PHY 320 - Assignment 4

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1 Problem 4

A particle of mass m moving in three dimensions under the potential energy function $V(x, y, z) = \alpha x + \beta y^2 + \gamma z^3$ has speed v_0 when it passes through the origin.

(a) What will its speed be if and when it passes through the point $(1, 1, 1)$?

Solution Conservation of mechanical energy.

$$
\begin{aligned}
T_0 + V_0 &= T + V \\
\frac{1}{2}mv_0^2 + V(0,0,0) &= \frac{1}{2}mv^2 + V(1,1,1) \\
\frac{1}{2}mv_0^2 + 0 &= \frac{1}{2}mv^2 + \alpha + \beta + \gamma \\
\frac{1}{2}mv^2 &= \frac{1}{2}mv_0^2 - \alpha - \beta - \gamma \\
v &= \sqrt{v_0^2 - \frac{2}{m}(\alpha + \beta + \gamma)}
\end{aligned}
\tag{1}
$$

(b) If the point $(1, 1, 1)$ is a turning point in the motion ($v = 0$), what is v_0?

Solution Using final equation from part a.

$$
\begin{aligned}
0 &= \sqrt{v_0^2 - \frac{2}{m}(\alpha + \beta + \gamma)} \\
v_0 &= \sqrt{\frac{2}{m}(\alpha + \beta + \gamma)}
\end{aligned}
\tag{2}
$$

(c) What are the component differential equations of motion of the particle?

Solution Gradient operator.

$$
\begin{gathered}
F = -\nabla V \\
F = -\left[\hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right] \\
F = -\hat{i}\alpha - \hat{j}2\beta y - \hat{k}3\gamma z^2 \\
F_x = -\alpha \quad F_y = -2\beta y \quad F_z = -3\gamma z^2
\end{gathered}
\tag{3}
$$

2 Problem 8

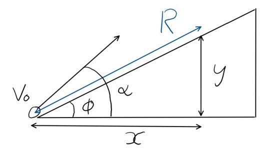
A gun is located at the bottom of a hill of constant slope ϕ. Show that the range of the gun measured up the slope of the hill is

$$\frac{2v_0^2 \cos\alpha \sin(\alpha - \phi)}{g \cos^2\phi} \tag{4}$$

where α is the angle of elevation of the gun, and that the maximum value of the slope range is

$$\frac{v_0^2}{g(1 + \sin\phi)} \tag{5}$$

Solution Equating parabola to slope. Playing with trigonometric identities from Appendix B to reach given form of equations.



$$x = R\cos\phi$$
$$y = R\sin\phi$$

$$\dot{x}_0 = V_0 \cos\alpha$$
$$\dot{y}_0 = V_0 \sin\alpha$$
$$t = \frac{x}{\dot{x}_0} = \frac{R\cos\phi}{V_0\cos\alpha}$$

$$\ddot{x} = 0 \qquad \ddot{y} = -g$$
$$\dot{x} = \dot{x}_0 \qquad \dot{y} = \dot{y}_0 - gt$$
$$x = \dot{x}_0 t \qquad y = \dot{y}_0 - \frac{1}{2}gt^2$$

$$\begin{aligned}
R\sin\phi &= \dot{y}_0 t - \frac{gt^2}{2} \\
R\sin\phi &= \frac{\dot{y}_0 x}{\dot{x}_0} - \frac{gx^2}{2\dot{x}_0^2} \\
R\sin\phi &= \frac{v_0 \sin\alpha R\cos\phi}{v_0\cos\alpha} - \frac{gR^2\cos^2\phi}{2v_0^2\cos^2\alpha} \\
\sin\phi &= \tan\alpha\cos\phi - \frac{gR\cos^2\phi}{2v_0^2\cos^2\alpha} \\
\frac{gR\cos^2\phi}{2v_0^2\cos^2\alpha} &= \tan\alpha\cos\phi - \sin\phi \\
R &= \frac{2v_0^2\cos^2\alpha}{g\cos^2\phi}(\tan\alpha\cos\phi - \sin\phi) \\
R &= \frac{2v_0^2\cos\alpha}{g\cos^2\phi}(\sin\alpha\cos\phi - \sin\phi\cos\alpha) \\
R &= \frac{2v_0^2\cos\alpha\sin(\alpha-\phi)}{g\cos^2\phi}
\end{aligned} \tag{6}$$

Maximum value of range with respect to α comes from $\frac{dR}{d\alpha} = 0$.

