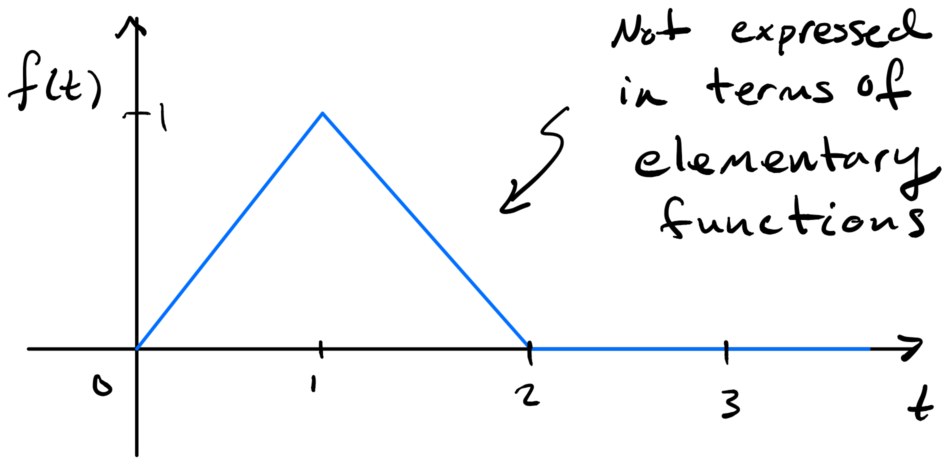
Lecture 15

Laplace Transforms (part 4)

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt$$

$$a \frac{d^2y}{dt^2} + b \frac{dy}{dt} + cy = f(t)$$

$$(t \geq 0)$$



Not expressed in terms of elementary functions

We call this a "piece-wise" function

$$f(t) = \begin{cases} t & : 0 \leq t < 1 \\ 2-t & : 1 \leq t < 2 \\ 0 & : t \geq 2 \end{cases}$$

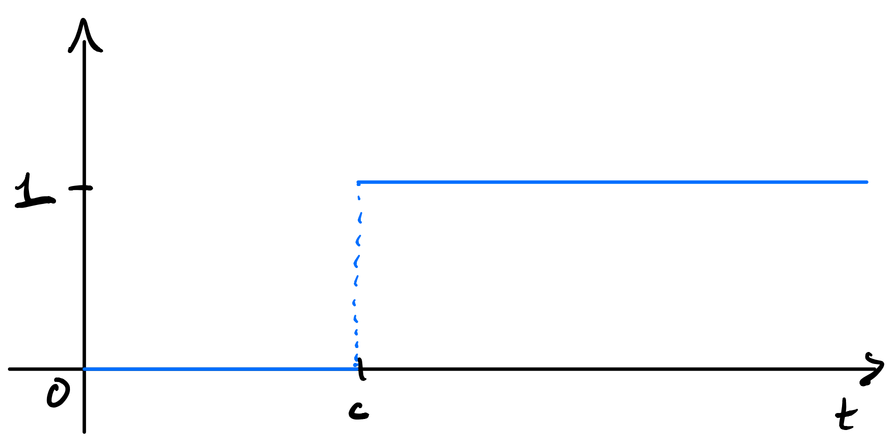
$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$$

$$= \int_0^1 f(t)e^{-st}\,dt + \int_1^2 f(t)e^{-st}\,dt + \int_2^\infty f(t)e^{-st}\,dt$$

$$= \int_0^1 t e^{-st}\,dt + \int_1^2 (2-t)e^{-st}\,dt + \cancel{\int_2^\infty (0)e^{-st}\,dt}$$

$$= \frac{1}{s^2}\left[1 - 2e^{-s} + e^{-2s}\right]$$

Rather than applying the LT definition every time, we can develop simple rules for piecewise function in terms of the <u>Heaviside Function</u>: $u_c(t)$



