Math 133A: Weeks 6 & 7
Second-Order Differential Equations

Section 1: Second-Order Systems

Consider the following class of differential equations.

Definition 1

A **second-order linear** differential equation is given by the form

$$y'' + p(x)y' + q(x)y = g(x) \tag{1}$$

where $y = y(x)$. The equation (5) is furthermore said to be:

1. **constant coefficient** if $p(x)$ and $q(x)$ are constants:
$$ay'' + by' + cy = g(x), \tag{2}$$
2. **homogeneous** if $g(x) = 0$:
$$y'' + p(x)y' + q(x)y = 0. \tag{3}$$

We can think of second-order linear differential equations as an extension of first-order DEs of the form

$$y' + p(x)y = q(x). \tag{4}$$

In this case, however, we allow the second-order derivative y'' to appear in the expression. While this may seem like a small generalization, in fact, it will render most of the methods relevant to analyzing the first-order DE (4) moot. In particular, the integrating factor solution method will not apply, and we will not be able to build a slope field diagram to visualize solutions prior to consideration of the solutions. Second-order differential equations are nevertheless among the most commonly encountered form of differential equations in the sciences. For example, they arise directly as a consequence of Newton's second law of motion $F = ma$ in classical physics.

We start our consideration of second-order differential equations with the easiest possible case: **linear**, **homogeneous**, and **constant coefficient**. Such a differential equation can be written in the general form

$$ay'' + by' + cy = 0. \tag{5}$$

We already know that such a differential equation can be rewritten as a linear system in two variables and then solved, but suppose we did not know that. We cannot separate the variables, or find an integrating factor, or find an obvious substitution which will reduce the differential equation to first-order. This exhausts our list of solution methods.

In fact, we will simply take an **educated guess**. We know that the first-order system

$$y' = ry$$

has the exponential solution $y(x) = e^{rx}$, so we will guess that a solution of (5) has the form

$$y(x) = e^{rx}$$

for some $r \in \mathbb{R}$. In the worst case scenario, even if this does not work out, we have not lost a great deal of time—taking derivatives of exponentials is easy.

Consider the following example.

Example 1

Find a solution of the following second-order differential equation in the form $y(x) = e^{rx}$:

$$y'' - 5y' + 4y = 0. \tag{6}$$

Solution: We will guess that the solution has the form $y(x) = e^{rx}$. This gives

$$y = e^{rx}, \quad y' = re^{rx}, \quad y'' = r^2e^{rx}$$

so that

$$\begin{aligned} y'' - 5y' + 4y &= r^2e^{rx} - 5re^{rx} + 4e^{rx} \\ &= e^{rx}(r^2 - 5r + 4) \\ &= e^{rx}(r-1)(r-4) = 0. \end{aligned}$$

The exponential is always positive, so we have either $r = 1$ or $r = 4$. It follows that

$$y_1(x) = e^x \quad \text{and} \quad y_2(x) = e^{4x}$$

are solutions of the differential equation.

In fact, we can easily verify this. For $y_1(x) = e^x$, we have

$$y_1'' - 5y_1' + 4y = e^x - 5e^x + 4e^x = 0$$

and for $y_2(x) = e^{4x}$, we have

$$y_2'' - 5y_2' + 4y = 16e^{4x} - 5(4e^{4x}) + 4e^{4x} = 0.$$

We might be initially surprised that we have two solutions to the equation, but we should not be. Since we have two initial conditions for second-order equations, we will also have two independent solutions, and each one will come with an undetermined constant.

In fact, this is a general property which applies to *all* homogeneous linear systems. We have the following result, which is known as the **Principle of Superposition**.

Theorem 1

Suppose that $y_1(x)$ and $y_2(x)$ are solutions of

$$y'' + p(x)y' + q(x)y = 0. \tag{7}$$

Then $y = C_1y_1 + C_2y_2$ where $C_1, C_2 \in \mathbb{R}$ is a solution of (7).

Proof

Since $y_1(x)$ and $y_2(x)$ are solutions of (7), it follows that

$$y_1'' + p(x)y_1' + q(x)y_1 = 0 \quad \text{and} \quad y_2'' + p(x)y_2' + q(x)y_2 = 0. \tag{8}$$

We will now check if $y(x) = C_1y_1(x) + C_2y_2(x)$ is a solution of (7). Note first of all that we have

$$y'(x) = C_1y_1'(x) + C_2y_2'(x) \quad \text{and} \quad y''(x) = C_1y_1''(x) + C_2y_2''(x).$$

On the left-hand side of (7), we therefore have

$$\begin{aligned}
&y'' + p(x)y' + q(x)y \\
&= \left[C_1 y_1'' + C_2 y_2''\right] + p(x)\left[C_1 y_1' + C_2 y_2'\right] + q(x)\left[C_1 y_1 + C_2 y_2\right] \\
&= C_1\left[y_1'' + p(x)y_1' + q(x)y_1\right] + C_2\left[y_2'' + p(x)y_2' + q(x)y_2\right] \\
&= 0
\end{aligned}$$

where the last line follows from (8). Connecting the first line to the final one, we have that $y'' + p(x)y' + q(x)y = 0$ so that $y(x)$ is a solution of (7), and we are done.

Note: The principle of superposition does not require the equation to have **constant coefficients**. It is easy, however, to find violations of Theorem 1 for equations which are not **linear** or not **homogeneous**. For example, the nonlinear (but homogeneous) DE $y' - 2y^{1/2} = 0$ has the solution $y(x) = x^2$ but $\tilde{y}(x) = Cx^2$ is not a solution for $C \neq \pm 1$. Similarly, the nonhomogeneous (but linear) DE $y' - y = e^x$ has the solution $y(x) = xe^x$ but $\tilde{y}(x) = Cxe^x$ is not a solution for $C \neq 1$.

