PHY 310 — Mathematical Methods in Physics

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Problem 1: Classical Mechanics

Bead Sliding on the External Surface of a Rough Sphere. Use of Acceleration Vector in Plane Polar Coordinates

A bead is placed at the highest of a fixed sphere of radius $\boldsymbol{R}$. The bead is then given a tiny kick, so that it essentially starts from rest and moves on the external surface of the sphere. The coefficient of kinetic friction between the bead and the sphere is $\boldsymbol{\mu_k}$. The angular position of the bead at time $\boldsymbol{t}$ is $\boldsymbol{\phi}$.

(a) Show that

$$\boldsymbol{\dot{\phi}^2 = \frac{2g}{(4\mu_k^2+1)R}\left\{(2\mu_k^2-1)\left[\sin\phi - e^{\mu_k(\pi-2\phi)}\right] - 3\mu_k\cos\phi\right\}}$$

Solution Plane polar coordinates are used as there is no variation in the azimuthal angle, the ball does not deviate from a set longitude at the start because of azimuthal symmetry. Therefore, if we start from spherical coordinates, the conditions force it to mirror a polar coordinate system. Starting by finding the radial and tangential components of the force acting on the ball.

$$\overrightarrow{a} = \left(\ddot{\rho} - \rho\dot{\phi}^2\right)\hat{\rho} + \left(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}\right)\hat{\phi}$$

$$\rho = R \quad \dot{\rho} = 0 \quad \ddot{\rho} = 0$$

$$\text{Radial Direction} \quad \hat{\rho}: \quad m\left(-R\dot{\phi}^2\right) = -mg\sin\phi + N$$

$$\text{Tangential Direction} \quad \hat{\phi}: \quad m\left(R\ddot{\phi}\right) = \mu N - mg\cos\phi$$

$$-mR\dot{\phi}^2 - \frac{mR\ddot{\phi}}{\mu_k} = -mg\sin\phi + N - N + \frac{mg\cos\phi}{\mu_k}$$

$$-mR\mu_k\dot{\phi}^2 - mR\ddot{\phi} = -\mu_k mg\sin\phi + mg\cos\phi$$

$$R\ddot{\phi} + R\mu_k\dot{\phi}^2 = \mu_k g\sin\phi - g\cos\phi$$

$$\ddot{\phi} + \mu_k\dot{\phi}^2 = \frac{g}{R}\left(\mu_k\sin\phi - \cos\phi\right)$$

This is a second order, non-linear, nonhomogeneous differential equation. One way to solve these is to reduce the DE down to a first order by a convenient substitution. Let $z = \dot{\phi}^2$

$$\frac{dz}{d\phi} = \frac{dz}{dt}\frac{dt}{d\phi} = \left(2\dot{\phi}\frac{d\dot{\phi}}{dt}\right)\left(\frac{1}{\dot{\phi}}\right) = 2\ddot{\phi} \quad \Rightarrow \quad \ddot{\phi} = \frac{1}{2}\frac{dz}{d\phi}$$

$$\frac{1}{2}\frac{dz}{d\phi} + \mu_k z = \frac{g}{R}\left(\mu_k\sin\phi - \cos\phi\right)$$

$$\frac{dz}{d\phi} + 2\mu_k z = \frac{2g}{R}\left(\mu_k\sin\phi - \cos\phi\right)$$

Using the integrating factor method to find the solution. $e^{\int 2\mu_k d\phi} = e^{2\mu_k\phi}$

$$e^{2\mu_k\phi}\frac{dz}{d\phi} + 2\mu_k e^{2\mu_k\phi}z = \frac{2g}{R}e^{2\mu_k\phi}\left(\mu_k\sin\phi - \cos\phi\right)$$

$$\frac{d}{d\phi}\left[ze^{2\mu_k\phi}\right] = \frac{2g}{R}e^{2\mu_k\phi}\left(\mu_k\sin\phi - \cos\phi\right)$$

$$ze^{2\mu_k\phi} = \frac{2g}{R}\left[\mu_k\int e^{2\mu_k\phi}\sin\phi\,d\phi - \int e^{2\mu_k\phi}\cos\phi\,d\phi\right]$$

Using the following general formulas for integrals of products of trigonometric functions and exponential.

