PHY 310 — Mathematical Methods in Physics

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Problem 2: Mathematical Physics

Fourier-Hermite Expansion of a Function

A function $\boldsymbol{f(x)}$ is expanded in a Hermite series:

$$\boldsymbol{f(x) = \sum_{n=0}^{\infty} a_n H_n(x).}$$

(a) Show that the coefficients $\boldsymbol{a_n}$ are given by

$$\boldsymbol{\frac{1}{\sqrt{\pi}2^n n!} \int_{-\infty}^{\infty} f(t) H_n(t) e^{-t^2}\, dt.}$$

Solution The Hermite polynomials are known to be orthonormal from the following equation

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) = \begin{cases} 0 & if\ n \neq m \\ 2^n n! \sqrt{\pi} & if\ n = m \end{cases}$$

Clearly the fraction in the question resembles the case when $n = m$, therefore transforming the question to one where the integral exists by multiplying by $H_n(x)$ and e^{-x^2}.

$$f(x) H_n(x) e^{-x^2} = \sum_{n=0}^{\infty} a_n e^{-x^2} H_n(x) H_n(x)$$

Multiplying by an integral from $-\infty$ to ∞, and adding a dummy variable t to differentiate the equality.

$$\int_{-\infty}^{\infty} f(t) H_n(t) e^{-t^2}\, dt = \sum_{n=0}^{\infty} a_n \int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_n(x)\, dx$$

Finally.

$$a_n = \frac{1}{\sqrt{\pi}2^n n!} \int_{-\infty}^{\infty} f(t) H_n(t) e^{-t^2}\, dt$$

(b) i. Expand the function

$$\boldsymbol{f(x) = e^{2bx}}$$

in a Hermite series.

Solution Using the generating function.

$$e^{2tx - t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

t can be any real number. Set $t = b$.

$$e^{2bx} e^{-b^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} b^n$$

$$f(x) = e^{2bx} = e^{b^2} \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} b^n$$

ii. Use the result of part(b)i to deduce

$$\int_{-\infty}^{\infty} e^{-x^2 + 2bx} H_n(x)\, dx = \sqrt{\pi} (2b)^n e^{b^2}$$

Solution

$$\int_{-\infty}^{\infty} e^{-x^2+2bx} H_n(x)\, dx = \int_{-\infty}^{\infty} e^{-x^2} e^{2bx} H_n(x)\, dx = \int_{-\infty}^{\infty} f(x) H_n(x) e^{-x^2}\, dx$$

$$\int_{-\infty}^{\infty} f(x) H_n(x) e^{-x^2}\, dx = \int_{-\infty}^{\infty} e^{b^2} \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} b^n H_n(x) e^{-x^2}\, dx$$
$$= \sum_{n=0}^{\infty} \frac{e^{b^2} b^n}{n!} \int_{-\infty}^{\infty} H_n(x) H_n(x) e^{-x^2}\, dx$$

Applying the orthonormality condition.

$$= \sum_{n=0}^{\infty} \frac{e^{b^2} b^n}{n!} 2^n n! \sqrt{\pi} = \sum_{n=0}^{\infty} e^{b^2} b^n 2^n \sqrt{\pi} = \sqrt{\pi}(2b)^n e^{b^2}$$

Problem 3: Quantum Mechanics

Raising and Lowering Operators of Quantum Harmonic Oscillator

An operator $\boldsymbol{\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left(x + \frac{i}{m\omega}\hat{p}\right)}$ and its adjoint $\boldsymbol{\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(x - \frac{i}{m\omega}\hat{p}\right)}$, with $\boldsymbol{\hat{p} = -i\hbar\frac{d}{dx}}$, act on a harmonic oscillator wavefunction,

$$\boldsymbol{\phi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}} H_n\left(\sqrt{\frac{m\omega}{\hbar}} x\right)}.$$