$$\begin{aligned}
\frac{dR}{d\alpha} &= \frac{2v_0^2}{g\cos^2\phi}\frac{d}{d\alpha}\left[\cos\alpha\sin(\alpha-\phi)\right] \\
\frac{dR}{d\alpha} &= \frac{2v_0^2}{g\cos^2\phi}\left[-\sin\alpha\sin(\alpha-\phi) + \cos\alpha\cos(\alpha-\phi)\right] = 0 \\
0 &= cos(2\alpha - \phi) \\
\alpha &= \frac{\pi}{4} + \frac{\phi}{2}
\end{aligned} \tag{7}$$

$$\begin{aligned}
R &= \frac{2v_0^2\cos(\frac{\pi}{4}+\frac{\phi}{2})\sin(\frac{\pi}{4}-\frac{\phi}{2})}{g\cos^2\phi} \\
R &= \frac{2v_0^2\cos(\frac{\pi}{4}+\frac{\phi}{2})\cos(\frac{\pi}{4}+\frac{\phi}{2})}{g\cos^2\phi} \\
R &= \frac{2v_0^2}{g\cos^2\phi}\cos^2(\frac{\pi}{4}+\frac{\phi}{2}) \\
R &= \frac{2v_0^2}{g\cos^2\phi}\frac{\cos 2(\frac{\pi}{4}+\frac{\phi}{2})+1}{2} \\
R &= \frac{v_0^2}{g(1-\sin^2\phi)}(\cos(\frac{\pi}{2}+\phi)+1) \\
R &= \frac{v_0^2}{g(1-\sin\phi)(1+\sin\phi)}(1-\sin\phi) \\
R &= \frac{v_0^2}{g(1+\sin\phi)}
\end{aligned} \tag{8}$$

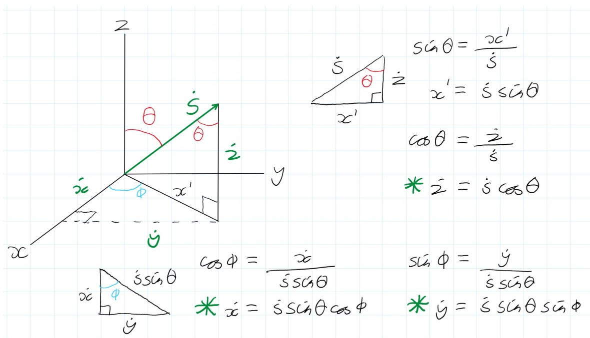
3 Problem 14

Write down the component form of the differential equations of motion of a projectile if the air resistance is proportional to the square of the speed. Are the equations separated? Show that the x component of the velocity is given by

$$\dot{x} = \dot{x}_0 e^{-\gamma s} \tag{9}$$

where s is the distance the projectile has traveled along the path of motion, and $\gamma = c_2/m$.

Solution Assuming spherical coordinate system due to s as distance. Therefore taking $\dot{s}$ as velocity, and c_2 as the proportionality constant.



$$\dot{x} = \dot{s}\sin\theta\cos\phi \quad \dot{y} = \dot{s}\sin\theta\sin\phi \quad \dot{z} = \dot{s}\cos\theta \tag{10}$$

$$\begin{aligned} F_s &\,\alpha\, \dot{s}^2 \\ F_s &= -c_2\dot{s}^2 \\ \ddot{s} &= -\gamma\dot{s}^2 = -\gamma(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\ \frac{d^2 s}{dt^2} &= -\gamma\left[\frac{dx^2}{dt} + \frac{dy^2}{dt} + \frac{dz^2}{dt}\right] \end{aligned} \tag{11}$$

Equation above clearly shows that this is not a separable differential equation. Now for the equations of motion, similar approach with spherical coordinates where instead of $\dot{s}$ the resultant vector is F_s.

$$F_x = F_s\sin\theta\cos\phi \quad F_y = F_s\sin\theta\sin\phi \quad F_z = F_s\cos\theta - mg \tag{12}$$

$$\begin{aligned} m\ddot{x} &= -c_2\dot{s}^2\sin\theta\cos\phi \\ \frac{d\dot{x}}{dt} &= -\gamma\dot{s}\dot{x} \\ \frac{d\dot{x}}{ds}\frac{ds}{dt} &= -\gamma\dot{s}\dot{x} \\ \int\frac{d\dot{x}}{\dot{x}} &= \int -\gamma\, ds \\ \ln\dot{x} - \ln\dot{x}_0 &= -\gamma s \\ \ln\frac{\dot{x}}{\dot{x}_0} &= -\gamma s \\ \dot{x} &= \dot{x}_0 e^{-\gamma s} \end{aligned} \tag{13}$$

