Lecture 23 — Eigenvalues Continued...

Last time, in Example 2:

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$$

Let's say we had

$$\frac{dx_1}{dt} = x_1 + 2x_2 - x_3$$

$$\frac{dx_2}{dt} = x_1 + x_3$$

$$\frac{dx_3}{dt} = 4x_1 - 4x_2 + 5x_3$$

$$\Rightarrow \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longrightarrow \frac{d\vec{x}}{dt} = \underline{\underline{A}}\,\vec{x}$$

1) Find the eigenvalues:

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \begin{vmatrix} 1-\lambda & 2 & -1 \\ 1 & -\lambda & 1 \\ 4 & -4 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(\lambda-2)(\lambda-3) = 0 \Rightarrow \lambda = \{1, 2, 3\}$$

2) Find eigenvectors for $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$

Example: pick $\lambda_1 = 1$

Find $\vec{v}_1 = \begin{bmatrix} A_1 \\ B_1 \\ C_1 \end{bmatrix}$ from $(\underline{\underline{A}} - \lambda_1 \underline{\underline{I}})\vec{v}_1 = \vec{0}$

$$\Rightarrow (\underline{\underline{A}} - \underline{\underline{I}})\vec{v}_1 = \begin{bmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 4 & -4 & 4 \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \\ C_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\cancel{4A_1 - 4B_1 + 4C_1 = 0}$

$$\Rightarrow 2B_1 - C_1 = 0, \quad A_1 - B_1 + C_1 = 0$$

$$\Rightarrow C_1 = 2B_1, \quad A_1 = B_1 - C_1 = B_1 - 2B_1 = -B_1$$

$$\Rightarrow \vec{v}_1 = \begin{bmatrix} A_1 \\ B_1 \\ C_1 \end{bmatrix} = \begin{bmatrix} -B_1 \\ B_1 \\ 2B_1 \end{bmatrix} = B_1 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \sim \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

(chose $B_1 = 1$)

Similarly: $\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$

$(\lambda_2 = 2)$ $\qquad$ $(\lambda_3 = 3)$

3) General Solution

Recall: $ay'' + by' + cy = 0$

$\rightarrow y(t) = c_1 y_1(t) + c_2 y_2(t)$

Now: $\vec{x}(t) = c_1 \underbrace{\vec{v}_1 e^{\lambda_1 t}}_{\vec{x}_1(t)} + c_2 \underbrace{\vec{v}_2 e^{\lambda_2 t}}_{\vec{x}_2(t)} + c_3 \underbrace{\vec{v}_3 e^{\lambda_3 t}}_{\vec{x}_3(t)}$

Answer $\rightarrow$ $= c_1 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} e^{3t}$

$$= \underbrace{\begin{bmatrix} -e^{t} & -2e^{2t} & -e^{3t} \\ e^{t} & e^{2t} & e^{3t} \\ 2e^{t} & 4e^{2t} & 4e^{3t} \end{bmatrix}}_{\underline{\underline{X}}(t) = \text{Fundamental matrix}} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

4) What if we had an initial condition:

$$x_1(0) = -1 \quad x_2(0) = 0 \quad x_3(0) = 0 \longrightarrow \vec{x}(0) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \vec{x}_0$$

$$\Rightarrow \vec{x}(0) = \begin{bmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} -1 & -2 & -1 & -1 \\ 1 & 1 & 1 & 0 \\ 2 & 4 & 4 & 0 \end{array}\right] \overset{R_2+R_1,\ R_3+2R_1}{\sim} \left[\begin{array}{ccc|c} -1 & -2 & -1 & -1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 2 & -2 \end{array}\right]$$

$$\overset{R_1 \to -R_1,\ R_2 \to -R_2,\ R_3 \to \frac{1}{2}R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] \overset{R_1 - 2R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\overset{R_1 - R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] \Rightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{x}(t) = \begin{bmatrix} -e^{t} & -2e^{2t} & -e^{3t} \\ e^{t} & e^{2t} & e^{3t} \\ 2e^{t} & 4e^{2t} & 4e^{3t} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

or

$$x_1(t) = -2e^{2t} + e^{3t}$$

$$x_2(t) = e^{2t} - e^{3t}$$

$$x_3(t) = 4e^{2t} - 4e^{3t}$$

Definition

If $\vec{x}(t) = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t} + \dots + c_n \vec{v}_n e^{\lambda_n t}$

