Lecture Notes: Differential Equations (Session 7)

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In This Session : First-Order Linear Equations, Superposition, and Integrating Factors

1 Context: Where This Fits In

The immediate previous topics established the necessity of rigorous classification and analysis of first-order equations. Specifically, we covered the formal conditions for recognizing a Separable Equation (an ODE $y' = R(x, y)$ where $R(x, y)$ can be factored into $f(x)g(y)$), and we formally introduced the Standard Form of the First-Order Linear Equation as $\dot{x}+P(t)x = Q(t)$. We also examined the calculation of the Domain of the Solution for an Initial Value Problem (IVP), emphasizing that the solution is defined on the largest connected interval containing the initial point.

Today's topic uses the structure of the linear equation ($\dot{x} + P(t)x = Q(t)$) to define the powerful Principle of Superposition and then rigorously derive the complete structure of the general solution for this type of equation using the method of the Integrating Factor.

2 Core Concepts & Definitions

Session 7 provides the theoretical framework for solving all first-order linear equations, introducing concepts vital for all subsequent linear systems.

First-Order Linear Equation (Standard Form) A differential equation that can be written as $\mathbf{\dot{x} + P(t)x = Q(t)}$. Here, x is the dependent variable, t is the independent variable, $P(t)$ is the coefficient function, and $Q(t)$ is the "input" or "forcing" function.

Homogeneous Linear Equation The simplest case, where the external input term $Q(t)$ is zero: $\mathbf{\dot{x} + P(t)x = 0}$. This is also called the associated homogeneous equation or the reduced equation.

Non-Homogeneous Linear Equation The general case where the external input term $\mathbf{Q(t)}$ is not zero: $\dot{x} + P(t)x = Q(t)$. This is also referred to as the complete equation.

Homogeneous Solution ($\mathbf{x_h(t)}$) The general solution of the associated homogeneous equation ($\dot{x} + P(t)x = 0$). This solution contains the necessary arbitrary constant C. It represents the natural, unforced behavior of the system.

Principle of Superposition Applies ONLY to linear equations. This states that if a linear system is driven by multiple inputs, the total output (solution) is the linear combination (sum) of the individual outputs that result from each input acting separately. If $q_1 \to y_1$ and $q_2 \to y_2$, then $C_1q_1 + C_2q_2 \to C_1y_1 + C_2y_2$.

Integrating Factor ($\mathbf{u(t)}$ or $\mu(\mathbf{t})$) A function defined as $\mathbf{u(t) = e^{\int P(t)dt}}$. When the entire non-homogeneous linear equation is multiplied by this factor, the left side transforms into the exact derivative of a product, $\frac{d}{dt}(u(t)x)$.

General Solution (Linear Non-Homogeneous) The complete solution is the sum of the homogeneous solution ($\mathbf{x_h}$) and a particular solution ($\mathbf{x_p}$) corresponding to the input $Q(t)$: $\mathbf{x(t) = x_h(t) + x_p(t)}$. The superposition principle justifies this structure.

Analogy for Superposition and Solution Structure

Imagine a ringing bell ($\dot{x} + P(t)x = Q(t)$).

- **Homogeneous Solution (x_h):** The sound the bell makes when struck once and left to ring naturally. It contains the arbitrary constant C, as the loudness depends on the initial strike.
- **Particular Solution (x_p):** The sound the bell makes if you continuously push it with a specific pattern ($Q(t)$). It models the specific effect of the external forcing.
- **General Solution ($x_h + x_p$):** The total sound is the combination (superposition) of the natural ring (transient response) and the forced motion (steady-state response).

3 Prerequisite Skill Refresh

Solving first-order linear equations requires solid skills in separation of variables and recognizing the reverse product rule.

3.1 Separation of Variables & Exponential Solutions

The solution of the homogeneous part ($\dot{x} + P(t)x = 0$) is derived from separation, using logarithms and exponentiation.

Separation and Logarithms The technique involves rearranging $\frac{dx}{dt}$ into $\frac{dx}{x} = -P(t)\,dt$ and integrating to yield $\ln|x| = -\int P(t)\,dt + C_1$.

