Demo quiz over week 6 and 7 materials - Solutions

Question 1 :

Use iteration to guess an explicit formula for the sequence -

$d_k = 2d_{k-1} + 3$, for all integers $k \geq 2$ $d_1 = 2$

Answer:

$d_k = 2d_{k-1} + 3$, for all integers $k \geq 2$

$d_1 = 2$

$d_2 = 2(d_1) + 3 = 2(2) + 3 = 2^2 + 3$

$d_3 = 2(d_2) + 3 = 2(2^2 + 3) + 3 = 2^3 + 2\cdot 3 + 3$

$d_4 = 2(d_3) + 3 = 2(2^3 + 2\cdot 3 + 3) + 3 = 2^4 + 2^2\cdot 3 + 2\cdot 3 + 3$

$d_5 = 2(d_4) + 3 = 2(2^4 + 2^2\cdot 3 + 2\cdot 3 + 3) + 3 = 2^5 + 2^3\cdot 3 + 2^2\cdot 3 + 2\cdot 3 + 3$

Guess:

$$\begin{aligned}
d_n &= 2^n + 2^{n-2}\cdot 3 + 2^{n-3}\cdot 3 + \cdots + 2^2\cdot 3 + 2\cdot 3 + 3 \\
&= 2^n + 3\cdot \sum_{i=0}^{n-2} 2^i \\
&= 2^n + 3\cdot \frac{2^{n-1} - 1}{2 - 1} \\
&= 2^n + 3\cdot (2^{n-1} - 1) \\
&= 2^n + 3\cdot 2^{n-1} - 3 \\
&= 2\cdot 2^{n-1} + 3\cdot 2^{n-1} - 3 \\
&= 5\cdot 2^{n-1} - 3
\end{aligned}$$

Question 2:

a) Give a recursive definition of the set of positive even integers that are greater than 8.

b) Give a recursive definition of the set of binary strings that have odd length.

Answer:

a) Basis step : $10 \in S$
   Recursive step: If $x \in S$, then $x + 2 \in S$.
b) Basis step: $0 \in S$, $1 \in S$.
   Recursive step: If $w \in S$, then $00w \in S$, $01w \in S$, $10w \in S$ and $11w \in S$.

Question 3:

Consider the following recursive definition of a set S of ordered pairs of integers.

Base Case: $(0,1) \in S, (1,0) \in S$

Recursive Case: If $(a,b) \in S$, then $(a+1,b+1) \in S$

Use structural induction to prove that for any $(a,b) \in S$, it is the case that $a+b$ is odd.

Answer:

Let's use induction to prove this.

Show base cases are true: (0 + 1, 1 + 1) = (1, 2), 1 + 2 = 3, which is odd

(1 + 1, 0 + 1) = (2, 1), 2 + 1 = 3, which is odd

Show a way to get additional elements recursively and show that they also follow a + b is odd:

By the recursive case, If (a ,b)∈ S, then (a+1,b+1)∈ S by the recursive case, If (a ,b)∈ S ,then (a+1,b+1)∈ S.

So for any (a,b) , for which a + b is odd, if we add 1 to each part in the pair (switching parity of each part in the pair), then we are adding 2 to the total (changing both parities results in no change in the overall parity) and we still have the same property of oddness in the total number (a+1 + b + 1 = a + b + 2, where a + b is odd, 2 is even, and an even + an odd is, so for any (a,b), for which a + b is odd.

Question 4:

There are 50 baskets of apples. Each basket contains no more than 24 apples. Show that there are at least 3 baskets containing the same number of apples.

Answer:

The baskets are the pigeons, and we place each of them in one of 24 pigeonholes according to how many apples are in it. Thus the ratio n/k of pigeons to pigeonholes is 50/24 = 2 1⁄2. By Generalized Pigeon Hole Principle, there are at least this many baskets with the same number of apples, so there must be at least 3.

Question 5:

Answer the following problems -

a) A box contains 14 red balls and 10 blue balls. It is dark. How many balls must you take out of the box to be sure of having a pair the same color?

b) What is the minimum number of students required in a discrete mathematics class to be sure that at least five will receive the same grade, if there are seven possible grades, A, A-, B+, B, B-, C and F.

Answer:

a) To get at least one pair of balls you must draw three balls. In worst case, you will draw one red ball and then one blue ball, the last one, as there are only two color choices, must be red or blue creating a pair no matter what.

b) By using the Pigeonhole Principle that states that n objects can be placed in k boxes, there is at least one box containing at least ceiling (n/k) objects. We can see the grades as being boxes, and we get ceiling (n/7) = 5. Therefore, we get 29 boxes.

Question 6:

a) How many integers from 1 through 1000 are multiples of 2 or multiples of 9?

b) A bit string is a finite sequence of 0's and 1's. How many bit strings of length 7 end with a 1?

Answer:

a) Every second integer from 2 to 1000 is a multiple of 2, and each can be represented in the form 2j for some integer j from 1 to 500. So there are 500 multiples of 2 in this range.
Every ninth integer from 9 to 999 is a multiple of 9, and each can be represented in the form 9k for some integer k from 1 to 111. There are 111 multiples of 9 in this range.
However, whenever k is even, the multiple of 9 is also a multiple of 2 (example: k=2 and 9k = 18, a multiple of 2). k will be even in the range 1 to 111 exactly 55 times.

Therefore the answer to this question by the inclusion/exclusion rule is 500 + 111 – 55 = 556 integers.

b) Consider all the 7-bit bit strings trailing with 1. Because the last digit is fixed at 1, only the first 6 digits can vary. Since there are 6 digits with two choices of digits, the total number of strings is 2^6.