CS 225_400: Discrete Structures in CS

Midterm Exam Solutions

Winter 2017

1. Determine the contrapositive of the following two statements:

(a) Being alive is a necessary condition for having a right to life.

if-then form:

Answer: If you have a right to life, then you are alive.

Contrapositive:

Answer: If you are not alive, then you don't have a right to life.

(b) If it rains, the picnic will be cancelled.

Contrapositive:

If the picnic will not be cancelled, then it will not rain.

2. Write the negation of each following statements:

(a) You can solve it by either factoring or with the quadratic formula.

Let,

p: You can solve it by factoring

q: You can solve it with the quadratic formula

$\sim(p \vee q) = \sim p \wedge \sim q$

Answer: You can't solve it by factoring and you can't solve it with the quadratic formula.

Or

You can solve it neither by factoring nor with the quadratic equation.

(b) Productivity increases only if wages rise.

Let,

p: Productivity increases

q: wages rise

$\sim(p \rightarrow q) = p \wedge \sim q$

Answer: Productivity increases and wages do not rise.

2. Let U be the set of all problems on a comprehensive list of problems in science. Define four predicates over U by -

P (x): x is a mathematics problem

Q(x): x is time consuming

R(x): x is easy

S(x): x is solvable

Translate each of the following formulas into English sentences :

1. $\forall x\ (P(x) \rightarrow \sim (S(x) \vee R(x))$
2. $\sim\forall x\ (S(x) \rightarrow\ R(x))$
3. $\exists x\ (P(x) \wedge \sim Q(x))$
4. $\sim\exists x\ \ (Q(x)\ \wedge S(x))$

1. For all problems, if the problem is a mathematics problem then it is not solvable and it is not easy. Or

   For all problems, if the problem is a mathematics problem then it is neither solvable nor easy. Or

   All mathematics problems are neither solvable nor easy.

2. Not all solvable problems are easy. Or

   There exists a problem that is a solvable problem and it is not easy.

3. There exists a problem that is a mathematics problem and it is not time consuming. Or

   Some mathematics problems are not time consuming.

4. There is not a problem that is both time consuming and solvable. Or

   All problems are either not time consuming or not solvable. Or

   No time consuming problem is solvable.

3. Let B(x), W(x), and S(x) be the predicates
B(x): x is a female
W(x): x is a good athlete
S(x): x is young
Express each of the following English sentences in terms of B(x), W(x), S(x), quantifiers, and logical connectives. Assume the domain is all people.

(You may need to use these symbols: : ≥ ≤ ≠ ¬ ∧ ∨ ⨁ ≡ → ↔ ∃ ∀)

a) It is not true that all young people are good athletes.

b) Some female are good athletes but they are not young.

c) Not all young or female people are good athletes.

d) There is someone who is a good athlete and female.

(a) $\sim\forall x\, (S(x) \rightarrow W(x))$ or $\exists x\, (S(x) \wedge \sim W(x))$

(b) $\exists x\, (B(x) \wedge W(x) \wedge \sim S(x))$

(c) $\sim\forall x\, ((S(x) \vee B(x)) \rightarrow W(x))$ or $\exists x\, ((S(x) \vee B(x)) \wedge \sim W(x))$

(d) $\exists x\, (W(x) \wedge B(x))$

4. Prove the statement by mathematical induction that n^3-7n+3 is divisible by 3, for each integer $n \geq 0$.

Answer:

Base Case: $P(0) = 0^3 - 7(0) + 3 = 3$ is divisible by 3.

Inductive case: Now for the inductive step, suppose that for some integer $k \geq 0$, $k^3 - 7k + 3$ is divisible by 3, which means that $k^3 - 7k + 3 = 3a$ for some integer a. To show that $(k + 1)^3 - 7(k + 1) + 3$ is also greater than or equal to 0, observe that

$(k+1)^3 - 7k + 3$

$= k^3 + 3k + 3k^2 + 1 - 7k - 7 + 3$

$= k^3 + 3k^2 - 4k - 3$

$= (k^3 - 7k + 3) + 3(k^2 + k - 2)$

$= 3a + 3(k^2 + k - 2)$

$= 3\,(a + (k^2 + k - 2)$

and since $a+k^2+k-2$ is an integer (any combination of sums, differences, and products of integers is also an integer), we see that $(k + 1)^3 - 7(k + 1) + 3$ is divisible by 3 as well. Since we have completed both the base and the inductive step, the inductive proof is complete.

5. Use truth tables to determine whether $((p \vee q) \wedge (\neg q \vee r)) \rightarrow (q \vee r)$ is a tautology.

Answer:

p	q	r	-q	p v q	~q v r	q v r	(p v q) ∧ (~q v r)	(p v q) ∧ (~q v r) → (q v r)
T	T	T	F	T	T	T	T	T
T	T	F	F	T	F	T	F	T
T	F	T	T	T	T	T	T	T
T	F	F	T	T	T	F	T	F
F	T	T	F	T	T	T	T	T
F	T	F	F	T	F	T	F	T
F	F	T	T	F	T	T	F	T
F	F	F	T	F	T	F	F	T

Because not all values are true in the final column, $(p \vee q) \wedge (\sim q \vee r) \rightarrow (q \vee r)$ is not a tautology.

