Midterm Exam – FALL 2015

Question 1:

1.

Determine the negation following statements:

(a) I wear glasses, or I can't read the chalkboard.

(b) Having a ticket in a lottery is necessary for winning that lottery.

2.

Let P, Q, and R be the propositions
P : John is wealthy
Q : John is healthy
R : John is rich
Express each of the following compound propositions as an English sentence.

a) $R \longleftrightarrow (P \wedge \neg Q)$
b) $\neg R \vee \neg(Q \wedge P)$

Answer:

1a. I don't wear glasses, and I can read the chalkboard.

b. You win that lottery and you don't have the ticket.

2a. John is rich if and only if John is wealthy and John is not healthy.

b. John is not rich or it is not the case that John is healthy and John is wealthy.

Or

John is not rich or John is not healthy or John is not wealthy.

Question 2:

Determine whether $(p \vee q) \wedge (\neg q \vee r)) \rightarrow (q \vee r)$ is a tautology. (Hint: Use the truth table method)

Answer:

The statement is false. We will show it through the truth table.

p q r	$(p \vee q)$	$(\neg q \vee r)$	$(q \vee r)$	$(p \vee q) \wedge (\neg q \vee r))$	$(p \vee q) \wedge (\neg q \vee r)) \rightarrow (q \vee r)$
T T T	T	T	T	T	T
T T F	T	F	T	F	T
T F T	T	T	T	T	T
T F F	T	T	F	T	F
F T T	T	T	T	T	T
F T F	T	F	T	F	T
F F T	F	T	T	F	T
F F F	F	T	F	F	T

It is not always true. So, $(p \vee q) \wedge (\neg q \vee r)) \rightarrow (q \vee r)$ is not a tautology.

Question 3:

In the domain of all students, we define predicates
M(x) : x is a math major
C(x) : x is a computer science major
A(x) : x is required to take CS 225 .
Express each of the following English sentences in terms of M(x), C(x), A(x), quantifiers, and logical connectives.

(a) Some math majors are computer science majors.
(b) Not all math majors are required to take CS 225.
(c) No computer science students are required to take CS 225.
(d) There is a student who is both math and computer major but is not required to take CS 225.

Answer:

(a) $\exists x\ (\ M(x) \wedge C(x)\ )$
(b) $\neg\forall x(M(x)\rightarrow A(x))$ or $\exists x\ (M(x) \wedge \neg A(x))$
(c) $\neg\exists x(C(x) \wedge A(x))$ or $\forall x\ (C(x) \rightarrow \neg A(x))$
(d) $\exists x(\ M(x) \wedge C(x) \wedge \neg A(x)\ )$

Question 4:

Use the direct proof method to prove that If r is any rational number, then $3r^2 - 2r + 4$ is rational. (Instructions: You can use the facts that sum, difference and product of integers are also integers. Please mention the variables and their restrictions correctly)

Answer:

Given r is a rational number, by the definition of rational number, we have r = p/q, where p, q are integers and **q != 0.**
Then $3r^2 - 2r + 4 = 3(p/q)^2 - 2(p/q) + 4 = (3p^2 - 2pq + 4q^2) / q^2$
Where $3p^2 - 2pq+4q^2$ and q^2 are integers (sum, difference, product and square of integers are integers), q^2 != 0. By the definition of rational number, $(3p^2 - 2pq + 4q^2) / q^2$ is rational. Therefore, we conclude that $3r^2 - 2r + 4$ is rational.

Question 5:

Show that if $n^3 + 5$ is even, then n is odd for all natural numbers. (Direction: You have to use proof by contraposition)

Answer:

Let p be the proposition that $n^3 +5$ is even and q is the proposition that n is an odd natural number. The contrapositive of the implication is "if n is even then $n^3 +5$ is odd". By the definition of even integer, n= 2k for some k. Substituting the value of n, we find that $(2k)^3 +5= 8k^3 +4+1= 2(4k^3 +2)+1= 2k'+1$ where k′ is a natural number, is an odd number. This satisfies the contrapositive proposition we wished to solve, and thus solves the original proposition that if $n^3 +5$ is even then n is odd for natural numbers. This concludes our proof.

Question 6:

Prove by contradiction that the product of a nonzero rational number and an irrational number is irrational.