$$\int e^{bx}\sin ax\,dx = \frac{e^{bx}}{a^2+b^2}\left(b\sin ax - a\cos ax\right) \qquad \int e^{bx}\cos ax\,dx = \frac{e^{bx}}{a^2+b^2}\left(a\sin ax + b\cos ax\right)$$

$$ze^{2\mu_k\phi} = \frac{2g}{R}\left[\mu_k\left(\frac{e^{2\mu_k\phi}}{1^2+4\mu_k^2}\left(2\mu_k\sin\phi-\cos\phi\right)\right)-\left(\frac{e^{2\mu_k\phi}}{1^2+4\mu_k^2}\left(\sin\phi+2\mu_k\cos\phi\right)\right)\right]+C$$

$$ze^{2\mu_k\phi} = \frac{2g}{R}\left[\frac{\mu e^{2\mu_k\phi}}{4\mu_k^2+1}\left(2\mu_k\sin\phi-\cos\phi\right)-\frac{e^{2\mu_k\phi}}{4\mu_k^2+1}\left(\sin\phi+2\mu_k\cos\phi\right)\right]+C$$

$$\dot{\phi}^2 = \frac{2g}{R\left(4\mu_k^2+1\right)}\left[2\mu_k^2\sin\phi-\mu_k\cos\phi-\sin\phi-2\mu_k\cos\phi\right]+Ce^{-2\mu_k\phi}$$

$$\dot{\phi}^2 = \frac{2g}{R\left(4\mu_k^2+1\right)}\left[\left(2\mu_k^2-1\right)\sin\phi-3\mu_k\cos\phi\right]+Ce^{-2\mu_k\phi}$$

Applying initial conditions at the top of the sphere to find C: $\dot{\phi}=0$, $\phi=\frac{\pi}{2}$.

$$0 = \frac{2g}{R\left(4\mu_k^2+1\right)}\left[\left(2\mu_k^2-1\right)\right]+Ce^{-\mu_k\pi}$$

$$C = -\frac{2g\left(2\mu_k^2-1\right)}{R\left(4\mu_k^2+1\right)}e^{\mu_k\pi}$$

$$\dot{\phi}^2 = \frac{2g}{R\left(4\mu_k^2+1\right)}\left[\left(2\mu_k^2-1\right)\sin\phi-3\mu_k\cos\phi\right]-\frac{2g\left(2\mu_k^2-1\right)}{R\left(4\mu_k^2+1\right)}e^{\mu_k\pi}e^{-2\mu_k\phi}$$

$$\dot{\phi}^2 = \frac{2g}{R\left(4\mu_k^2+1\right)}\left[\left(2\mu_k^2-1\right)\sin\phi-3\mu_k\cos\phi-\left(2\mu_k^2-1\right)e^{\mu_k(\pi-2\phi)}\right]$$

$$\dot{\phi}^2 = \frac{2g}{R\left(4\mu_k^2+1\right)}\left[\left(2\mu_k^2-1\right)\left(\sin\phi-e^{\mu_k(\pi-2\phi)}\right)-3\mu_k\cos\phi\right]$$

(b) i. Show that the bead will leave the surface of the sphere, when $\boldsymbol{\phi=\alpha}$, where $\boldsymbol{\alpha}$ is determined from

$$\mathbf{3(\sin\alpha+2\mu_k\cos\alpha)=2(1-2\mu_k^2)e^{\mu_k(\pi-2\alpha)}.}$$

Solution When the bead leaves the surface of the sphere at $\boldsymbol{\phi=\alpha}$, it is no longer in contact and so the normal force $\boldsymbol{N=0}$.

$$-mR\dot{\phi}^2 = -mg\sin\alpha$$

$$\dot{\phi}^2 = \frac{g}{R}\sin\alpha$$

$$\frac{g}{R}\sin\alpha = \frac{2g}{R\left(4\mu_k^2+1\right)}\left[\left(2\mu_k^2-1\right)\left(\sin\alpha-e^{\mu_k(\pi-2\alpha)}\right)-3\mu_k\cos\alpha\right]$$

$$\sin\alpha = \frac{2}{4\mu_k^2+1}\left[\left(2\mu_k^2-1\right)\sin\alpha-\left(2\mu_k^2-1\right)e^{\mu_k(\pi-2\alpha)}-3\mu_k\cos\alpha\right]$$

$$\left(4\mu_k^2+1\right)\sin\alpha = 2\left(2\mu_k^2-1\right)\sin\alpha-2\left(2\mu_k^2-1\right)e^{\mu_k(\pi-2\alpha)}-6\mu_k\cos\alpha$$

