Экзамен
УМФ
Билет №15

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РЛ2-31
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09:00–11:00

1 $U_{tt} = 36 U_{xx}$ на $0<x<3{,}5$, $0<t<\infty$

$U(x,0) = 17\sin 9\pi x$

$U_t(x,0)=0$

$U(0,t)=0$

$U_x(3{,}5;t)=0$

Воспользуемся методом разделённых переменных: $U = T(t)\cdot X(x) \not\equiv 0$

$\Rightarrow U_{tt} = 36U_{xx} \to T''X = 36X''T \Rightarrow$

$\frac{T''}{T} = 36\frac{X''}{X} = -\lambda$

I $36\frac{X''}{X} = -\lambda \Rightarrow \begin{cases} X'' + \frac{1}{36}\lambda X = 0 \\ X(0)=0 \\ X'(3{,}5)=0 \end{cases}$

$U(0,t)=0 \quad X(0)\,\overset{const}{T(t)}=0 \Rightarrow X(0)=0$

аналогично $U_x(3{,}5;t)=0 \quad X'(3{,}5)=0$

① $\lambda = 0 \Rightarrow$

$\begin{cases} X''=0 \\ X(0)=0 \\ X'(3{,}5)=0 \end{cases}$

$\begin{cases} X = C_1x + C_2 \\ X' = C_1 \end{cases}$

$\begin{cases} 0 = 0\cdot C_1 + C_2 \\ 0 = C_1 \end{cases}$

$C_1 = C_2 = 0 \Rightarrow$

$X = 0$ – не с.ф $\Rightarrow$

$\lambda = 0$ не с.з

② $\lambda < 0$

$\lambda = -\omega^2$

$X'' - \frac{1}{36}\omega^2 X = 0$

$\begin{cases} X = C_1e^{\frac{1}{6}\omega x} + C_2e^{-\frac{1}{6}\omega x} \\ X' = \frac{1}{6}\omega C_1 e^{\frac{1}{6}\omega x} - \frac{1}{6}\omega C_2 e^{-\frac{1}{6}\omega x} \end{cases}$

$\begin{cases} 0 = C_1 + C_2 \\ 0 = \cancel{\tfrac{1}{6}\omega} C_1 e^{\frac{3,5}{6}\omega} - \cancel{\tfrac{1}{6}\omega} C_2 e^{-\frac{3,5}{6}\omega} \end{cases}$ $(\omega \neq 0)$

$\Rightarrow \varphi \neq 0 \quad C_1 \neq C_2 \Rightarrow$

$\begin{pmatrix} 1 & 1 \\ e^{\frac{3,5}{6}\omega} & -e^{-\frac{3,5}{6}\omega} \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0$

где $\begin{vmatrix} 1 & 1 \\ e^{\frac{3,5}{6}\omega} & -e^{-\frac{3,5}{6}\omega} \end{vmatrix} = 0$

$-e^{\frac{3,5}{6}\omega} - e^{\frac{3,5}{6}\omega} = 0$

ни при каких ω не выполн

$\Rightarrow C_1 = C_2 = 0 \Rightarrow \varphi = 0$ не с.ф

$\Rightarrow \lambda < 0$ не с.з

③ $\lambda > 0 \quad \lambda = \omega^2$

$X'' + \frac{1}{36}\omega^2 X = 0$

$\begin{cases} X = C_1\cos\frac{\omega}{6}x + C_2\sin\frac{\omega}{6}x \\ X' = -\frac{\omega}{6}C_1\sin\frac{\omega}{6}x + \frac{\omega}{6}C_2\cos\frac{\omega}{6}x \end{cases}$

$\begin{cases} 0 = C_1 \\ 0 = -\frac{\omega}{6}\overset{0}{C_1}\sin\frac{3,5}{6}\omega + \frac{\omega}{6}C_2\cos\frac{3,5}{6}\omega \end{cases}$

$\frac{\omega}{6}C_2\cos\frac{3,5}{6}\omega = 0$

$C_2 \neq 0 \quad \omega \neq 2 \Rightarrow$

$\cos\frac{3,5}{6}\omega = 0 \quad \frac{\pi + 2\pi n}{2}$

$\frac{3,5}{6}\omega = \frac{\pi}{2} + \pi n. \quad n = \overline{0,\infty}$

$\omega = \frac{3\pi + 6\pi n}{3,5} = \frac{3}{3,5}\pi(1+2n)$

$\lambda_n = \left(\frac{3}{3,5}\pi(1+2n)\right)^2$ – с.з

$n = \overline{0,\infty}$

$X_n = \sin\frac{\pi(1+2n)}{7}x$ – с.ф

$\|X\|^2 = \frac{3,5}{2}$

II $\frac{T''}{T} = -\lambda$

$T'' + \lambda T = 0$

$T'' + \left(\frac{3\pi}{3,5}(1+2n)\right)^2 T = 0$

$T = A_n\cos\frac{3\pi}{3,5}(1+2n)t + B_n\sin\frac{3\pi}{3,5}(1+2n)t$

$U = \sum_{n=0}^{\infty} X_n(x)T_n(t) \Rightarrow U_n = \sum_{n=0}^{\infty}\sin\frac{\pi(1+2n)}{7}x\cdot\left[A_n\cos\left(\frac{3\pi}{3,5}(1+2n)\right)t + B_n\sin\left(\frac{3\pi}{3,5}(1+2n)\right)t\right]$

