PHY 350 — Quantum Mechanics

Name: Khalifa Salem Almatrooshi
Student Number: @00090847

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Assignment: 4

1 Problem 1: Exercise 6.2

A particle of mass m moves in the xy plane in the potential

$$V(x,y) = \begin{cases} \frac{1}{2}m\omega^2 x^2, & \text{for all y and } 0 < x < a \\ +\infty, & \text{elsewhere} \end{cases} \tag{1.1}$$

(a) Write down the time-independent Schrödinger equation for this particle and reduce it to a set of familiar one-dimensional equations.

Solution The particle is in an infinite square well potential bounded in the x-direction from 0 to a, and unbounded in the y-direction. The particle also experiences a harmonic oscillator potential $\frac{1}{2}m\omega^2 x^2$ in this region. To reduce it to a set of familiar one-dimensional equations, we have to first assume a separable solution, $\psi(x,y) = X(x)Y(y)$.

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi(x,y) + V(x,y)\psi(x,y) = E\psi(x,y), \tag{1.2}$$

$$-\frac{\hbar^2}{2m}\left(Y\frac{\partial^2}{\partial x^2}X + X\frac{\partial^2}{\partial y^2}Y\right) + \left(\frac{1}{2}m\omega^2 x^2\right)XY = EXY, \tag{1.3}$$

$$-\frac{\hbar^2}{2m}\left(\frac{1}{X}\frac{\partial^2}{\partial x^2}X + \frac{1}{Y}\frac{\partial^2}{\partial y^2}Y\right) + \left(\frac{1}{2}m\omega^2 x^2\right) = E, \tag{1.4}$$

$$\left[-\frac{\hbar^2}{2m}\frac{1}{X}\frac{\partial^2}{\partial x^2}X + \frac{1}{2}m\omega^2 x^2\right] + \left[-\frac{\hbar^2}{2m}\frac{1}{Y}\frac{\partial^2}{\partial y^2}Y\right] = E. \tag{1.5}$$

We now have a separation constant, $E = E_x + E_y$.

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}X + \frac{1}{2}m\omega^2 x^2 X = E_x X}, \tag{1.6}$$

$$\mathcal{H}_x X = E_x X, \tag{1.7}$$

where $\mathcal{H}_x = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2$.

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2}{dy^2}Y = E_y Y}, \tag{1.8}$$

$$\mathcal{H}_y Y = E_y Y, \tag{1.9}$$

where $\mathcal{H}_y = -\frac{\hbar^2}{2m}\frac{d^2}{dy^2}$.

(b) Find the normalized eigenfunctions and eigenenergies.

Solution The eigenfunctions X and Y correspond to the one dimensional harmonic oscillator and the free particle wave functions, respectively. Similarly, with the eigenenergies E_x and E_y. X is bounded in the x-direction, so we have to find the normalization constant in this case. Rearranging the full equation.

$$\frac{d^2X}{dx^2} + \left[\frac{2mE}{\hbar^2} - \left(\frac{m\omega}{\hbar}\right)^2 x^2\right] X = 0 \tag{1.10}$$

A helpful substitution is $\alpha = \sqrt{m\omega/\hbar}\, x$.

$$\frac{d}{dx} = \frac{d\alpha}{dx}\frac{d}{d\alpha} = \sqrt{\frac{m\omega}{\hbar}}\frac{d}{d\alpha} \tag{1.11}$$

$$\frac{dX}{dx} = \sqrt{\frac{m\omega}{\hbar}}\frac{dX}{d\alpha} \tag{1.12}$$

$$\frac{d}{dx}\left(\frac{dX}{dx}\right) = \sqrt{\frac{m\omega}{\hbar}}\frac{d}{dx}\left(\frac{dX}{d\alpha}\right) = \frac{m\omega}{\hbar}\frac{d^2X}{d\alpha^2} \tag{1.13}$$

$$\frac{m\omega}{\hbar}\frac{d^2X}{d\alpha^2} + \left[\frac{2mE_x}{\hbar^2} - \frac{m\omega}{\hbar}\alpha^2\right] X = 0 \tag{1.14}$$

$$\frac{d^2X}{d\alpha^2} + \left[\frac{2E_x}{\hbar\omega} - \alpha^2\right] X = 0 \tag{1.15}$$

We attempt a Gaussian type solution of the form: $X(\alpha) = f(\alpha)e^{-\alpha^2/2}$.