The principle of superposition clarifies our earlier concern about solutions to the second-order equation (5). We are not completely done, however. Consider guessing the solution form $y(x) = e^{rx}$ for the general form (5). We have

$$ay'' + by' + cy = 0 \implies e^{rx}(ar^2 + br + c) = 0 \implies ar^2 + br + c = 0.$$

If we cannot factor this expression, we need to use the quadratic formula. This gives

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{9}$$

If there are two real-valued roots, we obtain the two fundamental solutions $y_1(x) = e^{r_1 x}$ and $y_2(x) = e^{r_2 x}$. If the roots are complex or repeated, however, we will need alternative solution forms. We have the following result.

Theorem 2

The general solution of the second-order DE (5) is given by the following:

1. If $b^2-4ac>0$ the general solution is $y(x)=C_1e^{r_1x}+C_2e^{r_2x}$ where r_1 and r_2 are the two distinct roots of (9).
2. If $b^2-4ac=0$ the general solution is $y(x)=C_1e^{rx}+C_2xe^{rx}$ where r is the single repeated root of (9).
3. If $b^2-4ac<0$ the general solution is

$$y(x)=e^{\alpha x}(C_1\cos(\beta x)+C_2\sin(\beta x))$$

   where $r=\alpha+\beta i$.

Consider the following examples.

Example 2

Find the general solution of $4y''+12y'+9y=0$. Then find the particular solution for $y(0)=2$ and $y'(0)=0$.

Solution: We guess the solution form $y(x)=e^{rx}$. This gives

$$4y''+12y'+9y=e^{rx}(4r^2+12r+9)=e^{rx}(2r+3)^2=0.$$

It follows that we only have a solution if $r=-3/2$. Since this is a repeated root, we are in Case 2 and the general solution is given by

$$y(x)=C_1e^{-(3/2)x}+C_2xe^{-(3/2)x}.$$

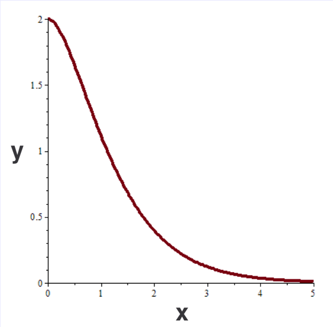
To solve for the particular solution, we compute

$$y'(x)=-\frac{3}{2}C_1e^{-(3/2)x}+C_2e^{-(3/2)x}-\frac{3}{2}C_2xe^{-(3/2)x}.$$

The conditions $y(0)=3$ and $y'(0)=0$ give the system

$$\begin{aligned} C_1 \qquad &= 2 \\ -\frac{3}{2}C_1+C_2 &= 0. \end{aligned}$$

We can quickly solve this to get $C_1=2$ and $C_2=3$. It follows that the particular solution is $y(x)=2e^{-(3/2)x}+3xe^{-(3/2)x}$:



Example 3

Find the general solution of $y'' - 2y' + 5y = 0$. Then find the particular solution for $y(0) = 1$ and $y'(0) = -1$.

Solution: We guess the solution $y(x) = e^{rx}$. This gives

$$y'' - 2y' + 5y = e^{rx}(r^2 - 2r + 5) = 0.$$

The quadratic formula gives the solution

$$r = \frac{2 \pm \sqrt{(2)^2 - 4(1)(5)}}{2} = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i.$$

Since this a complex root, we are in case 3 with $\alpha = 1$ and $\beta = 2$. The general solution is

$$y(x) = C_1 e^x \cos(2x) + C_2 e^x \sin(2x).$$

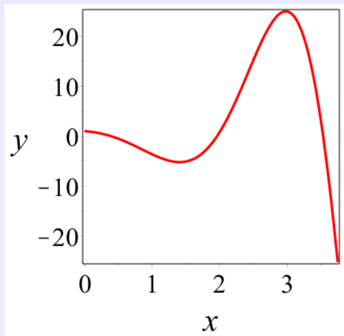
To solve for the particular solution, we compute

$$\begin{aligned} y'(x) &= C_1 e^x \cos(2x) - 2C_2 e^x \sin(2x) + C_1 e^x \sin(2x) + 2C_2 e^x \cos(2x) \\ &= C_1 e^x (\cos(2x) + \sin(2x)) + 2C_2 e^x (\cos(2x) - \sin(2x)). \end{aligned}$$

The conditions $y(0) = 1$ and $y(0) = -1$ give the system

$$\begin{aligned} C_1 \qquad &= 1 \\ C_1 + 2C_2 &= -1. \end{aligned}$$

It follows immediately that $C_1 = 1$ and $C_2 = -1$ so that the particular solution is $y(x) = e^x(\cos(2x) - \sin(2x))$:

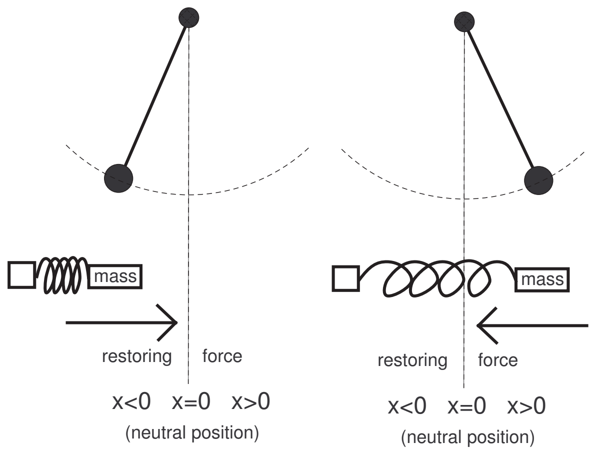


We can observe the components the solution in the features of the graph. The solution grows because of the exponential term e^x, and oscillates because of the trigonometric terms $\sin(2x)$ and $\cos(2x)$. Terms like these, and the associated qualitative behaviors, will be key features of solutions of this type of differential equation.

