PHY 320 - Assignment 5

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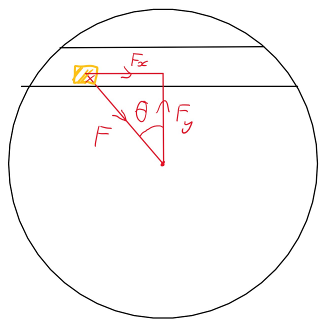
1 Problem 4

Show that the motion is simple harmonic with the same period as the previous problem for a particle sliding in a straight, smooth tube passing obliquely through Earth.

Solution Previous problem shows that $F_g = -\frac{GmM}{r^2}\,\hat{e}_r = -\frac{4}{3}G\pi\rho mr\,\hat{e}_r = -kr\,\hat{e}_r$. A particle dropped into a straight hole drilled from pole to pole executes simple harmonic motion. With period $T = 2\pi\sqrt{\frac{3}{4G\pi\rho}} = 1.4\,hrs$. This problem has a particle sliding down obliquely, off-center, through the Earth. Therefore the force would be multiplied by an angle, assuming a central force at the center of the Earth.



$$F_y = N$$
$$F_x = F\sin\theta$$
$$= -\frac{4}{3}G\pi\rho m r\,\frac{x}{r}$$
$$= -\frac{4}{3}G\rho m x\,\hat{\imath}$$
$$= -kx\,\hat{\imath}$$

$$F_g = F_x + F_y \tag{1}$$

F_y is constantly balanced by the normal force from the particle as it is sliding across. Clearly, the net force is F_x which also exhibits simple harmonic motion. The period is the same, $T = 1.4\,hrs$, as it only depends on the density, and we are assuming a constant density.

2 Problem 10

A particle moving in a central field describes the spiral orbit $r = r_0 e^{k\theta}$. Show that the force law is inverse cube and that θ varies logarithmically with t.

Solution Using the orbit equation (6.5.10b) and $u = \frac{1}{r} = \frac{1}{r_0}e^{-k\theta}$.

$$\begin{aligned}
\frac{d^2u}{d\theta^2} + u &= -\frac{f(u^{-1})}{ml^2u^2} \\
\frac{k^2}{r_0}e^{k\theta} + \frac{1}{r_0}e^{k\theta} &= -\frac{f(u^{-1})}{ml^2u^2} \\
-(k^2u + u)(ml^2u^2) &= f(u^{-1}) \\
f(r) &= -\frac{ml^2(k^2+1)}{r^3}
\end{aligned} \tag{2}$$

$\frac{d\theta}{dt}$ to find how θ varies with t, from $l = r^2\dot{\theta}$.

$$\begin{aligned}
\frac{d\theta}{dt} &= \frac{l}{r^2} = \frac{l}{r_0^2}e^{-2k\theta} \\
e^{2k\theta}\,d\theta &= \frac{l}{r_0^2}\,dt \\
\frac{1}{2k}e^{2k\theta} &= \frac{l}{r_0^2}t + C \\
\theta &= \frac{1}{2k}\ln\left[2k\left(\frac{l}{r_0^2}t + C\right)\right]
\end{aligned} \tag{3}$$

3 Problem 14

A particle of unit mass is projected with a velocity v_0 at right angles to the radius vector at a distance a from the origin of a center of attractive force, given by

$$f(r) = -k\left(\frac{4}{r^3} + \frac{a^2}{r^5}\right) \quad v_0^2 = \frac{9k}{2a^2} \tag{4}$$

(a) Find the polar equation of the resulting orbit.

Solution Energy equation of an orbit (6.9.2). Unit mass, $m = 1\,kg$.

$$E = T_0 + V_0 = \frac{1}{2}mv_0^2 + V(a) = \frac{9k}{4a^2} - k\left(\frac{2}{a^2} + \frac{1}{4a^2}\right) = 0 \tag{5}$$

From conservation of angular momentum, $l^2 = r^4\dot{\theta}^2 = a^2v_0^2 = \frac{9k}{2}$

$$\begin{aligned}
\frac{1}{2}m\left(\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right) + V(r) = 0 \\
\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2\left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right) - k\left(\frac{2}{r^2} + \frac{a^2}{4r^2}\right) = 0 \\
\left(\frac{dr}{d\theta}\right)^2 + r^2 = 2k\frac{2r^4}{9k}\left(\frac{2}{r^2} + \frac{a^2}{4r^2}\right) = \frac{8r^2}{9} + \frac{a^2}{9} \\
\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{9}(a^2 - r^2)
\end{aligned} \tag{6}$$

