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$$\begin{cases} \Delta U + U = 0 \quad 0 \leq r < 2 \quad 0 \leq \varphi < 2\pi \\ U(2,\varphi) = \cos^2\varphi + 5\sin\varphi \end{cases}$$

$U(r,\varphi) \equiv U(r,\varphi+2\pi)$, $\Phi(\varphi) \equiv \Phi(\varphi+2\pi)$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \varphi^2} + k^2 U = 0$$

$U = R(r)\,\Phi(\varphi) \not\equiv 0$

$$\frac{\Phi}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{R}{r^2}\Phi'' + k^2 R\Phi = 0$$

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + k^2 r^2 + \frac{\Phi''}{\Phi} = 0$$

$$-\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) - k^2 r^2 = \frac{\Phi''}{\Phi} = -\lambda$$

I $\begin{cases} \Phi'' + \lambda\Phi = 0 \\ \Phi(0) = \Phi(2\pi) \\ \Phi'(0) = \Phi'(2\pi) \end{cases}$

1) $\lambda = 0$ $\quad \Phi'' = 0 \quad \Phi = C_1\varphi + C_2$, $\Phi' = C_1$ $\Rightarrow \begin{cases} C_2 = C_1 \cdot 2\pi + C_2 \\ C_1 = C_1 \end{cases}$ $C_1 = 0 \Rightarrow \forall C_2 \neq 0$

$\Phi = C$ – С.Ф

$\lambda = 0$ – С.З

2) $\lambda = -w^2 < 0$

$\Phi'' - w^2\Phi = 0$

$\begin{cases} \Phi = C_1 e^{w\varphi} + C_2 e^{-w\varphi} \\ \Phi' = C_1 w e^{w\varphi} - C_2 w e^{-w\varphi} \end{cases}$ $\quad \begin{cases} C_1 + C_2 = C_1 e^{2\pi w} - C_2 e^{-2\pi w} \\ C_1 w - C_2 w = C_1 w e^{2\pi w} - C_2 w e^{-2\pi w} \end{cases}$

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$$\begin{pmatrix} 1-e^{2\pi w} & 1-e^{-2\pi w} \\ 1-e^{2\pi w} & -(1-e^{-2\pi w}) \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} 1-e^{2\pi w} & 1-e^{-2\pi w} \\ 1-e^{2\pi w} & -(1-e^{-2\pi w}) \end{vmatrix} = -2(1-e^{2\pi w})(1-e^{-2\pi w}) = 0$$

$C_1 = C_2 = 0 \Rightarrow \varphi$ не с.ф, $\lambda < 0$ не с.з.

3) $\lambda = w^2 > 0$

$\varphi'' + w^2\varphi = 0$

$$\begin{cases} \varphi = C_1 \cos w\varphi + C_2 \sin w\varphi \\ \varphi' = -C_1 w \sin w\varphi + C_2 w \cos w\varphi \end{cases}$$

$$\begin{cases} C_1 = C_1 \cos 2\pi w + C_2 \sin 2\pi w \\ C_2 w = -C_1 w \sin 2\pi w + C_2 w \cos 2\pi w \end{cases}$$

$$\begin{vmatrix} 1-\cos 2\pi w & -\sin 2\pi w \\ \sin 2\pi w & 1-\cos 2\pi w \end{vmatrix} = 0$$

$(1-\cos 2\pi w)^2 + \sin^2 \pi w = 0$

$1 - 2\cos 2\pi w + \cos^2 2\pi w + \sin^2 2\pi w = 0$

$1 - 2\cos 2\pi w + 1 = 0$

$\cos 2\pi w = 2\pi n, \quad n = 1, 2, \ldots$

$w = n, \quad \lambda = (n)^2$

$$\begin{cases} C_1 = C_1 \\ C_2 = C_2 \end{cases}$$

$\varphi = \begin{cases} \cos n\varphi \\ \sin n\varphi \end{cases}, \quad n = 1, 2 \ldots$

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$\|\varphi_n\|^2 = \pi$

$\varphi_0 = C$, $\lambda_0 = 0$; $\|\varphi_0\|^2 = 2\pi$

$\varphi_n = \begin{cases} \cos n\varphi \\ \sin n\varphi \end{cases}$ $\lambda_n = n^2$; $n = \overline{0,\infty}$

II $\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + (kr)^2 - n^2 = 0$

$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left((kr)^2 - n^2\right)R = 0$ ур-е Бесселя n-ого порядка