$\frac{\partial U}{\partial t} = \sum_{n=0}^{\infty}\sin\frac{\pi(1+2n)}{7}x\left[-\frac{3\pi}{3,5}(1+2n)A_n\sin\left(\frac{3\pi}{3,5}(1+2n)\right)t + \frac{3\pi}{3,5}(1+2n)B_n\cos\left(\frac{3\pi}{3,5}(1+2n)\right)t\right]$

при $U(x,0) = 17\sin 9\pi x$:

$U_t(x,0) = 0$

①

• $17\sin 9\pi x = \sum_{n=0}^{\infty} \sin\frac{\pi(1+2n)}{7}x \left[\cdot A_n \cos 0^{\,1}\right. \quad |\cdot \sin\frac{\pi(1+2n)}{7}x$ $\qquad \frac{63-1+2n}{2} \quad 38 = \frac{62-2n}{2}$

$$17\int_0^{3,5} \sin 9\pi x \cdot \sin\frac{\pi(1+2n)}{7}x\,dx = \overbrace{\int_0^{3,5} \sin^2\frac{\pi(1+2n)}{7}x\,dx}^{\|X_n\|^2} \cdot A_n$$

$$* \int_0^{3,5} \sin 9\pi x \cdot \sin\frac{\pi(1+2n)}{7}x\,dx = \frac{1}{2}\int_0^{3,5}\left(\cos\left(9\pi - \frac{\pi(1+2n)}{7}\right)x - \cos\left(9\pi + \frac{\pi(1+2n)}{7}\right)x\right)dx =$$

по 1 замеч. предел, при $\forall n$ $\to 0$

$$\frac{1}{2}\left[\frac{\sin\pi\left(9 - \left(\frac{1+2n}{7}\right)\right)x}{\pi\left(9 - \frac{1+2n}{7}\right)} - \frac{\sin\pi\left(9 + \frac{1+2n}{7}\right)x}{\pi\left(9 + \frac{1+2n}{7}\right)}\right]\Bigg|_0^{3,5} = \frac{3,5\cdot 1}{2}\left[\frac{\sin\pi\left(9 - \frac{1+2n}{7}\right)3,5}{\pi\left(9 - \frac{1+2n}{7}\right)3,5} - \frac{\sin\pi\left(9 + \frac{1+2n}{7}\right)3,5}{\pi\left(9 + \frac{1+2n}{7}\right)}\right] =$$

$$= \frac{3,5}{2}\begin{cases}1, & n = 31\\ 0, & n \ne 31\end{cases} \Rightarrow 17\cdot\frac{3,5}{2}\begin{cases}1, & n = 31\\ 0, & n \ne 31\end{cases} = A_n \cdot \|X_n\| \qquad \|X_{31}\| = \frac{3,5}{2}$$

$$17\cdot\frac{3,5}{2} = A_n\cdot\frac{3,5}{2} \Rightarrow A_n = 17$$

• $0 = \sum_{n=0}^{\infty} \sin\frac{\pi(1+2n)}{7}x \quad \frac{3\pi}{3,5}(1+2n)\,B_n \quad \Big|\cdot \sin\frac{\pi(1+2n)}{7}x$

$$0 = \|X_n\|^2\cdot\frac{3\pi}{3,5}(1+2n)\,B_n \quad \left\{\|X_n\| \ne 0 \quad \frac{3\pi}{3,5}(1+2n) \ne 0\right\} \Rightarrow B_n = 0 \text{ при } \forall n \Rightarrow$$

$$U_n = A_n \cdot \cos\frac{3\pi}{3,5}(1+2n)t \cdot \sin\frac{\pi(1+2n)}{7}x \qquad U_{31} = 17\cos 64\pi t \cdot \sin 9\pi x = 17\sin 9\pi x \cos 54\pi t$$

$\cancel{U_{31} = 17\cos 64\pi t \cdot \sin 9\pi x}$

Ответ: $17\sin 9\pi x \cdot \cos 54\pi t$

№2 Найти решение Лапласа в круговом секторе

$\Delta U = 0 \qquad 0 \le r \le 1$

$U(1,\varphi) = \sin 6\varphi \qquad 0 \le \varphi \le \frac{\pi}{3}$

$U(r,0) = 0$

$U(r,\frac{\pi}{3}) = 0$

$$U = R(r)\cdot\Phi(\varphi) \ne 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial\varphi^2} = 0$$

$$\frac{\Phi}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{R}{r^2}\frac{\partial^2\Phi}{\partial\varphi^2} = 0 \quad \Big|\cdot\frac{r^2}{R\Phi}$$

