1. Problem 3-14

Show that real projective space $\mathbb{P}^n$ is an n-manifold. [Hint: consider the subsets $U_i \subseteq \mathbb{R}^{n+1}$ where $x_i = 1$.]

Solution:
Let $V_i \subseteq \mathbb{R}^{n+1}$ be $V_i = \{x \in \mathbb{R}^{n+1} : x_i = 1\}$.

Lemma: *The set V_i is homeomorphic to $\mathbb{R}^n$*
Define $i : V_i \to \mathbb{R}^n$ by $i(x_1, ..., x_{n+1}) = (x_1, ..., \hat{x}_i, ..., x_{n+1})$, where x_i is omitted, and $x_i = 1$ because $x \in V_i$. Clearly i is bijective and continuous, since each coordinate function is continuous.

Also, i^{-1} is the map $i^{-1} : (x_1, ..., x_n) = (x_1, ..., 1, ..., x_{n+1})$, where there is a 1 inserted in slot i. Each component function is continuous so it is continuous and hence i is a homeomorphism. □

We will denote the quotient map from $\mathbb{R}^{n+1}$ to $\mathbb{P}^n$ by q.

Let $U_i \subseteq \mathbb{R}^{n+1}$ be $U_i = \{x \in \mathbb{R}^{n+1} : x_i \neq 0\}$.

Define $\phi_i : U_i \to V_i$ be the map $\phi_i(x) = \frac{1}{x_i}(x_1, ..., x_{n+1})$.

Then ϕ_i is constant on the fibers of $\mathbb{P}^n$, because if $x \sim y$ then $x_k = \lambda y_k$ for $1 \leq k \leq n+1$, and:

$$\phi_i(x) = \frac{1}{x_i}(x_1, ..., x_{n+1}) = \frac{1}{\lambda y_i}(\lambda y_1, ..., \lambda y_{n+1}) = \frac{1}{y_i}(y_1, ..., y_{n+1}) = \phi_i(y).$$

Also observe that ϕ_i is continuous, in class we discussed that the map "scalar multiplication" is continuous.

So, in the diagram

$$\begin{tikzcd} U_i \arrow[d, "q"'] \arrow[dr, "\phi_i"] & \\ \mathbb{P}^n \arrow[r, "\tilde{\phi}_i"'] & V_i \end{tikzcd}$$

the function $\tilde{\phi}_i$ exists, and furthermore it is continuous since ϕ_i is continuous and q is a quotient map.

Next consider this diagram

$$\begin{tikzcd} V_i \arrow[r, hook, "i"] \arrow[dr, "\tilde{\psi}_i"'] & \mathbb{R}^n \arrow[d, "q"] \\ & \mathbb{P}^n \end{tikzcd}$$

The topological embedding from V_i to $\mathbb{R}^n$ is continuous from the Lemma, and q is a quotient map, hence $\tilde{\psi}_i$ is continuous.

Note that $\phi_i(x) = x$ if $x \in V_i$, and that $i(x) = x$ is the identity map when $x \in V_i$. Then

$\tilde{\phi}_i^{-1} = \psi_i$, because

$$\begin{aligned} x \in V_i : \quad & \tilde{\phi}_i(\tilde{\psi}_i(x)) = \tilde{\phi}_i(q(i(x))) = \phi_i(i(x)) = \phi_i(x) = x. \\ x \in \mathbb{P}^n : \quad & \tilde{\psi}_i(\tilde{\phi}_i(x)) = q(i(\tilde{\phi}_i(x))) = q(\tilde{\phi}_i(x)) = \phi_i(x) = x. \end{aligned}$$

Thus $\tilde{\phi}_i$ is a bijection from $\mathbb{P}^n$ to V_i and hence a homeomorphism. In the Lemma we showed that V_i is homeomorphic to $\mathbb{R}^n$, so $\mathbb{P}^n$ is homeomorphic to $\mathbb{R}^n$

The functions $\tilde{\phi}_i$ define charts on $\mathbb{P}^n$ to V_i. Observe that every $x \in \mathbb{P}^n$ is in the domain of at least one chart. Hence $\mathbb{P}^n$ is locally Euclidean.

$\mathbb{P}^n$ is then second countable because U_i is second countable. Further, $\mathbb{P}^n$ is Hausdorff hence $\mathbb{P}^n$ is an n dimensional manifold.

2. Problem 3-16

Let X be the subset $(\mathbb{R} \times \{0\}) \cup (\mathbb{R} \times \{1\}) \subseteq \mathbb{R}^2$. Define an equivalence relation on X by declaring $(x, 0) \sim (x, 1)$ if $x \neq 0$. Show that the quotient space $X/\sim$ is locally Euclidean and second countable, but not Hausdorff. (This space is called the line with two origins.)

