Math 133A, Week 4: Substitution Methods, Exact DEs

Section 1: Substitution Methods

Most first-order differential equations, even those with simple solutions, do not fall directly within the two classes we have studied so far (separable and first-order linear). Consider the following example.

Example 1

Find the general solution of

$$y' = \frac{x^2 + y^2}{2xy}.$$

Solution: This differential equation is neither separable nor first-order linear (check!). In order to solve it, we will take a page from our Calculus courses. Recall that very few integrals naturally appeared in a form where they could be integrated directly, rather, we had to first perform a **variable substitution**.

The study of differential equations is no different! What we will do is introduce a new variable $v(x, y)$ and rewrite the DE in x and y as a DE in terms of v and x. For this example, consider the variable substitution

$$v(x, y) = \frac{y}{x}.$$

(We do not need to know why we chose this substitution—yet.) We can now transform the differential equation by using the product rule

$$y = xv \quad \Longrightarrow \quad \frac{dy}{dx} = x\frac{dv}{dx} + v.$$

The differential equation can be rewritten in terms of v and x as

$$2x(xv)\left(x\frac{dv}{dx}+v\right)=x^2+(xv)^2$$

$$\implies \quad 2x^3v\frac{dv}{dx}=x^2+x^2v^2-2x^2v^2$$

$$\implies \quad 2x^3v\frac{dv}{dx}=x^2(1-v^2)$$

$$\implies \quad \frac{2v}{1-v^2}\,dv=\frac{1}{x}\,dx.$$

After looking at this equation for a moment, we realize something amazing has happened—this DE is *separable* even though the original DE was not. We can integrate to obtain

$$\int\frac{2v}{1-v^2}\,dv=\int\frac{1}{x}\,dx$$

$$\implies -\ln(1-v^2)=\ln(x)+C,$$

$$\implies 1-v^2=\frac{\tilde{C}}{x}.$$

We can now to return to the original variables x and y. We started with $v=y/x$, so we now have

$$1-\left(\frac{y}{x}\right)^2=\frac{\tilde{C}}{x}$$

$$\implies \quad x^2-y^2=\tilde{C}x.$$

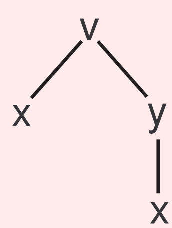
This yields the final solution

$$y=\pm\sqrt{x^2-\tilde{C}x}.$$

This was a little work, but the end result is very satisfying. The real question is how we knew to use the substitution $v=y/x$. In fact, this DE belongs to a general class of systems known as **(power) homogeneous equations** for which this substitution will *always* yield a separable equation in the variables v and x. We will also consider a general class of systems known as **Bernoulli equations** for which an appropriate variable substitution always yields a first-order linear DE.

Note: It is hoped that it is the *method* which is memorized, not the end formula. In other words, remember the required variable substitutions for the types of equations rather than the final expression for the solution.

Note: As with any problem involving layers of variable dependences, it is helpful to write out the tree of variable dependences. For the differential equations we are looking at, where we are looking for a function $y = y(x)$ (i.e. y as a function of x) and using a variable transformation $v = v(x, y)$, we have the tree



which gives the following derivative (by the multivariate chain rule):

$$\frac{dv}{dx} = \frac{\partial v}{\partial y}\frac{dy}{dx} + \frac{\partial v}{\partial x}.$$

Section 2: (Power) Homogeneous DEs

We start with the following definition.

Definition 1

A first-order differential equation is called **(power) homogeneous** if it has the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right).$$

A substitution of the form $v = \dfrac{y}{x}$ produces a *separable* differential equation in v and x of the form

$$x\frac{dv}{dx} = F(v) - v.$$

Note: (Power) homogeneous equations are more easily recognized by the observation that the combined powers of x and y in each term (i.e. the sum of the powers) must be the same for all terms. For instance

$$y' = \frac{x^{1/2}y^{3/2} + y^2 + xy}{x^{3/2}y^{1/2} + x^2}$$

is (power) homogeneous because every term has the effective power of 2. Note that this must hold for both the numerator and the denominator in the RHS. For example, $y' = x^2 + y^2$ is *not* (power) homogeneous because the denominator (not shown) has the effective power of zero.

