Introduction

Prime numbers are natural numbers greater than 1 that are divisible only by 1 and themselves. They are considered the "building blocks" of arithmetic. However, to this day, their distribution remains one of the most mysterious and fascinating topics in mathematics.

Similarly, accurately estimating the number of prime numbers less than or equal to a positive integer m, i.e., the function $\pi(m)$, is a central problem in number theory with significant applications in cryptography, algorithm analysis, and other fields. However, calculating $\pi(m)$ accurately remains a major open question.

In Chapter I of this study, I will identify the patterns of prime numbers, or in other words, how we can predict whether a given number might be prime.

In Chapter II of this study, I will present a method for accurately calculating $\pi(m)$\.

I. Primes Rule

In this section, we will present the distribution of prime numbers in a tabular format, allowing for a clearer visualization of the patterns inherent in their distribution. By systematically organizing the prime numbers and examining their positions, we aim to identify underlying regularities and trends. This approach will provide us with a more structured and accessible view of prime number behavior.

Subsequently, based on the observed patterns, we will develop an algorithm designed to accurately identify these regularities. The proposed algorithm will be capable of predicting prime number distributions, enhancing both our understanding and ability to compute prime numbers more efficiently. The goal is to create an approach that can be generalized and applied to a broad range of prime-related problems in number theory and related fields.

I.1. Primes 1 to 1000

2	3	5	7	11	13	17	19	23	29
31	37	41	43	47	53	59	61	67	71
73	79	83	89	97	101	103	107	109	113
127	131	137	139	149	151	157	163	167	173
179	181	191	193	197	199	211	223	227	229
233	239	241	251	257	263	269	271	277	281
283	293	307	311	313	317	331	337	347	349
353	359	367	373	379	383	389	397	401	409
419	421	431	433	439	443	449	457	461	463
467	479	487	491	499	503	509	521	523	541
547	557	563	569	571	577	587	593	599	601
607	613	617	619	631	641	643	647	653	659
661	673	677	683	691	701	709	719	727	733
739	743	751	757	761	769	773	787	797	809
811	821	823	827	829	839	853	857	859	863
877	881	883	887	907	911	919	929	937	941
947	953	967	971	977	983	991	997		

In order to clearly discern the underlying structure, it is necessary to partition the set into the following subsets:

	2	3		5
7				11
13				17
19				23
				29

	2	3		5		7				11		13				17		19				23						29	
31						37				41		43				47						53						59	
61						67				71		73						79				83						89	
						97				101		103				107		109				113							
						127				131						137		139										149	
151						157						163				167						173						179	
181										191		193				197		199											

I.2. Definitions

P(n): The set of all prime numbers up to the *n*-th level.

R(n): The set of multiples of p_n that aren't divisible by any of $p_0, p_1, \dots, p_{n-1}$.

X(n): The base prime table of level n; this contains the reduced residues modulo *A(n)*, where all primes greater than p_n are congruent to some $x_i \in X(n)$ modulo *A(n).*

A(n): The prime coefficient product of level n, defined as:

$$A(n) = \prod_{i=0}^{n} p_i$$

I.3. Procedure

We intialize :

$$P(0) = \{2\}$$

$$R(0) = \{2\}$$

$$X(0) = \{1,\}$$

$$A(0) = 2$$

Step 1: Constructing *X(n+1)*

- Repeat the set $X(n)$ p_{n+1}-1 times to form a candidate list.
- For each $j > n$ where $p_j^2 \leq A(n+1)$, define:

$$R(j) = p_j \cdot \{x_1, x_2, \dots\dots x_k\},\ x_i \in X(j-1)$$

Where:

$$p_j \cdot x_k \leq A(n+1) < p_j \cdot x_{k+1}$$

Step 2: Update Core Structures

- Define:

$$X(n+1) = p_{n+1}X(n) - R(p_{n+1}) - \{p_{n+1}\} \qquad (1)$$

- Update:

$$A(n+1) = A(n).\ p_{n+1}$$

Step 3: Prime |Set Construction

The update prime set is given by:

$$P(n+1) = X(n+1) + \{p_0, p_{1,}, \dots, p_{n+1,}\} - \sum_{i=n+1}^{j} R(i) - \{1\}$$

Where:

$$p_j^2 \leq A(n+1) < p_{j+1}^2$$

I.4. *Example Computations*

Case n = 1

P(0)= {2}, R(0)= {2}, X(0)= {1,}, A(0)= 2 .

