Theory of characteristics for first order partial differential equations

Anders Kock
University of Aarhus

Dabei sehen wir von unendlich kleinen Grössen höhere Ordnung ab.
Lie [5] p. 523

Introduction

The present note makes no claim of originality; it is a "conspectus" of some of the classical theory of characteristics for 1st order PDEs, as expounded geometrically by Lie and elaborated by Klein. These authors use extensively a synthetic geometric language, but ultimately describe notions rigourously only by presenting them in analytic terms. Our approach describe the notions (like "united position" ("vereignigte Lage") and "characteristic") rigourously in pure synthetic coordinate free terms, and introduce coordinates only at a later point, when it comes to proving some of the relations between the notions introduced.

So we are not claiming that describing the notions synthetically is an effective tool for *proving*; usually, coordinates are better suited for this. The virtue of the synthetic descriptions are, as also appears from the work of Monge, Lie, Klein, ..., that it gives a geometric language to speak about geometric entities, and in particular, make them coordinate free from the outset.

The particular version of synthetic language that we use is that of Synthetic Differential Geometry, as in [2], say, and notably as in [4], where the main synthetic relation is the first and second order neighbour relation, as first considered in French algebraic geometry in the 1950s. We denote these relations by the symbol $\sim_1$ (or just $\sim$) and $\sim_2$, respectively. They are reflexive symmetric relations on the set of points of a manifold. The set of kth order neighbours of a point x in a manifold M is denoted $\mathfrak{M}_k(x)$, i.e. $\mathfrak{M}_k(x) = \{y \in M \mid y \sim_k x\}$. One has that $x \sim_1 y \sim_1 z$ implies $x \sim_2 z$. The axiomatics used for these neighbourhoods is essentially the "Kock-Lawvere" (KL) axiom scheme, which we shall quote when needed. The basic manifold is the number line R; here $x \sim_k y$ iff $(y-x)^{k+1} = 0$. In R^n, the set $\mathfrak{M}_1(0)$ is also denoted $D(n)$, and $\mathfrak{M}_k(0)$ is denoted $D_k(n)$.

1 Surface elements and calottes

Let M be a 3-dimensional manifold.

A *surface element* at x is a set $P \subset M$ of the form $\mathfrak{M}_1(x) \cap F$, where $F \subset M$ is a surface (2-dimensional submanifold) containing x. Similarly a *calotte* at x is a set $K \subset M$ of the form $\mathfrak{M}_2(x) \cap F$ where $F \subset M$ is a surface containing x. (The notion of calotte is from [1] p. 281.) If K is a calotte at x, it is clear that $\mathfrak{M}_1(x) \cap K$ is a surface element P at x, called the *restriction* of K, and similarly, K is an *extension* calotte of P, or a calotte *through* P.

It follows from Proposition 7 in the Appendix that the base point x of a surface element P can be reconstructed[1] from P (viewed as a subset), and similarly, the restriction of a calotte K (and hence also the base point of the calotte) can be reconstructed from K (viewed as a subset).

One could also use the terms "1-jet (resp. 2-jet) of a surface" for surface elements, respectively calottes, in M. Therefore, and for uniformity, we denote the manifold of surface elements, respectively the manifold of calottes, by the symbols $S_1(M)$, respectively $S_2(M)$. We have surjective submersions

$$S_2(M) \to S_1(M) \to M.$$

The dimensions of these manifolds are 8, 5, and 3, respectively, cf. Section 6. The manifold $S_1(M)$ may be described as the projectivization $P(T^*M)$ of the cotangent bundle $T^*(M) \to M$.

A calotte K at x defines a family of surface elements namely the family of sets $\mathfrak{M}_1(y) \cap K$ for y ranging over $\mathfrak{M}_1(x)$. The surface elements P' coming about from K in this way are said to *belong to* K, or be *contained in* K, written $P' \subseteq K$. Note that the restriction of K belongs to K; a surface element which belongs to K is the restriction of K iff its base point is x.

If $F \subset M$ is a surface, there is a map $F \to S_1(M)$, associating to $x \in F$ the surface element $\mathfrak{M}_1(x) \cap F$.

[1] It is possible that the synthetic "combinatorics" presented here makes sense in other contexts than SDG; in that case, one might probably have to consider the base point x of a surface element P as part of the data of it.

