Consumer default with complete markets: default-based pricing and finite punishment*

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Abstract

This paper studies economies with complete markets where there is positive default on consumer debt. In a simple tractable two-period model, households can default partially, at a finite punishment cost, and competitive intermediaries price loans of different sizes separately. This environment yields only partial insurance. The default-based pricing of debt makes it too costly for the borrower to achieve full insurance and there is too little trade in securities. This framework is in contrast with existing literature. Unlike the literature with default, there are no restrictions on the set of state contingent securities that are issued. Unlike the literature on lack of commitment, limited trade arises without need of debt constraints that rule default out. Compared with the latter, the present approach appears to imply more consumption inequality. An extended model with an infinite horizon, idiosyncratic risk and more realistic assumptions is used to demonstrate the general validity of this approach and its main implications.

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1 Introduction

This paper studies consumer credit default in an economy with complete financial markets. In spite of the evidence of consumer credit delinquency and bankruptcy, the existence of positive default within this elementary framework has received little attention in the literature on limited commitment. Works that consider complete markets rule out positive default; works that study equilibrium bankruptcy restrict the set of assets that can be traded. In this paper, positive default coexists with complete markets. This environment provides a new explanation of imperfect insurance (or incomplete trade) with distinct implications for consumption inequality.

The first objective of this paper is to put forward a sparse tractable general equilibrium model for analysing the basics of default with complete markets.[1] The approach rests on the existence of partial default and individualized default-based pricing. A second objective is to discuss, in that context, the reasons why default matters for the allocation of resources, even in a frictionless competitive environment. We will study the consequences of default for risk sharing (i.e., insurance) and intertemporal consumption smoothing. The third objective is to explore the potential significance of this default-pricing approach. We will compare its implications for consumption inequality with those from a model which – in the vein of much literature on endogenous incomplete markets – has debt constraints that rule out default. We will finally develop an extended dynamic model to demonstrate numerically the idea's broader general validity. It will also be used to start exploring quantitative implications for households' debt and default.

One motivation for entertaining partial default is that in the U.S. default outside formal bankruptcy procedures is substantial and might account for the bulk of loan write-offs.[2] Unlike the full discharge of unsecured debts in formal procedures like Chapter 7, informal bankruptcy is best seen as involving debtors failing to repay a chosen portion of their liabilities. Similarly, in the context of sovereign debt, partial default seems to be the norm.[3]

[1] An early word on terminology is in order. Here, as in much literature on limited commitment (e.g., Kehoe and Levine (1993), Alvarez and Jermann (2000)) we speak of complete markets as the availability of securities spanning the space of states of nature. This is a narrower notion than in Dubey, Geanakoplos, and Shubik (2005) where tradeable assets are defined also over the level of default punishments, possibly including infinite penalties. Another difference is that that paper associates each type asset with a specific debt constraint; in our case, each type of asset is associated with one level of debt. See below for further discussion.

[2] This is the theme of recent empirical and theoretical studies in Dawsey and Ausubel (2004), Dawsey, Hynes, and Ausubel (2008), and Benjamin and Mateos-Planas (2012), Chatterjee (2010) and Herkenhoff (2012). In the Survey of Consumer Finances 2007, about 1% of the U.S. population had filed under Chapter 7, whereas over 5% held delinquent loans. See Díaz-Giménez, Glover, and Ríos-Rull (2011).

[3] A point taken up in Arellano, Mateos-Planas, and Ríos-Rull (2013). Yet previous analyses of sovereign debt since Eaton and Gersovitz (1981), including more recently Arellano (2008), Benjamin and Wright (2009), and Yue (2010), appear to consider default only as a binary choice.

The simple general-equilibrium model consists of a two-period endowment economy populated by two types of households who are subject to endowment risk in the second period. Regarding financial markets, households can borrow and lend freely through perfectly competitive financial intermediaries, and have access to a full set of securities that span the space of promised deliveries. Borrowers can default fully or partially on their promised deliveries. Default carries a finite utility penalty to the household which depends on the scale of default. The same penalties apply in all contracts. Because different levels of debt are associated with a different default rate, loans of different sizes are different assets and command a different price. In order to characterize the behavior of intermediaries and households, one needs to account for the price schedule associated with all possible contracts, including those that are not traded in equilibrium. Using specific functional assumptions, we characterise the equilibrium and draw analytical and numerical results.

How can positive default arise with complete assets? The common view is that default happens as the result of a contingency against which a contract cannot be written. Lenders are still willing to trade since ex-ante they can write off the bankruptcy losses against gains in other states. With one security available for each contingency, however, there is no room for such compensation across states on any one asset. If default is on the totality of debt then the asset will not be traded. That default can be partial is therefore essential for default to arise in this paper. It involves an ex-ante known proportion in each particular state – rather than a probability distribution over unknown states – of debt going unpaid.[4]

One key result is that default on its own has real consequences for the allocation of consumption. The distinctive mechanism is that prices reflect the default incidence for different contract sizes in a way that affects the borrowing and lending decisions in equilibrium. Specifically, the price of securities declines with the value of promised deliveries. Since a borrower's marginal gain in terms of current consumption from issuing debt is then less than its price – a price wedge – an economy with endogenous bankruptcy will feature less trade in assets. Specifically, the possibility of bankruptcy implies incomplete consumption risk sharing and suboptimal intertemporal consumption smoothing. Across states of nature in the second period, individual consumption varies positively with income. If, as it is arguably the case, penalty levels vary positively with household income, the model can also account for default decreasing with income.[5]

[4]Since the outcome under each contingency is always perfectly anticipated, a possible question is whether this default does really represent a failure to fulfill the contract. The implications of the theory for measured observable default might thus need to be interpreted with caution. Here we simply note that incomplete-markets models with rational expectations would not be totally immune to this observation. In any event, it would not detract from the significance of the theory for observable variables like consumption, a central point of this paper.

[5]In models with a complete set of contingent assets like the present one, as the agent attempts to insure consumption, income and debt labilities tend to be positively associated ex-post. The emergence of something resembling standard debt contracts would in principle require asymmetric information and monitoring costs as in Diamond (1984). An open question which we will not pursue here is whether the present model could deliver a similar outcome via a specific structure of default penalties. We thank a referee for suggesting this

It is critical that the default decisions are endogenous and that contracts of different sizes are priced individually. If default rates for each individual type and contingency were exogenous then the price of claims would be given to households and allocations would be optimal, featuring full risk sharing.[6] If contracts for each contingent state were anonymous pools, with the same price for asset sales of different size, then the price faced by an individual borrower would not vary with the amount of securities sold and there would be no price wedge, yielding full risk sharing.

In order to assess the implications for consumption inequality in this default-pricing model, we consider a debt-constrained version of the economy à la Kehoe and Levine (1993) as a comparison benchmark. This obtains naturally if the penalty for defaulting is fixed regardless of its scale. Default, which becomes in effect an all-or-nothing choice, is ruled out and liabilities cannot exceed a certain endogenous debt limit. The source of imperfect risk sharing is the same lack of commitment in the two models, yet the implications can be different. For a class of symmetric economies, the default-pricing model encompasses less risk sharing and more consumption inequality than the debt-constrained model. Illustrative numerical examples support this result more generally.

Specific assumptions of the simple model – two periods, two individual types, exogenous ex-ante heterogeneity, convex non-pecuniary punishment – are relaxed in an infinite-horizon idiosyncratic-risk version of the model and shown not to be critical for the results. This extended model is also used to demonstrate that this can be a quantitatively reasonable story of household debt and default.

In considering complete markets, this paper is related to the large and influential literature on limited enforcement with complete markets. This includes Kehoe and Levine (1993), Alvarez and Jermann (2000), Kehoe and Levine (2001), Krueger and Perri (2006), and Kehoe and Perri (2002).[7] Like those works, we also derive endogenous incomplete insurance given a full set of securities, but this is done without imposing an exogenous participation constraint or not-too-tight borrowing limits. Furthermore, in the present paper there is positive default in equilibrium, whereas these other papers rule out positive bankruptcy as an equilibrium outcome.[8] The existing literature has used that framework to address empirical facts on consumption and wealth inequality. As said, the approach here seems to have novel implications and should be relevant for understanding that evidence.

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avenue.

[6]In this case, default would not be meaningful, as the model would be equivalent to a model without default and debt determined by the portion repaid in the model with default.

[7]In an interesting paper, Koeppl (2007) endogenizes the level of enforcement in dynamic risk-sharing problems.

[8]There is also a literature on endogenous limits with incomplete markets that similarly rules out positive default in equilibrium. This includes Zhang (1997), Mateos-Planas and Seccia (2006), Ábrahám and Cárceles-Poveda (2010), Andolfatto and Gervais (2008), and Wang (2011).

This paper is also related to the recent quantitative literature on consumer credit and bankruptcy. The works that undertake a quantitative general equilibrium analysis of incomplete markets include Chatterjee, Corbae, Nakajima, and Ríos-Rull (2007), and Livshits, MacGee, and Tertilt (2007), Mateos-Planas (2013), Mateos-Planas and Ríos-Rull (2013), Benjamin and Mateos-Planas (2011), and Athreya, Tam, and Young (2009). The present paper deals in a similar way with the pricing of default but does not rule out trade opportunities arbitrarily by restricting the set of assets available. Unlike our paper, these works also consider all-or-nothing default decisions.[9]

Dubey, Geanakoplos, and Shubik (2005) (DGS) present a model of default accommodating various specifications of the tradeable set of assets. In that paper, the first best allocation obtains when the tradeable set consists of assets that span the space of promised deliveries, with each asset serving any loan size and prescribing an infinite non-pecuniary default penalty.[10] In the present paper, the set of tradeable assets also spans promised deliveries fully, but it has a different asset serving each specific debt amount and specifies one finite non-linear default penalty. Therefore, as departures from their first-best benchmark, we rule out both assets with infinite punishment and assets that pool loans of various sizes. That is, while the present paper has default-based pricing of assets of different face value, the first-best benchmark in DGS treats assets as pools. As already noted, with pooling, the first best full insurance would also obtain in our model, even with finite penalties and positive default. Another departure from DGS is that, when finite, the default penalty in our case is non-linear rather than linear, which allows for partial default at the individual level and, as noted, is essential for positive default to occur at all. In their paper DGS examine numerical examples of departures from the benchmark first-best tradeable set of assets, but do not accommodate the class of economies with full span of promises, full defaut-based pricing, and finite non-linear penalties studied here.[11] With the adaptation of the specification of penalties, the general setting in DGS could probably accommodate the present case as long as borrowing always happens at the debt limit and different loan sizes can therefore be priced separately.

In the general equilibrium literature, default has been typically studied in models with incomplete financial markets. The emphasis has been on the role of default in providing some degree of insurance against individual risk. This is the case, for example, in Zame (1993) and, again, Dubey, Geanakoplos, and Shubik (2005). This paper studies instead situations where there is a complete set of securities and insurance implications of default are of a

[9]Kehoe and Levine (2006) brings together the two streams by studying an economy with incomplete markets and collateral constraints in a way that reconciles the outcomes of a debt-constrained model with complete markets.

