# Introduction

Prime numbers are natural numbers greater than 1 that are divisible only by 1 and themselves. They are considered the "building blocks" of arithmetic. However, to this day, their distribution remains one of the most mysterious and fascinating topics in mathematics.

Similarly, accurately estimating the number of prime numbers less than or equal to a positive integer m, i.e., the function π(m), is a central problem in number theory with significant applications in cryptography, algorithm analysis, and other fields. However, calculating π(m) accurately remains a major open question.

In Part 2 of this study, I will identify the patterns of prime numbers, or in other words, how we can predict whether a given number might be prime.

In Part 3 of this study, I will present a method for accurately calculating π(m)\.

# I. Primes Rule

In this section, we will present the distribution of prime numbers in a tabular format, allowing for a clearer visualization of the patterns inherent in their distribution. By systematically organizing the prime numbers and examining their positions, we aim to identify underlying regularities and trends. This approach will provide us with a more structured and accessible view of prime number behavior.

Subsequently, based on the observed patterns, we will develop an algorithm designed to accurately identify these regularities. The proposed algorithm will be capable of predicting prime number distributions, enhancing both our understanding and ability to compute prime numbers more efficiently. The goal is to create an approach that can be generalized and applied to a broad range of prime-related problems in number theory and related fields.

## *I.1. Primes 1 to 1000*

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 |
| 31 | 37 | 41 | 43 | 47 | 53 | 59 | 61 | 67 | 71 |
| 73 | 79 | 83 | 89 | 97 | 101 | 103 | 107 | 109 | 113 |
| 127 | 131 | 137 | 139 | 149 | 151 | 157 | 163 | 167 | 173 |
| 179 | 181 | 191 | 193 | 197 | 199 | 211 | 223 | 227 | 229 |
| 233 | 239 | 241 | 251 | 257 | 263 | 269 | 271 | 277 | 281 |
| 283 | 293 | 307 | 311 | 313 | 317 | 331 | 337 | 347 | 349 |
| 353 | 359 | 367 | 373 | 379 | 383 | 389 | 397 | 401 | 409 |
| 419 | 421 | 431 | 433 | 439 | 443 | 449 | 457 | 461 | 463 |
| 467 | 479 | 487 | 491 | 499 | 503 | 509 | 521 | 523 | 541 |
| 547 | 557 | 563 | 569 | 571 | 577 | 587 | 593 | 599 | 601 |
| 607 | 613 | 617 | 619 | 631 | 641 | 643 | 647 | 653 | 659 |
| 661 | 673 | 677 | 683 | 691 | 701 | 709 | 719 | 727 | 733 |
| 739 | 743 | 751 | 757 | 761 | 769 | 773 | 787 | 797 | 809 |
| 811 | 821 | 823 | 827 | 829 | 839 | 853 | 857 | 859 | 863 |
| 877 | 881 | 883 | 887 | 907 | 911 | 919 | 929 | 937 | 941 |
| 947 | 953 | 967 | 971 | 977 | 983 | 991 | 997 | | |

In order to clearly discern the underlying structure, it is necessary to partition the set into the following subsets:

| | | | | |
|---|---|---|---|---|
| | 2 | 3 | | 5 |
| 7 | | | | 11 |
| 13 | | | | 17 |
| 19 | | | | 23 |
| | | | | 29 |

| | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 2 | 3 | | 5 | | 7 | | | | 11 | | 13 | | | | 17 | | 19 | | | | 23 | | | | | | 29 | |
| 31 | | | | | | 37 | | | | 41 | | 43 | | | | 47 | | | | | | 53 | | | | | | 59 | |
| 61 | | | | | | 67 | | | | 71 | | 73 | | | | | | 79 | | | | 83 | | | | | | 89 | |
| | | | | | | 97 | | | | 101 | | 103 | | | | 107 | | 109 | | | | 113 | | | | | | | |
| | | | | | | 127 | | | | 131 | | | | | | 137 | | 139 | | | | | | | | | | 149 | |
| 151 | | | | | | 157 | | | | | | 163 | | | | 167 | | | | | | 173 | | | | | | 179 | |
| 181 | | | | | | | | | | 191 | | 193 | | | | 197 | | 199 | | | | | | | | | | | |

## *I.2. Definitions*

*P(n)*: The set of all prime numbers up to the *n*-th level.

