Math 133A, Week 14: Systems of Differential Equations

Section 1: Motivating Example

Imagine a being asked to build mathematical model of the chemical reaction $X \to Y$ occurring in a mixing tank subject to continuous inflow and outflow of both species. For modeling purposes, we will make the following assumptions:

1. The concentrations of X and Y in the tank at time t are denoted by $x(t)$ and $y(t)$, respectively.
2. Inflow occurs at constant rate and outflow rate proportional to either $x(t)$ or $y(t)$.
3. The reaction $X \to Y$ negatively affects X, positively affects Y, and occurs at a rate proportional to $x(t)$.

These assumptions can be accommodated by the following differential equation model:

$$\begin{cases} \dfrac{dx}{dt} = 1 - x \\ \dfrac{dy}{dt} = 1 + x - 2y \\ x(0) = 0, \quad y(0) = 1. \end{cases} \tag{1}$$

Without dwelling too long on the physical motivation, we can image the positive terms as contributing to an increase in the amount of the associated variable, and the negative terms corresponding to a decrease in the corresponding amounts.

This is an example of a first-order *system* of differential equations (with initial conditions). We formally define the following.

Definition 1

The general form of a **first-order system of differential equations** in two variables is

$$\begin{cases} \dfrac{dx}{dt} = f(x, y), & x(t_0) = x_0 \\ \dfrac{dy}{dt} = g(x, y), & y(t_0) = y_0. \end{cases} \tag{2}$$

A **solution** of such a system of differential equations is a set of functions $\{x(t), y(t)\}$ which satisfy (2).

This should seem familiar from our studies to date. The difference is that we have *two* variables which change over time, and that both of these variables may influence *each other's* rate of change. It is common to say that such systems are **coupled** because the behavior of x influences the behavior of y, and y influences x. Such a situation is common in real-world systems. For example, the behavior of a predator in the wilderness depends upon the behavior of its prey, and vice-versa. Similarly, competitors for consumers in a competitive marketplace must careful react to their the policies, successes, and failures of their rivals, just as their rivals react to their own.

Even though the generalization from a single first-order equation to a coupled system of two equations may seem like a small one, we will see that the approaches taken will be significantly different. We start with the following.

Example 1

Show that the initial value problem (10) has the following solution:

$$\begin{aligned} x(t) &= 1 - e^{-t} \\ y(t) &= 1 - e^{-t} + e^{-2t}. \end{aligned}$$

Describe the long-term behavior of the system.

Solution: Even though there are two functions, $x(t)$ and $y(t)$, we can check solutions by simply evaluating on the left-hand and right-hand sides of the equations as before. We can easily check $x(0) = 1-e^{-(0)} = 0$,

$y(0) = 1 - e^{-(0)} + e^{-2(0)} = 1$, and

$$\frac{dx}{dt} = \frac{d}{dt}\left[1 - e^{-t}\right] = e^{-t} = 1 - [1 - e^{-t}] = 1 - x(t)$$

and

$$\begin{aligned} \frac{dy}{dt} &= \frac{d}{dt}\left[1 - e^{-t} + e^{-2t}\right] \\ &= e^{-t} - 2e^{-2t} \\ &= 1 + [1 - e^{-t}] - 2[1 - e^{-t} + e^{-2t}] \\ &= 1 + x(t) - 2y(t). \end{aligned}$$

It follows that the pair of functions $\{x(t), y(t)\}$ is a solution.

To check the long-term behavior, we evaluate the limit as $t \to \infty$. We have

$$\begin{aligned} \lim_{t\to\infty} x(t) &= \lim_{t\to\infty} \left[1 - e^{-t}\right] = 1 \\ \lim_{t\to\infty} y(t) &= \lim_{t\to\infty} \left[1 - e^{-t} + e^{-2t}\right] = 1. \end{aligned}$$

It follows that the system asymptotically approaches a state of equilibrium where x and y have the constant concentrations $x^* = 1$ and $y^* = 1$. Note that this is a *dynamic* equilibrium in the sense that the reaction $X \to Y$ is still occurring in the tank. This process is, however, balanced by the inflow and outflow processes.

This example is somewhat of an outlier. It is not common for systems of first-order differential equations to have solutions which may be expressed in terms of simple functions. We will investigate alternate methods for analyzing such systems shortly.

