Math 133A: Week 8
Modified Trial Forms, Resonance

Section 1: Modified Trial Forms

It turns out that we have only partially answered the question of how to solve a general second-order system with constant coefficients.

Consider finding the general solution of the differential equation

$$y'' + 4y = \cos(2x) \tag{1}$$

with the method given by the Method of Undetermined Coefficients Algorithm. We confidently guess the trial function

$$y_p(x) = A\cos(2x) + B\sin(2x).$$

This gives

$$\begin{aligned} y_p'(x) &= -2A\sin(2x) + 2B\cos(2x) \\ y_p''(x) &= -4A\cos(2x) + 4B\sin(2x). \end{aligned}$$

However, we can easily check that

$$y_p''(x) + 4y_p(x) = -4A\cos(2x) + 4B\sin(2x) + 4(A\cos(2x) + B\sin(2x)) = 0.$$

We need to match constants so that this equals $g(x) = \sin(2x)$ but the terms on the LHS have vanished. There are no constants left to solve for!

Something has gone seriously wrong, but after a moment of thought we realize that we should have expected this. The complementary function is $y_c(x) = C_1\cos(2x) + C_2\sin(2x)$ so that the combination of functions in the given trial function *had to* vanish when it was substituted into the LHS of (1). The Method of Undetermined Coefficients Algorithm will never work for differential equations where the complementary function $y_c(x)$ contains terms requires of the proposed trial form $y_p(x)$!

For such differential equations, the fix is to choose a different trial function. We have the following modification of step 2..

Modified Method of Undetermined Coefficients

2.* If the complementary solution $y_c(x)$ contains common terms with the required trial functions $y_p(x)$ given in step 2. of the Method of Undetermined Coefficients Algorithm, then select the corresponding portion of the trial function $y_p(x)$ according to the following:

(a*) $y_p(x) = A_n x^{n+s} + A_{n-1} x^{n+s-1} + \cdots + A_1 x^{s+1} + A_0 x^s$

(b*) $y_p(x) = Bx^s e^{rx}$

(c*) $y_p(x) = Ax^s \cos(ax) + Bx^s \sin(ax)$

where s is the lowest power which produces a term which is independent of those in $y_c(x)$. Importantly, the number of undetermined coefficients must be the same in the modified form as if it would have been if there were no conflict with the complementary function $y_c(x)$.

Note: We will not offer a rigorous proofs of the forms in the modified algorithm in this course. It should be noted that the notion of multiplying by the independent variable x to generate independent solutions is a common technique, and was used previously to generate the solutions $y(x) = C_1 e^{rx} + C_2 x e^{rx}$ for differential equations with repeated roots.

Example 1

Find the general solution of

$$y'' + 4y = \cos(2x).$$

Solution: The complementary function was $y_c(x) = C_1 \cos(2x) + C_2 \sin(2x)$ so we are not allowed to use $y_p(x) = A\cos(2x) + B\sin(2x)$ as a trial function. Instead, we must use

$$y_p(x) = Ax\cos(2x) + Bx\sin(2x).$$

This gives

$$y_p'(x) = A\cos(2x) + B\sin(2x) - 2Ax\sin(2x) + 2Bx\cos(2x)$$
$$y_p''(x) = 4B\cos(2x) - 4A\sin(2x) - 4Ax\cos(2x) - 4Bx\sin(2x).$$

Plugging into the DE gives

$$\begin{aligned}
&y_p'' + 4y_p \\
&= 4B\cos(2x) - 4A\sin(2x) - 4Ax\cos(2x) - 4Bx\sin(2x) \\
&\quad + 4(Ax\cos(2x) + Bx\sin(2x)) \\
&= 4B\cos(2x) - 4A\sin(2x) \\
&= \cos(2x).
\end{aligned}$$