*R(n)*: The set of multiples of $p_n$ that aren't divisible by any of $p_0, p_1, \ldots, p_{n-1}$.

*X(n)*: The base prime table of level n; this contains the reduced residues modulo *A(n)*, where all primes greater than $p_n$ are congruent to some $x_i \in X(n)$ modulo *A(n).*

*A(n)*: The prime coefficient product of level n, defined as:

$$A(n) = \prod_{i=0}^{n} p_i$$

### *I.3. Procedure*

We intialize :

P(0)= {2}

R(0)= {2}

X(0)= {1,}

A(0) = 2

**Step 1: Constructing** *X(n+1)*

- Repeat the set $X(n)$ $p_{n+1}-1$ times to form a candidate list.
- For each $j > n$ where $p_j^2 \leq A(n+1)$, define:

$$R(j) = p_j \cdot \{x_1, x_2, \ldots\ldots x_k\},\ x_i \in X(j-1)$$

Where:

$$p_j \cdot x_k \leq A(n+1) < p_j \cdot x_{k+1}$$

**Step 2: Update Core Structures**

- Define:

$$X(n+1) = p_{n+1}X(n) - R(p_{n+1}) - \{p_{n+1}\} \qquad (1)$$

- Update:

$$A(n+1) = A(n).\ p_{n+1}$$

**Step 3: Prime |Set Construction**

The update prime set is given by:

$$P(n+1) = X(n+1) + \{p_0, p_{1,}, \ldots, p_{n+1,}\} - \sum_{i=n+1}^{j} R(i) - \{1\}$$

Where:

$$p_j^2 \leq A(n+1) < p_{j+1}^2$$

### *I.4. Example Computations*

**Case** n = 1

P(0)= {2}, R(0)= {2}, X(0)= {1,}, A(0)= 2 .

Repeat X(0) 2 times:

| | |
|---|---|
| 1 | |
| 3 | |
| 5 | |

X(1) = {1, , , ,5, }

P(1) = {2,3,5}

A(1) = 6

**Case** n = 2

Repeat X(1) 4 times :

| | | | | | |
|---|---|---|---|---|---|
| 1 | | | | 5 | |
| 7 | | | | 11 | |
| 13 | | | | 17 | |
| 19 | | | | 23 | |
| 25 | | | | 29 | |

Since $p_2^2 = 5^2 = 25 < 5 \cdot 6 = 30$, we construct

$$R(2) = 5 \cdot \{5\} = \{25\}$$

Then:

$$X(2) = 5(X(1)) - R(2) - \{5\}$$

= {1, , , ,5, ,7, , , ,11, ,13, , , ,17, ,19, , , ,23, ,25, , , ,29, }-{25} –{5}

= {1, , , , ,7, , , ,11, ,13, , , ,17, ,19, , , ,23, , , , ,29, }

P(2) = {2, 3, 5, 7, 11, 13, 17, 19, 23, 29}

$$A(2) = 5 \cdot 6 = 30$$

**Case** n=3: Repeat X(2) 6 times to form a table

| | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | 7 | | | | 11 | | 13 | | | | 17 | | 19 | | | | 23 | | | | | | 29 | |
| 31 | | | | | | 37 | | | | 41 | | 43 | | | | 47 | | 49 | | | | 53 | | | | | | 59 | |
| 61 | | | | | | 67 | | | | 71 | | 73 | | | | 77 | | 79 | | | | 83 | | | | | | 89 | |
| 91 | | | | | | 97 | | | | 101 | | 103 | | | | 107 | | 109 | | | | 113 | | | | | | 119 | |
| 121 | | | | | | 127 | | | | 131 | | 133 | | | | 137 | | 139 | | | | 143 | | | | | | 149 | |
| 151 | | | | | | 157 | | | | 161 | | 163 | | | | 167 | | 169 | | | | 173 | | | | | | 179 | |
| 181 | | | | | | 187 | | | | 191 | | 193 | | | | 197 | | 199 | | | | 203 | | | | | | 209 | |