Section 2: Reduction to Systems

We might wonder how common the form (2) is for systems of differential equations. Consider our earlier differential equation for the motion of an undamped pendulum from the first week of class:

$$x''(t) + \frac{k}{m}x(t) = 0. \tag{3}$$

where $k > 0$ was the **restoring force constant** and $m > 0$ was the object's **mass**.

We might be tempted to think that (3) may be manipulated into the form (2) by rearranging the terms. It is important to notice, however, that (2) only permits *first-order derivatives*. A second-order derivative like $x''(t)$ is not permitted in (2) and, in fact, the techniques we develop over the next few weeks will not be applicable to such a system. Even the intuition for vectors field diagrams (or slope field diagrams) will not be applicable since these depended on interpreting first-order derivatives as the slope of the solution!

Before we abandon hope, however, consider the following argument. We will make the substitutions

$$\left\{ \begin{array}{r} x_1(t) = x(t) \\ x_2(t) = x'(t) \end{array} \right. \tag{4}$$

This is undoubtedly an odd thing to do, but we wish to carry through with it. We wish to rewrite (3) in terms of these new variables $x_1(t)$ and $x_2(t)$, which requires obtaining at the very least a term $x''(t)$. To accomplish this, we differentiate (4) with respect to t. We have

$$\left\{ \begin{array}{r} x_1'(t) = x'(t) \\ x_2'(t) = x''(t). \end{array} \right. \tag{5}$$

We now notice two things. First of all, the first equation in (5) can be simplified by observing that $x_1'(t) = x'(t) = x_2(t)$. Secondly, the second equation in (5) relates to $x''(t)$ in (5). Rewriting (5) we obtain $x''(t) = -\frac{k}{m}x(t) = -\frac{k}{m}x_1(t)$. Combining everything, we obtain the following first-order system of differential equations in x_1 and x_2:

$$\left\{ \begin{array}{l} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = -\dfrac{k}{m}x_1. \end{array} \right. \tag{6}$$

We should pause to consider exactly what we have done. We have rewritten what was originally a second-order differential equation as a first-order system of differential equations in two variables. Any solution of (3) corresponds to a solution of (6), and vice versa, where we interpret $x_1(t)$ as the position of the pendulum and $x_2(t)$ as the velocity. Being able to represent this system as a first-order system will have significant advantages when analyzing the behavior of the system!

Reduction Algorithm:
Almost all higher-order ordinary differential equation can be reduced to a system of first-order differential equations. The general algorithm is the following:

S.1 Solve for the highest-order derivative in the expression to obtain

$$x^{(n)}(t) = \text{everything else.} \tag{7}$$

S.2 Assign an independent variable to $x(t)$ and then to each derivative of $x(t)$ up to the $(n-1)^{st}$ order. That is, we set

$$x_1(t) = x(t), \quad x_2(t) = x'(t), \quad \ldots, \quad x_n(t) = x^{(n-1)}(t).$$

S.3 Differentiate the first $n-1$ terms in the expression in S.2 with respect to t and substitute to remove the original variable x. This yields

$$x_1'(t) = x_2(t), \quad x_2'(t) = x_3(t), \quad \ldots, \quad x_{n-1}'(t) = x_n(t).$$

S.4 Differentiate the final nth term in the expression in S.2 with respect to t then substitute into (9) to obtain

$$x_n'(t) = f(t, x_1, \ldots, x_{n-1}).$$

S.5 If applicable, substitute the initial conditions $x_1(t_0) = x(t_0) = x_0$, $x_2(t_0) = x'(t_0) = x_0'$, $\ldots$, $x_n(t_0) = x^{(n-1)}(t_0) = x_0^{(n-1)}$.