It follows that we need $A = 0$ and $B = 1/4$ so that we have the particular solution

$$y_p(x) = \frac{1}{4}x\sin(2x).$$

The general solution of the differential equation is therefore

$$y(x) = C_1\cos(2x) + C_2\sin(2x) + \frac{1}{4}x\sin(2x).$$

Example 2

Find the general solution of

$$y'' - 2y' + y = 2e^x + 4e^{-x}.$$

Solution: We first solve the complementary problem

$$y_c'' - 2y_c' + y_c = 0.$$

We guess the solution $y_c(x) = e^{rt}$ to get

$$e^{rx}(r^2 - 2r + 1) = e^{rx}(r-1)^2 = 0$$

so that we have the repeated root $r = 1$. It follows that the complementary solution is

$$y_c(x) = C_1e^x + C_2xe^x.$$

We now consider trial forms for the particular solution. Notice, however, that the natural trial form

$$y_p(x) = Ae^x + Be^{-x}$$

overlaps with the complementary solution $y_c(x)$ in the term e^x. This choice of $y_p(x)$ therefore does not have enough degrees of freedom to solve for all of the undetermined constants. We attempt instead to multiply this problematic portion of this trial form by x. (Notice that we do not need to multiplify the term e^{-x} by x since we have enough degrees of freedom to solve for B.) This gives

$$y_p^*(x) = Axe^x + Be^{-x}.$$

We notice, however, that this form *still* overlaps with $y_c(x)$ (in the term xe^x this time). It follows that we need to multiply by x again to get

$$y_p^{**}(x) = Ax^2e^x + Be^{-x}.$$

It is only now, after multiplying by x twice, that we can check that the trial form $y_p(x)$ does not contain any overlap with $y_c(x)$.

We now plug $y_p^{**}(x)$ (which we relabel $y_p(x)$), into the differential equation. We have

$$\begin{aligned} y_p'(x) &= Ax^2e^x + 2Axe^x - Be^{-x} \\ y_p''(x) &= Ax^2e^x + 4Axe^x + 2Ae^x + Be^{-x}. \end{aligned}$$

Substituting in the differential equation gives:

$$\begin{aligned} y_p'' - 5y_p' + 6y_p &= [Ax^2e^x + 4Axe^x + 2Ae^x + Be^{-x}] \\ &\quad - 2[Ax^2e^x + 2Axe^x - Be^{-x}] \\ &\quad + [Ax^2e^x + Be^{-x}] \\ &= 2Ae^x + 4Be^{-x}. \end{aligned}$$

Since the RHS of the differential equation was $2e^x + 4e^{-x}$, we can easily see that we need $A = 1$ and $B = 1$. It follows that the particular solution is

$$y_p(x) = x^2e^x + e^{-x}.$$

The general solution is therefore

$$y(x) = y_c(x) + y_p(x) = C_1 e^x + C_2 x e^x + x^2 e^x + e^{-x}.$$

Section 2: Resonance

We now focus our attention on the physical pendulum/spring models which motivated our study of second-order equations:

$$mx'' + cx' + kx = g(t)$$

where m was the **mass**, c was the **damping constant**, k was the **restoring constant**, and $g(t)$ was the **forcing function**. We want to focus, in particular, on the effect of the forcing term $g(t)$ on the behavior of the solution.

It will be useful to represent the solutions in a simplified form which consists of a single term. In this case, we will have multiple trigonometric functions, one corresponding to a *fast mode* of oscillation, and one corresponding to a *slow mode* of oscillation. We have the following result.

Lemma 1

For all constants ω_0 and ω, we have

$$\cos(\omega t) - \cos(\omega_0 t) = 2 \sin(\alpha t) \sin(\beta t)$$

where

$$\alpha = \frac{1}{2}(\omega_0 + \omega), \quad \text{and} \quad \beta = \frac{1}{2}(\omega_0 - \omega).$$

Proof

The trigonometric identities $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$ and $\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$ can be subtracted from one another to give $2\sin(A)\sin(B) = \cos(A-B) - \cos(A+B)$. If we take $A = \frac{1}{2}(\omega_0 + \omega)t$ and $B = \frac{1}{2}(\omega_0 - \omega)t$ we have the desired result.

Example 3

Consider the initial value problem

$$\begin{cases} x'' + 4x = \cos(\omega t) \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$$

where $\omega \neq 2$. Write the solution in the form $x(t) = A\sin(\alpha t)\sin(\beta t)$. Comment on the behavior of the corresponding pendulum-spring system as $\omega \to 2$.

