Lecture Notes: Differential Equations - Session 13

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1 Context: Where This Fits In

In Session 12, we covered the solution of non-homogeneous linear ODEs with constant coefficients ($m\ddot{x} + b\dot{x} + Kx = Be^{at}$) using the **Exponential Response Formula (ERF)**. We also introduced the concept of resonance ($p(a) = 0$).

Session 13 completes the development of methods for finding particular solutions (x_p). The focus shifts from simple exponential forcing to the handling of trigonometric inputs ($\cos(\omega t)$ or $\sin(\omega t)$). This connects directly to the ERF by using the mathematical identity:

$$\cos(\omega t) = \mathrm{Re}(e^{i\omega t})$$

We replace the real trigonometric input with a complex exponential input ($e^{i\omega t}$), apply the ERF using a complex exponent $a = i\omega$, solve the complex ODE, and then take the real part of the result to find the physical solution $x_p(t)$.

2 Core Concepts & Definitions

2.1 The Exponential Response Formula (ERF)

A formula to find the particular solution x_p for the ODE $p(D)x = Be^{at}$. If $p(a) \neq 0$, the solution is:

$$\boxed{x_p(t) = \frac{Be^{at}}{p(a)}} \tag{1}$$

where $p(r) = mr^2 + br + K$ is the characteristic polynomial.

2.2 Complex Replacement for Trigonometric Input

To solve $m\ddot{x} + b\dot{x} + Kx = B\cos(\omega t)$, we define a complex auxiliary equation:

$$m\ddot{Z} + b\dot{Z} + KZ = Be^{i\omega t}$$

We use the ERF with the complex exponent $a = i\omega$ to find the complex particular solution $Z_p(t)$. The required real solution is:

$$x_p(t) = \text{Re}(Z_p) \tag{2}$$

Note: If the input is $B\sin(\omega t)$, we would take $Im(Z_p)$.

2.3 Generalized Exponential Response Formula (Resonance)

This formula provides x_p when the input exponent a is a root of the characteristic polynomial $p(r)$ (i.e., $p(a) = 0$, the **resonance** case). If a is a root of multiplicity s (meaning $p(a) = p'(a) = \cdots = p^{(s-1)}(a) = 0$, but $p^{(s)}(a) \neq 0$), the particular solution is:

$$\boxed{x_p = \frac{Bt^s e^{at}}{p^{(s)}(a)}} \tag{3}$$

3 Prerequisite Skill Refresh

The methods of Session 13 rely heavily on two core skills: differentiating polynomials (for the Generalized ERF) and complex number manipulation (for trigonometric inputs).

3.1 Complex Number Manipulation: Polar Form

[Image of complex plane polar coordinates]

When solving for x_p with a trigonometric input, the result involves a complex fraction $\frac{B}{p(i\omega)}$. To extract the real part and identify the amplitude and phase shift, the denominator $p(i\omega) = \alpha + i\beta$ must be converted to polar form $W = ||W||e^{i\phi}$.

- **Magnitude ($||W||$):** Represents the amplitude of the resulting oscillation.

$$||W|| = \sqrt{\alpha^2 + \beta^2} \tag{4}$$

- **Phase Angle (ϕ):** Represents the time delay or phase shift relative to the input.

$$\phi = \arctan\left(\frac{\beta}{\alpha}\right) \tag{5}$$

3.2 Differentiation of Polynomials (For Generalized ERF)

To determine the multiplicity s of a root a, we calculate derivatives of $p(r)$ until the result is non-zero at a.

- Reminder: The derivative of r^n is nr^{n-1}.

4 Key Examples: A Step-by-Step Walkthrough

4.1 Example 1: Sinusoidal Forcing with Resonance (Complex Replacement)

Problem: Find a particular solution (x_p) for the ODE:

$$\ddot{x} + 2\dot{x} + 2x = e^{-t}\cos t$$

Step 1: Perform Complex Replacement Using Euler's formula, we note that $e^{-t}\cos t = \mathrm{Re}(e^{-t}e^{it}) = \mathrm{Re}(e^{(-1+i)t})$. We define the complex equation in Z:

$$\ddot{Z} + 2\dot{Z} + 2Z = e^{(-1+i)t}$$

Step 2: Identify Parameters Characteristic Polynomial: $p(r) = r^2 + 2r + 2$. Input Coefficient: $B = 1$. Complex Exponent: $a = -1 + i$.

Step 3: Check for Resonance ($p(a) \stackrel{?}{=} 0$) Evaluate $p(r)$ at $a = -1 + i$:

$$\begin{aligned} p(-1+i) &= (-1+i)^2 + 2(-1+i) + 2 \\ &= (1 - 2i + i^2) - 2 + 2i + 2 \\ &= (1 - 2i - 1) + 2i = 0 \end{aligned}$$

Since $p(a) = 0$, resonance occurs. We must use the Generalized ERF (Eq. 3).

