Project 0 - Proof

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Proof. Hypothesis: A set S with cardinality $n \geq 1$ has exactly 2^n unique subsets

Base case: When $n = 1$, or when we are dealing with the set with one element, the set has two subsets: the set itself and $\{\emptyset\}$. So, a set with size $n = 1$ has two unique subset, and $2^1 = 2$, thus the base case $P(1)$ holds.

Inductive step: Fix some $k \geq 1$. Assume that the hypothesis holds true for $n = k$. This means that the size of the power set of set M with size k is 2^k, or set M with size k has 2^k unique subsets. We want to prove that set M with size $k+1$ has $2^k + 1$ unique subsets, or the size of the power set of set M with size $k+1$ is 2^{k+1}.

Assume for set M with size $k+1$, we have that the the set includes all elements up to and including a_{k+1}, where the power set of M is made up of all subsets of M:

$$|M| = \{k+1\} \text{ which is } M = \{a_1, a_2, \ldots, a_k, a_{k+1}\}$$

Thus, the power set of M will be:

$$P(M) = \{N | N \subseteq M\} = \{N | N \subseteq \{a_1, a_2, \ldots, a_{k+1}\}\}$$

Let A be the power set, or set of subsets up to and including a_k:

$$A = \{N | N \subseteq \{a_1, a_2, \ldots, a_k\}\} = P(\{a_1, a_2, \ldots, a_k\})$$

According to our assumption and induction hypothesis, we know that it has a size of 2^k, which can be written as $|A| = 2^k$.

Let B is the power set, or set of subsets up to and including a_{k+1}:

$$B = \{N \cup \{a_{k+1}\} | N \in A\}$$

Since the power set A is the power set of the set $\{a_1, a_2, \ldots, a_{k+1}\}$, the power set B is basically the power set A, but each set in the power set has an extra element of 2^{k+1}. Therefore, the size of the power set B is 2^k, which means that set B has 2^k unique subsets.

Thus, we can write the power set of M as the union of A and B, and we can find its cardinality by adding the sizes of A and B:

$$|P(M)| = |A| + |B| = 2^k + 2^k = 2^{k+1}$$

From this we can say that for the set of $k+1$ elements, there are 2^{k+1} unique subsets. Therefore, the invariant holds for the set of $k+1$ elements.

Conclusion: We know that $P(1)$ is true because of the base case. Since we know that $P(1)$ is true and we showed that $P(1)$ implies $P(2)$ in the inductive step (when $n \leq 1$), this means that $P(2)$ is true. Since we know that $P(2)$ is true and we showed that $P(2)$ implies $P(3)$ in the inductive step, we know that $P(3)$ is true. Continuing in this manner, we can show that $P(n)$ is true for all $n \geq 1$.

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Proof. Hypothesis: A set S with cardinality $n \geq 2$ has exactly $\frac{n \times (n-1)}{2}$ unique subsets with cardinality 2.

Base case: When $n = 2$, or when we are dealing with the set with two elements, there are one subset of cardinality 2, which is the set itself. So, a set with size $n = 2$ has one unique subset of cardinality 2, and $\frac{2 \times (2-1)}{2} = 1$, thus the base case $P(2)$ holds.

Inductive step: Fix some $k \geq 2$. Assume that the hypothesis holds true for $n = k$. This means that the the set M with size k has $\frac{k \times (k-1)}{2}$ unique subsets with cardinality 2. We want to prove that set M with size $k+1$ has $\frac{k \times (k+1)}{2}$ unique subsets with cardinality 2.

Assume for set M with size $k+1$, we have that the the set includes all elements up to and including a_k (k many elements) has a total of $\frac{k \times (k-1)}{2}$ subsets with cardinality 2, according to our inductive hypothesis.

When we include the a_{k+1} element, we are accounting for $k+1$ many elements in the set. To find the subsets of set M with this new element, with cardinality 2, we can simply add the element of a_{k+1} to every single one of the element up to and including a_k. This will gives us an additional k subsets of cardinality 2.

