Math 133A, Week 3: Separable and First-Order Linear Differential Equations

Section 1: Separable DEs

We are finally to the point in the course where we can consider how to find solutions to differential equations. We start with a motivating example.

Example 1

Consider the first-order ODE

$$y' = \frac{1-y}{x}.$$

The only method we have discussed so far for finding solutions $y(x)$ is direct integration. As written, however, the RHS of this DE depends on *both* x and y and so cannot be integrated directly to yield a solution y. We will need another method of approach.

Our trick will to notice that we can make the problem *look like* an integration problem with respect to x by removing the y from the right-hand side and moving the differential dx to the other side. This leaves us with

$$\frac{dy}{1-y} = \frac{dx}{x}.$$

Not only does the right-hand side look like an integral question (with respect to x), but the left-hand side looks like an integral question as well (with respect to y). This is exactly how we will treat the equation.

We integrate (wrt y on the left and x on the right) to obtain

$$\begin{aligned}
\int \frac{1}{1-y}\, dy &= \int \frac{1}{x}\, dx \\
\implies -\ln(1-y) &= \ln(x) + C \\
\implies \ln(1-y) &= -\ln(x) - C \\
\implies \ln(1-y) &= \ln\left(\frac{1}{x}\right) - C \\
\implies 1-y &= e^{\ln\left(\frac{1}{x}\right)-C} \\
\implies 1-y &= \frac{\tilde{C}}{x}
\end{aligned}$$

where $\tilde{C} = e^{-C}$. The general solution is therefore

$$y(x) = 1 - \frac{\tilde{C}}{x}, \quad \tilde{C} \in \mathbb{R}.$$

We will usually just replace the modified constant $\tilde{C}$ with simply C, since on arbitrary constant is as good as another. We can verify that

$$y(x) = 1 - \frac{C}{x}$$

is a solution of the differential equation $y' = \frac{1-y}{x}$ since we have

$$\begin{aligned}
\text{LHS} &= \frac{dy}{dx} = \frac{d}{dx}\left[1 - \frac{C}{x}\right] = \frac{C}{x^2} \\
\text{RHS} &= \frac{1-y}{x} = \frac{1-\left(1-\frac{C}{x}\right)}{t} = \frac{\frac{C}{x}}{x} = \frac{C}{x^2}.
\end{aligned}$$

Since LHS=RHS, we have that $x(t)$ satisfies the differential equation.

What we have discovered in this example is a class of DEs for which integration is sufficient to determine the solution $y(x)$. We formally introduce the following.

Definition 1

The first order DE $y' = f(x, y)$ is called **separable** if it can be written

in the form
$$g(y)\frac{dy}{dx} = h(x) \quad \text{or} \quad g(y)\,dy = h(x)\,dx$$
for some function g of y and h of x.

That is, a DE is separable if the dependence on x and y in $f(x, y)$ may be separated. In such cases, we may then integrate the sides separately and rearrange to obtain a solution $y(x)$. A rigorous justification of the validity of this separate and integrate method follows from the chain rule but will not be covered here.

Note: Integration will factor significant in the analysis of separable equations. Every time we evaluate such a DE, we will have to integrate not just once, but *twice*.

Example 2

Solve the IVP
$$y' = -y^2\frac{(1 + 2x^2)}{x}, \quad y(1) = 1.$$

Solution: Notice that, if we divide the equation by y^2 and move the differential dx to the RHS, we have
$$\frac{1}{y^2}\,dy = -\frac{(1 + 2x^2)}{x}\,dx.$$

This is perfect! We have isolated the dependence on y on the LHS, and the dependence on x on the RHS. Now it is only a matter of integrating (twice!). We have
$$\begin{aligned} \int \frac{1}{y^2}\,dy &= -\int (\frac{1}{x} + 2x)\,dx \\ \implies -\frac{1}{y} &= -\ln(x) - x^2 + C \\ \implies y &= \frac{1}{\ln(x) + x^2 - C}. \end{aligned}$$

The initial condition $y(1) = 1$ gives

$$1 = \frac{1}{1 - C}$$

so that $C = 0$. It follows that the particular solution is

$$y = \frac{1}{\ln(x) + x^2}.$$

Again, we can check that the solution is valid by substituting it into the DE directly:

$$\begin{aligned}
\text{LHS} &= \frac{dx}{dt} = \frac{d}{dx}\left[\frac{1}{\ln(x) + x^2}\right] \\
&= -\frac{\frac{1}{x} + 2x}{(\ln(x) + x^2)^2} = -\frac{1 + 2x^2}{x(\ln(x) + x^2)^2} \\
\text{RHS} &= -y^2 \frac{(1 + 2x^2)}{x} = -\left(\frac{1}{\ln(x) + x^2}\right)^2 \left(\frac{1 + 2x^2}{x}\right) \\
&= -\frac{1 + 2x^2}{x(\ln(x) + x^2)^2}
\end{aligned}$$

Note that not it is not always reasonable to find the solution for a separable equation in the explicit form $y = \cdots$. In such cases, solutions may be left in implicit form. Consider the following example.

