- 1. Prove the following variation of Exercise 3.32.
  - a) The space  $(X_1 \times X_2) \times X_3$  is homeomorphic to  $X_1 \times X_2 \times X_3$ . You may not use the words "open" or "closed" at any point in your proof. (*Hint*: Use the Characteristic Property, Luke!)
  - b) A projection map from an arbitrary product space is an open map.
  - c) An arbitrary product of Hausdorff spaces is Hausdorff.
  - d) A countable product of second countable spaces is second countable.

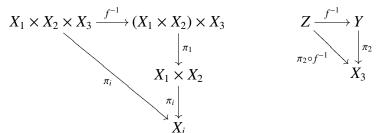
## Solution, part a:

Let  $Y = (X_1 \times X_2) \times X_3$  and  $Z = X_1 \times X_2 \times X_3$  and define  $f : Y \to Z$  as  $f((x_1, x_2), x_3) = (x_1, x_2, x_3)$ . Evidently f is a bijection.

Observe that these diagrams commute:

The first diagram implies that  $\pi_i \circ \pi_1 = f \circ \pi_1$ . Since a composition of projection functions is continuous, by the characteristic property of the product topology f is continuous in its first component. The second diagram shows that f is continuous in its second component, so f is continuous.

Now  $f^{-1}: Z \to Y$  is  $f^{-1}(x_1, x_2, x_3) = ((x_1, x_2), x_3)$ , and we have these diagrams that commute:



From the left diagram, we have  $\pi_i \circ \pi_1 \circ f^{-1} = \pi_i$ , and since  $\pi_i \circ \pi_1$  and  $\pi_i$  are continuous,  $f^{-1}$  is, too, in its first two components. From the right diagram, the diagonal is  $\pi_2 \circ f^{-1} = \pi_3$ , which is continuous, so  $f^{-1}$  is continuous.

Hence f is a homeomorphism, therefore Y and Z are homeomorphic.

## Solution, part b:

Let  $\pi: \prod_{\alpha \in I} X_{\alpha} \to X_0$  be a projection map from the arbitrary product space  $\prod_{\alpha \in I} X_{\alpha}$ , where I is an index collection, to  $X_0 \in \{X_{\alpha}\}_{\alpha \in I}$ .

Let U be a basic open set in  $\prod_{\alpha \in I} X_{\alpha}$ , so  $U = \prod_{\alpha \in I} U_{\alpha}$ , where  $U_{\alpha}$  is open in  $X_{\alpha}$ , and all but finitely many are  $X_{\alpha}$ . Then,  $\pi(U) = U_0$  where  $U_0$  is open in  $X_0$ . Thus  $\pi$  is an open map.

## Solution, part c:

Let  $\prod_{\alpha \in I} X_{\alpha}$  be an arbitrary product space, where  $X_{\alpha}$  is Hausdorff for each  $\alpha \in I$ . Let  $x, y \in \prod_{\alpha \in I} X_{\alpha}$  with  $x \neq y$ . Then for some  $\beta \in I$ , we will have  $x_{\beta} \neq y_{\beta}$  where  $x_{\beta}, y_{\beta} \in X_{\beta} \in \{X_{\alpha}\}_{\alpha \in I}$ .

The topological space  $X_{\beta}$  is Hausdorff so there exist sets  $U_1$  and  $U_2$  open in  $X_{\beta}$  with  $U_1 \cap U_2 = \emptyset$ .

Let  $\pi_{\beta}: \prod_{\alpha \in I} \to X_{\beta}$  be the projection function to  $X_{\beta}$  and define  $V_1 = \pi_{\beta}^{-1}(U_1)$ , and  $V_2 = \pi_{\beta}^{-1}(U_2)$ . These are open sets in  $\prod_{\alpha \in I} X_{\alpha}$  and  $V_1 \cap V_2 = \emptyset$ . Further,  $x \in V_1$  and  $y \in V_2$ , so  $\prod_{\alpha \in X_{\alpha}}$  is Hausdorff.

## Solution, part d:

Let  $X = \prod_{n=1}^{\infty} X_n$  be a product of countably many spaces  $X_n$ , where each  $X_n$  is second countable. Hence for each  $X_n$  we have a countable basis  $\mathcal{B}_n = \{B_k\}_{k=1}^{\infty}$ .

Define  $S = \{\pi_n^{-1}(B_k) : B_k \in \mathcal{B}_n \text{ and } k, n \in \mathbb{N}\}$ , where  $\pi_n : X \to X_n$  is the projection function onto the *n*-th component. Note that S is countable.

Now take any  $x = (x_1, x_2, ...) \in X$ . Then  $x_1 \in B_k$  for some  $B_k \in \mathcal{B}_1$ . So,  $\pi_1^{-1}(B_k) = B_k \times \prod_{n=2}^{\infty} X_n$ , so  $x \in \pi_1^{-1}(B_k) \in \mathcal{S}$ . Hence  $\mathcal{S}$  covers X and so it is a sub-basis for X.

We can then form a basis for X from S as follows:

$$\mathcal{B} = \left\{ \bigcap_{k=1}^{N} S_k : S_k \in \mathcal{S} \right\}.$$

Observe that  $\mathcal{B}$  is countable, as it is composed of all finite intersections of elements of the countable set  $\mathcal{S}$ .