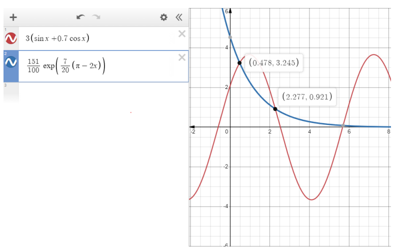
$$3\sin\alpha = -2\left(2\mu_k^2-1\right)e^{\mu_k(\pi-2\alpha)}-6\mu_k\cos\alpha$$

$$3(\sin\alpha+2\mu_k\cos\alpha) = 2(1-2\mu_k^2)e^{\mu_k(\pi-2\alpha)}$$

ii. Take $\boldsymbol{\mu_k=0.35}$. Solve the transcendental equation for $\boldsymbol{\alpha}$ in part (b)i.

Solution Using desmos.com/calculator to plot the graph.

$$3(\sin\alpha+0.7\cos\alpha) = \frac{151}{100}e^{\frac{7}{20}(\pi-2\alpha)}$$



iii. What is $\boldsymbol{\alpha}$ for a perfectly smooth sphere?

Solution For a perfectly smooth sphere, $\boldsymbol{\mu_k=0}$.

$$3\sin\alpha = 2$$

$$\alpha = \arcsin\left(\frac{3}{2}\right)\approx 41.8^\circ$$

Problem 3: Electromagnetism

Derivation of Continuity Equation and Wave Equations for the Electric & Magnetic Field from Maxwell's Equation

(a) Using only Kronecker delta $\boldsymbol{\delta_{ij}}$ and Levi-Civita tensor $\boldsymbol{\epsilon_{ijk}}$, show that

i. $\vec{\boldsymbol{\nabla}} \cdot \left(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{V}}\right) = \mathbf{0}$

Solution Using the property that the Levi-Civita Tensor is anti symmetric and the symmetry of partial derivatives. $\boldsymbol{\epsilon_{ijk} = -\epsilon_{jik}}$ and $\boldsymbol{\partial_i \partial_j = \partial_j \partial_i}$

$$\begin{aligned}
\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{V}\right) &= \vec{\nabla}_i \left(\vec{\nabla} \times \vec{V}\right)_i = \partial_i \epsilon_{ijk} \partial_j A_k \\
&= \epsilon_{ijk} \partial_i \partial_j A_k \\
&= \frac{1}{2} \epsilon_{ijk} \partial_i \partial_j A_k + \frac{1}{2} \epsilon_{ijk} \partial_i \partial_j A_k \\
&= \frac{1}{2} \epsilon_{ijk} \partial_i \partial_j A_k + \frac{1}{2} \epsilon_{jik} \partial_j \partial_i A_k \\
&= \frac{1}{2} \epsilon_{ijk} \partial_i \partial_j A_k - \frac{1}{2} \epsilon_{ijk} \partial_i \partial_j A_k \\
&= 0
\end{aligned}$$

ii. $\vec{\boldsymbol{\nabla}} \times \left(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{V}}\right) = \vec{\boldsymbol{\nabla}} \left(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{V}}\right) - \boldsymbol{\nabla^2} \vec{\boldsymbol{V}}$

Solution

$$\begin{aligned}
\left[\vec{\nabla} \times \left(\vec{\nabla} \times \vec{V}\right)\right]_i &= \epsilon_{ijk} \partial_j \left(\vec{\nabla} \times \vec{V}\right)_k = \epsilon_{ijk} \partial_j \epsilon_{klm} \partial_l V_m \\
&= \epsilon_{ijk} \epsilon_{klm} \partial_j \partial_l V_m = \left(\delta il \delta_{jm} - \delta_{im} \delta_{jl}\right) \partial_j \partial_l V_m \\
&= \partial_i \partial_j V_j - \partial_j \partial_j V_i = \nabla_i \left(\vec{\nabla} \cdot \vec{V}\right) - \vec{\nabla} \cdot \vec{\nabla} V_i \\
&= \vec{\nabla} \left(\vec{\nabla} \cdot \vec{V}\right) - \nabla^2 \vec{V}
\end{aligned}$$

