AI VIETNAM
AI Insight Course

Softmax Regression

(Draft)

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Ph.D. in Computer Science

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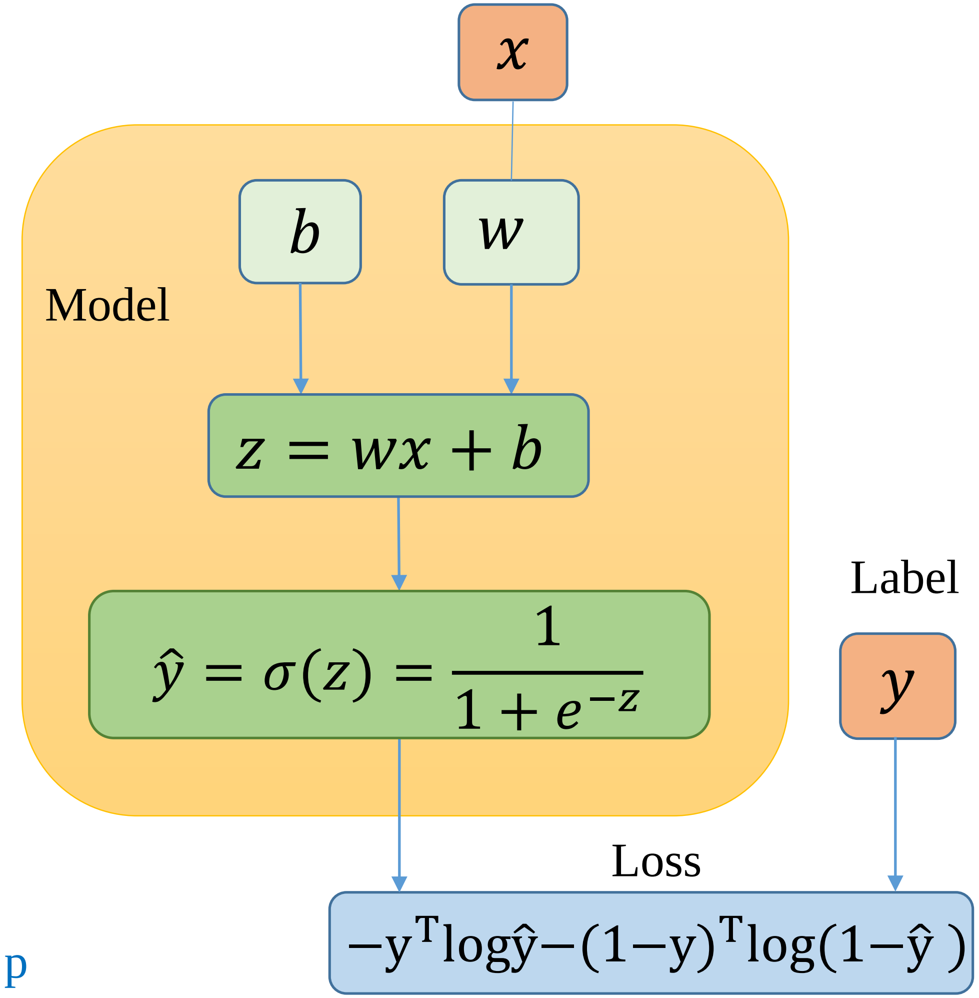
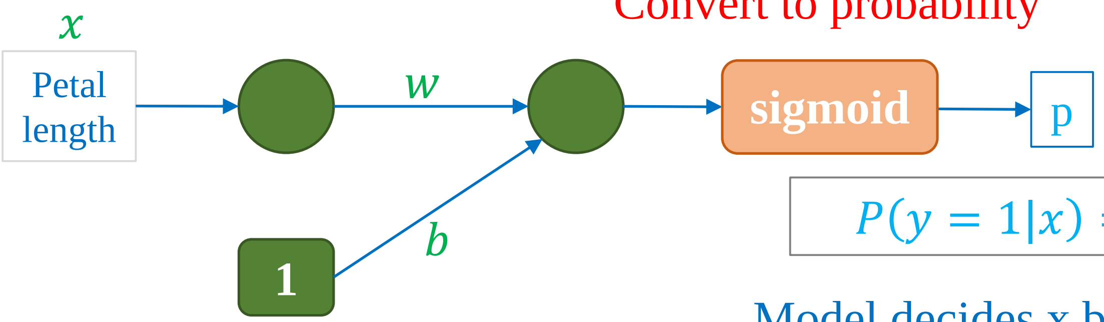
Outline

- Motivation
- Model Construction
- Loss Function
- Simple Example and Generalization
- Examples - Stochastic and Batch
- Another Approach

Motivation

Petal_Length	Label
1.4	0
1.3	0
1.5	0
4.5	1
4.1	1
4.6	1





$P(y = 1|x) = p$

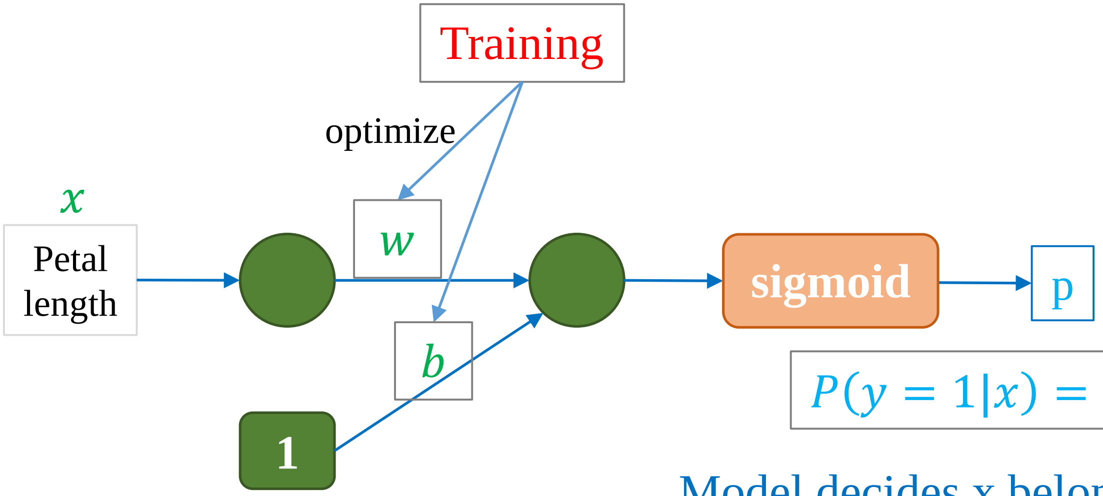
Model decides x belongs to the category (label=1) with the belief of p

Implicitly conclude that $P(y = 0|x) = 1 - p$

Motivation

Problem!

Feature	Label
Petal_Length	Label
1.4	0
1.3	0
1.5	0
4.5	1
4.1	1
4.6	1



$P(y = 1|x) = p$

Model decides x belongs to category (y=1) with the belief of p

Implicitly extract that $P(label = 0|x) = 1 - p$

Optimize w and b for $P(label = 1|x)$ affects $P(label = 0|x)$ and vice versa

How to have explicitly $P(y = 0|x)$?

Motivation

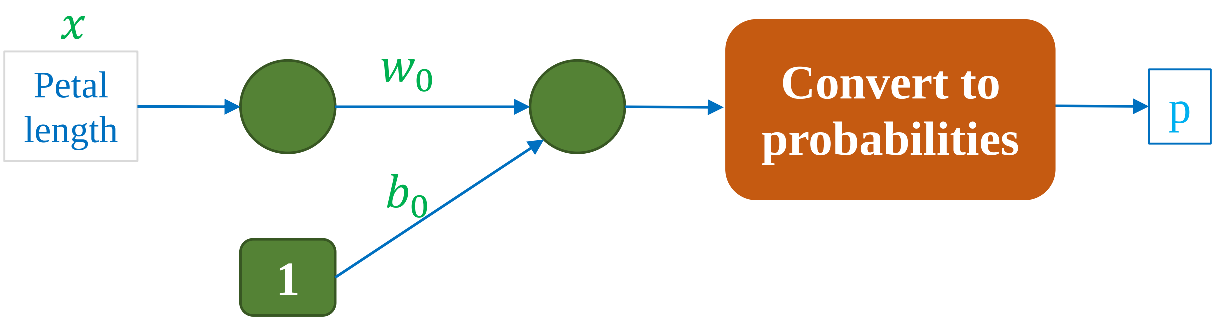
Problem!

Feature **Label**

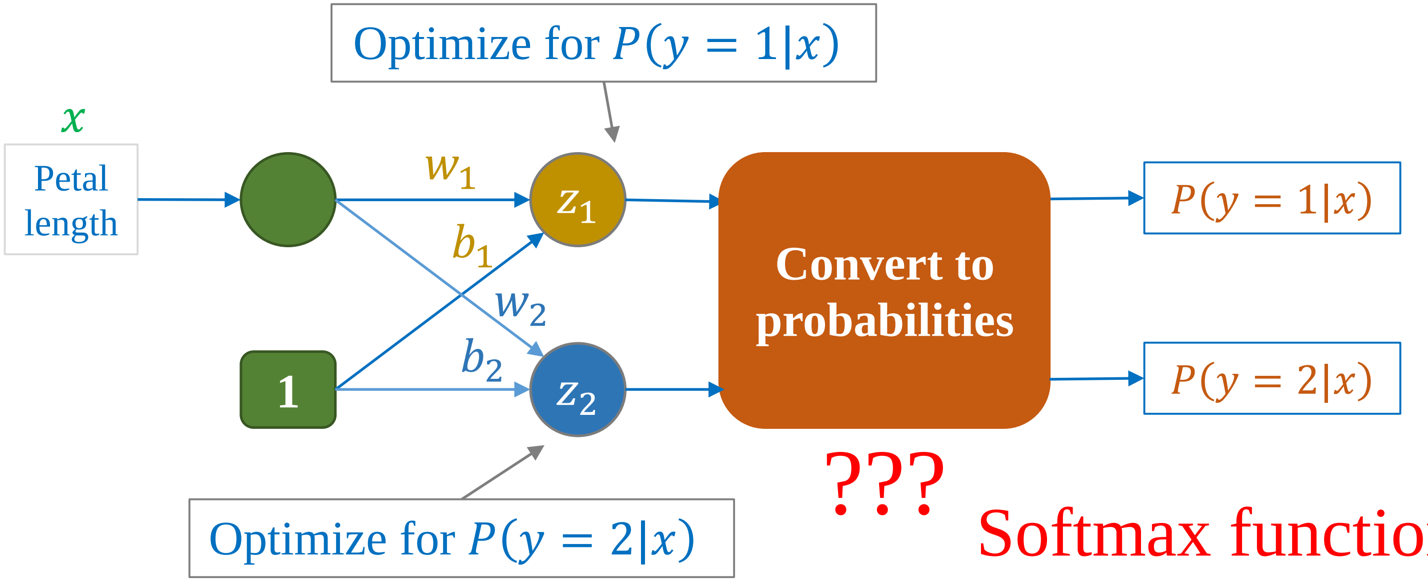
Petal_Length	Label
1.4	1
1.3	1
1.5	1
4.5	2
4.1	2
4.6	2

* Indices is from 1

Change notation a little bit



Explicitly output $P(y=1|x)$ and $P(y=2|x)$



??? Softmax function

Motivation

Softmax function

$$P_i = f(z_i) = \frac{e^{z_i}}{\sum_j e^{z_j}}$$

$$0 \leq f(z_i) \leq 1$$

$$\sum_i f(z_i) = 1$$

Input → Softmax → Probability

$z_1 = 1.0$ → $f(z_1) = 0.12$

$z_2 = 3.0$ → $f(z_2) = 0.88$

Input → Softmax → Probability

$z_1 = 1.0$ → $f(z_1) = 0.09$

$z_2 = 2.0$ → $f(z_2) = 0.24$

$z_3 = 3.0$ → $f(z_3) = 0.67$

Softmax function

Chuyển các giá trị của một vector thành các giá trị xác suất

Formula

$$f(x_i) = \frac{e^{x_i}}{\sum_j e^{x_j}}$$

$$0 \leq f(x_i) \leq 1$$

$$\sum_i f(x_i) = 1$$

Input				Probability
$x_1 = 1.0$				$f(x_1) = 0.09$
$x_2 = 2.0$	→	**Softmax**	→	$f(x_2) = 0.24$
$x_3 = 3.0$				$f(x_3) = 0.67$

Implementation (straightforward)

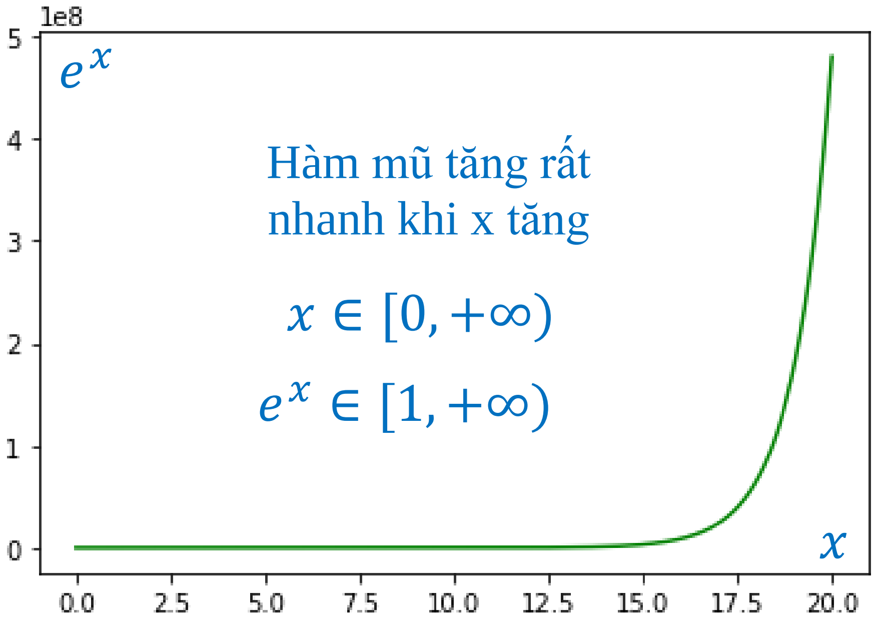
```
import numpy as np

def softmax(X):
    exps = np.exp(X)
    return exps / np.sum(exps)
```

```
X = np.array([1.0, 2.0, 3.0])
f = softmax(X)
print(f)
```

```
[0.09003057 0.24472847 0.66524096]
```

```
X = np.array([1000.0, 1001.0, 1002.0])
f = softmax(X)
print(f)
```

```
[nan nan nan]
```



Giá trị nan vì e^x vượt giới hạn lưu trữ của biến

Softmax function (stable)

(Stable) Formula

$$m = \max(\boldsymbol{x})$$

$$f(x_i) = \frac{e^{(x_i - m)}}{\sum_j e^{(x_j - m)}}$$

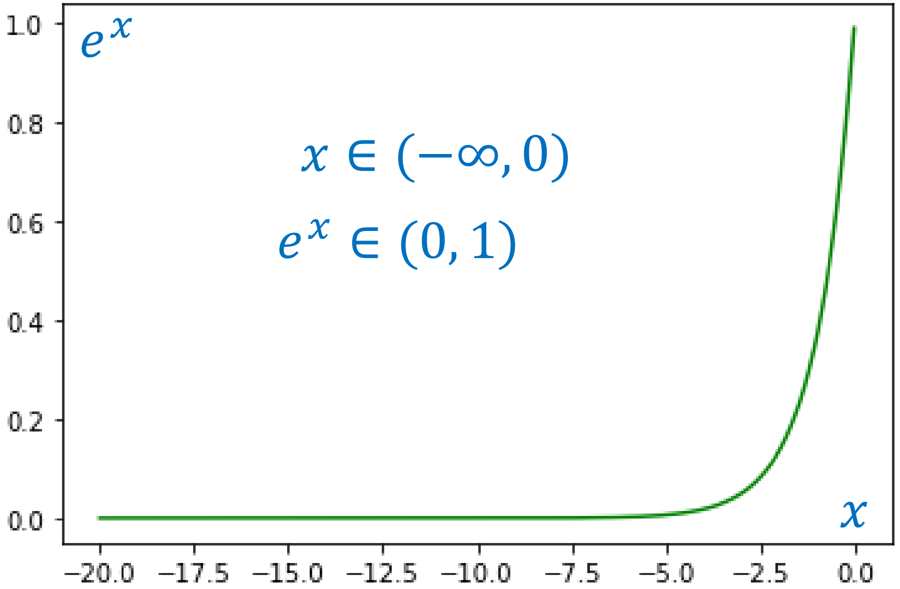
X	X-m		Probability
$x_1 = 1.0$	$x_1 = -2.0$		$f(x_1) = 0.09$
$x_2 = 2.0$ ➡	$x_2 = -1.0$ ➡	**Softmax** ➡	$f(x_2) = 0.24$
$x_3 = 3.0$	$x_3 = 0$		$f(x_3) = 0.67$

