Definition 1 *A preference relation $\succcurlyeq$ is said to be represented by a <u>utility function</u> $u : \mathcal{N} \to \mathbb{R}$ if and only if*

$$x \succcurlyeq y \iff u(x) \geq u(y)$$

Utility functions map preference relations over items in the universe to an ordered relation defined over real numbers. At this point, we haven't required that $\succcurlyeq$ is a rational preference relation; we also haven't required that such a function actually exists.

If a utility function exists, it induces a choice rule defined as:

$$C(\mathcal{B}, u(\cdot)) = \{x \in \mathcal{B} \mid x \text{ is a solution of } \max_{y \in \mathcal{B}} u(y)\}$$

When can a preference relation be represented by a utility function? It appears that rationality is necessary for the existence of a utility function ... Is it sufficient?

Theorem 1 *If $\mathcal{N}$ is finite, then any complete and transitive preference relation can be represented by a utility function $u : \mathcal{N} \to \mathbb{R}$.*

Proof Sketch: Proof by induction (recursive approach). Suppose $|\mathcal{N}| = n$; then, we can construct a mapping from $\mathcal{N}$ to $\{1, 2, \ldots, n\}$. Consider $C(\mathcal{N}; \succcurlyeq)$, the set of most preferred elements in $\mathcal{N}$. Since C is a choice function induced by weak preference, this set is nonempty. Now, take all of the elements in $C(\mathcal{N}; \succcurlyeq)$ and map them to n. Remove to obtain $\mathcal{N}^* = \mathcal{N} \setminus C(\mathcal{N}; \succcurlyeq)$, and repeat the same steps on $\mathcal{N}^*$, assigning the value $n-1$ to the most preferred elements in $\mathcal{N}^*$. Repeat until there are no elements left. $\quad \square$

So, finiteness and rationality are sufficient to guarantee the existence of a utility function to represent the preference relation $\succcurlyeq$.

$$C(\cdot) \xrightarrow{\text{HARP}} \text{ rational choice rule } \xrightarrow[\text{of } \mathcal{N}]{\text{finiteness}} \text{ utility representation}$$

Next, we introduce the idea of probabilistic or stochastic choice. Why?

1. Relax the rationality assumption. For example, suppose there exists a set of items $\{a, b\}$ for which a consumer sometimes likes a and sometimes likes b. We can model the consumer as drawing from a distribution of preferences, where the consumer's preferences are only rational within the span of a given day.

2. Model choices of populations. For example, if a is chosen 10 times and b is chosen 10 times, we can model a set of 20 people with heterogeneous preferences.

To proceed, we assume that $\mathcal{N}$ is finite, and in each choice instance, only one element is chosen.

Definition 2 *A <u>stochastic choice rule</u> is specified as $\mathbb{P}(\cdot|\cdot)$, where $\mathbb{P}(x|\mathcal{B})$ denotes the probability that x is chosen from $\mathcal{B}$.*

We assume that $\mathbb{P}(\cdot|\cdot)$ satisfies the usual probability rules:

1. $0 \leq \mathbb{P}(x|\mathcal{B}) \leq 1$, $\hspace{2cm} \forall\ x \in \mathcal{B},\ \forall\ \mathcal{B}$ (Probabilities between 0 and 1)

2. $\sum_{x \in \mathcal{B}} \mathbb{P}(x|\mathcal{B}) = 1$, $\hspace{2cm} \forall\ \mathcal{B}$ (Normalization condition)

3. $\mathbb{P}(x \text{ or } y|\mathcal{B}) = \mathbb{P}(x|\mathcal{B}) + P(y|\mathcal{B})$ (Only one item chosen)
$\mathbb{P}(x \text{ and } y|\mathcal{B}) = 0$

Remark: Deterministic choice rules are a special case of stochastic choice rules as defined here.

To capture a deterministic univalent rational choice rule, it is sufficient to have one preference order (a.k.a. preference list, rank list, permutation). To capture a stochastic choice rule, an order is not sufficient.