Section 2: Pendulum Model

Consider a pendulum acting under the force of gravity (alternatively, an elongated spring obeying Hooke's law). Suppose the rest position is $x = 0$ so that any displacement to the right corresponds to $x > 0$ while displacement to the left corresponds to $x < 0$.

If we move the pendulum to the right ($x > 0$), gravity acts against the pendulum to force it left ($F < 0$); conversely, if we move the pendulum to the left ($x > 0$), gravity acts against the pendulum to force it right ($F > 0$).



This gives rise to what is known as a **restoring force** $F_{restoring}$. While it is clear that this force should act in the opposite direction of the displacement, it is not so clear what exact form it should take. It is common to assume that the strength of the restoring force is proportional to the displacement. That is, we assume there is some $k > 0$ such that

$$F_{restoring} = -kx.$$

In most realistic situations, of course, our pendulum would encounter more that simply a restoring force. The most obvious further force to add is a **damping force** $F_{damping}$ corresponding to air resistance and/or friction from the pendulum's hinge. If we imagine that $x' = 0$ corresponds to no velocity, $x' > 0$ corresponds to movement to the right, and $x' < 0$ corresponds to movement to the left, we imagine that the damping force should always acts in the opposite direction as the velocity. It will once again be convenient to assume that the strength of damping force is proportional to the velocity. That is, we assume there is some $c > 0$ such that

$$F_{damping} = -cx'.$$

We now want to incorporate these assumptions into a differential equation model for the pendulum. We will invoke Newton's famous second law of motion $F = ma$. We have

$$\begin{cases} ma = mx'' \\ F = F_{restoring} + F_{damping} = -kx - cx'. \end{cases}$$

Our resulting initial value problem is

$$\begin{cases} mx'' + cx' + kx = 0, \\ x(0) = x_0, \\ x'(0) = v_0. \end{cases} \tag{10}$$

We could derive the same differential equations, with a slightly different interpretation of the constants involved, by considering a mass-spring example obeying Hooke's law. Surprisingly, the same model also arises for the current in an RLC electrical circuit.

Note: Even though (10) is a single equation, we require *two* initial conditions. We can realize this by considering the pendulum example. Consider looking at a snapshot of a pendulum at the resting position $x = 0$ and asking how the pendulum will move in the next instant. We should quickly realize that there are three possibilities:

1. The pendulum could truly have been **at rest** (i.e. it was not moving), in which case it will stay there.

2. The pendulum could have been **swinging to the right**, in which case it will continue to the right, lose speed, and eventually reverse (or swing over the top).

3. The pendulum could have been **swinging to the left**, in which case it will contnue to the left, lose speed, and eventually reverse (or swing over the top).

In any case, we see that it is very important to consider not only the **position** of the pendulum at the time the snapshot was taken, but also the **velocity**—that is, we need two initial conditions.

We recognize that the initial value problem (10) is exactly the type of differential equation we have been studying: **second-order**, **linear**, **constant coefficient**, and **homogeneous**. We can perform this analysis directly by guessing $x(t) = e^{rt}$ as usual. This gives

$$mx'' + cx' + kx = e^{rt}(mr^2 + cr + k) = 0.$$

It follows that we have

$$mr^2 + cr + k = 0.$$

so that

$$r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}.$$

Note that the form of the solution is exactly the three cases we have already discussed. In particular, we may have two distinct real roots ($c^2 - 4mk > 0$), one repeated real room ($c^2 - 4mk = 0$), or a complex conjugate pair of roots ($c^2 - 4mk < 0$). We should be careful with our interpretation, however. Notice that oscillations are only possible if $c^2 - 4mk < 0$, since this is the condition for obtaining sines and cosines in the solution. For $c^2 - 4mk > 0$, then the solution will consist of exponentials. The somewhat surprising conclusion is that we may eliminate oscillations in our system by **increasing the damping**!

In general, we have the following classifications.

Classifications: The system (10) is said to be:

1. **undamped** if $c = 0$,
2. **underdamped** if $c^2 < 4mk$,
3. **critically damped** if $c^2 = 4mk$, and
4. **overdamped** if $c^2 > 4mk$.

The important thing to note is that we can determine the qualitative behavior of the solution without performing every step of computing a solution. Consider the following.

Example 4

Determine whether the following systems are underdamped, critically damped, or overdamped:

$$\begin{aligned} &\text{(a)} \;\; 3x'' + 5x' + 2x = 0 \\ &\text{(b)} \;\; 2x'' + 8x' + 8x = 0 \\ &\text{(c)} \;\; x'' + cx' + 9x = 0. \end{aligned}$$

Solution: For (a) we have that $m = 3$, $c = 5$, and $k = 2$. It follows that $c^2 = (5)^2 = 25$ and $4mk = 4(3)(2) = 24$. Since $c^2 > 4mk$ it follows that the system is overdamped, and will consequently consistent of exponential solutions.