Implementation (stable)

```python
import numpy as np

def stable_softmax(X):
    exps = np.exp(X-np.max(X))
    return exps / np.sum(exps)
```

```python
X = np.array([1.0, 2.0, 3.0])
f = stable_softmax(X)
print(f)
```

```
[0.09003057 0.24472847 0.66524096]
```




```python
X = np.array([1000.0, 1001.0, 1002.0])
f = stable_softmax(X)
print(f)
```

```
[0.09003057 0.24472847 0.66524096]
```

```python
X = np.array([1.0, 1001.0, 1002.0])
f = stable_softmax(X)
print(f)
```

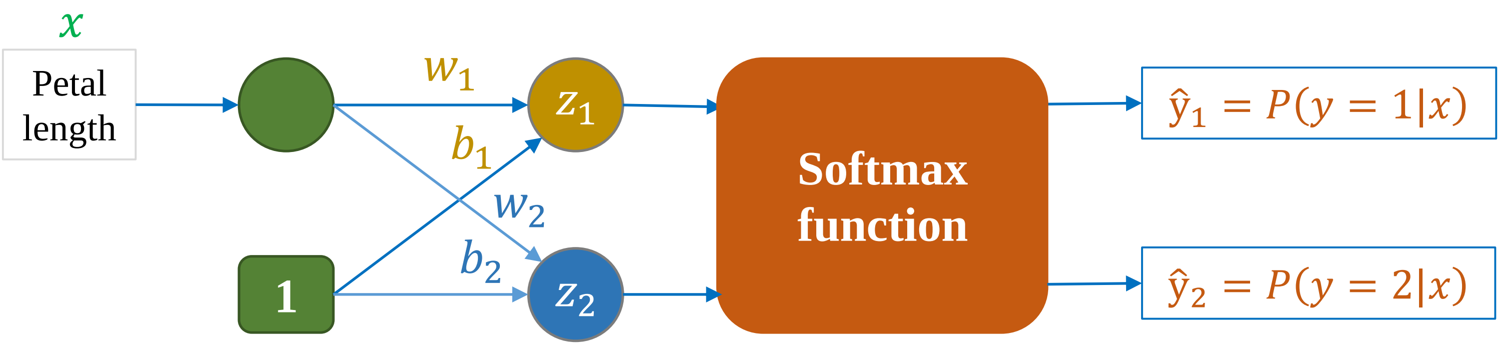
```
[0.         0.26894142 0.73105858]
```

Motivation

Feature **Label**

Petal_Length	Label
1.4	1
1.3	1
1.5	1
4.5	2
4.1	2
4.6	2

Explicitly output $P(y=1|x)$ and $P(y=0|x)$



Softmax function

$$P_i = f(z_i) = \frac{e^{z_i}}{\sum_j e^{z_j}}$$

$$0 \leq f(z_i) \leq 1$$

$$\sum_i f(z_i) = 1$$

How about loss function?

Index from 0 $\quad L(\boldsymbol{\theta}) = -y\log\hat{y} - (1-y)\log(1-\hat{y})$

Otherwise $\quad L(\boldsymbol{\theta}) = -\delta(y,1)\log\hat{y}_1 - \delta(y,2)\log\hat{y}_2$

$$\delta(i,j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Outline

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Model Construction

❖ 1-D Feature and two classes

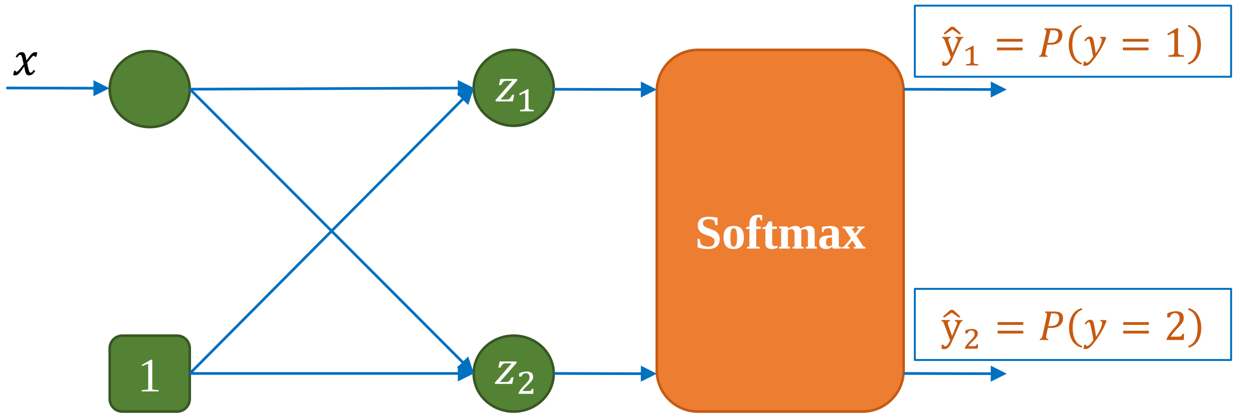
Feature Label

Petal_Length	Label
1.4	1
1.3	1
1.5	1
4.5	2
4.1	2
4.6	2

#class=2

#feature=1

Model



Feature is with one dimension

→ Need one node for input

Two categories

→ Need two node for output

Model Construction

❖ 1-D Feature and three classes

Feature	Label
Petal_Length	Label
1.4	1
1.3	1
1.5	1
4.5	2
4.1	2
4.6	2
5.2	3
5.6	3
5.9	3

#class=3

#feature=1

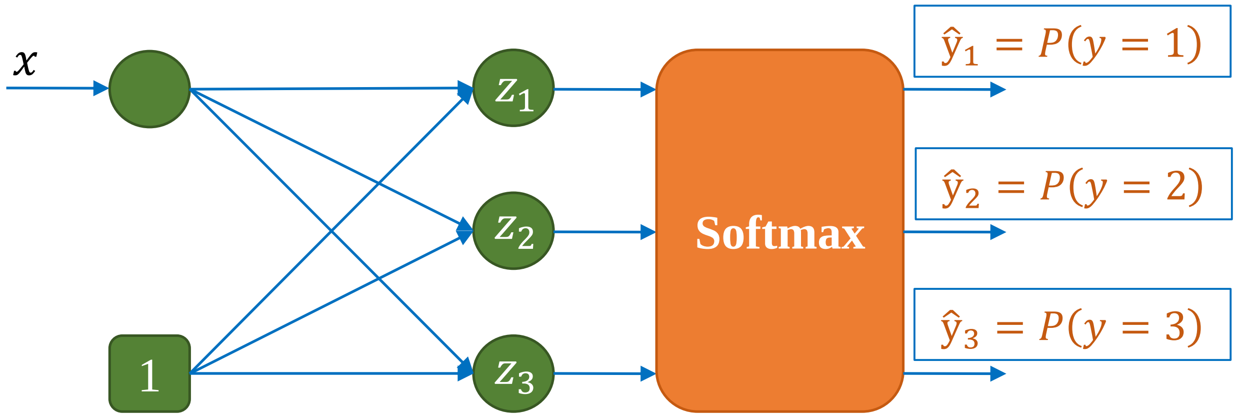
Feature is with one dimension

→ Need one node for input

Three categories

→ Need three nodes for output

Model



Model Construction

❖ 4-D Feature and three classes

Feature | Label

Petal_Length	Petal_Width	Label
1.5	0.2	1
1.4	0.2	1
1.6	0.2	1
4.7	1.6	2
3.3	1.1	2
4.6	1.3	2
5.6	2.2	3
5.1	1.5	3
5.6	1.4	3

#class=3

#feature=2

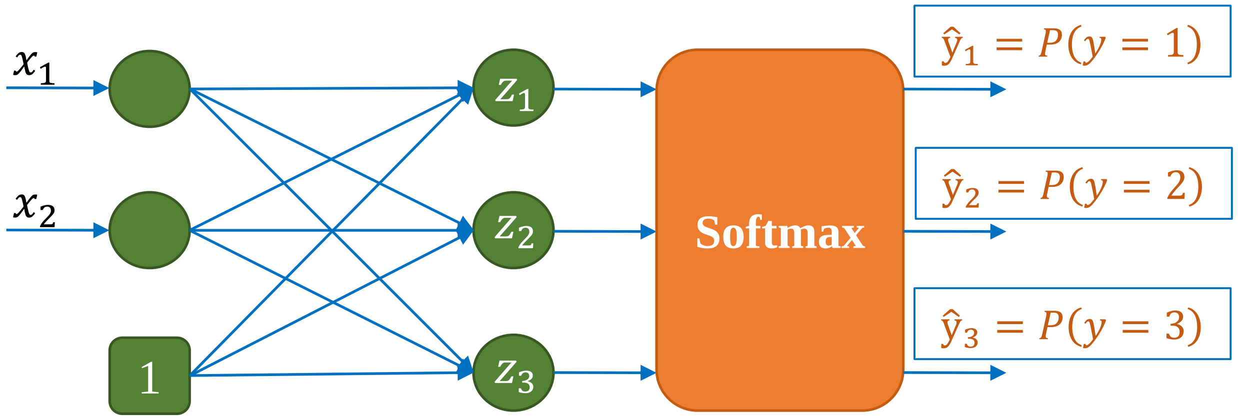
Feature is with two dimensions

→ Need two nodes for input

Three categories

→ Need three nodes for output

Model



Model Construction

❖ 4-D Feature and three classes

Feature				Label
Sepal_Length	Sepal_Width	Petal_Length	Petal_Width	Label
5.2	3.5	1.5	0.2	1
5.2	3.4	1.4	0.2	1
4.7	3.2	1.6	0.2	1
6.3	3.3	4.7	1.6	2
4.9	2.4	3.3	1.1	2
6.6	2.9	4.6	1.3	2
6.4	2.8	5.6	2.2	3
6.3	2.8	5.1	1.5	3
6.1	2.6	5.6	1.4	3

Feature is with four dimensions

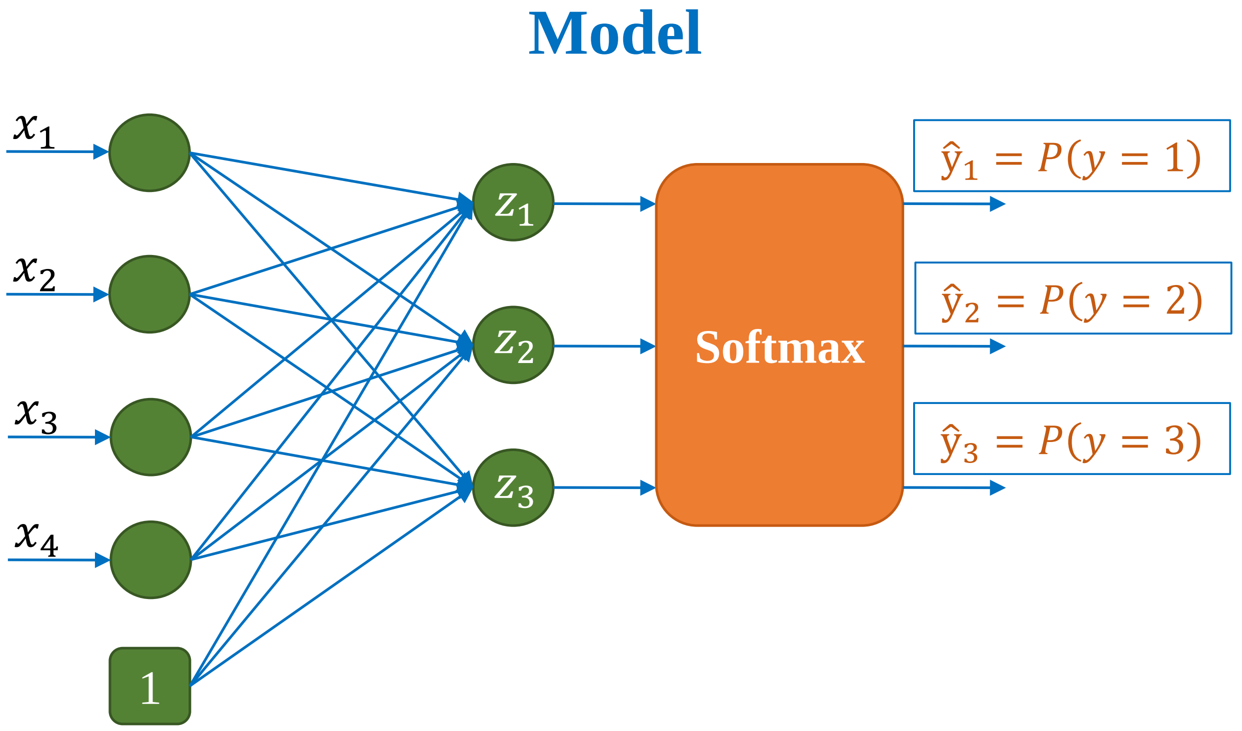
→ Need four nodes for input

Three categories

→ Need three nodes for output

#class=3

#feature=4



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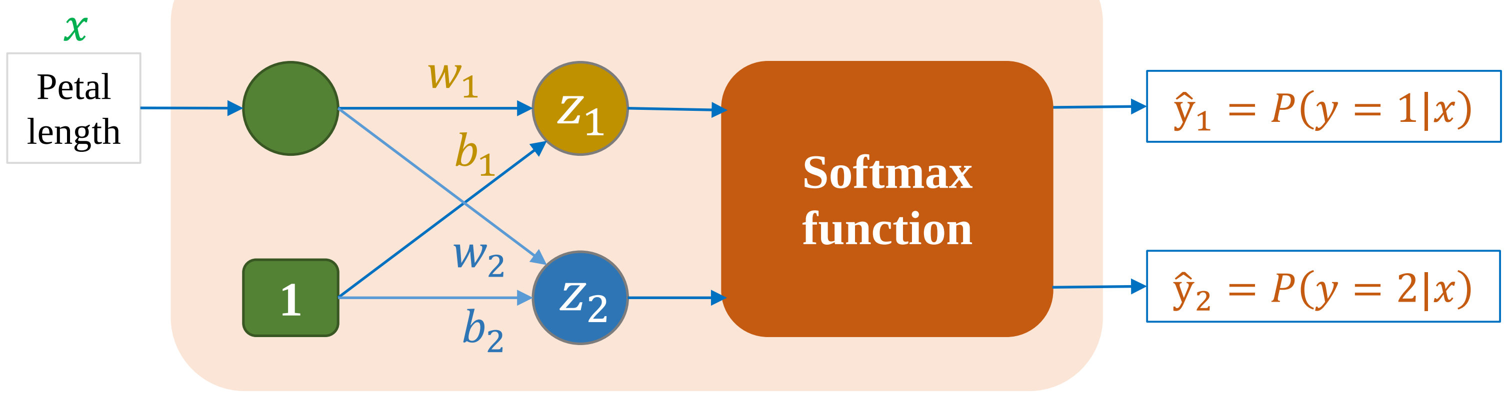
Loss function

❖ Simple illustration

Feature **Label**

Petal_Length	Label
1.4	1
1.3	1
1.5	1
4.5	2
4.1	2
4.6	2

Model



$$z_1 = xw_1 + b_1$$

$$z_2 = xw_2 + b_2$$

$$\hat{y}_1 = \frac{e^{z_1}}{\sum_{j=1}^{2} e^{z_j}}$$

$$\hat{y}_2 = \frac{e^{z_1}}{\sum_{j=1}^{2} e^{z_j}}$$

$$\boldsymbol{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} b_1 & w_1 \\ b_2 & w_2 \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} \boldsymbol{\theta}_1^T \\ \boldsymbol{\theta}_2^T \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} = \boldsymbol{\theta}^T \boldsymbol{x}$$

$$\hat{\mathbf{y}} = \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \end{bmatrix} = \frac{1}{\sum_{j=1}^{2} e^{z_j}} \begin{bmatrix} e^{z_1} \\ e^{z_2} \end{bmatrix} = \frac{e^{\boldsymbol{z}}}{\sum_{j=1}^{2} e^{z_j}}$$

