Practice Midterm -1 : Answer Keys

Question 1

Determine the negation of each of the following statements:

(a) If it snows, then they do not drive the car.

Let p be the statement "It snows"

Let q be the statement "They drive the car."

The statement above can be expressed as $p \to \neg q = \neg p \vee \neg q$

The negation is $\neg(\neg p \vee \neg q)$. By De Morgan's laws, this is equivalent to $p \wedge q$.

In English, "It snows and they drive the car".

(b) Only if Susan studies well she will pass the test.

Rewrite this statement: If Susan passes the test, then she studies well.

Let p be the statement "Susan passes the test"

Let q be the statement "She studies well"

The statement above can be written $p \to q$.
The negation is $\neg(\neg p \vee q)$. By De Morgan's laws, this is equivalent to $p \wedge \neg q$

In English, "Susan passes the test but she does not study well".

(c) My car is in the repair shop and I can't get to class.

Let p be the statement "My car is in the repair shop." Let q be the statement "I can get to class."

The statement above can be expressed as $p \wedge \neg q$. The negation will be $\neg(p \wedge \neg q) = (\neg p \vee q)$

In English, "My car is not in the repair shop, or I can get to class."

(d) He swims if and only if the water is warm.

Let p be the statement "He swims"

Let q be the statement "The water is warm"

The above statement is a bi-conditional that can be expressed as $p \leftrightarrow q$, which is equivalent to $(p \to q) \wedge (q \to p)$

The negation is $\neg((p \to q) \wedge (q \to p))$

By the definition of implication, $\neg(\neg p \vee q) \vee (\neg(\neg q \vee p))$

By De Morgan's law again, $(p \wedge \neg q) \vee (\neg p \wedge q)$

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In English, "He swims and the water is not warm, or the water is warm and he does not swim." To clarify this, "He swims if and only if the water is not warm."

Question 2:

Let P, Q, and R be the propositions -
P : The sun is shining.
Q : It is Friday.
R : The basketball game will take place .
Express each of the following compound propositions as an English sentence.

1) $(P \land Q) \rightarrow \neg R$
2) $R \rightarrow (P \lor Q)$
3) $\neg(P \lor Q)$
4) $\neg R \leftrightarrow \neg P$

Answer:

1) If the sun is shining and it is Friday, then the basketball game will not take place.

2) If the basketball game will take place, then the sun is shining or it is Friday.

3) $\neg (P \lor Q) = \neg P \land \neg Q$, the sun is not shining and it is not Friday.

4) The basketball game will not take place if and only if the sun is not shining.

Question 3:

Use truth tables to prove or disprove that the two compound propositions $(P \rightarrow (Q \rightarrow R))$ and $((P \land Q) \rightarrow R)$ are logically equivalent.

Answer:

P	Q	R	$(Q \rightarrow R)$	$P \rightarrow (Q \rightarrow R)$	$(P \land Q)$	$(P \land Q) \rightarrow R$
T	T	T	T	T	T	T
T	T	F	F	F	T	F
T	F	T	T	T	F	T
T	F	F	T	T	F	T
F	T	T	T	T	F	T
F	T	F	F	T	F	T
F	F	T	T	T	F	T
F	F	F	T	T	F	T

Because 2 compound propositions share the same truth table, they are logically equivalent.

Question 4:

Let B(x), W(x), and S(x) be the predicates -

B(x) : x is a female

W(x) : x is a good athlete

S(x) : x is young

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Express each of the following English sentences in terms of B(x), W(x), S(x), quantifiers, and logical connectives. Assume the domain is all people.

(You may need to use these symbols : : $\geq \leq \neq \neg \wedge \vee \oplus \equiv \rightarrow \leftrightarrow \exists \forall$)

a) Not all female are good athletes.

b) Some female are not good athletes unless they are young.

c) If someone is a female then she is young or a good athlete.

d) There is someone who is a good athlete and female.

Answer:

a) $\exists(x)[(B(x) \wedge \neg W(x))]$ *OR * $\neg \forall(x)[(B(x) \rightarrow W(x)]$
b) $\exists(x) ((B(x) \wedge (\neg W(x) \wedge S(x)))]$
c) $\forall(x)(B(x) \rightarrow (S(x) \vee W(x))$
d) $\exists(x)(W(x) \wedge B(x))$

Question 5:

Use a direct proof to show that if two integers have opposite parity, then their product is even (Hint: Parity means whether it is even or odd).

Answer:

Let the two integers be x and y. Since they have opposite parity, let's assume that x is even and y is odd. Let x=2m and y=2n+1, where m and n are integers. Their product xy = 2m (2n+1) = 2(2mn+m). Because m and n are both integers, xy is even. (QED)

Question 6:

Use a proof by contraposition to show that if m + n is an irrational number then either m is irrational or n is irrational.