2 The contact distribution $\approx$

We consider a general 3-dimensional manifold M, and the corresponding 5-dimensional manifold $S_1(M)$ of surface elements.

Being a manifold, $S_1(M)$ carries a (1st order) neighbour relation $\sim$. It carries a further structure, namely a reflexive symmetric relation $\approx$ refining $\sim$, and called "united position" ("vereinigte Lage", in the terminology og Lie and Klein): if P and Q are neigbour surface elements with base points p and q, respectively, we say that

$$P \approx Q \quad \text{if} \quad q \in P.$$

This is almost a literal translation of the definition in Lie [5] p. 523: "a surface element is in united position with another one if the point of the latter lies in the plane of the former". It is not immediate from the definition that $\approx$ is a *symmetric* relation, but this can be proved (see Section 6) if we, in Lie's verbal rendering ([5] p. 523), *"ignore infinitesimally small quantities of higher order"*. In our context, the "ignored quantities" are not only ignored, but are *equal* to 0, using $P \sim Q$, as the coordinate calculation below (beginning of Section 6) will reveal.

Let F be a surface in M. Since the passage from points x in F to the corresponding surface elements $\mathfrak{M}_1(x) \cap F$ is a function, it follows from general principles that the surface elements of F at x and y in F (with $x \sim y$) are neighbours in $S_1(M)$. Furthermore, $y \in \mathfrak{M}_1(x) \cap F$; so y belongs to the surface element of F at x. Thus we see that if F is a surface, the surface elements at neighbouring points of F are in united position. This is the motivation for the notion.

It follows that if K is a calotte at x, and P is a surface element belonging to the calotte, then $P \approx K_1$, where K_1 is the surface element obtained by restriction of K. Consider namely some surface F such that $K = \mathfrak{M}_2(x) \cap F$, and apply the above reasoning to F.

(In modern treatments, the structure "united position" is presented as subordinate to the canonical *contact manifold* structure which the cotangent bundle T^*M carries – a certain canonical 1-form. However, $P(T^*M)$ does not carry a canonical 1-form (only "modulo a scalar factor"), and our description (i.e. Lie's) of $\approx$ is purely geometric.)

3 First order PDEs

By a *first order PDE* on a 3-dimensional manifold M, one understands a 4-dimensional submanifold Ψ of the 5-dimensional manifold $S_1(M)$ of surface elements in M. The *solutions* of Ψ are then the surfaces F in M such that all surface elements of F belong to Ψ.

This geometric formulation of the anlytic notion of "first order partial differential equation" goes back to Monge, Lie, and other 19th century geometers, cf. classical texts like [5], [1], [6],....

By a *solution calotte* of Ψ, we mean a calotte all of whose surface elements belong to Ψ. A necessary condition that a calotte K at x is a solution calotte is of course that its restriction belongs to Ψ, i.e. $\mathfrak{M}_1(x) \cap K \in \Psi$. We ask the converse question: let $P \in \Psi$. How many solution calottes through P are there, i.e. how many solution calottes are there with restriction P? We shall prove that the set of such calottes form a 1-dimensional manifold, see Section 6.

4 Characteristic neighbours

A neighbour surface element P' of P belonging to *some* solution calotte of Ψ through P (is in united position with P, and belongs to Ψ, but) may not belong to *all* solution calottes through P.

We ask: given $P \in \Psi$, how many neighbour surface elements P' of P have the property that they belong to *all* these ∞^1 solution calottes through P? We pose:

Definition 1 *Let Ψ be a PDE, and let $P \sim P'$ be neighbour surface elements in Ψ. If P' belongs to* all *solution calottes through P, we say that P' is a* characteristic *neighbour of P, written $P \approx_\Psi P'$.*

Thus, if F is a solution surface of Ψ and contains P, then F will also contain P'. In particular, if two solution surfaces F_1 and F_2 are tangent to each other at x, meaning that $x \in F_1 \cap F_2$ and $\mathfrak{M}_1(x) \cap F_1 = \mathfrak{M}_1(x) \cap F_2$ $(= P$, say), and if P' is a characteristic neighbour of P, then F_1 and F_2 both contain P', equivalently, the surfaces are tangent to each other at the base point of P'.