Using the transformation $\boldsymbol{\xi \equiv \sqrt{\frac{m\omega}{\hbar}} x}$ so that,

$$\boldsymbol{\hat{a} = \frac{1}{\sqrt{2}}\left(\xi + \frac{d}{d\xi}\right), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\xi - \frac{d}{d\xi}\right) \quad \text{and} \quad \phi_n(\xi) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2} H_n(\xi),}$$

show that

(a) $\boldsymbol{\hat{a}\,\phi_n(\xi) = \sqrt{n}\,\phi_{n-1}(\xi)},$

$$\left[\frac{1}{\sqrt{2}}\left(\xi + \frac{d}{d\xi}\right)\right] \left[\frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2} H_n(\xi)\right]$$

$$\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \left[\left(\xi e^{-\frac{1}{2}\xi^2} H_n(\xi) + \frac{d}{d\xi}\left(e^{-\frac{1}{2}\xi^2} H_n(\xi)\right)\right)\right]$$

$$\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \left[\left(\xi e^{-\frac{1}{2}\xi^2} H_n(\xi) + \left(-\xi e^{-\frac{1}{2}\xi^2} H_n(\xi) + e^{-\frac{1}{2}\xi^2} H_n'(\xi)\right)\right)\right]$$

$$\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2} \left[H_n'(\xi)\right]$$

Using the recurrence relation $H_n'(x) = 2nH_{n-1}(x)$

$$\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2} \left[2nH_{n-1}(\xi)\right]$$

$$\frac{2}{\sqrt{2}\sqrt{2^n}} \frac{n}{\sqrt{n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2} \left[H_{n-1}(\xi)\right]$$

Utilizing properties of $\boldsymbol{n!}$, $\boldsymbol{\frac{1}{n!} = \frac{1}{n(n-1)!}}$.

$$\frac{\sqrt{2}}{\sqrt{2^n}} \frac{n}{\sqrt{n(n-1)!}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2} \left[H_{n-1}(\xi)\right]$$

$$\frac{\sqrt{n}}{\sqrt{2^{n-1}(n-1)!}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2} \left[H_{n-1}(\xi)\right]$$

$$\sqrt{n}\,\phi_{n-1}(\xi) = \frac{\sqrt{n}}{\sqrt{2^{n-1}(n-1)!}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2} H_{n-1}(\xi)$$

(b) $\boldsymbol{\hat{a}^\dagger\, \phi_n(\xi) = \sqrt{n+1}\,\phi_{n+1}(\xi)}$,

$$\left[\frac{1}{\sqrt{2}}\left(\xi - \frac{d}{d\xi}\right)\right]\left[\frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2} H_n(\xi)\right]$$

$$\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\left[\left(\xi e^{-\frac{1}{2}\xi^2} H_n(\xi) - \frac{d}{d\xi}\left(e^{-\frac{1}{2}\xi^2} H_n(\xi)\right)\right)\right]$$

$$\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\left[\left(\xi e^{-\frac{1}{2}\xi^2} H_n(\xi) - \left(-\xi e^{-\frac{1}{2}\xi^2} H_n(\xi) + e^{-\frac{1}{2}\xi^2} H_n'(\xi)\right)\right)\right]$$

$$\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2}\left[2\xi H_n(\xi) - H_n'(\xi)\right]$$

Using the recurrence relation $H_n'(x) = 2nH_{n-1}(x)$

$$\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2}\left[2\xi H_n(\xi) - 2nH_{n-1}(\xi)\right]$$

Using the recurrence relation $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$

$$\frac{1}{\sqrt{2^{n+1} n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2} H_{n+1}(\xi)$$

$$\frac{\sqrt{n+1}}{\sqrt{2^{n+1}(n+1)n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2} H_{n+1}(\xi)$$

Utilizing properties of $\boldsymbol{n!}$, $\boldsymbol{\frac{1}{(n+1)n!} = \frac{1}{(n+1)!}}$.

$$\sqrt{n+1}\,\phi_{n+1}(\xi) = \frac{\sqrt{n+1}}{\sqrt{2^{n+1}(n+1)!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2} H_{n+1}(\xi)$$

