Discrete Mathematics

BASIC STRUCTURES:
SETS, FUNCTIONS, SEQUENCES & SUMS

FPT University

Department of Mathematics

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Outline of Lecture

1. **Sets & Set Operations**
2. **Introduction to Functions**
3. **Inverse & Composition Functions**
4. **Sequences**
5. **Summations**

Textbook: Discrete Mathematics and Its Applications, Seventh edition, K.Rosen.

Upcoming ...

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Introduction to Sets

- A **set** is an unordered collection of objects.
- An object of a set is called an **element** or a **member**, of that set.
- The **cardinality** of the set A is the number of distinct elements of A, denoted by $|A|$.
- The **empty set**, denoted by $\emptyset$, is the set whose cardinality is 0.

Example.

1. The set $\{a, cat, catches, a, mouse\}$ has 4 elements.
2. The set $\{a, b, c, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}, \emptyset\}$ has 8 elements.

Subset

- If x is an element of A, we write $x \in A$. Otherwise, we write $x \notin A$.
- If all elements of A are also elements of B, we write $A \subseteq B$, and A is called a **subset** of B.
- If A is a **proper subset** of B, meaning $A \subseteq B$ and $A \neq B$, we write $A \subset B$.
- The empty set $\emptyset$ is a subset of any set; and the set A is a subset of itself, $A \subseteq A$.

Example. Which of the following statements are true?

1. $x \in \{x\}$ True
2. $x \subseteq \{x\}$ False
3. $\{a, b\} \subseteq \{a, b, \{a, b\}, c\}$ True
4. $\{a, b\} \in \{a, b, \{a, b\}, c\}$ True
5. $\emptyset \in \{\emptyset\}$ True
6. $\emptyset \subset \{\emptyset\}$ True
7. $\{a, b, c\} \subseteq \{a, b, c\}$ True
8. $\{a, b, c\} \in \{a, b, c\}$ False

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Cartesian Product

The **Cartesian product** of two sets A and B, denoted $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

In general, the Cartesian product of n sets $A_1, A_2, \ldots, A_n$ is defined as

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i, i = \overline{1, n}\}.$$

Example. Let $A = \{a, b\}$ and $B = \{1, 2, 3\}$. Then,

$$A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}.$$

Question. Given $A_1 = \{a, b\}, A_2 = \{1, 2, 3\}$ and $A_3 = \{x, y\}$. Determine all elements of the set $A_1 \times A_2 \times A_3$.

Power Set

The **power set** of the set A, denoted by $P(A)$, is the set of all subsets of A.

Example. Let $S = \{a, b, c\}$. Then

$$P(S) = \{a, b, c, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}, \emptyset\}.$$

Theorem

1. If $|A| = m$ and $|B| = n$ then $|A \times B| = mn$.
2. If $|A_1| = k_1, |A_2| = k_2, ..., |A_n| = k_n$ then $|A_1 \times A_2 \times \cdots \times A_n| = k_1 k_2 \ldots k_n$.
3. If $|A| = n$ then $|P(A)| = 2^n$.

Set Operations

Let A and B be two sets.

1. **Union** of A and B is

$$A \cup B = \{x \mid (x \in A) \vee (x \in B)\}.$$

2. **Intersection** of A and B is

$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}.$$

3. **Difference** of A and B is

$$A - B = \{x \mid (x \in A) \wedge (x \notin B)\}.$$

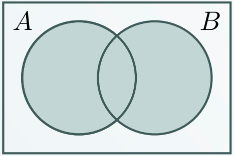
4. **Symmetric difference** of A and B is

$$A \oplus B = \{x \mid (x \in A) \oplus (x \in B)\}$$
$$A \oplus B = \{x \mid (x \in A \cup B) \wedge (x \notin A \cap B)\}.$$