We shall prove that the characteristic neighbour relation $\approx_\Psi$ defines a 1-dimensional distribution on the manifold Ψ, and hence can be integrated into curves. They are the classical "characteristic stripes" of the PDE Ψ.

Thus, if if two solution surfaces F_1 and F_2 are tangent to each other at x, then they are tangent to each other along the characteristic stripe through P.

If P and P' are characteristic neighbours, and x' is the base point of P', then P' can be reconstructed from x' and P. For, take any solution calotte K through P (such calottes do exist - there are in fact ∞^1 of them). Since $P \approx P'$, we have that $x' \in P \subseteq K$, and since P' belongs to all such solution calottes by assumptions, $P' = \mathfrak{M}_1(x') \cap K$ (and this is independent of the choice of K).

A point x' which appears as the base point of a characteristic neigbour P' of P may be called a characteristic neighbour *point* of P "in the calotte sense". There is another, older, notion of characteristic neighbour point of P, going back to Monge, Lagrange, ..., namely, it is a point x' of P, on the line along which P is tangent to the "Monge cone" at x (where x is the base point of P). We shall describe these notions in synthetic form in Section 5, and prove that x' is a characteristic neighbour point of P in the "calotte" sense iff it is so in the "Monge" sense. This we have been unable to prove from the purely synthetic data, and we prove it by establishing the differential equations that analytically express the synthetic notions of "characteristic".

5 Monge cone

In the classical treatment, the manifold M is R^3, and the surface elements in R^3 are called *plane* elements, since a surface element at $\underline{x} \in R^3$ may be given by a *plane* through $\underline{x}$. The plane elements of a PDE Ψ through a fixed point $\underline{x}$ have an enveloping surface, which is a cone, called the Monge cone at $\underline{x}$; each individual plane element $P \in \Psi$ through $\underline{x}$ is tangent to the Monge cone at $\underline{x}$ along a generator of the cone, and this generator $l \subseteq P$ is the *characteristic line* of the plane element. Paraphrasing, we then arrive at the provisional definition that $\underline{x}'$ is a characteristic neighbour (in the "Monge sense") of the plane element $P \in \Psi$ through $\underline{x}$ if $\underline{x}' \in \mathfrak{M}_1(\underline{x}) \cap l$.

However, as argued in [3], the relationship between enveloping surfaces and characteristics is that the characteristics are logically prior to the enveloping surface (which is made up of the characteristics). From this conception, it is therefore a detour to define the characteristic lines l in terms of the Monge cones. In fact, we define directly the notion of characteristic neighbour ("in the Monge sense"), and applicable in any 3-dimensional manifold M. (The set of characteristic neighbours of P, as P ranges over surface elements in Ψ and with base point x is then an infinitesimal version of the classical Monge cone.)

Definition 2 *Let Ψ be a PDE on e 3-dimensional manifold M, and let $P \in \Psi$ with base point x. Then $x' \in P$ is a characteristic neighbour for P if for all $P' \sim P$ with $x \in P' \in \Psi$, we have $x' \in P'$.*

This may be seen as a rigourous formulation of the description of Lie, [5] p. 510: "–... *so hat man im Punkte (x,y,z) die Schnittlinie der Ebenen zweier solcher unmittelbar banachbarte Flächenelemente ... zu suchen ...* " (he is talking about two plane elements through (x,y,z). So instead of intersecting P with "an immediate neighbour" P', we intersect it with *all* its "immediate" (first order) neigbours; this is the key idea in the conception of [3].)

6 Coordinate calculations

We consider the case where $M = R^3$. A function $f : R^2 \to R$ gives rise to a surface F in R^3, namely its graph. Not all surfaces in R^3 come about this way (they may contain vertical surface elements), but since our considerations are local, it suffices to consider such "graph"-surfaces.

If $(x,y) \in R^2$ and $f : R^2 \to R$ is a function with graph F, the 1-jet of f at (x,y), i.e. the restriction of f to $\mathfrak{M}_1(x,y)$, is a surface element of F at $(x,y,f(x,y))$. Similarly, the 2-jet of f at (x,y) is a calotte at (x,y). The 1-jet of f at (x,y) is determined by the 5-tuple (x,y,z,p,q), where $z = f(x,y)$, $p = \partial f/\partial x(x,y)$, $q = \partial f/\partial y(x,y)$. Similary, the 2-jet of f at (x,y) is determined by the 8-tuple (x,y,z,p,q,r,s,t) with z,p,q as before, and with r,s,t the second order partial derivatives of f at (x,y), $r = \partial^2 f/\partial x^2$, $s = \partial^2 f/\partial x \partial y$, and $t = \partial^2 f/\partial y^2$ (evaluated at (x,y)).

