Flow Through Porous Media

Introduction to Porous Media. Many materials (ex. soil, sand, packed catalyst beds) consist of a large number of particles or fibers packed closely together. In between the solid particles or fibers there is open space, giving rise to pores through which fluids can flow. An object does not have to consist of many particles to be porous; for instance, it could simply be composed of a single continuos solid body that has many pores (or holes) in it. Such is the case with certain rocks and filters. Regardless of how the porous medium is constructed, because of the irregular, tortuous nature of the pores it is exceedingly difficult to model fluid flow through such materials exactly. In this handout some simple models will be introduced that, when combined with experimental data, can be used to calculate quantities of interest such as frictional dissipation (head loss) or pressure drop across porous media during fluid flow. This information is useful in modeling several chemical engineering operations, including filtration units, packed beds, and certain types of chemical reactors. An example of a porous medium is illustrated in Fig. 1, which depicts a cylindrical packed bed filled with particles of diverse shape and size.

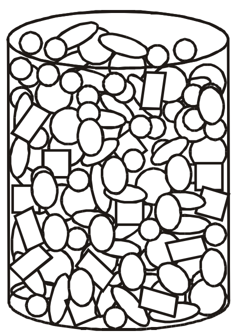

Figure 1. A "packed bed" made of irregular particles.

Definition of the Friction Factor for Porous Media. We first recall the definition of the friction factor for flow through pipes and channels (see handout #13):

$$f_{\text{pipe}} = \frac{W_f}{\frac{1}{2}V_o^2}\left(\frac{D'}{L}\right) \tag{1}$$

We will use this same definition of f for porous media after modifying it as discussed below. In equation (1), W_{f} is the mechanical energy dissipated to internal energy per unit mass of fluid flowing through a pipe or channel. W_{f} is an average over all streamlines. In other words, if the total flowrate of mass through the pipe is M units of mass per second, then the amount of mechanical energy dissipated per second would be $M\,W_{\text{f}}$. L is the length of the pipe or channel. V_{o} is the *average* fluid velocity. D' is called the "**hydraulic radius.**" The hydraulic radius D' is defined as (see handout #13)

$$D' = 4\,A/P_{\text{w}} \tag{2}$$

where A is the cross-sectional area available for flow and P_w is called the "wetted perimeter". The wetted perimeter is the perimeter of the area in which the fluid is in direct contact with the confining solid walls. If fluid fills the entire cross-sectional area A, then P_w is the perimeter of A. For instance, for a circular pipe that is fully filled with a flowing fluid, $A = \pi R^2$ and $P_w = 2\pi R$, where R is the pipe radius. For this case

$$D'_{\text{pipe}} = 4\ \pi R^2/2\pi R = 2R \tag{3}$$

Note the somewhat confusing convention of referring to D' as the hydraulic *radius* despite the fact that it equals the *diameter* (for a circular pipe).

How do we define the hydraulic radius for a porous medium? First, two parameters describing the characteristics of the porous medium need to be introduced:

a_W = wetted area per unit volume of the porous medium

ε = "void fraction" = volume of empty space (occupied by the pores) per unit volume of the porous medium

a_W and ε can both be measured. To define D', according to equation (2) we need A and P_W. The cross-sectional area A available for flow of fluid through a porous body can be obtained from the total cross-sectional area A_T of the porous medium (Fig. 2) by multiplying A_T by the fraction of space ε occupied by the pores,

