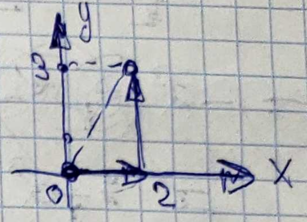
№1

$$\int\limits_{(0,0)}^{(2,3)} \frac{xdx+ydy}{\sqrt{x^2+y^2+2}} =$$ ф-ла Ньютона–Лейбница

$$\int\limits_{(x_0,y_0)}^{(x_1,y_1)} P(x,y)dx + Q(x,y)dy = U(x_1,y_1) - U(x_0,y_0)$$

1) $P = \frac{x}{\sqrt{x^2+y^2+2}}$ $Q = \frac{y}{\sqrt{x^2+y^2+2}}$



$$\underbrace{\frac{\partial P}{\partial y} = -\frac{xy}{(x^2+y^2+2)^{3/2}} \quad \frac{\partial Q}{\partial x} = -\frac{xy}{(x^2+y^2+2)^{3/2}}}$$

$\Rightarrow$ частные производ равны

2) $C_1: \begin{cases} x = t, & dx = dt \\ y = 0, & dy = 0 \end{cases}$ $0 \le t \le 2$

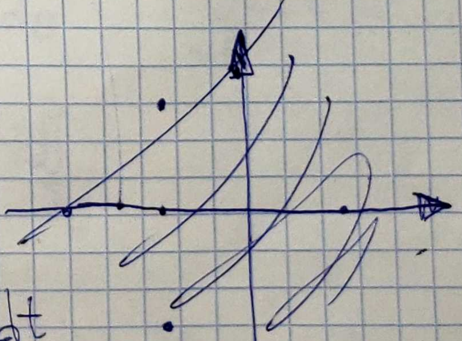
$C_2: \begin{cases} x = 2, & dx = 0 \\ y = t, & dy = dt \end{cases}$ $0 \le t \le 3$

3) $= \int\limits_0^2 \frac{t\,dt}{\sqrt{t^2+2}} + \int\limits_0^3 \frac{t\,dt}{\sqrt{t^2+6}} = \left[\sqrt{t^2+2}\right]_0^2 + \left[\sqrt{t^2+6}\right]_0^3 =$

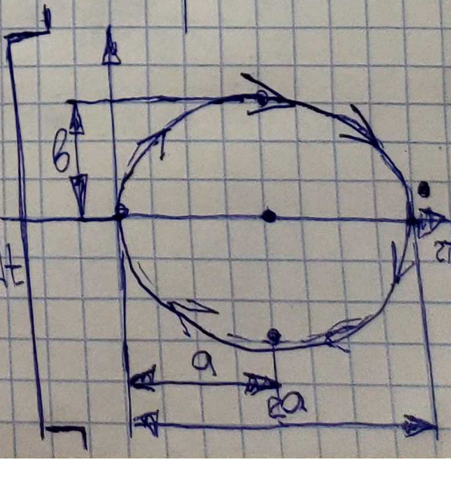
$= \cancel{\sqrt{6}} - \sqrt{2} + \sqrt{15} - \cancel{\sqrt{6}} = \sqrt{15} - \sqrt{2} > 0$

№2

$$\oint\limits_L (x^2+2y)dx - xydy \qquad \frac{(x-a)^2}{a^2} + \frac{y^2}{b^2} = 1$$



$\begin{cases} x = a + a\cos t \\ y = b\sin t \end{cases} \Rightarrow \begin{matrix} dx = (-a\sin t)dt \\ dy = b\cos t\,dt \end{matrix}$



проходить будем от 2π до 0

$(x^2+2y)dx - xydy = -a^3(1+2\cos t+\cos^2 t$
$+ 2b\sin t)\cdot \sin t\,dt - ab^2(1+\cos t)\sin t\cos t\,dt$
$= -a^3(\sin t + \sin 2t + \cos^2 t\sin t + 2b\sin^2 t)dt -$
$- ab^2(\sin t + \frac{1}{2}\sin 2t)dt =$

$$= -a^3\int_{2\pi}^{0}(\sin t+\sin 2t+\cos^2 t\sin t+2b\sin^2 t)\,dt - ab^2\int_{2\pi}^{0}\left(\sin t+\frac{1}{2}\sin 2t\right)dt =$$

