PHY 310 — Mathematical Methods in Physics

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Problem 1: Electromagnetism

Parallel Plate Capacitor

(a) Consider a general nonhomogeneous BVP for a $\mathbf{2^{nd}}$ order ODE:

$$\frac{d^2y(x)}{dx^2} = f(x), \qquad y(a) = \alpha,\ y(b) = \beta,\ \text{and } x \in [a, b]\,.$$

Show that the solution to this BVP is

$$y(x) = \alpha + \frac{\beta - \alpha}{b - a} + \int_a^b G(x, x')f(x')\,dx',$$

where the Green's function of the differential operator is given by,

$$G(x, x') = \frac{1}{b - a}\begin{cases}(x' - a)(x - b) & \text{for } x' \leq x,\\ (x - a)(x' - b) & \text{for } x \leq x'.\end{cases}$$

Solution A nonhomogeneous BVP with nonhomogeneous BC's is generally solved in the following way: the solution is the sum of the nonhomogeneous BVP with homogeneous BCs and the homogeneous BVP with nonhomogeneous BC s. In this case:

$$y = y_c + y_p \quad \text{where} \quad y_c'' = 0 \begin{cases} y_c(a) = \alpha \\ y_c(b) = \beta \end{cases} \quad \text{and} \quad y_p'' = f(x) \begin{cases} y_p(a) = 0 \\ y_p(b) = 0 \end{cases}$$

Starting with y_p. Solving for the homogeneous BVP to find the Green's function. Where the Green's function is defined as

$$G(x, x') = \begin{cases} \dfrac{y_1(x')y_2(x)}{W(y_1, y_2)(x')} \ ; \ a \leq x' \leq x \\ \dfrac{y_1(x)y_2(x')}{W(y_1, y_2)(x')} \ ; \ x \leq x' \leq b \end{cases}$$

$$\text{Simple integration} \quad y_p'' = 0 \quad y_p' = c_1 \quad y_p = c_1x + c_2$$

$$\begin{aligned} y_p(a) &= c_1a + c_2 = 0 \quad c_2 = -c_1a \\ y_p(x) &= c_1x - c_1a = c_1(x - a) \quad c_1 \text{ is arbitrary, choose } c_1 = 1 \\ y_{p_1}(x) &= x - a \end{aligned}$$

$$\begin{aligned} y_p(b) &= c_1b + c_2 = 0 \quad c_2 = -c_1b \\ y_p(x) &= c_1x - c_1b = c_1(x - b) \quad c_1 \text{ is arbitrary, choose } c_1 = 1 \\ y_{p_2}(x) &= x - b \end{aligned}$$

$$\begin{aligned} W(y_{p_1}, y_{p_2})(x') = \begin{vmatrix} x' - a & x' - b \\ 1 & 1 \end{vmatrix} &= (x' - a)(1) - (x' - b)(1) \\ &= b - a \end{aligned}$$

$$G(x, x') = \begin{cases} \dfrac{(x' - a)(x - b)}{b - a} \ ; \ a \leq x' \leq x \\ \dfrac{(x - a)(x' - b)}{b - a} \ ; \ x \leq x' \leq b \end{cases}$$

$$G(x, x') = \frac{1}{b - a}\begin{cases} (x' - a)(x - b) \ ; \ a \leq x' \leq x \\ (x - a)(x' - b) \ ; \ x \leq x' \leq b \end{cases}$$

Solving for y_c. Similar start to y_p.

$$\text{Simple integration} \quad y_p'' = 0 \quad y_p' = c_1 \quad y_p = c_1 x + c_2$$

$$\begin{matrix} y_p(a) = c_1 a + c_2 = \alpha \\ y_p(b) = c_1 b + c_2 = \beta \end{matrix} \quad \Rightarrow \quad \begin{matrix} -c_1 a - c_2 = -\alpha \\ c_1 b + c_2 = \beta \end{matrix} \quad \Rightarrow \quad \begin{aligned} c_1(b-a) &= \beta - \alpha \\ c_1 &= \frac{\beta-\alpha}{b-a} \\ c_2 &= \alpha - \left(\frac{\beta-\alpha}{b-a}\right) a \end{aligned}$$

$$\begin{aligned} y_c(x) &= \left(\frac{\beta-\alpha}{b-a}\right) x + \alpha - \left(\frac{\beta-\alpha}{b-a}\right) a \\ &= \alpha - \left(\frac{\beta-\alpha}{b-a}\right)(x-a) \end{aligned}$$