(b) Maxwell's equations describe the spatial $\left(\vec{\boldsymbol{r}}\right)$ and temporal $\boldsymbol{(t)}$ variation of electric field $\vec{\boldsymbol{E}}\left(\vec{\boldsymbol{r}}, \boldsymbol{t}\right)$ and magnetic field $\vec{\boldsymbol{B}}\left(\vec{\boldsymbol{r}}, \boldsymbol{t}\right)$ arising from the electric charge $\boldsymbol{\rho}\left(\vec{\boldsymbol{r}}, \boldsymbol{t}\right)$ and current $\vec{\boldsymbol{J}}\left(\vec{\boldsymbol{r}}, \boldsymbol{t}\right)$ distributions. They take the form

$$\begin{aligned}
\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}} &= \frac{\boldsymbol{4\pi\rho}}{\boldsymbol{\epsilon}}, \\
\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{B}} &= \mathbf{0}, \\
\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{E}} &= -\frac{\mathbf{1}}{\boldsymbol{c}} \frac{\boldsymbol{\partial} \vec{\boldsymbol{B}}}{\boldsymbol{\partial t}}, \\
\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}} &= \frac{\boldsymbol{4\pi\mu}}{\boldsymbol{c}} \vec{\boldsymbol{J}} + \frac{\boldsymbol{\mu\epsilon}}{\boldsymbol{c}} \frac{\boldsymbol{\partial} \vec{\boldsymbol{E}}}{\boldsymbol{\partial t}},
\end{aligned}$$

where $\boldsymbol{\epsilon}$, $\boldsymbol{\mu}$ are dielectric and permittivity constants of the medium, while $\boldsymbol{c}$ is the speed of light.

i. Show that the 'continuity equation' $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{J}} + \frac{\boldsymbol{\partial\rho}}{\boldsymbol{\partial t}} = \mathbf{0}$ follows directly from Maxwell's equations.

Solution

$$\begin{aligned}
\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{B}\right) &= \frac{4\pi\mu}{c} \vec{\nabla} \cdot \vec{J} + \frac{\mu\epsilon}{c} \frac{\partial}{\partial t} \left(\vec{\nabla} \cdot \vec{E}\right) \\
0 &= \frac{4\pi\mu}{c} \vec{\nabla} \cdot \vec{J} + \frac{\mu\epsilon}{c} \frac{\partial}{\partial t} \left(\frac{4\pi\rho}{\epsilon}\right) \\
0 &= \frac{4\pi\mu}{c} \vec{\nabla} \cdot \vec{J} + \frac{4\pi\mu}{c} \frac{\partial\rho}{\partial t} \\
\vec{\nabla} \cdot \vec{J} + \frac{\partial\rho}{\partial t} &= 0
\end{aligned}$$

ii. Derive the wave equation for the electric & magnetic fields:

$$\boldsymbol{\nabla^2 \vec{E} - \frac{\mu\epsilon}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \frac{4\pi\mu}{c^2}\frac{\partial \vec{J}}{\partial t} + \frac{4\pi}{\epsilon}\vec{\nabla}\rho, \qquad \nabla^2 \vec{B} - \frac{\mu\epsilon}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} = -\frac{4\pi\mu}{c}\vec{\nabla} \times J.}$$

Solution

$$\begin{aligned}
\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) &= -\frac{1}{c}\frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{B}\right) \\
\vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) - \nabla^2 \vec{E} &= -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{4\pi\mu}{c}\vec{J} + \frac{\mu\epsilon}{c}\frac{\partial \vec{E}}{\partial t}\right) \\
\nabla^2 \vec{E} - \vec{\nabla}\left(\frac{4\pi\rho}{\epsilon}\right) &= \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{4\pi\mu}{c}\vec{J} + \frac{\mu\epsilon}{c}\frac{\partial \vec{E}}{\partial t}\right) \\
\nabla^2 \vec{E} - \frac{4\pi}{\epsilon}\vec{\nabla}\rho &= \frac{4\pi\mu}{c^2}\frac{\partial \vec{J}}{\partial t} + \frac{\mu\epsilon}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} \\
\nabla^2 \vec{E} - \frac{\mu\epsilon}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} &= \frac{4\pi\mu}{c^2}\frac{\partial \vec{J}}{\partial t} + \frac{4\pi}{\epsilon}\vec{\nabla}\rho
\end{aligned}$$

