Lecture Notes: Differential Equations (Session 18)

Dr. Khajeh Salehani - University of Tehran

1 Context: Where This Fits In

In previous sessions (16 and 17), we focused on the mechanics of constructing Fourier Series. We established that a periodic function $f(t)$ can be represented as an infinite sum of sines and cosines:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L}t\right) + b_n \sin\left(\frac{n\pi}{L}t\right) \right)$$

Previously, we assumed this equality held true without rigorous justification. In Session 18, we transition from calculation to theory to answer a critical question: **Does the infinite series actually equal the function $f(t)$?**

Specifically, we investigate what happens at points where the function $f(t)$ has a "break" or "jump" (discontinuity). The answer is provided by Dirichlet's Theorem.

2 Core Concepts & Definitions

2.1 Pointwise Convergence

We say a Fourier series converges *pointwise* to $f(t)$ at a specific point t_0 if, when t_0 is substituted into the infinite sum, the result equals the value $f(t_0)$.

- **Analogy:** Imagine approximating a smooth curve with rectangular Lego blocks. Pointwise convergence implies that at a specific horizontal position, the height of the block stack exactly matches the height of the curve.

2.2 Piecewise Continuity

A function is **piecewise continuous** on an interval if it is continuous everywhere except for a finite number of "jumps." At these jump points, the function must have finite limits from the left and right (it cannot shoot off to infinity).

- **Visual:**

  [Image of piecewise continuous function with jump discontinuities] You can draw the graph without lifting your pencil, except at a few specific points where you "hop" to a new level. Examples include square waves or sawtooth waves.

2.3 Dirichlet's Theorem (The Convergence Theorem)

This is the central theorem of the session. If a periodic function $f(t)$ is **piecewise smooth** (meaning $f(t)$ and $f'(t)$ are piecewise continuous), the Fourier series converges as follows:

1. **At points of continuity:** The series converges exactly to $f(t)$.

2. **At points of discontinuity (jumps):** The series converges to the **average of the jump**.

Mathematically, at a discontinuity t_0, the series converges to:

$$\frac{f(t_0^+) + f(t_0^-)}{2}$$

where $f(t_0^+)$ is the limit from the right and $f(t_0^-)$ is the limit from the left.

2.4 Gibbs Phenomenon

When approximating a jump discontinuity with partial sums of sines and cosines, the series tends to "overshoot" the corner before settling down. This "ringing" effect near discontinuities is known as the Gibbs Phenomenon.

3 Prerequisite Skill Refresh

3.1 Skill 1: Limits at a Discontinuity

At a jump, the function approaches different values from different directions.

- **Right Limit** $(t \to t_0^+)$**:** Approaching from values larger than t_0.

- **Left Limit** $(t \to t_0^-)$**:** Approaching from values smaller than t_0.

Example: If $f(t) = -1$ for $t < 0$ and $f(t) = 1$ for $t > 0$:

$$\text{Average at } t = 0 \implies \frac{1 + (-1)}{2} = 0$$

3.2 Skill 2: Integration by Parts

Required for calculating coefficients, typically for $\int t \sin(nt)\, dt$ or $\int t \cos(nt)\, dt$.

$$\int u\, dv = uv - \int v\, du$$

3.3 Skill 3: Even and Odd Symmetry

Exploiting symmetry simplifies coefficient calculation:

- **Even Functions** $(f(-t) = f(t))$**:** $b_n = 0$ (only cosine terms).

- **Odd Functions** $(f(-t) = -f(t))$**:** $a_n = 0$ (only sine terms).

4 Key Examples: A Step-by-Step Walkthrough

4.1 Example 1: Determining Convergence at a Discontinuity

Problem: Let $f(t)$ be a square wave where $f(t) = 1$ for $0 < t < \pi$ and $f(t) = -1$ for $-\pi < t < 0$. The Fourier series is given by $S(t) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(nt)}{n}$. To what value does this series converge at $t = 0$?

