Lecture 6 : Integrating Factors

Review Separation of Variables

$$\frac{dx}{dt} = 3xt^4$$

1) $\frac{1}{x}\,dx = 3t^4\,dt$

2) $\int \frac{1}{x}\,dx = \int 3t^4\,dt$

3) $\ln(|x|) = \frac{3}{5}t^5 + c$

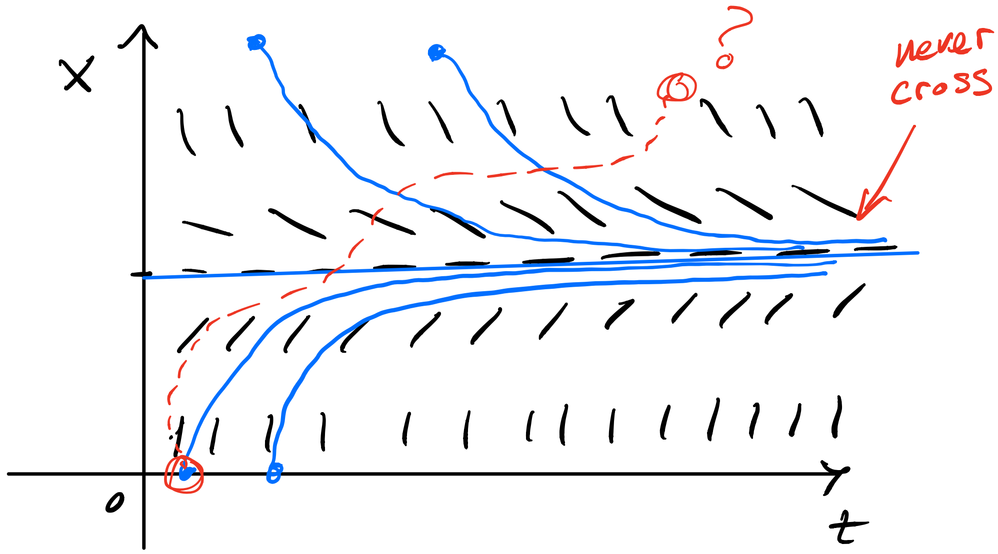
$$\boxed{\ln(|x|) - \frac{3}{5}t^5 = c}$$

"implicit solution"

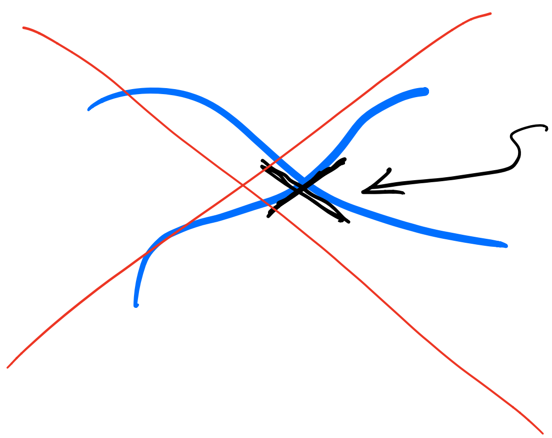
Generally : $F(x,t) = c$

or $F(y,x) = 0$, ...

Existence / Uniqueness $\Rightarrow$ Solutions don't cross each other



If two solutions crossed:



$$\frac{dx}{dt} = f(x,t)$$

need to have 2 values for solutions to cross

☆ This won't happen if $f(x,t)$ is a normal function.

Back to Integrating Factors

Ex 1

$$\frac{dy}{dx} = \cos(x)$$

$$\int \frac{dy}{dx}\,dx = \int \cos(x)\,dx$$

$$\Rightarrow \boxed{y = \sin(x) + C}$$

Ex 2

$$\frac{dy}{dx} = f(x)$$

$$\int \frac{dy}{dx}\,dx = \int f(x)\,dx$$

$$\Rightarrow \boxed{\begin{array}{c} y = F(x) + C \\ \left(\frac{dF}{dx} = f \right) \end{array}}$$

Ex 3

$$\underbrace{x\frac{dy}{dx} + y}_{} = x$$

Remember the product Rule

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + \frac{df}{dx}g$$

$$\frac{d}{dx}\left(x\,y(x)\right) = x\frac{dy}{dx} + y$$

$$\Rightarrow \underbrace{\frac{d}{dx}(xy)}_{\text{derivative of something}} = \underbrace{x}_{\text{function of } x}$$

Integrate

$$\int \frac{d}{dx}(xy)\,dx = \int x\,dx$$

$$\Rightarrow xy = \frac{1}{2}x^2 + C$$

$$xy - \frac{1}{2}x^2 = c \leftarrow \text{"implicit form"}$$

or divide by x:

$$\boxed{\underbrace{y = \frac{1}{2}x + \frac{c}{x}}_{\text{general solution}}}$$

* Assume $x \neq 0$

* c is an arbitrary number, so it cannot absorb a function of x.

Rules

- $2c \to c$
- $c^2 \to c$
- $e^c \to c$
- $\cancel{cx \to c}$

Ex 4

$$x \frac{dy}{dx} + 2y = x$$

Can we write the left hand side as the derivative of something?

$$x \frac{dy}{dx} + 2y = \frac{d}{dx}(???)$$

NOPE!