Solution: We have already seen that the complementary function for this differential equation is

$$x_c(t) = C_1 \cos(2t) + C_2 \sin(2t).$$

Since $\omega \neq 2$, we use the trial function $x_p(t) = A\cos(\omega t) + B\sin(\omega t)$. This gives

$$x_p''(t) = -A\omega^2 \cos(\omega t) - B\omega^2 \sin(\omega t)$$

so that we have

$$\begin{aligned} x_p'' + 4x_p &= \left[-A\omega^2 \cos(\omega t) - B\omega^2 \sin(\omega t)\right] + 4\left[A\cos(\omega t) + B\sin(\omega t)\right] \\ &= (4-\omega^2)(A\cos(\omega t) + B\sin(\omega t)) \\ &= \cos(\omega t). \end{aligned}$$

Since $\omega \neq 2$ implies $\omega^2 \neq 4$, it follows that $A = 1/(4-\omega^2)$ and $B = 0$ so that we have the general solution

$$x(t) = C_1 \cos(2t) + C_2 \sin(2t) + \frac{1}{4-\omega^2}\cos(\omega t).$$

We now use the initial conditions $x(0) = x'(0) = 0$ to solve for C_1 and C_2. We have

$$x'(t) = -2C_1 \sin(2t) + 2C_2 \cos(2t) - \frac{\omega}{4-\omega^2}\sin(\omega t)$$

so that, at $t = 0$, we have

$$\begin{aligned} C_1 &= -\frac{1}{4-\omega^2} \\ 2C_2 &= 0 \end{aligned}$$

which implies $C_1 = -1/(4-\omega^2)$ and $C_2 = 0$. It follows that the particular solution is

$$\begin{aligned} x(t) &= -\frac{1}{4-\omega^2}\cos(2t) + \frac{1}{4-\omega^2}\cos(\omega t) \\ &= \frac{1}{4-\omega^2}\left(\cos(\omega t) - \cos(2t)\right). \end{aligned}$$

In terms of simplification, this is pretty good, but we want to express the solution in the form $x(t) = A\sin(\alpha t)\sin(\beta t)$. We therefore use Lemma 1 with $\alpha = \frac{1}{2}(2+\omega)$ and $\beta = \frac{1}{2}(2-\omega)$ to write the solution as

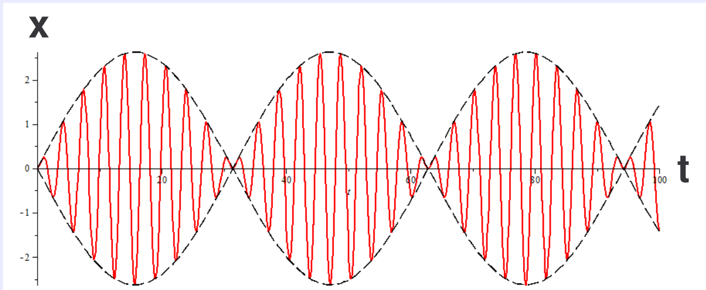
$$x(t) = \frac{2}{4-\omega^2}\sin\left(\frac{1}{2}(2+\omega)\,t\right)\sin\left(\frac{1}{2}(2-\omega)\,t\right).$$

This was a lot of algebra, but we have obtained an incredibly insightful form of the solution. We now have the solution decomposed into two sine functions with different frequencies, corresponding to the difference in the **natural** and **forcing frequencies**. If ω is near 2, there is a separation of time-scales in the two modes. We have that

1. The **slow oscillatory mode** has wavelength $4\pi/(2-\omega)$. This mode can be thought of as an envelop which restricts all other modes. (All other modes must multiply through this function, so can only be as big as this slow mode allows it to be.)

2. The **fast oscillatory mode** has wavelength $4\pi/(2+\omega)$. This mode oscillates faster than the other mode but is restricted through each period by its slower counterpart.

