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РЛ2-31
билет №8

① Решить первую смешанную задачу для волнового уравнения $u_{tt} = u_{xx}$ на отрезке $0<x<\frac{3}{2}$, $0<t<\infty$ с начальными и граничными условиями $u(x,0)=x(x-\frac{3}{2})$, $u_t(x,0)=0$, $u(0,t)=0$, $u(\frac{3}{2},t)=0$

$$\begin{cases} u_{tt}=u_{xx} \\ u_t(x,0)=0, \quad u(x,0)=x(x-\frac{3}{2}) \\ u(0,t)=u(\frac{3}{2},t)=0 \end{cases}$$

$$u(x,t)=X(x)T(t)\not\equiv 0$$

$$XT''=X''T$$

$$\frac{T''}{T}=\frac{X''}{X}=-\lambda=const$$

$$\begin{cases} X''=-\lambda X \\ T''=-\lambda T \end{cases} \rightarrow \begin{cases} X''+\lambda X=0 & (1) \\ T''+\lambda T=0 & (2) \end{cases}$$

задача Ш-Л.

(1): 1) $\begin{cases} \lambda=0 \\ X''+\lambda X=0 \\ u(0,t)=u(\frac{3}{2},t)=0 \end{cases} \rightarrow X''=0 \rightarrow X=C_1x+C_2$

$$\begin{cases} C_2=0 \\ \frac{3}{2}C_1=0 \end{cases} \rightarrow C_1=C_2=0 \rightarrow$$

$X=C_1x+C_2$ не с. ф
$\lambda=0$ - не с. зн.

2) $\lambda=-\omega^2<0$

$$\begin{cases} X''-\omega^2X=0 \\ u(0,t)=u(\frac{3}{2},t)=0 \end{cases}$$

$$X=C_1e^{\omega x}+C_2e^{-\omega x}$$

$$\begin{cases} C_1+C_2=0 \\ C_1e^{3/2\omega}+C_2e^{-3/2\omega}=0 \end{cases} \quad \begin{pmatrix} 1 & 1 \\ e^{3/2\omega} & e^{-3/2\omega} \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}=0$$

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$$\begin{vmatrix} 1 & 1 \\ e^{3/2w} & e^{-3/2w} \end{vmatrix} = 0$$

$$e^{-3/2w} - e^{3/2w} = 0$$

Не выполняется ни при каких w

$\downarrow$

$X = C_1 e^{wx} + C_2 e^{-wx}$ не с.ф

$\lambda = -w^2$ не с.зн.

3) $\lambda = w^2 > 0$

$$\begin{cases} X'' + w^2 X = 0 \\ u(0,t) = u(\frac{3}{2}, t) = 0 \end{cases}$$

$X = C_1 \cos wx + C_2 \sin wx$

$$\begin{cases} C_1 = 0 \\ C_2 \sin \frac{3}{2} w = 0 \end{cases}$$

$\sin \frac{3}{2} w = 0$

$\frac{3}{2} w = \pi n$

$w = \frac{2}{3} \pi n \qquad n = \overline{1, \infty}$

$\lambda = \frac{4}{9} \pi^2 n^2$ – с. зн.

$X = \sin \frac{2}{3} \pi n x$ – с. ф.

$\|X_n\| = \frac{3}{4} \qquad n = \overline{1, \infty}.$

Подставим λ в (2): $T'' + \frac{4}{9} \pi^2 n^2 T = 0$

$T = A_n \cos \frac{2}{3} \pi n t + B_n \sin \frac{2}{3} \pi n t$

$$u(x,t) = \sum_{n=1}^{\infty} (A_n \cos \frac{2}{3} \pi n t + B_n \sin \frac{2}{3} \pi n t) \cdot \sin \frac{2}{3} \pi n x$$

