Integer programming formulations of vehicle routing problems

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Abstract. Various mathematical formulations are available for situations represented by vehicle routing problems. The assignment-based integer programming formulations of these problems are more common and easy to understand. Such formulations are discussed in this paper and a much simpler formulation for the vechicle routing problem is presented for the case, when all the vehicles have the same load capacity and maximum allowable cost per route.

Keywords: Combinatorial analysis, optimization, assignment programming, integer programming

1. Introduction

The vehicle routing problem (VRP) [1–4] in the most general sense can be defined as follows: A set of customers each with a known location and a known requirement for some commodity, is to be supplied from a set of depots by a set of delivery vehicles of known capacity. The objective of the solution may be stated as cost minimization (distribution costs and vehicle or depot acquisition costs), or service improvement (increasing distribution capacities, reducing distribution time and related network design issues). It is required to design the routes for the vehicles subject to the following constraints:

(a) The requirements of all the customers must be met.

(b) The node constraints of vehicles must not be violated (i.e. the number of nodes/customers allocated to each vehicle must not exceed some predetermined number).

(c) The cost (alternatively time or distance) constraints of vehicles must not be violated (i.e. the total cost for each vehicle to complete its tour must not exceed some predetermined level).

(d) The load constraints of the vehicle must not be violated (i.e. the total load allocated to each vehicle must not exceed its capacity).

Depending on the number of depots the problem may be classified as a single depot or a multi-depot problem. Several subproblems and their variations may be formulated by considering various possible combinations of the constraints discussed earlier in (a) to (d). In order to simplify the problem, if it is assumed that the fleet consists of M vehicles of sufficiently large capacity, so that the constraints (b), (c) and (d) can be ignored, then the problem reduces to the M-Travelling Salesman Problem (M-TSP) [5]. In its simplest version, if it is further assumed that there is only one vehicle of very large capacity, then the problem reduces to the wellknown Travelling Salesman Problem (TSP). It is for this reason that the

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formulation for TSP is taken as a core model for the development of the mathematical formulations for more complicated cases. Therefore, most of the mathematical formulations of VRP are variants and/or extensions of the well known TSP [6].

VRP can be formulated as a dynamic programming problem or as an Integer Linear Programming (ILP) problem [7,8], however the ILP formulation is more used because of its inherent simplicity. In this paper, only the ILP formulations are discussed for various problems and also simpler ILP formulations for $M-$ TSP, single depot VRP and multi depot VRP are presented.

2. The travelling salesman problem

If c_{ij} denotes the cost of travelling from city i to city j, N denotes the number of cities and Z denotes the total cost of the solution, then setting $c_{ii}=\infty$, an assignment based ILP formulation for TSP can be written as:

$$\text{Minimize} \quad Z=\sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij}x_{ij}, \tag{1}$$

$$\text{subject to} \quad x_{ij}=1 \text{ or } 0 \quad \text{for all } i, j, \tag{2}$$

$$\sum_{i=1}^{N} x_{ij}=1 \quad \text{for } j=1, 2, \ldots N, \tag{3}$$

$$\sum_{j=1}^{N} x_{ij}=1 \quad \text{for } i=1, 2, \ldots N, \tag{4}$$

$$x_{ij} \in S, \tag{5}$$

where $x_{ij}=1$ indicates that the pair (i, j) is in the tour and $x_{ij}=0$ indicates that the pair (i, j) is not in the tour. The set S in (5) is selected to prohibit subtour solutions satisfying the assignment constraints (2), (3) and (4). Several alternatives have been proposed for S. Some of these alternatives are:

$$\text{(i)} \quad S=\left\{(x_{ij}):\ \sum_{i\in B}\sum_{j\notin B} x_{ij} \geqslant 1 \quad \text{for every nonempty proper subset } B \text{ of } E\right\},$$

where E is a set of nodes, i.e. $E=\{1, 2, \ldots, N\}$.

$$\text{(ii)} \quad S=\left\{(x_{ij}):\ \sum_{i\in B}\sum_{j\in B} x_{ij} \geqslant |B|-1 \quad \text{for every nonempty subset } B \text{ of } \{1, 2, \ldots, N-1\}\right\},$$

where city N represents the home city.

$$\text{(iii)} \quad S=\left\{(x_{ij}): y_i-y_j+Nx_{ij} \leqslant N-1 \quad \text{for } 1 \leqslant i \neq j \leqslant N-1\right\},$$

where y_i is an arbitrary real number.

