Math 133A, Week 9: Power Series Methods

Section 1: Power Series Methods

Consider being asked to solve the differential equation

$$(x-1)y''(x) - xy'(x) + y(x) = 0. \tag{1}$$

The key difference between this second-order differential equation and the previous ones we have considered is that the coefficients are not constants—rather, they vary with x. While this may seem like an insignificant change, it will prove disastruous for our previous intuition of guessing specific functional forms and cleaning things up. *There is no general method for finding the explicit solution of second-order differential equations with variable coefficients.*

For differential equations with variable coefficients, we will have to rely on approximation methods. A particular common choice is a **power series method**, which relies on the following intuition:

1. Suppose the solution $y(x)$ has a power series expansion
$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$
2. If it does, we can differentiate this form term-by-term to obtain series expansions for $y'(x)$, $y''(x)$, and so on.
3. We can then plug these power series forms into the differential equation, rearrange, and solve for the coefficients a_n.
4. Wherever the power series converges, we can obtain an estimate for the value of the solution at that point by computing a sufficient number of terms in the expansion.

We will see that, in addition to providing a reasonable approximation to the true solution, this *power series method* can also on occasion lead to a general analytic solution as well. Let's reconsider our previous example.

Example 1

Consider the differential equation (1). Perform the following tasks:

1. Determine up to the x^4 term in the power series representation centered at $x_0 = 0$ of the solution with the initial conditions $y(0) = 1$ and $y'(0) = 0$.

2. Determine the general power series representation of the solution centered at $x_0 = 0$ (i.e. in summation form) as a function of a_0 and a_1.

3. Use the general power series representation to determine the closed-form general solution of the (1) (i.e. a solution involving simple functions, like e^x, $\ln(x)$, $\sin(x)$, etc., rather than infinite summations).

Solution (a) & (b): We are looking for a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

Differentiating term-by-term yields

$$\begin{aligned} y'(x) &= \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1} \\ &= a_1 + 2a_2 x + 3a_3 x^2 + \cdots \end{aligned}$$

and

$$\begin{aligned} y''(x) &= \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \\ &= 2a_2 + 6a_3 x + 12a_4 x^2 + \cdots \end{aligned}$$

where we have removed from the sum the terms which evaluate to zero. We can now plug this into the left-hand side of the differential equation to get

$$
\begin{aligned}
&(x-1)y''(x) - xy'(x) + y(x) \\
&= (x-1)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n
\end{aligned}
$$

We will have to be a little bit careful at the point with our indexing, and how we split our sums. We want to move the terms $x-1$ and x inside the summations. We notice that they multiply *every* term in the associated sums, so that we can actually just float them into the summations directly. We have

$$
\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n
$$

We now want to collect like terms according to their powers of x. This cannot be done directly, since two of the sums are with respect to x^{n-1} and two are with respect to x^n. To resolve this, we will need to reindex two of the summations to match the others. It may not be obvious that we should be able to do this at all, but we can easily check that

$$
\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} = 2a_2x + 6a_3x^2 + 12a_4x^3 + \cdots
$$

and

$$
\sum_{n=1}^{\infty} (n+1)na_{n+1} x^n = 2a_2x + 6a_3x^2 + 12a_4x^3 + \cdots
$$

give *exactly* the same series. It should also be now clear how they were correspond. If we want to shift the index inside the summation (i.e. the n) *up* we need to shift the starting point for the summation *down*. A similar rule applies for shifting interior indices *down* (the external bounds must be shifted *up*).

The series can be written as

$$
\sum_{n=1}^{\infty} (n+1)na_{n+1} x^n - \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n
$$

We are not done yet. The summations do not begin at the same index, so we cannot yet combine them. To resolve this (small) problem all we need to do is *take out* all of the terms below the lowest common starting point for the sums. That is to say, because our sums start at either $n = 0$ and $n = 1$, we simply take out all of the terms in the sums corresponding to $n = 0$. (Note that only two of the sums have terms corresponding to $n = 0$!) This gives us

$$-2a_2 + a_0 + \sum_{n=1}^{\infty}(n+1)na_{n+1}x^n - \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n \\ - \sum_{n=1}^{\infty} na_n x^n + \sum_{n=1}^{\infty} a_n x^n$$

