# An introduction to Bayesian modeling using R and JAGS


Instructors

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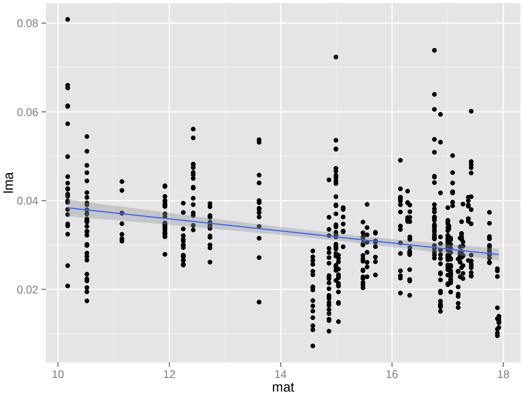
# Overview of the Workshop

- Focus on using R and JAGS for Bayesian analysis
- Linear regression including mixed modeling – Kent Holsinger
  - Simple linear regression
  - Multiple regression (including random effects)
- Multicollinearity – Xiaojing Wang
  - Hierarchical independent prior distributions
  - Variable selection

# Linear Regression

One of the most common statistical procedures in ecology and evolution. For example,

- Data on LMA from 535 individuals in the genus *Protea* (42 species, 48 sites, 142 unique site/species combinations)
- Data on mean annual temperature for each of those sites



# Linear Regression

## In R

```
> summary(lm(lma ~ mat, data=tmp))

Call:
lm(formula = lma ~ mat, data = tmp)

Residuals:
      Min        1Q    Median        3Q       Max
-0.025126 -0.005781 -0.000785  0.004647  0.044444

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.0521895  0.0027116  19.246  < 2e-16 ***
mat         -0.0013587  0.0001785  -7.611 1.24e-13 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 0.009815 on 533 degrees of freedom
Multiple R-squared:  0.09803, Adjusted R-squared:  0.09634
F-statistic: 57.93 on 1 and 533 DF,  p-value: 1.241e-13
```

# Linear Regression

Remember basic assumptions of simple linear regression

$$
\begin{aligned}
y_i &= \beta_0 + \beta_1 x_i + \epsilon_i \\
\epsilon_i &\sim \mathrm{N}(0, \sigma^2)
\end{aligned}
$$

Here's another way to write that

$$
\begin{aligned}
y_i &\sim \mathrm{N}(\mu_i, \sigma^2) \\
\mu_i &= \beta_0 + \beta_1 x_i
\end{aligned}
$$

The second way of writing the model will be more convenient for us, so that's the approach we'll use.

# Statistical Analysis

Statistical inference is the process of learning about the general characteristics of a population from a sample.

- Characteristics often expressed in terms of parameters $\theta$.
- Measurements on the subset of members given by numerical values $Y$.
- Before the data are observed, both $Y$ and $\theta$ are unknown.
- A probability model is assumed for observed data if we knew $\theta$ is the truth.
- What if we have prior information about $\theta$?

# Bayesian Inference

Bayesian inference allows us to update prior beliefs with the observed data to quantify uncertainty about $\theta$.

- Prior Distribution: $p(\theta)$
- Sampling Model (likelihood): $p(y \mid \theta)$
- Posterior Distribution

$$p(\theta \mid y) = \frac{p(y \mid \theta)p(\theta)}{p(y)}$$

- Calculating $p(y)$ is typically very challenging. Use MCMC (implemented in JAGS) to estimate $p(\theta \mid y)$.

# Metropolis-Hastings Algorithm

For $\theta_j$

- Propose a new $\theta_j^* \sim q(\theta_j^{(t)}|\theta_j^t)$
- Calculate Metropolis-Hastings ratio

$$\alpha = \frac{p(Y|\theta_j^*)p(\theta_j^*)/q(\theta_j^*|\theta_j^{(t)})}{p(Y|\theta_j^{(t)})p(\theta_j^{(t)})/q(\theta_j^{(t)}|\theta_j^*)}$$

- if $\alpha < 1$ set

$$\theta_j^{(t+1)} = \begin{cases} \theta_j^* & \text{with probability } \alpha \\ \theta_j^{(t)} & \text{with probability } 1-\alpha \end{cases}$$

  If $\alpha > 1$ set $\theta_j^{(t+1)} = \theta_j^*$
- Repeat for $j = 1, \ldots, J$
- Repeat for $t = 1, \ldots, T$