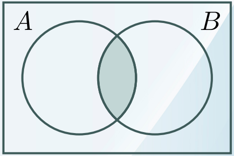
5. **Complement** of A with respect to the universal set U is

$$\bar{A} = U - A = \{x \mid (x \in U) \wedge (x \notin A)\}.$$

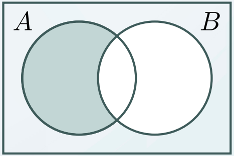
Set Operations: Venn Diagram



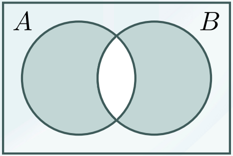
$A \cup B$



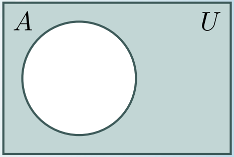
$A \cap B$



$A - B$



$A \oplus B$



$\bar{A}$

Set Identities

Complementation law	$\overline{\overline{A}} = A$
Identity laws	$A \cup \emptyset = A, \quad A \cap U = A$
Domination laws	$A \cup U = U, \quad A \cap \emptyset = \emptyset$
Complement laws	$A \cup \overline{A} = U, \quad A \cap \overline{A} = \emptyset$
Idempotent laws	$A \cup A = A, \quad A \cap A = A$
Commutative laws	$A \cup B = B \cup A$ $A \cap B = B \cap A$
Associative laws	$(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = A \cap (B \cap C)$
Distributive laws	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Absorption laws	$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$
De Morgan's laws	$\overline{A \cup B} = \overline{A} \cap \overline{B},$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$

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Computer Representation of Sets

Let U be a universal set. Fix an ordering of elements of U as $a_1, a_2, ..., a_n$. If A is a subset of U, represent A with a bit string of length n, where the ith bit is 1 if $a_i \in A$ and 0 if $a_i \notin A$.

Example. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

- The subset $A = \{1, 3, 5, 7, 9\}$ is represented as the bit string 1010101010.
- The subset $B = \{1, 8, 9\}$ is represented as the bit string 1000000110.
- We have

$$A \cup B = 1010101010 \vee 1000000110 = 1010101110$$

  which implies $A \cup B = \{1, 3, 5, 7, 8, 9\}$.
- We have

$$A \cap B = 1010101010 \wedge 1000000110 = 1000000010$$

  which implies $A \cap B = \{1, 9\}$.

Upcoming ...

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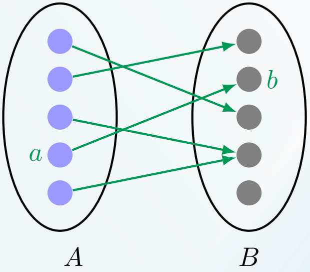
Functions



A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B.

The set A is called **domain** and B is called **codomain** of f.

If $f(a) = b$, we say b is the **image** of a and a is a **preimage** of b.

Let S be a subset of A. The set

$$f(S) = \{b \in B \mid \exists a \in A,\ f(a) = b\}$$

is called the **image of** S, and the set

$$f^{-1}(S) = \{a \in A \mid f(a) \in S\}$$

is called the **preimage of** S. The set $f(A)$ is called the **range** of f.

Some Important Functions

Given a real number x.

- **Floor function** is the function that gives as output the greatest integer less than or equal to x, denoted $\lfloor x \rfloor$ or **floor**(x).

$$\lfloor x \rfloor = \max\{m \in \mathbb{Z} \mid m \leq x\}.$$

- **Ceil function** is the function that maps to the least integer greater than or equal to x, denoted $\lceil x \rceil$ or **ceil**(x).

$$\lceil x \rceil = \min\{n \in \mathbb{Z} \mid n \geq x\}.$$

For all real numbers x,

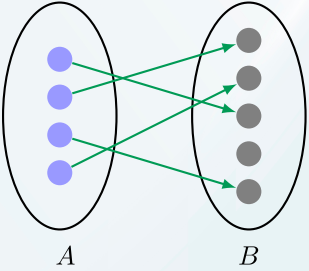
$$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1.$$

Example. Which statements are true for all real numbers x, y and all integers n?

