Lecture 5

Review: Separable ODEs

$$\frac{dy}{dx} = g(x)\,p(y)$$

1) Separate variables $\qquad \frac{1}{p(y)}\,dy = g(x)\,dx$

2) Integrate $\qquad \int \frac{1}{p(y)}\,dy = \int g(x)\,dx$

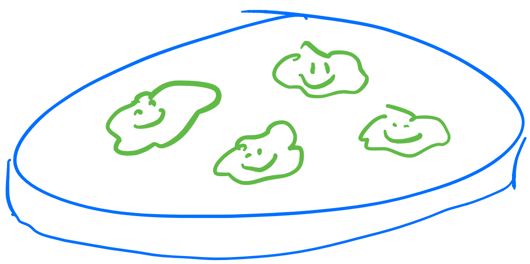
3) Solve for y (if you can)

ex. $\qquad y^2 + e^y + \cos(x) = c$

↑ can't solve for y
(just leave it this way)

"implicit equation"

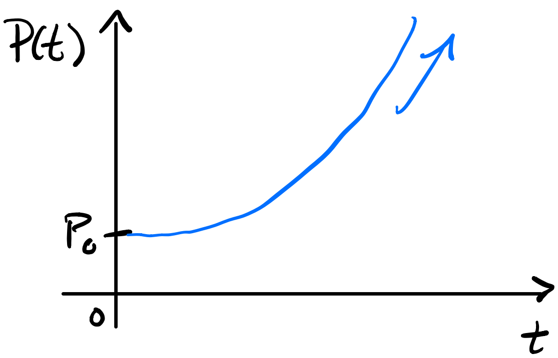
Population Models & Stability Analysis



In general:

$$\frac{dP}{dt} = rP, \quad P(0) = P_0$$

$$\Rightarrow P(t) = P_0 e^{rt}$$



Exponential growth can't go on forever, so how do we fix this model?

Given a general form: $\dfrac{dP}{dt} = f(t, P)$

$f(t,P) > 0 \Rightarrow P(t)$ grows

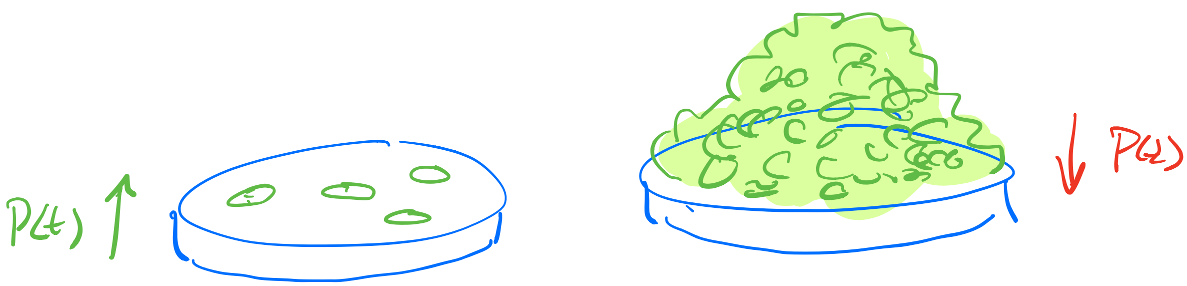
$f(t,P) < 0 \Rightarrow P(t)$ decays

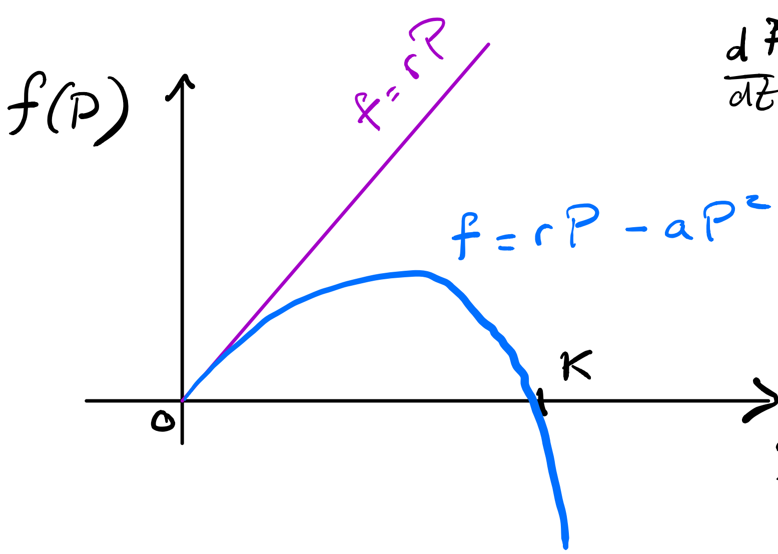
Example: $\frac{dP}{dt} = rP \Rightarrow f = rP$

