Lecture 20 — Matrix Inverses

Let's look at row reduction again

Example 1

$$2x_1 + 6x_2 + 8x_3 = 16$$
$$4x_1 + 15x_2 + 19x_3 = 38$$
$$2x_1 + 3x_3 = 6$$

we can rewrite in terms of vectors and matrices:

$$\begin{bmatrix} 2 & 6 & 8 \\ 4 & 15 & 19 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 16 \\ 38 \\ 6 \end{bmatrix}$$

To solve, we can write this as an "augmented matrix"

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 6 & 8 & 16 \\ 4 & 15 & 19 & 38 \\ 2 & 0 & 3 & 6 \end{array}\right]$$

(short hand notation)

Goal: get matrix to look like $\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$

$R_2 - 2R_1$, $R_3 - R_1$, $R_1 \to \frac{1}{2}R_1$

$$\left[\begin{array}{ccc|c} 2 & 6 & 8 & 16 \\ 4 & 15 & 19 & 38 \\ 2 & 0 & 3 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 4 & 8 \\ 0 & 3 & 3 & 6 \\ 0 & -6 & -5 & -10 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$R_1 - R_2$, $R_3 + 2R_2$, $R_2 \to \frac{1}{3}R_2$ $\qquad$ $R_1 - R_3$, $R_2 - R_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$\Rightarrow \quad x_1 = 0, \quad x_2 = 0, \quad x_3 = 2$$

* Any <u>row</u> can be multiplied by a number (<u>not</u> any column)
* Any <u>row</u> can be swapped (<u>not</u> columns)

Example 2

$$
\begin{aligned}
x_1 - x_2 + 2x_3 + 2x_4 &= 0 \\
2x_1 - 2x_2 + 4x_3 + 3x_4 &= 1 \\
3x_1 - 3x_2 + 6x_3 + 9x_4 &= -3 \\
4x_1 - 4x_2 + 8x_3 + 8x_4 &= 0
\end{aligned}
$$

$$
\Rightarrow \begin{bmatrix} 1 & -1 & 2 & 2 \\ 2 & -2 & 4 & 3 \\ 3 & -3 & 6 & 9 \\ 4 & -4 & 8 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -3 \\ 0 \end{bmatrix}
$$

$R_2 - 2R_1 \quad R_3 - 3R_1 \quad R_4 - 4R_1$

$$
\Rightarrow \left[\begin{array}{cccc:c} 1 & -1 & 2 & 2 & 0 \\ 2 & -2 & 4 & 3 & 1 \\ 3 & -3 & 6 & 9 & -3 \\ 4 & -4 & 8 & 8 & 0 \end{array}\right] \sim \left[\begin{array}{cccc:c} 1 & -1 & 2 & 2 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]
$$

$$
\sim \left[\begin{array}{cccc:c} 1 & -1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]
$$

$R_3 + 3R_2, \quad R_1 + 2R_2, \quad R_2 \to -R_2$

$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 0$

$\boxed{\text{Infinite Solutions}}$

Example 3

$$x_1 - 3x_2 = 1 \quad , \quad -7x_1 + 21x_2 = 0$$

$$\Rightarrow \begin{aligned} x_1 - 3x_2 &= 1 \\ -7x_1 + 21x_2 &= 0 \end{aligned} \Rightarrow \begin{bmatrix} 1 & -3 \\ -7 & 21 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ -7 & 21 & 0 \end{array}\right] \overset{R_2 + 7R_1}{\sim} \left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 0 & 7 \end{array}\right]$$

$$0 \cdot x_1 + 0 \cdot x_2 = 7 \quad \Rightarrow \quad \boxed{\text{No Solution}}$$

General Goal:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & c_1 \\ 0 & 1 & \cdots & 0 & c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & c_n \end{array}\right]$$

($N \times N$ matrix)

Note:

$$\underline{\underline{I}} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

is known as the "Identity Matrix"

$\Rightarrow$ $\underline{\underline{I}}$ is the matrix version of "1".

$$\underline{\underline{I}}\,\vec{v} = \vec{v}, \quad \underline{\underline{I}}\,\underline{\underline{A}} = \underline{\underline{A}}$$

Example 4

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\underline{\underline{I}}\,\underline{\underline{A}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

(Similarly, $\underline{\underline{A}}\,\underline{\underline{I}} = \underline{\underline{A}}$)

- In highschool, $ax = b \Rightarrow x = \frac{b}{a}$

  $(x = a^{-1} b)$

- Now, we're solving $\underline{\underline{A}}\vec{x} = \vec{b}$, but you <u>can't</u> divide by a matrix!

  But, there are matrix inverses

- The "inverse" of a number is defined by: $a^{-1} a = 1$

  $2x = 1 \Rightarrow x = 0.5$

- We can extend this idea to (square) matrices!

Back to Example 4: $\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

Is there a matrix such that $\underline{\underline{B}}\,\underline{\underline{A}} = \underline{\underline{I}}$?

Let $\underline{\underline{B}} = \begin{bmatrix} 3/2 & -1 & 1/2 \\ 1/2 & 0 & -1/2 \\ -3/2 & 1 & 1/2 \end{bmatrix}$

$$\underline{\underline{B}}\,\underline{\underline{A}} = \begin{bmatrix} 3/2 & -1 & 1/2 \\ 1/2 & 0 & -1/2 \\ -3/2 & 1 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow \underline{\underline{B}}\,\underline{\underline{A}} = \underline{\underline{I}}$ (similarly, $\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{I}}$)

★ $\underline{\underline{B}}$ is known as the <u>inverse</u> of $\underline{\underline{A}}$

- We denote this as $\underline{\underline{A}}^{-1}$
- It is unique
- $\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{B}}\,\underline{\underline{A}} = \underline{\underline{I}}$

<u>Example 5</u>

$\underline{\underline{A}}\,\vec{x} = \vec{b}$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ d \end{bmatrix}$$

$$\Rightarrow \quad \underline{\underline{B}}\,\underline{\underline{A}}\,\vec{x} = \underline{\underline{B}}\,\vec{b}$$

$$\Rightarrow \quad \underline{\underline{I}}\,\vec{x} = \underline{\underline{B}}\,\vec{b}$$

$$\Rightarrow \quad \vec{x} = \underline{\underline{B}}\,\vec{b}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underline{\underline{B}}\,\vec{b} = \begin{bmatrix} 3/2 & -1 & 1/2 \\ 1/2 & 0 & -1/2 \\ -3/2 & 1 & 1/2 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -5 \end{bmatrix}$$

$$\Rightarrow \quad x_1 = 5, \quad x_2 = 1, \quad x_3 = -5$$

* Not all matrices have inverses (see examples 2 & 3)

* How do we calculate $\underline{\underline{B}}$ $(\underline{\underline{A}}^{-1})$?

$\rightarrow$ know $\underline{\underline{A}}\,\underline{\underline{A}}^{-1} = \underline{\underline{I}}$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solve This