Definition 3 *A stochastic choice rule $\mathbb{P}(\cdot|\cdot)$ is <u>stochastically rational</u> if there exists a probability measure π on $\mathcal{S}_n$ such that:*

$$P(x|\mathcal{B}) = \pi(\succ\ \in \mathcal{S}_n : x => \succ \mathcal{B})$$

where either $x \succ y \in \mathcal{B}$, $\forall y \neq x$, or x is the unique $\succ$-maximal element in $\mathcal{B}$. Here, $x \Rightarrow_\succ \mathcal{B}$ means "x is the most preferred element in set $\mathcal{B}$", and $\mathcal{S}_n$ is the set of all possible linear (complete, transitive, and antisymmetric) preference relations $\succ$ on the set $\mathcal{N}$.

Example. Let $\mathcal{N} = \{x, y, z\}$. Suppose:

$$\mathbb{P}(x|\{x,y\}) = \tfrac{3}{4} \qquad \mathbb{P}(y|\{x,y\}) = \tfrac{1}{4}$$

Is this stochastically rational? To answer this question, we need to construct a probability mass over all possible preference orderings:

$$
\begin{array}{lll}
x \succ y \succ z \quad (\sigma_1) & y \succ x \succ z \quad (\sigma_3) & z \succ x \succ y \quad (\sigma_5) \\
x \succ z \succ y \quad (\sigma_2) & y \succ z \succ x \quad (\sigma_4) & z \succ y \succ x \quad (\sigma_6)
\end{array}
$$

Then, we have:

$Pr(x \text{ chosen from } \mathcal{B} \text{ under probability distribution } \pi) = \pi(\sigma_1) + \pi(\sigma_2) + \pi(\sigma_5)$
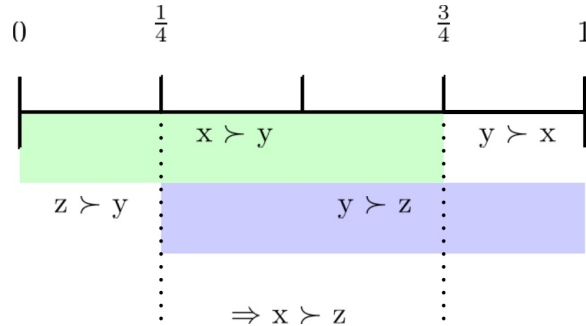
So, for example, we could have:

$$\pi(\sigma_1) = \tfrac{3}{4} \qquad \pi(\sigma_3) = \tfrac{1}{4} \qquad \pi(\sigma_i) = 0 \text{ otherwise}$$

Remark: Not every stochastic choice rule is stochastically rational.

Example. Let $\mathcal{N} = \{x, y, z\}$. Suppose:

$$
\begin{aligned}
\mathbb{P}(x|\{x,y\}) &= \tfrac{3}{4} & \mathbb{P}(y|\{x,y\}) &= \tfrac{1}{4} \\
\mathbb{P}(y|\{y,z\}) &= \tfrac{3}{4} & \mathbb{P}(z|\{y,z\}) &= \tfrac{1}{4} \\
\mathbb{P}(z|\{x,z\}) &= \tfrac{3}{4} & \mathbb{P}(x|\{x,z\}) &= \tfrac{1}{4}
\end{aligned}
$$

This is unsolvable. These probabilities imply that $\mathbb{P}(x \text{ is chosen from } \{x,z\}) \geq \tfrac{1}{2}$, which is a contradiction.

So, under what conditions is a rule stochastically rational? Such conditions exist, but they are hard to verify.