It will be noted that S contains nearly 2^N subtour breaking constraints in (i) and (ii), but only (N^2-3N+2) constraints in (iii) proposed by Miller et al. [9]. For the formulations that follow, the constraints given by (iii) are used.

3. The M-travelling salesman problem

The M-TSP is a generalization of the TSP. Several papers [10–14] derived equivalent TSP formulations and consequently showed that the M-TSP is no more difficult than a TSP. The ILP formulation of the M-TSP using the earlier notations, as proposed by Svestka [10] and later on modified by Gavish [15,16] is as follows ($R = M + N - 1$):

$$\text{Minimize} \quad Z = \sum_{i=1}^{R} \sum_{j=1}^{R} d_{ij} x_{ij}, \tag{6}$$

$$\text{subject to} \quad x_{ij} = 0 \text{ or } 1, \tag{7}$$

$$\sum_{i=1}^{R} x_{ij} = 1 \quad \text{for } j = 1, 2, \ldots, R, \tag{8}$$

$$\sum_{j=1}^{R} x_{ij} = 1 \quad \text{for } i = 1, 2, \ldots, R, \tag{9}$$

$$y_i - y_j + (N - M) x_{ij} \leqslant N - M - 1 \quad \text{for } 1 \leqslant i \neq j \leqslant N - 1, \tag{10}$$

where M is the number of salesmen. The augmented cost matrix d_{ij} is obtained by adding $(M - 1)$ rows and columns to the original cost matrix c_{ij} such that each new row or column is a duplicate of the last row or column of the matrix c_{ij} (it is assumed here that the last city corresponds to the home city) and setting all other elements of the augmented matrix equal to infinity.

4. The vehicle routing problem

The VRP was considered for the first time by Dantzig and Ramser [17], who developed a heuristic approach using linear programming ideas and aggregation of nodes. For a single depot VRP, if the vehicles have only the capacity and maximum cost (time or distance) constraints, then the formulation for this problem [18,19] can be given as follows:

$$\text{Minimize} \quad Z = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{V} c_{ij} x_{ijk}, \tag{11}$$

$$\text{subject to} \quad \sum_{i=1}^{N} \sum_{k=1}^{V} x_{ijk} = 1 \quad \text{for } j = 1, 2, \ldots, N - 1, \tag{12}$$

$$\sum_{j=1}^{N} \sum_{k=1}^{V} x_{ijk} = 1 \quad \text{for } i = 1, 2, \ldots, N - 1, \tag{13}$$

$$\sum_{i=1}^{N} x_{ihk} - \sum_{j=1}^{N} x_{hjk} = 0 \quad \text{for } k = 1, 2, \ldots V, \quad h = 1, 2, \ldots N, \tag{14}$$

$$\sum_{i=1}^{N} Q_i \sum_{j=1}^{N} x_{ijk} \leqslant P_k \quad \text{for } k = 1, 2, \ldots V, \tag{15}$$

$$\sum_{i=1}^{N} \sum_{j=1}^{N} c_{ij} x_{ijk} \leqslant T_k \quad \text{for } k = 1, 2, \ldots V, \tag{16}$$

$$\sum_{j=1}^{N-1} x_{Njk} \leqslant 1 \quad \text{for } k = 1, 2, \ldots V, \tag{17}$$

$$\sum_{i=1}^{N-1} x_{iNk} \leqslant 1 \quad \text{for } k = 1, 2, \ldots V, \tag{18}$$

$$x_{ijk} = 0 \text{ or } 1 \quad \text{for all } i, j, k, \tag{19}$$

$$y_i - y_j + Nx_{ijk} \leqslant N - 1 \quad \text{for } 1 \leqslant i \neq j \leqslant N-1,\ 1 \leqslant k \leqslant V, \tag{20}$$

where

V = Number of vehicles,
P_k = Capacity of vehicle k,
T_k = Maximum cost allowed for a route of vehicle k,
Q_i = Demand at node i, $Q_N = 0$,
x_{ijh} = 1 if pair i, j is in the route of vehicle k, 0 otherwise.