$R(r) = C_1 J_n(kr) + C_2 N_n(kr)$

$N_n(kr)$ не огр при $kr \to 0 \Rightarrow C_2 = 0$

$J_n(kr)$ огр при $kr \to 0 \Rightarrow C_1 = 1$

$R(r) = J_n(kr)$

$U = R\Phi = \sum_{n=0}^{\infty} R_n(r)\Phi_n(\varphi) = \sum_{n=0}^{\infty} J_n(kr)\left(A_n\cos n\varphi + B_n \sin n\varphi\right)$

$U(r,\varphi) = A_0 J_0(u) + \sum_{n=1}^{\infty} J_n(2)(A_n\cos n\varphi + B_n\sin n\varphi) =$

$= \cos^2\varphi + 5\sin\varphi = \frac{1}{2} + \frac{1}{2}\cos 2\varphi + 5\sin\varphi$

Найдем A_n

$A_0 J_0(u)\int_0^{2\pi} \cos 0\varphi\, d\varphi + J_n(u) A_n \int_0^{2\pi} \cos n\varphi \cos 0\varphi\, d\varphi +$

$+ J_n(u) B_n \int_0^{2\pi} \sin n\varphi \cos 0\varphi\, d\varphi = \int_0^{2\pi}\left(\frac{1}{2} + \frac{1}{2}\cos 2\varphi + 5\sin\varphi\right)\cos 0\varphi\, d\varphi$

$2\pi A_0 J_0(u) = \int_0^{2\pi}\left(\frac{1}{2} + \frac{1}{2}\cos 2\varphi + 5\sin\varphi\right) d\varphi$

$2\pi A_0 J(2) = \pi$

$A_0 = \frac{1}{\ldots}$

$$A_0 S_0(u) \int_0^{2\pi} \cos n\varphi \, d\varphi + y_n(u) A_n \int_0^{2\pi} \cos n\varphi \cos n\varphi \, d\varphi + y_n(u) A_n \int_0^{2\pi} \sin n\varphi \cos n\varphi \, d\varphi = A_0 y_0(2) \cdot 0 + y_n(u) A_n \|\varphi_n\|^2 + \frac{1}{2} y_n(2) A_n \cdot \int_0^{2\pi} \sin 2\varphi d\varphi$$

$$= y_n(2) A_n \cdot \pi + \frac{1}{2} y_n(2) A_n \cdot 0 = \int_0^{2\pi} \frac{1}{2} \cos n\varphi \, d\varphi + \int_0^{2\pi} \frac{1}{2} \cos 2\varphi \cdot \cos n\varphi \, d\varphi +$$

I, II

$$+ \int_0^{2\pi} 5 \sin \varphi \cdot \cos n\varphi \, d\varphi$$

III

$$\mathrm{I} \quad \int_0^{2\pi} \frac{1}{2} \cos n\varphi \, d\varphi = \frac{1}{2} \left(\frac{\sin n\varphi}{n} \right) \Big|_0^{2\pi} = 0$$

$$\mathrm{II} \quad \int_0^{2\pi} \frac{1}{2} \cos 2\varphi \cdot \cos n\varphi \, d\varphi = \frac{1}{2} \cdot \frac{1}{2} \int_0^{2\pi} (\cos(2-n)\varphi + \cos(2+n)\varphi) d\varphi =$$

$$= \frac{1}{4} \left(\frac{\sin(2-n)\varphi}{2-n} + \frac{\sin(2+n)\varphi}{2+n} \right) \Bigg|_0^{2\pi} = \begin{cases} \frac{\pi}{2}, & n = 2 \\ 0, & n \neq 2 \end{cases}$$

$$\mathrm{III} \quad \int_0^{2\pi} 5 \sin \varphi \cdot \cos n\varphi \, d\varphi = \frac{5}{2} \int_0^{2\pi} (\sin(1-n)\varphi + \sin(1+n)\varphi) d\varphi =$$

$$= \frac{5}{2} \left(-\frac{\cos(1-n)\varphi}{1-n} - \frac{\cos(1+n)\varphi}{1+n} \right) \Big|_0^{2\pi} = 0$$

$$y_2(2) A_2 \pi = \frac{\pi}{2}$$

$$A_2 = \frac{1}{2 y_2(2)}$$

Найдём B_n:

$$A_0 y_0(2) \int_0^{2\pi} \sin n\varphi \, d\varphi + A_n y_n(2) \int_0^{2\pi} \cos n\varphi \cdot \sin n\varphi \, d\varphi + B_n y_n(2) \cdot \|\varphi_n\|^2 =$$

