1. The problem setup

Let M be a real n-dimensional manifold and $x_0 \in M$. We consider a formal complex-valued function

$$\varphi = \nu^{-1}\varphi_{-1} + \varphi_0 + \nu\varphi_1 + \dots$$

and a formal complex-valued density

$$\rho = \rho_0 + \nu\rho_1 + \dots$$

on M such that x_0 is a nondegenerate critical point of φ_{-1} with zero critical value, $\varphi_{-1}(x_0) = 0$, and $\rho_0(x_0) \neq 0$. We want to relate to the formal oscillatory integral

$$f \mapsto \nu^{-\frac{n}{2}} \int_{(x_0)} e^{\varphi} f\, \rho,$$

where f is an amplitude (say, $f \in C^\infty(M)[[\nu]]$), a formal distribution supported at x_0,

$$f \mapsto \Lambda(f),$$

where $\Lambda = \Lambda_0 + \nu\Lambda_1 + \dots$. The assignment should be based on a number of formal properties of the formal integral (see details in my most recent preprint).

The answer is as follows. Choose local coordinates $\{x^i\}$ around x_0 such that $x^i(x_0) = 0$ for all i. The Hessian matrix of φ_{-1} at x_0 is denoted by h_{ij},

$$h_{ij} := \frac{\partial\varphi_{-1}}{\partial x^i \partial x^j}(x_0).$$

It is a complex symmetric nondegenerate matrix with constant coefficients. We denote by h^{ij} its inverse matrix and introduce the Laplace operator

$$\Delta := -\frac{1}{2} h^{ij} \frac{\partial^2}{\partial x^i \partial x^j}.$$

We use the following model formal Gaussian integral,

$$\nu^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{\frac{1}{2\nu} h_{ij} x^i x^j} dx^1 \dots dx^n = \pm\sqrt{\frac{(-2\pi)^n}{\det(h_{ij})}}\, e^{\nu\Delta} f \Big|_{x=0}.$$

We do not specify the sign on the right-hand side.

Assume that locally

$$\rho = e^u\, dx^1 \dots dx^n,$$

where $u = u_0 + \nu u_1 + \dots$. Set

$$\chi(x) := \varphi(x) - \frac{1}{2\nu} h_{ij} x^i x^j - \varphi_0(0) + u(x) - u_0(0).$$

Then

$$\nu^{-\frac{n}{2}} \int_{(x_0)} e^{\varphi} f\, \rho = \nu^{-\frac{n}{2}} e^{\varphi_0(0)+u_0(0)} \int_{(x_0)} e^{\frac{1}{2\nu} h_{ij} x^i x^j} (e^{\chi} f)\, dx^1 \ldots dx^n =$$
$$\pm \sqrt{\frac{(-2\pi)^n}{\det(h_{ij})}}\, e^{\varphi_0(0)+u_0(0)} e^{\nu\Delta}(e^{\chi} f)|_{x=0}.$$

This expression is coordinate-independent, which follows, say, from my axiomatic description of formal oscillatory integrals. Consider the functional

$$K(f) := e^{\nu\Delta}(e^{\chi} f)|_{x=0}.$$

It is well-defined (see the Appendix to my preprint). Our task is to identify all such functionals and to recover the phase remainder χ from K. In my preprint I proved that the full jet of χ at $x_0 = 0$ is determined uniquely by the functional K. Now I will describe all functionals K without referring to χ and recover χ from K via a constructive procedure.

2. A description of the functional K

Let $N = N_0 + \nu N_1 + \ldots$ be a formal differential operator on a neighborhood U. I call it natural if the order of each differential operator N_r is not greater than r. The natural operators on U form an algebra. The operators $\nu^{-1}N$, where N is natural, form a Lie algebra with respect to the commutator.