A vector is by default a column vector $\boldsymbol{\theta}_1 = \begin{bmatrix} b_1 \\ w_1 \end{bmatrix}$

vector transpose $\boldsymbol{\theta}_1^T = [b_1 \quad w_1]$

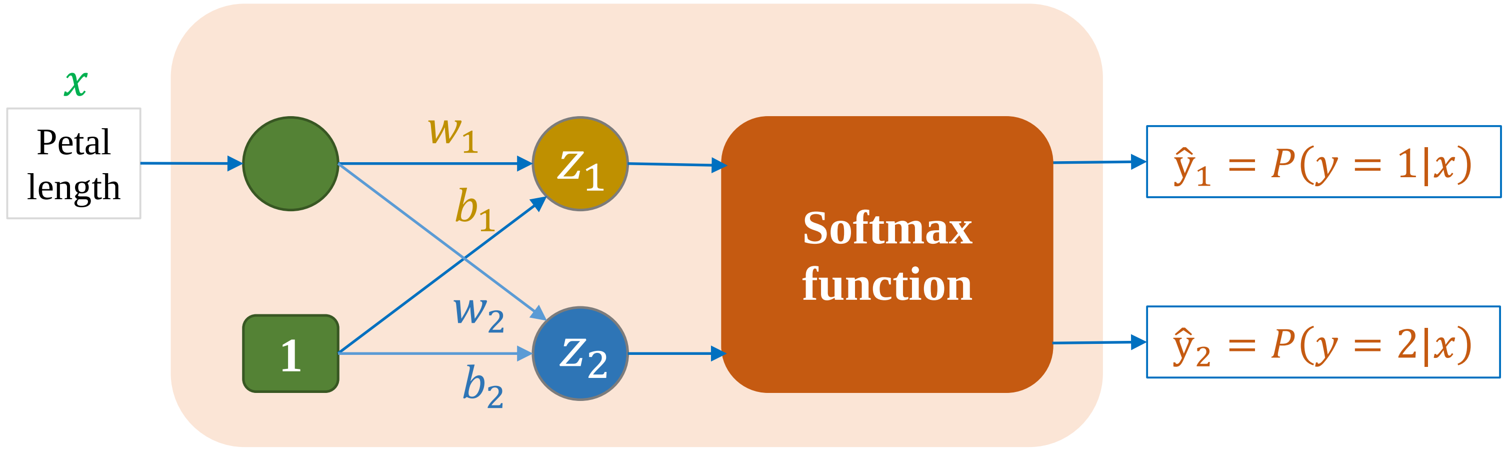
Loss function

❖ Simple illustration

Feature | **Label**

Petal_Length	Label
1.4	1
1.3	1
1.5	1
4.5	2
4.1	2
4.6	2

Model



$$\delta(i,j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$z_1 = xw_1 + b_1$$

$$z_2 = xw_2 + b_2$$

$$\hat{y}_1 = \frac{e^{z_1}}{\sum_{j=1}^{2} e^{z_j}}$$

$$\hat{y}_2 = \frac{e^{z_1}}{\sum_{j=1}^{2} e^{z_j}}$$

$$\boldsymbol{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} b_1 & w_1 \\ b_2 & w_2 \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} \boldsymbol{\theta}_1^T \\ \boldsymbol{\theta}_2^T \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} = \boldsymbol{\theta}^T \boldsymbol{x}$$

$$\hat{\mathbf{y}} = \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \end{bmatrix} = \frac{1}{\sum_{j=1}^{2} e^{z_j}} \begin{bmatrix} e^{z_1} \\ e^{z_2} \end{bmatrix} = \frac{e^{\boldsymbol{z}}}{\sum_{j=1}^{2} e^{z_j}}$$

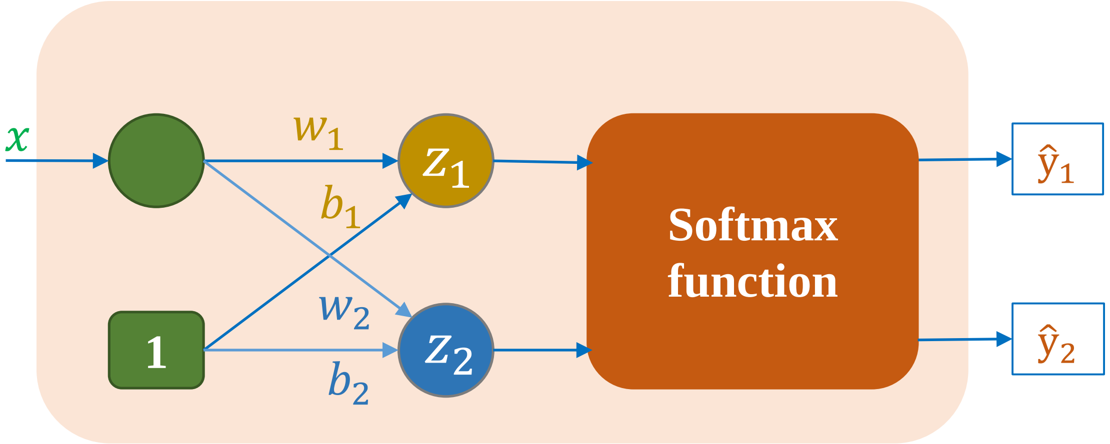
$$L(\boldsymbol{\theta}) = -\delta(i,j)\log\hat{y}_1 - \delta(i,j)\log\hat{y}_2$$

$$= -\sum_{i=1}^{2} \delta(i,y)\log\hat{y}_i$$

Loss function

$$L(\boldsymbol{\theta}) = -\sum_{i=1}^{2} \delta(i,y)\log\hat{y}_i$$

Model



$$\hat{y}_1 = \frac{e^{z_1}}{\sum_{j=1}^{2} e^{z_j}}$$

$$\hat{y}_2 = \frac{e^{z_2}}{\sum_{j=1}^{2} e^{z_j}}$$

Derivative

$$\frac{\partial \hat{y}_i}{\partial z_j} = \hat{y}_i(\delta(i,j) - \hat{y}_i)$$

$$\frac{\partial L}{\partial z_i} = \hat{y}_i - \delta(i,y)$$

Loss function

One-hot encoding for label

$$\begin{array}{ll} & \quad\quad\quad\quad\quad\quad\;\; y_0 \;\; y_1 \\ y = 0 \rightarrow \boldsymbol{y} = & [1 \quad 0] \\ y = 1 \rightarrow \boldsymbol{y} = & [0 \quad 1] \end{array}$$

scalar   vector

$$z_1 = xw_1 + b_1$$

$$z_2 = xw_2 + b_2$$

$$\hat{y}_1 = \frac{e^{z_1}}{\sum_{j=1}^{2} e^{z_j}}$$

$$\hat{y}_2 = \frac{e^{z_1}}{\sum_{j=1}^{2} e^{z_j}}$$

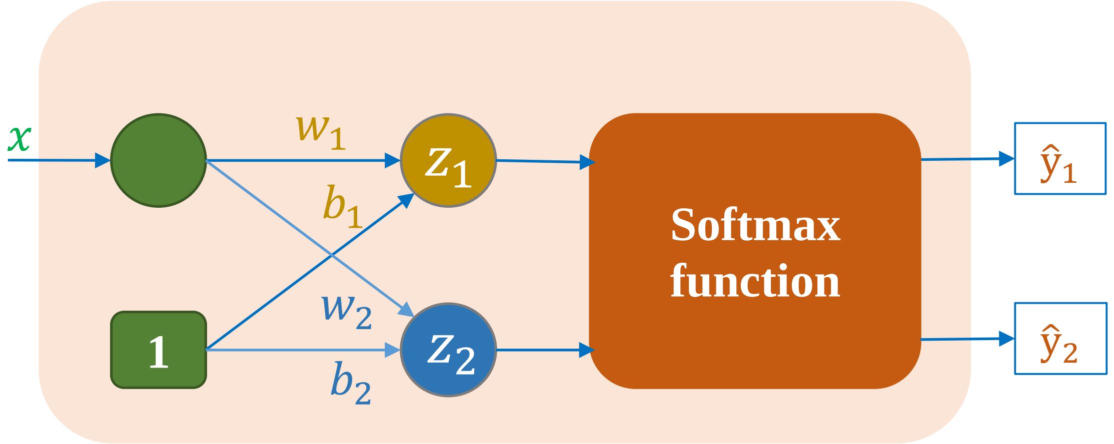
$$\boldsymbol{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} b_1 & w_1 \\ b_2 & w_2 \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} \boldsymbol{\theta}_1^T \\ \boldsymbol{\theta}_2^T \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} = \boldsymbol{\theta}^T \boldsymbol{x}$$

$$\hat{\mathbf{y}} = \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \end{bmatrix} = \frac{1}{\sum_{j=1}^{2} e^{z_j}} \begin{bmatrix} e^{z_1} \\ e^{z_2} \end{bmatrix} = \frac{e^{\boldsymbol{z}}}{\sum_{j=1}^{2} e^{z_j}}$$

$$L(\boldsymbol{\theta}) = -\delta(i,j)\log\hat{y}_1 - \delta(i,j)\log\hat{y}_2$$

$$= -\sum_{i=1}^{2} \delta(i,y)\log\hat{y}_i$$

Model



Derivative

$$\frac{\partial \hat{y}_i}{\partial z_j} = \hat{y}_i(\delta(i,j) - \hat{y}_i)$$

$$\frac{\partial L}{\partial z_i} = \hat{y}_i - \delta(i,y)$$

$$\frac{\partial L}{\partial w_i} = x\big(\hat{y}_i - \delta(i,y)\big)$$

$$\frac{\partial L}{\partial b_i} = \hat{y}_i - \delta(i,y)$$

Simple Illustration - Summary

Feature **Label**

Petal_Length	Label
1.4	1
1.3	1
1.5	1
4.5	2
4.1	2
4.6	2

* Label indices are from 1

$$\boldsymbol{\theta} = \begin{bmatrix} b_1 & b_2 \\ w_1 & w_2 \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ x \end{bmatrix}$$

Input with one example
$(x, y) = (1.4, 1)$

Forward computation

$$\boldsymbol{z} = \boldsymbol{\theta}^T \boldsymbol{x}$$

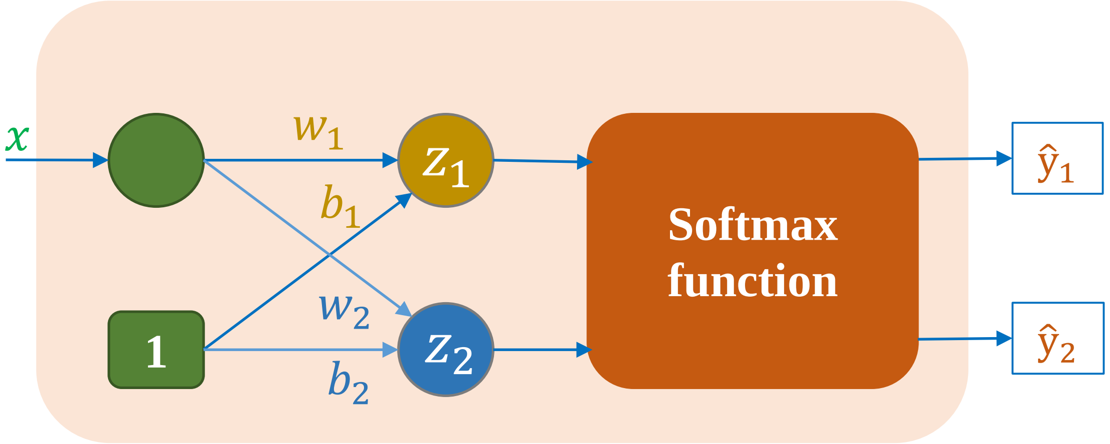
$$\hat{\mathbf{y}} = \frac{e^{\boldsymbol{z}}}{\sum_{j=1}^{2} e^{z_j}}$$

Loss function

$$L(\boldsymbol{\theta}) = -\sum_{i=1}^{2} \delta(i, y) \log \hat{y}_i$$

$$\delta(i, j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Model



Derivative

$$\frac{\partial \hat{y}_i}{\partial z_j} = \hat{y}_i(\delta(i, j) - \hat{y}_i)$$

$$\frac{\partial L}{\partial z_i} = \hat{y}_i - \delta(i, y)$$

$$\frac{\partial L}{\partial w_i} = x(\hat{y}_i - \delta(i, y))$$

$$\frac{\partial L}{\partial b_i} = \hat{y}_i - \delta(i, y)$$

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Simple Example

Training data

Feature **Label**

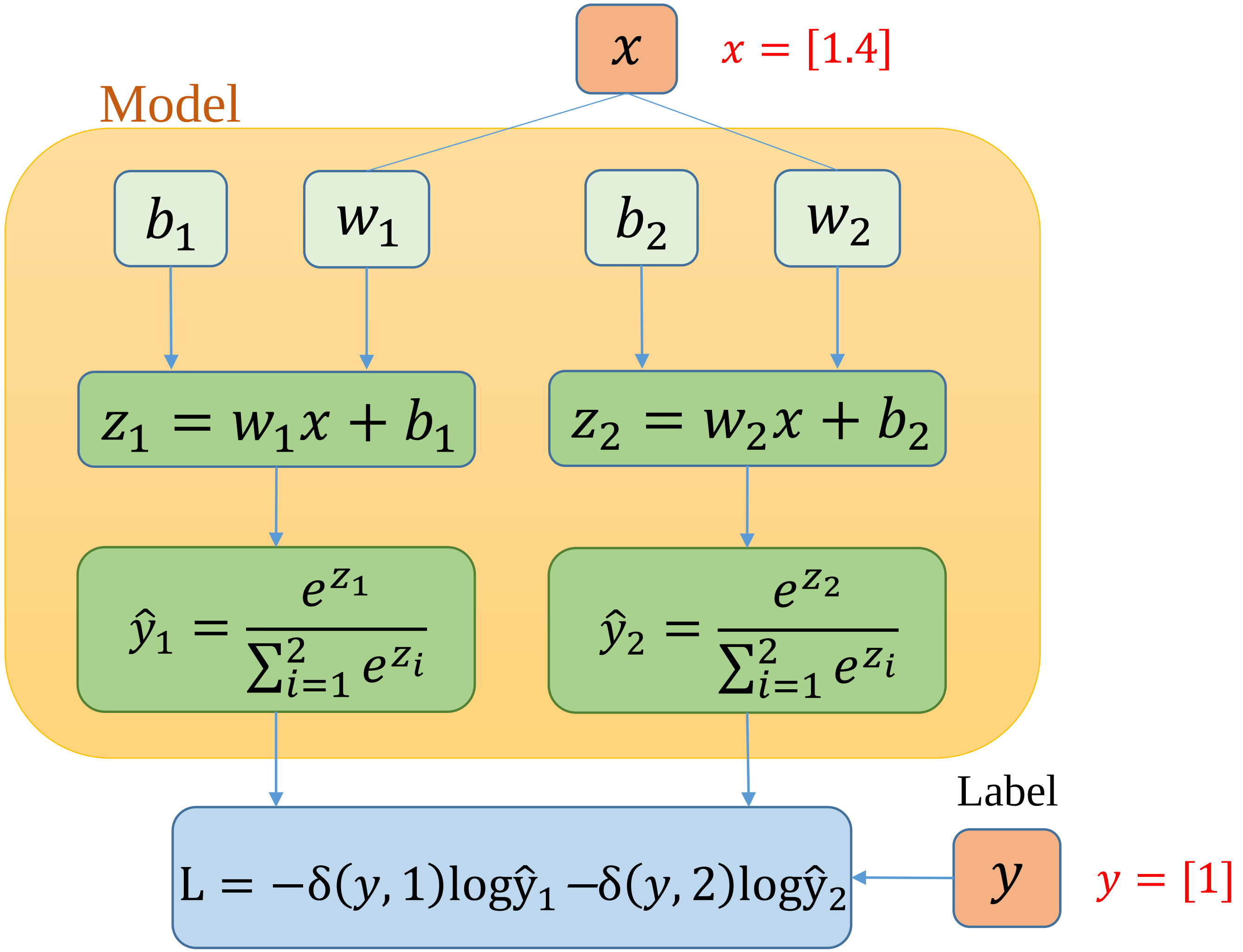
Petal_Length	Label
1.4	1
1.3	1
1.5	1
4.5	2
4.1	2
4.6	2