Note: Within the study of differential equations there are *two* accepted definitions of what constitutes a homogeneous differential equation, and these definitions are very different. Usually context will dictate which meaning is implied, but just to be clear we will use *power* homogeneous to refer to the class of differential equations we just introduced.

Example 2

Determine the general solution of $\quad y' = \dfrac{y + 2\sqrt{xy + x^2}}{x}.$

Solution: We can quickly check that this DE is neither separable nor first-order linear. We can, however, observe that the effective power of each term is one. This is obvious for the terms y and x, but it also holds for the terms xy and x^2 under the square root. If we prefer, we can divide x through the RHS explicit to obtain

$$y' = \frac{y}{x} + 2\sqrt{\frac{y}{x} + 1} = F\left(\frac{y}{x}\right)$$

where $F(v) = v + 2\sqrt{v+1}$.

We may now solve the DE with the substitution $v = y/x$. This gives $y = xv$ so that $y' = v + xv'$. It follows that we have

$$\begin{aligned} & v + xv' = v + 2\sqrt{v+1} \\ \implies & xv' = 2\sqrt{v+1} \\ \implies & \int \frac{1}{2\sqrt{v+1}}\, dv = \int \frac{1}{x}\, dx \\ \implies & \sqrt{v+1} = \ln(x) + C. \end{aligned}$$

We now return to our original variables y and x. We have

$$\sqrt{\frac{y}{x} + 1} = \ln(x) + C$$

so that

$$y(x) = x\left[(\ln(x) + C)^2 - 1\right].$$

Section 3: Bernoulli DEs

We now consider the following class of systems.

Definition 2

A first-order differential equation is **Bernoulli** if it has the form

$$y' + P(x)y = Q(x)y^n$$

for $n \neq 0, 1$. A substitution of the form $v = y^{1-n}$ produces a *first-order linear* differential equation in v and x of the form

$$v' + (1-n)P(x)v = (1-n)Q(x).$$

Bernoulli equations are close to first-order linear in the sense that, if we rearrange the equation to the standard form for first-linear linear DEs, the only difference is a single additional term which is a power of y. Rearranging to standard form will be our typical method for determining whether a DE in of Bernoulli form.

Note: It is worth noting that the substitution $v = y^{1-n}$ holds for *all* values of n other than $n = 0$ and $n = 1$. That is to say, we can consider fractional powers (e.g. $n = 1/2$, $n = 7/5$, $n = 92/13$, etc.) and negative powers ($n = -3$, $n = -4/9$, $n = -103$, etc.). The values $n = 0$ and $n = 1$ are degenerate for the substitution, but correspond to trivial cases for the original DE since it is already first-order linear.

Example 3

Determine the solution of

$$3xy^2y' = 3x^4 + y^3.$$

Solution: It should again not take much argument to convince ourselves that this equation is not separable, is not first-order linear, and is not even power homogeneous (although it is close). To check whether it is Bernoulli, we rearrange this equation to get it as close to the first-order linear form as possible. We have

$$y' = x^3y^{-2} + \frac{1}{3x}y \quad \Longrightarrow \quad y' - \frac{1}{3x}y = x^3y^{-2}.$$

The only term which distinguishes this from first-order linear is the one on the right-hand side. We are not happy with the y^{-2} and would like to make it go away.

We use the substitution $v = y^{1-n} = y^3$ where $n = -2$. We want to rewrite this differential equation in y and x as a differential equation in v and x. We have

$$y = v^{1/3} \quad \Longrightarrow \quad \frac{dy}{dx} = \left(\frac{1}{3}v^{-2/3}\right)\frac{dv}{dx}$$

and $y^{-2} = v^{-2/3}$. It follows that the differential equation can be rewritten as

$$\left(\frac{1}{3}v^{-2/3}\right)v' - \frac{1}{3x}v^{1/3} = x^3 v^{-2/3}.$$

Multiplying across by $3v^{2/3}$ we arrive at

$$v' - \frac{1}{x}v = 3x^3.$$

Remarkably, the trouble term y^{-2} has disappeared. In fact, this is now a linear equation in v and x which we can solve. We have the integrating factor

$$\mu(x) = e^{-\int \frac{1}{x}\,dx} = e^{-\ln(x)} = \frac{1}{x}.$$

This gives us

$$\frac{1}{x}v' - \frac{1}{x^2}v = 3x^2 \quad \Longrightarrow \quad \frac{d}{dx}\left[\frac{1}{x}v\right] = 3x^2$$

$$\Longrightarrow \quad \frac{1}{x}v = \int 3x^2\,dx = x^3 + C \quad \Longrightarrow \quad v = x^4 + Cx.$$