Step 1: Analyze the limits at $t = 0$. There is a jump at $t = 0$:

$$\lim_{t \to 0^-} f(t) = -1$$

$$\lim_{t \to 0^+} f(t) = 1$$

Step 2: Apply Dirichlet's Theorem. The series converges to the average of the limits:

$$\text{Sum} = \frac{f(0^+) + f(0^-)}{2} = \frac{1 + (-1)}{2} = 0$$

Step 3: Verify with the series. Substituting $t = 0$ into $S(t) = \frac{4}{\pi}(\sin t + \frac{1}{3}\sin 3t + \dots)$:

$$S(0) = \frac{4}{\pi}(0 + 0 + \dots) = 0$$

Conclusion: The theorem holds; the series converges to 0, the midpoint of the jump.

4.2 Example 2: Using Fourier Series to Sum Numerical Series

Problem: The Fourier series for $f(t) = |t|$ on $[-\pi, \pi]$ is $f(t) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos(nt)}{n^2}$. Use this to find the sum $S = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$.

Step 1: Choose a strategic value for t. We need $\cos(nt)$ to become 1. Let $t = 0$.

Step 2: Equate Function and Series. At $t = 0$, $f(0) = |0| = 0$. Since $f(t)$ is continuous at 0, the series equals the function.

$$0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\cos(0)}{n^2}$$

$$0 = \frac{\pi}{2} - \frac{4}{\pi} \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right)$$

Step 3: Solve for the sum.

$$\frac{4}{\pi} S = \frac{\pi}{2} \implies S = \frac{\pi}{2} \cdot \frac{\pi}{4} = \frac{\pi^2}{8}$$

4.3 Example 3: Solving ODEs with Periodic Forcing

Problem: Find a particular solution to $\ddot{x} + ax = f(t)$, where a is not an integer square, and $f(t) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(nt)}{n}$.

Step 1: Principle of Superposition. Because the ODE is linear, we can solve for each term in the sine series individually and sum the results. We solve:

$$\ddot{x}_n + ax_n = \frac{4}{\pi n} \sin(nt)$$

Step 2: Solve for a single term. Assume $x_n(t) = A_n \sin(nt)$. Then $\ddot{x}_n = -n^2 A_n \sin(nt)$. Substituting back:

$$(-n^2 + a)A_n \sin(nt) = \frac{4}{\pi n} \sin(nt)$$

$$A_n = \frac{4}{\pi n(a - n^2)}$$

Step 3: Construct full solution.

$$x_p(t) = \sum_{n \text{ odd}} \frac{4}{\pi n(a - n^2)} \sin(nt)$$

5 Conceptual Understanding

The Big Picture: Smooth functions (sines and cosines) struggle to replicate sharp corners. When we use them to approximate a "rough" signal like a square wave, Dirichlet's Theorem assures us that the approximation is "good enough" for engineering.

- Everywhere except the jump, the match is perfect.

- At the jump, the series predictably hits the midpoint.

In the context of ODEs (Example 3), this solution represents the **steady-state response** of a system to a periodic force. The series reveals how the system resonates with each specific frequency component of the input. If $a = n^2$, the denominator vanishes, indicating **resonance**.

6 Common Mistakes to Avoid

- **The Endpoints:** Remember that Fourier series represent the *periodic extension* of the function. If $f(t)$ is defined on $[0, 2\pi]$ and $f(0) \neq f(2\pi)$, the periodic extension creates a new jump discontinuity at the endpoints. The series will converge to the average value there, not necessarily the function's endpoint value.

- **Term-by-Term Differentiation:** You cannot always differentiate a Fourier series term-by-term. If $f(t)$ is discontinuous (like a square wave), its derivative involves Dirac delta "spikes," and the differentiated series may not converge. Integration is generally safer as it smooths the function.

7 Summary & What's Next

Key Takeaways:

- **Dirichlet's Theorem:** The series converges to $f(t)$ where continuous, and to the average $\frac{f(t^+) + f(t^-)}{2}$ at discontinuities.

- **Periodic Extension:** The series automatically repeats the pattern outside the defined interval. Always check boundaries for hidden jumps.

- **ODE Application:** Linear differential equations driven by periodic forces can be solved by expanding the force into a Fourier series and applying superposition.

Note: This concludes the module on Fourier Series. The upcoming exam will cover all topics up to and including Session 18.