$$\frac{dX}{d\alpha} = -\alpha e^{-\alpha^2/2} f(\alpha) + e^{-\alpha^2/2} f'(\alpha) \tag{1.16}$$

$$= (f' - \alpha f)e^{-\alpha^2/2} \tag{1.17}$$

$$\frac{d^2X}{d\alpha^2} = (f'' - f - \alpha f')e^{-\alpha^2/2} + (f' - \alpha f)(-\alpha e^{-\alpha^2/2}) \tag{1.18}$$

$$= \left[f'' - 2\alpha f' + (\alpha^2 - 1)f\right] e^{-\alpha^2/2} \tag{1.19}$$

$$\left[f'' - 2\alpha f' + (\alpha^2 - 1)f\right] e^{-\alpha^2/2} + \frac{2E_x}{\hbar\omega} f e^{-\alpha^2/2} - \alpha^2 f e^{-\alpha^2/2} = 0 \tag{1.20}$$

$$f'' - 2\alpha f' + \left(\frac{2E_x}{\hbar\omega} - 1\right) f = 0 \tag{1.21}$$

This is Hermite's differential equation, $y'' - 2xy' + 2\lambda y = 0$, where λ is typically a non-negative integer. Now we can find the eigenenergies.

$$\frac{2E_x}{\hbar\omega} - 1 = 2n_x \tag{1.22}$$

$$\boxed{E_x = \hbar\omega\left(n_x + \frac{1}{2}\right)} \tag{1.23}$$

For the eigenfunction, since we have found the system to follow Hermite's differential equation, $f(\alpha)$ takes the form $H_n(\alpha)$, with it already being quantized.

$$X_n(\alpha) = e^{-\alpha^2/2} f_n(\alpha) = e^{-\alpha^2/2} H_n(\alpha) \tag{1.24}$$

Tracing back to $\alpha = \sqrt{m\omega/\hbar}\, x$, let $x_0 = \sqrt{\hbar/m\omega}$, so $\alpha = x/x_0$.

$$X_n(x) = e^{-x^2/2x_0^2} H_n(x/x_0) \tag{1.25}$$

We have to normalize the wave function within the bounds, so introduce an overall multiplicative constant N associated with $X_n(x)$.

$$1 = \int_0^a |N|^2 \left[e^{-x^2/2x_0^2}\right]^2 \left[H_n(x/x_0)\right]^2\, dx, \qquad \begin{cases} u &= x/x_0 \\ du &= dx/x_0 \end{cases} \tag{1.26}$$

$$1 = |N|^2 x_0 \int_0^{a/x_0} e^{-u^2} \left[H_n(u)\right]^2\, du \tag{1.27}$$

The orthonormality condition of Hermite polynomials over $(-\infty, \infty)$ does not directly translate to normalization over finite intervals, so numerical methods must be used. Let's examine the ground state $X_0(x)$.

$$X_0(x) = Ne^{-x^2/2x_0^2} H_0(x/x_0) = Ne^{-x^2/2x_0^2} \tag{1.28}$$

$$|X_0(x)|^2 = \int_0^a |N|^2 e^{-x^2/x_0^2}\, dx = 1, \qquad \begin{cases} u &= x/x_0 \\ du &= dx/x_0 \end{cases} \tag{1.29}$$

$$1 = |N|^2 x_0 \int_0^{a/x_0} e^{-u^2}\, du \tag{1.30}$$

This integral is the error function $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt$.

$$1 = |N|^2 x_0 \frac{\sqrt{\pi}}{2} \mathrm{erf}(a/x_0) \tag{1.31}$$

$$N = \sqrt{\frac{2}{\sqrt{\pi} x_0 \,\mathrm{erf}(a/x_0)}} \tag{1.32}$$

$$X_0(x) = \sqrt{\frac{2}{\sqrt{\pi} x_0 \,\mathrm{erf}(a/x_0)}} e^{-x^2/2x_0^2} \tag{1.33}$$

This normalization constant is specific to the ground state wave function. To find the wave function of any excited state we apply the creation operator $a^\dagger$ on the ground state n times.