We are not completely done. The original DE was with respect to y and x, so we need to change our solution back to these original variables. We have

$$y^3 = x^4 + Cx \quad \Longrightarrow \quad y(x) = \sqrt[3]{x^4 + Cx}.$$

Section 4: Exact DEs

The trick to solving first-order linear differential equations was to recognize that the *product rule* yields a form which looks like a first-order linear equation. Specifically, we had

$$\frac{d}{dx}\left[f(x)\,y\right] = f(x)y' + f'(x)y.$$

After rearranging, this begins to look like the LHS of a first-order linear differential equation

$$y' + p(x)y = q(x).$$

The technique for solving this class of DEs was to reverse the product rule, which we were always able to do after multiplying by an appropriate *integrating factor* $\mu(x)$.

We might wonder if there is another classical differentiation rule which yields first-order differential equations. If we consider a general multivariate function $F(x, y)$, where $y = y(x)$ is a function of x, application of the *chain rule* gives

$$\frac{d}{dx}\left[F(x, y)\right] = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}.$$

Once again, we have obtained something which looks like a first-order differential equation! This suggests that there may be a "chain rule in reverse" which allows us to take the form above and integrate to obtain an implicit solution $F(x, y)$.

We formally define the following.

Definition 3

A first-order differential equation is said to be **exact** if it has the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{1}$$

where $M(x, y) = \dfrac{\partial F}{\partial x}$ and $N(x, y) = \dfrac{\partial F}{\partial y}$ for some function $F(x, y)$.

Notes on exact equations:

(1) Exact differential equations are commonly written in the form

$$M(x, y)\ dx + N(x, y)\ dy = 0.$$

(2) Unlike first-order linear DEs, in exact equations F_x and F_y are allowed to be functions of *both* x and y, and they may be *nonlinear* functions of y. That is, we may have terms like y^2, $\sin(y)$, $1/y$, etc.

(3) Unlike first-order linear DEs, the solution obtained is always in the *implicit form* $F(x, y) = C$.

The question becomes how we determine whether a DE is exact. The answer comes from the equality of mixed-order partial derivatives. For a general twice differentiable function $F(x, y)$, we have

$$\begin{aligned} & \frac{\partial^2}{\partial y \partial x} F(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y) \\ \Longrightarrow \quad & \frac{\partial}{\partial y} F_x(x, y) = \frac{\partial}{\partial x} F_y(x, y) \\ \Longrightarrow \quad & \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \end{aligned}$$

It can be shown that this is a necessary and sufficient condition for exactness. To see how the method works in practice to produce a solution $F(x, y) = C$, we consider an example.

Example 4

Show that the following differential equation is exact and then find the general solution:

$$(4xy^{1/2})dx + \left(\frac{x^2}{y^{1/2}} + 2\right) dy = 0.$$

Solution: We have $M(x, y) = 4xy^{1/2}$ and $N(x, y) = \dfrac{x^2}{y^{1/2}} + 2$. The required condition for exactness is easy to check:

$$\frac{\partial M}{\partial y} = \frac{2x}{y^{1/2}} = \frac{\partial N}{\partial x}.$$

It follows that the equation is exact and, consequently, that there is a solution of the form $F(x, y) = C$. It remains to find this solution.

The key to determining $F(x, y)$ is to notice that we still have not used all of the information about exact equations. We still have the system of equations

$$\begin{aligned} \frac{\partial F}{\partial x} &= M(x, y) = 4xy^{1/2} \\ \frac{\partial F}{\partial y} &= N(x, y) = \frac{x^2}{y^{1/2}} + 2. \end{aligned} \tag{2}$$

Showing the system is exact guarantees that this can be solved. We can find $F(x, y)$ by integrating either expression by the respective variable of the partial derivative. The first expression gives

$$F(x,y) = \int F(x,y)\, dx = \int 4xy^{1/2}\, dx = 2x^2y^{1/2} + g(y)$$

where we have to include an arbitrary function of y (i.e. the $g(y)$) because partial differentiation with respect to x would eliminate such a term. We now solve for $g(y)$ by taking the derivative of F with respect to the *other* variable, y. We have

$$\frac{\partial F}{\partial y} = \frac{x^2}{y^{1/2}} + g'(y).$$

We can see by comparing this equation to (2) that we need to have $g'(y) = 2$. It follows that $g(y) = 2y + C$ so that the general solution is

$$F(x,y) = 2x^2y^{1/2} + 2y = C.$$

Note: In general, it does not matter the order in which integration and differentiation is performed to obtain the solution. It is very important, however, to be do the operations carefully with respect to the correct variable.