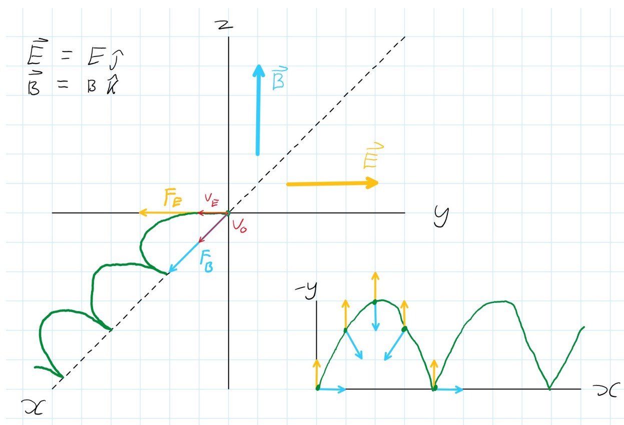
4 Problem 20

An electron moves in a force field due to a uniform electric field $\mathbf{E}$ and a uniform magnetic field $\mathbf{B}$ that is at right angles to $\mathbf{E}$. Let $\mathbf{E} = \hat{j}E$ and $\mathbf{B} = \hat{k}B$. Take the initial position of the electron at the origin with initial velocity $\mathbf{v}_0 = \hat{i}v_0$ in the x direction. Find the resulting motion of the particle. Show that the path of motion is a cycloid:

$$\begin{aligned} x &= a \sin \omega t + bt \\ y &= a(1 - \cos \omega t) \\ z &= 0 \end{aligned} \tag{14}$$

Cycloidal motion of electrons is used in an electronic tube called a magnetron to produce microwaves in a microwave oven.

Solution The Lorentz force describes the force on a charged particle due to the crossed electric and magnetic fields. Figure below is for an electron: at the origin, the force due to the electric field should be on the $-\hat{j}$ direction due to the negative charge of the electron, and the force due to the magnetic field should be along the $\hat{i}$ direction. Therefore constraining the motion of the particle on the xy plane and traveling in the $\hat{i}$ direction.



$$\begin{aligned} \mathbf{F} &= \mathbf{F}_E + \mathbf{F}_B \\ \mathbf{F} &= q\mathbf{E} + q(\mathbf{v} \times \mathbf{B}) \\ \mathbf{F}_E &= q\mathbf{E} = qE\hat{j} \\ \mathbf{F}_B &= q(\mathbf{v} \times \mathbf{B}) = q \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dot{x} & \dot{y} & \dot{z} \\ 0 & 0 & B \end{vmatrix} = qB\dot{y}\hat{i} - qB\dot{x}\hat{j} \\ \mathbf{F} &= qE\hat{j} + q\dot{y}B\hat{i} - q\dot{x}B\hat{j} \end{aligned} \tag{15}$$

This equation is for a general case as we do not know the trajectory of the particle. A $q = -e$ for an electron produces forces that correspond to the figure above.

$$m\ddot{x} = q\dot{y}B \quad m\ddot{y} = qE - q\dot{x}B \quad m\ddot{z} = 0 \tag{16}$$

$$\begin{aligned} \dddot{x} &= \frac{qB}{m}\ddot{y} \\ \ddot{y} &= \frac{m}{qB}\dddot{x} = \frac{qE}{m} - \frac{qB}{m}\dot{x} \\ \dddot{x} + \left(\frac{qB}{m}\right)^2 \dot{x} &= \left(\frac{q}{m}\right)^2 EB \\ \frac{d^2\dot{x}}{dt^2} + \left(\frac{qB}{m}\right)^2 \dot{x} &= \left(\frac{q}{m}\right)^2 EB \end{aligned} \tag{17}$$

A nonhomogenous differential equation with constant coefficients. Using the auxiliary polynomial to find the complementary solution.

$$\begin{aligned} a(r) &= r^2 + \left(\frac{qB}{m}\right)^2 = 0 \\ r &= \pm\frac{qB}{m}i = \pm\omega i \\ \dot{x}(t) &= \dot{x}_c(t) + \dot{x}_p(t) \\ \dot{x}_c(t) &= c_1\cos\omega t + c_2\sin\omega t \end{aligned} \tag{18}$$

Now for the particular solution. Note that the right hand side is just a constant.

$$\begin{gathered} g(r) = \left(\frac{q}{m}\right)^2 EBe^{0r} \\ \text{Since 0 is not a root} \end{gathered} \tag{19}$$

$$\dot{x}_p = C \quad \ddot{x}_p = 0 \quad \dddot{x}_p = 0 \tag{20}$$

Inputting these solutions into the differential equation.

