Math 133A: Week 10
Laplace Transforms

Section 1: Laplace Transform

We now introduce a particularly important transformation in the theory and application of ordinary differential equations.

Definition 1

The **Laplace transform** of a function $f(t)$ is given by

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

Note: The Laplace transform of a function $f(t)$ integrates out t (often called the **time domain**) in exchange for a introducing a new dependence on a variable s (often called the **Laplace domain** or **frequency domain**). For this reason, the Laplace transform is often also represented as $F(s)$.

The Laplace transform allows us to rewrite many differential equations in an alternative, but equivalent, form which is often easier to solve. There are several advantages to the method, but the main ones are the following:

1. The transformed problem may be solved by **algebraic methods alone**. There are no derivatives or integrals at all. (Algebraic methods such as factoring and partial fractions, in particular, factor prominently.)

2. The transformed problem is a **one-step solution method**. In particular, it incorporates initial conditions and nonhomogeneities into the method directly.

3. Laplace transform methods are particularly well-suited to handle problems with **discontinuous forcing functions**. Previously, if we had

a discontinuous forcing function, we had to solve the problems independently in each interval and then match boundary conditions.

Section 2: Elementary Functions

We will start by building a catalogue of Laplace transforms. Our first observation is to notice that the Laplace transform is an *indefinite integral*. In order to evaluate even the most basic of transforms, we will have to recall that

$$\int_0^\infty e^{-st} f(t)\, dt = \lim_{A\to\infty} \int_0^A e^{-st} f(t)\, dt.$$

Notice also that this integral may converge or diverge. The Laplace transform of a function only exists if the integral converges.

Example 1

Compute the Laplace transform of $f(t) = c$ where c is some constant.

Solution: We just need to apply the definition. We have

$$\mathcal{L}\{c\} = \int_0^\infty ce^{-st}\, dt = -c \lim_{A\to\infty} \left.\frac{e^{-st}}{s}\right|_0^A = -c\left[\lim_{A\to\infty} \frac{e^{-sA}}{s} - \frac{1}{s}\right].$$

We can see that $\lim_{A\to\infty} e^{-sA}/s$ diverges to infinity if $s < 0$ (and the form was invalid for $s = 0$, which clearly diverges). Otherwise, it converges to zero. It follows that we have convergence for $s > 0$ only, so that

$$\mathcal{L}\{c\} = \frac{c}{s}, \quad s > 0.$$

That was pretty easy, but we would be understandably skeptical if we did not believe the Laplace transform would work out as clean as that for more complicated functions. Let's try a few more.

Example 2

Compute the Laplace transform of $f(t) = e^{at}$ where $a \neq 0$.

Solution: We have

$$\begin{aligned}\mathcal{L}\left\{e^{at}\right\} &= \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{(a-s)t} \\ &= \lim_{A\to\infty} \left.\frac{e^{(a-s)t}}{a-s}\right|_0^A = \lim_{A\to\infty} \frac{e^{(a-s)A}}{a-s} - \frac{1}{a-s}.\end{aligned}$$

Again, the limit does not converge everywhere, but we can see that it does for $s > a$. It follows that we have

$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}, \quad s > a.$$

Example 3

Compute the Laplace transform of $f(t) = t$.

Solution: We have

$$\begin{aligned}\mathcal{L}\left\{t\right\} = \int_0^\infty t e^{-st}\, dt &= \lim_{A\to\infty} \left[-t \left.\frac{e^{-st}}{s}\right|_0^A + \frac{1}{s}\int_0^A e^{-st} \right] \\ &= \lim_{A\to\infty} \left[-A\frac{e^{-sA}}{s} - \left.\frac{e^{-st}}{s^2}\right|_0^A \right] \\ &= \lim_{A\to\infty} \left[-A\frac{e^{-sA}}{s} - \frac{e^{-sA}}{s^2} + \frac{1}{s^2} \right]\end{aligned}$$

by integration by parts. As we have seen before, these integrals converge to zero if $s > 0$ so that we have

$$\mathcal{L}\left\{t\right\} = \frac{1}{s^2}.$$

It is a common property of the derivation of Laplace transforms result in integrals which require integration by parts. The good news is that, with a little bit of effort, we are capable of handling such integrals for most standard functions. Combining our results so far with several forms we can easily check, we have the following:

Laplace Transforms

$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}, \quad s > a$$

$$\mathcal{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}}, \quad s > 0 \quad (n = 0, 1, 2, 3, \ldots)$$

$$\mathcal{L}\left\{\sin(bt)\right\} = \frac{b}{s^2+b^2}, \quad s > 0$$

$$\mathcal{L}\left\{\cos(bt)\right\} = \frac{s}{s^2+b^2}, \quad s > 0$$

It is also important to note that the Laplace transform can easily be applied to linear combinations of these functions. We have the following.