Definition 4 *Axiom of Revealed Stochastic Preference (ARSP). A stochastic choice rule $\mathbb{P}(\cdot|\cdot)$ defined over $\mathcal{N}$ satisfies ARSP if, for every finite collection of tuples $(x_1, \mathcal{B}_1), (x_2, \mathcal{B}_2), \ldots, (x_m, \mathcal{B}_m)$ such that $x_i \in \mathcal{B}_i$ and $\mathcal{B}_i \subseteq \mathcal{N}$, $\forall\, i$, we have that:*

$$\sum_{i=1}^{m} k_i\ \mathbb{P}(x_i|\mathcal{B}_i) \quad \leq \quad \max_{\succ \in \mathcal{S}_n} \sum_{i=1}^{m} k_i\ \mathbb{1}[\succ, x_i, \mathcal{B}_i]$$

for any collection of nonnegative integers $k_i, i = 1, \ldots, m$ and where

$$\mathbb{1}[\succ, x, \mathcal{B}] = \begin{cases} 1 & \text{if } x \in C(\mathcal{B}) \\ 0 & \text{otherwise} \end{cases}$$

Theorem 2 (McFadden and Richter '90) *A stochastic choice rule $\mathbb{P}(\cdot|\cdot)$ is stochastically rationalizable iff it satisfies ARSP.*

Proof:
1. Stochastically rational $\implies$ ARSP. Idea: average $\leq$ max.
Because $\mathbb{P}(\cdot|\cdot)$ is stochastically rational, there exists a probability mass function $\pi : \mathcal{S}_n \to [0, 1]$ such that

$$\mathbb{P}(x|\mathcal{B}) = \sum_{\succ \in \mathcal{S}_n} \pi(\succ)\ \mathbb{1}[\succ, x, \mathcal{B}] = \sum_{\substack{\succ \in \mathcal{S}_n\ :\ x \succ y, \\ \forall\ y \in \mathcal{B},\ x \neq y}} \pi(\succ), \quad \forall\ x, \mathcal{B} \qquad (*)$$

Now, for any collection of nonnegative integers k_i, $1 \leq i \leq m$, we must have:

$$
\begin{aligned}
LHS = \sum_{i=1}^{m} k_i\ \mathbb{P}(x_i|\mathcal{B}_i) \quad &= \quad \sum_{i=1}^{m} k_i \sum_{\succ \in \mathcal{S}_n} \pi(\succ)\ \mathbb{1}[\succ, x_i, \mathcal{B}_i] \qquad \text{by } (*) \\
&= \quad \sum_{\succ \in \mathcal{S}_n} \pi(\succ) \sum_{i=1}^{m} k_i\ \mathbb{1}[\succ, x_i, \mathcal{B}_i] \\
&= \quad \sum_{\succ \in \mathcal{S}_n} \pi(\succ) M(\succ) \\
&= \quad \mathbb{E}_\pi[M(\succ)] \\
&\leq \quad \max_{\succ \in \mathcal{S}_n} M(\succ) = RHS
\end{aligned}
$$

where $M(\succ) = \sum_{i=1}^{m} k_i\ \mathbb{1}[\succ, x_i, \mathcal{B}_i]$.

2. ARSP $\implies$ stochastic rationality. We prove the contrapositive, that is, $\neg$ stochastically rational $\implies$ $\neg$ ARSP.
We need to show that in the absence of stochastic rationality, there is no PMF $\pi : \mathcal{S}_n \to [0, 1]$ s.t. $\mathbb{P}(x|\mathcal{B}) = \sum_{\succ \in \mathcal{S}_n} \pi(\succ)\ \mathbb{1}[\succ, x, \mathcal{B}]$. We need to show the existence of at least one set of k_i such that $\sum_{i=1}^{m} k_i\ \mathbb{P}(x_i|\mathcal{B}_i) > \max_{\succ \in \mathcal{S}_n} \sum_{k=1}^{m} k_i\ \mathbb{1}[\succ, x_i, \mathcal{B}_i]$.

