

# Lecture Notes: Differential Equations - Session 16

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## 1 Context: Where This Fits In

In previous sessions (Sessions 13-15), we developed methods to solve non-homogeneous linear ODEs ( $P(D)x = f(t)$ ) for specific input types: exponentials, polynomials, and single sinusoids. We primarily used the **Exponential Response Formula (ERF)** and the **Method of Undetermined Coefficients (MUC)**.

Session 16 introduces **Fourier Series**, a powerful tool that allows us to solve ODEs where the input  $f(t)$  is an **arbitrary periodic function** (e.g., square waves, triangular waves). By decomposing a complex periodic function into an infinite sum of simple sine and cosine terms, we can leverage the **Principle of Superposition** to solve the differential equation for each term individually and sum the results.

## 2 Core Concepts & Definitions

### 2.1 Periodic Function

A function is periodic if it repeats its pattern exactly after a fixed interval  $T$ , called the period:

$$f(t) = f(t + T)$$

### 2.2 Fourier Series (Period $T = 2\pi$ )

If  $f(t)$  has a period  $T = 2\pi$ , it can be represented as an infinite sum:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt)) \quad (1)$$

The coefficients are calculated using:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt \quad (\text{DC Component}) \quad (2)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt \quad (3)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt \quad (4)$$

## 2.3 Symmetry Properties

- **Even Function (Function Zoj):**  $f(-t) = f(t)$ .
- **Odd Function (Function Fard):**  $f(-t) = -f(t)$ .

**Implications for Coefficients:**

- If  $f(t)$  is **Odd**:  $a_0 = 0$  and  $a_n = 0$ . The series contains only sine terms.
- If  $f(t)$  is **Even**:  $b_n = 0$ . The series contains only cosine terms and the constant  $a_0$ .

## 2.4 General Fourier Series (Arbitrary Period $T = 2L$ )

For a general period  $T = 2L$ , we normalize the variable using  $\frac{\pi t}{L}$ :

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right) \right) \quad (5)$$

The coefficients are:

$$a_n = \frac{1}{L} \int_{-L}^L f(t) \cos\left(\frac{n\pi t}{L}\right) dt, \quad b_n = \frac{1}{L} \int_{-L}^L f(t) \sin\left(\frac{n\pi t}{L}\right) dt \quad (6)$$

# 3 Prerequisite Skill Refresh

## 3.1 Integration of Trigonometric Functions

Standard integrals used repeatedly:

- $\int \cos(nt) dt = \frac{\sin(nt)}{n}$
- $\int \sin(nt) dt = -\frac{\cos(nt)}{n}$

Note that  $\sin(n\pi) = 0$  and  $\cos(n\pi) = (-1)^n$  for integer  $n$ .

## 3.2 Integration by Parts

Required when  $f(t)$  involves  $t$  (e.g.,  $f(t) = t$  or  $f(t) = |t|$ ).

$$\int u dv = uv - \int v du$$

**Example:** To integrate  $\int t \cos(nt) dt$ : Let  $u = t \implies du = dt$ , and  $dv = \cos(nt) dt \implies v = \frac{\sin(nt)}{n}$ .

## 4 Key Examples: A Step-by-Step Walkthrough

### 4.1 Example 1: Square Wave ( $T = 2\pi$ , Odd Function)

**Problem:** Find the Fourier coefficients for  $f(t)$  on  $[-\pi, \pi]$ :

$$f(t) = \begin{cases} -1 & -\pi < t \leq 0 \\ +1 & 0 < t \leq \pi \end{cases}$$

**Step 1: Check Symmetry**  $f(-t) = -f(t)$ , so  $f(t)$  is **Odd**. Immediately,  $a_0 = 0$  and  $a_n = 0$ .

**Step 2: Calculate  $b_n$**  Since  $f(t)$  is odd,  $f(t) \sin(nt)$  is even. We integrate over  $[0, \pi]$  and multiply by 2:

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^\pi (1) \sin(nt) dt \\ &= \frac{2}{\pi} \left[ -\frac{\cos(nt)}{n} \right]_0^\pi \\ &= \frac{2}{n\pi} [-\cos(n\pi) - (-\cos(0))] \\ &= \frac{2}{n\pi} [1 - (-1)^n] \end{aligned}$$

If  $n$  is even,  $b_n = 0$ . If  $n$  is odd,  $b_n = \frac{4}{n\pi}$ .