$$\begin{aligned}
\vec{\nabla} \times \left(\vec{\nabla} \times \vec{B}\right) &= \frac{4\pi\mu}{c}\left(\vec{\nabla} \times \vec{J}\right) + \frac{\mu\epsilon}{c}\frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{E}\right) \\
\vec{\nabla}\left(\vec{\nabla} \cdot \vec{B}\right) - \nabla^2 \vec{B} &= \frac{4\pi\mu}{c}\left(\vec{\nabla} \times \vec{J}\right) + \frac{\mu\epsilon}{c}\frac{\partial}{\partial t}\left(-\frac{1}{c}\frac{\partial \vec{B}}{\partial t}\right) \\
-\nabla^2 \vec{B} &= \frac{4\pi\mu}{c}\left(\vec{\nabla} \times \vec{J}\right) - \frac{\mu\epsilon}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} \\
\nabla^2 \vec{B} - \frac{\mu\epsilon}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} &= -\frac{4\pi\mu}{c}\vec{\nabla} \times J
\end{aligned}$$

Problem 6: Quantum Mechanics

Important Commutators of Quantum Vector Operators

(a) For any two vector operators $\vec{A}$ and $\vec{B}$ that do not commute, show that

i. $\vec{\mathbb{A}} \cdot \vec{\mathbb{B}} = \vec{\mathbb{B}} \cdot \vec{\mathbb{A}} + [\mathbb{A}_j, \mathbb{B}_i]$

Solution

$$\begin{aligned}
\vec{A} \cdot \vec{B} &= A_i B_i = A_i B_i - B_i A_i + B_i A_i = [A_i, B_i] + B_i A_i \\
&= \vec{A} \cdot \vec{B} + [A_i, B_i]
\end{aligned}$$

ii. $\left(\vec{\mathbb{A}} \times \vec{\mathbb{B}}\right)_i = -\left(\vec{\mathbb{B}} \times \vec{\mathbb{A}}\right)_i + \epsilon_{ijk}[\mathbb{A}_j, \mathbb{B}_k]$

Solution Adding $-B_k A_j + B_k A_j$ does nothing to the overall equation as it is essentially zero.

$$\begin{aligned}
\left(\vec{A} \times \vec{B}\right)_i &= \epsilon_{ijk} A_j B_k = \epsilon_{ijk}\left(A_j B_k - B_k A_j + B_k A_j\right) \\
&= \epsilon_{ijk}[A_j, B_k] + \epsilon_{ijk} B_k A_j \\
&= -\epsilon_{ikj} B_k A_j + \epsilon_{ijk}[A_j, B_k] \\
&= -\left(\vec{B} \times \vec{A}\right)_i + \epsilon_{ijk}[A_j, B_k]
\end{aligned}$$

(a) Define linear and angular momentum vector operators, respectively as $\vec{\mathbb{p}} \equiv -i\hbar\vec{\nabla}$ and $\vec{\mathbb{L}} = \vec{\mathbb{r}} \times \vec{\mathbb{p}}$. Show that

i. $[\mathbb{r}_i, \mathbb{p}_j] = i\hbar\delta_{ij}$,

Solution It is important to note that $\partial_i r_j = \delta_{ij}$ because the derivative of r_j is either **1** or **0** depending on the index of the partial derivative operator and r.

