Lecture Notes: Differential Equations (Session 17)

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1 Context: Where This Fits In

In the previous session (Session 16), we introduced the Fourier Series, learning that a periodic function $f(t)$ can be represented as an infinite sum of sine and cosine functions. At that stage, we accepted the formulas for calculating the coefficients (a_n and b_n) on faith.

In this session (Session 17), we establish the mathematical foundation for *why* those formulas work. The mechanism relies on a property called **orthogonality**, which allows trigonometric functions to act like perpendicular coordinate axes for infinite-dimensional function spaces.

2 Core Concepts & Definitions

2.1 Orthogonal Functions

Two functions, $u(t)$ and $v(t)$, are said to be orthogonal on an interval $[a, b]$ if the integral of their product equals zero.

$$\int_a^b u(t)v(t)\,dt = 0$$

Analogy: Consider vectors in physics. Two vectors are perpendicular (orthogonal) if their dot product is zero ($\vec{A} \cdot \vec{B} = 0$). In function space, the integral acts as the dot product. If this "function dot product" is zero, the functions are effectively "perpendicular" to each other.

2.2 Orthogonal Set (or System)

A collection of functions is an **orthogonal set** if every distinct pair of functions in the list is orthogonal to one another.

- **Example:** The standard basis of Euclidean space $(\hat{i}, \hat{j}, \hat{k})$ is an orthogonal set.
- **Function Space Example:** The set of trigonometric functions $\{1, \cos(\frac{n\pi}{L}t), \sin(\frac{n\pi}{L}t)\}$ forms an orthogonal set on the interval $[-L, L]$.

2.3 The Inner Product

The quantity $\int_a^b f(x)g(x)\,dx$ is formally defined as the **inner product** of the functions f and g, often denoted as (f, g). This notation reinforces the connection to the vector dot product.

2.4 Norm of a Function

The "length" or "size" of a function on an interval is defined using the inner product, analogous to the magnitude of a vector.

$$||f|| = \sqrt{\int_a^b [f(x)]^2 \, dx}$$

Normalizing a function (creating a "unit function") involves dividing the function by its norm.

3 Prerequisite Skill Refresh

To verify orthogonality and derive coefficients, specific integration techniques are required.

3.1 Skill 1: Product-to-Sum Identities

Integrals of the form $\int \cos(mt)\cos(nt)\,dt$ cannot be solved directly. We must convert multiplication into addition using trigonometric identities:

$$\cos(A)\cos(B) = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$$
$$\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$
$$\sin(A)\cos(B) = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$$

3.2 Skill 2: Evaluating Definite Trig Integrals

Recall that the sine of any integer multiple of π is zero:

$$\sin(n\pi) = 0 \quad \text{for } n = 1, 2, 3, \dots$$

3.3 Skill 3: Integration of Odd and Even Functions

[Image of even and odd function symmetry]

Exploiting symmetry is a massive shortcut in Fourier analysis.

- **Even Function:** Symmetric across the y-axis (e.g., $x^2, \cos x$).

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

- **Odd Function:** Symmetric about the origin (e.g., $x^3, \sin x$).

$$\int_{-a}^{a} f(x)\,dx = 0$$

Note: The product of an Even function and an Odd function is always **Odd**.

4 Key Examples: A Step-by-Step Walkthrough

4.1 Example 1: Checking Orthogonality of Mixed Trig Functions

Problem: Verify that $u(t) = \cos\left(\frac{m\pi}{L}t\right)$ and $v(t) = \sin\left(\frac{n\pi}{L}t\right)$ are orthogonal on $[-L, L]$.

Step 1: Set up the orthogonality integral.

$$I = \int_{-L}^{L} \cos\left(\frac{m\pi}{L}t\right) \sin\left(\frac{n\pi}{L}t\right)\, dt$$

Step 2: Analyze symmetry.

- Cosine is Even: $\cos(-x) = \cos(x)$.
- Sine is Odd: $\sin(-x) = -\sin(x)$.
- Product: Even $\times$ Odd = **Odd**.

