

Lecture Notes: Differential Equations - Session 10

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1 Context: Where This Fits In

The lecture began with a recap of the previous session's topic: **Bernoulli Differential Equations**. These are non-linear first-order ODEs of the form $\frac{dy}{dx} + P(x)y = Q(x)y^\alpha$ that can be linearized using the substitution $v = y^{1-\alpha}$.

The primary focus of this session (Session 10) is the introduction of **Exact Differential Equations**. Unlike previous methods (Separable, Linear, or Bernoulli) which rely on algebraic manipulation or specific substitutions, Exact Equations rely on the properties of multivariable calculus—specifically, the total differential of a potential function $F(x, y)$.

2 Core Concepts & Definitions

This session establishes the criteria and solution method for equations that can be derived from a potential function.

2.1 Exact Differential Equations

A first-order differential equation written in the differential form:

$$M(x, y)dx + N(x, y)dy = 0 \tag{S10.1}$$

is called **exact** if there exists a scalar function $F(x, y)$ (called the Fundamental or Potential Function) whose total differential matches the equation.

2.2 The Total Differential

The total differential of a function $F(x, y)$ is defined as:

$$dF(x, y) = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy \tag{S10.2}$$

For the ODE (S10.1) to be exact, we must have:

$$M(x, y) = \frac{\partial F}{\partial x} \quad \text{and} \quad N(x, y) = \frac{\partial F}{\partial y}$$

2.3 Criterion for Exactness (Theorem)

A necessary and sufficient condition for the equation $Mdx + Ndy = 0$ to be exact on a simply connected region is that the mixed partial derivatives are equal:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad (\text{S10.3})$$

2.4 The General Solution

If the equation is exact, the general solution is defined implicitly by the level curves of the potential function:

$$F(x, y) = C \quad (\text{S10.4})$$

3 Prerequisite Skill Refresh

Solving exact equations requires proficiency in two inverse operations from multivariable calculus.

3.1 Partial Differentiation

To verify exactness, one must calculate partial derivatives.

- $\frac{\partial M}{\partial y}$: Differentiate M with respect to y , treating x as a constant.
- $\frac{\partial N}{\partial x}$: Differentiate N with respect to x , treating y as a constant.

Example: If $F(x, y) = 3x^2y - y^3x$:

$$\frac{\partial F}{\partial x} = 6xy - y^3 \quad \text{and} \quad \frac{\partial F}{\partial y} = 3x^2 - 3y^2x$$

$$\frac{\partial}{\partial y}(6xy - y^3) = 6x - 3y^2 \quad \text{and} \quad \frac{\partial}{\partial x}(3x^2 - 3y^2x) = 6x - 3y^2$$

3.2 Partial Integration

To find $F(x, y)$, one must integrate the partial derivatives.

- Integrating $M = \partial F / \partial x$ with respect to x introduces an arbitrary constant that is a function of y :

$$F(x, y) = \int M(x, y)dx + g(y)$$

- The unknown function $g(y)$ is determined by differentiating this result with respect to y and comparing it to $N(x, y)$.

4 Key Examples: A Step-by-Step Walkthrough

4.1 Example 1: Solving an Exact Differential Equation

Problem: Solve the differential equation:

$$(4xy - y^3)dx + (2x^2 - 3y^2x + 4y)dy = 0$$

Step 1: Identify M and N

$$M(x, y) = 4xy - y^3$$

$$N(x, y) = 2x^2 - 3y^2x + 4y$$

Step 2: Test the Exactness Criterion Calculate the mixed partial derivatives:

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(4xy - y^3) = 4x - 3y^2 \\ \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2x^2 - 3y^2x + 4y) = 4x - 3y^2\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Step 3: Integrate M to find F(x,y)

$$F(x, y) = \int Mdx + g(y) = \int (4xy - y^3)dx + g(y)$$

$$F(x, y) = 2x^2y - xy^3 + g(y)$$

Step 4: Determine g(y) Differentiate the result from Step 3 with respect to y and set it equal to $N(x, y)$:

$$\frac{\partial F}{\partial y} = 2x^2 - 3xy^2 + g'(y)$$

Set equal to N :

$$2x^2 - 3xy^2 + g'(y) = 2x^2 - 3y^2x + 4y$$

Cancel common terms:

$$g'(y) = 4y$$

Integrate to find $g(y)$:

$$g(y) = \int 4ydy = 2y^2$$

Step 5: Write the General Solution Substitute $g(y)$ back into $F(x, y)$ and set equal to C :

$$2x^2y - xy^3 + 2y^2 = C$$

4.2 Example 2: Bernoulli Equation Review

Problem: Solve the differential equation $2xyy' = x^2 + y^2$ (assuming $x > 0$).

Step 1: Convert to Standard Bernoulli Form Divide by $2xy$:

$$y' = \frac{x}{2y} + \frac{y}{2x} \implies y' - \frac{1}{2x}y = \frac{x}{2}y^{-1}$$

Identify $\alpha = -1$.

Step 2: Substitution Let $v = y^{1-\alpha} = y^2$. Then $v' = 2yy'$. Multiply the standard ODE by $2y$:

$$2yy' - \frac{1}{x}y^2 = x$$

Substitute v and v' :

$$v' - \frac{1}{x}v = x$$

Step 3: Solve Linear ODE Integrating Factor $\mu(x) = e^{\int -1/xdx} = 1/x$.

$$\frac{d}{dx} \left(\frac{1}{x}v \right) = 1$$

$$\frac{1}{x}v = x + C \implies v = x^2 + Cx$$

Step 4: Back-Substitute

$$y(x) = \sqrt{x^2 + Cx}$$

5 Conceptual Understanding

Why the Exact Method Works: The method reverses the chain rule. If a potential function $F(x, y) = C$ exists, its total derivative is zero. The differential equation $Mdx + Ndy = 0$ is simply the expanded form of $dF = 0$. The criterion $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (Clairaut's Theorem) guarantees that the vector field $\langle M, N \rangle$ is conservative, meaning a potential function F exists.

Geometric Interpretation: The solution $F(x, y) = C$ describes the **level curves** (contours) of the surface $z = F(x, y)$. The solution curves are the paths along which the function's value remains constant.

6 Common Mistakes to Avoid

- **Incorrect Partial Differentiation:** A common error is differentiating x terms when computing $\frac{\partial}{\partial y}$ (or vice versa). Remember to treat the other variable as a constant coefficient.
- **Forgetting the Function of Integration:** When integrating M with respect to x , you must add $+g(y)$, not just $+C$. If you add a simple constant, you will fail to account for terms in N that depend solely on y .
- **Final Solution Format:** The solution is the equation $F(x, y) = C$. Do not stop at finding $F(x, y)$.

7 Summary & What's Next

Key Takeaways:

1. An ODE is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.
2. The solution is found by integrating M w.r.t. x and N w.r.t. y and matching the terms to form $F(x, y) = C$.

Next Session Topic: The course moves to **Second Order Linear Equations**. The upcoming notes cover homogeneous equations with constant coefficients ($m\ddot{x} + b\dot{x} + Kx = 0$) and the characteristic equation method ($mr^2 + br + K = 0$).