Lecture 17 Systems of Equations

Example
$$\frac{dx}{dt} = -y + t^2$$
$$\frac{dy}{dt} = x + 1$$

$$x(0) = 0 \ , \quad y(0) = 1$$

(i)
$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(-y + t^2\right)$$
$$= -\frac{dy}{dt} + 2t$$
$$= -\left(x+1\right) + 2t$$
$$= -x - 1 + 2t$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} + x = 2t - 1 \ , \quad x(0) = 0$$

A) Homogeneous equation

$$\frac{d^2x_h}{dt^2} + x_h = 0$$

$$r^2 + 1 = 0 \leftarrow$$ Characteristic Equation

$$\Rightarrow r^2 = -1 \Rightarrow \boxed{r = \pm i}$$

Case II : $r = \alpha \pm i\beta \quad (\alpha = 0, \beta = 1)$

$$x_h(t) = e^{\alpha t}\left(c_1 \cos(\beta t) + c_2 \sin(\beta t)\right)$$

$$= c_1 \cos(t) + c_2 \sin(t)$$

B) Particular Solution

Trial solution: $x_p(t) = At + B$

$$\frac{d^2x_p}{dt^2} + x_p = 2t - 1$$

$$\frac{dx_p}{dt} = A, \quad \frac{d^2x_p}{dt^2} = 0$$

$$(0) + (At + B) = 2t - 1$$

$$\Rightarrow At + B = 2t - 1$$

$$\Rightarrow A = 2, \quad B = -1$$

$$x_p(t) = 2t - 1$$

c) General Solution:

$$\boxed{\begin{aligned} x(t) &= x_h(t) + x_p(t) \\ &= c_1 \cos(t) + c_2 \sin(t) + 2t - 1 \end{aligned}}$$

Next, use $\frac{dx}{dt} = -y + t^2$ to find $y(t)$

$$\Rightarrow y(t) = -\frac{dx}{dt} + t^2$$

$$\frac{dx}{dt} = -c_1 \sin(t) + c_2 \cos(t) + 2$$

$$\boxed{\begin{aligned} y(t) &= -\left(-c_1 \sin(t) + c_2 \cos(t) + 2\right) + t^2 \\ &= c_1 \sin(t) - c_2 \cos(t) + t^2 - 2 \end{aligned}}$$

Solve for c_1 & c_2 using the initial conditions: $x(0) = 0$, $y(0) = 1$

$$x(0) = c_1 \cos(0) + \cancel{c_2 \sin(0)} + \cancel{2(0)} - 1$$

$$= c_1 - 1 = 0 \Rightarrow \boxed{c_1 = 1}$$

$$y(0) = \cancel{c_1 \sin(0)} - c_2 \cos(0) + \cancel{0^2} - 2$$

$$= -c_2 - 2 = 1$$

$$\Rightarrow -c_2 = 3 \Rightarrow \boxed{c_2 = -3}$$

$$\boxed{\begin{aligned} x(t) &= \cos(t) - 3\sin(t) + 2t - 1 \\ y(t) &= \sin(t) + 3\cos(t) + t^2 - 2 \end{aligned}}$$

What happens for larger systems

$$\frac{dx}{dt} = x + 2y + 3z$$

$$\frac{dy}{dt} = 4x - 3y + 2z$$

$$\frac{dz}{dt} = x + 5y - z$$

The substitution approach, while still working, gets messy quickly!

ENTER: LINEAR ALGEBRA

Review of Linear Algebraic Equations.