- $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$
- $\lfloor x + n \rfloor = \lfloor x \rfloor + \lfloor n \rfloor$
- $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$
- $\lceil x + n \rceil = \lceil x \rceil + \lceil n \rceil$

One-to-one

The function $f : A \to B$ is **one-to-one** if $f(a_1) \neq f(a_2)$ for all $a_1 \neq a_2$ in A.



- A function $f : A \to B$ is said to be one-to-one if $f(x_1) = f(x_2) \Longrightarrow x_1 = x_2$ for all $x_1, x_2 \in A$.
- A one-to-one function is also called an **injection**.
- We call a function **injective** if it is one-to-one.
- A function that is NOT one-to-one is referred to as **many-to-one**.

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Problem Solving: One-to-one or Not?

Prove a Function is One-to-one

To conclude that the function $f : A \to B$ is one-to-one, we proceed as follows:

1. Assume $f(x_1) = f(x_2)$.
2. Show that it must be true that $x_1 = x_2$.

Prove a Function is not One-to-one

To conclude that the function $f : A \to B$ is not one-to-one, we take a counterexample where $x_1 \neq x_2$ and $f(x_1) = f(x_2)$.

Example. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x - 1$ is one-to-one. Indeed, assume that $f(x_1) = f(x_2)$ which means

$$2x_1 - 1 = 2x_2 - 1.$$

Therefore, $x_1 = x_2$. Hence, the function f is one-to-one.

Question

Which functions are one-to-one?

1. $f:\ \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2.$
2. $f:\ \mathbb{R}^+ \to \mathbb{R}, \quad f(x) = x^2.$
3. $f:\ \mathbb{Z} \to \mathbb{Z}, \quad f(n) = \left\lfloor \dfrac{n+1}{2} \right\rfloor.$
4. $f:\ \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad f(m,n) = m + n.$
5. $f:\ \mathbb{Z} \to \mathbb{Z}, \quad f(n) = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even} \\ \dfrac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$

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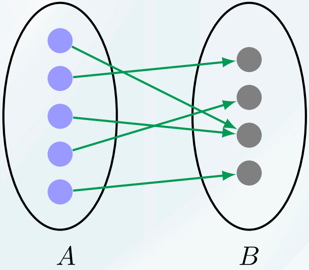
Onto

The function $f : A \to B$ is **onto** if for each b in B, there is a in A such that $f(a) = b$. In other words, the function $f : A \to B$ is onto if $f(A) = B$.



Note. An onto function is also called **surjection**, and we say it is **surjective**.

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Problem Solving: Onto or Not?

Prove a Function is Onto

To conclude that the function $f : A \to B$ is onto, we proceed as follows:

1. Let y be any element in the codomain B.
2. Figure out an element in the domain A that is a preimage of y.
3. Choose x equal to the value you found.
4. Demonstrate that x is indeed an element of the domain A.
5. Show that $f(x) = y$.

Example. Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 5x + 1$. Show that f is onto.

Solution. Let y be any element of $\mathbb{R}$. Choose $x = \frac{y-1}{5}$. It is easy to see that the real numbers are closed under subtraction and non-zero division, i.e., $x \in \mathbb{R}$. Also, $f(x) = f(\frac{y-1}{5}) = 5.\frac{y-1}{5} + 1 = y$.

Therefore, we found an $x \in \mathbb{R}$ such that $f(x) = y$. In other words, given an arbitrary element of the codomain, we have shown a preimage in the domain. We conclude that f is onto.

Question

1. Check whether the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by

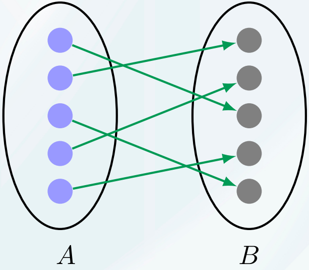
$$f(n) = \begin{cases} 2n & \text{if } n \geq 0 \\ -n & \text{if } n < 0 \end{cases}$$

   is one-to-one or onto?

