Lecture 12 — LAPLACE TRANSFORMS

First...

One more Example of the Method of Undetermined Coefficients

$$y'' + 2y' + y = t^2 + 8e^t$$

$$y(0) = 1, \quad y'(0) = 0$$

A) Solve Homogeneous Problem

$$y_h'' + 2y_h' + y_h = 0$$

$$\longrightarrow \quad r^2 + 2r + 1 = 0 \longleftarrow$$

Characteristic (or "auxiliary") equation

$$r = \frac{-2 \pm \sqrt{4-4}}{2} = -1 \quad (\text{Case III})$$

$$\Rightarrow \quad y_h = c_1 e^{-t} + c_2 t e^{-t}$$

B) Find the particular solution

Trial solution: $y_p(t) = At^2 + Bt + C + De^t$

$$\rightarrow \; y_p'(t) = 2At + B + De^t$$

$$\rightarrow \; y_p''(t) = 2A + De^t$$

$$y_p'' + 2y_p' + y_p = 2A + De^t$$
$$+ 2\left(2At + B + De^t\right)$$
$$+ At^2 + Bt + C + De^t$$

$$= At^2 + (4A + B)t + (2A + 2B + C) + 4De^t$$

$$= t^2 + 8e^t$$

$$\Rightarrow \boxed{A = 1}, \quad 4A + B = 0, \quad 2A + 2B + C = 0, \; 4D = 8$$

$$\hookrightarrow 4 + B = 0 \qquad \hookrightarrow 2(1) + 2(-4) + C = 0$$

$$\hookrightarrow \boxed{B = -4} \qquad \hookrightarrow 2 - 8 + C = 0$$

$$\hookrightarrow \boxed{C = 6}$$

$$4D = 8 \rightarrow \boxed{D = 2}$$

$$\Rightarrow \quad y_p(t) = t^2 - 4t + 6 + 2e^t$$

c) General Solution: $y(t) = y_h(t) + y_p(t)$

$$\Rightarrow \quad y(t) = c_1 e^{-t} + c_2 t e^{-t} + t^2 - 4t + 6 + 2e^t$$

D) Use $y(0) = 1$, $y'(0) = 0$ to solve $c_{1,2}$

$$\to \quad y'(t) = -c_1 e^{-t} + c_2 e^{-t} - c_2 t e^{-t} + 2t - 4 + 2e^t$$

- $y(0) = c_1 + 6 + 2 = c_1 + 8 = 1$

$$\Rightarrow \boxed{c_1 = -7}$$

- $y'(0) = -c_1 + c_2 - 4 + 2$

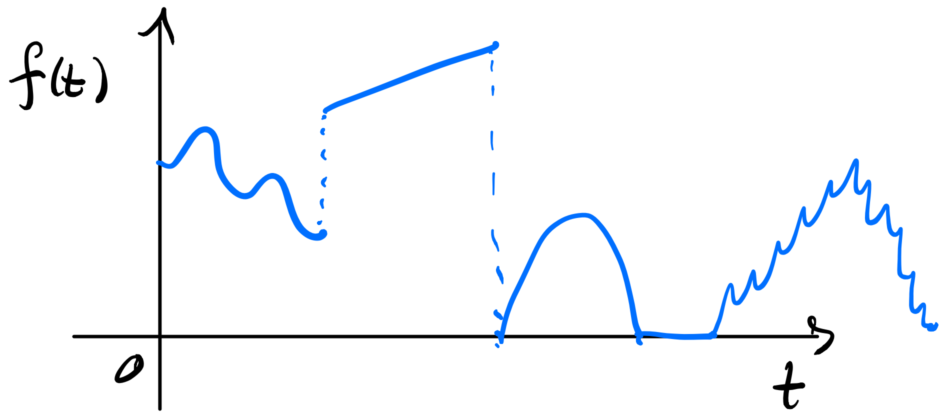
$$= 7 + c_2 - 4 + 2$$

$$= 5 + c_2 = 0 \Rightarrow \boxed{c_2 = -5}$$

$$\boxed{y(t) = -7e^{-t} - 5te^{-t} + t^2 - 4t + 6 + 2e^t}$$

The solution

So far, this seems to work (even if it's tedious). What happens if we have: $ay'' + by' + cy = f(t)$