In total, the set M with $k+1$ elements will have:

$$\begin{aligned}
\frac{k \times (k-1)}{2} + k &= \frac{k \times (k-1) + 2k}{2} \\
&= \frac{k \times (k-1+2)}{2} \\
&= \frac{k \times (k+1)}{2}
\end{aligned}$$

subsets of cardinality 2. Therefore, the invariant holds for the set of $k+1$ elements.

Conclusion: We know that $P(2)$ is true because of the base case. Since we know that $P(2)$ is true and we showed that $P(2)$ implies $P(3)$ in the inductive step (when $n \leq 2$), this means that $P(3)$ is true. Since we know that $P(3)$ is true and we showed that $P(3)$ implies $P(4)$ in the inductive step, we know that $P(4)$ is true. Continuing in this manner, we can show that $P(n)$ is true for all $n \geq 2$.

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Proof. Hypothesis: The complement of the union of any n sets $S_1, S_2, \ldots, S_n$ is equivalent to the intersection of each of their individual complements (i.e., that $\overline{S_1 \cup S_2 \cup \ldots \cup S_n} = \overline{S_1} \cap \overline{S_2} \cap \ldots \cap \overline{S_n}$) for all $n \geq 1$.

Base case: When $n = 1$, the compliment of the union of one set and the intersection of its complement is basically the set itself. When $n = 2$, $\overline{S_1 \cup S_2} = \overline{S_1} \cap \overline{S_2}$, according to De Morgan's Law. Thus, the base case holds for $n = 1$ and $n = 2$.

Inductive step: Fix some $k \geq 2$. Assume that the hypothesis holds true for $n = k$. This means that the complement of the union of any k sets $S_1, S_2, \ldots, S_k$ is equivalent to the intersection of each of their individual complements, or $\overline{S_1 \cup S_2 \cup \ldots \cup S_k} = \overline{S_1} \cap \overline{S_2} \cap \ldots \cap \overline{S_k}$. We want to show that $\overline{S_1 \cup S_2 \cup \ldots \cup S_k \cup S_{k+1}} = \overline{S_1} \cap \overline{S_2} \cap \ldots \cap \overline{S_k} \cap \overline{S_{k+1}}$.

Assume that we have $k + 1$ sets, which gives us:

$$\begin{aligned}
\overline{S_1} \cap \overline{S_2} \cap \ldots \cap \overline{S_k} \cap \overline{S_{k+1}}. &= \overline{S_1 \cup S_2 \cup \ldots \cup S_k} \cap \overline{S_{k+1}} \text{ (inductive hypothesis)} \\
&= \overline{S_1 \cup S_2 \cup \ldots \cup S_k \cup S_{k+1}} \text{ (De Morgan's Law)}
\end{aligned}$$

Therefore, the invariant holds for $k + 1$ sets.

Conclusion: We know that $P(1)$ and $P(2)$ is true because of the base case. Since we know that $P(1)$ and $P(2)$ is true and we showed that $P(1)$ and $P(2)$ implies $P(3)$ in the inductive step (when $n \leq 2$), this means that $P(3)$ is true. Since we know that $P(3)$ is true and we showed that $P(3)$ implies $P(4)$ in the inductive step, we know that $P(4)$ is true. Continuing in this manner, we can show that $P(n)$ is true for all $n \geq 1$.

□

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Proof. Hypothesis: The intersection of any set S_1 with the difference of any set S_2 and S_1 is the empty set ($S_1 \cap (S_2 \setminus S_1) = \varnothing$).

Suppose for the sake of contradiction that the intersection of any set S_1 with the difference of any set S_2 and S_1 is not the empty set. Assume that $x \in S_1 \cap (S_2 \setminus S_1)$. This is only possible when $x \in S_1$ and $x \in (S_2 \setminus S_1)$.

Considering $x \in (S_2 \setminus S_1)$, we have that $x \in S_2$ and $x \notin S_1$. However, $x \in S_1$, as stated assumption above. This poses a contradiction, so $S_1 \cap (S_2 \setminus S_1) = \varnothing$. Therefore, the intersection of any set S_1 with the difference of any set S_2 and S_1 is the empty set.

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