Math 133A, Weeks 1 & 2: First-Order Differential Equations

Section 1: Course Structure

In this course, we will study a very important modeling tool used throughout the sciences: **differential equations**. Differential equations are used to model many real-world phenomena, from the motion of physical objects obeying Newton's laws of motion, to the interactions between competing species in an ecosystem, to the evolution of financial markets. As computational power has exploded over the past half century, the analysis of even extremely complicated systems has become manageable through numerical simulation. Differential equation modeling remains a very active area of research and application to date.

Our exploration of differential equations will focus on solution techniques and applications, although we will occasionally require some mathematical theory to guide our exploration. The rough outline of the course is the following:

- First-Order Differential Equations (Chapters 1 & 2) (Four Weeks)
- Second-Order Differential Equations (Chapter 3) (Three Weeks)
- Power Series Methods (Chapter 5) (One Week)
- Laplace Transforms (Chapter 6) (Two Weeks)
- Linear Systems of Differential Equations (Chapter 7) (Four Weeks)

We will cover roughly Chapters 1, 2, 3, 5, 6 & 7 of Boyce & DiPrima, although not always in the order given in the text. I will post my comprehensive lecture notes online.

Note: A functional knowledge of the basic concepts from **Math 30**, **31**, and **32** will be assumed throughout this course. In particular, application of basic differentiation rules and integration techniques such as integration by substitution and partial fraction decomposition will be frequently required when studying differential equations. **A review of these topics will not be given!** A course in linear algebra (e.g. **Math 129A**) is helpful but not required.

Section 2: Differential Equations

Differential equations are everywhere. They are used across the natural and social sciences to model many real-world phenomena, from the motion of physical objects obeying Newton's laws, to the interactions between competing species in an ecosystem, to the evolution of financial markets. As computational power has exploded over the past half century, the analysis of even extremely complicated systems has become manageable.

It should also be pointed out that differential equations are interesting mathematical objects of study in their own right, and can be studied with no reference to application at all. The theoretical study of differential equations will be touched upon in this course, but *will not* be a primary focus of study. We will see at the end of the course that the tools we will learn when studying linear algebra will frequently appear in the understanding of differential equations.

We had better stop for a moment and make sure we understand a **differential equation actually is**. We should also motivate some common avenues by which they arise.

Definition 1

A **differential equation** (DE) is any equation (i.e. algebraic expression with an equal sign) which involves functions and at least one of their derivatives. A differential equation is said to be an **ordinary differential equation** (ODE) if all derivatives are with respect to a single variable.

That's it—if you see an equation and it has a derivative in it, it is a differential equation. So an equation like

$$y'' + 7y = \cos(x)$$

(where y is a function of x, i.e. $y = y(x)$) is a differential equation, but the linear system

$$\begin{cases} 2x + y = -3 \\ x - 4y = 2 \end{cases}$$

is not. Similarly, an equation like

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

(where u is a function of both x and y, i.e. $u = u(x, y)$) is a differential equation, but *not* an ordinary differential equation. This is an example of a **partial differential equation** (PDE) which is the object of study of **Math 133B** rather than this course. In general, PDEs are much harder to analyze than ODEs.

Of course, not all ordinary differential equations are created equal. The solution and analysis methods we will employ throughout this course will depend on many factors. We will be primarily interested in the following characteristics.

Definition 2

Consider a differential equation $y^{(n)} = f(x, y, \ldots, y^{(n-1)})$ where y is a function of x (i.e. $y = y(x)$) and $y^{(n)}$ denotes the n^{th} order derivative with respect to x. Then:

- the **order** of the differential equation is the order of the highest-order derivative (in the general case above, it is n).
- the differential equation is said to be **linear** if it is linear in all of its dependent variables and derivatives.

Note: By linear we mean that we cannot modify the quantities of interest by anything more complicated than addition and scalar multiplication. That is, we do not allow terms like y^2, $\sin(y'')$, $\frac{1}{y'}$, etc. We do, however, allow nonlinearities in the *independent* variable. For example, we allow terms like x^2, $\sin(x)$, $\frac{1}{x}$, etc. The general form for an n^{th}-order linear ordinary differential equation is

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = g(x).$$

Example 1

Classify the following differential equations:

(a) $y'' - \left(\dfrac{4}{x}\right) y = \sin(x)$

(b) $y' + y^2 = x$

(c) $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$

Solution: We have that: (a) is a second-order linear ordinary differential equation; (b) is a first-order nonlinear ordinary differential equation (nonlinear because of the y^2!); and (c) is second-order linear partial differential equation. The latter PDE is a model for heat conduction but will not be studied in this course (see **Math 133B**).

