Assignment 5 *Part 1:* <u>5.2</u>: 9, 27, 35

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Prove each statement using mathematical induction.

9. For all integers $n \geq 3$,

$$4^3 + 4^4 + 4^5 + \ldots + 4^n = \frac{4(4^n - 16)}{3}$$

Base case: $n = 3$

$$64 = 64$$

Induction Case:

Assume that $n = k$ is true

$$4^3 + 4^4 + 4^5 + \ldots + 4^k = \frac{4(4^k - 16)}{3}$$

Show that it is true $n = k+1$

$$4^3 + 4^4 + 4^5 + \ldots + 4^k = \frac{4(4^k - 16)}{3}$$

$$\underbrace{4^3 + 4^4 + 4^5 + \ldots + 4^k} + 4^{k+1} = \frac{4(4^{k+1} - 16)}{3}$$

$$\frac{4(4^k - 16)}{3} + 4^{k+1} =$$

$$\frac{(4 \cdot 4^k) - (4 \cdot 16) + (3 \cdot 4^k \cdot 4)}{3} =$$

$$\frac{4((1 \cdot 4^k) - (1 \cdot 16) + (3 \cdot 4^k))}{3} =$$

$$\frac{4(4 \cdot 4^k - 16)}{3} =$$

$$\boxed{4\left(\frac{4^{k+1} - 16}{3}\right) = \frac{4(4^{k+1} - 16)}{3}}$$

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27. Use the formula for the sum of the first n integers and/or the formulas for the sum of a geometric sequence to evaluate the sums

$5^3 + 5^4 + 5^5 + \dots + 5^k$, where k is any integer with $k \geq 3$

$\frac{(n+1)\cdot n}{2}$ $\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1}$

$k \geq 3$

$5^3 + 5^4 + 5^5 + \dots + 5^k = \frac{5^k - 1}{4}$

Base Case: $k = 3$ $5^3 = 5^3$ ✓

Induction Case. Assume $k = m$

$k = m - 3$

$$5^3\left[\underbrace{5^3 + 5^4 + 5^5 + \dots + 5^m} + 5^{m-3}\right] = 5^3\left[\frac{5^{(m-3)+1} - 1}{4}\right]$$

$$5^3\left[\frac{5^m - 1}{4} + \frac{4 \cdot 5^{m-3}}{4}\right] = 5^3\left[\frac{5^{m-2} - 1}{4}\right]$$

$$5^3\left[\frac{5 \cdot 5^m \cdot 5^{m-3} - 1}{4}\right] =$$

$$5^3\left[\frac{5 \cdot 5^m \cdot 5^{-3} \cdot 5^m - 1}{4}\right] =$$

$$5^3\left[\frac{5^{-2} \cdot 5^m - 1}{4}\right] =$$

$$5^3\left[\frac{5^{m-2} - 1}{4}\right] = 5^3\left[\frac{5^{m-2} - 1}{4}\right]$$

□

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35. Find the mistake in proof.

Theorem: For any integer $n \geq 1$,

$$\sum_{i=1}^{n} i(i!) = (n+1)! - 1$$

Proof (by Mathematical induction): Let the property

$P(n)$ be $\sum_{i=1}^{n} i(i!) = (n+1)! - 1$

Show that $P(1)$ is true: when $n=1$

$$\sum_{i=1}^{1} i(i!) = (1+1)! - 1$$

So $1(1!) = 2! - 1$

and $1 = 1$

Thus, $P(1)$ is true

the mistake is proving $P(1)$ to be true by prematurely considering $n=1$ and solving by substitution. We need to prove that the LHS, $\sum_{i=1}^{n} i(i!)$ is equal to $(n+1)!-1$, the RHS, through mathematical induction instead, starting with the assumption, $n=m$ then $m+1$

Assignment 5 *Part 1:* 5.3: 10, 18, 23.b

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10. Prove each statement by mathematical induction.

$n^3 - 7n + 3$ is divisible by 3, for each integer $n \geq 0$

Base case: show it is true, $n = 0$

$0^3 - 7(0) + 3 = 3$, 3 is divisible by 3 ✓

Induction Case:

Assume that $n = k$

$k^3 - 7k + 3 = 3r$ $r =$ some integer r

Show that $n = k+1$, $(k+1)^3 + 7(k+1) - 3$ is divisible by 3

$$\begin{aligned}(k+1)^3 - 7(k+1) + 3 &= (k+1)(k+1)(k+1) - 7k - 7 + 3 \\ &= (k^2+2k+1)(k+1) - 7k - 7 + 3 \\ &= k^3 + 2k^2 + k + k^2 + 2k + 1 - 7k - 7 + 3 \\ &= (k^3 - 7k + 3) + 3k^2 + 3k - 3 \\ &= 3r + 3k^2 + 3k - 3 \\ &= 3(r + k^2 + k - 1)\end{aligned}$$

But $(r + k^2 + k - 1)$ is a sum of products of integers and so by definition of divisibility, $n^3 - 7n + 3$ is divisble by 3

Assignment 5 *Part 1:* 5.3: 10, 18, 23.b

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18. $5^n + 9 < 6^n$ for all integer $n \geq 2$

Base Case: Show that $n = 2$

$5^2 + 9 < 6^2 = 34 < 36$ ✓

Induction Case:

Assume $n = k$ $k \geq 2$

Show that $n = k+1$ $5^k + 9 < 6^k$

$5^{k+1} + 9 < 6^{k+1}$

But $5^{k+1} + 9 < 6^k + 6^k$

$\therefore 5^{k+1} + 9 < 6^{k+1}$

By mathematical induction, $5^n + 9 < 6^n$ for all $n \geq 2$

23b. $n! > n^2$, for all integers $n \geq 4$

Base Case: $n = 4$

$n! > n^2 = 4! > 4^2 = 4 \cdot 3 \cdot 2 \cdot 1 > 16 = \underline{24 > 16}$ ✓

Induction: $n = k$.

Assume $k! > k^2$

Show that $n = k+1$,

$(k+1)! > (k+1)^2$

By Assumption.

$k! > k^2$

$k(k+1)! > k^2(k+1)$

but $k^2 > k+1$
since $n \geq 4$

$\frac{k^2}{k} > \frac{k+1}{k} = k > 1 + \frac{1}{k}$

since $n \geq 4$, $4 > 1 + \frac{1}{4}$

$4 > 1\frac{1}{4}$ ✓

Since $k^2 > k+1$ we can assume it is true that by substituting

$(k+1)! > (k+1)(k+1)$

$(k+1)! > (k+1)^2$

By mathematical induction it is true that

$n! > n^2$ for all $n > 4$ $\square$