$$X_n(x) = \langle x|n\rangle = \frac{1}{\sqrt{n!}} \langle x| \left(a^\dagger\right)^n |0\rangle = \frac{1}{\sqrt{n!}} \left(\frac{1}{\sqrt{2}x_0}\right)^n \left(x - x_0^2 \frac{d}{dx}\right)^n X_0(x) \tag{1.34}$$

$$X_n(x) = \frac{1}{\sqrt{n!}} \left(\frac{1}{\sqrt{2}x_0}\right)^n \left(x - x_0^2 \frac{d}{dx}\right)^n \sqrt{\frac{2}{\sqrt{\pi} x_0 \,\mathrm{erf}(a/x_0)}} e^{-x^2/2x_0^2} \tag{1.35}$$

$$= \sqrt{\frac{2}{\sqrt{\pi} 2^n n! \,\mathrm{erf}(a/x_0)}} \frac{1}{x_0^{n+1/2}} \left(x - x_0^2 \frac{d}{dx}\right)^n e^{-x^2/2x_0^2} \tag{1.36}$$

$$\boxed{X_n(x) = \sqrt{\frac{2}{\sqrt{\pi} 2^n n! x_0 \,\mathrm{erf}(a/x_0)}} e^{-x^2/2x_0^2} H_n\left(\frac{x}{x_0}\right)} \tag{1.37}$$

Now for Y and E_y, a free particle as it is unbounded in the y-direction.

$$\frac{d^2 Y}{dy^2} + \frac{2mE}{\hbar^2} Y = 0 \tag{1.38}$$

$$\frac{d^2 Y}{dy^2} + k^2 Y = 0 \tag{1.39}$$

$$Y(y) = Ae^{iky} + Be^{-iky} \tag{1.40}$$

$$\boxed{E_y = \frac{\hbar^2 k^2}{2m}} \tag{1.41}$$

To normalize the free particle we consider a single plane wave, instead of a superposition, for simplicity. This forces the momentum to travel in one direction. In this case, we use delta function normalization.

$$\int_{-\infty}^{\infty} Y(y) Y^\star(y')\, dy = \delta(y - y') \tag{1.42}$$

$$\int_{-\infty}^{\infty} Ae^{iky} A^\star e^{-iky'}\, dy = \delta(y - y') \tag{1.43}$$

$$|A|^2 \int_{-\infty}^{\infty} e^{ik(y-y')}\, dy = \delta(y - y') \tag{1.44}$$

Using the Fourier transform of $\delta(x)$ in Appendix A.

$$|A|^2 \left[2\pi\delta(y - y')\right] = \delta(y - y') \tag{1.45}$$

$$A = \frac{1}{\sqrt{2\pi}} \tag{1.46}$$

$$\boxed{Y(y) = \frac{1}{\sqrt{2\pi}} e^{iky}} \tag{1.47}$$

2 Problem 2: Exercise 6.8

Consider a muonic atom which consists of a nucleus that has Z protons (no neutrons) and a negative muon moving around it; the muon's charge is $-e$ and its mass is 207 times the mass of the electron, $m_{\mu^-} = 207 m_e$. For a muonic atom with $Z = 6$, calculate

(a) the radius of the first Bohr orbit,

Solution Starting with a Hydrogen atom, we can easily derive the radius of the Bohr orbits with two assumptions: the electron orbits the nucles, the angular momentum of the electron is quantized.

$$m_e \frac{v^2}{r} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2} \tag{2.1}$$

$$r = \frac{1}{4\pi\epsilon_0}\frac{e^2}{m_e}\frac{m_e^2 r^2}{n^2\hbar^2} \tag{2.2}$$

$$r_n = \left(\frac{4\pi\epsilon_0\hbar^2}{m_e e^2}\right) n^2 = a_0 n^2 \tag{2.3}$$

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2} \approx 0.529\,18\,\text{Å} \tag{2.4}$$

$$L = m_e v r = n\hbar \tag{2.5}$$

$$m_e v_n^2 = \frac{n^2\hbar^2}{m_e r_n^2} \tag{2.6}$$

$$m_e v_n^2 = \frac{n^2\hbar^2}{m_e}\left(\frac{m_e e^2}{4\pi\epsilon_0\hbar^2}\right)^2 \frac{1}{n^4} \tag{2.7}$$

$$m_e v_n^2 = \frac{m_e}{\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{1}{n^2} \tag{2.8}$$