We may now (finally!) combine the summations. We have

$$\begin{aligned} &a_0 - 2a_2 + \sum_{n=1}^{\infty}\left[(n+1)na_{n+1} - (n+2)(n+1)a_{n+2} - na_n + a_n\right]x^n \\ &= a_0 - 2a_2 + \sum_{n=1}^{\infty}\left[-(n+2)(n+1)a_{n+2} + (n+1)na_{n+1} + (1-n)a_n\right]x^n \\ &= 0. \end{aligned} \tag{2}$$

By expanding the summation (slightly), we can see that what this expression is really telling us is that

$$(a_0 - 2a_2) + (-6a_3 + 2a_2)x + (12a_4 + 6a_3 - a_2)x^2 + \cdots = 0.$$

The only way for this to hold is if *every coefficient is equal to zero*. That is to say, we need to have

$$a_0 - 2a_2 = 0, \ -6a_3 + 2a_2 = 0, \ 12a_4 + 6a_3 - a_2 = 0, \text{ etc.}$$

It would be a lot of work to do this for each term individually. We notice, however, that our general form (2) gives a far more general form corresponding to the coefficients being equation to zero. We have that

$$a_0 - 2a_2 = 0 \implies a_2 = \frac{1}{2}a_0$$

and

$$
\begin{aligned}
&-(n+2)(n+1)a_{n+2} + (n+1)na_{n+1} + (1-n)a_n = 0 \\
&\implies \quad a_{n+2} = \frac{n}{n+2}a_{n+1} + \frac{1-n}{(n+2)(n+1)}a_n
\end{aligned}
$$

for $n \geq 1$. In other words, we can explicitly relate each coefficient in the power series expansion of the original function $y(x)$ to the previous one by a *recurrence relation.* For indexing reasons, it is typical to adjust the index from $n+2$ to n so that we have

$$
a_n = \frac{n-2}{n}a_{n-1} + \frac{3-n}{n(n-1)}a_{n-2}
$$

For part **(a)**, we notice that, given an initial condition, we can now successively solve for the terms in the power series expansion. We clearly have from $y(x) = \sum_{n=0}^{\infty} a_n x^n$ that $y(0) = 1$ implies $a_0 = 1$ and $y'(0) = 0$ implies $a_1 = 0$. The recurrence relationship then gives

$$
\begin{aligned}
a_2 &= \frac{1}{2}a_0 = \frac{1}{2} \\
a_3 &= \frac{1}{3}a_2 = \frac{1}{3}\left(\frac{1}{2}\right) = \frac{1}{6} \\
a_4 &= \frac{1}{2}a_3 - \frac{1}{12}a_2 = \frac{1}{2}\left(\frac{1}{6}\right) - \frac{1}{12}\left(\frac{1}{2}\right) = \frac{1}{24}.
\end{aligned}
$$

It follows that, up to the x^4 term (i.e. a_4), we have

$$
\begin{aligned}
y(x) &\approx a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \\
&= 1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4.
\end{aligned}
$$

For part **(b)**, we try to find the general form for a_n. That is to say, we want to solve the recurrence relation. Consider the general recurrence relation. We can see that, increasing n, we have

$$
\begin{aligned}
n = 2 \quad &\Longrightarrow \quad a_2 = \frac{1}{2}a_0 \\
n = 3 \quad &\Longrightarrow \quad a_3 = \frac{1}{3}a_2 = \frac{1}{2 \cdot 3}a_0 \\
n = 4 \quad &\Longrightarrow \quad a_4 = \frac{2}{4}a_3 - \frac{1}{4 \cdot 3}a_2 = \frac{2}{2 \cdot 3 \cdot 4}a_0 - \frac{1}{2 \cdot 3 \cdot 4}a_0 = \frac{1}{2 \cdot 3 \cdot 4}a_0 \\
n = 5 \quad &\Longrightarrow \quad a_5 = \frac{3}{5}a_4 - \frac{2}{5 \cdot 4}a_3 = \frac{3}{2 \cdot 3 \cdot 4 \cdot 5}a_0 - \frac{2}{2 \cdot 3 \cdot 4 \cdot 5}a_0 = \frac{1}{2 \cdot 3 \cdot 4 \cdot 5}a_0 \\
&\qquad\quad \vdots
\end{aligned}
$$

This is, of course, an unnatural way to write these coefficient. It does, however, allow us to conjecture at that the general form of the term is

$$a_n = \frac{a_0}{n!}$$

for $n \geq 2$. This can be proved by induction, but we will not perform this task. This gives the general form for the solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_0 \sum_{n=2}^{\infty} \frac{x^n}{n!}.$$

This is the power series solution representation of $y(x)$.