#class=2

#feature=1

Training example

$(x, y) = (1.4, 1)$



Simple Example

Training data

Feature **Label**

Petal_Length	Label
1.4	1
1.3	1
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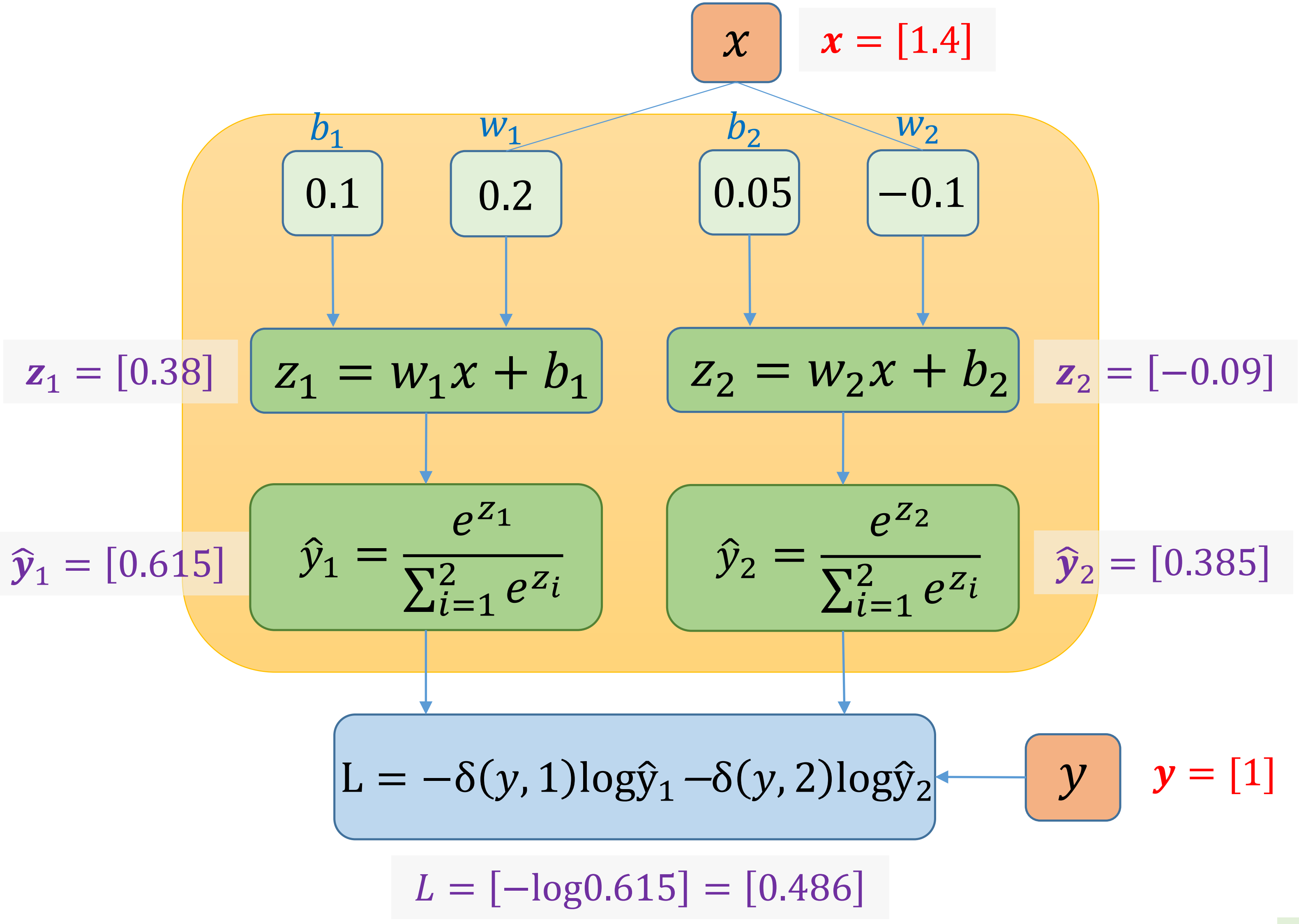
#class=2

#feature=1

Training example

$(x, y) = (1.4, 1)$



Simple Example

Training example

$$(x, y) = (1.4, 1)$$

Derivative

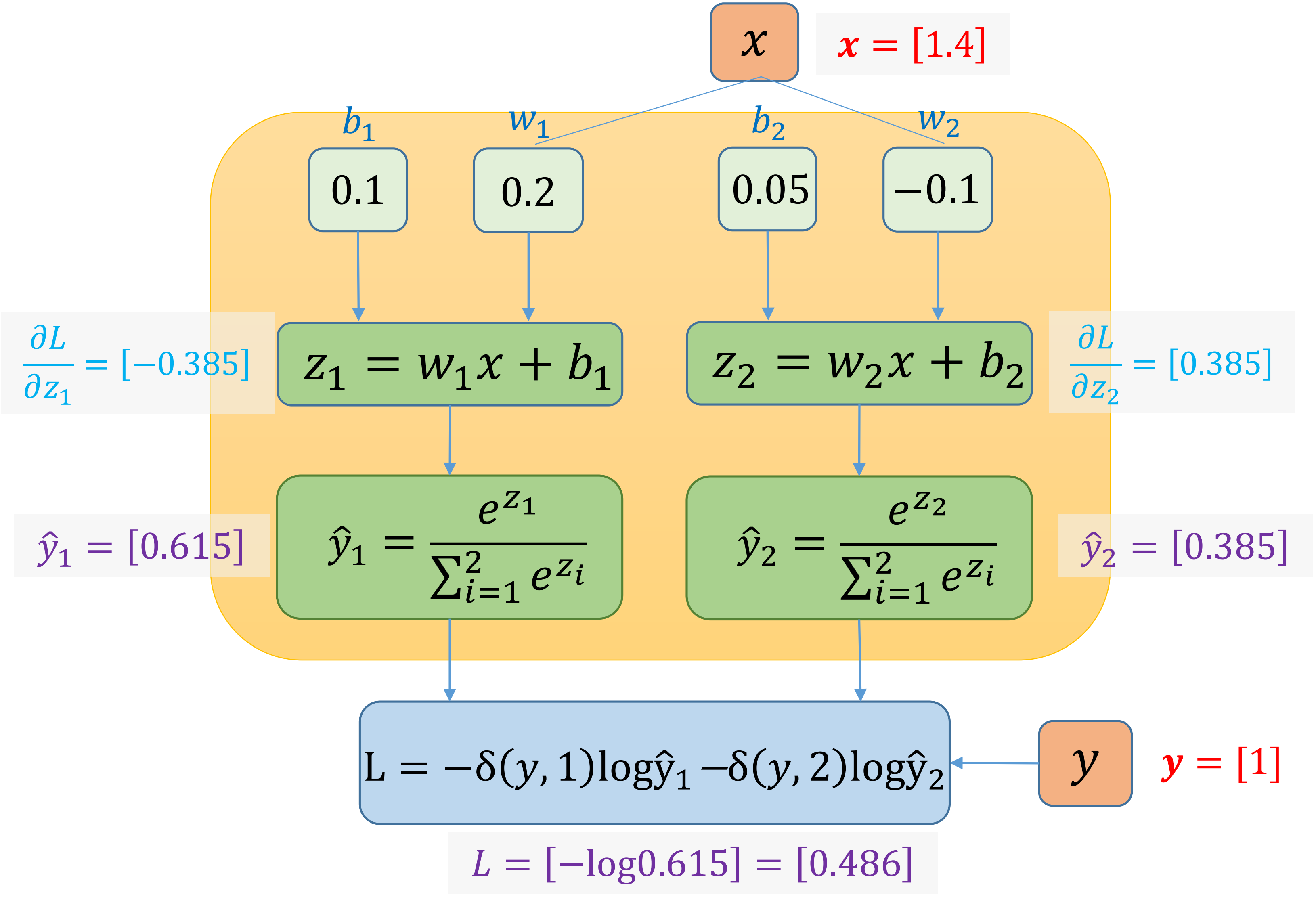
$$\frac{\partial L}{\partial z_i} = \hat{y}_i - \delta(i, y)$$

$$\frac{\partial L}{\partial w_i} = x\big(\hat{y}_i - \delta(i, y)\big)$$

$$\frac{\partial L}{\partial b_i} = \hat{y}_i - \delta(i, y)$$

$$\frac{\partial L}{\partial z_1} = \hat{y}_1 - \delta(1, y)$$
$$= 0.615 - 1 = -0.385$$

$$\frac{\partial L}{\partial z_2} = \hat{y}_2 - \delta(2, y)$$
$$= 0.385 - 0 = 0.385$$



Simple Example

Training example

$(x, y) = (1.4, 1)$

Derivative

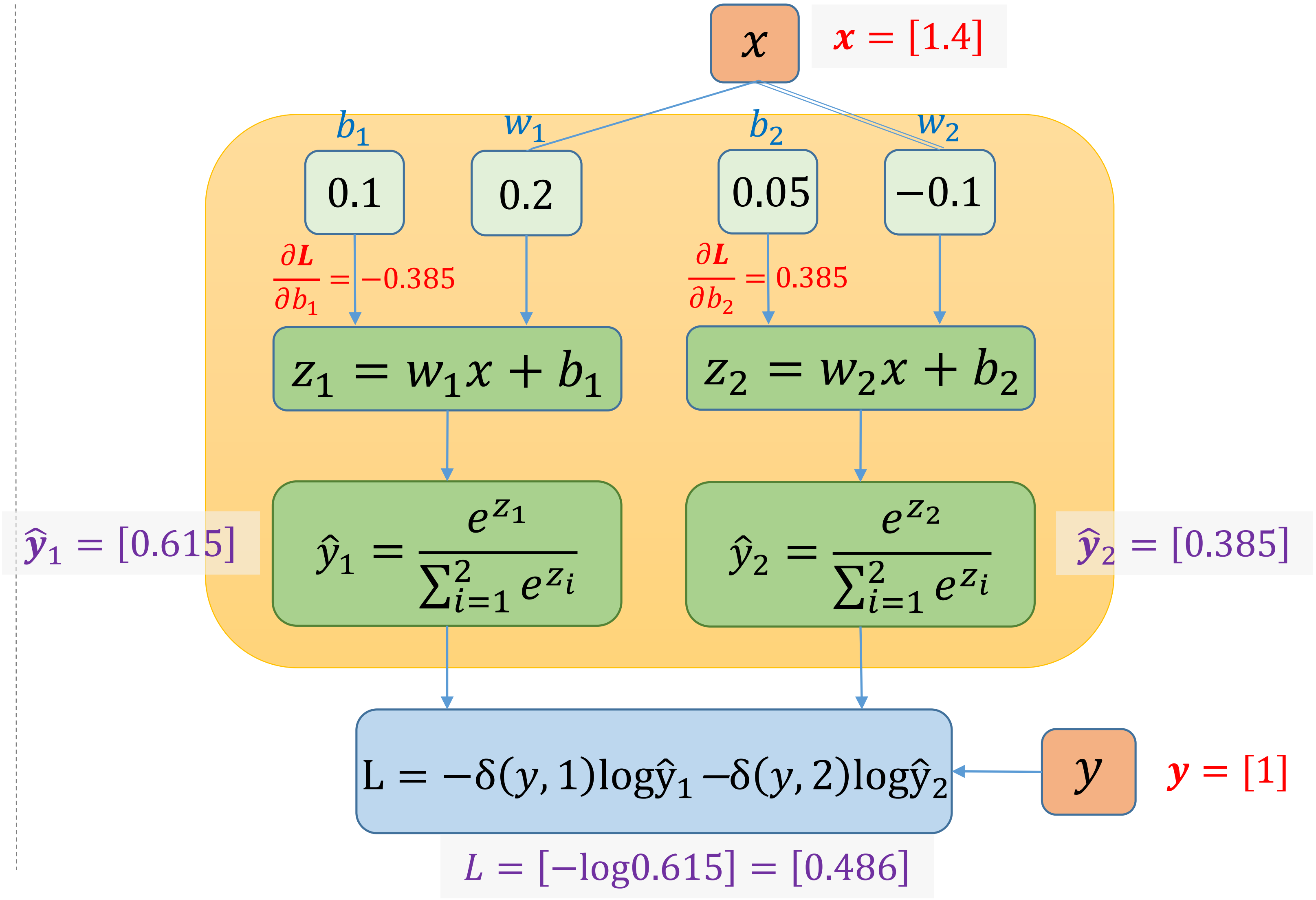
$$\frac{\partial L}{\partial z_i} = \hat{y}_i - \delta(i, y)$$

$$\frac{\partial L}{\partial w_i} = x\big(\hat{y}_i - \delta(i, y)\big)$$

$$\frac{\partial L}{\partial b_i} = \hat{y}_i - \delta(i, y)$$

$$\frac{\partial L}{\partial b_1} = \hat{y}_1 - \delta(1, y)$$
$$= 0.615 - 1 = -0.385$$

$$\frac{\partial L}{\partial b_2} = \hat{y}_2 - \delta(2, y)$$
$$= 0.385 - 0 = 0.385$$



Simple Example

Training example

$(x, y) = (1.4, 1)$

Derivative

$$\frac{\partial L}{\partial z_i} = \hat{y}_i - \delta(i, y)$$

$$\frac{\partial L}{\partial w_i} = x\big(\hat{y}_i - \delta(i, y)\big)$$

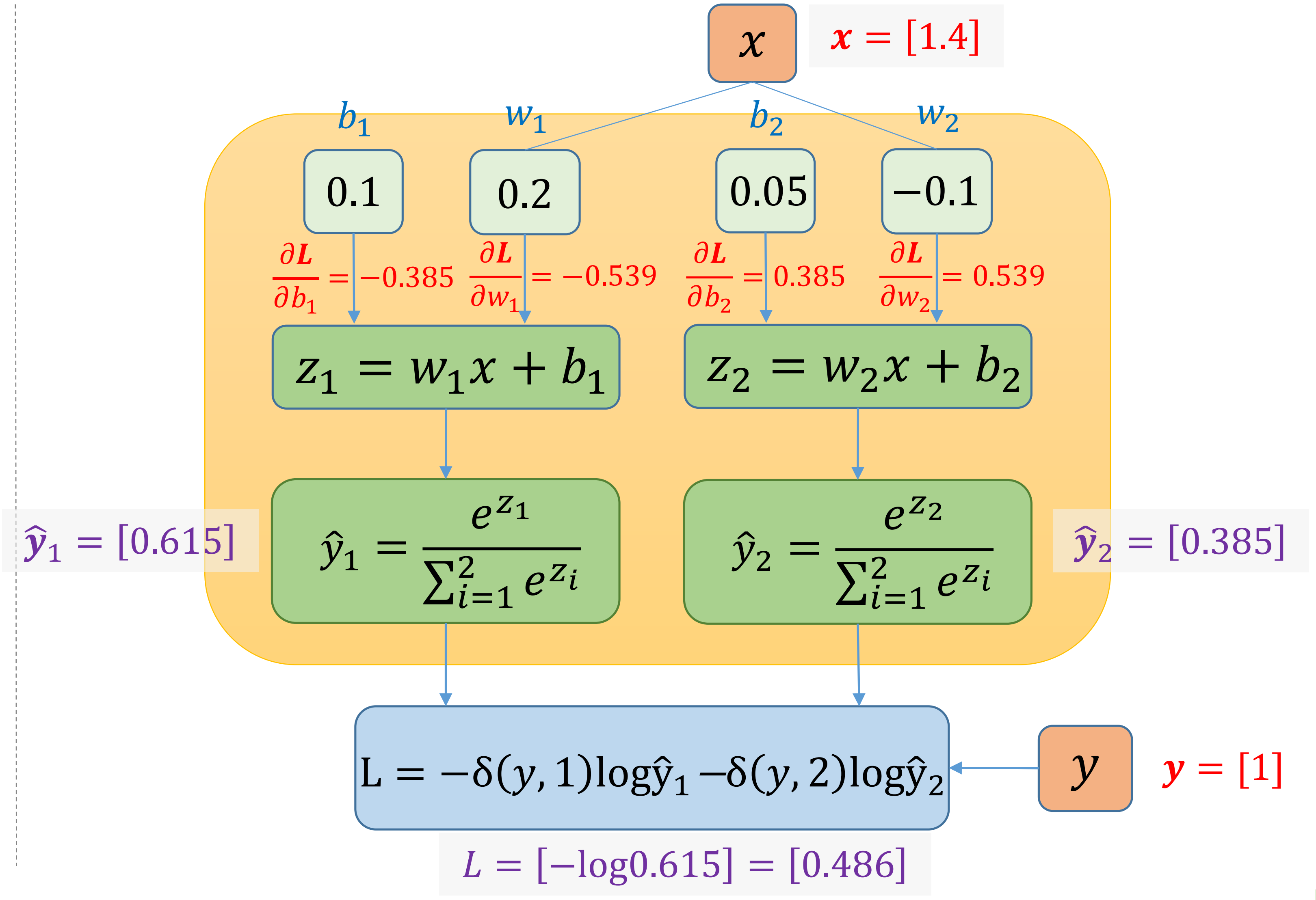
$$\frac{\partial L}{\partial b_i} = \hat{y}_i - \delta(i, y)$$

