Math 133A, Week 15: Nonhomogeneous Systems

Section 1: Fundamental Matrices

Consider the problem of determining the particular solution for an *ensemble* of initial conditions. For instance, suppose we are considering the differential equation

$$\frac{dx}{dt} = -x + 3y$$
$$\frac{dy}{dt} = 3x - y$$

(1)

which we know has the general solution

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-4t} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}.$$

Suppose, however, that instead of determining the particular solution for a single initial condition $x(0) = x_0$, $y(0) = y_0$, we wish to determine an ensemble of particular solutions for a variety conditions. As things stand now, we are required to solve for C_1 and C_2 individually *each time* we wish to apply an initial condition. This could be a lot of work!

Fortunately, this will not be required, but we will have to make use of a little linear algebra in order to see how to get around the problem. First of all, we should recognize that we can write the solution in the equivalent form

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -e^{-4t} & e^{2t} \\ e^{-4t} & e^{2t} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}.$$

In other words, we can write the equation in the matrix form

$$\mathbf{x}(t) = \Psi(t)\mathbf{c}$$

(2)

where $\Psi(t)$ is the 2×2 matrix with the fundamental solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ along the columns.

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Our goal is to relate the undetermined coefficients $\mathbf{c} \in \mathbb{R}^2$ to the initial conditions $\mathbf{x}(0)$ in general. This equation tells us exactly how to do that! We have

$$\mathbf{x}(0) = \Psi(0)\mathbf{c} \implies \mathbf{c} = \Psi(0)^{-1}\mathbf{x}(0) = \Psi^{-1}(0)\mathbf{x}_0 \tag{3}$$

where $\Psi(0) \in \mathbb{R}^{2\times 2}$ is the matrix $\Psi(t)$ evaluated at zero and $\Psi^{-1}(0)$ is the inverse of this matrix. We can now plug (3) into (2) to get

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}_0 \tag{4}$$

where $\Phi(t) = \Psi(t)\Psi^{-1}(0) \in \mathbb{R}^{2\times 2}$. This is exactly what was wanted! Once we have determined the matrix $\Phi(t)$, we have a relationship which immediately gives the solution $\mathbf{x}(t)$ for an arbitrary initial vector $\mathbf{x}_0$.

Definition 1

Consider the initial value problem $\dot{\mathbf{x}} = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$. A matrix $\Psi(t) \in \mathbb{R}^{2\times 2}$ is called a **fundamental matrix** of the system if it satisfies $\mathbf{x}(t) = \Psi(t)\mathbf{c}$ where $\mathbf{c} \in \mathbb{R}^2$ and $\mathbf{x}(t)$ is a solution of the system. The matrix $\Phi(t)$ is the fundamental matrix with the property $\mathbf{x}(t) = \Phi(t)\mathbf{x}_0$.

Note: The fundamental matrix $\Phi(t)$ has the following properties: (1) $\Phi(0) = I$; and (2) $\Phi'(t) = A\Phi(t)$. The first property is an important check for the correctness of our results. The second property follows from the observation that each column is a solution of $\dot{\mathbf{x}} = A\mathbf{x}$.

Example 1

Determine the fundamental matrix $\Phi(t)$ for the system (1).

Solution: We have

$$\Psi(t) = \begin{bmatrix} -e^{-4t} & e^{2t} \\ e^{-4t} & e^{2t} \end{bmatrix} \implies \Psi(0) = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

which implies that

$$\Psi^{-1}(0) = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

It follows that

$$\Phi(t) = \Psi(t)\Psi^{-1}(0) = \frac{1}{2} \begin{bmatrix} -e^{-4t} & e^{2t} \\ e^{-4t} & e^{2t} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} e^{-4t} + e^{2t} & -e^{-4t} + e^{2t} \\ -e^{-4t} + e^{2t} & e^{-4t} + e^{2t} \end{bmatrix}$$

and therefore the solution can be written in the form $\mathbf{x}(t) = \Phi(t)\mathbf{x}_0$ as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{-4t} + e^{2t} & -e^{-4t} + e^{2t} \\ -e^{-4t} + e^{2t} & e^{-4t} + e^{2t} \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

We can easily check that $\Phi(0) = I$ by computing

$$\Phi(0) = \frac{1}{2} \begin{bmatrix} e^{-4(0)} + e^{2(0)} & -e^{-4(0)} + e^{2(0)} \\ -e^{-4(0)} + e^{2(0)} & e^{-4(0)} + e^{2(0)} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

As expected, the result relates the solutions directly to the initial conditions. We no longer have to compute the constants C_1 and C_2 at every iteration!