$\rightarrow$ $\underline{\underline{X}}(t) = \left[\vec{v}_1 e^{\lambda_1 t} \quad \vec{v}_2 e^{\lambda_2 t} \quad \dots \quad \vec{v}_n e^{\lambda_n t} \right]$

is known as the "fundamental matrix"

In our last example:

$$\underline{\underline{X}}(t) = \begin{bmatrix} -e^{t} & -2e^{2t} & -e^{3t} \\ e^{t} & e^{2t} & e^{3t} \\ 2e^{t} & 4e^{2t} & 4e^{3t} \end{bmatrix}$$

Some basic uses:

General solution can be written as: $\vec{x}(t) = \underline{\underline{X}}(t)\vec{c}$

where $\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

Initial condition: $\vec{x}(0) = \vec{x}_0$

$\vec{x}(0) = \underline{\underline{X}}(0)\vec{c} = \vec{x}_0 \Rightarrow \vec{c} = \underline{\underline{X}}^{-1}(0)\vec{x}_0$

$$\Rightarrow \boxed{\vec{x}(t) = \underline{\underline{X}}(t)\,\underline{\underline{X}}^{-1}(0)\,\vec{x}_0}$$

What about Complex eigenvalues?

<u>Example</u> $\underline{\underline{A}} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

Eigenvalues:

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)^2 - (1)(-1) = \lambda^2 - 2\lambda + 1 + 1$$

$$= \lambda^2 - 2\lambda + 2 = 0$$

Quadratic Formula

$$\lambda = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2}$$

$$= \frac{2 \pm 2i}{2} = 1 \pm i = \alpha \pm i\beta$$

$$\lambda_1 = 1 + i, \quad \lambda_2 = 1 - i$$

Eigenvectors: Pick $\lambda_1 = 1+i$

$$(\underline{\underline{A}} - \lambda_1 \underline{\underline{I}})\vec{v}_1 = \vec{0}, \quad \text{where } \vec{v}_1 = \begin{bmatrix} A_1 \\ B_1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1-(1+i) & -1 \\ 1 & 1-(1+i) \end{bmatrix}\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} -iA_1 - B_1 = 0 \\ A_1 - iB_1 = 0 \end{cases}$$

Note $i(-iA_1 - B_1) = i(0)$

$$-i^2 A_1 - iB_1 = 0$$

$$\rightarrow A_1 - iB_1 = 0$$

$$\Rightarrow A_1 = iB_1$$

$$\Rightarrow \vec{v}_1 = \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} iB_1 \\ B_1 \end{bmatrix} = B_1 \begin{bmatrix} i \\ 1 \end{bmatrix} \sim \begin{bmatrix} i \\ 1 \end{bmatrix}$$

(chose $B_1 = 1$)

Good news:

$$\boxed{\begin{array}{l} \lambda_1 = 1+i \Rightarrow \vec{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix} \\ \lambda_2 = 1-i \Rightarrow \vec{v}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix} \end{array}}$$

Note: It's common to express complex eigenvectors as:

$$\vec{v} = \vec{a} + i\,\vec{b}$$

Example:

$$\vec{v} = \begin{bmatrix} 1+i \\ 2+3i \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} i \\ 3i \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} + i \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\Rightarrow \vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Example

$$\vec{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 + i \\ 1 + 0i \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} i \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Here $\vec{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

If an eigenvalues are of the form $\lambda = \alpha \pm i\beta$, the corresponding eigenvectors will be of the form $\vec{v} = \vec{a} \pm i\vec{b}$

In our last example:

$$\lambda = 1 \pm i, \quad \vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pm i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$