Conversely, the (non-vertical) surface element P in R^3 may be given by 5-tuples (x,y,z,p,q); this 5-tuple denotes the graph of the 1-jet at (x,y) of the affine function $f_1 : R^2 \to R$

$$f_1(\xi,\eta) = z + p(\xi - x) + q(\eta - y). \tag{1}$$

Also, the calottes with non-vertical restriction may be given by 8-tuples (x,y,z,p,q,r,s,t); this 8-tuple denotes the graph of the 2-jet at (x,y) of the quadratic function $f_2 : R^2 \to R$ given by

$$f_2(\xi,\eta) = z + p(\xi - x) + q(\eta - y) + \tfrac{1}{2}(\xi - x)^2 + s(\xi - x)(\eta - y) + \tfrac{1}{2}(\eta - y)^2. \tag{2}$$

(Note that the 1-jet of the function f_2 at (x,y) agrees with the 1-jet at (x,y) of the function f_1, since on $\mathfrak{M}_1(x,y)$, the second order terms vanish.)

The points belonging to the surface element (x,y,z,p,q) are the points of the form

$$(x + dx, y + dy, z + p\,dx + q\,dy)$$

with $(dx,dy) \in D(2)^2$. The base point of this surface element is (x,y,z).

The points belonging to the calotte (x,y,z,p,q,r,s,t) are the points of the form

$$(x + \delta x, y + \delta y, z + p\,\delta x + q\,\delta y + \tfrac{1}{2}r(\delta x)^2 + s\,\delta x \delta y + \tfrac{1}{2}t(\delta y)^2)$$

with $(\delta x, \delta y) \in D_2(2)$.

[2] We follow Klein and Lie in this notation for "first order infinitesimal elements", i.e. for elements in $D, D(2), \ldots$, but we want to emphasize that $dx, dy, \ldots$ are *not* differential forms (which behave contravariantly), but rather, dx and dy are elements of R ("numbers"), behaving in a certain sense *covariantly*; more precisely, the neighbour relation $\sim$ is preserved by any map between manifolds.

For two surface elements P and P' to be in united position, $P \approx P'$, they must first of all be neighbours, $P \sim P'$, so they are of the form

$$(x,y,z,p,q) \quad \text{and} \quad (x+dx, y+dy, z+dz, p+dp, q+dq)$$

respectively, with $(dx,dy,dz,dp,dq) \in D(5)$; and then

$$P \approx P' \quad \text{iff} \quad dz = p\,dx + q\,dy. \tag{3}$$

To prove symmetry of the relation $\approx$, we should from $dz = p\,dx + q\,dy$ deduce that

$$-dz = (p+dp)(-dx) + (q+dq)(-dy);$$

but this follows because $dp \cdot dx = 0$ and $dq \cdot dy = 0$ since $(dx, \dots, dq) \in D(5)$. (Lie puts it this way, p. 523: "*here, we ignore infinitely small quantities of higher order*"; in our formalism, the "higher order quantities" to be ignored are $dp \cdot dx$ and $dq \cdot dy$; they are both 0.)

We next consider a calotte $K = (x,y,z,p,q,r,s,t)$ with restriction $P = (x,y,z,p,q)$. We ask which surface elements P' belong to K. A necessary condition is that the base point (x',y',z') of P' is in P, so that $(x',y',z') = (x+dx, y+dy, z+p\,dx+q\,dy)$ for some $(dx,dy) \in D(2)$.

Proposition 1 *A surface element $P' = (x',y',z',p',q')$ (as above) with base point in P is contained in K iff*

$$p' = p + r\,dx + s\,dy \quad \text{and} \quad q' = q + s\,dx + t\,dy.$$

Proof. Consider the function $f = f_2$ from (2), whose 2-jet at (x,y,z) has K as graph. Its first partial derivatives at $(x+dx, y+dy)$ are $p + r\,dx + s\,dy$ and $q + s\,dx + t\,dy$, respectively.