Repeat X(0) 2 times:

1	
3	
5	

X(1) = {1, , , ,5, }

P(1) = {2,3,5}

A(1) = 6

Case n = 2

Repeat X(1) 4 times :

1				5	
7				11	
13				17	
19				23	
25				29	

Since $p_2^2 = 5^2 = 25 < 5 \cdot 6 = 30$, we construct

$$R(2) = 5 \cdot \{5\} = \{25\}$$

Then:

$$X(2) = 5(X(1)) - R(2) - \{5\}$$

= {1, , , ,5, ,7, , , ,11, ,13, , , ,17, ,19, , , ,23, ,25, , , ,29, }-{25} –{5}

= {1, , , , ,7, , , ,11, ,13, , , ,17, ,19, , , ,23, , , , ,29, }

P(2) = {2, 3, 5, 7, 11, 13, 17, 19, 23, 29}

$$A(2) = 5 \cdot 6 = 30$$

Case n=3: Repeat X(2) 6 times to form a table

1						7				11		13				17		19				23						29	
31						37				41		43				47		49				53						59	
61						67				71		73				77		79				83						89	
91						97				101		103				107		109				113						119	
121						127				131		133				137		139				143						149	
151						157				161		163				167		169				173						179	
181						187				191		193				197		199				203						209	

Since: $p_3^2 = 7^2 = 49 < 7.\ 30 = 210$

$R(3) = 7 \cdot \{7,11,13,17,19,23,29\}$

= {49, 77, 91, 119, 133, 161, 203}

$P_4^2 = 11^2 = 121 < 7.30 = 210$

$R(4) = 11 \cdot \{11,13,17,19\}$

={121, 143, 187, 209}

$P_5^2 = 13^2 = 169 < 7.30 = 210$

$R(5) = 13 \cdot \{13\} = \{169\}$

X(3) = 7.X(2) – R(3) - 7={1, , , , ,7, , , ,11, ,13, , , ,17, ,19, , , ,23, , , , ,29, , 31, , , , ,37, , , ,41, ,43, , , ,47, ,49, , , ,53, , , , ,59, , 61, , , , ,67, , , ,71, ,73, , , ,77, ,79, , , ,83, , , , ,89, , 91, , , , ,97, , , ,101, ,103, , , ,107, ,109, , , ,113, , , , ,119, , 121, , , , ,127, , , ,131, ,133, , , ,137, ,139, , , ,143, , , , ,149, ,151, , , , ,157, , , ,161, ,163, , , ,167, ,169, , , ,173, , , , ,179, , 181, , , , ,187, , , ,191, ,193, , , ,197, ,199, , , ,203, , , , ,209,} –{49, 77, 91, 119, 133, 161, 203}-{7}

={1, , , , , , , , , ,11, ,13, , , ,17, ,19, , , ,23, , , , ,29, 31, , , , , ,37, , , ,41, ,43, , , ,47, , , , , ,53, , , , , ,59, , 61, , , , , ,67, , , ,71, ,73, , , , , ,79, , , ,83, , , , , ,89, , , , , , ,97, , , ,101, ,103, , , ,107, ,109, , , ,113, , , , , , , 121, , , , ,127, , , ,131, , , , , , ,137, ,139, , , ,143, , , , ,149, ,151, , , , ,157, , , , , , ,163, , , ,167, ,169, , , ,173, , , , ,179, , 181, , , , ,187, , , ,191, ,193, , , ,197, ,199, , , , , , , , , ,209,}

1										11		13				17		19				23						29	
31						37				41		43				47						53						59	
61						67				71		73						79				83						89	
						97				101		103				107		109				113							
121						127				131						137		139				143						149	
151						157						163				167		169				173						179	
181						187				191		193				197		199										209	

P(3) = X(3) + {2,3,5,7}- R(4) – R(5) –{1}

={ 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199}

	2	3		5		7				11		13				17		19				23						29	
31						37				41		43				47						53						59	
61						67				71		73						79				83						89	
						97				101		103				107		109				113							
						127				131						137		139										149	
151						157						163				167						173						179	
181										191		193				197		199											

II. $\pi(m)$

The prime-counting function π(m) denotes the number of prime numbers less than or equal to a given number *m*. In this section, we do not delve into the theoretical background of π(m), but instead focus on an algorithmic approach to compute its exact value.