Your solution should not be longer than a page. Extra credit for the shortest correct solution.

Solution:
Denote the quotient map by q and define the sets $X_0 = X \setminus \{(0, 1)\}$ and $X_1 = X \setminus \{(0, 0)\}$, and the maps

$$\begin{aligned} \phi_0 &: X_0 \to \mathbb{R} \text{ by } \phi_0(x, a) = x \\ \phi_1 &: X_1 \to \mathbb{R} \text{ by } \phi_1(x, a) = x. \end{aligned}$$

Each of ϕ_i is a projection and hence continuous. The sets X_0 and X_1 are both open in X as removing a single point results in an open set.

Claim: ϕ_i is constant on the fibers of q. To see this, let $q(x, a) = q(y, b)$, where (x, a) and $(y, b) \in X_i$. Then, we must have $x = y$, and hence $\phi_i(x) = \phi_i(y)$. Thus, in the diagram

$$\begin{tikzcd} X_i \arrow[d, "q"'] \arrow[dr, "\phi_i"] & \\ X/\sim \arrow[r, "\tilde{\phi}_i"'] & \mathbb{R} \end{tikzcd}$$

the map $\tilde{\phi}_i$ exists, and it is continuous. In fact we can write $\tilde{\phi}_i([x]) = x$, and evidently $\tilde{\phi}_i$ is bijective.

Now consider the diagram

$$\begin{tikzcd} \mathbb{R} \arrow[r, "f_i"] \arrow[dr, "\tilde{\psi}_i"'] & X_i \arrow[d, "q"] \\ & X/\sim \end{tikzcd}$$

Where the map $f_i : \mathbb{R} \to X_i$ is $f_i(x) = (x, i)$. Note that $(x, i) \in X_i$, and that f_i is continuous because each component is. Then by the characteristic property, $\tilde{\psi}_i$ is continuous. Moreover, $\tilde{\psi}_i = \tilde{\phi}_i^{-1}$ because

$$\begin{aligned} \tilde{\phi}_i(\tilde{\psi}_i(x)) &= \tilde{\phi}_i(q(f_i(x))) = \phi_i(f_i(x)) = \phi_i(x, i) = x \\ \tilde{\psi}_i(\tilde{\phi}_i([x])) &= q(f_i(\tilde{\phi}_i([x]))) = q(f_i(x)) = q(x, i) = [x]. \end{aligned}$$

Therefore $\tilde{\phi}_i$ is a homeomorphism. Hence the pair $\tilde{\phi}_i$ are two charts on $X/\sim$ and therefore $X/\sim$ is locally Euclidean. Then because X is second countable, $X/\sim$ is second countable.

Finally, $X/\sim$ is not Hausdorff. To see this, let $U_i = (a_i, b_i) \times \{i\}$ be basic open neighborhoods of $(0, i) \in X$, for $i = 0, 1$. Take any $(x, 0) \in U_0$, and WLOG suppose $x > 0$, so $0 < x < b_0$. If $(x, 1) \in U_1$ we are done, so suppose $(x, 1) \notin U_1$. Then $0 < b_1 < x$, and choose c such that $0 < c < b_1$. But then $0 < c < b_0$, so $(c, 0) \in U_0$ and $(c, 1) \in U_1$, so $q(U_0) \cap q(U_1) \neq \emptyset$.

3. Exercise 4.4

Prove that a topological space X is disconnected if and only if there exists a nonconstant continuous function from X to the discrete space $\{0, 1\}$.

Solution:
$(\implies)$:
Suppose X is disconnected. Then there exist disjoint open sets U, V in X where U and V are nonempty, are not X, and with $U \cup V = X$.

Define $f : X \to \{0, 1\}$ by $f(U) = 0$ and $f(V) = 1$. Since U and V are both nonempty, f is not constant.

There are only 4 open sets in $\{0, 1\}$, and the preimage of all of them is open:

$$f^{-1}(\{0\}) = U, \quad f^{-1}(\{1\}) = V, \quad f^{-1}(\{0, 1\}) = X, \quad f^{-1}(\emptyset) = \emptyset.$$

So f is continuous.

$(\impliedby)$:
The contrapositive of the reverse implication can be interpreted as: If X is connected, then any continuous function from X to $\{0, 1\}$ must be constant. But this is precisely Proposition 4.2 in the text.

