Final Review Exam Solutions

Question 1:

Give a direct proof to show that if $a,b \in \mathbb{N}$ are perfect squares , so is ab.

Answer: By Definition, let $a= m^2$ and $b= n^2$, $ab= m^2 * n^2 = (mn)^2$ for $m, n \in \mathbb{N}$.

Since, $m, n \in \mathbb{N}$, so by definition ab is a perfect square.

Question 2:

Use proof by induction to show that 5^{2k} - 1 is divisible by 4 for all k that belongs to N (set of natural numbers).

Answer:

Basis step: k=1. For k=1, $5^{2.1} - 1= 24$, which is divisible by 4.

Inductive step: We assume that 5^{2k} - 1 is divisible by 4 for some k = n, and use this assumption to show that this implies it is also true for k=n+1.

If $5^{2n} - 1$ is divisible by 4 it means that there is some integer p for which $4p=5^{2n} - 1$.

We now try to determine if $5^{2(n+1)} - 1$ is divisible by 4.

$5^{2(n+1)} - 1$
$= 5^{2n+2} - 1$
$= 5^2 * 5^{2n} - 1$
$= 25 * 5^{2n} - 1.$

$= 24 * 5^{2n} + 5^{2n} - 1$

From the inductive hypothesis we already know that $5^{2n} - 1$ is divisible by 4 and the term (6*4) 5^{2n} is also divisible by 4,

= 4q + 4p. Since $5^{2n} - 1 = 4p$ and let $q = 6* 5^{2n}$ where q is an integer. So, the inductive step is complete. Therefore, $5^{2k} - 1$ is divisible by 4 for all $k \in \mathbb{N}$, which concludes the proof.

Question 3:

For any set A, B and C, prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Answer:

Answer: First we show $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. We consider the two cases, $x \in B$ or $x \in C$:
Case 1: $x \in A$ and $x \in B$. This gives us $x \in A \cap B$. Therefore $x \in (A \cap B) \cup (A \cap C)$.
Case 2: $x \in A$ and $x \in C$. Similarly, this gives us $x \in A \cap C$. Therefore again $x \in (A \cap B) \cup (A \cap C)$.
In both cases we got $x \in (A\cap B)\cup(A\cap C)$. This proves that $A\cap(B\cup C) \subseteq (A\cap B)\cup(A\cap C)$.
Second we show $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. Let $x \in (A \cap B) \cup (A \cap C)$. There are (again) two cases: $x \in A \cap B$ or $x \in A \cap C$.
Case 1: $x \in A \cap B$. This gives us $x \in A$ and $x \in B$. From $x \in B$ we get $x \in B \cup C$. So $x \in A$ and $x \in B \cup C$. Therefore $x \in A \cap (B \cup C)$.
Case 2: $x \in A\cap C$. Similarly, this gives us $x \in A$ and $x \in C$. From $x \in C$ we get $x \in B \cup C$. So $x \in A$ and $x \in B \cup C$. Therefore $x \in A \cap (B \cup C)$.
In both cases we got $x \in A\cap(B \cup C)$. This proves that $(A\cap B)\cup(A\cap C) \subseteq A\cap(B \cup C)$.

Question 4:

a) Give a recursive definition of the sequence $\{a_n\}$ "where" $a_n= n^2$

Answer:

Base case, $a_0=0^2=0$

Recursive case: $a_n=n^2$, so therefore $a_{n+1}= (n+1)^2 = n^2+2n+1 = a_n +2n+1$. So $a_{n+1} =a_n +2n+1$.

Or the solution provided in demo quiz. $a_n =a_{n-1} + 2n -1$.

b) Give a recursive definition of the set of binary strings that have even length. Recall that the empty string λ has length 0 and hence has even length.

Answer:

Base case: $\lambda \in S$

Recursive case: If $b \in S$, then $b00, b01, b10, b11 \in S$.

Question 5:

Consider the following recursive definition of a set S of ordered pairs of integers.

Base Case: $(0,0) \in S$

Recursive Case: "If" $(a,b) \in S$,"then" $(a,b+1) \in S, (a+1,b+1) \in S$,"and" $(a+2,b+1) \in S$.

Use structural induction to prove that for any $(a,b) \in S$, it is the case that $a \leq 2b$.

Answer:

Basis Step: This holds for the basis step because (0, 0) holds for $a \leq 2b$.
Recursive Step: In the recursive step, we need to show that if $a \leq 2b$ holds, then this also holds for the elements obtained from (a, b). Suppose $a \leq 2b$. Then we consider all the three cases.