The differences between the DEs above may seem small at first glance, but we will see that the methods required to analyze these systems vary greatly as we cross the various lines of classification. In this course, we will only consider *ordinary* differential equations and will start by considering equations which are *low order* and *linear*. As we expand our focus (e.g. increasing the order, consideration of nonlinear equations) we will see that we will need to consider new and sometimes very sophisticated tools.

Section 3: Derivation and Solutions

We now consider the question of how differential equations arise in practice. In fact, we have already seen many examples from basic calculus. For instance, we know that

$$y = \sin(x) \tag{1}$$

can be differentiated (with respect to x) to give

$$y' = \cos(x). \tag{2}$$

After a moment of thought, we notice that (2) fits into the definition we have just given. This is an example of a differential equation, albeit a slightly trivial one. We can also easily classify it is first-order, linear, and ordinary, which is our simplest possible case.

This example also gives us some sense of what it means to *solve* a differential equation.

Definition 3

A **solution** to a differential equation $y^{(n)} = f(x, y, \ldots, y^{(n-1)})$ is any function $y(x)$ which satisfies the given equation.

Note: It is important to notice that a solution to a DE is always a *function*. This is different from equations encountered in algebra, which typically have *numbers* as their solutions, e.g. $x^2 - 5x + 4 = 0$ has the solutions $x = 1$ and $x = 4$.

It should be clear that (1) is a solution of (2) since we can differentiate (1) to directly obtain (2). Furthermore, if we were only given (1), we know that we could obtain (2) by simply integrating. This gives our first example of a method for solving a DE and one which will work for any DE of the form $y' = f(x)$.

The First Fundamental Theorem of Calculus (which guarantees that integration *undoes* differentiation) so that, for this example, we have

$$\int \frac{dy}{dx}\, dx = \int \cos(x)\, dx \quad \Longrightarrow \quad y(x) = \sin(x) + C$$

so that our solution is $y(x) = \sin(x) + C$ for any $C \in \mathbb{R}$.

This is an example of a **general solution** since it contains a family of solutions curves which depends upon the choice of C. If we choose $C = 0$ we obtain the **particular solution** obtained earlier. We will discuss the importance of the undetermined constant C for physical systems shortly.

Of course, not all differential equations are in the form $y' = f(x)$; in fact, most are not. We should note, however, that regardless of the form we are always able to check whether a proposed function $y(x)$ is, in fact, a solution of a given differential equation. Consider the following example.

Example 2

Show that $y = \tan(x - C)$ is a solution to the differential equation $y' = 1 + y^2$ for any value of $C \in \mathbb{R}$.

Solution: All we need to do is plug y into the right-hand and left-hand sides of the equation to see if they are equal. We have

$$\text{LHS} = \frac{dy}{dx} = \frac{d}{dx}\left[\tan(x - C)\right] = \sec^2(x - C)$$

and

$$\text{RHS} = 1 + y^2 = 1 + \tan^2(x - C) = \sec^2(x - C)$$

where the last line follows from the well-known trigonometric identity $1 + \tan^2(x) = \sec^2(x)$. Since we have LHS = RHS, we are done!

Note that this example could not be obtained by simply integrating because the RHS is not a function of x alone. That is, we **cannot** solve the system by attempting to evaluate

$$\int \frac{dy}{dx}\, dx = \int (1 + y^2)\, dx \quad \Longrightarrow \quad y(x) = \int (1 + y^2)\, dx$$

because the right hand side depends upon y, which is an unknown function of x. That is, we cannot evaluate the integral because it is with respect to x while the term to be integrated is with respect to y. The method of direct integration can go no farther than this (without breaking a mathematical rule).