Example 3

Determine the solution to the following initial value problem:

$$\begin{cases} y' = \dfrac{y}{1 + y} \\ y(0) = 1. \end{cases}$$

Solution: The equation is separable since we can write it as

$$\frac{1 + y}{y}\, dy = \, dx.$$

We therefore must integrate

$$\int \left(\frac{1}{y} + 1 \right) \, dy = \int 1 \, dx$$

which gives

$$\ln(y) + y = x + C. \tag{1}$$

We expect at this point that we should attempt to solve for y; however, such an exercise would prove futile in this case. The solution has no simple expression of the form $y = \cdots$. Consequently, we will leave the solution in the implicit form (1).

Regardless of whether we have the solution is in explicit or implicit form, however, we will still be able to use initial conditions to solve for C. We have that $y(0) = 1$ gives

$$\ln(1) + (1) = (0) + C \implies C = 1$$

so that the solution is

$$\ln(y) + y = x + 1.$$

Section 2: First-Order Linear DEs

We now introduce another common solution method for first-order DEs. We start by re-considering the original DE

$$y' = \frac{1 - y}{x}.$$

Supposing we were unfamiliar with the method of separation, we might notice that we can rewrite the expression as:

$$xy' + y = 1. \tag{2}$$

There is nothing in the expression dictating that we would want to represent this equation in this form, but we can at the very least notice one nice thing about this form: it was easy to classify! It is a **first-order linear differential equation**.

There is a little bit of cheating that has been done in rearranging the expression this way, but it is a suggestive bit of cheating. Let's focus on the

LHS of the above expression:

$$xy' + y.$$

If we stare this for long enough, or were born with unparalleled mathematical powers, we might notice that this can be written in a more compact form. Without justifying, for a moment, why we would *want* to do this, we might notice that this expression is the end result of the product rule for differentiation on the term xy. We have

$$\frac{d}{dx}[xy] = xy' + y.$$

In other words, we can take the two terms on the left-hand side and condense them into a single term, at the expense of having to recall the product rule for differentiation. We can now rewrite (2) as

$$\frac{d}{dx}[xy] = 1.$$

This is a *huge* improvement over our previous expression. The reason should be clear: we can integrate it! If we integrate the left-hand and right-hand sides by x, the Fundamental Theorem of Calculus tells us the differential on the left-hand side disappears, and the right-hand side can be evaluated as long as we know an anti-derivative of whatever the term there happens to be. That is, we have

$$\int \frac{d}{dx}[xy]\ dx = \int 1\ dx$$

$$\implies \quad xy = x + C, \quad C \in \mathbb{R}$$

which, after dividing by x, implies that we have the general solution

$$y(x) = 1 + \frac{C}{x}, \quad C \in \mathbb{R}.$$

This coincides with the solution obtained by separating the variables and integrating directly.

To see whether this method is sufficient to solve all first-order linear DEs, consider the example

$$xy' + 2y = 1. \tag{3}$$

This is only subtly different that the previous example—in fact, the only difference is the coefficient of the y term is now two. This subtle difference, however, is enough to sabotage our earlier intuition with regards to a solution method, since there is no function $f(x)$ such that

$$\frac{d}{dx}\left[f(x)\,y\right] = xy' + 2y.$$

So what can we do?

Let's consider changing the expression (again!) but in a different way. Let's *multiply* (3) by a single term that is a function of x. In this case, we will choose the function to be x itself. This gives us

$$x^2y' + 2xy = x.$$

If there were any questions with regards to *why* we would want to do that, they have now been answered. We have that

$$\frac{d}{dx}\left[x^2y\right] = x^2y' + 2xy = x.$$

Again, we can integrate to get the solution. We have

$$\int \frac{d}{dx}\left[x^2y\right]\,dx = \int x\,dx$$

$$\implies \quad x^2y = \frac{x^2}{2} + C, \quad C \in \mathbb{R}$$

so that the desired solution is

$$y(x) = \frac{1}{2} + \frac{C}{x^2}, \quad C \in \mathbb{R}.$$

The difference with this example was that we had to *multiply* by some factor before we could use the product rule trick that we just discovered to get to a form we could integrate. This multiplicative factor is called an **integrating factor** and is generally denoted $\mu(x)$. We still have to wonder how we could find integrating factors. After all, how did we know to multiply by the factor x?