We consider formal differential operators acting on formal jets at zero. We introduce a descending filtration on the space of formal jets at zero by assigning filtration degree 2 to ν and filtration degree r to a jet at zero (which does not depend on ν) that has zero of order r. This filtration induces a filtration on the space of formal differential operators on U. For example, in local coordinates, the degree of the operator

$$\nu x^1 \frac{\partial}{\partial x^2}$$

is 2+1-1=2. Our main object is the Lie algebra $\mathfrak{g}$ of formal differential operators on the formal neighborhood of zero of the form $\nu^{-1}N$, where N is natural and $\nu^{-1}N$ has a positive filtration degree. This is a pronilpotent Lie algebra. The multiplication operator by the phase remainder $\chi = \nu^{-1}\chi_{-1} + \chi_0 + \ldots$ lies in $\mathfrak{g}$, because the multiplication operator $\nu\chi$ is natural, the order of zero of χ_{-1} is at least 3, and the order of zero of χ_0 is at least one. The algebra $\mathfrak{g}$ acts on the formal distributions supported at zero from the right:

$$\mathfrak{g} \ni A : u \mapsto \langle u|A,$$

so that

$$\langle u|A|f\rangle = \langle u, Af\rangle,$$

where f is a full jet at zero. Here we use the Dirac bra-ket notation.

Denote by $\mathfrak{b}$ the annihilator of $\delta(x)$ in $\mathfrak{g}$. For $A \in \mathfrak{b}$,

$$\langle \delta|A|f\rangle = (Af)(0) = 0.$$

The subalgebra Lie $\mathfrak{b}$ does not depend on the choice of local coordinates. Now fix local coordinates around zero. Denote by $\mathfrak{c}$ the subalgebra of $\mathfrak{g}$ of formal differential operators with constant coefficients. Clearly,

$$\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{c}. \tag{1}$$

Writing $A \in \mathfrak{g}$ in the local coordinates in the normal form, $A = A(x, \frac{\partial}{\partial x})$, we decompose it according to (1) as

$$A = \left(A\left(x, \frac{\partial}{\partial x}\right) - A\left(0, \frac{\partial}{\partial x}\right)\right) + A\left(0, \frac{\partial}{\partial x}\right).$$

The functional $f \mapsto \langle \delta|A|f\rangle$ depends only on the $\mathfrak{c}$-component of A. Moreover, the $\mathfrak{c}$-component of A is completely determined by this functional.

An important remark: *An operator $C \in \mathfrak{c}$ has constant coefficients and positive filtration degree. Also, νC is natural,*

$$\nu C = N_0 + \nu N_1 + \nu^2 N_2 + \ldots,$$

where N_r is a differential operator with constant coefficients of order not greater than r. The filtration degree of N_r is at least $-r$. Now,

$$C = \nu^{-1} N_0 + N_1 + \nu N_2 + \ldots.$$

The filtration degree of $\nu^{r-1} N_r$ is at least $2(r-1) - r = r - 2$. Since the filtration degree of C is positive, we see that $N_0 = 0, N_1 = 0$, and N_2 is of order not greater than one.

Now consider the pronilpotent Lie group $\exp \mathfrak{g}$. We will show that it can be realized as a group of formal differential operators

$$A = \sum_{r=-\infty}^{\infty} \nu^r A_r$$

of filtration degree zero on the space $\mathcal{F}$ of formal jets

$$f = \sum_{r=-\infty}^{\infty} \nu^r f_r$$

at zero of finite filtration degree, i.e., for every $f \in \mathcal{F}$ there exists an integer k such that the order of zero of f_r is at least $k - 2r$ for every $r \in \mathbb{Z}$.

First we want to show that any natural operator leaves $\mathcal{F}$ invariant. Let $N = N_0 + \nu N_1 + \dots$ be a natural operator and $f \in \mathcal{F}$. Assume for the sake of simplicity that the filtration degree of f is zero. Then the filtration degree of f_r is at least $-2r$. The coefficient at ν^k of Nf is

$$\sum_{r=0}^{\infty} N_r f_{k-r} \tag{2}$$

Since the order of N_r is at most r, its filtration degree is at least $-r$. The filtration degree of f_{k-r} is at least $-2(k-r) = -2k+2r$. Therefore, the filtration degree of $N_r f_{k-r}$ is at least $-r - 2k + 2r = r - 2k$. It follows that the series (2) is convergent in the topology induced by the filtration and defines a formal jet at zero. The filtration degree of this formal jet is at least zero, because the filtration degree of a natural operator is at least zero.

