Lecture Notes: Differential Equations (Session 12)

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1 Context: Where This Fits In

In Session 11, we covered the solution methodology for the homogeneous second order linear ODE with constant coefficients ($\mathbf{m\ddot{x} + b\dot{x} + Kx = 0}$), specifically addressing Case 1 (Distinct Real Roots) and Case 2 (Complex Conjugate Roots).

Session 12 completes the analysis of homogeneous equations by examining **Case 3: Repeated Real Roots**. Additionally, this session introduces the concept of **System Stability** and the **Exponential Response Formula (ERF)** for solving non-homogeneous equations with exponential inputs.

2 Core Concepts & Definitions

Homogeneous Equations: Case 3

Repeated Real Roots

Occurs when the discriminant of the characteristic equation ($mr^2 + br + K = 0$) is exactly zero ($\Delta = b^2 - 4mK = 0$). This yields a single unique real root, r_1.

Linearly Independent Solutions

A second-order equation requires two linearly independent solutions to form the general solution. Since we only have one distinct root r_1, the first solution is $x_1(t) = e^{r_1 t}$. To ensure independence (i.e., not a scalar multiple), the second solution is constructed by multiplying the first by t:

$$\mathbf{x_2(t) = te^{r_1 t}}$$

General Solution (Repeated Roots)

By the Principle of Superposition, the general solution is:

$$\mathbf{x(t) = C_1 e^{r_1 t} + C_2 t e^{r_1 t} = e^{r_1 t}(C_1 + C_2 t)}$$

Non-Homogeneous Equations & Stability

For a forced system $m\ddot{x} + b\dot{x} + Kx = F(t)$, the general solution is $x(t) = x_h(t) + x_p(t)$.

Homogeneous Solution (x_h)
Also called the **Transient Solution**. It contains the arbitrary constants (C_1, C_2) determined by initial conditions. It represents the system's natural behavior.

Particular Solution (x_p)
Also called the **Steady State Solution**. It represents the response driven by the external forcing function $F(t)$.

Stability Criteria
A system is **stable** (or asymptotically stable) if $x_h(t) \to 0$ as $t \to \infty$. This occurs if and only if the **real part of all roots** of the characteristic polynomial is strictly negative.

Exponential Response Formula (ERF)
A shortcut to find x_p when $F(t) = Be^{at}$:

$$\mathbf{x_p(t)} = \frac{\mathbf{Be^{at}}}{\mathbf{p(a)}} \quad \text{provided } p(a) \neq 0$$

If $p(a) = 0$ (Resonance), use the Generalized ERF:

$$\mathbf{x_p(t)} = \frac{\mathbf{Bte^{at}}}{\mathbf{p'(a)}} \quad \text{provided } p'(a) \neq 0$$

3 Prerequisite Skill Refresh

A. Perfect Square Polynomials

Case 3 occurs when the characteristic equation is a perfect square: $m(r - r_1)^2 = 0$. Recall that $Ar^2 + Br + C$ is a perfect square if $B^2 = 4AC$.

- Example: $r^2 + 4r + 4 = 0 \implies (r + 2)^2 = 0 \implies r = -2$.

B. The Product Rule (Leibniz Rule)

To verify that $x_2(t) = te^{rt}$ is a solution, we must differentiate a product $u(t)v(t)$:

$$\frac{d}{dt}(uv) = u'v + uv'$$

First Derivative of te^{rt}:

$$\dot{x} = (1)e^{rt} + t(re^{rt}) = e^{rt}(1 + rt)$$

Second Derivative:

$$\ddot{x} = re^{rt}(1 + rt) + e^{rt}(r) = e^{rt}(r + r^2t + r) = e^{rt}(2r + r^2t)$$

This derivation is essential for proving the validity of the second solution in Case 3.

4 Key Examples: A Step-by-Step Walkthrough

4.1 Example 1: Case 3 - Repeated Real Roots

Problem: Solve the homogeneous equation $\mathbf{\ddot{x} + 4\dot{x} + 4x = 0}$.

Step 1: Derive Characteristic Equation Assume $x(t) = e^{rt}$. Substituting into the ODE yields:

$$r^2 + 4r + 4 = 0$$

Step 2: Find the Roots Factor the quadratic:

$$(r+2)^2 = 0$$

This yields a single, repeated real root $\mathbf{r_1 = -2}$.