Step 3: Evaluate. The integral of an odd function over the symmetric interval $[-L, L]$ is zero.

$$I = 0$$

Conclusion: The functions are orthogonal for any integers m and n.

4.2 Example 2: Calculating the Norm (Squared) of Cosine

Problem: Evaluate $I = \int_{-L}^{L} \cos^2\left(\frac{n\pi}{L}t\right)\, dt$ for $n \neq 0$.

Step 1: Apply half-angle identity.

$$I = \int_{-L}^{L} \frac{1 + \cos(\frac{2n\pi}{L}t)}{2}\, dt$$

Step 2: Split and Integrate.

$$I = \frac{1}{2}\int_{-L}^{L} 1\, dt + \frac{1}{2}\int_{-L}^{L} \cos\left(\frac{2n\pi}{L}t\right)\, dt$$

The first integral yields $\frac{1}{2}[L - (-L)] = L$. The second integral is the integration of a cosine over full periods, which sums to zero.

$$I = L + 0 = L$$

Conclusion: The squared norm is L. This explains the factor of $1/L$ in the Fourier coefficient formula a_n.

4.3 Example 3: Deriving the Fourier Coefficient a_n

Problem: Derive the formula for a_m in the Fourier series expansion.

Step 1: Multiply by the "Probe" Function. Multiply the Fourier series equation by $\cos\left(\frac{m\pi}{L}t\right)$:

$$f(t)\cos\left(\frac{m\pi}{L}t\right) = \left[\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi}{L}t\right) + \dots\right)\right]\cos\left(\frac{m\pi}{L}t\right)$$

Step 2: Integrate over $[-L, L]$. Apply the integral to both sides. By the property of orthogonality:

- The integral of the a_0 term vanishes.
- The integrals of all sine terms vanish (Example 1).
- The integrals of all cosine terms vanish, **except** when $n = m$.

Step 3: Solve for a_m. The infinite sum collapses to a single term:

$$\int_{-L}^{L} f(t)\cos\left(\frac{m\pi}{L}t\right)\,dt = a_m \int_{-L}^{L} \cos^2\left(\frac{m\pi}{L}t\right)\,dt$$

Using the result from Example 2 (the integral on the right is L):

$$\int_{-L}^{L} f(t)\cos\left(\frac{m\pi}{L}t\right)\,dt = a_m(L)$$

$$a_m = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{m\pi}{L}t\right)\,dt$$

5 Conceptual Understanding

[Image of prism splitting light analogy]

The "Filter" Analogy: The method of Orthogonal Functions works like a prism or a filter. Just as a prism splits white light into distinct colors, orthogonality allows us to decompose a complex function $f(t)$ into pure frequencies.

When we multiply a series by a specific "probe" function (like $\cos(mt)$) and integrate, orthogonality acts as a sieve. Every term in the infinite series integrates to zero except the one term that matches the probe. This isolates the specific coefficient we need.

6 Common Mistakes to Avoid

- **The $a_0/2$ Factor:** Do not forget that the constant term is $\frac{a_0}{2}$. This adjustment allows the general formula for a_n (dividing by L) to remain valid for $n = 0$. Without the $1/2$, the $n = 0$ case would require dividing by $2L$.
- **Indices Confusion:** In derivations, distinguish between the summation index (n) and the fixed probe index (m). The integral is non-zero only when $n = m$.
- **Interval Limits:** These formulas rely on symmetric intervals $[-L, L]$. While orthogonality holds for $[0, 2L]$ or $[0, \pi]$, one must adjust integration limits carefully if the problem setup changes.

7 Summary & What's Next

Key Takeaways:

- **Orthogonality:** Defined by $\int uv = 0$. It is the "engine" of Fourier Series.
- **Orthogonal System:** The set $\{1, \cos, \sin\}$ is orthogonal on $[-L, L]$.
- **Coefficient Extraction:** We isolate coefficients by multiplying by a trig function and integrating; orthogonality ensures only the matching term survives.

Next Session (Session 18): We will examine the **Convergence of Fourier Series**, specifically Dirichlet's Theorem, which outlines the conditions (continuity, boundedness) required for the infinite series to accurately equal $f(t)$.