Занятие 4. Интегрирование рациональных дробей.

Рациональная дробь $\dfrac{P_m(x)}{Q_n(x)} = \dfrac{a_m x^m + \dots + a_1 x + a_0}{b_n x^n + \dots + b_1 x + b_0}$ называется *правильной*, если $m < n$. Из любой неправильной рациональной дроби можно выделить целую часть, то есть представить её в виде $\dfrac{P_m(x)}{Q_n(x)} = M_{m-n}(x) + \dfrac{R_r(x)}{Q_n(x)}$, где M и R — многочлены, $r < n$. Таким образом, задача интегрирования рациональной дроби сводится к задаче интегрирования правильной рациональной дроби.

Пусть $Q_n(x) = b_n x^n + \dots + b_1 x + b_0 = b_n (x-\alpha_1)^{s_1} \dots (x-\alpha_l)^{s_l} (x^2+p_1x+q_1)^{t_1} \dots (x^2+p_kx+q_k)^{t_k}$. Тогда для правильной дроби $\dfrac{P_m(x)}{Q_n(x)}$ справедливо разложение в сумму *простейших дробей*

$$\frac{P_m(x)}{Q_n(x)} = \frac{A_1^{(1)}}{x-\alpha_1} + \dots + \frac{A_{s_1}^{(1)}}{(x-\alpha_1)^{s_1}} + \dots + \frac{A_1^{(1)}}{x-\alpha_l} + \dots + \frac{A_{s_l}^{(l)}}{(x-\alpha_l)^{s_l}} +$$

$$+ \frac{B_1^{(1)}x + C_1^{(1)}}{x^2+p_1x+q_1} + \dots + \frac{B_{t_1}^{(1)}x + C_{t_1}^{(1)}}{(x^2+p_1x+q_1)^{t_1}} + \frac{B_1^{(k)}x + C_1^{(k)}}{x^2+p_kx+q_k} + \dots + \frac{B_{t_k}^{(k)}x + C_{t_k}^{(k)}}{(x^2+p_kx+q_k)^{t_k}},$$

где $A_i^{(j)}$, $B_i^{(j)}$, $C_i^{(j)}$ — некоторые числа.

6.167.

$$\triangleleft \int \frac{2x^2-1}{x^3-5x^2+6x}\,dx = \int \frac{2x^2-1}{x(x-2)(x-3)}\,dx = \int \frac{A_1}{x}\,dx + \int \frac{A_2}{x-2}\,dx + \int \frac{A_3}{x-3}\,dx =$$

$$= \left| \begin{cases} A_1 + A_2 + A_3 & = \ 2 \qquad A_1 = -1/6 \\ -5A_1 - 3A_2 - 2A_3 & = \ 0 \qquad A_2 = -7/2 \\ 6A_1 & = \ -1 \qquad A_3 = 17/3 \end{cases} \right| = -\frac{1}{6}\int \frac{dx}{x} - \frac{7}{2}\int \frac{dx}{x-2} + \frac{17}{3}\int \frac{dx}{x-3} =$$

$$= -\frac{1}{6}\ln|x| - \frac{7}{2}\ln|x-2| + \frac{17}{3}\ln|x-3| + C. \triangleright$$

6.168.

$$\triangleleft \int \frac{x^3+2}{x^3-4x}\,dx = \int \frac{x^3-4x+4x+2}{x^3-4x}\,dx = \int dx + \int \frac{4x+2}{x^3-4x}\,dx;$$

$$\int \frac{4x+2}{x^3-4x}\,dx = \int \frac{4x+2}{x(x-2)(x+2)}\,dx = \int \frac{A_1}{x}\,dx + \int \frac{A_2}{x-2}\,dx + \int \frac{A_3}{x+2}\,dx =$$

$$= \left| \begin{cases} A_1 + A_2 + A_3 & = \ 0 \qquad A_1 = -1/2 \\ 2A_2 - 2A_3 & = \ 4 \qquad A_2 = 5/4 \\ -4A_1 & = \ 2 \qquad A_3 = -3/4 \end{cases} \right| = -\frac{1}{2}\int \frac{dx}{x} + \frac{5}{4}\int \frac{dx}{x-2} - \frac{3}{4}\int \frac{dx}{x+2} =$$

$$= -\frac{1}{2}\ln|x| + \frac{5}{4}\ln|x-2| - \frac{3}{4}\ln|x+2| + C;$$

$$\int \frac{x^3+2}{x^3-4x}\,dx = x - \frac{1}{2}\ln|x| + \frac{5}{4}\ln|x-2| - \frac{3}{4}\ln|x+2| + C. \triangleright$$

6.169.

