Lecture 10

Linear 2^{nd}-order ODEs

* Review of characteristic Equations

Linear

Given $$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = \underline{\underline{0}}$$

constant coefficient

homogeneous

Let $y(t) = ce^{rt} \Rightarrow$ $\boxed{ar^2 + br + c = 0}$

"characteristic equation"

Roots are given by $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$b^2 > 4ac \Rightarrow$ <u>real</u> roots, and both roots are valid solutions

$$\Rightarrow \boxed{y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}}$$

(If there are initial conditions, then you can solve for c_1 and c_2)

What about complex roots?

Example 1 $\quad \dfrac{d^2y}{dt^2} + y = 0$

Solution: $y(t) = c_1 \cos(t) + c_2 \sin(t)$

Could we have used a characteristic equation to find the solution?

Let $y = ce^{rt}$, $y' = cre^{rt}$, $y'' = cr^2e^{rt}$

$$y'' + y = 0 \Rightarrow cr^2e^{rt} + ce^{rt} = 0$$

$$\Rightarrow ce^{rt}(r^2+1) = 0$$

$$\Rightarrow r^2 + 1 = 0$$

$$r^2 = -1 \Rightarrow r = \pm\sqrt{-1} = \pm i \;(j)$$

$$\Rightarrow \text{Solution: } y(t) = c_1 e^{it} + c_2 e^{-it} ?$$

Is this true?

INTRODUCING: EULER'S FORMULA

$$e^{ix} = \cos(x) + i \sin(x)$$

$i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, ...

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} i^n \frac{x^n}{n!}$$

$$= \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}(-1)^n + i \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}(-1)^k$$

$$= \cos(x) + i \sin(x)$$

Back to our solution:

$$y(t) = c_1 e^{it} + c_2 e^{-it}$$

$$= c_1 \left(\cos(t) + i\sin(t)\right)$$

$$+ c_2 \left(\cos(t) - i\sin(t)\right)$$

$$= \underbrace{(c_1+c_2)}_{d_1 = c_1+c_2}\cos(t) + \underbrace{i(c_1-c_2)}_{d_2 = i(c_1-c_2)}\sin(t)$$

$$= \boxed{d_1 \cos(t) + d_2 \sin(t)}$$

If $e^{ix} = \cos(x) + i\sin(x)$

$\Rightarrow e^{-ix} = \cos(x) - i\sin(x)$

Example 2 $\quad \dfrac{d^2y}{dt^2} + 9y = 0$

Characteristic Equation: $\quad \overset{1}{a}r^2 + \overset{0}{b}r + \overset{9}{c} = 0$

$$\Rightarrow \boxed{r^2 + 9 = 0}$$

$$r^2 = -9 \Rightarrow r = \pm\sqrt{-9} = \pm\sqrt{9}\sqrt{-1} = \pm 3i$$

Solution: $y = c_1 e^{3it} + c_2 e^{-3it}$

$$ay'' + by' + cy = 0$$

$$\hookrightarrow ar^2 + br + c = 0$$

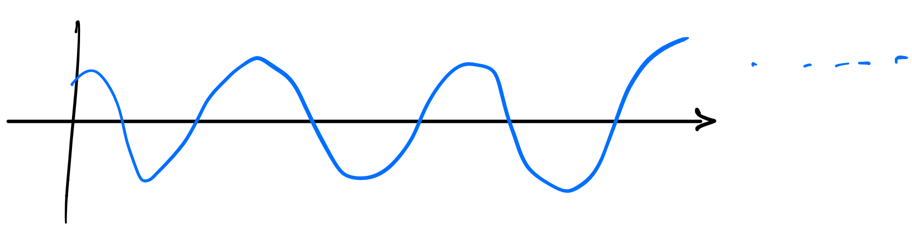
$a = 1 \qquad b = 0 \qquad c = 9$

$$= c_1\big(\cos(3t) + i\sin(3t)\big) + c_2\big(\cos(3t) - i\sin(3t)\big)$$

$$= (c_1 + c_2)\cos(3t) + i(c_1 - c_2)\sin(3t)$$

$$= \boxed{d_1\cos(3t) + d_2\sin(3t)}$$

Same as before, only with $3t$



Example 3

$$1\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 0$$

Characteristic Equation: (Let $y = e^{rt}$)

$$1 r^2 + 2r + 5 = 0$$

Roots: $$r = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm \sqrt{-16}}{2}$$

$$= \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

Solution: $$y(t) = c_1 e^{(-1+2i)t} + c_2 e^{(-1-2i)t}$$

$$e^{(-1+2i)t} = e^{-t+2it} = e^{-t}e^{2it}, \quad e^{(-1-2i)t} = e^{-t}e^{-2it}$$

$$\Rightarrow y(t) = c_1 e^{-t}e^{2it} + c_2 e^{-t}e^{-2it}$$

$$= e^{-t}\left(c_1 e^{2it} + c_2 e^{-2it}\right)$$

$$= e^{-t}\left(d_1 \cos(2t) + d_2 \sin(2t)\right)$$

$d_1 = c_1 + c_2$, $d_2 = i(c_1 - c_2)$

If $r = -1 \pm 2i$

$$y = e^{-t}\left(d_1 \cos(2t) + d_2 \sin(2t)\right)$$

General Rule (complex roots)

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \alpha \pm i\beta$$

$(b^2 - 4ac < 0)$

where $\alpha = -\frac{b}{2a}$, $\beta = \frac{\sqrt{4ac - b^2}}{2a}$

$$\Rightarrow \boxed{y(t) = e^{\alpha t}\left(d_1 \cos(\beta t) + d_2 \sin(\beta t)\right)}$$

What is the third case?

