PHY 310 — Mathematical Methods in Physics

Name: Khalifa Salem Almatrooshi **Due Date:** 22 Feb 2023
Student Number: @00090847 **Assignment:** HW 2

Problem 1

(a) **Oscillating Piston In a Gas Chamber** A frictionless closed cylinder, fitted with a piston of mass $\boldsymbol{m}$ and containing an ideal gas, has cross-sectional area $\boldsymbol{A}$ and length $\boldsymbol{2L}$. At time $\boldsymbol{t=0}$, when the piston is at the midpoint of the cylinder $\boldsymbol{x=0}$ and the pressure on either side of the piston is $\boldsymbol{p_0}$, the piston is displaced from this equilibrium position.

i. Assuming the process to be adiabatic, show that the position of the piston as a function of time, $\boldsymbol{x(t)}$ satisfies the equation

$$\frac{d^2x}{dt^2} + \frac{p_0AL^\gamma}{m}\left[\frac{1}{(L-x)^\gamma} - \frac{1}{(L+x)^\gamma}\right] = 0$$

Where $\boldsymbol{\gamma}$ is the ratio of specific heats at constant pressure and at constant volume for the gas.
Let $\alpha = \dfrac{p_0AL^\gamma}{m}$

$$\begin{aligned}
\text{Initially}\quad p_0V^\gamma &= p_0V^\gamma = k = p_0(AL)^\gamma \\
\text{After displacement}\quad p_1V_1^\gamma &= p_2V_2^\gamma = k = p_0(AL)^\gamma \\
p_1(A(L+x))^\gamma &= p_2(A(L-x))^\gamma = p_0(AL)^\gamma \\
p_1 &= \frac{p_0(AL)^\gamma}{(A(L+x))^\gamma} = \frac{p_0L^\gamma}{(L+x)^\gamma} \\
p_2 &= \frac{p_0(AL)^\gamma}{(A(L-x))^\gamma} = \frac{p_0L^\gamma}{(L-x)^\gamma}
\end{aligned}$$

$$\begin{aligned}
m\ddot{x} &= F_1 - F_2 \\
m\ddot{x} &= \frac{p_1}{A} - \frac{p_2}{A} \\
m\ddot{x} &= \frac{\frac{p_0L^\gamma}{(L+x)^\gamma}}{A} - \frac{\frac{p_0L^\gamma}{(L-x)^\gamma}}{A} \\
m\ddot{x} &= \frac{p_0AL^\gamma}{(L+x)^\gamma} - \frac{p_0AL^\gamma}{(L-x)^\gamma} \\
\frac{d^2x}{dt^2} &+ \frac{p_0AL^\gamma}{m}\left[\frac{1}{(L-x)^\gamma} - \frac{1}{(L+x)^\gamma}\right] = 0 \\
\text{At x = 0}\quad &\frac{d^2x}{dt^2} = 0 \\
\text{At x = L}\quad &\frac{d^2x}{dt^2} + \frac{p_0AL^\gamma}{m}\left[-\frac{1}{(2L)^\gamma}\right] = 0 \\
&m\frac{d^2x}{dt^2} = \frac{p_0A}{2^\gamma}
\end{aligned}$$

ii. Show that in the limit $\boldsymbol{x \ll L}$, the piston oscillates harmonically and find its angular frequency of small oscillations.

