PHY 320 - Assignment 3

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09/10/2022

1 Problem 3

A particle undergoes simple harmonic motion with a frequency of 10 Hz. Find the displacement x at any time t for the following initial conditions:

$$t = 0,\ x = 0.25\,\text{m},\ \dot{x} = 0.1\,\text{m/s} \tag{1}$$

Solution Another way to solve homogeneous differential equations is through the auxiliary polynomial. Where the complex root $m = \alpha \pm \beta i$ corresponds to $x(t) = e^{\alpha t}\left(A\cos(+\beta t) + B\sin(+\beta t)\right)$.

$$\begin{aligned}
\omega &= 2\pi f = 20\pi \\
m\ddot{x} + kx &= 0 \\
\ddot{x} + \omega^2 x &= 0 \\
a(r) = r^2 + \omega^2 &= 0 \\
r &= 0 \pm \omega i \\
x(t) &= A\cos(\omega t) + B\sin(\omega t) \\
\dot{x}(t) &= -A\omega\sin(\omega t) + B\omega\cos(\omega t)
\end{aligned} \tag{2}$$

$$\begin{aligned}
x(0) = A + 0 &= 0.25 \\
A &= \frac{1}{4} \\
\dot{x}(0) = 0 + B\omega &= 0.1 \\
B = \frac{0.1}{\omega} &= \frac{1}{200\pi}
\end{aligned} \tag{3}$$

$$x(t) = \frac{1}{4}\cos(20\pi t) + \frac{1}{200\pi}\sin(20\pi t) \tag{4}$$

2 Problem 5

A particle undergoing simple harmonic motion has a velocity $\dot{x}_1$ when the displacement is x_1 and a velocity $\dot{x}_2$ when the displacement is x_2. Find the angular frequency and the amplitude of the motion in terms of the given quantities.

Solution Through the conservation of mechanical energy, $T_0 + V_0 = T + V$.

$$\begin{aligned}
T_1 + V_1 &= T_2 + V_2 \\
\frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}kx_1^2 &= \frac{1}{2}m\dot{x}_2^2 + \frac{1}{2}kx_2^2 \\
m\dot{x}_1^2 + kx_1^2 &= m\dot{x}_2^2 + kx_2^2 \\
m\left(\dot{x}_1^2 - \dot{x}_2^2\right) &= k\left(x_2^2 - x_1^2\right) \\
\sqrt{\frac{k}{m}} &= \left(\frac{\dot{x}_1^2 - \dot{x}_2^2}{x_2^2 - x_1^2}\right)^{\frac{1}{2}} = \omega
\end{aligned} \tag{5}$$

$$
\begin{aligned}
A^2 &= x_1^2 + \frac{\dot{x}_1^2}{\omega^2} \\
A^2 &= x_1^2 + \dot{x}_1^2\left(\frac{x_2^2 - x_1^2}{\dot{x}_1^2 - \dot{x}_2^2}\right) \\
A &= \left(x_1^2 + \frac{x_2^2\dot{x}_1^2 - x_1^2\dot{x}_1^2}{\dot{x}_1^2 - \dot{x}_2^2}\right)^{\frac{1}{2}} \\
A &= \left(\frac{x_1^2\dot{x}_1^2 - x_1^2\dot{x}_2^2 + x_2^2\dot{x}_1^2 - x_1^2\dot{x}_1^2}{\dot{x}_1^2 - \dot{x}_2^2}\right)^{\frac{1}{2}} \\
A &= \left(\frac{x_2^2\dot{x}_1^2 - x_1^2\dot{x}_2^2}{\dot{x}_1^2 - \dot{x}_2^2}\right)^{\frac{1}{2}}
\end{aligned}
\tag{6}
$$

3 Problem 11

A mass m moves along the x-axis subject to an attractive force given by $17\beta^2 mx/2$ and a retarding force given by $3\beta m\dot{x}$, where x is its distance from the origin and β is a constant. A driving force given by $mA\cos\omega t$, where A is a constant, is applied to the particle along the x-axis.

(a) What value of ω results in steady-state oscillations about the origin with maximum amplitude?