$$\frac{\partial \boldsymbol{L}}{\partial w_1} = x(\hat{y}_1 - \delta(1, y))$$

$$= -0.385 * 1.4 = -0.539$$

$$\frac{\partial \boldsymbol{L}}{\partial w_2} = x(\hat{y}_2 - \delta(2, y))$$

$$= 0.385 * 1.4 = 0.539$$



Simple Example

Update parameters

$$\boldsymbol{\theta} = \boldsymbol{\theta} - \eta L'_{\boldsymbol{\theta}}$$

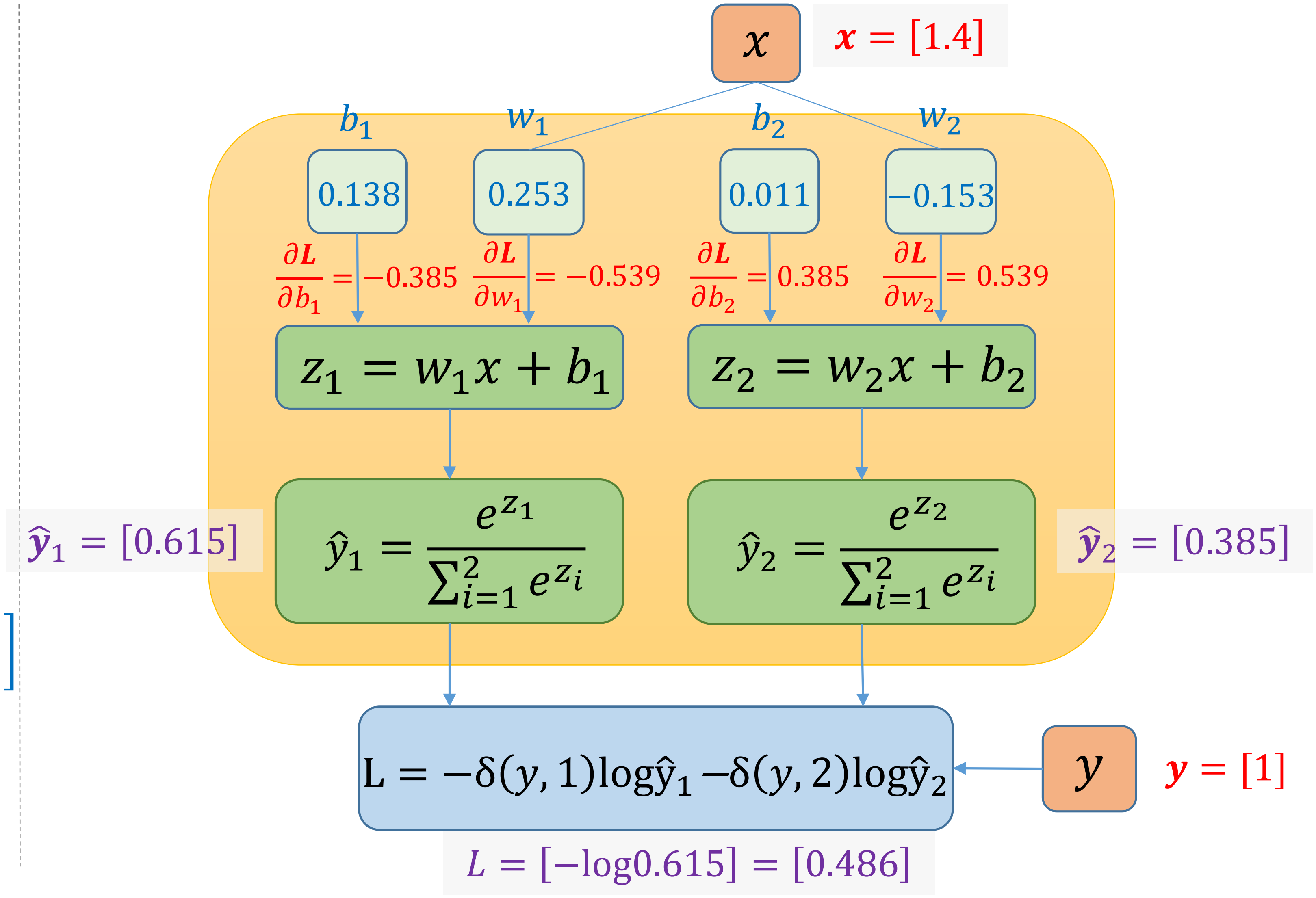
η is learning rate

$$\boldsymbol{\theta} = \begin{bmatrix} b_1 & b_2 \\ w_1 & w_2 \end{bmatrix} \qquad L'_{\boldsymbol{\theta}} = \begin{bmatrix} \frac{\partial L}{\partial b_1} & \frac{\partial L}{\partial b_2} \\ \frac{\partial L}{\partial w_1} & \frac{\partial L}{\partial w_2} \end{bmatrix}$$

$$\eta = 0.1$$

$$\boldsymbol{\theta} = \begin{bmatrix} 0.1 & 0.05 \\ 0.2 & -0.1 \end{bmatrix} - 0.1 \begin{bmatrix} -0.385 & 0.385 \\ -0.539 & 0.539 \end{bmatrix}$$

$$= \begin{bmatrix} 0.138 & 0.011 \\ 0.253 & -0.153 \end{bmatrix}$$



Generalization

Feature **Label**

Petal_Length	Label
1.4	1
1.3	1
1.5	1
4.5	2
4.1	2
4.6	2

$$\boldsymbol{\theta} = [\boldsymbol{\theta}_1 \quad \boldsymbol{\theta}_2] = \begin{bmatrix} b_1 & b_2 \\ w_1 & w_2 \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ x \end{bmatrix}$$

#feature n=1 #class k=2

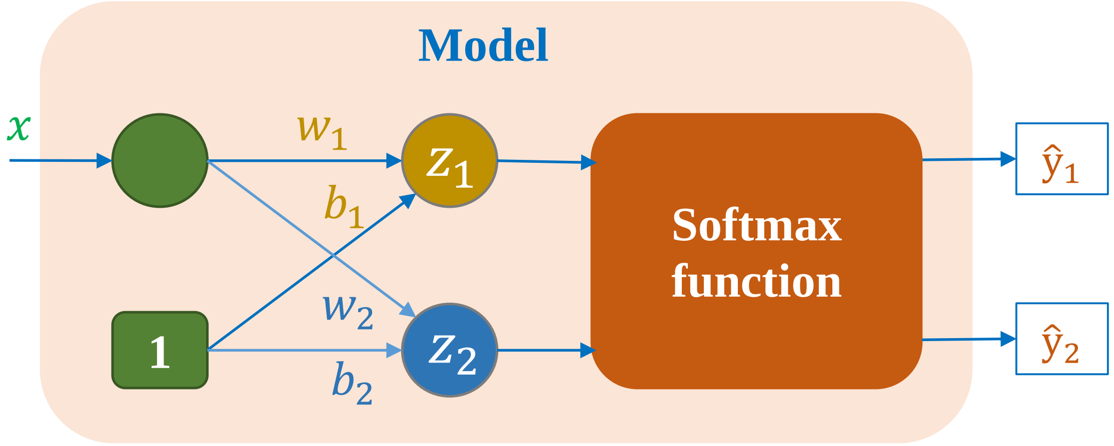
Petal_Length	Label
1.4	1
1.3	1
1.5	1
4.5	2
4.1	2
4.6	2
5.2	3
5.6	3
5.9	3

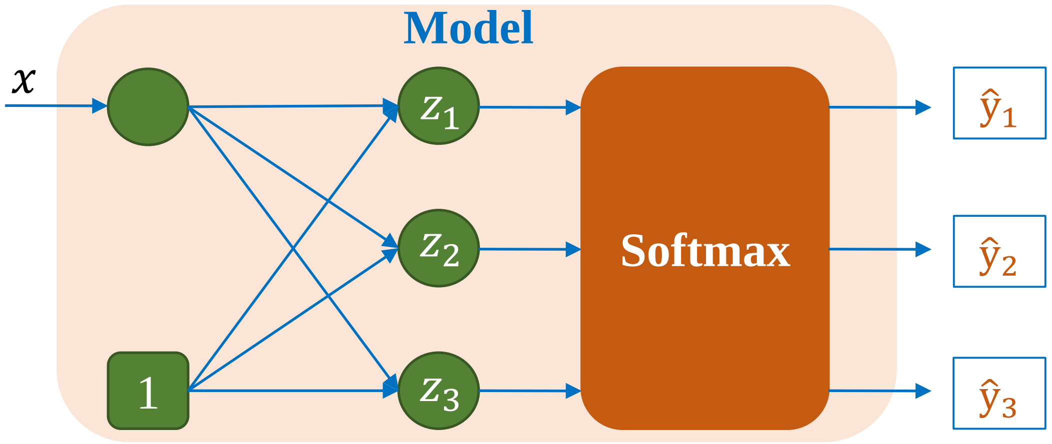
$$\boldsymbol{\theta} = [\boldsymbol{\theta}_1 \quad \boldsymbol{\theta}_2 \quad \boldsymbol{\theta}_3]$$

$$= \begin{bmatrix} b_1 & b_2 & b_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ x \end{bmatrix}$$

#feature n=1 #class K=3



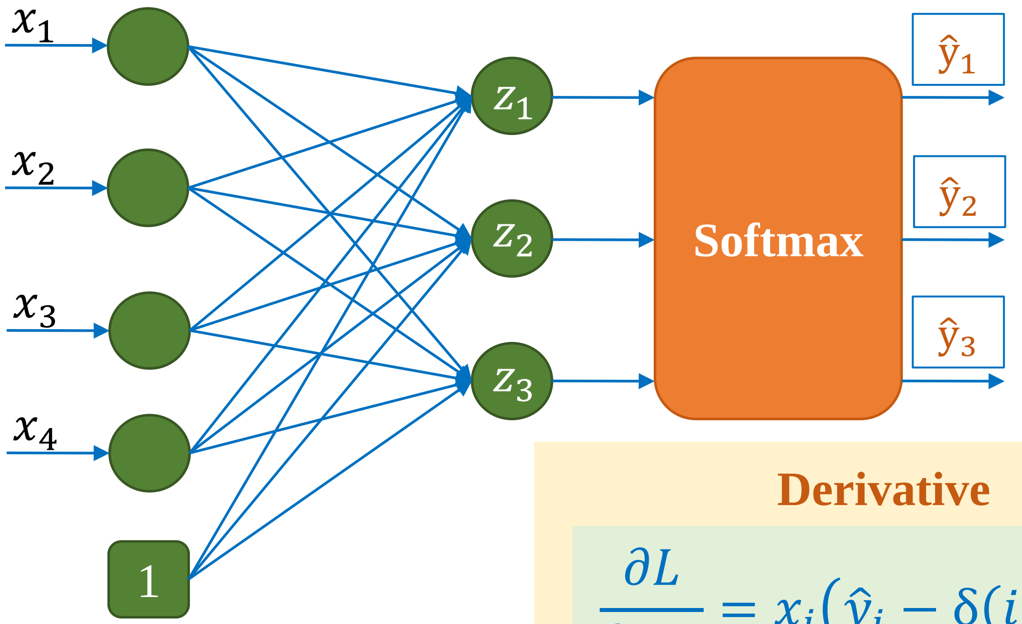


Generalization - Stochastic

Sepal_Length	Sepal_Width	Petal_Length	Petal_Width	Label
5.2	3.5	1.5	0.2	1
5.2	3.4	1.4	0.2	1
4.7	3.2	1.6	0.2	1
6.3	3.3	4.7	1.6	2
4.9	2.4	3.3	1.1	2
6.6	2.9	4.6	1.3	2
6.4	2.8	5.6	2.2	3
6.3	2.8	5.1	1.5	3
6.1	2.6	5.6	1.4	3

Feature: Sepal_Length, Sepal_Width, Petal_Length, Petal_Width; Label: Label

#feature n=4 #class k=3



$$x_j \xrightarrow{w_{ij}} z_i$$

$$\boldsymbol{\theta} = [\boldsymbol{\theta}_1 \ \ \dots \ \ \boldsymbol{\theta}_k] = \begin{bmatrix} b_1 & \dots & b_k \\ w_{11} & \dots & w_{k1} \\ & \dots & \\ w_{1n} & \dots & w_{kn} \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} x_0 \\ x_1 \\ \dots \\ x_n \end{bmatrix} \qquad x_0 = 1$$

Forward computation

$$\boldsymbol{z} = \boldsymbol{\theta}^T \boldsymbol{x} \qquad \hat{\mathbf{y}} = \frac{e^{\boldsymbol{z}}}{\sum_{i=1}^{k} e^{z_i}}$$

Loss function

$$L(\boldsymbol{\theta}) = -\sum_{i=1}^{k} \delta(i, y) \log \hat{y}_i$$

Derivative

$$\frac{\partial L}{\partial w_{ij}} = x_j\big(\hat{y}_i - \delta(i, y)\big)$$

$$\frac{\partial L}{\partial b_i} = \hat{y}_i - \delta(i, y)$$

$$\frac{\partial L}{\partial \boldsymbol{\theta}_i} = \boldsymbol{x}\big(\hat{y}_i - \delta(i, y)\big)$$

Generalization - Batch

Feature | **Label**

Sepal_Length	Sepal_Width	Petal_Length	Petal_Width	Label
5.2	3.5	1.5	0.2	1
5.2	3.4	1.4	0.2	1
4.7	3.2	1.6	0.2	1
6.3	3.3	4.7	1.6	2
4.9	2.4	3.3	1.1	2
6.6	2.9	4.6	1.3	2
6.4	2.8	5.6	2.2	3
6.3	2.8	5.1	1.5	3
6.1	2.6	5.6	1.4	3

#feature n=4 #class k=3 #example m=9

$$\boldsymbol{\theta} = [\boldsymbol{\theta}_1 \quad \dots \quad \boldsymbol{\theta}_k] = \begin{bmatrix} b_1 & \dots & b_k \\ w_{11} & \dots & w_{k1} \\ & \dots & \\ w_{1n} & \dots & w_{kn} \end{bmatrix}$$

$$\boldsymbol{x} = [\boldsymbol{x}^{(1)} \quad \dots \quad \boldsymbol{x}^{(m)}]$$

$$\boldsymbol{x} = \begin{bmatrix} x_0^{(1)} & \dots & x_0^{(m)} \\ x_1^{(1)} & \dots & x_1^{(m)} \\ & \dots & \\ x_n^{(1)} & \dots & x_n^{(m)} \end{bmatrix}$$

Forward computation

$$\boldsymbol{z} = \boldsymbol{\theta}^T \boldsymbol{x} = \begin{bmatrix} z_0^{(1)} & \dots & z_0^{(m)} \\ z_1^{(1)} & \dots & z_1^{(m)} \\ & \dots & \\ z_k^{(1)} & \dots & z_k^{(m)} \end{bmatrix}$$

$$\hat{\boldsymbol{y}} = \frac{e^{\boldsymbol{z}}}{\sum_{i=1}^{k} e^{z_i}} = \begin{bmatrix} \hat{y}_0^{(1)} & \dots & \hat{y}_0^{(m)} \\ \hat{y}_1^{(1)} & \dots & \hat{y}_1^{(m)} \\ & \dots & \\ \hat{y}_k^{(1)} & \dots & \hat{y}_k^{(m)} \end{bmatrix}$$

Derivative

$$\frac{\partial L^{(u)}}{\partial w_{ij}} = x_j \left(\hat{y}_i^{(u)} - \delta(i, y^{(u)}) \right)$$

$$\frac{\partial L^{(u)}}{\partial b_i} = \hat{y}_i^{(u)} - \delta(i, y^{(u)})$$

$$\frac{\partial L}{\partial \boldsymbol{\theta}_i} = \frac{1}{m} \sum_{u=1}^{m} \boldsymbol{x} \left(\hat{y}_i^{(u)} - \delta(i, y^{(u)}) \right)$$