2. Which of the following functions are onto?
   1. $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2.$
   2. $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^3.$
   3. $f : \mathbb{R} \to \mathbb{Z}, \quad f(x) = 2\lfloor x \rfloor.$
   4. $f : \mathbb{R} \to \mathbb{Z}, \quad f(x) = \lfloor 2x \rfloor.$
   5. $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad f(m, n) = m + n.$

Bijection

The function $f : A \to B$ is a **bijection** (or one-to-one correspondence) if it is both one-to-one and onto.



Note. If a function is a bijection, we say that it is **bijective**.

Problem Solving: Bijection or Not?

Prove a Function is Bijective

To conclude that the function $f : A \to B$ is bijective, we proceed as follows:

1. f is injective.
2. f is surjective.

Example. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x - 5$ is bijective.

Solution. We proceed as follows:

1. Show that f is injective.
   Assume that $f(x_1) = f(x_2)$. Then $3x_1 - 5 = 3x_2 - 5$ which implies $x_1 = x_2$.
2. Show that f is surjective.
   Let $y \in \mathbb{R}$ be an arbitrary. Choose $x = \frac{y+5}{3} \in \mathbb{R}$. Then

$$f(x) = f(\frac{y+5}{3}) = 3 \cdot \frac{y+5}{3} - 5 = y.$$

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Question

1. Which of the following functions are bijection?
   1. $f: \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2.$
   2. $f: \mathbb{R} \to \mathbb{R}, \quad f(x) = x^3.$
   3. $f: \mathbb{R} \to \mathbb{Z}, \quad f(x) = \lfloor 2x \rfloor.$
   4. $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad f(m, n) = m + n.$
   5. $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}, \quad f(m, n) = (m, m + n).$

2. Does there exist
   1. a bijection/ one-to-one/ onto function from a set of 7 elements to a set of 5 elements? from a set of 5 elements to a set of 7 elements?
   2. a bijection from the set of even integers to the set of odd integers?
   3. a bijection from the set of odd inetegers to the set of all integers?
   4. a bijection from the set of all real numbers to the set of positive real numbers?

Upcoming ...

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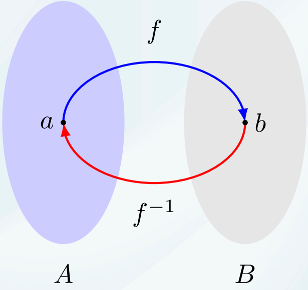
Inverse function

Let $f : A \to B$ be a bijective function. Then, its **inverse function** is the function

$$\begin{aligned} f^{-1} : B &\to A \\ b &\mapsto f^{-1}(b) = a. \end{aligned}$$

Note. In the above definition,

$$f^{-1}(b) = a \quad \text{equivalent} \quad b = f(a).$$



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Find f^{-1}

We can find the inverse function f^{-1} by following these steps:

1. Check if the function f is a bijective function.
   - If f is not a bijective function, stop, f^{-1} does not exist.
   - If f is a bijective function, we continue.
2. Since $f^{-1}(y) = x \Leftrightarrow y = f(x)$, then we solve for x and express x in terms of y.
3. The resulting expression is $f^{-1}(y)$.

Example. Find the inverse function of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x - 5$.

Solution. Since f is a bijective function, there exists f^{-1}. Put $y = 3x - 5$ which implies $x = \frac{y+5}{3}$.
Therefore,

$$\begin{aligned} f^{-1} : \mathbb{R} &\to \mathbb{R} \\ y \mapsto f^{-1}(y) &= \frac{y+5}{3}. \end{aligned}$$

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Question

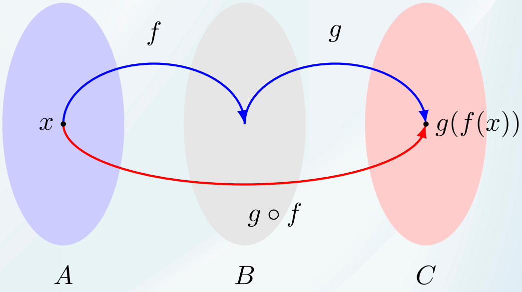
Find the inverse functions of the following functions:

1. $f: [-3, +\infty) \to [0, +\infty), \quad f(x) = \sqrt{x+3}.$
2. $g: \mathbb{R} \to (0, +\infty), \quad g(x) = e^x.$
3. $h: \mathbb{R} \to \mathbb{R}, \quad h(x) = \begin{cases} 3x & \text{if } x \leq 1 \\ 2x+1 & \text{if } x > 1. \end{cases}$

Composite function

Let $f : A \to B_1$ and $g : B \to C$ where $B_1 \subseteq B$. Then, the **composite function**, denoted as $g \circ f$, is defined by

$$\begin{aligned} g \circ f : A &\to C \\ x &\mapsto (g \circ f)(x) = g(f(x)). \end{aligned}$$



Note. In general,

$$f \circ g \neq g \circ f.$$

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Example

Example. Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x^{2023}$ and $g(x) = 2023x + 1$. Find $g \circ f$ and $f \circ g$.

Solution. We have

$$g \circ f : \mathbb{R} \to \mathbb{R}, (g \circ f)(x) = g(f(x)) = 2023f(x) + 1 = 2023x^{2023} + 1,$$

$$f \circ g : \mathbb{R} \to \mathbb{R}, (f \circ g)(x) = f(g(x)) = [g(x)]^{2023} = (2023x + 1)^{2023}.$$

Question. Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 2x + 1, & \text{if } x \geq 0 \\ 1 - 2x, & \text{otherwise.} \end{cases} \qquad \text{and } g(x) = x - 3.$$

Find $f \circ g$ and $g \circ f$.

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Upcoming ...

1. Sets & Set Operations
2. Introduction to Functions
3. Inverse & Composition Functions
4. **Sequences**
5. Summations

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Sequences

A **sequence** is a function from a subset of the set of integers (usually either the set $\{0, 1, 2, ...\}$ or the set $\{1, 2, 3, ...\}$) to a set S. We use the notation an to denote the image of the integer n. We call a_n a **term** of the sequence.

Example. Consider the sequence $\{a_n\}$, where $a_n = \dfrac{1}{n}$. The list of the terms of this sequence, beginning with a_1, namely,

$$a_1, a_2, a_3, a_4, \ldots,$$

starts with

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots.$$

Geometric & Arithmetic Progressions

A **geometric progression** is a sequence of the form

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

where the initial term a and the common ratio r are real numbers.

An **arithmetic progression** is a sequence of the form

$$a, a+d, a+2d, \ldots, a+nd, \ldots$$

where the initial term a and the common difference d are real numbers.

Example.

- *Geometric progression.* $1, 2, 4, 8, ...$
- *Arithmetic progression.* $1, 3, 5, 7, ...$

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Recurrence relation

A **recurrence relation** for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, ..., a_{n1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer. A sequence is called a **solution of a recurrence relation** if its terms satisfy the recurrence relation.

Example. The **Fibonacci sequence**, $f_0, f_1, f_2, ...$, is defined by the initial conditions $f_0 = 0, f_1 = 1$, and the recurrence relation

$$f_n = f_{n-1} + f_{n-2} \text{ for } n = 2, 3, 4, ...$$

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Student's Work

1. Find the first six terms of the sequence defined by each of these recurrence relations and initial conditions.

   - $a_n = 2a_{n-1},\ a_0 = 1$
   - $a_n = a_{n-1}a_{n-2},\ a_0 = 2, a_1 = 1$
   - $a_n = 3a_{n_1}^2 - 1,\ a_0 = 1$
   - $a_n = na_{n-1} + a_{n-2}^2,\ a_0 = 1, a_1 = 0$
   - $a_n = a_{n-1}a_{n-2} + a_{n-3},\ a_0 = 1, a_1 = 1, a_2 = 2$

