Lecture Notes: Differential Equations (Sessions 3 and 4)

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1 Context: Where This Fits In

The immediate previous topics were the foundational language of differential equations (ODEs), including independent and dependent variables, parameters, and differentiation notations (Leibniz: $\frac{dy}{dx}$, Lagrange: y', Newton: $\dot{x}$). We also reviewed the properties of the exponential function $x(t) = e^{at}$, which is the solution form for the fundamental ODE $\dot{y} = ay$.

Today's topic takes this basic language and applies it to formally define and classify differential equations. Most importantly, we introduce the procedure for finding unique, particular solutions using Initial Value Problems (IVPs).

2 Core Concepts & Definitions

These sessions formally introduce the classification of differential equations and the conceptual framework for their solutions.

Differential Equation (Formal) An equation that establishes a relationship between a function, the independent variable, and the function's derivatives (rates of change).

Order of an ODE Determined by the **highest** derivative present in the equation. For example, $y'' + 4y = 0$ is a second-order equation.

General Solution The family of all functions that satisfy the differential equation. The number of arbitrary constants (parameters) in the general solution of an ODE is equal to its order.

Solving a Differential Equation The objective is to find the function(s) that, when substituted along with its derivatives into the equation, make the equation true.

Parameter A constant, often C, C_1, C_2, etc., introduced during integration. These parameters define the specific members within the general family of solutions.

Initial Value Problem (IVP) A differential equation paired with auxiliary conditions (initial conditions) that specify the value of the function and/or its derivatives at a single point, x_0. An IVP's purpose is to find a unique particular solution.

Particular Solution A single function obtained from the general solution when the arbitrary constants are assigned specific, numerical values, typically by solving an IVP.

Separable Equation An ODE that can be expressed in the form $\frac{dy}{dx} = f(x)g(y)$, allowing algebraic separation of variables for integration.

Trivial Solution The specific solution $y(t) = 0$ that satisfies some ODEs, such as $\dot{y} = 3y$.

Conceptual Check: Order and Parameters

The relationship between order and parameters is fundamental:

- An ODE of the n-th order will have a general solution containing n arbitrary parameters $(C_1, C_2, \ldots, C_n)$.
- For example, the second-order ODE $\ddot{x} = 3t$ has a solution $x(t) = \frac{t^3}{2} + C_1 t + C_2$, containing two parameters (C_1 and C_2).

3 Prerequisite Skill Refresh

Solving ODEs and IVPs relies on core calculus and algebra techniques.

3.1 Indefinite Integration (Antidifferentiation)

This is the fundamental step for finding general solutions. Indefinite integration always introduces an arbitrary constant.

Basic Integration $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$, for $n \neq -1$.

Integration of $1/u$ $\int \frac{1}{u} \, du = \ln |u| + C$.

Iterative Integration To solve an n-th order ODE by direct integration, you must integrate n times, introducing a new, independent constant at each step.

Example of Iterative Integration (to solve $\ddot{x} = 3t$):

$$\int 3t \, dt = \frac{3t^2}{2} + C_1 \tag{1}$$

$$\int \left(\frac{3t^2}{2} + C_1 \right) dt = \frac{3t^3}{6} + C_1 t + C_2 = \frac{t^3}{2} + C_1 t + C_2 \tag{2}$$

3.2 Algebraic Manipulation of Constants

You must be comfortable combining and renaming arbitrary constants, especially with logarithms and exponentials.

Combining Exponents $e^{A+B} = e^A \cdot e^B$.

Consolidating Parameters This is a critical step for simplifying solutions.

Example of Consolidating Constants:

$$\begin{aligned}
\ln|y| &= f(x) + C_1 \\
e^{\ln|y|} &= e^{f(x)+C_1} \\
|y| &= e^{C_1}e^{f(x)} \\
y &= (\pm e^{C_1})e^{f(x)} \\
y &= Ce^{f(x)} && \text{(where } C = \pm e^{C_1}\text{, an arbitrary non-zero constant)}
\end{aligned}$$

3.3 Solving Simultaneous Linear Equations

This skill is necessary for solving IVPs to find the values of $C_1, C_2, \ldots$.

Reminder An IVP provides constraints at a specific point. Each constraint generates a linear equation involving the unknown constants.

Example: Suppose the general solution is $x(t) = C_1 t + C_2$ and the initial conditions are $x(0) = 5$ and $x(1) = 8$.

- Use (1): $x(0) = C_1(0) + C_2 = 5 \implies C_2 = 5$.
- Use (2): $x(1) = C_1(1) + C_2 = 8$.
- Solve: $C_1 + 5 = 8 \implies C_1 = 3$.

The particular solution is $x(t) = 3t + 5$.

4 Key Examples: A Step-by-Step Walkthrough

These examples demonstrate solving higher-order ODEs and applying initial conditions.

4.1 Example 1: Solving a Second-Order ODE by Successive Integration

Problem

Find the general solution $x(t)$ of the differential equation $\ddot{x} = 3t$, where $\ddot{x} = \frac{d^2x}{dt^2}$.

Solution

This is a second-order equation, so we integrate twice, introducing a constant at each step.

$$\frac{d^2x}{dt^2} = 3t \tag{3}$$

$$\dot{x}(t) = \int 3t\,dt = \frac{3t^2}{2} + C_1 \qquad \text{(First integration)} \tag{4}$$

$$x(t) = \int \left(\frac{3t^2}{2} + C_1\right) dt = \frac{3t^3}{6} + C_1 t + C_2 \qquad \text{(Second integration)} \tag{5}$$

$$\mathbf{x(t) = \frac{t^3}{2} + C_1 t + C_2} \tag{6}$$

This is the general solution, with two parameters (C_1, C_2) as expected.

