Lecture 4

Direction Fields

A direction/slope field is a plot in the (x,y) plane that shows the tangent slopes of every possible solution $y(x)$ given the ODE

$$\frac{dy}{dx} = f(x,y)$$

Example 1

$$m\frac{dv}{dt} = mg - \gamma v$$

$$m = 1, \quad \gamma = 2, \quad g = 10$$

$$\Rightarrow \quad \frac{dv}{dt} = 10 - 2v$$

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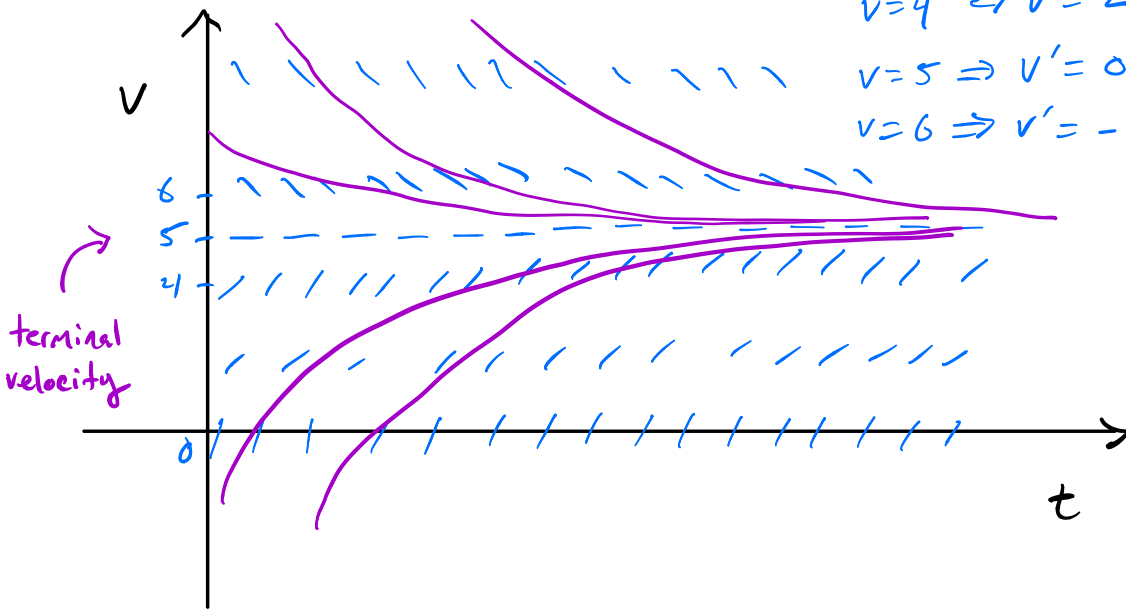
$v = 0 \Rightarrow v' = 10$

$v = 4 \Rightarrow v' = 2$

$v = 5 \Rightarrow v' = 0$

$v = 6 \Rightarrow v' = -2$



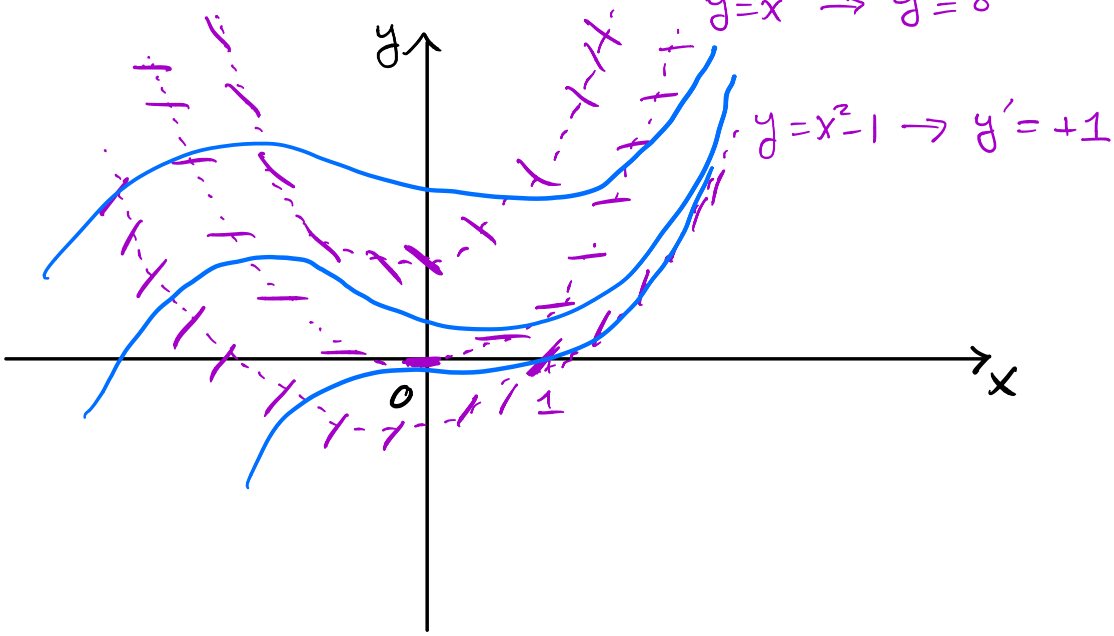
General Solution

$$v(t) = 5 + C e^{-2t}$$

C determined from the initial condition $v(0)$

Example 2

$$\frac{dy}{dx} = x^2 - y$$



$\begin{matrix} x=0 \\ y=0 \end{matrix} \Rightarrow y' = 0$ $\qquad$ $\begin{matrix} x=1 \\ y=0 \end{matrix} \Rightarrow y' = 1$ $\qquad$ $\begin{matrix} x=0 \\ y=1 \end{matrix} \Rightarrow y' = -1$

