

# Lecture Notes: Differential Equations - Session 10

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## 1 Context: Where This Fits In

The lecture began with a recap of the previous session's topic: **Bernoulli Differential Equations**. These are non-linear first-order ODEs of the form  $\frac{dy}{dx} + P(x)y = Q(x)y^\alpha$  that can be linearized using the substitution  $v = y^{1-\alpha}$ .

The primary focus of this session (Session 10) is the introduction of **Exact Differential Equations**. Unlike previous methods (Separable, Linear, or Bernoulli) which rely on algebraic manipulation or specific substitutions, Exact Equations rely on the properties of multivariable calculus—specifically, the total differential of a potential function  $F(x, y)$ .

## 2 Core Concepts & Definitions

This session establishes the criteria and solution method for equations that can be derived from a potential function.

### 2.1 Exact Differential Equations

A first-order differential equation written in the differential form:

$$M(x, y)dx + N(x, y)dy = 0 \quad (\text{S10.1})$$

is called **exact** if there exists a scalar function  $F(x, y)$  (called the Fundamental or Potential Function) whose total differential matches the equation.

### 2.2 The Total Differential

The total differential of a function  $F(x, y)$  is defined as:

$$dF(x, y) = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy \quad (\text{S10.2})$$

For the ODE (S10.1) to be exact, we must have:

$$M(x, y) = \frac{\partial F}{\partial x} \quad \text{and} \quad N(x, y) = \frac{\partial F}{\partial y}$$

### 2.3 Criterion for Exactness (Theorem)

A necessary and sufficient condition for the equation  $Mdx + Ndy = 0$  to be exact on a simply connected region is that the mixed partial derivatives are equal:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad (\text{S10.3})$$

### 2.4 The General Solution

If the equation is exact, the general solution is defined implicitly by the level curves of the potential function:

$$F(x, y) = C \quad (\text{S10.4})$$

## 3 Prerequisite Skill Refresh

Solving exact equations requires proficiency in two inverse operations from multivariable calculus.

### 3.1 Partial Differentiation

To verify exactness, one must calculate partial derivatives.

- $\frac{\partial M}{\partial y}$ : Differentiate  $M$  with respect to  $y$ , treating  $x$  as a constant.
- $\frac{\partial N}{\partial x}$ : Differentiate  $N$  with respect to  $x$ , treating  $y$  as a constant.

**Example:** If  $F(x, y) = 3x^2y - y^3x$ :

$$\frac{\partial F}{\partial x} = 6xy - y^3 \quad \text{and} \quad \frac{\partial F}{\partial y} = 3x^2 - 3y^2x$$

$$\frac{\partial}{\partial y}(6xy - y^3) = 6x - 3y^2 \quad \text{and} \quad \frac{\partial}{\partial x}(3x^2 - 3y^2x) = 6x - 3y^2$$

### 3.2 Partial Integration

To find  $F(x, y)$ , one must integrate the partial derivatives.

- Integrating  $M = \partial F / \partial x$  with respect to  $x$  introduces an arbitrary constant that is a function of  $y$ :

$$F(x, y) = \int M(x, y)dx + g(y)$$

- The unknown function  $g(y)$  is determined by differentiating this result with respect to  $y$  and comparing it to  $N(x, y)$ .

## 4 Key Examples: A Step-by-Step Walkthrough

### 4.1 Example 1: Solving an Exact Differential Equation

**Problem:** Solve the differential equation:

$$(4xy - y^3)dx + (2x^2 - 3y^2x + 4y)dy = 0$$

**Step 1: Identify M and N**

$$M(x, y) = 4xy - y^3$$

$$N(x, y) = 2x^2 - 3y^2x + 4y$$

**Step 2: Test the Exactness Criterion** Calculate the mixed partial derivatives:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(4xy - y^3) = 4x - 3y^2$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(2x^2 - 3y^2x + 4y) = 4x - 3y^2$$

Since  $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ , the equation is exact.