Finally.

$$\begin{aligned} y &= y_c + y_p \\ y &= \alpha - \left(\frac{\beta-\alpha}{b-a}\right)(x-a) + \int_a^b G(x,x')f(x')\,dx' \end{aligned}$$

$$\text{where } G(x,x') = \frac{1}{b-a}\begin{cases} (x'-a)(x-b)\ ;\ a \le x' \le x \\ (x-a)(x'-b)\ ;\ x \le x' \le b \end{cases} \quad \text{and } f(x') = f(x)$$

(b) Two very large parallel conducting plates are separated by a distance $\boldsymbol{d}$ and maintained at potentials $\mathbf{0}$ and $\boldsymbol{V_0}$. The region between the plates is filled with a continuous distribution of electrons having a volume charge density $\boldsymbol{p_c(z) = -\frac{\rho_0}{d} z}$ ($\boldsymbol{p_0}$ constant). Assume negligible fringing effect at the edges. Using only the result of part (a), determine the electrostatic potential $\boldsymbol{\Phi(z)}$ at any point between the plates, where $\boldsymbol{\Phi(z)}$, in the Cartesian coordinate system, satisfies the one-dimensional Poisson's equation

$$\frac{d^2\Phi(z)}{dz^2} = -\frac{p_c(z)}{\epsilon_0} = \frac{\rho_0}{\epsilon_0 d} z$$

Solution The solution found in part a is the general solution to any system that satisfies the one-dimensional Poisson's equation. BCs are $\Phi(0) = 0$ and $\Phi(d) = V_0$

$$\begin{aligned} y &= \alpha - \left(\frac{\beta-\alpha}{b-a}\right)(x-a) + \int_a^b G(x,x')f(x')\,dx' \\ \Phi(z) &= 0 - \left(\frac{V_0-0}{d-0}\right)(z-0) + \int_0^d G(z,z')\left(\frac{\rho_0}{\epsilon_0 d}z'\right)dz' \qquad (1) \\ \Phi(z) &= -\frac{V_0}{d}z + \frac{\rho_0}{\epsilon_0 d}\int_0^d G(z,z')z'\,dz' \end{aligned}$$

$$G(z,z') = \frac{1}{d}\begin{cases} z'(z-d)\ ;\ 0 \le z' \le z \\ z(z'-d)\ ;\ z \le z' \le d \end{cases}$$

$$\Phi_p(z) = \frac{\rho_0}{\epsilon_0 d}\left[\frac{1}{d}\int_0^z z'(z-d)z'\,dz' + \frac{1}{d}\int_z^d z(z'-d)z'\,dz'\right] = \frac{\rho_0}{\epsilon_0 d^2}\left[\int_0^z z'^2 z - z'^2 d\,dz' + \int_z^d z'^2 z - zz'd\,dz'\right]$$

$$\Phi_p(z) = \frac{\rho_0}{\epsilon_0 d^2}\left[\left[\frac{z'^3 z}{3}\right]_0^z - \left[\frac{z'^3 d}{3}\right]_0^z + \left[\frac{z'^3 z}{3}\right]_z^d - \left[\frac{zz'^2 d}{2}\right]_z^d\right] = \frac{\rho_0}{\epsilon_0 d^2}\left[\frac{z^4}{3} - \frac{z^3 d}{3} + \left[\frac{d^3 z}{3} - \frac{z^4}{3}\right] - \left[\frac{zd^3}{2} - \frac{z^3 d}{2}\right]\right]$$

$$\Phi_p(z) = \frac{\rho_0}{\epsilon_0 d^2}\left[\frac{z^3 d}{6} - \frac{d^3 z}{6}\right] = \frac{\rho_0}{\epsilon_0 d}\left[\frac{z^3}{6} - \frac{d^2 z}{6}\right]$$

$$\Phi(z) = -\frac{V_0}{d}z + \frac{\rho_0}{\epsilon_0 d}z\left[\frac{z^2 - d^2}{6}\right]$$