Example 2

Determine the fundamental matrix $\Phi(t) \in \mathbb{R}^{2\times 2}$ for the system of differential equations

$$\frac{dx}{dt} = x - 4y$$

$$\frac{dy}{dt} = x - 3y.$$

Solution: We determined last week that this system of differential equations has the general solution

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \left(C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \left(t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \right) e^{-t}.$$

We now want to determine the fundamental matrix $\Phi(t) = \Psi(t)\Psi^{-1}(0)$ where $\Psi(t)$ is the matrix with the fundamental solutions above along the columns. We will have to be a little careful when establishing our fundamental solutions—they are component-wise coefficients of the un-determined constants C_1 and C_2. In this case, we have that

$$\Psi(t) = \begin{bmatrix} 2e^{-t} & e^{-t} + 2te^{-t} \\ e^{-t} & te^{-t} \end{bmatrix}.$$

It follows that

$$\Psi(0) = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \implies \Psi^{-1}(0) = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}.$$

Consequently, we have

$$\Phi(t) = \Psi(t)\Psi^{-1}(0) = \begin{bmatrix} 2e^{-t} & e^{-t} + 2te^{-t} \\ e^{-t} & te^{-t} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} + 2te^{-t} & -4te^{-t} \\ te^{-t} & e^{-t} - 2te^{-t} \end{bmatrix}.$$

Section 2: Matrix Exponentials

The fundamental matrix turns out to be very important when constructing solutions of more complicated linear system. In order to take this step forward, however, we will first have to take a step backward. Reconsider the general linear and homogeneous system

$$\mathbf{x}'(t) = A\mathbf{x}(t). \tag{5}$$

If we carry through with our earlier analogy with a single first order equation of the form $x'(t) = ax(t)$, we would recognize the general form of the solution as $x(t) = x_0 e^{at}$. So perhaps there is some sense in which we could write

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0$$

as the solution for our system (5). (Note that this is very close to the solution $\mathbf{x}(t) = \Phi(t)\mathbf{x}_0$ we just discovered!)

The question then becomes: how might we define e^{At}? It should be clear that we cannot simply take the exponential of each term in the matrix

as this does not generally lead a function $\mathbf{x}(t)$ which is a solution of (5). Instead, we recall that the exponential e^{at} can be expanded as the Taylor series

$$e^{at} = \sum_{n=0}^{\infty} \frac{(at)^n}{n!} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \cdots$$

This suggests that, instead of defining the matrix exponent e^{At} directly, we define it as the infinite series

$$e^{At} = \sum_{n=0}^{\infty} \frac{1}{n!} A^n t^n = I + At + \frac{1}{2!} A^2 t^2 + \frac{1}{3!} A^3 t^3 + \cdots \tag{6}$$

(Notice here that $A^2 = A \cdot A$, $A^3 = A \cdot A \cdot A$, etc.)

We should have some healthy skepticism about this formula. After all, it is an infinite sum, which we have no hope of computing explicitly. The best we might hope for is some result regarding convergence. But before we give up hope on this interpretation entirely, consider the following observation: *It is a solution of the system of differential equation!* To check this, we can simply evaluate. We clearly have

$$\mathbf{x}(0) = e^{A(0)} \mathbf{x}_0 = (I + A(0) + \frac{1}{2!} A^2(0)^2 + \cdots)\mathbf{x}_0 = \mathbf{x}_0$$

so that it satisfies the initial condition. On the left-hand side of the system, we have

$$\frac{d}{dt} \left(e^{At} \mathbf{x}_0 \right) = \frac{d}{dt} \left(I + At + \frac{1}{2!} A^2 t^2 + \frac{1}{3!} A^3 t^3 + \cdots \right) \mathbf{x}_0$$
$$= \left(A + A^2 t + \frac{1}{2!} A^3 t^2 + \cdots \right) \mathbf{x}_0$$

and on the right-hand side we have

$$A \left(e^{At} \mathbf{x}_0 \right) = A \left(I + At + \frac{1}{2!} A^2 t^2 + \frac{1}{3!} A^3 t^3 + \cdots \right) \mathbf{x}_0$$
$$= \left(A + A^2 t + \frac{1}{2!} A^3 t^2 + \cdots \right) \mathbf{x}_0$$

so it satisfies the differential equation. No matter how uncomfortable (6) may seem, we cannot escape the implication that it is meaningful! (Whether it is *useful* is another discussion entirely.)