Theorem 1

Suppose $f(t)$ and $g(t)$ are functions which have Laplace transforms on intervals $s > s_1$ and $s > s_2$, respectively. Then

$$\mathcal{L}\left\{C_1 f(t) + C_2 g(t)\right\} = C_1 \mathcal{L}\left\{f(t)\right\} + C_2 \mathcal{L}\left\{g(t)\right\}$$

and this exists on the interval $s > \max\left\{s_1, s_2\right\}$.

Proof

We have

$$\begin{aligned}\mathcal{L}\left\{C_1 f(t) + C_2 g(t)\right\} &= \int_0^\infty e^{-st}(C_1 f(t) + C_2 g(t))\, dt \\ &= C_1 \int_0^\infty e^{-st} f(t)\, dt + C_2 \int_0^\infty e^{-st} g(t)\, dt \\ &= C_1 \mathcal{L}\left\{f(t)\right\} + C_2 \mathcal{L}\left\{g(t)\right\}.\end{aligned}$$

Since this will only converge if both integrals do, we have that this exists on $s > \max\{s_1, s_2\}$, and we are done.

Example 4

Determine the Laplace transform of

$$f(t) = e^{-2t} + 3t^2 - \sin(5t).$$

Solution: We can decompose this into smaller problems by the linearity property just discovered. We have

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{e^{-2t}\right\} + 3\mathcal{L}\left\{t^2\right\} - \mathcal{L}\{\sin(5t)\}.$$

We can now easily compute that

$$\mathcal{L}\{f(t)\} = \frac{1}{s+2} + \frac{6}{s^3} - \frac{5}{s^2+25}.$$

Also notice that the first transformation is valid for $s > -2$ while the second two are valid for $s > 0$. It follows that the above transformation is valid for $s > 0$.

Section 3: Derivatives

Suppose now that we are asked to take the Laplace transform of a *derivative*. We have the following result.

Theorem 2

Suppose $f(t)$ has the Laplace transform $F(s)$ which exists on $s > s^*$ and that

$$\lim_{A\to\infty} e^{-sA} f(A) = 0.$$

Then

$$\mathcal{L}\left\{f'(t)\right\} = sF(s) - f(0)$$

and this exists on the interval $s > s^*$.

Proof

By definition, we have

$$\begin{aligned}
\mathcal{L}\left\{f'(t)\right\} &= \int_0^\infty e^{-st} f'(t)\ dt \\
&= \lim_{A\to\infty} e^{-st} f(t)\Big|_0^A + s\int_0^\infty e^{-st} f(t)\ dt \\
&= \lim_{A\to\infty} e^{-sA} f(A) - f(0) + sF(s)
\end{aligned}$$

where

$$F(s) = \mathcal{L}\left\{f(t)\right\} = \int_0^\infty e^{-st} f(t)\ dt.$$

We have to be somewhat careful when evaluating the remaining limit. We would like to say that the limit converges in the limit to zero for $s > 0$, since the term e^{-sA} does, but that will only be true if $f(A)$ does not grow faster than exponential in the limit as $A \to \infty$ (in math language, that is to say, we required that $|f(t)| \leq ke^{at}$ where k and a are some constants). That is to say, we could *not* allow something like $f(t) = e^{t^2}$, which grows very, *very* quickly relative to a standard exponential function (in fact, *any* standard exponential function).