In the above formulation, (12) and (13) ensure that each customer is served by one and only one vehicle. Route continuity is represented by (14). (15) represents the vehicle capacity constraints. (16) represents the total route constraints. (17) and (18) make certain that vehicle availability is not exceeded. Finally, (20) are the subtourbreaking constraints. Since (12) and (14) imply constraints (13) and (14) and (17) imply constraints (18), (13) and (18) are redundant constraints and these may be dropped from the formulation without altering the final result. The subtour breaking constraints (20) can also be written as a more compact inequality set [2]:

$$y_i - y_j + N \sum_{k=1}^{V} x_{ijk} \leqslant N - 1 \quad \text{for } 1 \leqslant i \neq j \leqslant N-1 \tag{20a}$$

It has been assumed here that for $1 \leqslant i \leqslant N$ and $1 \leqslant k \leqslant V$,

$$\max(Q_i) \leqslant \min(P_k),$$

i.e. the demand at each node is less than or at the most equal to the capacity of each vehicle.

In this model, it has been assumed that whenever a customer is serviced, his requirements are satisfied. A more complicated mixed ILP heterogeneous fleet problem formulation was given by Garvin [20], in which this assumption is relaxed. Foster and Ryan [21] developed an approach due to Pierce [22–24] who in turn adopted a zero–one ILP formulation of Balinski and Quandt [25] for the routing problem.

5. The multi-depot vehcile routing problem

The ILP formulations of single depot VRP can be modified to incorporate multiple depots by making minor changes in the earlier formulations [19]. If the nodes (N + 1, … N + M) represent the M depots, then the new formulation would be:

$$\text{Minimize} \quad Z = \sum_{i=1}^{N+M} \sum_{j=1}^{N+M} \sum_{k=1}^{V} c_{ij} x_{ijk}, \tag{21}$$

$$\text{subject to} \quad \sum_{i=1}^{N+M} \sum_{k=1}^{V} x_{ijk} = 1 \quad \text{for } j = 1, 2, \ldots, N, \tag{22}$$

$$\sum_{j=1}^{N+M} \sum_{k=1}^{V} x_{ijk} = 1 \quad \text{for } i = 1, 2, \ldots, N, \tag{23}$$

$$\sum_{i=1}^{N+M} x_{ihk} - \sum_{j=1}^{N+M} x_{hjk} = 0 \quad \text{for } k = 1, 2, \ldots V, \quad h = 1, 2, \ldots, N+M \tag{24}$$

$$\sum_{i=1}^{N+M} Q_i \sum_{j=1}^{N+M} x_{ijk} \leqslant P_k \quad \text{for } k = 1, 2, \ldots, V, \tag{25}$$

$$\sum_{i=1}^{N+M} \sum_{j=1}^{N+M} c_{ij} x_{ijk} \leqslant T_k \quad \text{for } k = 1, 2, \ldots, V, \tag{26}$$

$$\sum_{i=N+1}^{N+M} \sum_{j=1}^{N} x_{ijk} \leqslant 1 \quad \text{for } k = 1, 2, \ldots, V, \tag{27}$$

$$\sum_{j=N+1}^{N+M} \sum_{i=1}^{N} x_{ijk} \leqslant 1 \quad \text{for } k = 1, 2, \ldots, V, \tag{28}$$

$$x_{ijk} = 0 \text{ or } 1 \quad \text{for all } i, j, k, \tag{29}$$

$$y_i - y_j + (M+N) x_{ijk} \leqslant N + M - 1 \quad \text{for } 1 \leqslant i \neq j \leqslant N \text{ and } 1 \leqslant k \leqslant V. \tag{30}$$