Solution As the equation of motion seems to be a damped system, the value for omega to result in steady-state oscillations is the resonant frequency of the system. $\omega_r^2 = \omega_0^2 + 2\gamma^2$.

$$
\begin{aligned}
m\ddot{x} &= -\frac{17}{2}\beta^2 mx - 3\beta m\dot{x} + mA\cos\omega t \\
m\ddot{x} + 3\beta m\dot{x} + \frac{17}{2}\beta^2 mx &= mA\cos\omega t \\
\gamma &= \frac{3\beta m}{2m} = \frac{3\beta}{2} \\
\omega_0^2 &= \frac{17\beta^2 m}{2m} = \frac{17\beta^2}{2} \\
\omega_r^2 &= \frac{17\beta^2}{2} + 2\left(\frac{3\beta}{2}\right)^2 = 4\beta^2 \\
\omega_r &= 2\beta
\end{aligned}
\tag{7}
$$

(b) What is the maximum amplitude?

Solution Using the equation for amplitude as a function of angular frequency, with an input of the resonant frequency of the system to get the maximum amplitude.

$$
\begin{aligned}
A(\omega) &= \frac{F_0/m}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \left(2\gamma\omega\right)^2}} \\
A(2\beta) &= \frac{F_0/m}{\sqrt{\left(\frac{17\beta^2}{2} - 4\beta^2\right)^2 + \left(2\left(\frac{3\beta}{2}\right)(2\beta)\right)^2}} \\
&= \frac{F_0/m}{\sqrt{\frac{81\beta^4}{4} + 36\beta^4}} \\
&= \frac{F_0/m}{\frac{15\beta^2}{2}} = \frac{2F_0}{15\beta^2 m}
\end{aligned}
\tag{8}
$$

4 Problem 13

Given: The amplitude of a damped harmonic oscillator drops to $1/e$ of its initial value after n complete cycles. Show that the ratio of period of the oscillation to the period of the same oscillator with no damping is given by

$$\frac{T_d}{T_0} = \left(1 + \frac{1}{4\pi^2 n^2}\right)^{\frac{1}{2}} \approx 1 + \frac{1}{8\pi^2 n^2} \tag{9}$$

Solution According to the Analytical mechanics book in page 101, for a damped harmonic oscillator: "In one complete period the amplitude diminishes by a factor $e^{-\gamma T_d}$ ".

$$\begin{aligned}
x(t) &= e^{-\gamma t} A \sin(\omega_d t + \phi_0) \\
x(T_d) &= e^{-\gamma T_d} A \sin(\omega_d T_d + \phi_0) \\
x(T_d) &= e^{-\gamma T_d} A \sin(\phi_0) = e^{-\gamma T_d} A \\
A\left(e^{-\gamma T_d}\right)^n &= Ae^{-1} \\
e^{-\gamma T_d n} &= e^{-1} \\
\gamma T_d n &= 1 \\
\gamma &= \frac{1}{T_d n} = \frac{\omega_d}{2\pi n}
\end{aligned} \tag{10}$$

Now with a value for gamma in terms of ω_d: $\omega_d^2 = \omega_0^2 - \gamma^2$.

$$\begin{aligned}
\omega_0 = \left(\omega_d^2 + \gamma^2\right)^{1/2} &= \left(\omega_d^2 + \frac{\omega_d^2}{4\pi^2 n^2}\right)^{1/2} \\
&= \omega_d \left(1 + \frac{1}{4\pi^2 n^2}\right)^{1/2} \\
\frac{T_d}{T_0} = \frac{\frac{2\pi}{w_d}}{\frac{2\pi}{w_0}} = \frac{\omega_0}{\omega_d} &= \frac{\omega_d\left(1 + \frac{1}{4\pi^2 n^2}\right)^{1/2}}{\omega_d} = \left(1 + \frac{1}{4\pi^2 n^2}\right)^{1/2}
\end{aligned} \tag{11}$$

As n gets larger, $\frac{1}{4\pi^2 n^2}$ gets smaller. A useful approximation is given in Appendix D.

$$\begin{aligned}
(1+x)^{1/2} &\approx 1 + \frac{1}{2}x \\
\left(1 + \frac{1}{4\pi^2 n^2}\right)^{1/2} &\approx 1 + \frac{1}{8\pi^2 n^2}
\end{aligned} \tag{12}$$

5 Problem 18

Solve the differential equation of motion of the damped harmonic oscillator driven by a damped harmonic force: (Hint: $e^{-\alpha t}\cos\omega t = Re(e^{-\alpha t + i\omega t}) = Re(e^{\beta t})$, where $\beta = -\alpha + i\omega$. Assume a solution of the form $Ae^{\beta t - i\phi}$)

$$F_{ext}(t) = F_0 e^{-\alpha t}\cos\omega t \tag{13}$$

Solution As a general approach to solving differential equations: Assuming a solution of the form $Ae^{\beta t - i\phi}$, which will give the particular solution $x_p(t)$ of the general solution; generally referred to as the steady state solution. However, the new concept here is a harmonic driving force that is damped, which means both terms will be transient due to the limiting exponential. Using the hint.