ENTER: LAPLACE TRANSFORM

$$\boxed{\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)}$$

* The Laplace transform (LT) takes a $f(t)$ and returns a new function $F(s)$

★ $f(t)$ is in the "time domain", and $F(s)$ is in the "frequency domain".

★ It is common notation to denote the LT of a function $f(t), y(t), i(t), \ldots$ with a capital letter: $F(s), Y(s), I(s), \ldots$

Named after Pierre-Simon Laplace

Example 1 $f(t) = 3t$

$$\mathcal{L}\{3t\} = \int_0^\infty e^{-st}(3t)\,dt$$

$$= \cancel{-\frac{3}{s}te^{-st}\Big|_0^\infty}^{\,0} + \frac{3}{s}\int_0^\infty e^{-st}\,dt$$

$(s>0)$

$$= -\frac{3}{s^2}e^{-st}\Big|_0^\infty = 0 - \left(-\frac{3}{s^2}e^0\right) = \boxed{\frac{3}{s^2} \quad (s>0)}$$

Example 2 $\quad f(t) = c$

$$\mathcal{L}\{c\} = \int_0^\infty e^{-st} c \, dt = -\frac{c}{s} e^{-st} \Big|_0^\infty$$

$$= 0 - \left(-\frac{c}{s} e^0\right) = \frac{c}{s}$$

$$\Rightarrow \boxed{F(s) = \frac{c}{s}, \quad s > 0}$$

Example 3 $\quad f(t) = e^{2t}$

$$\mathcal{L}\{e^{2t}\} = \int_0^\infty e^{-st}(e^{2t})dt = \int_0^\infty e^{(2-s)t} dt$$

$$= \frac{1}{2-s} e^{(2-s)t} \Big|_0^\infty = \underset{(s>2)}{0} - \frac{1}{2-s} e^0 = \frac{1}{s-2}$$

$$\Rightarrow \boxed{F(s) = \frac{1}{s-2}, \quad s > 2}$$

If you're bad at integrals, don't worry! Tables are extensively used for Laplace Transforms. (see Laplace_Table.pdf)

★ You won't always find an exact match on the Table, so it's important to know a few properties.

A) Linearity:

- $\mathcal{L}\{c\,f(t)\} = c\,\mathcal{L}\{f(t)\}$
- $\mathcal{L}\{f(t)+g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$

<u>Example 4</u>

$$f(t) = e^{-2t} + 3t^2 - \sin(5t)$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{-2t} + 3t^2 - \sin(5t)\}$$

$$= \mathcal{L}\{e^{-2t}\} + 3\,\mathcal{L}\{t^2\} - \mathcal{L}\{\sin(5t)\}$$

#2 $a=-2$ #3 $n=2$ #7 $b=5$

$$= \frac{1}{s+a} + 3\left(\frac{2}{s^3}\right) - \frac{5}{s^2+25}$$

B) Shifts:

If $\mathcal{L}\{f(t)\} = F(s)$

Then $\mathcal{L}\{e^{bt} f(t)\} = F(s-b)$

<u>Examples</u> 5 & 6

$$\mathcal{L}\{e^{t} \sin(5t)\} = \frac{5}{(s-1)^2 + 25}$$

#7, $b=5$: $\mathcal{L}\{\sin(5t)\} = \frac{5}{s^2+25}$

$$\mathcal{L}\{e^{-4t} t^2\} = \frac{2}{(s+4)^3}$$

#3, $n=2$: $\mathcal{L}\{t^2\} = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$

Canvas

↳ Files

↳ Course Documents

↳ Laplace_Table.pdf