Since $P > 0$: $r > 0 \Rightarrow P$ grows

$r < 0 \Rightarrow P$ decays

We need an equation, where P grows in some range $0 < P < K$ and decays if $P > K$





$\frac{dP}{dt} = f(P)$

$$\Rightarrow \quad f(P) = rP - aP^2$$
$$= rP\left(1 - \frac{P}{K}\right)$$

Logistic Equation: $\boxed{\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)}$

r = growth rate

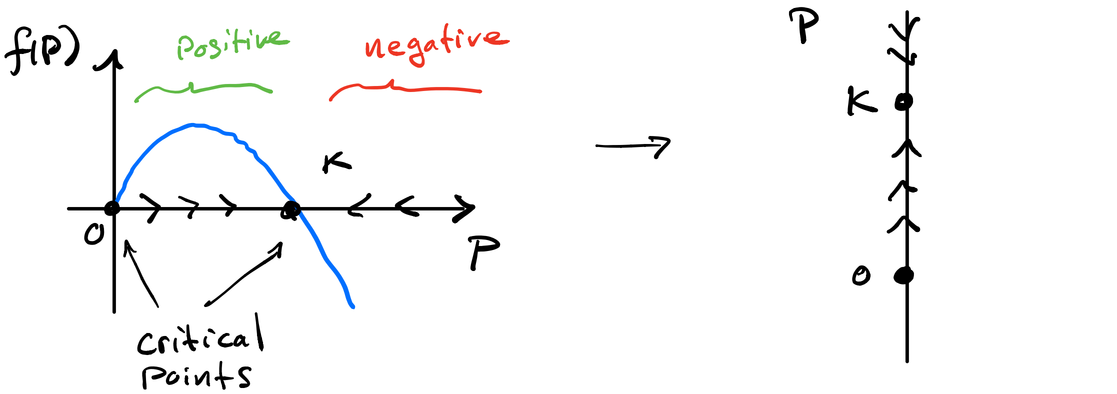
K = max capacity (Kapazität)

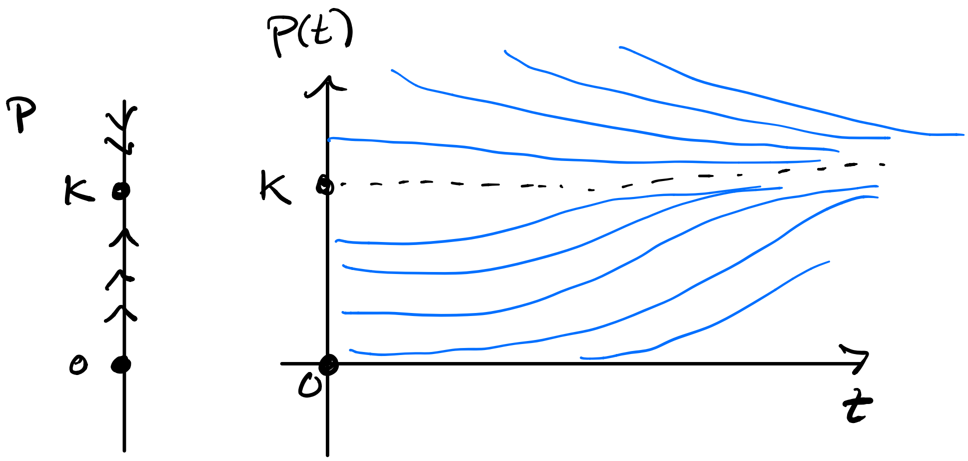
The long-term behavior of the solutions can be investigated by looking at "equilibrium" solutions: $\frac{dP}{dt} = 0$

$$0 = rP\left(1 - \frac{P}{K}\right) \Rightarrow P = 0 \text{ or } K$$

★ These equilibrium solutions are known as "critical points".

★ Solutions will usually tend towards or away from critical points.