$$\begin{aligned}
[\mathbb{r}_i, \mathbb{p}_j]\,\psi(r) &= r_i p_j \psi - p_j r_i \psi = r_i\left(-i\hbar\partial_j\right)\psi - \left(-i\hbar\partial_j\right) r_i \psi = -i\hbar\left[r_i \partial_j \psi - \partial_j\left(r_i \psi\right)\right] \\
&= -i\hbar\left[r_i \partial_j \psi - r_i \partial_j \psi + \psi \partial_i r_j\right] = -i\hbar\left[\psi \partial_i r_j\right] \\
[\mathbb{r}_i, \mathbb{p}_j]\,\psi(r) &= -i\hbar\psi\delta_{ij} \\
[\mathbb{r}_i, \mathbb{p}_j] &= -i\hbar\delta_{ij}
\end{aligned}$$

ii. $[\mathbb{L}_i, \mathbb{p}_j] = i\hbar\epsilon_{ijk}\mathbb{p}_k$,

Solution A common trick is to flip the commutator to make use of the $[A, BC]$ formula. $[r_i, r_j] = 0$, $[p_i, p_j] = 0$.

$$\begin{aligned}
[\mathbb{L}_i, \mathbb{p}_j] &= [\epsilon_{ilm}\, r_l\, p_m, p_j] = \epsilon_{ilm}\,[r_l\, p_m, p_j] = -\epsilon_{ilm}\,[p_j, r_l\, p_m] \\
&= -\epsilon_{ilm}\left([p_j, r_l]\, p_m + r_l\,[p_j, p_m]\right) = -\epsilon_{ilm}\left(-i\hbar\delta_{lj}\right) p_m \\
&= i\hbar\epsilon_{ijm}p_m = i\hbar\epsilon_{ijk}p_k
\end{aligned}$$

iii. $[\mathbb{L}_i, \mathbb{L}_j] = i\hbar\epsilon_{ijk}\mathbb{L}_k$,

Solution

$$\begin{aligned}
[\mathbb{L}_i, \mathbb{L}_j] &= [\epsilon_{ixy}\, r_x\, p_y, \epsilon_{jlm}\, r_l\, p_m] = \epsilon_{ixy}\epsilon_{jlm}\,[r_x\, p_y, r_l\, p_m] \\
&= \epsilon_{ixy}\epsilon_{jlm}\left([r_x\, p_y, r_l]\, p_m + r_l\,[r_x\, p_y, p_m]\right) \\
&= -\epsilon_{ixy}\epsilon_{jlm}\left([r_l, r_x\, p_y]\, p_m + r_l\,[p_m, r_x\, p_y]\right) \\
&= -\epsilon_{ixy}\epsilon_{jlm}\left\{\left([r_l, r_x]\, p_y + r_x\,[r_l, p_y]\right) p_m + r_l\left([p_m, r_x]\, p_y + r_x\,[p_m, p_y]\right)\right\} \\
&= -\epsilon_{ixy}\epsilon_{jlm}\left\{i\hbar\delta_{ly}r_xp_m - i\hbar\delta_{mx}r_lp_y\right\} \\
&= -i\hbar\left\{\epsilon_{ixy}\epsilon_{jlm}\delta_{ly}r_xp_m - \epsilon_{ixy}\epsilon_{jlm}\delta_{mx}r_lp_y\right\} \\
&= -i\hbar\left\{\epsilon_{ixy}\epsilon_{jym}r_xp_m - \epsilon_{ixy}\epsilon_{jlx}r_lp_y\right\} \\
&= i\hbar\left\{\epsilon_{yix}\epsilon_{yjm}r_xp_m - \epsilon_{xiy}\epsilon_{xjl}r_lp_y\right\} \\
&= i\hbar\left\{\left(\delta_{ij}\delta_{xm} - \delta_{im}\delta_{xj}\right) r_xp_m - \left(\delta_{ij}\delta_{yl} - \delta_{il}\delta_{yj}\right) r_lp_y\right\} \\
&= i\hbar\left\{\delta_{ij}r_xp_x - r_jp_i - \delta_{ij}r_lp_l + r_ip_j\right\} \\
&= i\hbar\left(r_ip_j - r_jp_i\right) \\
&= i\hbar\epsilon_{ijk}L_k \\
\epsilon_{ijk}L_k &= \epsilon_{ijk}\left(\epsilon_{kmn}r_mp_n\right) = \epsilon_{kij}\epsilon_{kmn}r_mp_n \\
&= \left(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\right) r_mp_n = r_ip_j - r_jp_i
\end{aligned}$$

iv. $\mathbb{L}^2 = \mathbb{r}^2\mathbb{p}^2 - \left(\overrightarrow{\mathbb{r}} \cdot \overrightarrow{\mathbb{p}}\right)^2 + i\hbar\overrightarrow{\mathbb{r}} \cdot \overrightarrow{\mathbb{p}}$,

