

Lecture Notes: Differential Equations - Session 16

Dr. Khajeh Salehani - University of Tehran

These notes were collaboratively gathered and compiled. We warmly welcome your feedback and suggestions at: k.ghanbari@ut.ac.ir and hamidrezahosseini@ut.ac.ir

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1 Context: Where This Fits In

In previous sessions (Sessions 13-15), we developed methods to solve non-homogeneous linear ODEs ($P(D)x = f(t)$) for specific input types: exponentials, polynomials, and single sinusoids. We primarily used the **Exponential Response Formula (ERF)** and the **Method of Undetermined Coefficients (MUC)**.

Session 16 introduces **Fourier Series**, a powerful tool that allows us to solve ODEs where the input $f(t)$ is an **arbitrary periodic function** (e.g., square waves, triangular waves). By decomposing a complex periodic function into an infinite sum of simple sine and cosine terms, we can leverage the **Principle of Superposition** to solve the differential equation for each term individually and sum the results.

2 Core Concepts & Definitions

2.1 Periodic Function

A function is periodic if it repeats its pattern exactly after a fixed interval T , called the period:

$$f(t) = f(t + T)$$

2.2 Fourier Series (Period $T = 2\pi$)

If $f(t)$ has a period $T = 2\pi$, it can be represented as an infinite sum:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt)) \quad (1)$$

The coefficients are calculated using:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt \quad (\text{DC Component}) \quad (2)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt \quad (3)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt \quad (4)$$

2.3 Symmetry Properties

- **Even Function (Function Zoj):** $f(-t) = f(t)$.
- **Odd Function (Function Fard):** $f(-t) = -f(t)$.

Implications for Coefficients:

- If $f(t)$ is **Odd**: $a_0 = 0$ and $a_n = 0$. The series contains only sine terms.
- If $f(t)$ is **Even**: $b_n = 0$. The series contains only cosine terms and the constant a_0 .

2.4 General Fourier Series (Arbitrary Period $T = 2L$)

For a general period $T = 2L$, we normalize the variable using $\frac{\pi t}{L}$:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \left(\frac{n\pi t}{L} \right) + b_n \sin \left(\frac{n\pi t}{L} \right) \right) \quad (5)$$

The coefficients are:

$$a_n = \frac{1}{L} \int_{-L}^L f(t) \cos \left(\frac{n\pi t}{L} \right) dt, \quad b_n = \frac{1}{L} \int_{-L}^L f(t) \sin \left(\frac{n\pi t}{L} \right) dt \quad (6)$$

3 Prerequisite Skill Refresh

3.1 Integration of Trigonometric Functions

Standard integrals used repeatedly:

- $\int \cos(nt) dt = \frac{\sin(nt)}{n}$
- $\int \sin(nt) dt = -\frac{\cos(nt)}{n}$

Note that $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$ for integer n .

3.2 Integration by Parts

Required when $f(t)$ involves t (e.g., $f(t) = t$ or $f(t) = |t|$).

$$\int u \, dv = uv - \int v \, du$$

Example: To integrate $\int t \cos(nt) dt$: Let $u = t \implies du = dt$, and $dv = \cos(nt) dt \implies v = \frac{\sin(nt)}{n}$.

4 Key Examples: A Step-by-Step Walkthrough

4.1 Example 1: Square Wave ($T = 2\pi$, Odd Function)

Problem: Find the Fourier coefficients for $f(t)$ on $[-\pi, \pi]$:

$$f(t) = \begin{cases} -1 & -\pi < t \leq 0 \\ +1 & 0 < t \leq \pi \end{cases}$$

Step 1: Check Symmetry $f(-t) = -f(t)$, so $f(t)$ is **Odd**. Immediately, $a_0 = 0$ and $a_n = 0$.

Step 2: Calculate b_n Since $f(t)$ is odd, $f(t) \sin(nt)$ is even. We integrate over $[0, \pi]$ and multiply by 2:

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^\pi (1) \sin(nt) dt \\ &= \frac{2}{\pi} \left[-\frac{\cos(nt)}{n} \right]_0^\pi \\ &= \frac{2}{n\pi} [-\cos(n\pi) - (-\cos(0))] \\ &= \frac{2}{n\pi} [1 - (-1)^n] \end{aligned}$$

If n is even, $b_n = 0$. If n is odd, $b_n = \frac{4}{n\pi}$.