We will sweep this technicality aside, since we will typically be dealing with functions like exponentials, polynomials, sines, and cosines. For these functions, we may certainly say the growth is at most exponential, and therefore that

$$\mathcal{L}\left\{f'(t)\right\} = sF(s) - f(0).$$

At first glance, this does not seem to be a useful result. After all, we may not necessarily know anything about $f(t)$ or its Laplace transform $F(s)$. Context will play a key role here. When we begin evaluating differential equations, we will have information regarding the derivatives of $f(t)$ but not about $f(t)$ itself. When taking Laplace transforms, we will be satisfied to simply relate the Laplace transform of $f'(t)$ (or higher derivatives) to the Laplace transform of $f(t)$.

In fact, we can go quite a bit further. This was just the *first* derivative, and we often have equations with higher-order derivatives as well. We have the following result, the proof of which we leave as an exercise.

Theorem 3

Suppose $f(t)$ has the Laplace transform $F(s)$ which exists on $s > s^*$ and that

$$\lim_{A\to\infty} e^{-sA} f(A) = 0.$$

Then

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1} f(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

and this exists for $s > s^*$.

Note: The Laplace transform of a derivative can be expressed in terms of the Laplace transform of the base function *together with* the values of the function at the first point of integration (usually taken to be $t = 0$). In terms of differential equations, this will be a very useful property, since the values at $t = 0$ will correspond to the initial conditions in our differential equation problem.

Example 5

Verify the formula $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$ for the function $f(t) = e^{rt}$ where r is a constant.

Solution: For $f(t) = e^{rt}$, we have that $f'(t) = re^{rt}$ which has the Laplace transform

$$\mathcal{L}\left\{f'(t)\right\} = r\mathcal{L}\left\{e^{rt}\right\} = \frac{r}{s-r}.$$

We also have that

$$F(s) = \mathcal{L}\left\{e^{rt}\right\} = \frac{1}{s-r}$$

and $f(0) = 1$ so that

$$sF(s) - f(0) = s\frac{1}{s-r} - 1 = \frac{s-(s-r)}{s-r} = \frac{r}{s-r}.$$

Section 4: Inverse Laplace Transforms

A key portion of what we will doing is converting back from the Laplace tranform world to standard functions of t. This naturally means we will have to *invert* the Laplace transform, i.e. we will have to compute

$$\mathcal{L}^{-1}\left\{F(s)\right\} = f(t).$$

Formally defining this operation would be incredibly tedious, but it is also going to be unnecessary. We already know the basic forms of a number of Laplace transforms. From these known forms, we can begin to form a catalogue of inverse Laplace transforms.

Inverse Laplace Transforms

$$\begin{aligned}
\mathcal{L}\left\{e^{at}\right\} &= \frac{1}{s-a} &\implies&& \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} &= e^{at} \\
\mathcal{L}\left\{t^{n}\right\} &= \frac{n!}{s^{n+1}} &\implies&& \mathcal{L}^{-1}\left\{\frac{(n-1)!}{s^{n}}\right\} &= t^{n-1} \\
\mathcal{L}\left\{\sin(bt)\right\} &= \frac{b}{s^2+b^2} &\implies&& \mathcal{L}^{-1}\left\{\frac{b}{s^2+b^2}\right\} &= \sin(bt) \\
\mathcal{L}\left\{\cos(bt)\right\} &= \frac{s}{s^2+b^2} &\implies&& \mathcal{L}^{-1}\left\{\frac{s}{s^2+b^2}\right\} &= \cos(bt)
\end{aligned}$$

We will add to this list later, but it will be important to go through a few examples now to see how these examples proceed.

Example 6

Determine the inverse Laplace transform of

$$F(s) = \frac{24}{s^4} - \frac{9}{s^2+9}.$$

Solution: It is important first of all to recognize the linearity of the Laplace transform also applies to the inverse Laplace transforms. That is to say, we have

$$\mathcal{L}^{-1}\left\{F(s)+G(s)\right\}=\mathcal{L}^{-1}\left\{F(s)\right\}+\mathcal{L}^{-1}\left\{G(s)\right\}.$$

It follows from this observation that, for the original problem, we have

$$\mathcal{L}^{-1}\left\{F(s)\right\}=\mathcal{L}^{-1}\left\{\frac{24}{s^4}\right\}-\mathcal{L}^{-1}\left\{\frac{9}{s^2+9}\right\}.$$

We have to be careful at this point. We need to recognize that the inverse Laplace transfer forms *depend explicitly on constants*. For instance, we recognize that the first term corresponds structure to the form required for

$$\mathcal{L}^{-1}\left\{\frac{(n-1)!}{s^n}\right\}=t^{n-1}$$

with $n=4$. We will have to make sure that, however, that the constant in the numerator is the correct one! If we have $n=4$, we must have $(n-1)!=3!=6$ absorbed by the inverse Laplace transform.

