Lecture 18 Linear Algebra

Last time, we looked at linear algebraic equations:

Ex Unknowns: x_1, x_2, x_3

$$x_1 + 2x_2 + 5x_3 = 7$$
$$2x_1 + 4x_2 + x_3 = 13$$
$$x_2 + 9x_3 = 10$$

For large systems $(x_1, x_2, x_3, \ldots, x_n)$, we need a better way to talk about linear systems.

VECTORS

Given a list of numbers

$$\{2, 3, 7, 5\}$$

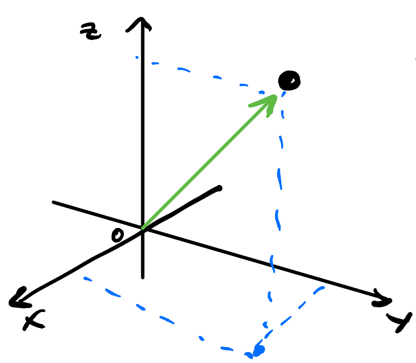
We can express them in terms of a <u>vector</u>:

$$\vec{v} = \begin{bmatrix} 2 \\ 3 \\ 7 \\ 5 \end{bmatrix}$$ "column vector"

$$\vec{w} = \begin{bmatrix} 2 & 3 & 7 & 5 \end{bmatrix}$$ "row vector"

(columns are usually the default)

Example: Position



$$\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

written in bold in text

Notation: $\vec{v} = \underline{v} = \mathbf{v} = v$

Also $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

★ Basic Operations

1) Addition

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1+4 \\ 2+5 \\ 3+6 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 8 \end{bmatrix}$$

Must be same size!!

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 & 5 & 1 \end{bmatrix} = ???$$

$$\begin{bmatrix} 1 & 2 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 \end{bmatrix}$$

In general:

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1+b_1 \\ a_1+b_2 \\ \vdots \\ a_n+b_n \end{bmatrix}$$

2) Multiplication

Given: $\vec{a}$ and $\vec{b}$, $\vec{a}\vec{b} = ???$

↑ Meaningless

Dot product:

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6$$
$$= 4 + 10 + 18$$
$$= 32$$

Side note: Transpose (swap rows & columns)

$$\vec{V} = \begin{bmatrix} 2 \\ 7 \\ 9 \end{bmatrix}$$

$$\vec{V}^T = \begin{bmatrix} 2 & 7 & 9 \end{bmatrix} \leftarrow \text{transpose}$$

the dot product can be written in terms of the transpose:

$$\vec{a}, \vec{b} \Rightarrow \text{Dot product } \vec{a} \cdot \vec{b} = \vec{a}^T \vec{b}$$
$$= \vec{b}^T \vec{a}$$

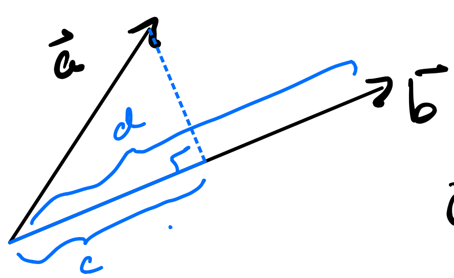
In general:

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\vec{a} \cdot \vec{b} = \vec{a}^T \vec{b} = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

$$= \sum_{k=1}^{n} a_k b_k$$

Geometric Interpretation



$$\vec{a} \cdot \vec{b} = cd$$

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos(\theta)$$

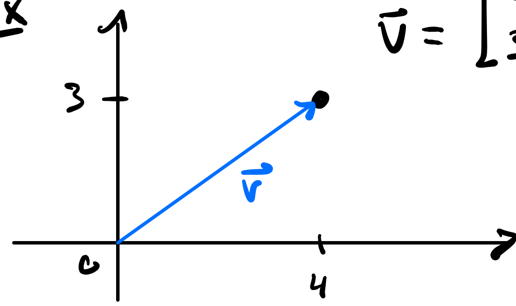
$|\vec{a}|$ = length of $\vec{a}$

$|\vec{b}|$ = length of $\vec{b}$

$$\Rightarrow \vec{a} \cdot \vec{a} = |\vec{a}|^2 \Rightarrow |\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}}$$

Notation: $|\vec{a}| = a$

Ex



$\vec{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

$$\text{length} = v = |\vec{v}| = \sqrt{\vec{v}\cdot\vec{v}}$$
$$= \sqrt{4^2+3^2} = \sqrt{25} = 5$$

Matrices

- A matrix is basically a list of vectors (lists)

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \leftarrow \text{Matrix}$$

Matrices are $N \times M$ arrays of numbers

$$\begin{bmatrix} 1 & 5 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \qquad \begin{bmatrix} 5 & 1 & 7 & 1 \\ 13 & 2 & 2 & 0 \end{bmatrix}$$

(3x2 matrix) (2x4 matrix)

In general:

$$\underline{\underline{A}} = \overbrace{\left.\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1M} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2M} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3M} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{N1} & a_{N2} & a_{N3} & \cdots & a_{NM} \end{bmatrix}\right\}\text{N tall}}^{\text{M wide}}$$

If $N=M \Rightarrow$ "square matrix"

Notation: $\underline{\underline{A}} = \mathbf{A} = A$

bold capital letter in text

Operations

1) Addition: (same with vectors)

$$\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \underline{\underline{B}} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\Rightarrow \underline{\underline{A}} + \underline{\underline{B}} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$$

Example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 1 \\ 2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1+7 & 2+0 & 3+1 \\ 4+2 & 5+2 & 6+3 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 2 & 4 \\ 6 & 7 & 9 \end{bmatrix}$$

Again, $\underline{\underline{A}}$ and $\underline{\underline{B}}$ must be the same size for $\underline{\underline{A}} + \underline{\underline{B}}$ to exist.

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} = ???$$

Multiplication

★ Treat each row as a vector and perform a dot product on the corresponding column.

$$\underbrace{\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \end{bmatrix}}_{2\times 3 \text{ Matrix}} \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{\substack{3\times 1 \text{ column vector} \\ \text{(matrix)}}} = \begin{bmatrix} 1\cdot 1 + 0\cdot 2 + 3\cdot 3 \\ 2\cdot 1 + 1\cdot 2 + 1\cdot 3 \end{bmatrix} = \underbrace{\begin{bmatrix} 10 \\ 7 \end{bmatrix}}_{2\times 1 \text{ column vector}}$$

★ If $\underline{\underline{A}}$ is $N \times M$ matrix, and $\underline{\underline{B}}$ is $M \times P$ matrix, then $\underline{\underline{A}}\,\underline{\underline{B}}$ is $N \times P$

Example

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{2\times 2}\underbrace{\begin{bmatrix} 1 & 0 & 3 \\ 2 & 2 & 0 \end{bmatrix}}_{2\times 3} = \begin{bmatrix} 1\cdot 1+2\cdot 2 & 1\cdot 0+2\cdot 2 & 1\cdot 3+2\cdot 0 \\ 3\cdot 1+4\cdot 2 & 3\cdot 0+4\cdot 2 & 3\cdot 3+4\cdot 0 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 5 & 4 & 3 \\ 11 & 8 & 9 \end{bmatrix}}_{2\times 3}$$

If $\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{C}} \Rightarrow c_{ij} = \sum_k a_{ik} b_{kj}$

↑ width of $\underline{\underline{A}}$ must be the same as the height of $\underline{\underline{B}}$.

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = ???$$

Can't be Done!