Step 3: Final Series

$$f(t) = \sum_{n \text{ odd}} \frac{4}{n\pi} \sin(nt) = \frac{4}{\pi} \left(\sin t + \frac{1}{3} \sin 3t + \dots \right)$$

4.2 Example 2: Triangular Wave ($T = 2$, Even Function)

Problem: Find the Fourier series for $f(t) = |t|$ on $[-1, 1]$. Here $2L = 2$, so $L = 1$.

Step 1: Check Symmetry $f(-t) = |-t| = |t| = f(t)$, so $f(t)$ is **Even**. Immediately, $b_n = 0$.

Step 2: Calculate a_0 Using symmetry (integrate 0 to 1 and double):

$$a_0 = \frac{2}{1} \int_0^1 t dt = 2 \left[\frac{t^2}{2} \right]_0^1 = 1$$

Step 3: Calculate a_n

$$a_n = 2 \int_0^1 t \cos(n\pi t) dt$$

Using integration by parts ($u = t, dv = \cos(n\pi t)dt$):

$$\begin{aligned} a_n &= 2 \left(\left[\frac{t \sin(n\pi t)}{n\pi} \right]_0^1 - \int_0^1 \frac{\sin(n\pi t)}{n\pi} dt \right) \\ &= 2 \left(0 - \left[\frac{-\cos(n\pi t)}{(n\pi)^2} \right]_0^1 \right) \\ &= \frac{2}{(n\pi)^2} [\cos(n\pi) - \cos(0)] \\ &= \frac{2}{(n\pi)^2} [(-1)^n - 1] \end{aligned}$$

If n is even, $a_n = 0$. If n is odd, $a_n = \frac{-4}{(n\pi)^2}$.

Step 4: Final Series

$$f(t) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^2} \cos(n\pi t)$$

4.3 Example 3: Application to ODEs (Conceptual)

Problem: Solve $P(D)x = f(t)$ where $f(t)$ is periodic.

Step 1: Decompose Input Replace $f(t)$ with its Fourier Series:

$$P(D)x = \frac{a_0}{2} + \sum (a_n \cos(\omega_n t) + b_n \sin(\omega_n t))$$

Step 2: Solve Homogeneous Equation Find x_h such that $P(D)x_h = 0$.

Step 3: Solve Particular Equations (Superposition) Find $x_{p,0}$ for the constant term $\frac{a_0}{2}$. Find $x_{p,n}$ for each $\cos(\omega_n t)$ and $\sin(\omega_n t)$ term using ERF.

Step 4: Sum Solutions

$$x(t) = x_h(t) + x_{p,0} + \sum_{n=1}^{\infty} x_{p,n}(t)$$

5 Conceptual Understanding

Why This Works:

- **Orthogonality:** The functions $\{1, \cos(nt), \sin(nt)\}$ are orthogonal. This means we can isolate each frequency component of a complex signal independently using integrals.
- **Superposition:** Linear systems respond to a sum of inputs as the sum of the individual responses. We break a "hard" periodic input into many "easy" sinusoidal inputs.

Physical Interpretation: x_h is the *transient* response (decays to zero for stable systems). x_p is the *steady-state* response (the system oscillating at the frequencies present in the input).

6 Common Mistakes to Avoid

- **Symmetry Check:** Always check for Even/Odd symmetry first. It can reduce your work by half.
- **Period Confusion:** Do not confuse π with L . If $T = 2$, then $L = 1$, and the terms are $\cos(n\pi t)$. If $T = 2\pi$, then $L = \pi$, and terms are $\cos(nt)$.
- **Resonance:** If one of the Fourier frequencies $in\omega$ is a root of the characteristic polynomial $P(r)$, resonance occurs for that specific term. You must modify the particular solution for that term (multiply by t).

7 Summary & What's Next

Summary: Fourier Series allows us to decompose any periodic $f(t)$ into sines and cosines. We calculate coefficients a_0, a_n, b_n via integration, exploiting symmetry where possible. We then solve the ODE for each term and sum the results.

Next Session: Session 17 will cover **Differentiation and Integration of Fourier Series**. We will discuss when it is mathematically valid to differentiate a series term-by-term, which is crucial for applying these series directly inside differential equations.