Final Exam Review

- Open Everything (Notes, browser, ...)
- Longer version of Exams 1 & 2
  (9 questions - 15 min/question)
- 135 min time limit
- Available 15 min before (9:30 AM)
  and available 15 min after (12:15 pm)
- Upload work to Canvas for re-grades.
- Zoom meeting for questions
  - Send chats to "everyone"
  - Alana (TA) will be there
- Normal office hours

Topics (HW1 - HW11 , Ch. 1-5, 7 , 9)

- 1^{st}-order ODEs (separable, integrating factors, ...)
- Numerical Methods (Euler method)
- 2^{nd}-order ODEs (homogeneous, undetermined coefficients)
- Laplace Transforms
  - ODE $y'' + \dots = f(t)$
    $\rightarrow s^2 Y(s) + \dots = F(s)$
  - Given $F(s) \rightarrow f(t)$
- Linear Algebra (Matrices, Vectors)
  - ODE $\rightarrow$ Matrix problem
  - Operations, determinants, inverses
  - Eigenvalues & Eigenvectors
  - $\frac{d\vec{x}}{dt} = \underline{\underline{A}}\,\vec{x} + \vec{f}$

Example 1

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 2 \\ 3 & t \end{bmatrix}, \quad \underline{\underline{B}} = \begin{bmatrix} t^2 & t \\ t^3 & 2t^2 \end{bmatrix}$$

$$\frac{d}{dt}\left(\underline{\underline{A}}\,\underline{\underline{B}}\right) = \left(\frac{d}{dt}\underline{\underline{A}}\right)\underline{\underline{B}} + \underline{\underline{A}}\left(\frac{d}{dt}\underline{\underline{B}}\right)$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} t^2 & t \\ t^3 & 2t^2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & t \end{bmatrix}\begin{bmatrix} 2t & 1 \\ 3t^2 & 4t \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ t^3 & 2t^2 \end{bmatrix} + \begin{bmatrix} 2t+6t^2 & 1+8t \\ 6t+3t^3 & 3+4t^2 \end{bmatrix}$$

$$= \begin{bmatrix} 2t+6t^2 & 1+8t \\ 6t+4t^3 & 3+6t^2 \end{bmatrix}$$

OR

$$\frac{d}{dt}\left(\underline{\underline{A}}\,\underline{\underline{B}}\right) = \frac{d}{dt}\left(\begin{bmatrix} 1 & 2 \\ 3 & t \end{bmatrix}\begin{bmatrix} t^2 & t \\ t^3 & 2t^2 \end{bmatrix}\right)$$

$$= \frac{d}{dt}\begin{bmatrix} t^2+2t^3 & t+4t^2 \\ 3t^2+t^4 & 3t+2t^3 \end{bmatrix}$$

$$= \begin{bmatrix} 2t+6t^2 & 1+8t \\ 6t+4t^3 & 3+6t^2 \end{bmatrix}$$

Example 2

$$\underline{\underline{A}} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & 5 \\ 1 & 1 & 10 \end{bmatrix}$$

$$\det(\underline{\underline{A}}) = 6\begin{vmatrix} 6 & 5 \\ 1 & 10 \end{vmatrix} - (-5)\begin{vmatrix} -5 & 5 \\ 1 & 10 \end{vmatrix} + 5\begin{vmatrix} -5 & 6 \\ 1 & 1 \end{vmatrix}$$

$$= 6(60-5) + 5(-50-5) + 5(-5-6)$$

$$= 0 \iff \underline{\underline{A}} \text{ is singular}$$

$$\underline{\underline{A}}\vec{x} = \vec{b}, \quad \vec{b} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$$

1) $$\left[\begin{array}{ccc|c} 6 & -5 & 5 & 3 \\ -5 & 6 & 5 & 0 \\ 1 & 1 & 10 & 3 \end{array}\right] \overset{R2+R1}{\sim} \left[\begin{array}{ccc|c} 6 & -5 & 5 & 3 \\ 1 & 1 & 10 & 3 \\ 1 & 1 & 10 & 3 \end{array}\right]$$

(For matrix inverse formula → Cramer's Rule)

$$\overset{R3-R2}{\sim} \left[\begin{array}{ccc|c} 6 & -5 & 5 & 3 \\ 1 & 1 & 10 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \overset{R1-6R2}{\sim} \left[\begin{array}{ccc|c} 0 & -11 & -55 & -15 \\ 1 & 1 & 10 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\overset{\text{Swap}, \div -11}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 10 & 3 \\ 0 & 1 & 5 & 15/11 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & 18/11 \\ 0 & 1 & 5 & 15/11 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Last Row $\Rightarrow$ $0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 0$

True for any x_1, x_2 & x_3

$\Rightarrow$ Infinite number of solutions

(If $0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 7$

$\Rightarrow$ No solutions)

Example 3

$$\frac{dx}{dt} = x + y - 1, \quad \frac{dy}{dt} = 4x + y + e^t$$

$$\text{Let } \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \vec{f} = \begin{bmatrix} -1 \\ e^t \end{bmatrix}$$

$$\Rightarrow \quad \frac{dx}{dt} = x + y - 1$$
$$\frac{dy}{dt} = 4x + y + e^t$$

$$\Rightarrow \underbrace{\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix}}_{\frac{d}{dt}\vec{x}} = \underbrace{\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}}_{\underline{\underline{A}}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}} + \underbrace{\begin{bmatrix} -1 \\ e^t \end{bmatrix}}_{\vec{f}}$$

$$\Rightarrow \frac{d\vec{x}}{dt} = \underline{\underline{A}}\,\vec{x} + \vec{f}$$