$$\frac{d^2x}{dt^2} = \alpha\left[(L+x)^{-\gamma} - (L-x)^{-\gamma}\right] = \alpha\left[\frac{1}{L^\gamma}\left(1+\frac{x}{L}\right)^{-\gamma} - \frac{1}{L^\gamma}\left(1-\frac{x}{L}\right)^{-\gamma}\right] = \frac{\alpha}{L^\gamma}\left[(1-\frac{\gamma x}{L}) - (1+\frac{\gamma x}{L})\right]$$

$$= -\frac{2\alpha\gamma x}{L^\gamma L} = -\frac{p_0AL^\gamma}{m}\frac{2\gamma x}{L^\gamma L} = -\frac{2p_0A\gamma x}{mL}$$

$$\frac{d^2x}{dt^2} + \left(\frac{2p_0A\gamma}{mL}\right)x = 0$$

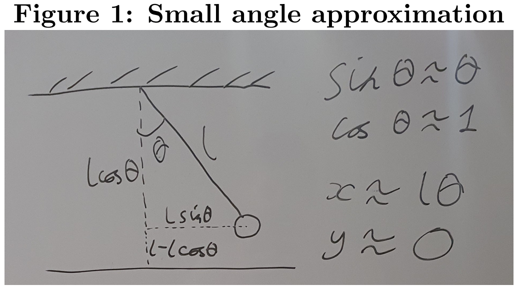
$$\omega = \sqrt{\frac{mL}{2p_0A\gamma}}$$

(b) **Oscillating Charge in Electric and Gravitational Fields** A pendulum with a massless string of length $\boldsymbol{l}$ has on its end a small sphere with charge $+\boldsymbol{q}$ and mass $\boldsymbol{m}$. A distance $\boldsymbol{d}$ on either side of the pendulum are two fixed small spheres each carrying a charge $+\boldsymbol{Q}$. The pendulum is now given a tiny displacement $(\ll \boldsymbol{l}, \boldsymbol{d})$ from its equilibrium position. Show that the time period of small oscillations of the pendulum is

$$2\pi\sqrt{\frac{\pi\epsilon_0 mld^3}{qQl + \pi\epsilon_0 mgd^3}}$$

where $\boldsymbol{\epsilon_0}$ is permittivity of free space.

Figure 1: Small angle approximation



For small oscillations, the net force acting on the pendulum is nearly horizontal, as apparent by figure 1. Therefore, it is appropriate to only take the net force along the x axis.

$$\begin{aligned} m\ddot{y} &= mg\cos\theta - T \\ &= mg - T \end{aligned}$$

$$\begin{aligned} m\ddot{x} &= F_1 - F_2 - mg\sin\theta \\ m\ddot{x} &= \frac{kQq}{(d+x)^2} - \frac{kQq}{(d-x)^2} - mg\theta \\ m\ddot{x} &- \left[\frac{kQq}{(d+x)^2} - \frac{kQq}{(d-x)^2} - \frac{mgx}{l}\right] = 0 \\ m\ddot{x} &- \left[\frac{kQq}{d^2}(1+\tfrac{x}{d})^{-2} - \frac{kQq}{d^2}(1-\tfrac{x}{d})^{-2} - \frac{mgx}{l}\right] = 0 \end{aligned}$$

Using binomial expansion $(1+x)^n = 1 + nx + \ldots$, and taking appropriate approximations.

$$\begin{aligned} m\ddot{x} &- \left[\frac{kQq}{d^2}(1-\tfrac{2x}{d}) - \frac{kQq}{d^2}(1+\tfrac{2x}{d}) - \frac{mgx}{l}\right] = 0 \\ m\ddot{x} &- \left[-\tfrac{4kQqx}{d^3} - \frac{mgx}{l}\right] = 0 \\ m\ddot{x} &+ \left[\tfrac{Qq}{\pi\epsilon_0 d^3} + \frac{mg}{l}\right]x = 0 \\ \ddot{x} &+ \left[\tfrac{Qql + \pi\epsilon_0 mgd^3}{\pi\epsilon_0 mld^3}\right]x = 0 \end{aligned}$$

$$\omega^2 = \frac{qQl + \pi\epsilon_0 mgd^3}{\pi\epsilon_0 mld^3}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\pi\epsilon_0 mld^3}{qQl + \pi\epsilon_0 mgd^3}}$$

Problem 2

An LCR Circuit with a Variable Potential Difference Source The terminals of a generator producing a time varying voltage $\boldsymbol{V(t)}$ are connected through a capacitor of capacitance $\boldsymbol{C}$, a wire of resistance $\boldsymbol{R}$ and a coil of self-inductance $\boldsymbol{L}$ in series.