We may give this Proposition a different twist, by asking: given two surface elements in united position,

$$P = (x,y,z,p,q), \quad P' = (x+dx, y+dy, z+p\,dx+q\,dy),$$

what is the condition on r,s,t that the calotte $K = (x,y,z,p,q,r,s,t)$ contains P'?

Proposition 2 *The calotte K contains P' iff (r,s,t) is a solution of a certain linear equation system (two equations in three unknowns), namely the linear system with augmented matrix*

$$\left[\begin{array}{ccc|c} dx & dy & & dp \\ & dx & dy & dq \end{array}\right]. \tag{4}$$

Proof. This is just a rewrite of the relation between p' and q' on the one hand, and the r, s, t on the other, given by Proposition 1.

Now we bring in the PDE Ψ, a 4-dimensional submanifold of the 5-dimensional manifold $S_1(R^3)$ of surface elements in R^3. Our considerations are local, so we may assume that Ψ is given as the zero set of a certain function $\psi : R^5 \to R$, in other words, $(x,y,z,p,q) \in \Psi$ iff $\psi(x,y,z,p,q) = 0$. The graph of a function $f : R^2 \to R$ is then a solution surface iff for all (x,y,z)

$$\psi(x,y,z,\tfrac{\partial f}{\partial x}(x,y),\tfrac{\partial f}{\partial y}(x,y)) = 0$$

which is a partial differential equation of order 1 (and this is the justification for our more general use of the term "PDE"). We assume that "p and q really occur in the function ψ" (or "ψ is not free of p and q"): we assume that $\partial\psi/\partial p$ and $\partial\psi/\partial q$ do not vanish simultaneously at any point (x,y,z,p,q) (more precisely: at least one of them is invertible).

We proceed to describe the solution calottes for Ψ in analytic terms. A necessary condition that a calotte $K = (x,y,z,p,q,r,s,t)$ is a solution calotte is of course that its restriction (x,y,z,p,q) is in Ψ.

Proposition 3 *Assume $(x,y,z,p,q) \in \Psi$. Then the calotte (x,y,z,p,q,r,s,t) is a solution calotte for Ψ iff (r,s,t) is a solution of the linear equation system (two equations in three unknowns), with augmented matrix*

$$\left[\begin{array}{ccc|c} \psi_p & \psi_q & & -\psi_x - p\cdot\psi_z \\ & \psi_p & \psi_q & -\psi_y - q\cdot\psi_z \end{array}\right]. \tag{5}$$

where ψ_x denotes $\frac{\partial\psi}{\partial x}(x,y,z,p,q)$, and similarly for $\psi_y, \psi_z, \psi_p, \psi_q$.

Proof. Let $f = f_2 : R^2 \to R$ be the quadratic function given by (2). The calotte in question is then a solution calotte iff for all $(dx,dy) \in D(2)$

$$\psi(x+dx, y+dy, z+p\,dx+q\,dy, \partial f/\partial x, \partial f/\partial y) = 0$$

where the partial derivatives are to be evaluated at $(x+dx, y+dy)$; these partial derivatives are $r\cdot dx + s\cdot dy$ and $s\cdot dx + t\cdot dy$, respectively, so K is a solution calotte iff

$$\psi(x+dx, y+dy, z+p\,dx+q\,dy, r\,dx+s\,dy, s\,dx+t\,dy) = 0. \tag{6}$$

We Taylor expand ψ and use $\psi(x,y,z,p,q) = 0$; then we see that (6) is equivalent to

$$\frac{\partial\psi}{\partial x}\cdot dx + \frac{\partial\psi}{\partial y}\cdot dy + \frac{\partial\psi}{\partial z}\cdot(z+p\,dx+q\,dy)$$
$$+\frac{\partial\psi}{\partial p}\cdot(r\,dx+s\,dy) + \frac{\partial\psi}{\partial q}\cdot(s\,dx+t\,dy) = 0$$

where the partial derivatives are to be evaluated at (x,y,z,p,q). Reorganizing, we see that this is a linear equation system (two equations in the three unknowns), and that its augmented matrix is the one provided.