$$\begin{aligned} E &= K + U && (2.9)\\ &= \frac{1}{2} m_e v_n^2 - \frac{1}{4\pi\epsilon_0}\frac{e^2}{r_n} && (2.10)\\ &= \frac{1}{2}\frac{m_e}{\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\frac{1}{n^2} - \frac{1}{4\pi\epsilon_0}e^2\left(\frac{m_e e^2}{4\pi\epsilon_0\hbar^2}\right)\frac{1}{n^2} && (2.11)\\ &= \frac{1}{2}\frac{m_e}{\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\frac{1}{n^2} - \frac{m_e}{\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\frac{1}{n^2} && (2.12)\\ E_n &= -\frac{m_e}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\frac{1}{n^2} = -\frac{\mathcal{R}}{n^2} \approx -\frac{13.606\,\text{eV}}{n^2} && (2.13) \end{aligned}$$

where $\mathcal{R}$ is the Rydberg constant:

$$\mathcal{R} = \frac{m_e}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \approx 13.606\,\text{eV} \tag{2.14}$$

For a hydrogen-like atom, we introduce two assumptions: include the mass of the protons M with the reduced mass μ, replace e^2 by Ze^2 where Z is the number of protons.

$$\mu = \frac{M m_e}{M + m_e} = \frac{m_e}{1 + m_e/M} \tag{2.15}$$

$$r_n = \left(\frac{4\pi\epsilon_0\hbar^2}{\mu Z e^2}\right) n^2 = (1 + m_e/M)\frac{a_0}{Z} n^2 \tag{2.16}$$

$$E_n = -\frac{\mu}{2\hbar^2}\left(\frac{Ze^2}{4\pi\epsilon_0}\right)^2\frac{1}{n^2} = -\frac{Z^2}{1 + m_e/M}\frac{\mathcal{R}}{n^2} \tag{2.17}$$

In this case, we have $Z = 6$ and a muon instead of an electron where $m_{\mu^-} = 207 m_e$. Therefore the reduced mass μ becomes

$$\mu = 1 + \frac{207 m_e}{6 m_p} = 1 + \frac{207 \times 9.1094 \times 10^{-31}\,\text{kg}}{6 \times 1.6726 \times 10^{-27}\,\text{kg}} = 1.0188. \tag{2.18}$$

This is not a negligible ratio so we have to keep it in the calculation.

$$r_1 = 1.0188 \times \frac{0.529\,18\,\text{Å}}{6} = \boxed{0.089\,855\,\text{Å}} \tag{2.19}$$

(b) the energy of the ground, first, and second excited states, and

Solution

$$E_1 = -\frac{6^2}{1.0188} \times \frac{13.606\,\mathrm{eV}}{1^2} = \boxed{-480.78\,\mathrm{eV}} \tag{2.20}$$

$$E_2 = -\frac{6^2}{1.0188} \times \frac{13.606\,\mathrm{eV}}{2^2} = \boxed{-120.19\,\mathrm{eV}} \tag{2.21}$$

$$E_3 = -\frac{6^2}{1.0188} \times \frac{13.606\,\mathrm{eV}}{3^2} = \boxed{-53.420\,\mathrm{eV}} \tag{2.22}$$

$$\tag{2.23}$$

(c) the frequency associated with the transitions $n_i = 2 \to n_f = 1$, $n_i = 3 \to n_f = 1$, and $n_i = 3 \to n_f = 2$.

Solution $hf = \Delta E = E_f - E_i$.

$$f_{2\to1} = \frac{E_2 - E_1}{h} = \frac{-120.19\,\mathrm{eV} - (-480.78\,\mathrm{eV})}{4.1357 \times 10^{-15}\,\mathrm{eV\,Hz}^{-1}} = \boxed{87.190 \times 10^{15}\,\mathrm{Hz}} \tag{2.24}$$

$$f_{3\to1} = \frac{E_3 - E_1}{h} = \frac{-53.420\,\mathrm{eV} - (-480.78\,\mathrm{eV})}{4.1357 \times 10^{-15}\,\mathrm{eV\,Hz}^{-1}} = \boxed{103.33 \times 10^{15}\,\mathrm{Hz}} \tag{2.25}$$

$$f_{3\to2} = \frac{E_3 - E_2}{h} = \frac{-53.420\,\mathrm{eV} - (-120.19\,\mathrm{eV})}{4.1357 \times 10^{-15}\,\mathrm{eV\,Hz}^{-1}} = \boxed{16.145 \times 10^{15}\,\mathrm{Hz}} \tag{2.26}$$

3 Problem 3: Exercise 6.13

The wave function of an electron in a hydrogen atom is given by

$$\psi_{2,1,m_l,m_s}(r,\theta,\varphi) = R_{2,1}\left[\frac{1}{\sqrt{3}}Y_{1,0}(\theta,\varphi)\left|\frac{1}{2},\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}Y_{1,1}(\theta,\varphi)\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right], \tag{3.1}$$

where $\left|\frac{1}{2},\pm\frac{1}{2}\right\rangle$ are the spin state vectors.

