Chapter 2

Euclidean transformations

We begin with general definitions in n dimensions, but our main interest is of course in 2 and 3 dimensions.

Definition 2.1. A ***Euclidean transformation***, or ***isometry***, of $\mathbb{R}^n$ is any map $f : \mathbb{R}^n \to \mathbb{R}^n$ which preserves distance:

$$|f(\mathbf{x}) - f(\mathbf{y})| \;=\; |\mathbf{x} - \mathbf{y}|.$$

☆

Definition 2.2. (1) An $n \times n$ matrix is ***orthogonal*** if $A^T A = I$. The set of all orthogonal $n \times n$ matrices is denoted $\mathsf{O}(n)$. The set of ***special orthogonal*** matrices is

$$\mathsf{SO}(n) = \{A \in \mathsf{O}(n) \mid \det A = 1\}.$$

These are also called ***rotation matrices***.
(2) A ***translation*** on $\mathbb{R}^n$ is a map determined by a single vector. For $\mathbf{u} \in \mathbb{R}^n$, the corresponding translation is the map

$$T_{\mathbf{u}} : \mathbb{R}^n \to \mathbb{R}^n, \quad T_{\mathbf{u}}(\mathbf{x}) = \mathbf{x} + \mathbf{u}.$$

☆

Notice that since $A^T A = I$, it follows that $\det(AA^T) = 1$, and since $\det(A^T) = \det(A)$ we conclude that $\det(A)^2 = 1$ or $\det(A) = \pm 1$. In particular, every orthogonal matrix is invertible. In fact (see Problem 2.2) the set of orthogonal matrices forms a group.

The relation between these two definitions is given by the following two results, of which we only prove the first.

Proposition 2.3. *Every composition of an orthogonal matrix and a translation is a Euclidean transformation.*

Proof: Let $A \in \mathsf{O}(n)$ and $\mathbf{u} \in \mathbb{R}^n$, and write $f(\mathbf{x}) = A\mathbf{x} + \mathbf{u}$. Then

$$\begin{aligned} f(\mathbf{x}) - f(\mathbf{y}) &= (A\mathbf{x} + \mathbf{u}) - (A\mathbf{y} + \mathbf{u}) \\ &= A(\mathbf{x} - \mathbf{y}). \end{aligned}$$

To show f is an isometry we therefore need only show that $A(\mathbf{x}-\mathbf{y})$ has the same magnitude as $\mathbf{x}-\mathbf{y}$. Let $\mathbf{z}=\mathbf{x}-\mathbf{y}$; we want to show $|A\mathbf{z}| = |\mathbf{z}|$ for all $\mathbf{z}\in\mathbb{R}^n$. We use the easy fact that $\mathbf{u}\cdot\mathbf{v}=\mathbf{u}^T\mathbf{v}$ (both sides are equal to $\sum_{i=1}^n u_i v_i$). Then

$$\begin{aligned} |A\mathbf{z}|^2 &= (A\mathbf{z})^T(A\mathbf{z}) \\ &= \mathbf{z}^T A^T A\mathbf{z} \\ &= \mathbf{z}^T\mathbf{z} = |\mathbf{z}|^2. \end{aligned}$$

Here we used that $A^TA=I$. It follows that $|A\mathbf{z}| = |\mathbf{z}|$, as required. ❐

More difficult to prove is the converse of the proposition, which is a famous theorem from Geometry.

Theorem 2.4. *Any Euclidean transformation of $\mathbb{R}^n$ is equal to the composition of an orthogonal matrix and a translation.*

What the theorem says is that given any Euclidean transformation f, there is an orthogonal matrix $A\in\mathsf{O}(n)$ and a vector $\mathbf{u}\in\mathbb{R}^n$ such that

$$f(\mathbf{x}) = A\mathbf{x}+\mathbf{u}. \tag{2.1}$$

A proof in the case $n=2$ can be found in the book by Armstrong (Chapter 24).