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№3

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$$= B_n \cdot J_n(2) \cdot \pi = \int_0^{2\pi} \frac{1}{2} \sin n\varphi + \int_0^{2\pi} \frac{1}{2} \cos 2\varphi \cdot \sin n\varphi \, d\varphi +$$

$$+ \int_0^{2\pi} 5 \sin\varphi \sin n\varphi \, d\varphi = \frac{5}{2} \int_0^{2\pi} (\cos(1-n)\varphi - \cos(1+n)\varphi) \, d\varphi = \frac{5}{2} \cdot$$

$$\cdot \left(\frac{\sin(1-n)\varphi}{1-n} - \frac{\sin(1+n)\varphi}{1+n} \right) \Bigg|_0^{2\pi} = \begin{cases} 5\pi, & n = 1 \\ 0, & n \neq 1 \end{cases}$$

$$B_1 J_1(2)\pi = 5\pi$$

$$B_1 = \frac{5}{J_1(2)}$$

$$U(r,\varphi) = \frac{1}{2J_0(0)} J_0(r) + \frac{1}{2J_2(2)} J_2(r) \cos 2\varphi + \frac{5}{J_1(2)} J_1(r) \sin\varphi$$

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№1

Решить первую смешанную задачу для волнового уравнения

$U_{tt}=U_{xx}$

$0<x<3$

$0<t<\infty$

$U(x,0)=0$

$U_t(x,0)=5\pi\cos 5\pi x$

$U_x(0,t)=0$

$U_x(3,t)=0$

$$\begin{cases} U_{tt}=U_{xx} \\ U(x,t)=X(x)T(t)\not\equiv 0 \\ XT''=TX'' \end{cases}$$

$$\frac{T''}{T}=\frac{X''}{X}=-\lambda$$

$$\text{I}\ \begin{cases} X''+\lambda X=0 \\ U_x(0,t)=0 \\ U_x(3,t)=0 \end{cases}$$

1) $\lambda=0$

$X''=0$

$\left.\begin{matrix} X=C_1x+C_2 \\ X'=C_1 \end{matrix}\right\} \Rightarrow$

$\Rightarrow \begin{cases} C_2=0 \\ C_1=0 \end{cases} \Rightarrow$ $\lambda=0$ – не с.з

$X=0$ – не с.ф

$\cancel{2)\ \lambda=-\omega^2<0}$

$\cancel{X=C_1\cos\omega x+C_2\sin\omega x}$

$\cancel{X'=-\omega C_1\sin\omega x+\omega C_2\sin\omega x}$

$\cancel{0=C_1\sin 0\omega+C_2\sin 0\omega}$

$\cancel{0=C_1\sin 3\omega \cdot C_2\sin 3\omega}$

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2) $\lambda = -w^2 < 0$

$$\begin{cases} x' = -C_1 w e^{w\cancel{x}} + C_2 w e^{-w\cancel{x}} \\ x' = -C_1 w e^{w\cancel{x}} + C_2 w e^{-wx} \end{cases} \quad \begin{cases} 0 = -C_1 w e^{0w} + C_2 w e^{-0w} \\ 0 = -C_1 w e^{3w} + C_2 w e^{-3w} \end{cases}$$

$$\begin{pmatrix} we^{0w} & +we^{-0w} \\ we^{3w} & +we^{-3w} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} -1 & +1 \\ -we^{3w} & +we^{-3w} \end{vmatrix} = \cancel{}\, 0 \quad -we^{3w} + we^{-3w} = 0 \quad C_1 = C_2$$

$\lambda = -w^2 < 0$ – не с.з $\quad x = 0$ не с.ф

3) $\lambda = w^2 > 0$

$x'' + w^2 x = 0$

$x = C_1 \cos wx + C_2 \sin wx$

$x' = -C_1 w \sin wx + C_2 w \cos wx$

$$\begin{cases} 0 = -C_1 w \sin 0w + C_2 w \cos 0w \\ 0 = -C_1 w \sin 3w + C_2 w \cos 3w \end{cases} \Rightarrow \begin{cases} 0 = C_2 w \cos 0w \\ 0 = -C_1 w \sin 3w + C_2 w \cos 3w \end{cases}$$

$$\begin{cases} C_2 = 0 \\ \cancel{C_1 \neq} 0 = C_1 \sin 3w \end{cases}$$

$\sin 3w = 0$

$3w = \pi n, \ n = \overline{1, \infty}$

$w = \frac{\pi}{3} n, \ n = \overline{1, \infty}$

$\lambda = \left(\frac{\pi}{3} n\right)^2$ – с.з $\qquad x = \sin \frac{\pi}{3} nx$ – с.ф, $n = \overline{1, \infty}$

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$$\|x\|^2 = \int_0^3 \sin^2 \frac{\pi}{3} n x \, dx = \frac{1}{2} \int_0^3 (1 - \cos \frac{2\pi}{3} n x) dx =$$

$$= \frac{1}{2} \cdot \left(x \Big|_0^3 - \frac{1}{2} \frac{\overset{\sim 0 \, \forall x}{\sin \frac{2}{3} \pi n}}{\frac{2}{3} \pi n} \Bigg|_0^3 \right) = \frac{1}{2} \cdot 3 = \frac{3}{2}$$