[10]This is their Theorem 3.

[11]Their Example 1 considers the effect of finite penalties with full span of payoffs but assumes pooling. Examples 2 and 3 consider pooling and a restricted set of tradeable promises. Example 4 studies assets with different debt limits but with restrictions on the set of tradeable promises.

different nature. That general-equilibrium stream of literature includes extensions to DGS in Araujo, Monteiro, and Páscoa (1998), and recent papers that introduce more realistic bankruptcy institutions and penalties such as collateral in Araujo and Páscoa (2002), and securitised contracts in Poblete-Cazenave and Torres-Martínez (2010). These works deal with more general commodity, endowments and agent spaces than our current paper and derive important existence results; unlike our paper, their object of study consists mainly of economies with an incomplete set of assets, in the form of debt contracts, and risk pooling. This literature has also developed sparser models which, in our spirit, are used to address more applied questions. Peiris and Vardoulakis (2011) consider the role of liquid storage for precautionary savings through its effect on default expectations and access to credit; the setting however is one where the only financial asset is a bond and there is pooling. Kilenthong (2011) shares with the present paper a setting with complete contracts, to which it adds collateral requirements; imperfect risk sharing arises due to limited or scarce collateral, and the existence of positive default is not important for an equilibrium that is constrained efficient (like in Alvarez and Jermann (2000)), whereas our friction works through security prices where default matters greatly.

Section 2 and 3 present the simple model, define the equilibrium, and derive a key property for the pricing of loans. Section 4 examines the mechanism for incomplete insurance and intertemporal smoothing in this model. Section 5 compares the insurance implications of this model with those of the debt-constrained model. Section 6 illustrates properties of the quantitative extended model. Section 7 concludes.

2 A basic model

The economy lasts for two periods. The possible states are $s = 0, 1, , , S$. In the first period the state is 0; in the second period the state of nature can be $s = 1, ..., S$. The probability of a state s in the second period is denoted as π_s. There are two types $i = A, B$ of individual households. The two groups are of the same size. These agents receive endowments of goods that depend on their type i and the state of nature $s = 0, 1, ..., S$, and are denoted by y_s^i. In the first period, households can borrow or save against second-period states $s = 1, ..., S$ using securities traded through banks. Let a_s^i denote the risk-free delivery values of the assets that pay in state s held by an individual of type i. Let l_s^i denote the promised delivery values of the loans held by an individual of type i that pay in state s. Let p_s denote the price of assets at 0. As for debts, since different levels of debt may carry a different risk of failure, the price at 0 of debt held by agent i with promised delivery value l in state s is a function of its face value $Q_s^i(l)$. A borrower can default on a fraction of their debt. Let d_s^i denote the fraction of promised repayments l_s^i that borrower i fails to deliver in state s. Individuals consume in each period their net available resources, and c_s^i denotes consumption in state s by an individual of type i.

Individual preferences are represented by a function of consumption and default. A borrower who defaults experiences a utility stigma loss z that depends on the rate of default incurred d_s^i. More specifically, the utility of an individual of type i is represented by

$$u(c_0^i) + \beta \sum_{s=1,\dots,S} \pi_s[u(c_s^i) - z(d_s^i)],$$

where $u(.)$ is the period utility function, $z(.)$ describes the penalty for default, and β is a discount parameter. We will often use a specific functional form for preferences, with

$$\begin{aligned} u(c) &= \log c \\ z(d) &= \eta d^\gamma \end{aligned} \tag{1}$$

where η and γ are positive parameters accounting for the level and curvature of the penalty, respectively. Strict convexity of the penalty will be assumed so $\gamma > 1$.

Banks intermediate the trade in assets and debts. There are separate intermediaries for potentially each borrower type, loan size, and contingency. Their lending activity carries a degree of default. Let $D_s^i(l)$ denote the default rate associated with a value l of promised deliveries in state s by agent i. The revenues to the bank for lending l claims against state s to agent i are then $(1 - D_s^i(l))l$. The costs to the bank consist of the delivery value of the deposits taken a. The bank's book balancing requires the value of deposits taken to equate the value of loans made, that is $Q_s^i(l)l = p_s a$. Therefore the present value of bank's profits is

$$\left(1 - D_s^i(l) - \frac{Q_s^i(l)}{p_s}\right) l.$$

There is a market open for loans of every possible size. All markets clear under perfect competition and free entry in intermediation.

3 Equilibrium

The equilibrium determines the price of assets p_s and, for all tradeable debt sizes l, the price schedule $Q_s^i(l)$ and default schedule $D_s^i(l)$. There will be only one type of borrower i under each state s and, since all agents of a given type behave in the same way, all traded loans are of a particular size l_s^i and carry a single effective default rate $d_s^i = D_s^i(l_s^i)$ and price $q_s^i = Q_s^i(l_s^i)$, if $l_s^i > 0$.

In order to determine these specific realizations, one will need to understand the values associated with off equilibrium allocations for contracts that are not traded, as described by the mappings $Q_s^i(.)$ and $D_s^i(.)$. On one hand, consumers make their borrowing and default plans bearing in mind how the interest charged changes with the liabilities through $Q_s^i(.)$. On the other hand, banks bidding entry into the industry take into account the variation in the degree of default associated with loans of different size through $D_s^i(.)$. In this model, these off equilibrium expectations will be consistent with the assumption made that markets for non traded contracts are open.[12] We turn now to defining the equilibrium more precisely.

One remark about the equilibrium is that borrowing or savings will be positive for only one agent type $i \in \{A, B\}$ in a particular state $s = 1, ..., S$. That is, if $l_s^i > 0$ then $a_s^i = 0$, and for the other type i_-, $l_s^{i-} = 0$ and $a_s^{i-} > 0$. This allows us to use a sparse notation.

3.1 Definition

An equilibrium for the above economy consists of the following objects for each state $s = 1, ..., S$, and for all agents $i \in \{A, B\}$: Traded prices q_s^i and p_s, price menus for debt $Q_s^i(.)$, default functions $D_s^i(.)$, portfolios l_s^i, and consumption allocations c_s^i, and default rates d_s^i. They satisfy the following conditions:

(i) Optimal portfolios: For each $i \in \{A, B\}$, given prices p_s and $Q_s^i(.)$, and the default mapping $D_s^i(.)$, the first-period choices $a_s^i \geq 0$ and $l_s^i \geq 0$ maximise utility

$$u(c_0^i) + \beta \sum_{s \geq 1} \pi_s [u(c_s^i) - z(D_s^i(l_s^i))]$$

subject to

$$c_0^i = y_0^i - \sum_{s \geq 1} p_s a_s^i + \sum_{s \geq 1} Q_s^i(l_s^i) l_s^i$$

and, for $s = 1, ..., S$, if $l_s^i > 0$,

$$c_s^i = y_s^i - (1 - D_s^i(l_s^i)) l_s^i$$

or, if $a_s^i > 0$,

$$c_s^i = y_s^i + a_s^i.$$

(ii) Optimal default: For a given loan size $l > 0$, the value of the default function schedule $D_s^i(l)$ is the $\tilde{d}$ that maximises second-period agent i's utility in state s, $u(c_s^i) - z(\tilde{d})$, subject to

$$c_s^i = \quad y_s^i - (1 - \tilde{d}) l,$$

for $s = 1, ..., S$.

(iii) Bank competition: Given the default schedule $D_s^i(.)$, the price menu $Q_s^i(.)$ satisfies zero-profit condition on any potential credit contract size l.

[12] This is not unlike Dubey, Geanakoplos, and Shubik (2005)'s refinement to prevent the existence of markets being ruled out by excessively pessimistic expectations.

(iv) Consistency: For $s = 1, ..., S$ and all $i \in \{A, B\}$, the traded price corresponds to the traded contracts

$$q_s^i = Q_s^i(l_s^i) \text{ if } l_s^i > 0$$

and the default rate corresponds to the traded contract

$$d_s^i = D_s^i(l_s^i) \text{ if } l_s^i > 0.$$

(v) Market clearing: For $s = 1, ..., S$,

$$\sum_i c_0^i = \sum_i y_0^i,$$

$$\sum_i c_s^i = \sum_i y_s^i,$$

$$\sum_i q_s^i l_s^i = \sum_i p_s a_s^i.$$

We now translate this definition into more specific usable conditions. We are supposing for now that optimal choices are interior and can be characterized by first order conditions. The next subsection will justify this supposition. In point(i) the budget constraint shows that the borrower bears in mind the effect of the level of debt on the price through $Q_s^i(.)$. More specifically, if $l_s^i > 0$, for the borrower the optimality condition for debt is

$$u'(c_0^i)\left[q_s^i + Q_s^{i\prime}(l_s^i)l_s^i\right] = \beta u'(c_s^i)(1 - d_s^i)\pi_s, \;\; l_s^i > 0 \tag{2}$$

where use has been made of the condition in (iv). For the lender with $l_s^i = 0$, the standard conditions for optimal savings holds, so

$$u'(c_0^i)p_s = \beta u'(c_s^i)\pi_s, \;\; l_s^i = 0 \tag{3}$$

Point (ii) describes default behavior for arbitrary levels of debt so $D_s^i(l)$ solves

$$u'(y_s^i - (1 - D_s^i(l))l)l = z'(D_s^i(l)) \quad \text{all} \quad l > 0. \tag{4}$$

Note that, because of separability of the utility function, the default menu has the convenient property that it does not depend on the price schedule $Q_s^i(.)$. It follows that, using point (iv), the corresponding condition for the equilibrium allocation is

$$u'(c_s^i)l_s^i = z'(d_s^i), \;\; l_s^i > 0 \tag{5}$$

Point (iii) describes the zero-profit condition for all loan sizes. Given the expression for the bank's cash-flow presented earlier, this can be written more explicitly as:

$$1 - D_s^i(l) - \frac{Q_s^i(l)}{p_s} = 0 \quad \text{all} \quad l > 0. \tag{6}$$

The definition of the price of debt traded in equilibrium in point (iv) along with the fact just discussed that the realised default $d_s^i = D_s^i(l_s^i)$ if $l_s^i > 0$, permits writing an analogous expression specifically for the active intermediaries:[13]

$$1 - d_s^i - \frac{q_s^i}{p_s} = 0 \tag{7}$$

Point (v) describes clearing in the goods markets and financial markets, the latter implying $q_s^i l_s^i = p_s a_s^{i-}$ when $l_s^i > 0$, where i_- denotes the complementary to type type i. Combining these market clearing conditions with the budget constraints in point (i) and the zero-profit condition (7), one can write the consumption allocation in terms of debts l_s^i as follows:

$$\begin{array}{rcl} c_0^i & = & y_0^i + \sum_{s\geq 1} p_s(y_s^i - c_s^i) \\ c_s^i & = & y_s^i - (1 - d_s^i)l_s^i \;\text{ for }\; l_s^i > 0 \\ c_s^i & = & y_s^i + (q_s^{i-}/p_s)l_s^{i-} \;\text{ for }\; l_s^i = 0. \end{array} \tag{8}$$

Note that (7) implies that the last equality can be written as $c_s^i = y_s^i + (1 - d_s^{i-})l_s^{i-}$.