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Н.у:

$$\begin{cases} u(x,0) = x(x-\frac{3}{2}) \\ u_t(x,0) = 0 \end{cases}$$

$$u_t = \sum_{n=1}^{\infty} [-A_n \cdot \frac{2}{3}\pi n \cdot \sin\frac{2}{3}\pi n t + \frac{2}{3}\pi n \cdot B_n \cos\frac{2}{3}\pi n t] \cdot \sin\frac{2}{3}\pi n x$$

$$u_t(x,0) = \sum_{n=1}^{\infty} [0 + \frac{2}{3}\pi n \cdot B_n] \cdot \sin\frac{2}{3}\pi n x = 0 \Rightarrow B_n = 0$$

$$x(x-\frac{3}{2}) = \sum_{n=1}^{\infty} A_n \underbrace{\cos\frac{2}{3}\pi n t}_{1 (t=0)} \cdot \sin\frac{2}{3}\pi n x = \sum_{n=1}^{\infty} A_n \cdot \sin\frac{2}{3}\pi n x$$

$$x(x-\frac{3}{2}) = \sum_{n=1}^{\infty} A_n \cdot \sin\frac{2}{3}\pi n x \quad \Big| \cdot \sin\frac{2}{3}\pi n x$$

$$\int_0^{3/2} x(x-\frac{3}{2}) \cdot \sin\frac{2}{3}\pi n x \, dx = A_n \int_0^{3/2} \sin\frac{2}{3}\pi n x$$

$$\int_0^{3/2} x(x-\frac{3}{2}) \cdot \sin\frac{2}{3}\pi n x \, dx = \left| \begin{matrix} u = x^2 - \frac{3}{2}x & dV = \sin\frac{2}{3}\pi n x \\ du = (2x-\frac{3}{2})dx & V = -\frac{3}{2\pi n}\cos\frac{2}{3}\pi n x \end{matrix} \right| =$$

$$= \underbrace{-\frac{3}{2\pi n}(x^2 - \frac{3}{2}x)\cos\frac{2}{3}\pi n x \Big|_0^{3/2}}_{0} + \frac{3}{2\pi n}\int (2x-\frac{3}{2})\cos\frac{2}{3}\pi n x \, dx =$$

$$= \frac{3}{2\pi n}\int_0^{3/2} (2x-\frac{3}{2})\cos\frac{2}{3}\pi n x \, dx = \left| \begin{matrix} u = 2x-\frac{3}{2} & dv = \cos\frac{2}{3}\pi n x \, dx \\ du = 2dx & v = \frac{3}{2\pi n}\sin\frac{2}{3}\pi n x \end{matrix} \right| =$$

$$= \frac{3}{2\pi n}\Big[\underbrace{\frac{3}{2\pi n}(2x-\frac{3}{2})\sin\frac{2}{3}\pi n x \Big|_0^{3/2}}_{0} - \frac{3 \cdot 2}{2\pi n}\int_0^{3/2} \sin\frac{2}{3}\pi n x \, dx\Big] =$$

$$= \frac{3}{2\pi n}\left(-\frac{6}{2\pi n}\int_0^{3/2} \sin\frac{2}{3}\pi n x \, dx\right) =$$

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$$\ominus \frac{3}{2\pi n}\left[-\frac{6}{2\pi n}\left(-\frac{3}{2\pi n}\cos\frac{2}{3}\pi n x\Big|_0^{3/2}\right)\right] =$$

$$= + \frac{18}{4(\pi n)^2}\cdot\frac{3}{2\pi n}(\cos\pi n - 1) = \frac{27}{4(\pi n)^3}(\cos\pi n - 1) =$$

$$= -\frac{27}{4(\pi n)^3}(1-\cos\pi n) = \frac{-27}{4}\cdot\frac{1-(-1)^n}{(\pi n)^3}$$

$$u(x,t) = -\frac{27}{4}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{(\pi n)^3}\cos\frac{2}{3}\pi n t\cdot\sin\frac{2}{3}\pi n x\right)\cdot\frac{1}{\|X_n\|^2}$$

$$\|X_n\|^2 = \frac{3}{4}$$

$$u(x,t) = -\frac{27}{4}\cdot\frac{4}{3}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{(\pi n)^3}\cos\frac{2}{3}\pi n t\cdot\sin\frac{2}{3}\pi n x =$$

$$= -9\sum_{n=1}^{\infty}\frac{1-(-1)^n}{(\pi n)^3}\cos\frac{2}{3}\pi n t\cdot\sin\frac{2}{3}\pi n x$$

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② Решить краевую задачу для уравнения Гельмгольца $\Delta u + u = 0$ в круге $0 \le r < 1$, $0 \le \varphi < 2\pi$, на границе которого искомая ф-ция $u(r,\varphi)$ удовлетворяет условию: $u(1,\varphi) = \cos^2\varphi - 3\sin\varphi$

$$\frac{1}{r}\left(\frac{\partial}{\partial r}\right)\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial u^2} + k^2 u = 0$$

$$u = R(r)\,\Phi(\varphi) \not\equiv 0$$

$$\frac{\Phi}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{R}{r^2}\Phi'' + k^2 R\Phi = 0$$