Problem 2: Electromagnetism

Conical Capacitor Two coaxial conducting cones of semi-vertical angles $\boldsymbol{\alpha_1}$ and $\boldsymbol{\alpha_2}$ and of large extent have their vertices separated by an infinitesimal gap. The inner cone is grounded while the outer is maintained at potential $\boldsymbol{V_0}$. In the spherical polar coordinate system, the electrostatic potential $\boldsymbol{\Phi}$ in between the cones depends only on a single spatial coordinate, namely, the polar angle $\boldsymbol{\theta}$ which satisfies the Poisson's equation

$$\frac{1}{r^2 \sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{d\Phi(\theta)}{d\theta}\right] = -\frac{p_c(\theta)}{\epsilon_0},$$

Where $p_c(\theta)$ is volume charge density in the region between the cones. Show that

$$\Phi(\theta) = V_0\frac{\ln\left[\frac{\tan\frac{\theta}{2}}{\tan\frac{\alpha_1}{2}}\right]}{\ln\left[\frac{\tan\frac{\alpha_2}{2}}{\tan\frac{\alpha_1}{2}}\right]} - \frac{r^2}{\epsilon_0}\int_{\alpha_1}^{\alpha_2} G(\theta,\theta')\rho_c(\theta')\sin\theta'\,d\theta'$$

and precisely identity the Green's function $\boldsymbol{G(\theta,\theta')}$ for the given BVP.

Solution Following a similar process as in problem 1, splitting the final solution into two manageable BVPs. The BCs are given in the statement of the problem. The $\frac{1}{r^2\sin\theta}$ is taken to the RHS.

$$\Phi(\theta) = \Phi_c(\theta) + \Phi_p(\theta)$$

$$\frac{d}{d\theta}\left[\sin\theta\frac{d\Phi_c(\theta)}{d\theta}\right] = 0 \begin{cases}\Phi_c(\alpha_1) = 0\\ \Phi_c(\alpha_2) = V_0\end{cases} \quad \text{and} \quad \frac{d}{d\theta}\left[\sin\theta\frac{d\Phi_p(\theta)}{d\theta}\right] = -\frac{p_c(\theta)r^2\sin\theta}{\epsilon_0} \begin{cases}\Phi_p(\alpha_1) = 0\\ \Phi_p(\alpha_2) = 0\end{cases}$$

Starting with $\Phi_c(\theta)$.

$$\int \frac{d}{d\theta}\left[\sin\theta\frac{d\Phi_c(\theta)}{d\theta}\right]d\theta = \int 0\,d\theta$$
$$\sin\theta\frac{d\Phi_c(\theta)}{d\theta} = c_1$$
$$\int d\Phi_c(\theta) = c_1\int\frac{d\theta}{\sin\theta}$$
$$\Phi_c(\theta) = c_1\ln\left(\tan\tfrac{\theta}{2}\right) + c_2$$

Applying initial conditions to find c_1 and c_2.

$$\begin{aligned}\Phi_c(\alpha_1) &= c_1\ln\left(\tan\tfrac{\alpha_1}{2}\right) + c_2 = 0\\ \Phi_c(\alpha_2) &= c_1\ln\left(\tan\tfrac{\alpha_2}{2}\right) + c_2 = V_0\end{aligned} \Rightarrow \Phi_c(\alpha_2) - \Phi_c(\alpha_1) \Rightarrow c_1\left[\ln\left(\tan\tfrac{\alpha_2}{2}\right) - \ln\left(\tan\tfrac{\alpha_1}{2}\right)\right] = V_0$$

$$c_1 = \frac{V_0}{\ln\left[\frac{\tan\frac{\alpha_2}{2}}{\tan\frac{\alpha_1}{2}}\right]} \qquad c_2 = -\frac{V_0}{\ln\left[\frac{\tan\frac{\alpha_2}{2}}{\tan\frac{\alpha_1}{2}}\right]}\ln\left[\tan\tfrac{\alpha_1}{2}\right]$$

$$\Phi_c(\theta) = \left(\frac{V_0}{\ln\left[\frac{\tan\frac{\alpha_2}{2}}{\tan\frac{\alpha_1}{2}}\right]}\right)\ln\left[\tan\tfrac{\theta}{2}\right] + \left(-\frac{V_0}{\ln\left[\frac{\tan\frac{\alpha_2}{2}}{\tan\frac{\alpha_1}{2}}\right]}\ln\left[\tan\tfrac{\alpha_1}{2}\right]\right)$$

Factor V_0, take denominator as common factor, apply log rule to numerator.