(c) $\boldsymbol{\left[\hat{a}, \hat{a}^\dagger\right]\phi_n(\xi) \equiv \left(\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\right)\phi_n(\xi) = \phi_n(\xi)}$,

$$\left[\frac{1}{\sqrt{2}}\left(\xi + \frac{d}{d\xi}\right)\frac{1}{\sqrt{2}}\left(\xi - \frac{d}{d\xi}\right) - \frac{1}{\sqrt{2}}\left(\xi - \frac{d}{d\xi}\right)\frac{1}{\sqrt{2}}\left(\xi + \frac{d}{d\xi}\right)\right]\phi_n(\xi)$$

$$\frac{1}{2}\left[\left(\xi + \frac{d}{d\xi}\right)\left(\xi - \frac{d}{d\xi}\right) - \left(\xi - \frac{d}{d\xi}\right)\left(\xi + \frac{d}{d\xi}\right)\right]\phi_n(\xi)$$

An important property of operators is that they do not necessarily commute. $\hat{a}\hat{a}^\dagger \neq \hat{a}^\dagger\hat{a}$.

$$\frac{1}{2}\left[\left(\xi + \frac{d}{d\xi}\right)\left(\xi - \frac{d}{d\xi}\right) - \left(\xi - \frac{d}{d\xi}\right)\left(\xi + \frac{d}{d\xi}\right)\right]\phi_n(\xi)$$

$$\frac{1}{2}\left[\left(\xi + 1 - \frac{d^2}{d\xi^2}\right) - \left(\xi - 1 - \frac{d^2}{d\xi^2}\right)\right]\phi_n(\xi)$$

$$\frac{1}{2}\left[\xi + 1 - \frac{d^2}{d\xi^2} - \xi + 1 + \frac{d^2}{d\xi^2}\right]\phi_n(\xi)$$

$$\phi_n(\xi)$$

(d) $\boldsymbol{\phi_n(\xi) = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}\phi_0(\xi) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\left(\xi - \frac{d}{d\xi}\right)^n e^{-\frac{1}{2}\xi^2}}$,

$$\frac{1}{\sqrt{n!}}\left(\frac{1}{\sqrt{2}}\left(\xi - \frac{d}{d\xi}\right)\right)^n\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2} H_0(\xi)$$

$$H_0(\xi) = 1$$

$$\phi_n(\xi) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\left(\xi - \frac{d}{d\xi}\right)^n e^{-\frac{1}{2}\xi^2}$$

(e) $\boldsymbol{\xi\,\phi_n(\xi) = \frac{1}{\sqrt{2}}\left[\sqrt{n+1}\,\phi_{n+1}(\xi) + \sqrt{n}\,\phi_{n-1}(\xi)\right]}$,

$$\frac{\xi}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2} H_n(\xi)$$

Using the recurrence relation $H_n(x) = \frac{1}{2x}\left[H_{n+1}(x) + 2nH_{n-1}(x)\right]$

$$\frac{\xi}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2}\left[\frac{1}{2\xi}\left[H_{n+1}(\xi) + 2nH_{n-1}(\xi)\right]\right]$$

$$\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2^{n+1}n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2}\left[H_{n+1}(\xi) + 2nH_{n-1}(\xi)\right]$$

$$\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2^{n+1}n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2}H_{n+1}(\xi) + \frac{1}{\sqrt{2}}\frac{2n}{\sqrt{2^{n+1}n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2}H_{n-1}(\xi)$$

$$\frac{1}{\sqrt{2}}\left[\frac{\sqrt{n+1}}{\sqrt{2^{n+1}(n+1)n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2}H_{n+1}(\xi) + \frac{2n}{\sqrt{2^{n+1}n(n-1)!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2}H_{n-1}(\xi)\right]$$

$$\frac{1}{\sqrt{2}}\left[\frac{\sqrt{n+1}}{\sqrt{2^{n+1}(n+1)!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2}H_{n+1}(\xi) + \frac{\sqrt{n}}{\sqrt{2^{n-1}(n-1)!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2}H_{n-1}(\xi)\right]$$

$$\xi\,\phi_n(\xi) = \frac{1}{\sqrt{2}}\left[\sqrt{n+1}\,\phi_{n+1}(\xi) + \sqrt{n}\,\phi_{n-1}(\xi)\right]$$

