Quiz over week 6 and 7 materials: Solutions

Question 1:

Use iteration to guess an explicit formula for the sequence

$d_k = 5d_{k-1} + 2$, for all integers $k \geq 1$

$d_0 = 2$

Answer to question 1:

$d_k = 5d_{k-1} + 2$, for all integers $k \geq 1$

$d_0 = 2$

$d_1 = 5(d_0) + 2 = 5\cdot 2 + 2$

$d_2 = 5(d_1) + 2 = 5(5\cdot 2 + 2) + 2 = 5^2\cdot 2 + 5\cdot 2 + 2$

$d_3 = 5(d_2) + 2 = 5(5^2\cdot 2 + 5\cdot 2 + 2) + 2 = 5^3\cdot 2 + 5^2\cdot 2 + 5\cdot 2 + 2$

$d_4 = 5(d_3) + 2 = 5(5^3\cdot 2 + 5^2\cdot 2 + 5\cdot 2 + 2) + 2 = 5^4\cdot 2 + 5^3\cdot 2 + 5^2\cdot 2 + 5\cdot 2 + 2$

$d_n = 5^n \cdot 2 + 5^{n-1}\cdot 2 + 5^{n-2}\cdot + \ldots\ldots + 5^2 \cdot 2 + 5^1\cdot 2 + 5^0 \cdot 2$

$= 2. \sum_{i=0}^{n} 5^i$

$= 2. ((5^{n+1} - 1) /5-1) = (5. 5^n - 1)/ 2$

(Ans)

Question 2:

a) Give a recursive definition of the set of positive integers that are multiple of 3.

b) Give a recursive definition of the set of binary strings that have even length. Please remember that empty string λ has length 0 and hence has even length.

Answer to question 2:

a)

- Basis Step: $3 \in S$
- Recursive Step: If $x \in S$, then $x+3 \in S$

b)

- Base Step: $\lambda \in S$
- Recursive Step : If $x \in S$, then $00x \in S$, $01x \in S$, $10\,x \in S$, $11x \in S$

Question 3:

Define a set S recursively as follows:

I. BASE: $0 \in S$, $5 \in S$

II. RECURSION: If $s \in S$, and $t \in S$ then

a. $s + t \in S$ b. $s - t \in S$

III. RESTRICTION: Nothing is in S other than objects defined in I and II above.
Use structural induction to prove that every integer in S is divisible by 5.

Answer to question 3:

- Base case: $0 \in S$, $5 \in S$. Since 0 and 5 are multiple of 5, the base case is true.
- Inductive Case: Assume $s \in S$ and $t \in S$ and both s and t are divisible by 5. By definition, $s = 5a$ and $t = 5b$. Then $s + t = 5a + 5b = 5(a+b)$. Thus, $s + t$ is divisible by 5. Again, $s - t = 5a - 5b = 5(a - b)$. Thus, $s - t$ is divisible by 5. Therefore, every integer in S is divisible by 5.

Question 4:
a) A small town has only 500 residents. Must there be 2 residents who have the same birthday? Why?

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b) Show that if seven integers are selected from the first 10 positive integers there must be at least two pairs of these integers with the sum 11.

Answer to question 4:

a) Yes. This follows from the generalized pigeonhole principle with 500 pigeons, 365 pigeonholes, using the fact that $500 > 365$, must there be 2 residents who have the same birthday.

b)
Let A= {1, 2, 3, 4, 5, 6, 7} is the set of seven selected from the first 10 positive integers.
There can be 5 subsets:
{1,10} {2,9} {3,8} {4,7} {6,5} , each consisting of two integers whose sum is 11. If 7 integers are selected from A, Then by the Pigeonhole Principle at least two must be from the same subset and the sum of these two integers is 11.

Question 5:
a) According to the Inclusion/Exclusion Rule for Two Sets
"If A and B are finite sets, then $N(A \cup B) = N(A) + N(B) - N(A \cap B)$"
Now, how many integers from 1 through 100 are neither multiples of 2 nor multiples of 9?
b) How many strings are there of four lowercase letters that have the letter 'd' in them?

Answer to question 5:

Let A = the set of all integers from 1 through 100 that are multiples of 2.
Let B = the set of all integers from 1 through 100 that are multiples of 9.
Then
$A \cup B$ = the set of all integers from 1 through 100 that are multiples of 2 or multiples of 9
and
$A \cap B$ = the set of all integers from 1 through 100 that are multiples of both 2 and 9
= the set of all integers from 1 through 100 that are multiples of 18.
[Now calculate N(A), N(B), and $N(A \cap B)$ and use the inclusion/exclusion rule to solve for $N(A \cup B)$.]

Because every other integer from 2 through 100 is a multiple of 2, each can be represented in the form 2k, for some integer k from 1 through 50. Hence there are 50 multiples of 2 from 1 through 100, and so N(A) = 50.

Similarly, each multiple of 9 from 1 through 99 has the form 9k, for some integer k from 1 through 11. So, N(B) =11.

Finally, each multiple of 18 from 1 through 100 has the form 18k, for some integer k from 1 through 5 (since $90 = 5\cdot 18$). So, $N(A \cap B) = 5$
So, It follows by the inclusion/exclusion rule that
$N(A \cup B) = N(A) + N(B) - N(A \cap B) = 50 + 11 - 5 = 56$

There are 100 integers from 1 through 100, and 56 of these are multiples of 2 or multiples of 9. Thus, by the set difference rule, there are $100 - 56 = 44$ that are neither multiples of 2 nor multiples of 9. (Ans)

b)
Total number of possible strings (26 letters in each of the four positions) = 26^4
Total number of strings with no 'd' (25 letters in each of the four positions) = 25^4
Total number of strings that have the letter 'd' = $26^4 - 25^4 = 66351$