$$A = A_T\ \varepsilon \tag{4}$$

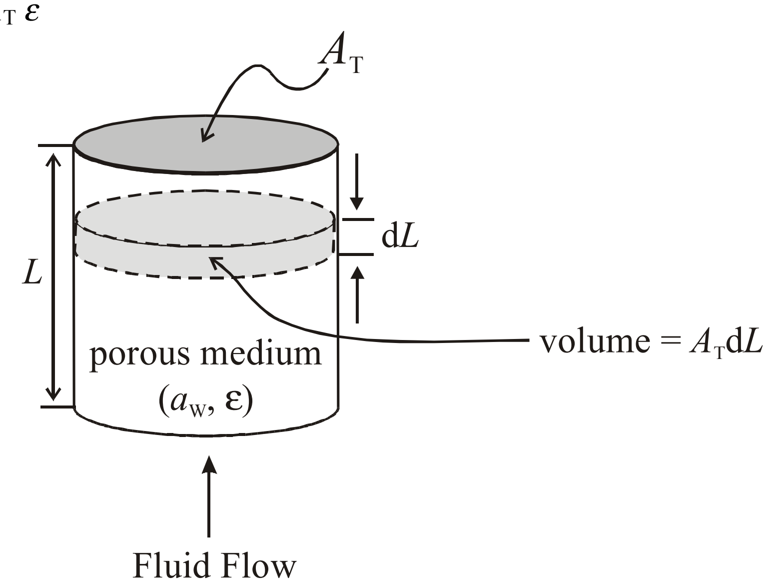

Figure 2.

Furthermore, the total wetted area inside a differential volume A_T dL (Fig. 2) is equal to $a_w A_T$ dL. Therefore, the wetted perimeter P_w can be determined as

$$\textit{Total Wetted Area Inside Differential Volume } A_T\, \mathrm{d}L = a_w A_T\, \mathrm{d}L = P_w\, \mathrm{d}L \tag{5}$$

Thus,

$$P_w = a_w A_T \tag{6}$$

Inserting equations (4) and (6) into the definition of hydraulic radius in equation (2) yields the desired expression for D' of a porous medium:

$$D' = 4 A_T \varepsilon / (a_w A_T) = 4 \varepsilon / a_w \tag{7}$$

To define the friction factor for a porous medium we also need V_o, the average velocity of fluid inside the pores of the medium. V_o would be difficult to measure directly, but is straightforwardly related to the volumetric flowrate Q of fluid through the medium as follows:

$$Q = V_o A = V_o \varepsilon A_T = V_S A_T \tag{8}$$

where

$$V_S = V_o \varepsilon \qquad \text{so that} \qquad V_o = V_S / \varepsilon \tag{9}$$

V_S is called the "**superficial velocity**," and is easily calculated from the volumetric flowrate Q by dividing it by the total cross-sectional area A_T:

$$V_S = Q/A_T \tag{10}$$

With the above expressions, we now return to considering the friction factor f. In defining the friction factor for a porous medium, it is customary to substitute equation (9) for V_o. Inserting equations (9) and (7) into the friction factor definition (1),

$$f = \frac{8 W_f \varepsilon^3}{V_S^2 a_W L} \tag{11}$$

In equation (11), W_f is the energy dissipated per unit mass of fluid flowing through length L of the porous medium. For flow of fluids through porous media consisting of small particles packed together (such as packed beds of sand or catalyst pellets) a_W, the wetted area per volume of porous medium, can be related to the characteristics of the particles from which the porous medium is constructed. First, we assume that all of the particles have identical shape and size, and define the **surface area per particle volume** a_V:

$$a_V = (\text{particle surface area}) / (\text{volume occupied by particle}) \tag{12}$$

For example, for a spherical particle

$$a_V = \frac{4\pi\left(\frac{D_p}{2}\right)^2}{\frac{4}{3}\pi\left(\frac{D_p}{2}\right)^3} = 6 / D_P \tag{13}$$

where D_P is the particle diameter. For non-spherical particles, an "**equivalent particle diameter**" D_{PE} is *defined* by rearranging equation (13):

$$D_{PE} = 6 / a_V \tag{14}$$

where a_V is calculated from equation (12). For example, D_{PE} for cubic particles with sides of length B is:

$$D_{PE} = 6 / (6B^2 / B^3) = B$$

Returning to a_W, we see that it can be rewritten in terms of a_V and ε:

$$a_W = \frac{wetted\ surface}{volume\ of\ porous\ media} = \overbrace{\frac{wetted\ surface}{volume\ occupied\ by\ particles}}^{=a_V} \times \overbrace{\frac{volume\ occupied\ by\ particles}{volume\ of\ porous\ media}}^{=(1-\varepsilon)}$$