In the above formulation. (22) and (23) ensure that each customer is served by one and only one vehcile. Route continuity is represented by (24). (25) and (26) are the vehcile capacity and total route cost constraints. Vehicle availability is verified by (27) and (28) and subtour breaking constraints are provided by (30). Since (22) and (24) imply constraints (23) and (24) and (27) imply constraints (28), (23) and (28) are redundant constraints and these may be dropped from the formulation without altering the final result. It has been assumed here that the demand at each node is either less than or at the most equal to the capacity of each vehicle. Similar to single depot VRP, the subtour breaking constraint (30) can be rewritten in a more compact form:

$$y_i - y_j + (M+N) \sum_{k=1}^{V} x_{ijk} \leqslant M + N - 1 \quad \text{for } i \leqslant i \neq j \leqslant M + N - 1. \tag{30a}$$

6. A new formulation for M-TSP

A new formulation for the M-TSP is presented here without using the augmented cost matrix discussed earlier in Section 3. This model is based on the work by Miller et al. [9], which incorporates a constraint on the number of cities visited by the salesman. Thus, if L is the maximum number of cities that can be assigned to a salesman; then the formulation using the earlier notation is:

$$\text{Minimize} \quad Z = \sum_{i=1}^{N} \sum_{j=1}^{N} c_{ij} x_{ij}, \tag{31}$$

$$\text{subject to} \quad \sum_{i=1}^{N} x_{ij} = 1 \quad \text{for } j = 1, 2, \ldots, N-1, \tag{32}$$

$$\sum_{j=1}^{N} x_{ij} = 1 \quad \text{for } i = 1, 2, \ldots, N-1 \tag{33}$$

$$\sum_{i=1}^{N} x_{iN} = M, \tag{34}$$

$$\sum_{j=1}^{N} x_{Nj} = M, \tag{35}$$

$$x_{ij} = 0 \text{ or } 1 \quad \text{for all } i, j, \tag{36}$$

$$y_i - y_j + Lx_{ij} \leqslant L - 1 \quad \text{for } 1 \leqslant i \neq j \leqslant N - 1. \tag{37}$$

In the above formulation, (32) and (33) ensure that each node is being served only once whereas (34) and (35) ensure that all the M salesmen are assigned the job. Constraints given by (37) ensure that each salesman is starting his journey from the depot (home city N) and also the number of cities visited by him is always less than or at the most equal to L. This can be very easily proved by contradiction [9]. In particular, for $L = N - M$, the unrestricted M-TSP of Section 3 is obtained.

7. A new formulation for VRP

A simple formulation on similar lines is presented for the case when the limiting capacity and maximum cost (time or distance) constraints are the same for all the vehicles. If P is the capacity of all the vehicles and T is the maximum cost allowed for a route for the vehicle, then for a single depot VRP the formulation can be written as:

$$\text{Minimize} \quad Z = \sum_{i=1}^{N} \sum_{j=1}^{N} c_{ij} x_{ij}, \tag{38}$$

$$\text{subject to} \quad \sum_{i=1}^{N} x_{ij} = 1 \quad \text{for } j = 1, 2, \ldots N - 1, \tag{39}$$

$$\sum_{j=1}^{N} x_{ij} = 1 \quad \text{for } i = 1, 2, \ldots N - 1, \tag{40}$$

$$\sum_{i=1}^{N} x_{iN} = V, \tag{41}$$

$$\sum_{j=1}^{N} x_{Nj} = V, \tag{42}$$

$$x_{ij} = 0 \text{ or } 1 \quad \text{for all } i, j, \tag{43}$$

$$y_i - y_j + Lx_{ij} \leqslant L - 1 \quad \text{for } 1 \leqslant i \neq j \leqslant N - 1, \tag{44}$$

$$u_i - u_j + Px_{ij} \leqslant P - Q_i \quad \text{for } 1 \leqslant i \neq j \leqslant N - 1, \tag{45}$$

$$v_i - v_j + Tx_{ij} \leqslant T - c_{ij} \quad \text{for } 1 \leqslant i \neq j \leqslant N - 1, \tag{46}$$

where y_i, u_i and v_i are all arbitrary real numbers.