For (b) we have that $m = 2$, $c = 8$, and $k = 8$ we have $c^2 = (8)^2 = 64$ and $4mk = 4(2)(8) = 64$. Since $c^2 = 4mk$ is critically damped. It follows that the system is exactly at the balancing between where, if there were any more damping, the solutions would be strictly exponential, while if there was any less damping, the solutions would begin to oscillate.

For (c) we have that $m = 1$, c is undetermined, and $k = 9$ so that $4mk = 4(1)(9) = 36$. It follows that the system is

$$\begin{aligned} \text{underdamped if } 0 < c^2 < 36 &\implies 0 < c < 6 \\ \text{critically damped if } c^2 = 36 &\implies c = 6 \\ \text{overdamped if } c^2 > 36 &\implies c > 6. \end{aligned}$$

Section 3: Harmonic Oscillator

We now consider specific cases of the pendulum model (10). We start with the simplest case when there is no damping (i.e. $c = 0$), which is often referred to as the *harmonic oscillator* (we will see why!).

Consider the following example.

Example 5

Consider a 2 kg weight attached to the end of a spring which requires a force of 8 Newtons to stretch one meter. Suppose the spring does not experience any damping. If the mass is initially stretched 2 meters to the right and released with an initial velocity of 2 meters per second to the right, find the solution describing the position of the mass as a function of time.

Solution: The given information implies that $m = 2$, $k = 8$ and $c = 0$. This gives the model

$$2x'' + 8x = 0$$

with initial conditions $x(0) = 2$ and $x'(0) = 2$. The guess $y(x) = e^{rx}$ gives

$$e^{rx}(2r^2 + 8) = 2e^{rx}(r^2 + 4) = 0$$

so that $r = \pm 2i$. It follows that the general solution has the form

$$x(t) = C_1 \cos(2t) + C_2 \sin(2t).$$

To find the particular solution satisfying the initial conditions, we must compute

$$x'(t) = -2C_1 \sin(2t) + 2C_2 \cos(2t).$$

The initial conditions give

$$\begin{aligned} x(0) = 2 \quad &\Longrightarrow \quad C_1 = 2 \\ x'(0) = 2 \quad &\Longrightarrow \quad 2C_2 = 2 \quad \Longrightarrow \quad C_2 = 1. \end{aligned}$$

It follows that the particular solution is

$$x(t) = 2\cos(2t) + \sin(2t). \tag{11}$$

While the equation (11) is nice in many respects, it does not answer many questions we might be interested in regarding the physical pendulum model. For example, what is the maximum *amplitude* of the pendulum? We could apply techniques from calculus to determine a local maximum, but the answer is certainly not apparent from the form of (11).

In fact, cases where we have functions of the form $C_1 \cos(\omega t) + C_2 \sin(\omega t)$, it is often convenient to write them in an alternative *phase-shifted cosine form*. We have the following result.

Lemma 1

For any constants C_1, C_2, and ω, we have

$$C_1 \cos(\omega t) + C_2 \sin(\omega t) = A \cos(\omega t - \alpha) \tag{12}$$

where

$$A = \sqrt{C_1^2 + C_2^2}, \qquad \alpha = \begin{cases} \arctan\left(\dfrac{C_2}{C_1}\right), & \text{if } C_1 \geq 0 \\ \arctan\left(\dfrac{C_2}{C_1}\right) + \pi, & \text{if } C_1 \leq 0. \end{cases}$$

Proof

From basic trignometric identifies, we have

$$A\cos(\omega t - \alpha) = A\cos(\alpha)\cos(\omega t) + A\sin(\alpha)\sin(\omega t).$$

In order to satisfy (12), we need to satisfy

$$\begin{aligned} A\cos(\alpha) &= C_1 \\ A\sin(\alpha) &= C_2. \end{aligned} \tag{13}$$

Squaring and adding the equations gives

$$A^2(\cos^2(\alpha) + \sin^2(\alpha)) = C_1^2 + C_2^2 \implies A = \sqrt{C_1^2 + C_2^2}.$$

This gives the required value of the amplitude A. To determine α, we divide the second equation of (12) by the first to get

$$\frac{A\sin(\alpha)}{A\cos(\alpha)} = \frac{C_2}{C_1} \implies \tan(\alpha) = \frac{C_2}{C_1} \implies \alpha = \arctan\left(\frac{C_2}{C_1}\right).$$

Notice, however, that the range of $\arctan(\cdot)$ is $(-\pi/2, \pi/2)$. We cannot obtain angles outside of this range, which includes everything in quadrant II and quadrant III. We notice that the point (C_1, C_2) is in these quadrants only if $C_1 \leq 0$. We must therefore correct these angles by a factor of π if $C_1 \leq 0$, and we are done.

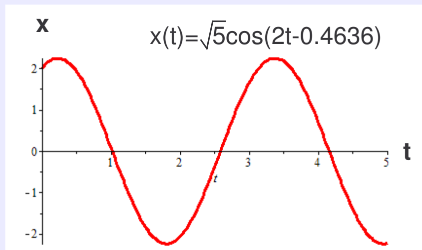
The advantage of the form $A\cos(\omega t - \alpha)$ is that the amplitude and phase-shift of oscillation is clearly identified. Consider the following example.

Example 6

Reconsider the model derived in Example 6. Write the solution in the phase-shifted cosine form $x(t) = A\cos(\omega t + \alpha)$.