4

Let:

$$A_{m\times n!} = \begin{bmatrix} \ddots & & \vdots & \\ \cdots & \mathbb{1}[\succ, x_i, \mathcal{B}_i] & \cdots \\ & & \vdots & \ddots \end{bmatrix}, \quad \vec{\pi}_{n!\times 1} = \begin{bmatrix} \vdots \\ \pi(\succ) \\ \vdots \end{bmatrix}, \quad \vec{p}_{m\times 1} = \begin{bmatrix} \vdots \\ \mathbb{P}(x_i|\mathcal{B}_i) \\ \vdots \end{bmatrix}$$

where the columns of A represent different preference lists $\succ$ and the rows contain binary indicators of whether or not a given x_i is chosen from those preference lists. The entries of π represent the probability of picking a given preference list. Then, we have:

$$A\vec{\pi} = \vec{p} \qquad (**)$$

where $A \in \{0,1\}^{m\times n!}$, $\vec{\pi} \in \{0,1\}^{n!\times 1}$, and $\vec{p} \in \{0,1\}^m$, subject to $\sum_{\succ\in\mathcal{S}_n} \pi(\succ) = 1$ and $\pi(\succ) \geq 0, \ \forall \ \succ$.

Both A and $\vec{p}$ are known to us, so if stochastically rational, this system has a feasible solution. But, because $\mathbb{P}(\cdot|\cdot)$ is not rational, the system $(**)$ is infeasible.

We rewrite the above system as:
$$\tilde{A}\vec{\pi} = \tilde{p}$$

where

$$\tilde{A} = \begin{bmatrix} A \\ \hline 1\ldots \end{bmatrix}, \qquad \tilde{p} = \begin{bmatrix} p \\ \hline 1 \end{bmatrix}, \qquad \vec{\pi} \geq 0$$

Since the system is infeasible, we invoke Farkas' Lemma. There exists $\vec{z} \in \mathbb{R}^{m+1}$ such that $\vec{z}^{\top}\tilde{A} \leq 0$ and $\vec{z}^{\top}\tilde{p} > 0$. That is,

$$\begin{bmatrix} z_1 & z_2 & \cdots & z_m & z_0 \end{bmatrix} \begin{bmatrix} \ddots & & \vdots & & \\ \cdots & \mathbb{1}[\succ, x_i, \mathcal{B}_i] & \cdots \\ & & \vdots & & \ddots \\ \hline 1 & \cdots & 1 & \cdots & 1 \end{bmatrix} \leq 0$$

and

$$\begin{bmatrix} z_1 & z_2 & \cdots & z_m & z_0 \end{bmatrix} \begin{bmatrix} \vdots \\ \mathbb{P}(x_i|\mathcal{B}_i) \\ \vdots \\ \hline 1 \end{bmatrix} > 0$$

We can rewrite the above as:

$$\sum_{i=1}^{m} z_i \, \mathbb{1}[\succ, x_i, \mathcal{B}_i] + z_0 \leq 0, \quad \forall \ \succ\in \mathcal{S}_n$$

and

$$\sum_{i=1}^{m} z_i \mathbb{P}(x_i|\mathcal{B}_i) + z_0 > 0$$

So far, we've shown that:

$$\sum_{i=1}^{m} z_i \; \mathbb{P}(x_i|\mathcal{B}_i) + z_0 \;\; > \;\; \sum_{i=1}^{m} z_i \; \mathbb{1}[\succ, x_i, \mathcal{B}_i] + z_0, \;\; \forall \; \succ \;\; \in S_n$$

$$LHS \;\; > \;\; \max_{\succ \in \mathcal{S}_n} \sum_{i=1}^{m} z_i \; \mathbb{1}[\succ, x_i, \mathcal{B}_i]$$

So, we've shown that there exists a $\vec{z}$ that violates the ARSP condition. We still need to show that the k_i are nonnegative. Sketch of the remaining proof:

- We go from the existence of real z's to rational q to integers. Because the above inequality is strict, $\exists \; \varepsilon_i > 0$, for $1 \leq i \leq m$, such that:

$$\sum_{i=1}^{m}(z_i + \varepsilon_i) \; \mathbb{P}(x_i|\mathcal{B}_i) > \max_{\succ \in \mathcal{S}_n} \sum_{i=1}^{m}(z_i + \varepsilon_i) \; \mathbb{1}[\succ, x_i, \mathcal{B}_i]$$

- To go from integers to nonnegative integers, we split into negative and positive integers. We exploit the fact that everything is normalized to 1. A note will be posted to complete the proof.

□