$$\Phi_c(\theta) = V_0\frac{\ln\left[\frac{\tan\frac{\theta}{2}}{\tan\frac{\alpha_1}{2}}\right]}{\ln\left[\frac{\tan\frac{\alpha_2}{2}}{\tan\frac{\alpha_1}{2}}\right]}$$

Now for $\Phi_p(\theta)$.

$$\frac{d}{d\theta}\left[\sin\theta\frac{d\Phi(\theta)}{d\theta}\right] = -\frac{r^2}{\epsilon_0}p_c(\theta)\sin(\theta) = f(\theta),$$

The Green's function is defined as

$$\begin{aligned}
\Phi_p(\theta) &= \int_a^b G(\theta,\theta')f(\theta')\,d\theta' \\
&= \int_{\alpha_1}^{\alpha_2} G(\theta,\theta')\left(-\frac{r^2}{\epsilon_0}p_c(\theta')\sin(\theta')\right)\,d\theta' \\
&= -\frac{r^2}{\epsilon_0}\int_{\alpha_1}^{\alpha_2} G(\theta,\theta')p_c(\theta')\sin(\theta')\,d\theta'
\end{aligned}$$

$$G(\theta,\theta') = \begin{cases} \dfrac{\Phi_1(\theta')\Phi_2(\theta)}{W(\Phi_1,\Phi_2)(\theta')} \ ; \ \alpha_1 \le \theta' \le \theta \\ \dfrac{\Phi_1(\theta)\Phi_2(\theta')}{W(\Phi_1,\Phi_2)(\theta')} \ ; \ \theta \le \theta' \le \alpha_2 \end{cases}$$

To find the Green's function, the homogeneous BVP has already been solved in the previous part.

$$\begin{aligned}
\Phi_p(\alpha_1) &= c_1\ln\bigl(\tan\tfrac{\alpha_1}{2}\bigr) + c_2 = 0, \quad c_2 = -c_1\ln\bigl(\tan\tfrac{\alpha_1}{2}\bigr) \\
&c_1 \text{ is arbitrary, choose } c_1 = -1, \text{ then } c_2 = \ln\bigl(\tan\tfrac{\alpha_1}{2}\bigr) \\
\Phi_{p_1}(\theta) &= -\ln\bigl(\tan\tfrac{\theta}{2}\bigr) + \ln\bigl(\tan\tfrac{\alpha_1}{2}\bigr) = ln\left[\frac{\tan\frac{\alpha_1}{2}}{\tan\frac{\theta}{2}}\right]
\end{aligned}$$

$$\begin{aligned}
\Phi_p(\alpha_2) &= c_1\ln\bigl(\tan\tfrac{\alpha_2}{2}\bigr) + c_2 = 0, \quad c_2 = -c_1\ln\bigl(\tan\tfrac{\alpha_2}{2}\bigr) \\
&c_1 \text{ is arbitrary, choose } c_1 = -1, \text{ then } c_2 = \ln\bigl(\tan\tfrac{\alpha_2}{2}\bigr) \\
\Phi_{p_2}(\theta) &= -\ln\bigl(\tan\tfrac{\theta}{2}\bigr) + \ln\bigl(\tan\tfrac{\alpha_2}{2}\bigr) = ln\left[\frac{\tan\frac{\alpha_2}{2}}{\tan\frac{\theta}{2}}\right]
\end{aligned}$$

$$\begin{aligned}
W(\Phi_1,\Phi_2)(\theta') &= \begin{vmatrix} ln\dfrac{\tan\frac{\alpha_1}{2}}{\tan\frac{\theta'}{2}} & ln\dfrac{\tan\frac{\alpha_2}{2}}{\tan\frac{\theta'}{2}} \\ -\dfrac{\tan\frac{\theta'}{2}}{2\sin^2\frac{\theta'}{2}} & -\dfrac{\tan\frac{\theta'}{2}}{2\sin^2\frac{\theta'}{2}} \end{vmatrix} \\
&= ln\left[\frac{\tan\frac{\alpha_1}{2}}{\tan\frac{\theta'}{2}}\right]\left[-\frac{\tan\frac{\theta'}{2}}{2\sin^2\frac{\theta'}{2}}\right] - ln\left[\frac{\tan\frac{\alpha_2}{2}}{\tan\frac{\theta'}{2}}\right]\left[-\frac{\tan\frac{\theta'}{2}}{2\sin^2\frac{\theta'}{2}}\right] \\
&= \left[-\frac{\tan\frac{\theta'}{2}}{2\sin^2\frac{\theta'}{2}}\right]\left[ln\left[\frac{\tan\frac{\alpha_1}{2}}{\tan\frac{\theta'}{2}}\right] - ln\left[\frac{\tan\frac{\alpha_2}{2}}{\tan\frac{\theta'}{2}}\right]\right] \\
&= \left[\frac{\tan\frac{\theta'}{2}}{2\sin^2\frac{\theta'}{2}}\right]\left[ln\left[\frac{\tan\frac{\alpha_2}{2}}{\tan\frac{\alpha_1}{2}}\right]\right]
\end{aligned}$$