$$F_{ext}(t) = Re(F_0 e^{\beta t}) \tag{14}$$

Now by demanding the real part of the equation of motion of the damped harmonic oscillator. Following section 3.6 in the book.

$$\begin{aligned} m\ddot{x} + b\dot{x} + kx &= F_0 e^{\beta t} \\ x(t) &= x_c(t) + x_p(t) \end{aligned} \tag{15}$$

$$\begin{aligned} x_p(t) &= Ae^{\beta t - i\phi} \\ \dot{x}_p(t) &= A\beta e^{\beta t - i\phi} \\ \ddot{x}_p(t) &= A\beta^2 e^{\beta t - i\phi} \end{aligned} \tag{16}$$

$$\begin{aligned} mA\beta^2 e^{\beta t - i\phi} + bA\beta e^{\beta t - i\phi} + kAe^{\beta t - i\phi} &= F_0 e^{\beta t} \\ m\beta^2 + b\beta + k &= \frac{F_0}{A} e^{i\phi} \\ m(-\alpha + i\omega)^2 + b(-\alpha + i\omega) + k &= \frac{F_0}{A}(\cos\phi + i\sin\phi) \\ m\alpha^2 - 2im\alpha\omega - m\omega^2 - b\alpha + ib\omega + k &= \frac{F_0}{A}\cos\phi + \frac{F_0}{A} i \sin\phi \end{aligned} \tag{17}$$

Equating the real and imaginary parts of the equation.

$$\begin{aligned} m\alpha^2 - m\omega^2 - b\alpha + k &= \frac{F_0}{A}\cos\phi \\ -2m\alpha\omega + b\omega &= \frac{F_0}{A}\sin\phi \end{aligned} \tag{18}$$

2 equations and 2 unknowns. ϕ from $\frac{\sin\phi}{\cos\phi} = \tan\phi$, and A from $\sin^2 + \cos^2 = 1$.

$$\begin{aligned} \frac{\frac{F_0}{A}\sin\phi}{\frac{F_0}{A}\cos\phi} &= \frac{b\omega - 2m\alpha\omega}{m\alpha^2 - m\omega^2 - b\alpha + k} \\ \phi &= \arctan\left(\frac{b\omega - 2m\alpha\omega}{m\alpha^2 - m\omega^2 - b\alpha + k}\right) \end{aligned} \tag{19}$$

$$\begin{aligned} \left(\frac{F_0}{A}\sin\phi\right)^2 + \left(\frac{F_0}{A}\cos\phi\right)^2 &= (m\alpha^2 - m\omega^2 - b\alpha + k)^2 + (b\omega - 2m\alpha\omega)^2 \\ \frac{F_0^2}{A^2}(\sin^2\phi + \cos^2\phi) &= (m\alpha^2 - m\omega^2 - b\alpha + k)^2 + (b\omega - 2m\alpha\omega)^2 \\ A &= \frac{F_0}{\left((m\alpha^2 - m\omega^2 - b\alpha + k)^2 + (b\omega - 2m\alpha\omega)^2\right)^{1/2}} \end{aligned} \tag{20}$$

Finally with these expressions for ϕ and A.

$$\begin{aligned} Re(x_p(t)) = Re(Ae^{\beta t - i\phi}) &= Ae^{-\alpha t + i\omega t - i\phi} = Ae^{-\alpha t} e^{i(\omega t - \phi)} \\ &= Ae^{-\alpha t}(\cos(\omega t - \phi) + i\sin(\omega t - \phi)) \\ x_p(t) &= Ae^{-\alpha t}\cos(\omega t - \phi) \end{aligned} \tag{21}$$

Another transient term would come from solving the homogeneous part of the equation of motion for the complementary solution, which should just be the general solution for a damped harmonic oscillator.

$$\begin{aligned} x(t) &= Ae^{-\alpha t}\cos(\omega t - \phi) + x_c(t) \\ x(t) &= A_p e^{-\alpha t}\cos(\omega_p t - \phi_p) + e^{-\gamma t}\left(A_1 e^{\sqrt{\gamma^2 - \omega_0^2}t} + A_2 e^{-\sqrt{\gamma^2 - \omega_0^2}t}\right) \end{aligned} \tag{22}$$