Lecture 21 Inverses, Determinants and Calculus on Matrices

Identity matrix: $\underline{\underline{I}} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$

$(N \times N)$

If $\underline{\underline{A}}$ is a square matrix, then its inverse is defined as:

$$\underline{\underline{A}}\,\underline{\underline{A}}^{-1} = \underline{\underline{I}} \quad \left(\text{or } \underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{I}}\right)$$

Example: $\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

what is $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Matlab $A(i,j)$, Python $A[i][j]$?

Use an augmented matrix:

$$\left[\begin{array}{ccc:ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 3 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \overset{R2-R1,\ R3-R1}{\sim} \left[\begin{array}{ccc:ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\overset{R1-2R2,\ R3+2R2}{\sim} \left[\begin{array}{ccc:ccc} 1 & 0 & -1 & 3 & -2 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & -3 & 2 & 1 \end{array}\right] \overset{R3 \to \frac{1}{2}R3}{\sim} \left[\begin{array}{ccc:ccc} 1 & 0 & -1 & 3 & -2 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -3/2 & 1 & 1/2 \end{array}\right]$$

$$\overset{R1+R3,\ R2-R3}{\sim} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 3/2 & -1 & 1/2 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 1 & -3/2 & 1 & 1/2 \end{array}\right]$$

$$\Rightarrow \boxed{\underline{\underline{A}}^{-1} = \begin{bmatrix} 3/2 & -1 & 1/2 \\ 1/2 & 0 & -1/2 \\ -3/2 & 1 & 1/2 \end{bmatrix}}$$

For 2x2 matrices, there's a simple formula:

$$\underline{\underline{A}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \underline{\underline{A}}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

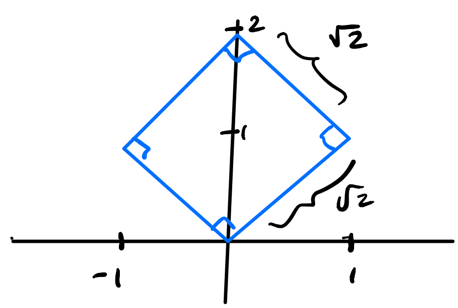
Here, $\underline{\underline{A}}^{-1}$ only exists if $ad-bc \neq 0$

★ $ad-bc$ is known as the <u>Determinant</u>

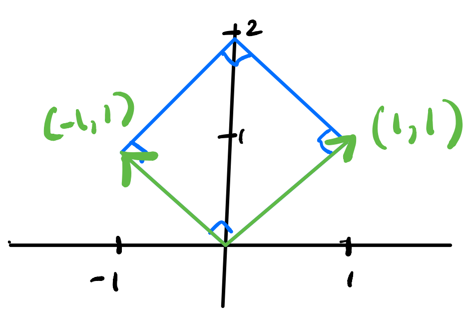
$$\det(\underline{\underline{A}}) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

vertical lines

Determinants represent areas/volumes



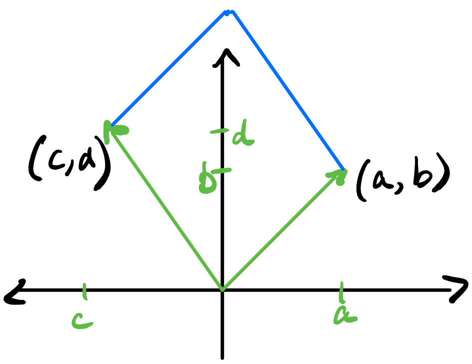
Area = 2



$$\text{Area} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= (1)(1) - (-1)(1)$$

$$= 2$$



$$\text{Area} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$$

How do we take determinants for $N \times N$ matrices?

- First we have to define the "minor" of a matrix.

$$\text{If } \underline{\underline{A}} = \begin{array}{cl} \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix} & \begin{array}{l} \\ \\ \leftarrow i^{th} \text{ Row} \\ \\ \\ \end{array} \\ \quad\quad\quad\quad\quad\quad \uparrow j^{th} \text{ column} & \end{array}$$

$$\text{The minor } M_{ij} = \begin{vmatrix} a_{11} & \cdots & a_{1j-1} & a_{1j+1} & \cdots & a_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{i-1\,1} & \cdots & & & \cdots & a_{i-1\,n} \\ a_{i+1\,1} & \cdots & & & & a_{i+1\,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj-1} & a_{nj+1} & \cdots & a_{nn} \end{vmatrix}$$

Determinant of $\underline{\underline{A}}$ with row i and column j removed

Ex. $\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 5 \\ 2 & 1 & -1 \end{bmatrix}$

$$M_{23} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = (1)(1) - (2)(2) = -3$$

$$\det(\underline{\underline{A}}) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

<u>**Ex**</u> What is $\begin{vmatrix} 1 & 2 & 1 \\ 0 & 3 & 5 \\ 2 & 1 & -1 \end{vmatrix}$?