Finally.

$$G(\theta,\theta') = \begin{cases} \dfrac{\left[ln\dfrac{\tan\frac{\alpha_1}{2}}{\tan\frac{\theta'}{2}}\right]\left[ln\dfrac{\tan\frac{\alpha_2}{2}}{\tan\frac{\theta}{2}}\right]}{\left[\dfrac{\tan\frac{\theta'}{2}}{2\sin^2\frac{\theta'}{2}}\right]\left[ln\dfrac{\tan\frac{\alpha_2}{2}}{\tan\frac{\alpha_1}{2}}\right]} & ;\ \alpha_1 \le \theta' \le \theta \\ \dfrac{\left[ln\dfrac{\tan\frac{\alpha_1}{2}}{\tan\frac{\theta}{2}}\right]\left[ln\dfrac{\tan\frac{\alpha_2}{2}}{\tan\frac{\theta'}{2}}\right]}{\left[\dfrac{\tan\frac{\theta'}{2}}{2\sin^2\frac{\theta'}{2}}\right]\left[ln\dfrac{\tan\frac{\alpha_2}{2}}{\tan\frac{\alpha_1}{2}}\right]} & ;\ \theta \le \theta' \le \alpha_2 \end{cases}$$

$$G(\theta,\theta') = \frac{1}{\left[ln\dfrac{\tan\frac{\alpha_2}{2}}{\tan\frac{\alpha_1}{2}}\right]} \begin{cases} \dfrac{\left[ln\dfrac{\tan\frac{\alpha_1}{2}}{\tan\frac{\theta'}{2}}\right]\left[ln\dfrac{\tan\frac{\alpha_2}{2}}{\tan\frac{\theta}{2}}\right]}{\left[\dfrac{\tan\frac{\theta'}{2}}{2\sin^2\frac{\theta'}{2}}\right]} & ;\ \alpha_1 \le \theta' \le \theta \\ \dfrac{\left[ln\dfrac{\tan\frac{\alpha_1}{2}}{\tan\frac{\theta}{2}}\right]\left[ln\dfrac{\tan\frac{\alpha_2}{2}}{\tan\frac{\theta'}{2}}\right]}{\left[\dfrac{\tan\frac{\theta'}{2}}{2\sin^2\frac{\theta'}{2}}\right]} & ;\ \theta \le \theta' \le \alpha_2 \end{cases}$$

This will satisfy the final expression. Combining Φ_c and Φ_p.

$$\Phi(\theta) = V_0 \frac{\ln\left[\dfrac{\tan\frac{\theta}{2}}{\tan\frac{\alpha_1}{2}}\right]}{\ln\left[\dfrac{\tan\frac{\alpha_2}{2}}{\tan\frac{\alpha_1}{2}}\right]} - \frac{r^2}{\epsilon_0}\int_{\alpha_1}^{\alpha_2} G(\theta,\theta')\rho_c(\theta')\sin\theta'\,d\theta'$$

Problem 5: Quantum Mechanics

Particle Trapped in a One-Dimensional Potential Well of Infinite Depth A particle of mass $\boldsymbol{m}$ with potential energy

$$V(x) = \begin{cases} 0 & \text{for } a \leq x \leq b, \\ \infty & \text{otherwise,} \end{cases}$$

satisfies the BVP for the Schrodinger equation:

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\phi(x) = E\phi(x), \quad \phi(a) = 0,\ \phi(b) = 0,$$

where $\boldsymbol{\hbar}$, $\boldsymbol{E}$ and $\boldsymbol{\phi(x)}$ are the reduced Planck's constant, energy eigenvalues and the wavefunction, respectively.