Consolidating Parameters The integration constant C_1 is converted into a multiplicative parameter C. From $\ln|x| = A + C_1$, we get $x = \pm e^{C_1}e^A$. We define $C = \pm e^{C_1}$, resulting in $x = Ce^A$.

3.2 The Product Rule (Calculus)

The integrating factor method succeeds by transforming the equation into the derivative of a product.

Reminder The derivative of a product $u \cdot x$ is:

$$\frac{d}{dt}(ux) = u\frac{dx}{dt} + x\frac{du}{dt} \tag{1}$$

Core Idea The integrating factor $u(t)$ is designed to make the left side of the ODE ($u\frac{dx}{dt} + uPx$) match the product rule by ensuring that $\frac{du}{dt} = uP(t)$.

3.3 Integration by Parts (IBP) (Calculus)

When solving, the final step often requires calculating $\int u(t)Q(t)\,dt$.

IBP Formula $\int u\,dv = uv - \int v\,du$

Example Calculate $\int xe^{3x}\,dx$. Choose $u = x$ and $dv = e^{3x}\,dx$. Then $du = dx$ and $v = \frac{1}{3}e^{3x}$.

$$\int xe^{3x}\,dx = x\left(\frac{1}{3}e^{3x}\right) - \int \frac{1}{3}e^{3x}\,dx \tag{2}$$

$$= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$$

4 Key Examples: A Step-by-Step Walkthrough

These examples focus on the domain of a solution and the application of the Principle of Superposition and the Integrating Factor.

4.1 Example 1: Solving an IVP and Determining the Domain

Problem

Solve the IVP and determine the domain of the solution:

$$y' = y^2 \quad \text{with initial condition} \quad y(0) = 1.$$

Solution

First, we solve the separable equation.

$$\begin{aligned}
\frac{dy}{dx} &= y^2 \\
\frac{dy}{y^2} &= dx && \text{(Separate variables)} \\
\int y^{-2}\, dy &= \int dx && \text{(Integrate both sides)} \\
-\frac{1}{y} &= x + C && \text{(General solution)}
\end{aligned}$$

Apply the initial condition $y(0) = 1$:

$$-\frac{1}{1} = 0 + C \implies C = -1$$

State the particular solution by substituting $C = -1$:

$$-\frac{1}{y} = x - 1 \implies \mathbf{y(x) = \frac{1}{1-x}}$$

Determine the domain: The function $y(x) = \frac{1}{1-x}$ has a vertical asymptote at $x = 1$. Since the initial point is $x_0 = 0$, the largest connected interval that includes $x = 0$ is $(-\infty, \mathbf{1})$.

4.2 Example 2: Applying the Principle of Superposition

Problem

Given the following known solutions for the linear ODE $\dot{x} + 2x = Q(t)$:

- $x_1(t) = \frac{1}{2}$ is a solution for $\dot{x} + 2x = 1$. (Input $Q_1 = 1$)
- $x_2(t) = te^{-2t}$ is a solution for $\dot{x} + 2x = e^{-2t}$. (Input $Q_2 = e^{-2t}$)
- $x_h(t) = Ce^{-2t}$ is the general solution for $\dot{x} + 2x = 0$.

Find the General Solution $x(t)$ for the complete equation: $\dot{x} + 2x = 1 + e^{-2t}$.

Solution

Decompose the input: $Q(t) = Q_1(t) + Q_2(t) = 1 + e^{-2t}$.

Combine Particular Solutions (x_p) By the superposition principle, the particular solution is the sum of the individual particular solutions:

$$x_p(t) = x_1(t) + x_2(t) = \frac{1}{2} + te^{-2t} \tag{3}$$

Construct General Solution The full general solution is $x(t) = x_h(t) + x_p(t)$.

$$\mathbf{x(t) = Ce^{-2t} + \frac{1}{2} + te^{-2t}} \tag{4}$$

The arbitrary constant C is carried by the homogeneous solution.

4.3 Example 3: Finding the Homogeneous Solution $x_h(t)$

Problem

Derive the homogeneous solution $x_h(t)$ for the standard homogeneous linear equation: $\dot{x} + P(t)x = 0$.