Solution (c): We might notice that the final sum is very close to the Taylor series expansion for e^x. In fact, it only differs in the second term. We can be somewhat creative, therefore, and *complete* the sum. To do this, we need to add and subtract $a_0 x$ to the sum, which gives

$$
\begin{aligned}
y(x) &= (a_1 - a_0)x + a_0 + a_0 x + \sum_{n=2}^{\infty} \frac{x^n}{n!} \\
&= (a_1 - a_0)x + a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!}.
\end{aligned}
$$

This may not seem like much, but it is actually pretty remarkable. The terms a_0 and a_1 are undetermined constants, and the summation corresponds to the Taylor series expansion of e^x. It follows that we have

$$y(x) = C_1 x + C_2 e^x$$

where $C_1 = a_1 - a_0$ and $C_2 = a_0$. (This answer can be easily checked.) That is to say, we have successfully used the power series method to find the *general solution* of the differential equation. This should be remarkable, since we had no direct method for solving this particular differential equation.

We should stop here to make a few notes:

- It is not generally the case that we will be able to correspond our final power series solution to analytic solutions of the form e^x, $\ln(x)$, etc. In general, having the answer in series form may be the best we can do.
- Notice that, as a result of the recurrence relationship, we will have our series written in terms of some (two, in our case) baseline constants which will not be solved for numerical. As in the example above, these will correspond to our undetermined constants in the general solution, and can be solved for numerically by appropriate initial conditions.
- It may be very difficult to determine an explicit form for the general terms a_n. In such cases, it is more common to seek out the first three or four terms, and disregard the rest.

Section 2: General Properties

We should pause to review some general properties about power series. The first thing we should probably recall is that all of the elementary functions have expansions in terms of a special kind of power series known as a *Taylor series*, i.e. a series of the form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

That is to say, the power series with $a_n = f^{(n)}(x_0)/n!$. We have the following well-known Taylor series expansions:

$$
\begin{aligned}
e^x &= \sum_{n=0}^{n} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots \\
\sin(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots \\
\cos(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots \\
\ln(1-x) &= -\sum_{n=1}^{\infty} \frac{1}{n} x^n = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \cdots, \qquad -1 \le x < 1 \\
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots, \qquad |x| < 1.
\end{aligned}
$$

We should also remind ourselves that power series are not guaranteed to converge for all $x \in \mathbb{R}$. They may have a limited *radius of convergence*, $|x - x_0| < \rho$, outside of which the series does not settle down as $n \to \infty$ (i.e. as we take more terms). The most common test for the convergence of power series is the ratio test, which says, if we evaluate

$$
\lim_{n\to\infty} \left| \frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n} \right| = |x - x_0| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = |x - x_0| L
$$

then the series converges if $|x - x_0|L < 1$ and diverges if $|x - x_0|L > 1$. The endpoints, where $|x - x_0|L = 1$, have to be considered separately since the test does not apply to them.

Example 2

Show that the Taylor series expansion given above for e^x converges for all x while the expansion for $\ln(1-x)$ converges only for $-1 \le x < 1$.