Loss function

$$L(\boldsymbol{\theta}) = -\frac{1}{m} \sum_{u=1}^{m} \sum_{i=1}^{k} \delta(i, y^{(u)}) \log \hat{y}_i^{(u)}$$

Outline

- Motivation
- Model Construction
- Loss Function
- Simple Example and Generalization
- Examples - Stochastic and Batch
- Another Approach

Softmax Regression - Stochastic

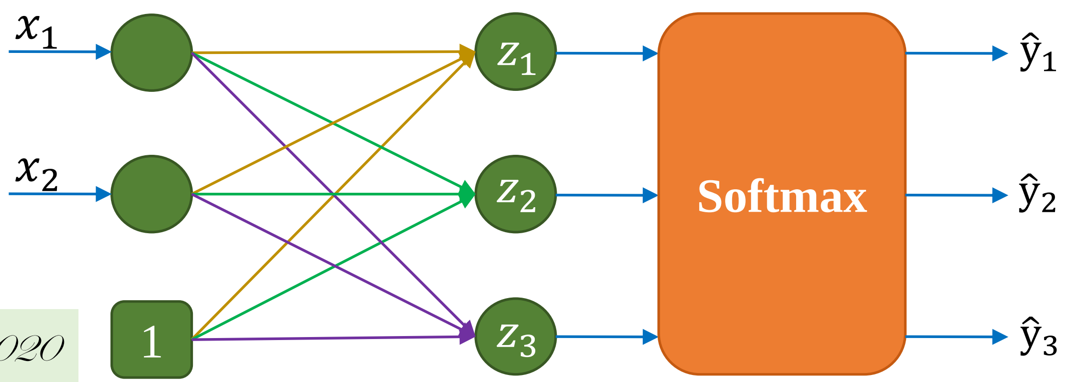
Petal_Length	Petal_Width	Label
1.5	0.2	1
1.4	0.2	1
1.6	0.2	1
4.7	1.6	2
3.3	1.1	2
4.6	1.3	2
5.6	2.2	3
5.1	1.5	3
5.6	1.4	3

$$\boldsymbol{\theta} = [\boldsymbol{\theta}_1 \quad \boldsymbol{\theta}_2 \quad \boldsymbol{\theta}_3] = \begin{bmatrix} b_1 & b_2 & b_3 \\ w_{11} & w_{21} & w_{31} \\ w_{12} & w_{22} & w_{32} \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} \qquad x_0 = 1$$

#feature n=2 #example m=9 #class k=3

Model



1) Pick a sample (x, y) from training data

2) Tính output $\hat{\boldsymbol{y}}$

$$\boldsymbol{z} = \boldsymbol{\theta}^T \boldsymbol{x}$$

$$\hat{\boldsymbol{y}} = \frac{e^{\boldsymbol{z}}}{\sum_{i=1}^{k} e^{z_i}}$$

3) Tính loss (cross-entropy)

$$L(\boldsymbol{\theta}) = -\sum_{i=1}^{k} \delta(i, y) \log \hat{y}_i$$

4) Tính đạo hàm

$$\frac{\partial L}{\partial \boldsymbol{\theta}_i} = \boldsymbol{x}\big(\hat{y}_i - \delta(i, y)\big)$$

5) Cập nhật tham số

$$\boldsymbol{\theta} = \boldsymbol{\theta} - \eta L'_{\boldsymbol{\theta}}$$

η is learning rate

Softmax Regression - Stochastic

Petal_Length	Petal_Width	Label
1.5	0.2	1
1.4	0.2	1
1.6	0.2	1
4.7	1.6	2
3.3	1.1	2
4.6	1.3	2
5.6	2.2	3
5.1	1.5	3
5.6	1.4	3

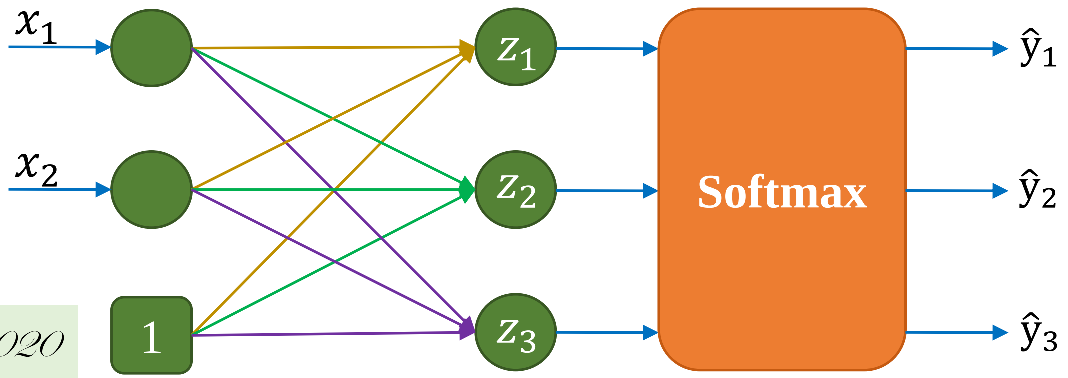
$$\boldsymbol{\theta} = [\boldsymbol{\theta}_1 \quad \boldsymbol{\theta}_2 \quad \boldsymbol{\theta}_3] = \begin{bmatrix} 0.1 & 0.05 & -0.1 \\ 0.1 & -0.1 & 0.1 \\ 0.2 & 0.2 & -0.1 \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 1.4 \\ 0.2 \end{bmatrix} \qquad y = 1$$

#feature n=2 #example m=9 #class k=3

1) Pick a sample (x, y) from training data

Model



Softmax Regression - Stochastic

$$\boldsymbol{\theta} = [\boldsymbol{\theta}_1 \quad \boldsymbol{\theta}_2 \quad \boldsymbol{\theta}_3]$$

$$= \begin{bmatrix} 0.1 & 0.05 & -0.1 \\ 0.1 & -0.1 & 0.1 \\ 0.2 & 0.2 & -0.1 \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 1.4 \\ 0.2 \end{bmatrix} \qquad y = 1$$

2) Tính output $\hat{\boldsymbol{y}}$

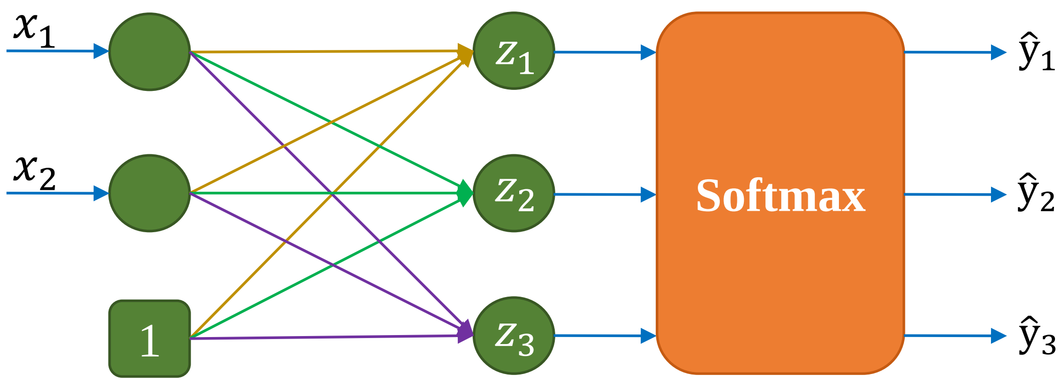
$$\boldsymbol{z} = \boldsymbol{\theta}^T \boldsymbol{x}$$

$$\hat{\mathbf{y}} = \frac{e^{\mathbf{z}}}{\sum_{i=1}^{k} e^{z_i}}$$

3) Tính loss (cross-entropy)

$$L(\boldsymbol{\theta}) = -\sum_{i=1}^{k} \delta(i, y)\log\hat{y}_i$$

$$\boldsymbol{z} = \begin{bmatrix} 0.1 & 0.1 & 0.2 \\ 0.05 & -0.1 & 0.2 \\ -0.1 & 0.1 & -0.1 \end{bmatrix} \begin{bmatrix} 1 \\ 1.4 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.28 \\ -0.05 \\ 0.02 \end{bmatrix} \qquad \hat{\mathbf{y}} = \begin{bmatrix} 0.4016 \\ 0.2887 \\ 0.3096 \end{bmatrix}$$



#class k=3

$$L(\boldsymbol{\theta}) = -\sum_{i=1}^{3} \delta(i, 1)\log\hat{y}_i$$

$$= -\delta(1,1)\log\hat{y}_1$$

$$= -\log 0.4016 = 0.9122$$

Softmax Regression - Stochastic

$$\boldsymbol{\theta} = [\boldsymbol{\theta}_1 \quad \boldsymbol{\theta}_2 \quad \boldsymbol{\theta}_3]$$

$$= \begin{bmatrix} 0.1 & 0.05 & -0.1 \\ 0.1 & -0.1 & 0.1 \\ 0.2 & 0.2 & -0.1 \end{bmatrix}$$

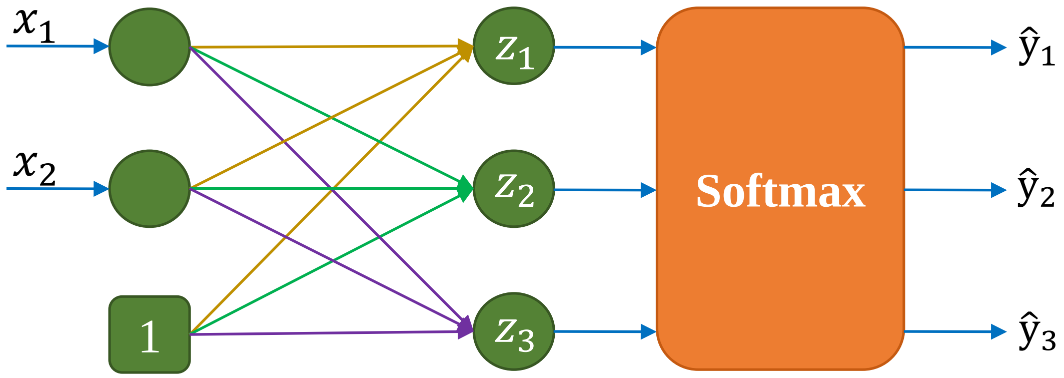
$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 1.4 \\ 0.2 \end{bmatrix} \qquad y = 1$$

4) Tính đạo hàm

$$\frac{\partial L}{\partial \boldsymbol{\theta}_i} = \boldsymbol{x}\big(\hat{y}_i - \delta(i, y)\big)$$

$$\boldsymbol{z} = \begin{bmatrix} 0.1 & 0.1 & 0.2 \\ 0.05 & -0.1 & 0.2 \\ -0.1 & 0.1 & -0.1 \end{bmatrix} \begin{bmatrix} 1 \\ 1.4 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.28 \\ -0.05 \\ 0.02 \end{bmatrix}$$

$$\hat{\mathbf{y}} = \begin{bmatrix} 0.4016 \\ 0.2887 \\ 0.3096 \end{bmatrix}$$



$$L(\boldsymbol{\theta}) = 0.9122$$

$$\frac{\partial L}{\partial \boldsymbol{\theta}_3} = \begin{bmatrix} 0.309 \\ 0.433 \\ 0.061 \end{bmatrix}$$

$$\frac{\partial L}{\partial \boldsymbol{\theta}_1} = \boldsymbol{x}\big(\hat{y}_1 - \delta(1, y)\big) = \begin{bmatrix} 1 \\ 1.4 \\ 0.2 \end{bmatrix} (0.4016 - 1) = \begin{bmatrix} -0.598 \\ -0.837 \\ -0.119 \end{bmatrix}$$

$$\frac{\partial L}{\partial \boldsymbol{\theta}_2} = \boldsymbol{x}\big(\hat{y}_2 - \delta(2, y)\big) = \begin{bmatrix} 1 \\ 1.4 \\ 0.2 \end{bmatrix} 0.2887 = \begin{bmatrix} 0.288 \\ 0.404 \\ 0.057 \end{bmatrix}$$

Softmax Regression - Stochastic

$$\boldsymbol{\theta} = [\boldsymbol{\theta}_1 \quad \boldsymbol{\theta}_2 \quad \boldsymbol{\theta}_3]$$

$$= \begin{bmatrix} 0.1 & 0.05 & -0.1 \\ 0.1 & -0.1 & 0.1 \\ 0.2 & 0.2 & -0.1 \end{bmatrix}$$

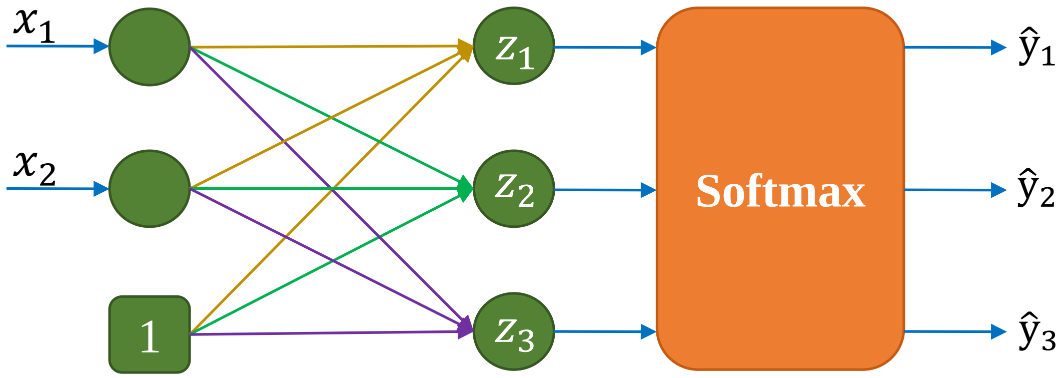
$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 1.4 \\ 0.2 \end{bmatrix} \qquad y = 1$$

5) Cập nhật tham số

$$\boldsymbol{\theta} = \boldsymbol{\theta} - \eta L'_{\boldsymbol{\theta}}$$

$$\eta = 0.1$$

$$\boldsymbol{z} = \begin{bmatrix} 0.1 & 0.1 & 0.2 \\ 0.05 & -0.1 & 0.2 \\ -0.1 & 0.1 & -0.1 \end{bmatrix} \begin{bmatrix} 1 \\ 1.4 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.28 \\ -0.05 \\ 0.02 \end{bmatrix} \qquad \hat{\mathbf{y}} = \begin{bmatrix} 0.4016 \\ 0.2887 \\ 0.3096 \end{bmatrix}$$



$$L(\boldsymbol{\theta}) = 0.9122$$

$$\frac{\partial L}{\partial \boldsymbol{\theta}_1} = \begin{bmatrix} -0.598 \\ -0.837 \\ -0.119 \end{bmatrix} \quad \frac{\partial L}{\partial \boldsymbol{\theta}_2} = \begin{bmatrix} 0.288 \\ 0.404 \\ 0.057 \end{bmatrix} \quad \frac{\partial L}{\partial \boldsymbol{\theta}_3} = \begin{bmatrix} 0.309 \\ 0.433 \\ 0.061 \end{bmatrix}$$

$$\boldsymbol{\theta} = \begin{bmatrix} 0.1 & 0.05 & -0.1 \\ 0.1 & -0.1 & 0.1 \\ 0.2 & 0.2 & -0.1 \end{bmatrix} - 0.1 \begin{bmatrix} -0.598 & 0.288 & 0.309 \\ -0.837 & 0.404 & 0.433 \\ -0.119 & 0.057 & 0.061 \end{bmatrix}$$

$$= \begin{bmatrix} 0.159 & 0.021 & -0.13 \\ 0.183 & -0.14 & 0.056 \\ 0.211 & 0.194 & -0.11 \end{bmatrix}$$

Outline

- Motivation
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Softmax Regression - Minibatch

Petal_Length	Petal_Width	Label
1.5	0.2	1
1.4	0.2	1
1.6	0.2	1
4.7	1.6	2
3.3	1.1	2
4.6	1.3	2
5.6	2.2	3
5.1	1.5	3
5.6	1.4	3