Solution: We want to put the solution $x(t) = 2\cos(2t) + \sin(2t)$ in the form $x(t) = A\cos(\omega t - \alpha)$. From Lemma 1, we have $C_1 = 2$ and $C_2 = 1$ so that $A = \sqrt{2^2 + 1^2} = \sqrt{5}$ and $\alpha = \tan^{-1}(1/2) \approx 0.4636$. We can check that $(C_1, C_2) = (2, 1)$ is the first quadrant so that we do not need to adjust by a factor of π. It follows that the solution can be written

$$x(t) = 2\cos(2t) + \sin(2t) = \sqrt{5}\cos(2t - 0.4636) :$$



In particular, we can immediately identify from the phase-shifted cosine form that the maximum displacement of the pendulum is $\sqrt{5}$ units (since cosine is bounded between 1 and -1).

Section 4: Damping

Now consider a pendulum also subject to damping.

Example 7

Reconsider the set-up provided in Example 5, but assume there is a damping of 4 Newtons for each meter/second of velocity. Find the solution describing the position of the mass as a function of time. Write the solution in the phase-shifted cosine form $x(t) = A(t)\cos(\omega_0 t + \alpha)$.

Solution: The given information tells us that we have $m = 2$, $k = 8$, and $c = 4$. This gives the model

$$2x'' + 4x' + 8x = 0$$

with initial conditions $x(0) = 2$ and $x'(0) = 2$. The guess $x(t) = e^{rt}$ gives

$$e^{rt}(2r^2 + 4r + 8) = 2e^{rt}(r^2 + 2r + 4) = 0$$

which implies

$$r_{1,2} = \frac{-2 \pm \sqrt{4-16}}{2} = -1 \pm \sqrt{3}i.$$

The general solution is given by

$$x(t) = e^{-t}(C_1 \cos(\sqrt{3}t) + C_2 \sin(\sqrt{3}t).$$

In order to determine the particular solution, we must find $x'(t)$. We have

$$\begin{aligned} x'(t) = &-e^{-t}\left(C_1 \cos(\sqrt{3}t) + C_2 \sin(\sqrt{3}t)\right) \\ &+ \sqrt{3}e^{-t}\left(-C_1 \sin(\sqrt{3}t) + C_2 \cos(\sqrt{3}t)\right). \end{aligned}$$

The initial conditions result in the system

$$\begin{aligned} C_1 \qquad\quad &= 2 \\ -C_1 + \sqrt{3}C_2 &= 2. \end{aligned}$$

It follows that $C_1 = 2$ and $C_2 = \frac{4}{\sqrt{3}}$. It follows that the particular solution is

$$x(t) = e^{-t}\left(2\cos(\sqrt{3}t) + \frac{4}{\sqrt{3}}\sin(\sqrt{3}t)\right).$$

Notice that the second (bracketed) portion is example the form required of Lemma 1. We can therefore write the solution as

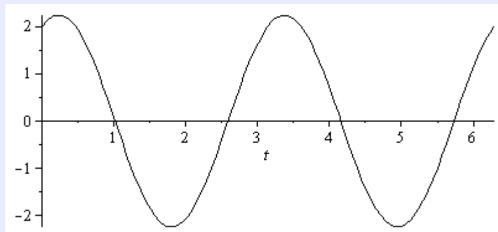
$$x(t) = A(t)\cos(\omega_0 t + \alpha)$$

where $A(t) = Ae^{\alpha t}$. As before, we have $A = \sqrt{C_1^2 + C_2^2} = \sqrt{2^2 + (4/\sqrt{3})^2}$ ≈ 3.0551 and $\alpha = \tan^{-1}(C_2/C_1) = \tan^{-1}((4/\sqrt{3})/2) = \tan^{-1}(2/\sqrt{3}) =$ 0.8571. It follows that the solution can be written as

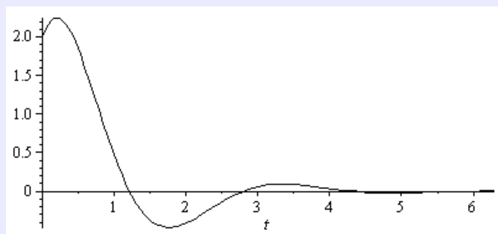
$$x(t) = 3.0551e^{-t}\cos\left(\sqrt{3}t - 0.8571\right).$$

In figure (a) below, we see the undamped ($c = 0$) solution while in figure (b) below we see the damped solution ($c = 4$) derived above:

(a)



(b)



We should not grow too attached, however, to solutions to the damped pendulum model (10) always having oscillations. Consider the following example, which lies in the overdamped region $c^2 > 4mk$.

Example 8

Consider a pendulum with a mass of 3kg, subject to a restoring force of 4N/m, and a frictional force of 8N/(m/s). Set up and a solve a differential equation for the motion of the pendulum.

Solution: We have the governing system of equations

$$3x'' + 8x' + 4x = 0.$$

We guess $x(t) = e^{rt}$ so that

$$e^{rt}\left(3r^2 + 8r + 4\right) = 0 \implies 3r^2 + 8r + 4 = 0.$$

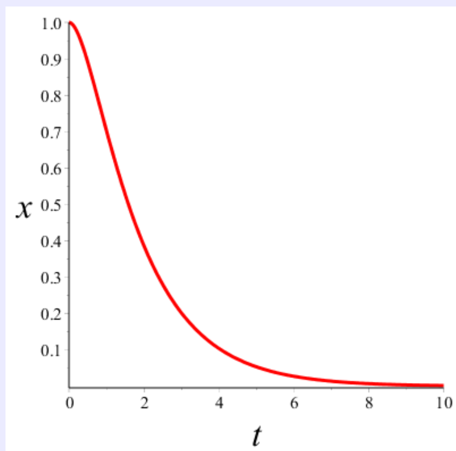
We have

$$r = \frac{-8 \pm \sqrt{64 - 4(4)(3)}}{6} = \frac{-4 \pm 2}{3}$$

so that $r = -2$ or $r = -2/3$. It follows that the general solution is

$$x(t) = C_1 e^{-2t} + C_2 e^{-2/3t}.$$

We can see that the pendulum experiences exponential decay toward the equilibrium state $x = 0$ (since $t \to \infty$ implies $x \to 0$). The system is overdamped so the pendulum does not experience any oscillations. For example, the solution with $x(0) = 1$ and $x'(0) = -1$ looks like the following:



We can clearly see that the pendulum converges monotonically (i.e. from one side) toward its resting position $x = 0$.