Therefore,

$$a_W = a_V\,(1 - \varepsilon) \tag{15}$$

Using the definition of the equivalent particle diameter D_{PE} (equation (14)) results in

$$a_W = 6\,(1 - \varepsilon) / D_{PE} \tag{16}$$

Inserting expression (16) into expression (11) for the friction factor produces

$$f = \frac{4 W_f\, D_{PE}\, \varepsilon^3}{3 V_S^2\, (1-\varepsilon) L}$$

The 4/3 prefactor is usually excluded, so that the friction factor is simply written

$$f = \frac{W_f\, D_{PE}\, \varepsilon^3}{V_S^2\, (1-\varepsilon) L} \tag{17}$$

Equation (17) is the final form of the friction factor that we are after. Later, we will see how it can be used to calculate pressure drop across a porous medium. Recall that in deriving equation (17) it was assumed that the particles possess a uniform shape and size, what allowed a_W in equation (11) to be replaced by equation (16).

Definition of the Reynolds Number for Porous Media. The Reynolds number *Re* was previously defined as (see handout #12)

$$Re = \rho V_o D'/\mu \tag{18}$$

Here ρ is the fluid density, μ is the fluid viscosity, and V_o and D' are a characteristic velocity and a characteristic dimension that depend on the flow geometry being considered. For porous media, V_o and D' are defined by analogy to flow in pipes and channels; that is, V_o is the average fluid velocity in the porous medium (given by equation (9), $V_o = V_S / \varepsilon$) and D' is the hydraulic radius (given by equation (7), $D' = 4\,\varepsilon / a_W$). Using equation (16) for a_W, the expression for the hydraulic radius can be rearranged into

$$D' = 4\,\varepsilon / a_W = \frac{2\varepsilon D_{PE}}{3(1-\varepsilon)} \tag{19}$$

Equation (19) is for a porous medium consisting of particles of uniform shape and size, and characterized by an equivalent particle diameter D_{PE}. Inserting equation (19) for D' and equation (9) for V_o into the definition for Re

$$Re = \frac{2\rho V_S\, D_{PE}}{3\mu(1-\varepsilon)}$$

The factor of 2/3 is generally omitted, so that

$$Re = \frac{\rho V_S\, D_{PE}}{\mu(1-\varepsilon)} \tag{20}$$

Equation (20) is the most common definition of the Reynolds number for porous media. It applies when the medium is constructed from identical particles with an equivalent particle diameter D_{PE}.

Calculation of the Friction Factor for Porous Media. As for the case of pipe flow, when estimating the friction factor for porous media we will distinguish between laminar and turbulent flows. Furthermore, we specialize to the case when the porous medium is constructed from uniformly-sized, spherical particles with diameter $D_{PE} = D_P$.

First, we discuss the case of laminar flow. As a simplest estimate, the porous medium might be modeled as a complex assembly of twisted and tortuous pipes that represent the pore spaces through which the fluid flows. Admittedly, such a model embodies rather strong assumptions, but it does provide a starting point for considering the very complex nature of porous media. For laminar pipe flow, it was shown in handout #13 that

$$f_{pipe} = 64 / Re \tag{21}$$

Since we are conceptualizing the porous medium as an assembly of tortuous pipes, we might guess that the friction factor for porous media might obey a similar relationship. Inserting equation (17) for f and equation (20) for Re into equation (21) then leads to the following prediction

$$f = \frac{W_f\ D_P \varepsilon^3}{V_S^2\ (1-\varepsilon) L} = 64\ \frac{\mu(1-\varepsilon)}{\rho V_S D_P} \tag{22}$$

The remarkable outcome is that, aside from the numerical prefactor, equation (22) turns out to agree very well with experimental observations. In other words, the dependence of W_f on ε, μ, D_P and the other parameters is captured by the functional form of equation (22). The only adjustment needed to optimize agreement with experimental measurements is to replace the factor 64 by 150,

$$f = \frac{W_f\ D_P \varepsilon^3}{V_S^2\ (1-\varepsilon) L} = 150\ \frac{\mu(1-\varepsilon)}{\rho V_S D_P} = 150 / Re \qquad \text{(Blake-Kozeny equation, } Re < 10) \tag{23}$$

Equation (23) is known as the "Blake-Kozeny" equation, and reproduces experimental data for porous media of uniform spheres rather well up to Reynolds numbers of about 10 (see attached figure). Sometimes, an empirical coefficient of 180 instead of 150 is recommended, and equation (23) is then usually referred to as the Carman-Kozeny equation.