# Linear Regression - as a Bayesian

We start with the sampling model $p(y \mid \theta)$, where $\theta = (\beta_0, \beta_1, \sigma^2)'$,

$$\begin{aligned} y_i &\sim \mathrm{N}(\mu_i, \sigma^2)\ , \\ \mu_i &= \beta_0 + \beta_1 x_i\ , \end{aligned}$$

and $x_i$ is the value of the covariate in individual $i$. Then we add prior distributions $p(\theta)$,

$$\begin{aligned} \beta_0 &\sim \mathrm{N}(0, \tau^{-1}), \\ \beta_1 &\sim \mathrm{N}(0, \tau^{-1}), \\ \sigma^2 &= \frac{1}{\tau_{resid}}, \\ \tau_{resid} &\sim \mathrm{Gamma}(1, \phi) \end{aligned}$$

# Linear Regression - in R+JAGS[1]

- Rescale all variables to mean of 0, standard deviation of 1

```
Inference for Bugs model at "simple-linear-regression.jags", fit using jags,
 5 chains, each with 10000 iterations (first 5000 discarded), n.thin = 5
 n.sims = 5000 iterations saved
             mu.vect sd.vect   2.5%    25%    50%    75%  97.5% Rhat n.eff
beta.0         0.000   0.041 -0.082 -0.028  0.000  0.028  0.080 1.001  5000
beta.mat      -0.313   0.041 -0.393 -0.341 -0.314 -0.285 -0.231 1.002  1900
sigma.resid    0.953   0.029  0.897  0.933  0.952  0.972  1.012 1.001  3700
```

- Compare with lm() results from R

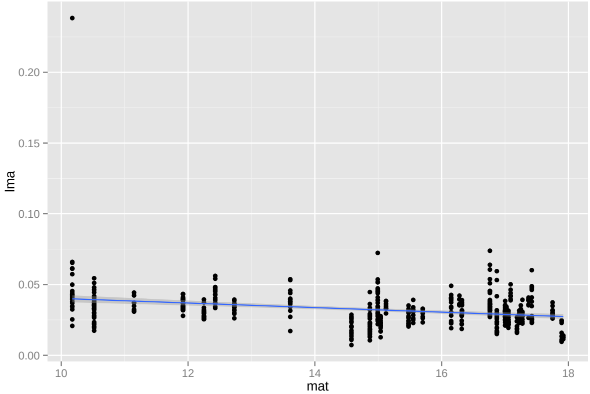
```
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  8.614e-17  4.110e-02   0.000        1
mat         -3.131e-01  4.114e-02  -7.611 1.24e-13 ***

Residual standard error: 0.9506 on 533 degrees of freedom
```

[1] simple-linear-regression.jags

# Robust Regression

- What do we do with outliers?




- Could delete the observation if we have good evidence that it's an error.
- Probably better to use an approach that leaves the data in and gives less weight to outliers.

# Robust Regression[3]

- Use a Cauchy distribution ("fat tails") instead of normal for prior on regression coefficients and residual error.[2]

| | | Coefficient estimates | |
|---|---|---|---|
| Model | Data set | $\beta_0$ | $\beta_{MAT}$ |
| Simple | Original | 0.000 | -0.313 |
| | Modified | 0.028 | -0.372 |
| Robust | Original | -0.078 | -0.294 |
| | Modified | -0.079 | -0.293 |

- Estimates not affected by extreme outlier when using Cauchy prior.

[2] t distribution with 1 degree of freedom in JAGS

[3] robust-linear-regression.jags

# Multiple Linear Regression

Simple generalization of what we've already seen

$$
\begin{aligned}
y_i &\sim \mathrm{N}(\mu_i, \sigma^2) \ , \\
\mu_i &= \beta_0 + \sum_{k=1}^{K} \beta_k x_{ik} \ ,
\end{aligned}
$$

where $x_{ik}$ is the value of the $k$th covariate in individual $i$. The priors are

$$
\begin{aligned}
\beta_k &\sim \mathrm{N}(0, \tau^{-1}), \quad k = 0, \ldots, K \ , \\
\sigma^2 &= \frac{1}{\tau_{resid}} \ , \\
\tau_{resid} &\sim \mathrm{Gamma}(1, \phi).
\end{aligned}
$$