Section 5: Integrating Factors

Now consider being asked to solve the differential equation

$$(4xy)dx + (x^2 + 2y^{1/2})dy = 0.$$

This is the previous example multiplied through by $y^{1/2}$. Since this equation has the same solutions, we suspect that the same methods will apply, but we can see that

$$\frac{\partial M}{\partial y} = 4x \neq 2x = \frac{\partial N}{\partial x}.$$

In other words, the equation is not exact! We only know how to solve equations of this form if they are exact. We seem to be stuck.

The resolution comes by recognizing where the difference between the two equations came. We can change this expression into the standard exact form by dividing through by $y^{1/2}$ (or multiplying through by $y^{-1/2}$, if you prefer). Just as with first-order linear equations, sometimes we will need to multiply through by an **integrating factor** in order to get the equation in the form we can use.

We might wonder if *all* equations of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{3}$$

can be made exact by multiplication by an integrating factor. This was what happened for first-order linear differential equations, so it is not an unfair question. The answer in this case, however, is unfortunately a pronounced no. There are many differential equations of the form (3) which cannot be manipulated so that they are exact. The question then becomes which differential equations can be made exact. We have the following result.

Theorem 1

Consider a first-order differential equation of the form (3). Then:

1. If $R(x) = \left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)/N$ is a function of x alone, then the integrating factor
$$\mu(x) = e^{\int R(x)\,dx}$$
will make (3) exact.

2. If $R(y) = \left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)/M$ is a function of y alone, then the integrating factor
$$\mu(y) = e^{\int R(y)\,dy}$$
will make (3) exact.

We will not justify these forms, although it is a good exercise. Let's reconsider our earlier example.

Example 5

Show that

$$(4xy)dx + (x^2 + 2y^{1/2})dy = 0.$$

can be made exact according to Proposition 1.

Solution: We need to check one or the other of the above conditions. We have

$$\frac{\partial M}{\partial y} = 4x \quad \text{and} \quad \frac{\partial N}{\partial x} = 2x.$$

To check whether the first condition is satisfied, we compute

$$\left(\frac{M_y - N_x}{N}\right) = \left(\frac{4x - 2x}{x^2 + 2y^{1/2}}\right) = \left(\frac{2x}{x^2 + 2y^{1/2}}\right).$$

Since this is not a function of x alone, the first condition fails and we are not allowed to construct an integrating factor out depending on x.

Now consider the second condition. We have

$$\left(\frac{N_x - M_y}{M}\right) = \left(\frac{2x - 4x}{4xy}\right) = -\frac{2x}{4xy} = -\frac{1}{2y}.$$

Since this is a function of y alone, we are allow to construct an integrating factor out of it. Setting $R(y) = -1/(2y)$, we have

$$\mu(y) = e^{\int R(y)\, dy} = e^{-\int \frac{1}{2y}\, dy} = e^{-\frac{1}{2}\ln(y)} = y^{-1/2}.$$

This is exactly integrating factor we expected. Multiplying through the expression by $\mu(x) = y^{-1/2}$ gives

$$(4xy^{1/2})dx + \left(\frac{x^2}{y^{1/2}} + 2\right) dy = 0.$$

This is the earlier expression, which we have already shown in exact, and for which we already know has the solution

$$F(x, y) = 2x^2y^{1/2} + 2y = C.$$

Note: It will be very important to keep the conditions on the variables

x and y straight (though practice!). The key terms are M_y and N_x, so that the coefficient of dx has a y derivative taken, and the coefficient of dy has an x derivative taken. If the wrong derivatives are evaluated, the methods will fail and likely lead to more complicated and time-consuming work.