Above Re of about 10 the flow begins to depart from laminar. For highly turbulent flows, characterized by $Re > 1000$, experiment shows that the friction factor becomes nearly independent of Re (see attached figure). In this regime of high Reynolds number and turbulent flow the friction factor is well represented by an expression known as the "Burke-Plummer" equation

$$f = \frac{W_f\ D_P \varepsilon^3}{V_S^2\ (1-\varepsilon) L} = \text{constant} = 1.75 \qquad \text{(Burke-Plummer equation, } Re > 1000) \tag{24}$$

To derive an expression for f that applies over intermediate Reynolds numbers, $10 < Re < 1000$, the common (simplest) approach is to add the laminar (equation (23)) and turbulent (equation (24)) contributions to f. This is not rigorous, but turns out to work reasonably well:

$$f = \frac{W_f\ D_P \varepsilon^3}{V_S^2\ (1-\varepsilon) L} = 150\ \frac{\mu(1-\varepsilon)}{\rho V_S D_P} + 1.75 = 150 / Re + 1.75 \tag{25}$$

(Ergun equation, $10 < Re < 1000$)

Equation (25) is commonly referred to as the "Ergun" equation. All three expressions for f, equations 23 to 25, are compared with experimental data in the attached figure. The agreement with experiment is quite good. The caution must be raised, however, that these expressions and data are strictly applicable only to flow through porous media consisting of identical, spherical particles packed together. Nevertheless, the data and equations are also often applied to packed particle beds constructed from nonspherical particles. For the situation of nonspherical but otherwise identical particles, the sphere diameter D_P is replaced by the equivalent particle diameter D_{PE}. D_{PE} is calculated from equation (14). With this substitution, equations 23 to 25 and the attached figure of f vs Re can be employed to provide a crude estimate of W_f for packed beds of nonspherical particles; however, this approach is best supplemented by direct experimental measurements for the particle geometry of interest.

Calculating the Pressure Drop Across A Porous Medium. The previous section introduced the approach for calculating the rate of frictional dissipation for fluid flows through porous media. Often, we may be more interested in other quantities, such as the pressure drop across a packed bed. To this end, it is useful to recall the Extended Bernoulli Equation averaged over all streamlines passing through a macroscopic control volume (equation (15), handout # 9):

$$m^* \left(\frac{v_2^{\,2}}{2} + g z_2 + \frac{p_2}{\rho} - \frac{v_1^{\,2}}{2} - g z_1 - \frac{p_1}{\rho} \right) = -\frac{\mathrm{d}W_\mathrm{f}}{\mathrm{d}t} - \frac{\mathrm{d}W\mathrm{s}}{\mathrm{d}t} \tag{26}$$

The control volume of interest could a packed bed, for example (Fig. 3). Dividing all terms in equation (26) by the mass flowrate m^*, and assuming no shaft work is occurring,

$$\frac{v_2^{\,2} - v_1^{\,2}}{2} + g(z_2 - z_1) + \frac{p_2 - p_1}{\rho} = -W_\mathrm{f} \tag{27}$$

Note that division of $\mathrm{d}W_\mathrm{f}/\mathrm{d}t$ (frictional dissipation per time) by m^* (fluid mass transport per time) yields W_f (frictional dissipation per mass of fluid transported). In equation (27), the subscript 2 could refer to the point at which fluid exits the porous medium while the subscript 1 could refer to the entry port.