Solution

$$\begin{aligned}
\mathbb{L}^2 &= \left(\overrightarrow{r} \times \overrightarrow{p}\right) \cdot \left(\overrightarrow{r} \times \overrightarrow{p}\right) = \left(\overrightarrow{r} \times \overrightarrow{p}\right)_i \left(\overrightarrow{r} \times \overrightarrow{p}\right)_i = \left(\epsilon_{ijk}r_jp_k\right)\left(\epsilon_{ilm}r_lp_m\right) \\
&= \epsilon_{ijk}\epsilon_{ilm}r_jp_kr_lp_m = \left(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\right) r_jp_kr_lp_m \\
&= r_jp_kr_jp_k - r_jp_kr_kp_j = r_jp_kr_jp_k - r_jp_kr_kp_j \\
&= r_j\left(-i\hbar\delta_{jk} + r_jp_k\right) p_k - r_jp_k\left(i\hbar\delta_{jk} + p_jr_k\right) \\
&= -i\hbar r_j\delta_{jk}p_k + r_jr_jp_kp_k - i\hbar r_j\delta_{jk}p_k - r_jp_kp_jr_k \\
&= -i\hbar r_jp_j + r^2p^2 - i\hbar r_jp_j - r_jp_kp_jr_k \\
&= -2i\hbar\overrightarrow{r} \cdot \overrightarrow{p} + r^2p^2 - r_jp_jr_kp_k \\
&= -2i\hbar\overrightarrow{r} \cdot \overrightarrow{p} + r^2p^2 - \overrightarrow{r} \cdot \overrightarrow{p}\left(i\hbar\delta_{kk} + r_kp_k\right) \\
&= -2i\hbar\overrightarrow{r} \cdot \overrightarrow{p} + r^2p^2 - 3i\hbar\overrightarrow{r} \cdot \overrightarrow{p} - \overrightarrow{r} \cdot \overrightarrow{p}r_kp_k \\
&= i\hbar\overrightarrow{r} \cdot \overrightarrow{p} + r^2p^2 - \left(\overrightarrow{r} \cdot \overrightarrow{p}\right)\left(\overrightarrow{r} \cdot \overrightarrow{p}\right) \\
&= i\hbar\overrightarrow{r} \cdot \overrightarrow{p} + r^2p^2 - \left(\overrightarrow{r} \cdot \overrightarrow{p}\right)^2
\end{aligned}$$

v. $\overrightarrow{\mathbb{r}} \cdot \overrightarrow{\mathbb{p}} = \overrightarrow{\mathbb{p}} \cdot \overrightarrow{\mathbb{r}} + 3i\hbar$,

Solution

$$\begin{aligned}
\overrightarrow{\mathbb{r}} \cdot \overrightarrow{\mathbb{p}} &= \left(\overrightarrow{p} \cdot \overrightarrow{r}\right) + [r_i, p_i] \\
&= \overrightarrow{p} \cdot \overrightarrow{r} + i\hbar\delta_{ii} \\
&= \overrightarrow{p} \cdot \overrightarrow{r} + 3i\hbar
\end{aligned}$$

vi. $\overrightarrow{\mathbb{L}} \times \overrightarrow{\mathbb{L}} = i\hbar\,\overrightarrow{\mathbb{L}}$,

Solution

$$\begin{aligned}
\overrightarrow{\mathbb{L}} \times \overrightarrow{\mathbb{L}} &= -\left(\overrightarrow{L} \times \overrightarrow{L}\right) + \epsilon_{ijk}\left[L_j, L_k\right] \\
\left(\overrightarrow{L} \times \overrightarrow{L}\right)_i &= -\left(\overrightarrow{L} \times \overrightarrow{L}\right)_i + \epsilon_{ijk}\left(i\hbar\epsilon_{jkl}L_l\right) \\
2\left(\overrightarrow{L} \times \overrightarrow{L}\right)_i &= i\hbar 2\delta_{il}L_l \\
\left(\overrightarrow{L} \times \overrightarrow{L}\right)_i &= i\hbar L_i \\
\overrightarrow{L} \times \overrightarrow{L} &= i\hbar\overrightarrow{L}
\end{aligned}$$

vii. $\overrightarrow{\mathbb{p}} \times \overrightarrow{\mathbb{L}} + \overrightarrow{\mathbb{L}} \times \overrightarrow{\mathbb{p}} = \boldsymbol{2i\hbar}\overrightarrow{\mathbb{p}}$.

