# Lecture 11

## Method of Undetermined Coefficients

Linear, $2^{nd}$-order, constant-coefficient, <u>non</u>homogeneous ODEs:

$$a \frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t)$$

Nonhomogeneous $\Rightarrow f(t) \neq 0$

<u>Example 1</u>: $y'' + 4y = 8t$

$y(0) = 0, \; y'(0) = 0$

Can we still use a characteristic equation?

Let $y = ce^{rt} \Rightarrow cr^2e^{rt} + 4ce^{rt} = 8t$

No choice of r will make this true for all $t$!

Homogeneous version: $y'' + 4y = 0$

$\rightarrow$ characteristic eqn: $r^2 + 4 = 0$

Solve for $r$: $r = \pm\sqrt{-4} = \pm 2i$

Using the Table $\rightarrow$ Case II: $r = \alpha \pm \beta i$

$\Rightarrow \alpha = 0, \beta = 2$

$$y(t) = e^{\alpha t}\left[c_1 \cos(\beta t) + c_2 \sin(\beta t)\right]$$
$$= c_1 \cos(2t) + c_2 \sin(2t)$$

$(e^{0 \cdot t} = 1)$

Back to $y'' + 4y = 8t$

- Since $8t$ is a linear function, maybe $y(t)$ is one too?

$\Rightarrow$ Guess $y(t) = \underbrace{At + B}_{\text{Generic linear function}}$

If $y = At + B \Rightarrow y' = A, \; y'' = 0$

Plugging in to the ODE:

$$y'' + 4y = (0) + 4(At+B) = 8t$$

$$\Rightarrow 4At + 4B = 8t + 0$$

$$4A = 8 \Rightarrow A = 2, \quad 4B = 0 \Rightarrow B = 0$$

$$\boxed{y(t) = 2t \text{ solves the ODE!}}$$

What about the unknown constants?

→ Try to apply initial conditions

$$y(0) = 0, \quad y'(0) = 0$$

$$y(0) = 2(0) = 0 \quad ✓$$

$$y'(0) = 2 \neq 0 \quad ✗$$

While $y(t) = 2t$ "solves" the ODE, it's not the most <u>general</u> form

★ We would like to solve the ODE but still have unknown coefficients like with the homogeneous case...

... Can we use both solutions?

YES! Just <u>add</u> the two solutions.

$$y(t) = \underbrace{c_1 \cos(2t) + c_2 \sin(2t)}_{\text{homogeneous solution}} + \underbrace{2t}_{\text{particular solution}}$$

$$y(0) = c_1 \cos(0) + c_2 \cancel{\sin(0)} + 2\cancel{(0)} = \boxed{c_1 = 0}$$

$$y'(t) = -2c_1 \sin(2t) + 2c_2 \cos(2t) + 2$$

$$y'(0) = -2\cancel{c_1}\,\cancel{\sin(0)} + 2c_2 \cos(0) + 2 = 2c_2 + 2$$

$$2c_2 + 2 = 0 \Rightarrow 2c_2 = -2 \Rightarrow \boxed{c_2 = -1}$$

$$\boxed{\text{Answer: } y(t) = 2t - \sin(2t)}$$

## General Strategy

Given $ay'' + by' + cy = f(t)$

1) Solve homogeneous problem:

$$ay_h'' + by_h' + cy_h = 0$$

(use Table for characteristic equations)

$$\Rightarrow y_h(t) = c_1 y_1(t) + c_2 y_2(t)$$

2) Find the particular solution that produces $f(t)$.

$$ay_p'' + by_p' + cy_p = f(t) \rightarrow y_p(t)$$

3) The general solution will be the sum: $y(t) = y_h(t) + y_p(t)$

(Solve for $c_1$ & $c_2$ using initial conditions)

★ How do we find $y_p(t)$?? ★

- In the last example, we guessed a "generic" version of $f(t)$
- We expect this to work based on the following observations:

▷ $\frac{d}{dt}$[exponentials] = [more exponentials]

▷ $\frac{d}{dt}$[polynomials] = [more polynomials]

▷ $\frac{d}{dt}$[trig functions] = [more trig functions]

sin, cos

To guess $y_p(t)$, introduce a "trial" function that has undetermined coeffs.