# Multiple Linear Regression[4]

## From JAGS

```
Inference for Bugs model at "multiple-linear-regression.jags", fit using jags,
 5 chains, each with 10000 iterations (first 5000 discarded), n.thin = 5
 n.sims = 5000 iterations saved
            mu.vect sd.vect   2.5%    25%    50%    75%  97.5%  Rhat n.eff
beta.0        0.000   0.040 -0.078 -0.027  0.000  0.028  0.078 1.001  4800
beta.cdd      0.106   0.073 -0.036  0.058  0.105  0.154  0.251 1.001  5000
beta.elev    -0.319   0.094 -0.500 -0.384 -0.319 -0.256 -0.136 1.001  5000
beta.inso     0.054   0.053 -0.048  0.018  0.054  0.091  0.157 1.001  5000
beta.map     -0.022   0.078 -0.178 -0.074 -0.020  0.031  0.129 1.001  5000
beta.mat     -0.463   0.114 -0.688 -0.541 -0.461 -0.385 -0.243 1.001  5000
beta.ratio   -0.016   0.079 -0.172 -0.069 -0.013  0.039  0.134 1.001  5000
sigma.resid   0.939   0.029  0.884  0.919  0.939  0.958  1.000 1.001  5000
```

## Compare to lm() from R

```
              Estimate Std. Error t value Pr(>|t|)
(Intercept) -3.520e-17  4.052e-02   0.000 1.000000
cdd          1.051e-01  7.315e-02   1.437 0.151291
elev        -3.184e-01  9.534e-02  -3.339 0.000899 ***
inso         5.288e-02  5.287e-02   1.000 0.317702
map         -2.309e-02  7.749e-02  -0.298 0.765836
mat         -4.629e-01  1.147e-01  -4.037  6.2e-05 ***
ratio       -1.481e-02  7.944e-02  -0.186 0.852218

Residual standard error: 0.9373 on 528 degrees of freedom
```

[4] multiple-linear-regression.jags

# Multiple Linear Regression with Species Random Effect

$\gamma_i^{(s)}$ denotes the mean for species $s$ to which inidividual $i$ belongs

$$
\begin{aligned}
y_i &\sim \mathrm{N}(\mu_i, \sigma^2_{resid}) \ , \\
\mu_i &= \beta_0 + \sum_{k=1}^{K} \beta_k x_{ik} + \gamma_i^{(s)} \ , \\
\beta_k &\sim \mathrm{N}(0, \tau^{-1}), \quad k = 0, \ldots, K \ , \\
\sigma^2_{resid} &= \frac{1}{\tau_{resid}} \ , \\
\tau_{resid} &\sim \mathrm{Gamma}(1, \phi) \ , \\
\gamma_i^{(s)} &\sim \mathrm{N}(0, \sigma^2_{species}) \ , \\
\sigma^2_{species} &= \frac{1}{\tau_{species}} \ , \\
\tau_{species} &\sim \mathrm{Gamma}(1, \phi) \ .
\end{aligned}
$$

# Multiple Linear Regression with Species Random Effect

Alternatively

$$
\begin{aligned}
y_i &\sim \mathrm{N}(\mu_i, \sigma^2_{resid}) \ , \\
\mu_i &= \beta^{(s)}_{0i} + \sum_{k=1}^{K} \beta_k x_{ik} \ , \\
\sigma^2_{resid} &= \frac{1}{\tau_{resid}} \ , \\
\tau_{resid} &\sim \mathrm{Gamma}(1, \phi) \ , \\
\beta^{(s)}_{0i} &\sim \mathrm{N}(\beta_0, \sigma^2_{species}) \ , \\
\sigma^2_{species} &= \frac{1}{\tau_{species}} \ , \\
\tau_{species} &\sim \mathrm{Gamma}(1, \phi) \ , \\
\beta_i &\sim \mathrm{N}(0, \tau), \quad i = 0, \ldots, K \ .
\end{aligned}
$$