3.2 Formal properties

The assumption that $\gamma > 1$, which imparts convexity into the cost function $z(.)$, will conveniently imply the concavity of the objective function of the household. One can then establish that, as long as debt relative to output is not too large, the optimal default rate is interior and unique, and satisfies the first order condition (4); the solution is full default otherwise. The pricing schedule reflects these properties of the default function by arbitrage in (6). Furthermore, it has the desirable property that the nominal value of the loan $Q_s^i(l) \times l$ is concave in l, and increasing when default does not exceed $(\gamma - 1)/\gamma$.[14]

Proposition 1 *(DEFAULT) Assume $\gamma > 1$. Consider the default choice problem in point (ii) of the definition. If $l < y_s^i\eta\gamma$, there is a unique interior solution characterized by equation (4). Otherwise, the unique solution is $D_s^i(l) = 1$. The default policy function $D_s^i(l)$ is increasing, continuous, and differentiable except at $l = y_s^i\eta\gamma$.*

Proposition 2 *(PRICE FUNCTION) Assume $\gamma > 1$. The price schedule of liabilities $Q_s^i(l)$ is a continuous decreasing function of debt, differentiable except at $l = y_s^i\eta\gamma$ where it*

[13] Portfolios for an individual could not have both l_s^i and a_s^i positive. For the two (interior) conditions (2) and (3) to hold, it is required that $q_s^i/p_s > 1 - d_s^i$, a contradiction with (7).

[14] Proofs of propositions 1,2 and 3 are in the Appendix.

attains a value of zero. The nominal value $Q_s^i(l)l$ is a concave function of l, and increasing for and only for l such that $D_s^i(l) < (\gamma - 1)/\gamma$.

Given the properties of the pricing function, optimality will imply less than full default and choices that are always interior as in (2) and (5). Specifically, that issuing more cannot optimally reduce the nominal value of resources borrowed, $Q_s^i(l)l$, implies a bound for default rates. The next proposition states this result. Condition (3) is standard and will not be discussed here.

Proposition 3 *(INDIVIDUAL DECISIONS) Assume $\gamma > 1$. Consider an equilibrium with trade where agent i issues debt against state s. The optimal choice in point (i) of the definition is unique and characterized by the first-order conditions in* (2) *and* (5)*. There $d_s^i < (\gamma - 1)/\gamma < 1$.*

3.3 Characterisation

The set of equations (2), (5), (3), (7) and (8) forms a system in the endogenous variables consisting of prices p_s and q_s^i, debt promised deliveries l_s^i, default d_s^i, and consumption allocations c_0^i and c_s^i, for $i = A, B$ and $s = 1, ..., S$. However this system does not fully determine the outcomes as it does not specify the response of the price of debt to changes in the borrowing decision expressed in the derivative of the price schedule $Q_s^{i\prime}$ in (2). The two remaining equations (4) and (6) precisely describe the patterns for default and pricing which are needed to pin down this derivative. Therefore, in order to completely characterise the equilibrium, we will study the properties of the price menu implied by (4) and (6).

To be specific, we will consider the functional forms for utility and stigma costs in (1). The default mapping $D_s^i(l)$ is determined from (4) as the solution to

$$\frac{l}{y_s^i - (1 - D_s^i(l))l} = \gamma\eta D_s^i(l)^{\gamma-1}.$$

Computing its derivative $D_s^{i\prime}(l)$ and, using condition (5) with (8), evaluating it at the equilibrium l_s^i and d_s^i, one obtains:

$$D_s^{i\prime}(l_s^i) = \frac{1 - d_s^i + \frac{1}{\gamma\eta} {d_s^i}^{1-\gamma}}{\left(\frac{1}{\gamma\eta} {d_s^i}^{-\gamma} - 1\right)(\gamma - 1) + \gamma} \frac{1}{l_s^i}$$

Now the zero-profit condition (6), with (7), shows that the change in the price of debt with liabilities has the opposite sign, $Q_s^{i\prime}(l) = -q_s^i(1 - d_s^i)^{-1} D_s^{i\prime}(l)$. With this information, one can find an explicit expression for the change in the amount that one can borrow by incurring extra debt:

$$q_s^i + Q_s^{i\prime}(l_s^i) l_s^i = q_s^i \frac{1}{1 - d_s^i} \frac{{d_s^i}^{-\gamma}}{(\gamma - 1){d_s^i}^{-\gamma} + \gamma\eta}((1 - d_s^i)(\gamma - 1) - d_s^i) \qquad (9)$$

This term (9) has to be positive in an equilibrium with positive borrowing and lending where (2) holds. By incurring more liabilities for tomorrow, the borrower must be able to raise more consumption today. Note that our maintained assumption that the default penalty is steep enough, or $\gamma > 1$, is a necessary condition for this to happen. On the other hand, note that this condition guarantees that, as stated in propositions 1 and 2, the default rate increases and hence the debt price decreases with the value of promised debt repayments, or ${D_s^i}'(l_s^i) > 0$ and ${Q_s^i}'(l_s^i) < 0$.

4 Risk sharing, smoothing, and default

In this section, we discuss the properties of the equilibrium regarding risk sharing and the intertemporal allocations of consumption. We first consider the case where default is exogenous as the standard benchmark where there is full risk sharing and default does not matter. Then we study the model with endogenous default in order to establish that those properties do not hold. This section will conclude with a discussion of the pattern of default, some remarks on the specification of penalties assumed, and a comparison with an economy with pooling of assets.

To facilitate the discussion we will assume there is no aggregate risk so $\sum_{i=A,B} y_s^i$ is a constant y for all $s = 1, ..., S$.

4.1 Risk sharing

Consider first the situation where the default rates d_s^i are exogenous. An equilibrium is still characterized by the above conditions except (5), (4) and (6), and with the property that risk does not depend on loan size or ${Q_s^i}'(l) = 0$.

Denote by j the type who borrows against state s (i.e., $l_s^j > 0$), and by j_- the complementary lender type. In this setting with exogenous default, the first-order conditions (2) and (3), alongside the no-arbitrage condition (7) that $p_s = q_s^j/(1 - d_s^j)$, imply the equalization of the intertemporal marginal rates of substitution of the two agents in each state, that is $u'(c_s^j)/u'(c_0^j) = u'(c_s^{j-})/u'(c_0^{j-})$. On the other hand, the market clearing/budget constraint condition (8) implies that $c_s^j + c_s^{j-} = y$. Combining these two properties,

$$\frac{u'(c_s^j)}{u'(y - c_s^j)} = \frac{u'(c_0^j)}{u'(c_0^{j-})}.$$

It follows that consumption of any agent is invariant to the state s. There is, in other words, complete risk sharing. The intuition for this result is standard. With default rates given exogenously, no-arbitrage prices fully account for the incomplete repayment of promised deliveries. No-arbitrage prices lead to the equalization of marginal rates of substitution across agents.

We turn now to the case with endogenous default. The equilibrium is described by equations (2), (5), (3), (7) and (8) with (9). Complete risk sharing fails to hold in this case. Although the same no-arbitrage condition stands, there is no equalization of marginal rates of substitution across agents. The reason is that the borrower household takes into account the endogenous response of the price to their loan choice. More formally, let again j represent the agent that borrows against a particular state s. First, from intertemporal optimality in (2) and (3) we obtain, for the borrower,

$$\frac{u'(c_s^j)}{u'(c_0^j)} = \frac{q_s^j + Q_s^{j\prime}(l_s^j)l_s^j}{\beta(1-d_s^j)\pi_s},$$

and, for the lender,

$$\frac{u'(c_s^{j-})}{u'(c_0^{j-})} = \frac{p_s}{\beta\pi_s}.$$

Now the no-arbitrage condition (7) that $p_s = q_s^j/(1-d_s^j)$ does not imply equalization of marginal rates of substitution as long as $Q_s^{j\prime} \neq 0$. Combining these expressions with market clearing (8), one obtains:

$$\frac{u'(c_s^j)}{u'(y-c_s^j)} = \frac{q_s^j + Q_s^{j\prime}(l_s^j)l_s^j}{q_s^j}\frac{u'(c_0^j)}{u'(c_0^{j-})}.$$

The gap between the marginal rate of substitution of the borrower and the lender in a particular state is given by the term $(q_s^j + Q_s^{j\prime}(l_s^j)l_s^j)/q_s^j < 1$, to be compared with a value of 1 in the risk sharing case above. This price ratio is the current marginal gain to borrowing relative to the marginal cost to saving. It turns out that it is a negative function of the default rate d_s^j: higher default is associated with steeper borrowing costs. Therefore in general full risk sharing will fail to hold as long as default varies across states s. To see this more explicitly, using (9), we can write this expression as

$$\frac{u'(c_s^j)}{u'(y-c_s^j)} = \frac{1}{(\gamma-1)+\gamma\eta {d_s^j}^{\gamma}}\left[\gamma - 1 - \frac{d_s^j}{1-d_s^j}\right]\frac{u'(c_0^j)}{u'(c_0^{j-})}. \tag{10}$$

So individual consumption levels vary with the default rate d_s^j in a given state s. More specifically, the borrower's consumption c_s^j increases with the default rate. With higher default and the resulting increased costs of borrowing, this individual will have borrowed less against s and can consume more in that state. The implications of (10) can be summarized graphically by the positively sloped curve in Figure 1 associated with intertemporal optimality.[15]

[15]Note this figure holds the variables c_0^{j-} and c_0^j fixed. This is correct as the analysis here considers differences across states within the same equilibrium and, therefore, the same initial consumption allocations.

The optimal choice of default provides another key relation between the default rate and consumption of an individual who is in debt. The optimality condition for default (5), using the functional assumptions in (1) and market clearing (8), can be written as

$$\frac{y_s^j - c_s^j}{c_s^j} = \gamma\eta(1 - d_s^j){d_s^j}^{\gamma-1}. \tag{11}$$

This expression relates consumption, default and the endowment in a particular state. It will be important to know how the right-hand side of this expression changes with the default rate. Its derivative has the same sign as $(\gamma - 1)/\gamma - d_s^j$ so the RHS of (11) is hump shaped. Now, the fact stated in Proposition 3 that $(\gamma-1)/\gamma - d_s^j > 0$ implies that the RHS of (11) is increasing in the default rate d_s^j. This implies that more consumption leads to a lower default rate. With a lower marginal utility, the household is in less need to prop up consumption by means of shunning promised deliveries. Graphically, this relationship (11) is represented as the negatively sloped optimal default curve in Figure 1.