Example 2

Convert the following third-order differential equation into a system of first-order differential equations:

$$\begin{cases} xy''' - y' + y^2 = x\sin(x), \\ y(0) = 2,\ y'(0) = -1,\ y''(0) = 0. \end{cases}$$

Solution: We apply the Reduction Algorithm. Since the equation is third-order, we need to introduce three new variables corresponding all derivatives of y up to one short of the highest order. We have

$$\begin{cases} y_1 = y \\ y_2 = y' \\ y_3 = y''. \end{cases}$$

Notice that we include the variable $y_3 = y''$ even though y'' does not appear in the differential equation. In order to obtain a system in the variables y_1, y_2, and y_3, we need to rearrange the differential equation in terms of the highest order derivative (y''', in this case). We have

$$y''' = \frac{1}{x}y' - \frac{1}{x}y^2 + \sin(x).$$

Taking the derivative of the substitutions gives

$$\begin{cases} y_1' = y' = y_2 \\ y_2' = y'' = y_3 \\ y_3' = y''' = \dfrac{1}{x}y' - \dfrac{1}{x}y^2 + \sin(x) = \dfrac{1}{x}y_2 - \dfrac{1}{x}y_1^2 + \sin(x). \end{cases}$$

The initial conditions can also be quick determined to be

$$\begin{cases} y_1(0) = y(0) = 2 \\ y_2(0) = y'(0) = -1 \\ y_3(0) = y''(0) = 0. \end{cases}$$

The final system of equations is:

$$\begin{cases} y_1' = y_2, & y_1(0) = 2 \\ y_2' = y_3, & y_2(0) = -1 \\ y_3' = \frac{1}{x}y_2 - \frac{1}{x}y_1^2 + \sin(x), & y_3(0) = 0. \end{cases}$$

Section 3: Linear Systems of DEs

Solving coupled systems of differential equations in even just two variables may be very difficult (or even impossible). To start our discussion of such systems, we will first consider the following.

Definition 2

A **first-order homogeneous linear system of differential equations** in two variables has the form

$$\begin{cases} \dfrac{dx}{dt} = ax + by \\ \dfrac{dy}{dt} = cx + dy. \end{cases} \tag{8}$$

Before we attempt to find the analytic solution $\{x(t), y(t)\}$ of (8), we consider the problem from a geometrical point of view. In other words, we will try to draw a *picture*. This will be similar to drawing slope fields for one-dimensional systems, but will have important distinctions.

We make the following observations:

- The system is first-order so that, at every point (x, y) in the (x, y)-plane we know whether any solution through the point (x, y) is pointed right or left ($x'(t) > 0$ or $x'(t) < 0$) or up or down ($y'(t) > 0$ or $y'(t) < 0$).
- We know the equation $x'(t) = 0$ corresponds to $y = -\frac{a}{b}x$ and $y'(t) = 0$ corresponds to $y = -\frac{c}{d}x$. (Note that the slopes may be positive, since the constants a, b, c, and d may be negative.) In other words, we know where the the solution $\{x(t), y(t)\}$ is flat ($y'(t) = 0$) and vertical ($x'(t) = 0$). These curves are called **nullclines** (alternatively, **isoclines**). Also note that nullclines generalize to nonlinear differential equations (2) by considering $f(x, y) = 0$ and $g(x, y) = 0$.

All told, at each point in the (x, y) plane solutions may tend toward one of *eight* possible directions, which are perhaps most easily identified with their compass points: North ($\uparrow$), Northeast ($\nearrow$), East ($\rightarrow$), Southeast ($\searrow$), South ($\downarrow$), Southwest ($\swarrow$), West ($\leftarrow$), and Northwest ($\nwarrow$). The resulting picture is called a **vector field diagram** (alternatively, **direction field diagram**).

Example 3

Sketch the vector field diagram for

$$\frac{dx}{dt} = -x + 3y$$
$$\frac{dy}{dt} = 3x - y.$$

Solution: We first determine that

$$\frac{dx}{dt} = 0 \quad \Longrightarrow \quad y = \frac{1}{3}x$$

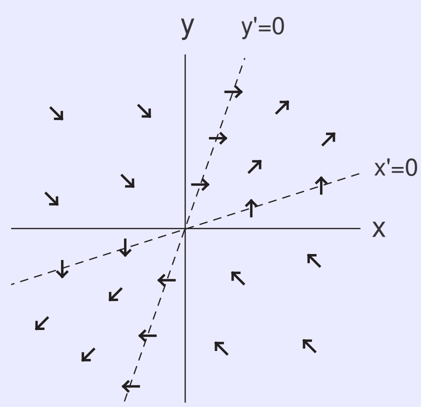
and

$$\frac{dy}{dt} = 0 \quad \Longrightarrow \quad y = 3x.$$

The question then becomes what happens in the regions between these two lines. We can see that, for instance, that if we want to determine where $x' > 0$ and $y' > 0$, we have

$$x' > 0 \quad \Longrightarrow \quad -x + 3y > 0 \quad \Longrightarrow \quad y > \frac{1}{3}x$$
$$y' > 0 \quad \Longrightarrow \quad 3x - y > 0 \quad \Longrightarrow \quad y < 3x$$

so that the direction of flow is northeast ($\nearrow$) if we are *above* the line $y = \frac{1}{3}x$ and *below* the line $y = 3x$. Eventually, we arrive at the following picture:



Example 4

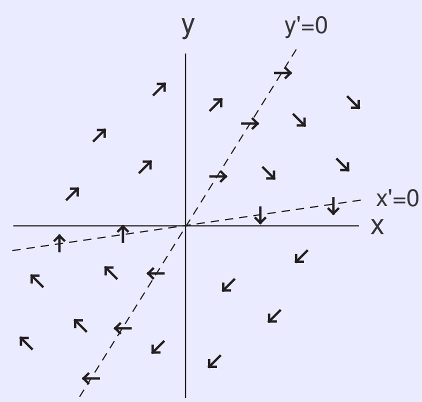
Sketch the vector field diagram for

$$\frac{dx}{dt} = -x + 5y$$
$$\frac{dy}{dt} = -2x + y.$$

Solution: We can easily determine that $x' = 0$ implies $y = \frac{1}{5}x$, and that $y' = 0$ implies $y = 2x$. We can also determine what happens in the four regions. If we want $x' > 0$ and $y' > 0$, for instance, we have

$$x' > 0 \implies -x + 5y > 0 \implies y > \frac{1}{5}x$$
$$y' > 0 \implies -2x + y \implies y > 2x.$$

So we have $\nearrow$ when we are *above* both of the lines. Considering the rest of the regions, we arrive at the following picture:



Without even attempting to solve the system of differential equations, we can tell very important things about the types of behaviors we might encounter. It looks like the solutions of Example 3 originate somewhere in the top-left or bottom-right, pool together, then travel toward either the top-right or the bottom-left. Solutions of Example 4, by contrast, appear to spiral around $(0, 0)$, although it is unclear whether they approach $(0, 0)$ or drift away.

Section 4: Solutions / Real Eigenvalues

To investigate how we might find a solution to these systems, we rewrite (8) in an alternative form. In particular, we can use our newfound knowledge of linear algebra to rewrite the system (8) in **matrix form** as

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} \tag{9}$$

where

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix}, \ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \ \text{ and } \ \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

This form suggests that results from linear algebra will be relevant in general for solving systems of linear differential equations, and this intuition is completely justified.

To motivate the general solution method, consider the similarities to the first-order linear differential equation in one variable:

$$\frac{dx}{dt} = ax, \quad x(0) = x_0$$

We know this has solution $x(t) = x_0 e^{at}$. The question is how to extend this result to something applicable to (9). It seems natural to write $\mathbf{x}(t) = e^{At}\mathbf{x}_0$, but we have nothing (yet!) to justify such notation. After all, we do not know what we mean by the exponential of a matrix A.

Consider instead the following argument. We will guess that the solution $\mathbf{x}(t) = (x(t), y(t))$ has the exponential form e^{at}, but we will allow the components of $\mathbf{x}(t)$ to vary according to some constant vector. In other words, we write $\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$ for some $\lambda \in \mathbb{R}$. This keeps the general intuition that the solution is exponential while allowing that each component may be weighted independently.

We now check how this solution performs in the LHS and RHS of (9). We have

$$\frac{d\mathbf{x}}{dt} = \frac{d}{dt}[\mathbf{v}e^{\lambda t}] = [\lambda \mathbf{v}]e^{\lambda t}$$

and

$$A\mathbf{x} = A[\mathbf{v}e^{\lambda t}] = [A\mathbf{v}]e^{\lambda t}.$$

It follows that we have

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} \iff [\lambda\mathbf{v}]e^{\lambda t} = [A\mathbf{v}]e^{\lambda t} \iff A\mathbf{v} = \lambda\mathbf{v}$$

where we could divide by $e^{\lambda t}$ because it is never zero.