Note that the proof above of the proposition shows a useful property of orthogonal transformations and hence of Euclidean transformations: not only do they preserve the magnitude of a vector, they preserve angles between vectors. This follows from the following,

$$(A\mathbf{x})\cdot(A\mathbf{y}) = \mathbf{x}^TA^TA\mathbf{y} = \mathbf{x}^T\mathbf{y} = \mathbf{x}\cdot\mathbf{y}.$$

Seitz symbol The fact that every Euclidean transformation can be written as $\mathbf{x}\mapsto A\mathbf{x}+\mathbf{u}$ allows a neat notation for Euclidean transformations. The Seitz symbol for the transformation $\mathbf{x}\mapsto A\mathbf{x}+\mathbf{u}$ is simply $(A\,|\,\mathbf{u})$. Thus,

$$(A\,|\,\mathbf{u})\cdot\mathbf{x} = A\mathbf{x}+\mathbf{u}. \tag{2.2}$$

For example, the identity transformation $\mathbf{x}\mapsto\mathbf{x}$ has Seitz symbol $(I\,|\,\mathbf{0})$. Notice that, in the Seitz symbol $(A\,|\,\mathbf{u})$, A is applied first and then the translation by $\mathbf{u}$.

It is easy to see from their definition that every Euclidean transformation is injective. One important consequence of the theorem above is that they are also surjective. This is seen as follows (using the Seitz symbol). Let $(A\,|\,\mathbf{u})$ be a Euclidean transformation, and let $\mathbf{y}\in\mathbb{R}^n$. We want to show there is an $\mathbf{x}\in\mathbb{R}^n$ for which $(A\,|\,\mathbf{u})\cdot\mathbf{x}=\mathbf{y}$, that is, such that $A\mathbf{x}+\mathbf{u}=\mathbf{y}$.

Multiplying both sides by A^{-1} $(=A^T)$, we get $A^{-1}\mathbf{y}=\mathbf{x}+A^{-1}\mathbf{u}$. Rearranging this gives $\mathbf{x}=A^{-1}\mathbf{y}-A^{-1}\mathbf{u}$, which shows $(A\,|\,\mathbf{u})$ is surjective. Notice that the inverse of $(A\,|\,\mathbf{u})$ is $\mathbf{y}\mapsto A^{-1}\mathbf{y}-A^{-1}\mathbf{u}$, or in Seitz symbols,

$$(A\,|\,\mathbf{u})^{-1} = (A^{-1}\,|\,-A^{-1}\mathbf{u}). \tag{2.3}$$

This gives an expression for the inverse of any Euclidean transformation.

Proposition 2.5. *The set of Euclidean transformations forms a group under composition, denoted* $\mathsf{E}(n)$.

Proof: (1) Clearly the identity transformation is an isometry.
(2) We showed above that the inverse of an isometry is also an isometry (since all Seitz symbols represent isometries).
(3) Now we want to show that the composite of two isometries is another isometry. But this is clear: if f, g are two isometries, then

$$|f(\mathbf{x}) - f(\mathbf{y})| = |\mathbf{x} - \mathbf{y}|, \quad \text{and} \quad |g(\mathbf{u}) - g(\mathbf{v})| = |\mathbf{u} - \mathbf{v}|,$$

then putting $\mathbf{u} = f(\mathbf{x})$ and $\mathbf{v} = g(\mathbf{y})$ shows that

$$|g(f(\mathbf{x})) - g(f(\mathbf{y}))| = |f(\mathbf{x}) - f(\mathbf{y})| = |\mathbf{x} - \mathbf{y}|,$$

showing that $g \circ f$ is also an isometry. ❐

It is useful to derive the formula for the product (composite) of two isometries using the Seitz symbols. Consider $(A \mid \mathbf{u})$ and $(B \mid \mathbf{v})$ and calculate their composite:

$$\begin{aligned} (A \mid \mathbf{u})(B \mid \mathbf{v}) \cdot \mathbf{x} &= (A \mid \mathbf{u}) \cdot ((B \mid \mathbf{v}) \cdot \mathbf{x}) \\ &= (A \mid \mathbf{u}) \cdot (B\mathbf{x} + \mathbf{v}) \\ &= A(B\mathbf{x} + \mathbf{v}) + \mathbf{u} \\ &= AB\,\mathbf{x} + (A\mathbf{v} + \mathbf{u}). \end{aligned} \tag{2.4}$$

This then is the Euclidean transformation consisting of multiplying by the matrix AB (which is necessarily orthogonal since A and B are) and then translating by $\mathbf{u} + A\mathbf{v}$, and thus

$$(A \mid \mathbf{u})(B \mid \mathbf{v}) = (AB \mid \mathbf{u} + A\mathbf{v}). \tag{2.5}$$

Before proceeding with studying Euclidean transformations in the plane, we point out one property of the special orthogonal group.

Proposition 2.6. $\mathsf{SO}(n)$ *is a normal subgroup of* $\mathsf{O}(n)$.