We discuss now how individual consumption changes across states s. We consider first changes where household j remains the debtor so (10) and (11) apply throughout. Suppose that the income endowment y_s^j increases across two states s. Condition (11) can be rewritten $y_s^j = c_s^j + c_s^j\gamma\eta(1 - d_s^j){d_s^j}^{\gamma-1}$. Given the positive relationship between consumption and default from (10), the state with higher income will have a higher consumption if the RHS of (11) is increasing in d_s^j, a property we have just established. More intuitively, for given consumption, a larger endowment means the household must have higher debts and defaults more (i.e., (11)); markets read steeper interest rates into this and households borrow less in the first place and consume more in the second period (i.e., (10)). Graphically in Figure 1, consider the equilibrium allocation in state s as the intersection of the two solid curves. For a state with a larger endowment, the optimal default curve lies further to the right, implying a higher level of consumption.

Proposition 4 *(Imperfect income risk sharing with constant roles) Across states s where type j is the debtor, its consumption c_s^j and income endowment y_s^j are positively correlated.*

Note that the response of consumption to the rise in idiosyncratic income is mediated by a rise in default which, by increasing the cost of further borrowing in equilibrium, deters the accumulation of debt. However, this positive association between income and default is not a necessary implication of the theory if there is some other factors affecting the cost of borrowing that also changes with individual income. This will be discussed below.

Consider now comparisons across states where a particular household changes role. That is, suppose an equilibrium where agent j holds debt in state s and holds assets in another state s'. Again, optimal borrowing against state s is characterized using (2) by

$$\frac{u'(c_s^j)}{u'(c_0^j)} = \frac{q_s^j + Q_s^{j\prime}(l_s^j)l_s^j}{\beta(1-d_s^j)\pi_s},$$

and optimal saving against state s' reads from (3) as

$$\frac{u'(c_{s'}^j)}{u'(c_0^j)} = \frac{p_{s'}}{\beta\pi_{s'}} = \frac{q_{s'}^{j-}}{\beta(1-d_{s'}^{j-})\pi_{s'}},$$

where the last equality follows from the zero-profit condition (7). Now perfect risk sharing amounts to having an equality between the two RHS terms in these two expressions. We show this cannot be true by way of contradiction. Assuming this equality, the property that $Q_s^{j\prime}(.) < 0$ implies that $q_s^j/(\beta(1-d_s^j)\pi_s) > q_{s'}^{j-}/(\beta(1-d_{s'}^{j-})\pi_{s'})$. From market clearing in (8), there must also be risk sharing for agent j, which, by an analogous argument, means that the contrary inequality must hold. The same argument can be used to establish that risk-sharing fails to hold in the sense that

$$\frac{q_s^i + Q_s^{j\prime}(l_s^j)l_s^j}{\beta(1-d_s^j)\pi_s} < \frac{q_{s'}^{j-}}{\beta(1-d_{s'}^{j-})\pi_{s'}}. \tag{12}$$

The marginal cost to debt exceeds the marginal return to savings. This property together with the preceding two expressions implies that $c_s^j > c_{s'}^j$. That is, a household's consumption is higher in the states where they hold debt than in the states where they hold savings. Given this we can also establish how income correlates with consumption. Consumption levels for household j are determined from (8) as $c_s^j = y_s^j - (1-d_s^j)l_s^j$ and $c_{s'}^j = y_{s'}^j + (1-d_{s'}^{j-})l_{s'}^{j-}$. The fact that $c_s^j > c_{s'}^j$ clearly requires that $y_s^j > y_{s'}^j$ so consumption and income are positively related.

Proposition 5 *(Default, debt/saving and consumption with changing roles) Across states s where household j switches between debtor and saver, its consumption c_s^j, debt level l_s^j and income endowment y_s^j are positively correlated.*

Having established that risk sharing fails, some further intuition might be helpful. One way to gain this intuition is by comparison with the outcomes for an economy where default rates are exogenous but coincide with the equilibrium default rates of our economy. Consider the constraints that determine consumption in (8). In both economies, consumption changes less than income as the household holds more debt liabilities in states where income is higher. In the exogenous-default case the portfolio composition varies across states to an extent that neutralizes the effect of income on consumption. In the endogenous-default case, portfolios also adjust but to a lesser extent thus making consumption responsive to income.[16] In this case, the interest rate faced by the households rises gradually with the level of debt issued so they stop short of taking the large debt positions needed to fully insure consumption.

4.2 Consumption smoothing

We also would like to discuss how, for the same default rates, the economy with endogenous default and the economy with exogenous default differ in terms of allocations and prices. By construction, the risk premium is the same in the two economies. Therefore, one needs to figure out how the level of interest rates and hence consumption differ between them.

The simplest way to make this point is to consider a deterministic version of the model, with only one state in the second period. One can now compare this economy with the analogous case where default is exogenous, and the default rate is the same. The main difference between these two economies is found in the intertemporal optimality condition for the borrower in equation (2). For given prices of debt, with endogenous default the borrower faces a steeper interest rate, or a lower marginal benefit to issuing debt. She will issue less debt and consume less in the first period compared to the exogenous case. In equilibrium, a higher price of debt brings about the corresponding reduction in lending from the lender according to (3). Therefore, there is less consumption smoothing in the case where default is endogenous.

4.3 The relationship between income and default

In the previous analysis of Figure 1, higher income leads to more default. If one concedes that default in this model can be empirically meaningful, this might be a problematic implication. However, it is not a fundamental requirement of the model. It comes about only because of the simplifying maintained assumption that the punishment parameter η is the same for all individuals in all states. The punishment could plausibly be increasing in the level of individual income. Creditors can collect more from higher earnings; legal bankruptcy protection is often means tested.[17] If this is the case, higher income can cause the default rate to decrease and still have consumption increasing.

To see this define η_s^j as the default penalty for an individual of type j in state s, and substitute it for η in (10) and (11). Consider a change in the state s such that income y_s^j goes up. Suppose this also implies a rise in η_s^j. The graphic analysis in Figure 2 will suffice to make the point. As seen before, the increase in the endowment of income y_s^j shifts the default optimality curve to the right. The associated increase in the parameter for the cost of default η_s^j reduces default and shifts this curve back to the left. On the other

[16] Also a feature of models with debt constraints in the vein of Kehoe and Levine (1993).

[17] State-dependent penalties is also typical of models of sovereign debt like Arellano (2008) or Chatterjee and Eyigungor (2012).

hand, the higher cost of default reduces borrowing and increases consumption thus shifting the intertemporal optimality curve upwards. Note that the change in the punishment η_s^j can always be chosen to overturn the effect of income and leave the optimal default curve unchanged, in which case higher consumption and lower default follow as income rises. The same type of outcome still obtains when, as depicted in Figure 2, the reaction of the default cost is smaller.

4.4 Fixed penalty component

In the model discussed so far it is always optimal for the agent in debt to default on some amount of their promises. This might be at odds with observations on individual default's rates. The introduction of a fixed penalty component, say z_0, has the effect of inducing agents to default only in the states where the benefits from defaulting are large enough. The optimal default level, if positive, will still be determined by equation (5). For a positive z_0, there will in general be a range of debt positions low enough that the agent will choose not to default. However, if she chooses to default, all equilibrium values will be determined as in the main model.

4.5 Pooling

Dubey, Geanakoplos, and Shubik (2005) have studied specific examples of situations where contracts of different size are pooled together.[18] In the context of our model, pooling means that there is a single price q_s^i, rather than a menu of prices, for debt sold by agent i with delivery in state s, irrespective of the amount sold l_s^i. In other words, the menu of prices $Q_s^i(.)$ is a flat function. Formally, the equilibrium is characterized as before except that in the intertemporal optimality condition for the borrower (2) the term ${Q_s^i}'(l_s^i) = 0$. A consequence is that this economy achieves the first best allocation with full risk sharing. Although, by virtue of (5), default rates are endogenous, market prices account for the event of default efficiently. This observation underscores the conclusion that default-based pricing of loans is an essential ingredient in our explanation of endogenous financial frictions. With pooling, further exogenous degrees of market incompleteness are required in order to account for imperfect insurance.

5 Consumption inequality: default pricing or debt constraints

This section discusses the differences between the present default-pricing model and the debt-constrained model regarding the implications for consumption inequality. The ability of different models for understanding the evidence on risk-sharing has been the subject of an important body of quantitative literature (e.g., Krueger and Perri (2006), Kaplan and Violante (2010)). The conclusion that seems to emerge is that the debt-constrained model in the Kehoe and Levine (1993) tradition underestimates inequality. It is thus important to

[18] See their examples 1, 2 and 3.

understand how the default-pricing model could comparatively perform on this dimension.[19]

The debt-constrained economy obtains by changing the specification of the punishment technology z in (1). It suffices to assume that the utility penalty is a fixed amount, equivalent to the punishment for fully defaulting η, independently of the scale of default.[20] This assumption would naturally lead defaulters to default on the totality of their debts. This being so will cause intermediaries not to offer loans that induce any default (i.e., to command a price of zero). This results in a situation where loans traded carry no default risk, so lending rates are risk free and (7) implies $q_s^i = p_s$, but the size of loans is constrained by the maximum debt that would not trigger default. Formally, individuals of type i face a debt limit against state s that we denote $\bar{l}_s^i$. This is determined as the value that makes the individual indifferent between defaulting or not:

$$u(y_s^i) - \eta = u(y_s^i - \bar{l}_s^i). \tag{13}$$

Faced with such a limit to the value of debt that can be sold, the consumer optimality condition that replaces (2) and (3) is

$$p_s u'(c_0^i) \geq \beta \pi_s u'(c_s^i), \tag{14}$$

with equality if the debt constraint does not bind, and generically an inequality when the constraint binds $l_s^i = \bar{l}_s^i$. This economy features imperfect risk sharing if in some states and for some households the debt constraint binds.

One could argue that, at least for some level of the default penalty, the default-pricing model brings about more inequality than the debt-constrained economy. We have discussed that there is imperfect risk sharing in the default-pricing economy as long as there is positive default in equilibrium. For any finite penalty, there will always be some default in that setting. On the other hand, in the debt-constrained economy one can surely find a large enough punishment that makes consumption inequality zero. By a continuity argument, there will also be a region where the debt-constrained economy has binding debt limits and incomplete insurance but still displays less inequality than the default-pricing economy. While some especial assumption on the punishment technology made in this paper might play a part, it betrays the sense that the default-pricing economy might have less risk-sharing more generally. If default can be partial, there will be more states where default and hence the distortion that follows is positive.

[19]See Cordoba (2008) and Broer (2011) for further analysis of the evidence.

[20]Therefore, if default happens in this case, it will be at a rate of 100 per cent. Therefore using the cost parameter $\eta 1^{\gamma} = \eta$ is defensible as the choice congruent with the punishment technology assumed in the default-pricing model. Choosing a smaller cost for the debt-constrained economy, while interesting, is more arbitrary. On the other hand, this debt-constrained economy can also be seen as a special case of the model where we are merely setting the elasticity of the punishment γ to zero.