- Let's say you have n variables:

$$\{x_1, x_2, x_3, \dots, x_n\}$$

- To solve for n unknowns, you need n equations.
- n linear equations can be expressed as:

$$
n \text{ equations} \left\{
\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3 \\
\quad \vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\qquad\qquad \vdots \qquad\quad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{nn}x_n = b_n
\end{array}
\right.
$$

Simplest Approach: Gaussian Elimination (Gauss-Jordan)

▷ Solve for $x_1, x_2, \dots x_n$ using row addition/multiplication

Example 1

$$2x_1 + 6x_2 + 8x_3 = 16 \quad (R1)$$
$$4x_1 + 15x_2 + 19x_3 = 38 \quad (R2)$$
$$2x_1 \qquad + 3x_3 = 6 \quad (R3)$$

1) Use R1 to eliminate x_1 from the other rows

2) Use R2 to eliminate x_2 from the other rows

3) Use R3 to eliminate x_3 from the other rows

- Subtract 2R1 from R2:

$$2x_1 + 6x_2 + 8x_3 = 16 \quad (R1)$$
$$3x_2 + 3x_3 = 6 \quad (R2)$$
$$2x_1 \qquad + 3x_3 = 6 \quad (R3)$$

- Subtract R1 from R3

$$2x_1 + 6x_2 + 8x_3 = 16 \quad (R1)$$
$$3x_2 + 3x_3 = 6 \quad (R2)$$
$$-6x_2 - 5x_3 = -10 \quad (R3)$$

- Subract 2·R2 from R1

$$2x_1 + 2x_3 = 4 \quad (R1)$$
$$3x_2 + 3x_3 = 6 \quad (R2)$$
$$-6x_2 - 5x_3 = -10 \quad (R3)$$

- Add 2·R2 to R3

$$2x_1 + 2x_3 = 4 \quad (R1)$$
$$3x_2 + 3x_3 = 6 \quad (R2)$$
$$x_3 = 2 \quad (R3)$$

- Subtract $2 \cdot R3$ from $R1$

$$
\begin{aligned}
2x_1 \qquad\qquad\qquad &= 0 \quad \text{(R1)} \\
3x_2 + 3x_3 &= 6 \quad \text{(R2)} \\
x_3 &= 2 \quad \text{(R3)}
\end{aligned}
$$

- Subtract $3 \cdot R3$ from $R2$

$$
\begin{aligned}
2x_1 \qquad\qquad &= 0 \quad \text{(R1)} \\
3x_2 \qquad &= 0 \quad \text{(R2)} \\
x_3 &= 2 \quad \text{(R3)}
\end{aligned}
$$

$\Rightarrow$ Solve: $2x_1 = 0$, $3x_2 = 0$, $x_3 = 2$

$$\boxed{\Rightarrow \quad x_1 = 0,\ x_2 = 0,\ x_3 = 2}$$

Two "issues" can come up.

Example 2 (issue 1)

$$
\begin{aligned}
x_1 + 2x_2 + 4x_3 + x_4 &= 0 \quad (R1)\\
-x_1 - 2x_2 - 2x_3 \phantom{{}+x_4} &= 1 \quad (R2)\\
-2x_1 - 4x_2 - 8x_3 + 2x_4 &= 4 \quad (R3)\\
x_1 + 4x_2 + 2x_3 \phantom{{}+x_4} &= -3 \quad (R4)
\end{aligned}
$$

1) Eliminate x_1 from R2, R3 & R4

Add R1 to R2, Add 2R1 to R3, Subtract R1 from R4

$$
\begin{aligned}
x_1 + 2x_2 + 4x_3 + x_4 &= 0 \quad (R1)\\
2x_3 + x_4 &= 1 \quad (R2)\\
4x_4 &= 4 \quad (R3)\\
2x_2 - 2x_3 - x_4 &= -3 \quad (R4)
\end{aligned}
$$

Swap

No x_2 in R2

New order:

$$x_1 + 2x_2 + 4x_3 + x_4 = 0 \quad (R1)$$
$$2x_2 - 2x_3 - x_4 = -3 \quad (R2)$$
$$4x_4 = 4 \quad (R3)$$
$$2x_3 + x_4 = 1 \quad (R4)$$

Subtract R2 from R1

$$x_1 + 6x_3 + 2x_4 = 3 \quad (R1)$$
$$2x_2 - 2x_3 - x_4 = -3 \quad (R2)$$
$$4x_4 = 4 \quad (R3)$$
$$2x_3 + x_4 = 1 \quad (R4)$$