# Multiple Linear Regression with Species Random Effect[5]

## From JAGS

```
beta.cdd        0.081   0.055  -0.024   0.045   0.082   0.118   0.188 1.001  3100
beta.elev      -0.201   0.108  -0.408  -0.274  -0.201  -0.128   0.010 1.002  2400
beta.inso      -0.073   0.058  -0.187  -0.112  -0.073  -0.034   0.040 1.001  3400
beta.map       -0.418   0.083  -0.581  -0.474  -0.419  -0.362  -0.257 1.001  5000
beta.mat        0.093   0.110  -0.117   0.020   0.092   0.169   0.307 1.001  5000
beta.ratio      0.427   0.078   0.278   0.374   0.427   0.478   0.578 1.001  4800
beta.zero      -0.064   0.154  -0.369  -0.167  -0.062   0.037   0.241 1.001  5000
sigma.resid     0.545   0.023   0.511   0.532   0.544   0.556   0.581 1.001  3100
sigma.species   0.966   0.117   0.767   0.884   0.960   1.036   1.218 1.001  5000
```

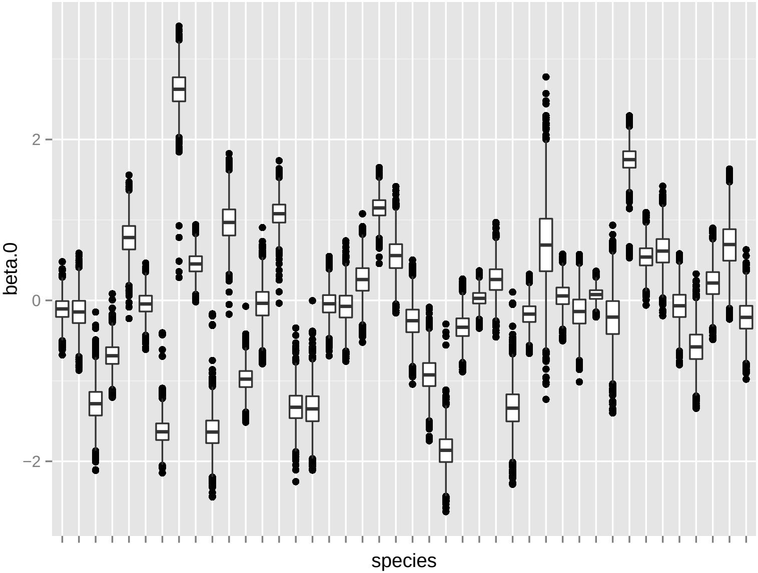
## Compare to lmer() From R

```
Random effects:
 Groups   Name        Variance Std.Dev.
 species  (Intercept) 0.8951   0.9461
 Residual             0.2924   0.5408
Number of obs: 535, groups:  species, 42

Fixed effects:
            Estimate Std. Error t value
(Intercept) -0.06289    0.14898  -0.422
cdd          0.07941    0.05576   1.424
elev        -0.19835    0.10946  -1.812
inso        -0.07259    0.05791  -1.254
map         -0.42009    0.08176  -5.138
mat          0.09811    0.10893   0.901
ratio        0.43022    0.07776   5.533
```

[5] random-effect-multiple-regression.jags

# Multiple Linear Regression with Species Random Effect



# Problems with Multicollinearity

- Variables may appear to be unimportant (when they are).
- Coefficient estimates are unstable and hard to interpret (can estimate combinations of coefficients but not individual coefficients).

Alternative Bayesian solutions:

- Independent Prior Distributions
- Variable Selection

# Hierarichal Model with Independent Priors

Hierarchical Model:

$$\begin{aligned} \beta_j \mid \lambda_j, \sigma^2 &\sim \mathsf{N}(0, \sigma^2/\lambda_j) \\ \lambda_j \mid \sigma^2 &\sim \mathsf{Gamma}(1/2, 1/2) \\ 1/\sigma^2 &\sim \mathsf{Gamma}(\nu_0/2, \nu_0\sigma_0^2/2) \end{aligned}$$

- Leads to nice conjugate updates for all full conditionals
- Easy to code in JAGS
- Allows each parameter to have own precision with mean 1

# Cauchy Prior

First two equations imply that $\beta_j|\sigma^2 \sim \text{Cauchy}(0, \sigma^2)$

$$p(\beta) = \frac{1}{\pi\sigma}\left(1 + \frac{\beta^2}{\sigma^2}\right)^{-1},$$

leading to a collapsed model

$$\begin{aligned} Y|\boldsymbol{\beta}, \sigma^2 &\sim \text{N}(X\boldsymbol{\beta}, \sigma^2 I_n)\ , \\ \beta_j|\sigma^2 &\sim \text{Cauchy}(0, \sigma^2)\ , \\ 1/\sigma^2 &\sim \text{Gamma}(\nu_0/2, \nu_0\sigma_0^2/2)\ . \end{aligned}$$

No nice full conditional for $\beta_j$.

