Calibration of the Heston model

Under the risk-neutral measure $\mathbb{Q}$, the evolution of the underlying asset S_t is given by the following stochastic differential equation:

$$dS_t = rS_t dt + \sqrt{v_t} S_t dW_t^x$$

where

- r is the risk free rate
- v_t is the variance of the asset, and it is modeled by the so-called CIR process i.e.

$$dv_t = \kappa(\bar{v} - v_t)dt + \gamma\sqrt{v_t}dW_t^v$$

with dW_t^v is correlated with dW_t^x, i.e. $dW_t^x dW_t^v = \rho dt$.

Next, the first we will need is bring the the model in the affine diffusion (AD) class so we can drive the discounted characteristic function for the Heston model.

Put $X_t = \ln(S_t)$, and by Itô's formula we get

$$\begin{cases} dX_t = \left(r - \frac{1}{2}v_t\right)dt + \sqrt{v_t}dW_t^x \\ dv_t = \kappa(\bar{v} - v_t)dt + \gamma\sqrt{v_t}dW_t^v \end{cases}$$

Which belongs to the AD class. Hence, the characteristic function has the following form

$$\phi_X(u; X_0, t_0, T) = e^{iuX_0}.\exp\left[iur(T - t_0) + \frac{v_0\left(1 - e^{-D(T-t_0)}\right)}{\gamma^2\left(1 - ge^{-D(T-t_0)}\right)}(\kappa - i\rho\gamma u - D)\right]$$
$$\times \exp\left[\frac{\kappa\bar{v}}{\gamma^2}\left\{(T - t_0)(\kappa - i\rho\gamma u - D) - 2\ln\left(\frac{1 - ge^{-D(T-t_0)}}{1 - g}\right)\right\}\right]$$

with

$$D = \sqrt{(\kappa - i\rho\gamma u)^2 + (u^2 + iu)\gamma^2} \quad \text{and} \quad g = \frac{\kappa - i\rho\gamma u - D}{\kappa - i\rho\gamma u + D}.$$

Following the COS formula we get using $X_t = \frac{S_t}{K}$ with K being the strike price

$$V(t_0, X_0) \approx Ke^{-r(T-t_0)} Re\left\{\frac{F_0 U_0}{2} + \sum_{k=1}^{N-1} F_k U_k\right\}$$

where

- $x = X_0$
- $F_k(x) = \phi_X(\frac{k\pi}{b-a}; x, t_0, T)\exp\left(\frac{-ika\pi}{b-a}\right)$
- the coefficients U_k are as follows:

$$U_k = \begin{cases} \frac{2}{b-a}(\psi_k(0, b) - \varphi_k(0, b)) \text{ for call} \\ \frac{2}{b-a}(\varphi_k(a, 0) - \psi_k(a, 0)) \text{ for put} \end{cases}$$

where

$$\psi_k(c,d) = \frac{1}{1+\left(\frac{k\pi}{b-a}\right)^2}\left[cos\left(k\pi\frac{d-a}{b-a}\right)e^d - cos\left(k\pi\frac{c-a}{b-a}\right)e^c + \frac{k\pi}{b-a}sin\left(k\pi\frac{d-a}{b-a}\right)e^d - \frac{k\pi}{b-a}sin\left(k\pi\frac{c-a}{b-a}\right)e^c\right]$$

and

$$\varphi_k(c,d) = \frac{b-a}{k\pi}\left[sin\left(k\pi\frac{d-a}{b-a}\right) - sin\left(k\pi\frac{c-a}{b-a}\right)\right]1_{k\neq 0} + (d-c)1_{k=0}.$$

The error function.

One of the mots common choices for the error function to calibrate the model is:

$$\min_{\Omega} \sqrt{\sum_i \sum_j \omega_{ij}(V_c^{market}(t_0, S_0; r_i, K_j, \tau_i) - V_c(t_0, S_0; r_i, K_j, \tau_i))^2}.$$

The parameters to be optimized are :v_0, κ, $\bar{v}$, γ and ρ.

[1]- Oosterlee, Cornelis W., and Lech A. Grzelak. Mathematical modeling and computation in finance: with exercises and Python and MATLAB computer codes. World Scientific, 2019.

[2]- https://quantpy.com.au/stochastic-volatility-models/heston-model-calibration-to-option-prices/