Swap

No x_3 in R3

$$x_1 + 6x_3 + 2x_4 = 3 \quad (R1)$$
$$2x_2 - 2x_3 - x_4 = -3 \quad (R2)$$
$$2x_3 + x_4 = 1 \quad (R3)$$
$$4x_4 = 4 \quad (R4)$$

Add R3 to R2, Subtract 3R3 from R1

$$
\begin{aligned}
x_1 \quad\quad\quad\quad\quad\quad - x_4 &= 0 \quad (R1) \\
2x_2 \quad\quad\quad\quad &= -2 \quad (R2) \\
2x_3 + x_4 &= 1 \quad (R3) \\
\cancel{4}\, x_4 &= \cancel{4} \quad (R4)
\end{aligned}
$$

Divide R4 by 4

$$
\begin{aligned}
x_1 \quad\quad\quad\quad\quad\quad - x_4 &= 0 \quad (R1) \\
2x_2 \quad\quad\quad\quad &= -2 \quad (R2) \\
2x_3 + x_4 &= 1 \quad (R3) \\
x_4 &= 1 \quad (R4)
\end{aligned}
$$

Add R4 to R1, Subtract R4 from R3

$$
\begin{aligned}
x_1 \quad\quad\quad\quad\quad\quad &= 1 \quad \text{(R1)} \\
2x_2 \quad\quad\quad\quad &= -2 \quad \text{(R2)} \\
2x_3 \quad\quad &= 0 \quad \text{(R3)} \\
x_4 &= 1 \quad \text{(R4)}
\end{aligned}
$$

Solve: $x_1 = 1$, $2x_2 = -2$, $2x_3 = 0$, $x_4 = 1$

$$\boxed{\Rightarrow \quad x_1 = 1, \quad x_2 = -1, \quad x_3 = 0, \quad x_4 = 1}$$

The second issue is more serious

Example 3 (issue 2)

$$
\begin{aligned}
2x_1 + 4x_2 + x_3 &= 8 \quad \text{(R1)} \\
2x_1 + 4x_2 \quad\quad &= 6 \quad \text{(R2)} \\
-4x_1 - 8x_2 + x_3 &= -10 \quad \text{(R3)}
\end{aligned}
$$

Subtract R1 from R2, Add 2·R1 to R3

$$\begin{aligned} 2x_1 + 4x_2 + x_3 &= 8 \quad \text{(R1)} \\ -x_3 &= -2 \quad \text{(R2)} \\ 3x_3 &= 6 \quad \text{(R3)} \end{aligned}$$

No x_2 in R2 or R3! Use R2 to eliminate x_3?

Add R2 to R1, Add 3·R2 to R3

$$\begin{aligned} 2x_1 + 4x_2 &= 6 \quad \text{(R1)} \\ -x_3 &= -2 \quad \text{(R2)} \\ 0 &= 0 \quad \text{(R3)} \end{aligned}$$

$$\Rightarrow \quad 2x_1 + 4x_2 = 6, \quad -x_3 = -2$$

$$\Rightarrow \quad \boxed{x_1 + 2x_2 = 3, \quad x_3 = 2}$$

There are an infinite number of solutions, as long as

$x_3 = 2$ and $x_1 + 2x_2 = 3$

Example: $(x_1 = 1,\ x_2 = 1,\ x_3 = 2)$

$(x_1 = -1,\ x_2 = 2,\ x_3 = 2)$

are both solutions to Ex 3

★ This happens when one (or more) of the equations is <u>redundant</u>.

$$
\begin{aligned}
2x_1 + 4x_2 + x_3 &= 8 \quad (R1)\\
2x_1 + 4x_2 &= 6 \quad (R2)\\
-4x_1 - 8x_2 + x_3 &= -10 \quad (R3)
\end{aligned}
$$

↑ Add $3 \cdot R2$ to R3

$$2x_1 + 4x_2 + x_3 = 8 \quad (R1)$$
$$2x_1 + 4x_2 = 6 \quad (R2)$$
$$2x_1 + 4x_2 + x_3 = 8 \quad (R3)$$

R1 and R3 are the same!

Need Linear Algebra!

To be Continued...