Lecture 8

Numerical Methods
(Part 2)

Upcoming Dates:

Sept. 21st - Review Day
☆ Sept. 23rd - Exam 1 ☆
Sept. 30th - Homework #4 Due

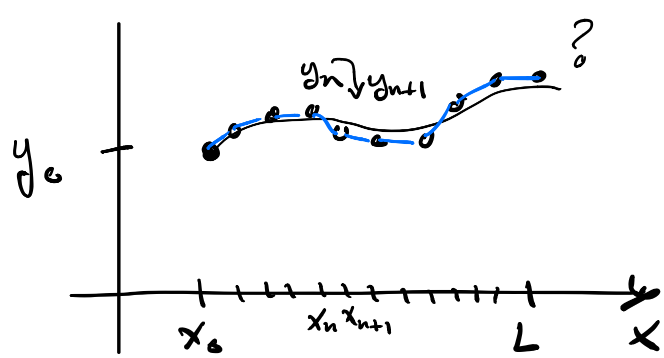
Exam is on HW 1-3

Review

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$

Euler's Method
(Forward Euler)

$$\boxed{y_{n+1} = y_n + h f(x_n, y_n)}$$



Main Example: $\frac{dy}{dx} = -y \, , \; y(0) = 1$

$x_0 = 0 \, , \; y_0 = 1 \, , \; f(x, y) = -y$

(Exact Solution: $\phi(x) = e^{-x}$)

Euler: $$\frac{y(x+h) - y(x)}{h} \approx f(x, y)$$

Alternatively: $$\frac{y(x) - y(x-h)}{h} \approx f(x, y(x))$$

$$\Rightarrow \quad \frac{y(x+h) - y(x)}{h} \approx f(x+h, y(x+h))$$

Let $x_{n+1} = x_n + h \, , \; y_n = y(x_n)$

$$\Rightarrow \quad \frac{y_{n+1} - y_n}{h} = f(x_{n+1}, y_{n+1})$$

OR $$\boxed{y_{n+1} = y_n + h f(x_{n+1}, y_{n+1})}$$

"implicit algorithm"

Known as "Backward Euler Method"

<u>Revisit Example</u> $\dfrac{dy}{dx} = -y \quad , \quad y(0) = 1$

Exact solution: $\phi(x) = e^{-x}$

$$y_{n+1} = y_n + h f(x_{n+1}, y_{n+1})$$

$$y_{n+1} = y_n + h(-y_{n+1})$$

$$= y_n - h y_{n+1}$$

$$y_{n+1} + h y_{n+1} = y_n$$

$$\Rightarrow (1+h) y_{n+1} = y_n$$

Backward Euler

$$\boxed{y_{n+1} = \frac{y_n}{1+h}}$$

Reminder:

Euler $\rightarrow$ $y_{n+1} = (1-h)y_n$

What is $y(0.1)$?

$x_0 = 0$, $y_0 = 1$, $h = 0.1$

$$y(0.1) = y_1 \approx \frac{y_0}{1+h} = \frac{1}{1+0.1} \approx \underline{\underline{0.9091}}$$

over

Euler: $y(0.1) \approx (1-h)y_0 = (0.9)\cdot 1 = \underline{\underline{0.9}}$

under

Exact Answer $= \phi(0.1) = e^{-0.1} \approx 0.9048$

What is $y(1)$?

$$y(0.1) = y_1 \approx \frac{y_0}{1+h}$$

$$y(0.2) = y_2 \approx \frac{y_1}{1+h} = \frac{y_0}{(1+h)^2}$$

$$y(0.3) = y_3 \approx \frac{y_2}{1+h} = \frac{y_0}{(1+h)^3}$$

$$\vdots$$

$$y(1) = y_{10} \approx \frac{y_9}{1+h} = \frac{y_0}{(1+h)^{10}}$$

$$y_0 = 1,\ h = 0.1 \Rightarrow y(1) \approx \frac{1}{(1.1)^{10}}$$

If $h = 0.01$

$$y_{100} = y(1) = \frac{1}{(1.01)^{100}}$$

For better accuracy, what if we took the average??

(forward) Euler: $y_{n+1} = y_n + h\,f(x_n, y_n)$

$+$ Backward Euler: $y_{n+1} = y_n + h\,f(x_{n+1}, y_{n+1})$

divide by 2

$$\Rightarrow \boxed{y_{n+1} = y_n + \frac{h}{2}\Big(f(x_n, y_n) + f(x_{n+1}, y_{n+1})\Big)}$$

Trapezoid Method

(a.k.a. semi-implicit Euler Method)

