Бушуев Денис РЛ2-31 Билет 2 21.01.21.

№1.

$u_{tt} = 4u_{xx}$ $0<x<3$, $0<t<\infty$ $u(x,0) = 8\sin \pi x$, $u_t(x,0) = 8\pi \sin 4\pi x$, $u(0,t)=0$

$u(3,t)=0$

$\ast$ Пусть $u(x,t) = X(x)T(t) \not\equiv 0$

$XT'' = 4TX'' \Rightarrow \dfrac{T''}{T} = 4\dfrac{X''}{X} = -\lambda$

разделяющиеся переменные

(I)

$\begin{cases} X'' + \frac{1}{4}X = 0 \\ X(0)=0 \\ X(3)=0 \end{cases}$ Решим задачу Штурма - Лиувилля.

1) $\lambda=0$ $\begin{cases} X''=0 \\ X(0)=0 \\ X(3)=0 \end{cases}$

$X = c_1 x + c_2$

$\begin{cases} c_2=0 \\ 3c_1=0 \end{cases}$

$\begin{cases} c_2=0 \\ c_1=0 \end{cases} \Rightarrow \lambda=0$ - не С.З., $X=0$ - не С.Ф.

2) $\lambda = -\omega^2 < 0$

$\begin{cases} X'' - \frac{1}{4}\omega^2 X = 0 \\ X(0)=0 \\ X(3)=0 \end{cases}$

$X = c_1 e^{\frac{\omega}{2}x} + c_2 e^{-\frac{\omega}{2}x}$

$\begin{cases} c_1+c_2=0 \\ c_1 e^{\frac{3}{2}\omega} + c_2 e^{-\frac{3}{2}\omega} = 0 \end{cases}$

$\begin{pmatrix} 1 & 1 \\ e^{\frac{3}{2}\omega} & e^{-\frac{3}{2}\omega} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\begin{vmatrix} 1 & 1 \\ e^{\frac{3}{2}\omega} & e^{-\frac{3}{2}\omega} \end{vmatrix} = \underbrace{e^{-\frac{3}{2}\omega} - e^{\frac{3}{2}\omega} = 0}_{\text{дисперсионное ур-е.}} \Rightarrow c_1=c_2=0$

$\lambda<0$ - не С.З., $X=0$ - не С.Ф.

3) $\lambda = \omega^2 > 0$

$\begin{cases} X'' + \frac{\omega^2}{4}X = 0 \\ X(0)=0 \\ X(3)=0 \end{cases}$

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$X = c_1 \cos\frac{\omega}{2}x + c_2 \sin\frac{\omega}{2}x$

$$\begin{cases} c_1 = 0 \\ c_2 \sin\frac{3\omega}{2} = 0 \end{cases} \Rightarrow c_2 \neq 0 \Rightarrow \sin\frac{3\omega}{2} = 0$$

$\frac{3\omega}{2} = \pi n, \quad n = \overline{1,\infty}$

$\omega = \frac{2\pi n}{3}, \quad n = \overline{1,\infty}$

$\lambda_n = \left(\frac{2\pi n}{3}\right)^2$ - С.З. $\quad n = \overline{1,\infty}$

$X = \cancel{\frac{8\pi}{3}} \sin\frac{\pi n}{3}x$ - С.Ф. $\quad n = \overline{1,\infty}$

$$\|X_n\|^2 = \int_0^3 \sin^2\frac{\pi n}{3}x\,dx = \frac{1}{2}\left[\int_0^3 dx - \int_0^3 \cos\frac{2\pi n}{3}x\,dx\right] = \frac{3}{2} - \frac{3}{2\pi n}\sin\frac{2\pi n}{3}x\Big|_0^3 = \frac{3}{2}$$

(II) $\frac{T''}{T} = -\lambda \Rightarrow T'' + \left(\frac{2\pi n}{3}\right)^2 T = 0$

$T_n = c_1 \cos\frac{2\pi n}{3}t + c_2 \sin\frac{2\pi n}{3}t$

(III) $U = X \cdot T$ Общее решение

$U_n = X_n \cdot T_n = \sin\frac{\pi n}{3}x\left(A_n \cos\frac{2\pi n}{3}t + B_n \sin\frac{2\pi n t}{3}\right)$

$$U = \sum_{n=0}^{\infty} \sin\frac{\pi n}{3}x\left(A_n \cos\frac{2\pi n}{3}t + B_n \sin\frac{2\pi n t}{3}\right)$$

(IV) Г.У.