Now recall our earlier discussion, where we defined the fundamental matrix $\Phi(t) = \Psi(t)\Psi^{-1}(0)$ and noted that the function $\mathbf{x}(t) = \Phi(t)\mathbf{x}_0$ was a

solution of our initial value problem. In other words, we really have *two* functions which we are claiming are solutions of the initial value problem

$$\mathbf{x}'(t) = A\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{7}$$

specifically, $\mathbf{x}(t) = e^{At}\mathbf{x}_0$ and $\mathbf{x}(t) = \Phi(t)\mathbf{x}_0$. But how can this be? How can an initial value problem have two solutions? The answer is that it *can't*. It is a well-known fact that the initial value problem (7) always has a *unique* solution. We are inescapably drawn to conclude that

$$e^{At} = \Phi(t). \tag{8}$$

That is the matrix exponential coincides with the fundamental matrix $\Phi(t)$ with the property $\Phi(0) = I$.

> This equation should be surprising! What it will allow us to do is interpret the fundamental matrix $\Phi(t)$ in two ways: a solution of a linear system of differential equations (right-hand side of (8)), or as a matrix exponential with many of the properties and intuitions which come from the standard exponential function (left-hand side of (8)).

Example 3

Determine the matrix exponential e^{At} for the matrix

$$A = \begin{bmatrix} -1 & 5 \\ -2 & 1 \end{bmatrix}.$$

Solution: Without the preceding motivation, we might be a little lost. By the definition, we have

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

which, beyond a few terms, we have no hope of computing explicitly. Nevertheless, this *is* the definition—we are now allowed to take the exponential of A component-wise as this will produce an incorrect result.

What we realize given the previous discussion is that $e^{At} = \Phi(t)$ where $\Phi(t)$ is the particular fundamental solution of the differential

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equation

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

with the property $\Phi(0) = I$. In other words, all we need to do is find $\Phi(t)$ for the first-order system

$$\frac{dx}{dt} = -x + 5y$$
$$\frac{dy}{dt} = -2x + y.$$

We found last week that this system had the general solution

$$\mathbf{x}(t) = C_1 \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cos(3t) - \begin{bmatrix} -3 \\ 0 \end{bmatrix} \sin(3t) \right)$$
$$+ C_2 \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \sin(3t) + \begin{bmatrix} -3 \\ 0 \end{bmatrix} \cos(3t) \right).$$

The fundamental matrix $\Psi(t)$ is therefore given by

$$\Psi(t) = \begin{bmatrix} \cos(3t) + 3\sin(3t) & -3\cos(3t) + \sin(3t) \\ 2\cos(3t) & 2\sin(3t) \end{bmatrix}$$

so that

$$\Psi(0) = \begin{bmatrix} 1 & -3 \\ 2 & 0 \end{bmatrix} \implies \Psi(0)^{-1} = \frac{1}{6}\begin{bmatrix} 0 & 3 \\ -2 & 1 \end{bmatrix}.$$

It follows that

$$e^{At} = \Phi(t) = \frac{1}{6}\begin{bmatrix} \cos(3t) + 3\sin(3t) & -3\cos(3t) + \sin(3t) \\ 2\cos(3t) & 2\sin(3t) \end{bmatrix}\begin{bmatrix} 0 & 3 \\ -2 & 1 \end{bmatrix}$$
$$= \frac{1}{6}\begin{bmatrix} 6\cos(3t) - 2\sin(3t) & 10\sin(3t) \\ -4\sin(3t) & 6\cos(3t) + 2\sin(3t) \end{bmatrix}$$
$$= \begin{bmatrix} \cos(3t) - \frac{1}{3}\sin(3t) & \frac{5}{3}\sin(3t) \\ -\frac{2}{3}\sin(3t) & \cos(3t) + \frac{1}{3}\sin(3t) \end{bmatrix}$$

Notice that, as expected, we have $e^{A(0)} = \Phi(0) = I$. We have accomplished our task, without having to ever directly consider properties of the matrix exponential e^{At}!