6. By direct proof method show that the sum of any three consecutive integers is divisible by three.

Answer:

If x is an integer, then x, x+1, and x + 2 are three consecutive integers. A number N is divisible by 3 if it can be represented as a multiple of 3: N = 3k, where k is an integer.

Hence, we want to prove that the statement "$\forall$ integer x, $\exists$ integer k, such that $x + (x + 1) + (x + 2) = 3k$ is true.

Let x be an integer.

So, $x + (x + 1) + (x + 2) = 3x + 3 = 3(x + 1)$

Since x is an integer, x + 1 would also be an integer. Let k = x + 1. Hence, $x + (x + 1) + (x + 2) = 3k$. x was chosen arbitrary, hence the sum of any three consecutive integers is divisible by 3.

7. Suppose $x, y \in \mathbb{Z}$. Proof by contraposition that If $x^2(y + 3)$ is even, then x is even or y is odd.

Answer:

contraposition: ~q −> ~p: If x is odd and y is even, then $x^2(y+3)$ is odd.

Let by definition, x = 2k+1 and y = 2l , where k and l are arbitrary integers. Then

$(2k + 1)^2 (2l + 3)$

$= (4k^2 + 4k + 1) (2l + 3)$

$= 8k^2l + 8kl + 2l + 12k^2 + 12k + 3$

$= 2(4k^2l + 4kl + l + 6k^2 + 6k + 1) + 1$

$(4k^2l + 4kl + l + 6k^2 + 6k + 1)$ is the sum and product of integers which is also an integer and $2(4k^2l + 4kl + l + 6k^2 + 6k + 1)$ is an even integer by definition. Taken altogether, 2(4k^2m + 4km + m + 6k^2 + 6k + 1) + 1 is odd, which proves ~p and concludes the proof.

8. Let A and B be sets.

Prove that for any sets A and B, if $A \subseteq B$, then $A \cup B = B$

Answer:

Suppose A and B are sets with $A \subseteq B$. We must show both that $A \cup B \subseteq B$ and that $B \subseteq A \cup B$. To show that $A \cup B \subseteq B$, let $x \in A \cup B$ (So, $x \in A$ or $x \in B$). Since $A \subseteq B$, all the elements of set A are in B also. Thus, $A \cup B \subseteq B$. For the other direction, suppose $x \in B$. Then by definition of unions, $x \in A \cup B$, so clearly $B \subseteq A \cup B$. Therefore, we have proven that $A \cup B = B$, which is what we needed to show.

9. Construct an algebraic proof that for all sets A and B,

$A - (A \cap B) = A - B$.

Cite a property from Theorem 6.2.2 (7).png for every step of the proof.

Answer:

L.H.S=

$A - (A \cap B)$

- ⇨ $A \cap (A \cap B)^c$ By Set difference law
- ⇨ $A \cap A^c \cup B^c$ By De Morgan's law
- ⇨ $(A \cap A^c) \cup (A \cap B^c)$ By Distributive law
- ⇨ $\emptyset \cup (A \cap B^c)$ By Complement law
- ⇨ $A \cap B^c$ By Identity law
- ⇨ $A - B$ By Set difference law

= R.H.S (Proved)

10. Compute the value of the following sums.

You do need to show the intermediate steps to obtain a final number.
It is sufficient for you to produce a closed form expression for the answer that could be easily evaluated with a calculator such as: **6 * 2^16 + 5**.
The notes included near the top of this exam includes summation formulae that must be used to help you compute the sums for this question.

1.

$$\sum_{4}^{6}(3i + (-1)^i) = 3\sum_{4}^{6} i + \sum_{4}^{6}(-1)^i = 3\left(\sum_{1}^{6} i - \sum_{1}^{3} i\right) + \left(\sum_{0}^{6}(-1)^i - \sum_{1}^{3} i\right)$$

$$= 3\left(\frac{6(6+1)}{2} - \frac{3(3+1)}{2}\right) + \left(\frac{((-1)^{6+1} - 1)}{-2} - \frac{((-1)^{3+1} - 1)}{-2}\right) = 3(21 - 6) + 1 - 0$$

$$= 46\ (Ans)$$

2. $\sum_0^5(4^{j+2} + 6) = \sum_0^5 \; 4^2.4^j + \sum_0^5 6 = 16\left(\frac{(4^{5+1}-1)}{4-1}\right) + 6.6 = \mathbf{16}\left(\frac{(\mathbf{4^6-1})}{\mathbf{3}}\right) + \mathbf{36}$

$$= 16 * 1365 + 36 = 21876\ (Ans)$$