If N is a natural operator, then $\nu^{-1} N$ also leaves $\mathcal{F}$ invariant.

Given $f \in \mathcal{F}$ and $A \in \mathfrak{g}$, $e^A f = \sum_{r=0}^{\infty} \frac{1}{r!} A^r f$ is a convergent series in $\mathcal{F}$, because A leaves $\mathcal{F}$ invariant and the filtration degree of A is positive. The group $\exp \mathfrak{g}$ respects the filtration on $\mathcal{F}$, because its elements have zero filtration degree.

Given $A \in \mathfrak{g}$, one can write uniquely

$$e^A = e^B e^C,$$

where $B \in \mathfrak{b}$ and $C \in \mathfrak{c}$. This is a constructive procedure. First decompose $A = A_1 = B_1 + C_1$. Then iterate

$$e^{A_{r+1}} = e^{-B_r} e^{A_r} e^{-C_r},$$

where $B_r \in \mathfrak{b}, C_r \in \mathfrak{c}$, and $A_r = B_r + C_r$. The filtration degree of A_r goes to infinity as $r \to \infty$, which means that $e^{A_r} \to 1$. We get that

$$e^A = e^{B_1} e^{A_2} e^{C_1} = e^{B_1} e^{B_2} e^{A_3} e^{C_2} e^{C_1} = \dots$$

Therefore,

$$e^A = e^B e^C,$$

where

$$e^B = e^{B_1} e^{B_2} e^{B_3} \cdots \in \exp \mathfrak{b} \text{ and } e^C = \dots e^{C_3} e^{C_2} e^{C_1} \in \mathfrak{c}$$

are convergent products in the topology induced by the filtration.

The formal differential operator $\exp(\nu\Delta)$ has filtration degree zero and does not lie in $\exp \mathfrak{g}$, but it acts upon it by conjugations. Since

$\chi \in \mathfrak{g}$, we have

$$A := e^{\nu\Delta}\chi e^{-\nu\Delta} \in \mathfrak{g}.$$

Now, decompose

$$e^{\nu\Delta}e^{\chi}e^{-\nu\Delta} = e^{A} = e^{B}e^{C}, \tag{3}$$

where $B \in \mathfrak{b}$ and $C \in \mathfrak{c}$. Denote by $\mathcal{F}_0$ the subspace of $\mathcal{F}$ of elements of filtration degree at least zero.

Example:

$$\nu^{-1}x^1x^2 \in \mathcal{F}_0.$$

Observe that if $f \in \mathcal{F}_0$, then $f(0) \in \mathbb{C}[[\nu]]$, i.e., $\delta : \mathcal{F}_0 \to \mathbb{C}[[\nu]]$.

Denote by $\mathcal{F}_+$ the space of formal jets at zero of the form

$$f = \sum_{r=0}^{\infty} \nu^r f_r.$$

Then $\mathcal{F}_+ \subset \mathcal{F}_0$ (a proper subspace). The functional K can be written as

$$K(f) = \langle \delta | (e^{\nu\Delta}e^{\chi}e^{-\nu\Delta}) | e^{\nu\Delta} f\rangle.$$

We want to show that K is defined on $\mathcal{F}_+$. The operator $e^{\nu\Delta}$ leaves $\mathcal{F}_+$ invariant. We have shown that $\mathcal{F}_+ \subset \mathcal{F}_0$ and that the element $e^{\nu\Delta}e^{\chi}e^{-\nu\Delta} \in \exp\mathfrak{g}$ leaves $\mathcal{F}_0$ invariant. Also, $\delta : \mathcal{F}_0 \to \mathbb{C}[[\nu]]$. It follows that $K : \mathcal{F}_+ \to \mathbb{C}[[\nu]]$, which means that

$$K : C^{\infty}(M)[[\nu]] \to \mathbb{C}[[\nu]].$$

Using (3), we can write

$$K(f) = \langle \delta | e^{\nu\Delta}e^{\chi} | f\rangle = \langle \delta | e^{B}e^{C}e^{\nu\Delta} | f\rangle = \langle \delta | e^{\nu\Delta + C} | f\rangle.$$

Thus, the functional K is given by the formal differential operator with constant coefficients $\exp(\nu\Delta + C)$.