2. Find a general formula for a_n of each sequence:

   - $1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, \ldots$
   - $1, 2, 2, 3, 4, 4, 5, 6, 6, 7, 8, 8, \ldots$
   - $1, 0, 2, 0, 4, 0, 8, 0, 16, 0, \ldots$
   - $3, 6, 12, 24, 48, 96, 192, \ldots$
   - $15, 8, 1, 6, 13, 20, 27, \ldots$
   - $3, 5, 8, 12, 17, 23, 30, 38, 47, \ldots$
   - $2, 16, 54, 128, 250, 432, 686, \ldots$
   - $2, 3, 7, 25, 121, 721, 5041, 40321, \ldots$

3. Let a_n be the nth term of the sequence

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6, 6, 6, 6, 6, 6, \ldots,$$

   constructed by including the integer k exactly k times.
   Show that $a_n = \left\lfloor \sqrt{2n} - \frac{1}{2} \right\rfloor$.

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Upcoming ...

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Summations Σ

To **express the sum** of the terms $a_m, a_{m+1}, \ldots, a_n$ from the sequence $\{a_n\}$. We use the notation

$$\sum_{j=m}^{n} a_j, \text{ or } \sum_{m \le j \le n} a_j.$$

Note. Here, the variable j is called the **index of summation**, and the choice of the letter j as the variable is arbitrary.

$$\sum_{j=m}^{n} a_j = \sum_{i=m}^{n} a_i = \sum_{k=m}^{n} a_k$$

Example.

- $\displaystyle\sum_{i=1}^{10} ia_i = a_1 + 2a_2 + \cdots + 10a_{10}$
- $\displaystyle\sum_{i=1}^{10} a_{2i-1} = a_1 + a_3 + \cdots + a_9$
- $\displaystyle\sum_{i=1}^{10} a_{2n}^2 = a_2^2 + a_4^2 + \cdots + a_{10}^2$
- $\displaystyle\sum_{i=1}^{10} 1 = \underbrace{1 + 1 + \cdots + 1}_{10 \text{ terms}}$

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Properties of Summations

Basic properties

1. $$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$
2. $$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i,\ c \text{ is a constant}$$
3. $$\sum_{i=1}^{n} a_i = \sum_{i=1}^{k} a_i + \sum_{i=k+1}^{n} a_i,\ 1 < k < n$$
4. $$\sum_{i=1}^{n} a_i = \sum_{i=j+1}^{j+k} a_{i-j},\ j \in \mathbb{Z}$$

Some Useful Summation Formulae

- $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$
- $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$
- $$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$
- $$\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}\ r \neq 1$$
- $$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x},\ |x| < 1$$
- $$\sum_{i=1}^{\infty} ix^{i-1} = \frac{1}{(x-1)^2},\ |x| < 1$$

Student's Work

1. What are the values of these sums?
   - $\displaystyle\sum_{i=1}^{5}(2i+i^2)$
   - $\displaystyle\sum_{i=1}^{5}\frac{1}{i(i+1)}$
   - $\displaystyle\sum_{i=1}^{5}\frac{3^i}{4^{i+1}}$
   - $\displaystyle\sum_{i=1}^{5}(2i-1)$
2. Compute each of these double sums.
   - $\displaystyle\sum_{i=1}^{5}\sum_{j=1}^{5}(2i+j)$
   - $\displaystyle\sum_{i=1}^{5}\sum_{j=1}^{5}(i-j)$
   - $\displaystyle\sum_{i=1}^{5}\sum_{j=1}^{5}ij$
   - $\displaystyle\sum_{i=1}^{5}\sum_{j=1}^{5}i^2j^3$
3. Find a formula for $\displaystyle\sum_{k=1}^{n}\left\lfloor\sqrt{k}\right\rfloor$.

Thank you!

Call it a day

"The irreducible price of learning is realizing that you do not know."
JAMES BALDWIN

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