Since: $p_3^2 = 7^2 = 49 < 7.\ 30 = 210$

$R(3) = 7 \cdot \{7,11,13,17,19,23,29\}$

$= \{49, 77, 91, 119, 133, 161, 203\}$

$P_4^2 = 11^2 = 121 < 7.30 = 210$

$R(4) = 11 \cdot \{11,13,17,19\}$

$= \{121, 143, 187, 209\}$

$P_5^2 = 13^2 = 169 < 7.30 = 210$

$R(5) = 13 \cdot \{13\} = \{169\}$

X(3) = 7.X(2) – R(3) - 7={1, , , , ,7, , , ,11, ,13, , , ,17, ,19, , , ,23, , , , ,29, , 31, , , , ,37, , , ,41, ,43, , , ,47, ,49, , , ,53, , , , ,59, , 61, , , , ,67, , , ,71, ,73, , , ,77, ,79, , , ,83, , , , ,89, , 91, , , , ,97, , , ,101, ,103, , , ,107, ,109, , , ,113, , , , ,119, , 121, , , , ,127, , , ,131, ,133, , , ,137, ,139, , , ,143, , , , ,149, ,151, , , , ,157, , , ,161, ,163, , , ,167, ,169, , , ,173, , , , ,179, , 181, , , , ,187, , , ,191, ,193, , , ,197, ,199, , , ,203, , , , ,209,} –{49, 77, 91, 119, 133, 161, 203}-{7}

={1, , , , , , , , , , ,11, ,13, , , ,17, ,19, , , ,23, , , , ,29, 31, , , , , ,37, , , ,41, ,43, , , ,47, , , , , ,53, , , , , ,59, , 61, , , , , ,67, , , ,71, ,73, , , , , ,79, , , ,83, , , , , ,89, , ,

, , , ,97, , , ,101, ,103, , , ,107, ,109, , , ,113, , , , , , , 121, , , , ,127, , , ,131, , , , , ,137, ,139, , , ,143, , , , ,149, ,151, , , , ,157, , , , , ,163, , , ,167, ,169, , , ,173, , , , ,179, , 181, , , , ,187, , , ,191, ,193, , , ,197, ,199, , , , , , , , ,209,}

| | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | | 11 | | 13 | | | | 17 | | 19 | | | | 23 | | | | | | 29 | |
| 31 | | | | | | 37 | | | | 41 | | 43 | | | | 47 | | | | | | 53 | | | | | | 59 | |
| 61 | | | | | | 67 | | | | 71 | | 73 | | | | | | 79 | | | | 83 | | | | | | 89 | |
| | | | | | | 97 | | | | 101 | | 103 | | | | 107 | | 109 | | | | 113 | | | | | | | |
| 121 | | | | | | 127 | | | | 131 | | | | | | 137 | | 139 | | | | 143 | | | | | | 149 | |
| 151 | | | | | | 157 | | | | | | 163 | | | | 167 | | 169 | | | | 173 | | | | | | 179 | |
| 181 | | | | | | 187 | | | | 191 | | 193 | | | | 197 | | 199 | | | | | | | | | | 209 | |

P(3) = X(3) + {2,3,5,7}- R(4) – R(5) -1

={ 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199}

| | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 2 | 3 | | 5 | | 7 | | | | 11 | | 13 | | | | 17 | | 19 | | | | 23 | | | | | | 29 | |
| 31 | | | | | | 37 | | | | 41 | | 43 | | | | 47 | | | | | | 53 | | | | | | 59 | |
| 61 | | | | | | 67 | | | | 71 | | 73 | | | | | | 79 | | | | 83 | | | | | | 89 | |
| | | | | | | 97 | | | | 101 | | 103 | | | | 107 | | 109 | | | | 113 | | | | | | | |
| | | | | | | 127 | | | | 131 | | | | | | 137 | | 139 | | | | | | | | | | 149 | |
| 151 | | | | | | 157 | | | | | | 163 | | | | 167 | | | | | | 173 | | | | | | 179 | |
| 181 | | | | | | | | | | 191 | | 193 | | | | 197 | | 199 | | | | | | | | | | | |

# *II.* $\pi(m)$

The prime-counting function π(m) denotes the number of prime numbers less than or equal to a given number *m*. In this section, we do not delve into the theoretical background of π(m), but instead focus on an algorithmic approach to compute its exact value.