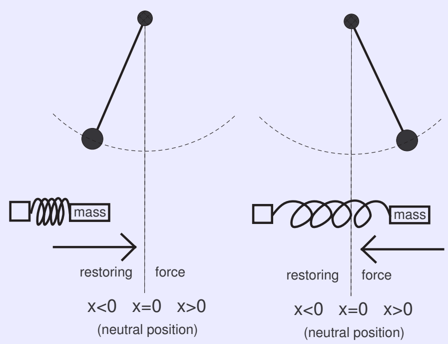
Note: Whenever we have a differential equation and a solution and are asked to check whether the solution works, it is sufficient to **plug the function into the equation**. This can be a time-saver on exams and quizzes, since it is easier to check answers than to derive them!

We will see that all of the integration techniques considered in previous courses (integration by substitution, integration by parts, trigonometric substitution, integration of rational functions, etc.) will be very important when solving differential equations. **These topics will be considered background knowledge and will not be reviewed in this course!** If you struggled with those topics in your previous calculus courses, it is very important to review them sooner rather than later.

Now consider a more physical example, motivated by Newton's second law of motion from classical mechanics.

Example 3

Consider the motion of a rigid pendulum suspended from a stationary pivot as below (alternatively, consider a mass attached to a stiff spring):



We will assume that the resting position is $x = 0$ (left is $x < 0$, right is $x > 0$). Newton's second law of motion guarantees that the force exerted on the pendulum is equal to its mass times it acceleration, i.e.

$$F = ma. \tag{3}$$

The pendulum's acceleration corresponds to the second derivative of its position x so that

$$a = x''.$$

Since we have already introduced a derivative into the equation, we are well on our way toward a complete differential equation modeling the motion of the pendulum!

To complete the model, we need to make an assumption on the *forces* which act on the pendulum. We will make several assumptions in this course, depending on the application, but the simplest is to assume that there is a **restoring force** proportional to the object's distance from its resting position. This arises from gravity in low-amplitude pendulum models, and *Hooke's law* for springs. The restoring force is commonly given by $F(x) = -kx$ where $k > 0$.

Notice that we can check if this is reasonable by consider the signs. If $x > 0$ (i.e. object is to the right of its resting position) then there is a restoring force pushing to the left; conversely, if $x < 0$ (i.e. object is to the left of its resting position), then there is a restoring force pushing to the right.

Combining everything into (3), we have

$$mx'' = -kx \quad \Longrightarrow \quad mx'' + kx = 0. \tag{4}$$

This is certainly a differential equation (it involves the function $x(t)$ and one of its derivatives, in this case the second derivative) but it **cannot be solved directly by integration**.

Nevertheless, if we are given a function $x(t)$ we can check to see if it is in fact a solution. For example, consider the function

$$\begin{aligned} x_1(t) &= \sin\left(\sqrt{\frac{k}{m}}t\right) \\ \Longrightarrow\ x_1'(t) &= \sqrt{\frac{k}{m}}\cos\left(\sqrt{\frac{k}{m}}t\right) \\ \Longrightarrow\ x_1''(t) &= -\frac{k}{m}\sin\left(\sqrt{\frac{k}{m}}t\right) \end{aligned}$$

so that

$$mx_1'' + kx_1 = m\left(-\frac{k}{m}\sin\left(\sqrt{\frac{k}{m}}t\right)\right) + k\sin\left(\sqrt{\frac{k}{m}}t\right) = 0.$$

It follows that $x_1(t)$ is a solution of the differential equation. In fact, it is a very illustrative one, since it tells us that solutions of the given differential equation oscillate continually regardless of the mass $m > 0$ or restoring force $k > 0$. This is a phenomenon we have probably observed before with (well-oiled) pendulums. This is an example of **harmonic motion** which we will study later in this course.

This function is not, however, alone in describing solutions to the equation. We can also quickly check that $x_2(t) = \cos\left(\sqrt{\frac{k}{m}}t\right)$ also satisfies the differential equation. In fact, any function of the form

$$x(t) = C_1 \sin\left(\sqrt{\frac{k}{m}}t\right) + C_2 \cos\left(\sqrt{\frac{k}{m}}t\right)$$

where $C_1, C_2 \in \mathbb{R}$ will work. It is interesting to see that there are *two* arbitrary constant, and we should note that they did not arise from integration like we are used to seeing. We will get to the general method which was used for this example in a few weeks.