Section 5: Nonhomogeneous Systems

Before returning to consideration of the pendulum model, consider the following second-order differential equation

$$y'' + 4y = 12x. \tag{14}$$

The only difference between this and the type of equations we have been considering so far is the term $12x$ on the right-hand side. This extra term makes the differential equation **nonhomogeneous** ($g(x) \neq 0$ in the standard form).

Since the equation is still linear and constant coefficent, we might think that the technique used to solve homogeneous equations might work. Guessing that the solution has the exponential form $y(x) = e^{rx}$, however, gives

$$y'' + 4y = e^{rx}\left(r^2 + 4\right) = 12x.$$

There is no value of r which satisfies this equation for all x. The guess we used for homogeneous linear equations will not work for nonhomogeneous equations.

As unsatisfying as it is, we are simply going to *guess something else.* In particular, we are going to guess a function which, when substituted in the left-hand side, gives the non-homogeneous term $12x$ on the right-hand side. This turns out to be easier than we probably suspect. We can see very quickly that $y(x) = 3x$ works since we have $y''(x) = 0$ so that

$$y'' + 4y = 4(3x) + 4(0) = 12x.$$

Probably the only way we would not have seen this would have been to overthink the problem!

We will say that we have found a **particular solution** $y_p(x) = 3x$ since it satisfies the differential equation and has no undetermined constants. We might wonder, however, how to build a **general solution** to (14) out of this observation. It is not as easy as multiplying the found solution by a constant, since $y(x) = Cx$ is not a solution for all $C \in \mathbb{R}$. For example, we can easily check that $y(x) = 4x$ is *not* a particular solution of the equation.

Instead, consider the homogeneous equation

$$y_c'' + 4y_c = 0. \tag{15}$$

This has the solution $y_c(x) = C_1 \cos(2x) + C_2 \sin(2x)$. At first this seems unimportant, since (15) and (14) are not the same differential equation, but consider *adding* (15) to the particular solution $3x$ of (14). That is, consider

$$y(x) = C_1 \cos(2x) + C_2 \sin(2x) + 3x.$$

We can directly check that is also a solution of (14)! We have

$$y''(x) = -4C_1 \cos(2x) - 4C_2 \sin(2x)$$

so that

$$\begin{aligned} y'' + 4y &= [-4C_1 \cos(2x) - 4C_2 \sin(2x)] \\ &\qquad\qquad + 4[C_1 \cos(2x) + C_2 \sin(2x) + 3x] \\ &= 12x. \end{aligned}$$

We have actually stumbled upon the general solution method for solving second-order (and higher) linear differential equations. The trick will be to find two parts of the solution: one which equals zero when substituted into the LHS, and one which equals the term $g(x)$.

Theorem 3

Any solution of a differential equation of the form

$$y''(x) + p(x)y'(x) + q(x)y(x) = g(x). \tag{16}$$

can be written

$$y(x) = y_c(x) + y_p(x)$$

where $y_p(x)$ is any **particular solution** of (18) and the **complementary function** $y_c(x) = C_1y_1(x) + C_2y_2(x)$ is the general solution of the homogeneous system

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0. \tag{17}$$

Proof

We will prove that a function $y(x) = y_c(x) + y_p(x)$ is a solution of (18). We will omit the proof that this is the *only* form of such a solution.

Suppose that $y_p(x)$ is a particular solution of (18) and $y_c(x) = C_1y_1(x) + C_2y_2(x)$ is a general solution of (17). That is, suppose that

$$\begin{aligned} y_p''(x) + p(x)y_p'(x) + q(x)y_p(x) &= g(x) \\ y_c''(x) + p(x)y_c'(x) + q(x)y_c(x) &= 0. \end{aligned}$$

It follows that we have

$$\begin{aligned} &y''(x) + p(x)y'(x) + q(x)y(x) \\ &= \left[y_c''(x) + y_p''(x)\right] + p(x)\left[y_c'(x) + y_p'(x)\right] + q(x)\left[y_c(x) + y_p(x)\right] \\ &= \left[y_c''(x) + p(x)y_c'(x) + q(x)y_c(x)\right] + \left[y_p''(x) + p(x)y_p'(x) + q(x)y_p(x)\right] \\ &= g(x). \end{aligned}$$

It follows that $y(x) = y_c(x) + y_p(x)$ is a solution of (18) and we are done.

Note: Since we already know how to solve homogeneous second-order linear DEs with constant coefficients, this result tells us that we need only worry about finding $y_p(x)$. In general, however, it may be very difficult to determine the complementary solution $y_c(x)$ if the coefficients are allowed to vary with x.

Section 6: Undetermined Coefficients

The question now becomes how we find the particular solution $y_p(x)$. There are two primary methods for such a problem. The first is called the **method of undetermined coefficients**, and will be an intuitive extension of what we have been doing to date—that is, we will be *guessing* the solution form.