Solution

This equation is separable.

$$
\begin{aligned}
\frac{dx}{dt} &= -P(t)x \\
\frac{dx}{x} &= -P(t)\,dt && \text{(Separate variables)} \\
\int \frac{dx}{x} &= \int -P(t)\,dt && \text{(Integrate)} \\
\ln|x| &= -\int P(t)\,dt + C_1 \\
|x| &= e^{-\int P(t)\,dt + C_1} = e^{C_1} e^{-\int P(t)\,dt} && \text{(Exponentiate)} \\
x(t) &= (\pm e^{C_1}) e^{-\int P(t)\,dt} && \text{(Remove absolute value)}
\end{aligned}
$$

Consolidate the parameter by defining $C = \pm e^{C_1}$ (allowing $C = 0$ for the trivial solution $x = 0$). The homogeneous solution is:

$$\mathbf{x_h(t) = Ce^{-\int P(t)dt}}$$

5 Conceptual Understanding

5.1 Why Does the $x = x_h + x_p$ Structure Work?

This structure relies on the Principle of Superposition, a consequence of the linearity of differentiation.

- x_h solves $\dot{x} + P(t)x = 0$.
- x_p solves $\dot{x} + P(t)x = Q(t)$.

If we substitute the sum $(x_h + x_p)$ into the non-homogeneous equation:

$$
\begin{aligned}
\frac{d}{dt}(x_h + x_p) + P(t)(x_h + x_p) &= \left[\frac{d}{dt}x_h + P(t)x_h\right] + \left[\frac{d}{dt}x_p + P(t)x_p\right] \\
&= [0] + [Q(t)] \\
&= Q(t)
\end{aligned}
$$

This shows the sum is indeed the general solution.

5.2 What Does the Solution Represent? (Physical Significance)

In physical models, the two parts of $x(t) = x_h(t) + x_p(t)$ describe different behaviors.

$x_h(t)$ **(Homogeneous Solution / Transient Response)** This term contains the arbitrary constant C (determined by initial conditions) and typically involves exponential decay (e.g., e^{-kt}). This models the system's initial, transient response, which fades away to zero as $t \to \infty$.

$x_p(t)$ **(Particular Solution / Steady-State Response)** This term is free of arbitrary constants and is determined by the forcing function $Q(t)$. It models the long-term or steady-state behavior of the system, which is sustained by the external input $Q(t)$.

6 Common Mistakes to Avoid

- **Domain Errors:** Failing to check for singularities. The solution's domain is restricted to the largest *connected interval* that includes the initial condition.
- **Forgetting Standard Linear Form:** The integrating factor $\mu(t) = e^{\int P(t)dt}$ is ONLY valid if the linear equation is first written in the standard form $\dot{x} + P(t)x = Q(t)$. The coefficient of $\dot{x}$ must be 1. If you start with $A(t)y' + B(t)y = C(t)$, you must first divide by $A(t)$ to find $P(t) = B(t)/A(t)$.
- **Mechanical Application of Formula:** Do not blindly memorize the final solution formula. Understand the *procedure* (multiply by μ, recognize product rule, integrate) to correctly handle different $P(t)$ and $Q(t)$ and to incorporate the constant C.

7 Summary & What's Next

7.1 Key Takeaways

- **Superposition and Structure:** For linear ODEs ($\dot{x} + P(t)x = Q(t)$), the General Solution is always the sum of the homogeneous solution $x_h(t)$ (with constant C) and a particular solution $x_p(t)$ (driven by $Q(t)$).
- **Integrating Factor:** The formal method to solve linear equations uses $\mu(t) = e^{\int P(t)dt}$, which transforms the equation into the exact derivative of the product $(\mu(t)x)$.
- **Domain of Solutions:** For an IVP, the solution is only valid on the largest connected, continuous interval that includes the initial point (x_0, y_0).

7.2 Next Time

The next topics will likely focus on First-Order Linear Equations with Constant Coefficients, $\dot{y} + ky = q(t)$, and the method of Undetermined Coefficients (assuming solutions like $x_p(t) = Ae^{rt}$) to find the particular solution x_p. We may also begin exploring Homogeneous Equations (which require substitution).