(f) $\boldsymbol{\hat{P}\,\phi_n(\xi) = \frac{i}{\sqrt{2}}\left[\sqrt{n+1}\,\phi_{n+1}(\xi) - \sqrt{n}\,\phi_{n-1}(\xi)\right], \quad \hat{P} \equiv \frac{\hat{p}}{\sqrt{m\omega\hbar}}}$.

$$-\frac{i\hbar}{\sqrt{m\omega\hbar}}\sqrt{\frac{m\omega}{\hbar}}\frac{d}{d\xi}\left[\frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2}H_n(\xi)\right]$$

$$-\frac{i}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\left[-\xi e^{-\frac{1}{2}\xi^2}H_n(\xi) + e^{-\frac{1}{2}\xi^2}H_n'(\xi)\right]$$

Using the recurrence relations $H_n'(x) = 2nH_{n-1}(x)$ and $H_n(x) = \frac{1}{2x}\left[H_{n+1}(x) + 2nH_{n-1}(x)\right]$.

$$-\frac{i}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2}\left[-\xi\left(\frac{1}{2\xi}\left[H_{n+1}(x) + 2nH_{n-1}(x)\right]\right) + (2nH_{n-1}(x))\right]$$

$$\frac{i}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2}\left[\frac{1}{2}H_{n+1}(x) - nH_{n-1}(x)\right]$$

$$i\left[\frac{1}{\sqrt{2^{n+2}n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2}H_{n+1}(x) - \frac{n}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2}H_{n-1}(x)\right]$$

$$\frac{i}{\sqrt{2}}\left[\frac{\sqrt{n+1}}{\sqrt{2^{n+1}(n+1)n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2}H_{n+1}(x) - \frac{n}{\sqrt{2^{n-1}n(n-1)!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2}H_{n-1}(x)\right]$$

$$\frac{i}{\sqrt{2}}\left[\frac{\sqrt{n+1}}{\sqrt{2^{n+1}(n+1)!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2}H_{n+1}(x) - \frac{\sqrt{n}}{\sqrt{2^{n-1}(n-1)!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2}H_{n-1}(x)\right]$$

$$\hat{P}\,\phi_n(\xi) = \frac{i}{\sqrt{2}}\left[\sqrt{n+1}\,\phi_{n+1}(\xi) - \sqrt{n}\,\phi_{n-1}(\xi)\right]$$

Problem 5: Quantum Mechanics

Coupled Harmonic Oscillators II. Separation of Center of Mass and Relative Motions

Particles with mass $\boldsymbol{m_1}$ and $\boldsymbol{m_2}$ are each bound in a one-dimensional harmonic oscillator potential $\frac{1}{2}\boldsymbol{m_i\omega_0^2 x_i^2}$ $(\boldsymbol{i=1,2})$ and interact with each other via the potential $\frac{1}{2}\boldsymbol{\lambda(x_1-x_2)^2}$. Thus the Hamiltonian of the system takes the form

$$\hat{H} = -\frac{\hbar^2}{2m_1}\frac{\partial^2}{\partial x_1^2} + \frac{1}{2}m_1\omega_0^2 x_1^2 - \frac{\hbar^2}{2m_2}\frac{\partial^2}{\partial x_2^2} + \frac{1}{2}m_2\omega_0^2 x_2^2 + \frac{1}{2}\lambda(x_1-x_2)^2.$$

