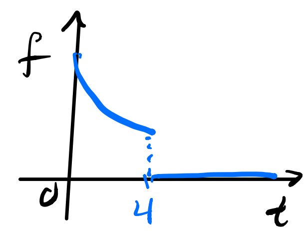
Lecture 16 Systems of ODEs

Example 5

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = f(t)$$

$$x(0) = 1, \quad \frac{dx}{dt}(0) = 0$$

$$f(t) = \begin{cases} e^{-3t} & : 0 \le t < 4 \\ 0 & : 4 \le t < \infty \end{cases}$$



$$f(t) = e^{-3t}\left[u_0(t) - u_4(t)\right] + \cancel{0 \cdot \left[u_4(t) - u_\infty(t)\right]}$$

Remember: $u_0(t) = 1$ for $t \geq 0$

$u_\infty(t) = 0$ since $t < \infty$

$$\Rightarrow f(t) = e^{-3t}\left[1 - u_4(t)\right]$$

$u_4(t) = u(t-4)$ ← same thing

Step 1 Take Laplace Transform of ODE

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = f(t)$$

#36 #35

$x'(0)=0$

$$\left[s^2X(s) - s\,x(0) - \cancel{x'(0)}\right] + 6\left[s\,X(s) - x(0)\right] + 9X(s) = F(s)$$

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\left\{e^{-3t}(1-u_4(t))\right\}$$

$$= \mathcal{L}\{e^{-3t}\} - \mathcal{L}\{e^{-3t}u_4(t)\}$$

#2 $a=-3$ #28 $(c=4)$

$$= \frac{1}{s+3} - e^{-4s}\mathcal{L}\left\{e^{-3(t+4)}\right\}$$

#28 $\mathcal{L}\{u_c(t)g(t)\} = e^{-cs}\mathcal{L}\{g(t+c)\}$

$$= \frac{1}{s+3} - e^{-4s}e^{-12}\mathcal{L}\{e^{-3t}\}$$

$$= \frac{1}{s+3} - \frac{e^{-4(s+3)}}{s+3} = \frac{1-e^{-4(s+3)}}{s+3}$$

$$s^2 X(s) - s + 6s X(s) - 6 + 9X(s) = \frac{1-e^{-4(s+3)}}{s+3}$$

Step 2) Solve for $X(s)$

$$(s^2 + 6s + 9) X(s) - s - 6 = \frac{1-e^{-4(s+3)}}{s+3}$$

$$\underbrace{(s^2 + 6s + 9)}_{= (s+3)^2} X(s) = s + 6 + \frac{1-e^{-4(s+3)}}{s+3}$$

$$\Rightarrow \boxed{X(s) = \frac{s+6}{(s+3)^2} + \frac{1-e^{-4(s+3)}}{(s+3)^3}}$$

Note: $s+6 = (s+3) + 3$

$$X(s) = \frac{s+3}{(s+3)^2} + \frac{3}{(s+3)^2} + \frac{1-e^{-4(s+3)}}{(s+3)^3}$$

$$= \frac{1}{s+3} + \frac{3}{(s+3)^2} + \frac{1-e^{-4(s+3)}}{(s+3)^3}$$

Use #29 : $\mathcal{L}\{e^{ct}f(t)\} = F(s-c)$

$$\Rightarrow \quad e^{-ct}f(t) = \mathcal{L}^{-1}\{F(s+c)\}$$

Step 3) Invert $x(t) = \mathcal{L}^{-1}\{X(s)\}$

$$x(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+3} + \frac{3}{(s+3)^2} + \frac{1-e^{-4(s+3)}}{(s+3)^3}\right\}$$

$$= e^{-3t}\mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{3}{s^2} + \frac{1}{s^3} - \frac{e^{-4s}}{s^3}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1 \qquad (\#1)$$

$$\mathcal{L}^{-1}\left\{\frac{3}{s^2}\right\} = 3\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = 3t \qquad (\#3)\ n=1$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = \frac{1}{2}t^2 \qquad (\#3)\ n=2$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s^3}\right\} = \frac{1}{2}(t-4)^2 u_4(t)$$