Example 6

Determine the solution of

$$y\cos(x)dx + (1-y^2)\sin(x)dy = 0.$$

Solution: We might notice that this equation is separable, but ignoring that for the time-being, we will treat as an exact (or nearly exact) equation. To check for exactness, we compute

$$M_y = \cos(x) \neq (1-y^2)\cos(x) = N_x.$$

So that differential equation is not exact. In order to check for an integrating factor, we compute

$$\left(\frac{M_y - N_x}{N}\right) = \left(\frac{\cos(x) - (1-y^2)\cos(x)}{(1-y^2)\sin(x)}\right) = \left(\frac{y^2\cos(x)}{(1-y^2)\sin(x)}\right).$$

This is clearly not a function of x alone, so we may remove it from consideration. The other condition gives

$$\left(\frac{N_x - M_y}{M}\right) = \left(\frac{(1-y^2)\cos(x) - \cos(x)}{y\cos(x)}\right)\left(\frac{-y^2\cos(x)}{y\cos(x)}\right) = -y.$$

Since this is a function of y alone, we set $R(y) = -y$ and evaluate the integrating factor

$$\mu(y) = e^{\int R(y)\,dy} = e^{-\int y\,dy} = e^{-\frac{y^2}{2}}.$$

We now multiply the expression through by this term. We have

$$ye^{-\frac{y^2}{2}}\cos(x)dx + (1-y^2)e^{-\frac{y^2}{2}}\sin(x)dy = 0.$$

We can quickly verify that

$$M_y = (1-y^2)e^{-\frac{y^2}{2}}\cos(x) = N_x$$

so that the integrating factor has been chosen correctly. We now have the system of equations

$$\begin{aligned}\frac{\partial F}{\partial x} &= M(x,y) = ye^{-\frac{y^2}{2}}\cos(x)\\ \frac{\partial F}{\partial y} &= N(x,y) = (1-y^2)e^{-\frac{y^2}{2}}\sin(x).\end{aligned}$$

The obvious choice (I hope!) is to integrate the first expression with respect to x. We have

$$F(x,y) = \int \frac{\partial F}{\partial x}\,dx = ye^{-\frac{y^2}{2}}\sin(x) + g(y).$$

Taking the derivative of this with respect to y yields

$$\frac{\partial F}{\partial y} = e^{-\frac{y^2}{2}}\sin(x) - y^2e^{-\frac{y^2}{2}}\sin(x) + g'(y) = (1-y^2)e^{\frac{y^2}{2}}\sin(x) + g'(y).$$

Comparing this with the second equation gives $g'(y) = 0$ so that $g(y) = C$. This gives the general (implicit) solution

$$F(x,y) = ye^{-\frac{y^2}{2}}\sin(x) = C.$$

Suggested Problems:

1. Use a substitution method to find the general solution of the following DEs:

   (a) $y' = \dfrac{5y^4 - 2xy}{x^2}$

   (b) $y' = \dfrac{y+x}{y-x}$

   (c) $y' = \dfrac{y^2}{x^2+2xy}$

   (d) $y' = \dfrac{3xy^{4/3} - 6y}{x}$

   (e) $y' = \dfrac{4xy^{4/3} + y}{x}$

   (f) $y' = -\dfrac{y + x^2y^2e^{2x}}{x}$

(g) $y' = \dfrac{y + 2\sqrt{xy}}{x}$ (h) $y' = \dfrac{2y^2 + x\sqrt{x^2+y^2}}{2xy}$

2. Find the general solution of the following DEs:

   (a) $(2x - 5y)\ dx + (8y - 5x)\ dy = 0$

   (b) $\dfrac{y}{(x+y)^2}\ dx - \dfrac{x}{(x+y)^2}\ dy = 0$

   (c) $(y - x^2)\ dx + 2x\ dy = 0$

   (d) $\ln(y^2+1)\ dx + \dfrac{2xy+1}{y^2+1}\ dy = 0$

3. Solve the following IVPs:

   (a) $\begin{cases} (2x^3 - y)\ dx + x\ dy = 0 \\ y(0) = 1 \end{cases}$

   (b) $\begin{cases} y\ dx + (2xy - e^{-2y})\ dy = 0 \\ y(1) = 0 \end{cases}$

   (c) $\begin{cases} \dfrac{x}{(x^2+y^2)^{3/2}}\ dx + \dfrac{y}{(x^2+y^2)^{3/2}}\ dy = 0 \\ y(1) = -1 \end{cases}$