Solution

$$\begin{aligned}
\left(\overrightarrow{p} \times \overrightarrow{L}\right)_i + \left(\overrightarrow{L} \times \overrightarrow{p}\right)_i &= \left[-\left(\overrightarrow{L} \times \overrightarrow{p}\right)_i + \epsilon_{ijk}\left[p_j, L_k\right]\right] + \left[-\left(\overrightarrow{p} \times \overrightarrow{L}\right)_i + \epsilon_{ilm}\left[L_l, p_m\right]\right] \\
2\left[\left(\overrightarrow{p} \times \overrightarrow{L}\right)_i + \left(\overrightarrow{L} \times \overrightarrow{p}\right)_i\right] &= \epsilon_{ijk}\left[p_j, L_k\right] + \epsilon_{ilm}\left[L_l, p_m\right] \\
&= -\epsilon_{ijk}\left[L_k, p_j\right] + \epsilon_{ilm}\left[L_l, p_m\right] \\
&= \epsilon_{ikj}\left(i\hbar\epsilon_{kjx}p_x\right) + \epsilon_{ilm}\left(i\hbar\epsilon_{lmy}p_y\right) \\
&= \epsilon_{kji}\epsilon_{kjx}\left(i\hbar p_x\right) + \epsilon_{lmi}\epsilon_{lmy}\left(i\hbar p_y\right) \\
&= 2i\hbar\delta_{ix}p_x + 2i\hbar\delta_{iy}p_y \\
&= 2i\hbar p_i + 2i\hbar p_i \\
2\left[\left(\overrightarrow{p} \times \overrightarrow{L}\right)_i + \left(\overrightarrow{L} \times \overrightarrow{p}\right)_i\right] &= 4i\hbar p_i \\
\left(\overrightarrow{p} \times \overrightarrow{L}\right)_i + \left(\overrightarrow{L} \times \overrightarrow{p}\right)_i &= 2i\hbar p_i \\
\left(\overrightarrow{p} \times \overrightarrow{L}\right) + \left(\overrightarrow{L} \times \overrightarrow{p}\right) &= 2i\hbar\overrightarrow{p}
\end{aligned}$$

Formulas and identities

$$\delta_{ij} = \begin{cases} 1 & ; \quad i = j \\ 0 & ; \quad i \neq j \end{cases}$$

$$\epsilon_{ijk} = \begin{cases} +1 & ; \quad \text{if ijk is cyclic when read from left to right} \\ -1 & ; \quad \text{if ijk is not cyclic when read from left to right} \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$$\overrightarrow{A} \cdot \overrightarrow{B} = \overrightarrow{B} \cdot \overrightarrow{A} + \left[A_j, B_i\right]$$

$$\left(\overrightarrow{A} \times \overrightarrow{B}\right)_i = -\left(\overrightarrow{B} \times \overrightarrow{A}\right)_i + \epsilon_{ijk}\left[A_j, B_k\right]$$

$$[A, B] = AB - BA \neq 0$$

$$[A, B] = -[B, A]$$

$$[A, kB] = k[A, B]$$

$$[A, BC] = A(BC) - (BC)A$$

$$V_k\delta_{kj} = V_j$$

$$\left(\overrightarrow{A} \times \overrightarrow{B}\right)_i = \epsilon_{ijk}A_jB_k$$

$$\left(\overrightarrow{\nabla} \times \overrightarrow{A}\right)_i = \epsilon_{ijk}\partial_jA_k$$

$$\overrightarrow{A} \cdot \overrightarrow{B} = A_kB_k$$

$$\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$\epsilon_{ijk}\epsilon_{ljk} = 2\delta_{il}$$

$$\epsilon_{ijk}\epsilon_{ijk} = 3$$