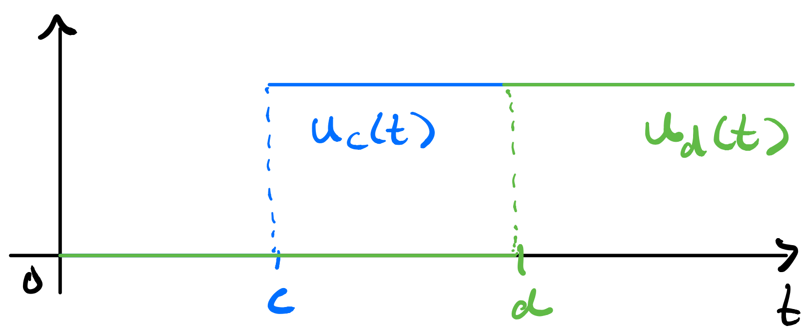
Oliver Heaviside

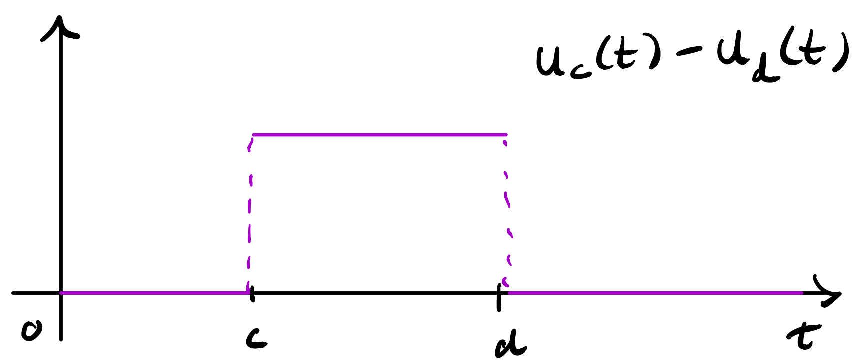
$$u_c(t) = \begin{cases} 0 & : \ 0 \le t < c \\ 1 & : \ c \le t \end{cases}$$

(or $t \geq c$)

- $u_c(t)$ acts like a "switch" at $t=c$

- If you wanted to switch a function on in a particular region, you can use the difference of two Heaviside functions





Example 1

$$f(t) = \begin{cases} t & : \ 0 \le t < 1 \\ 2-t & : \ 1 \le t < 2 \\ 0 & : \ 2 \le t < \infty \end{cases}$$

Note: $t \geq 0 \Rightarrow u_0(t) = 1$

$t < \infty \Rightarrow u_\infty(t) = 0$

$$f(t) = t\left[u_0(t) - u_1(t)\right]$$

$$+ (2-t)\left[u_1(t) - u_2(t)\right]$$

$$\cancel{+ \ 0 \cdot \left[u_2(t) - u_\infty(t)\right]}$$

$$= t u_0(t) - t u_1(t) + (2-t)u_1(t)$$

$$- (2-t)u_2(t)$$

$$= t + 2(1-t)u_1(t) - (2-t)u_2(t)$$

★ Laplace Transforms on functions with $u_c(t)$ have simple rules:

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$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}$$

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OR $$\mathcal{L}\{u_c(t)f(t)\} = e^{-cs}\mathcal{L}\{f(t+c)\}$$

Consider: $f(t) = t + 2(1-t)u_1(t) - (2-t)u_2(t)$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t\} - 2\mathcal{L}\{u_1(t)(t-1)\}$$
$$+ \mathcal{L}\{(t-2)u_2(t)\}$$
$$= \mathcal{L}\{t\} - 2e^{-s}\mathcal{L}\{t\} + e^{-2s}\mathcal{L}\{t\}$$
$$= \boxed{\frac{1}{s^2} - 2e^{-s}\frac{1}{s^2} + e^{-2s}\frac{1}{s^2}}$$

Example 2

What is $\mathcal{L}\{t^2 u_1(t)\}$?

$$\mathcal{L}\{t^2 u_1(t)\} = e^{-s}\mathcal{L}\{(t+1)^2\}$$

$$= e^{-s}\mathcal{L}\{t^2 + 2t + 1\}$$

$$= e^{-s}\left(\mathcal{L}\{t^2\} + 2\mathcal{L}\{t\} + \mathcal{L}\{1\} \right)$$

#3, n=2   #3, n=1   #1

$$= e^{-s}\left(\frac{2}{s^3} + 2\frac{1}{s^2} + \frac{1}{s} \right)$$

Example 3

What is $\mathcal{L}^{-1}\left\{ e^{-s}\frac{4}{s^2+16} \right\}$

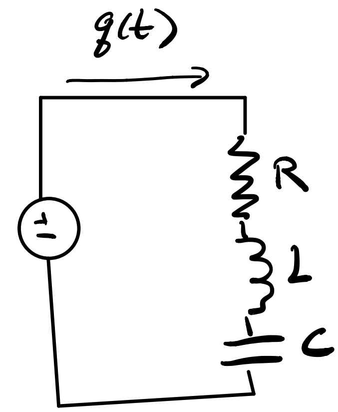
use $\mathcal{L}^{-1}\left\{ \frac{4}{s^2+16} \right\} = \sin(4t)$ (#7) $b=4$

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\}$$

$$\Rightarrow \quad u_c(t)f(t-c) = \mathcal{L}^{-1}\{e^{-cs}F(s)\}$$

$$\Rightarrow \quad \mathcal{L}^{-1}\left\{ e^{-s} \frac{4}{s^2+16} \right\} = u_1(t)\sin(4(t-1))$$

Example 4



Series RLC circuit with a resistence of 2 Ω, inductance of 1 H, and a capacitance of ½ F.

Initially, there is no current or charge.