$U(x,0) = 8\sin\pi x$

$U_t(x,0) = 8\pi \sin 4\pi x$

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№2.

$\Delta U=0 \qquad 0\le r<4 \qquad 0\le \varphi\le 2\pi \qquad U(4,\varphi)=\cos^3\varphi+4\sin^3\varphi+\sin^2\varphi$

Пусть $U=\Phi(\varphi)\cdot R(r)\not\equiv 0$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 U}{\partial\varphi^2}=0$$

$$\cancel{\frac{\Phi}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)+\frac{R}{r^2}\frac{\partial\Phi}{\partial\varphi^2}=0}$$

$$\frac{\Phi}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right)+\frac{R}{r^2}\frac{d^2\Phi}{d\varphi^2}=0 \quad \Big|\cdot\frac{r^2}{R\Phi}$$

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right)+\frac{\Phi''}{\Phi}=-\lambda$$

(I) $\begin{cases}\Phi''+\lambda\Phi=0\\ \Phi(0)=\Phi(2\pi)\\ \Phi'(0)=\Phi'(2\pi)\end{cases}$

(1) $\lambda=0 \quad \Phi''=0 \qquad \Phi=C_1\varphi+C_2,\ \Phi'=C_1 \qquad \begin{cases}C_2=2\pi C_1+C_2\\ C_1=C_1\end{cases} \Rightarrow \begin{matrix}C_1=0\\ C_2\neq 0\end{matrix}$

$\Phi=C$ - С.Ф.

$\lambda=0$ - С.З.

(2) $\lambda=-\omega^2<0$

$\begin{cases}\Phi''-\omega^2\Phi=0\\ \Phi(0)=\Phi(2\pi)\\ \Phi'(0)=\Phi'(2\pi)\end{cases}$

$\Phi=C_1e^{\omega\varphi}+C_2e^{-\omega\varphi}$

$\Phi'=C_1\omega e^{\omega\varphi}-C_2\omega e^{-\omega\varphi}$

$\begin{cases}C_1+C_2=C_1e^{2\pi\omega}+C_2e^{-2\pi\omega}\\ C_1\omega-C_2\omega=C_1\omega e^{2\pi\omega}-C_2\omega e^{-2\pi\omega}\ \Big|:\omega\end{cases}$

$$\begin{pmatrix}1-e^{2\pi\omega} & 1-e^{-2\pi\omega}\\ 1-e^{2\pi\omega} & -(1-e^{-2\pi\omega})\end{pmatrix}\begin{pmatrix}C_1\\ C_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$\begin{vmatrix}1-e^{2\pi\omega} & 1-e^{-2\pi\omega}\\ 1-e^{2\pi\omega} & -(1-e^{-2\pi\omega})\end{vmatrix}=-2(1-e^{2\pi\omega})(1-e^{-2\pi\omega})\neq 0 \Rightarrow C_1=C_2=0 \Rightarrow \lambda<0$ - не С.З.

$\Phi=0$ - не С.Ф.