Section 3: Nonhomogeneous Systems

Consider being asked to solve the differential equation

$$\frac{dx}{dt} = 1 - x$$
$$\frac{dy}{dt} = 1 + x - 2y \tag{9}$$

by using the matrix algebra methods we have been employing over the past two weeks.

We can rewrite the left-hand side as a vector derivative $\mathbf{x}'(t) = (x'(t), y'(t))$ and collect most of the right-hand side as

$$A\mathbf{x} = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

This, however, does not incorporate all of terms in the equation since the 1's do not fit into either of these forms. The best we can do is write them as a separate vector, so that we have

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

This is a new class of systems of differential equations, which we now formally define.

Definition 2

A **linear nonhomogeneous system of differential equations** is one which can be written in the form

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{g}(t) \tag{10}$$

where $A \in \mathbb{R}^{2\times 2}$ is a matrix of constants and $\mathbf{g}(t) \in \mathbb{R}^2$ may depend upon t.

To motivate how we might solve such an equation as (10), consider one of the first integration techniques we learned for first-order differential equations. When we considered

$$\frac{dx}{dt} = ax(t) + g(t)$$

we recognized that we could re-arrange the expression as

$$\frac{dx}{dt} - ax(t) = g(t)$$

and then obtain the integrating factor

$$\mu(t) = e^{\int -a\,dt} = e^{-at}$$

so that we could write

$$\frac{d}{dt}\left[e^{-at}x(t)\right] = e^{-at}g(t).$$

We could then integrate and solve for $x(t)$ to get

$$x(t) = Ce^{at} + e^{at}\int e^{-at}g(t)\,dt.$$

There is nothing stopping us from attempting this technique with our linear system (10)! We may write the expression as

$$\frac{d\mathbf{x}}{dt} - A\mathbf{x}(t) = \mathbf{g}(t)$$

and then obtain the integrating factor $\mu(t) = e^{-At}$. We then obtain

$$\frac{d}{dt}\left[e^{-At}\mathbf{x}(t)\right] = e^{-At}\mathbf{g}(t).$$

We can integrate and solve for $\mathbf{x}(t)$ to obtain

$$\mathbf{x}(t) = e^{At}\mathbf{c} + e^{At}\int e^{-At}\mathbf{g}(t)\,dt. \tag{11}$$

If we had encountered this solution before our previous discussion of matrix exponentials, we would be very distressed. We now know, however, that $e^{At} = \Phi(t)$ where $\Phi(t)$ is the fundamental matrix of the corresponding **homogeneous system**

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

which satisfies also $\Phi(0) = I$. We also know $\Phi(t)$ can be computed from an arbitrary fundamental matrix $\Psi(t)$ by the formula $\Phi(t) = \Psi(t)\Psi^{-1}(0)$.

We can take advantage of this intuition to rewrite (10) in a computable form. In order to do this, we will use the property $e^{-At} = [e^{At}]^{-1} = \Phi^{-1}(t)$ (true, but beyond the scope of this course), and substitute back from $\Phi(t)$ to an arbitrary fundamental matrix $\Psi(t)$. We arrive at the following formula:

$$\mathbf{x}(t) = \Psi(t)\mathbf{c} + \Psi(t) \int \Psi^{-1}(s)\mathbf{g}(s)\, ds \tag{12}$$

where $\mathbf{c} = (C_1, C_2)$.

Example 4

Use the system formulation to solve the following initial value problem:

$$\begin{aligned} \frac{dx}{dt} &= 1 - x, & x(0) &= 0 \\ \frac{dy}{dt} &= 1 + x - 2y, & y(0) &= 1. \end{aligned} \tag{13}$$

Solution: We start by solving the linear system of differential equations $\mathbf{x}'(t) = A\mathbf{x}(t)$ with the matrix

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}.$$

We can quickly compute that the characteristic polynomial is $(-1 - \lambda)(-2 - \lambda) = 0$ so that $\lambda_1 = -1$ and $\lambda_2 = -2$. For $\lambda_1 = -1$ we have the eigenvector $\mathbf{v}_1 = (1, 1)$ and for $\lambda_2 = -2$ we have the eigenvector $\mathbf{v}_2 = (0, 1)$. It follows that the general solution is

