On-Manifold Preintegration Cheat-Sheet (Equations & Jacobians)

This cheat-sheet summarizes the core mathematical equations and Jacobians required to implement the on-manifold IMU preintegration method from Forster et al. (2016).

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1. Notation

- Rotation: $R \in SO(3)$
- Exponential map: $\exp(\cdot) : \mathfrak{so}(3) \to SO(3)$
- Log map: $\log(\cdot) : SO(3) \to \mathfrak{so}(3)$
- Skew operator: $[v]_\times$
- IMU measurements:
- Gyro: $\tilde{\omega} = \omega + b^g + \eta^g$
- Accel: $\tilde{a} = R^T(a - g) + b^a + \eta^a$

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2. Preintegrated Measurements

Preintegrated terms express relative motion between keyframes i and j:

2.1 Preintegrated Rotation

Definition

$$\Delta\tilde{R}_{ij} = \prod_{k=i}^{j-1} \exp\left((\tilde{\omega}_k - b_i^g)\,\Delta t\right)$$

Noise Model

$$\delta\phi_{ij} \approx \sum F_R\,\eta^g$$

where $\delta\phi$ is the rotation perturbation in the tangent space.

2.2 Preintegrated Velocity

Definition

$$\Delta\tilde{v}_{ij} = \sum_{k=i}^{j-1} \Delta\tilde{R}_{ik}(\tilde{a}_k - b_i^a)\,\Delta t$$

Noise Propagation

$$\delta v = J_{v,\phi}\,\delta\phi + J_{v,ba}\,\delta b^a + J_{v,bg}\,\delta b^g + n_v$$

2.3 Preintegrated Position

Definition

$$\Delta\tilde{p}_{ij} = \sum_{k=i}^{j-1} \left[\Delta\tilde{v}_{ik}\,\Delta t + \frac{1}{2}\Delta\tilde{R}_{ik}(\tilde{a}_k - b_i^a)\,\Delta t^2\right]$$

Noise Propagation

$$\delta p = J_{p,\phi}\,\delta\phi + J_{p,ba}\,\delta b^a + J_{p,bg}\,\delta b^g + n_p$$

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3. Residuals (Factor Graph)

The IMU factor between states x_i and x_j has three residual blocks.

3.1 Rotation Residual

$$r_R = \log\left((\Delta\tilde{R}_{ij}\cdot\exp(J_{R,bg}\,\delta b^g))^T R_i^T R_j\right)$$

3.2 Velocity Residual

$$r_v = R_i^T(v_j - v_i - g\,\Delta t_{ij}) - (\Delta\tilde{v}_{ij} + J_{v,bg}\,\delta b^g + J_{v,ba}\,\delta b^a)$$

3.3 Position Residual

$$r_p = R_i^T(p_j - p_i - v_i\,\Delta t_{ij} - \tfrac{1}{2}g\,\Delta t_{ij}^2) - (\Delta\tilde{p}_{ij} + J_{p,bg}\,\delta b^g + J_{p,ba}\,\delta b^a)$$

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4. Jacobians

The IMU factor connects variables:

$$x = [R_i, p_i, v_i, b_i^g, b_i^a, R_j, p_j, v_j]^T$$

Below are the key Jacobians.

4.1 Rotation Residual Jacobians

wrt initial pose rotation R_i

$$\frac{\partial r_R}{\partial R_i} = -J_r \cdot \mathrm{Ad}_{R_j^T R_i}$$

where J_r is the right Jacobian of SO(3).

wrt final pose rotation R_j

$$\frac{\partial r_R}{\partial R_j} = J_r$$

wrt gyro bias

$$\frac{\partial r_R}{\partial b^g} = J_r \, J_{R,bg}$$

4.2 Velocity Residual Jacobians

wrt initial rotation R_i

$$\frac{\partial r_v}{\partial R_i} = -R_i^T [v_j - v_i - g\Delta t]_\times$$

wrt initial velocity v_i

$$\frac{\partial r_v}{\partial v_i} = -R_i^T$$

wrt final velocity v_j

$$\frac{\partial r_v}{\partial v_j} = R_i^T$$

wrt gyro & accel bias

$$\frac{\partial r_v}{\partial b^g} = -J_{v,bg}, \qquad \frac{\partial r_v}{\partial b^a} = -J_{v,ba}$$

4.3 Position Residual Jacobians

wrt initial rotation

$$\frac{\partial r_p}{\partial R_i} = -R_i^T [p_j - p_i - v_i \Delta t - \tfrac{1}{2} g \Delta t^2]_\times$$

wrt initial position

$$\frac{\partial r_p}{\partial p_i} = -R_i^T$$

wrt final position

$$\frac{\partial r_p}{\partial p_j} = R_i^T$$

wrt initial velocity

$$\frac{\partial r_p}{\partial v_i} = -R_i^T \Delta t$$

wrt biases

$$\frac{\partial r_p}{\partial b^g} = -J_{p,bg}, \qquad \frac{\partial r_p}{\partial b^a} = -J_{p,ba}$$

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5. Bias Correction Terms

Let biases change during optimization by: $\delta b^g, \delta b^a$

Rotation

$$\Delta R_{ij}(b) \approx \Delta R_{ij}(b_0) \cdot \exp(J_{R,bg}\, \delta b^g)$$

Velocity

$$\Delta v_{ij}(b) \approx \Delta v_{ij}(b_0) + J_{v,bg}\, \delta b^g + J_{v,ba}\, \delta b^a$$

Position

$$\Delta p_{ij}(b) \approx \Delta p_{ij}(b_0) + J_{p,bg}\, \delta b^g + J_{p,ba}\, \delta b^a$$

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6. Covariance Propagation

A continuous-time noise covariance is integrated via linear approximation:

$$\Sigma_{k+1} = F_k\, \Sigma_k\, F_k^T + G_k\, Q\, G_k^T$$

Where: - F_k is the state-transition Jacobian - G_k embeds IMU noise into the state - Q is IMU noise covariance

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7. Summary of Required Components

To fully implement the on-manifold preintegration factor, you need: - SO(3) exponential/log maps - Preintegrated increments: $\Delta R, \Delta v, \Delta p$ - Jacobians wrt bias: $J_{R,bg}, J_{v,bg}, J_{v,ba}, J_{p,bg}, J_{p,ba}$ - Covariance propagation across IMU samples - Factor residuals (r_R, r_v, r_p) - Complete Jacobians wrt all state variables

This is everything required to build the GTSAM-compatible IMU factor from scratch.