37 \times 10^6\,\mathrm{m} + 36\,000 \times 10^3\,\mathrm{m})} \\
&= 3.05 \times 10^{10}\,\mathrm{J}
\end{aligned}$$

Also an order of magnitude of difference.

$$\text{Dry rocket: } E_{\text{total}} = 3.45 \times 10^{11}\,\mathrm{J} + 3.05 \times 10^{10}\,\mathrm{J} = 3.76 \times 10^{11}\,\mathrm{J} = 1.10 \times 10^4 \text{ Gasoline liters}$$

$$\text{1 stage rocket: } E_{\text{total}} = 6.42 \times 10^{12}\,\mathrm{J} + 5.68 \times 10^{11}\,\mathrm{J} = 6.99 \times 10^{12}\,\mathrm{J} = 2.04 \times 10^5 \text{ Gasoline liters}$$

For further stages, I found that the initial mass of the rocket is lower, but it would have no practical effect because of the small difference. The only significant difference is that mass would be expended at each stage at a certain altitude, so the energy from reaching the altitude is split into stages with different masses and distances similar to the rocket equation. This would eventually lead to a smaller K_{altitude}. I can safely say that for further stages the following inequality applies.

$$3.76 \times 10^{11}\,\mathrm{J} \le E_{\text{total}} \le 6.99 \times 10^{12}\,\mathrm{J}$$

$$1.10 \times 10^4 \text{ Gasoline liters} \le E_{\text{total}} \le 2.04 \times 10^5 \text{ Gasoline liters}$$

Problem 2:

Consider a satellite orbiting at 500 km, carrying solar arrays of 20% efficiency, which decrease by 2.5% each year, and non-rechargeable batteries to operate during eclipses only. The satellite needs a continuous supply of 1.25 kW of power.

(a) Determine the energy needed from the batteries.

Solution To get the total energy needed from the batteries during the eclipse, we need to find how long the satellite is eclipsed by the Earth.

$$\theta = \arcsin\left(\frac{R_{\mathrm{E}}}{R_{\mathrm{E}} + h}\right) = \arcsin\left(\frac{6.37 \times 10^6\,\mathrm{m}}{(6.37 \times 10^6\,\mathrm{m}) + (500 \times 10^3\,\mathrm{m})}\right) = 68^\circ$$

$$\text{Time spent eclipsed} = \frac{2\theta}{360^\circ} \cdot T_{\mathrm{orbit}}$$

$$T_{\mathrm{orbit}} = \sqrt{\frac{4\pi^2}{GM}a^3} = \sqrt{\frac{4\pi^2}{GM}(6871 \times 10^3\,\mathrm{m})^3} = 5670\,\mathrm{s}$$

$$\text{Time spent eclipsed} = \frac{2(68^\circ)}{360^\circ} \cdot (5670\,\mathrm{s}) = 2142\,\mathrm{s}$$

$$\text{Energy needed during eclipse } = P \cdot t = (1.25\,\mathrm{kW})(2142\,\mathrm{s}) = 2.68 \times 10^6\ \mathrm{J}$$

(b) Determine the total area needed for the solar arrays, if they make (on average) an angle of 30° with the sun.

Solution Simple equation.

$$P = S_{\mathrm{E}} \cdot A \cdot e \cdot \cos\theta$$

$$A = \frac{P}{S_{\mathrm{E}} \cdot e \cdot \cos\theta} = \frac{1.25\,\mathrm{kW}}{(1.37\,\mathrm{kW\,m^{-2}})(0.2)(\cos 30^\circ)} = 5.27\,\mathrm{m}^2$$

(c) How long will the solar arrays last?

Solution Arithmetic series.

$$0.2 - 0.025n = 0$$

$$n = 8\ \mathrm{y}$$

As the solar array degrades, a larger area would be needed to maintain the continuous supply power. Therefore it is beneficial to choose a solar array area that would last the whole lifetime.

$$A(7) = \frac{1.25\,\mathrm{kW}}{(1.37\,\mathrm{kW\,m^{-2}})(0.2 - 0.025(7))(\cos 30^\circ)} = 42.1\,\mathrm{m}^2$$

$$A(7.5) = 84.3\,\mathrm{m}^2 \qquad A(7.9) = 421\,\mathrm{m}^2$$

Problem 3:

A remote-sensing satellite orbiting at 250 km above Earth's surface is tasked with imaging a forest fire with an area of $40000\,\mathrm{m}^2$. The temperature of that fire is $\approx 800\,^\circ\mathrm{C}$.

(a) Determine the most appropriate wavelength of imaging.

Solution Wien's Law.

$$\lambda_{\mathrm{peak}} T = 2.898 \times 10^{-3}\,\mathrm{m\,K}$$

$$\lambda_{\mathrm{peak}} = \frac{2.898 \times 10^{-3}\,\mathrm{m\,K}}{800 + 273.15} = 2.7 \times 10^{-6}\,\mathrm{m}$$

(b) If the detector is a PbSe CCD of 1024×1024 pixels and has a radius of 1 cm, determine: the minimal linear resolution; the swath width; the focal length; the field of view; and the minimum sensor aperture to achieve the required resolution.

Solution Assume that the fire covers a square area for simplicity. Therefore the minimal linear resolution would be 200 m, meaning that at least one pixel can resolve the square area. Also, small angle approximations apply here. The swath width is the total width of the area imaged on the ground, so with 1024 pixels.

$$\text{Swath width} = 1024 \cdot 200\,\mathrm{m} = 204.8\,\mathrm{km}$$

$$\text{Angular Resolution: } \tan\theta \approx \theta = \frac{\mathrm{Res}/2}{h} = \frac{(200\,\mathrm{m})/2}{250 \times 10^3\,\mathrm{m}} = 4 \times 10^{-4}\,\mathrm{rad}$$

$$\text{Field of View} = 2\theta = 2(4 \times 10^{-4}\,\mathrm{rad}) = 8 \times 10^{-4}\,\mathrm{rad}$$

$$\text{Focal length: } FOV = 2\arctan\left(\frac{r_d}{fl}\right) = 2\theta$$

$$fl = \frac{r_d}{\tan\theta} = \frac{1 \times 10^{-2}\,\mathrm{m}}{\tan(4 \times 10^{-4}\,\mathrm{rad})} = 25\,\mathrm{m}$$

$$\text{Sensor Aperture, D} = \frac{1.22\lambda}{\theta} = \frac{1.22(2.7 \times 10^{-6}\,\mathrm{m})}{4 \times 10^{-4}\,\mathrm{rad}} = 8.235 \times 10^{-3}\,\mathrm{m}$$