We begin with the following relation:

$$P(n+1) = X(n+1) + \{p_0, p_{1,}, \dots, p_{n+1,}\} - \{1\} - \bigcup_{i=n+1}^{j} R(i)$$

*where:*

$$p_j^2 \leq A(n+1) < p_{j+1}^2$$

This shows that the exact value of π(m) depends primarily on two components: the set X(n+1) and the sets R(i).

From relation (1), we have:

$$X(n+1) = p_{n+1}X(n) - R(p_{n+1}) - p_{n+1} \qquad (a)$$

where $$R(n+1) = p_{n+1} \cdot (X(n) - \{1\}) \qquad (b)$$

To compute π(m), we follow these steps:

**Step 1:**

- Find the index *k* such that:

$$\prod_{i=0}^{k} p_i \leq m < \prod_{i=0}^{k+1} p_i$$

- Find the index *t* such that:

$$p_t^2 \leq m < p_{t+1}^2$$

- Initialize: a = 0 d = 0

**Step 2: Construct X(k) and R(i) for i ≤ t**

From *(a) and (b)*:

(a) $X(n+1) = p_{n+1}X(n) - R(p_{n+1}) - p_{n+1}$

*(b)* $R(n+1) = p_{n+1} \cdot (X(n) - \{1\})$

$\Rightarrow$ $X(n+1) = p_{n+1}X(n) - p_{n+1} \cdot X(n)$

Since the base cases X(0), X(1) and X(2) are easy to determine explicitly, it follows that we can also easily compute (3) using

$$X(3) = p_3X(2) - p_3 \cdot X(2)$$

By iterating the above steps, we can systematically determine X(k)

**Step 3:** Iteratively Compute R(i) for *i=k+1* to t

Compute:

$R(k+1) = p_{k+1} \cdot \{x_1, x_2, .....,x_i\}$ such that $p_{k+1} \cdot x_i \leq m < p_{k+1} \cdot x_{i+1}, x_i \in X(k)$

Let $R(k+1) = p_{k+1} \cdot \{r_0,r_1,...,r_i\}, \ a \leftarrow a + i +1$

$R(k+2) = p_{k+2}.(\{r_1,r_2,...,r_i\} - p_{k+1} \cdot \{r_0,r_1,...,r_j\})$

such that

$p_{k+2} \cdot r_i \leq m < p_{k+2} \cdot r_{i+1}, \quad p_{k+1}. p_{k+2}.r_j \leq m < p_{k+1}. p_{k+2}.r_{j+1}, \quad r_i, r_j \in R(k+1)$

Let $R(k+2)= p_{k+2} \cdot \{r_0,r_1,...,r_t\}, \quad a \leftarrow a + j +1$

Repeat this procedure until we obtain R(t)

Final step:

$$R(t) = p_t \cdot (\{r_1,r_2,...,r_i\} - p_{t-1} \cdot \{r_0,r_1,...,r_j\})$$

with conditions:

$$r_i \leq \frac{m}{p_t} < r_{i+1}, \ r_j \leq \frac{m}{p_t * p_{t-1}} < r_{j+1}, \qquad r_i, r_j \in R(t-1)$$

then $R(t)=p_t.\{ r_0,r_1,...,r_x\}, \qquad a \leftarrow a + x +1$

We define $X(k)= \{ x_0,x_1,x_2, .....,x_i \}$ and set $d= i+1.$

For $m=b \cdot A(k)+ c$, identify the index j such that:

$$x_j \leq c < x_{j+1}, \quad e = j+1$$

Final Formula for π(m):

$$\pi(m) = b \cdot d + e + (k+1) - 1 - a$$

**Example: Compute π(2218)**

We observe that:

$$2 \cdot 3 \cdot 5 \cdot 7=210<2218<2310=2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \Rightarrow k=3$$

Also, since

$\sqrt{2218} = 47.0956 = p_{14}$. We compute X(3), then successively compute R(4) through R(14) using the above method.