(a) Show that the charge $\boldsymbol{q(t)}$ on the capacitor satisfies the equation

$$LC\frac{d^2q}{dt^2} + RC\frac{dq}{dt} + q = CV(t)$$

One way to show that the charge $\boldsymbol{q(t)}$ on the capacitor satisfies the differential equation is by using Kirchoff's Voltage Rule, starting clockwise from the positive terminal of the generator.

$$-V(t) + L\frac{dI}{dt} + RI + \frac{q}{C} = 0 \qquad I = \frac{dq}{dt}$$
$$LC\frac{d^2q}{dt^2} + RC\frac{dq}{dt} + q = CV(t)$$

(b) Assume that $\boldsymbol{L = \frac{1}{2}CR^2}$ and

$$V(t) = \begin{cases} 0 & \text{for } t < 0, \\ V_0 & \text{for } 0 < t < \pi RC \\ 0 & \text{for } t > \pi RC \end{cases}$$

where $\boldsymbol{V_0}$ is constant.

i. Show that the current flowing round the circuit is given by

$$\frac{2V_0}{R}\exp\left(-\frac{t}{RC}\right)\sin\left(\frac{t}{RC}\right) \quad \text{for } 0 < t < \pi RC$$

Since $I(t) = \frac{dq(t)}{dt}$, we have to first solve the differential equation.

$$LC\frac{d^2q}{dt^2} + RC\frac{dq}{dt} + q = CV(t)$$

A non-homogeneous differential equation with constant coefficients. This means that the final expression is in the form.

$$q(t) = q_c(t) + q_p(t) \quad \text{where} \begin{cases} q_c(t) & \text{complementary solution} \\ q_p(t) & \text{particular solution} \end{cases}$$

To find $q_c(t)$.

$$LCq'' + RCq' + q = 0 \quad L = \frac{1}{2}CR^2$$
$$q'' + \frac{R}{L}q' + \frac{1}{LC}q = 0$$
$$q'' + \frac{2}{RC}q' + \frac{2}{(RC)^2}q = 0 \quad k = \frac{1}{RC}$$
$$q'' + 2kq' + 2k^2q = 0$$

Now the auxiliary polynomial.

$$a(m) = m^2 + 2km + 2k^2 = 0$$
$$m = \frac{-2k \pm \sqrt{4k^2 - 8k^2}}{2} = \frac{-2k \pm \sqrt{-4k^2}}{2} = -k \pm ki$$

$$\begin{aligned}
q(t) &= e^{-kt}(A\cos kt + B\sin kt) \\
I(t) = q'(t) &= (-ke^{-kt})(A\cos kt + B\sin kt) + (e^{-kt})(-Ak\sin kt + Bk\cos kt) \\
&= ke^{-kt}(-A\cos kt - B\sin kt - A\sin kt + B\cos kt) \\
q'(t) &= ke^{-kt}((B-A)\cos kt - (B+A)\sin kt)
\end{aligned}$$

To find $q_p(t)$. Note that the RHS is just a constant in terms of q.

$$q'' + 2kq' + 2k^2 q = \frac{2V}{CR^2}$$

$$\begin{aligned}
q_p(t) = \alpha \quad q_p'(t) &= 0 \quad q_p''(t) = 0 \\
2k^2\alpha &= \frac{2V}{CR^2} \\
\alpha &= \frac{V}{CR^2(\frac{1}{R^2C^2})} = CV(t) = q_p(t)
\end{aligned}$$

Assuming some initial conditions to find the constants A and B. $q(0) = Q_0$ and $q'(0) = 0$.

