Lecture 9 — 2^{nd}-order ODEs

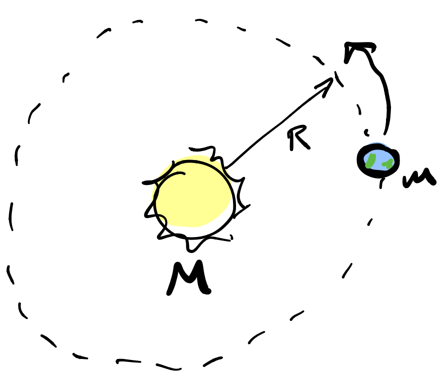
2nd-order

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$$

- These ODEs are of <u>huge</u> importance for many reasons - namely Newton's 2^{nd} Law.

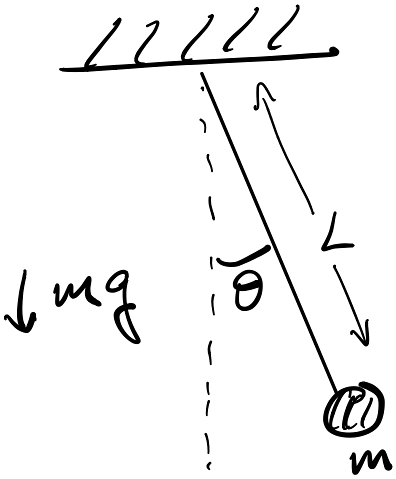
$$F = ma, \quad a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

$$\Rightarrow m\frac{d^2x}{dt^2} = F(x)$$



$$m\frac{d^2x}{dt^2} = -\frac{mMG}{R^3}x$$

(celestial mechanics)



$\theta = \theta(t)$

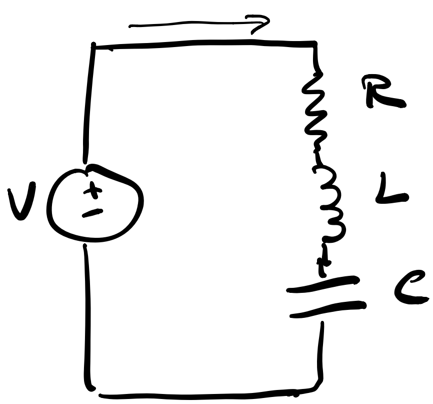
Pendulum

$$mL \frac{d^2\theta}{dt^2} = -mg\sin(\theta)$$

$$\Rightarrow \frac{d^2\theta}{dt^2} + \frac{g}{L}\underbrace{\sin(\theta)}_{\text{nonlinear}} = 0$$

(Linear) Circuit Theory

$i(t)$ (current)



$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = F(t)$$

$$F(t) = \frac{dV}{dt}$$

Some Definitions

external input

Linear: $a(t)\dfrac{d^2y}{dt^2} + b(t)\dfrac{dy}{dt} + c(t)y = f(t)$

Homogeneous: $a(t)\dfrac{d^2y}{dt^2} + b(t)\dfrac{dy}{dt} + c(t)y = \underline{\underline{0}}$

Constant Coefficient: $a\dfrac{d^2y}{dt^2} + b\dfrac{dy}{dt} + cy = f(t)$

a, b, c = constant

Let's start with the simplest case:

Linear

$$\underbrace{a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy}_{\text{constant coefficient}} = 0$$

Homogeneous

- **None** of the methods for 1st-order ODEs will work, but maybe the 1st-order version of this problem will give us a hint.

$$\underbrace{a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy}_{} = 0$$

look at these terms

$$b\frac{dy}{dt} + cy = 0 \implies y = Ae^{-\frac{c}{b}t}$$

★ The solution is an exponential, which should make sense, because the derivative of an exponential is another exponential times a constant.

▷ Let's try guessing an exponential

Let $y(t) = c e^{rt}$

(c, r are unknown)

$$\frac{dy}{dt} = c r e^{rt} \quad , \quad \frac{d^2y}{dt^2} = c r^2 e^{rt}$$

<u>Example 1</u>: $\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 4y = 0$

Let $y(t) = c e^{rt}$

$$\Rightarrow \quad c r^2 e^{rt} - 5 c r e^{rt} + 4 c e^{rt} = 0$$

$$\Rightarrow \quad c e^{rt} \left[r^2 - 5r + 4 \right] = 0$$

Assuming $y \neq 0$ $\Rightarrow$ $\boxed{r^2 - 5r + 4 = 0}$

We must have $r = \dfrac{5 \pm \sqrt{25-16}}{2} = \dfrac{5 \pm 3}{2} = \{1, 4\}$

There are $\underline{\underline{2}}$ solutions!

$$y_1(t) = c_1 e^{t} \quad \text{\&} \quad y_2(t) = c_2 e^{4t}$$

Check for $y_1(t)$:

$$y'' - 5y' + 4y = c_1 e^{t} - 5c_1 e^{t} + 4c_1 e^{t}$$

$$= c_1 e^{t}(\cancel{1} - \cancel{5} + \cancel{4}) = 0 \quad \checkmark$$

This should make sense:

- 1^{st}-order ODE $\rightarrow$ 1 integration $\rightarrow$ 1 constant
- 2^{nd}-order ODE $\rightarrow$ 2 integrations $\rightarrow$ 2 constants

Which solution should you use? Both!

$$\boxed{y(t) = c_1 e^{t} + c_2 e^{4t}}$$

This is due to the "Superposition Principle"

★ If $y_1(t)$ and $y_2(t)$ are <u>both</u> solutions to a <u>linear</u> ODE, then so is $y_1(t) + y_2(t)$.

