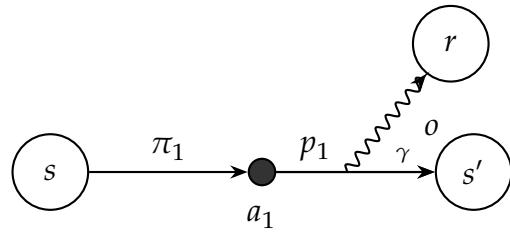


# Bellman Equation as Dot Products



$$\begin{aligned}
 g &= \vec{\gamma} \cdot \vec{r} \\
 \vec{o} &= \vec{r} + \gamma \vec{v}' && \text{recursive} \\
 q &= \vec{p} \cdot \vec{o} && = E [ g | s, a ] \\
 v &= \vec{\pi} \cdot \vec{q} && = E [ g | s ]
 \end{aligned}$$

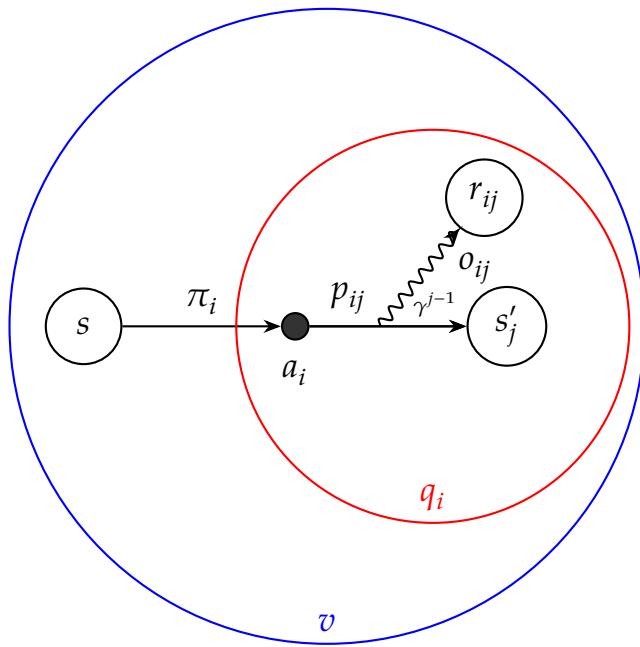
$$\begin{aligned}
 \vec{\gamma} &= \begin{bmatrix} \gamma^0 \\ \gamma^1 \\ \vdots \end{bmatrix} & \vec{p} &= \begin{bmatrix} p_1 \\ p_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} p(s'_1 | a, s) \\ p(s'_2 | a, s) \\ \vdots \end{bmatrix} \\
 \vec{r} &= \begin{bmatrix} r_1 \\ r_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} r(s, a, s'_1) \\ r(s, a, s'_2) \\ \vdots \end{bmatrix} & \vec{q} &= \begin{bmatrix} q_1 \\ q_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \vec{p}_1 \cdot \vec{o}_1 \\ \vec{p}_2 \cdot \vec{o}_2 \\ \vdots \end{bmatrix} \\
 \vec{v} &= \begin{bmatrix} v_0 \\ v_1 \\ \vdots \end{bmatrix} = \begin{bmatrix} v(s_0) \\ v(s_1) \\ \vdots \end{bmatrix} = \begin{bmatrix} \vec{\pi}_1 \cdot \vec{q}_1 \\ \vec{\pi}_2 \cdot \vec{q}_2 \\ \vdots \end{bmatrix} & \vec{\pi} &= \begin{bmatrix} \pi_1 \\ \pi_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} p(a_1 | s) \\ p(a_2 | s) \\ \vdots \end{bmatrix} \\
 \vec{v}' &= \begin{bmatrix} v'_0 \\ v'_1 \\ \vdots \end{bmatrix} = \begin{bmatrix} v(s'_0) \\ v(s'_1) \\ \vdots \end{bmatrix} \\
 \vec{o} &= \begin{bmatrix} o_1 \\ o_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} r_1 + \gamma v'_1 \\ r_2 + \gamma v'_2 \\ \vdots \end{bmatrix}
 \end{aligned}$$

# Einstein Notation

(implicit summation over repeated indices)

$i = \text{actions}$

$j = \text{next states}$



$$g = \gamma^{j-1} r_j \quad (\text{discounted sum over trajectory steps})$$

$$o_{ij} = r_{ij} + \gamma^{j-1} v'_j \quad (\text{elementwise — no summation})$$

$$q_i = p_{ij} o_{ij} \quad (\text{sum over next states } j)$$

$$v = \pi_i q_i \quad (\text{sum over actions } i)$$

$$v = \pi_i p_{ij} (r_{ij} + \gamma^{j-1} v'_j) \quad (\text{full Bellman equation — sum over } i \text{ and } j)$$

*Bellman's genius recursive insight:*

$$v = E[g | s] = E[o | s]$$

*The infinite future folds into one step —  $v'$  summarizes everything beyond*

## GLOSSARY:

$r$  = reward *(rewarded **after** action  $a$  is taken)*

$\gamma$  = discounting factor

$g = g_0 = r_1 + \gamma r_2 + \gamma^2 r_3 + \dots$  = return *(reward now plus discounted future rewards)*

$o = r + \gamma v'$  = outcome

$s = s_0$  = current state

$s' = s_1$  = next state

$v = v(s) = v(s_0)$

$v' = v(s') = v(s_1)$

$a$  = action

$p$  = transition probability, dynamics

$v$  = state value function

$q$  = state-action value function

$\pi$  = policy