③ $\lambda = \omega^2 > 0$

$\Phi'' + \omega^2 \Phi = 0$

$\Phi = c_1 \cos \omega\varphi + c_2 \sin \omega\varphi$

$\Phi' = -c_1\omega \sin \omega\varphi + c_2 \omega \cos \omega\varphi$

$$\begin{cases} c_1 = c_1 \cos 2\pi\omega + c_2 \sin 2\pi\omega \\ c_2\omega = -c_1\omega \sin 2\pi\omega + c_2\omega \cos 2\pi\omega \end{cases}$$

$$\begin{vmatrix} 1-\cos 2\pi\omega & -\sin 2\pi\omega \\ \sin 2\pi\omega & 1-\cos 2\pi\omega \end{vmatrix} = 0$$

$(1-\cos 2\pi\omega)^2 + \sin^2 2\pi\omega = 0$

$1 - 2\cos 2\pi\omega + \cos^2 2\pi\omega + \sin^2 2\pi\omega = 0$

$1 - 2\cos 2\pi\omega + 1 = 0$

$\cos 2\pi\omega = 1$

$2\pi\omega = 2\pi n$, $n = \overline{1,\infty}$

$\omega = n$; $\lambda_n = n^2$ – С.З.

$$\begin{cases} c_1 = c_1 \\ c_2 = c_2 \end{cases}$$

$\Phi = \begin{cases} \cos n\varphi \\ \sin n\varphi \end{cases}$ $n = \overline{1,\infty}$

$\|\Phi_n\|^2 = \pi$

$\Phi_0 = C$, $\lambda_0 = 0$; $\|\Phi_0\|^2 = 2\pi$

(II) $\frac{r}{R} \frac{d}{dr}\left(r \frac{dR}{dr}\right) = \lambda \ | \cdot R$

$r \frac{d}{dr}\left(r \frac{dR}{dr}\right) - n^2 R = 0$

$r^2 R'' + rR' - n^2 R = 0$ – уравнение Эйлера.

Пусть $r = e^t$ $R(r) \to y(t)$

$e^t \frac{d}{de^t}\left(e^t \frac{dR}{de^t}\right) - n^2 R = 0$

$\frac{d}{dt}\left(\frac{dR}{dt}\right) - n^2 R = 0$

$y'' - n^2 y = 0$

1) $n \neq 0$

$y = c_1 e^{nt} + c_2 e^{-nt}$

$\lambda_n = n^2$, $n = \overline{0,\infty}$

2) $n = 0$

$y'' = 0$ $y = c_1 t + c_2$

$n \neq 0$ $R(r) = c_1 r^n + c_2 \frac{1}{r^n}$

$n = 0$ $R(r) = c_1 \ln r + c_2$

нас интересуют решения при $r \to 0$. $\ln r$ при $r \to 0$ расходится.

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$$\begin{cases} R(r) = C_2 \\ R(r) = C_1 r^n \end{cases}$$

$R(r) = r^n, \quad n = \overline{0,\infty}$

(III) $u = R(r)\Phi(\varphi)$

$$\cancel{u = \sum_{n=0}^{\infty} r^n \{A_n \cos\varphi n + B_n \sin\varphi n\}} \quad u = A_0 + \sum_{n=1}^{\infty} r^n \{A_n \cos n\varphi + B_n \sin n\varphi\}$$

$$u(4,\varphi) = \cos^3\varphi + 4\sin^3\varphi + \sin^2\varphi$$

$$u(4,\varphi) = \frac{\cos\varphi}{2} + \frac{\cos\varphi\cos 2\varphi}{2} + 2\sin\varphi - 2\sin\varphi\cos 2\varphi + \frac{1}{2} - \frac{\cos 2\varphi}{2} = \frac{\cos\varphi}{2} + \frac{\cos\varphi}{4} + \frac{\cos 3\varphi}{4} + 2\sin\varphi$$

$$+ \sin\varphi - \sin 3\varphi + \frac{1}{2} - \frac{\cos 2\varphi}{2} = \frac{3\cos\varphi}{4} + \frac{\cos 3\varphi}{4} + 3\sin\varphi - \sin 3\varphi - \frac{\cos 2\varphi}{2} + \frac{1}{2}$$