**Step 3: Integrate M to find F(x,y)**

$$F(x, y) = \int M dx + g(y) = \int (4xy - y^3) dx + g(y)$$

$$F(x, y) = 2x^2y - xy^3 + g(y)$$

**Step 4: Determine g(y)** Differentiate the result from Step 3 with respect to  $y$  and set it equal to  $N(x, y)$ :

$$\frac{\partial F}{\partial y} = 2x^2 - 3xy^2 + g'(y)$$

Set equal to  $N$ :

$$2x^2 - 3xy^2 + g'(y) = 2x^2 - 3y^2x + 4y$$

Cancel common terms:

$$g'(y) = 4y$$

Integrate to find  $g(y)$ :

$$g(y) = \int 4y dy = 2y^2$$

**Step 5: Write the General Solution** Substitute  $g(y)$  back into  $F(x, y)$  and set equal to  $C$ :

$$2x^2y - xy^3 + 2y^2 = C$$

## 4.2 Example 2: Bernoulli Equation Review

**Problem:** Solve the differential equation  $2xyy' = x^2 + y^2$  (assuming  $x > 0$ ).

**Step 1: Convert to Standard Bernoulli Form** Divide by  $2xy$ :

$$y' = \frac{x}{2y} + \frac{y}{2x} \implies y' - \frac{1}{2x}y = \frac{x}{2}y^{-1}$$

Identify  $\alpha = -1$ .

**Step 2: Substitution** Let  $v = y^{1-\alpha} = y^2$ . Then  $v' = 2yy'$ . Multiply the standard ODE by  $2y$ :

$$2yy' - \frac{1}{x}y^2 = x$$

Substitute  $v$  and  $v'$ :

$$v' - \frac{1}{x}v = x$$

**Step 3: Solve Linear ODE** Integrating Factor  $\mu(x) = e^{\int -1/x dx} = 1/x$ .

$$\frac{d}{dx} \left( \frac{1}{x}v \right) = 1$$

$$\frac{1}{x}v = x + C \implies v = x^2 + Cx$$

**Step 4: Back-Substitute**

$$y(x) = \sqrt{x^2 + Cx}$$

## 5 Conceptual Understanding

**Why the Exact Method Works:** The method reverses the chain rule. If a potential function  $F(x, y) = C$  exists, its total derivative is zero. The differential equation  $Mdx + Ndy = 0$  is simply the expanded form of  $dF = 0$ . The criterion  $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$  (Clairaut's Theorem) guarantees that the vector field  $\langle M, N \rangle$  is conservative, meaning a potential function  $F$  exists.

**Geometric Interpretation:** The solution  $F(x, y) = C$  describes the **level curves** (contours) of the surface  $z = F(x, y)$ . The solution curves are the paths along which the function's value remains constant.

## 6 Common Mistakes to Avoid

- **Incorrect Partial Differentiation:** A common error is differentiating  $x$  terms when computing  $\frac{\partial}{\partial y}$  (or vice versa). Remember to treat the other variable as a constant coefficient.
- **Forgetting the Function of Integration:** When integrating  $M$  with respect to  $x$ , you must add  $+g(y)$ , not just  $+C$ . If you add a simple constant, you will fail to account for terms in  $N$  that depend solely on  $y$ .
- **Final Solution Format:** The solution is the equation  $F(x, y) = C$ . Do not stop at finding  $F(x, y)$ .

## 7 Summary & What's Next

### Key Takeaways:

1. An ODE is exact if  $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ .
2. The solution is found by integrating  $M$  w.r.t.  $x$  and  $N$  w.r.t.  $y$  and matching the terms to form  $F(x, y) = C$ .

**Next Session Topic:** The course moves to **Second Order Linear Equations**. The upcoming notes cover homogeneous equations with constant coefficients ( $m\ddot{x} + b\dot{x} + Kx = 0$ ) and the characteristic equation method ( $mr^2 + br + K = 0$ ).