If you get the sense that we have seen this equation before, it is because we have. This is exactly the eigenvalue/eigenvector equation for the matrix A! In this context of differential equations, this tells us that eigenvalues and eigenvectors pairs $\{\lambda_1, \mathbf{v}_1\}$ and $\{\lambda_2, \mathbf{v}_2\}$ give us solutions to (9) of the form $\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda_1 t}$ and $\mathbf{x}_2(t) = \mathbf{v}_2 e^{\lambda_2 t}$.

In fact, we have the following result:

Theorem 1

Consider a first-order linear system of differential equations of the form (9) with solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$. Then

$$\mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t)$$

is also a solution of (9), where $C_1, C_2 \in \mathbb{R}$ are arbitrary constants.

Proof

Suppose $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are solutions of (9) and define $\mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t)$ where $C_1, C_2 \in \mathbb{R}$. Then we have

$$\begin{aligned} \mathbf{x}'(t) &= C_1\mathbf{x}_1'(t) + C_2\mathbf{x}_2'(t) \\ &= C_1 A\mathbf{x}_1(t) + C_2 A\mathbf{x}_2(t) \\ &= A(C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t)) = A\mathbf{x}(t). \end{aligned}$$

It follows that $\mathbf{x}(t)$ is a solution of (9).

It is easy to extend our previous eigenvector argument into the following, which tells us exactly what the solutions of (9) are under certain conditions. We will omit the proof (which requires a little more linear algebra than we will have time to justify).

Theorem 2

Consider the system (9) and suppose that A has eigenvalue/eigenvector pairs $\{\lambda_1, \mathbf{v}_1\}$ and $\{\lambda_2, \mathbf{v}_2\}$ where λ_1 and λ_2 are real-valued and distinct (i.e. $\lambda_1 \neq \lambda_2$). Then the solution of (9) has the form

$$\mathbf{x}(t) = C_1 \mathbf{v}_1 e^{\lambda_1 t} + C_2 \mathbf{v}_2 e^{\lambda_2 t}.$$

Example 5

Find the general solution of the following initial value problem:

$$\begin{cases} \dfrac{dx}{dt} = -x + 3y \\ \dfrac{dy}{dt} = 3x - y \\ x(0) = 2, \ y(0) = 0. \end{cases}$$

Solution: Notice that we have

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad \text{with} \quad A = \begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix}.$$

We can quickly compute that the eigenvalues are given by $(-1-\lambda)(-1-\lambda)-9 = \lambda^2+2\lambda-8 = (\lambda+4)(\lambda-2) = 0$ so that $\lambda_1 = -4$ and $\lambda_2 = 2$. The corresponding eigenvectors are $\mathbf{v}_1 = (1, -1)$ and $\mathbf{v}_2 = (1, 1)$. It follows that we have two solutions of the form $\mathbf{x}_1(t) = \mathbf{v}_1 e^{\lambda_1 t}$ and $\mathbf{x}_2(t) = \mathbf{v}_2 e^{\lambda_2 t}$. It follows that the general solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}.$$

To solve for the constants C_1 and C_2, we use the initial conditions $x(0) = 2$ and $y(0) = 0$. We have

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4(0)} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2(0)}.$$

It follows that we need to satisfy

$$\begin{aligned} C_1 + C_2 &= 2 \\ -C_1 + C_2 &= 0. \end{aligned}$$

This is a system of two equations on two unknowns which may be solved in a number of ways. For our purposes, we will notice the lines may be added to obtain $2C_2 = 2$, or $C_2 = 1$. This may then be substituted in either equation to give $C_1 = 1$. It follows that our particular solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}.$$

Section 5: Complex Eigenvalues

We saw that the key to solving a linear systems of differential equations was determining their eigenvalue and eigenvector pairs, $\{\lambda_i, \mathbf{v}_i\}$, to generate solutions of the form

$$x_i(t) = \mathbf{v}_i e^{\lambda_i t}. \tag{10}$$

If this seemed too simple to be believed, your hesitation is not entirely unwarranted. Reconsider the characteristic polynomial for the eigenvalues of a 2×2 matrix. We have:

$$\lambda^2 + (a+d)\lambda + (ad - bc) = 0$$

which, after applying the quadratic formula, gave

$$\lambda = \frac{-(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2a}.$$

We should quickly realize that we do not always have two distinct roots to this expression. In general, we can have two other cases: *complex roots* (i.e. $\lambda = \alpha + \beta i$) or *repeated roots* (i.e. $\lambda := \lambda_1 = \lambda_2$).

