1. The problem setup

Let M be a real n-dimensional manifold and $x_0 \in M$. We consider a formal complex-valued function

$$\varphi = \nu^{-1}\varphi_{-1} + \varphi_0 + \nu\varphi_1 + \ldots$$

and a formal complex-valued density

$$\rho = \rho_0 + \nu\rho_1 + \ldots$$

on M such that x_0 is a nondegenerate critical point of φ_{-1} with zero critical value, $\varphi_{-1}(x_0) = 0$, and $\rho_0(x_0) \neq 0$. We want to relate to the formal oscillatory integral

$$f \mapsto \nu^{-\frac{n}{2}} \int_{(x_0)} e^{\varphi} f\, \rho,$$

where f is an amplitude (say, $f \in C^\infty(M)[[\nu]]$), a formal distribution

$$\Lambda = \Lambda_0 + \nu\Lambda_1 + \ldots$$

supported at x_0. The assignment should be based on a number of formal properties of the formal integral (see details in my most recent preprint).

The answer is as follows. Choose local coordinates $\{x^i\}$ around x_0 such that $x^i(x_0) = 0$ for all i. The Hessian matrix of φ_{-1} at x_0 is denoted by h_{ij},

$$h_{ij} := \frac{\partial\varphi_{-1}}{\partial x^i \partial x^j}(x_0).$$

It is a complex symmetric nondegenerate matrix with constant coefficients. We denote by h^{ij} its inverse matrix and introduce the Laplace operator

$$\Delta := -\frac{1}{2} h^{ij} \frac{\partial^2}{\partial x^i \partial x^j}.$$

We use the following model Gaussian integral,

$$\nu^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{\frac{1}{2\nu} h_{ij} x^i x^j} dx^1 \ldots dx^n = \pm \sqrt{\frac{(-2\pi)^n}{\det(h_{ij})}}\, e^{\nu\Delta} f|_{x=0}.$$

We do not specify the sign on the right-hand side.

Assume that locally

$$\rho = e^u\, dx^1 \ldots dx^n,$$

where $u = u_0 + \nu u_1 + \ldots$. Set

$$\chi(x) := \varphi(x) - \frac{1}{2\nu} h_{ij} x^i x^j - \varphi_0(0) + u(x) - u_0(0).$$

Then

$$\nu^{-\frac{n}{2}} \int_{(x_0)} e^{\varphi} f \, \rho = \nu^{-\frac{n}{2}} e^{\varphi_0(0)+u_0(0)} \int_{(x_0)} e^{\frac{1}{2\nu} h_{ij} x^i x^j} (e^{\chi} f) \, dx^1 \dots dx^n =$$

$$\pm \sqrt{\frac{(-2\pi)^n}{\det(h_{ij})}} \, e^{\varphi_0(0)+u_0(0)} e^{\nu\Delta} (e^{\chi} f)|_{x=0}.$$

Consider the functional

$$K(f) := e^{\nu\Delta}(e^{\chi} f)|_{x=0}.$$

It is well-defined (see the Appendix to my preprint). Our task is to identify all such functionals and to recover the phase remainder χ from K. In my preprint I proved that the full jet of χ at $x_0 = 0$ is determined uniquely by the functional K. Now I will describe all functionals K without referring to χ and recover χ from K via a constructive procedure.

2. The description of the operator K

Let $A = A_0 + \nu A_1 + \dots$ be a formal differential operator on a neighborhood of zero, U. I call it natural if the order of each differential operator A_r is not greater than r. The natural operators on U form an algebra. The operators $\nu^{-1} A$, where A is natural, form a Lie algebra with respect to the commutator.

We consider formal differential operators acting on formal jets at zero. We introduce a descending filtration on the space of formal jets at zero by assigning filtration degree 2 to ν and filtration degree r to a jet at zero (which does not depend on ν) that has zero of order r. This filtration induces a filtration on the space of formal differential operators on U. For example, in local coordinates, the degree of the operator

$$\nu x^1 \frac{\partial}{\partial x^2}$$

is two. Our main object is the Lie algebra $\mathfrak{g}$ of formal differential operators on the formal neighborhood of zero of the form ν^{-1}, where A is natural and such that ν^{-1} has a positive filtration degree. This is a pronilpotent Lie algebra. The multiplication operator by the phase remainder $\chi = \nu^{-1}\chi_{-1} + \chi_0 + \dots$ lies in $\mathfrak{g}$, because the multiplication operator $\nu\chi$ is natural, the order of zero of χ_{-1} is at least 3, and the order of zero of χ_0 is at least one. The algebra $\mathfrak{g}$ acts on the formal distributions supported at zero from the right:

$$\mathfrak{g} \ni A : u \mapsto \langle u | A.$$

Denote by $\mathfrak{s}$ the stabilizer of $\delta(x)$ in $\mathfrak{g}$. For $A \in \mathfrak{s}$,

$$\langle \delta | A | f \rangle = (Af)(0) = 0.$$

The subalgebra Lie $\mathfrak{s}$ does not depend on the choice of local coordinates. Now fix local coordinates around zero. Denote by $\mathfrak{c}$ the subalgebra of $\mathfrak{g}$ of formal differential operators with constant coefficients. Clearly,

$$\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{c}. \tag{1}$$

Writing $A \in \mathfrak{g}$ in the local coordinates in the normal form, $A = A(x, \frac{\partial}{\partial x})$, we decompose it according to (1) as

$$A = \left(A\left(x, \frac{\partial}{\partial x}\right) - A\left(0, \frac{\partial}{\partial x}\right)\right) + A\left(0, \frac{\partial}{\partial x}\right).$$

The functional $f \mapsto \langle \delta | A | f \rangle$ depends only on the $\mathfrak{c}$-component of A.