(a) Is this wave function an eigenfunction of $\hat{J}_z$ the z-component of the electron's total angular momentum? If yes, find the eigenvalue. (*Hint*: For this, you need to calculate $\hat{J}_z\psi_{2,1,m_l,m_s}$)

Solution The total angular momentum $\hat{J}$ is the sum of the orbital angular momentum $\hat{L}$ and the spin angular momentum $\hat{S}$, so $\hat{J} = \hat{L}+\hat{S}$. Therefore the z-component, $\hat{J}_z = \hat{L}_z+\hat{S}_z$. For ease of computation, Let α and β correspond to the two terms inside the square bracket, where $|2,1,m_l,m_s\rangle = R_{2,1}[\alpha+\beta]$

$$\begin{aligned}\hat{J}_z|2,1,m_l,m_s\rangle &= \hat{L}_z|2,1,m_l,m_s\rangle + \hat{S}_z|2,1,m_l,m_s\rangle && (3.2)\\ &= m_l\hbar|2,1,m_l,m_s\rangle + m_s\hbar|2,1,m_l,m_s\rangle && (3.3)\end{aligned}$$

$$\begin{aligned}\hat{L}_z|2,1,m_l,m_s\rangle &= R_{2,1}[(0\times\hbar)\alpha + (1\times\hbar)\beta] && (3.4)\\ &= \hbar R_{2,1}\beta && (3.5)\end{aligned}$$

$$\begin{aligned}\hat{S}_z|2,1,m_l,m_s\rangle &= R_{2,1}\left[(\tfrac{1}{2}\times\hbar)\alpha + (-\tfrac{1}{2}\times\hbar)\beta\right] && (3.6)\\ &= \frac{\hbar}{2}R_{2,1}[\alpha-\beta] && (3.7)\end{aligned}$$

$$\begin{aligned}\hat{J}_z|2,1,m_l,m_s\rangle &= \hbar R_{2,1}\beta + \frac{\hbar}{2}R_{2,1}[\alpha-\beta] && (3.8)\\ &= \frac{\hbar}{2}R_{2,1}[\alpha+\beta] && (3.9)\\ &= \boxed{\frac{\hbar}{2}|2,1,m_l,m_s\rangle} && (3.10)\end{aligned}$$

(b) If you measure the z-component of the electron's spin angular momentum, what values will you obtain? What are the corresponding probabilities?

Solution From the previous part, we know that the values we obtain are $\hbar/2$ or $-\hbar/2$, with probabilities $P_{\hbar/2}$ and $P_{-\hbar/2}$.

$$P_{\hbar/2} = \left|\left\langle\tfrac{1}{2},\tfrac{1}{2}\middle|\psi\right\rangle\right|^2 = \boxed{\frac{1}{3}} \tag{3.11}$$

$$P_{\hbar/2} = \left|\left\langle\tfrac{1}{2},-\tfrac{1}{2}\middle|\psi\right\rangle\right|^2 = \boxed{\frac{2}{3}} \tag{3.12}$$

(c) If you measure $\hat{J}_z^2$, what values will you obtain? What are the corresponding probabilities?

Solution $\hat{J}_z^2 = (\hat{L}_z+\hat{S}_z)^2 = \hat{L}_z^2 + \hat{L}_z\hat{S}_z + \hat{S}_z\hat{L}_z + \hat{S}_z^2$

$$\begin{aligned}\hat{L}_z^2|2,1,m_l,m_s\rangle &= \hat{L}_z[\hbar R_{2,1}\beta] = \hbar^2R_{2,1}\beta && (3.13)\\ \hat{L}_z\hat{S}_z|2,1,m_l,m_s\rangle &= \hat{L}_z\left[\tfrac{\hbar}{2}R_{2,1}(\alpha-\beta)\right] = -\tfrac{\hbar}{2}R_{2,1}\beta && (3.14)\\ \hat{S}_z\hat{L}_z|2,1,m_l,m_s\rangle &= \hat{S}_z[\hbar R_{2,1}\beta] = -\tfrac{\hbar}{2}R_{2,1}\beta && (3.15)\\ \hat{S}_z^2|2,1,m_l,m_s\rangle &= \hat{S}_z\left[\frac{\hbar}{2}R_{2,1}[\alpha-\beta]\right] = \frac{\hbar^2}{4}R_{2,1}[\alpha+\beta] && (3.16)\end{aligned}$$