4. Problem 4-4

Show that the following topological spaces are not manifolds:

a) the union of the x-axis and the y-axis in $\mathbb{R}^2$

b) the conical surface $C \subseteq \mathbb{R}^3$ defined by

$$C = \{(x, y, z) : z^2 = x^2 + y^2\}$$

Solution, part a:

Solution, part b:

5. Problem 4-5

Let $M = \mathbb{S}^1 \times \mathbb{R}$, and let $A = \mathbb{S}^1 \times \{0\}$. Show that the space M/A obtained by collapsing A to a point is homeomorphic to the space C of Problem 4-4(b), and thus is Hausdorff and second countable but not locally Euclidean.

Solution:
In this problem we will work in cylindrical coordinates. Then a point $(r, \theta, z) \in \mathbb{R}^3$, where $r > 0$, $\theta \in [0, 2\pi)$, and $z \in \mathbb{R}$, is in M if $r = 1$, and it is in C if $r = |z|$.

Define a function $f : M \to C$ by $f(1, \theta, z) = (|z|, \theta, z)$. Then from the above the codomain of M is C. Also note that f is clearly surjective, as any $(|z|, \theta, z) \in C$ is evidently mapped to by $(1, \theta, z) \in M$.

Now consider the diagram

$$\begin{tikzcd} M \arrow[d, "q"'] \arrow[dr, "f"] & \\ M/A \arrow[r, dotted, "\tilde{f}"'] & C \end{tikzcd}$$

We've used q to denote the quotient map from M to M/A. The diagram will show that $\tilde{f}$ exists and is continuous, provided that we can show f is constant on the fibers of q, and that f is continuous.

The function f is continuous if its component functions are, and each component function here, $f_1(x, y, z) = |z|$, $f_2(x, y, z) = y$, and $f_3(x, y, z) = z$ is continuous.

Suppose that $q(1, \theta_1, z_1) = q(1, \theta_2, z_2)$. This means either (i) $\theta_1 = \theta_2$ and $z_1 = z_2$, or (ii) $z_1 = z_2 = 0$. In case (i), then clearly $f(1, \theta_1, z_1) = f(1, \theta_2, z_2)$. In case (ii), we have $f(1, \theta_1, 0) = (0, \theta_1, 0)$ and $f(1, \theta_2, 0) = (0, \theta_2, 0)$, which is the same point in cylindrical coordinates, so f is constant on the fibers of q.

Therefore f descends to the quotient and $\tilde{f}$ exists and is continuous. Further, since f is surjective, $\tilde{f}$ is bijective.

Next, consider the map $p : C \to M$ defined by $p(r, \theta, z) = (1, \theta, z)$. Observe that p is continuous, since each component is continuous.

Then in this diagram:

$$\begin{tikzcd} C \arrow[r, "p"] \arrow[dr, "g"'] & M \arrow[d, "q"] \\ & M/A \end{tikzcd}$$

We have that the function g is continuous, since p is and q is a quotient map and hence continuous.

Also, we can now write:

$$(\tilde{f} \circ g)(r, \theta, z) = \tilde{f}(q(p(r, \theta, z))) = f(p(r, \theta, z) = f(1, \theta, z) = (|z|, \theta, z).$$

Since the original point (r, θ, z) was in C, it satisfied $r = |z|$, hence $\tilde{f} = g^{-1}$. Since $\tilde{f}$ is bijective, we also have $\tilde{f}^{-1} = g$.

Therefore $\tilde{f}$ is a homeomorphism, and M/A is homeomorphic to C. Thus, because C is Hausdorff and second countable but not locally Euclidean, M/A is as well.

6. Let X be a topological space with components $\{C_\alpha\}$. Show that X has the disjoint union topology $\coprod C_\alpha$ if and only if each C_α is open.

Solution:
Suppose X has the disjoint union topology defined by $\coprod C_\alpha$, and choose any $x \in C_\alpha$ for some α. Let U be an open set in X with $x \in U$. Then by the disjoint union topology, $U \cap C_\alpha$ is open in C_α, hence C_α is open.

Suppose each C_α is open. Observe that for any $U \subseteq X$:

$$U = X \cap U = \left(\bigcup_{\alpha \in I} C_\alpha \right) \cap U = \bigcup_{\alpha \in I} (C_\alpha \cap U).$$

If U is open in X, each $C_\alpha \cap U$ is also open, hence U is an open set in the disjoint union topology $\coprod C_\alpha$. Conversely, if each $C_\alpha \cap U$ is open in C_α, then U is open in X. Hence, the open sets in the topology on X and the disjoint union topology are the same.