Now consider the pronilpotent Lie group $\exp \mathfrak{g}$. It can be realized as a group of formal differential operators on a space of formal jets

$$f = \sum_{r=-\infty}^{\infty} f_r$$

of finite filtration degree (I still have to verify it). Given $A \in \mathfrak{g}$, one can write uniquely

$$e^A = e^S e^C,$$

This is a constructive procedure. First decompose $A = A_1 = S_1 + C_1$. Then iterate

$$e^{A_r} = e^{S_r} e^{A_{r+1}} e^{C_r},$$

where $S_r \in \mathfrak{s}$ and $C_r \in \mathfrak{c}$. The filtration degree of A_r goes to infinity as $r \to \infty$, $S_r \to S$, and $C_r \to C$.

An important remark: An operator $C \in \mathfrak{c}$ has constant coefficients and positive filtration degree. Also, νC is natural,

$$\nu C = A_0 + \nu A_1 + \nu^2 A_2 + \ldots,$$

where A_r is a differential operator with constant coefficients of order not greater than r. The filtration degree of A_r is at least $-r$. Now,

$$C = \nu^{-1} A_0 + A_1 + \nu A_2 + \ldots.$$

The filtration degree of $\nu^{r-1} A_r$ is at least $2(r-1) - r = r - 2$.

Since the filtration degree of C is positive, $A_0 = 0, A_1 = 0$, and A_2 is of order not greater than one.

The formal differential operator $\exp(\nu\Delta)$ has filtration degree zero and does not lie in $\exp \mathfrak{g}$, but it acts upon it by conjugations. Since $\chi \in \mathfrak{g}$, we have

$$A := e^{\nu\Delta} \chi e^{-\nu\Delta} \in \mathfrak{g}.$$

Now, decompose

$$e^{\nu\Delta}e^{\chi}e^{-\nu\Delta} = e^{A} = e^{S}e^{C}. \tag{2}$$

The rest still has to be justified:

$$K(f) = \langle\delta|e^{\nu\Delta}e^{\chi}|f\rangle = \langle\delta|e^{S}e^{C}e^{\nu\Delta}|f\rangle = \langle\delta|e^{\nu\Delta+C}|f\rangle.$$

Thus, the functional K is given by the formal differential operator with constant coefficients $\exp(\nu\Delta + C)$.

According to the remark above, the operator $\nu\Delta + C$ is of the form

$$\nu A_2 + \nu^2 A_3 + \dots,$$

where the order of A_r is not greater than r and the principal symbol of A_2 is $-\frac{1}{2}h^{ij}\xi_i\xi_j$, where h^{ij} is nondegenerate. We claim that **any** operator of this format is equal to the operator K for an appropriate phase remainder χ.

3. The rest of the proof

Now we need to define the action of differential operators on functions **from the right**. This is achieved by passing to the transpose,

$$(x^i)^t = x^i \text{ and } \left(\frac{\partial}{\partial x^i}\right)^t = -\frac{\partial}{\partial x^i},$$

so that

$$\langle f|A := A^t|f\rangle.$$

Denote by $\mathfrak{r}$ the subalgebra of $\mathfrak{g}$ of the operators which annihilate constants from the right and by $\mathfrak{f}$ the subalgebra of $\mathfrak{g}$ of the multiplication operators. Then

$$\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{f}. \tag{3}$$

Given $A \in \mathfrak{g}$, we expand it according to (3) as follows,

$$A = (A - \langle 1|A) + \langle 1|A,$$

where we interpret the function $\langle 1|A$ as a multiplication operator.

We will need the following calculation:

$$e^{-\nu\Delta}x^k e^{\nu\Delta} = x^k + \nu h^{kl}\frac{\partial}{\partial x^l}.$$

Given a nondegenerate symmetric complex matrix h^{ij} with constant coefficients, introduce the function

$$\psi = \frac{1}{2}h_{ij}x^i x^j.$$

Now,
$$e^{\nu^{-1}\psi}e^{-\nu\Delta}x^k e^{\nu\Delta}e^{-\nu^{-1}\psi} = \nu h^{kl}\frac{\partial}{\partial x^l}.$$
Therefore, if $S \in \mathfrak{s}$, then
$$R := e^{\nu^{-1}\psi}e^{-\nu\Delta}Se^{\nu\Delta}e^{-\nu^{-1}\psi} \in \mathfrak{r}.$$
Assume that, as in (2),
$$e^{\nu\Delta}e^{\chi}e^{-\nu\Delta} = e^S e^C.$$
Then
$$e^{\chi} = e^{-\nu\Delta}e^S e^{\nu\Delta}e^C,$$
from whence we get that
$$e^{\nu^{-1}\psi}e^{\chi}e^{-\nu^{-1}\psi} = e^{\nu^{-1}\psi}e^{-\nu\Delta}e^S e^{\nu\Delta}e^{-\nu^{-1}\psi}e^{\nu^{-1}\psi}e^C e^{-\nu^{-1}\psi}.$$
Therefore,
$$e^{\chi} = e^R e^{\nu^{-1}\psi}e^C e^{-\nu^{-1}\psi}.$$
Applying it to 1 from the right, we get
$$e^{\chi} = \langle 1|e^{\nu^{-1}\psi}e^C e^{-\nu^{-1}\psi}.$$
Thus, we have recovered χ from an arbitrary C.