The answer is given by the following result.

Theorem 1

Consider a first-order linear DE given in the standard form

$$y' + p(x)y = q(x). \tag{4}$$

Then the solution $y(x)$ is given by

$$y(x) = \frac{1}{\mu(x)} \int \mu(x)q(x)\ dx \tag{5}$$

where $\mu(x) = e^{\int p(x)\ dx}$ is the system's integrating factor.

Proof

By the Fundamental Theorem of Calculus and the chain rule we have that $\mu'(x) = p(x)\mu(x)$. If we multiply the entire expression (4) by $\mu(x)$, we have

$$\mu(x)y' + p(x)\mu(x)y = \mu(x)q(x).$$

The LHS can be simplified by noting that

$$\frac{d}{dx}\left[\mu(x)y\right] = \mu(x)y' + \mu' y = \mu(x)y' + p(x)\mu(x)y.$$

It follows that we have

$$\frac{d}{dx}\left[\mu(x)y\right] = \mu(x)q(x).$$

We can then integrate to get

$$\mu(x)y = \int \mu(x)q(x)\ dx$$

and isolate y to get the general solution

$$\begin{aligned} y(x) &= \frac{1}{\mu(x)} \int \mu(x)q(x)\ dx \\ &= e^{-\int p(x)\ dx} \int \left(e^{\int p(x)\ dx} q(x)\right)\ dx. \end{aligned}$$

This result is very encouraging! So long as we can evaluate the require integrals, we can always find the solution of a first-order linear DE.

Note: It is strongly recommended that you remember the solution method rather than the solution formula (5) (which is cumbersome to use in practice and difficult to remember). The general method is:

1. Write the DE in the standard form (4).
2. Determine the integrating factor $\mu(x) = e^{\int p(x)\, dx}$.
3. Multiply the entire expression through by $\mu(x)$.
4. Combine the LHS by applying the product rule in reverse.
5. Integrate with respect to x and isolate for y.

Note: It is important to have the equation in the standard form (4) where the coefficient of y' is one. Otherwise, the given integrating factor method will not work (and you will likely waste a great deal of time computing incorrect integrals!).

Example 4

Solve the following IVP:

$$\begin{cases} y' + \frac{1}{x} \cdot y = \frac{1}{x} \\ y(1) = 0. \end{cases}$$

Solution: This is already in standard form, so we are ready to determine the integrating factor. We have

$$\begin{aligned} \mu(x) &= e^{\int p(x)\, dx} \\ &= e^{\int \frac{1}{x}\, dx} \\ &= e^{\ln(x)} = x. \end{aligned}$$

We can ignored the normally required $|x|$ in the $\ln(x)$ term by noticing that the two absolute value cases ($x > 0$ and $x < 0$) amount to multiplying the whole differential equation by a negative, which does not change it. Multiplying the entire expression by $\mu(x) = x$ gives us

$$xy' + y = 1$$

which we have already seen. This was our original toy example. We already know that the general solution is

$$y(x) = 1 + \frac{C}{x}.$$

Substituting the intial value $y(1) = 1$ gives us

$$y(1) = 0 = 1 + C \quad \Longrightarrow \quad C = -1.$$

It follows that the particular solution is

$$y(x) = 1 - \frac{1}{x}.$$

Example 5

Solve the following IVP:

$$\begin{cases} y' + y = e^{-3x} \\ y(0) = 2. \end{cases}$$

Solution: This is already in standard form, so we are ready to determine the integrating factor. We have

$$\begin{aligned} \mu(x) &= e^{\int p(x)\,dx} \\ &= e^{\int 1\,dx} \\ &= e^{x}. \end{aligned}$$

