Lecture Notes: Differential Equations - Session 14

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1 Context: Where This Fits In

In previous sessions (specifically Session 13), we focused on finding the particular solution (x_p) for non-homogeneous second-order linear ODEs, $m\ddot{x} + b\dot{x} + Kx = F(t)$, often using the Complex Replacement Method and the Exponential Response Formula (ERF).

Session 14 shifts focus from solving for x_p to analyzing the long-term behavior of the **homogeneous solution** (x_h). The central question is: does the natural behavior of the system die out over time, leaving only the forced response? This defines the concept of **System Stability** (or Permanence).

2 Core Concepts & Definitions

2.1 System Permanence (Stability)

A system modeled by the ODE $m\ddot{x} + b\dot{x} + Kx = F(t)$ is called **permanent** (or stable) if the long-term behavior of the general solution, $x(t) = x_h + x_p$, is independent of the initial conditions. This occurs if and only if the homogeneous solution vanishes as time approaches infinity:

$$\lim_{t\to\infty} x_h(t) = 0 \tag{1}$$

2.2 Solution Components

- **Homogeneous Solution (x_h):** Also called the *transient solution*. It contains the arbitrary constants (C_1, C_2) and reflects the system's initial reaction. In a stable system, this decays to zero.
- **Particular Solution (x_p):** Also called the *steady-state solution*. It reflects the long-term response driven by $F(t)$ and is independent of initial conditions.

2.3 General Stability Criterion

The differential equation $p(D)x = F(t)$ is stable if and only if the real part of **all** roots (r_i) of the characteristic polynomial $p(r) = 0$ is negative.

$$\boxed{\text{Re}(r_i) < 0 \quad \text{for all } i} \tag{2}$$

2.4 Stability Criterion (Second Order Coefficient Test)

For a second-order ODE $m\ddot{x} + b\dot{x} + Kx = F(t)$ (where $m \neq 0$), the system is stable if and only if all coefficients (m, b, K) are non-zero and have the same sign.

2.5 Routh–Hurwitz Criterion

A method used for linear ODEs of order $n \geq 3$ to check stability without explicitly calculating the roots of high-order polynomials.

3 Prerequisite Skill Refresh

Stability analysis relies on finding characteristic roots. For the second-order equation $m\ddot{x} + b\dot{x} + Kx = 0$, the characteristic polynomial is $p(r) = mr^2 + br + K = 0$.

3.1 The Quadratic Formula

The roots are found using:

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4mK}}{2m} \tag{3}$$

The stability depends on the discriminant $\Delta = b^2 - 4mK$:

Case	Discriminant	Form of $x_h(t)$	Stability Condition
1. Real Distinct	$\Delta > 0$	$C_1e^{r_1t} + C_2e^{r_2t}$	$r_1 < 0$ and $r_2 < 0$
2. Complex	$\Delta < 0$	$e^{at}(C_1 \cos bt + C_2 \sin bt)$	$a = \text{Re}(r) < 0$
3. Repeated	$\Delta = 0$	$e^{r_1t}(C_1 + C_2t)$	$r_1 < 0$

3.2 Identifying the Real Part of Complex Roots

When $\Delta < 0$, the roots are $r_{1,2} = a \pm ib$. The real part a determines decay:

$$a = \text{Re}(r) = \frac{-b}{2m} \tag{4}$$

If $a < 0$, the factor e^{at} decays to zero.

4 Key Examples: A Step-by-Step Walkthrough

4.1 Example 1: Distinct Real Roots (Case 1)

Problem: Determine if the transient solution of $\ddot{x} + 8\dot{x} + 7x = F(t)$ is permanent.

Step 1: Solve Characteristic Equation

$$p(r) = r^2 + 8r + 7 = 0$$

Factoring the polynomial:

$$(r+1)(r+7) = 0 \implies r_1 = -1, \quad r_2 = -7$$

Step 2: Check Stability Criterion Both roots are real. We check if they are negative.

$$r_1 = -1 < 0 \quad \text{and} \quad r_2 = -7 < 0$$

Since $\text{Re}(r_i) < 0$ for both roots, the exponentials e^{-t} and e^{-7t} decay. **Conclusion:** The system is **Stable**.

4.2 Example 2: Complex Conjugate Roots (Case 2)

Problem: Determine if the system $\ddot{x} + 4\dot{x} + 5x = F(t)$ is stable.

Step 1: Solve Characteristic Equation

$$r^2 + 4r + 5 = 0$$

Using the quadratic formula:

$$r_{1,2} = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$$

Step 2: Check Real Part The roots are complex with real part $a = -2$.

$$a = -2 < 0$$

The solution contains the term e^{-2t}, which decays to zero. **Conclusion:** The system is **Stable**.

4.3 Example 3: Coefficient Test (Shortcut)

Problem: Determine if $2\ddot{x} + 5\dot{x} + 3x = F(t)$ is stable without calculating roots.

Step 1: Identify Coefficients

$$m = 2, \quad b = 5, \quad K = 3$$

Step 2: Apply Test All coefficients are non-zero. All coefficients are positive $(+2, +5, +3)$. Since signs are identical, stability is guaranteed. **Conclusion:** The system is **Stable**.

5 Conceptual Understanding

Why the Method Works: Stability hinges on the behavior of exponentials.

- If r is real, $e^{rt} \to 0$ only if r is negative.
- If $r = a + ib$ is complex, Euler's formula gives $e^{(a+ib)t} = e^{at}(\cos bt + i \sin bt)$. The trigonometric parts oscillate forever, so the decay depends entirely on the amplitude factor e^{at}. This decays only if $a < 0$.

[Image of exponential decay graph]
Physical Interpretation:

- **Transient (x_h):** Represents the system's memory of its initial "kick." In stable systems (like a shock absorber), this energy dissipates.
- **Steady State (x_p):** Represents the permanent oscillation driven by the external force.

6 Common Mistakes to Avoid

- **Zero Real Part:** If $\text{Re}(r) = 0$ (e.g., $r = \pm 3i$), the system is **not** asymptotically stable. It yields undamped oscillations ($\cos 3t$) that never decay. The condition is strictly $\text{Re}(r) < 0$.
- **Incomplete Root Check:** Every single root must have a negative real part. If a 3rd-order system has roots $-1, -2, +1$, the system is unstable due to the single positive root.
- **Misapplying Coefficient Test:** The rule "all coefficients same sign" is a sufficient condition for stability *only* for 2nd-order ODEs. Do not apply this blindly to 3rd-order or higher equations without further tools (like Routh-Hurwitz).

7 Summary & What's Next

Summary:

1. Stability requires $\lim_{t\to\infty} x_h(t) = 0$.
2. Mathematically, this requires $\text{Re}(r_i) < 0$ for all characteristic roots.
3. For $m\ddot{x} + b\dot{x} + Kx = 0$, if m, b, K share the same sign, the system is stable.

Next Session: Having mastered stability, we move to Session 15: **The Method of Undetermined Coefficients for Polynomial Inputs**. We will solve ODEs like $y'' + 3y' + 4y = 4x^2 - 2x$ by guessing polynomial trial solutions (e.g., $y_p = Ax^2 + Bx + C$).