$$\mathbf{x}(t) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t}.$$

This gives the fundamental matrix

$$\Psi(t) = \begin{bmatrix} e^{-t} & 0 \\ e^{-t} & e^{-2t} \end{bmatrix}.$$

We quickly derive that

$$\Psi^{-1}(t) = \frac{1}{e^{-3t}} \begin{bmatrix} e^{-2t} & 0 \\ -e^{-t} & e^{-t} \end{bmatrix} = \begin{bmatrix} e^{t} & 0 \\ -e^{2t} & e^{2t} \end{bmatrix}$$

where $\det(\Psi(t)) = e^{-t}e^{-2t} - (0)e^{-t} = e^{-3t}$. We then set up

$$\int \Psi^{-1}(t)\mathbf{g}(t) \, dt = \int \begin{bmatrix} e^{t} & 0 \\ -e^{2t} & e^{2t} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \, dt$$

$$= \int \begin{bmatrix} e^{t} \\ 0 \end{bmatrix} \, dt$$

$$= \begin{bmatrix} e^{t} \\ 0 \end{bmatrix}$$

where we have ignored the constants because they already appear in the final form (12). It follows that we have

$$\Psi(t) \int \Psi^{-1}(t)\mathbf{g}(t) \, dt = \begin{bmatrix} e^{-t} & 0 \\ e^{-t} & e^{-2t} \end{bmatrix} \begin{bmatrix} e^{t} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The general solution is therefore

$$\mathbf{x}(t) = \Psi(t)\mathbf{c} + \Psi(t) \int \Psi^{-1}(t)\mathbf{g}(t) \, dt$$

$$= \begin{bmatrix} e^{-t} & 0 \\ e^{-t} & e^{-2t} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The initial conditions $x(0) = 0$ and $y(0) = 1$ give the system

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

It follows that we have

$$0 = C_1 + 1$$
$$1 = C_1 + C_2 + 1$$

so that $C_1 = -1$ and $C_2 = 1$. We therefore obtain the solution

$$x(t) = 1 - e^{-t}$$
$$y(t) = 1 - e^{-t} + e^{-2t}$$

as we had before. Despite the amount of work we have done, we should be encouraged that none of the individual steps were particularly challenging, and that the result obtained is consistent with our earlier work!

Example 5

Find the solution of the following initial value problem

$$\frac{dx}{dt} = -4x - 3y + e^{-t}, \qquad x(0) = 0$$
$$\frac{dy}{dt} = 3x + 2y + e^{-t}, \qquad y(0) = 0.$$

Solution: We start by rewriting the system as

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -4 & -3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix}.$$

We first solve the homogeneous equation $\mathbf{x}'(t) = A\mathbf{x}(t)$ where

$$A = \begin{bmatrix} -4 & -3 \\ 3 & 2 \end{bmatrix}.$$

We have the characteristic polynomial $(-4-\lambda)(2-\lambda)+9 = \lambda^2+2\lambda+1 = (\lambda+1)^2 = 0$ so that $\lambda = -1$. Since we have a repeated root, we need to find a regular and generalized eigenvector. We can see that $(A+I)\mathbf{v} = \mathbf{0}$ gives

$$\begin{bmatrix} -3 & -3 & | & 0 \\ 3 & 3 & | & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

so that $v_1 + v_2 = 0$. Setting $v_2 = 1$, we have $\mathbf{v} = (-1, 1)$. Since we have only obtained a single eigenvector, we must solve the equation

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$(A + I)\mathbf{w} = \mathbf{v}$. This gives

$$
\left[\begin{array}{cc|c} -3 & -3 & -1 \\ 3 & 3 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & \frac{1}{3} \\ 0 & 0 & 0 \end{array}\right].
$$

It follows that we have $w_1 + w_2 = \frac{1}{3}$. Setting $w_2 = 0$ we have $w_1 = \frac{1}{3}$ so we have $\mathbf{w} = (\frac{1}{3}, 0)$. We therefore have the general solution to the homogeneous system

$$
\mathbf{x}(t) = \left(C_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 \left(t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix} \right) \right) e^{-t}.
$$

It follows that the fundamental matrix $\Psi(t)$ is given by

$$
\Psi(t) = \begin{bmatrix} -e^{-t} & \frac{1}{3}e^{-t} - te^{-t} \\ e^{-t} & te^{-t} \end{bmatrix}
$$

so that

$$
\Psi^{-1}(t) = \frac{1}{-\frac{1}{3}e^{-2t}} \begin{bmatrix} te^{-t} & -\frac{1}{3}e^{-t} + te^{-t} \\ -e^{-t} & -e^{-t} \end{bmatrix}
$$
$$
= \begin{bmatrix} -3te^{t} & e^{t} - 3te^{t} \\ 3e^{t} & 3e^{t} \end{bmatrix}.
$$