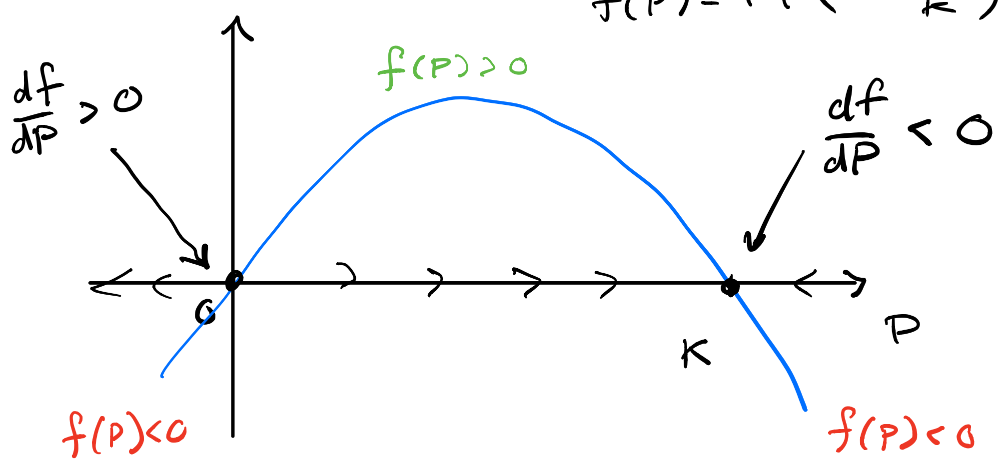


$P = K$ is a "stable" point

$P = 0$ is an "unstable" point

How could we tell if critical points are stable/unstable without solving the ODE?

$$f(P) = rP\left(1 - \frac{P}{K}\right)$$



Stable Point

$f > 0$ to left

$f < 0$ to right

$\Rightarrow \frac{df}{dP} < 0$ at that point

Unstable Point

$f < 0$ to left

$f > 0$ to right

$\Rightarrow \frac{df}{dP} > 0$ at that point

Example: $f(P) = rP\left(1 - \frac{P}{K}\right)$

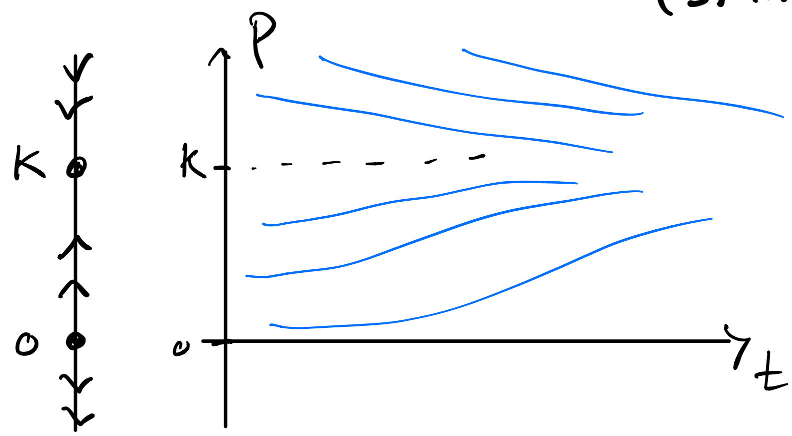
$$\frac{df}{dP} = r - \frac{2r}{K}P$$

Two critical points: $P = \{0, K\}$

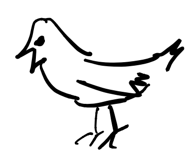
$$\frac{df}{dP}(0) = r - \cancel{\frac{2r}{K}}(0) = r > 0 \quad \text{(unstable)}$$

$$\frac{df}{dP}(K) = r - \frac{2r}{K}K = r - 2r = -r < 0$$

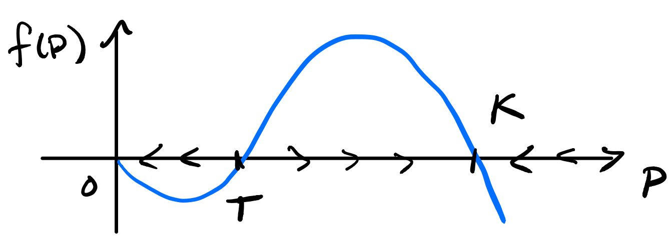
(stable)



Example: Passenger Pigeons



$$\frac{dP}{dt} = P\left(1 - \frac{P}{20}\right)\left(\frac{P}{100} - 1\right) = f(P)$$



