1. Prove the following variation of Exercise 3.32.

a) The space $(X_1 \times X_2) \times X_3$ is homeomorphic to $X_1 \times X_2 \times X_3$. You may not use the words "open" or "closed" at any point in your proof. (*Hint*: Use the Characteristic Property, Luke!)

b) A projection map from an arbitrary product space is an open map.

c) An arbitrary product of Hausdorff spaces is Hausdorff.

d) A countable product of second countable spaces is second countable.

Solution, part a:
Let $Y = (X_1 \times X_2) \times X_3$ and $Z = X_1 \times X_2 \times X_3$ and define $f : Y \to Z$ as $f((x_1, x_2), x_3) = (x_1, x_2, x_3)$. Evidently f is a bijection.

Observe that these diagrams commute:

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\pi_1"'] & Z \arrow[d, "\pi_i"] \\ X_1 \times X_2 \arrow[r, "\pi_i"] & X_i \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "\pi_3 \circ f"'] & Z \arrow[d, "\pi_3"] \\ & X_3 \end{tikzcd}$$

The first diagram implies that $\pi_i \circ \pi_1 = f \circ \pi_1$. Since a composition of projection functions is continuous, by the characteristic property of the product topology f is continuous in its first component. The second diagram shows that f is continuous in its second component, so f is continuous.

Now $f^{-1} : Z \to Y$ is $f^{-1}(x_1, x_2, x_3) = ((x_1, x_2), x_3)$, and we have these diagrams that commute:

$$\begin{tikzcd} X_1 \times X_2 \times X_3 \arrow[r, "f^{-1}"] \arrow[ddr, "\pi_i"'] & (X_1 \times X_2) \times X_3 \arrow[d, "\pi_1"] \\ & X_1 \times X_2 \arrow[d, "\pi_i"'] \\ & X_i \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[r, "f^{-1}"] \arrow[dr, "\pi_2 \circ f^{-1}"'] & Y \arrow[d, "\pi_2"] \\ & X_3 \end{tikzcd}$$

From the left diagram, we have $\pi_i \circ \pi_1 \circ f^{-1} = \pi_i$, and since $\pi_i \circ \pi_1$ and π_i are continuous, f^{-1} is, too, in its first two components. From the right diagram, the diagonal is $\pi_2 \circ f^{-1} = \pi_3$, which is continuous, so f^{-1} is continuous.

Hence f is a homeomorphism, therefore Y and Z are homeomorphic.

Solution, part b:
Let $\pi : \prod_{\alpha\in I} X_\alpha \to X_0$ be a projection map from the arbitrary product space $\prod_{\alpha\in I} X_\alpha$, where I is an index collection, to $X_0 \in \{X_\alpha\}_{\alpha\in I}$.

Let U be a basic open set in $\prod_{\alpha\in I} X_\alpha$, so $U = \prod_{\alpha\in I} U_\alpha$, where U_α is open in X_α, and all but finitely many are X_α. Then, $\pi(U) = U_0$ where U_0 is open in X_0. Thus π is an open map.

Solution, part c:
Let $\prod_{\alpha\in I} X_\alpha$ be an arbitrary product space, where X_α is Hausdorff for each $\alpha \in I$. Let $x, y \in \prod_{\alpha\in I} X_\alpha$ with $x \neq y$. Then for some $\beta \in I$, we will have $x_\beta \neq y_\beta$ where $x_\beta, y_\beta \in X_\beta \in \{X_\alpha\}_{\alpha\in I}$.

The topological space X_β is Hausdorff so there exist sets U_1 and U_2 open in X_β with $U_1 \cap U_2 = \emptyset$.

Let $\pi_\beta : \prod_{\alpha\in I} \to X_\beta$ be the projection function to X_β and define $V_1 = \pi_\beta^{-1}(U_1)$, and $V_2 = \pi_\beta^{-1}(U_2)$. These are open sets in $\prod_{\alpha\in I} X_\alpha$ and $V_1 \cap V_2 = \emptyset$. Further, $x \in V_1$ and $y \in V_2$, so $\prod_{\alpha\in X_\alpha}$ is Hausdorff.

Solution, part d:
Let $X = \prod_{n=1}^{\infty} X_n$ be a product of countably many spaces X_n, where each X_n is second countable. Hence for each X_n we have a countable basis $\mathcal{B}_n = \{B_k\}_{k=1}^{\infty}$.

Define $\mathcal{S} = \{\pi_n^{-1}(B_k) : B_k \in \mathcal{B}_n \text{ and } k, n \in \mathbb{N}\}$, where $\pi_n : X \to X_n$ is the projection function onto the n-th component. Note that $\mathcal{S}$ is countable.

Now take any $x = (x_1, x_2, ...) \in X$. Then $x_1 \in B_k$ for some $B_k \in \mathcal{B}_1$. So, $\pi_1^{-1}(B_k) = B_k \times \prod_{n=2}^{\infty} X_n$, so $x \in \pi_1^{-1}(B_k) \in \mathcal{S}$. Hence $\mathcal{S}$ covers X and so it is a sub-basis for X.

We can then form a basis for X from $\mathcal{S}$ as follows:

$$\mathcal{B} = \left\{ \bigcap_{k=1}^{N} S_k : S_k \in \mathcal{S} \right\}.$$

Observe that $\mathcal{B}$ is countable, as it is composed of all finite intersections of elements of the countable set $\mathcal{S}$.