$$\begin{aligned}
q(t) &= e^{-kt}(A\cos kt + B\sin kt) + CV(t) \\
q(0) &= A = Q_0 \\
A &= CV_0 \\
q'(t) &= ke^{-kt}((B-A)\cos kt - (B+A)\sin kt) \\
q'(0) &= kB - kA = 0 \\
B &= A = CV_0
\end{aligned}$$

Finally.

$$\begin{aligned}
q(t) &= e^{-kt}(CV_0\cos kt + CV_0\sin kt) + CV(t) \\
&= CV_0 e^{-\frac{t}{RC}}\left(\cos\left(\frac{t}{RC}\right) + \sin\left(\frac{t}{RC}\right)\right) + CV(t) \\
I(t) = q'(t) &= ke^{-kt}((CV_0 - CV_0)\cos kt - (CV_0 + CV_0)\sin kt) \\
&= ke^{-kt}(-2CV_0\sin kt) \\
&= -\frac{2V_0}{R}e^{-\frac{t}{RC}}\sin\frac{t}{RC}
\end{aligned}$$

ii. Find $\boldsymbol{q(t)}$ for $\boldsymbol{t > \pi RC}$.

$$\begin{aligned}
q(\pi RC) &= CV_0 e^{-\frac{\pi RC}{RC}}\left(\cos\left(\frac{\pi RC}{RC}\right) + \sin\left(\frac{\pi RC}{RC}\right)\right) + CV(\pi RC) \\
&= -CV_0 e^{-\pi}
\end{aligned}$$

Because $V(t) = 0$ at $t > \pi RC$, the capacitor reaches a maximum charge that would oscillate as t approaches infinity.

Problem 3

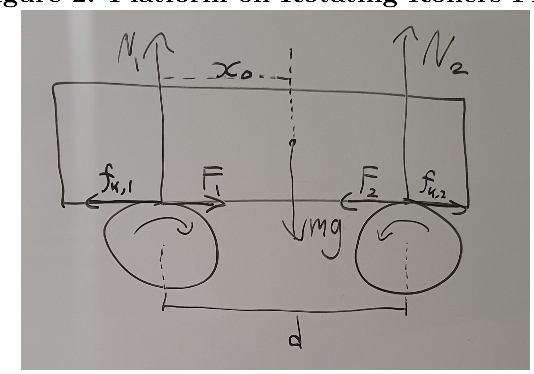
Platform on Rotation Rollers. Bounded vs. Unbounded Motion Two rapidly counter-rotating identical cylindrical rollers have axes separated by a fixed distance $\boldsymbol{d}$. At time $\boldsymbol{t = 0}$, a thin uniform bar of mass $\boldsymbol{m}$ is placed asymmetrically at rest across the top of the rollers, so that its center $\boldsymbol{C}$ is at a distance $\boldsymbol{x_0}$ from the left roller. Coefficient of kinetic friction between the bar and the rollers is $\boldsymbol{\mu_k}$ At time $\boldsymbol{t}$ the center $\boldsymbol{C}$ has made an unknown displacement $\boldsymbol{x(t)}$ from the left roller.

(a) Consider the situation where the wheels rotate inward to the center of the bar.

i. At time t,

A. draw the free-body diagram of the bar.

Figure 2: Platform on Rotating Rollers FBD



B. write down Newton's $\boldsymbol{2^{nd}}$ law for the bar in both horizontal and vertical directions.

$$m\ddot{y} = mg - N_1 - N_2 = 0$$
$$mg = N_1 + N_2$$

$$m\ddot{x} = F_1 - F_2 = f_{k,1} - f_{k,2}$$
$$m\ddot{x} = \mu_k N_1 - \mu_k N_2$$
$$\frac{m\ddot{x}}{\mu_k} = N_1 - N_2$$

C. take the torque about a convenient point on the bar.

A convenient point would be the center of the bar, as it takes into account $x(t)$ and removes torque from mg. Taking clockwise rotation as positive. $\alpha = 0$ as the bar is not rotating.