1) Find the critical points

$$f(P) = P\left(1 - \frac{P}{20}\right)\left(\frac{P}{100} - 1\right) = 0$$

what is P?

$$P = 0 \quad \text{or} \quad \left(1 - \frac{P}{20}\right) = 0 \Rightarrow P = 20$$

$$\text{or} \quad \left(\frac{P}{100} - 1\right) = 0 \Rightarrow P = 100$$

3 critical points: $P_c = \{\underline{0}, \underline{20}, \underline{100}\}$

2) Determine stability: look at the sign of $\frac{df}{dP}$ at P_c

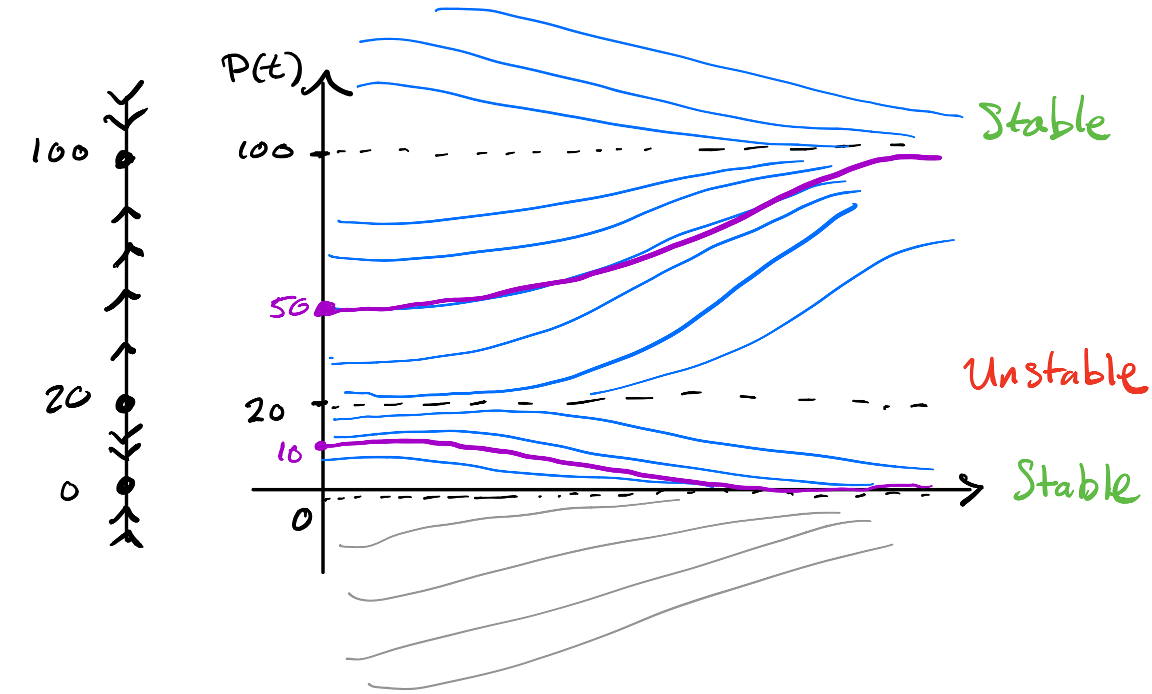
Product Rule

$$\frac{df}{dP} = \left(1 - \frac{P}{20}\right)\left(\frac{P}{100} - 1\right) - \frac{P}{20}\left(\frac{P}{100} - 1\right) + \frac{P}{100}\left(1 - \frac{P}{20}\right)$$

$$\frac{df}{dP}(0) = (1)(-1) - 0 + 0 = -1 < 0 \qquad \text{Stable}$$

$$\frac{df}{dP}(20) = 0 - \left(\frac{1}{5} - 1\right) + 0 = 4/5 > 0 \qquad \text{Unstable}$$

$$\frac{df}{dP}(100) = 0 - 0 + (1 - 5) = -4 < 0 \qquad \text{Stable}$$



A) If $P(0) = 50$, what happens as $t \to \infty$?

Answer: $\lim_{t \to \infty} P(t) = 100$

B) If $P(0) = 10$, what happens as $t \to \infty$?

Answer: $\lim_{t \to \infty} P(t) = 0$

(If $P(0) = 20 \Rightarrow \lim_{t \to \infty} P(t) = 20$)