Step 4: Apply Generalized ERF Calculate the derivative $p'(r) = 2r + 2$ and evaluate at a:

$$p'(-1+i) = 2(-1+i) + 2 = -2 + 2i + 2 = 2i$$

Since $p'(a) = 2i \neq 0$, the multiplicity is $s = 1$.

$$Z_p = \frac{Bt^1 e^{at}}{p'(a)} = \frac{te^{(-1+i)t}}{2i}$$

Step 5: Extract Real Solution To find $x_p = \mathrm{Re}(Z_p)$, we separate real and imaginary parts:

$$\begin{aligned} Z_p &= \frac{te^{-t}e^{it}}{2i} = \frac{te^{-t}}{2}\left(\frac{\cos t + i\sin t}{i}\right) \\ &= \frac{te^{-t}}{2}(-i\cos t + \sin t) \quad \text{(multiplying top/bottom by } -i) \\ &= \frac{te^{-t}}{2}(\sin t - i\cos t) \end{aligned}$$

Taking the real part:

$$\boxed{x_p(t) = \frac{te^{-t}}{2}\sin t}$$

4.2 Example 2: General Application (Non-Resonant)

Problem: Find the steady-state solution to:

$$\ddot{x} + 3\dot{x} + 10x = 20\cos(2t)$$

Step 1: Identify Parameters $p(r) = r^2 + 3r + 10$. Input Frequency $\omega = 2$, Coefficient $B = 20$. Complex Exponent $a = i\omega = 2i$.

Step 2: Evaluate $p(i\omega)$ and convert to Polar Form

$$\begin{aligned} W = p(2i) &= (2i)^2 + 3(2i) + 10 \\ &= -4 + 6i + 10 = 6 + 6i \end{aligned}$$

Calculate Magnitude and Phase (Eq. 4 and 5):

$$\begin{aligned} ||W|| &= \sqrt{6^2 + 6^2} = \sqrt{72} = 6\sqrt{2} \\ \phi &= \arctan\left(\frac{6}{6}\right) = \frac{\pi}{4} \end{aligned}$$

Step 3: Write Particular Solution Using the derived formula for non-resonant trigonometric input:

$$x_p(t) = \frac{B}{||p(i\omega)||}\cos(\omega t - \phi) \tag{6}$$

Substituting our values:

$$x_p(t) = \frac{20}{6\sqrt{2}}\cos\left(2t - \frac{\pi}{4}\right)$$

Simplifying the amplitude $\frac{20}{6\sqrt{2}} = \frac{5\sqrt{2}}{3}$:

$$\boxed{x_p(t) = \frac{5\sqrt{2}}{3}\cos\left(2t - \frac{\pi}{4}\right)}$$

5 Conceptual Understanding

Why the Method Works: The method relies on Euler's Formula ($B\cos(\omega t) = \mathrm{Re}(Be^{i\omega t})$). Since linear ODEs with constant coefficients map complex inputs to complex outputs linearly, the real part of the complex solution Z_p corresponds exactly to the real input. This transforms a calculus problem (solving differential equations) into an algebraic problem ($Z_p = \frac{Be^{i\omega t}}{p(i\omega)}$).

What the Solution Represents: The solution $x_p(t)$ represents the **Steady State Solution**. In stable systems, the homogeneous solution x_h decays to zero over time. The remaining x_p describes the system oscillating at the input frequency ω, but with a modified amplitude and a phase shift ϕ caused by the system's damping and inertia.

[Image of transient versus steady state solution]

6 Common Mistakes to Avoid

- **Complex Arithmetic Errors:** Errors are common when simplifying powers of i (e.g., forgetting $i^2 = -1$) or when substituting $i\omega$ into the polynomial.
- **Polar Conversion:** When calculating $\phi = \arctan(b/a)$, ensure you check the quadrant if a or b are negative.
- **Missing the Resonance Check:** Always check if $p(a) = 0$. If it is zero, the standard ERF (Eq. 1) fails (division by zero), and the Generalized ERF (Eq. 3) must be used.
- **Generalized ERF Multiplicity:** For higher-order resonance, ensure you find the correct multiplicity s (where the derivative stops vanishing).
- **Real vs. Imaginary Part:** If the input is $\cos(\omega t)$, take $\mathrm{Re}(Z_p)$. If the input is $\sin(\omega t)$, take $\mathrm{Im}(Z_p)$.

7 Summary & What's Next

Key Takeaways:

1. To solve for $B\cos(\omega t)$, use Complex Replacement ($Be^{i\omega t}$), apply ERF with $a = i\omega$, and take the real part.
2. The non-resonant solution is $x_p = \frac{B}{||p(i\omega)||}\cos(\omega t - \phi)$.
3. If $p(a) = 0$ (resonance), use the Generalized ERF: $x_p = \frac{Bt^s e^{at}}{p^{(s)}(a)}$.

Next Session: Session 14 will focus on the **Criteria for System Stability**. We will analyze the long-term behavior of systems to determine the conditions under which $\mathrm{Re}(r_i) < 0$ for all characteristic roots, ensuring the system is permanent (stable).