#feature n=2 #class k=3

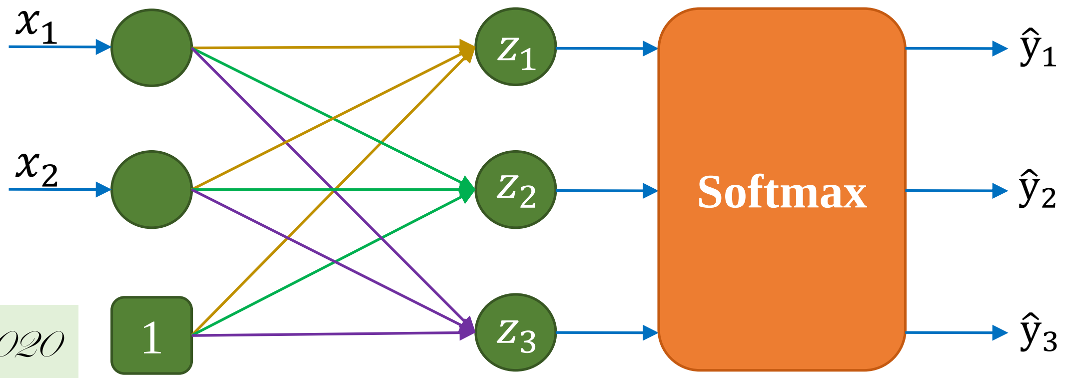
#example m=9 #minibatch s=3

$$\boldsymbol{\theta} = [\boldsymbol{\theta}_1 \quad \boldsymbol{\theta}_2 \quad \boldsymbol{\theta}_3] = \begin{bmatrix} b_1 & b_2 & b_3 \\ w_{11} & w_{21} & w_{31} \\ w_{12} & w_{22} & w_{32} \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} x_0^{(1)} & x_0^{(2)} & x_0^{(3)} \\ x_1^{(1)} & x_1^{(2)} & x_1^{(3)} \\ x_2^{(1)} & x_2^{(2)} & x_2^{(3)} \end{bmatrix}$$

$$x_0^{(1)} = 1$$

Model



1) Pick s samples $(\boldsymbol{x}, \boldsymbol{y})$

2) Tính output $\hat{\boldsymbol{y}}$

$$\boldsymbol{z} = \boldsymbol{\theta}^T \boldsymbol{x}$$

$$\hat{\boldsymbol{y}} = \frac{e^{\boldsymbol{z}}}{\sum_{i=1}^{k} e^{z_i}}$$

3) Tính loss (cross-entropy)

$$L(\boldsymbol{\theta}) = -\frac{1}{s}\sum_{u=1}^{s}\sum_{i=1}^{k} \delta(i, y^{(u)}) \log \hat{y}_i^{(u)}$$

4) Tính đạo hàm

$$\frac{\partial L}{\partial \boldsymbol{\theta}_i} = \frac{1}{s}\sum_{u=1}^{s} \boldsymbol{x}^{(i)}\left(\hat{y}_i^{(u)} - \delta(i, y^{(u)})\right)$$

5) Cập nhật tham số

$$\boldsymbol{\theta} = \boldsymbol{\theta} - \eta L'_{\boldsymbol{\theta}}$$

η is learning rate

Softmax Regression - Minibatch

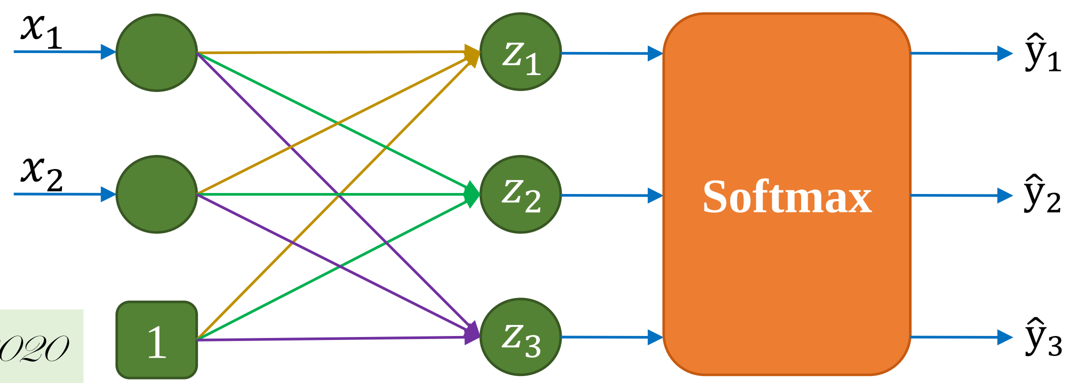
Petal_Length	Petal_Width	Label
1.5	0.2	1
1.4	0.2	1
1.6	0.2	1
4.7	1.6	2
3.3	1.1	2
4.6	1.3	2
5.6	2.2	3
5.1	1.5	3
5.6	1.4	3

$$\boldsymbol{\theta} = [\boldsymbol{\theta}_1 \quad \boldsymbol{\theta}_2 \quad \boldsymbol{\theta}_3]$$
$$= \begin{bmatrix} 0.1 & 0.05 & -0.1 \\ 0.1 & -0.1 & 0.1 \\ 0.2 & 0.2 & -0.1 \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} 1 & 1 & 1 \\ 1.5 & 4.7 & 5.6 \\ 0.2 & 1.6 & 2.2 \end{bmatrix}$$

$$\boldsymbol{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

1) Pick s samples $(\mathbf{x}, \boldsymbol{y})$



Softmax Regression - Minibatch

$$\boldsymbol{\theta} = [\boldsymbol{\theta}_1 \quad \boldsymbol{\theta}_2 \quad \boldsymbol{\theta}_3] = \begin{bmatrix} 0.1 & 0.05 & -0.1 \\ 0.1 & -0.1 & 0.1 \\ 0.2 & 0.2 & -0.1 \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} 1 & 1 & 1 \\ 1.5 & 4.7 & 5.6 \\ 0.2 & 1.6 & 2.2 \end{bmatrix} \quad \boldsymbol{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

2) Tính output $\hat{\boldsymbol{y}}$

$$\boldsymbol{z} = \boldsymbol{\theta}^T \boldsymbol{x}$$

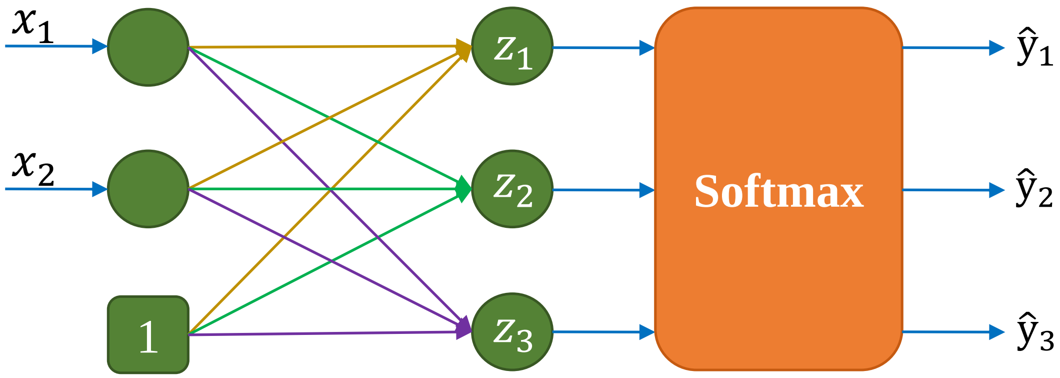
$$\hat{\mathbf{y}} = \frac{e^{\boldsymbol{z}}}{\sum_{i=1}^{k} e^{z_i}}$$

3) Tính loss (cross-entropy)

$$L(\boldsymbol{\theta}) = -\frac{1}{s}\sum_{u=1}^{s}\sum_{i=1}^{k} \delta\left(i, y^{(u)}\right)\log\hat{y}_i^{(u)}$$

#class k=3 #minibatch s=3

$$\boldsymbol{z} = \begin{bmatrix} 0.1 & 0.1 & 0.2 \\ 0.05 & -0.1 & 0.2 \\ -0.1 & 0.1 & -0.1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1.5 & 4.7 & 5.6 \\ 0.2 & 1.6 & 2.2 \end{bmatrix} = \begin{bmatrix} 0.29 & 0.89 & 1.1 \\ -0.06 & -0.1 & -0.07 \\ 0.03 & 0.21 & 0.24 \end{bmatrix}$$



$$\hat{\mathbf{y}} = \begin{bmatrix} 0.4039 & 0.5324 & 0.5768 \\ 0.2846 & 0.1978 & 0.1790 \\ 0.3114 & 0.2697 & 0.2441 \end{bmatrix}$$

$$L(\boldsymbol{\theta}) = -\frac{1}{3}\left[\sum_{i=1}^{k}\delta\left(i, y^{(1)}\right)\log\hat{y}_i^{(1)} + \sum_{i=1}^{k}\delta\left(i, y^{(2)}\right)\log\hat{y}_i^{(2)} + \sum_{i=1}^{k}\delta\left(i, y^{(3)}\right)\log\hat{y}_i^{(3)}\right]$$

$$= -\frac{1}{3}\left[\delta\left(1, y^{(1)}\right)\log\hat{y}_1^{(1)} + \delta\left(2, y^{(2)}\right)\log\hat{y}_2^{(2)} + \delta\left(3, y^{(3)}\right)\log\hat{y}_3^{(3)}\right]$$

$$= -\frac{1}{3}[\log 0.4039 + \log 0.1978 + \log 0.2441]$$

$$= -\frac{1}{3}[-0.90 - 1.62 - 1.41] = 1.31$$

Softmax Regression - Minibatch

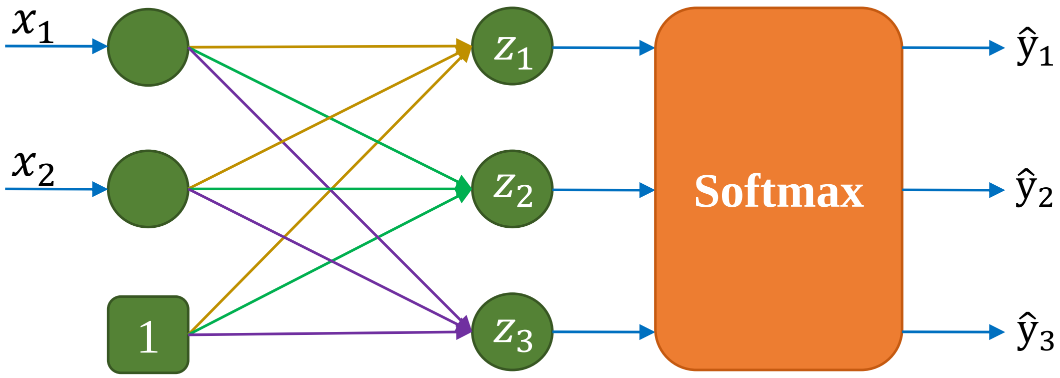
$$\boldsymbol{\theta} = [\boldsymbol{\theta}_1 \quad \boldsymbol{\theta}_2 \quad \boldsymbol{\theta}_3] = \begin{bmatrix} 0.1 & 0.05 & -0.1 \\ 0.1 & -0.1 & 0.1 \\ 0.2 & 0.2 & -0.1 \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} 1 & 1 & 1 \\ 1.5 & 4.7 & 5.6 \\ 0.2 & 1.6 & 2.2 \end{bmatrix} \quad \boldsymbol{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

4) Tính đạo hàm

$$\frac{\partial L}{\partial \boldsymbol{\theta}_i} = \frac{1}{s}\sum_{u=1}^{s} \boldsymbol{x}^{(i)}\left(\hat{y}_i^{(u)} - \delta\left(i, y^{(u)}\right)\right)$$

$$\boldsymbol{z} = \begin{bmatrix} 0.1 & 0.1 & 0.2 \\ 0.05 & -0.1 & 0.2 \\ -0.1 & 0.1 & -0.1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1.5 & 4.7 & 5.6 \\ 0.2 & 1.6 & 2.2 \end{bmatrix} = \begin{bmatrix} 0.29 & 0.89 & 1.1 \\ -0.06 & -0.1 & -0.07 \\ 0.03 & 0.21 & 0.24 \end{bmatrix}$$



$$\hat{\mathbf{y}} = \begin{bmatrix} 0.4039 & 0.5324 & 0.5768 \\ 0.2846 & 0.1978 & 0.1790 \\ 0.3114 & 0.2697 & 0.2441 \end{bmatrix}$$

$$L(\boldsymbol{\theta}) = 1.31$$

$$\frac{\partial L}{\partial \boldsymbol{\theta}_1} = -\frac{1}{3}\left[\boldsymbol{x}^{(1)}\left(\hat{y}_1^{(1)} - 1\right) + \boldsymbol{x}^{(1)}\left(\hat{y}_1^{(2)}\right) + \boldsymbol{x}^{(1)}\left(\hat{y}_1^{(3)}\right)\right]$$

$$= -\frac{1}{3}\left[\begin{bmatrix} 1 \\ 1.5 \\ 0.2 \end{bmatrix}(0.4039 - 1) + \begin{bmatrix} 1 \\ 1.5 \\ 0.2 \end{bmatrix} 0.1978 + \begin{bmatrix} 1 \\ 1.5 \\ 0.2 \end{bmatrix} 0.2441\right] = \begin{bmatrix} 0.171 \\ 1.612 \\ 0.667 \end{bmatrix}$$

$$\frac{\partial L}{\partial \boldsymbol{\theta}_2} = \begin{bmatrix} -0.112 \\ -0.780 \\ -0.277 \end{bmatrix} \qquad \frac{\partial L}{\partial \boldsymbol{\theta}_3} = \begin{bmatrix} -0.058 \\ -0.832 \\ -0.389 \end{bmatrix}$$

Softmax Regression - Minibatch

$$\boldsymbol{\theta} = [\boldsymbol{\theta}_1 \quad \boldsymbol{\theta}_2 \quad \boldsymbol{\theta}_3] = \begin{bmatrix} 0.1 & 0.05 & -0.1 \\ 0.1 & -0.1 & 0.1 \\ 0.2 & 0.2 & -0.1 \end{bmatrix}$$

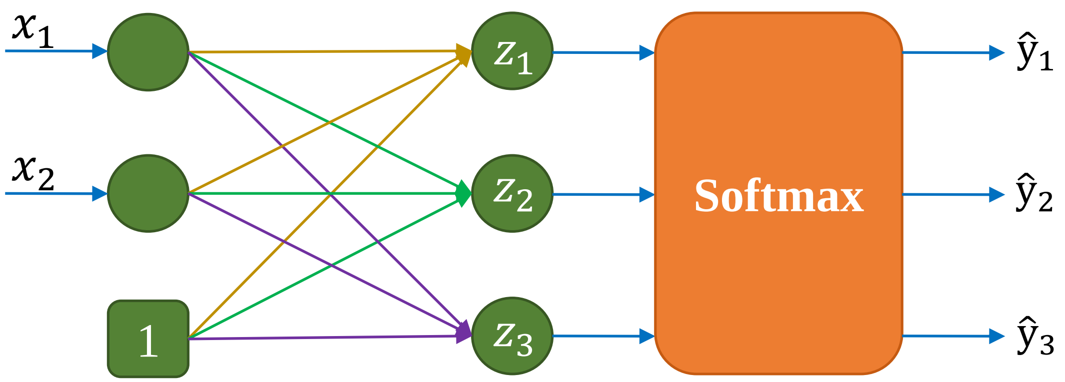
$$\boldsymbol{x} = \begin{bmatrix} 1 & 1 & 1 \\ 1.5 & 4.7 & 5.6 \\ 0.2 & 1.6 & 2.2 \end{bmatrix} \quad \boldsymbol{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