We will also have to be careful with the form

$$\mathcal{L}^{-1}\left\{\frac{b}{s^2+b^2}\right\}=\sin(bt).$$

We recognize that $b=3$ for our given form, so this much will be absorb by the inverse transformation. All told, we should recognize that we have

$$\mathcal{L}^{-1}\left\{F(s)\right\}=4\mathcal{L}^{-1}\left\{\frac{6}{s^4}\right\}-3\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}=4t^3-3\sin(3t).$$

Example 7

Find the inverse Laplace transform of

$$F(s)=\frac{8}{s^3+4s}.$$

Solution: Our first observation is that, not matter how we adjust the constant in the numerator, this does not fit readily into our given forms.

So what can we do?

The key, which will be a common feature of inverting Laplace transforms, is that we can factor the denominator into

$$F(s) = \frac{8}{s^3 + 4s} = \frac{8}{s(s^2 + 4)}.$$

This may seem like a modest gain, but it actually leads to a general method. We know from work on integrate that we can break terms like this into *separate* terms with simpler denominators. In this case, we know that we can write this as

$$\frac{8}{s(s^2 + 4)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4}$$

for some constants A, B, and C by using *partial fraction decomposition*. The forms on the right-hand side are easily recognized from your table of inverse transforms! So partial fraction decomposition is the key to resolving inverse transformations with complicated denominators.

In this case, we have

$$8 = A(s^2 + 4) + (Bs + C)s \quad \Longrightarrow \quad (A + B)s^2 + Cs + (4A - 8) = 0.$$

Equating coefficients on both sides gives the system

$$\begin{aligned} A + B &= 0 \\ C &= 0 \\ 4A - 8 &= 0. \end{aligned}$$

It follows from the final constraint that $A = 2$ so that $B = -2$ by the first. Also, clearly $C = 0$ by the second. It follows that we have

$$\begin{aligned}
&\mathcal{L}^{-1}\left\{\frac{8}{s(s^2+4)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{2}{s} - \frac{2s}{s^2+4}\right\} \\
&= 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} \\
&= 2 - 2\cos(2t).
\end{aligned}$$

Section 5: Shifted Laplace Transform

A particular nice property of Laplace transforms is that it very easily handles shifts in the domain variable s. For instance, imagine trying to determine the inverse Laplace transform

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\}.$$

While we can certainly identify elements of the Laplace transforms of $\sin(x)$, it is not quite in the necessary form because of the domain shift $s-1$ in the denominator.

The following result allows us to compute such inverse transformations.

Theorem 4

Suppose that $\mathcal{L}\left\{f(t)\right\} = F(s)$. Then $\mathcal{L}\left\{e^{ct}f(t)\right\} = F(s-c)$. Conversely, we have $\mathcal{L}^{-1}\left\{F(s-c)\right\} = e^{ct}f(t)$.

Proof

The proof is simpler than likely anticipated. By definition, we have

$$\mathcal{L}\left\{e^{ct}f(t)\right\} = \int_0^\infty e^{-st}e^{ct}f(t)\,dt = \int_0^\infty e^{-(s-c)t}f(t)\,dx = F(s-c).$$

An important consequence of Theorem 4 is that it very quickly allows us to expand our catalogue of Laplace transformations.

Inverse Laplace Transforms

$$\mathcal{L}\left\{e^{at}t^n\right\} = \frac{n!}{(s-a)^{n+1}} \implies \mathcal{L}^{-1}\left\{\frac{(n-1)!}{(s-a)^n}\right\} = e^{at}t^{n-1}$$

$$\mathcal{L}\left\{e^{at}\sin(bt)\right\} = \frac{b}{(s-a)^2+b^2} \implies \mathcal{L}^{-1}\left\{\frac{b}{(s-a)^2+b^2}\right\} = e^{at}\sin(bt)$$

$$\mathcal{L}\left\{e^{at}\cos(bt)\right\} = \frac{(s-a)}{(s-a)^2+b^2} \implies \mathcal{L}^{-1}\left\{\frac{(s-a)}{(s-a)^2+b^2}\right\} = e^{at}\cos(bt)$$

where $s > a$.

