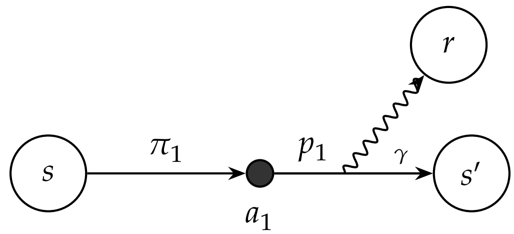
Bellman Equation as Dot Products



$$g = \vec{\gamma} \cdot \vec{r}$$

$$\vec{o} = \vec{r} + \gamma \vec{v}'$$

$$q = \vec{p} \cdot \vec{o} \qquad = E\,[\; g \mid s, a \;]$$

$$v = \vec{\pi} \cdot \vec{q} \qquad = E\,[\; g \mid s \;]$$

$$\vec{\gamma} = \begin{bmatrix} \gamma^0 \\ \gamma^1 \\ \vdots \end{bmatrix}$$

$$\vec{p} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} p(s'_1 \mid a, s) \\ p(s'_2 \mid a, s) \\ \vdots \end{bmatrix}$$

$$\vec{r} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} r(s, a, s'_1) \\ r(s, a, s'_2) \\ \vdots \end{bmatrix}$$

$$\vec{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \vec{p}_1 \cdot \vec{o}_1 \\ \vec{p}_2 \cdot \vec{o}_2 \\ \vdots \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \end{bmatrix} = \begin{bmatrix} v(s_0) \\ v(s_1) \\ \vdots \end{bmatrix} = \begin{bmatrix} \vec{\pi}_1 \cdot \vec{q}_1 \\ \vec{\pi}_2 \cdot \vec{q}_2 \\ \vdots \end{bmatrix}$$

$$\vec{\pi} = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} p(a_1 \mid s) \\ p(a_2 \mid s) \\ \vdots \end{bmatrix}$$

$$\vec{v}' = \begin{bmatrix} v'_0 \\ v'_1 \\ \vdots \end{bmatrix} = \begin{bmatrix} v(s'_0) \\ v(s'_1) \\ \vdots \end{bmatrix}$$

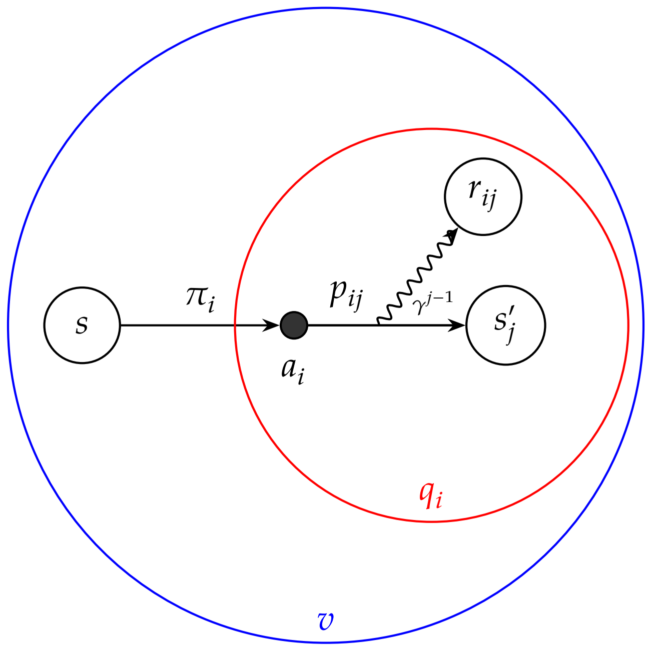
$$\vec{o} = \begin{bmatrix} o_1 \\ o_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} r_1 + \gamma v'_1 \\ r_2 + \gamma v'_2 \\ \vdots \end{bmatrix}$$

Einstein Notation

(implicit summation over repeated indices)

i = **actions**

j = **next states**



$$g = \gamma^{j-1} r_j$$ *(discounted sum over trajectory steps)*

$$o_{ij} = r_{ij} + \gamma v'_j$$ *(elementwise — no summation)*

$$q_i = p_{ij}\, o_{ij}$$ *(sum over next states j)*

$$v = \pi_i\, q_i$$ *(sum over actions i)*

$$v = \pi_i\, p_{ij}\left(r_{ij} + \gamma v'_j\right)$$ *(full Bellman equation — sum over i and j)*

Bellman's genius recursive insight:

$$v = E[\, g \mid s \,] = E[\, o \mid s \,]$$

The infinite future folds into one step — v' summarizes everything beyond

GLOSSARY:

r = reward (*rewarded* ***after*** *action a is taken*)

γ = discounting factor

$g = g_0 = r_1 + \gamma r_2 + \gamma^2 r_3 + \cdots$ = return (*reward now plus discounted future reward*)

$o = r + \gamma v'$ = outcome

$s = s_0$ = current state

$s' = s_1$ = next state

$v = v(s) = v(s_0)$

$v' = v(s') = v(s_1)$

a = action

p = transition probability, dynamics

v = state value function

q = state-action value function

π = policy

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