Greeks Derivations

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Shared Computations

$$
\begin{aligned}
N'(d_1) = \frac{\partial N(d_1)}{\partial d_1} =& \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \\
N'(d_2) = \frac{\partial N(d_2)}{\partial d_2} =& \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}} \\
=& \frac{1}{\sqrt{2\pi}} e^{-\frac{(d_1-\sigma\sqrt{T})^2}{2}} \\
=& \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \cdot e^{d_1\sigma\sqrt{T}-\frac{\sigma^2 T}{2}} \\
=& N'(d_1) \cdot e^{\ln\left(\frac{S_0}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T-\frac{\sigma^2 T}{2}} \\
=& N'(d_1) \cdot \frac{S_0}{K} e^{rT} \\
\therefore S_0 N'(d_1) =& K e^{-rT} N'(d_2)
\end{aligned}
$$

1 Delta

$$
\begin{aligned}
\frac{\partial d_1}{\partial S_0} =& \frac{\partial}{\partial S_0}\left(\frac{\ln\left(\frac{S_0}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) \\
=& \frac{1}{\frac{S_0}{K}\sigma\sqrt{T}}\cdot\frac{1}{K} \\
=& \frac{1}{S_0\sigma\sqrt{T}} \\
\frac{\partial d_2}{\partial S_0} =& \frac{\partial}{\partial S_0}\left(d_1-\sigma\sqrt{T}\right) \\
=& \frac{\partial d_1}{\partial S_0} \\
=& \frac{1}{S_0\sigma\sqrt{T}} \\
& \text{They cancel out below, so technically this wasn't necessary.} \\
\Delta_C =& \frac{\partial C}{\partial S_0} \\
=& \frac{\partial}{\partial S_0}\left(S_0 N\left(d_1\right)-Ke^{-rT}N\left(d_2\right)\right) \\
=& N\left(d_1\right)+S_0\frac{\partial N\left(d_1\right)}{\partial S_0}-Ke^{-rT}\frac{\partial N\left(d_2\right)}{\partial S_0} \\
=& N\left(d_1\right)+S_0\cdot N'\left(d_1\right)\cdot\frac{\partial d_1}{\partial S_0}-Ke^{-rT}N'\left(d_2\right)\cdot\frac{\partial d_2}{\partial S_0} \\
=& N\left(d_1\right)+S_0\cdot N'\left(d_1\right)\cdot\frac{1}{S_0\sigma\sqrt{T}}-S_0\cdot N'\left(d_1\right)\cdot\frac{1}{S_0\sigma\sqrt{T}} \\
=& \boxed{N\left(d_1\right)} \\
C+Ke^{-rT} =& P+S_0 \\
\therefore \frac{\partial}{\partial S_0}\left(C+Ke^{-rT}\right) =& \frac{\partial}{\partial S_0}\left(P+S_0\right) \\
\therefore \frac{\partial C}{\partial S_0} =& \frac{\partial P}{\partial S_0}+1 \\
\therefore \Delta_P =& \frac{\partial P}{\partial S_0} \\
=& \frac{\partial C}{\partial S_0}-1 \\
=& \boxed{N\left(d_1\right)-1}
\end{aligned}
$$

So,

$$\Delta_C = \frac{\partial C}{\partial S_0} = N\left(d_1\right)$$

$$\Delta_P = \frac{\partial P}{\partial S_0} = N\left(d_1\right) - 1$$

2 Gamma

$$
\begin{aligned}
\Gamma =& \frac{\partial^2 C}{\partial S_0^2} \\
=& \frac{\partial N(d_1)}{\partial S_0} \\
=& \frac{\partial N(d_1)}{\partial d_1} \cdot \frac{\partial d_1}{\partial S_0} \\
=& \boxed{N'(d_1) \cdot \frac{1}{S_0 \sigma \sqrt{T}}} \\
\Gamma =& \frac{\partial^2 P}{\partial S_0^2} \\
=& \frac{\partial}{\partial S_0} \left(N(d_1) - 1 \right) \\
=& \boxed{N'(d_1) \cdot \frac{1}{S_0 \sigma \sqrt{T}}}
\end{aligned}
$$

So,

$$
\Gamma = \frac{\partial^2 C}{\partial S_0^2} = \frac{\partial^2 P}{\partial S_0^2} = N'(d_1) \cdot \frac{1}{S_0 \sigma \sqrt{T}}
$$

