Lecture 24 — Nonhomogeneous Systems

Schedule

Today (11/30) – last regular lecture

Wed (12/2) – optional lecture on epidemiology

Fri (12/4) – last HW #11 due

Mon (12/7) – Review for final

Wed (12/9) – final exam (9:45–12:00)

Last time, we covered complex eigenvalues

Ex $\underline{A} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$\lambda_1 = 1+i, \quad \vec{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix} \quad ; \quad \lambda_2 = 1-i, \quad \vec{v}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$$\Rightarrow \quad \lambda = 1 \pm i \,, \quad \vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pm i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Complex pairs can always be written as:

$$\lambda = \alpha \pm i\beta \,, \quad \vec{v} = \vec{a} \pm i \vec{b}$$

Here: $\quad \alpha = 1 \,, \ \beta = 1 \,, \ \vec{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} , \ \vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

How does this apply to ODEs?

<u>Example</u>

$$\frac{d\vec{x}}{dt} = \underline{\underline{A}}\, \vec{x} \quad \rightarrow \quad \begin{aligned} \lambda &= \alpha \pm i\beta \\ \vec{v} &= \vec{a} \pm i\vec{b} \end{aligned}$$

$$\vec{x}(t) = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t}$$

$$= c_1 \left(\vec{a} + i\vec{b}\right) e^{(\alpha + i\beta)t} + c_2 \left(\vec{a} - i\vec{b}\right) e^{(\alpha - i\beta)t}$$

Euler's Formula

$\Rightarrow$ Formula for ODEs with complex λ

$$\boxed{\begin{aligned} \vec{w}_1(t) &= e^{\alpha t}\left(\vec{a}\cos(\beta t) - \vec{b}\sin(\beta t)\right) \\ \vec{w}_2(t) &= e^{\alpha t}\left(\vec{a}\sin(\beta t) + \vec{b}\cos(\beta t)\right) \\ \vec{x}(t) &= c_1\vec{w}_1(t) + c_2\vec{w}_2(t) \end{aligned}}$$

Example

$$\frac{dx}{dt} = x - y, \quad \frac{dy}{dt} = x + y$$

Let $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$$\Rightarrow \underline{\underline{A}} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$\lambda_1 = 1+i, \quad \vec{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}; \quad \lambda_2 = 1-i, \quad \vec{v}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$$\Rightarrow \quad \lambda = 1 \pm i, \quad \vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \pm i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\alpha = 1, \quad \beta = 1, \quad \vec{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \vec{w}_1(t) &= e^{\alpha t}\left(\vec{a}\cos(\beta t) - \vec{b}\sin(\beta t)\right) \\ &= e^{t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos(t) - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin(t)\right) \\ &= e^{t}\begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \vec{w}_2(t) &= e^{\alpha t}\left(\vec{a}\sin(\beta t) + \vec{b}\cos(\beta t)\right) \\ &= e^{t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos(t)\right) \\ &= e^{t}\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} \end{aligned}$$

$$\Rightarrow \quad \vec{x}(t) = c_1 \vec{w}_1(t) + c_2 \vec{w}_2(t)$$

$$= c_1 e^t \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} + c_2 e^t \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

$$= \begin{bmatrix} e^t(-c_1 \sin(t) + c_2 \cos(t)) \\ e^t(c_1 \cos(t) + c_2 \sin(t)) \end{bmatrix}$$

$$\Rightarrow \quad \boxed{\begin{aligned} x(t) &= e^t(-c_1 \sin(t) + c_2 \cos(t)) \\ y(t) &= e^t(c_1 \cos(t) + c_2 \sin(t)) \end{aligned}}$$

Non-homogeneous Problems

$$\frac{d\vec{x}}{dt} = \underline{\underline{A}}\,\vec{x} + \vec{f}(t)$$

Example

$$\frac{d}{dt}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}}_{\underline{\underline{A}}}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} + \underbrace{\begin{bmatrix} e^{2t} \\ 1 \end{bmatrix}}_{\vec{f}}$$

Remember:

$$\frac{dx}{dt} + p(t)x = g(t), \qquad \mu(t) = e^{\int p(t)\,dt}$$

$$\Rightarrow \quad x(t) = \mu^{-1}(t)\int \mu(t')\,g(t')\,dt'$$

$$= \mu^{-1}(t)\,c + \mu^{-1}(t)\int_0^t \mu(t')\,g(t')\,dt'$$

★ The Fundament Matrix $\underline{\underline{X}}(t)$ is like $\mu^{-1}(t)$

General Solution if $\vec{x}(0)$ is unknown

$$\Rightarrow \boxed{\begin{aligned}\vec{x}(t) &= \underline{\underline{X}}(t)\int \underline{\underline{X}}^{-1}(t')\,\vec{f}(t')\,dt' \\ &= \underline{\underline{X}}(t)\,\vec{c} + \underline{\underline{X}}(t)\int_0^t \underline{\underline{X}}^{-1}(t')\,\vec{f}(t')\,dt'\end{aligned}}$$

From last time, we saw that $\vec{c} = \underline{\underline{X}}^{-1}(0)\,\vec{x}(0)$

$$\Rightarrow \boxed{\vec{x}(t) = \underline{\underline{X}}(t)\,\underline{\underline{X}}^{-1}(0)\,\vec{x}(0) + \underline{\underline{X}}(t)\int_0^t \underline{\underline{X}}^{-1}(t')\,\vec{f}(t')\,dt'}$$

solution if $\vec{x}(0)$ is known

* For a real derivation, see "Variation of parameters" in Section (9.7)