$$\triangleleft \int \frac{x^4+3x^3+3x^2-5}{x^3+3x^2+3x+1}\,dx = \int \frac{x^4+3x^3+3x^2+x-x-5}{x^3+3x^2+3x+1}\,dx = \int x\,dx - \int \frac{x+5}{(x+1)^3}\,dx;$$

$$\int \frac{x+5}{(x+1)^3}\,dx = \int \frac{A_1}{x+1}\,dx + \int \frac{A_2}{(x+1)^2}\,dx + \int \frac{A_3}{(x+1)^3}\,dx =$$

$$= \left| \begin{cases} A_1 & = & 0 & A_1=0 \\ 2A_1+A_2 & = & 1 & A_2=1 \\ A_1+A_2+A_3 & = & 5 & A_3=4 \end{cases} \right| = \int \frac{dx}{(x+1)^2} + 4\int \frac{dx}{(x+1)^3} =$$

$$= \int \frac{d(x+1)}{(x+1)^2} + 4\int \frac{d(x+1)}{(x+1)^3} = -\frac{1}{x+1} - \frac{2}{(x+1)^2} + C;$$

$$\int \frac{x^4+3x^3+3x^2-5}{x^3+3x^2+3x+1}\,dx = \frac{x^2}{2} + \frac{1}{x+1} + \frac{2}{(x+1)^2} + C. \triangleright$$

6.172.

$$\triangleleft \int \frac{dx}{x(x^2+2)} = \int \frac{A}{x}\,dx + \int \frac{Bx+C}{x^2+2}\,dx = \left| \begin{cases} A+B & = & 0 & A=1/2 \\ C & = & 0 & B=-1/2 \\ 2A & = & 1 & C=0 \end{cases} \right| =$$

$$= \frac{1}{2}\int \frac{dx}{x} - \frac{1}{2}\int \frac{x}{x^2+2}\,dx = \frac{1}{2}\int \frac{dx}{x} - \frac{1}{4}\int \frac{d(x^2+2)}{x^2+2} = \frac{1}{2}\ln|x| - \frac{1}{4}\ln(x^2+2) + C. \triangleright$$

6.174.

$$\triangleleft \int \frac{(x-1)\,dx}{(x^2+1)^3} = \frac{1}{2}\int \frac{d(x^2+1)}{(x^2+1)^3} - \int \frac{dx}{(x^2+1)^3} = -\frac{1}{4(x^2+1)^2} - \int \frac{dx}{(x^2+1)^3};$$

$$\int \frac{dx}{(x^2+1)^3} = \int \frac{1+x^2-x^2}{(x^2+1)^3}\,dx = \int \frac{dx}{(x^2+1)^2} + \int x\frac{-x\,dx}{(x^2+1)^3} = \left| \begin{matrix} u = & x \\ v = & (x^2+1)^{-2}/4 \end{matrix} \right| =$$

$$= \int \frac{dx}{(x^2+1)^2} + \frac{x}{4(x^2+1)^2} - \int \frac{dx}{4(x^2+1)^2} = \frac{3}{4}\int \frac{dx}{(x^2+1)^2} + \frac{x}{4(x^2+1)^2};$$

$$\int \frac{dx}{(x^2+1)^2} = \int \frac{1+x^2-x^2}{(x^2+1)^2}\,dx = \int \frac{dx}{x^2+1} + \int x\frac{-x\,dx}{(x^2+1)^2} = \left| \begin{matrix} u = & x \\ v = & (x^2+1)^{-1}/2 \end{matrix} \right| =$$

$$= \int \frac{dx}{x^2+1} + \frac{x}{2(x^2+1)} - \frac{1}{2}\int \frac{dx}{x^2+1} = \frac{x}{2(x^2+1)} + \frac{1}{2}\int \frac{dx}{x^2+1} = \frac{x}{2(x^2+1)} + \frac{1}{2}\operatorname{arctg} x + C;$$

$$\int \frac{dx}{(x^2+1)^3} = \frac{3x}{8(x^2+1)} + \frac{3}{8}\operatorname{arctg} x + \frac{x}{4(x^2+1)^2} + C;$$

$$\int \frac{(x-1)\,dx}{(x^2+1)^3} = -\frac{1+x}{4(x^2+1)^2} - \frac{3x}{8(x^2+1)} - \frac{3}{8}\operatorname{arctg} x + C. \triangleright$$

6.177.