$$\begin{aligned} 0 + \left(\frac{qB}{m}\right)^2 C &= \left(\frac{q}{m}\right)^2 EB \\ C &= \frac{E}{B} = \dot{x}_p(t) \\ \dot{x}(t) &= c_1\cos\omega t + c_2\sin\omega t + \frac{E}{B} \\ \dot{y} &= \frac{m}{qb}\ddot{x} \\ \dot{y}(t) &= \frac{m}{qB}\left[\frac{qB}{m}\left(-c_1\sin\omega t + c_2\cos\omega t\right)\right] \\ \dot{y}(t) &= -c_1\sin\omega t + c_2\cos\omega t \end{aligned} \tag{21}$$

Applying initial conditions to find c_1 and c_2.

$$\begin{aligned} &\text{At } t = 0,\ \dot{x} = \dot{x}_0 \\ &\qquad \dot{x}_0 = c_1\cos 0 + c_2\sin 0 + \frac{E}{B} \\ &\qquad c_1 = \dot{x}_0 - \frac{E}{B} \\ &\text{At } t = 0,\ \dot{y} = 0 \\ &\qquad 0 = -c_1\sin 0 + c_2\cos 0 \\ &\qquad c_2 = 0 \\ &\qquad \dot{x}(t) = \left(\dot{x}_0 - \frac{E}{B}\right)\cos\omega t + \frac{E}{B} \\ &\qquad \dot{y}(t) = \left(\frac{E}{B} - \dot{x}_0\right)\sin\omega t \\ &\qquad \dot{z}(t) = 0 \end{aligned} \tag{22}$$

Now to find trajectory of the particle.

$$\begin{aligned} x(t) &= \int \dot{x}\,dt = \left(\frac{\dot{x}_0 - \frac{E}{B}}{\omega}\right)\sin\omega t + \frac{E}{B}t + x_0 \\ y(t) &= \int \dot{y}\,dt = \left(\frac{\dot{x}_0 - \frac{E}{B}}{\omega}\right)\cos\omega t + y_0 \end{aligned} \tag{23}$$

Applying initial conditions to find x_0 and y_0.

$$\begin{aligned}
&\text{At } t=0,\ x=0 \\
&0 = \left(\frac{\dot{x}_0 - \frac{E}{B}}{\omega}\right)\sin 0 + 0 + x_0 \\
&x_0 = 0 \\
&\text{At } t=0,\ y=0 \\
&0 = \left(\frac{\dot{x}_0 - \frac{E}{B}}{\omega}\right)\cos 0 + y_0 \\
&y_0 = -\left(\frac{\dot{x}_0 - \frac{E}{B}}{\omega}\right) \\
&x(t) = \left(\frac{\dot{x}_0 - \frac{E}{B}}{\omega}\right)\sin\omega t + \frac{E}{B}t \\
&y(t) = \left(\frac{\dot{x}_0 - \frac{E}{B}}{\omega}\right)\cos\omega t - \left(\frac{\dot{x}_0 - \frac{E}{B}}{\omega}\right)
\end{aligned} \tag{24}$$

Now for an electron, since $\omega = \frac{qB}{m} = \frac{-eB}{m}$.

$$\begin{aligned}
x(t) &= \left(\frac{\dot{x}_0 - \frac{E}{B}}{-\omega}\right)\sin(-\omega t) + \frac{E}{B}t \\
&= \left(\frac{\dot{x}_0 - \frac{E}{B}}{\omega}\right)\sin(\omega t) + \frac{E}{B}t \\
y(t) &= \left(\frac{\dot{x}_0 - \frac{E}{B}}{-\omega}\right)(\cos(-\omega t) - 1) \\
&= \left(\frac{\dot{x}_0 - \frac{E}{B}}{\omega}\right)(1 - \cos\omega t)
\end{aligned} \tag{25}$$

$$\begin{aligned}
x(t) &= a\sin\omega t + bt \\
y(t) &= a(1-\cos\omega t) \\
z(t) &= 0
\end{aligned} \tag{26}$$

5 Problem 23

Show that the period of the particle sliding in the cycloidal trough of Example 4.6.2 is $4\pi(A/g)^{1/2}$.

Solution The energy equation has the values for k. This works because a particle sliding in a smooth cycloidal trough exhibits harmonic motion. ($E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$)

$$\begin{aligned}
E &= \frac{1}{2}m\dot{s}^2 + \frac{1}{2}\left(\frac{mg}{4A}\right)s^2 \\
T_s &= \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{\frac{mg}{4A}}} = 4\pi\sqrt{\frac{A}{g}}
\end{aligned} \tag{27}$$