MTH 4851

Lecture No: 1

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Topic: Finite Difference Operator, steps in solving a problem with a computer.

Learning Outcome: After completing this lecture students will

(i) Be able to derive the definition of Numerical analysis.
(ii) Be able to introduce forward, backward and central difference operator.

Numerical Analysis:

Numerical Analysis is the branch of Mathematics which deals with approximate methods of calculation and also the methods of estimating the accuracy such calculations.

The underlying theory of numerical analysis is based on the principle of finite difference which we shall deal in this and next lectures. The principle of finite difference is the study of changes that take place in the value of the function $y = f(x)$ with respect to the finite changes in the independent variable x.

Forward Difference operator:

The operator Δ is called forward difference operator and defined as

$$\Delta f(x) = f(x+h) - f(x)$$

Backward Difference operator:

The operator ∇ is called backward difference operator and defined as

$$\nabla f(x) = f(x) - f(x-h)$$

Central Difference operator:

The central difference operator is defined by

$$\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$$

Or, $\delta y_x = y_{x+h/2} - y_{x-h/2}$

Shifting or displacement or translation operator:

Let $y = f(x)$ be a function of x. Then the shift operator E is defined by

$$Ef(x) = f(x+h)$$

Or, $\mathrm{Ey_x = y_{x+h}}$ [h is increment]

Hence $\mathrm{E^2f(x) = EEf(x)}$

$$= \mathrm{Ef(x+h)}$$

$$= \mathrm{f(x+2h)}$$

In general $\mathrm{E^nf(x) = f(x+nh)}, \quad \forall\ \mathrm{n \in N}$

In particular $\mathrm{Ey_0 = y_1}, \quad \mathrm{E^2y_0 = y_2}, \dots\dots\dots, \quad \mathrm{E^ny_0 = y_n}$

The inverse shift operator $\mathrm{E^{-1}}$ is defined by

$$\mathrm{E^{-1}f(x) = f(x-h)}$$

Averaging operator:

The averaging operator μ is defined by

$$\mathrm{\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right]}$$

Differential operator:

The differential operator D is defined by

$$\mathrm{Df(x) = \frac{d}{dx}\{f(x)\}}$$

$$D^2 f(x) = \frac{d^2}{dx^2}\{f(x)\}$$

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$$D^n f(x) = \frac{d^n}{dx^n}\{f(x)\}$$

<u>Unit operator</u>:

The unit operator 1 is defined by(

$$1.\, f(x) = f(x)$$

<u>Algebraic properties of operators E and Δ</u>

(i) Operators E and Δ are distributive. Let $F(x)$be a function which is the sum of the functions $f(x),\ g(x),\ h(x), \ldots.,$ So that

$$F(x) = f(x) + g(x) + h(x) + \cdots$$

Then, $EF(x) = Ef(x) + Eg(x) + Eh(x) + \cdots$

Similarly, $\Delta F(x) = \Delta f(x) + \Delta g(x) + \Delta h(x) + \cdots$

(ii) E and Δ are commutative with regard to a constant

$$E(\alpha F(x)) = \alpha EF(x)$$

and $\Delta(\alpha F(x)) = \alpha \Delta F(x)$

(iii) E and Δ are not commutative with respect to variables.

i.e. if $F(x) = f(x)\, g(x)$

Then $EF(x) \neq f(x)\, Eg(x)\ \ and\ \Delta F(x) \neq f(x)\, \Delta g(x)$

(iv) E and Δ are obey the law of indices

i.e. $E^m E^n F(x) = E^{m+n} F(x)$

and $\Delta^m \Delta^n F(x) = \Delta^{m+n} F(x)$

(v) $E(af(x) + bg(x)) = aEf(x) + bEg(x)$

(vi) $E^{-n} f(x) = f(x - nh)$

(vii) $E^2 f(x) \neq (EF(x))^2$

(viii) E and Δ cannot stand without the operand.