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{\Phi''}{\Phi} = 0 \Rightarrow -\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = \frac{\Phi''}{\Phi} = -\lambda \Rightarrow$$

① $\frac{\Phi''}{\Phi} = -\lambda \quad \frac{\Phi''}{\Phi} + \lambda = 0 \quad \begin{cases}\Phi'' + \lambda\Phi = 0\\ \Phi(0) = 0\\ \Phi(\frac{\pi}{3}) = 0\end{cases} \quad U(r,0) = 0 \quad U = \overset{const}{\cancel{R(r)}}\cdot\Phi(0) = 0$

$\Phi(0) = 0$

аналогично: $\Phi(\frac{\pi}{3}) = 0$

① $\lambda = 0$

$\Phi'' = 0$

$\Phi = C_1 x + C_2$

$\begin{cases}0 = C_1\cdot 0 + C_2\\ 0 = C_1\cdot\frac{\pi}{3} + C_2\end{cases}$

$\begin{cases}0 = C_2\\ 0 = C_1\end{cases} \Rightarrow \Phi = 0$ — не С.Ф

$\Rightarrow \lambda = 0$ не С.З

② $\lambda < 0 \quad \lambda = -\omega^2$

$\Phi'' - \omega^2\Phi = 0$

$\Phi = C_1 e^{\omega\varphi} + C_2 e^{-\omega\varphi}$

$\begin{cases}0 = C_1 + C_2\\ 0 = C_1 e^{3,5\omega} + C_2 e^{-3,5\omega}\end{cases}$

$C_1 \ne C_2 \ne 0 \quad \Phi \ne 0$

$\begin{pmatrix}1 & 1\\ e^{3,5\omega} & e^{-3,5\omega}\end{pmatrix}\begin{pmatrix}C_1\\ C_2\end{pmatrix} = 0$

$\begin{vmatrix}1 & 1\\ e^{3,5\omega} & e^{-3,5\omega}\end{vmatrix} = 0$

$e^{-3,5\omega} - e^{3,5\omega} = 0$

ни при каких ω не выпол

$\Rightarrow C_1 = C_2 = 0 \quad \Phi = 0$ не С.Ф $\quad \lambda < 0$ не С.З

③ $\lambda > 0 \quad \lambda = \omega^2$

$\Phi'' + \omega^2\Phi = 0$

$\Phi = C_1\cos\omega\varphi + C_2\sin\omega\varphi$

$\begin{cases}0 = C_1\\ 0 = C_2\sin\omega\frac{\pi}{3} \quad C_2 \ne 0\end{cases}$

$C_2\sin\omega\frac{\pi}{3} = 0$

$\sin\omega\frac{\pi}{3} = 0$

$\omega\frac{\pi}{3} = \pi n \qquad n = \overline{0,\infty}$

$\omega = 3n$

$\lambda = (3n)^2$ — С.З

$\Phi_n = \sin 3n\varphi$

$\|\Phi_n\| = \frac{\pi}{6}$

III $\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right)-\lambda=0 \quad | \cdot R$

$r\frac{d}{dr}\left(r\frac{dR}{dr}\right)-(3n)^2R=0 \quad n=\overline{0,\infty}$.

Уравнение Бесселя

$r=e^t \quad R(r)=y(t)$

$e^t\frac{d}{de^t}\left(e^t\frac{dy}{de^t}\right)-(3n)^2y=0$

$y''-(3n)^2y=0$

$y=C_1e^{3nt}+C_2e^{-3nt}$

$R(r)=C_1r^{3n}+C_2r^{-3n}$

подставим естественн. Г.У

При $r\to 0 \quad C_2\frac{1}{r^{3n}}\to\infty \Rightarrow C_2=0$

$\Rightarrow R(r)=C_1r^{3n} \quad C_1=1\Rightarrow R(r)=r^{3n}$

$\Rightarrow U_n=\sum_{n=0}^{\infty}R_n(r)\cdot\Phi_n(\varphi)$

$U_n=\sum_{n=0}^{\infty}r^{3n}\cdot\sin 3n\varphi$ - обобщ. ряд Фурье

IV подставим Г.У при $r=1$

$U(1,\varphi)=\sin 6\varphi$

$\sin 6\varphi=\sum_{n=0}^{\infty}A_n\sin 3n\varphi \quad |\cdot\sin 3n\varphi$

$\int_0^{\frac{\pi}{3}}\sin 6\varphi\cdot\sin 3n\varphi\,d\varphi=A_n\,\|\Phi_n\|^2$