(a) Show that the Green's function for the given BVP is

$$G(x, x') = \frac{1}{k_E \sin\left[k_E(b-a)\right]} \begin{cases} \sin\left[k_E(b-a)\right]\sin\left[k_E(x'-b)\right] & \text{for } x < x' \\ \sin\left[k_E(b-a)\right]\sin\left[k_E(x'-a)\right] & \text{for } x > x' \end{cases}$$

where $k_E \equiv \sqrt{\dfrac{2mE}{\hbar^2}}$.

Solution The Green's function is the particular solution ϕ_p, which is the nonhomogenous BVP with homogenous BCs. Solving for the homogeneous DE to find the Green's function. Where the Green's function is defined as

$$G(x, x') = \begin{cases} \dfrac{\phi_1(x')\phi_2(x)}{W(\phi_1, \phi_2)(x')} & ;\ a \leq x' \leq x \\ \dfrac{\phi_1(x)\phi_2(x')}{W(\phi_1, \phi_2)(x')} & ;\ x \leq x' \leq b \end{cases}$$

$$-\frac{\hbar^2}{2m}\phi'' + \left[V - E\right]\phi = 0$$

Take $V(x) = 0$, a provided condition between a and b, and reverse the signs.

$$\phi'' + \frac{2mE}{\hbar^2}\phi = 0$$

$$\phi'' + k_E^2\phi = 0 \quad \text{where } k_E = \sqrt{\frac{2mE}{\hbar^2}}$$

This is a homogeneous second order DE that governs the simple harmonic oscillator, with a known general solution.

$$\phi(x) = c_1\sin(k_E x) + c_2\cos(k_E x)$$

Following the same steps as in problem 1.

$$\phi(a) = c_1\sin(k_E a) + c_2\cos(k_E a) = 0, \quad c_1 = -\frac{c_2\cos(k_E a)}{\sin(k_E a)}$$

$$c_2 \text{ is arbitrary, choose } c_2 = -\sin(k_E a), \text{then } c_1 = \cos(k_E a)$$

$$\phi_1(x) = \cos(k_E a)\sin(k_E x) - \sin(k_E a)\cos(k_E x)$$

$$\phi_1(x) = \sin(k_E x - k_E a) = \sin(k_E(x-a))$$

$$\phi(b) = c_1\sin(k_E b) + c_2\cos(k_E b) = 0, \quad c_1 = -\frac{c_2\cos(k_E b)}{\sin(k_E b)}$$

$$c_2 \text{ is arbitrary, choose } c_2 = -\sin(k_E b), \text{then } c_1 = \cos(k_E b)$$

$$\phi_2(x) = \cos(k_E b)\sin(k_E x) - \sin(k_E b)\cos(k_E x)$$

$$\phi_2(x) = \sin(k_E x - k_E b) = \sin(k_E(x-b))$$

$$
\begin{aligned}
W(\phi_1,\phi_2)(x') &= \begin{vmatrix} \sin(k_E(x-a)) & \sin(k_E(x-b)) \\ k_E\cos(k_E(x-a)) & k_E\cos(k_E(x-b)) \end{vmatrix} \\
&= (\sin(k_E(x-a)))(k_E\cos(k_E(x-b))) - (\sin(k_E(x-b)))(k_E\cos(k_E(x-a))) \\
&= k_E\left[(\sin(k_E(x-a)))(\cos(k_E(x-b))) - (\sin(k_E(x-b)))(\cos(k_E(x-a)))\right] \\
&= k_E\sin((k_E(x-a)) - (k_E(x-b))) \\
&= k_E\sin(k_E(b-a))
\end{aligned}
$$

$$
G(x,x') = \begin{cases} \dfrac{\sin(k_E(x'-a))\sin(k_E(x-b))}{k_E\sin(k_E(b-a))} \ ; \ a \le x' \le x \\ \dfrac{\sin(k_E(x-a))\sin(k_E(x'-b))}{k_E\sin(k_E(b-a))} \ ; \ x \le x' \le b \end{cases}
$$

$$
G(x,x') = \frac{1}{k_E\sin(k_E(b-a))}\begin{cases} \sin(k_E(x'-a))\sin(k_E(x-b)) \ ; \ a \le x' \le x \\ \sin(k_E(x-a))\sin(k_E(x'-b)) \ ; \ x \le x' \le b \end{cases}
$$