The question then becomes how the solutions of the form (10) are modified for these particular cases. We will start by considering complex eigenvalues.

Theorem 3

If A has a complex pair of eigenvalues and eigenvectors of the form $\{\lambda = \alpha + \beta i, \mathbf{v} = \mathbf{a} + \mathbf{b}i\}$ then the solution of the corresponding linear system has the form

$$\mathbf{x}(t) = C_1 e^{\alpha t}\left(\mathbf{a}\cos(\beta t) - \mathbf{b}\sin(\beta t)\right) + C_2 e^{\alpha t}\left(\mathbf{a}\sin(\beta t) + \mathbf{b}\cos(\beta t)\right).$$

Example 6

Determine the solution of

$$\begin{aligned} \frac{dx}{dt} &= -x + 5y, \qquad && x(0) = 1 \\ \frac{dy}{dt} &= -2x + y, \qquad && y(0) = 1. \end{aligned}$$

Solution: To find the eigenvalues, we realize

$$A = \begin{bmatrix} -1 & 5 \\ -2 & 1 \end{bmatrix}, \quad \text{so} \quad A - \lambda I = \begin{bmatrix} -1-\lambda & 5 \\ -2 & 1-\lambda \end{bmatrix}.$$

The characteristic polynomial is given by

$$(-1-\lambda)(1-\lambda) + 10 = \lambda^2 + 9 = 0.$$

It follows that $\lambda = \pm 3i$. We need to find the eigenvectors corresponding to these values. We have

$$(A - (3i)I) = \begin{bmatrix} -1-3i & 5 \\ -2 & 1-3i \end{bmatrix}.$$

To find the corresponding eigenvector, we row reduce to get

$$\left[\begin{array}{cc|c} -1-3i & 5 & 0 \\ -2 & 1-3i & 0 \end{array}\right] \xrightarrow{(-1+3i)R_1} \left[\begin{array}{cc|c} (-1-3i)(-1+3i) & 5(-1+3i) & 0 \\ -2 & 1-3i & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|c} 10 & -5+15i & 0 \\ -2 & 1-3i & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -\frac{1}{2}+\frac{3}{2}i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so that $\mathbf{v} = (1-3i, 2)$. We rewrite this as

$$\mathbf{v} = \begin{bmatrix} 1-3i \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + i \begin{bmatrix} -3 \\ 0 \end{bmatrix}.$$

We set $\alpha = Re(\lambda) = 0$ and $\beta = Im(\lambda) = 3$ and $\mathbf{a} = Re(\mathbf{v}) = (1,2)$ and $\mathbf{b} = Im(\mathbf{v}) = (-3,0)$. It follows that the general solution is

$$\begin{aligned} \mathbf{x}(t) = C_1 &\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cos(3t) - \begin{bmatrix} -3 \\ 0 \end{bmatrix} \sin(3t) \right) \\ &+ C_2 \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \sin(3t) + \begin{bmatrix} -3 \\ 0 \end{bmatrix} \cos(3t) \right). \end{aligned}$$

To solve for C_1 and C_2, we utilize the initial conditions $x(0) = 1$ and $y(0) = 1$. At $t = 0$ we have

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

so that we have

$$\begin{aligned} C_1 - 3C_2 &= 1 \\ 2C_1 \qquad &= 1 \end{aligned}.$$

It follows immediately that $C_1 = 1/2$ and $C_2 = -1/6$ so we have

$$\begin{aligned} \mathbf{x}(t) &= \frac{1}{2} \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cos(3t) - \begin{bmatrix} -3 \\ 0 \end{bmatrix} \sin(3t) \right) \\ &\quad - \frac{1}{6} \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \sin(3t) + \begin{bmatrix} -3 \\ 0 \end{bmatrix} \cos(3t) \right) \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(3t) + \frac{1}{3} \begin{bmatrix} 4 \\ -1 \end{bmatrix} \sin(3t) \end{aligned}$$

Note: Although this was a lot of work, it should be noted that, for complex eigenvalue pairs, we only need to perform the eigenvector computation *once*. This is because, supposing $\lambda = \alpha + \beta i$ yields the eigenvector $\mathbf{v} = \mathbf{a} + \mathbf{b}i$, then $\lambda = \alpha - \beta i$ will necessarily yield $\mathbf{v} = \mathbf{a} - \mathbf{b}i$. So performing the computation with either pair suffices to completely solve the problem!