In the above formulation, (39)–(42) ensure that each node is being served only once and that all the V vehicles are being used. (44)–(46) are the subtour breaking constraints which also represent the node constraints, capacity constraints and cost constraints respectively. In words these equations ensure that all the tours are starting and ending at the home city N and further every route serves at the most L nodes and the laod and the cost on every route are less than or equal to the vehicle capacity P and the maximum allowable cost T respectively. The proof for (44) is available in Miller et al. [9]. A proof for (45) and (46) can be derived on similar lines.

Proof. Consider equation (45). If there exists a tour $(r_1, r_2, \ldots)$, then $x_{r_1r_2} = 1$ and unless $r_2 = N$ there is a unique r_3 with $x_{r_2r_3} = 1$ and so on, till we get some $r_j = N$. This must happen or else the other alternative is that we get $r_k = r_j, j+1 < k$. Since none of the r's is N, we have

$$u_{r_i} - u_{r_{(i+1)}} + Px_{r_ir_{(i+1)}} \leqslant p - Q_{r_i}$$

or

$$u_{r_i} - u_{r_{(i+1)}} \leqslant -Q_{r_i}.$$

Summing from $i = j$ to $k-1$, we have

$$u_{r_j} - u_{r_k} = 0 \leqslant -\left(Q_{r_j} + Q_{r_{(j+1)}} + \cdots + Q_{r_{(k-1)}}\right),$$

which is a contradiction. Thus all the tours include the home city N. Further, it has to be shown that no route has the total load exceeding the capacity P of the vehicle. Suppose such a tour exists, say $(N, r_1, r_2, \ldots, r_k, N)$ with all $r_i \neq N$, i.e.

$$Q_{r_1} + Q_{r_2} + \cdots + Q_{r_k} \geqslant P,$$

then we have as before

$$u_{r_i} - u_{r_{(i+1)}} \leqslant -Q_{r_i}.$$

Summing from $i = 1$ to $k-1$, we get

$$u_{r_1} - u_{r_k} \leqslant -\left[Q_{r_1} + Q_{r_2} + \cdots + Q_{r_{k-1}}\right]$$

or

$$u_{r_k} - u_{r_1} \geqslant \left[Q_{r_1} + Q_{r_2} + \cdots + Q_{r_{k-1}}\right].$$

Since

$$Q_{r_1} + Q_{r_2} + \cdots + Q_{r_{k-1}} \geqslant P - Q_{r_k}.$$

we can write the above equation as

$$u_{r_k} - u_{r_1} \geqslant P - Q_{r_k}.$$

But we have

$$u_{r_k} - u_{r_1} + Px_{r_kr_1} \leqslant P - Q_{r_k}$$

or

$$u_{r_k} - u_{r_1} \leqslant P\left[1 - x_{r_kr_1}\right] - Q_{r_k} \leqslant P - Q_{r_k},$$

which is a contradiction.

Conversely, if

$$(N, r_1, r_2, \ldots, r_j, \ldots, r_k, N)$$

is one of the routes of a feasible solution, then we have as before

$$u_{r_i} - u_{r_{(i+1)}} \leqslant -Q_{r_i}.$$

It is clear that the values of u_{r_i} can always be adjusted to have

$$u_{r_i} - u_{r_{(i+1)}} = -Q_{r_i}.$$

Summing from i to $j-1$, we get

$$u_{r_i} - u_{r_j} = -\left[Q_{r_i} + Q_{r_{(i+1)}} + \cdots + Q_{r_{(j-1)}}\right].$$

For any other combination (r_i, r_j) of nodes which do not form a link in the tour, we have

$$u_{r_i} - u_{r_j} + Px_{r_i r_j} \leqslant P - Q_{r_i}$$

or

$$u_{r_i} - u_{r_j} \leqslant P - Q_{r_i},$$

which is in agreement with the previous result.