5.1 A symmetric case

We use a specific version of the model to establish analytically the lower degree of risk sharing in the default-pricing economy compared to the debt-constrained economy. The two types of households have the same initial endowment y_0, with aggregate income $y = 2y_0$. There are only two states s in the second period. In one state, one type of household receives a high income y_l and the other type receives a low realization y_a (the choice of subscripts will become clear very soon); in the other state, the allocation of endowments across household types is the reverse. The total endowment is as in the initial period so $y_l + y_a = y$. The two states occur with the same probability $\pi = 1/2$. The two types of households are therefore ex-ante identical.

In an equilibrium with trade, the allocations of debt liabilities l, assets a and default d, and the prices of debt and assets q and p, are all independent of the state. Consumption in the first period c_0 is the same for the two types of households. In the second period, the household who receives the high endowment y_l owes promised deliveries of debt l and consumes c_l; the households with the low endowment y_a holds claims a and consumes c_a. These values of consumption are independent of the contingent state s. Appendix B discusses this characterisation in more detail. It is simple to define a measure of risk sharing, θ, as the difference between the high income endowment y_l and consumption in that state c_l:

$$\theta \equiv y_l - c_l \in [0, 1/2(y_l - y_a)]. \tag{15}$$

Zero risk sharing obtains when $\theta = 0$ and consumption matches the endowment; full risk sharing occurs when the total endowment is evenly split and $c_l = 1/2(y_l + y_a)$. A lower θ is associated with more consumption inequality.

Given the symmetry of outcomes, the equilibrium of the defaut-pricing economy can be written explicitly in terms of two relationships involving the default rate d and the degree of risk sharing θ. The first relation comes from the intertemporal optimal borrowing/lending conditions for the household (2) and (3). Having used market clearing $y = c_a + c_l$ from (8), no-arbitrage $p(1-d) = q$ from (7), the default-based price term from (9), and the definition in (15), we find the *cost-of-borrowing* condition:

$$\frac{u'(y_l - \theta)}{u'(y - y_l + \theta)} = \frac{1}{(\gamma - 1) + \gamma\eta d^\gamma}\left(\gamma - 1 - \frac{d}{1-d}\right) \tag{16}$$

This traces a negative relation between the default rate d and the extent of risk sharing θ.[21] The RHS term is the price term from (9) that captures the distortion or wedge that default-pricing brings about. As discussed earlier, this is a negative function of the default rate d as it makes it more costly for the household to borrow. The LHS is the relative marginal utility of the consumer in the good state which is positively related to risk sharing

[21] Naturally, we will be supposing that the condition for existence discussed earlier $\gamma - 1 - d/(1-d) > 0$ holds.

θ. A higher default rate reduces debt and hence increases consumption in the good state thus decreasing risk sharing. The second relationship between d and θ derives from the condition for optimal default (5). Using the budget relation $l(1-d) = y_l - c_l$ from (8), the *optimal-default* condition reads:

$$u'(y_l - \theta)\frac{\theta}{1-d} = \eta\gamma d^{\gamma-1} \tag{17}$$

It describes a positive relation between risk sharing and the default rate. More risk sharing means higher debt liabilities in the good state and consequently a bigger incentive to default. An equilibrium consists of values θ and d solving (16) and (17). Graphically, it is the intersection of the cost-of-borrowing curve and the optimal-default curve in the top panel of Figure 3.[22]

Consider now the equilibrium in the debt-constrained economy. In the case where the penalty η is large enough that the debt constraint is never binding the conclusion is straightforward. This economy has perfect insurance and, consequently, less inequality than the default-based pricing economy. We turn to the more interesting case where the debt limit binds. In the present symmetric environment, the value of consumer liabilities l coincides with the debt limit $\bar{l}$ which, by the consumer budget in (8), is related to the household's consumption in his good state by $\bar{l} = y_l - c_l$. The participation condition (13), with the definition in (15), allows us to characterize the degree of risk sharing in the debt-constrained economy as the value θ solving

$$u(y_l - \theta) - [u(y_l) - \eta] = 0 \tag{18}$$

The LHS is the gain to not defaulting, a decreasing function of θ. For a higher degree of risk sharing, the high-income household would be better off by choosing to default. Graphically, the equilibrium is the zero of the participation curve in the bottom panel of Figure 3.

We now compare the value of θ that solves (16) and (17) for the default-pricing economy, with the one that solves (18) for the debt-constrained economy. We will now use again the log utility assumption from (1). In the default-pricing model, we know the default rate cannot exceed the maximum consistent with existence, that is $d < \bar{d} \equiv (\gamma-1)/\gamma$. Then the default optimality condition (17) implies that the degree of risk sharing under default-pricing is bounded from above by

$$\frac{\eta y_l}{1/\bar{d}^{\gamma-1} + \eta.}$$

We now evaluate the participation condition on the LHS of (18) for the debt-constrained

[22] At high enough d, indicated as d max, the borrowing cost condition will not be well defined and existence would fail.

model at this level of risk sharing. This yields

$$-\log(1+\eta\overline{d}^{\gamma-1})+\eta>0,$$

implying that this level of risk sharing is too low to be an equilibrium for the debt-constrained economy. Figure 3 illustrates the argument graphically. We have thus established that there is more consumption inequality in our default-based pricing economy.

Proposition 6 *Consider the symmetric case. Given the same penalty parameter η, the default-pricing model generates less risk sharing and more consumption inequality than the debt-constrained model.*

Nonetheless, it is worth noting that, for a given penalty level in the default-pricing model, a sufficiently lower penalty level in the debt-constrained economy can be found – by shifting leftwards the participation curve on Figure 3 – so the two economies deliver the same degree of risk sharing.

5.2 Numerical illustration

At this point, we resort to numerical examples to illustrate this result as well as the fact that equilibrium existence is not a fiction. Only two random states s are assumed. The first example has symmetric consumers and the only difference between them is the state when they get the bad shock, but the probability is the same. This corresponds to the setting studied in Proposition 6. The parameters are $\beta=1$, $y_0^i=0.50$ all i, equal probabilities $\pi_s=0.50$, symmetric income processes y_s^i, $(0.60, 0.40)$, and punishment $\eta=0.12$ and $\gamma=3.0$. It suffices the consider outcomes for just one agent. Table 1 shows the results for the two economies and, as a benchmark reference, those under full risk-sharing for an economy with commitment. The default-pricing model delivers far more consumption volatility and less trade in assets. This is reflected in a lower risk-free lending rate and higher borrowing rates. This is a conservative example in that it sets a considerably low default penalty. Mild punishment parameters, needed in order to generate some inequality in the debt-constrained economy, are associated with very large default rates in the default-pricing economy.

The previous example focuses on within period insurance only. The second example involves effects on smoothing over time as well. This is an asymmetric economy in that household A, having a back-loaded endowment profile, borrows against all future states. The parameters are as follows: $\beta=1$, $\pi_s=0.50$ all s, $y_0^A=0.40$, $\{y_s^A\}=(0.54, 0.48)$, $y_0^B=0.60$, $\{y_s^B\}=(0.46, 0.52)$. The punishment parameters are the curvature $\gamma=3.0$, and the penalty level $\eta=0.15$. The results are displayed in Table 2. The debt-constrained model has the limit binding only in one of the states and individuals can achieve perfect intertemporal smoothing in the other state. We see that the default-pricing economy generates less smoothing both over time and across states since there is less trade.

6 Towards a more quantitative model

The model of household debt and partial default above contains very specific assumptions, including default penalties in the form of particular utility costs, the short two-period horizon, and only two types of households with exogenous ex-ante heterogeneity. This is an attempt to extend the model of the household and debt pricing by considering more realistic forms of default punishment and an infinite horizon. Furthermore, the model will have a continuum of heterogeneous households subject to individual idiosyncratic risk, and ex-post heterogeneity of income and wealth.[23]

The purpose is to begin to see how the ideas of this paper might work in a context more amenable to quantitative analysis. We demonstrate first that partial default in general equilibrium continues to arise and produce lack of insurance. Second, outcomes continue to support the finding that there is less risk sharing than in a comparable economy with debt constraints. Finally, we show that, at least in a partial-equilibrium sense, there are conditions such that the model of the household and debt pricing can deliver a reasonable profile of indebtedness and default in the population.

6.1 The model extended

There is a continuum of consumers. We adopt a recursive representation and consider stationary situations. The individual exogenous state is $s \in \mathcal{S} = \{1, 2, ..., S\}$. Each state is associated with a realization of income y_s for $s \in S$. Transition probabilities to state s' conditional on current state s are $\pi(s' \mid s)$ for $s, s' \in \mathcal{S}$. The endogenous states for each individual consist of credit status $h \in \{CC, NC\}$ and asset position a. The variable h is an indicator of whether the agent is credit constrained or not. Choices are over portfolios $(a'_{s'})_{s' \in \mathcal{S}}$ and default d. When not credit constrained, $h = NC$, and the default decision determines the probability $\delta(d, a)$ that the agent will remain unconstrained next period $h' = NC$. When credit constrained, $h = CC$, and the probability that the agent will become unconstrained next period is a given $\bar{\delta}$. In this constrained state, the household cannot default. (This is mainly to simplify the state space and computational burden.) There is a current non-pecuniary fixed penalty to defaulting z_0.[24] While default has not been forgiven, the household is in the constrained state and also experiences a financial deadweight loss equivalent to a proportion τ of the fraction of debt initially defaulted def.[25] Therefore, like in the literature on bankruptcy we have, besides a fixed utility cost, a pecuniary cost and a time exclusion cost from markets.

[23] In the vein of the literature studying risk sharing and default like Krueger and Perri (2006) or Chatterjee, Corbae, Nakajima, and Ríos-Rull (2007).

[24] We are dispensing with the convex smooth function of previous sections.

[25] Besides deadweight costs, Livshits, MacGee, and Tertilt (2010) also consider costs that recover loans for the lender which, for present purposes, we omit. A cost proportional to the total value of debt defaulted has been considered but not pursued further as it has some problematic implications for the pricing mappings.

There is a price $p_{s,s'}$ all $s, s' \in \mathcal{S}$ for positive positions (or savings) that will deliver in state s' given the current state s. For non-constrained household, the pricing kernel for debts specifies a price for the value of debt $(-a')$ in state s, for delivery in state s' next period. So it can be written as $Q_{s,s'}(a')$. For the timing, the household chooses the default variable before choosing the portfolio. The household decision can be represented recursively as follows.