The second method is called **variation of parameters** and does not require any guessing. It will, however, require a significantly greater amount of work and some potentially tricky integration—even when we could simply guess the solution. We will largely omit studying variation of parameters.

Consider a general differential equation of the form

$$ay'' + by' + cy = g(x) \tag{18}$$

where we want to determine the particular solution $y_p(x)$. Notice that the LHS of (18) involves just y and its derivatives, while the RHS contains a known functions of x. What we need is a form of $y_p(x)$ which can be differentiated to give a function of the form of $g(x)$. Notice that

$$\begin{aligned} \frac{d}{dx}[\text{polynomial}] &= \text{polynomial} \\ \frac{d}{dx}[\text{exponential}] &= \text{exponential} \\ \frac{d}{dx}[\text{sine and/or cosine}] &= \text{sine and/or cosine.} \end{aligned}$$

This suggests that, if $g(x)$ is a polynomial we should have a **polynomial** $y_p(x)$, if $g(x)$ is exponential we should have an **exponential** $y_p(x)$, and if $g(x)$ is trigonometric we should have a **trigonometric** $y_p(x)$. This suggests the following steps for solving a differential equation of the form (18):

Method of Undetermined Coefficients

1. Find the general solution $y_c(x) = C_1 y_1(x) + C_2 y_2(x)$ of the homogeneous equation
$$ay_c'' + by_c' + cy_c = 0.$$

2. Select a **trial function** $y_p(x)$ according to the following:

   (a) If $g(x)$ contains x^n then use
   $$y_p(x) = A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0.$$

   (b) If $g(x)$ contains e^{rx} then use
   $$y_p(x) = Be^{rx}.$$

   (c) If $g(x)$ contains $\sin(\alpha x)$ or $\cos(\alpha x)$ then use
   $$y_p(x) = A\cos(\alpha x) + B\sin(\alpha x).$$

3. Substitute the trial function $y_p(x)$ into
$$ay_p'' + by_p' + cy_p = g(x)$$
and solve for the undetermined coefficients in $y_p(x)$ (i.e. solve for $A, B, A_0, A_1, \ldots$)

4. If relevant, use the initial conditions to solve for the undetermined constants in the general solution $y(x) = y_c(x) + y_p(x)$.

Note: We may need to use **combinations** of these functions. For example, if we have $g(x) = e^x \sin(x)$, we need to use $y_p(x) = Ae^x\cos(x) + Be^x\sin(x)$. If we have $g(x) = x^2e^{-x}$, we would need to use $y_p(x) = (Ax^2+Bx+C)e^{-x}$, etc. Also note that the arguments inside the trigonometric and exponential terms are also important. For instance, the forcing term $g(x) = \sin(x) + \cos(2x)$ requires the trial function $y_p(x) = A\cos(x) + B\sin(x) + C\cos(2x) + D\sin(2x)$ while $g(x) = \sin(x) + \cos(x)$ requires only $y_p(x) = A\sin(x) + B\cos(x)$.

Example 9

Determine the correct trial function form $y_p(x)$ for the following second-order differential equations:

1. $y'' + 4y = x^2 - 5x + 2\cos(3x)$
2. $y'' + 4y = -xe^x + 20e^{-x}$
3. $y'' + 4y = 5x\sin(x)$

Solution: Although it may not seem relevant yet, we will first need to determine the general solution to the homogeneous DE $y'' + 4y = 0$. The guess $y = e^{rx}$ gives

$$e^{rx}(r^2 + 4) = 0$$

which has the solution $r = \pm 2i$ (i.e. $\alpha = 0$, $\beta = 2$). It follows that the general solution of the complementary problem is

$$y_c(x) = C_1\cos(2x) + C_2\sin(2x).$$

Solution (a): We have polynomials and trigonometric functions on the RHS, separated by addition. From Algorithm 1, the correct trial form is

$$y_p(x) = Ax^2 + Bx + C + D\cos(3x) + E\sin(3x).$$

Notice that we must *must* include all powers lower than x^2 in the trial form, including the constant (i.e. power zero). Also notice we do not include the constant coefficients or signs (these would be absorbed into the unknown constants A, B, etc.). Furthermore, although both x^2 and x appear on the RHS separated, it is sufficient to apply the rule only to x^2 since it also contains all of the terms given by x.

Solution (b): We have xe^x, which combines the polynomial and exponential rules. From Algorithm 1, the correct trial form is

$$y_p(x) = Axe^x + Be^x + Ce^{-x}.$$

Notice that we apply the rule separately to e^x and e^{-x}.

Solution (c): We have $x\sin(x)$, so that we have to apply the polynomial and trinometry rule simultaneously. By Algorithm 1, the correct trial form is

$$y_p(x) = Ax\sin(x) + Bx\cos(x) + C\sin(x) + D\cos(x).$$

Example 10

Find the general solution of the differential equation

$$y'' + 4y = e^{-x} - 3x^3.$$

Solution: We need to first solve the homogeneous equation

$$y'' + 4y = 0.$$

The guess $y_c(x) = e^{rx}$ gives $e^{rx}(r^2+4) = 0$ so that $r = \pm 2i$. It follows that

$$y_c(x) = C_1\cos(2x) + C_2\sin(2x).$$

We now need to use a trial function $y_p(x)$ with a suitable form that it could give $e^{-x} - 3x^3$ after differentiation. We try

$$\begin{aligned} & y_p(x) = Ae^{-x} + Bx^3 + Cx^2 + Dx + E \\ \implies \quad & y_p'(x) = -Ae^{-x} + 3Bx^2 + 2Cx + D \\ \implies \quad & y_p''(x) = Ae^{-x} + 6Bx + 2C. \end{aligned}$$