We begin with the following relation:

$$P(n+1) = X(n+1) + \{p_0, p_{1,}, \dots, p_{n+1,}\} - \{1\} - \bigcup_{i=n+1}^{j} R(i)$$

where:

$$p_j^2 \le A(n+1) < p_{j+1}^2$$

This shows that the exact value of π(m) depends primarily on two components: the set X(n+1) and the sets R(i).

From relation (1), we have:

$$X(n+1) = p_{n+1}X(n) - R(p_{n+1}) - p_{n+1} \qquad (a)$$

where $$R(n+1) = p_{n+1} \cdot (X(n) - \{1\}) \qquad (b)$$

To compute π(m), we follow these steps:

Step 1:

- Find the index *k* such that:

$$\prod_{i=0}^{k} p_i \le m < \prod_{i=0}^{k+1} p_i$$

- Find the index *t* such that:

$$p_t^2 \le m < p_{t+1}^2$$

- Initialize: a = 0 d = 0

Step 2: Construct X(k) and R(i) for i ≤ t

From *(a) and (b)*:

(a) $X(n+1) = p_{n+1}X(n) - R(p_{n+1}) - p_{n+1}$

(b) $R(n+1) = p_{n+1} \cdot (X(n) - \{1\})$

$$\Rightarrow \quad X(n+1) = p_{n+1}X(n) - p_{n+1} \cdot X(n)$$

Since the base cases X(0), X(1) and X(2) are easy to determine explicitly, it follows that we can also easily compute (3) using

$$X(3) = p_3X(2) - p_3 \cdot X(2)$$

By iterating the above steps, we can systematically determine X(k)

Step 3: Iteratively Compute R(i) for *i=k+1* to t

Compute:

$R(k+1) = p_{k+1} \cdot \{x_1, x_2, \ldots, x_i\}$ such that $p_{k+1} \cdot x_i \leq m < p_{k+1} \cdot x_{i+1},\ x_i \in X(k)$

Let $R(k+1) = p_{k+1} \cdot \{r_0, r_1, \ldots, r_i\},\ a \leftarrow a + i + 1$

$R(k+2) = p_{k+2}.(\{r_1, r_2, \ldots, r_i\} - p_{k+1} \cdot \{r_0, r_1, \ldots, r_j\})$

such that

$p_{k+2} \cdot r_i \leq m < p_{k+2} \cdot r_{i+1},\quad p_{k+1}.\ p_{k+2}.r_j \leq m < p_{k+1}.\ p_{k+2}.r_{j+1},\quad r_i,\ r_j \in R(k+1)$

Let $R(k+2) = p_{k+2} \cdot \{r_0, r_1, \ldots, r_t\},\quad a \leftarrow a + j + 1$

Repeat this procedure until we obtain R(t)

Final step:

$$R(t) = p_t \cdot (\{r_1, r_2, \ldots, r_i\} - p_{t-1} \cdot \{r_0, r_1, \ldots, r_j\})$$

with conditions:

$$r_i \leq \frac{m}{p_t} < r_{i+1},\ \ r_j \leq \frac{m}{p_t * p_{t-1}} < r_{j+1},\qquad r_i,\ r_j \in R(t-1)$$

then $R(t) = p_t.\{ r_0, r_1, \ldots, r_x\},\qquad a \leftarrow a + x + 1$

We define $X(k) = \{ x_0, x_1, x_2, \ldots, x_i \}$ and set $d = i+1.$

For $m = b \cdot A(k) + c$, identify the index j such that:

$$x_j \leq c < x_{j+1},\quad e = j+1$$

Final Formula for π(m):

$$\pi(m) = b \cdot d + e + (k+1) - 1 - a$$

Example: Compute π(2218)

We observe that:

$$2 \cdot 3 \cdot 5 \cdot 7 = 210 < 2218 < 2310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \Rightarrow k = 3$$

Also, since

$\sqrt{2218} = 47.0956 = p_{14}$. We compute X(3), then successively compute R(4) through R(14) using the above method.