(a) Introduce COM and relative coordinates, $\boldsymbol{R=\frac{m_1x_1+m_2x_2}{m_1+m_2}}$, $\boldsymbol{r=x_1-x_2}$, respectively, and show that

$$\hat{H} = -\frac{\hbar^2}{2M}\frac{\partial^2}{\partial R^2} + \frac{1}{2}M\omega_0^2 R^2 - \frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2} + \frac{1}{2}\mu\Omega^2 r^2, \quad \text{where,} \quad M \equiv m_1+m_2, \mu \equiv \frac{m_1m_2}{m_1+m_2}, \Omega^2 \equiv \omega_0^2 + \frac{\lambda}{\mu}.$$

Solution

$$-\frac{\hbar^2}{2m_1}\frac{\partial^2}{\partial x_1^2} + \frac{1}{2}m_1\omega_0^2x_1^2 - \frac{\hbar^2}{2m_2}\frac{\partial^2}{\partial x_2^2} + \frac{1}{2}m_2\omega_0^2x_2^2 + \frac{1}{2}\lambda(x_1-x_2)^2$$

$$-\frac{\hbar^2}{2m_1}\frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m_2}\frac{\partial^2}{\partial x_2^2} + \frac{1}{2}m_1\omega_0^2x_1^2 + \frac{1}{2}m_2\omega_0^2x_2^2 + \frac{1}{2}\lambda r^2$$

$$-\frac{\hbar^2}{2}\left[\frac{1}{m_1}\frac{\partial^2}{\partial x_1^2} + \frac{1}{m_2}\frac{\partial^2}{\partial x_2^2}\right] + \frac{1}{2}\omega_0^2\left[m_1x_1^2 + m_2x_2^2\right] + \frac{1}{2}\lambda r^2$$

Taking it step by step converting each term.

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial R}\frac{\partial R}{\partial x_1} + \frac{\partial}{\partial r}\frac{\partial r}{\partial x_1} = \frac{\partial}{\partial R}\left[\frac{1}{M}(m_1)\right] + \frac{\partial}{\partial r}[1]$$

$$\begin{aligned}
\frac{\partial^2}{\partial x_1^2} &= \frac{\partial}{\partial x_1}\left[\frac{\partial}{\partial R}\left[\frac{1}{M}(m_1)\right] + \frac{\partial}{\partial r}[1]\right] \\
&= \left[\frac{\partial}{\partial R}\left[\frac{1}{M}(m_1)\right] + \frac{\partial}{\partial r}[1]\right]\left[\frac{\partial}{\partial R}\left[\frac{1}{M}(m_1)\right] + \frac{\partial}{\partial r}[1]\right] \\
&= \frac{\partial^2}{\partial R^2}\left[\frac{1}{M}(m_1)\right]^2 + 2\frac{\partial^2}{\partial R\partial r}\left[\left(\frac{1}{M}(m_1)\right)(1)\right] + \frac{\partial^2}{\partial r^2}[1]^2 \\
\frac{1}{m_1}\frac{\partial^2}{\partial x_1^2} &= \frac{\partial^2}{\partial R^2}\left[\frac{m_1}{M^2}\right] + 2\frac{\partial^2}{\partial R\partial r}\left[\frac{1}{M}\right] + \frac{1}{m_1}\frac{\partial^2}{\partial r^2}
\end{aligned}$$

$$\frac{\partial}{\partial x_2} = \frac{\partial}{\partial R}\frac{\partial R}{\partial x_2} + \frac{\partial}{\partial r}\frac{\partial r}{\partial x_2} = \frac{\partial}{\partial R}\left[\frac{1}{M}(m_2)\right] + \frac{\partial}{\partial r}[-1]$$