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{3\cos\varphi}{4} + \frac{\cos 3\varphi}{4} + 3\sin\varphi - \sin 3\varphi - \frac{\cos 2\varphi}{2} + \frac{1}{2}\right]d\varphi = \frac{1}{2\pi}\left[\frac{3\sin\varphi}{4} + \frac{\sin 3\varphi}{12} - 3\cos\varphi + \frac{\cos 3\varphi}{3} +\right.$$

$$\left. + \frac{\sin 2\varphi}{4} + \frac{\varphi}{2}\right]\Bigg|_0^{2\pi} = \cancel{\frac{1}{2\pi}[0+0+3\pi-(6-3)+\tfrac{1}{3}]} = \frac{1}{2\pi}\left[0 + 0 - 3 + \frac{1}{3} + 0 + \pi - 0 - 0 + 3 - \frac{1}{3} + 0 + 0\right] = \frac{1}{2}$$

$$A_n = \frac{1}{\pi}\int_0^{2\pi}\left[\frac{3\cos\varphi}{4} + \frac{\cos 3\varphi}{4} + 3\sin\varphi - \sin 3\varphi - \frac{\cos 2\varphi}{2} + \frac{1}{2}\right]\cdot\cos n\varphi\, d\varphi$$

Расс-им 6 интегралов:

$$I_1 = \frac{3}{4\pi}\int_0^{2\pi}\cos\varphi\cos n\varphi\, d\varphi = \frac{3}{8\pi}\left[\frac{\sin(1-n)\varphi}{1-n} + \underbrace{\cancel{\frac{\sin(1+n)\varphi}{1+n}}}_{0}\right]\Bigg|_0^{2\pi} = \frac{3}{8\pi}\cdot\frac{2\pi\sin(1-n)2\pi}{(1-n)2\pi} = \frac{3}{4}\begin{cases}1, & n=1\\ 0, & n\neq 1\end{cases}$$

$$I_2 = \frac{1}{4\pi}\int_0^{2\pi}\cos 3\varphi\cos n\varphi\, d\varphi = \frac{1}{8\pi}\left[\frac{\sin(3-n)\varphi}{3-n} + \frac{\sin(3+n)\varphi}{3+n}\right]\Bigg|_0^{2\pi} = \frac{1}{8\pi}\cdot\frac{2\pi\sin(3-n)2\pi}{(3-n)2\pi} = \frac{1}{4}\begin{cases}1, & n=3\\ 0, & n\neq 3\end{cases}$$

$$I_3 = \frac{3}{\pi}\int_0^{2\pi}\sin\varphi\cos n\varphi\, d\varphi = -\frac{3}{2\pi}\left[\frac{\cos(1-n)\varphi}{1-n} + \frac{\cos(1+n)\varphi}{1+n}\right]_0^{2\pi} = -\frac{3}{2\pi}\left[\frac{\cos(1-n)2\pi}{1-n} + \frac{\cos(1+n)2\pi}{1+n} - \frac{1}{1-n} - \frac{1}{1+n}\right] = 0$$

$$I_4 = -\frac{1}{\pi}\int_0^{2\pi}\sin 3\varphi\cos n\varphi\, d\varphi = \frac{1}{2\pi}\left[\frac{\cos(3-n)\varphi}{3-n} + \frac{\cos(3+n)\varphi}{3+n}\right]_0^{2\pi} = 0, \quad \forall n$$

$$I_5 = \frac{1}{2\pi}\int_0^{2\pi}\cos 2\varphi\cos n\varphi\, d\varphi = \frac{1}{4\pi}\left[\frac{\sin(2-n)\varphi}{2-n} + \frac{\sin(2+n)\varphi}{2+n}\right]\Bigg|_0^{2\pi} = \frac{1}{4\pi}\frac{2\pi\sin(2-n)2\pi}{(2-n)2\pi} = \frac{1}{2}\begin{cases}1, & n=2\\ 0, & n\neq 2\end{cases}$$

$$I_6 = \frac{1}{2\pi}\int_0^{2\pi}\cos n\varphi\, d\varphi = \frac{1}{2\pi n}\sin n\varphi\Big|_0^{2\pi} = 0$$

$$B_n = \cancel{\tfrac{1}{2\pi}}\ \frac{1}{\pi}\int_0^{2\pi}\left[\frac{3\cos\varphi}{4} + \frac{\cos 3\varphi}{4} + 3\sin\varphi - \sin 3\varphi - \frac{\cos 2\varphi}{2} + \frac{1}{2}\right]\cdot\sin n\varphi\, d\varphi$$