#27 $\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t-c)$

$$e^{-4s},\ F(s) = \frac{1}{s^3} \Rightarrow f(t) = \frac{1}{2}t^2$$

$$\Rightarrow \boxed{x(t) = e^{-3t}\left(1 + 3t + \tfrac{1}{2}t^2 - \tfrac{1}{2}(t-4)^2 u_4(t)\right)}$$

or $u(t-4)$

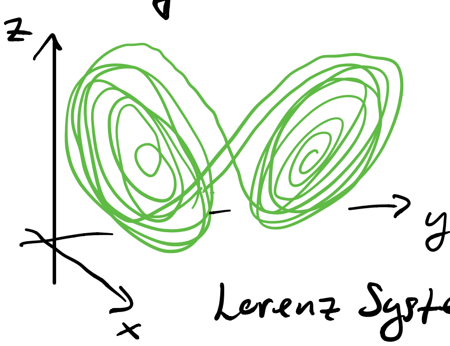
OK! Back to Systems of ODEs

So far, the ODEs have been in terms of 1 dependent variable

$$\frac{d^n y}{dt^n} = F\left(t, y, \frac{dy}{dt}, \dots\right)$$

However, problems usually more than 1.

$$x' = y - x$$
$$y' = x(1-z) - y$$
$$z' = xy - z$$



Lorenz System

Modeling Battle

$x(t)$ = size of army 1

$y(t)$ = sizer of army 2

$$\frac{dx}{dt} = -y \quad , \quad \frac{dy}{dt} = -x$$

Solve for $x(t)$ using a substitution

* Differentiate $x' = -y \Rightarrow x'' = -y'$

$$\Rightarrow x'' = -y' = -(-x) = x$$

or $x'' - x = 0$ $\quad (r^2 - 1 = 0$, $r = \pm 1)$

$$\boxed{x(t) = c_1 e^{t} + c_2 e^{-t}}$$

What is $y(t)$?

$$\boxed{y(t) = -\frac{dx}{dt} = -c_1 e^{t} + c_2 e^{-t}}$$

Example Predator-prey model

$x(t)$ = population of prey

$y(t)$ = population of predator

$$\frac{dx}{dt} = 3x - 4y \qquad (\text{Eq } 1)$$

$$\frac{dy}{dt} = 4x - 7y \qquad (\text{Eq } 2)$$

Let's find $x(t)$

Differentiate Eq 1 : $\dfrac{d^2x}{dt^2} = 3\dfrac{dx}{dt} - 4\dfrac{dy}{dt}$

$$\Rightarrow x'' = 3x' - 4y'$$

sub Eq 2

$$= 3x' - 4(4x - 7y)$$

$$= 3x' - 16x + 28y$$

$$x' = 3x - 4y \Rightarrow y = \frac{1}{4}(3x - x')$$

$$x'' = 3x' - 16x + \frac{28}{4}(3x - x')$$

$$= 3x' - 16x + 21x - 7x'$$

$$= -4x' + 5x$$

$$\Rightarrow \boxed{x'' + 4x' - 5x = 0}$$

$$r^2 + 4r - 5 = (r-1)(r+5) = 0$$

$$\Rightarrow \boxed{x(t) = c_1 e^{t} + c_2 e^{-5t}}$$

what about $y(t)$?

$$y(t) = \frac{3}{4}x(t) - \frac{1}{4}x'(t)$$

$$= \frac{3}{4}\left(c_1 e^{t} + c_2 e^{-5t}\right) - \frac{1}{4}\left(c_1 e^{t} - 5c_2 e^{-5t}\right)$$

$$= c_1\left(\frac{3}{4} - \frac{1}{4}\right)e^{t} + c_2\left(\frac{3}{4} + \frac{5}{4}\right)e^{-5t}$$

$$= \boxed{\frac{1}{2}c_1 e^{t} + 2c_2 e^{-5t}}$$

- Once you have the general solution, you can solve for c_1, c_2 using initial conditions.

Example $x(0) = 2$, $y(0) = 0$

$$x(0) = c_1 + c_2 = 2$$

$$y(0) = \frac{1}{2}c_1 + 2c_2 = 0$$

$$c_1 = -4c_2$$

$$-4c_2 + c_2 = -3c_2 = 2$$

$$\boxed{\begin{aligned} c_2 &= -2/3 \\ c_1 &= -4c_2 = 8/3 \end{aligned}}$$