Example 4

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0$$

Characteristic Equation: $\quad r^2 + 2r + 1 = 0$

$$r = \frac{-2 \pm \sqrt{4-4}}{2} = \frac{-2 \pm 0}{2} = -1$$

(only 1 root!)

$$\cancel{y(t) = c_1 e^{-t} + c_2 e^{-t}}$$

$$\cancel{= (c_1 + c_2) e^{-t} = d_1 e^{-t}}$$

Don't Do THIS!!!

If you know 1 solution $y_1(t) = c_1 e^{-t}$, you can find the other by letting

$$y(t) = V(t)\, y_1(t)$$

"Reduction of Order"

Let $y = v(t)\, c_1 e^{-t}$

$$y' = c_1 v' e^{-t} - c_1 v e^{-t}$$

$$y'' = c_1 v'' e^{-t} - 2c_1 v' e^{-t} + c_1 v e^{-t}$$

ODE: $y'' + 2y' + y = 0$

$$c_1 e^{-t}(v'' - 2v' + v) + 2c_1 e^{-t}(v' - v) + c_1 e^{-t} v = 0$$

$$= c_1 e^{-t}(v'' \cancel{-2v'} \cancel{+v} \cancel{+2v'} \cancel{-2v} \cancel{+v}) = 0$$

$$\Rightarrow v'' = 0 \Rightarrow v = at + b$$

$$y(t) = ac_1 e^{-t} t + c_1 b e^{-t}$$

$$= a e^{-t} t + b e^{-t}$$

Rule: Repeated Root $(b^2 - 4ac = 0)$

$r = -\frac{b}{2a}$: $\boxed{y(t) = c_1 e^{rt} + c_2 t e^{rt}}$

Summary

Given: $$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$$

(2nd-order, linear, constant coefficient, homogeneous)

Characteristic Equation: $ar^2 + br + c = 0$

Roots: $$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

★ CASE I: $b^2 > 4ac \rightarrow$ 2 real roots $r_{1,2}$

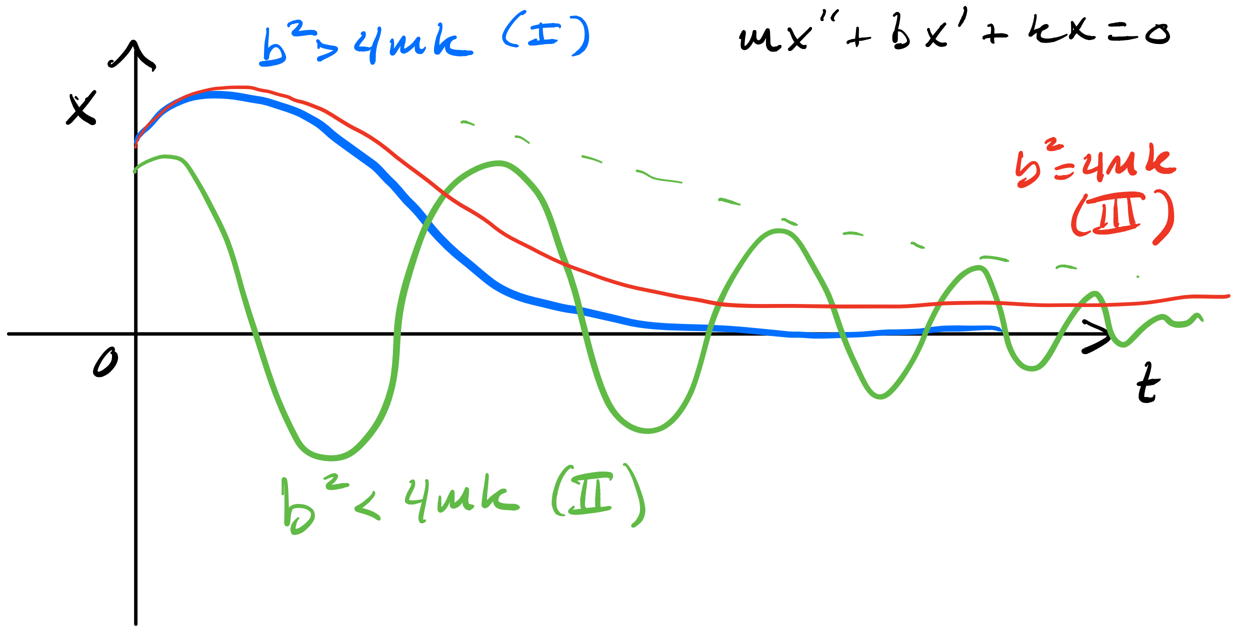
$$\Rightarrow y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

★ CASE II: $b^2 < 4ac \rightarrow$ Complex roots $r = \alpha \pm i\beta$

$$\Rightarrow y(t) = e^{\alpha t}\left(c_1 \cos(\beta t) + c_2 \sin(\beta t)\right)$$

★ CASE III: $b^2 = 4ac \rightarrow$ Repeated root $r = \frac{-b}{2a}$

$$\Rightarrow y(t) = c_1 e^{rt} + c_2 t e^{rt}$$


$b^2 > 4mk$ (I)
$mx'' + bx' + kx = 0$
x
$b^2 = 4mk$ (III)
0
t
$b^2 < 4mk$ (II)