Расс-им 6 интегралов:

$$I_7 = \frac{3}{4\pi}\int_0^{2\pi}\sin n\varphi\cos\varphi\, d\varphi = -\frac{3}{8\pi}\left[\frac{\cos(n-1)\varphi}{n-1} + \frac{\cos(n+1)\varphi}{n+1}\right]\Bigg|_0^{2\pi} = 0 \quad \forall n$$

$$I_8 = \frac{1}{4\pi}\int_0^{2\pi}\sin n\varphi\cos 3\varphi\, d\varphi = -\frac{1}{8\pi}\left[\frac{\cos(n-3)\varphi}{n-3} + \frac{\cos(n+3)\varphi}{n+3}\right]\Bigg|_0^{2\pi} = 0 \quad \forall n$$

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$$I_9 = \frac{3}{\pi}\int_0^{2\pi} \sin n\varphi \sin\varphi\, d\varphi = \frac{3}{2\pi}\left[\frac{\sin(n-1)\varphi}{n-1} - \frac{\sin(n+1)\varphi}{n+1}\right]\Big|_0^{2\pi} = \frac{3}{2\pi}\cdot\frac{2\pi\sin(n-1)2\pi}{(n-1)2\pi} = 3\begin{cases}1, & n=1\\ 0, & n\neq 1\end{cases}$$

$$I_{10} = -\frac{1}{\pi}\int_0^{2\pi} \sin n\varphi \sin 3\varphi\, d\varphi = -\frac{1}{2\pi}\left[\frac{\sin(n-3)\varphi}{n-3} - \frac{\sin(n+3)\varphi}{n+3}\right]\Big|_0^{2\pi} = -\frac{1}{2\pi}\cdot\frac{2\pi\sin(n-3)2\pi}{(n-3)2\pi} = -1\begin{cases}1, & n=3\\ 0, & n\neq 3\end{cases}$$

$$I_{11} = -\frac{1}{2\pi}\int_0^{2\pi} \sin n\varphi \cos 2\varphi\, d\varphi = \frac{1}{4\pi}\left[\frac{\cos(n-2)\varphi}{n-2} + \frac{\cos(n+2)\varphi}{n+2}\right]\Big|_0^{2\pi} = 0 \quad \forall n$$

$$I_{12} = \frac{1}{2\pi}\int_0^{2\pi} \sin n\varphi\, d\varphi = -\frac{1}{2\pi n}\cos n\varphi\Big|_0^{2\pi} = 0 \quad \forall n$$

$A_1 = \frac{3}{4}$ $\quad A_3 = \frac{1}{4}$ $\quad A_2 = \frac{1}{2}$

$B_1 = 3$ $\quad B_3 = -1$

$$U = \frac{1}{2} + r\cdot\left(\frac{3}{4}\cdot\cos\varphi + 3\sin\varphi\right) + r^2\cdot\frac{1}{2}\cos 2\varphi + r^3\left(\frac{1}{4}\cos 3\varphi - \sin 3\varphi\right)$$

№3.

a) ~~$\Delta<0$ $a_{12}^2 - a_{11}a$~~

$$a_{11}U_{xx} + 2a_{12}U_{xy} + a_{22}U_{yy} + F(x,y,U,U_x,U_y) = 0$$

$\Delta<0$ $\quad a_{12}^2 - a_{11}a_{12} < 0$

λ_1, λ_2 - будут различны и комплексно сопряженны

$\Downarrow$

$\xi(x,y) = c_1$ $\qquad \xi^*(x,y) = c_2$

$\Downarrow$

$\xi(x,y) = c_1$

$\alpha = \mathrm{Re}\, c_1$ $\qquad \beta = \mathrm{Im}\, c_1$

действительная часть $\qquad$ мнимая часть

$U_{\alpha\alpha} + U_{\beta\beta} = \bar{F}$ - канонический вид эллиптического типа.

