Lecture Notes: Differential Equations - Session 9

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1 Context: Where This Fits In

In the previous session, we covered the solution of First Order Linear Equations with Constant Coefficients where the non-homogeneous input function, $q(t)$, was an exponential function ($q(t) = Be^{rt}$).

The goal established for this session (Session 9) is to solve the same type of differential equation when the forcing function is trigonometric:

$$\dot{x} + Kx = B\cos(\omega t) \quad \text{or} \quad \dot{x} + Kx = B\sin(\omega t)$$

This session introduces the **Complex Variable Method** to handle these sinusoidal inputs and introduces a new family of non-linear equations known as **Bernoulli Equations**.

2 Core Concepts & Definitions

This section outlines the two primary mathematical frameworks introduced in this session.

2.1 The Complex Variable Method

This method is used to find the particular solution (x_p) for linear ODEs with sinusoidal forcing functions. Instead of using real-valued trigonometry, we utilize the algebraic simplicity of complex exponentials.

- **Euler's Formula:** The foundation connecting complex exponentials to trigonometry.

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{S9.1}$$

- **Complex Replacement Strategy:** Since $\cos(\omega t) = \mathrm{Re}(e^{i\omega t})$, we replace the real input with a complex input to form a complex ODE. The physical solution is the real part of the complex solution $Z(t)$:

$$x(t) = \mathrm{Re}(Z(t)) \tag{S9.2}$$

2.2 Bernoulli Differential Equations

A Bernoulli equation is a specific type of first-order non-linear differential equation that can be transformed into a linear equation via substitution.

- **Standard Bernoulli Form:**

$$\frac{dy}{dx} + P(x)y = Q(x)y^{\alpha} \quad (\text{where } \alpha \in \mathbb{R} \setminus \{0, 1\}) \tag{S9.3}$$

- **Bernoulli Substitution:** To linearize the equation, we define a new variable v:

$$v := y^{1-\alpha} \tag{S9.4}$$

This transforms the non-linear ODE into the linear form:

$$\frac{dv}{dx} + (1-\alpha)P(x)v = (1-\alpha)Q(x)$$

3 Prerequisite Skill Refresh

Successfully applying the methods in Session 9 requires mastery of two specific prerequisite skills.

3.1 Complex Number Operations (Polar Form)

When solving for coefficients in the Complex Variable Method, you must often convert Cartesian forms ($z = a+bi$) to Polar forms ($z = re^{i\phi}$) to handle division and multiplication easily.

- **Magnitude (r):** $r = |z| = \sqrt{a^2 + b^2}$
- **Phase Angle (ϕ):** $\phi = \arctan\left(\frac{b}{a}\right)$ (Adjust for quadrant).

3.2 Solving Linear ODEs (Integrating Factor)

The Bernoulli method reduces non-linear problems to standard linear problems. These are solved using the Integrating Factor method. For a standard linear equation $\frac{dy}{dx} + P(x)y = Q(x)$:

- **Step 1: Integrating Factor Formula**

$$\mu(x) = e^{\int P(x)dx} \tag{S9.5}$$

- **Step 2: General Solution Formula**

$$y(x) = \frac{1}{\mu(x)} \left(\int \mu(x)Q(x)dx + C \right) \tag{S9.6}$$

4 Key Examples: A Step-by-Step Walkthrough

4.1 Example 1: Linear ODE with Sinusoidal Input

Problem: Find the general solution for $\dot{x} + 2x = 2\cos(2t)$.

Solution: The general solution is $x(t) = x_h(t) + x_p(t)$.

Step 1: Homogeneous Solution (x_h)

$$\dot{x} + 2x = 0 \implies x_h(t) = Ce^{-2t}$$

Step 2: Particular Solution (x_p) via Complex Replacement Replace $2\cos(2t)$ with $2e^{i2t}$ to form the complex ODE:

$$\dot{Z} + 2Z = 2e^{i2t}$$

Assume particular solution $Z_p = Ae^{i2t}$. Substitute into the ODE:

$$\begin{aligned} (i2)Ae^{i2t} + 2Ae^{i2t} &= 2e^{i2t} \\ A(2+2i) &= 2 \\ A &= \frac{2}{2+2i} = \frac{1}{1+i} \end{aligned}$$

Convert coefficient A to Polar Form to simplify:

$$\begin{aligned} A &= \frac{1}{1+i} \cdot \frac{1-i}{1-i} = \frac{1-i}{2} = \frac{1}{2} - \frac{i}{2} \\ \text{Magnitude } r_A &= \sqrt{(1/2)^2 + (-1/2)^2} = \frac{1}{\sqrt{2}} \\ \text{Angle } \phi_A &= \arctan(-1) = -\frac{\pi}{4} \\ \implies A &= \frac{1}{\sqrt{2}} e^{-i\pi/4} \end{aligned}$$

Substitute A back into Z_p:

$$Z_p = \left(\frac{1}{\sqrt{2}} e^{-i\pi/4}\right) e^{i2t} = \frac{1}{\sqrt{2}} e^{i(2t-\pi/4)}$$

Apply Euler's formula (Eq S9.1) and take the Real Part (Eq S9.2):

$$x_p(t) = \mathrm{Re}(Z_p) = \frac{1}{\sqrt{2}} \cos\left(2t - \frac{\pi}{4}\right)$$

Final General Solution:

$$\mathbf{x(t) = \frac{1}{\sqrt{2}} \cos\left(2t - \frac{\pi}{4}\right) + Ce^{-2t}}$$

4.2 Example 2: Bernoulli Equation

Problem: Solve $2xyy' = x^2 + y^2$ (assuming $x > 0$).

