Lecture Notes: Differential Equations - Session 15

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1 Context: Where This Fits In

The preceding sessions (Sessions 13 and 14) focused on finding the particular solution (x_p) for non-homogeneous second-order linear ODEs with exponential (Be^{at}) or trigonometric ($B\cos(\omega t)$) inputs, primarily utilizing the Exponential Response Formula (ERF). Session 14 also established the necessary criteria for System Stability by analyzing the roots of the characteristic polynomial.

Session 15 completes the toolkit for standard forcing functions by addressing polynomial inputs (e.g., $4x^2 - 2x$). Instead of the ERF, we employ the **Method of Undetermined Coefficients (MUC)**. This involves hypothesizing a specific form for the solution based on the input function's structure. Additionally, this session formalizes the concept of Time Invariance for constant coefficient systems.

2 Core Concepts & Definitions

2.1 Method of Undetermined Coefficients (MUC)

A technique used to find the particular solution (y_p) of a non-homogeneous linear ODE with constant coefficients:

$$\mathbf{p}(\mathbf{D})\mathbf{y} = \mathbf{q}(\mathbf{x})$$

when the input function $q(x)$ belongs to a closed family of functions (polynomials, exponentials, sines, or cosines). The method involves guessing the form of y_p, substituting it into the ODE, and solving a system of algebraic equations to determine the unknown coefficients.

2.2 Polynomial Trial Solution (y_p)

If the input $q(x)$ is a polynomial of degree N, and if $r = 0$ is **not** a root of the characteristic polynomial $p(r)$ (i.e., $p(0) \neq 0$), the trial solution must be a complete polynomial of the same degree N:

$$y_p = A_N x^N + \cdots + A_1 x + A_0 \tag{1}$$

Note: Even if the input lacks certain terms (e.g., $q(x) = x^2$), the trial solution must include all lower-order terms ($Ax^2 + Bx + C$).

2.3 Resonance & MUC Multiplicity Rule

For polynomial inputs, resonance occurs if the constant input $q(x) = 1$ is a solution to the homogeneous equation. This happens if $p(0) = 0$. If $r = 0$ is a root of multiplicity s (meaning $p(0) = p'(0) = \cdots = p^{(s-1)}(0) = 0$, but $p^{(s)}(0) \neq 0$), the correct trial solution is the standard polynomial multiplied by x^s:

$$y_p = x^s(A_N x^N + \cdots + A_1 x + A_0) \tag{2}$$

2.4 Time Invariance

A property of linear ODEs with constant coefficients ($p(D)x(t) = q(t)$). It states that if $x_p(t)$ is a particular solution for the input $q(t)$, then the shifted function $x_p(t-c)$ is a particular solution for the shifted input $q(t-c)$:

$$p(D)x_p(t-c) = q(t-c) \tag{3}$$

where c is a constant time shift.

3 Prerequisite Skill Refresh

3.1 Differentiation of Polynomials

The core of MUC involves finding derivatives of the trial polynomial.

- Power Rule: $\frac{d}{dx}(Ax^n) = nAx^{n-1}$.
- The derivative of a constant is zero.

Example: If $y_p = Ax^2 + Bx + C$:

$$\begin{aligned} y_p' &= 2Ax + B \\ y_p'' &= 2A \\ y_p''' &= 0 \end{aligned}$$

3.2 Solving Systems by Coefficient Comparison

After substitution, we equate coefficients of corresponding powers of x. **Example:** Given $(4A)x^2 + (4B + 3A)x = 8x^2 - 1$.

$$\begin{aligned} x^2 &: \quad 4A = 8 \implies A = 2 \\ x^1 &: \quad 4B + 3A = 0 \implies 4B + 6 = 0 \implies B = -3/2 \end{aligned}$$

4 Key Examples: A Step-by-Step Walkthrough

4.1 Example 1: Standard MUC (Non-Resonant, $p(0) \neq 0$)

Problem: Find a particular solution (y_p) for $\mathbf{y'' + 3y' + 4y = 4x^2 - 2x}$.

Step 1: Check Resonance & Choose Trial Solution $p(r) = r^2 + 3r + 4$. $p(0) = 4 \neq 0$. No resonance. Input is degree 2, so we choose a complete quadratic polynomial:

$$y_p = Ax^2 + Bx + C$$

Step 2: Calculate Derivatives

$$y_p' = 2Ax + B, \quad y_p'' = 2A$$

Step 3: Substitute and Group Substitute into the ODE:

$$(2A) + 3(2Ax + B) + 4(Ax^2 + Bx + C) = 4x^2 - 2x$$

Group by powers of x:

$$(4A)x^2 + (6A + 4B)x + (2A + 3B + 4C) = 4x^2 - 2x + 0$$

Step 4: Solve System

$$\begin{aligned}
x^2 &: \quad 4A = 4 \implies \mathbf{A = 1} \\
x^1 &: \quad 6A + 4B = -2 \implies 6(1) + 4B = -2 \implies 4B = -8 \implies \mathbf{B = -2} \\
x^0 &: \quad 2A + 3B + 4C = 0 \implies 2(1) + 3(-2) + 4C = 0 \implies -4 + 4C = 0 \implies \mathbf{C = 1}
\end{aligned}$$

Step 5: Final Solution

$$\boxed{y_p = x^2 - 2x + 1}$$

4.2 Example 2: Resonance Case ($p(0) = 0$)

Problem: Find a particular solution (y_p) for $\mathbf{y''' + 3y'' + 2y' = x^2 + x}$.