3. Since sine is bounded by -1 and 1, the maximal amplitude of the solution is $2/(4-\omega^2)$.

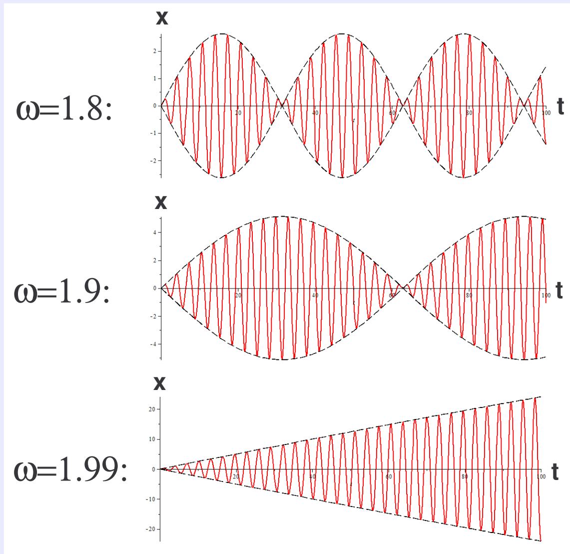


This raises a very interesting question: What happens as the forcing frequency is *changed* related to the fixed natural frequency of the system (i.e. the frequency the undamped pendulum or spring swings when left alone)? In particular, what happens as $\omega \to 2$?

We can consider this as ω approaches 2 from either side, since the separation of time-scales holds. We make the following observations:

1. The ω approachs 2, the wavelength of the slow mode *explodes* while the wavelength of the fast mode stays roughly the same. That is to say, the separation in time-scales intensifies in that the number of times the fast mode completes its cycle before the slow mode completes its cycle becomes unbounded.

2. The amplitude $\frac{2}{4-\omega^2}$ explodes. In fact, in the limit, we have that the amplitude is infinite.



Something seems to be going incredibly wrong in this example. We know that the solution oscillates with a fixed period so, as time goes on, the solution will make increasingly erratic jumps from the positive extreme to the negative extreme. In fact, the jumps approach infinite amplitude even as the period approaches a finite limit.

To understand what is happening, we will reconsider the physical motivaton. Recall that 2 is the natural frequency for the **underlying system**—that is, this is the frequency at which the body naturally oscillates if we simply let it go. Now imagine shaking the pendulum at a frequency which is completely **in phase** with the natural rhythm of the body. In this case, every time the pendulum naturally wants to swing left, we give it an extra push, and every time it wants to swing right, we give it an extra push in that direction, too. If we do this exactly in sync with the body's natural rhythm, the amplitude will grow. This is like pushing somebody on a swing. We get the most out of our effort if we wait to push when the swing is at the top of its arc, and push in the direction it is already traveling.

We have discovered an interesting phenomenon which is a concern in many applications—namely, that *forcing functions may yield large amplitude responses when the forcing frequency is close to the natural frequency of the system.* This phenomenon is called **resonance**. To complete the discussion, we consider the limiting case of $\omega = 2$.

Example 4

Solve the initial value problem

$$\begin{cases} x'' + 4x = \cos(2t) \\ x(0) = 0, \\ x'(0) = 0. \end{cases}$$

Comment on the behavior of the solution in the context of the physical set-up as a pendulum/spring system.

Solution: Notice that the complementary solution remains $x_c(t) = C_1 \cos(2t) + C_2 \sin(2t)$ but that the trial function $x_p(t) = A\cos(2t) + B\sin(2t)$ no longer works. Rather, we must use the trial function $x_p(t) = At\cos(2t) + Bt\sin(2t)$. This gives

$$x_p''(t) = 4B\cos(2t) - 4A\sin(2t) - 4At\cos(2t) - 4Bt\sin(2t).$$

We therefore have

$$\begin{aligned} x_p'' + 4x_p &= [4B\cos(2t) - 4A\sin(2t) - 4At\cos(2t) - 4Bt\sin(2t)] \\ &\qquad + 4\left[At\cos(2t) + Bt\sin(2t)\right] \\ &= 4B\cos(2t) - 4A\sin(2t) \\ &= \cos(2t) \end{aligned}$$

