Lecture 3

Review:

Order - highest derivative in an ordinary differential equation (ODE)

Linearity - An ODE is linear if all of the dependent variables show linearly

Examples

$$y''' + y' + y^3 = x$$

3rd order

non linear term

2nd order

$$yy' + e^y + \sin(y) + x^7 y + x^6 \frac{1}{y} = (y'')^2$$

NL NL NL Linear NL NL

y - dep. variable, x - indep. variable

1st-order ODE:

$$y' = f(x, y)$$

1st-order Linear ODE:

$$a_1(x)\, y' + a_2(x)\, y = b(x)$$

why is e^y nonlinear?

$$e^y = 1 + \underbrace{y}_{\text{linear}} + \underbrace{\tfrac{1}{2} y^2 + \tfrac{1}{6} y^3 + \tfrac{1}{24} y^4 + \cdots}_{\text{Non linear terms}}$$

"Solving" ODEs

Ex.

$$x(t) = 5e^{3t} \quad \text{solves} \quad \frac{dx}{dt} = 3x$$

Plug in, and make sure both sides match!

$$\left(\begin{array}{l} x=1 \quad \text{is a solution of} \quad x^2-2x+1=0 \\ \text{Check:} \quad 1^2-2\cdot 1+1 = 0 \ ✓ \end{array} \right)$$

Lefthand Side (LHS)

$$\frac{dx}{dt} = \frac{d}{dt}\, 5e^{3t} = 5\cdot 3e^{3t} = 15e^{3t}$$

$$3x = 3\cdot\left(5e^{3t}\right) = 15e^{3t}$$

Same! ✓

Right Hand Side (RHS)

Ex 2 Show that

$y(x) = x^2 - x^{-1}$ solves $\frac{d^2y}{dx^2} - \frac{2}{x^2}y = 0$

$$\frac{dy}{dx} = 2x + x^{-2}$$

$$\frac{d^2y}{dx^2} = 2 - 2x^{-3}$$

$$\frac{d^2y}{dx^2} - \frac{2}{x^2}y = \underbrace{\left(2-2x^{-3}\right)}_{\frac{d^2y}{dx^2}} - \frac{2}{x^2}\underbrace{\left(x^2-x^{-1}\right)}_{y(x)}$$

$$= \cancel{2} - \cancel{2x^{-3}} - \cancel{2} + \cancel{2x^{-3}} = 0 \ ✓$$

Ex 3

Show that

$$y = c_1 e^{-x} + c_2 e^{2x} \quad \text{solves} \quad y'' - y' - 2y = 0$$

where c_1 & c_2 are constants

$$y' = \frac{d}{dx}\left(c_1 e^{-x} + c_2 e^{2x}\right) = -c_1 e^{-x} + 2c_2 e^{2x}$$

$$y'' = \frac{d}{dx}\left(-c_1 e^{-x} + 2c_2 e^{2x}\right) = c_1 e^{-x} + 4c_2 e^{2x}$$

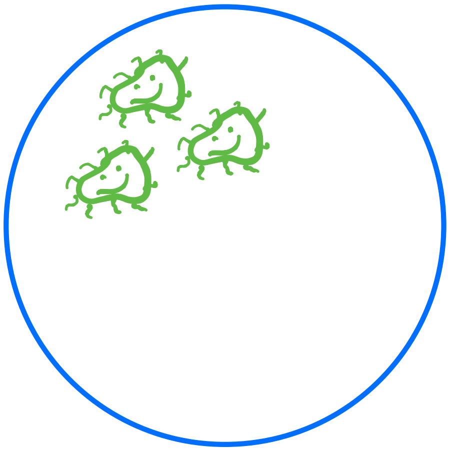
$$\Rightarrow y'' - y' - 2y = c_1 e^{-x} + 4c_2 e^{2x} - \left(-c_1 e^{-x} + 2c_2 e^{2x}\right) - 2\left(c_1 e^{-x} + c_2 e^{2x}\right)$$

$$= c_1 e^{-x} + 4c_2 e^{2x} + c_1 e^{-x} - 2c_2 e^{2x} - 2c_1 e^{-x} - 2c_2 e^{2x}$$

$$= \left(\cancel{c_1} + \cancel{c_1} - \cancel{2c_1}\right)e^{-x} + \left(\cancel{4c_2} - \cancel{2c_2} - \cancel{2c_2}\right)e^{2x}$$

$$= 0 \checkmark$$

Basic Mathematical Models



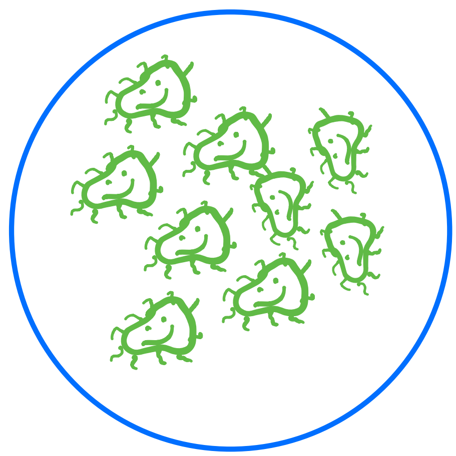
- Start with 1 bacterium
- After 1 hour, 2 more have grown.

