Lecture 13

Laplace Transforms (part 2)

Review:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

Linearity:

- $\mathcal{L}\{c f(t)\} = c\,\mathcal{L}\{f(t)\}$
- $\mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$

Shifts: If $\mathcal{L}\{f(t)\} = F(s)$

$$\Rightarrow \mathcal{L}\{e^{bt} f(t)\} = F(s-b)$$

Inverse Laplace Transform:

$$\mathcal{L}\{f(t)\} = F(s) \iff \mathcal{L}^{-1}\{F(s)\} = f(t)$$

Example 1 $\quad F(s) = \frac{24}{s^4} - \frac{9}{s^2+9}$, $f(t)$?

#3 $\quad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

#7 $\quad \mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2}$

Set $n=3$: $\mathcal{L}^{-1}\left\{\frac{6}{s^4}\right\} = t^3$

$a = 3$: $\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} = \sin(3t)$

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left\{\frac{24}{s^4} - \frac{9}{s^2+9}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{24}{s^4}\right\} - \mathcal{L}^{-1}\left\{\frac{9}{s^2+9}\right\} \\ &= 4\,\mathcal{L}^{-1}\left\{\frac{6}{s^4}\right\} - 3\,\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} \\ &= 4t^3 - 3\sin(3t) \end{aligned}$$

Derivatives

If $F(s) = \mathcal{L}\{f(t)\}$, what is $\mathcal{L}\{f'(t)\}$?

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

integration by parts

$$= e^{-st} f(t) \Big|_0^\infty - \int_0^\infty (-s e^{-st}) f(t)\,dt$$

$$= 0 - e^0 f(0) + s \underbrace{\int_0^\infty e^{-st} f(t)\,dt}_{F(s)}$$

$$= \boxed{s F(s) - f(0)}$$ (#35)

$$\boxed{\mathcal{L}\{f''(t)\} = s^2 F(s) - s f(0) - f'(0)}$$ (#36)

(#37)

$\vdots$

$$\boxed{\begin{aligned}\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \\ \dots - s f^{(n-2)}(0) - f^{(n-1)}(0)\end{aligned}}$$

Example 2

$$\frac{dy}{dt} = y, \quad y(0) = 1$$

1) Take Laplace Transform (LT) of both sides of the equation

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = \mathcal{L}\{y\}$$

(#35)

$$s\,Y(s) - y(0) = Y(s)$$

$$\Rightarrow \quad s\,Y(s) - 1 = Y(s)$$

since $y(0)=1$

ODE turns into an algebraic equation

2) Solve for $Y(s)$

$$\Rightarrow \quad s\,Y(s) - Y(s) = 1$$

$$\Rightarrow \quad (s-1)\,Y(s) = 1$$

$$\Rightarrow \quad Y(s) = \frac{1}{s-1}$$

3) Invert $Y(s)$ back to the time domain using your table

#2 $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ $a=1$

$$\Rightarrow \boxed{y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = e^{t}}$$

Example 3

$$\frac{d^2y}{dt^2} + 4y = 0, \quad y(0)=0, \quad y'(0)=1$$

1) Take LT of the ODE

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} + 4\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

#36

$$s^2Y(s) - \cancel{sy(0)} - y'(0) + 4Y = 0 \qquad y(0)=0$$

$$\Rightarrow s^2Y(s) - 1 + 4Y(s) = 0$$

2) Solve for $Y(s)$

$$s^2 Y(s) + 4Y(s) = 1$$

$$(s^2+4)\,Y(s) = 1$$

$$\Rightarrow Y(s) = \frac{1}{s^2+4}$$

3) Invert: $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

#7: $\mathcal{L}\{\sin(at)\} = \dfrac{a}{s^2+a^2}$ $a=2$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$= \frac{1}{2}\sin(2t)$$

General 2nd-order, constant coefficient, linear, ODE:

$$ay'' + by' + cy = f(t)$$

$$y(0) = \alpha, \quad y'(0) = \beta$$

1) Take LT of ODE:

$$a\,\mathcal{L}\{y''\} + b\,\mathcal{L}\{y'\} + c\,\mathcal{L}\{y\} = \mathcal{L}\{f\}$$

$$a\underbrace{\left(s^2Y - sy(0) - y'(0)\right)}_{\#36} + b\underbrace{\left(sY - y(0)\right)}_{\#35} + cY = F$$

$$\Rightarrow a\left(s^2Y - s\alpha - \beta\right) + b\left(sY - \alpha\right) + cY = F$$

2) Solve for $Y(s)$

$$\Rightarrow as^2Y(s) + bsY(s) + cY(s) = F + a(s\alpha+\beta) + b\alpha$$

$$\Rightarrow (as^2 + bs + c)Y(s) = F + a(s\alpha+\beta) + b\alpha$$

$$\Rightarrow Y(s) = \frac{\overbrace{F(s)}^{\text{external terms}} + \overbrace{a(s\alpha+\beta) + b\alpha}^{\text{initial conditions}}}{\underbrace{as^2 + bs + c}_{\text{response of system}}}$$

3) Invert: $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

★ Since there will always be a polynomial in the denominator, algebra tricks can be <u>useful</u>

(A) Completing the square

$$x^2 + bx + c = x^2 + bx + \frac{b^2}{4} - \frac{b^2}{4} + c$$

$$= \left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c$$

<u>Ex</u> #19: $\mathcal{L}\{e^{at}\sin(bt)\} = \dfrac{b}{(s-a)^2+b^2}$

(B) Partial Fractions

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} \; ?$$

$$\Rightarrow \quad 1 = \frac{A}{\cancel{x+1}}\cancel{(x+1)}(x+2) + \frac{B}{\cancel{x+2}}(x+1)\cancel{(x+2)}$$

$$= A(x+2) + B(x+1)$$

$$= (A+B)x + (2A+B)$$

$$\Rightarrow \quad A+B=0 \quad , \quad 2A+B=1$$

$$B=-A \longrightarrow 2A-A=1 \Rightarrow \boxed{A=1}$$

$$\boxed{B=-A=-1}$$

$$\Rightarrow \boxed{\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}}$$