Section 6: Repeated Eigenvalues

It remains to consider eigenvalues which are real but repeated. In this case, we have an eigenvalue and eigenvector pair $\{\lambda, \mathbf{v}\}$ which as before give us a solution $\mathbf{x}_1(t) = \mathbf{v}e^{\lambda t}$.

The problem with this solution is that it is *not complete*. As with the other cases for a system in two variables, we will need two independent solutions. We might suspect the completion of the solution has something to do with the generalized eigenvectors we derived earlier, and this intuition is justified. We have the following result.

Theorem 4

If A has a repeated eigenvalue λ and a single eigenvector $\mathbf{v}$ then the solution of the corresponding linear system has the form

$$\mathbf{x}(t) = (C_1\mathbf{v} + C_2(\mathbf{v}t + \mathbf{w}))e^{\lambda t}$$

where $\mathbf{w} \in \mathbb{R}^2$ is a **generalized eigenvector** of A.

Note: As expected, we have a fundamental solution of the form $\mathbf{x}_1(t) = \mathbf{v}e^{\lambda t}$, but we also have a fundamental solution of the form $\mathbf{x}_2(t) = (\mathbf{v}t + \mathbf{w})e^{\lambda t}$. The additional factor of t is a recurring motif when there are repeated roots in linear differential equations.

Example 7

Determine the solution of

$$\begin{aligned} \frac{dx}{dt} &= x - 4y, \qquad & x(0) = -1 \\ \frac{dy}{dt} &= x - 3y, \qquad & y(0) = 2. \end{aligned}$$

Solution: To find the eigenvalues, we realize

$$A = \begin{bmatrix} 1 & -4 \\ 1 & -3 \end{bmatrix}, \quad \text{so} \quad A - \lambda I = \begin{bmatrix} 1-\lambda & -4 \\ 1 & -3-\lambda \end{bmatrix}.$$

The characteristic polynomial is given by

$$(1-\lambda)(-3-\lambda) + 4 = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2 = 0$$

so that $\lambda = -1$ is a repeated eigenvector. To check for the eigenvector(s) corresponding to this value, we have

$$(A - (-1)I) = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}.$$

To find the corresponding eigenvector, we row reduce to get

$$\left[\begin{array}{cc|c} 2 & -4 & 0 \\ 1 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

so that $\mathbf{v} = (2,1)$. We notice that we have not obtained eigenvectors, so that we need to look for a generalized eigenvector $\mathbf{w}$. We have

$$(A - \lambda I)\mathbf{w} = \mathbf{v} \quad \Longrightarrow \quad \left[\begin{array}{cc|c} 2 & -4 & 2 \\ 1 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 0 & 0 \end{array}\right].$$

If we set $w_2 = t$, we see that $w_1 = 1 + 2t$ so that we have

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1+2t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Setting $t = 0$, we have $\mathbf{w} = (1,0)$.

It follows that the general solution is given by

$$\mathbf{x}(t) = \left(C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \left(t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \right) e^{-t}$$

To solve for C_1 and C_2, we utilize the initial conditions $x(0) = -1$ and $y(0) = 2$. At $t = 0$ we have

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix} = C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

which implies

$$\begin{aligned} 2C_1 + C_2 &= -1 \\ C_1 \qquad &= 2. \end{aligned}$$

It follows that $C_1 = 2$ and $C_2 = -5$. It follows that the solution is

$$\begin{aligned} \mathbf{x}(t) &= \left(2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 5 \left(t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \right) e^{-t} \\ &= \left(\begin{bmatrix} -1 \\ 2 \end{bmatrix} - t \begin{bmatrix} 10 \\ 5 \end{bmatrix} \right) e^{-t} \end{aligned}$$

Suggested Problems

1. Sketch the vector field for the following systems of first-order differential equations, and then verify that the equations have the given solution $\{x(t), y(t)\}$.