<u>Revisit Example</u> $\frac{dy}{dx} = -y$, $y(0) = 1$

$x_0 = 0$, $y_0 = 1$, $f(x, y) = -y$

$$y_{n+1} = y_n + \frac{h}{2}\left[(-y_n) + (-y_{n+1})\right]$$

$$\Rightarrow y_{n+1} = y_n - \frac{h}{2} y_n - \frac{h}{2} y_{n+1}$$

$$\left(1 + \frac{h}{2}\right) y_{n+1} = \left(1 - \frac{h}{2}\right) y_n$$

$$\Rightarrow \boxed{y_{n+1} = \frac{1 - h/2}{1 + h/2} y_n}$$

trapezoid method

Set $x = 0$, $h = 0.1$

$$y(0.1) \approx \frac{1 - \frac{1}{2}(0.1)}{1 + \frac{1}{2}(0.1)} y(0) = \frac{0.95}{1.05}(1)$$

$$\approx 0.90476$$

Exact answer: $\phi(0.1) = e^{-0.1} \approx 0.9048$

Not bad!

Example 2

$$\frac{dy}{dx} = \sin(xy), \quad y(0) = \pi$$

How would we set up the trapezoid method

$$\rightarrow \quad f(x,y) = \sin(xy)$$

Problem

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}) \right]$$

$$= y_n + \frac{h}{2}\left[\sin(x_n y_n) + \sin(x_{n+1} y_{n+1}) \right]$$

How do we solve for y_{n+1} !?

- Euler = easy to implement, inaccurate
  (explicit)

- Trapezoid = hard to implement, accurate
  (implicit)

Trick: Use Euler method to guess y_{n+1}

$$\tilde{y} = y_n + h f(x_n, y_n)$$

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, \tilde{y}) \right]$$

"Improved Euler Method"

Example 1 (again) $\frac{dy}{dx} = -y, \quad y(0) = 1$

Set $x_0 = 0, \quad y_0 = 1, \quad h = 0.1$

$f(x,y) = -y$

$$\tilde{y} = y_0 - h y_0 = 1 - (0.1)(1) = 0.9$$

$$y_1 = y_0 + \frac{h}{2}\left[(-y_0) + (-\tilde{y}) \right]$$

$$\Rightarrow \quad y(0.1) = y_1 = 1 + \frac{0.1}{2}\left[(-1) + (-0.9)\right]$$

$$= 1 - 0.095 = 0.905$$

Answer: $e^{-0.1} \approx 0.9048$

Summary

1) Euler : inaccurate but easy (explicit)

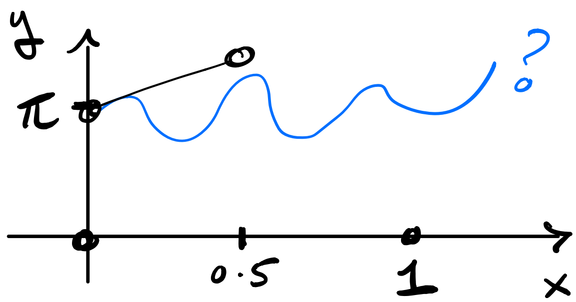
2) Trapezoid : accurate but difficult (implicit)

3) Improved Euler : accurate <u>and</u> easy (explicit)

Example 2 (again)

$$\frac{dy}{dx} = \sin(xy), \quad y(0) = \pi$$

What is $y(1)$?



Set $h = 0.5$

$f(x,y) = \sin(xy)$, $x_0 = 0$, $y_0 = \pi$

$x_1 = 0 + h = 0.5$

$y_1 \simeq y(0.5)$

$$\tilde{y} = y_0 + 0.5 \sin(x_0 y_0)$$

$$= \pi + \cancel{0.5 \sin(0 \cdot \pi)} = \pi$$

$$y_1 = y_0 + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, \tilde{y}_1)\right]$$

$$= y_0 + \frac{h}{2}\left[\sin(x_0 y_0) + \sin(x_1 \tilde{y})\right]$$

$$= \pi + \frac{1}{4}\left[\underbrace{\sin(0 \cdot \pi)}_{=0} + \underbrace{\sin(0.5\pi)}_{=1}\right]$$

$$= \pi + \frac{1}{4}(1) = \pi + \frac{1}{4}$$

Next step: $x_2 = x_1 + h = 0.5 + 0.5 = 1$

$$y_2 \approx y(1)$$

Guess: $\tilde{y} = y_1 + h\, f(x_1, y_1)$

$$= y_1 + h \sin(x_1 y_1)$$

$$= \left(\pi + \tfrac{1}{4}\right) + 0.5 \sin\left(0.5\left(\pi + \tfrac{1}{4}\right)\right)$$

$$\approx 3.88769\ldots$$

$$y_2 = y_1 + \frac{1}{4}\left[\sin\left(\frac{\pi}{2}+\frac{1}{8}\right) + \sin(3.88769)\right]$$

$$\approx 3.966$$

(Exact answer = 3.643...)

☆ Make sure your calculator is in <u>radians</u>

☆ Don't round answers until the end