There are many contexts where differential equations arise naturally from simple physical assumptions. For example, consider the following:

- **Exponential growth (populations)**

  $$P' = rP, \quad r > 0$$

  [Rate of population growth (P') is proportional to population size (P)]

- **Logistic growth (populations)**

  $$P' = rP(K - P), \quad r, K > 0$$

  [Rate of population growth (P') positive for small population (rKP dominates) and negative for large population ($-rP^2$) dominates]

- **Newton's law of cooling**

  $$T' = k(T_{ext} - T), \quad T_{ext} > 0$$

  [Rate of temperature change (T') proportional to difference between current temperature T and ambient temperature T_{ext}]

- **Restoring with friction and forcing (second-order linear)**

  $$mx'' + cx' + kx = g(t), \quad m, c, k > 0$$

  [Newton's second law with additional forcing terms cx' (damping) and $g(t)$ (forcing)]

We now have some sense of the kind of questions we are going to be interested in throughout this course, and also why they are important. We will continue this discussion with consideration of the general properties of differential equations.

Section 4: Initial Value Problems

A differential equation can, in general, given rise to *multiple solutions*. Given our understanding of differential equations as modeling physical phenomenon, we should be slightly disconcerted. When we throw a projectile (or release a pendulum, or connect an electrical circuit, etc.) we do not observe multiple solutions—rather, we observe exactly one.

We should recognize then that something is missing from our understanding of DEs to date. That missing piece is called an *initial condition*. Formally, we define the following.

Definition 4

The **initial value problem** (IVP) associated with a first-order differential equation is given by

$$\text{IVP} \quad \begin{cases} y' = f(x, y), \\ y(x_0) = y_0 \end{cases}$$

where $x_0, y_0 \in \mathbb{R}$.

A solution to an IVP corresponds to picking the specific trajectory in the slope field diagram which goes through the point (x_0, y_0). The terminology *initial value* was chosen to reflect the reality that we are usually interested in centering the problem at zero (i.e. setting $x_0 = 0$). We can, however, choose x_0 equal to any value we like (e.g. conditions like $y(3) = -7$ or $y(-1) = 10$).

For physical problems, the initial value corresponds to some initial data which we may incorporate in the model. For instance, when tracking a falling projectile, we may know whether it was initially released from rest or was initial moving in the upward or downward direction. This was previously not captured in our model, but it is clear that our ability to predict when the projectile hits the ground depends upon it.

Example 4

Determine the particular solution for the initial value problem

$$\begin{cases} y' = 1 + y, \\ y(0) = 1. \end{cases}$$

Solution: We already have the general solution $y = \tan(x - C)$, so it remains only to use the initial condition to solve for the value of C. From $y(0) = 1$, we have

$$\pi = \tan(-C) \implies \arctan(1) = -C \implies C = -\frac{\pi}{4}.$$

It follows that the particular solution is $y = \tan\left(x + \frac{\pi}{4}\right)$.

Example 5

Consider a projectile thrown up into the air from the top of a cliff which is 50 meters from the ground. Suppose the projectile is subject only to the force of gravity ($F = -mg = -9.8m$ kg·m/s^2) and suppose the initial upward velocity of the throw is 10 m/s. Solve the initial value problem. How long does it take the projectile to reach the bottom of the cliff?

Solution: From Newton's second law, we have that $F = ma$ so that

$$-mg = mx''.$$

With the given information, and removing the dimensions (which fortunately do match up) we can restate this as an initial value problem as

$$x'' = -9.8, \quad x(0) = 50, \ x'(0) = 10.$$

This can be directly integrated to get

$$x'(t) = \int \frac{d^2x}{dt^2}\, dt = -\int 9.8\, dt = -9.8t + C.$$

We can now use the first piece of initial information to get

$$x'(0) = 10 \quad \Longrightarrow \quad 10 = -(9.8)(0) + C \quad \Longrightarrow \quad C = 10.$$

It follows that we have

$$x'(t) = -9.8t + 10.$$

We can integrate this again to get

$$x(t) = \int \frac{dx}{dt}\, dt = \int (-9.8t + 10)\, dt = -4.9t^2 + 10t + D.$$

The other piece of initial information gives us

$$x(0) = 50 \quad \Longrightarrow \quad 50 = -4.9(0)^2 + 10(0) + D \quad \Longrightarrow \quad D = 50.$$