$$= -a^3\left[-\cos t-\frac{\cos 2t}{2}+\frac{b}{2}\sin 2t-\frac{1}{3}\cos^3 t+3t\right]_{2\pi}^{0} -$$

$$- ab^2\left[\cos t+\frac{\cos 2t}{4}\right]_{2\pi}^{0} = 0$$

$$\cancel{= -a^3\left(-\frac{8}{3}-\pi b\right)-ab^2(-2) =}$$

$$\cancel{= \frac{8}{3}a^3+\pi a^3 b+2ab^2 = \frac{8}{3}a^2+\pi a^2 b+2b^2}$$

По ф. Грина:

$$P = x^2+2y \Rightarrow P'_y = 2 \qquad \begin{cases} x = a+a\cos t \\ y = b\sin t \end{cases} \Rightarrow \begin{cases} dx = -a\sin t\,dt \\ dy = b\cos t\,dt \end{cases}$$

$$Q = xy \Rightarrow Q'_x = y$$

$$\oint_L (x^2+2y)\,dx - xy\,dy = \iint_D (y-2)\,dx\,dy =$$

$$= -\int_{2\pi}^{0}(b\sin t-2)\,a\cdot b\sin t\cos t\,dt = -ab^2\int_{2\pi}^{0}\overbrace{\sin^2 t\cos t\,dt}^{=0} + 2ab\int_{2\pi}^{0}\overbrace{\sin t\cos t\,dt}^{=0}$$

$$= 0$$

Задача №3

$F\{yx; y^3; z^3\}$

а) векторные линии поля: $\frac{d\vec{r}}{dt} = \vec{F}$

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = \{\vec{x}, \vec{y}, \vec{z}\} \Rightarrow d\vec{r} = \vec{i}dx + \vec{j}dy + \vec{k}dz$

$\left\{\frac{dx}{dt}; \frac{dy}{dt}; \frac{dz}{dt}\right\} = \{yx; y^3; z^3\}$

$$\begin{cases} \frac{dx}{dt} = yx \\ \frac{dy}{dt} = y^3 \\ \frac{dz}{dt} = z^3 \end{cases} \Rightarrow \frac{dx}{yx} = \frac{dy}{y^3} = \frac{dz}{z^3} \Rightarrow \begin{cases} \frac{dx}{x} = \frac{dy}{y^2} \\ \frac{dx}{yx} = \frac{dz}{z^3} \end{cases} \Rightarrow \begin{cases} \ln|x| = -\frac{1}{y} + C_1 \\ \frac{1}{y}\ln|x| = -\frac{1}{2z^2} + C_2 \end{cases}$$

$x = e^{-\frac{1}{y}} + C_1$

$x = e^{-\frac{y}{2z^2}} + C_2$

$e^{-\frac{1}{y}} + C_1 - e^{-\frac{y}{2z^2}} - C_2 = 0$

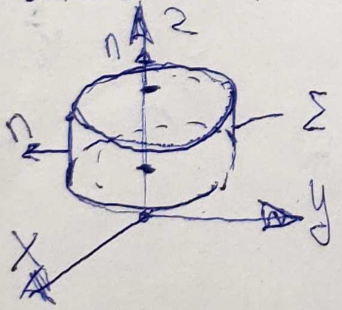
б) $\operatorname{div}\vec{F}$ и $\operatorname{div}\vec{F}$ в т. $M_0(7;1;-1)$

$\operatorname{div}\vec{F} = (\vec{\nabla}, \vec{F}) \quad F = \{yx; y^3; z^3\} \quad \nabla = +\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$

$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(yx) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3) = y + 3y^2 + 3z^2$

$\operatorname{div}\vec{F}\Big|_{M_0} = 1 + 3 + 3 = 7$

в) Найти поток векторного поля $\vec{F}$ через границу S трёхмерной области V конечного объема: $x^2+y^2=4$, $z=2$, $z=4$



1. Способ: (по определению)

$\Pi_\Sigma = \iint_\Sigma (\vec{F}, d\vec{\sigma}) = \iint_\Sigma (\vec{F}, \vec{n})\, d\sigma$

$\vec{n}$ - единичная нормаль к пов-ти

$\Sigma = \Sigma_1 + 2\Sigma_2 \Rightarrow \Pi_\Sigma = \iint_{\Sigma_1} + 2\iint_{\Sigma_2}$