**Step 3: Final Series**

$$f(t) = \sum_{n \text{ odd}} \frac{4}{n\pi} \sin(nt) = \frac{4}{\pi} \left( \sin t + \frac{1}{3} \sin 3t + \dots \right)$$

### 4.2 Example 2: Triangular Wave ( $T = 2$ , Even Function)

**Problem:** Find the Fourier series for  $f(t) = |t|$  on  $[-1, 1]$ . Here  $2L = 2$ , so  $L = 1$ .

**Step 1: Check Symmetry**  $f(-t) = |-t| = |t| = f(t)$ , so  $f(t)$  is **Even**. Immediately,  $b_n = 0$ .

**Step 2: Calculate  $a_0$**  Using symmetry (integrate 0 to 1 and double):

$$a_0 = \frac{2}{1} \int_0^1 t dt = 2 \left[ \frac{t^2}{2} \right]_0^1 = 1$$

**Step 3: Calculate  $a_n$**

$$a_n = 2 \int_0^1 t \cos(n\pi t) dt$$

Using integration by parts ( $u = t, dv = \cos(n\pi t)dt$ ):

$$\begin{aligned} a_n &= 2 \left( \left[ \frac{t \sin(n\pi t)}{n\pi} \right]_0^1 - \int_0^1 \frac{\sin(n\pi t)}{n\pi} dt \right) \\ &= 2 \left( 0 - \left[ \frac{-\cos(n\pi t)}{(n\pi)^2} \right]_0^1 \right) \\ &= \frac{2}{(n\pi)^2} [\cos(n\pi) - \cos(0)] \\ &= \frac{2}{(n\pi)^2} [(-1)^n - 1] \end{aligned}$$

If  $n$  is even,  $a_n = 0$ . If  $n$  is odd,  $a_n = \frac{-4}{(n\pi)^2}$ .

**Step 4: Final Series**

$$f(t) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^2} \cos(n\pi t)$$

### 4.3 Example 3: Application to ODEs (Conceptual)

**Problem:** Solve  $P(D)x = f(t)$  where  $f(t)$  is periodic.

**Step 1: Decompose Input** Replace  $f(t)$  with its Fourier Series:

$$P(D)x = \frac{a_0}{2} + \sum (a_n \cos(\omega_n t) + b_n \sin(\omega_n t))$$

**Step 2: Solve Homogeneous Equation** Find  $x_h$  such that  $P(D)x_h = 0$ .

**Step 3: Solve Particular Equations (Superposition)** Find  $x_{p,0}$  for the constant term  $\frac{a_0}{2}$ . Find  $x_{p,n}$  for each  $\cos(\omega_n t)$  and  $\sin(\omega_n t)$  term using ERF.

**Step 4: Sum Solutions**

$$x(t) = x_h(t) + x_{p,0} + \sum_{n=1}^{\infty} x_{p,n}(t)$$

## 5 Conceptual Understanding

**Why This Works:**

- **Orthogonality:** The functions  $\{1, \cos(nt), \sin(nt)\}$  are orthogonal. This means we can isolate each frequency component of a complex signal independently using integrals.
- **Superposition:** Linear systems respond to a sum of inputs as the sum of the individual responses. We break a "hard" periodic input into many "easy" sinusoidal inputs.

**Physical Interpretation:**  $x_h$  is the *transient* response (decays to zero for stable systems).  $x_p$  is the *steady-state* response (the system oscillating at the frequencies present in the input).

## 6 Common Mistakes to Avoid

- **Symmetry Check:** Always check for Even/Odd symmetry first. It can reduce your work by half.
- **Period Confusion:** Do not confuse  $\pi$  with  $L$ . If  $T = 2$ , then  $L = 1$ , and the terms are  $\cos(n\pi t)$ . If  $T = 2\pi$ , then  $L = \pi$ , and terms are  $\cos(nt)$ .
- **Resonance:** If one of the Fourier frequencies  $i n \omega$  is a root of the characteristic polynomial  $P(r)$ , resonance occurs for that specific term. You must modify the particular solution for that term (multiply by  $t$ ).

## 7 Summary & What's Next

**Summary:** Fourier Series allows us to decompose any periodic  $f(t)$  into sines and cosines. We calculate coefficients  $a_0, a_n, b_n$  via integration, exploiting symmetry where possible. We then solve the ODE for each term and sum the results.

**Next Session:** Session 17 will cover **Differentiation and Integration of Fourier Series**. We will discuss when it is mathematically valid to differentiate a series term-by-term, which is crucial for applying these series directly inside differential equations.