4.2 Example 2: Solving an Initial Value Problem (IVP)

Problem

Find the solution to the IVP: $\ddot{x} = 3t$ with initial conditions $x(1) = 1$ and $\dot{x}(1) = 2$.

Solution

From Example 1, we start with the general solutions for position and velocity:

$$x(t) = \frac{t^3}{2} + C_1 t + C_2$$
$$\dot{x}(t) = \frac{3t^2}{2} + C_1$$

First, apply the velocity condition $\dot{x}(1) = 2$:

$$\begin{aligned} \dot{x}(1) = \frac{3(1)^2}{2} + C_1 &= 2 \\ 1.5 + C_1 &= 2 \\ C_1 &= 0.5 \end{aligned}$$

Next, apply the position condition $x(1) = 1$, using our new value $C_1 = 0.5$:

$$\begin{aligned} x(1) = \frac{(1)^3}{2} + (0.5)(1) + C_2 &= 1 \\ 0.5 + 0.5 + C_2 &= 1 \\ 1.0 + C_2 &= 1 \\ C_2 &= 0 \end{aligned}$$

Substitute $C_1 = 1/2$ and $C_2 = 0$ back into the general solution:

$$\mathbf{x(t) = \frac{t^3}{2} + \frac{t}{2}}$$

This is the unique particular solution for the IVP.

4.3 Example 3: Solving a Separable Equation and Checking for Lost Solutions

Problem

Find all solutions of the ODE $y' = 2x(1-y)^2$.

Solution

Step 1: Check for constant solutions. Set $y = a$ (a constant), which means $y' = 0$.

$$0 = 2x(1-a)^2$$

For this to be true for all x, we must have $(1-a)^2 = 0$, so $a = 1$. Thus, $\mathbf{y(x) = 1}$ is a valid constant solution.

Step 2: Find non-constant solutions by separation. Assuming $y \neq 1$, we can divide by $(1-y)^2$.

$$\begin{aligned}
\frac{dy}{dx} &= 2x(1-y)^2 \\
\frac{dy}{(1-y)^2} &= 2x\,dx && \text{(Separate variables)} \\
\int \frac{dy}{(1-y)^2} &= \int 2x\,dx && \text{(Integrate both sides)} \\
\frac{1}{1-y} &= x^2 + C && \text{(LHS: } u = 1-y, du = -dy\text{)} \\
1 - y &= \frac{1}{x^2+C} && \text{(Solve for } y\text{)} \\
y(x) &= 1 - \frac{1}{x^2+C}
\end{aligned}$$

Step 3: State the full solution set. The solutions consist of the family of functions $\mathbf{y(x) = 1 - \frac{1}{x^2+C}}$ and the singular constant solution $\mathbf{y(x) = 1}$.

5 Conceptual Understanding

The Big Picture: Why We Use IVPs

The general solution of an ODE (e.g., $y = x^2 + C$) is a blueprint describing all possible states a system could be in (a family of curves). In reality, a physical system begins at a specific state.

The initial conditions (ICs) of an IVP (e.g., $y(x_0)$ and $y'(x_0)$) are data points specifying this starting state. Applying these ICs fixes the arbitrary constants ($C_1, C_2, \dots$), selecting the single, unique curve from the infinite family that matches the real-world initial state. This unique function is the particular solution.

Conceptual Insight: Order Determines Required ICs

For an ODE of order n, we require n independent initial conditions to determine the n arbitrary constants uniquely.

- **First-Order ODE** (e.g., $y' = 2x$): Needs 1 IC (e.g., $y(x_0) = y_0$) to find C_1.
- **Second-Order ODE** (e.g., $\ddot{x} = 3t$): Needs 2 ICs (e.g., $x(t_0) = x_0$ and $\dot{x}(t_0) = v_0$) to find C_1 and C_2.

6 Common Mistakes to Avoid

- **Forgetting or Mismanaging Constants:** Every time an indefinite integral is performed, a new arbitrary constant must be introduced. Forgetting these, or improperly consolidating them (like dropping the $\pm$ from $\ln|y|$ solutions), will lead to an incomplete solution.
- **Losing Constant Solutions:** When using separation of variables $\frac{dy}{g(y)} = f(x)dx$, if $g(y) = 0$ for a constant value $y = a$, that constant solution $y = a$ will be lost during division. It must be verified separately against the original ODE and noted in the final solution set.
- **Mixing Variables:** Failure to completely separate terms (e.g., having an x term in $\int M(y)dy$) will make the problem unsolvable by this method.

7 Summary & What's Next

Key Takeaways

- The **Order** of an ODE is the rank of its highest derivative, and this order dictates the number of arbitrary **parameters** in the **General Solution**.
- An **Initial Value Problem (IVP)** uses auxiliary conditions to find the precise values of these parameters, yielding a unique **Particular Solution**.
- **Separable Equations** ($\frac{dy}{dx} = f(x)g(y)$) are solved by separating and integrating, but always require checking for constant solutions lost during division.

Next Session

The next topics will likely explore standard forms of first-order equations that are not immediately separable, beginning with:

- Homogeneous Equations

Analogy for Initial Value Problems

Think of the **General Solution** as a map of every possible train track a train could take. The **Initial Value Problem (IVP)** is like marking a specific station (the initial condition $x(t_0)$) on the map and telling the engineer the exact speed and direction (the initial derivative $\dot{x}(t_0)$). Solving the IVP forces you to pick the single track (the **Particular Solution**) that passes through that starting station with precisely the required initial velocity.