Solved Problems

Problem – 1: Prove that $E \equiv 1 + \Delta$

Proof: Let $y = f(x)$ be a function of x

$$\because \ \Delta f(x) = f(x+h) - f(x)$$

$$= Ef(x) - f(x) \qquad [Ef(x) = f(x+h)]$$

$$= (E-1)f(x)$$

Thus $\Delta f(x) = (E-1)f(x). \qquad \forall f(x)$

Hence $\Delta \equiv E - 1$

$$\because E \equiv 1 + \Delta$$

Problem – 2: Show that $\nabla \equiv 1 - E^{-1}$

Proof: $\nabla f(x) = f(x) - f(x-h) \ldots \ldots \ldots . (i)$

But by definition of inverse shift operator E^{-1}, we get

$$E^{-1} f(x) = f(x-h) \ldots \ldots \ldots \ldots (ii)$$

Using (i) and (ii) we get

$$\nabla f(x) = f(x) - E^{-1} f(x)$$

$$\Rightarrow \nabla f(x) = (1 - E^{-1}) f(x). \qquad \forall f(x)$$

Hence $\nabla = 1 - E^{-1}$

Problem – 3: Show that $\delta = E^{1/2} - E^{-1/2}$

Proof: $\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$

$= f\left(x + \frac{1}{2}h\right) - f\left(x - \frac{1}{2}h\right)$

$= E^{1/2} f(x) - E^{-1/2} f(x)$

$= (E^{1/2} - E^{-1/2}) f(x)$

Thus $\delta f(x) = (E^{1/2} - E^{-1/2}) f(x)$

Hence $\delta = E^{1/2} - E^{-1/2}$

Problem – 4: Show that $E\nabla \equiv \nabla E \equiv \Delta$

Proof: $(E\nabla) f(x) = \mathrm{E}(\nabla f(x))$

$= \mathrm{E}(f(x) - f(x-h))$, where h is the interval of differencing

$= \mathrm{E} f(x) - E f(x-h)$

$= f(x+h) - f(x)$

$= \Delta \mathrm{f(x)}$

Thus $(E\nabla) f(x) = \Delta \mathrm{f(x)}$

Hence $E\nabla \equiv \Delta$ … … … … (i)

Again, $(\nabla E) f(x) = \nabla(E f(x)) = \nabla(f(x+h))$

$= \mathrm{f(x+h) - f(x)}$

$= \Delta \mathrm{f(x)}$

Thus, $(\nabla E) f(x) = \Delta \mathrm{f(x)}$

Hence, $\nabla E \equiv \Delta \ldots \ldots \ldots \ldots (ii)$

From (i) and (ii) we get

$E\nabla \equiv \nabla E \equiv \Delta$

Problem – 4: Find the population in 1981 from the following table:

Year (x)	1941	1951	1961	1971	1981	1991
Population Lakhs (y)	363	391	421	451	?	501

Solution: There are five given values. We can have unique 4th degree polynomial to satisfy the data.

Hence $\Delta^5 y_0 = 0$

$\Rightarrow (E-1)^5 y_0 = 0$

$\Rightarrow (E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)y_0 = 0$

$\Rightarrow y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 = 0$

$\Rightarrow 501 - 5y_4 + 10 \times 451 - 10 \times 421 + 5 \times 391 - 363 = 0$

$\Rightarrow 501 - 5y_4 + 4510 - 4210 + 1955 - 363 = 0$

$\Rightarrow -5y_4 + 6966 - 4573 = 0$

$\Rightarrow -5y_4 + 2393 = 0$

$\Rightarrow -5y_4 = -2393$

$\Rightarrow 5y_4 = 2393$

$\Rightarrow y_4 = \frac{2393}{5}$

$\therefore y_4 = 478.6$

Hence the population in 1981 is 478.6 lakhs.

Problem – 5: Find the missing terms in the following table:

X	16	18	20	22	24	26
Y	39	85	?	151	264	388

Solution: There are five given values. We can have unique 4th degree polynomial to satisfy the data.

Hence $\Delta^5 y_k = 0$

$\Rightarrow (E-1)^5 y_k = 0$

$\Rightarrow (E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)y_k = 0$

$\Rightarrow y_{k+5} - 5y_{k+4} + 10y_{k+3} - 10y_{k+2} + 5y_{k+1} - y_k = 0 \quad \dots \dots \dots \dots (i)$

Put k=0 in (i)

$\Rightarrow y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 = 0$

$\Rightarrow 388 - 5 \times 264 + 10 \times 151 - 10y_2 + 5 \times 85 - 39 = 0$

$\Rightarrow 388 - 1320 + 1510 - 10y_2 + 425 - 39 = 0$

$\Rightarrow -10y_2 + 2323 - 1359 = 0$

$\Rightarrow -10y_2 + 964 = 0$

$\Rightarrow -10y_2 = -964$

$\Rightarrow 10y_2 = 964$

$\Rightarrow y_2 = \frac{964}{10}$

$\therefore y_2 = 96.4$

Hence the missing term is 96.4

H.W

Problem – 1: Find the missing terms in the following table:

X	100	101	102	103	104
Y	2	2.0043	?	2.0128	2.0170

Ans. 2.0086

Problem – 2: Find the missing terms in the following table:

X	7	9	11	13	15	17
Y	32	78	?	144	257	381

Ans. 89.4