It follows that the differential equation gives

$$\begin{aligned} & y_p''(x) + 4y_p(x) \\ & = (Ae^{-x} + 6Bx + 2C) + 4(Ae^{-x} + Bx^3 + Cx^2 + Dx + E) \\ & = 5Ae^{-x} + 4Bx^3 + 4Cx^2 + (6B+4D)x + (2C+4E) \\ & = e^{-x} - 3x^3. \end{aligned}$$

It follows that we need to satisfy

$$\begin{aligned} 5A &= 1 \\ 4B &= -3 \\ 4C &= 0 \\ 6B + 4D &= 0 \\ 2C + 4E &= 0. \end{aligned}$$

It follows that we have $A = 1/5$, $B = -3/4$, $C = 0$, $D = 9/8$, and $E = 0$. The corresponding particular solution is

$$y_p(x) = \frac{1}{5}e^{-x} - \frac{3}{4}x^3 + \frac{9}{8}x.$$

The general solution is therefore

$$y(x) = y_c(x) + y_p(x) = C_1 \cos(2x) + C_2 \sin(2x) + \frac{1}{5}e^{-x} - \frac{3}{4}x^3 + \frac{9}{8}x.$$

Suggested Problems

1. Solve the following second-order linear differential equations (all derivatives with respect to x):

(a) $\begin{cases} y'' - 7y' + 10y = 0, \\ y(0) = 3, \\ y'(0) = 0 \end{cases}$

(b) $\begin{cases} 4y'' - 12y' + 9y = 0, \\ y(0) = 1, \\ y'(0) = -1/2 \end{cases}$

(c) $\begin{cases} y'' + 4y' + 5y = 0, \\ y(0) = 1, \\ y'(0) = 1 \end{cases}$

(d) $\begin{cases} 2y'' - 9y' + 7y = 0, \\ y(0) = 0, \\ y'(0) = 5 \end{cases}$

(e) $\begin{cases} 4y'' + 4y' + 5y = 0, \\ y(0) = -1, \\ y'(0) = \frac{3}{2} \end{cases}$

(f) $\begin{cases} y'' - 4y' + 4y = 0, \\ y(0) = -1, \\ y'(0) = 1 \end{cases}$

(g) $\begin{cases} 4y'' - 3y' - y = 0, \\ y(0) = 1, \\ y'(0) = -\frac{3}{2} \end{cases}$

(h) $\begin{cases} y'' + 4y' + 29y = 0, \\ y(0) = 2, \\ y'(0) = -2 \end{cases}$

2. Differential equations of the form

$$ax^2y'' + bxy' + cy = 0$$

can be transformed into a constant coefficient system by the variable transformation $v = \ln(x)$. Use this transformation to solve the following non-constant coefficient second-order systems.

(a) $\begin{cases} x^2y'' + 9xy' + 15y = 0, \\ y(1) = 1, \\ y'(1) = -3 \end{cases}$ (b) $\begin{cases} x^2y'' + xy' + 25y = 0, \\ y(1) = 1, \\ y'(1) = 5 \end{cases}$

3. Determine the values of m, c, or k for which the following systems are critically damped (all derivatives with respect to t):

(a) $mx'' + 4x' + 4x = 0$

(b) $9x'' + cx' + 36x = 0$

(c) $x'' + x' + kx = 0$

(d) $mx'' + 4x' + kx = 0$

(e) $4x'' + cx' + x = 0$

(f) $mx'' + cx' + 25x = 0$

4. A 4 kg mass is attached to a spring which has a restoring constant of 9 N/m. Suppose the pendulum is not influenced by damping or any outside forcing. Suppose the pendulum is initially displaced 2 m to the right and given an initially thrust of 3 m/s to the right. Set-up and solve the differential equation which governs the motion of the pendulum. What is the maximum amplitude of displacement from $x = 0$? Reconsider the solutions with the damping constants $c = 4$, $c = 12$, and $c = 15$. If possible, write the solutions in the phase-shifted cosine form.

5. Show that the following differential equations have the given particular solutions $y_p(x)$. Then determine general solution, and solution to the given initial value problem. (All derivatives with respect to x.)

(a) $y'' + 2y' + y = x$
$y_p(x) = x - 2$
$y(0) = 1,\ y'(0) = 0$

(b) $y'' - 4y = x^2 - \frac{1}{2}$
$y_p(x) = -\frac{1}{4}x^2$
$y(0) = 2,\ y'(0) = 0$

(c) $y'' + 4y = 3\sin(x)$
$y_p(x) = \sin(x)$
$y(0) = -1,\ y'(0) = 1$

(d) $y'' + 6y' + 10y = 5e^{-x} + 10e^{4x}$
$y_p(x) = e^{-x} + e^{4x}$
$y(0) = 0,\ y'(0) = 1$

6. Use Algorithm 1 to solve the following initial value problems (all derivatives with respect to x):

(a) $\begin{cases} y'' - 3y' + 2y = 2e^{3x}, \\ y(0) = -1, \\ y'(0) = 0 \end{cases}$

(b) $\begin{cases} 2y'' + 5y' - 3y = 10\sin(x), \\ y(0) = 1, \\ y'(0) = 0 \end{cases}$

(c) $\begin{cases} y'' - 2y' + 5y = 5x - 2, \\ y(0) = 1, \\ y'(0) = -2 \end{cases}$

(d) $\begin{cases} 3y'' + 7y' + 2y = 12e^{x} + 4x^2, \\ y(0) = 0, \\ y'(0) = 0 \end{cases}$