As the particle is in orbit in a polar coordinate system, r and a are related in this way: $r = a\cos(\omega\theta)$ where $\frac{dr}{d\theta} = -a\omega\sin(\omega\theta)$.

$$\begin{aligned}
(-a\omega\sin(\omega\theta))^2 &= \frac{1}{9}(a^2 - a^2\cos^2(\omega\theta)) \\
-a\omega\sin(\omega\theta) &= \frac{a}{3}\sqrt{1 - \cos^2(\omega\theta)} \\
\omega &= -\frac{1}{3} \\
r &= a\cos(-\frac{1}{3}\theta) = a\cos(\frac{1}{3}\theta)
\end{aligned} \tag{7}$$

(b) How long does it take the particle to travel through an angle $3\pi/2$? Where is the particle at that time?

Solution

$$\begin{aligned}
\frac{d\theta}{dt} = \frac{l}{r^2} = \frac{av_0}{a^2\cos^2(\frac{1}{3}\theta)} &= \sqrt{\frac{9k}{2a^2}}\frac{1}{a\cos^2(\frac{1}{3}\theta)} \\
dt &= \frac{a^2\sqrt{2}}{3\sqrt{k}}\cos^2(\frac{1}{3}\theta)\,d\theta \\
t &= \frac{a^2\sqrt{2}}{3\sqrt{k}}\int_0^{\frac{3\pi}{2}}\cos^2(\frac{1}{3}\theta)\,d\theta = \frac{a^2\sqrt{2}}{3\sqrt{k}}\frac{3\pi}{4} = \frac{\pi a^2}{4}\sqrt{\frac{2}{k}}
\end{aligned} \tag{8}$$

$$r = a\cos(\frac{1}{3}\frac{3\pi}{2}) = 0, \text{ the particle is at the origin of the central force.} \tag{9}$$

(c) What is the velocity of the particle at that time?

Solution Due to conservation of angular momentum, $l = rv = av_0 = constant$. As r approaches 0, v approaches ∞ to keep angular momentum constant.

4 Problem 25

Find the condition for which circular orbits are stable if the force function is of the form

$$f(r) = -\frac{k}{r^2} - \frac{\epsilon}{r^4} \tag{10}$$

Solution Equation 6.12.7 for stable orbits.

$$\begin{aligned}
f(a) + \frac{a}{3}f'(a) &< 0 \\
-ka^{-2} - \epsilon a^{-4} + \frac{a}{3}(2ka^{-3} + 4\epsilon a^{-5}) &< 0 \\
-\frac{1}{3}ka^{-2} + \frac{1}{3}\epsilon a^{-4} &< 0 \\
\epsilon a^{-2} &< k \\
a &> \sqrt{\frac{\epsilon}{k}}
\end{aligned} \tag{11}$$

5 Problem 33

Show that the differential scattering cross section for a particle of mass m subject to a central force field $f(r) = k/r^3$ is given by the expression

$$\sigma(\theta_s)\, d\Omega = 2\pi\, |bdb| = \frac{k\pi^3}{E}\left[\frac{\pi - \theta_s}{(2\pi - \theta_s)^2\theta_s^2}\right] d\theta_s \tag{12}$$

Solution This is a repulsive force with an inverse cube. First to find the orbit equation.

$$\begin{aligned}
\frac{d^2u}{d\theta^2} + u &= -\frac{f(u^{-1})}{ml^2u^2} = -\frac{ku}{ml^2} \\
\frac{d^2u}{d\theta^2} + u\left(1 + \frac{k}{ml^2}\right) &= 0 \\
\frac{1}{r} &= u = Asin(\omega\theta + \alpha)
\end{aligned} \tag{13}$$

Now to find constants at $r = \infty$.
At $r = \infty, u = 0$ and $\theta = 0$, then $\alpha = 0$. Also, $E = \frac{1}{2}m\dot{r}_\infty^2$ and $l = r^2\dot{\theta}$, defining $\dot{r}$ in terms of u.

$$\begin{aligned}
\dot{r}_\infty = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\frac{l}{r^2} = l\frac{du}{d\theta} = lA\omega\cos(\omega\theta) = lA\omega &= \sqrt{\frac{2E}{m}} \\
A &= \frac{1}{l\omega}\sqrt{\frac{2E}{m}}
\end{aligned} \tag{14}$$

$$r^{-1} = u = \frac{1}{l\omega}\sqrt{\frac{2E}{m}}sin(\omega\theta) \tag{15}$$

To find θ_s, u_{max} occurs when $\sin(\omega\theta_0) = 1$ at closest approach, so $\theta_0 = \frac{\pi}{2\omega}$. Where $\omega = \sqrt{1 + \frac{k}{ml^2}}$