What if we multiply both sides by x?

$$\Rightarrow \quad \underbrace{x^2 \frac{dy}{dx} + 2xy}_{= \frac{d}{dx}(x^2 y)} = x^2$$

$$\Rightarrow \frac{d}{dx}\left(x^2 y\right) = x^2$$

1) Integrate:

$$\int \frac{d}{dx}\left(x^2 y\right) dx = \int x^2 dx$$

$$\rightarrow \quad x^2 y = \frac{1}{3}x^3 + c$$

2) Divide by x^2:

$$\boxed{y = \frac{1}{3}x + \frac{c}{x^2}}$$

general solution

CAN I ALWAYS DO THIS?

YES.... for 1st-order, Linear ODEs

General form of a 1^{st}-order, linear ODE is given by:

$$\frac{dy}{dx} + p(x)\, y = g(x)$$

Let's multiply both sides by an arbitrary function $\mu(x)$.

$$\underbrace{\mu \frac{dy}{dx} + \mu p y}_{\text{we want this to be equal to } \frac{d}{dx}(\mu y)} = \mu g$$

$$\mu \frac{dy}{dx} + \mu p y$$

$$= \mu \frac{dy}{dx} + \frac{d\mu}{dx} y$$

equal

equal

We need $\frac{d\mu}{dx} = p\mu$!

Separate: $\frac{1}{\mu} d\mu = p(x)\, dx$

Integrate $\ln(\mu) = \int p(x)\, dx$

Solve for μ :

$$\boxed{\boxed{\mu(x) = e^{\int p(x) dx}}}$$

Integrating Factor

Procedure

1) Get in form: $\frac{dy}{dx} + p(x)\, y = g(x)$

2) Calculate integrating factor: $\mu(x) = e^{\int p(x) dx}$

3) Multiply the ODE with $\mu(x)$ and rewrite:

$$\mu \frac{dy}{dx} + \mu p y = \mu q$$

$$\frac{d}{dx}(\mu y) = \mu q$$

4) Integrate and Divide

$$\int \frac{d}{dx}(\mu y)\,dx = \int \mu q\,dx$$

$$\Rightarrow \mu y = \int \mu q\,dx$$

$$\Rightarrow \boxed{y = \frac{1}{\mu}\int \mu q\,dx}$$

Back to Ex 4

$$x \frac{dy}{dx} + 2y = x$$

1) Write ODE as $y' + py = g$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2}{x} y = 1$$

$P(x) = \frac{2}{x}$ $\quad$ $g(x) = 1$

2) Integrating factor

$$\mu = e^{\int p(x)dx} = e^{\int \frac{2}{x} dx}$$

$$= e^{2 \ln x} = e^{\ln x^2} = x^2$$

(Note: integration constant C will always cancel out in the solution)

3) Multiply both sides by $\mu = x^2$

$$x^2 \frac{dy}{dx} + 2xy = x^2$$

$$\frac{d}{dx}\left(x^2 y\right) = x^2$$

4) Integrate & Divide.

$$\int \frac{d}{dx}\left(x^2 y\right) dx = \int x^2 dx$$

$$\Rightarrow \quad x^2 y = \frac{1}{3}x^3 + C$$

$$\Rightarrow \quad \boxed{y = \frac{1}{3}x + \frac{C}{x^2}}$$

Ex 5

$$(t^2+1)\frac{dx}{dt} + tx - t = 0$$

$$x(0) = 2$$

1) Get in form $\frac{dx}{dt} + p(t)\,x = g(t)$

$$(t^2+1)\frac{dx}{dt} + tx = t$$

$$\frac{dx}{dt} + \underbrace{\frac{t}{t^2+1}}_{p(t)}\,x = \underbrace{\frac{t}{t^2+1}}_{g(t)}$$

2) $$\mu(t) = e^{\int \frac{t}{t^2+1}\,dt} = e^{\frac{1}{2}\ln(t^2+1)}$$

use substitution $u = t^2+1$

or Wolfram Alpha

$$= e^{\ln(t^2+1)^{1/2}} = (t^2+1)^{1/2}$$

3) Use Formula:

$$x(t) = \frac{1}{\mu(t)} \int \mu(t) q(t)\, dt$$

$$= \frac{1}{(t^2+1)^{1/2}} \int (t^2+1)^{1/2} \frac{t}{t^2+1}\, dt$$

$$= \frac{1}{(t^2+1)^{1/2}} \int \frac{t}{(t^2+1)^{1/2}}\, dt$$

$$= \frac{1}{(t^2+1)^{1/2}} \left[(t^2+1)^{1/2} + C \right]$$

$$= \boxed{1 + \frac{C}{(t^2+1)^{1/2}}}$$

Initial Condition: $x(0) = 2$

$$2 = 1 + \frac{C}{(0+1)^{1/2}} = 1 + C \Rightarrow C = 1$$

$$\boxed{x(t) = 1 + \frac{1}{(t^2+1)^{1/2}}}$$