Lecture Notes: Differential Equations (Session 11)

Dr. Khajeh Salehani - University of Tehran

These notes were collaboratively gathered and compiled. We warmly welcome your feedback and suggestions at: **k.ghanbari@ut.ac.ir** and **hamidrezahosseini@ut.ac.ir**

(Last Mod : Dec2/11Azar)

1 Context: Where This Fits In

The previous session (Session 10) concluded the study of First Order Differential Equations, covering Exact Differential Equations ($M(x, y)dx + N(x, y)dy = 0$) and Bernoulli equations.

Session 11 marks a dramatic shift in focus. We move from first-order equations to **Second Order Differential Equations**. We transition from techniques involving single integrations to solving algebraic polynomials (finding roots) to construct solutions for equations involving the second derivative ($\ddot{x}$).

2 Core Concepts & Definitions

This session introduces the fundamental solution method for the **Homogeneous Second Order Linear Equation with Constant Coefficients**.

Second Order Linear Homogeneous Equation

A differential equation where the highest derivative is the second derivative ($\ddot{x}$). "Linear" implies the dependent variable (x) and its derivatives appear only to the first power with constant coefficients. "Homogeneous" means the equation equals zero. **Standard Form:**

$$m\ddot{x} + b\dot{x} + Kx = 0$$

Exponential Solution Form ($x(t) = e^{rt}$)

We hypothesize that the solution is an exponential function because the derivatives of an exponential are multiples of the function itself ($\dot{x} = re^{rt}$, $\ddot{x} = r^2e^{rt}$). This allows us to transform the differential equation into an algebraic one.

Characteristic Equation (Auxiliary Equation)

Substituting $x(t) = e^{rt}$ into the ODE yields the algebraic quadratic equation:

$$mr^2 + br + K = 0$$

The roots of this polynomial (r) dictate the nature of the solution.

Principle of Superposition

If $x_1(t)$ and $x_2(t)$ are two linearly independent solutions to a homogeneous linear ODE, then any linear combination is also a general solution:

$$\mathbf{x(t) = C_1 x_1(t) + C_2 x_2(t)}$$

Since the equation is second order, the general solution *must* contain two arbitrary constants to satisfy two potential initial conditions.

Classification of Roots

The nature of the solution depends on the discriminant, $\Delta = b^2 - 4mK$, of the characteristic equation:

- **Case 1: Real and Distinct Roots ($\Delta > 0$).** Yields two different real numbers r_1, r_2.
- **Case 2: Complex Conjugate Roots ($\Delta < 0$).** Yields complex numbers $r = a \pm ib$.
- **Case 3: Repeated Real Roots ($\Delta = 0$).** Yields one real root (covered in Session 12).

3 Prerequisite Skill Refresh

Solving these equations requires mastery of three specific algebraic skills.

A. Differentiation of Exponentials

You must be able to apply the Chain Rule to $x(t) = e^{rt}$:

$$\dot{x} = \frac{d}{dt}(e^{rt}) = re^{rt}$$
$$\ddot{x} = \frac{d}{dt}(re^{rt}) = r^2 e^{rt}$$

B. The Quadratic Formula

To find the roots of $mr^2 + br + K = 0$:

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4mK}}{2m}$$

C. Euler's Formula (For Case 2)

To convert complex exponentials into real trigonometric functions, we use:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

This implies that $e^{(a+ib)t} = e^{at}(\cos(bt) + i\sin(bt))$.

4 Key Examples: A Step-by-Step Walkthrough

4.1 Example 1: Case 1 - Real and Distinct Roots

Problem: Find the general solution to $\ddot{x} + 8\dot{x} + 7x = 0$.

Step 1: Derive Characteristic Equation Assume $x(t) = e^{rt}$. Substitute $\dot{x} = re^{rt}$ and $\ddot{x} = r^2e^{rt}$ into the ODE:

$$\begin{aligned} r^2e^{rt} + 8re^{rt} + 7e^{rt} &= 0 \\ e^{rt}(r^2 + 8r + 7) &= 0 \end{aligned}$$

Since $e^{rt} \neq 0$, we solve $r^2 + 8r + 7 = 0$.

Step 2: Find Roots Factor the polynomial:

$$(r+1)(r+7) = 0 \quad \Longrightarrow \quad r_1 = -1, \quad r_2 = -7$$

These are real and distinct roots.

