PHY 320 - Assignment 2

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1 Problem 2c

Find the velocity $\dot{x}$ as a function of the displacement x for a particle of mass m, which starts from rest at $x = 0$, subject to the following force function: $F_x = F_0 \cos cx$, where F_0 and c are positive constants.

Solution Since $F_x = ma_x = m\ddot{x}$, the chain rule can be used on $\frac{d\dot{x}}{dt}$ to extract $\dot{x}$ as a function of dx.

$$\begin{aligned}
ma_x &= F_0 \cos cx \\
m\ddot{x} &= F_0 \cos cx \\
\frac{d\dot{x}}{dx}\frac{dx}{dt} &= \frac{F_0 \cos cx}{m} \\
\dot{x} d\dot{x} &= \frac{F_0 \cos cx}{m} dx \\
\int_0^{\dot{x}} \dot{x} d\dot{x} &= \frac{F_0}{m} \int_0^x \cos cx \; dx \\
\frac{\dot{x}^2}{2} &= \frac{F_0 \sin cx}{cm} \\
\dot{x}(x) &= \left(\frac{2F_0 \sin cx}{cm} \right)^{1/2}
\end{aligned} \tag{1}$$

2 Problem 5

A particle of mass m is constrained to lie along a frictionless, horizontal plane subject to a force given by the expression $F(x) = -kx + kx^3/A^2$, where k and A are positive constants.

(a) Find the potential energy function $V(x)$ for this force.

Solution With the definition $F(x) = -\frac{dV(x)}{dx}$, and assuming $x_0 = 0$.

$$\begin{aligned}
-kx + \frac{kx^3}{A^2} &= -\frac{dV(x)}{dx} \\
\int_0^x -kx + \frac{kx^3}{A^2} \; dx &= -\int_0^x dV(x) \\
-k \int_0^x x \; dx + \frac{k}{A^2} \int_0^x x^3 \; dx &= -V(x) \\
-k \left[\frac{x^2}{2} \right] + \frac{k}{A^2} \left[\frac{x^4}{4} \right] &= -V(x) \\
V(x) &= \frac{kx^2}{2} - \frac{kx^4}{4A^2}
\end{aligned} \tag{2}$$

(b) Find the kinetic energy.

Solution The kinetic energy is related to the potential energy function by the Conservation of Mechanical Energy.

$$\begin{aligned}
T + V(x) &= T_0 + V(x_0) \\
T &= T_0 - V(x) \\
T(x) &= \frac{mv_0^2}{2} - \frac{kx^2}{2} + \frac{kx^4}{4A^2}
\end{aligned} \tag{3}$$

(c) The total energy of the particle as a function of its position.

Solution Its total mechanical energy.

$$
\begin{aligned}
E &\equiv T + V(x) \\
E &= \frac{mv_0^2}{2} - \frac{kx^2}{2} + \frac{kx^4}{4A^2} + \frac{kx^2}{2} - \frac{kx^4}{4A^2} \\
E &= T_0 = T(0)
\end{aligned} \tag{4}
$$

(d) Find the turning points of the motion and the condition the total energy of the particle must satisfy if its motion is to exhibit turning points.

Solution The turning points occur where $dV(x)/dx = 0$.

$$
\begin{aligned}
kx - \frac{kx^3}{A^2} &= 0 \\
x\left(k - \frac{kx^2}{A^2}\right) &= 0 \\
\frac{kx^2}{A^2} &= k \\
x &= \pm A \\
V(x_{\max}) &= \frac{kA^2}{2} - \frac{kA^4}{4A^2} \\
V(x_{\max}) &= \frac{kA^2}{4}
\end{aligned} \tag{5}
$$

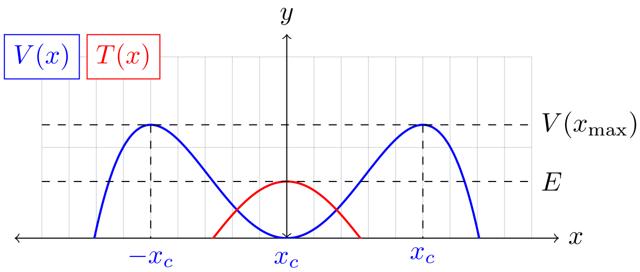
The condition on E for the motion to exhibit turning points is $E < V(x_{\max})$ as this satisfies the Law of Conservation of Mechanical Energy, also from equation 2.3.9 in the book. Now to find the turning points x_c through $T(x) = 0$ and the quadratic formula.

$$
\begin{aligned}
T(x) = E - \frac{kx^2}{2} + \frac{kx^4}{4A^2} &= 0 \\
kx^4 - 2A^2kx^2 + 4A^2E &= 0
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
x^2 &= \frac{-(-2A^2k) \pm \sqrt{(-2A^2k)^2 - 4(k)(4A^2E)}}{2k} \\
&= \frac{2A^2k \pm \sqrt{(4A^4k^2) - (16A^2Ek)}}{2k} \\
&= \frac{2A^2k \pm 2A^2k\sqrt{(1) - (\frac{4E}{A^2k})}}{2k} \\
x^2 &= A^2\left[1 \pm \left(1 - \left(\frac{4E}{A^2k}\right)\right)^{1/2}\right] \\
x_c &= \pm A\left[1 \pm \left(1 - \left(\frac{4E}{A^2k}\right)\right)^{1/2}\right]^{1/2}
\end{aligned} \tag{7}
$$

(e) Sketch the potential, kinetic, and total energy functions.