Step 1: Convert to Standard Form Divide by $2xy$:

$$y' = \frac{x}{2y} + \frac{y}{2x} \implies \frac{dy}{dx} - \frac{1}{2x}y = \frac{x}{2}y^{-1}$$

Identify components: $P(x) = -\frac{1}{2x}$, $Q(x) = \frac{x}{2}$, $\alpha = -1$.

Step 2: Bernoulli Substitution Using Eq S9.4, let $v = y^{1-(-1)} = y^2$. Differentiate with respect to x: $v' = 2yy'$. Multiply the standard form equation by $2y$ to facilitate substitution:

$$2yy' - \frac{1}{x}y^2 = x$$

Substitute v and v':

$$\frac{dv}{dx} - \frac{1}{x}v = x$$

Step 3: Solve Linear ODE for v Calculate Integrating Factor (Eq S9.5):

$$\mu(x) = e^{\int -1/x dx} = e^{-\ln x} = \frac{1}{x}$$

Apply formula S9.6:

$$v(x) = x\left(\int \frac{1}{x}(x)dx + C\right) = x(x + C) = x^2 + Cx$$

Step 4: Back-Substitute Since $v = y^2$:

$$y^2 = x^2 + Cx \implies \mathbf{y(x) = \pm\sqrt{x^2 + Cx}}$$

4.3 Example 3: Reducible to Linear (Exponential Substitution)

Problem: Solve $2xe^{2y}\frac{dy}{dx} = 3x^2 + e^{2y}$ (assuming $x > 0$).

Step 1: Transform Rearrange to isolate derivative terms:

$$2xe^{2y}y' - e^{2y} = 3x^2$$

Notice the recurrence of e^{2y}. Let $v = e^{2y}$.

$$v' = \frac{d}{dx}(e^{2y}) = 2e^{2y}y'$$

Substitute into the rearranged equation:

$$xv' - v = 3x^2 \implies v' - \frac{1}{x}v = 3x$$

Step 2: Solve Linear ODE Integrating Factor: $\mu(x) = e^{\int -1/x dx} = \frac{1}{x}$.

$$v(x) = x\left(\int \frac{1}{x}(3x)dx + C\right) = x(3x + C) = 3x^2 + Cx$$

Step 3: Back-Substitute

$$e^{2y} = 3x^2 + Cx \implies \mathbf{y(x) = \frac{1}{2}\ln(3x^2 + Cx)}$$

5 Conceptual Understanding

Why the Complex Method Works: The differential equation $\dot{x} + Kx = B\cos(\omega t)$ involves only real coefficients. Because $\cos(\omega t)$ is strictly the real part of $e^{i\omega t}$, the physical solution must be the real part of the complex solution. The complex domain allows us to turn differentiation into simple algebraic multiplication.

Why Bernoulli Works: The substitution $v = y^{1-\alpha}$ effectively "linearizes" the non-linearity y^α. It is a mathematical key that converts a difficult problem into a universally solvable Linear ODE.

Physical Interpretation of Solution Parts:

$$x(t) = \underbrace{x_h(t)}_{\text{Transient}} + \underbrace{x_p(t)}_{\text{Steady-State}}$$

- **Transient (x_h):** Natural behavior of the system. Decays to zero as $t \to \infty$.
- **Steady-State (x_p):** Long-term response driven by the external forcing function.

6 Common Mistakes to Avoid

- **Complex Algebra Errors:** When simplifying $A = \frac{B}{K+i\omega}$, ensure you multiply by the conjugate correctly or convert to polar form.
- **Forgetting the Real Part:** In the Complex Variable Method, $Z(t)$ is *not* the answer. You must take $x(t) = \mathrm{Re}(Z(t))$.
- **Bernoulli Chain Rule:** When calculating v', do not forget the chain rule: $\frac{dv}{dx} = (1-\alpha)y^{-\alpha}\frac{dy}{dx}$.
- **Omission of Back-Substitution:** Finding $v(x)$ is not the end. You must convert back to $y(x)$ using $y = v^{1/(1-\alpha)}$.

7 Summary & What's Next

Key Takeaways:

1. Use the **Complex Variable Method** for linear ODEs with sinusoidal inputs to avoid complex trigonometric algebra.
2. Use the **Bernoulli Substitution** ($v = y^{1-\alpha}$) to linearize ODEs of the form $y' + Py = Qy^\alpha$.

Next Session Topic: Session 10 will cover **Exact Differential Equations**. This involves equations of the form $M(x,y)dx + N(x,y)dy = 0$, where the criterion for exactness is:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$