General Problem

$$a \frac{d^2y}{dt^2} + b \frac{dy}{dt} + cy = 0$$

- Let $y = ce^{rt} \Rightarrow \boxed{ar^2 + br + c = 0}$

  "characteristic equation"

- Two roots: $r_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

- Solution: $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

- Any initial conditions can be used to solve for c_1 & c_2.

Example 2

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0$$

$$\text{IC: } y(0) = 2, \quad \frac{dy}{dt}(0) = 3$$

1) Let $y = ce^{rt}$

$$\Rightarrow \quad cr^2e^{rt} + 5cre^{rt} + 6ce^{rt} = 0$$

$$\Rightarrow \quad ce^{rt}\left[r^2 + 5r + 6\right] = 0$$

$$\Rightarrow \quad r^2 + 5r + 6 = 0$$

2) $$r = \frac{-5 \pm \sqrt{25-24}}{2} = \frac{-5 \pm 1}{2} = \{-2, -3\}$$

3) $$\underbrace{y(t) = c_1 e^{-2t} + c_2 e^{-3t}}_{\text{"General Solution"}}$$

4) Apply initial conditions

$$y(0) = 2 \ , \quad y'(0) = 3$$

$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}$$

$$\frac{dy}{dt}(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t}$$

$$\Rightarrow \quad y(0) = c_1 e^{0} + c_2 e^{0} = c_1 + c_2 = 2$$

$$\frac{dy}{dt}(0) = -2c_1 e^{0} - 3c_2 e^{0} = -2c_1 - 3c_2 = 3$$

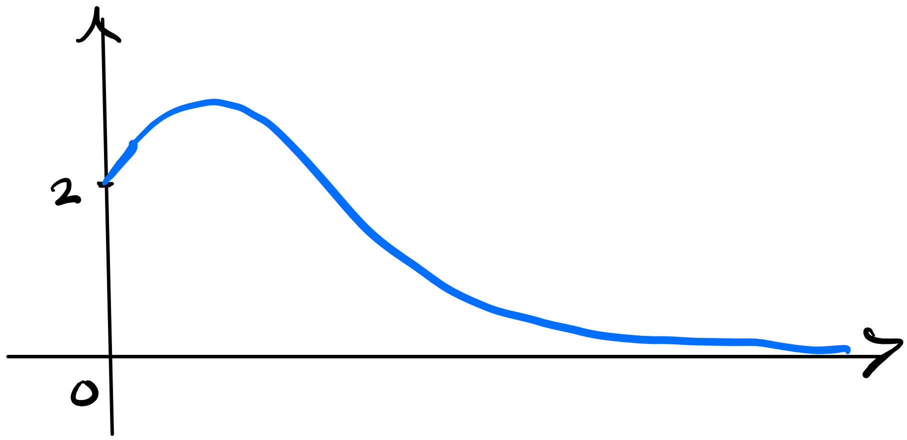
Solve:

$$\begin{array}{rl} c_1 + c_2 = 2 & R_1 \\ -2c_1 - 3c_2 = 3 & R_2 \end{array}$$

$$2R_1 + R_2 = 0 - c_2 = 7 \Rightarrow c_2 = -7$$

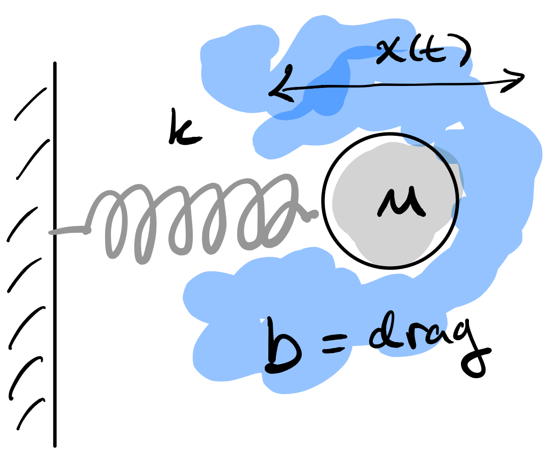
Substitute into R_1: $c_1 - 7 = 2 \Rightarrow c_1 = 9$

$$\Rightarrow \quad \boxed{y(t) = 9e^{-2t} - 7e^{-3t}}$$



What happens for $b^2 - 4ac \lesssim 0$

- To understand solutions, let's connect the ODE to a physical model.



$$m\frac{d^2x}{dt^2} = F$$

$$\approx -kx - b\frac{dx}{dt}$$

$$\Rightarrow \quad \underset{\substack{\uparrow \\ a}}{m}\frac{d^2x}{dt^2} + \underset{\substack{\uparrow \\ b}}{b}\frac{dx}{dt} + \underset{\substack{\uparrow \\ c}}{k}x = 0$$

Real roots: $b^2 > 4mk$ (strong damping)

Complex (imaginary) Roots: $b^2 < 4mk$ (weak damping)

→ Oscillations

Example 3 $\quad \frac{d^2x}{dt^2} + x = 0$

$(m=1,\ b=0,\ k=1)$

Solution:

$$x(t) = c_1 \cos(t) + c_2 \sin(t)$$

TO BE CONTINUED ....