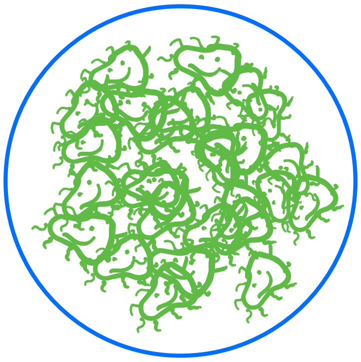
$$P(0) = 1 \leftarrow \text{Population}$$

$$\frac{dP}{dt} = 2 \ \frac{\text{bacteria}}{\text{hour}}$$

After another hour, the 3 bacteria have grown to 9 bacteria

$$\frac{dP}{dt} = 6 \ \frac{\text{bacteria}}{\text{hour}}$$





After another hour,

9 bacteria → 27 bacteria

$$\frac{dP}{dt} = 18 \ \frac{\text{bacteria}}{\text{hour}}$$

Collect the Data

$P(t)$	$P=1$	$P=3$	$P=9$	$P=27$
$\frac{dP}{dt}$	$P'=2$	$P'=6$	$P'=18$	$P'=54$

2nd row is <u>Double</u> the 1st row!

$$\Rightarrow \boxed{\frac{dP}{dt} = 2P}$$

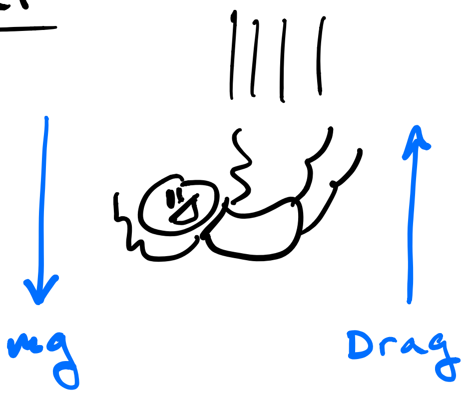
Solution $P(t) = Ce^{2t}$

LHS: $\frac{dP}{dt} = \frac{d}{dt} Ce^{2t} = 2Ce^{2t}$

RHS: $2P = 2Ce^{2t}$

Same! ✓

Falling Object



$\approx -\gamma v$

gamma

Newton's 2^{nd} Law

$$F = ma \quad , \quad a = \frac{dv}{dt}$$

$$F \approx mg - \gamma v$$

$$mg - \gamma v = m\frac{dv}{dt}$$

$$\boxed{\frac{dv}{dt} = -\frac{\gamma}{m}v + g}$$

$m = 1$ kg.

$\gamma = 2$ kg/s

$g = 10$ m/s²

$$v(t) = \frac{mg}{\gamma} + c\,e^{-\frac{\gamma}{m}t}$$

$$= 5 + c\,e^{-2t}$$

What about c? (constant of integration)

Need an <u>Initial Condition</u>

Example: $v(0) = 0$

Plug $t=0$ into $V(t) \Rightarrow$ solve for C

$$V(0) = 5 + Ce^{-2\cdot 0} = 5 + Ce^{0}$$

$$= 5 + C\cdot 1 = 5 + C = 0$$

$$\Rightarrow C = -5$$

$$\boxed{V(t) = 5 - 5e^{-2t}}$$

What happens for large t?

(i.e. what happens when $t \to \infty$)

$$\lim_{t\to\infty} V(t) = \lim_{t\to\infty} \left(\frac{mg}{\gamma} + Ce^{-\frac{\gamma}{m}t} \right)$$

$$= \frac{mg}{\gamma} + C \underbrace{\lim_{t\to\infty} e^{-\frac{\gamma}{m}t}}_{0}$$

$$= \frac{mg}{\gamma} \leftarrow \text{terminal velocity}$$

(Does it <u>not</u> depend on initial condition)

Exponentials come up a <u>lot</u> in ODEs

Basic Properties

Basics: $e^0 = 1$, $\ln(e^x) = x$

Products: $e^x e^y = e^{x+y}$, $e^x / e^y = e^{x-y}$

Powers: $e^{-x} = 1/e^x$, $(e^x)^y = e^{xy}$

Limits: $\lim_{x\to\infty} e^x = \infty$, $\lim_{x\to\infty} e^{-x} = 0$

Calculus: $\frac{d}{dx} e^{ax} = ae^{ax}$, $\int e^{ax}\,dx = \frac{e^{ax}}{a} + c$