Lecture Notes: Differential Equations (Sessions 1 and 2)

First-Order Linear Equations with Constant Coefficients

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1 Context: Where This Fits In

The immediate previous session focused on the theoretical structure of solutions for the general First-Order Linear Equation ($\dot{x} + P(t)x = Q(t)$). We rigorously established the Principle of Superposition, which proves that the General Solution is composed of the homogeneous solution and a particular solution: $\mathbf{x(t) = x_h(t) + x_p(t)}$. We also formally derived the Integrating Factor method (Euler's method) used to solve these equations.

Today's topic deepens this study by specializing in the case where the coefficient of the dependent variable is a constant ($\dot{y} + ky = q(t)$). This simplification allows us to rigorously define the physical significance of the two parts of the solution (y_h and y_p) as transient versus long-term responses, concepts which are fundamental for understanding higher-order linear equations.

2 Core Concepts & Definitions

This session focuses on the specialized structure and solution methods for linear ODEs where the coefficient is a constant, and introduces a critical new solving technique, Undetermined Coefficients, for specific input types.

First-Order Linear Equation with Constant Coefficients A differential equation in the standard form $\mathbf{\dot{y} + ky = q(t)}$. Here, the coefficient k is a fixed numerical constant, and $q(t)$ is the external input or forcing function.

Integrating Factor ($\mathbf{u(t)}$) For this specific equation, the integrating factor (using Euler's method) simplifies to: $\mathbf{u(t) = e^{\int kdt} = e^{kt}}$.

Homogeneous Solution ($\mathbf{y_h(t)}$) The solution to the associated homogeneous equation ($\dot{y} + ky = 0$). By separation of variables, this is found to be $\mathbf{y_h(t) = e^{-kt}}$ (or Ce^{-kt} when including the arbitrary constant C).

Particular Solution ($\mathbf{y_p(t)}$) The specific part of the solution determined by the input function $q(t)$. It is calculated as $y_p(t) = e^{-kt}\left(\int e^{kt}q(t)\,dt\right)$.

Transient Solution This term refers to the homogeneous solution, $\mathbf{Cy_h(t) = Ce^{-kt}}$, when the constant k is positive ($\mathbf{k > 0}$). In this case, $e^{-kt} \to 0$ as $t \to \infty$, meaning this part of the response decays and fades away over time.

Long-Term or Stable Solution If $k > 0$, the particular solution $\mathbf{y_p(t)}$ is the part of the general solution that remains after the transient component has decayed. This represents the stable behavior that the system ultimately approaches, regardless of the initial condition.

Method of Undetermined Coefficients A technique used to find $x_p(t)$ when $q(t)$ is of a specific form (e.g., exponential, sine, cosine, polynomial). The method involves guessing the form of $x_p(t)$ based on $q(t)$ (e.g., if $q(t) = e^{rt}$, we guess $x_p(t) = Ae^{rt}$), and then substituting the guess into the ODE to algebraically determine the unknown coefficients (like A).

Analogy for Transient and Long-Term Solutions ($k > 0$)

Imagine a boat in water ($\dot{y} + ky = q(t)$), where $k > 0$ represents the natural drag.

- **Transient Solution (Ce^{-kt}):** This is the movement caused by the initial push. Since there is drag ($k > 0$), this initial motion dies out over time.
- **Long-Term Solution (y_p):** This is the speed and direction the boat maintains while the engine ($q(t)$) is running. The boat settles into a stable speed determined solely by the engine's power, independent of the initial push.

If $k \leq 0$ (no drag), the system is unstable, and the initial condition remains important as $t \to \infty$.

3 Prerequisite Skill Refresh

The algebraic technique of Undetermined Coefficients relies heavily on differentiation and solving algebraic equations.

3.1 Basic Differentiation Rules (Calculus)

The method requires finding the derivative of your guessed solution $x_p(t)$.

Exponential Derivative If $x_p(t) = Ae^{rt}$, then $\dot{x}_p(t) = rAe^{rt}$.

Polynomial Derivative If $x_p(t) = At^n$, then $\dot{x}_p(t) = nAt^{n-1}$.

3.2 Algebraic Substitution and Equating Coefficients (Algebra)

This is the central skill for the Method of Undetermined Coefficients.

Example Solve $\dot{x} + 2x = 4e^{3t}$. We guess $x_p(t) = Ae^{3t}$.

- Differentiate: $\dot{x}_p(t) = 3Ae^{3t}$.
- Substitute into ODE: $(3Ae^{3t}) + 2(Ae^{3t}) = 4e^{3t}$.
- Combine and Equate Coefficients:

$$\begin{aligned} 5Ae^{3t} &= 4e^{3t} \\ 5A &= 4 \\ A &= 4/5 \end{aligned}$$

3.3 Indefinite Integration (Calculus)

The Integrating Factor method still requires integration.

Exponential Integration If $P(t) = k$, the integral for the integrating factor is:

$$\int k\,dt = kt + C_1 \tag{1}$$

This leads directly to the integrating factor $u(t) = e^{kt}$.

4 Key Examples: A Step-by-Step Walkthrough

These examples focus on solving the constant coefficient linear equation using the Integrating Factor and the Method of Undetermined Coefficients.

4.1 Example 1: Solving using the Integrating Factor (General Derivation)

Problem

Find the general solution to the first-order linear differential equation with a constant coefficient: $\dot{y} + ky = q(t)$.