b) $U_{xx} + 4U_{xy} + U_{yy} + U_x + U_y - x^2 y = 0$

$a_{11} = 1$ $\quad a_{12} = a_{21} = 2$ $\quad a_{22} = 1$

$\Delta = \begin{vmatrix}1 & 2\\ 2 & 1\end{vmatrix} = -4+1 = -3 < 0$ - эллиптич. тип.

$a_{11}\lambda^2 - 2a_{12}\lambda + a_{22} = 0$ $\qquad \lambda^2 - 4\lambda + 1 = 0$

~~$D=16$~~ $D = 16 - 4 = 12$

$\lambda_{1,2} = \frac{4\pm 2\sqrt{3}}{2}$

~~$\lambda_1 = 2+\sqrt{3}$~~
~~$\lambda_2 = 2-\sqrt{3}$~~

$\lambda_{1,2} = 2 \pm \sqrt{3}i$

$\lambda = \frac{dy}{dx}$

$\lambda = 2 - \sqrt{3}i$

$\frac{dy}{dx} = 2 - \sqrt{3}i$

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$y = 2x - x\sqrt{3}i + c$

$c = y - 2x + x\sqrt{3}i$

$\mathrm{Re}\, c = y - 2x = \xi$ $\qquad \xi_x = -2 \quad \eta_x = \sqrt{3}$

$\mathrm{Im}\, c = x\sqrt{3} = \eta$ $\qquad \xi_y = 1 \quad \eta_y = 0 \qquad x = \frac{\eta}{\sqrt{3}} \qquad y = \xi - \frac{2\eta}{\sqrt{3}}$

$u_x = \frac{\partial u}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial u}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial u}{\partial \eta} = -2u_\xi + \sqrt{3}u_\eta$

$u_y = \frac{\partial u}{\partial y} = u_\xi + 0 = u_\xi$

$u_{xx} = -2(-2u_{\xi\xi} + \sqrt{3}u_{\xi\eta}) + \sqrt{3}(-2u_{\xi\eta} + \sqrt{3}u_{\eta\eta}) = 4u_{\xi\xi} - 2\sqrt{3}u_{\xi\eta} - 2\sqrt{3}u_{\xi\eta} + 3u_{\eta\eta} = 4u_{\xi\xi} - 4\sqrt{3}u_{\xi\eta} + 3u_{\eta\eta}$

$u_{yy} = u_{\xi\xi}$

$u_{xy} = \frac{\partial(u_\xi)}{\partial x} = -2u_{\xi\xi} + \sqrt{3}u_{\xi\eta}$

$-x^2y = 0$

$4u_{\xi\xi} - 4\sqrt{3}u_{\xi\eta} + 3u_{\eta\eta} + \cancel{4u_{\xi\xi} - 4\sqrt{3}u_{\xi\eta} + u_{\xi\xi}}\ -8u_{\xi\xi} + 4\sqrt{3}u_{\xi\eta} + u_{\xi\xi} + 2u_\xi + \sqrt{3}u_\eta + u_\xi - \cancel{x^2y = 0}$

$-3u_{\xi\xi} + 3u_{\eta\eta} - u_\xi + \sqrt{3}u_\eta - \frac{\eta^2}{3}\cdot\left(\xi - \frac{2\eta}{\sqrt{3}}\right) = 0 \ |:-3$

$u_{\xi\xi} - u_{\eta\eta} + \frac{u_\xi}{3} - \frac{\sqrt{3}}{3}u_\eta + \frac{\eta^2}{9}\left(\xi - \frac{2\eta}{\sqrt{3}}\right) = 0$

Канонический вид: $u_{\xi\xi} + u_{\eta\eta} = 0$