For our above example, we may now quickly identify that the inverse transformation gives

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\} = e^t\sin(t).$$

Example 8

Determine the inverse Laplace transform of

$$F(s) = \frac{s-1}{s^2-4s+5}.$$

Solution: We need to determine

$$\mathcal{L}^{-1}\left\{\frac{s-1}{s^2-4s+5}\right\}$$

but do not recognize this immediately as being in one of our prescribed forms. Worst still, the bottom cannot be factored (over the real numbers, anyway) so that we cannot split the denominator.

Our only alternative is to *complete the square* in the denominator. This will be a general method, in fact. If we have an irreducible quadratic term, we must complete the square to get it in the standard for $(s -$

$a)^2 + b^2$. In this case, we have

$$s^2 - 4s + 5 = (s^2 - 4s + 4) - 4 + 5 = (s-2)^2 + 1.$$

Now we are getting somewhere! We have

$$\mathcal{L}^{-1}\left\{\frac{s-1}{s^2 - 4s + 5}\right\} = \mathcal{L}^{-1}\left\{\frac{s-1}{(s-2)^2 + 1}\right\}.$$

This is pretty good, but we are not out of the woods yet. The shifted sine and cosine forms require us to have some factor of either the shift (i.e. $s - c$) or the remaining term (i.e. b) in the numerator. We clearly have $c = 2$ and $b = 1$ but we do not have $s - 2$ or 1 by themselves in the numerator. Instead, we must *create* them. In this case, we can simply adjust to get what we need. If we subtract by a one in the numerator, we need to add by one. This gives

$$\mathcal{L}^{-1}\left\{\frac{s-1-1+1}{(s-2)^2 + 1}\right\} = \mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2 + 1} + \frac{1}{(s-2)^2 + 1}\right\}.$$

This is exactly we needed! We can immediately recognize these as the shift sine and cosine forms. After a little bit of work, we have been able to show that

$$\mathcal{L}^{-1}\left\{\frac{s-1}{s^2 - 4s + 5}\right\} = e^{2t}\cos(t) + e^{2t}\sin(t).$$

Section 6: Initial Value Problems

We may now combine all this groundwork toward our ultimate goal of solving differential equations. Consider the following example.

Example 9

Use the Laplace transform to solve the following initial value problem (all derivatives w.r.t. t):

$$x'' - 3x' + 2x = 0, \quad x(0) = 0, x'(0) = 1.$$

Solution: We already know how to solve this IVP by guessing $x(t) = e^{rt}$ and solving for r. After substituting the initial conditions, this would give us that answer

$$x(t) = -e^t + e^{2t}.$$

Our first step to solving this by the Laplace transform method is to transform the entire differential equation into the Laplace domain. We have

$$\begin{aligned}
& \mathcal{L}\left\{x'' - 3x' + 2x\right\} = 0 \\
\implies & \mathcal{L}\left\{x''\right\} - 3\mathcal{L}\left\{x'\right\} + 2\mathcal{L}\left\{x\right\} = 0 \\
\implies & \left[s^2X(s) - sx(0) - x'(0)\right] - 3\left[sX(s) - x(0)\right] + 2X(s) = 0
\end{aligned}$$

where $X(s)$ is the Laplace transform of $x(t)$. What is interesting here is that all of the derivatives have been absorbed into the initial conditions, and we know what these are. The only thing unsolved for here is $X(s)$, which we may isolate:

$$\begin{aligned}
\implies & \left[s^2X(s) - 1\right] - 3sX(s) + 2X(s) = 0 \\
\implies & (s^2 - 3s + 2)X(s) = 1 \\
\implies & X(s) = \frac{1}{s^2 - 3s + 2}.
\end{aligned}$$

This is great! We have now isolated the Laplace transform of the solution we want. It remains only to invert the transformation, and we have already performed this task several times. We will need to factor and perform partial fraction decomposition on the right-hand side. We have

$$X(s) = \frac{1}{s^2 - 3s + 2} = \frac{1}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}.$$