Solution: The series for e^x has $a_n = 1/n!$ so we compute

$$
\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| &= |x| \lim_{n\to\infty} \left| \frac{1/(n+1)!}{1/n!} \right| \\
&= |x| \lim_{n\to\infty} \frac{n!}{(n+1)!} = |x| \lim_{n\to\infty} \frac{1}{n+1} = 0.
\end{aligned}
$$

Since this is clearly less than 1 for any value of x we could happen to choose, we have that the series converges for all $x \in \mathbb{R}$. For the series $\ln(1-x)$, we have $a_n = -1/n$ so that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = |x| \lim_{n\to\infty} \left| \frac{(1/(n+1)}{1/n} \right| = |x| \lim_{n\to\infty} \frac{n}{n+1} = |x|.$$

Clearly, we have that this is only less than 1 if $|x| < 1$ so that the interval of convergence is 1. To check the endpoints, when $|x| = 1$, we have to the consider the series exactly. For $x = -1$ we have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

which converges. For $x = 1$ we have

$$-\sum_{n=1}^{\infty} \frac{1}{n}$$

which diverges (harmonic series). It follows that the interval of convergence is $-1 \leq x < 1$ and we are done.

For this course, we will not need to consider many more details of power series, except to recall that in order to add, subtract, or multiply two or more power series, or to differentiate or integrate a single power series, all we need to do is apply the operations *term-by-term*, and the resulting series will converge on the same interval (except perhaps the end points). That is to say, power series are very easy to manipulate!

Section 3: Ordinary Points

We will not investigate the theory underlying power series methods in depth, but it will be important to make one clarification about the general second-order differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0. \tag{3}$$

It is reasonable to ask what conditions are sufficient for the method to work. After all, we don't want to be wasting our valuable time performing operations we should have realized were doomed to fail. We have the following result.

Theorem 1

Suppose $P(x)$, $R(x)$, and $Q(x)$ are polynomials with no common factors and $P(x_0) \neq 0$. Then there is a neighborhood of x_0, $|x - x_0| < \rho$, in which a power series solution of (3) exists and converges.

The justification depends on the existence and uniqueness theorem for second-order linear differential equations, which we did not cover in class. We made the following notes:

- The points $x \in \mathbb{R}$ such that $P(x) \neq 0$ are called **ordinary points**. The points $x \in \mathbb{R}$ such that $P(x) = 0$ are called **singular points**.
- The important connection to make with Theorem 3.2.1 is with regards to *initial value problems*. This result tells us that, if x_0 is an ordinary point, then the initial value problem for the power series centered at x_0 can be solved. It will be important, therefore, to center our power series at ordinary points if we can!

Example 3

Find the first six non-zero terms in the power series solution (centered at $x_0 = 1$) of the following initial value problem

$$y''(x) - xy(x) - y = 0, \quad y(1) = 0, \quad y'(1) = 1.$$

Use this to estimate the value of $y(2)$.

Solution: We first of all note that $P(x) = 1$ so that $x_0 = 1$ (and in fact, any point) is an ordinary point so that we will be able to find a solution centering there. We assume the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n (x-1)^n$$

to get

$$y'(x) = \sum_{n=1}^{\infty} a_n n(x-1)^{n-1}$$

and

$$y''(x) = \sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2}.$$

Substituting this into the left-hand side of the differential equation gives

$$\sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2} - x\sum_{n=1}^{\infty} a_n n(x-1)^{n-1} - \sum_{n=0}^{\infty} a_n (x-1)^n.$$

It is tempting to immediately carry the stray x term into the corresponding summation; however, this would produce terms of the form $x(x-1)^{n-1}$ and *not* the desired form $(x-1)^n$. In order to resolve this, we must rewrite x in factors of the power term $x-1$. In this case, we can simply write $x = (x-1)+1$ to get

$$\begin{aligned}
&\sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2} - [(x-1)+1]\sum_{n=1}^{\infty} a_n n(x-1)^{n-1} - \sum_{n=0}^{\infty} a_n (x-1)^n \\
&= \sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-2} - \sum_{n=1}^{\infty} a_n n(x-1)^n \\
&\qquad - \sum_{n=1}^{\infty} a_n n(x-1)^{n-1} - \sum_{n=0}^{\infty} a_n (x-1)^n.
\end{aligned}$$

We now reindex the sums so that they have common terms $(x-1)^n$. This gives

$$\begin{aligned}
&\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x-1)^n - \sum_{n=1}^{\infty} a_n n(x-1)^n \\
&\qquad - \sum_{n=0}^{\infty} a_{n+1}(n+1)(x-1)^n - \sum_{n=0}^{\infty} a_n (x-1)^n.
\end{aligned}$$