Solution

Step 1: Find the Integrating Factor, $u(t)$. The equation is in standard linear form with $P(t) = k$.

$$u(t) = e^{\int P(t)\,dt} = e^{\int k\,dt} = \mathbf{e^{kt}} \tag{2}$$

Step 2: State the General Solution Formula. The general solution $y(t)$ is given by:

$$y(t) = \frac{1}{u(t)} \left(\int u(t)q(t)\,dt + C \right) \tag{3}$$

Step 3: Substitute $u(t)$ and Separate Components. Substitute $u(t) = e^{kt}$ and distribute e^{-kt}:

$$y(t) = e^{-kt}\left(\int e^{kt}q(t)\,dt\right) + Ce^{-kt} \tag{4}$$

Step 4: Identify the Solution Components. The solution is identified as $y(t) = y_p(t) + Cy_h(t)$, where:

- $\mathbf{y_p(t) = e^{-kt}\left(\int e^{kt}q(t)\,dt\right)}$ (Particular Solution)
- $\mathbf{y_h(t) = e^{-kt}}$ (Homogeneous Solution component)

4.2 Example 2: Finding the Particular Solution using MUC

Problem

Find the general solution for the differential equation $\dot{x} + 2x = 4e^{3t}$.

Solution

Step 1: Find the Homogeneous Solution (x_h). The homogeneous equation is $\dot{x}+2x = 0$. Since $k = 2$, the solution is $\mathbf{x_h(t) = Ce^{-2t}}$.

Step 2: Guess the form of the Particular Solution (x_p). The forcing function is $q(t) = 4e^{3t}$. We guess a solution of the same form: $\mathbf{x_p(t) = Ae^{3t}}$, where A is the undetermined coefficient.

Step 3: Calculate the Derivative of the Guess. $\mathbf{\dot{x}_p(t) = 3Ae^{3t}}$.

Step 4: Substitute x_p and $\dot{x}_p$ into the ODE. Substitute into $\dot{x} + 2x = 4e^{3t}$:

$$(3Ae^{3t}) + 2(Ae^{3t}) = 4e^{3t} \tag{5}$$

Step 5: Solve for the Undetermined Coefficient (A). Combine terms and equate coefficients:

$$5Ae^{3t} = 4e^{3t} \tag{6}$$

$$5A = 4 \implies \mathbf{A = 4/5} \tag{7}$$

Step 6: State the General Solution. Combine x_h and x_p using the Superposition Principle: $x(t) = x_h(t) + x_p(t)$.

$$\mathbf{x(t) = Ce^{-2t} + \frac{4}{5}e^{3t}} \tag{8}$$

5 Conceptual Understanding

5.1 Why Does the Method of Undetermined Coefficients (MUC) Work?

MUC is effective for linear ODEs when the input $q(t)$ has a form (like exponentials, sines, or polynomials) that is preserved under differentiation. Because linear ODEs involve only sums of the function and its derivatives, guessing x_p has the same form as $q(t)$ converts the entire differential relationship into a simple algebraic problem. We determine the coefficients by forcing the algebraic expression to equal $q(t)$ for all t.

5.2 What Does the Solution Represent (Transient vs. Long-Term)?

For $\dot{y} + ky = q(t)$, the solution $y(t) = y_p(t) + Cy_h(t)$ has specific physical meaning when $\mathbf{k > 0}$.

Transient Solution ($Cy_h(t) = Ce^{-kt}$) Since $k > 0$, the exponential term $\mathbf{e^{-kt} \to 0}$ as $t \to \infty$. This part of the response is temporary, or transient. It is determined by the arbitrary constant C, which is fixed by the initial condition. The effect of the initial state fades away.

Long-Term/Stable Solution ($y_p(t)$) As $t \to \infty$ (and $k > 0$), the total solution $\mathbf{y(t) \to y_p(t)}$. This component is independent of C and represents the steady, stable behavior driven solely by the external input $q(t)$, irrespective of how the system started.

If $k \leq 0$, the y_h component does not decay, and the long-term behavior remains dependent on the initial conditions.

6 Common Mistakes to Avoid

- **Handling the Constant k:** The interpretation of the solution as "transient" relies entirely on the sign of k. Failing to recognize the significance of $k > 0$ versus $k \leq 0$ is a common error in analysis.

- **Incorrect Guess Form (The Resonance Case):** A critical trap in MUC is if the guessed form $x_p(t)$ is already a solution to the homogeneous equation $x_h(t)$. (e.g., in $\dot{x} + 2x = e^{-2t}$, guessing $x_p = Ae^{-2t}$ will fail). In such "resonance" cases, the guess must be multiplied by t (e.g., $x_p = Ate^{-2t}$).

- **Algebraic Errors:** MUC requires precise algebraic comparison between the substituted left-hand side and the right-hand side $q(t)$. Any error in substitution or equating coefficients will result in an incorrect x_p.

7 Summary & What's Next

7.1 Key Takeaways

- **Constant Coefficient Structure:** For $\dot{y} + ky = q(t)$, the general solution is $y(t) = y_p(t) + Ce^{-kt}$, where $y_p(t)$ corresponds to the input $q(t)$.
- **Transient vs. Stable:** If $\mathbf{k > 0}$, Ce^{-kt} is the **transient solution** which decays to zero, leaving $y_p(t)$ as the **long-term or stable solution**.
- **Undetermined Coefficients:** MUC is a fast, algebraic way to find $y_p(t)$ for specific forcing functions (like $q(t) = 4e^{3t}$) by assuming y_p has the same functional form as $q(t)$.

7.2 Next Session

The next sessions will likely address solving the linear equation when the forcing function $q(t)$ involves sine or cosine terms (e.g., $\dot{x} + Kx = B\cos(wt)$). This typically involves the **Method of Complex Substitution**, which uses Euler's formula ($e^{i\theta} = \cos\theta + i\sin\theta$) to transform the trigonometric problem into an exponential one.