$$\begin{aligned}
r_{min}^{-1} = u_{max} = \frac{1}{l\omega}\sqrt{\frac{2E}{m}}sin(\omega(\frac{\pi}{2\omega})) &= \frac{1}{l\omega}\sqrt{\frac{2E}{m}} \\
\theta_s = \pi - 2\theta_0 = \pi\left(1 - \frac{1}{\omega}\right) &= \pi\left(1 - \frac{1}{\sqrt{1 + \frac{k}{ml^2}}}\right)
\end{aligned} \tag{16}$$

$$1 - \frac{\theta_s}{\pi} = \left(1 + \frac{k}{ml^2}\right)^{-\frac{1}{2}}$$

Due to conservation of angular momentum, $l = b\dot{r}_\infty = r_{min}v$, so $l^2 = b^2\dot{r}_\infty^2 = \frac{2b^2E}{m}$. b cannot be negative.

$$\begin{aligned} 1 - \frac{\theta_s}{\pi} &= \left(1 + \frac{k}{2b^2E}\right)^{-\frac{1}{2}} \\ b^2 &= \frac{k}{E}\frac{(\pi-\theta_s)^2}{(2\pi-\theta_s)2\theta_s} \\ b &= \left(\frac{k}{E}\frac{(\pi-\theta_s)^2}{(2\pi-\theta_s)2\theta_s}\right)^{\frac{1}{2}} \end{aligned} \tag{17}$$

$$\begin{aligned} \frac{db}{d\theta_s} = \sqrt{\frac{k}{2E}}\frac{d}{d\theta_s}\left[\left(\frac{(\pi-\theta_s)^2}{(2\pi-\theta_s)\theta_s}\right)^{\frac{1}{2}}\right] &= \sqrt{\frac{k}{2E}}\left[\frac{1}{2\left(\frac{(\pi-\theta_s)^2}{(2\pi-\theta_s)\theta_s}\right)^{\frac{1}{2}}}\frac{d}{d\theta}\left(\frac{(\pi-\theta_s)^2}{(2\pi-\theta_s)\theta_s}\right)\right] \\ \frac{d}{d\theta_s}\left(\frac{(\pi-\theta_s)^2}{(2\pi-\theta_s)\theta_s}\right) = \left(\frac{-2(\pi-\theta_s)((2\pi-\theta_s)\theta_s) - (\pi-\theta_s)^2(2\pi-2\theta_s)}{(2\pi-\theta_s)^2\theta_s^2}\right) &= \left(\frac{-2\pi^2(\pi-\theta_s)}{(2\pi-\theta_s)^2\theta_s^2}\right) \\ \frac{db}{d\theta_s} = \sqrt{\frac{k}{2E}}\left[\frac{1}{2\left(\frac{(\pi-\theta_s)^2}{(2\pi-\theta_s)\theta_s}\right)^{\frac{1}{2}}}\left(\frac{-2\pi^2(\pi-\theta_s)}{(2\pi-\theta_s)^2\theta_s^2}\right)\right] &= -\sqrt{\frac{k}{2E}}\left[\frac{\pi^2}{(2\pi-\theta_s)^{\frac{3}{2}}\theta_s^{\frac{3}{2}}}\right] \end{aligned} \tag{18}$$

$$\begin{aligned} \sigma(\theta_s) = \frac{b}{\sin\theta_s}\left|\frac{db}{d\theta_s}\right| &= \frac{\left(\frac{\sqrt{k}(\pi-\theta_s)}{\sqrt{2}\sqrt{E}(2\pi-\theta_s)^{\frac{1}{2}}\theta_s^{\frac{1}{2}}}\right)}{\sin\theta_s}\left[\frac{\sqrt{k}\pi^2}{\sqrt{2}\sqrt{E}(2\pi-\theta_s)^{\frac{3}{2}}\theta_s^{\frac{3}{2}}}\right] \\ &= \frac{1}{\sin\theta_s}\frac{k}{2E}\left(\frac{\pi^2(\pi-\theta_s)}{(2\pi-\theta_s)^2\theta_s^2}\right) \end{aligned} \tag{19}$$

$$\sigma(\theta_s)\,d\Omega = \frac{1}{\sin\theta_s}\frac{k}{2E}\left(\frac{\pi^2(\pi-\theta_s)}{(2\pi-\theta_s)^2\theta_s^2}\right)2\pi\sin\theta_s\,d\theta_s = \frac{k\pi^3}{E}\left[\frac{\pi-\theta_s}{(2\pi-\theta_s)^2\theta_s^2}\right]d\theta_s \tag{20}$$