$$\sum \tau = I\alpha = 0$$

$$\tau_1 - \tau_2 = 0$$
$$N_1\left(\tfrac{d}{2} + x\right) - N_2\left(\tfrac{d}{2} - x\right) = 0$$
$$N_1\tfrac{d}{2} + N_1 x - N_2\tfrac{d}{2} + N_2 x = 0$$
$$\frac{d}{2}(N_1 - N_2) + x(N_1 + N_2) = 0$$

ii. Hence, find $\boldsymbol{x(t)}$ explicitly and show that the bar oscillates with an angular frequency $\sqrt{\boldsymbol{\frac{2\mu_k g}{d}}}$.

$$\frac{d}{2}\left(\frac{m\ddot{x}}{\mu_k}\right) + x(mg) = 0$$

$$\ddot{x} + \left(\frac{2\mu_k g}{d}\right)x = 0 \qquad \omega = \sqrt{\frac{2\mu_k g}{d}}$$

This is a homogenous SHO differential equation, therefore we can assume a solution $x(t)$ of the form.

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$
$$x'(t) = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$$

Applying initial conditions. $x(0) = x_0$, $x'(0) = 0$

$$x(0) = A = x_0$$
$$x'(0) = B\omega = 0$$

$$x(t) = x_0\cos\left(\sqrt{\tfrac{2\mu_k g}{d}}t\right)$$

(b) Now, consider the situation where the wheels rotate outward to the ends of the bar. Repeat part **(*a*)** and show that in this case,

$$x(t) = \left(x_0 - \frac{d}{2}\right)\cosh\left(\sqrt{\frac{2\mu_k g}{d}}t\right) + \frac{d}{2}$$

In this case, the only difference is on the forces on the horizontal directions.

$$m\ddot{y} = mg - N_1 - N_2 = 0$$
$$mg = N_1 + N_2$$

$$m\ddot{x} = F_2 - F_1 = f_{k,2} - f_{k,21}$$
$$m\ddot{x} = \mu_k N_2 - \mu_k N_1$$
$$-\frac{m\ddot{x}}{\mu_k} = N_1 - N_2$$

$$\sum \tau = I\alpha = 0$$
$$\tau_1 - \tau_2 = 0$$
$$N_1\left(\tfrac{d}{2} + x\right) - N_2\left(\tfrac{d}{2} - x\right) = 0$$
$$N_1\tfrac{d}{2} + N_1 x - N_2\tfrac{d}{2} + N_2 x = 0$$
$$\frac{d}{2}(N_1 - N_2) + x(N_1 + N_2) = 0$$
$$\frac{d}{2}\left(-\frac{m\ddot{x}}{\mu_k}\right) + x(mg) = 0$$
$$\ddot{x} - \left(\frac{2\mu_k g}{d}\right)x = 0$$

This is different from the previous DE in part a. Using auxiliary polynomial.

$$a(m) = m^2 - \frac{2\mu_k g}{d} = 0$$
$$m = \pm\sqrt{\frac{2\mu_k g}{d}} = \pm\alpha$$

$$x(t) = Ae^{\alpha t} + Be^{-\alpha t}$$
$$x'(t) = A\alpha e^{\alpha t} - B\alpha e^{-\alpha t}$$

Applying initial conditions. $x(0) = x_0$, $x'(0) = 0$

$$\begin{aligned} x(0) &= A + B = x_0 \\ x'(0) &= A\alpha - B\alpha = 0 \\ A &= B = \frac{x_0}{2} \end{aligned}$$

$$\begin{aligned} x(t) &= \frac{x_0}{2} e^{\alpha t} + \frac{x_0}{2} e^{-\alpha t} \\ &= x_0 \left(\frac{e^{\alpha t} + e^{-\alpha t}}{2} \right) \\ &= x_0 \cosh\left(\sqrt{\frac{2\mu_k g}{d}} t \right) \end{aligned}$$