   (a) $\begin{cases} x' = -y \\ y' = x \\ x(t) = C_1 \cos(t) + C_2 \sin(t) \\ y(t) = C_1 \sin(t) - C_2 \cos(t) \end{cases}$

   (b) $\begin{cases} x' = x \\ y' = 2x - y \\ x(t) = C_1 e^t \\ y(t) = C_1 e^t + C_2 e^{-t} \end{cases}$

(c) $\begin{cases} x' = x - y \\ y' = x + y \\ x(t) = C_1 e^t \cos(t) + C_2 e^t \sin(t) \\ y(t) = C_1 e^t \sin(t) - C_2 e^t \cos(t) \end{cases}$

(d) $\begin{cases} x' = -2x - y \\ y' = x \\ x(t) = C_1(1-t)e^{-t} - C_2 e^{-t} \\ y(t) = C_1 t e^{-t} + C_1 e^{-t} \end{cases}$

2. Rewrite the following higher-order differential equations as a system of first-order differential equations. [**Note:** All derivatives are with respect to t.]

   (a) $x'' - x' + x = t$

   (b) $x'' + x' + \sin(x) = \sin(t)$

   (c) $\dfrac{x''}{x'} + \dfrac{x'}{x} + \dfrac{x}{t} = 0$

   (d) $a_n x^{(n)} + a_{n-1} x^{(n-1)} + \cdots + a_1 x' + a_0 x = f(t)$
   where $a_0, \ldots, a_n$ are constants, $x^{(i)}$ denotes the i^{th} derivative of $x(t)$, and $f(t)$ is an unknown function of t.

   (e) $\begin{cases} x'' + 3x' + 2x = 0 \\ x(0) = 2 \\ x'(0) = -1 \end{cases}$

3. For the following, determine the nullclines, sketch the vector field, and then solve the problem. (All derivatives are with respect to t.)

   (a) $\begin{cases} x' = -2x - 3y \\ y' = x + y \\ x(0) = 1,\ y(0) = 1 \end{cases}$

   (b) $\begin{cases} x' = -x + 3y \\ y' = x + y \\ x(0) = 1,\ y(0) = 3 \end{cases}$

   (c) $\begin{cases} x' = -x + 2y \\ y' = -x - 4y \\ x(0) = 3,\ y(0) = -3 \end{cases}$

   (d) $\begin{cases} x' = x - 2y \\ y' = -x + 2y \\ x(0) = 0,\ y(0) = 3 \end{cases}$

   (e) $\begin{cases} x' = -x + y \\ y' = -x + 3y \\ x(0) = 1,\ y(0) = 2 \end{cases}$

   (f) $\begin{cases} x' = 2x - 2y \\ y' = x \\ x(0) = 1,\ y(0) = 0 \end{cases}$

   (g) $\begin{cases} x' = -3x - 9y \\ y' = x + 3y \\ x(0) = 1,\ y(0) = 0 \end{cases}$

   (h) $\begin{cases} x' = -x + 2y \\ y' = -x + y \\ x(0) = 2,\ y(0) = -1 \end{cases}$

(i) $\begin{cases} x' = \sqrt{2}x + 3y \\ y' = -3x + \sqrt{2}y \\ x(0) = \sqrt{2},\ y(0) = \sqrt{2} \end{cases}$

(j) $\begin{cases} x' = -x + y \\ y' = -4x - 5y \\ x(0) = 0,\ y(0) = -2 \end{cases}$

(k) $\begin{cases} x' = x + 9y \\ y' = -4x + 13y \\ x(0) = 2,\ y(0) = -3 \end{cases}$

4. Use the substitutions $\tilde{x}(t) = x(t) - 1$ and $\tilde{y}(t) = y(t) - 1$ to rewrite the following initial value problem as a first-order linear system in $\tilde{x}(t)$ and $\tilde{y}(t)$. Solve this initial value problem in $\tilde{x}(t)$ and $\tilde{y}(t)$ and then invert the substitution to determine the solution for $x(t)$ and $y(t)$.

$$\begin{cases} \dfrac{dx}{dt} = 1 - x \\ \dfrac{dy}{dt} = 1 + x - 2y \\ x(0) = 0, \quad y(0) = 1. \end{cases}$$