Step 1: Check Resonance & Multiplicity $p(r) = r^3 + 3r^2 + 2r = r(r^2 + 3r + 2)$. $p(0) = 0$, so resonance occurs. Factor $p(r) = r(r+1)(r+2)$. Since r is a factor to the first power, multiplicity $s = 1$. Trial Solution: Multiply standard quadratic by x^1:

$$y_p = x(Ax^2 + Bx + C) = Ax^3 + Bx^2 + Cx$$

Step 2: Calculate Derivatives

$$\begin{aligned}
y_p' &= 3Ax^2 + 2Bx + C \\
y_p'' &= 6Ax + 2B \\
y_p''' &= 6A
\end{aligned}$$

Step 3: Substitute and Group

$$(6A) + 3(6Ax + 2B) + 2(3Ax^2 + 2Bx + C) = x^2 + x$$

Group by powers of x:

$$(6A)x^2 + (18A + 4B)x + (6A + 6B + 2C) = 1x^2 + 1x + 0$$

Step 4: Solve System

$$\begin{aligned}
x^2 :&\quad 6A = 1 \implies \mathbf{A = 1/6} \\
x^1 :&\quad 18A + 4B = 1 \implies 18(1/6) + 4B = 1 \implies 3 + 4B = 1 \implies \mathbf{B = -1/2} \\
x^0 :&\quad 6A + 6B + 2C = 0 \implies 1 - 3 + 2C = 0 \implies \mathbf{C = 1}
\end{aligned}$$

Step 5: Final Solution

$$\boxed{y_p = \frac{1}{6}x^3 - \frac{1}{2}x^2 + x}$$

4.3 Example 3: Application of Time Invariance

Problem: Given $x_p(t) = \sqrt{2}\sin(t - \frac{\pi}{4})$ solves $2\ddot{x} + \dot{x} + x = \sin(t)$, find x_p for $2\ddot{x} + \dot{x} + x = \sin(t - \frac{\pi}{3})$.

Step 1: Identify Shift The new input is the old input shifted by $c = \frac{\pi}{3}$.

Step 2: Apply Time Invariance Since the ODE has constant coefficients, we simply shift the known solution by c:

$$y_p(t) = x_p\left(t - \frac{\pi}{3}\right) = \sqrt{2}\sin\left(\left(t - \frac{\pi}{3}\right) - \frac{\pi}{4}\right)$$

Combine phase angles:

$$\frac{\pi}{3} + \frac{\pi}{4} = \frac{7\pi}{12}$$

Step 3: Final Solution

$$\boxed{y_p(t) = \sqrt{2}\sin\left(t - \frac{7\pi}{12}\right)}$$

5 Conceptual Understanding

Why MUC Works: MUC relies on the fact that polynomials, exponentials, and sinusoids form "closed families" under differentiation. When you substitute a polynomial into a linear differential operator with constant coefficients, the result is always another polynomial. By equating coefficients, we transform the differential equation into a solvable algebraic system.

Time Invariance Physical Meaning: This property reflects the physical reality that the "rules" of the system (m, b, K) do not change with time. If you push a swing now, it moves a certain way. If you push it exactly the same way 10 minutes later, it will move in the exact same pattern, just 10 minutes later.

6 Common Mistakes to Avoid

- **Missing Lower Order Terms:** If $q(x) = x^3$, your guess must still be $Ax^3 + Bx^2 + Cx + D$. Omitting terms like Bx^2 or D will often result in a system of equations that has no solution.
- **Ignoring Resonance ($p(0) = 0$):** This is the most critical error. Always check if the constant term is a solution to the homogeneous equation. If $p(0) = 0$, you **must** multiply your trial solution by x^s.
- **Algebraic Errors:** The systems of equations can get large. Verify your arithmetic, especially when substituting back values for A and B to find C.

7 Summary & What's Next

Key Takeaways:

1. Use MUC for polynomial inputs by guessing a trial polynomial of the same degree.
2. If $p(0) = 0$ (resonance), multiply the trial solution by x^s (where s is the multiplicity of the root $r = 0$).
3. Time Invariance allows us to solve shifted inputs $q(t-c)$ by simply shifting the solution $x_p(t-c)$.

Next Session: Session 16 introduces **Fourier Series**. We will learn to decompose general periodic forcing functions $f(t)$ into infinite sums of sines and cosines. This allows us to solve complex periodic problems by summing the solutions to simpler trigonometric problems using the principle of superposition.