7. Problem 4-9 (Just the manifold part)

Show that every n-manifold is homeomorphic to a disjoint union of countably many connected n-manifolds.

Solution:
Let M be an n-manifold. Let $\{C_\alpha\}_{\alpha \in I}$ be the connected components of M. Since M is a disjoint union of $\{C_\alpha\}$, each C_α is open by the previous excercise. Any open subset of an n-manifold is itself an n-manifold, hence each C_α is an n-manifold.

Manifolds are second countable, hence M has a countable basis. Let $\mathcal{B}$ be a countable basis for M. Choose any x_α in each C_α. Then there is a $B_\alpha \in \mathcal{B}$ with $x_\alpha \in B_\alpha \subseteq C_\alpha$, for each $\alpha \in I$.

Define the function $f : I \to I$ by $f(\alpha) = \beta$ where β is the β for which $B_\beta \subseteq C_\alpha$. Clearly f is surjective, as there is a B_β for each C_α.

For any $\alpha_1, \alpha_2 \in I$, where $\alpha_1 \neq \alpha_2$, we have $C_{\alpha_1} \cap C_{\alpha_2} = \emptyset$ hence $B_{f(\alpha_1)} \cap B_{f(\alpha_2)} = \emptyset$ too, and so $B_{f(\alpha_1)} \neq B_{f(\alpha_2)}$. Thus f is injective, and hence we have a bijection between a subset of $\mathcal{B}$ and $\{C_\alpha\}$. Since $\mathcal{B}$ is countable, $\{C_\alpha\}$ must be as well.

8. Problem 4-13

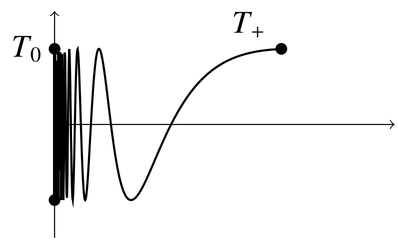
Define subsets of the plane by

$$T_0 = \{(x, y) : x = 0 \text{ and } y \in [-1, 1]\};$$
$$T_+ = \{(x, y) : x \in (0, 2\pi] \text{ and } y = \sin(1/x)\}.$$

Let $T = T_0 \cup T_+$. The space T is called the ***topologist's sine curve***.

a) Show that T is connected but not path-connected or locally connected.

b) Determine the components and the path components of T.



Lemma: $\overline{T_+} = T = T_0 \cup T_+$
Take any $P = (x, y) \in \mathbb{R}^2$. If $x < 0$, then $B_{-x}(x, y) \cap T = \emptyset$, so $P \notin \overline{T_+}$. If $x > 0$ and $y = \sin(1/x)$, then $P \in T_+$. If $x > 0$ and $y \neq \sin(1/x)$, let d be the distance from (x, y) to the curve $y = \sin(1/x)$, and then $B_d(x, y) \cap T = \emptyset$. If $|y| > 1$, then $B_r(x, y) \cap T = \emptyset$ if we take $r = (|y| - 1)/2$. The only points in $\mathbb{R}^2$ left are in T_0. We've just shown that every point in $\mathbb{R}^2$ is not in $\overline{T_+}$, or it is in $T_0 \cup T_+$, hence $\overline{T_+} = T$.

Solution, part a:
Observe that T_+ is connected, since it is the image of a continuous function of a connected set, $(0, 2\pi]$. Hence $\overline{T_+}$ is connected. From the Lemma, $\overline{T_+} = T$ so T is connected.

Suppose T is path connected. Let $a = (0, 0)$ and $b = (2/\pi, 1)$, and choose a path function $\gamma : [0, 1] \to T$, with $\gamma(0) = a$ and $\gamma(1) = b$. Let $B_r(a)$ be a ball of radius $r = 1/2$ around a. Since γ is continuous, $\gamma(t)$ must eventually be in $B_r(a)$, that is, for some $t_0 > 0$ we will have $\gamma(t) \in B_r(a)$ if $t < t_0$. Choose $t_1 < t_0$, and let $(x, y) = \gamma(t_1)$. Then for any $x' < x$, we must have $(x', \sin(1/x')) \in B_r(a)$. But, there are infinitely many points less than x for which $\sin(1/x) = 1$. Hence T is not path connected.

Further, T is not locally connected,

Solution, part b:
Since T is connected, it has only one connected component, T itself.

T_0 is path connected, it is a closed interval in $\mathbb{R}$ embedded in $\mathbb{R}^2$. T_+ is path connected, it is the image under a continuous function of a path connected set. But T is not path connected, hence it has 2 path components, T_0 and T_+.