Lecture Notes: Differential Equations (Sessions 5 and 6)

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In This Session : Separable Equations, Solution Domains, and First-Order Linear ODE

1 Context: Where This Fits In

The immediate previous topic was the formal classification of ODEs, including their order, the meaning of a General Solution (a family of curves), and solving Initial Value Problems (IVPs). We were also introduced to the technique for solving Separable Equations, $\frac{dy}{dx} = f(x)g(y)$.

Today's topic moves beyond the mechanics of separation to address fundamental conceptual issues: How do we rigorously determine if an equation is separable? What is the domain where a solution to an IVP is truly valid? We will also introduce the next major, more complex category of solvable ODEs: First-Order Linear Equations.

2 Core Concepts & Definitions

This lecture focuses on formalizing the treatment of first-order equations, setting conceptual limits, and classifying linear equations.

Separable Equation (Rigor) A first-order ODE $y' = R(x, y)$ is separable if $R(x, y)$ can be factored into a product of two single-variable functions, $f(x)$ and $g(y)$. If $R(x, y)$ contains mixed terms like $x + y$, it is generally not separable.

Domain of Solution (Connected Interval) When solving an IVP, the resulting particular solution may contain points where it is undefined (e.g., a vertical asymptote). The domain of the solution is the **largest connected interval** that contains the initial point (x_0, y_0) and over which the solution $y(x)$ is defined and differentiable.

ODE Modelling The use of differential equations to mathematically describe real-world phenomena involving rates of change, such as financial interest ($\dot{x} \propto x$) or physical systems.

First-Order Linear ODE A differential equation that can be written in the Standard Form:

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{1}$$

Crucially, the dependent variable (y) and its derivatives (y') must appear only to the first power.

Homogeneous Linear ODE A linear ODE in standard form where the right-hand side, $Q(x)$ (the input or forcing term), is identically zero: $\frac{dy}{dx} + P(x)y = 0$.

Non-homogeneous Linear ODE A linear ODE where $Q(x)$ is not zero. This models systems subject to an external input, $Q(x)$.

Integral Curve (Solution Curve) The graph of a particular solution $y = y(x)$. At every point (x, y) on this curve, the slope of the tangent line (y') is equal to $R(x, y)$.

Existence and Uniqueness Theorem (Intuitive) A theorem stating that if $R(x, y)$ and its partial derivative $\frac{\partial R}{\partial y}$ are continuous in a region containing the initial point (x_0, y_0), then **one and only one** solution (integral curve) passes through that point.

Analogy for Domain of Solution

Imagine your solution is $y(x) = \frac{1}{1-x}$ and your initial condition is $y(0) = 1$. You start at $x = 0$. There is a vertical wall (asymptote) at $x = 1$. Even though the function exists for $x > 1$, your solution path cannot jump over that wall. Therefore, the largest connected interval for your solution is $(-\infty, 1)$, as that is the continuous stretch of the graph that includes $x = 0$.

3 Prerequisite Skill Refresh

This section relies on advanced interpretation of integrals and fundamental calculus definitions.

3.1 Integration of Simple Rational Functions (Calculus)

Separation of variables frequently leads to these forms.

Integration of $1/u$ $\int \frac{1}{u}\,du = \ln|u| + C$.

Integration of u^n $(n \neq -1)$ $\int \frac{1}{y^2}\,dy = \int y^{-2}\,dy = \frac{y^{-1}}{-1} + C = -\frac{1}{y} + C$.

3.2 Algebraic Manipulation and Consolidation of Arbitrary Constants

This is essential when solving separable equations involving logarithms.

Exponential Separation $e^{A+B} = e^A e^B$. This separates the variable term from the constant term.

Constant Consolidation If $\ln|y| = f(x) + C_1$, then $|y| = e^{C_1} e^{f(x)}$. We define a new constant $C = \pm e^{C_1}$ to absorb the sign and the exponential, resulting in $y = Ce^{f(x)}$.