$$\begin{aligned}\hat{J}_z^2|2,1,m_l,m_s\rangle &= \hbar^2R_{2,1}\beta - \tfrac{\hbar}{2}R_{2,1}\beta - \tfrac{\hbar}{2}R_{2,1}\beta + \frac{\hbar^2}{4}R_{2,1}[\alpha+\beta] && (3.17)\\ &= \frac{\hbar^2}{4}|2,1,m_l,m_s\rangle && (3.18)\end{aligned}$$

Only one value, $\boxed{\hbar^2/4 \text{ with 100\% probability}}$.

4 Problem 4: Exercise 6.20

The wave function of a hydrogen-like atom at time $t = 0$ is

$$\Psi(\vec{r}, 0) = \frac{1}{\sqrt{11}} \left[\sqrt{3}\psi_{2,1,-1}(\vec{r}) - \psi_{2,1,0}(\vec{r}) + \sqrt{5}\psi_{2,1,1}(\vec{r}) + \sqrt{2}\psi_{3,1,1}(\vec{r}) \right], \tag{4.1}$$

where $\psi_{nlm}(\vec{r})$ is a normalized eigenfunction (i.e. $\psi_{nlm}(\vec{r}) = R_{nl}Y_{lm}(\theta, \varphi)$).

(a) What is the time-dependent wave function?

Solution The wave function at time t is described by $\Psi(\vec{r}, t) = \psi(\vec{r}) \exp\left(-\frac{i}{\hbar}E_n t\right)$. The energy of the hydrogen atom only depends on the principal quantum number n. Therefore, I expect this atom to have the sum of two energies, as we have $n = 2, 3$.

$$\Psi(\vec{r}, t) = \frac{1}{\sqrt{11}} \left[e^{-iE_2 t/\hbar} \left(\sqrt{3}\psi_{2,1,-1}(\vec{r}) - \psi_{2,1,0}(\vec{r}) + \sqrt{5}\psi_{2,1,1}(\vec{r}) \right) + e^{-iE_3 t/\hbar} \left(\sqrt{2}\psi_{3,1,1}(\vec{r}) \right) \right], \tag{4.2}$$

(b) If a measurement of energy is made, what values could be found and with what probabilities?

Solution By inspection, two values for energy are found, E_2 and E_3. With probabilities $P_{n,l,m}$.

$$P_{2,1,-1} = |\langle 2, 1, -1|\Psi\rangle|^2 = \left| \sqrt{\frac{3}{11}} e^{-iE_2 t/\hbar} \langle 2, 1, -1|2, 1, -1\rangle \right|^2 = \frac{3}{11} \tag{4.3}$$

Using the same logic with the rest.

$$P_{2,1,0} = |\langle 2, 1, 0|\Psi\rangle|^2 = \frac{1}{11} \tag{4.4}$$

$$P_{2,1,1} = |\langle 2, 1, 1|\Psi\rangle|^2 = \frac{5}{11} \tag{4.5}$$

$$P_{3,1,1} = |\langle 3, 1, 1|\Psi\rangle|^2 = \frac{2}{11} \tag{4.6}$$

$$\tag{4.7}$$

Therefore, $\boxed{\text{the probability } E_2 \text{ is measured equals } 9/11. \text{ The probability } E_3 \text{ is measured equals } 2/11}$.

(c) What is the probability for a measurement of $\hat{L}_z$ which yields $-\hbar$?

Solution

$$\hat{L}_z\Psi(\vec{r}, 0) = \frac{1}{\sqrt{11}} \left[\sqrt{3}(-1 \times \hbar)\psi_{2,1,-1}(\vec{r}) - (0 \times \hbar)\psi_{2,1,0}(\vec{r}) + \sqrt{5}(1 \times \hbar)\psi_{2,1,1}(\vec{r}) + \sqrt{2}(1 \times \hbar)\psi_{3,1,1}(\vec{r}) \right] \tag{4.8}$$

$$P_{-\hbar} = |\langle 2, 1, -1|\Psi\rangle|^2 = \boxed{\frac{3}{11}} \tag{4.9}$$