It follows that the solution to the initial value problem is

$$x(t) = -4.9t^2 + 10t + 50.$$

As we might have expected, this is a parabola opening down. The vertex corresponds to the maximum height before it starts its descent to the ground. To answer the final question, we recognize that reaching the ground corresponds to setting $x = 0$. It follows that we need to find a time such that

$$-4.9t^2 + 10t + 50 = 0.$$

The quadratic formula gives the solutions $t = -2.33$ and $t = 4.37$. We can reject the negative value since it occurs before we release the projectile and conclude that the projectile will reach the ground in 4.37 seconds.

Section 5: Slope Fields

We have seen examples of differential equations analyzed from an algebraic perspective. Even precise algebraic expression for solutions $y(x)$, however, may not give intuition into what the solution is doing. We now consider ways of *visualizing* solutions. An important feature of the approach we will take will be the construction of a **slope field** which can be constructed independently from determining a solution.

Consider the general first-order ODE:

$$y' = f(x, y). \tag{5}$$

We will suppose that we do not know the exact form of the solution $y(x)$ but are still interested in how it behaves.

Notice that, although we may not have information about $y(x)$, we do have information about $y'(x)$. Specifically, at any point $(x, y) \in \mathbb{R}^2$, we have that the *slope* of the solution $y(x)$ through (x, y) must correspond to the value of $f(x, y)$, which is known **at every point** (x, y) **in the plane**. The diagram produced by drawing lines with slope $f(x, y)$ for a sample of points (x, y) is called a **direction** or **slope field** of the differential equation.

Example 6

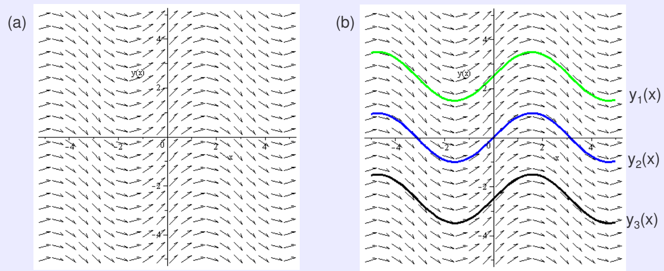
Construct a slope field for the differential equation

$$y' = \cos(x).$$

Solution: We already saw that this equation had solutions of the form $y(x) = \sin(x) + C$ for any value of $C \in \mathbb{R}$. We now want to construct a picture. We notice that we have $f(x, y) = \cos(x)$ so that the RHS of the equation only depends on x. In order to determine the slope values for the tangent lines, it is sufficient therefore to just consider a sample of values of x. We pick the easiest values we can. We have that

x	$\cos(x)$
$-\pi$	-1
$-\pi/2$	0
0	1
$\pi/2$	0
π	-1

and that the sequence of values repeats from there. When we consider the (x, y)-plane we arrive at a picture like



We can now see exactly how the solutions $y(x) = \sin(x) + C$ fit into the bigger picture! In (b), we have different solutions depending on which points we choose to have the solution pass through. Notice that we could have guessed the form (or at least the flavor) of the individual solutions just from the slope field.

Example 7

Construct a slope field for the differential equation

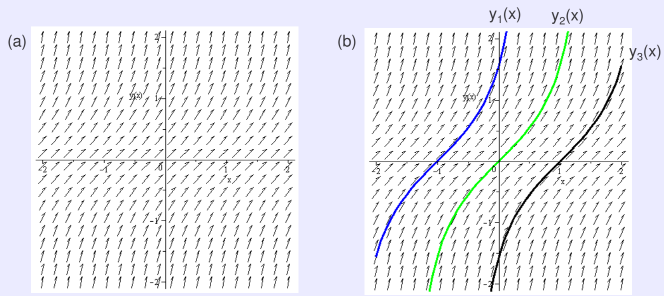
$$y' = 1 + y^2.$$

Solution: Again, we already know that this has the solution $y(x) = \tan(x + C)$ for any $C \in \mathbb{R}$. Now we want to construct the slope field.