$$\triangleleft \int \frac{x^2-x+4}{(x+1)(x-2)(x-3)}\,dx = \int \frac{A_1}{x+1}\,dx + \int \frac{A_2}{x-2}\,dx + \int \frac{A_3}{x-3}\,dx =$$

$$= \left| \begin{cases} A_1 + A_2 + A_3 & = 1 \\ -5A_1 - 2A_2 - A_3 & = -1 \\ 6A_1 - 3A_2 - 2A_3 & = 4 \end{cases} \quad \begin{matrix} A_1 = 1/2 \\ A_2 = -2 \\ A_3 = 5/2 \end{matrix} \right| = \frac{1}{2}\int \frac{dx}{x+1} - 2\int \frac{dx}{x-2} + \frac{5}{2}\int \frac{dx}{x-3} =$$

$$= \frac{1}{2}\ln|x+1| - 2\ln|x-2| + \frac{5}{2}\ln|x-3| + C. \triangleright$$

6.178.

$$\triangleleft \int \frac{dx}{x^3+8} = \int \frac{dx}{(x+2)(x^2-2x+4)} = \int \frac{A}{x+2}\,dx + \int \frac{Bx+C}{x^2-2x+4}\,dx =$$

$$= \left| \begin{cases} A + B & = 0 \\ -2A + 2B + C & = 0 \\ 4A + 2C & = 1 \end{cases} \quad \begin{matrix} A = 1/12 \\ B = -1/12 \\ C = 1/3 \end{matrix} \right| = \frac{1}{12}\int \frac{dx}{x+2} + \frac{1}{12}\int \frac{-x+4}{x^2-2x+4}\,dx =$$

$$= \frac{1}{12}\int \frac{dx}{x+2} - \frac{1}{24}\int \frac{2x-2-6}{x^2-2x+4}\,dx = \frac{1}{12}\int \frac{d(x+2)}{x+2} - \frac{1}{24}\int \frac{d(x^2-2x+4)}{x^2-2x+4} +$$

$$+ \frac{6}{24}\int \frac{d(x-1)}{(x-1)^2+3} = \frac{1}{12}\ln|x+2| - \frac{1}{24}\ln|x^2-2x+4| + \frac{1}{4\sqrt{3}}\operatorname{arctg}\frac{x-1}{\sqrt{3}} + C. \triangleright$$

6.179.

$$\triangleleft \int \frac{5x-13}{(x^2-5x+6)^2}\,dx = \int \frac{5x-13}{(x-2)^2(x-3)^2}\,dx = \int \frac{A_1}{x-2}\,dx + \int \frac{A_2}{(x-2)^2}\,dx +$$

$$+ \int \frac{A_3}{x-3}\,dx + \int \frac{A_4}{(x-3)^2}\,dx = \left| \begin{cases} A + B & = 0 \\ -2A + 2B + C & = 0 \\ 4A + 2C & = 1 \end{cases} \quad \begin{matrix} A = 1/12 \\ B = -1/12 \\ C = 1/3 \end{matrix} \right| =$$

Следующие интегралы вычислить, не применяя метода неопределённых коэффициентов.

6.186.

$$\triangleleft \int \frac{dx}{x^7+x^5} = \int \frac{x\,dx}{x^8+x^6} = \left| \begin{matrix} x = \sqrt{t} \\ t = x^2 \end{matrix} \right| = \frac{1}{2}\int \frac{dt}{t^4+t^3} = \frac{1}{2}\int \frac{dt}{t^3(t+1)} = \frac{1}{2}\int \frac{t^3+1-t^3}{t^3(t+1)}\,dt =$$

$$= \frac{1}{2}\int \frac{(t+1)(t^2-t+1)}{t^3(t+1)}\,dt - \frac{1}{2}\int \frac{t^3}{t^3(t+1)}\,dt = \frac{1}{2}\int \frac{t^2-t+1}{t^3}\,dt - \frac{1}{2}\int \frac{dt}{t+1} =$$

$$= \frac{1}{2}\int \frac{dt}{t} - \frac{1}{2}\int \frac{dt}{t^2} + \frac{1}{2}\int \frac{dt}{t^3} - \frac{1}{2}\int \frac{d(t+1)}{t+1} = \frac{1}{2}\ln|t| + \frac{1}{2t} - \frac{1}{4t^2} - \frac{1}{2}\ln|t+1| + C =$$

$$= \frac{1}{2}\ln\left|\frac{x^2}{x^2+1}\right| + \frac{1}{2x^2} - \frac{1}{4x^4} + C. \triangleright$$

6.188.

$$\triangleleft \int \frac{x^2-x}{(x+1)^9}\,dx = \int \frac{x^2+2x+1-3x-1}{(x+1)^9}\,dx = \int \frac{(x+1)^2-3(x+1)+2}{(x+1)^9}\,dx =$$

$$= \int \frac{d(x+1)}{(x+1)^7} - 3\int \frac{d(x+1)}{(x+1)^8} + 2\int \frac{d(x+1)}{(x+1)^9} = -\frac{1}{6(x+1)^6} + \frac{3}{7(x+1)^7} - \frac{1}{4(x+1)^8} + C. \triangleright$$