Solution

$$\vec{x}(t) = \underline{\underline{X}}(t)\,\vec{c} + \underline{\underline{X}}(t)\int_0^t \underline{\underline{X}}^{-1}(t)\,\vec{f}(t)\,dt$$

If $\vec{x}(0)$ is known $\Rightarrow$ $\vec{c} = \underline{\underline{X}}^{-1}(0)\,\vec{x}(0)$

(i) Eigenvalues

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = 0$$

$$= (1-\lambda)^2 - 4 = \underbrace{\lambda^2 - 2\lambda - 3}_{(\lambda+1)(\lambda-3)} = 0$$

$$\Rightarrow \lambda = \{-1, 3\}$$

(ii) Eigenvectors: $(\underline{\underline{A}} - \lambda_i \underline{\underline{I}})\vec{v}_i = \vec{0}$

$\lambda_1 = -1$

$$\begin{bmatrix} 1-(-1) & 1 \\ 4 & 1-(-1) \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad 2A_1 + B_1 = 0 \quad \Rightarrow \quad B_1 = -2A_1$$

$$\bar{v}_1 = \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} A_1 \\ -2A_1 \end{bmatrix} = A_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \sim \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Similarly $\bar{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ for $\lambda_2 = 3$

If $\vec{f} = \vec{0}$

$$\Rightarrow \quad \vec{x}(t) = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t}$$

$$= c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$$

$$= \underbrace{\begin{bmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{bmatrix}}_{\underline{\underline{X}}(t)} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\vec{c}}$$

(iii) Fundamental matrix $\underline{\underline{X}}(t)$ & $\underline{\underline{X}}^{-1}(t)$

$$\underline{\underline{X}}(t) = \begin{bmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{bmatrix}$$

$$\underline{\underline{X}}^{-1}(t) = \frac{1}{2e^{-t}e^{3t} + 2e^{-t}e^{3t}} \begin{bmatrix} 2e^{3t} & -e^{3t} \\ 2e^{-t} & e^{-t} \end{bmatrix}$$

$$= \frac{1}{4e^{2t}} \begin{bmatrix} 2e^{3t} & -e^{3t} \\ 2e^{-t} & e^{-t} \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 2e^{t} & -e^{t} \\ 2e^{-3t} & e^{-3t} \end{bmatrix}$$

(iv) Plug into formula

$$\vec{x}(t) = \underline{\underline{X}}(t)\,\vec{c} + \underline{\underline{X}}(t) \int_0^t \underline{\underline{X}}^{-1}(t)\,\vec{f}(t)\,dt$$

- $$\underline{\underline{X}}^{-1}(t)\vec{f}(t) = \frac{1}{4}\begin{bmatrix} 2e^{t} & -e^{t} \\ 2e^{-3t} & e^{-3t} \end{bmatrix}\begin{bmatrix} -1 \\ e^{t} \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -2e^{t}-e^{2t} \\ -2e^{-3t}+e^{-2t} \end{bmatrix}$$

- $$\int_0^t \underline{\underline{X}}^{-1}(t)\vec{f}(t)dt = \frac{1}{4}\int_0^t \begin{bmatrix} -2e^{t}-e^{2t} \\ -2e^{-3t}+e^{-2t} \end{bmatrix} dt$$

$$= \frac{1}{4}\begin{bmatrix} -2e^{t} - \frac{1}{2}e^{2t} \\ \frac{2}{3}e^{-3t} - \frac{1}{2}e^{-2t} \end{bmatrix}\Bigg|_0^t$$

$$= \frac{1}{4}\begin{bmatrix} -2e^{t} - \frac{1}{2}e^{2t} + 5/2 \\ \frac{2}{3}e^{-3t} - \frac{1}{2}e^{-2t} - 1/6 \end{bmatrix}$$

$$\underline{\underline{X}}(t)\int_0^t \underline{\underline{X}}^{-1}(t)\vec{f}(t)dt$$

$$= \frac{1}{4}\begin{bmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{bmatrix}\begin{bmatrix} -2e^{t} - \frac{1}{2}e^{2t} + 5/2 \\ \frac{2}{3}e^{-3t} - \frac{1}{2}e^{-2t} - 1/6 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -2 - \frac{1}{2}e^{t} + \frac{5}{2}e^{-t} + \frac{2}{3} - \frac{1}{2}e^{t} - \frac{1}{6}e^{3t} \\ 4 + \cancel{e^{t}} - 5e^{-t} + \frac{4}{3} - \cancel{e^{t}} - \frac{1}{3}e^{3t} \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -\frac{4}{3} - e^{t} + \frac{5}{2}e^{-t} - \frac{1}{6}e^{3t} \\ \frac{16}{3} - 5e^{-t} - \frac{1}{3}e^{3t} \end{bmatrix}$$

$$\boxed{\therefore \quad \vec{x}(t) = \begin{bmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} -\frac{4}{3} - e^{t} + \frac{5}{2}e^{-t} - \frac{1}{6}e^{3t} \\ \frac{16}{3} - 5e^{-t} - \frac{1}{3}e^{3t} \end{bmatrix}}$$

<u>DONE!</u>