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + k^2r^2 + \frac{\Phi''}{\Phi} = 0$$

$$-\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) - k^2r^2 = \frac{\Phi''}{\Phi} = -\lambda = \text{const}$$

1) $\begin{cases} \Phi'' + \lambda\Phi = 0 \\ \Phi(0) = \Phi(2\pi) \\ \Phi'(0) = \Phi'(2\pi) \end{cases}$ $\rightarrow$ Решение: $\lambda_n = n^2$; $n = 0, 1, 2, \dots$

$$\Phi_n = \begin{cases} \cos n\varphi \\ \sin n\varphi \end{cases} \quad n = 0, 1, 2, \dots$$

$$\|\Phi_0\|^2 = 2\pi$$
$$\|\Phi_n\|^2 = \pi, \quad n = \overline{1, \infty}$$

2) $\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + k^2r^2 - n^2 = 0 \quad | \cdot R$

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + (k^2r^2 - n^2)R = 0$$

Пусть $x = kr$; $y(x) = R(r) = R\left(\frac{x}{k}\right)$

$$r = \frac{x}{k}$$

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$$\frac{x}{k}\frac{d}{d\frac{x}{k}}\left(\frac{x}{k}\frac{dy}{d\frac{x}{k}}\right)+(x^2-n^2)y=0.$$

$$x\frac{d}{dx}\left(x\frac{dy}{dx}\right)+(x^2-n^2)y=0$$

$$y=C_1 J_n(x)+C_2 N_n(x).$$

$$R(r)=C_1 J_n(kr)+C_2 N_n(kr)$$

При $r\to 0$ ф-ции должны быть ограничены

$\underline{\underline{k=1}}$

$$R(r)=C_1 J_n(r)+C_2 N_n(r)$$

$N_n(r)$ не огр. при $r\to 0 \Rightarrow C_2=0 \quad C_1=1$

$$R(r)=J_n(r).$$

3) $u=R(r)\Phi(\varphi)=\sum\limits_{n=0}^{\infty} J_n(r)[A_n\cos n\varphi+B_n\sin n\varphi]$

$u(1,\varphi)=\cos^2\varphi-3\sin\varphi$

$\cos^2\varphi-3\sin\varphi=\sum\limits_{n=0}^{\infty} J_n(1)[A_n\cos n\varphi+B_n\sin n\varphi]\,|\cdot\cos n\varphi$

$$\int_0^{2\pi}\cos^2\varphi\cdot\cos n\varphi\,d\varphi-3\int_0^{2\pi}\sin\varphi\cdot\cos n\varphi\,d\varphi=\sum_{n=0}^{\infty}J_n(1)\cdot A_n\cdot\|\Phi_n\|^2$$

$$\frac{1}{2}\int_0^{2\pi}(1+\cos 2\varphi)\cdot\cos n\varphi\,d\varphi=\frac{1}{2}\int_0^{2\pi}\cos n\varphi\,d\varphi+\frac{1}{2}\int_0^{2\pi}\cos 2\varphi\cdot\cos n\varphi\,d\varphi=$$

$$=\frac{1}{2n}\underbrace{\sin n\varphi\Big|_0^{2\pi}}_{0}+\frac{1}{4}\int_0^{2\pi}[\cos(2+n)\varphi+\cos(2-n)\varphi]\,d\varphi=$$

$$=\frac{1}{4}\left(\frac{\sin(2+n)\varphi}{2+n}+\frac{\sin(2-n)\varphi}{2-n}\right)\Big|_0^{2\pi}=$$

$$=\frac{2\pi}{4}\frac{\sin(2-n)2\pi}{2\pi(2-n)}=\frac{\pi}{2}\cdot\begin{cases}1, & n=2\\ 0, & n\neq 2\end{cases}$$