We can multiply this across to get

$$1 = A(s-2) + B(s-1).$$

Setting $s = 1$ gives $A = -1$ and setting $s = 2$ gives $B = 1$. It follows that we have

$$X(s) = -\frac{1}{s-1} + \frac{1}{s-2}.$$

As expected, we can invert this to get

$$x(t) = -\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = -e^t + e^{2t}.$$

We should be very happy that we obtained the earlier expected answer, but we might wonder *why* we needed another method in the first place. After all, the standard method works just fine for this example. The answer will not be fully apparent just yet, but it is worth noting that the Laplace transform method is a *one-step* solution method. Our previous method required separate steps to hand homogeneities and initial conditions. The Laplace transform method directly incorporates these things into the method itself.

To see how non-homogeneities are incorporated in the method, consider the following problem.

Example 10

Use the Laplace transform method to solve

$$x'' + 4x' + 4x = 3te^{-2t}, \quad x(0) = 0,\ x'(0) = 1.$$

Solution: We have

$$\begin{aligned}
&\mathcal{L}\left\{x'' + 4x' + 4x\right\} = \mathcal{L}\left\{3te^{-2t}\right\} \\
\Longrightarrow\quad &[s^2X(s) - sx(0) - x'(0)] + 4[sX(s) - x(0)] + 4X(s) = \frac{3}{(s+2)^2} \\
\Longrightarrow\quad &(s+2)^2X(s) = 1 + \frac{3}{(s+2)^2} \\
\Longrightarrow\quad &X(s) = \frac{1}{(s+2)^2} + \frac{3}{(s+2)^4}.
\end{aligned}$$

We have

$$\begin{aligned}
x(t) &= \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} + \mathcal{L}\left\{\frac{3}{(s+2)^4}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{6}{(s+2)^4}\right\} \\
&= te^{-2t} + \frac{t^3}{2}e^{-2t}.
\end{aligned}$$

This form can be easily verified directly or by solving using the classical method, but requires separate steps to determine the complementary solution $x_c(t)$, the particular solution $x_p(t)$, and then to evaluate the intial conditions. By contrast, the Laplace transform method employed above was a *one-step* method.

Example 11

Solve the following differential equation using the Laplace transform method:

$$x'' + 4x' + 3x = 10\cos(t), \quad x(0) = 1,\ x'(0) = 2.$$

Solution: We take the Laplace transform of the entire equation to get

$$\mathcal{L}\left\{x''\right\} + 4\mathcal{L}\left\{x'\right\} + 3\mathcal{L}\left\{x\right\} = 10\mathcal{L}\left\{\cos(t)\right\}$$

$$\implies \quad [s^2X(s) - sx(0) - x'(0)] + 4[sX(s) - x(0)] + 3X(s) = \frac{10s}{s^2+1}$$

$$\implies \quad (s^2 + 4s + 3)X(s) = \frac{10s}{s^2+1} + s + 6$$

$$\implies \quad (s+3)(s+1)X(s) = \frac{10s}{s^2+1} + s + 6.$$

If we choose not to find a common denominator on the right-hand side, we encounter the equation

$$X(s) = \frac{10s}{(s+1)(s+3)(s^2+1)} + \frac{s+6}{(s+1)(s+3)}.$$

This is *correct* but it is undeniably *bad*. Neither of these terms are directly in a form where we can identify the inverse transform, and instead of performing partial fraction decomposition once, we will have to perform it *twice* (once for each term!). It is, in this case much easier to find a common denominator first. In this case, we have

$$(s+3)(s+1)X(s) = \frac{10s + (s+6)(s^2+1)}{s^2+1} = \frac{s^3 + 6s^2 + 11s + 6}{s^2+1}$$

$$\implies \quad X(s) = \frac{s^3 + 6s^2 + 11s + 6}{(s+1)(s+3)(s^2+1)} = \frac{A}{s+1} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1}.$$