# Cauchy Prior[6]

## Independent N(0, 1)

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| beta.cdd | 0.081 | 0.055 | -0.024 | 0.045 | 0.082 | 0.118 | 0.188 | 1.001 | 3100 |
| beta.elev | -0.201 | 0.108 | -0.408 | -0.274 | -0.201 | -0.128 | 0.010 | 1.002 | 2400 |
| beta.inso | -0.073 | 0.058 | -0.187 | -0.112 | -0.073 | -0.034 | 0.040 | 1.001 | 3400 |
| beta.map | -0.418 | 0.083 | -0.581 | -0.474 | -0.419 | -0.362 | -0.257 | 1.001 | 5000 |
| beta.mat | 0.093 | 0.110 | -0.117 | 0.020 | 0.092 | 0.169 | 0.307 | 1.001 | 5000 |
| beta.ratio | 0.427 | 0.078 | 0.278 | 0.374 | 0.427 | 0.478 | 0.578 | 1.001 | 4800 |
| beta.zero | -0.064 | 0.154 | -0.369 | -0.167 | -0.062 | 0.037 | 0.241 | 1.001 | 5000 |
| sigma.resid | 0.545 | 0.023 | 0.511 | 0.532 | 0.544 | 0.556 | 0.581 | 1.001 | 3100 |
| sigma.species | 0.966 | 0.117 | 0.767 | 0.884 | 0.960 | 1.036 | 1.218 | 1.001 | 5000 |

## Independent hierarchical

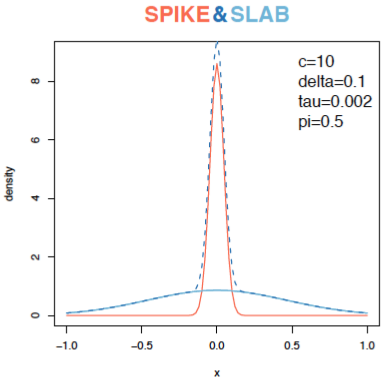
| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| beta.cdd | 0.084 | 0.055 | -0.026 | 0.048 | 0.085 | 0.122 | 0.188 | 1.003 | 1300 |
| beta.elev | -0.191 | 0.104 | -0.394 | -0.262 | -0.190 | -0.120 | 0.009 | 1.004 | 800 |
| beta.inso | -0.077 | 0.059 | -0.192 | -0.117 | -0.077 | -0.038 | 0.039 | 1.001 | 5000 |
| beta.map | -0.400 | 0.082 | -0.562 | -0.455 | -0.401 | -0.344 | -0.240 | 1.004 | 870 |
| beta.mat | 0.084 | 0.107 | -0.129 | 0.013 | 0.084 | 0.157 | 0.288 | 1.009 | 370 |
| beta.ratio | 0.416 | 0.079 | 0.262 | 0.363 | 0.417 | 0.469 | 0.570 | 1.007 | 510 |
| beta.zero | -0.060 | 0.142 | -0.338 | -0.154 | -0.061 | 0.035 | 0.224 | 1.002 | 2700 |
| sigma.resid | 0.545 | 0.018 | 0.512 | 0.533 | 0.544 | 0.556 | 0.580 | 1.001 | 5000 |
| sigma.species | 0.958 | 0.114 | 0.767 | 0.876 | 0.947 | 1.028 | 1.211 | 1.001 | 5000 |

[6] hierarchical-prior-multiple-regression.jags

# Stochastic Search Variable Selection

The Spike-and-Slab prior:

$$\begin{aligned} \beta_j \mid \gamma_j, c, \tau_j^{-1} &\sim (1-\gamma_j)\mathcal{N}(0, \tau_j^{-1}) + \gamma_j \mathcal{N}(0, \tau_j^{-1} c^2) \ , \\ \gamma_j \mid \pi_j &\sim \mathit{Bernoulli}(\pi_j) \ . \end{aligned}$$




- $\gamma_j = 0$: Variable not in the model;
- $\gamma_j = 1$: Variable in the model;
- Calibration of hyper-parameters $c$, $\tau_j^{-1}$ needed.