Solution With a value of 1 for k and A. $V(x_{\text{max}}) = \frac{1}{4}$ and $x_c = 1$. No value for E is given or can be found**?** so any arbitrary value for T_0 under $V(x_{\text{max}})$ is plausible.



3 Problem 14

A particle of mass m is released from rest a distance b from a fixed origin of force that attracts the particle according to the inverse square law: $F(x) = -kx^{-2}$. Show that the time required for the particle to reach the origin is $\pi(\frac{mb^3}{8k})^{1/2}$.

Solution By using the chain rule here, a separation of variables will help. The π in the solution must mean that a trig substitution is used. For the first integral, differential displacements are used. I used an integral calculator for the second integral.

$$\begin{aligned}
m\frac{d\dot{x}}{dx}\frac{dx}{dt} &= -kx^{-2} \\
\int_0^{\dot{x}} m\dot{x}\,d\dot{x} &= \int_b^x -kx^{-2}\,dx \\
\frac{m\dot{x}^2}{2} &= -k\left(-\frac{1}{x}+\frac{1}{b}\right) \\
\dot{x} &= \left(\frac{2k}{m}\left(\frac{1}{x}-\frac{1}{b}\right)\right)^{1/2} \\
\frac{dx}{dt} &= \left(\frac{2k}{m}\left(\frac{b-x}{bx}\right)\right)^{1/2} \\
\int_0^t dt &= \int_b^0 \left(\frac{m}{2k}\left(\frac{bx}{b-x}\right)\right)^{1/2} dx \\
t &= \left(\frac{bm}{2k}\right)^{1/2} \int_b^0 \left(\frac{x}{b-x}\right)^{1/2} dx \\
t &= \left(\frac{bm}{2k}\right)^{1/2} \left(\frac{\pi b}{2}\right) \\
t &= \pi\left(\frac{mb^3}{8k}\right)^{1/2}
\end{aligned} \tag{8}$$

4 Problem 8

Given that the velocity of a particle in rectilinear motion varies with the displacement x according to the equation $\dot{x} = bx^{-3}$, where b is a positive constant, find the force acting on the particle as a function of x. (Hint: $F = m\ddot{x} = m\dot{x}\, d\dot{x}/dx$)

Solution Using the hint, and since $\frac{d\dot{x}}{dx} = -3bx^{-4}$.

$$\begin{aligned} F(x) = m\frac{d\dot{x}}{dt} &= m\frac{d\dot{x}}{dx}\frac{dx}{dt} \\ F(x) &= m(-3bx^{-4})(bx^{-3}) \\ F(x) &= \frac{-3mb^2}{x^7} \end{aligned} \tag{9}$$

5 Problem 18

The force acting on a particle of mass m is given by $F = kvx$ in which k is a positive constant. The particle passes through the origin with speed v_0 at time $t = 0$. Find x as a function of t.

Solution Using the chain rule for a separation of variables. A substitution for the integral definition of arctan is useful, $u = \sqrt{\frac{k}{2mv_0}}x$.

$$\begin{aligned} ma &= kvx \\ m\frac{dv}{dt} &= k\frac{dx}{dt}x \\ m\frac{dv}{dx}\frac{dx}{dt} &= k\frac{dx}{dt}x \\ \int_{v_0}^{v} dv &= \int_0^x \frac{k}{m}x\,dx \\ v - v_0 &= \frac{kx^2}{2m} \\ \frac{dx}{dt} &= v_0 + \frac{kx^2}{2m} \\ \frac{dx}{dt} &= v_0\left(1 + \frac{kx^2}{2mv_0}\right) \\ \int_0^x \frac{dx}{1 + \frac{kx^2}{2mv_0}} &= \int_0^t v_0\,dt \\ \sqrt{\frac{2mv_0}{k}}\int_0^{\sqrt{\frac{k}{2mv_0}}x} \frac{du}{1+u^2} &= v_0 t \\ \sqrt{\frac{2mv_0}{k}}\arctan\left(\sqrt{\frac{k}{2mv_0}}x\right) &= v_0 t \\ x(t) &= \sqrt{\frac{2mv_0}{k}}\tan\left(\sqrt{\frac{kv_0}{2m}}t\right) \end{aligned} \tag{10}$$