Σ_1: $x^2+y^2=4 \quad \iint_{\Sigma_1} (\vec{F}, \vec{n})\, d\sigma = \iint_{\Sigma_1} (\vec{F}, \vec{n})\sqrt{|g|}\, dx\, dy$

$\vec{n} = \frac{\operatorname{grad} u}{|\operatorname{grad} u|} = \frac{\vec{\nabla} u}{|\vec{\nabla} u|}$, $\begin{matrix} \vec{\nabla} u = \{2x; 2y; 0\} \\ |\vec{\nabla} u| = \sqrt{4x^2+4y^2} \end{matrix} \Rightarrow \vec{n} = \left\{\frac{2x}{\sqrt{4x^2+4y^2}}; \frac{2y}{\sqrt{4x^2+4y^2}}; 0\right\}$

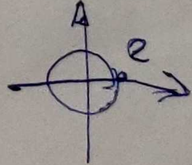
$u = x^2+y^2-4 = 0$

$(F, \vec{n}) = \frac{2yx^2 + 2y^4}{\sqrt{4x^2+4y^2}} = y\frac{(x^2+y^3)}{\sqrt{x^2+y^2}}$

$dS^2 = dx^2 + dy^2 + dz^2 \quad x^2+y^2=4$

$= dx^2 + dy^2$

$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$



$\Pi_{\Sigma_1} = \iint_{\Sigma} y\frac{(x^2+y^3)}{\sqrt{x^2+y^2}}\, dx dy = \begin{vmatrix} 0 \le \varphi \le 2\pi & \begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \\ 0 \le r \le 2 & \end{vmatrix} = \int_0^{2\pi} d\varphi \int_0^2 dr\,(\sin\varphi(r^2(\cos^2\varphi + r\sin^3\varphi)) =$

$$= \int_0^{2\pi} d\varphi \left(\underbrace{\frac{8}{3}\cdot\sin\varphi\cos^2\varphi}_{\mathrm{I}} + \underbrace{4\sin^4\varphi}_{\mathrm{II}}\right) = 3\pi$$

$$\mathrm{I} = \left|t=\cos\varphi\right| = -\int_0^{2\pi} t^2\,dt = -\frac{\cos^3\varphi}{3}\Big|_0^{2\pi} = -\frac{1}{3}+\frac{1}{3} = 0$$

$$\mathrm{II} = \int_0^{2\pi}(\sin^2\varphi)^2\,d\varphi = \int_0^{2\pi}\left(\frac{1-\cos2\varphi}{2}\right)^2 d\varphi = \int_0^{2\pi}\left(\frac{1}{4} - \frac{\cos2\varphi}{4} + \frac{1}{8} + \frac{\cos4\varphi}{32}\right)d\varphi =$$

$$= \left[\frac{3}{8}\varphi - \frac{\sin2\varphi}{4} + \frac{\sin4\varphi}{32}\right]_0^{2\pi} = \frac{3}{4}\pi$$

Π_{Σ_2}: $z=2$ $\quad n = \{0;0;0\} \Rightarrow \Pi_{\Sigma_2} = 0$

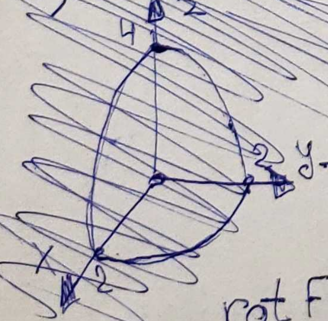
$$\Pi_\Sigma = \Pi_{\Sigma_1} + 2\Pi_{\Sigma_2} = 3\pi$$

~~Способ 2:~~ $\cancel{\Pi_\Sigma = \oiint_\Sigma (\vec F, d\vec r) = \iiint \operatorname{div}\vec F\,dV \quad \operatorname{div} f = y + 3y^2 + 3z^2}$

$\cancel{= 24\pi - \ldots 18\pi + 24\pi - 36\pi}$

~~2) Поток вектора через поверхн~~ $\cancel{\Sigma = x^2+y^2=4}$

$\cancel{\Pi_\Sigma = \iint_\Sigma (\vec F, d\vec\sigma)}$



д) ротор век. поля в произв точке

$$\operatorname{rot} F = [\nabla;\vec F] = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & y^3 & z^3 \end{vmatrix} = i(0-0) - j(0-0) + k(0-x) =$$

$$= \{0;0;-x\}$$