Step 3: General Solution Using the Principle of Superposition:

$$\mathbf{x(t) = C_1e^{-t} + C_2e^{-7t}}$$

4.2 Example 2: Case 2 - Complex Conjugate Roots

Problem: Solve the homogeneous equation $\ddot{x} + 4\dot{x} + 5x = 0$.

Step 1: Characteristic Equation and Roots The characteristic equation is $r^2 + 4r + 5 = 0$. Using the quadratic formula:

$$r_{1,2} = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$$

Here, the real part $a = -2$ and the imaginary part $b = 1$.

Step 2: Transform to Real Solution The complex solution is $e^{(-2\pm i)t}$. Using Euler's Formula:

$$x(t) = e^{-2t}(\cos(1t) + i\sin(1t))$$

The real and imaginary parts are linearly independent solutions.

Step 3: General Solution Combine the real and imaginary parts with arbitrary constants:

$$\mathbf{x(t) = e^{-2t}(C_1\cos t + C_2\sin t)}$$

4.3 Example 3: Initial Value Problem (IVP)

Problem: Find the particular solution to $\ddot{x} + 8\dot{x} + 7x = 0$ given $x(0) = 2$ and $\dot{x}(0) = -8$.

Step 1: General Solution and Derivative From Example 1, we have $x(t) = C_1e^{-t} + C_2e^{-7t}$. Differentiate with respect to t:

$$\dot{x}(t) = -C_1e^{-t} - 7C_2e^{-7t}$$

Step 2: Apply Initial Conditions At $t = 0$:

$$\begin{aligned} x(0) &= C_1 + C_2 = 2 && (I) \\ \dot{x}(0) &= -C_1 - 7C_2 = -8 && (II) \end{aligned}$$

Step 3: Solve System Add equations (I) and (II):

$$(C_1 - C_1) + (C_2 - 7C_2) = 2 - 8 \implies -6C_2 = -6 \implies C_2 = 1$$

Substitute $C_2 = 1$ into (I):

$$C_1 + 1 = 2 \implies C_1 = 1$$

Step 4: Particular Solution

$$\mathbf{x(t) = e^{-t} + e^{-7t}}$$

5 Conceptual Understanding

Why the Characteristic Equation Works: Differentiation "commutes" with exponential functions. Substituting e^{rt} reduces a calculus problem (derivatives) to an algebra problem (polynomials). The roots of the polynomial guarantee the structure of the solution.

Physical Interpretation of Solutions:

- **Case 1 (Real Roots):** Represents non-oscillatory motion. If roots are negative, it represents decay (stable behavior). If positive, it represents exponential growth (unstable).
- **Case 2 (Complex Roots):** Represents oscillatory motion (vibrations). The term e^{at} acts as an "envelope."
  - If $a < 0$: Damped oscillation (decaying vibration).
  - If $a > 0$: Growing oscillation.

6 Common Mistakes to Avoid

- **Discriminant Errors:** Incorrectly calculating $\Delta = b^2 - 4mK$ leads to the wrong Case selection (e.g., mistaking complex roots for real roots).
- **Complex Form Confusion:** When roots are $r = a \pm ib$, remember:
  - a goes in the exponential: e^{at} (damping/growth).
  - b goes in the trig functions: $\cos(bt), \sin(bt)$ (frequency).
- **Missing Constants:** Second-order equations must have **two** constants (C_1, C_2).
- **Variable Confusion:** Derivatives are w.r.t time (t), but the polynomial is in terms of r. Do not mix them during substitution.

7 Summary & What's Next

Session 11 Recap:

1. Replace the ODE $m\ddot{x} + b\dot{x} + Kx = 0$ with $mr^2 + br + K = 0$.

2. If roots are Real Distinct (r_1, r_2): $x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$.

3. If roots are Complex $(a \pm ib)$: $x(t) = e^{at}(C_1 \cos(bt) + C_2 \sin(bt))$.

Next Session (Session 12): We have covered $\Delta > 0$ and $\Delta < 0$. The next session will address the final case: **Repeated Real Roots** ($\Delta = 0$), where $r_1 = r_2$. We will learn how to generate the second linearly independent solution, which takes the form te^{rt}.