We notice that, once again, the RHS of the equation only depends on a single variable; however, in this case, we have that $f(x, y) = 1 + y^2$ only depends on y instead of x. We have that

y	$1 + y^2$
-2	5
-1	2
0	1
1	2
2	5

In general, we have that the tangent lines become steeper and steeper the farther away from $y = 0$ we are. This gives rise to the following picture:



In (b), we overlay several solutions $y(x) = \tan(x) + C$. Again, we can see how the analytic solutions fit into the bigger picture.

Example 8

Construct a slope field for the differential equation

$$y' = x^2 + y^2.$$

Solution: This example is different than the previous ones in two important ways. Firstly, the RHS depends on both x and y, so it will not be sufficient to consider an assortment of values of just x or y. We will need to consider an assortment of points in the whole (x, y)-plane. Secondly, we do not know the explicit solution of the DE; in fact, there is no explicit solution to this DE which involves only elementary functions (x^2, $\sin(x)$, $\ln(x)$, e^x, etc.)!

We could start by picking a variety of points in the (x, y) plane and computing the value of $f(x, y) = x^2 + y^2$. This is what your computer does; however, we cannot do computations as quickly as our computers. We will seek a simpler method. Consider the following logic:

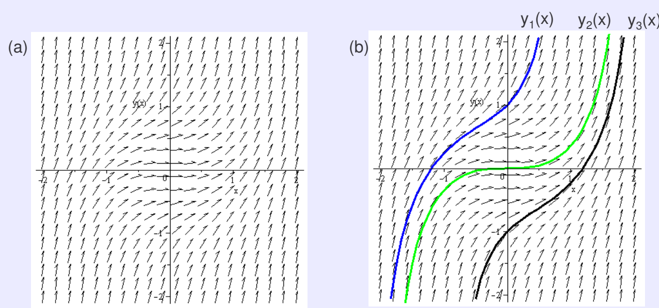
1. For fixed values $C^2 = x^2 + y^2$, the value of C corresponds to the radius of a circle centered at $(0, 0)$.

2. By the DE, the value of C^2 also corresponds to the slope of the solution through any corresponding point.

It follows that the slope of any solution through a point on the circle of radius C is C^2. For instance, we have that

$$\begin{aligned} \text{radius} &= 0 & \Longrightarrow & \quad \text{slope} = 0 \\ \text{radius} &= 1/2 & \Longrightarrow & \quad \text{slope} = 1/4 \\ \text{radius} &= 1 & \Longrightarrow & \quad \text{slope} = 1 \\ \text{radius} &= 3/2 & \Longrightarrow & \quad \text{slope} = 9/4 \\ \text{radius} &= 2 & \Longrightarrow & \quad \text{slope} = 4 \end{aligned}$$

Putting everything together gives the following picture:



The really important thing to notice about this example is that, even though we do not have access to an analytic solution $y(x)$, we can very clearly see what solutions look like. For example, we can see that all solutions approach infinity as x grows.

Section 6: Existence & Uniqueness

Before we begin our quest to find solutions to differential equations, it is worth considering the question of whether we should expect to find a solution at all and, if so, what properties it might have. In particular, we will focus on the **existence** and **uniqueness** of solutions to first-order IVPs.

The topic of existence is largely straight-forward. Most first-order IVPs have solutions and the ones which do not are typically pathological in an obvious way, e.g.

$$y' = \frac{y}{x}$$

has no solutions through any point (x_0, y_0) where $x_0 = 0$ since we cannot divide by zero, and

$$(y')^2 + y^2 = -1$$

has no solutions through any point (x_0, y_0) because a positive cannot equal a negative. In short, what is bad for an algebraic equation is also bad for a differential equation.

The topic of uniqueness, however, is more subtle. We have seen that, in general, differential equations have a **family of solutions**. In particular, we typically have a constant C which can take many values while satisfying the DE. What we mean by uniqueness is that a there is a unique solution through a point (x_0, y_0), i.e. the constant C can be chosen uniquely. While this may seem obtuse at first glance, consider the following example.