Isocline: curves of constant slope.

our example: $c = x^2 - y$

$c = 0 \Rightarrow y = x^2$, $\quad c = 1 \Rightarrow y = x^2 - 1$

$c = -1 \Rightarrow y = x^2 + 1$

How do we solve ODEs??

Example 3

Population model: $\frac{dP}{dt} = 2P$

$$\frac{dP}{\cancel{dt}}\cancel{dt} = 2P\,dt$$

$$\rightarrow \quad dP = 2P\,dt$$

$$\rightarrow \quad \frac{dP}{P} = 2\,dt \qquad\qquad \int \frac{dx}{x} = \ln|x| + c$$

$$\rightarrow \quad \int \frac{1}{P}\,dP = \int 2\,dt$$

$$\rightarrow \quad \ln|P| = 2t + c$$

$$\rightarrow \quad e^{\ln P} = e^{2t+c}$$

$$\rightarrow \quad P = e^{2t+c} = e^{2t}e^{c} = A\,e^{2t}$$

$A = e^{c}$

Integration constants

- $$\int x\,dx = \int y\,dy \rightarrow \tfrac{1}{2}x^2 + C_1 = \tfrac{1}{2}y^2 + C_2$$

  $$\Rightarrow \tfrac{1}{2}x^2 = \tfrac{1}{2}y^2 + (C_2 - C_1) = \tfrac{1}{2}y^2 + C$$

  $$\Rightarrow x^2 = y^2 + 2C \rightarrow y^2 + C$$

- $c + c' \rightarrow c$
- $2c \rightarrow c$
- $e^c \rightarrow c$
- $\cancel{cx \rightarrow c}$

Example 4

$$\frac{dy}{dx} + 2xy^2 = 0, \quad y(0) = 1$$

initial condition

$$\Rightarrow \quad \frac{dy}{dx} = -2xy^2$$

$\times \; dx$

$$\Rightarrow \quad dy = -2xy^2 dx$$

$\div \; y^2$

$$\Rightarrow \quad \frac{1}{y^2} dy = -2x \, dx$$

Integrate: $\displaystyle \int \frac{1}{y^2} dy = -2 \int x \, dx$

$$\rightarrow \quad -\frac{1}{y} = -x^2 + C$$

$$\rightarrow \quad \frac{1}{y} = x^2 - C$$

$$\rightarrow \quad y = \frac{1}{x^2 - C}$$

General Solution

use $y(0) = 1$ (initial condition)

$$1 = \frac{1}{0^2 - C} = -\frac{1}{C} \Rightarrow C = -1$$

$$\Rightarrow \quad \boxed{y = \frac{1}{x^2 + 1}}$$

or.... $\frac{1}{y} = x^2 + C$

$$\rightarrow \quad y = \frac{1}{x^2+C}$$

Use $y(0)=1 \Rightarrow 1 = \frac{1}{0^2+C} = \frac{1}{C} \Rightarrow C=1$

$$\Rightarrow \boxed{y = \frac{1}{x^2+1}}$$

Separable ODEs

$$\frac{dy}{dx} = g(x)\,p(y)$$

★ If an ODE can be expressed like this, it is called "separable".

separable

$$y' = x^2y^3$$

$$y' = \frac{a+bx}{c+dy}$$

$$y' = e^{x+y} = e^x e^y$$

Not separable

$$y' = x+y$$

$$y' = \sin(xy)$$

$$y' = x^2y^3+1$$

Example 5 $\qquad \frac{dx}{dt} = \frac{1-x}{t}, \quad x(2) = 0$

1) $dx = \frac{1-x}{t}\, dt$ $\qquad$ separate variables

$\rightarrow \frac{1}{1-x}\, dx = \frac{1}{t}\, dt$

2) $\int \frac{1}{1-x}\, dx = \int \frac{1}{t}\, dt$ $\qquad$ integrate

$\rightarrow -\ln(1-x) = \ln(t) + C$

3) $\ln(1-x) = -\ln(t) - C$ $\qquad$ $n \ln(x) = \ln(x^n)$

$\rightarrow e^{\ln(1-x)} = e^{-\ln(t) - C}$

$\rightarrow 1-x = e^{\ln(t^{-1}) - C} = t^{-1} e^{-C} = A t^{-1}$

$A = e^{-C}$

$\rightarrow \boxed{x = 1 - \frac{A}{t}}$ $\leftarrow$ General Solution

4) Use $x(2)=0$ to solve for A.

$$0 = 1 - \frac{A}{2} \Rightarrow A = 2$$

$$\Rightarrow \boxed{x(t) = 1 - \frac{2}{t}}$$

Done!

5) Check your answer!!