Lecture 22 — Eigenvalues & Eigenvectors

Recall: an n^{th}-order linear ODE can always be written as

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n$$

Let $\vec{x}(t) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

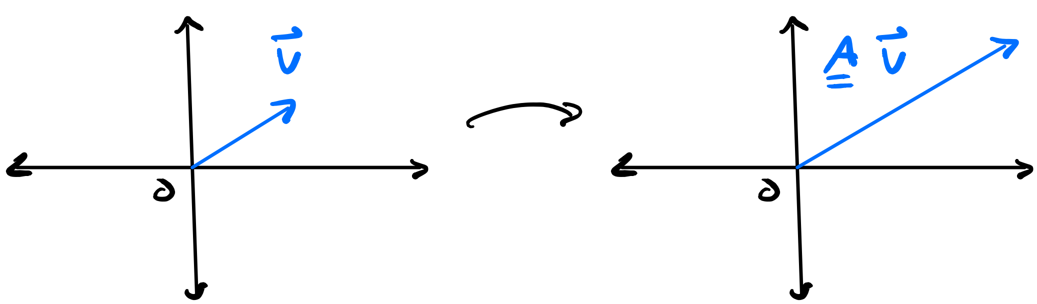
$$\Rightarrow \frac{d\vec{x}}{dt} = \underline{\underline{A}}\,\vec{x}$$

If all of the elements of $\underline{\underline{A}}$ are constant, we can then let

$$\vec{x}(t) = \vec{v}\,e^{\lambda t} \rightarrow \frac{d\vec{x}}{dt} = \lambda \vec{v} e^{\lambda t} = \lambda \vec{x}$$

$$\frac{d\vec{x}}{dt} = \underline{\underline{A}}\,\vec{x} \Rightarrow \boxed{\underline{\underline{A}}\,\vec{x} = \lambda \vec{x}}$$

- This is known as an "eigenvalue problem"
  - λ is the eigenvalue
  - $\vec{x}$ is the eigenvector



How do we calculate λ?

$$\underline{\underline{A}}\vec{x} = \lambda \vec{x}$$

$$\rightarrow \underline{\underline{A}}\vec{x} - \lambda\vec{x} = \vec{0}$$

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\rightarrow \underline{\underline{A}}\vec{x} - \lambda \underline{\underline{I}}\vec{x} = \vec{0}$$

$$\rightarrow \underbrace{(\underline{\underline{A}} - \lambda\underline{\underline{I}})}_{\text{Don't want this to be invertable}}\vec{x} = \vec{0}$$

$$\Rightarrow \boxed{\det(\underline{\underline{A}} - \lambda\underline{\underline{I}}) = 0}$$

Example 1

$$\underline{\underline{A}} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$$

$$\det(\underline{\underline{A}} - \lambda\underline{\underline{I}}) = \det\left(\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right)$$

$$\det\left(\begin{bmatrix} 2-\lambda & -3 \\ 1 & -2-\lambda \end{bmatrix}\right) = \begin{vmatrix} 2-\lambda & -3 \\ 1 & -2-\lambda \end{vmatrix} = 0$$

characteristic equation

$$\Rightarrow \quad (2-\lambda)(-2-\lambda) - (1)(-3) = 0$$

$$\lambda^2 + \cancel{2\lambda} - \cancel{2\lambda} - 4 + 3 = \boxed{\lambda^2 - 1 = 0}$$

$$\Rightarrow \boxed{\lambda = \{1, -1\}}$$

Example 2

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$$

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \begin{vmatrix} 1-\lambda & 2 & -1 \\ 1 & -\lambda & 1 \\ 4 & -4 & 5-\lambda \end{vmatrix} = 0$$

$$= (1-\lambda)\begin{vmatrix} -\lambda & 1 \\ -4 & 5-\lambda \end{vmatrix} - (2)\begin{vmatrix} 1 & 1 \\ 4 & 5-\lambda \end{vmatrix} + (-1)\begin{vmatrix} 1 & -\lambda \\ 4 & -4 \end{vmatrix}$$

$$= (1-\lambda)(\lambda^2 - 5\lambda + 4) - 2(5 - \lambda - 4) - (-4 + 4\lambda)$$

$$= (1-\lambda)(\lambda^2-5\lambda+4) - 2(1-\lambda) + 4(1-\lambda)$$

$$= (1-\lambda)(\lambda^2 - 5\lambda + 4 - 2 + 4) = 0$$

$$= (1-\lambda)(\lambda^2 - 5\lambda + 6)$$

Characteristic equation

$$= (1-\lambda)(\lambda-2)(\lambda-3) = 0$$

$$\Rightarrow \boxed{\lambda = \{1, 2, 3\}}$$

What about the eigen<u>vectors</u>?