Expanding this out, we have

$$s^3+6s^2+11s+6 = A(s+3)(s^2+1)+B(s+1)(s^2+1)+(Cs+D)(s+1)(s+3).$$

It's debatable which method is quicker for solving this method. I will choose to simplify by solving for the simple roots $s = -1$ and $s = -3$ first, then expand whatever is left over. We have

$$s = -1 \quad \implies \quad 0 = A(-4) \quad \implies \quad A = 0$$

and

$$s = -3 \quad \implies \quad 0 = B(-20) \quad \implies \quad B = 0.$$

This simplifies our life significantly! We now have

$$\begin{aligned} &s^3 + 6s^2 + 11s + 6 \\ &\quad = (Cs + D)(s^2 + 4s + 3) = Cs^3 + (4C + D)s^2 + (3C + 4D)s + 3D. \end{aligned}$$

It follows that $C = 1$ and $D = 2$ (from the first and last coefficients) so that we have

$$X(s) = \frac{s+2}{s^2+1} = \frac{s}{s^2+1} + \frac{2}{s^2+1}.$$

It follows that the solution is

$$x(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \cos(t) + 2\sin(t).$$

Suggested Problems

1. Use the definition of the Laplace transform to compute the Laplace transform $F(s)$ of the following:

   (a) $f(t) = e^{-7t}$
   (b) $f(t) = t^2$
   (c) $f(t) = te^{2t}$
   (d) $f(t) = \sin(4t)$

2. Use a table of Laplace transforms to compute the Laplace transform of the following functions:

   (a) $f(t) = e^t - 2t^2$
   (b) $f(t) = 3\sin(3t) - 4\cos(3t)$
   (c) $f(t) = t + t^2 + t^3$
   (d) $f(t) = \sinh(t)$

3. Verify the formula $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$ for the following functions:

   (a) $f(t) = e^t$
   (b) $f(t) = t^a, \quad a = 1, 2, \ldots$
   (c) $f(t) = \sin(2t)$
   (d) $f(t) = \sin(bt) + \cos(bt), \quad b \neq 0$

4. Use the definition of the Laplace Transform to prove Theorem 3.

5. Determine the Laplace transform of the following functions:

   (a) $f(t) = te^t$
   (b) $f(t) = t(1+t)e^{3t}$
   (c) $f(t) = e^{-2t}\sin(t)$
   (d) $f(t) = e^t\cos(\pi t)$

6. Determine the inverse Laplace transform $f(t) = \mathcal{L}^{-1}\{F(s)\}$ of the following functions:

   (a) $F(s) = \dfrac{1}{s^2}$
   (b) $F(s) = \dfrac{9}{s^2+16}$
   (c) $F(s) = \dfrac{1}{s^2-s}$
   (d) $F(s) = \dfrac{2}{s^2+2s+2}$
   (e) $F(s) = \dfrac{2}{(s+5)^4}$
   (f) $F(s) = \dfrac{3}{4s^3+5s^2+s}$
   (g) $F(s) = \dfrac{s}{s^2-4}$
   (h) $F(s) = \dfrac{s^2+5}{s^3-2s^2+5s}$

7. Use the Laplace Transform method to solve the following differential equations:

(a) $\begin{cases} x' - x = e^t, \\ x(0) = 1 \end{cases}$

(b) $\begin{cases} x' + 2x = 5\sin(t), \\ x(0) = 1 \end{cases}$

(c) $\begin{cases} x'' + 6x' + 5x = 0, \\ x(0) = 4, \\ x'(0) = 0 \end{cases}$

(d) $\begin{cases} x'' - 7x' + 12x = 0, \\ x(0) = 1, \\ x'(0) = 1 \end{cases}$

(e) $\begin{cases} x'' - 4x' + 12x = e^{2t}, \\ x(0) = 0, \\ x'(0) = 0 \end{cases}$

(f) $\begin{cases} 4x'' - 4x' + x = e^t, \\ x(0) = -1, \\ x'(0) = 0 \end{cases}$

(g) $\begin{cases} x'' + 2x' + 5x = 16te^t, \\ x(0) = 0, \\ x'(0) = 0 \end{cases}$

(h) $\begin{cases} x''' - x'' - x' + x = 0, \\ x(0) = 4, \\ x'(0) = 0, \\ x''(0) = 0 \end{cases}$

(i) $\begin{cases} x''' - 4x'' + 5x' = 0, \\ x(0) = 5, \\ x'(0) = 5, \\ x''(0) = 0 \end{cases}$