Step 3: Construct General Solution Since we have a repeated root, the two linearly independent solutions are $x_1 = e^{-2t}$ and $x_2 = te^{-2t}$. Using Superposition:

$$\mathbf{x(t) = C_1e^{-2t} + C_2te^{-2t} = e^{-2t}(C_1 + C_2t)}$$

4.2 Example 2: ERF (Non-Resonant Case)

Problem: Find a particular solution to $\mathbf{\ddot{x} + 2\dot{x} + 15x = e^{-5t}}$.

Step 1: Identify $p(r)$ and Parameters Characteristic polynomial: $p(r) = r^2 + 2r + 15$. Input forcing function: $F(t) = 1 \cdot e^{-5t}$, so $B = 1$ and $a = -5$.

Step 2: Check for Resonance Evaluate $p(r)$ at $r = a = -5$:

$$p(-5) = (-5)^2 + 2(-5) + 15 = 25 - 10 + 15 = 30$$

Since $p(-5) = 30 \neq 0$, there is no resonance.

Step 3: Apply ERF

$$x_p(t) = \frac{Be^{at}}{p(a)} = \frac{1 \cdot e^{-5t}}{30}$$

$$\mathbf{x_p(t) = \frac{1}{30}e^{-5t}}$$

4.3 Example 3: Generalized ERF (Resonance Case)

Problem: Find a particular solution to $\mathbf{\ddot{x} + 8\dot{x} + 15x = e^{-5t}}$.

Step 1: Identify $p(r)$ and Parameters Characteristic polynomial: $p(r) = r^2 + 8r + 15$. Input parameters: $B = 1, a = -5$.

Step 2: Check for Resonance Evaluate $p(r)$ at $a = -5$:

$$p(-5) = (-5)^2 + 8(-5) + 15 = 25 - 40 + 15 = 0$$

Since $p(-5) = 0$, the standard ERF fails (division by zero). This is **resonance**.

Step 3: Apply Generalized ERF We need the derivative of the polynomial, $p'(r)$:

$$p'(r) = 2r + 8$$

Evaluate at $a = -5$:

$$p'(-5) = 2(-5) + 8 = -2$$

The formula is $x_p = \frac{Bte^{at}}{p'(a)}$:

$$x_p(t) = \frac{1 \cdot te^{-5t}}{-2}$$

$$\mathbf{x_p(t) = -\frac{1}{2}te^{-5t}}$$

5 Conceptual Understanding

Why Multiply by t? When the characteristic equation has a repeated root r_1, the function $e^{r_1 t}$ is a solution. However, $C_1 e^{r_1 t} + C_2 e^{r_1 t}$ collapses into a single term $Ce^{r_1 t}$, which cannot satisfy two initial conditions. The function $x_2(t) = te^{r_1 t}$ is mathematically derived (via Reduction of Order) to be the necessary second solution. It is linearly independent from $e^{r_1 t}$ and satisfies the ODE specifically when the discriminant is zero.

Interpretation of Stability

- **Stable (Permanent):** If the real parts of all roots are negative, the homogeneous solution x_h decays to zero ($e^{-at} \to 0$). The system eventually settles into the behavior dictated by the forcing function (x_p).
- **Unstable:** If any root has a positive real part, the homogeneous solution grows exponentially ($e^{at} \to \infty$), and the system output diverges regardless of the input.

6 Common Mistakes to Avoid

- **Case 3 Solution Form:** Do not write $x(t) = C_1 e^{rt} + C_2 e^{rt}$. You must include the t in the second term: $x(t) = C_1 e^{rt} + C_2 te^{rt}$.
- **ERF Division by Zero:** Always calculate $p(a)$ first. If $p(a) = 0$, you cannot use the simple formula $Be^{at}/p(a)$. You must switch to the generalized formula involving t and $p'(a)$.
- **Stability Confusion:** Stability depends on the **real part** of the roots. Pure imaginary roots ($\text{Re}(r) = 0$) imply neutral stability (oscillations that do not decay), meaning the system is not asymptotically stable.

7 Summary & What's Next

Session 12 Key Takeaways:

1. **Repeated Roots:** If $\Delta = 0$, the general solution is $x(t) = e^{rt}(C_1 + C_2 t)$.
2. **ERF:** To solve $p(D)x = Be^{at}$, use $x_p = \frac{Be^{at}}{p(a)}$. If $p(a) = 0$, use $x_p = \frac{Bte^{at}}{p'(a)}$.
3. **Stability:** A system is stable iff $\text{Re}(r) < 0$ for all roots.

Next Session (Session 13): We have covered exponential inputs. The next session will extend these methods to **Trigonometric Input Functions** ($F(t) = \cos(\omega t)$ or $\sin(\omega t)$) using the Complex Replacement Method.