Case 1: (a, b+1)

If $a \leq 2b$ then $a \leq 2b +1 \leq 2b +2$. Since $2b +2 = 2(b +1)$, $a \leq 2(b +1)$ and therefore the hypothesis holds for this case.

Case 2: (a+1, b+1)

If $a \leq 2b$ then $a +1 \leq 2b+1 \leq 2b+2$. Since $2b +2 = 2(b +1)$, $a +1 \leq 2(b +1)$ and therefore the hypothesis holds for this case.

Case 3: (a+2,b+1)

If $a \leq 2b$ then $a +2 \leq 2b+2$. Since $2b +2 = 2(b +1)$, $a +2 \leq 2(b +1)$ and therefore the hypothesis holds for this case.

This completes our structural induction proof.

Question 6:

Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$.
a. If five integers are selected from A, must at least one pair of the integers have a sum of 9?
b. If four integers are selected from A, must at least one pair of the integers have a sum of 9?

a. Partition the set A into 4 subsets: $\{1, 8\}$, $\{2, 7\}$, $\{3, 6\}$, and $\{4, 5\}$,
each consisting of two integers whose sum is 9.
If 5 integers are selected from A, then by the Pigeonhole Principle at least two must be from the same subset. But then the sum of these two integers is 9.

b. The answer is no. This is a case where the pigeonhole principle does not apply;
the number of pigeons is not larger than the number of pigeonholes.
For instance, if you select the numbers 1, 2, 3, and 4, then since the largest sum of any two of these numbers is 7, no two of them add up to 9.

Question 7:

a) How many different ways can six letters of the word COMPUTER can be chosen and written in a row ?

b) How many different ways this can be done if it must begin with C and end with R ?

Answer:

a)The answer equals the number of six-permutations of a set of eight elements. This equals

$$P(8, 6) = 8!/(8-6)!$$

b) Since the first letter must be C and the last letter must be R there are effectively only four letters to be chosen and placed in the other four positions. And since the C is usedin the first position and R in the last, there are six other letters available to fill the remaining four positions. This equals

$$P(6, 4) = 6!/(6-4)!$$

Question 8:

How many solutions does the equation have, $x1 + x2 + x3 = 10$, where x1, x2, and x3 are non-negative integers?

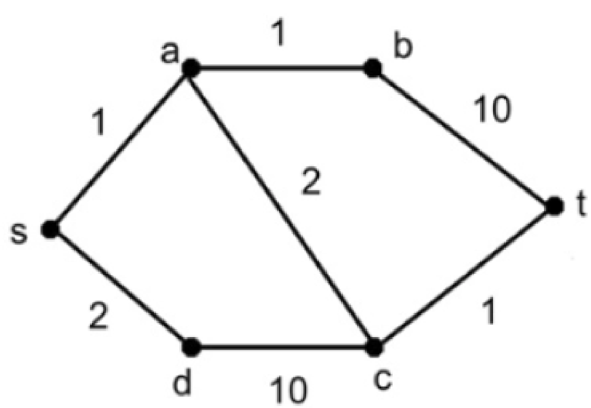
Answer: To count the number of solutions, we note that a solution corresponds to the ways of selecting 10 items from a set with three elements so that x_1 items of type one, x_2 items of type two, x3 items of type three. Hence, the number of solutions is equal to the number of 10-

combinations with repetitions allowed from a set with three elements. From Theorem 2, the equation has

$C(3 + 10 - 1, 10) = C(12, 10) = C(12, 2) = 66$ solutions.

Question 9:

For the following graph consider running Dijkstra's algorithm to find the shortest path between vertices s and t. Give the order that the vertices will be added to the "solved vertex set" S during during the run of the algorithm. Also give the labels for each of the vertices when they are added to the set S. (You do not need to give the label of each vertex in the graph at each iteration, just the vertex that is added to S.)



Answer:

Let's start with S{s =0}.

1st iteration: S{s=0, a=1}.

2nd iteration: S{s=0, a=1, b =2}.

3rd iteration: S{s=0, a=1, b=2, d=2}.

4th iteration: S{s=0, a=1, b=2, d=2, c=3}.

5th iteration: S{s=0, a=1, b=2, d=2, c=3, t=4}

Since t is now solved, this completes the algorithm. The shortest path is s-a-c-t with length = 4.

Question 10:

If a graph has an Euler circuit, then every vertex of the graph has positive even degree.

Answer:

Theorem 10.2.2 , Page 648 - 649