According to the remark above, the operator $\nu\Delta + C$ is of the form

$$\nu A_2 + \nu^2 A_3 + \dots,$$

where the order of A_r is not greater than r and the principal symbol of A_2 is $-\frac{1}{2}h^{ij}\xi_i\xi_j$, where h^{ij} is nondegenerate. We claim that **any** operator of this format is equal to the operator K for an appropriate phase remainder χ.

3. The rest of the proof

Now we need to define the action of differential operators on functions **from the right**. This is achieved by passing to the transpose,

$$(x^i)^t = x^i \text{ and } \left(\frac{\partial}{\partial x^i}\right)^t = -\frac{\partial}{\partial x^i},$$

so that

$$\langle f|A := A^t|f\rangle.$$

For example,

$$\left\langle 1 \middle| x^1 \frac{\partial}{\partial x^1} = \left\langle x^1 \middle| \frac{\partial}{\partial x^1} = -1.\right.\right.$$

This action depends on the choice of coordinates.

Denote by $\mathfrak{r}$ the subalgebra of $\mathfrak{g}$ of the operators which annihilate constants from the right and by $\mathfrak{f}$ the subalgebra of $\mathfrak{g}$ of the multiplication operators. Then

$$\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{f}. \tag{4}$$

Given $A \in \mathfrak{g}$, we expand it according to (4) as follows,

$$A = (A - \langle 1|A) + \langle 1|A,$$

where we interpret the function $\langle 1|A$ as a multiplication operator.

We will need the following calculation:

$$e^{-\nu\Delta} x^k e^{\nu\Delta} = x^k + \nu h^{kl} \frac{\partial}{\partial x^l}.$$

Given a nondegenerate symmetric complex matrix h^{ij} with constant coefficients, introduce the function

$$\psi = \frac{1}{2} h_{ij} x^i x^j.$$

Now,

$$e^{\nu^{-1}\psi} e^{-\nu\Delta} x^k e^{\nu\Delta} e^{-\nu^{-1}\psi} = \nu h^{kl} \frac{\partial}{\partial x^l}.$$

Therefore, if $B \in \mathfrak{b}$, i.e., $B = x^i A_i$, for some $A_i \in \mathfrak{g}$, then

$$R := e^{\nu^{-1}\psi} e^{-\nu\Delta} B e^{\nu\Delta} e^{-\nu^{-1}\psi} \in \mathfrak{r}. \tag{5}$$

Assume that, as in (3),

$$e^{\nu\Delta} e^{X} e^{-\nu\Delta} = e^B e^C.$$

for some $B \in \mathfrak{b}$ and $C \in \mathfrak{c}$. Then

$$e^X = e^{-\nu\Delta} e^B e^{\nu\Delta} e^C,$$

from whence we get that

$$e^{\nu^{-1}\psi}e^{\chi}e^{-\nu^{-1}\psi} = \left(e^{\nu^{-1}\psi}e^{-\nu\Delta}e^{B}e^{\nu\Delta}e^{-\nu^{-1}\psi}\right)e^{\nu^{-1}\psi}e^{C}e^{-\nu^{-1}\psi}.$$

Therefore,

$$e^{\chi} = e^{R}e^{\nu^{-1}\psi}e^{C}e^{-\nu^{-1}\psi}, \tag{6}$$

where R is as in (5). Applying (6) to 1 from the right, we get

$$e^{\chi} = \langle 1|e^{\nu^{-1}\psi}e^{C}e^{-\nu^{-1}\psi}.$$

Thus, we have recovered χ from an arbitrary $C \in \mathfrak{c}$.