$\int_0^{\frac{\pi}{3}}\sin 6\varphi\cdot\sin 3n\varphi\,d\varphi=\frac{1}{2}\int_0^{\frac{\pi}{3}}(\cos(6-3n)\varphi-\cos(6+3n)\varphi)d\varphi=$

$=\frac{1}{2}\left[\frac{\sin 3(2-n)\varphi}{6-3n}-\frac{\sin(6+3n)\varphi}{6+3n}\right]\Big|_0^{\frac{\pi}{3}}=$ при $\forall n$

$=\frac{1}{2}\left[\frac{\sin(6-3n)\frac{\pi}{3}}{(6-3n)\frac{\pi}{3}}\frac{\pi}{3}-\frac{\sin(6+3n)\frac{\pi}{3}}{6+3n}\right]=\frac{1}{2}\cdot\frac{\pi}{3}\begin{cases}1, n=2\\0, n\neq 2\end{cases}$

$\Rightarrow \frac{1}{2}\cdot\frac{\pi}{3}\begin{cases}1, n=2\\0, n\neq 2\end{cases}=A_n\|\Phi_n\|^2 \quad \|\Phi_2\|^2=\frac{\pi}{6}$

$\frac{\pi}{6}=A_2\cdot\frac{\pi}{6}\Rightarrow A_2=1\Rightarrow U_1=r^6\cdot\sin 6\varphi$

Ответ: $U_1=r^6\cdot\sin 6\varphi$

N3. б) $u_{xx}-2u_{xy}+u_{yy}-3u_x+12u_y=0$

$a_{11}=1 \quad a_{12}=-1 \quad a_{22}=1$

$\Delta=1-1=0$ - ур-е параболич. типа

$\lambda^2+2\lambda+1=0$ - ($a_{11}\lambda^2-2a_{12}\lambda+a_{22}=0$ - характеристич. ур-е)

$D/4=1-1=0$

ур-е характеристик: $\lambda(x,y)=\frac{dy}{dx}$

$\lambda_{12}=-1 \quad \frac{dy}{dx}=\lambda_{12}$

характеристика: $\lambda(x,y)$

$\frac{dy}{dx}=-1$

$dy=-dx$

$y=-x+C\Rightarrow$

$C=y+x=\xi$

$y=\eta$

Якобиан перехода: $\begin{vmatrix}\frac{\partial\xi}{\partial x} & \frac{\partial\xi}{\partial y}\\ \frac{\partial\eta}{\partial x} & \frac{\partial\eta}{\partial y}\end{vmatrix}\neq 0 \Rightarrow$

ξ,η - новые переменные

$\begin{vmatrix}1 & 1\\0 & 1\end{vmatrix}\neq 0$

$\xi_x=1 \quad \eta_x=0$

$\xi_y=1 \quad \eta_y=0$

$\Rightarrow u_x=\frac{\partial\xi}{\partial x}\frac{\partial u}{\partial\xi}+\frac{\partial\eta}{\partial x}\frac{\partial u}{\partial\eta}=u_\xi$

$u_y=\frac{\partial\xi}{\partial y}\frac{\partial u}{\partial\xi}+\frac{\partial\eta}{\partial y}\cdot\frac{\partial u}{\partial\eta}=u_\xi+u_\eta$

$u_{xx}=u_{\xi\xi}$

$u_{yy}=u_{\xi\xi}+2u_{\xi\eta}+u_{\eta\eta}$

$u_{xy}=u_{\xi\xi}+u_{\xi\eta}$

Подставляя $u_{xx}, u_{xy}, u_{yy}, u_y, u_x$

$$u_{\xi\xi} + 2(u_{\xi\xi} + u_{\xi\eta}) + u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} - 3u_{\xi} + 12u_{\xi} + 12u_{\eta} = 0$$

$$\cancel{u_{\xi\xi}} - \cancel{2u_{\xi\xi}} - \cancel{2u_{\xi\eta}} + \cancel{u_{\xi\xi}} + \cancel{2u_{\xi\eta}} + u_{\eta\eta} + 9u_{\xi} + 12u_{\eta} = 0$$

$u_{\eta\eta} = -9u_{\xi} - 12u_{\eta}$ -канонический вид

Ответ: $u_{\eta\eta} = -9u_{\xi} - 12u_{\eta}$

2) Уравнения Бесселя - ур-я цилиндрических функций

$$\frac{1}{x}\frac{d}{dx}\left(x\frac{dy}{dx}\right) + \left(1 - \frac{\nu^2}{x^2}\right)y(x) = 0 \Leftarrow \begin{cases} k(x) = x \\ q(x) = -x + \frac{\nu^2}{x^2} \end{cases}$$

Любое ненулевое решение ур-я Бесселя явл. цилиндрической функцией

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$