(For more numerical methods, see MATH 143C & MATH 143M)

Example

$$\underbrace{\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}}_{\underline{\underline{A}}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} + \underbrace{\begin{bmatrix} e^{2t} \\ 1 \end{bmatrix}}_{\vec{f}}, \quad \vec{x}(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Need
- $\lambda_{1,2} \rightarrow \vec{v}_{1,2}$
- $\vec{v}_{1,2} \rightarrow \underline{\underline{X}}(t)$
- $\underline{\underline{X}}(t) \rightarrow \underline{\underline{X}}^{-1}(t)$

The eigenvalues and eigenvectors of $\underline{\underline{A}} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$ are:

- $\lambda_1 = 1, \quad \vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$
- $\lambda_2 = -1, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(See Lecture 22)

$$\Rightarrow \underline{\underline{X}}(t) = \begin{bmatrix} \vec{v}_1 e^{\lambda_1 t} & \vec{v}_2 e^{\lambda_2 t} \end{bmatrix}$$

$$= \begin{bmatrix} 3e^{t} & e^{-t} \\ e^{t} & e^{-t} \end{bmatrix}$$

Remember:

$$\underline{\underline{A}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \underline{\underline{A}}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\underline{\underline{X}}^{-1}(t) = \frac{1}{(3e^{t})e^{-t} - (e^{t})e^{-t}} \begin{bmatrix} e^{-t} & -e^{-t} \\ -e^{t} & 3e^{t} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} e^{-t} & -e^{-t} \\ -e^{t} & 3e^{t} \end{bmatrix}$$

$$3e^{t}e^{-t} - e^{t}e^{-t} = 3 \cdot 1 - 1 = 2$$

$$\text{or} \quad e^{t}e^{-t} = e^{t-t} = e^{0} = 1$$

$$\text{or} \quad e^{t}e^{-t} = \frac{e^{t}}{e^{t}} = 1$$

$$\vec{x}(t) = \underbrace{\underline{\underline{X}}(t)\,\underline{\underline{X}}^{-1}(0)\,\vec{x}(0)}_{(A)} + \underbrace{\underline{\underline{X}}(t)\int_0^t \underline{\underline{X}}^{-1}(t')\,\vec{f}(t')\,dt'}_{(B)}$$

(A)

$$\underline{\underline{X}}(t)\,\underline{\underline{X}}^{-1}(0)\,\vec{x}(0) = \underline{\underline{X}}(t)\begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix}\underbrace{\begin{bmatrix} -1 \\ 0 \end{bmatrix}}_{\vec{x}(0)}$$

$$= \underline{\underline{X}}(t)\begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 3e^{t} & e^{-t} \\ e^{t} & e^{-t} \end{bmatrix}\begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{2}e^{t} + \frac{1}{2}e^{-t} \\ -\frac{1}{2}e^{t} + \frac{1}{2}e^{-t} \end{bmatrix}$$

(B)

$$\underline{\underline{X}}(t)\int_0^t \underline{\underline{X}}^{-1}(t)\,\vec{f}(t)\,dt = \underline{\underline{X}}(t)\int_0^t \begin{bmatrix} \frac{1}{2}e^{-t} & -\frac{1}{2}e^{-t} \\ -\frac{1}{2}e^{t} & \frac{3}{2}e^{t} \end{bmatrix}\begin{bmatrix} e^{2t} \\ 1 \end{bmatrix} dt$$

$$= \underline{\underline{X}}(t)\int_0^t \begin{bmatrix} \frac{1}{2}e^{t} - \frac{1}{2}e^{-t} \\ -\frac{1}{2}e^{3t} + \frac{3}{2}e^{t} \end{bmatrix} dt$$

$$= \underline{\underline{X}}(t)\begin{bmatrix} \frac{1}{2}e^{t} + \frac{1}{2}e^{-t} - 1 \\ -\frac{1}{6}e^{3t} + \frac{3}{2}e^{t} - 4/3 \end{bmatrix}$$

$$= \begin{bmatrix} 3e^{t} & e^{-t} \\ e^{t} & e^{-t} \end{bmatrix} \begin{bmatrix} \frac{1}{2}e^{t} + \frac{1}{2}e^{-t} - 1 \\ -\frac{1}{6}e^{3t} + \frac{3}{2}e^{t} - 4/3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2}e^{2t} + \frac{3}{2} - 3e^{t} - \frac{1}{6}e^{2t} + \frac{3}{2} - \frac{4}{3}e^{-t} \\ \frac{1}{2}e^{2t} + \frac{1}{2} - e^{t} - \frac{1}{6}e^{2t} + \frac{3}{2} - \frac{4}{3}e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{3}e^{2t} - 3e^{t} + 3 - \frac{4}{3}e^{-t} \\ \frac{1}{3}e^{2t} - e^{t} + 2 - \frac{4}{3}e^{-t} \end{bmatrix}$$

Adding all together

$$\vec{x}(t) = \begin{bmatrix} -\frac{3}{2}e^{t} + \frac{1}{2}e^{-t} \\ -\frac{1}{2}e^{t} + \frac{1}{2}e^{-t} \end{bmatrix} + \begin{bmatrix} \frac{4}{3}e^{2t} - 3e^{t} + 3 - \frac{4}{3}e^{-t} \\ \frac{1}{3}e^{2t} - e^{t} + 2 - \frac{4}{3}e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{9}{2}e^{t} - 5/6\, e^{-t} + 4/3\, e^{2t} + 3 \\ -3/2\, e^{t} - 5/6\, e^{-t} + 1/3\, e^{2t} + 2 \end{bmatrix}$$

DONE