The value function associated with the default decision for the unconstrained household

$$V_s^{NC}(a) = \max_d \left\{ \delta(d,a) W_s^{NC,NC}(a,d) + (1-\delta(d,a)) W_s^{NC,CC}(a,d) \right\},$$

which yields the policy function for default $D_s(a)$. The values $W_s^{NC,NC}$ and $W_s^{NC,CC}$ occur conditional on the realization of the credit status shock and are associated with the corresponding portfolio choices. When the NC household remains unconstrained NC,

$$W_s^{NC,NC}(a,d) = \max_{(a'_{s'})} \left\{ u(c) - z_0 + \beta \sum_{s' \in \mathcal{S}} \pi(s' \mid s) V_{s'}^{NC}(a'_{s'}) \right\}$$

subject to

$$c = y_s + a(1-d) - \sum_{s' \in \mathcal{S}} [1_{a'_{s'} \geq 0} a'_{s'} p_{s'} + 1_{a'_{s'} < 0} a'_{s'} Q_{s,s'}(a'_{s'})].$$

If she instead becomes constrained CC next period

$$W_s^{NC,CC}(a,d) = \max_{(a'_{s'})} \left\{ u(c) - z_0 + \beta \sum_{s' \in \mathcal{S}} \pi(s' \mid s) V_{s'}^{CC}(a'_{s'}, d) \right\}$$

subject to the above budget constraint and also that the burden carried from defaulting is given by d. The value of the credit constrained household is given by

$$V_s^{CC}(a,d) = \bar{\delta} W_s^{CC,NC}(a,d) + (1-\bar{\delta}) W_s^{CC,CC}(a,d),$$

where the value if the household is to regain access to credit

$$W_s^{CC,NC}(a,d) = \max_{(a'_{s'})} \left\{ u(c) + \beta \sum_{s' \in \mathcal{S}} \pi(s' \mid s) V_{s'}^{NC}(a'_{s'}) \right\}$$

subject to

$$c = y_s^i - \tau d + a - \sum_{s' \in \mathcal{S}} a'_{s'} p_{s'}.$$

and the credit constraint

$$a'_{s'} \geq 0,$$

and the value if she is to stay constrained

$$W_s^{CC,CC}(a,d) = \max_{(a'_{s'})} \left\{ u(c) + \beta \sum_{s' \in \mathcal{S}} \pi(s' \mid s) V_{s'}^{CC}(a'_{s'}, d) \right\},$$

subject to the same constraints.

One can think of trade in assets as occurring through intermediaries who sell insurance to and buy debts from individuals in each particular state.[26] By no-arbitrage (or zero profits in intermediation), debt prices must satisfy

$$p_{s,s'}(1 - D_{s'}(a')) = Q_{s,s'}(a').$$

Given prices $p_{s,s'}$, via standard iterations a stationary distribution over states $x(s,h,a,def)$ obtains which is used to characterise aggregate outcomes for consumption and assets demand. The model laid out so far keeps the price of positive securities $p_{s,s'}$ unexplained. We will discuss their determination in the specific context of the examples that follow.

6.2 Specification and basic parameters

We specify the probability of remaining unconstrained as a function of default only $\delta(d,a) \equiv \exp(-\gamma_\delta d)$, with γ_δ a positive parameter. The period utility function is the standard $u(c) = c^{1-\sigma}/(1-\sigma)$, with $\sigma > 0$.

The parameters related to preferences and the earnings process are set the same for all the experiments. For preferences we choose conventional σ and β. For income, we assume two realizations such that $y_1 > y_2$ and iid transition probabilities such that $\pi_1 > \pi_2$. This is a simple representation of the idea that there is a bad state which occurs with low probability and is temporary.[27] The i.i.d. property simplifies the state for the pricing schedule since the current state drops from the pricing condition which becomes $Q_{s'}(a') = (1 - D_{s'}(a'))p_{s'}$, and only two values for $p_{s'}$ will need to be pinned down. The parameter for redemption from the credit constrained state $\bar{\delta}$ will also be set, and equivalent to it taking one year, one usual choice in the literature.[28] Table 3 displays these chosen parameters.

The remaining parameters, related to default penalties, z_0, γ_δ, τ, and prices p_1 and p_2, will vary across the different illustrative examples.

[26] One could equivalently split insurers and lenders into separate operations who lend or borrow in a riskless bond. In any event, since individual risk is uncorrelated, we have to assume intermediaries can pool risks.

[27] This results in about 0.10 variance of the log of earnings, close to estimates of the transitory component of residual wages for the most recent periods in Heathcote, Storesletten, and Violante (2010), and of the same order as the 0.14 of the 3-state process for earnings in Fernandez-Villaverde and Krueger (2011).

[28] For example, Livshits, MacGee, and Tertilt (2010).

6.3 Risk sharing

We first show that partial default arises and is associated with lack of insurance. We consider the case where security prices satisfy the no arbitrage condition that p_s/π_s is constant across states $s = 1, ..., S$, and clear the market for goods.[29]

Our object here is to see how outcomes vary with the default penalties. Table 4 displays illustrative results. The first three rows show the chosen parameters, then the market clearing security prices $p_{s'}$, and the variance of log consumption in the cross section. The following rows describe aspects of the distribution including the proportion of households with positive debts, the proportion who do default, asset position of debtors, the average income of debtors and defaulters, and average proportion of debt defaulted by debtors and defaulters. (Asset and income are normalised by average earnings.)

We consider first the full enforcement case - in the sense that default $D_s(a) = 0$ all s and a - by assuming a large enough value of the financial cost of default parameter τ. It is enough to assume $\tau = 50.0$, with $z_0 = 0$ and $\gamma_\delta = 1$. Since there is full risk sharing and perfect smoothing, prices in this case satisfy $p_s = \beta\pi_s$ all s. As shown in the first column of Table 4, the variance of log consumption is zero and there is full insurance. There is positive debts, at a level roughly equivalent to 7 per cent of average income. Households buy insurance against the bad state and borrow against the good state, so the debtors and high earners coincide as indicated by the above-average earnings of debtors.

The default penalty parameter τ is now decreased. The second column of Table 4 shows outcomes for a lower cost parameter $\tau = 0.60$. At this level, there is incentive to default, with debt prices declining with the level of debt taken. At the initial prices for positive securities, households would save more and borrow less for precautionary reasons. Security prices therefore increase in the market-clearing equilibrium. These changes lead to an increase in consumption inequality as the variance of log consumption is now up to 0.043 per cent. The proportion of debtors declines slightly and the average value of debts decreases. Their earnings is above the average. All debtors default and the proportion of debt defaulted is 1.7 per cent.

This example still presents the anomaly that all debtors default. As shown in the third column of Table 4, introducing a positive fixed cost of defaulting z_0 brings down the proportion of defaulters. Now only relatively large debts default, as shown by the higher default rate among defaulters, but there is a reduction of the write off rate.[30]

[29] In this way p_s/π_s is the inverse of the risk-free rate on a redundant bond. Like in Krueger and Perri (2006), this will be making idiosyncratic risk fully insured, rather than uninsurable, in the case when there is perfect enforcement.

[30] This fixed cost is non-pecuniary. A financial cost works to exactly the same effect but will require additional notation.

The message is that with an infinite horizon, an endogenous wealth distribution, and realistic pecuniary costs of default, the model continues to produces partial default in equilibrium, with only a subset of debtors defaulting, and accounts for lack of insurance. The final two columns os Table 4 shows similar outcomes for a lower cost parameter $\tau = 0.40$.

6.4 Comparison with debt-constrained economies

Using the extended model, we demonstrate that the default-pricing approach implies higher volatility of consumption and less risk sharing than the debt-constrained approach. This will thus support the claims in the same direction made earlier in Section 5 based on the simpler two-period economy.

As discussed, the debt-constrained economy is one where default becomes a binary choice $d \in \{0, 1\}$. This obtains by assuming that the penalties for defaulting are all fixed, independently of the proportion defaulted. In the present quantitative model, such penalties are represented by the probability of not becoming credit constrained after default $\delta(d, a) = \exp(-\gamma_\delta d)$, and the pecuniary cost $\tau \times d$. For our purpose here, we can fix them at the values corresponding to the only possible positive default level of $d = 1$, that is $\delta(d, a) = \exp(-\gamma_\delta)$ and $\tau \times d = \tau$, if $d > 0$; otherwise, if $d = 0$, there is no penalty so $\delta(d, a) = 1$ and $\tau \times d = 0$.

For alternative pairs of the parameters τ and γ_δ, Table 5 displays the outcomes for the percentage standard deviation of log consumption, market-clearing contingent asset prices, the proportion of individuals with positive debts, and the average level of debt. The first row is the benchmark for the comparison, and reproduces values in our default-pricing economy from the last column of Table 4 seen earlier. The remaining rows are for the debt-constrained economy with fixed penalties. The second row corresponds to the debt-constrained economy for the same parameters τ and γ_δ as the benchmark. In this case, the fixed penalties for default are large enough that the economy behaves exactly as an economy with full enforcement and zero volatility of consumption. This conveys the message in its starkest form.

In order to generate imperfect insurance in the debt constrained economy, one will have to decrease the fixed default penalties. Will this raise consumption volatility in the debt-constrained economy above that of the default-pricing economy? The third row in Table 5 is for a ten-fold reduction in the pecuniary cost to defaulting τ. As reported, this economy shows now imperfect insurance yet, in spite of a highly conservative reduced default penalty, the volatility of consumption is still below the benchmark economy's. Among theoretically comparable economies (i.e., same structural parameters) there is more insurance in the debt-constrained economy than in the default-pricing economy. This is congruent with the previous analysis in Section 5 above. Figure 4 contains the policy functions for the proportion of debt defaulted in the benchmark and the debt-constrained economy. For the high income state, against which debts are held, default rises gradually with debt in the benchmark well before it surges in the debt-constrained economy.

Be that as it may, a more applied approach would like to compare outcomes of economies calibrated to similar targets, thereby acknowledging that parameter values may be different across them. The fourth row presents a case in point where the penalty parameter τ is chosen - or, in a limited sense, calibrated - so the debt-constrained economy delivers the same price of assets (i.e., the risk-free interest) as the default-pricing benchmark. This requires a somehow lower cost τ than in the preceding row therefore implying an increased consumption volatility which, nonetheless, continues to fall short of that in the default-pricing benchmark.

The last row in Table 5 shows a case in the same "calibration" vein but with also a reduced probability of credit exclusion from defaulting as γ_δ is halved.[31] The much lower cost of default in this case leads to a level of consumption volatility still below but close to the benchmark economy.

In sum, the examples discussed here illustrate numerical explorations suggesting that for a reasonable broad range of possibilities there is more insurance and less consumption volatility in the debt-constrained model than in the default-pricing model. Establishing the practical ramifications of this will nonetheless require a richer quantitative implementation of the model.