Answer:

Suppose not, for the sake of contradiction. Then suppose that the product of a nonzero rational number and an irrational number is rational.
Assume that x is a particular but arbitrarily chosen nonzero rational number, and y is an irrational number. Their product, xy, is a rational number z.
Now, it is known that a rational number can be expressed as the ratio of two integers. So let x = (a/b), where a and b are integers and $b \neq 0$. Also, let z = (m/n), where m and n are integers and $n \neq 0$.
So (a/b)*y = (m/n). If we solve for y, we have the quantity:
y = (bm/an).
Now, bm is the product of integers (since b and m are integers by assumption). Also, an is the product of integers (since a and n are integers by assumption). Therefore y is the ratio of two integers. Therefore y is rational. But according to our supposition, y is irrational. This is a contradiction. Thus the product of a nonzero rational number and an irrational number cannot be rational, and must therefore be irrational.

Q.E.D.

Question 7:

Let $A = \{x \in Z \mid x = 18a - 2$ for some integer a $\}$ and $B = \{y \in Z \mid y = 18b + 16$ for some integer b$\}$.

Prove or disprove that $B \subseteq A$

Answer:

By definition of subset, B ⊆ A holds if every element of B is also an element of A. For every element y in B, y = $18b + 16$ for some integer b. Let $b = a - 1$ for some integer a. Then $y = 18 \cdot (a - 1) + 16$, $y = 18a - 18 + 16$, $y = 18a - 2$ and $y \in A$. But y ∈ B is defined as y = 18b + 16. Therefore, y ∈ A and y ∈ B for all integers y.

Thus, B ⊆ A.
Q.E.D.

Question 8:

Answer the following question-

Let A, B and C be sets with A ⊆ B. Prove that A ∩ C ⊆ B ∩ C.

Answer:

Let x ∈ A∩C. Then by definition of intersection, x ∈ A and x ∈ C. Moreover, x ∈ A implies that x ∈ B (Since A ⊆ B). So x ∈B and x ∈C, hence x ∈B∩C. So for any arbitrary x in A ∩C , we conclude that A ∩C ⊆ B ∩C as required.

Question 9:

Use strong induction to prove that any postage of n cents (n≥18) can be formed using only 3-cent and 10-cent stamps. (Hint : Base case:- P(18) = Six 3-cent stamps , P(19) = , ...)

Answer:

Let P(n) be the statement that we can form n cents of postage using just 3-cent and 10-cent stamps. We want to prove that P (n) is true for all n ≥18. For the basis step, 18 = 3+ 3+3 +3 +3 +3 (Six 3-cent stamps). 19 = 3+ 3+3 +10 (Three 3-cent and one 10-cent stamp) 20 = 10+10 (two 10-cent stamp) . Assume that we can form k > =20 cents of postage (the inductive hypothesis); **Inductive step, for any value of k +1 greater than or equal to 20 we can form k + 1 cents worth of postage from only 3-cent and 10- cent stamps we can take k - 2 and add a 3-cent stamp.**

Question 10:

Compute the value of the following sums.

You must show the intermediate steps to obtain a final number.
It is sufficient for you to produce a closed form expression for the answer that could be easily evaluated with a calculator such as: **6 * 2^16 + 5**.
The notes included near the top of this exam includes summation formulae that must be used to help you compute the sums for this question.

1) Sigma j =2 to 6 (j - $(-1)^j$)
2) Sigma i =3 to 5 3^{i+3}

1) $\sum_{j=2}^{6} (j - (-1)^j)$

2) $\sum_{i=3}^{5} 3^{i+3}$

Answer:

1) . $\sum_{j=2}^{6}(j - (-1)^j) = \sum_{j=2}^{6} j - \sum_{j=2}^{6}(-1)^j = \sum_{j=1}^{6} j - j - (\sum_{j=0}^{6}(-1)^j - \sum_{j=0}^{1}(-1)^j) =$
$(\frac{6(6+1)}{2} - 1) - (\frac{(-1)^{6+1}-1}{(-1)-1} - \frac{(-1)^{1+1}-1}{(-1)-1}) = 20 - 1 = 19\ (Ans)$

2) $\sum_{i=3}^{5} 3^{i+3} = \sum_{i=3}^{5} 3^3 * 3^i = 27 * (\sum_{i=0}^{5} 3^i - \sum_{i=0}^{2} 3^i) = 27 * (\frac{3^{5+1}-1}{3-1}) - (\frac{3^{2+1}-1}{3-1}))\ (Ans)$