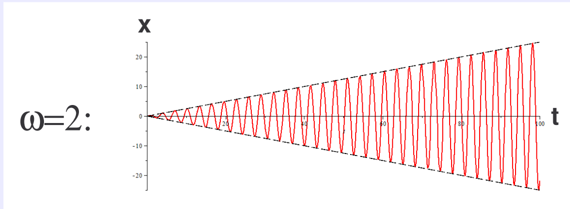
so that $A = 0$ and $B = 1/4$. It follows that

$$x(t) = C_1\cos(2t) + C_2\sin(2t) + \frac{1}{4}t\sin(2t).$$

The initial conditions $x(0) = 0$ and $x'(0) = 0$ give $C_1 = C_2 = 0$ so that we have the simple solution

$$x(t) = \frac{1}{4}t\sin(2t).$$

Just as we expected, we have a solution which oscillates with increasing amplitude (as t grows). We have filled in the gap in our previous physical reasoning. Even though the solution methods were completely different, the limit of the previous solution approaches this resonate solution as $\omega \to 2$.



Before we carry this example too far, we should recognize that there are physical constraints which we have no considered in this model. In particular, we have not considered any damping, which would certainly become a significant concern as the pendulum begins to pick up speed. We have also neglected that a pendulum may swing over the top, and that a spring may simply break it it is compressed or overextended to a significant degree. To incorporate these effects would require introducing nonlinearities which are beyond the scope of this course to handle.

Suggested Problems

1. Use the Method of Undetermined Coefficients Algorithm with the modified step 2* to solve the following initial value problems (all derivatives with respect to x):

(a) $\begin{cases} y'' + 3y' + 2y = e^x, \\ y(0) = 0, \\ y'(0) = 0 \end{cases}$

(b) $\begin{cases} 4y'' + 9y = 12\sin\left(\frac{3}{2}x\right), \\ y(0) = 3, \\ y'(0) = -1 \end{cases}$

(c) $\begin{cases} y'' - 6y' + 9y = 6xe^{3x}, \\ y(0) = 1, \\ y'(0) = -1 \end{cases}$

(d) $\begin{cases} y'' + 4y' + 5y = e^{-2x}\sin(x), \\ y(0) = 0, \\ y'(0) = 0 \end{cases}$

2. Consider a 9 kg weight attached to the end of a spring which requires a force of 4 Newtons to stretch one meter. Suppose the spring does not experience any damping. If the mass is initially stretched 1 meters to the right and released from rest, find the solution describing the position of the mass as a function of time.

3. Determine the value of ω_0 for which the following systems experience pure resonance, and then determine the solution $x(t)$ to the initial value problem with that choice for ω_0:

(a) $\begin{cases} x'' + 4x = 2\cos(\omega_0 t) \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$

(b) $\begin{cases} 9x'' + 4x = \sin(\omega_0 t) \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$

(c) $\begin{cases} 25x'' + 16x = 40\sin(\omega_0 t) + 40\cos(\omega_0 t) \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$

(d) $\begin{cases} 4x'' + 16x = 16\sin(\omega_0 t) \\ x(0) = 2 \\ x'(0) = -1 \end{cases}$

4. Consider the following example:

$$x'' + x' + x = \cos(\omega t). \tag{2}$$

where ω is as yet undetermined. That is to say, suppose we have $m = 1$ kg, $c = 1$ $N/(m/s)$ and $k = 1$ N/m.

(a) Find the general solution of (2). [**Hint:** Note that we do not need to consider cases for ω!]

(b) By considering the limit as $t \to \infty$, divide the solution from part (a) into two parts: a **transient solution** $x_{tr}(t)$ which goes to zero in the limit, and a **steady periodic** solution $x_{sp}(t)$ which does not. (In other words, write $x(t) = x_{tr}(t) + x_{sp}(t)$.)

(c) Find the amplitude of the steady periodic function $x_{sp}(t)$ found in part (c). [**Hint:** Consider writing the portion $x_{sp}(t)$ in the form $A\cos(\omega t - \alpha)$ but only find A.]

(d) At which value of ω does A achieve its maximum? Interpret this value in terms of the physical system. In particular, how does it compare to the quasi frequency of the unforced system? [**Hint:** Take the derivative of A with respect to ω!]