II $\frac{T''}{T} = -\lambda \Rightarrow T'' + \left(\frac{\pi}{3} n\right)^2 T = 0$

$$T_n = C_1 \cos \frac{\pi}{3} n t + C_2 \sin \frac{\pi n}{3} t$$

III $U = X \cdot T$

$$U_n = X_n T_n = \sin \frac{\pi n x}{3} \left(A_n \cos \frac{\pi}{3} n t + B_n \sin \frac{\pi}{3} n t\right)$$

$$U = \sum_{n=0}^{\infty} \sin \frac{\pi n x}{3} \left(A_n \cos \frac{\pi n t}{3} + B_n \sin \frac{\pi n t}{3}\right)$$

$U_t(x,0) = 5\pi \cos 5\pi x$

$U(x,0) = 0$

$$0 = \sum_{n=0}^{\infty} \sin \frac{\pi n x}{3} \cdot A_n \Rightarrow A_n = 0$$

$U_t(x,0) = 5\pi \cos 5\pi x$

$$U = \sum_{n=0}^{\infty} \sin \frac{\pi n x}{3} \left(A_n \left(-\frac{\pi n}{3}\right) \cdot \frac{\pi n}{3} t + B_n \frac{\pi n}{3} \cos \frac{\pi n t}{3}\right)$$

$$5\pi \cos 5\pi x = \sin \frac{\pi n x}{3} \cdot B_n \frac{\pi n}{3}$$

$$5\pi \int_0^3 \cos 5\pi x \cdot \sin \frac{\pi n x}{3} = \|x\|^2 \cdot B_n \cdot \frac{\pi n}{3}$$

$$B_n = \frac{5\pi \, 3}{\pi n \cdot \frac{3}{2}} \int_0^3 \cos 5\pi x \cdot \sin \frac{\pi n x}{3} dx = \frac{5}{2n} \int_0^3 \cos 5\pi x \cdot$$

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$$\sin\frac{5nx}{3}dx = \frac{5}{2n}\int_0^3 \frac{1}{2}\left(\sin\left(\frac{\pi n x}{3} - 5\pi x\right) + \sin\left(\frac{\pi n x}{3} + 5\pi x\right)\right) =$$

$$= \frac{5}{4n}\left[\frac{-\cos\left(\frac{\pi n}{3} - 5\pi\right)x}{\frac{\pi n}{3} - 5\pi} + \frac{\overset{=0}{\cos\left(\frac{\pi n}{3} + 5\pi\right)x}}{\frac{\pi n}{3} - 5\pi}\right]\Bigg|_0^3 =$$

$$= \frac{5}{4n}\left(\frac{-\cos\left(\frac{\pi n}{3} - 5\pi\right)x}{\frac{\pi n}{3} - 5\pi}\right)\Bigg|_0^3 = \cancel{\frac{5}{4n}\left(\frac{-\cos(\pi - 5\pi)x}{\pi - 5\pi} + \frac{\cos(-5\pi)x}{-5\pi}\right)}$$

$$= \cancel{\frac{5}{4n}\left(\frac{1}{-4n} + \frac{1}{5\pi}\right)} = \quad 5$$

$B_n = 0 \quad n \neq 5$

$B_n = 1 \quad n = 5$

$$U = \sin\frac{\pi n x}{3}\left(\sin\frac{\pi n t}{3}\right)$$

$$U = \sin\frac{\pi 5 x}{3}\left(\sin\frac{\pi 5 t}{3}\right)$$

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№3

в) $2U_{xx}+2U_{xy}+1U_{yy}+4U_x+4U_y=0$

$a_{11}=2$; $a_{12}=1$; $a_{22}=1$

$\Delta=1-1=0$ — параболический тип

канонический вид

$U_{\eta\eta}+\beta_1 U_\xi+\beta_2 U_\eta+\gamma U=g(\xi,\eta)$

$2\lambda^2-2\lambda+1=0$

$\lambda=\frac{dy}{dx}$

$\lambda_1=\frac{dy}{dx}=\frac{1\pm\sqrt{0}}{2}=\frac{1}{2}$

$\lambda_2=\frac{dx}{dy}=\frac{1\pm\sqrt{0}}{2}=\frac{1}{2}$

$dy=\frac{1}{2}dx$

$y=\frac{1}{2}x+C_1$

$$\begin{cases} \xi=-\frac{1}{2}x+y \\ \eta=y \\ y=\begin{vmatrix} -\frac{1}{2} & 1 \\ 0 & 1 \end{vmatrix}\neq 0 \end{cases}$$