3.3 Partial Differentiation (Calculus)

The Existence and Uniqueness Theorem hinges on the partial derivative.

Partial Derivative $\partial R/\partial y$ When taking the partial derivative with respect to y, treat the variable x as if it were a constant.

Example If $R(x, y) = x^2 y + y^3$, we treat x^2 as a constant coefficient:

$$\frac{\partial R}{\partial y} = x^2(1) + 3y^2 = x^2 + 3y^2 \tag{2}$$

Checking Continuity For the theorem to hold, this resulting partial derivative must be continuous near the initial point.

4 Key Examples: A Step-by-Step Walkthrough

These examples focus on rigorous domain analysis and the new technique for First-Order Linear Equations.

4.1 Example 1: Determining the Domain of a Particular Solution (IVP)

Problem

Solve the IVP and determine the domain of the solution:

$$y' = y^2 \quad \text{with initial condition} \quad y(0) = 1. \tag{3}$$

Solution

First, we solve the ODE using separation of variables.

$$\begin{aligned}
\frac{dy}{dx} &= y^2 \\
\frac{dy}{y^2} &= dx && \text{(Separate variables)} \\
\int y^{-2}\,dy &= \int dx && \text{(Integrate both sides)} \\
-\frac{1}{y} &= x + C && \text{(General solution, implicit)} \\
y &= -\frac{1}{x+C} && \text{(General solution, explicit)}
\end{aligned}$$

Next, apply the initial condition $y(0) = 1$ to find C.

$$1 = -\frac{1}{0+C} \implies 1 = -\frac{1}{C} \implies C = -1$$

This gives the particular solution:

$$y(x) = -\frac{1}{x-1} \quad \text{or} \quad \mathbf{y(x) = \frac{1}{1-x}}$$

Finally, determine the domain. The function $y = \frac{1}{1-x}$ has a vertical asymptote (singularity) at $x = 1$. The initial condition is at $x_0 = 0$. The largest connected interval containing $x_0 = 0$ where the solution is defined is $(-\infty, \mathbf{1})$.

4.2 Example 2: General Solution of a First-Order Homogeneous Linear ODE

Problem

Find the general solution of the homogeneous linear ODE: $\dot{x} + P(t)x = 0$, where $\dot{x} = \frac{dx}{dt}$.

Solution

This linear equation is also separable.

$$\begin{aligned}
\frac{dx}{dt} &= -P(t)x \\
\frac{dx}{x} &= -P(t)\,dt && \text{(Separate variables)} \\
\int \frac{dx}{x} &= \int -P(t)\,dt && \text{(Integrate both sides)} \\
\ln|x| &= -\int P(t)\,dt + C_1 \\
|x| &= e^{-\int P(t)\,dt + C_1} = e^{C_1} e^{-\int P(t)\,dt} && \text{(Exponentiate)} \\
x(t) &= (\pm e^{C_1}) e^{-\int P(t)\,dt} && \text{(Remove absolute value)}
\end{aligned}$$

We consolidate the parameter by letting $C = \pm e^{C_1}$ (and including the trivial solution $x = 0$, obtained when $C = 0$). The General Solution is:

$$\mathbf{x(t) = Ce^{-\int P(t)dt}}$$

4.3 Example 3: Solving a First-Order Non-Homogeneous Linear ODE

Problem

Solve the linear ODE: $\frac{dy}{dx} + \frac{1}{x}y = x^3$.

Solution

This equation is in the standard linear form $\frac{dy}{dx} + P(x)y = Q(x)$, with $P(x) = \frac{1}{x}$ and $Q(x) = x^3$.

Step 1: Calculate the Integrating Factor ($\mu(x)$).

$$\mu(x) = e^{\int P(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = |x| \tag{4}$$

We choose the simplest form, x (assuming $x > 0$ for this domain). So, $\mu(\mathbf{x}) = \mathbf{x}$.