X(k) = X(3)
={1,11,13,17,19,23,29,31,37,41,43,47,53,59,61,67,71,73,79,83,89,97,101,103,107,109,113,121,127,131,137,139,143,149,151,157,163,167,169,173,179,181,187,191,193,197,199,209}

we have: $\frac{2218}{11}= 201.63$

Thus:

**R(4)**=11·{11,13,17,19,23,29,31,37,41,43,47,53,59,…,199}

={121, 143, 187, 209, 253, 319, 341, 407, 451, 473, 517, 583, 649,…,2189}

Then:

a= 46

We have: $\frac{2218}{13}=170.61, \frac{2218}{13*11} =15.51$

Thus:

$$R(5) = 13 \cdot (\{13,17,19,23,29,31,37,41,\ldots,169\}-11.\{11,13\})$$

$$=13\cdot\{13,17,19,23,29,31,37,41,\ldots,169\}$$

$$= \{169, 221, 247, 299, 377, 403, 481, 533,\ldots, 2197\}$$

Then:

$$a=a+35=46+35=81$$

We have $\frac{2218}{17} = 130.47, \frac{2218}{13*17} = 10.03$

Thus:

$$R(6)=17\cdot\{17,19,23,29,31,37,41,43,\ldots,127\}$$

$$= \{289, 323, 391, 493, 527, 629, 697, 731,\ldots,2159\}$$

Then:

$$a= a+25 = 81+25 =106$$

We have: $\frac{2218}{19} = 116.73$

Thus:

$$R(7)=19 \cdot\{19,23,29,31,37,41,43,\ldots,113\}$$

$$= \{361, 437, 551, 589, 703, 779, 817,\ldots, 2147\}$$

Then:

$$a=a+23= 106+23= 129$$

We have: $\frac{2218}{23}= 96.43$

Thus:

$$R(8)=23 \cdot \{23,29,31,37,41,43,47,53,59,61,67,71,73,79,83,89\}$$

$$=\{529, 667, 713, 851, 943, 989, 1081, 1219, 1357, 1403, 1541, 1633, 1679, 1817, 1909, 2047\}$$

Then:

$$a= a+ 16 = 129+16 = 145$$

We have : $\frac{2218}{29}=76.48$

Thus:

$$\boldsymbol{R}(\mathbf{9})=29 \cdot \{29, 31,37,41,43,47,53,59,61,67,71,73\}$$

$$=\{841, 899, 1073, 1189, 1247, 1363, 1537, 1711, 1769, 1943, 2059, 2117\}$$

Then:

$$a= a+ 12= 145+12 = 157$$

We have: $\frac{2218}{31}=71.54$

Thus:

$$R(10) = 31 \cdot \{31,37,41,43,47,53,59,61,67,71\}$$

$= \{961, 1147, 1271, 1333, 1457, 1643, 1829, 1891, 2077, 2201\}$

Then:

$$a = a+10 = 157+10 = 167$$

we have: $\frac{2218}{37}=59.94$

Thus:

$$R(11) = 37 \cdot \{37,41,43,47,53,59\}$$

$$= \{1369, 1517, 1591, 1739, 1961, 2183\}$$

$$a = a+6 = 167+6 = 173$$

We have $\frac{2218}{41}=54.09$

Thus:

$$R(12) = 41\cdot\{41,43,47,53\} = \{1681, 1763, 1927, 2173\}$$

Then:

$$a = a+4 = 173+4 = 177$$

We have $\frac{2218}{43}=51.58$

Thus:

$$R(13) = 43\cdot\{43,47\} = \{1849, 2021\}$$

Then:

$$a = a+2 = 177+2 = 179$$

We have $\frac{2218}{47}=47.19$

Thus:

$$R(14) = 47.\{47\} = 2209$$

Then:

$$a = a+1 = 179+1 = 180$$

we have: $2218 = 10\cdot 210 + 118,\ x_{26}=113<118<121=x_{27}$

$$d = 48,\ b = 10,\ e = 27,\ k = 3$$

Finally, we evaluate:

$$\pi(2218) = 10 \cdot 48+27+3+1-1-180 = 330$$

# Conclusion

In Chapter II of this study, a clear understanding emerges of the distribution of prime numbers, which is encapsulated by the sequence X(n). In other words, any arbitrary prime number can be represented in the form $k \cdot A(n) + x_i$ where $x_i \in X(k)$.

Furthermore, Chapter III establishes that the precise calculation of π(m) is not only theoretically sound but also computationally feasible, with methodologies that ensure high accuracy in the prime counting function.

While the proposed method has proven to be feasible, there remain challenges that need further investigation and resolution, such as optimizing the computation algorithms for handling large values of mmm. In the future, methods based on the properties of prime number sequences may be developed to achieve higher computational efficiency and open new research directions in number theory and its applications.