Since at least one of $\frac{\partial \psi}{\partial p}$ and $\frac{\partial \psi}{\partial q}$ is invertible, we see that the rank of the matrix to the left of the augmentation bar is 2, whence it represents a surjective linear map $R^3 \to R^2$; the solution set of the equation system is therefore a 1-dimensional (and affine) subspace of the (r,s,t)-space. So also for a general (sufficiently non-singular) "abstract" PDE $\Psi \subseteq S_1(M)$, there are ∞^1 solution calottes extending a given $P \in \Psi$.

Given $P = (x,y,z,p,q) \in \Psi$. The condition on a neighbour P' that it is contained in a calotte K is given by a condition on the (r,s,t) of the calotte, namely that it is a solution of the equation system (4) in Proposition 2; the condition that a calotte through P is a solution calotte is that (r,s,t) is a solution of the equation system (5) in Proposition 3. To say that P' is a characteristic neighbour of P is therefore to say that whenever (r,s,t) solves (5), it also solves (4). From the "elementary linear algebra" in the Appendix therefore follows that P' is a characteristic neihbour of P iff the augmented matrix in (4) is a scalar multiple of the one in (5).

Therefore we have

Theorem 1 *Assume $P = (x,y,z,p,q)$ is in Ψ. For $P' = (x+dx, y+dy, z+p\,dx+q\,dy, p+dp, q+dq)$ to be a characteristic neighbour, it is necessary and sufficient that there exists a scalar λ such that*

$$(dx,dy,dp,dq) = \lambda \cdot (\psi_p, \psi_q, -\psi_x - p\cdot\psi_z, -\psi_y - q\cdot\psi_z) \tag{7}$$

or equivalently, that there exists a scalar λ such that

$$(dx,dy,dz,dp,dq) = \lambda \cdot (\psi_p, \psi_q, p\cdot\psi_p + q\cdot\psi_q, -\psi_x - p\cdot\psi_z, -\psi_y - q\cdot\psi_z) \tag{8}$$

Here, ψ_p denotes $\partial\psi/\partial p$ evaluated at $P = (x,y,z,p,q)$, and simlarly for ψ_q, ψ_x etc. Note that our assumption that at least one of ψ_p and ψ_q is invertible implies that the scalar λ is uniquely determined.

From the Theorem follows in particular that for $(x+dx, y+dy)$ to be a characteristic neighbour point of P (in the calotte sense), it is necessary that

$$(dx,dy) = \lambda \cdot (\psi_p, \psi_q) \tag{9}$$

In fact, it is also sufficient, since the relevant dp and dq then can be reconstructed from λ and the partial derivatives of ψ, using (7).

Now that we have an analytic expression (7) for P and P' being characteristic neighbours, we can also prove that this relationship is a symmetric one. The proof is much in the spirit of the proof that the relation $\approx$ ("united position") is symmetric, namely "ignoring infinitesimals of higher order":

To say that $P = (x,y,z,p,q)$ and $P' = (x+dx, y+dy, z+dz, p+dp, q+dq) = P+dP$ satisfy $P \approx_\Psi P'$ is, by (8) equivalent to saying that

$$dP = \lambda(P; dP) \cdot \tilde{\psi}(P), \tag{10}$$

where we in the λ-factor record the dependence of the scalar λ on P as well as on dP, and where $\tilde{\psi} : R^5 \to R^5$ is the function in the parenthesis on the right hand side of (8), but where we now explicitly record the $P = (x,y,z,p,q)$ where the various partial derivatives ψ_p etc. are to be taken. Similarly, to say that $P' \approx_\Psi P$ is by (8) equivalent to saying that

$$-dP = \lambda(P'; -dP) \cdot \tilde{\psi}(P'). \tag{11}$$

We note that for fixed P, and for $dP = 0$, we have $\lambda(P; dP) = 0$. By KL, the function $\lambda(P; -) : D(5) \to R$ extends uniquely to a linear function $R^5 \to R$. Since in the expressions on the right hand side of (11), we have that dP occurs linearily, it follows by the "Taylor principle" (cf. [4] p. 19) – essentially just Taylor expansion in the direction of dP – that we may replace $P' = P + dP$ by P in both the occurrences of P', so that (11) may be written

$$-dP = \lambda(P; -dP) \cdot \tilde{\psi}(P),$$

which is equivalent to (10) in view of the linearity of $\lambda(P; -)$. This proves the symmetry of $\approx_\Psi$.