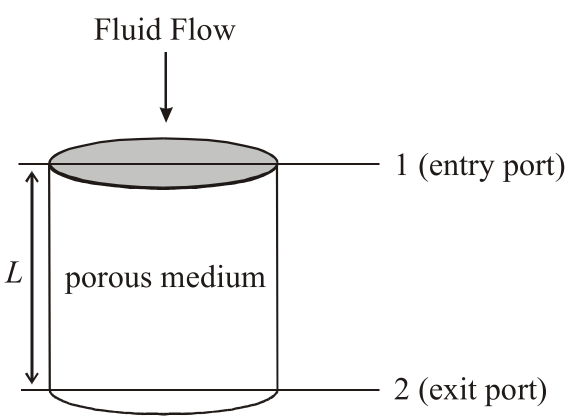

Figure 3.

The pressure drop $p_2 - p_1$ across a porous medium can be calculated from equation (27) if W_f is calculated from friction factor data and if the other quantities in equation (27) are known. In particular, if the change $(v_2^{\,2} - v_1^{\,2})/2$ in fluid kinetic energy and the change $g(z_2 - z_1)$ in fluid gravitational potential energy are small relative to W_f, then the approximation

$$W_\mathrm{f} = (p_1 - p_2)/\rho = \Delta p / \rho \tag{28}$$

can be made. Equation (28) is generally an excellent approximation for porous media because the enormous wetted area of contact between the solid medium and the flowing fluid leads to extensive frictional dissipation. This means that the flow work performed on the fluid (the flow work is represented by the $\Delta p / \rho$ term) can be assumed to be essentially fully dissipated in overcoming frictional resistances, with changes in the fluid kinetic or gravitational potential energy being negligible in comparison. If we substitute equation (28) into the definition of the friction factor for porous media (equation (17)) we get

$$f = \frac{\Delta p \, D_{PE} \varepsilon^3}{\rho V_S^2 \, (1-\varepsilon) L} \tag{29}$$

Substitution of equation (29) into the Blake-Kozeny, Burke-Plummer, or Ergun equations (equations 23 to 25) allows for direct estimation of the pressure drop Δp.

A final topic that we will mention concerns an empirical observation known as **Darcy's Law**. Darcy's Law is commonly employed to model *laminar* flow through porous media of all types,

$$V_S = K_D \, \Delta p / \mu L \qquad \text{(Darcy's Law, laminar flow)} \tag{30}$$

In equation (30) V_S is the superficial velocity as defined by equation (10), Δp is the pressure drop across the porous medium, μ is the viscosity of the fluid, and L is the length of the porous medium in the direction of fluid flow (Fig. 3). $\Delta p/L$ is the pressure gradient across the porous medium. K_D is termed the "**permeability**" of the porous medium, and is a function of the medium's geometry such as the connectivity and dimensions of the pores. In general, K_D needs to be determined experimentally by measuring flowrate Q (from which V_S can be calculated) as a function of Δp across the porous medium of interest. Note that Darcy's Law is consistent with the Blake-Kozeny equation (equation (23)), which applies for the specialized case of laminar flow through a packed bed of uniformly-sized, spherical particles. In particular, inspection of equations (30), (28) and (23) shows that $K_D = D_P^2 \varepsilon^3 / [150 \, (1 - \varepsilon)^2]$.

Darcy's Law can be expressed in differential form by writing the pressure gradient $\Delta p/L = (p_1 - p_2)/L$ as $-\nabla p$,

$$\mathbf{V}_S = -\frac{K_D}{\mu} \nabla p \tag{31}$$

An empirical value of K_D can be used in conjunction with equation (31) and the differential equation of mass conservation (i.e. differential equation of continuity) to model flow through the porous medium. For instance, if the density of fluid does not change appreciably within the porous medium (i.e. the flow can be modeled as incompressible) and assuming K_D and μ do not depend on position, then the differential continuity equation can be written

$$\nabla \bullet \mathbf{V}_S = -\frac{K_D}{\mu} \nabla^2 p = 0 \qquad \text{so that} \qquad \nabla^2 p = 0 \tag{32}$$

Equation (32), together with appropriate boundary conditions, could be solved to obtain the pressure distribution inside the porous medium.