## Inference for Variable Selection

- Highest posterior model (HPM): Select a model that has been visited most often.
- Median probability model (MPM): Select variables that appear at least in 50% of visited models.

## Alternative spike and slab models

- Popular approach in genomic research;
- Variants:
  - Conjugate version:

$$\beta_j \mid \gamma_j, c, \tau_j^{-1} \sim (1-\gamma_j)\mathcal{N}(0, \sigma^2\tau_j^{-1}) + \gamma_j\mathcal{N}(0, \sigma^2\tau_j^{-1}c^2).$$

  - Replace the spike normal in Spike-and-Slab prior by Dirac, i.e.,

$$\beta_j \mid \gamma_j, \tau_j^{-1} \sim (1-\gamma_j)\delta_0 + \gamma_j\mathcal{N}(0, \tau_j^{-1}).$$

# Variable selection – Dirac + Normal[7]

## Independent N(0, 1)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| beta.cdd | 0.081 | 0.055 | -0.024 | 0.045 | 0.082 | 0.118 | 0.188 | 1.001 | 3100 |
| beta.elev | -0.201 | 0.108 | -0.408 | -0.274 | -0.201 | -0.128 | 0.010 | 1.002 | 2400 |
| beta.inso | -0.073 | 0.058 | -0.187 | -0.112 | -0.073 | -0.034 | 0.040 | 1.001 | 3400 |
| beta.map | -0.418 | 0.083 | -0.581 | -0.474 | -0.419 | -0.362 | -0.257 | 1.001 | 5000 |
| beta.mat | 0.093 | 0.110 | -0.117 | 0.020 | 0.092 | 0.169 | 0.307 | 1.001 | 5000 |
| beta.ratio | 0.427 | 0.078 | 0.278 | 0.374 | 0.427 | 0.478 | 0.578 | 1.001 | 4800 |
| beta.zero | -0.064 | 0.154 | -0.369 | -0.167 | -0.062 | 0.037 | 0.241 | 1.001 | 5000 |
| sigma.resid | 0.545 | 0.023 | 0.511 | 0.532 | 0.544 | 0.556 | 0.581 | 1.001 | 3100 |
| sigma.species | 0.966 | 0.117 | 0.767 | 0.884 | 0.960 | 1.036 | 1.218 | 1.001 | 5000 |

## Dirac + N(0, 1)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| beta.cdd | 0.002 | 0.015 | 0.000 | 0.000 | 0.000 | 0.000 | 0.039 | 1.029 | 1100 |
| beta.elev | -0.277 | 0.112 | -0.457 | -0.349 | -0.292 | -0.224 | 0.000 | 1.016 | 310 |
| beta.inso | -0.003 | 0.019 | -0.052 | 0.000 | 0.000 | 0.000 | 0.000 | 1.054 | 460 |
| beta.map | -0.473 | 0.075 | -0.609 | -0.523 | -0.478 | -0.429 | -0.306 | 1.012 | 440 |
| beta.mat | 0.015 | 0.060 | 0.000 | 0.000 | 0.000 | 0.000 | 0.240 | 1.009 | 1100 |
| beta.ratio | 0.378 | 0.057 | 0.282 | 0.343 | 0.374 | 0.407 | 0.516 | 1.002 | 1800 |
| beta.zero | -0.061 | 0.151 | -0.355 | -0.160 | -0.060 | 0.039 | 0.240 | 1.001 | 5000 |
| sigma.resid | 0.546 | 0.023 | 0.512 | 0.533 | 0.545 | 0.557 | 0.581 | 1.001 | 5000 |
| sigma.species | 0.948 | 0.116 | 0.750 | 0.868 | 0.939 | 1.019 | 1.193 | 1.002 | 2100 |

