Choice Models in Operations

Lecture 1. : Preference Theory

Instructor: Srikanth Jagabathula *Scribe: Sherry Wang*

1 Notation

Table 1: Notation

Calliqraphic symbols	Sets	$\mathcal{X}, \mathcal{N}, \mathcal{M}$
Sans seril font	Random variables	$\mathsf{U}, \mathsf{X}, \mathsf{Y}$
Capital letter	Matrices	A, B
Greek and small letters	cardinality, indexing	i, j, m, n

2 Preference Theory

$\mathcal{N}$: universe of n items
$2^{\mathcal{N}}$: power set of $\mathcal{N}$, the set of collection of all subsets of $\mathcal{N}$.

2.1 Choice rule

Definition 1 *A function C: from $2^{\mathcal{N}}$ to $2^{\mathcal{N}}$, s.t. $C(\mathcal{B}) \subseteq \mathcal{B}, \forall \mathcal{B} \in 2^{\mathcal{N}}$.*
If $x \in C(\mathcal{B})$, then we say that "x is chosen from $\mathcal{B}$".
If $y \in \mathcal{B}$, then we say that "y could have been chosen from $\mathcal{B}$"

In classical economics, $\mathcal{B}$ is often called "budget set". In operation and marketing, $\mathcal{B}$ is an "offer set" or "choice set".
Remarks:

1. $C(\mathcal{B})$ doesn't have to be a singleton.
2. $C(\mathcal{B})$ may be empty.
3. If $C(\mathcal{B}) \neq \phi, \forall \mathcal{B} \in 2^{\mathcal{N}}$, then $C(.)$ is "decisive".
4. $C(.)$ is a singleton $\forall \mathcal{B}$, then $C(.)$ is "univalent".

2.2 Preference rule

Definition 2 *We define a weak preference $\succeq$ of a customer over the set $\mathcal{N}$ as follows: $x \succeq y \iff x$ is at least as good as y. "Making elements of $\mathcal{N}$ comparable to each other"*

Definition 3 *We say that x is strictly preferred to y if $x \succeq y$ but $y \not\succeq x$.*

We make 2 assumptions:

1. Completeness: A weak preference $\succeq$ is complete iff $\forall x, y \in \mathcal{N}$ we have either $x \succeq y$ or $y \succeq x$, or both.

2. Transitivity: A preference relation is transitive if whenever $x \succeq y$ and $y \succeq z$ then $x \succeq z, \forall x, y, z \in \mathcal{N}$

Representation: we can represent a rational preference relation as a graph with the items as nodes and a directed edge from x to y iff $x \succeq y$.

$$\begin{tikzcd}
1 \arrow[r] & 2 \arrow[d] \\
3 \arrow[u] & 4 \arrow[l]
\end{tikzcd}$$

In this graph, you can cycle among all the products. Kahneman and Tversky Eg: Participants want to buy \$125 stereo and \$15 calculator.
Q1: You learn that there is a \$5 discount on calculator at another store. Will you go to the other store?
Q2: You learn that there is a \$5 discount on stereo at another store. Will you go to the other store?
Q3: Both are out of stock. You are offered a \$5 discount to go to the other store. Do you care which item is discounted?
Most people say YES to Q1, NO to Q2, INDIFFERENT to Q3.
x: travel to the other store for discount on calculator.
y: travel to the other store for discount on stereo.
z: stay at the store.
Q1: YES implies $x \succ z$
Q2: NO implies $z \succ y$
By rationality, $x \succ y$
Q3: INDIFFERENCE implies $x \sim y$

Definition 4 *Every preference relation $\succeq$ that is rational induces a choice rule:* $C(\mathcal{B}; \succeq) = \{x \in \mathcal{B} | x \succeq y, \forall y \in \mathcal{B}\}$

Remarks:

1. $C(\mathcal{B}; \succeq)$ may contain more than one element.

2. $C(\mathcal{B}; \succeq)$ may be empty, only if the sets are infinite.
   e.g. $\succeq$ is $\geq$, let $\mathcal{N} = R, \mathcal{B} = [0, 1), then \{x \in \mathcal{B} : x \succeq y, \forall y \in \mathcal{B}\} = \phi$
   If $\mathcal{N}$ is finite, then $C(\mathcal{B}; \succeq)$ is always nonempty.