Example 9

Verify that the following is the general solution to the given differential equation, and then solve the given initial value problem:

$$y' = -2\sqrt{y}, \quad y(0) = 0,$$

$$y(x) = \begin{cases} (x - C)^2, & x < C \\ 0, & x \geq C. \end{cases}$$

Solution: To the first half of the proposed solution $[y = (x - C)^2]$, we have

$$\text{LHS} = y' = 2(x - C)$$

and

$$\text{RHS} = -2\sqrt{y} = -2\sqrt{(x - C)^2} = -2|x - C| = 2(x - C)$$

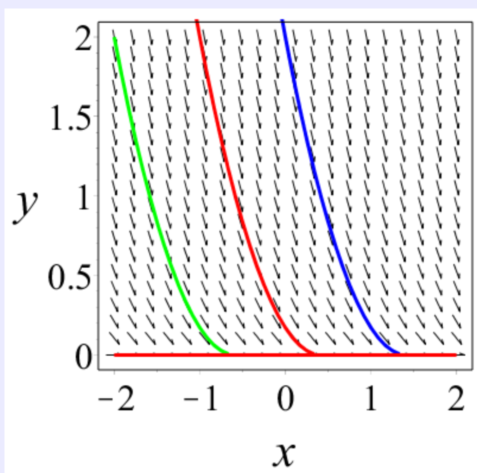
where we have used the fact that, for $x < C$, we have $|x-C| = -(x-C)$.

To the second half of the proposed solution $[y = 0]$, we have $y = 0$ implies $\text{LHS} = y' = 0$ and $\text{RHS} = -2\sqrt{y} = 0$ trivially. It follows that both halves of the expression satisfy the differential equation, and since we have continuity and equality in derivatives at $x = C$, the function is smoothly defined at the transition. It follows that the solution is defined as a parabola to the left of $x = C$, and zero to the right.

Now consider the initial condition $y(0) = 0$. We have $0 = ((0) - C)^2$ which implies that $C = 0$. That is, we have the following particular solution:

$$y(x) = \begin{cases} x^2, & x < 0 \\ 0, & x \geq 0. \end{cases}$$

In this case, we have both a well-defined general solution and a well-defined particular solution, but we *still* have to be careful. Consider constructing the slope field. We can easily see that $-2\sqrt{y}$ is only defined for $y \geq 0$, is non-positive, and is increasingly negative as y increases. We have the following:



Now ask the question of whether the solution just defined is the *only* solution which travels through the point $(x, y) = (0, 0)$. We can quickly see that it is not, since any parabola with a vertex to the left of $(0, 0)$ joins the solution $y = 0$ and then passes through $(0, 0)$. That is, many solutions not found by the previous argument pass through $(0, 0)$. **Differential equations are not guaranteed to have a unique solution through a given initial condition** $y(x_0) = y_0$.

Lest we think uniqueness is just a mathematical curiosity, consider that this differential equation is often use to model the behavior of a leaky bucket. The idea is very simple: more water will drain from the bucket if the amount of water is higher to begin with, which is captured by the term $-2\sqrt{y}$. This model also captures the notion that the bucket will completely drain in finite time, and then stay empty.

The lack of uniqueness of solutions corresponds to asking the question "Given that the bucket is empty now, when was it full?" As with our example, it could have just drained, drained several minutes ago, or never had any water in it to begin with. We have no way of knowing!

This raises an interesting question: Without prior knowledge about a solution, can be guaranteed a solution exists and/or that it is unique? Fortunately, the answer is *yes* and is the content of the following theorem.

Theorem 1

Consider the first-order ODE $y' = f(x, y)$ and an initial point (x_0, y_0). Let $\mathcal{R}$ denote a non-empty region around the point (x_0, y_0). Then:

1. if $f(x, y)$ is continuous in $\mathcal{R}$, then there is a subregion $\mathcal{R}' \subseteq \mathcal{R}$ also containing (x_0, y_0) in which there is a solution of the DE through (x_0, y_0);

2. if, furthermore, $\partial f/\partial y$ is continuous in $\mathcal{R}$, then there is a subregion $\mathcal{R}'' \subseteq \mathcal{R}'(\subseteq \mathcal{R})$ also containing (x_0, y_0) in which there is a *unique* solution of the DE through (x_0, y_0).