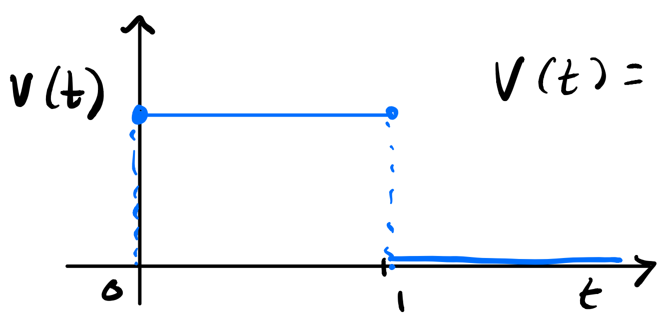
$$i(t) = \frac{dq}{dt}$$

$$\frac{d^2q}{dt^2} + 2\frac{dq}{dt} + 2q = V(t)$$

$L=1$ $\quad R=2$ $\quad 1/C = 2$

$q(0) = 0$

$\frac{dq}{dt}(0) = 0$



$$V(t) = u_0(t) - u_1(t)$$

1) Take LT of the whole ODE

$$\mathcal{L}\left\{ \frac{d^2q}{dt^2} + 2\frac{dq}{dt} + 2q \right\} = \mathcal{L}\left\{u_0(t) - u_1(t)\right\}$$

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$$\left[s^2 Q(s) - s\cancel{q(0)} - \cancel{q'(0)}\right] + 2\left[sQ(s) - \cancel{q(0)}\right] + 2Q(s)$$

$$= \frac{1}{s} - \frac{e^{-s}}{s}$$

$$\#25 \quad \mathcal{L}\left\{u_c(t)\right\} = \frac{e^{-cs}}{s}$$

$$\Rightarrow \quad s^2 Q(s) + 2sQ(s) + 2Q(s) = \frac{1}{s}\left(1 - e^{-s}\right)$$

2) Solve for $Q(s)$

$$\rightarrow \quad (s^2 + 2s + 2)\,Q(s) = \frac{1}{s}\left(1 - e^{-s}\right)$$

$$Q(s) = \frac{1 - e^{-s}}{s\,(s^2 + 2s + 2)}$$

3) Invert $q(t) = \mathcal{L}^{-1}\{Q(s)\}$

4) $$\frac{1}{s(s^2+2s+2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+2}$$

$$\Rightarrow \quad 1 = A(s^2+2s+2) + (Bs+C)s$$

set $s=0$: $\quad 1 = 2A + 0 \quad \Rightarrow \boxed{A = 1/2}$

set $s=1$: $\quad 1 = \frac{1}{2}(1+2+2) + (B+C)$

$$= \frac{5}{2} + B + C \rightarrow \quad B+C = -3/2$$

Set $s=-1$: $\quad 1 = \frac{1}{2}(1-2+2) - (-B+C)$

$$= \frac{1}{2} + B - C \rightarrow \quad B - C = 1/2$$

R1 $\quad B + C = -3/2$

R2 $\quad B - C = 1/2$

$$R1+R2 = 2B = -1 \Rightarrow \boxed{B = -\tfrac{1}{2}}$$

$$R1-R2 = 2C = -2 \Rightarrow \boxed{C = -1}$$

$$\frac{1}{s(s^2+2s+2)} = \frac{1}{2s} + \frac{-\frac{1}{2}s - 1}{s^2+2s+2}$$

$$= \frac{1}{2s} - \frac{1}{2}\,\frac{s+2}{s^2+2s+2}$$

$$\Rightarrow \quad Q(s) = \frac{1-e^{-s}}{s(s^2+2s+2)}$$

$$= \frac{1}{2s} - \frac{1}{2}\,\frac{s+2}{s^2+2s+2}$$

$$- \frac{e^{-s}}{2s} + \frac{1}{2}\,\frac{(s+2)e^{-s}}{s^2+2s+2}$$

Note: $s^2+2s+2 = (s+1)^2 + 1$

$$s+2 = (s+1) + 1$$

$$\Rightarrow Q(s) = \frac{1}{2s} - \frac{1}{2}\frac{(s+1)}{(s+1)^2+1} - \frac{1}{2}\frac{1}{(s+1)^2+1}$$

$$- \frac{e^{-s}}{2s} + \frac{1}{2}\frac{(s+1)e^{-s}}{(s+1)^2+1} + \frac{1}{2}\frac{e^{-s}}{(s+1)^2+1}$$

Inverting each term then gives the solution:

$$q(t) = \frac{1}{2} - \frac{1}{2}e^{-t}\cos(t) - \frac{1}{2}e^{-t}\sin(t)$$

$$- \frac{1}{2}u_1(t) + \frac{1}{2}e^{-(t-1)}\cos(t-1)u_1(t) + \frac{1}{2}e^{-(t-1)}\sin(t-1)u_1(t)$$