- If $f(t)$ contains $e^{\alpha t}$

  $\Rightarrow y_p(t) = A e^{\alpha t}$

- If $f(t)$ contains $t^n$

  $\Rightarrow y_p(t) = c_n t^n + c_{n-1} t^{n-1} + \dots + c_1 t + c_0$

- If $f(t)$ contains $\cos(\beta t)$ <u>or</u> $\sin(\beta t)$

  $\Rightarrow y_p(t) = A \cos(\beta t) + B \sin(\beta t)$

▷ Sums of these functions result in sums of trial functions

▷ Products of these functions result in products of trial functions

☆ Example Trial functions

$$ay'' + by' + cy = f(t)$$

- $f(t) = t^2 - 5t + 2\cos(3t)$

$\Rightarrow y_p(t) = At^2 + Bt + C + D\cos(3t) + E\sin(3t)$

- $f(t) = -te^{t} + 20e^{-t}$

$\Rightarrow y_p(t) = (At+B)e^{t} + Ce^{-t}$

- $f(t) = 5t\sin(t)$

$\Rightarrow y_p(t) = (At+B)\cos(t) + (Ct+D)\sin(t)$

---

<u>Example 2</u> $\quad y'' + 4y' + 3y = 5e^{2t}$

1) Solve homogeneous problem

$$y_h'' + 4y_h' + 3y_h = 0$$

Characteristic Equation: $r^2 + 4r + 3 = 0$

$$r = \frac{-4 \pm \sqrt{16-12}}{2} = \frac{-4 \pm \sqrt{4}}{2} = \frac{-4 \pm 2}{2}$$

$$= \{-3, -1\} \leftarrow$$ Two real roots $\Rightarrow$ Case I on the Table

$$\boxed{\begin{aligned} y_h(t) &= c_1 e^{r_1 t} + c_2 e^{r_2 t} \\ &= c_1 e^{-t} + c_2 e^{-3t} \end{aligned}}$$

2) Trial Solution for $f(t) = 5e^{2t}$

Guess: $y_p(t) = Ae^{2t}$

$$y_p'' + 4y_p' + 3y_p = 5e^{2t}$$

$y_p = Ae^{2t}$, $\quad y_p' = 2Ae^{2t}$, $\quad y_p'' = 4Ae^{2t}$

$$4Ae^{2t} + 4\cdot\left(2Ae^{2t}\right) + 3Ae^{2t} = 5e^{2t}$$

$$\Rightarrow \quad 4Ae^{2t} + 8Ae^{2t} + 3Ae^{2t} = 5e^{2t}$$

$$\Rightarrow \quad 15Ae^{2t} = 5e^{2t}$$

$$\Rightarrow \quad 15A = 5 \quad \Rightarrow \quad A = 1/3$$

$$\Rightarrow \quad \boxed{y_p(t) = Ae^{2t} = \frac{1}{3}e^{2t}}$$

3) The General Solution will be

$$\boxed{\begin{aligned} y(t) &= y_h(t) + y_p(t) \\ &= c_1 e^{-t} + c_2 e^{-3t} + \frac{1}{3}e^{2t} \end{aligned}}$$

Example 3 $\quad y'' + 2y' + 5y = 13e^{t}\sin(t)$

1) Homogeneous problem:

$$y_h'' + 2y_h' + 5y_h = 0$$

Characteristic Equation: $\quad r^2 + 2r + 5 = 0$

$$r = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2}$$

$$= -1 \pm 2i$$

In Case II, $r = \alpha \pm \beta i \implies \begin{matrix} \alpha = -1 \\ \beta = 2 \end{matrix}$

$$\boxed{y_h(t) = e^{-t}\left[c_1\cos(2t) + c_2\sin(2t)\right]}$$

2) Trial solution for $f(t) = 13e^{t}\sin(t)$

Guess $y_p(t) = e^{t}\left(A\cos(t) + B\sin(t)\right)$

Plug into: $y_p'' + 2y_p' + 5y_p = 13e^t \sin(t)$

$$y_p'(t) = (A+B)e^t \cos(t) + (B-A)e^t \sin(t)$$

$$y_p''(t) = 2Be^t \cos(t) - 2Ae^t \sin(t)$$

$$\Rightarrow (7A+4B)e^t \cos(t) + (7B-4A)e^t \sin(t) = 13e^t \sin(t)$$

$$\Rightarrow \begin{aligned} 7B - 4A &= 13 \\ 7A + 4B &= 0 \rightarrow B = -\tfrac{7}{4}A \end{aligned}$$

Solving: $A = -4/5$, $B = 7/5$

$$\boxed{y_p(t) = e^t\left(-\tfrac{4}{5}\cos(t) + \tfrac{7}{5}\sin(t)\right)}$$

3) General Solution:

$$\boxed{\begin{aligned} y(t) &= y_h(t) + y_p(t) \\ &= e^{-t}(c_1 \cos(2t) + c_2 \sin(2t)) \\ &\quad + e^t\left(-\tfrac{4}{5}\cos(t) + \tfrac{7}{5}\sin(t)\right) \end{aligned}}$$