$$\begin{aligned}
\frac{\partial^2}{\partial x_2^2} &= \frac{\partial}{\partial x_2}\left[\frac{\partial}{\partial R}\left[\frac{1}{M}(m_2)\right] + \frac{\partial}{\partial r}[-1]\right] \\
&= \left[\frac{\partial}{\partial R}\left[\frac{1}{M}(m_2)\right] + \frac{\partial}{\partial r}[-1]\right]\left[\frac{\partial}{\partial R}\left[\frac{1}{M}(m_2)\right] + \frac{\partial}{\partial r}[-1]\right] \\
&= \frac{\partial^2}{\partial R^2}\left[\frac{1}{M}(m_2)\right]^2 + 2\frac{\partial^2}{\partial R\partial r}\left[\left(\frac{1}{M}(m_2)\right)(-1)\right] + \frac{\partial^2}{\partial r^2}[-1]^2 \\
\frac{1}{m_2}\frac{\partial^2}{\partial x_2^2} &= \frac{\partial^2}{\partial R^2}\left[\frac{m_2}{M^2}\right] - 2\frac{\partial^2}{\partial R\partial r}\left[\frac{1}{M}\right] + \frac{1}{m_2}\frac{\partial^2}{\partial r^2}
\end{aligned}$$

$$\frac{\partial^2}{\partial R^2}\left[\frac{m_1}{M^2}\right] + 2\frac{\partial^2}{\partial R\partial r}\left[\frac{1}{M}\right] + \frac{1}{m_1}\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial R^2}\left[\frac{m_2}{M^2}\right] - 2\frac{\partial^2}{\partial R\partial r}\left[\frac{1}{M}\right] + \frac{1}{m_2}\frac{\partial^2}{\partial r^2}$$

$$\frac{\partial^2}{\partial R^2}\left[\frac{m_1}{M^2}\right] + \frac{1}{m_1}\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial R^2}\left[\frac{m_2}{M^2}\right] + \frac{1}{m_2}\frac{\partial^2}{\partial r^2}$$

$$\frac{\partial^2}{\partial R^2}\left[\frac{m_1}{M^2} + \frac{m_2}{M^2}\right] + \frac{\partial^2}{\partial r^2}\left[\frac{1}{m_1} + \frac{1}{m_2}\right]$$

$$\frac{\partial^2}{\partial R^2}\left[\frac{m_1+m_2}{M^2}\right] + \frac{\partial^2}{\partial r^2}\left[\frac{m_2+m_1}{m_1m_2}\right]$$

$$\frac{1}{M}\frac{\partial^2}{\partial R^2} + \frac{1}{\mu}\frac{\partial^2}{\partial r^2}$$

$$m_1x_1^2 + m_2x_2^2$$
$$x_1(m_1x_1) + x_2(m_2x_2)$$
$$x_1\left(M\mu\frac{x_1}{m_2}\right) + x_2\left(M\mu\frac{x_2}{m_1}\right)$$
$$M\mu\left[\frac{x_1^2}{m_2} + \frac{x_2^2}{m_1}\right]$$
$$M\mu\left[\frac{x_1^2}{m_2} + \frac{x_2^2}{m_1}\right]$$
$$M\mu\left[\frac{m_1x_1^2 + m_2x_2^2}{m_1m_2}\right]$$

(b) With $\boldsymbol{\hat{H}}$ given in part (a), show that the Schrodinger equation for the whole system

$$\boldsymbol{\hat{H}\phi(R,r) = E\phi(R,r)} \quad \text{can be decoupled into}$$

$$\boldsymbol{\left[-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial R^2} + \frac{1}{2}M\omega_0^2R^2\right]\phi_R(R) = E_R\phi_R(R)} \quad \text{and} \quad \boldsymbol{\left[-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2} + \frac{1}{2}\mu\Omega^2r^2\right]\phi_r(r) = E_r\phi_r(r)}$$

$$\text{if} \quad \boldsymbol{\phi(R,r) = \phi_R(R)\cdot\phi_r(r)} \quad \text{and} \quad \boldsymbol{E = E_R + E_r}$$

Solution Following posted solution.