Differential equation for Monge characteristics

We consider the surface element $P = (x,y,z,p,q)$ in Ψ, so $\psi(x,y,z,p,q) = 0$. A neighbour surface element with same base point is of the form $(x,y,z,p+\delta p, q+\delta q)$ with $(\delta p, \delta q) \in D(2)$, and this element is in Ψ if $\psi(x,y,z,p+\delta p, q+\delta q) = 0$; by Taylor expansion, and the previous equation, this is equivalent to

$$(\partial \psi / \partial p) \cdot \delta p + (\partial \psi / \partial q) \cdot \delta q = 0, \tag{12}$$

where the partial derivatives are to be evaluated at (x,y,z,p,q).

A point in P is of the form $(x+dx, y+dy, z+p\,dx+q\,dy)$ with $(dx,dy) \in D(2)$, and this point is in the surface element $(x,y,z,p+\delta p, q+\delta q)$ iff $p\,dx + q\,dy = (p+\delta p) \cdot dx + (q+\delta q) \cdot dy$, that is, iff[3]

$$dx \cdot \delta p + dy \cdot \delta q = 0. \tag{13}$$

[3] The reason we did not write dp and dq, rather than δp and δq is that notation (dp, dq) might lead one to think that e.g. $dx \cdot dp = 0$, which we have not assumed ; dx and δp are what Lie would call *independent* infinitesimals: $dx \cdot \delta p$ is not assumed to be 0.

So to say that $(x+dx, y+dy)$ is a Monge-characteristic neighbour of P is to say that all $(\delta p, \delta q) \in D(2)$ which satisfy (12) also satisfy (13). Assuming, as before, that $\partial\psi/\partial p$ and $\partial\psi/\partial q$ do not vanish simultaneously, this property is equivalent to: (dx, dy) is of the form $\lambda \cdot (\partial\psi/\partial p, \partial\psi/\partial q)$, see Remark after Proposition 4. Thus, $(x+dx, y+dy, z+p\,dx+q\,dy)$ is a Monge-characteristic neighbour of $P = (x, y, z, p, q)$ iff

$$(dx, dy) = \lambda \cdot (\partial\psi/\partial p, \partial\psi/\partial q).$$

We see that this is just the equation (9), which is the equation for characteristic neighbour point in the calotte sense. We conclude that the two notions of "characteristic neighbour point" agree.

Appendix

Since the linear algebra in question is over the commutative ring R which is not a field, only a local ring, we need to elaborate a little on the linear algebra/matrix theory over R. We use "vector space" and "linear" as synonyms for "R-module" and "R-linear.

Any linear map $R \to R$ is multiplication by a unique $\lambda \in R$. From this follows:

Proposition 4 *Let $p : A \to R$ be a surjective linear map, and let $q : A \to R$ be any linear map. If the kernel of p is contained in the kernel of q, then $q = \lambda \cdot p$ for a unique $\lambda \in R$.*

Proof. Contemplate the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & Ker(p) \arrow[r] \arrow[d, "incl"'] & A \arrow[r, "p"] \arrow[d, "id"] & R \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Ker(q) \arrow[r] & A \arrow[r, "q"'] & R &
\end{tikzcd} .$$

The right hand vertical map exists by exactness of top row, and is multiplication by a unique scalar.

Remark. An immediate consequence is that $\underline{a} = (a_1, \ldots, a_n) \in R^n$ is a proper vector (meaning: at least one of th coordinates a_i invertible), and if $\underline{b} \in R^n$ is a vector such that $\underline{a} \bullet \underline{\delta} = 0$ implies $\underline{b} \bullet \underline{\delta} = 0$ for all $\underline{\delta} \in R^n$ (where $\bullet$ is the standard dot product of coordinate vectors), then $\underline{b} = \lambda \cdot \underline{a}$ for some $\lambda \in R$. For this conclusion, it even suffices that the implication

$$\underline{a} \bullet \underline{\delta} = 0 \Rightarrow \underline{b} \bullet \underline{\delta} = 0$$

holds for all $\underline{\delta} \in D(n)$; for, by the KL axiom, a linear map $R^n \to R$ is completely determined by its value on $D(n)$.

