Exam 2 Review

Rules

- Open whatever
- Basically the same as Exam 1
- 75 min time limit
- Exam available 10 minutes before (11:50 AM) until 10 minutes after (1:25 pm)
- Zoom meeting for questions
- Option: upload work to Canvas (until 1:30 pm)
- Normal Office Hours
- Alana (TA) - additional Office Hours
- AEC people will be notified

Tips :
- $\ln|x|$
- Independent vs. Dependent Variables: $y(t)$, $z(x)$, $x(\theta)$, ...
- Heaviside functions: $u_c(t) = u(t-c)$

Topics (HW 4-8, Ch. 1, 3, 4, 5, 7)

- 5 Questions
- Numerical Methods
  - Forward Euler
- 2^{nd}-order ODEs

$$ay'' + by' + cy = f(t)$$

  - characteristic equations
  - Method of undetermined coefficients
  - Laplace Transforms

$$\boxed{f(t)}$$

$f(t) = 0$

- Characteristic Equation

$f(t) =$ elementary function

e^{at}, $\sin(bt)$, ct^n, ...

- Characteristic Equation
- Method of Undetermined coefficients

$y(t) = y_h(t) + y_p(t)$

General $f(t)$

- Laplace Transform

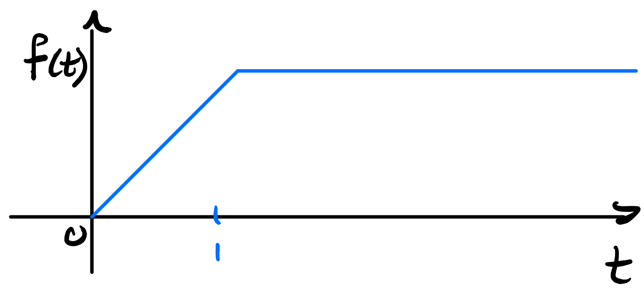
- Systems of ODEs
  - Elimination Method

$$x' = ax + by$$
$$y' = cx + dy$$

Example 1

$$y'' + 16y = f(t), \quad y(0)=1, \quad y'(0)=2$$

$$f(t) = \begin{cases} t & : 0 \le t < 1 \\ 1 & : 1 \le t < \infty \end{cases}$$



Find $Y(s)$

0) Write $f(t)$ in terms of Heaviside functions

$$f(t) = t \cdot [u_0(t) - u_1(t)] + 1 \cdot [u_1(t) - u_\infty(t)]$$

$$= t \cdot [\;1 \; - u_1(t)] + 1 \cdot [u_1(t) - \;0\;]$$

$$= t - t\,u_1(t) + 1 \cdot u_1(t)$$

$$= t - (t-1)u_1(t)$$

1) Take Laplace Transform of ODE

$$y'' + 16y = t - (t-1)u_1(t)$$

#36   #3 $n=1$ #27 $c=1$

$$\left[s^2 Y(s) - s y(0) - y'(0)\right] + 16Y(s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2}$$

$$\mathcal{L}\{(t-1)u_1(t)\} = e^{-s}\mathcal{L}\{t\} = \frac{e^{-s}}{s^2}$$

$$\left[s^2 Y(s) - s - 2\right] + 16Y(s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2}$$

$y(0)=1, \quad y'(0)=2$

$$\Rightarrow \quad s^2 Y(s) + 16Y(s) = s + 2 + \frac{1-e^{-s}}{s^2}$$

$$\Rightarrow \quad (s^2+16)\,Y(s) = \frac{s^3+2s^2+1-e^{-s}}{s^2}$$

$$\Rightarrow \quad \boxed{Y(s) = \frac{s^3+2s^2+1-e^{-s}}{s^2(s^2+16)}}$$

Example 2

$$Y(s) = \frac{3s+2}{s^2+2s+10}$$

Ask: Can $s^2+2s+10$ be factorized?

Yes

$(s+r_1)(s+r_2)$

$\rightarrow$ Partial fraction expansion

$\Rightarrow$ exponentials (Case I)

No

Complete the square

$$\left(s+\frac{b}{2}\right)^2 + c - \frac{b^2}{4}$$

$\Rightarrow$ trig functions (Case II)

$s^2+2s+10$ is <u>not</u> factorizable

$$\Rightarrow \quad (s+1)^2+9$$

$$\Rightarrow \quad Y(s) = \frac{3s+2}{(s+1)^2+9}$$

Goal: write $Y(s)$ in terms of $(s+1)$

$$Y(s) = \frac{3(s+1)-1}{(s+1)^2+9}$$

$$= \frac{3(s+1)}{(s+1)^2+9} - \frac{1}{(s+1)^2+9}$$

#19: $\mathcal{L}\{e^{at}\sin(bt)\} = \frac{b}{(s-a)^2+b^2}$

#20: $\mathcal{L}\{e^{at}\cos(bt)\} = \frac{s-a}{(s-a)^2+b^2}$

$$a=-1, \quad b=3$$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{3(s+1)}{(s+1)^2 + 9} - \frac{1}{(s+1)^2+9} \right\}$$

$$= 3\mathcal{L}^{-1}\left\{ \frac{(s+1)}{(s+1)^2+9} \right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{ \frac{3}{(s+1)^2+9} \right\}$$