5) Cập nhật tham số

$$\boldsymbol{\theta} = \boldsymbol{\theta} - \eta L'_{\boldsymbol{\theta}}$$

$$\eta = 0.1$$

$$\boldsymbol{z} = \begin{bmatrix} 0.1 & 0.1 & 0.2 \\ 0.05 & -0.1 & 0.2 \\ -0.1 & 0.1 & -0.1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1.5 & 4.7 & 5.6 \\ 0.2 & 1.6 & 2.2 \end{bmatrix} = \begin{bmatrix} 0.29 & 0.89 & 1.1 \\ -0.06 & -0.1 & -0.07 \\ 0.03 & 0.21 & 0.24 \end{bmatrix}$$



$$\hat{\mathbf{y}} = \begin{bmatrix} 0.4039 & 0.5324 & 0.5768 \\ 0.2846 & 0.1978 & 0.1790 \\ 0.3114 & 0.2697 & 0.2441 \end{bmatrix}$$

$$L(\boldsymbol{\theta}) = 1.31$$

$$\frac{\partial L}{\partial \boldsymbol{\theta}_1} = \begin{bmatrix} 0.171 \\ 1.612 \\ 0.667 \end{bmatrix} \quad \frac{\partial L}{\partial \boldsymbol{\theta}_2} = \begin{bmatrix} -0.112 \\ -0.780 \\ -0.277 \end{bmatrix} \quad \frac{\partial L}{\partial \boldsymbol{\theta}_3} = \begin{bmatrix} -0.058 \\ -0.832 \\ -0.389 \end{bmatrix}$$

$$\boldsymbol{\theta} = \begin{bmatrix} 0.1 & 0.05 & -0.1 \\ 0.1 & -0.1 & 0.1 \\ 0.2 & 0.2 & -0.1 \end{bmatrix} - 0.1 \begin{bmatrix} 0.171 & -0.112 & -0.058 \\ 1.612 & -0.780 & -0.832 \\ 0.667 & -0.277 & -0.389 \end{bmatrix}$$

$$= \begin{bmatrix} 0.083 & 0.061 & -0.094 \\ -0.061 & -0.022 & 0.183 \\ 0.133 & 0.228 & -0.061 \end{bmatrix}$$

Softmax Regression

❖ Demo

```
Python 3.7.3 (default, Apr 24 2019, 15:29:51) [MSC v.1915 64 bit (AMD64)] ::
Type "help", "copyright", "credits" or "license" for more information.
>>>
>>>
>>>
>>>
>>>
>>>
>>>
>>> for epoch in range(n_epochs):
...     sum_of_losses = 0
...     gradients = np.zeros((2,1))
...
...     for index in range(4):
...         xi = X_b[index:index+1]
...         yi = y[index:index+1]
```

Outline

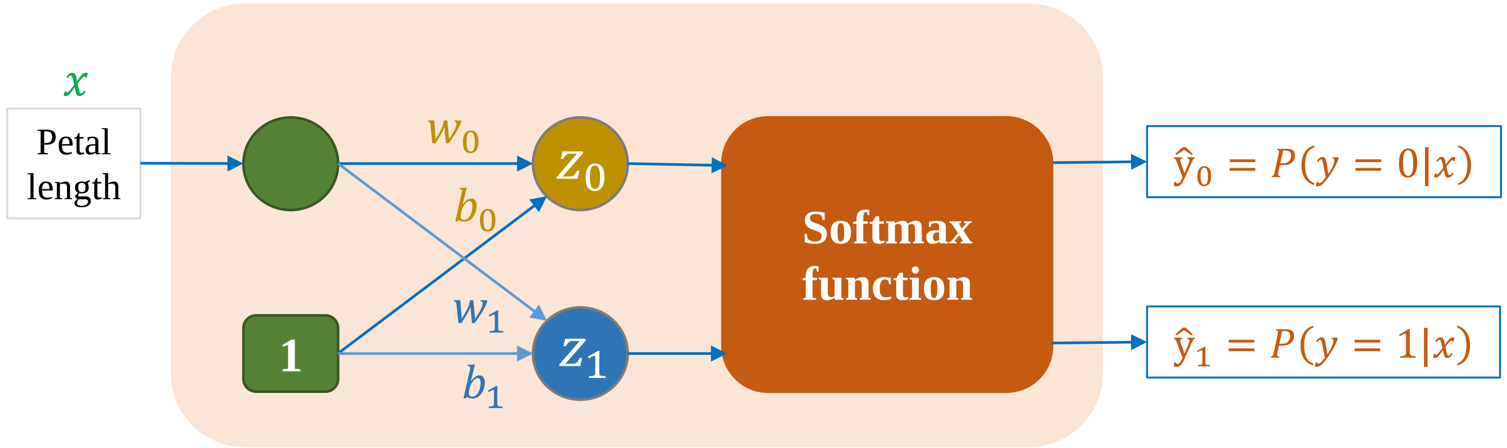
- Motivation
- Model Construction
- Loss Function
- Simple Example and Generalization
- Examples - Stochastic and Batch
- Another Approach

Another Approach

❖ Simple illustration

Feature	Label
Petal_Length	Category
1.4	0
1	0
1.5	0
3	1
3.8	1
4.1	1

Model



One-hot encoding for label

$$\begin{array}{l} \quad\quad\quad\quad\quad\quad\quad\quad\quad y_0 \quad y_1 \\ y = 0 \rightarrow \boldsymbol{y} = [1 \quad 0] \\ y = 1 \rightarrow \boldsymbol{y} = [0 \quad 1] \end{array}$$

scalar　vector

$$z_0 = xw_0 + b_0$$

$$z_1 = xw_1 + b_1$$

$$\hat{y}_0 = \frac{e^{z_0}}{\sum_{j=0}^{1} e^{z_j}}$$

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$$\boldsymbol{z} = \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \begin{bmatrix} b_0 & w_0 \\ b_1 & w_1 \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} \boldsymbol{\theta}_0^T \\ \boldsymbol{\theta}_1^T \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} = \boldsymbol{\theta}^T \boldsymbol{x}$$

$$\hat{\boldsymbol{y}} = \begin{bmatrix} \hat{y}_0 \\ \hat{y}_1 \end{bmatrix} = \frac{1}{\sum_{j=0}^{1} e^{z_j}} \begin{bmatrix} e^{z_0} \\ e^{z_1} \end{bmatrix} = \frac{e^{\boldsymbol{z}}}{\sum_{j=0}^{1} e^{z_j}}$$

A vector is by default a column vector $\boldsymbol{\theta}_0 = \begin{bmatrix} b_0 \\ w_0 \end{bmatrix}$

vector transpose $\boldsymbol{\theta}_0^T = [b_0 \quad w_0]$

Another Approach

❖ Simple illustration

Feature	Label
Petal_Length	Category
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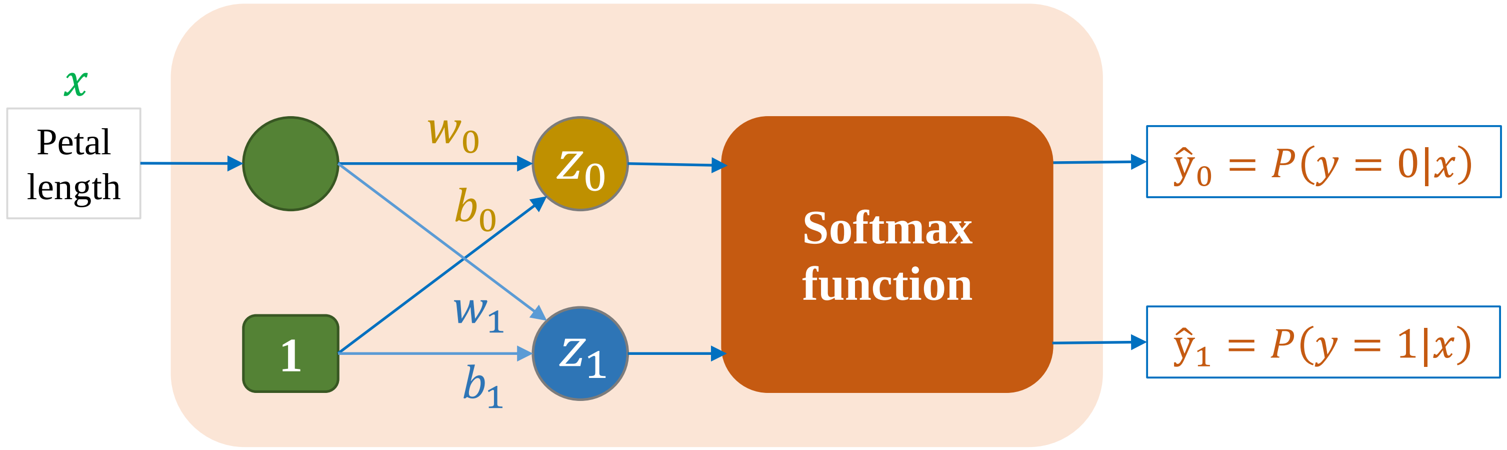
1	0
1.5	0
3	1
3.8	1
4.1	1

Model



One-hot encoding for label

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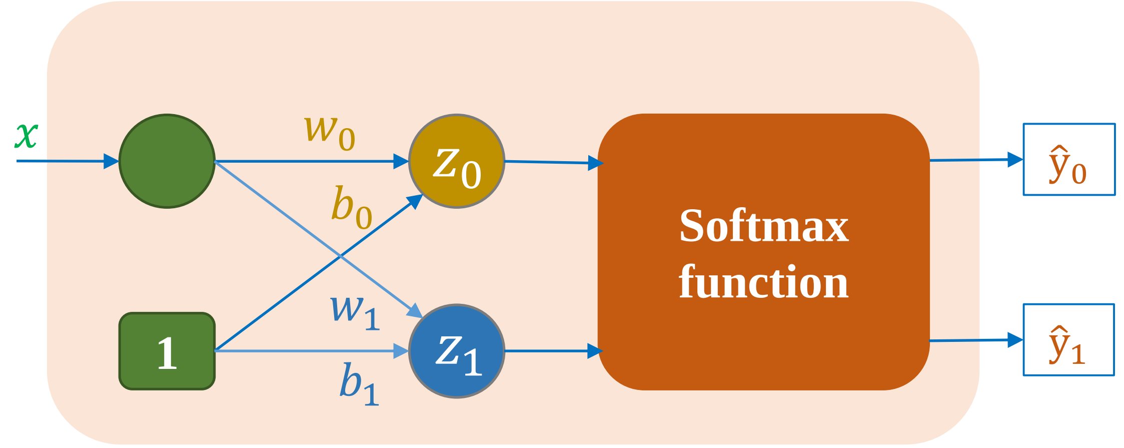
$$L(\boldsymbol{\theta}) = -y_0 \log \hat{y}_0 - y_1 \log \hat{y}_1 = -\sum_{i=0}^{1} y_i \log \hat{y}_i$$

Another Approach

Model



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$$\hat{y}_1 = \frac{e^{z_1}}{\sum_{j=0}^{1} e^{z_j}}$$

Derivative

$$\frac{\partial \hat{y}_i}{\partial z_i} = \begin{cases} \hat{y}_i(1-\hat{y}_i) & if\ i = j \\ -\hat{y}_i\hat{y}_j & if\ i \neq j \end{cases}$$

$$\frac{\partial L}{\partial z_i} = -\sum\nolimits_k y_k \frac{\partial \log(\hat{y}_k)}{\partial z_i} = -\sum\nolimits_k y_k \frac{\partial \log(\hat{y}_k)}{\partial \hat{y}_k}\frac{\partial \hat{y}_k}{\partial z_i} = -\sum\nolimits_k y_k \frac{1}{\hat{y}_k}\frac{\partial \hat{y}_k}{\partial z_i}$$

$$\frac{\partial L}{\partial z_i} = -y_i(1-\hat{y}_i) - \sum\nolimits_{k\neq i} y_k \frac{1}{\hat{y}_k} - \hat{y}_k\hat{y}_i$$

$$= -y_i(1-\hat{y}_i) - \sum\nolimits_{k\neq i} y_k\hat{y}_i$$

$$= -y_i + y_i\hat{y}_i - \sum\nolimits_{k\neq i} y_k\hat{y}_i$$

$$= \hat{y}_i\left(y_i - \sum\nolimits_{k\neq i} y_k\right) - y_i$$

$$= \hat{y}_i - y_i$$

Another Approach

One-hot encoding for label

$$\begin{array}{l} \\ y = 0 \rightarrow \boldsymbol{y} = \\ y = 1 \rightarrow \boldsymbol{y} = \end{array} \begin{array}{cc} y_0 & y_1 \\ [1 & 0] \\ [0 & 1] \end{array}$$

scalar vector

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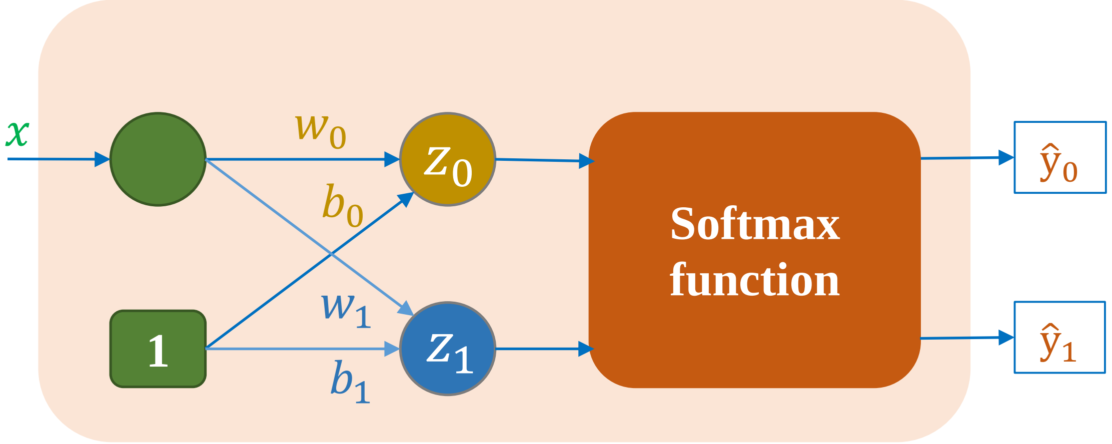
Model



Derivative

$$\frac{\partial L}{\partial \hat{y}_i} = \frac{y_i}{\hat{y}_i}$$

$$\frac{\partial \hat{y}_i}{\partial z_j} = \begin{cases} \hat{y}_i(1 - \hat{y}_i) & if\ i = j \\ -\hat{y}_i \hat{y}_j & if\ i \neq j \end{cases}$$

$$\frac{\partial L}{\partial z_i} = \hat{y}_i - y_i$$

$$\frac{\partial L}{\partial w_i} = x(\hat{y}_i - y_i)$$

$$\frac{\partial L}{\partial b_i} = \hat{y}_i - y_i$$

Summary

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One-hot encoding for label

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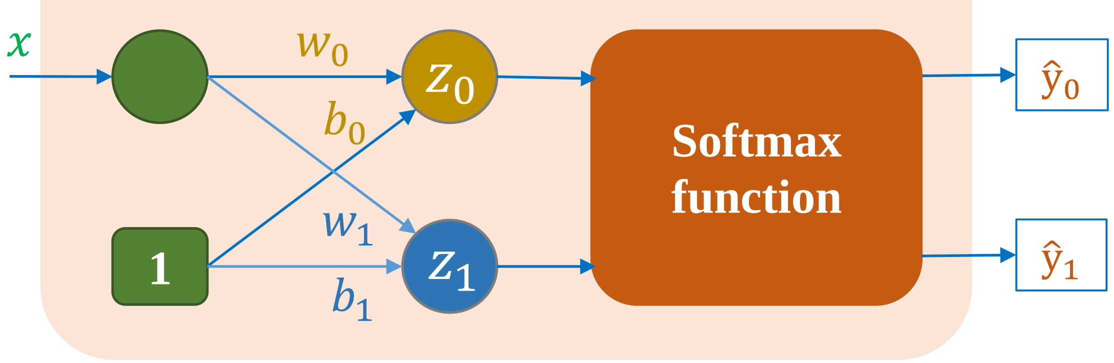
scalar vector

Model



$$\boldsymbol{\theta} = \begin{bmatrix} b_0 & b_1 \\ w_0 & w_1 \end{bmatrix} \qquad \boldsymbol{x} = [1 \quad x]$$

Forward computation

$$\boldsymbol{z} = \boldsymbol{\theta}^T \boldsymbol{x}$$

$$\hat{\mathbf{y}} = \frac{e^{\mathbf{z}}}{\sum_{j=0}^{1} e^{z_j}}$$

Loss function

$$L(\boldsymbol{\theta}) = -\sum_{i=0}^{1} y_i \log \hat{y}_i$$

Derivative

$$\frac{\partial L}{\partial \hat{y}_i} = \frac{y_i}{\hat{y}_i}$$

$$\frac{\partial \hat{y}_i}{\partial z_j} = \begin{cases} \hat{y}_i(1 - \hat{y}_i) & if\ i = j \\ -\hat{y}_i \hat{y}_j & if\ i \neq j \end{cases}$$

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$$\frac{\partial L}{\partial w_i} = x(\hat{y}_i - y_i)$$

$$\frac{\partial L}{\partial b_i} = \hat{y}_i - y_i$$

References

Softmax Regression

http://deeplearning.stanford.edu/tutorial/supervised/SoftmaxRegression/

Thank you!