3 Vega

$$
\begin{aligned}
x =& \ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T \\
\frac{\partial d_1}{\partial \sigma} =& \frac{\partial}{\partial \sigma}\left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) \\
=& \frac{\sigma T \cdot \sigma\sqrt{T} - x\sqrt{T}}{\sigma^2 T} \\
=& \sqrt{T} - \frac{1}{\sigma} \cdot \frac{x}{\sigma\sqrt{T}} \\
=& \sqrt{T} - \frac{d_1}{\sigma} \\
\frac{\partial d_2}{\partial \sigma} =& \frac{\partial}{\partial \sigma}\left(d_1 - \sigma\sqrt{T}\right) \\
=& \sqrt{T} - \frac{d_1}{\sigma} - \sqrt{T} \\
=& -\frac{d_1}{\sigma} \\
\frac{\partial C}{\partial \sigma} =& \frac{\partial}{\partial \sigma}\left(S_0 N\left(d_1\right) - K e^{-rT} N\left(d_2\right)\right) \\
=& S_0 \cdot \frac{\partial N\left(d_1\right)}{\partial d_1} \cdot \frac{\partial d_1}{\partial \sigma} - K e^{-rT} \frac{\partial N\left(d_2\right)}{\partial d_2} \cdot \frac{\partial d_2}{\partial \sigma} \\
=& S_0 \cdot N'\left(d_1\right) \cdot \left(-\frac{d_1}{\sigma} + \sqrt{T}\right) - K e^{-rT} N'\left(d_2\right) \cdot \left(-\frac{d_1}{\sigma}\right) \\
=& S_0 \cdot N'\left(d_1\right) \cdot \left(-\frac{d_1}{\sigma} + \sqrt{T}\right) - S_0 \cdot N'\left(d_1\right) \cdot \left(-\frac{d_1}{\sigma}\right) \\
=& \boxed{S_0 \cdot N'(d_1) \cdot \sqrt{T}} \\
C + K e^{-rT} =& P + S_0 \\
\therefore \frac{\partial}{\partial \sigma}\left(C + K e^{-rT}\right) =& \frac{\partial}{\partial \sigma}\left(P + S_0\right) \\
\therefore \frac{\partial C}{\partial \sigma} =& \frac{\partial P}{\partial \sigma} = \boxed{S_0 \cdot N'(d_1) \cdot \sqrt{T}}
\end{aligned}
$$

So,

$$
\nu = 1\% \cdot \frac{\partial C}{\partial \sigma} = 1\% \cdot \frac{\partial P}{\partial \sigma} = 1\% \cdot S_0 \cdot N'(d_1) \cdot \sqrt{T}
$$

4 Theta

$$
\begin{aligned}
x =& \ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) T \\
\frac{\partial d_1}{\partial T} =& \frac{\partial}{\partial T}\left(\frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}\right) \\
=& \frac{\left(r + \frac{\sigma^2}{2}\right) \cdot \sigma\sqrt{T} - \frac{\sigma}{2\sqrt{T}} x}{\sigma^2 T} \\
=& \frac{r + \frac{\sigma^2}{2}}{\sigma\sqrt{T}} - \frac{d_1}{2T} \\
\frac{\partial d_2}{\partial T} =& \frac{\partial}{\partial T}\left(d_1 - \sigma\sqrt{T}\right) \\
=& \frac{r + \frac{\sigma^2}{2}}{\sigma\sqrt{T}} - \frac{d_1}{2T} - \frac{\sigma}{2\sqrt{T}} \\
=& \frac{r}{\sigma\sqrt{T}} - \frac{d_1}{2T} \\
\frac{\partial C}{\partial T} =& \frac{\partial}{\partial T}\left(S_0 N\left(d_1\right) - K e^{-rT} N\left(d_2\right)\right) \\
=& S_0 \cdot \frac{dN\left(d_1\right)}{dd_1} \cdot \frac{dd_1}{dT} - K e^{-rT} \cdot \frac{dN\left(d_1\right)}{dd_2} \cdot \frac{dd_2}{dT} + rKe^{-rT} N\left(d_2\right) \\
=& S_0 \cdot N'\left(d_1\right) \cdot \left(\frac{r + \frac{\sigma^2}{2}}{\sigma\sqrt{T}} - \frac{d_1}{2T}\right) - K e^{-rT} \cdot N'\left(d_2\right) \cdot \left(\frac{r}{\sigma\sqrt{T}} - \frac{d_1}{2T}\right) + rKe^{-rT} N\left(d_2\right) \\
=& S_0 \cdot N'\left(d_1\right) \cdot \left(\frac{r + \frac{\sigma^2}{2}}{\sigma\sqrt{T}} - \frac{d_1}{2T}\right) - S_0 \cdot N'\left(d_1\right) \cdot \left(\frac{r}{\sigma\sqrt{T}} - \frac{d_1}{2T}\right) + rKe^{-rT} N\left(d_2\right) \\
=& \boxed{S_0 \cdot N'\left(d_1\right) \cdot \frac{\sigma}{2\sqrt{T}} + rKe^{-rT} N\left(d_2\right)} \\
C + Ke^{-rT} =& P + S_0 \\
\therefore \frac{\partial}{\partial \theta}\left(C + Ke^{-rT}\right) =& \frac{\partial}{\partial \theta}\left(P + S_0\right) \\
\therefore \frac{\partial C}{\partial \theta} + Ke^{-rT} \cdot (-r) =& \frac{\partial P}{\partial \theta} \\
\therefore \frac{\partial P}{\partial \theta} =& S_0 \cdot N'\left(d_1\right) \cdot \frac{\sigma}{2\sqrt{T}} - rKe^{-rT}\left(1 - N\left(d_2\right)\right) \\
=& \boxed{S_0 \cdot N'\left(d_1\right) \cdot \frac{\sigma}{2\sqrt{T}} - rKe^{-rT} \cdot N\left(-d_2\right)}
\end{aligned}
$$