6.5 Default and debt

The examples in Section 6.3 however do not quite illustrate quantitative possibilities of the model. Although figures for default rates and debt are arguably not off the mark, the implication that debts and hence default only happen among high earners, while perfectly logical, would be problematic for certain applications. At these actuarially fair prices for securities, consumers tend to buy insurance as an asset for the bad times and take debt to due in the good times.[32]

For a meaningful discussion of the quantitative implications for default and debt, the model should stand a chance of accounting for some debts being held in the bad state. In order to explore these possibilities, we depart here from market-clearing general equilibrium and consider exogenous conditions on security prices. A modification in this direction is to make saving (or insuring) against the bad state relatively less attractive, with a lower return for the bad-state security and/or a higher return for the good-state's. A simple way to do this is to have a single uniform price $p = p_{s'}$ for all securities s'. We choose this p as the average across the first-best prices weighted by the frequency of each state. In this example

[31] The probability of credit exclusion when defaulting $1 - \exp(-\gamma_\delta)$ goes from 63.2% in the previous cases down to 39.3%.

[32] A similar point is made in the recent quantitative analysis of a model à la Kehoe-Levine in Broer (2011).

therefore, we use $p = 0.70 \times 0.651 + 0.30 \times 0.279 = 0.5394$.

Under these conditions, we can now illustrate default behaviour and debt properties with a more quantitative approach.[33] We will consider parameters for the cost of default, z_0, γ_δ and τ, that proximately match observed data from the SCF 2007 for measures of assets, debt and default. We take a figure of (unsecured) debt as a proportion of average income to be around 12 per cent, and a ratio of net worth to income somehow above 2.[34] Regarding default, we take a view broader than formal bankruptcy so that it includes also informal default. On that count the data suggest formal bankruptcy of around 1.5 per cent and a proportion of 2-month+ late payers at 5-6 per cent.[35] Given some due latitude in interpreting the measure of late payers, a figure for the average default rate not far from the 5-7 per cent range will be reasonable.

The question of interest is whether the model delivers reasonable outcomes regarding the earnings position and default behavior of debtors and defaulters. The data indicates that debtors are not disproportionately income richer than non-debtors, and that defaulters are considerably income poorer.[36] Table 6 reports three examples in point, each for a different choice of γ_δ. The first three rows contain the default cost parameters. The next three rows report the corresponding loosely targeted variables, proportion of households who default, average asset position, and the average value of debt. They show a reasonable approximation to the data observations. The remaining two entries show the implied average income for debtors and for defaulters respectively. In all cases, debtors have lower earnings than the average individual, and defaulters have even lower earnings. Finally, for information, the proportion of debt defaulted by defaulters is in the last row.[37] Figure 5 contains the policy functions for the proportion of debt defaulted for the example where $\gamma_\delta = 1$. Interestingly, for a given level of debt, low-earnings individuals default more.

As a quantitatively relevant representation of the individual household this model has therefore potential. The specific maintained assumption of strict equality of security prices is a crude and merely illustrative one. We cannot rule out that a richer income process with persistence and a larger number of states could deliver in a more natural way, but this goes beyond the simple setting considered here.

[33]This setting for the prices of securities will cause lack of insurance for reasons unrelated to the default mechanism emphasized in this paper. The interest here however is not in risk sharing but rather in how the model stands as an account of household default and debt, leaving aside this time the issue of market clearing equilibrium.

[34]See for example, Benjamin and Mateos-Planas (2012) and Mateos-Planas and Ríos-Rull (2013). Livshits, MacGee, and Tertilt (2010) and Chatterjee, Corbae, Nakajima, and Ríos-Rull (2007) consider also figures for debt around 10 per cent.

[35]See Benjamin and Mateos-Planas (2012) or Díaz-Giménez, Glover, and Ríos-Rull (2011).

[36]See Díaz-Giménez, Glover, and Ríos-Rull (2011), Table 15 and Table 7.

[37]Additionally, in these examples the proportion of debtors is around 20 per cent so reassuringly only a fraction of debtors default. On the other hand, the measured proportion of debt defaulted varies between 4 and 7 per cent, not far from the charge-off rate used in, for example, Livshits, MacGee, and Tertilt (2010).

7 Concluding remarks

Default-based pricing and partial default in consumer credit can help to account for limited trade even when there is a complete set of securities. The specific mechanism differs from the existing literature on debt-constrained economies. The individualized pricing of default in this paper brings about price wedges that distort the borrowing decisions. This model seems to imply more consumption inequality than the debt-constrained model typical of much applied literature on the subject. This property might prove significant for understanding the evidence on consumption inequality.

The basic model in this paper is deliberately simple and, in the interest of analytical tractability, exploits very specific assumptions. An extended model nonetheless demonstrates the broader validity of this approach and its implications in a more quantitative setting. This model also appears to have potential to quantitatively account for features of the data on household default and debt. However, in order to draw firmer practical conclusions, a more thorough quantitative investigation is called for. This is being pursued in ongoing research.

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A Propositions 1, 2 and 3

Proof of proposition 1: The problem is $\max_d W(d) \equiv u(y_s^i - (1-d)l) - \eta d^\gamma$ subject to $d \in [0,1]$ and $y_s^i - (1-d)l \geq 0$, for some $l > 0$. The objective is continuous and the choice set is compact, so a solution exists. The objective is differentiable in the interior of the choice set, with the first and second derivatives $W'(d) = lu'(y_s^i - (1-d)l) - z'(d)$ and $W''(d) = l^2 u''(y_s^i - (1-d)l) - z''(d)$. Given $\gamma > 1$, the objective is thus strictly concave and the solution is unique. The derivative of the objective at the lowest feasible value is positive: if $y_s^i > l$, then $W'(0) = l/(y_s^i - l) > 0$; otherwise, $W'(1 - y_s^i/l) = +\infty$. At the other extreme, $W'(1) = l/y_s^i - \eta\gamma$. Therefore, if $l < y_s^i \eta\gamma 0$ the solution is interior and given by $W'(d) = 0$. Otherwise, if $l/y_s^i - \eta\gamma < 0$, the solution is at the corner with $d = 1$.

In an interior solution, $W'(d)$ increases with l; concavity implies that it decreases with d. Therefore, the optimal d rises with l. As l approaches $l/y_s^i - \eta\gamma$ from the left, the optimal d approaches 1. So d is continuous in l, and differentiable except at $l = y\eta\gamma$.

Proof of proposition 2: The first part is a direct implication of the properties of $D_s^i(.)$ in the previous Proposition and the determination of the price schedule in equation (6). Using (6) and (4) to differentiate implicitly $Q_s^i(l)l$, the resulting derivative is

$$
\begin{aligned}
Q_s^{i\prime}(l)l + Q_s^i(l) &= p_s \frac{\gamma\eta(\gamma-1)D_s^i(l)^{-1} - \gamma\eta\gamma}{\gamma\eta(\gamma-1)D_s^i(l)^{-1} + \gamma\eta l/(y_s^i - l(1 - D_s^i(l)))} \\
&= p_s \frac{\gamma\eta(\gamma-1)D_s^i(l)^{-1} - \gamma\eta\gamma}{\gamma\eta(\gamma-1)D_s^i(l)^{-1} + \gamma\eta\gamma\eta D_s^i(l)^{\gamma-1}},
\end{aligned}
$$

where the equality uses (4) again. This shows that $Q_s^i(l)l$ is increasing only when l is such that $D_s^i(l) < (\gamma-1)/\gamma$; that $D_s^i(.)$ is increasing implies the slope result. Concavity follows from the fact that, as $D_s^i(.)$ is increasing, the value $Q_s^{i\prime}(l)l + Q_s^i(l)$ decreases with l.

Proof of proposition 3: The shape of the pricing schedule means that in equilibrium the household will only choose $l_s^i < y_s^i \eta\gamma$. Therefore, default is less than full and the default and pricing functions are differentiable. Then condition (5) holds because, by proposition

1, (5) is satisfied and point (iv) of the definition holds. Consider the household's utility as a function of l_s^i with default determined by $D_s^i(l_s^i)$. Calculate the first and second derivatives, using the envelope property and second order condition on default $D(l)$. Because $Q_s^i(l)l_s^i$ is concave from proposition 2, this objective is also concave. That l_s^i is positive means the derivative is initially positive; that l_s^i eventually leads to full default and zero price means the derivative becomes negative for l_s^i large enough, and the unique solution is given by the first order condition (2). Note that this optimal solution requires $q_s^i + Q_s^{i\prime}(l_s^i)l_s^i$ to be positive which, by proposition 2, implies the upper bound for default.

B Characterisation of the equilibrium in Section 5.1

Symmetric fundamentals means $y_1^A = y_2^B$ and $y_1^B = y_2^A$. We can suppose, without any loss of generality, that $y_1^A = y_2^B > y_1^B = y_2^A$. We want to argue first that debts are held in the good state so $l_1^A > 0$ ad $l_2^B > 0$, that an equilibrium can be characterised by symmetric allocations, and that aggregate variables are then state independent. We consider only equilibria where there is some trade.

Debts held in high-income states. One can first argue that consistency with consumer constraints and market clearing in (8) and savers' optimality (3) requires that debts are held either in the individual bad states or in the good states. That is, either $l_1^A > 0$ and $l_2^B > 0$ or $l_2^A > 0$ and $l_1^B > 0$. For example, suppose, by way of contradiction, $l_1^A > 0$ and $l_2^B = 0$. Then (8b) implies $c_1^A < y_1^A$ and thus market clearing requires $c_1^B > y_1^B$. That $l_2^B = 0$ implies, by (8b), $c_2^A = y_2^A$ hence, by market clearing, $c_2^B = y_2^B$, and, by (8b), $l_2^A = 0$. By (3), $c_0^A = c_0^B$. But then, the constraints in (8a) imply $c_0^A > c_0^B$. A contradiction. Specifically, we can establish that $l_1^A > 0$ and $l_2^B > 0$ will be the case. By (8), this amounts to showing that $c_1^A < y_1^A$ and $c_2^B < y_2^B$. Suppose, by way of contradiction and given the first result above, that $c_1^A > y_1^A$ and $c_2^B > y_2^B$. By market clearing in (8) and (7) it follows that $c_1^B < y_1^B$ and $c_2^A < y_2^A$. Therefore $c_2^A < c_1^A$ and then $u(c_2^A)/u(c_1^A) > [q_2^A + Q_2^{A\prime}(l_2^A)l_2^A]/(p_2(1-d_2^A)$, which violates optimality conditions (2) and (3). A contradiction.

Symmetric allocations. Suppose $p_1 = p_2$, then (4) and (6) imply $Q_1^A(.) = Q_2^B(.)$. Suppose also $q_1^A = q_2^B$. Now consider the optimality conditions (2), (3), (5), (7), (8a) and (8b) for $i = A$, involving c_0^A, c_1^A, c_2^A, l_1^A and d_1^A. Given the symmetry of prices, conditions (2), (3), (5), (7), (8a) and (8b) for $i = B$ are satisfied for c_0^B, c_2^B, c_1^B, l_2^B and d_2^B equating, respectively, c_0^A, c_1^A, c_2^A, l_1^A and d_1^A. Finally, as the equilibrium condition (8c) holds for c_2^A, symmetric allocations also imply that it holds for c_1^B. So the equalization of prices assumed $p_1 = p_2$ and $q_1^A = q_2^B$ satisfies all the equilibrium conditions.