Step 2: Multiply the standard-form ODE by $\mu(x)$.

$$x\left(\frac{dy}{dx} + \frac{1}{x}y\right) = x(x^3) \tag{5}$$

$$x\frac{dy}{dx} + y = x^4 \tag{6}$$

Step 3: Condense the left side. The left side is now the result of the product rule for $\frac{d}{dx}(\mu(x)y)$.

$$\frac{d}{dx}(xy) = x^4 \tag{7}$$

Step 4: Integrate both sides.

$$\int \frac{d}{dx}(xy)\,dx = \int x^4\,dx \tag{8}$$

$$xy = \frac{x^5}{5} + C \tag{9}$$

Step 5: State the General Solution. Solve for y:

$$\mathbf{y(x) = \frac{x^4}{5} + Cx^{-1}} \tag{10}$$

5 Conceptual Understanding

5.1 Why Does the Integrating Factor Method Work?

The integrating factor $\mu(x) = e^{\int P(x)dx}$ is precisely the function needed to transform the left side of the linear equation, $\frac{dy}{dx} + P(x)y$, into the derivative of a product.

Recall that the derivative of $\mu(x)$ is $\mu'(x) = P(x)e^{\int P(x)dx} = P(x)\mu(x)$. When we multiply the left side by $\mu(x)$, we get:

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y \tag{11}$$

Substituting $\mu'(x)$ for $\mu(x)P(x)$, this becomes:

$$\mu(x)\frac{dy}{dx} + \mu'(x)y \tag{12}$$

By the product rule, this is exactly $\frac{d}{dx}(\mu(x)y)$. Once the left side is in this form, we can solve the equation $\frac{d}{dx}(\mu(x)y) = \mu(x)Q(x)$ by simple integration.

5.2 What Does the Solution Represent? (Linearity and Superposition)

The general solution of a non-homogeneous linear equation is the sum of two components:

$$\mathbf{y(x) = y_g(x) + y_p(x)} \tag{13}$$

$y_g(x)$ **(Homogeneous Solution)** The general solution of the reduced homogeneous equation ($Q(x) = 0$). This part contains the arbitrary constant C and represents the system's natural or transient behavior.

$y_p(x)$ **(Particular Solution)** Any specific solution of the full non-homogeneous equation. This part represents the system's response to the external input $Q(x)$.

This structure is a consequence of the Principle of Superposition, which applies only to linear equations.

6 Common Mistakes to Avoid

- **Standard Form Failure:** The integrating factor method *only* works if the ODE is first in standard form: $\frac{dy}{dx} + P(x)y = Q(x)$. If the y' term has a coefficient (e.g., $x\frac{dy}{dx} \dots$), you must divide the entire equation by it before identifying $P(x)$.

- **Forgetting the Integral:** The integrating factor is $\mu(x) = e^{\int P(x)dx}$. Forgetting the integral sign (using $e^{P(x)}$) or the exponential (using $\int P(x)dx$) is a common error.

- **Mechanical Memorization:** Do not just memorize the final complex formula for the solution. It is much safer and more effective to remember the *procedure*: (1) Find $\mu(x)$, (2) Multiply by $\mu(x)$, (3) Condense the left side to $\frac{d}{dx}(\mu y)$, (4) Integrate.

7 Summary & What's Next

7.1 Key Takeaways

- An ODE $y' = R(x, y)$ is **separable** if $R(x, y)$ can be factored into $f(x)g(y)$, which excludes simple sums like $x + y$.
- The **domain** of a solution to an IVP is the largest continuous interval containing the initial point (x_0, y_0) where the solution function is defined.
- **First-order linear ODEs** ($\frac{dy}{dx} + P(x)y = Q(x)$) are solved using the **Integrating Factor** $\mu(x) = e^{\int P(x)dx}$, which turns the left side into $\frac{d}{dx}(\mu y)$.

7.2 Next Session

We will formalize the Integrating Factor Method and the Principle of Superposition. Following that, we will move on to the next major standard type of first-order equation:

- **Homogeneous Equations**, which are not separable but can be made separable using the substitution $y = vx$.