X(k) = X(3) ={1,11,13,17,19,23,29,31,37,41,43,47,53,59,61,67,71,73,79,83,89,97,101,103,107,109,113,121,127,131,137,139,143,149,151,157,163,167,169,173,179,181,187,191,193,197,199,209}

we have: $\frac{2218}{11} = 201.63$

Thus:

R(4)=11·{11,13,17,19,23,29,31,37,41,43,47,53,59,…,199}

={121, 143, 187, 209, 253, 319, 341, 407, 451, 473, 517, 583, 649,…,2189}

Then:

a= 46

We have: $\frac{2218}{13} = 170.61,\ \frac{2218}{13*11} = 15.51$

Thus:

$$R(5) = 13 \cdot (\{13,17,19,23,29,31,37,41,\ldots,169\}-11.\{11,13\})$$

$$=13\cdot\{13,17,19,23,29,31,37,41,\ldots,169\}$$

$$= \{169, 221, 247, 299, 377, 403, 481, 533,\ldots, 2197\}$$

Then:

$$a=a+35=46+35=81$$

We have $\frac{2218}{17} = 130.47, \frac{2218}{13*17} = 10.03$

Thus:

$$R(6)=17\cdot\{17,19,23,29,31,37,41,43,\ldots,127\}$$

$$= \{289, 323, 391, 493, 527, 629, 697, 731,\ldots,2159\}$$

Then:

$$a= a+25 = 81+25 =106$$

We have: $\frac{2218}{19} = 116.73$

Thus:

$$R(7)=19 \cdot\{19,23,29,31,37,41,43,\ldots,113\}$$

$$= \{361, 437, 551, 589, 703, 779, 817,\ldots, 2147\}$$

Then:

$$a=a+23= 106+23= 129$$

We have: $\frac{2218}{23}= 96.43$

Thus:

$$R(8)=23 \cdot \{23,29,31,37,41,43,47,53,59,61,67,71,73,79,83,89\}$$

$$=\{529, 667, 713, 851, 943, 989, 1081, 1219, 1357, 1403, 1541, 1633, 1679, 1817, 1909, 2047\}$$

Then:

$$a= a+ 16 = 129+16 = 145$$

We have : $\frac{2218}{29}=76.48$

Thus:

$$\boldsymbol{R}(\mathbf{9})=29 \cdot \{29, 31,37,41,43,47,53,59,61,67,71,73\}$$

$$=\{841, 899, 1073, 1189, 1247, 1363, 1537, 1711, 1769, 1943, 2059, 2117\}$$

Then:

$$a= a+ 12= 145+12 = 157$$

We have: $\frac{2218}{31}=71.54$

Thus:

$$R(10) = 31 \cdot \{31,37,41,43,47,53,59,61,67,71\}$$

$= \{961, 1147, 1271, 1333, 1457, 1643, 1829, 1891, 2077, 2201\}$

Then:

$$a = a+10 = 157+10 = 167$$

$$\text{we have:} \quad \frac{2218}{37} = 59.94$$

Thus:

$$R(11) = 37 \cdot \{37, 41, 43, 47, 53, 59\}$$

$$= \{1369, 1517, 1591, 1739, 1961, 2183\}$$

$$a = a+6 = 167+6 = 173$$

$$\text{We have} \quad \frac{2218}{41} = 54.09$$

Thus:

$$R(12) = 41 \cdot \{41, 43, 47, 53\} = \{1681, 1763, 1927, 2173\}$$

Then:

$$a = a+4 = 173+4 = 177$$

$$\text{We have} \quad \frac{2218}{43} = 51.58$$

Thus:

$$R(13) = 43 \cdot \{43, 47\} = \{1849, 2021\}$$

Then:

$$a = a+2 = 177+2 = 179$$

$$\text{We have} \quad \frac{2218}{47} = 47.19$$

Thus:

$$R(14) = 47.\{47\} = 2209$$

Then:

$$a = a+1 = 179+1 = 180$$

we have: $2218 = 10 \cdot 210 + 118$, $x_{26} = 113 < 118 < 121 = x_{27}$

$$d = 48,\ b = 10,\ e = 27,\ k = 3$$

Finally, we evaluate:

$$\pi(2218) = 10 \cdot 48 + 27 + 3 + 1 - 1 - 180 = 330$$

Conclusion

In Chapter I of this study, a clear understanding emerges of the distribution of prime numbers, which is encapsulated by the sequence X(n). In other words, any arbitrary prime number can be represented in the form $k \cdot A(n) + x_i$ where $x_i \in X(k)$.

Furthermore, Chapter II establishes that the precise calculation of π(m) is not only theoretically sound but also computationally feasible, with methodologies that ensure high accuracy in the prime counting function.

While the proposed method has proven to be feasible, there remain challenges that need further investigation and resolution, such as optimizing the computation algorithms for handling large values of mmm. In the future, methods based on the properties of prime number sequences may be developed to achieve higher computational efficiency and open new research directions in number theory and its applications.