[7] dirac-plus-normal-multiple-regression.jags

# Variable selection – Dirac + Normal

## Dirac + $N(0, 1)$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| beta.cdd | 0.002 | 0.015 | 0.000 | 0.000 | 0.000 | 0.000 | 0.039 | 1.029 | 1100 |
| beta.elev | -0.277 | 0.112 | -0.457 | -0.349 | -0.292 | -0.224 | 0.000 | 1.016 | 310 |
| beta.inso | -0.003 | 0.019 | -0.052 | 0.000 | 0.000 | 0.000 | 0.000 | 1.054 | 460 |
| beta.map | -0.473 | 0.075 | -0.609 | -0.523 | -0.478 | -0.429 | -0.306 | 1.012 | 440 |
| beta.mat | 0.015 | 0.060 | 0.000 | 0.000 | 0.000 | 0.000 | 0.240 | 1.009 | 1100 |
| beta.ratio | 0.378 | 0.057 | 0.282 | 0.343 | 0.374 | 0.407 | 0.516 | 1.002 | 1800 |
| beta.zero | -0.061 | 0.151 | -0.355 | -0.160 | -0.060 | 0.039 | 0.240 | 1.001 | 5000 |
| sigma.resid | 0.546 | 0.023 | 0.512 | 0.533 | 0.545 | 0.557 | 0.581 | 1.001 | 5000 |
| sigma.species | 0.948 | 0.116 | 0.750 | 0.868 | 0.939 | 1.019 | 1.193 | 1.002 | 2100 |

## Posterior conditioned on $\gamma_i > 0$

```
  beta.cdd:  0.04      0.057 (-0.042,  0.168)
 beta.elev:  0.93     -0.298 (-0.459, -0.134)*
 beta.inso:  0.05     -0.061 (-0.188,  0.059)
  beta.map:  1.00     -0.473 (-0.609, -0.306)*
  beta.mat:  0.11      0.142 (-0.111,  0.367)
beta.ratio:  1.00      0.379 ( 0.283,  0.516)*
```

# Model Selection

Selection of a single model has the following problems

- When the criteria suggest that several models are equally good, what should we report? Still pick only one model?
- What do we report for our uncertainty after selecting a model?

Typical analysis ignores model uncertainty!

# Bayesian Model Choice

- Models for the variable selection problem are based on a subset of the $x_1, \cdots, x_p$ variables.
- Encode models with a vector $\boldsymbol{\gamma} = (\gamma_1, \cdots, \gamma_p)'$ where $\gamma_j \in \{0, 1\}$ is an indicator for whether variable $x_j$ should be included in the model $M_{\gamma}$. Notice $\gamma_j = 0 \Leftrightarrow \beta_j = 0$.
- Each value of $\boldsymbol{\gamma}$ represents one of the $2^p$ models.
- Under model $M_{\gamma}$:

$$Y \mid \boldsymbol{\beta}, \boldsymbol{\gamma}, \tau \sim \mathcal{N}(X_{\gamma}\boldsymbol{\beta}_{\gamma}, \tau^{-1}\mathbf{I})$$

where $X_{\gamma}$ is the design matrix using the columns in $X$ where $\gamma_j = 1$ and $\boldsymbol{\beta}_{\gamma}$ is the subset of $\boldsymbol{\beta}$ that are non-zero.

# Bayesian Model Averaging

Rather than use a single model, BMA uses all (or potentially a lot) models, but weights model predictions by their posterior probabilities (measure of how much each model is supported by the data).

- Posterior model probabilities

$$\mathrm{P}(M_j \mid Y) = \frac{\mathrm{P}(Y \mid M_j)\mathrm{P}(M_j)}{\sum_j \mathrm{P}(Y \mid M_j)\mathrm{P}(M_j)},$$

  Marginal likelihod of a model is

$$\mathrm{P}(Y \mid M_\gamma) = \int\int \mathrm{P}(Y \mid \boldsymbol{\beta}_\gamma, \tau)\mathrm{P}(\boldsymbol{\beta}_\gamma \mid \boldsymbol{\gamma}, \tau)\mathrm{P}(\tau \mid \boldsymbol{\gamma}) d\boldsymbol{\beta}_\gamma d\tau.$$

- Probability $\beta_j \neq 0$: $\sum_{M_j : \beta_j \neq 0} \mathrm{P}(M_j \mid Y)$.

# Bayesian Model Averaging (Continued)

- Predictions

$$\mathrm{P}(Y^{new} \mid Y) = \sum_j \mathrm{P}(Y^{new} \mid Y, M_j)\mathrm{P}(M_j \mid Y),$$

where

$$\mathrm{P}(Y^{new} \mid Y, M_\gamma) = \int \mathrm{P}(Y^{new} \mid Y, \boldsymbol{\beta}_\gamma, \tau)\mathrm{P}(\boldsymbol{\beta}_\gamma, \tau \mid Y) d\boldsymbol{\beta}_\gamma d\tau.$$