To combine the terms, we may start no lower than $n=1$, so we much extra the terms corresponding to $n=0$ from the sums. After a little rearranging, this gives

$$2a_2 - a_1 - a_0 + \sum_{n=1}^{\infty} \left[a_{n+2}(n+2)(n+1) - (n+1)(a_{n+1} + a_n)\right](x-1)^n = 0.$$

Equating coefficients on the left-hand and right-hand sides gives

$$2a_2 - a_1 - a_0 = 0 \quad \text{and} \quad a_{n+2}(n+2)(n+1) - (n+1)(a_{n+1} + a_n) = 0.$$

This simplifies to the general recursion relationship

$$a_n = \frac{a_{n-1} + a_{n-2}}{n}$$

for $n \geq 2$.

We are probably not going to be able to immediately identify a general solution for the terms a_n, but we can still (and will always be able to!) determine the first few terms in the series. We have

$$\begin{aligned}
a_2 &= \frac{a_1 + a_0}{2} = \frac{a_1}{2} + \frac{a_0}{2} \\
a_3 &= \frac{a_2 + a_1}{3} = \frac{\left(\frac{a_1}{2} + \frac{a_0}{2}\right) + a_1}{3} \\
&= \frac{a_1}{2} + \frac{a_0}{6} \\
a_4 &= \frac{a_3 + a_2}{4} = \frac{\left(\frac{a_1}{2} + \frac{a_0}{6}\right) + \left(\frac{a_1}{2} + \frac{a_0}{2}\right)}{4} \\
&= \frac{a_1}{4} + \frac{a_0}{6} \\
a_5 &= \frac{a_4 + a_3}{5} = \frac{\left(\frac{a_1}{4} + \frac{a_0}{6}\right) + \left(\frac{a_1}{2} + \frac{a_0}{6}\right)}{5} \\
&= \frac{3}{20}a_1 + \frac{a_0}{15}.
\end{aligned}$$

In other words, we can determine (with a little work!) how each coefficient of the power series solutions depends on a_0 and/or a_1. In the end, we have the series

$$
\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n (x-1)^n \\
&= a_0 + a_1(x-1) + \left(\frac{a_1}{2} + \frac{a_0}{2}\right)(x-1)^2 + \left(\frac{a_1}{2} + \frac{a_0}{6}\right)(x-1)^3 \\
&\quad + \left(\frac{a_1}{4} + \frac{a_0}{6}\right)(x-1)^4 + \left(\frac{3}{20}a_1 + \frac{a_0}{15}\right)(x-1)^5 + \cdots \\
&= a_0\left[1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{6}(x-1)^4 + \frac{1}{15}(x-1)^5 + \cdots\right] \\
&\quad + a_1\left[(x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{4}(x-1)^4 + \frac{3}{20}(x-1)^5 + \cdots\right]
\end{aligned}
$$

Factoring the solution by a_0 and a_1 allows us to make the immediate correspondence between the power series form and the general form

$$y(x) = C_1 y_1(x) + C_2 y_2(x).$$

That is to say, it allows us to extract a fundamental solution set in the form of independent series!

This form also allows us to quickly evaluate the initial conditions: $y(1) = 0$ and $y'(1) = 1$. In general, this could be a substantial task. The solution is composed of infinite sums, but we suspect that we may have to evaluate an infinite sum when computing a_0 or a_1. We notice, however, that evaluating $x = 1$ immediately eliminates all of the factored forms $(x-1)^n$ from the summation! Once we remove these terms, we are left with

$$y(1) = 0 \quad \Longrightarrow \quad a_0 = 0$$

and

$$y'(1) = 1 \quad \Longrightarrow \quad a_1 = 1.$$

It follows that the particular solution is

$$y(x) = (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{4}(x-1)^4 + \frac{3}{20}(x-1)^5 + \cdots$$

We can now easily estimate the value of $y(2)$ by evaluating

$$y(2) \approx (1) + \frac{1}{2}(1)^2 + \frac{1}{2}(1)^3 + \frac{1}{4}(1)^4 + \frac{3}{20}(5)^5 = \frac{12}{5} = 2.4.$$