So,

$$\theta_C = -\frac{1}{360} \cdot \frac{\partial C}{\partial T} = \frac{1}{360} \cdot \left(-S_0 \cdot N'\left(d_1\right) \cdot \frac{\sigma}{2\sqrt{T}} - rKe^{-rT} N\left(d_2\right) \right)$$

$$\theta_P = -\frac{1}{360} \cdot \frac{\partial P}{\partial T} = \frac{1}{360} \cdot \left(-S_0 \cdot N'\left(d_1\right) \cdot \frac{\sigma}{2\sqrt{T}} + rKe^{-rT} \cdot N\left(-d_2\right) \right)$$

Negative sign by convention, since YTE decreases as time passes.

5 Rho

$$\begin{aligned}
\frac{\partial d_2}{\partial r} =& \frac{\partial}{\partial r}\left(d_1 - \sigma\sqrt{T}\right) \\
=& \frac{\partial d_1}{\partial r} \\
\frac{\partial C}{\partial r} =& \frac{\partial}{\partial r}\left(S_0 N\left(d_1\right) - Ke^{-rT} N\left(d_2\right)\right) \\
=& S_0 \cdot \frac{\partial N\left(d_1\right)}{\partial d_1} \cdot \frac{\partial d_1}{\partial r} + TKe^{-rT} \cdot N\left(d_2\right) - Ke^{rT} \cdot \frac{\partial N\left(d_2\right)}{\partial d_2} \cdot \frac{\partial d_2}{\partial r} \\
=& S_0 \cdot N'\left(d_1\right) \cdot \frac{\partial d_1}{\partial r} + TKe^{-rT} \cdot N\left(d_2\right) - Ke^{rT} \cdot N'\left(d_2\right) \cdot \frac{\partial d_2}{\partial r} \\
=& S_0 \cdot N'\left(d_1\right) \cdot \frac{\partial d_1}{\partial r} + TKe^{-rT} \cdot N\left(d_2\right) - S_0 \cdot N'\left(d_1\right) \cdot \frac{\partial d_1}{\partial r} \\
=& \boxed{TKe^{-rT} \cdot N\left(d_2\right)} \\
C + Ke^{-rT} =& P + S_0 \\
\therefore \frac{\partial}{\partial r}\left(C + Ke^{-rT}\right) =& \frac{\partial}{\partial r}\left(P + S_0\right) \\
\therefore \frac{\partial C}{\partial r} + Ke^{-rT} \cdot (-T) =& \frac{\partial P}{\partial r} \\
\therefore \frac{\partial P}{\partial r} =& TKe^{-rT}\left(N\left(d_2\right) - 1\right) \\
=& \boxed{-TKe^{-rT} \cdot N\left(-d_2\right)}
\end{aligned}$$

So,

$$\rho_C = 1\% \cdot \frac{\partial C}{\partial r} = 1\% \cdot TKe^{-rT} \cdot N\left(d_2\right)$$

$$\rho_P = 1\% \cdot \frac{\partial P}{\partial r} = 1\% \cdot \left(-TKe^{-rT} \cdot N\left(-d_2\right)\right)$$