Let $p : A \to R$ be a linear map, and let $r \in R$. If $p(x_0) = r$, then the solution set of the equation $p(x) = r$ is the coset $x_0 + Ker(p)$. As a Corollary of the above Propostion, we then have

Proposition 5 *Let $p : A \to R$ be a surjective linear map, and $q : A \to R$ any linear map. Let $r, s \in R$. If the solution set of $p(x) = r$ is contained in the solution set of $q(x) = s$, then there is a unique $\lambda \in R$ so that $q = \lambda \cdot p$ and $s = \lambda \cdot r$.*

Proof. Take some $x_0 \in A$ such that $p(x_0) = r$, using p surjective. The assumption then implies that $q(x_0) = s$. The solution sets of the two equations are, respectively, $x_0 + \ker(p)$, and $x_0 + \ker(q)$, and the assumed inclusion relation then clearly implies $\ker(p) \subseteq \ker(q)$. By the previous Proposition, there is a unique $\lambda \in R$ with $q = \lambda \cdot p$. We then have

$$\lambda \cdot r = \lambda \cdot p(x_0) = q(x_0) = s.$$

Proposition 6 *Consider two linear equation systems given by the augmented matrices*

$$\left[\begin{array}{ccc|c} p_1 & p_2 & & r_1 \\ & p_1 & p_2 & r_2 \end{array}\right] \tag{14}$$

and

$$\left[\begin{array}{ccc|c} q_1 & q_2 & & s_1 \\ & q_1 & q_2 & s_2 \end{array}\right] \tag{15}$$

respectively, and assume that at least one of the p_is is invertible. Assume that the solution set of the first is contained in he solution set of the second. Then there exists a unique $\lambda \in R$ with

$$\lambda \cdot (q_1, q_2, s_1, s_2) = (p_1, p_2, r_1, r_2).$$

Proof. The linear map $R^2 \to R$ with matrix $[p_1, p_2]$ is surjective, by invertibility of one of the p_is. Assume (x, y) is a solution of the linear equation given by the first row of (14), one sees that there exists a unique z so that (y, z) is a solution of the equation given by the second row of (14). So (x, y, z) is a solution of the equation system given by (14), hence by assumption, it is also a solution of the equation system given by (15). We conclude that if (x, y) is a solution of the equation given by the first row in (14), then it is also a solution of the equation given by the first row of (15), and therefore, by the previous Proposition, there exists a unique $\lambda \in R$ such that

$$q_1 = \lambda p_1,\ q_2 = \lambda p_2,\ s_1 = \lambda r_1.$$

A similar reasoning with the second rows of (14) and (15) gives that there exists a unique $\mu \in R$ such that

$$q_1 = \mu p_1,\ q_2 = \mu p_2,\ s_2 = \mu r_2.$$

Since either p_1 or p_2 is invertible, one concludes by comparison that $\lambda = \mu$. This proves the Proposition.

If x is a point i a manifold M, the set $\mathfrak{M}_1(x) \subseteq M$ of first order neighbours of x has a natural "base" point, namely x. This point can be reconstructed from the subset; we claim

Proposition 7 *The point $x \in \mathfrak{M}_1(x)$ is the only point z with the property that for all $y \in M$ with $y \sim z$, we have $y \in \mathfrak{M}_1(x)$.*

Proof. Since the assertion is coordinate free, it suffices to prove it for the case where $M = R^n$ and $x = 0 \in R^n$. Note that now $\mathfrak{M}_1(x) = D(n)$. Then the assertion of the Proposition amounts to the assertion: if $z \in D(n)$ has the property that $z + u \in D(n)$ for all $u \in D(n)$, then $z = 0$. To prove that the first coordinate z_1 of z is $0 \in R$, we uset that $z + (d, 0, \dots, 0) \in D(n)$ for all $d \in D$, which implies $(z_1 + d)^2 = 0$ for all $d \in D$. Now

$$0 = (z_1 + d)^2 = z_1^2 + d^2 + 2z_1 d = 2z_1 d.$$

Since this holds for all $d \in D$, it follows from KL that $2z_1 = 0$, hence $z_1 = 0$. Similarly for the other coordinates $z_2, \dots, z_n$.

There are similar characterizations of $\mathfrak{M}_k(x)$ as as a subset of $\mathfrak{M}_{k+1}(x) \subseteq M$ for $k = 2, \dots$. One then need that the integer $k + 2$ is invertible in R.

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