A similar proof can be established for equations (46). The above ILP will involve $(N^2 - 3N + 2)$ constraints for each of the equations (44), (45) and (46). Each set of these equations ensure that all the tours are starting and ending at the home city N and further that the node, laod and cost constraints are satisfied. This is all the more necessary because, in a problem, at a time only one or more constraints might be applicable.

8. A new formulation for multi depot VRP

The above ILP formulation of single depot VRP can be easily extended to incorporate multiple depots. If the nodes $(N+1, \ldots, N+M)$ represent the M depots, then the new formulation can be written as

$$\text{Minimize} \quad Z = \sum_{i=1}^{N+M} \sum_{j=1}^{N+M} c_{ij} x_{ij}, \tag{47}$$

$$\text{subject to} \quad \sum_{i=1}^{N+M} x_{ij} = 1 \quad \text{for } j = 1, 2, \ldots, N, \tag{48}$$

$$\sum_{j=1}^{N+M} x_{ij} = 1 \quad \text{for } i = 1, 2, \ldots, N, \tag{49}$$

$$\sum_{i=N+1}^{N+M} \sum_{j=1}^{N+M} x_{ij} = V, \tag{50}$$

$$\sum_{j=N+1}^{N+M} \sum_{i=1}^{N+M} x_{ij} = V, \tag{51}$$

$$x_{ij} = 0 \text{ or } 1 \quad \text{for all } i, j, \tag{52}$$

$$y_i - y_j + Lx_{ij} \leqslant L - 1 \quad \text{for } 1 \leqslant i \neq j \leqslant N, \tag{53}$$

$$u_i - u_j + Px_{ij} \leqslant P - Q_i \quad \text{for } 1 \leqslant i \neq j \leqslant N, \tag{54}$$

$$v_i - v_j + Tx_{ij} \leqslant T - c_{ij} \quad \text{for } 1 \leqslant i \neq j \leqslant N, \tag{55}$$

where V is the total number of vehicles available.

In the above formulations, (48)–(51) ensure that each node is being served only once and that all the V vehicles are being used. Subtour breaking constraints are represented by (53)–(55). These equations also represent the node constraints, laod constraints and cost constraints respectively.

Instead of the total number of vehicles, if the number of vehicles available with various depots is known then the above formulation with minor changes in (50) and (51) can still be used. The changed equations would be

$$\sum_{j=1}^{N+M} x_{ij} = V_i \quad \text{for } i = N+1, N+2, \ldots, N+M, \tag{50a}$$

$$\sum_{i=1}^{N+M} x_{ij} = V_j \quad \text{for } j = N+1, N+2, \ldots, N+M, \tag{51a}$$

where V_i represents the number of vehicles available at node i.

9. Conclusions

The formulations presented in the earlier sections provide a basis for viewing a VRP as a generalised TSP. A comparison of formulations presented in Sections 6, 7 and 8 for M-TSP, VRP and multi depot VRP with their earlier counterparts in Sections 3, 4 and 5 clearly shows that the models developed here are simpler and easier to understand and are more flexible to incorporate various types of constraints. The principal advantage of these models is the drastic reduction, by a factor V, in the number of variables. The limitations of the models are (i) the number of the non-assignment constraints increases; and (ii) the capacity and the maximum permissible cost (time or distance) of each vehicle are the same.

The models presented here have an academic importance, since ILP methods are not yet fully developed to attempt any of the real life problems in reasonable computing time. It is for this reason that all the practical problems of this class are attempted by using heuristic procedures [18, 19, 26]. In any case, the models discussed above do provide a basis for viewing a VRP or a multi depot VRP as a generalised TSP and also illustrate how problems of this sort may be usefully formulated in ILP terms.

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