$$\hat{H}\phi(R,r) = E\phi(R,r)$$
$$\left[-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial R^2} + \frac{1}{2}M\omega_0^2R^2 - \frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2} + \frac{1}{2}\mu\Omega^2r^2\right]\phi(R,r) = E\phi(R,r)$$
$$\left[-\frac{\hbar^2}{2M}\phi_r\frac{\partial^2\phi_R}{\partial R^2} + \frac{1}{2}M\omega_0^2R^2\phi_R\phi_r - \frac{\hbar^2}{2\mu}\phi_R\frac{\partial^2\phi_r}{\partial r^2} + \frac{1}{2}\mu\Omega^2r^2\phi_R\phi_r\right] = E\phi_R\phi_r$$
$$\left[-\frac{\hbar^2}{2M}\frac{1}{\phi_R}\frac{\partial^2\phi_R}{\partial R^2} + \frac{1}{2}M\omega_0^2R^2 - \frac{\hbar^2}{2\mu}\frac{1}{\phi_r}\frac{\partial^2\phi_r}{\partial r^2} + \frac{1}{2}\mu\Omega^2r^2\right] = E = E_R + E_r$$

Because of the imposed separable condition, the solution can be decoupled.

$$\left[-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial R^2} + \frac{1}{2}M\omega_0^2R^2\right]\phi_R(R) = E_R\phi_R(R) \quad \text{and} \quad \left[-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial r^2} + \frac{1}{2}\mu\Omega^2r^2\right]\phi_r(r) = E_r\phi_r(r)$$

(c) Write down explicit expressions for the quantized $\boldsymbol{\phi_{n_R}(R), \phi_{n_r}(r), \phi_{n_Rn_r}(R,r), E_{n_R}, E_{n_r},}$ and $\boldsymbol{E_{n_Rn_r}}$

Solution Following posted solution. For a harmonic oscillator:

$$\text{If } \hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2$$
$$\text{Then } \phi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)$$
$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega, \quad n = 0, 1, 2, \ldots$$

Applying this to part (b) solution.

$$\phi_{n_R}(R) = \frac{1}{\sqrt{2^{n_R}n_R!}}\left(\frac{M\omega_0}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{M\omega_0R^2}{2\hbar}} H_{n_R}\left(\sqrt{\frac{M\omega_0}{\hbar}}R\right)$$

$$\phi_{n_r}(r) = \frac{1}{\sqrt{2^{n_r}n_r!}}\left(\frac{\mu\Omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{\mu\Omega r^2}{2\hbar}} H_{n_r}\left(\sqrt{\frac{\mu\Omega}{\hbar}}r\right)$$

$$\phi_{n_Rn_r}(R,r) = \left(\frac{M\omega_0}{\pi\hbar}\right)^{\frac{1}{4}}\left(\frac{\mu\Omega}{\pi\hbar}\right)^{\frac{1}{4}}\frac{1}{\sqrt{2^{n_R}n_R!}}\frac{1}{\sqrt{2^{n_r}n_r!}}e^{-\frac{M\omega_0R^2}{2\hbar}}e^{-\frac{\mu\Omega r^2}{2\hbar}}H_{n_R}\left(\sqrt{\frac{M\omega_0}{\hbar}}R\right)H_{n_r}\left(\sqrt{\frac{\mu\Omega}{\hbar}}r\right)$$
$$= (M\omega_0\mu\Omega)^{\frac{1}{4}}\frac{1}{\sqrt{\pi\hbar 2^{n_R+n_r}n_R!n_r!}}e^{-\frac{1}{2\hbar}(M\omega_0R^2 + \mu\Omega r^2)}H_{n_R}\left(\sqrt{\frac{M\omega_0}{\hbar}}R\right)H_{n_r}\left(\sqrt{\frac{\mu\Omega}{\hbar}}r\right)$$

$$E_{n_R} = \left(n_r + \frac{1}{2}\right)\hbar\omega_0, \quad E_{n_R} = \left(n_R + \frac{1}{2}\right)\hbar\Omega, \quad E_{n_Rn_r} = E_{n_R} + E_{n_r} = \left(n_r + \frac{1}{2}\right)\hbar\omega_0 + \left(n_R + \frac{1}{2}\right)\hbar\Omega$$