Constant values. Given the symmetric allocations, aggregate variables in section 5.1 are labeled as $y_l = y_1^A = y_2^B$, $c_l = c_1^A = c_2^B$, $y_a = y_2^A = y_1^B$, $c_a = c_2^A = c_1^B$, $l = l_1^A = l_2^B > 0$ and $d = d_1^A = d_2^B > 0$, and $p = p_1 = p_2$ and $q = q_1^A = q_2^B$. Clearly these values remain invariant to the state $s = 1, 2$.

Table 1: **Symmetric case**

	RISK-SHARING	DEFAULT-PRICING	DEBT-CONSTRAINED
c_0^i	0.500	0.500	0.500
$\{c_s^i\}$	(0.500, 0.500)	(0.583, 0.417)	(0.532, 0.468)
l_s^i	0.100	0.0268	0.0678
p_s	0.500	0.599	0.534
d_s	–	0.357	0.000
q_s	0.500	0.385	0.534
$\bar{l}$	–	–	0.0678

Table 2: **Asymmetric case**

	RISK-SHARING	DEFAULT-PRICING	DEBT-CONSTRAINED
c_0^A	0.4550	0.4190	0.4523
$\{c_s^A\}$	(0.4550, 0.4550)	(0.5183, 0.4691)	(0.4647, 0.4523)
c_0^B	0.5450	0.5810	0.5477
$\{c_s^B\}$	(0.5450, 0.5450)	(0.4818, 0.5309)	(0.5353, 0.5477)
l_s^A	(0.085, 0.025)	(0.0356, 0.0148)	(0.0753, 0.0277)
p_s	(0.500, 0.500)	(0.6029, 0.5471)	(0.5116, 0.5000)
q_s^A	(0.500, 0.500)	(0.3674, 0.4022)	(0.5116, 0.5000)
d_s	–	(0.3906, 0.2650)	–
$\bar{l}_s^A$	–	–	(0.0753, 0.0669)
$\bar{l}_s^B$	–	–	(0.0641, 0.0725)

Table 3: **Parameters set directly**

Description	Parameter	value
Risk aversion	σ	2.00
Discount	β	0.93
Earnings values	y_1, y_2	0.50 0.25
Transition probabilities	π_1, π_2	0.70 0.30
Redemption probability	$\bar{\delta}$	1.00

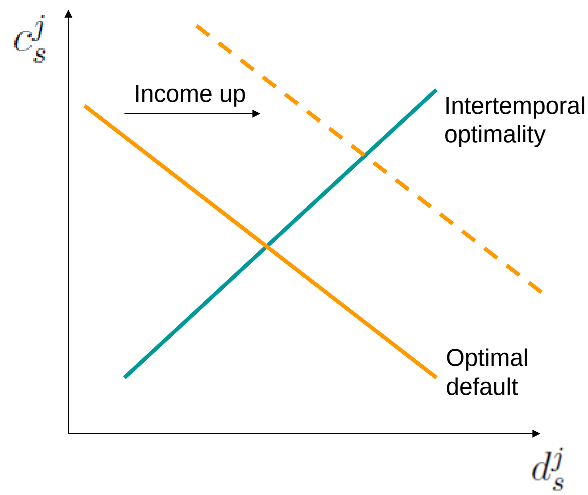

Figure 1: Equilibrium d_s^j and c_s^j when type j is a debtor in s.

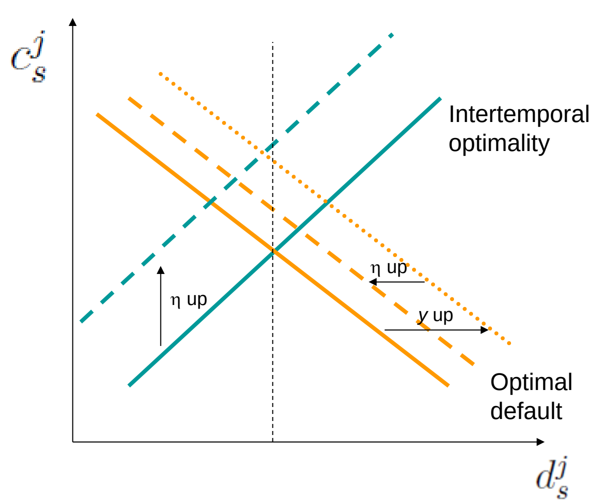

Figure 2: Equilibrium d_s^j and c_s^j when type j is a debtor in s. A rise in income y_s^j and η_s^j increases consumption and lowers default.

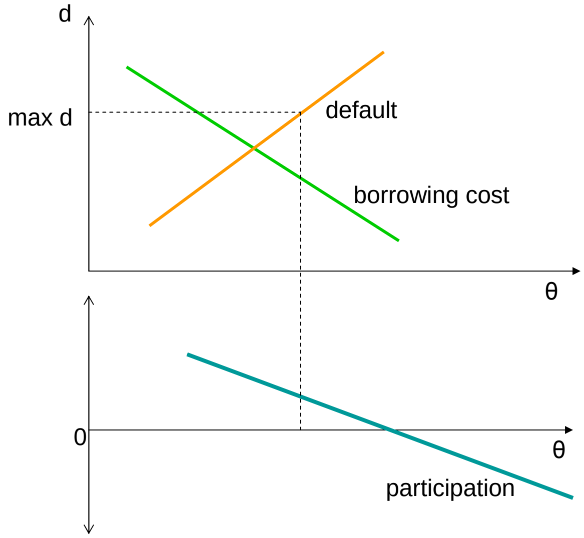

Figure 3: Equilibrium in default-pricing model (top) and in debt-constraint model (bottom).

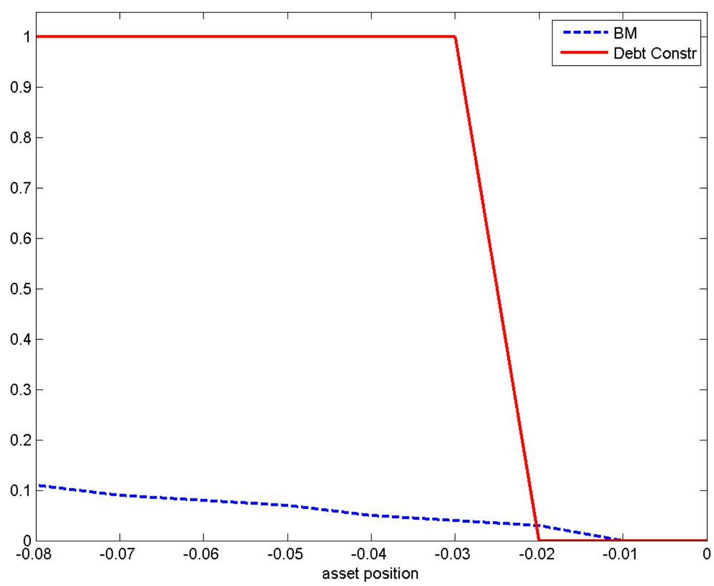

Figure 4: Default policy functions. High income state. Default-pricing benchmark: $\gamma_\delta = 1.0$, $\tau = 0.40$. Debt-constrained: $\gamma_\delta = 1.0$, $\tau = 0.04$.

Table 4: **Insurance and default**

Parameters					
τ	50.0	0.600	0.600	0.400	0.400
z_0	0.00	0.000	0.001	0.000	0.001
γ_δ	1.00	1.000	1.000	1.000	1.000
Variables					
p_1	0.651	0.659	0.661	0.663	0.665
p_2	0.279	0.282	0.283	0.284	0.285
% var log cons	0.000	0.043	0.046	0.040	0.041
proportion in debt	0.700	0.666	0.698	0.683	0.695
proportion defaulting	0.000	0.666	0.0342	0.683	0.0921
debt of debtors	0.0706	0.0447	0.0496	0.0360	0.0303
inc of debtors	1.170	1.170	1.170	1.170	1.170
inc of defaulters		1.170	1.170	1.170	1.170
d debtors	0.000	0.0171	0.0016	0.0181	0.0056
d defaulters		0.0171	0.0366	0.0181	0.0423

Table 5: **Insurance with debt constraints**

	Model	γ_δ	τ	% var cons	p_1	p_2	% in debt	debt
(1)	Default pricing	1.00	0.40	0.0413	0.665	0.285	0.695	0.0303
(2)	Debt constraints	1.00	0.40	0.0000	0.651	0.279	0.700	0.0706
(3)		1.00	0.04	0.0201	0.653	0.280	0.700	0.0471
(4)		1.00	0.03	0.0286	0.665	0.285	0.700	0.0235
(5)		0.50	0.035	0.0392	0.665	0.285	0.700	0.0235

Table 6: **Debt and default. Given $p = 0.5394$.**

Parameters			
γ_δ	2.00	1.00	0.50
τ	0.10	0.21	0.30
z_0	0.01	0.01	0.02
Variables			
proportion defaulting	0.0614	0.0797	0.0476
net asset	2.3106	2.2871	2.3082
debt of debtors	0.1099	0.1336	0.1169
inc of debtors	0.9186	0.9129	0.9129
inc of defaulters	0.7972	0.8071	0.6659
d defaulters	0.1340	0.1874	0.2640

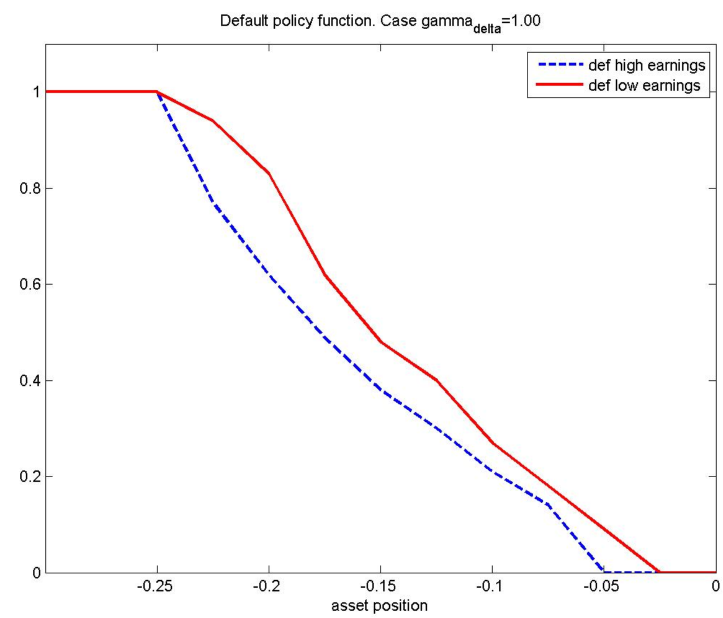

Figure 5: Default policy functions: $\gamma_\delta = 1.0$, $\tau = 0.21$, $z_0 = 0.02$.