This is close to the true value of $y(2) = 2.517182610$ but we should not be surprised that we will have to take significant more terms in order to get a truly "good" estimate. If we go up to the x^{10} term, we obtain the estimate $y(2) = 2.516316138$ which is accurate to two decimal places. Going up to the x^{20} term gives the estimate $y(2) = 2.517182608$, which is accurate to seven decimal places. And so on. We can see that the trade off being this and the earlier numerical methods is roughly the same: *the greater the accuracy desired, the greater the computation resources required.*

Example 4

Determine up to the x^5 term in the power series form centered at $x_0 = 0$ for the *Airy equation*

$$\begin{cases} y'' - xy = 0 \\ y(0) = 1 \\ y'(0) = 1. \end{cases}$$

Solution: We have

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

so that

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Substituting into the differential equation gives

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x \sum_{n=0}^{\infty} a_n x^n = 0.$$

Combining all terms of x gives

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0.$$

Adjusting the summations so that all terms are in terms of x^n gives

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty}a_{n-1}x^n = 0.$$

Removing the $n = 0$ term and combining for $n \geq 1$ gives

$$2a_2 + \sum_{n=1}^{\infty}\left((n+2)(n+1)a_{n+2} - a_{n-1}\right)x^n = 0.$$

After resetting the indices, it follows that

$$\begin{cases} a_2 = 0 \\ a_n = \dfrac{a_{n-3}}{n(n-1)}, \quad \text{for } n \geq 3. \end{cases}$$

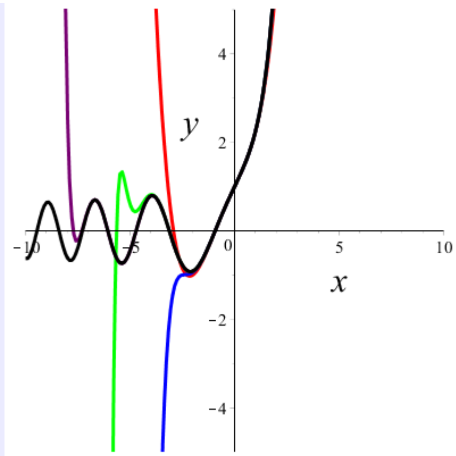
From the initial conditions, we have $a_0 = 1$ and $a_1 = 1$, and from above we have $a_2 = 0$. It follows from the recurrence relation for $n \geq 3$ that we have

$$\begin{aligned} a_3 &= \frac{a_0}{3(2)} = \frac{1}{6} \\ a_4 &= \frac{a_1}{4(3)} = \frac{1}{12} \\ a_5 &= \frac{a_2}{5(4)} = 0 \end{aligned}$$

so that the power series solution up to the x^5 term is

$$\begin{aligned} y(x) &\approx a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 \\ &= 1 + x + \frac{1}{6}x^3 + \frac{1}{12}x^4. \end{aligned}$$

Below is a plot of the solution with $n = 5$ (red), $n = 10$ (blue), $n = 20$ (green), $n = 50$ (purple), and $n = 200$ (black):



We can see that the behavior becomes more refined as we take higher and higher values of n, although clearly computing more than the first handful of terms in the series expansion requires the assistance of a computer.

Suggested Problems

1. For the following initial value problems, determine up to the x^5 term in the power series solutions centered at $x_0 = 0$:

   (a) $\begin{cases} y' - xy = 1 \\ y(0) = 0 \end{cases}$

   (b) $\begin{cases} y'' + x^2 y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$

   (c) $\begin{cases} y'' - 2xy' + y = 0 \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$

   (d) $\begin{cases} (x+1)y'' + xy' - y = 0 \\ y(0) = 1 \\ y'(0) = 1 \end{cases}$

2. For the following initial value problems, determine up to the x^5 term in the power series solutions centered at $x_0 = 1$: (remember to shift by setting $x = (x-1) + 1$!)

   (a) $\begin{cases} xy'' - y = 0 \\ y(0) = -1 \\ y'(0) = 1 \end{cases}$

   (b) $\begin{cases} x^2 y'' + xy' + y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$