Multiplying the entire expression by $\mu(x) = e^x$ gives us

$$e^x y' + e^x y = e^x \cdot e^{-3x} = e^{-2x}.$$

Recognizing that the left-hand side now must be the product rule form (expanded out), we have

$$\frac{d}{dx}\left[e^x y\right] = e^{-2x}.$$

We could jump right to this if we wanted to, but it is important to recognize the intermediate step to check that we have determined the correct integrating factor. We can integrate this to get

$$\int \frac{d}{dx}\left[e^x y\right]\, dx = \int e^{-2x}\, dx$$

$$\implies \quad e^x y = -\frac{e^{-2x}}{2} + C$$

$$\implies \quad y(x) = -\frac{e^{-3x}}{2} + Ce^{-x}.$$

Using the initial condition $y(0) = 2$ gives

$$y(0) = 2 = -\frac{1}{2} + C \quad \implies \quad C = \frac{5}{2}.$$

The particular solution is therefore

$$y(x) = -\frac{e^{-3x}}{2} + \frac{5e^{-x}}{2}.$$

Example 6

Solve the following IVP:

$$\begin{cases} (x+1)y' - xy = e^x \\ y(1) = 0. \end{cases}$$

Solution: This is not in standard form, so we need to do a little work. Dividing by $(x+1)$ we arrive at

$$y' - \frac{x}{x+1}y = \frac{e^x}{x+1}.$$

In order to determine the integrating factor, we will need to determine the integral of $-x/(x+1)$. Using the substitution $u = x+1$, we have

$$\begin{aligned} -\int \frac{x}{x+1}\, dx &= \int \frac{1-u}{u}\, du = \int \left(\frac{1}{u} - 1\right)\, du \\ &= \ln(u) - u = \ln(x+1) - (x+1). \end{aligned}$$

Recognizing that constants (i.e. the -1) do not matter for integrating factors, we arrive at

$$\mu(x) = e^{\ln(x+1)-x} = (x+1)e^{-x}.$$

Multiplying the entire expression by $\mu(x) = (x+1)e^{-x}$ gives us

$$(x+1)e^{-x}y' - xe^{-x}y = 1.$$

It follows that we have

$$\frac{d}{dx}\left[(x+1)e^{-x}y\right] = 1$$

which can be checked. Integrating with respect to x gives

$$(x+1)e^{-x}y = x + C$$

so that the general solution is

$$y(x) = \frac{e^x}{x+1}(x+C).$$

The initial condition $y(1) = 0$ gives

$$y(1) = 0 = \frac{e}{2}(1+C) \quad \Longrightarrow \quad C = -1.$$

It follows that the particular solution is

$$y(x) = e^x \left(\frac{x-1}{x+1}\right).$$

Suggested Problems:

1. Solve the following first-order IVPs, which are either separable or linear: (If it is possible to solve as both separable and first-order linear, consider solving by both methods!)

(a) $\begin{cases} y' = y^2 - 5y + 4 \\ y(0) = 1 \end{cases}$

(b) $\begin{cases} y' = x(y-1) \\ y(1) = 2 \end{cases}$

(c) $\begin{cases} y' = e^{x+y} \\ y(0) = 0 \end{cases}$

(d) $\begin{cases} y' = ay + b, \ a, b \in \mathbb{R} \\ y(x_0) = y_0, \ x_0, y_0 \in \mathbb{R} \end{cases}$

(e) $\begin{cases} y' = -\sin(x)y + \sin(2x) \\ y(\pi) = 0 \end{cases}$

(f) $\begin{cases} y' = \frac{y+x}{1+x} \\ y(0) = 1 \end{cases}$

(g) $\begin{cases} y' = \frac{x+1}{y+1} \\ y(0) = -1 \end{cases}$

(h) $\begin{cases} y' = -2\frac{y}{x} + \ln(x) \\ y(0) = 0 \end{cases}$

2. Newton's Law of Cooling states the rate of change of body's temperature is roughly proportional to the difference between the body's current temperature and the external forcing temperature. Consider the temperature (in Celsius) of a lake which is driven by the season variation in air temperature. Suppose the lake's temperature is drive according to the equation

$$\frac{dT}{dt} = \cos\left(\frac{\pi}{6}t\right) - \frac{\pi}{6}T \tag{6}$$

where the time t is in months, T is the current temperature of the lake, and $\cos\left(\frac{\pi}{6}t\right)$ is the season variation in the air temperature. (Note that air temperature complete a full period every 12 months).

(a) Find the general solution of (6). (Note: You will have to compute a tricky integral!)

(b) Write the solution $T(t)$ in the form $T(t) = T_{tr}(t) + T_{sp}(t)$ where $T_{tr}(t)$ is the *transient* portion of the solution (decays to zero as $t \to \infty$) and $T_{sp}(t)$ is the steady state periodic portion of the solution. What can be said about the long-term temperature of the lake? (Hint: consider $T_{sp}(t)$!)

(c) Is the variation in temperature of the lake in phase with the variation in the air temperature (which comes from the term $\cos\left(\frac{\pi}{6}t\right)$)? If you observe a difference, what might explain this phenomenon?