Note: The subregion of existence and/or uniqueness $\mathcal{R}$ is the *bare minimum* guaranteed by the Theorem. It is quite possible that solutions exist for a much broader region of the (x, y)-plane, even though the Theorem does not guarantee it. It is normally the case that solutions only cease to exist in very small regions.

Example 10

Use Theorem 1 to comment on the existence of uniqueness of solutions to the DE in Example 9.

Solution for Example 9: For the differential equation

$$y' = -2\sqrt{y}$$

we have that $f(x, y) = -2\sqrt{y}$ and $\frac{\partial f}{\partial y} = -\frac{1}{\sqrt{y}}$. We can see that $f(x, y)$ is continuous for all points (x, y) such that $y \geq 0$ while $\frac{\partial f}{\partial y}$ is continuous only if $y > 0$. We are therefore guaranteed that a solution exists if $y \geq 0$ but only guaranteed uniqueness if $y > 0$. In fact, we know that $y = 0$ is exactly where uniqueness of solutions breaks down.

Suggested Problems:

1. Verify that the following functions $y(x)$ are solutions to the given differential equation (in all cases, $C, C_1, C_2 \in \mathbb{R}$ are arbitrary constants). If initial conditions are given, use them to solve for the constants:

   (a) $y' = ky^2$;
   $y(x) = \dfrac{1}{C - kx}$

   (b) $y' = \dfrac{x^2 + 3y^2}{2xy}$;
   $y(x) = \sqrt{Cx^3 - x^2}$

   (c) $y'' + y' - 2y = 2x$;
   $y(x) = C_1 e^x + C_2 e^{-2x} - x - \frac{1}{2}$

   (d) $x^2 y'' - 2xy' + 2y = x^2$;
   $y(x) = C_1 x + C_2 x^2 + x^2 \ln(x)$

   (e) $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$;
   $u(t, x) = t^{-1/2} e^{-x^2/4t}$

   (f) $\begin{cases} y' = 5y(1-y) \\ y(1) = 5 \end{cases}$
   $y = \dfrac{e^{5x}}{C + e^{5x}}$

   (g) $\begin{cases} y' = y^2 - 1 \\ y(0) = 0 \end{cases}$
   $y = -\tanh(x) = \dfrac{e^{-x} - e^x}{e^{-x} + e^x}$

   (h) $\begin{cases} y'' + 2y' + y = 0 \\ y(0) = 1 \\ y'(0) = 5 \end{cases}$
   $y = C_1 e^{-x} + C_2 x e^{-x}$

2. Construct the slope field for the following first-order differential equations:

(a) $y' = y(1-y)$,

(b) $y' = x^2 - 1$,

(c) $y' = 2x - y$,

(d) $y' = \sin(x) + y$,

(e) $y' = \dfrac{x}{1+x^2} - y$,

(f) $y' = x^2 - y^2$,

3. Apply Theorem 1 to comment on the existence and uniqueness through various initial conditions $y(x_0)$ of the following first-order differential equations:

(a) $(y')^2 = -y^2$

(b) $y' = 1 + y^2$

(c) $y' = 2\sqrt{y}$

(d) $y' = \sqrt{y^2 - 1}$

(e) $y' = \sqrt{y^2 + x^2}$

(f) $y' = x/y$

4. Consider a skydiver in freefall. Suppose that the only force acting on the diver is air resistence.

   (a) Set up a model for the velocity of the diver v which captures the following reasonable assumption: "The rate of change of the velocity is proportion to the difference between the diver's velocity and some terminal velocity $v_{term} > 0$."

   (b) Can you solve the resulting differential equation for the diver's position $x(t)$?

   (c) If we knew the skydiver was released from 14000 feet above the ground and the terminal velocity is $v_{term} = 150$ ft/s, can we calculate how long it will take the skydiver to reach the ground (assuming he or she tragically forgets to pull the parachute cord)?

5. Consider a rubber ball dropped from rest at a height of 10 meters above ground. Suppose the projectile is subject only to the force of gravity ($F = -mg = -9.8m$ kg·m/s^2).

   (a) How long does it take the ball to hit the ground?

   (b) Suppose that ball bounces straight back into the air with half of the velocity with which it hits the ground. How long does it take to hit the ground twice?

   (c) Suppose the ball will stop bouncing when its velocity drops below 0.1 m/s. How many bounces will the ball take before stopping?