Lecture 19 — Intro to Vectors and Matrices (pt. 2)

Review

Vectors: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Matrices: $\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$

$N \times M$

OPERATIONS

- Transpose (swap rows & columns)

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \vec{x}^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$\underline{\underline{A}} = \underbrace{\begin{bmatrix} 1 & 5 & 7 \\ 2 & 2 & 0 \end{bmatrix}}_{(2\times3)} \Rightarrow \underline{\underline{A}}^T = \underbrace{\begin{bmatrix} 1 & 2 \\ 5 & 2 \\ 7 & 0 \end{bmatrix}}_{(3\times2)}$$

- Addition

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

Same with matrices.

Ex

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

have to be same size (both $N \times M$)

- Multiplication

If c is a scalar (not a vector or matrix)

$$\Rightarrow \quad c\vec{x} = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}, \quad c\underline{\underline{A}} = \begin{bmatrix} ca_{11} & \cdots & ca_{1m} \\ \vdots & \ddots & \vdots \\ ca_{n1} & \cdots & ca_{nm} \end{bmatrix}$$

Ex

$$2\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} + 3\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}2 & 4\\6 & 8\end{bmatrix} + \begin{bmatrix}3 & 0\\0 & 3\end{bmatrix} = \begin{bmatrix}5 & 4\\6 & 11\end{bmatrix}$$

- Dot products

$$\vec{x} \cdot \vec{y} = \vec{x}^T \vec{y} = \begin{bmatrix}x_1 & x_2 & \cdots & x_n\end{bmatrix}\begin{bmatrix}y_1\\y_2\\\vdots\\y_n\end{bmatrix}$$

$$= x_1y_1 + x_2y_2 + \cdots + x_ny_n \quad \leftarrow \text{scalar}$$

$$= \sum_{k=1}^{n} x_k y_k$$

In Python (not Linear Algebra)

$$\vec{x} * \vec{y} = \begin{bmatrix}x_1y_1\\x_2y_2\\\vdots\\x_ny_n\end{bmatrix} \leftarrow \text{Not a dot product}$$

Ex

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6$$

$$= 4 + 10 + 18$$

$$= 32$$

Notation

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \langle 1, 2, 3 \rangle = (1, 2, 3)$$

$$\vec{x}^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \langle 1 \quad 2 \quad 3 \rangle$$

- Matrix Multiplication

$$\underline{\underline{A}}\,\underline{\underline{B}} = \underset{(N \times M)}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}} \underset{(M \times P)}{\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{bmatrix}}$$

$$= \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{np} \end{bmatrix} \quad (N \times P)$$

The elements of $\underline{\underline{C}}$ are : $c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$

$= a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{im} b_{mj}$

Another way to look at this:

$$\begin{array}{cc} & \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{bmatrix} \\ \underline{\underline{A}}\,\underline{\underline{B}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} & \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & \boxed{c_{22}} & \ddots & c_{2p} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{np} \end{bmatrix} \end{array}$$

Ex

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1\cdot 3 + 2\cdot 2 & 1\cdot 1 + 2\cdot 1 \\ 1\cdot 3 + 1\cdot 2 & 1\cdot 1 + 1\cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 3 \\ 5 & 2 \end{bmatrix}$$

other basic properties:

- Associative: $$\underset{N\times M}{(\underline{\underline{A}}}\ \underset{M\times P}{\underline{\underline{B}}})\ \underset{P\times L}{\underline{\underline{C}}} = \underline{\underline{A}}(\underline{\underline{B}}\,\underline{\underline{C}})$$

- Distributive: $$(\underline{\underline{A}} + \underline{\underline{B}})\underline{\underline{C}} = \underline{\underline{A}}\,\underline{\underline{C}} + \underline{\underline{B}}\,\underline{\underline{C}}$$

Commutative? $\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{B}}\,\underline{\underline{A}}$?

<u>Usually</u> <u>NOT</u>

<u>Ex</u>

$$\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{2\times 3}{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}\underset{2\times 2}{\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}} = \text{Makes no sense}$$

not equal

Ex

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3+1 & 6+1 \\ 2+1 & 4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 7 \\ 3 & 5 \end{bmatrix}$$

Systems of Equations

$$x_1 + 2x_2 + 4x_3 = 7$$

$$\begin{bmatrix} 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 7$$

$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 7 \\ 3x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 + x_2 - 5x_3 &= -4 \end{aligned}$$

$$\Rightarrow \quad \underbrace{\begin{bmatrix} 1 & 2 & 4 \\ 3 & 2 & 1 \\ 2 & 1 & -5 \end{bmatrix}}_{\underline{\underline{A}}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 7 \\ 1 \\ -4 \end{bmatrix}}_{\vec{b}}$$

$$\Rightarrow \quad \underline{\underline{A}}\,\vec{x} = \vec{b}$$

<u>Ex</u>

$$\frac{dx}{dt} = x + 2y \quad , \quad \frac{dy}{dt} = -x + 3y$$

Let $x = x_1(t)$, $y = x_2(t)$

$$\Rightarrow \quad \begin{aligned} \frac{dx_1}{dt} &= x_1 + 2x_2 \\ \frac{dx_2}{dt} &= -x_1 + 3x_2 \end{aligned}$$

Let $\vec{x}(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \dfrac{d\vec{x}}{dt} = \begin{bmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{bmatrix}$

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}}_{\underline{\underline{A}}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \quad \frac{d\vec{x}}{dt} = \underline{\underline{A}}\,\vec{x}, \quad \underline{\underline{A}} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

<u>Ex</u> $\quad y'' + 2y' + y = 0$

Let $y = x_1(t)$, $y' = x_2(t)$

$$\Rightarrow \quad y'' = x_2'$$

$$\Rightarrow \quad x_1' = x_2, \quad x_2' + 2x_2 + x_1 = 0$$

Rewrite:

$$\boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 - 2x_2 \end{aligned}}$$

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\boxed{\Rightarrow \frac{d\vec{x}}{dt} = \underline{\underline{A}}\,\vec{x}, \quad \underline{\underline{A}} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}}$$

Ex

$$x'' + 2x' + y = 0$$
$$y'' + 3x - y = 0$$

Let $x = x_1$, $x' = x_2$, $y = x_3$, $y' = x_4$

$$x_2' + 2x_2 + x_3 = 0$$

$$\Rightarrow$$

$$x_4' + 3x_1 - x_3 = 0$$

Need $x_1' = x' = x_2$

$$x_3' = y' = x_4$$

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$$\Rightarrow \begin{aligned} x_1' &= x_2 \\ x_2' &= -2x_2 - x_3 \\ x_3' &= x_4 \\ x_4' &= -3x_1 + x_3 \end{aligned}$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\Rightarrow \boxed{\frac{d\vec{x}}{dt} = \underline{\underline{A}}\,\vec{x} \quad , \quad \underline{\underline{A}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 0 & 1 & 0 \end{bmatrix}}$$