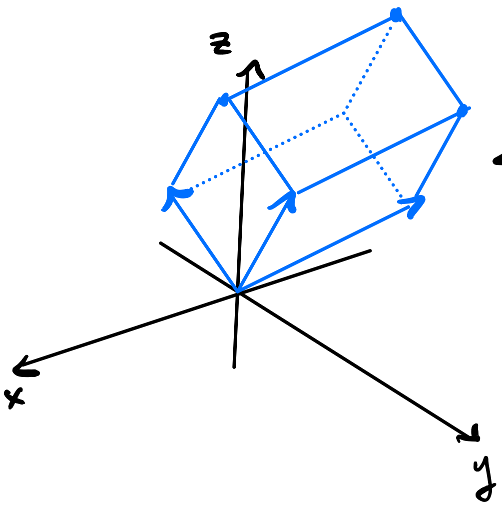
$$\det(\underline{\underline{A}}) = a_{11} M_{11} - a_{12} M_{12} + a_{13} M_{13}$$

$$= (1) \begin{vmatrix} 3 & 5 \\ 1 & -1 \end{vmatrix} - 2 \begin{vmatrix} 0 & 5 \\ 2 & -1 \end{vmatrix} + (1) \begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix}$$

$$= (1)(-3 - 5) - 2(0 - 10) + (1)(0 - 6)$$

$$= -8 + 20 - 6 = 6$$

For 3×3, $\det(\underline{\underline{A}})$ is a "volume"



$\leftarrow$ Volume $= \left|\det(\underline{\underline{A}})\right|$

General formula for 3×3:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

★ If $\det(\underline{\underline{A}}) = 0$, then

- $\underline{\underline{A}}$ is "singular"
- $\underline{\underline{A}}^{-1}$ doesn't exist
- Columns/rows are linearly dependent

- Either there are no solutions or infinitely many when solving $\underline{\underline{A}}\vec{x} = \vec{b}$

  (i.e. there is no $\vec{x} = \underline{\underline{A}}^{-1}\vec{b}$)

Example

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 3 & 0 & 1 & 1 \\ 2 & 0 & 1 & 3 \\ 2 & 1 & 5 & 5 \end{vmatrix} = -(2)\begin{vmatrix} 3 & 1 & 1 \\ 2 & 1 & 3 \\ 2 & 5 & 5 \end{vmatrix} + \cancel{(0)\begin{vmatrix} 1 & 3 & 4 \\ 2 & 1 & 3 \\ 2 & 5 & 5 \end{vmatrix}}$$

$$\cancel{-(0)\begin{vmatrix} 1 & 3 & 4 \\ 3 & 1 & 1 \\ 2 & 5 & 5 \end{vmatrix}} + (1)\begin{vmatrix} 1 & 3 & 4 \\ 3 & 1 & 1 \\ 2 & 1 & 3 \end{vmatrix}$$

$$= -2\begin{vmatrix} 3 & 1 & 1 \\ 2 & 1 & 3 \\ 2 & 5 & 5 \end{vmatrix} + \begin{vmatrix} 1 & 3 & 4 \\ 3 & 1 & 1 \\ 2 & 1 & 3 \end{vmatrix}$$

* For an $N \times N$ matrix, $\det(\underline{\underline{A}})$ requires $N!$ calculations ☹

The +/- Pattern is:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ & & & \cdots & + \end{bmatrix}$$

Calculus of Matrices

$$\vec{x}(t) = \begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix}, \quad \underline{\underline{A}}(t) = \begin{bmatrix} t & e^t \\ \sin(t) & 3 \end{bmatrix}$$

* Derivatives / integrals are applied to each element.

Ex

$$\frac{d}{dt}\vec{x} = \frac{d}{dt}\begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2t \end{bmatrix}$$

$$\frac{d}{dt}\underline{\underline{A}} = \frac{d}{dt}\begin{bmatrix} t & e^t \\ \sin t & 3 \end{bmatrix} = \begin{bmatrix} 1 & e^t \\ \cos t & 0 \end{bmatrix}$$

$$\int_0^1 \vec{x}(t)\,dt = \int_0^1 \begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix} dt$$

definite integral

$$= \begin{bmatrix} t \\ \frac{1}{2}t^2 \\ \frac{1}{3}t^3 \end{bmatrix}\Bigg|_0^1 = \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \end{bmatrix}$$

$$\int \underline{\underline{A}}(t)\,dt = \int \begin{bmatrix} t & e^t \\ \sin t & 3 \end{bmatrix} dt$$

indefinite integral

$$= \begin{bmatrix} \frac{1}{2}t^2 + c_1 & e^t + c_2 \\ -\cos t + c_3 & 3t + c_4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}t^2 & e^t \\ -\cos t & 3t \end{bmatrix} + \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$$

Finally:

$$\frac{d}{dt}\left(\underline{\underline{A}} + \underline{\underline{B}}\right) = \frac{d}{dt}\underline{\underline{A}} + \frac{d}{dt}\underline{\underline{B}}$$

$$\frac{d}{dt}\left(\underline{\underline{A}}\,\underline{\underline{B}}\right) = \underbrace{\left(\frac{d}{dt}\underline{\underline{A}}\right)\underline{\underline{B}} + \underline{\underline{A}}\left(\frac{d}{dt}\underline{\underline{B}}\right)}_{\substack{\text{order matters} \\ \text{since } \underline{\underline{A}}\,\underline{\underline{B}} \neq \underline{\underline{B}}\,\underline{\underline{A}}}}$$