- Each $\{\lambda_i\}$ has its own $\{\vec{v}_i\}$

$$(\underline{\underline{A}} - \lambda_i \underline{\underline{I}})\vec{v}_i = \vec{0} \quad \text{for each } \lambda_i$$

Example 1 (again)

$$\underline{\underline{A}} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}, \quad \lambda_1 = 1, \quad \lambda_2 = -1$$

$$\text{Let} \quad \vec{v}_1 = \begin{bmatrix} A_1 \\ B_1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} A_2 \\ B_2 \end{bmatrix}$$

$\underline{\lambda_1 = 1}$:

$$(\underline{\underline{A}} - \lambda_1 \underline{\underline{I}})\vec{v}_1 = (\underline{\underline{A}} - \underline{\underline{I}})\vec{v}_1$$

$$= \left(\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} A_1 - 3B_1 = 0 \\ A_1 - 3B_1 = 0 \end{cases} \Rightarrow A_1 = 3B_1$$

$$\vec{v}_1 = \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 3B_1 \\ B_1 \end{bmatrix} = B_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} \sim \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Choose $B_1 = 1$

(<u>any</u> nonzero value is fine)

$\underline{\lambda_2 = -1}$:

$$(\underline{\underline{A}} - \lambda_2 \underline{\underline{I}})\vec{v}_2 = (\underline{\underline{A}} + \underline{\underline{I}})\vec{v}_2$$

$$= \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} A_2 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 3A_2 - 3B_2 = 0 \\ A_2 - B_2 = 0 \end{cases} \Rightarrow A_2 = B_2$$

$$\vec{v}_2 = \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} = \begin{bmatrix} A_2 \\ A_2 \end{bmatrix} = A_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Choose $A_2 = 1$

Summary

- $\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is the eigenvector for $\lambda_1 = 1$
- $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the eigenvector for $\lambda_2 = -1$

Example 2 (again)

$$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 3$$

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix}$$

$\lambda_1 = 1$:

$$(\underline{\underline{A}} - \lambda_1 \underline{\underline{I}})\vec{v}_1 = (\underline{\underline{A}} - \underline{\underline{I}})\vec{v}_1 = \vec{0}$$

$$= \begin{bmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 4 & -4 & 4 \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \\ C_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad 2B_1 - C_1 = 0, \quad A_1 - B_1 + C_1 = 0$$

$$\Rightarrow \quad C_1 = 2B_1, \quad A_1 - B_1 + 2B_1 = A_1 + B_1 = 0$$

$$A_1 = -B_1$$

$$\Rightarrow \quad \vec{v}_1 = \begin{bmatrix} A_1 \\ B_1 \\ C_1 \end{bmatrix} = \begin{bmatrix} -B_1 \\ B_1 \\ 2B_1 \end{bmatrix} = B_1 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \sim \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

Chose $B_1 = 1$

- $\vec{v}_2$ & $\vec{v}_3$ follow similarly...

★ If $\frac{d\vec{x}}{dt} = \underline{\underline{A}}\,\vec{x}$ and $\underline{\underline{A}}$ has distinct eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ with eigenvectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$, then the general solution is given by:

$$\vec{x}(t) = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t} + \dots + c_n \vec{v}_n e^{\lambda_n t}$$

Example:

$$\frac{dx}{dt} = 2x - 3y \, , \quad \frac{dy}{dt} = x - 2y$$

Let $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\underline{\underline{A}} = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$

See Example 1

$$\Rightarrow \frac{d\vec{x}}{dt} = \underline{\underline{A}} \, \vec{x}$$

$\lambda_1 = 1$, $\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$; $\lambda_2 = -1$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\boxed{\begin{aligned} &\Rightarrow \vec{x}(t) = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} \\ &\Rightarrow \begin{aligned} x(t) &= 3c_1 e^{t} + c_2 e^{-t} \\ y(t) &= c_1 e^{t} + c_2 e^{-t} \end{aligned} \end{aligned}}$$