$\downarrow$ #20 $\qquad\qquad$ $\downarrow$ #19

$$= 3e^{-t}\cos(3t) - \frac{1}{3}e^{-t}\sin(3t)$$

Example 3 $\qquad \frac{dy}{dt} = t y^2, \quad y(0) = 2$

Forward Euler, $\quad h = 0.1$ (time step)

$$\frac{dy}{dt} = f(t, y)$$

Let: $\quad t_n = nh, \quad y_n = y(t_n)$

FE Algorithm: $\quad y_{n+1} = y_n + h f(t_n, y_n)$

1) $f(t, y) = t y^2$

2) $y_{n+1} = y_n + h t_n y_n^2$

3) Initial conditions: $t_0 = 0$, $y_0 = 2$

$y_1 = y_0 + h t_0 y_0^2$

$= 2 + \cancel{(0.1)(0)(2^2)}$

$= 2$

$y_2 = y_1 + h t_1 y_1^2$

$= 2 + (0.1)(0.1)(2^2)$

$= 2.04$

$y_3 = y_2 + h t_2 y_2^2$

$= 2.04 + (0.1)(0.2)(2.04)^2$

$= 2.123232$

★ Tip: Only round at the <u>end</u> when you're entering answers.

Example 4

$$x' = x + 3y \quad \leftarrow \quad y = \tfrac{1}{3}(x' - x)$$

$$y' = 3x + y + 1$$

$$x(0) = 0, \quad y(0) = 1$$

1) Pick an equation and differentiate

$$x'' = x' + 3y'$$

2) Get rid of y terms

$$\begin{aligned} x'' &= x' + 3y' \\ &= x' + 3(3x + y + 1) \\ &= x' + 9x + 3y + 3 \\ &= x' + 9x + 3\tfrac{1}{3}(x' - x) + 3 \\ &= x' + 9x + x' - x + 3 \end{aligned}$$

$$\Rightarrow \quad x'' - 2x' - 8x = 3$$

A) $x_h'' - 2x_h' - 8x_h = 0$

$$r^2 - 2r - 8 = 0$$

$$r = \frac{2 \pm \sqrt{4+32}}{2} = 1 \pm 3 = \{-2, 4\}$$

$$x_h(t) = c_1 e^{-2t} + c_2 e^{4t} \quad (\text{Case I})$$

B) Let $x_p = A \Rightarrow x_p = -3/8$

c) $$\boxed{\begin{aligned} x(t) &= x_h(t) + x_p(t) \\ &= c_1 e^{-2t} + c_2 e^{4t} - 3/8 \end{aligned}}$$

use: $y = \frac{1}{3}(x' - x)$

$$\Rightarrow y(t) = \frac{1}{3}(x' - x)$$

$$= \frac{1}{3}\Big(-2c_1 e^{-2t} + 4c_2 e^{4t}$$

$$- c_1 e^{-2t} - c_2 e^{4t} + \frac{3}{8}\Big)$$

$$= \frac{1}{3}\left(-3c_1 e^{-2t} + 3c_2 e^{4t} + \frac{3}{8}\right)$$

$$= \boxed{-c_1 e^{-2t} + c_2 e^{4t} + \frac{1}{8}}$$

★ Find c_1 & c_2 using $x(0) = 0$
$y(0) = 1$

$$x(0) = c_1 + c_2 - \frac{3}{8} = 0 \quad (R1)$$

$$y(0) = -c_1 + c_2 + \frac{1}{8} = 1 \quad (R2)$$

Add $R1 + R2$: $2c_2 - \frac{1}{4} = 1$

$$\Rightarrow c_2 = \frac{5}{8}$$

Subtract $R1 - R2$: $2c_1 - \frac{1}{2} = -1$

$$\Rightarrow c_1 = -1/4$$

$$\Rightarrow \boxed{c_1 = -1/4, \quad c_2 = 5/8}$$

The final answer is:

$$\boxed{\begin{aligned} x(t) &= -\frac{1}{4}e^{-2t} + \frac{5}{8}e^{4t} - \frac{3}{8} \\ y(t) &= \frac{1}{4}e^{-2t} + \frac{5}{8}e^{4t} + \frac{1}{8} \end{aligned}}$$