It follows that we have

$$
\Psi^{-1}(t)\mathbf{g}(t) = \begin{bmatrix} -3te^{t} & e^{t} - 3te^{t} \\ 3e^{t} & 3e^{t} \end{bmatrix} \begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix}
$$
$$
= \begin{bmatrix} -6t + 1 \\ 6 \end{bmatrix}
$$

We therefore have

$$
\int \Psi(t)^{-1}\mathbf{g}(t) \, dt = \int \begin{bmatrix} -6t + 1 \\ 6 \end{bmatrix} dt = \begin{bmatrix} -3t^2 + t \\ 6t \end{bmatrix}
$$

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so that

$$\Psi(t)\int \Psi(s)^{-1}\mathbf{g}(s)\,ds = \begin{bmatrix} -e^{-t} & \frac{1}{3}e^{-t} - te^{-t} \\ e^{-t} & te^{-t} \end{bmatrix}\begin{bmatrix} -3t^2 + t \\ 6t \end{bmatrix}$$

$$= \begin{bmatrix} -3t^2 e^{-t} + te^{-t} \\ 3t^2 e^{-t} + te^{-t} \end{bmatrix}.$$

It follows that the general solution is given by

$$\mathbf{x}(t) = \Psi(t)\mathbf{c} + \Psi(t)\int \Psi^{-1}(s)\mathbf{g}(s)\,ds$$

$$\implies \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -e^{-t} & \frac{1}{3}e^{-t} - te^{-t} \\ e^{-t} & te^{-t} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} + \begin{bmatrix} -3t^2 e^{-t} + te^{-t} \\ 3t^2 e^{-t} + te^{-t} \end{bmatrix}.$$

The initial conditions $x(0) = 0$ and $y(0) = 0$ give

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & \frac{1}{3} \\ 1 & 0 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

It follows that $C_1 = C_2 = 0$. The particular solution is therefore

$$x(t) = -3t^2 e^{-t} + te^{-t}$$
$$y(t) = 3t^2 e^{-t} + te^{-t}.$$

Suggested Problems

1. Determine the fundamental matrix $\Phi(t)$ for the following systems of differential equations (all derivatives with respect to t):

   (a) $\begin{cases} x' = x - 4y \\ y' = -y \end{cases}$

   (c) $\begin{cases} x' = -x + 9y \\ y' = -x - 7y \end{cases}$

   (b) $\begin{cases} x' = 7x + 4y \\ y' = -6x - 3y \end{cases}$

   (d) $\begin{cases} x' = 3x - 5y \\ y' = x + y \end{cases}$

2. Use the system formulation to solve the following initial value problems:

$$\text{(a)} \begin{cases} \dfrac{dx}{dt} = 2 - 2x, & x(0) = 1 \\[2mm] \dfrac{dy}{dt} = x - y, & y(0) = 1. \end{cases}$$

$$\text{(b)} \begin{cases} \dfrac{dx}{dt} = -3x + e^t, & x(0) = \dfrac{1}{4} \\[2mm] \dfrac{dy}{dt} = -4x - y, & y(0) = -\dfrac{1}{2}. \end{cases}$$

$$\text{(c)} \begin{cases} \dfrac{dx}{dt} = -2x + 4y + 2, & x(0) = 0 \\[2mm] \dfrac{dy}{dt} = -x + 2y + 1, & y(0) = -1. \end{cases}$$

$$\text{(d)} \begin{cases} \dfrac{dx}{dt} = 3x - 2y + t, & x(0) = 0 \\[2mm] \dfrac{dy}{dt} = 5x - 3y, & y(0) = 0. \end{cases}$$

3. Consider the reversible chemical reaction $X \rightleftarrows Y$ with inflow $g(t)$ of X outflow of Y proportional to its concentration. Assume this reaction system can be modeled by the following system of differential equations:

$$\begin{cases} \dfrac{dx}{dt} = 3 - 4x + y \\[2mm] \dfrac{dy}{dt} = 4x - 4y. \end{cases} \tag{14}$$

Determine the solution of the system for the initial conditions $x(0) = 1$ and $y(0) = 1$. Interpret the solution in terms of the chemical system described.