(Recall that a number is irrational if and only if is it is not rational.)

Answer:

Let p be "m + n is irrational" and q be "m is irrational" and r be "n is irrational". The statement is $p \rightarrow (q \vee r)$. The contrapositive is $\neg (q\vee r) \rightarrow \neg p \equiv (\neg q) \wedge (\neg r) \rightarrow \neg p$, which means: If m is rational and n is rational, then m + n is rational. Let m=a/b and n=c/d, where a, b, c, d are all integers and $b\neq0$ and $d\neq0$. Then, m + n=a/b + c/d= (ad + bc) / bc. Therefore, m + n is rational.

Question 7:

Show by contradiction that Suppose $a \in Z$. If a^2 is even, then a is even.

Answer:
For the sake of contradiction, suppose a^2 is even and a is not even. Then a^2 is even, and a is odd. Since a is odd, there is an integer c for which a = 2c +1. Then $a^2 = (2c+1)^2 = 4c^2 + 4c + 1 = 2(2c^2 + 2c) + 1$, so $a^2$is odd. Thus a^2 is even and a^2 is not even, a contradiction.

Question 8:

Answer the following question-

Let A, B and C be sets.

Prove that $(A-B) - C \subseteq A - (B-C)$

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Answer:

Proof: We want to prove that (A-B) - C ⊆ A - (B-C). By the definition of difference we know that x ∈ A and x ∉ B and x ∉ C. Therefore, A - B results in A, which will then be applied towards A - C. Since we know the definition of difference we know then that x ∈ A and not C. The right hand side is similar in that B-C tells us that x∈B and not C, resulting in A-B. The definition of difference tells us then that x∈A and not B. Since x∈A on both the left hand side and right hand side we can conclude then that (A-B) - C ⊆ A - (B-C), and our proof is complete.

Question 9:

Compute the value of the following sum.

You do need to show the intermediate steps to obtain a final number.

It is sufficient for you to produce a closed form expression for the answer that could be easily evaluated with a calculator such as: **6 * 2^16 + 5**.

a) $\sum_{j=1}^{5}\left(\frac{1}{j(j+1)}\right)$

b) $\sum_{i=2}^{5} 4 * 2^i$

Answer:

a) $\sum_{j=1}^{5}(\frac{1}{j}-\frac{1}{j+1}) = (\frac{1}{1}-\frac{1}{1+1}) + (\frac{1}{2}-\frac{1}{2+1}) + \left(\frac{1}{3}-\frac{1}{3+1}\right) + \left(\frac{1}{4}-\frac{1}{4+1}\right) + \left(\frac{1}{5}-\frac{1}{5+1}\right) = 1 - \frac{1}{5+1} = \frac{5}{6}$

b) $\sum_{i=2}^{5} 4 * 2^i = \sum_{i=0}^{5} 4 * 2^i - \sum_{i=0}^{1} 4 * 2^i = 4 * \frac{2^6-1}{2-1} - 4 * \frac{2^2-1}{2-1} = 4 * (63 - 3) = 4 * 60 = 240$

Question 10:

Using weak induction prove that For all n (n>=0), $n(n^2 + 5)$ is a multiple of 6.

Answer:

Proof:

Basis: $0(0^2+ 5) = 0 = 6 *0$.

Induction:
Inductive hypothesis: $n(n^2 + 5)$ is a multiple of 6.

Inductive step: Now we have to show that $(n + 1)((n + 1)^2 + 5)$ is a multiple of 6.
$(n + 1)((n + 1)^2 + 5) = (n + 1)(n^2 + 2n + 1+5)$
$= (n + 1)(n^2 + 5) + (n + 1)(2n + 1)$ (To factorize and use the inductive step we have separated the terms)
$= n(n^2 + 5) + (n^2 + 5) + (n + 1)(2n + 1)$ (factorized the first portion)
$= n(n^2 + 5) + (n^2 + 5) + (2n^2+ 3n + 1)$ (factorized the second portion)
$= n(n^2 + 5) + (3n^2 + 3n + 6)$ (simplified)
$= n(n^2 + 5) + 3(n^2 + n) + 6$
Now, to show that the rhs sum is a multiple of 6, we show that all three summands are. By inductive hypothesis, $n(n^2 + 5)$ is multiple of 6. Of course 6 is also. To show that $3(n^2 + n)$ is a multiple of 6, it is enough to show that $n^2 + n$ is even, which follows easily from the fact that $n^2 + n = n(n + 1)$ (for example 2*3, 4*5, 6*7 etc.) and so is the product of an odd and an even number. So, $n^2 + n = 2k$ (where k is a natural number). 3*2.k is also a multiple of 6. (QED)