Lecture 14

Laplace Transforms (part 3)

Algebra Tricks

(I) Completing the square

$$s^2 + bs + c = s^2 + bs + \frac{b^2}{4} - \frac{b^2}{4} + c$$

$$= \left(s + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c$$

Example 1

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 5y = 0$$

$$y(0) = 1, \quad y'(0) = 3$$

1) Take the Laplace Transform (LT) of the ODE:

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} - 4\,\mathcal{L}\left\{\frac{dy}{dt}\right\} + 5\,\mathcal{L}\{y\} = 0$$

$$\underbrace{s^2 Y(s) - s y(0) - y'(0)}_{\#36} - 4\left(\underbrace{s Y(s) - y(0)}_{\#35}\right) + 5 Y(s) = 0$$

$$\Rightarrow \quad s^2 Y(s) - s - 3 - 4 s Y(s) + 4 + 5 Y(s) = 0$$

2) Solve for $Y(s)$

$$\Rightarrow \left(s^2 - 4s + 5\right) Y(s) - s + 1 = 0$$

$$Y(s) = \frac{s - 1}{s^2 - 4s + 5}$$

$$\left(H(s) = \frac{1}{s^2 - 4s + 5} = \frac{Y(s)}{X_{input}(s)} \leftarrow \text{Transfer function} \right)$$

3) Invert $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s - 1}{s^2 - 4s + 5}\right\}$$

$s^2 - 4s + 5$ doesn't have real roots.

$$s^2 - 4s + 4 - 4 + 5 = (s-2)^2 + 1$$

"completing the square"

$$Y(s) = \frac{s-1}{s^2-4s+5} = \frac{s-1}{(s-2)^2+1}$$

#20: $\mathcal{L}\{e^{at}\cos(bt)\} = \frac{s-a}{(s-a)^2+b^2}$

$$Y(s) = \frac{s-1}{(s-2)^2+1} = \frac{(s-2)+1}{(s-2)^2+1}$$

$$= \frac{s-2}{(s-2)^2+1} + \frac{1}{(s-2)^2+1}$$

#19: $\mathcal{L}\{e^{at}\sin(bt)\} = \frac{b}{(s-a)^2+b^2}$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+1}\right\}$$

#20 $a=2, b=1$ #19 $a=2, b=1$

$$= e^{2t}\cos(t) + e^{2t}\sin(t)$$

(II) Partial Fractions

Given a ratio of two polynomials : $\dfrac{P(s)}{Q(s)}$

(Q is higher order than P)

IIA) Q(s) has distinct real roots:

$$Q(s) = (s-r_1)(s-r_2)\cdots(s-r_n)$$

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s-r_1} + \frac{A_2}{s-r_2} + \cdots + \frac{A_n}{s-r_n}$$

<u>Example 2</u>

$$\frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$= \frac{1}{s+1} - \frac{1}{s+2}$$

(see last lecture)

II B) $Q(s)$ has repeated roots

$$Q(s) = (s-r_1)(s-r_2)(s-r_3)(s-r_4)^3$$

$$\Rightarrow \frac{P(s)}{Q(s)} = \frac{A_1}{s-r_1} + \frac{A_2}{s-r_2} + \frac{A_3}{s-r_3}$$

$$+ \frac{B_1}{s-r_4} + \frac{B_2}{(s-r_4)^2} + \frac{B_3}{(s-r_4)^3}$$

<u>Example 3</u>

$$Y(s) = \frac{s^2+9s+2}{(s-1)^2(s+3)}$$

$$\frac{s^2+9s+2}{(s-1)^2(s+3)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+3}$$

$$\Rightarrow s^2+9s+2 = \frac{A}{\cancel{s-1}}(s-1)^{\cancel{2}}(s+3) + \frac{B}{\cancel{(s-1)^2}}\cancel{(s-1)^2}(s+3)$$

$$+ \frac{C}{\cancel{s+3}}(s-1)^2\cancel{(s+3)}$$

$$s^2 + 9s + 2 = A(s-1)(s+3) + B(s+3) + C(s-1)^2$$

Trick: LHS = RHS for all s, so set s to various values to solve for A, B and C

Set $s = 1$:

$$1 + 9 + 2 = 0 + B(1+3) + 0$$

$$\Rightarrow \quad 12 = 4B \quad \Rightarrow \quad \boxed{B = 3}$$

Set $s = -3$:

$$9 - 27 + 2 = 0 + 0 + C(-3-1)^2$$

$$\Rightarrow \quad -16 = 16C \quad \Rightarrow \quad \boxed{C = -1}$$

Set $s = 0$ (?):

$$2 = -3A + 3B + C$$

$$= -3A + 9 - 1$$

$$\Rightarrow \quad 2 = -3A + 8 \quad \Rightarrow \quad -6 = -3A$$

$$\Rightarrow \boxed{A = 2}$$

$$\boxed{Y(s) = \frac{2}{s-1} + \frac{3}{(s-1)^2} - \frac{1}{s+3}}$$

#2: $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

#23: $\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$

$$y(t) = 2\,\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 3\,\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

#2, $a=1$ #23, $a=1$, $n=1$ #2, $a=-3$

$$= 2e^{t} + 3te^{t} - e^{-3t}$$

IIC) Irreducible quadratics

$$Q(s) = (s-r_1)(s-r_2)(s-r_3)\underbrace{(s^2+bs+c)}_{\substack{\text{can't reduce} \\ \text{(complex roots)}}}$$

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s-r_1} + \frac{A_2}{s-r_2} + \frac{A_3}{s-r_3} + \frac{B_1 s + B_0}{s^2+bs+c}$$

Example 4

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = -8e^{-t}$$

$$y(0) = 2, \quad y'(0) = 12$$

1) Take the LT of the ODE

$$\underbrace{s^2 Y - s y(0) - y'(0)}_{\#36} - 2\underbrace{\left(s Y(s) - y(0)\right)}_{\#35} + 5Y(s) = \frac{-8}{\underbrace{s+1}_{\#2}}$$

$$\Rightarrow s^2 Y - 2s - 12 - 2sY + 4 + 5Y = \frac{-8}{s+1}$$

2) Solve for $Y(s)$

$$(s^2 - 2s + 5)Y(s) - 2s - 8 = \frac{-8}{s+1}$$

$$(s^2 - 2s + 5)Y(s) = 2s + 8 - \frac{8}{s+1}$$

$$= \frac{(2s+8)(s+1) - 8}{s+1}$$

$$= \frac{2s^2 + 10s + \cancel{8} - \cancel{8}}{s+1} = \frac{2s^2+10s}{s+1}$$

$$\boxed{Y(s) = \frac{2s^2 + 10s}{(s+1)(s^2-2s+5)}}$$

$$\frac{2s^2 + 10s}{(s+1)(s^2-2s+5)} = \frac{As + B}{s^2-2s+5} + \frac{C}{s+1}$$

multiply through by denominator

$$2s^2 + 10s = \frac{As+B}{\cancel{s^2-2s+5}}(s+1)\cancel{(s^2-2s+5)}$$

$$+ \frac{C}{\cancel{s+1}}\cancel{(s+1)}(s^2-2s+5)$$

$$= (As+B)(s+1) + C(s^2-2s+5)$$

set $s=-1$:

$$2 - 10 = 0 + C(1+2+5)$$

$$\Rightarrow -8 = 8C \Rightarrow \boxed{C=-1}$$

set $s=0$:

$$0 = B(0+1) + C(0+0+5)$$

$$= B + 5C = B - 5 \Rightarrow \boxed{B=5}$$

set $s=1$:

$$12 = (A + \overset{B=5}{5})(2) \overset{C=-1}{-}(1-2+5)$$

$$= 2A + 10 - 4$$

$$12 = 2A + 6 \Rightarrow 6 = 2A$$

$$\Rightarrow \boxed{A = 3}$$

$$Y(s) = \frac{3s+5}{s^2-2s+5} - \frac{1}{s+1}$$

$$s^2-2s+5 = s^2-2s+1-1+5 = (s-1)^2+4$$

$$Y(s) = \frac{3s+5}{(s-1)^2+4} - \frac{1}{s+1}$$

$$= \frac{3(s-1)+8}{(s-1)^2+4} - \frac{1}{s+1}$$

$$= 3\frac{(s-1)}{(s-1)^2+4} + 4\frac{2}{(s-1)^2+4} - \frac{1}{s+1}$$

#20 $a=1, b=2$

#19 $a=1, b=2$

#2 $a=-1$

$$\boxed{y(t) = 3e^{t}\cos(2t) + 4e^{t}\sin(2t) - e^{-t}}$$