Definition 5 *HARP:Houthaker's Axiom of Revealed Preference*
A choice function satisfies HARP if whenever $x, y \in \mathcal{A} \cap \mathcal{B}$ and $x \in C(\mathcal{A}), y \in C(\mathcal{B})$, then we must have $x \in C(\mathcal{B}), y \in C(\mathcal{A})$

Theorem 1 *$C(.)$ is non empty and satisfies HARP iff $\exists$ a complete and transitive $\succeq$ s.t. $C(\mathcal{B}) = C(\mathcal{B}; \succeq), \forall \mathcal{B}$*

Proof: $\Longleftarrow$
We first prove that $C(.)$ is not empty.
We use induction on the size of $\mathcal{B}$ to prove that $C(\mathcal{B})$ is not empty.
Base case: Suppose $\mathcal{B} = x$, then $C(\mathcal{B}) = x$, which is not empty.
Inductive hypothesis: Suppose HARP is true for any $\mathcal{B}$ of size at most n.
Now we show the result for sets of size $n + 1$.
For any set $\mathcal{A}$ of size $n + 1$, we claim that $C(\mathcal{A}) = C(\mathcal{A}; \succeq)$
$\mathcal{A} = \{x\} \cup \mathcal{B}$ for some $x \notin \mathcal{B}$ and $\mathcal{B} s.t. |\mathcal{B}| = n$
By inductive hypothesis, we know that $C(\mathcal{B}) \neq \phi$
Suppose $y \in C(\mathcal{B})$, because $\succeq$ is complete, one of the following must be true.

1. $x \succeq y$: By transitivity, we must have that $x \succeq z, \forall z \in \mathcal{B}$, so $x \in C(\mathcal{A})$.

2. $y \succeq x$: Then $y \in C(\mathcal{A})$ by definition.

It thus follows by induction that $C(\mathcal{B}) = C(\mathcal{B}; \succeq) \neq \phi$.
Then we agree that $C(.)$ must satisfy HARP.
$\Longrightarrow$
If $C(.)$ is non-empty and HARP, we need to show $\exists$ a complete and transitive $\succeq$ s.t. $C(\mathcal{B}) = C(\mathcal{B}; \succeq), \forall \mathcal{B}$.
First, we construct a preference relation $\succeq$ as follows:
$x \succeq y$ whenever $\exists$ a subset $\mathcal{B}$ s.t. $x, y \in \mathcal{B}$ and $x \in C(\mathcal{B})$.
First we need to show $\succeq$ is complete.
Consider $\mathcal{B} = \{x, y\}$. Since $C(.)$ is non-empty, one of the following must be true:

- $x \in C(\{x, y\}) \Rightarrow x \succeq y$

- $y \in C(\{x, y\}) \Rightarrow y \succeq x$

- $x, y \in C(\{x, y\}) \Rightarrow y \succeq x, x \succeq y$

In all cases, we have $y \succeq x$ or $x \succeq y$, so $\succeq$ is complete.
Then we need to show $\succeq$ is transitive: we need to show that if $x \succeq y$ and $y \succeq z$, then $x \succeq z$.
By definition of $\succeq$, we know that $\exists \mathcal{A}_1$ and $\mathcal{A}_2$ s.t.
$x, y \in \mathcal{A}_1$ and $x \in C(\mathcal{A}_1)$
$y, z \in \mathcal{A}_2$ and $y \in C(\mathcal{A}_2)$
Let's consider $\mathcal{B} = \{x, y, z\}$, there are three cases.

1. $x \in C(\mathcal{B})$, then by definition, $x \succeq z$

2. $y \in C(\mathcal{B})$. Now note that $x, y \in \mathcal{A}_1 \cap \mathcal{B}, x \in C(\mathcal{A}_1), y \in C(\mathcal{B})$, by HARP, $x \in C(\mathcal{B}) \Longrightarrow x \succeq z$

3. $z \in C(\mathcal{B})$ Note that $y, z \in \mathcal{A}_2 \cap \mathcal{B}$ and $y \in C(\mathcal{A}_2), z \in C(\mathcal{B})$. By HARP, $y \in C(\mathcal{B})$. And then it reduced to 2.

Then we need to show $C(.) = C(.; \succeq)$

1. $C(\mathcal{B}) \subseteq C(\mathcal{B}; \succeq), \forall \mathcal{B}$
   $x \in C(\mathcal{B})$, by definition of $\succeq$, $x \succeq y, \forall y \in \mathcal{B}$. By definition of $C(.; \succeq), x \in C(\mathcal{B}; \succeq)$

2. $C(\mathcal{B}) \supseteq C(\mathcal{B}; \succeq), \forall \mathcal{B}$
   $x \in C(\mathcal{B}; \succeq)$ Consider an element $y \in C(\mathcal{B})$. Such a y exists because $C(\mathcal{B}) \neq \phi$
   Because $x \in C(\mathcal{B}; \succeq) \implies x \succeq y$.
   Since $x \succeq y$, $\exists$ a set $\mathcal{A}$ s.t. $x \in C(\mathcal{A})$ and $x, y \in \mathcal{A}$
   Because $y \in C(\mathcal{B})$, we have that $x, y \in \mathcal{A} \cap \mathcal{B}, x \in C(\mathcal{A}), y \in C(\mathcal{B})$, by HARP, $x \in C(\mathcal{B})$.

It thus follows that $C(\mathcal{B}) = C(\mathcal{B}; \succeq), \forall \mathcal{B} \quad \square$