(b) Specialize the answer to part (a) for a potential well with infinite width $\boldsymbol{(b-a)\to\infty}$. This result is known as the free-particle propagator.

$$
G(x,x') = \frac{1}{k_E\left[\dfrac{e^{ik_E(b-a)} - e^{-ik_E(b-a)}}{2i}\right]}\begin{cases} \left[\dfrac{e^{ik_E(x'-a)} - e^{-ik_E(x'-a)}}{2i}\right]\left[\dfrac{e^{ik_E(x-b)} - e^{-ik_E(x-b)}}{2i}\right] \ ; \ a \le x' \le x \\ \left[\dfrac{e^{ik_E(x-a)} - e^{-ik_E(x-a)}}{2i}\right]\left[\dfrac{e^{ik_E(x'-b)} - e^{-ik_E(x'-b)}}{2i}\right] \ ; \ x \le x' \le b \end{cases}
$$

We want to drop the outgoing waves traveling to the "infinite" walls: for $\boldsymbol{a}$ we drop waves going from right to left, generally e^{-ikx}. For $\boldsymbol{b}$ we drop waves going left to right, generally e^{ikx}.

$$
G(x,x') = \frac{1}{k_E\left[\dfrac{e^{ik_E(b-a)} - e^{-ik_E(b-a)}}{2i}\right]}\begin{cases} \left[\dfrac{e^{ik_E(x'-a)}}{2i}\right]\left[\dfrac{-e^{-ik_E(x-b)}}{2i}\right] \ ; \ a \le x' \le x \\ \left[\dfrac{e^{ik_E(x-a)}}{2i}\right]\left[\dfrac{-e^{-ik_E(x'-b)}}{2i}\right] \ ; \ x \le x' \le b \end{cases}
$$

$$
G(x,x') = \frac{1}{4k_E\left[\dfrac{e^{ik_E(b-a)} - e^{-ik_E(b-a)}}{2i}\right]}\begin{cases} \left[e^{ik_E(x'-a)}\right]\left[e^{-ik_E(x-b)}\right] \ ; \ a \le x' \le x \\ \left[e^{ik_E(x-a)}\right]\left[e^{-ik_E(x'-b)}\right] \ ; \ x \le x' \le b \end{cases}
$$

$$
G(x,x') = \frac{i}{2k_E\left[e^{ik_E(b-a)} - e^{-ik_E(b-a)}\right]}\begin{cases} \left[e^{ik_E(x'-a-x+b)}\right] \ ; \ a \le x' \le x \\ \left[e^{ik_E(x-a-x'+b)}\right] \ ; \ x \le x' \le b \end{cases}
$$

$$
G(x,x') = \frac{i}{2k_E\left[e^{ik_E(b-a)} - e^{-ik_E(b-a)}\right]}\begin{cases} \left[e^{ik_E(x'-x)}\right]\left[e^{ik_E(b-a)}\right] \ ; \ a \le x' \le x \\ \left[e^{ik_E(x-x')}\right]\left[e^{ik_E(b-a)}\right] \ ; \ x \le x' \le b \end{cases}
$$

$$
G(x,x') = \frac{i}{2k_E}\frac{\left[e^{ik_E(b-a)}\right]}{\left[e^{ik_E(b-a)} - e^{-ik_E(b-a)}\right]}\begin{cases} \left[e^{ik_E(x'-x)}\right] \ ; \ x \ge x' \\ \left[e^{ik_E(x-x')}\right] \ ; \ x \le x' \end{cases}
$$

Now to take the limit as $\boldsymbol{(b-a)\to\infty}$. Let $x = b-a$

$$
\lim_{x\to\infty}\left[\frac{e^{ik_Ex}}{e^{ik_Ex} - e^{-ik_Ex}}\right] = \lim_{x\to\infty}\left[\frac{1}{1-\dfrac{1}{e^{2ik_Ex}}}\right] = \frac{1}{1-\frac{1}{e^{\infty}}} = 1
$$

Finally.

$$
G(x,x') = \frac{i}{2k_E}\begin{cases} \left[e^{ik_E(x'-x)}\right] \ ; \ x \ge x' \\ \left[e^{ik_E(x-x')}\right] \ ; \ x \le x' \end{cases}
$$

$$
G(x,x') = \frac{i}{2k_E}e^{ik_E|x-x'|}
$$