6

$$3\int_0^{2\pi} \sin\varphi \cdot \cos m\varphi \, d\varphi = 0.$$

$$\frac{\pi}{2} = J_n(1) \cdot A_2 \cdot \pi \Rightarrow A_2 = \frac{1}{2J_n(1)}$$

$$\sin n\varphi(\cos^2\varphi - 3\sin\varphi) = \sum_{n=0}^{\infty} J_n(1) B_n \|\varphi_n\|^2$$

$$\int_0^{2\pi}\cos^2\varphi \cdot \sin m\varphi d\varphi = \frac{1}{2}\int_0^{2\pi}(1+\cos 2\varphi)\sin n\varphi \, d\varphi = \frac{1}{2}\int_0^{2\pi}\sin n\varphi \, d\varphi +$$

$$+ \frac{1}{2}\int_0^{2\pi}\cos 2\varphi \sin n\varphi \, d\varphi = -\frac{1}{2}\left(\underbrace{\cos 2\pi n - 1}_{0}\right) +$$

$$+ \frac{1}{2}\cdot\frac{1}{2}\int_0^{2\pi}[\sin(2+n)\varphi + \sin(n-2)\varphi]\, d\varphi = \frac{1}{4}\left(-\frac{\cos(2+n)\varphi}{2+n} +\right.$$

$$\left. + \frac{\cos(2-n)\varphi}{2-n}\right)\Big|_0^{2\pi} = 0.$$

$$3\int_0^{2\pi}\sin\varphi \cdot \sin m\varphi \, d\varphi = \frac{3}{2}\int_0^{2\pi}[\cos(1-n)\varphi - \cos(1+n)\varphi]\, d\varphi =$$

$$= \frac{3}{2}\left(\frac{\sin(1-n)\varphi}{1-n} - \frac{\sin(1+n)\varphi}{(1+n)}\right)\Big|_0^{2\pi} = \frac{3}{2}\cdot 2\pi \frac{\sin(1-n)\cdot 2\pi}{(1-n)\cdot 2\pi} =$$

$$= 3\pi \begin{cases} 1, n=1 \\ 0, n\neq 0. \end{cases}$$

$$-3\pi = J_n(1) B_1 \cdot \pi \Rightarrow B_1 = -\frac{3}{J_n(1)}$$

$$u = J_n(1)\cdot\frac{1}{2J_n(1)}\cdot\cos 2\varphi - J_n(1)\frac{3}{J_n(1)}\cdot\sin\varphi$$

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(3)

6. Определить тип уравнения

$u_{xx} + 4u_{xy} + 13u_{yy} + 3u_x + 24u_y + 9x + 9y = 0.$

Привести его к каноническому виду.

$a_{11} = 1$

$a_{22} = 13$

$a_{12} = 2$

$\Delta = 4 - 13 < 0 \Rightarrow$ эллиптический тип.

$\lambda^2 - 4\lambda + 13 = 0$

$\lambda_{1,2} = \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = 4 \pm 6i$

$\frac{\partial y}{\partial x} = 4 - 6i$

$y = 4x - 6xi + C$

$\xi = y + \frac{2}{3}\eta \quad y = \xi - \frac{2}{3}\eta$

$\begin{cases} \xi = y - 4x \\ \eta = -6x \end{cases} \quad x = -\frac{\eta}{6}$

$\xi_x = -4 \quad \eta_y = 0$

$\xi_y = 1 \quad \eta_x = -6$

$u_x = \frac{\partial u}{\partial x} = -4u_\xi - 6u_\eta$

$u_y = u_\xi$

$u_{xx} = -4(-4u_{\xi\xi} - 6u_{\xi\eta}) - 6(-4u_{\xi\eta} - 6u_{\eta\eta})$

$u_{xy} = -4u_{\xi\xi} - 6u_{\eta\xi}$

$u_{yy} = u_{\xi\xi}$

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$\cancel{16u_{\xi\xi}} + \cancel{24u_{\xi\eta}} + 24u_{\xi\eta} + 36u_{\eta\eta} - \cancel{16u_{\xi\xi}} - \cancel{24u_{\xi\eta}} +$

$+13u_{\xi\xi} + 24u_{\xi} - 12u_{\xi} - 18u_{\eta} + 9\left(-\frac{\eta}{6}\right) + 9\left(\xi - \frac{2}{3}\eta\right) = 0$

$13u_{\xi\xi} + 36u_{\eta\eta} + 24u_{\xi\eta} + 12u_{\xi} - 18u_{\eta} - \frac{3}{2}\eta + 9\xi - 6\eta = 0.$

$\cancel{13u_{\xi\xi} + 36u_{\eta\eta} + 12u_{\xi} - 18u_{\eta} - \frac{13}{2}\eta + 9\xi = 0}.$

$\underline{13u_{\xi\xi} + 36u_{\eta\eta} + 12u_{\xi} - 18u_{\eta} - \frac{13}{2}\eta + 9\xi = 0}.$