Lecture 2

What is a differential Equation?

	operations	equation
Algebra:	$+,\ \times$	$\longrightarrow\ ax^2+bx+c=0$
Calculus:	$\frac{d}{dx}$	$\longrightarrow\ \underbrace{a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0}_{\text{example of differential equation}}$

Before, you solved for a number x

Now, you will solve for a function $y(x)$

Examples

1) $\frac{dy}{dx}=0 \implies y(x) = A$

2) $\frac{dy}{dx} = x$

↳ Integrate both sides

$$\int \frac{dy}{dx}\,dx = \int x\,dx$$

↓

$$y(x) = \frac{1}{2}x^2 + A$$

3) $\frac{d^2y}{dx^2} = x$

$$\rightarrow \int \frac{d^2y}{dx^2}\,dx = \int x\,dx$$

$$\frac{dy}{dx} = \frac{1}{2}x^2 + A$$

$$\int \frac{dy}{dx}\,dx = \int \left(\frac{1}{2}x^2 + A\right)dx$$

$$y(x) = \frac{1}{6}x^3 + Ax + B$$

Things get a <u>lot</u> harder when $y(x)$ shows up more than once in an equation.

Ex. $$\frac{dy}{dx} + y = 0$$

Dealing with this is the subject of this class!

Review of independent/dependent variables

	Ind. Var.	Dep. Var.
$y = y(x)$	x	y
$x = x(t)$	t	x
$u = u(x,y,t)$	x,y,t	u

In differential equations, derivatives are taken with respect to the <u>independent variables</u>, and you are trying to solve for the <u>dependent variables</u>.

★ Ordinary vs. Partial Differential Eqns.

- Ordinary Differential Equations (ODEs) have 1 independent variable.

Ex.

$$\frac{dy}{dx} + y^2 = 2 - x \quad , \quad \frac{d^2x}{dt^2} + e^t x = \sin(t)$$

- Partial Differential Equations (PDEs) have more than 1 independent variable.

Ex. $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}$

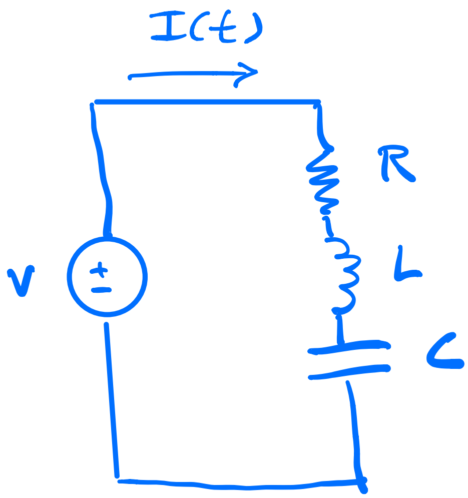
(if interested: MATH 133B)

WHO CARES?

ODEs + PDEs play an important role in modeling virtually <u>every</u> physical/technical/biological process with many applications:

- planetary motion
- chemical reactions
- neuron interactions
- stock option price
- etc....

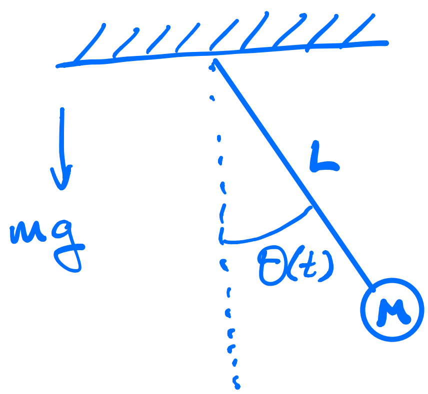
Example Models



RLC circuit

$$L \frac{d^2 I}{dt^2} + R \frac{dI}{dt} + \frac{1}{C} I = \frac{dV}{dt}$$

(circuit theory)



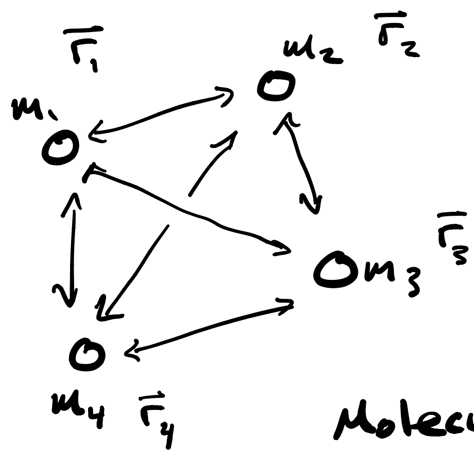
$$\frac{d^2\theta}{dt^2} + \frac{g}{L} \sin(\theta) = 0$$

(pendulum)

Newton's 2nd Law : $F = ma$

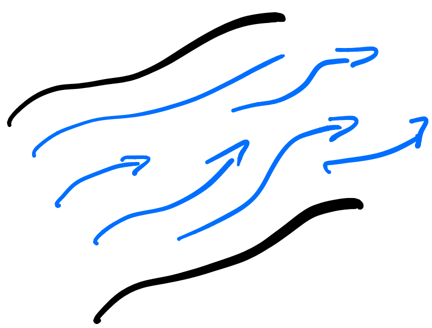
$$a = \frac{dv}{dt}, \quad v = \frac{dx}{dt} \Rightarrow a = \frac{d^2 x}{dt^2}$$

$$\Rightarrow \quad m\frac{d^2x}{dt^2} = F(x)$$



$$m_i \frac{d^2\vec{r}_i}{dt^2} = \sum_{j\neq i} \vec{F}_{ij}(\vec{r}_i, \vec{r}_j)$$

Molecular Dynamics



Density: $\rho(x,y,z,t)$

Velocity: $\vec{v}(x,y,z,t)$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0$$

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right) = -\nabla P + \mu\nabla^2\vec{v}$$

"Navier-Stokes Equation"

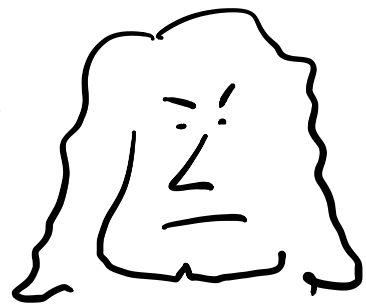
NOTATION + Terminology

Leibniz

$$\frac{dy}{dx} = y'(x)$$

$$\frac{d^2y}{dx^2} = y''(x)$$

$$\frac{d^ny}{dx^n} = y^{(n)}(x)$$



Newton

For time derivatives

$$\dot{x} = \frac{dx}{dt}, \quad \ddot{x} = \frac{d^2x}{dt^2}$$

<u>ORDER</u> : highest derivative present.

Ex.

$$y'' - e^x y = \sin(x)$$

2nd-order ODE

$$\frac{dy}{dx} + y^2 = 2 - x$$

1st-order ODE

A general n^{th}-order ODE is of the form:

$$y^{(n)} = F(x, y, y', y'', \ldots, y^{(n-1)})$$

<u>**LINEARITY**</u>: An ODE is <u>linear</u> if all of the dependent variable terms are linear.

$$y'' + e^x y = \sin(x)$$

both linear $\Rightarrow$ linear ODE

$$\frac{dy}{dx} + y^2 = 2 - x$$

nonlinear $\Rightarrow$ nonlinear ODE

General n^{th}-order linear ODE is of the form:

$$a_n(x)\frac{d^n y}{dx^n} + \dots + a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x)$$