Proof: We use the determinant

$$\begin{aligned} \det : \mathsf{O}(n) &\to \mathbb{Z}_2 := \{\pm 1\} \\ A &\mapsto \det(A). \end{aligned}$$

From linear algebra we know this is a homomorphism: $\det(AB) = \det(A)\det(B)$. Here $\mathbb{Z}_2 = \{1, -1\}$ with multiplication as binary operation.[1] The kernel of this homomorphism is,

$$\ker(\det) = \{A \in \mathsf{O}(n) \mid \det(A) = 1\} = \mathsf{SO}(n).$$

[1] There is just one group of order 2 (up to isomorphism). It is usually denoted $\mathbb{Z}_2$, whether it is written additively ($\mathbb{Z}_2 = \{0, 1\}$ with addition mod 2) or multiplicatively ($\mathbb{Z}_2 = \{1, -1\}$ with multiplication), or abstractly ($\mathbb{Z}_2 = \{e, a\}$ with $a^2 = e$). See the Appendix for details.

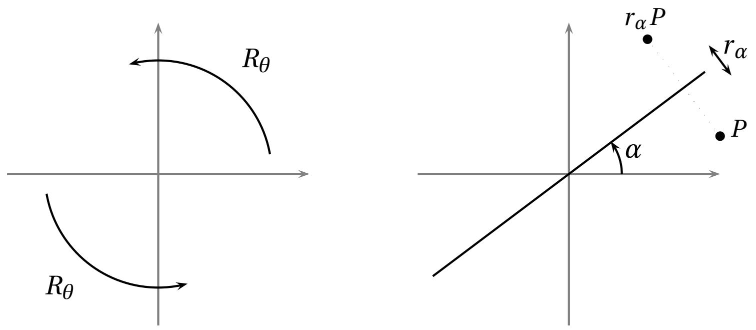


FIGURE 2.1: Rotations and reflections in the plane

It follows that $\mathsf{SO}(n)$ is a normal subgroup of $\mathsf{O}(n)$, because the kernel of any homomorphism is a normal subgroup (see the appendix). ❒

Remark 2.7. While we have that *as sets*, $\mathsf{E}(n) = \mathsf{O}(n) \times \mathbb{R}^n$ (since elements are ordered pairs $(A, \mathbf{v})$), *as a group* $\mathsf{E}(n)$ is not the Cartesian product. In the Cartesian product, the group operation is $(A, \mathbf{u})(B, \mathbf{v}) = (AB, \mathbf{u} + \mathbf{v})$ which is different from (2.5).

2.1 Rotations and reflections in the plane

Orthogonal transformations in the plane are particularly easy to understand: they are either rotations about the origin or reflections in a line through the origin. The rotations form the normal subgroup $\mathsf{SO}(2)$ of $\mathsf{O}(2)$. First we will discuss in some details what these rotations and reflections are and how they combine, and then show at the end of this section that these are the only possibilities for an orthogonal transformation.

Rotations We denote by R_θ the rotation about the origin through an angle θ (anticlockwise). In order to find the matrix associated to R_θ, pick the standard basis $\{e_1 = (1,0)^T, e_2 = (0,1)^T\}$ of $\mathbb{R}^2$. Applying R_θ to e_1 will give you the first column of A, and applying A to e_2 will give you the second column. Applying elementary geometry/trigonometry, the vectors are $R_\theta e_1 = (\cos\theta, \sin\theta)^T$ and $R_\theta e_2 = (-\sin\theta, \cos\theta)^T$. We deduce that

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{2.6}$$

It can be checked that $R_\theta^T R_\theta = \mathrm{Id}$. In addition, $\det(R_\theta) = \cos^2\theta + \sin^2\theta = 1$; that is, $R_\theta \in \mathsf{SO}(2)$.

Reflections Let r_α be the reflection in the line subtending an angle α with the x-axis (anticlockwise). As before, compute $r_\alpha e_1 = (\cos 2\alpha, \sin 2\alpha)$ and $r_\alpha e_2 = (\sin 2\alpha, -\cos 2\alpha)$. Thus,

$$r_\alpha = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}. \tag{2.7}$$

Note that $\det(r_\alpha) = -\cos^2 2\alpha - \sin^2 2\alpha = -1$. Since, in addition, $r_\alpha^T r_\alpha = I$, it follows that $r_\alpha \in \mathsf{O}(2) \setminus \mathsf{SO}(2)$.

It is useful to record the geometric results of applying two of these operations in succession:

$$\begin{cases} R_\theta R_\phi = R_{\theta+\phi}, & R_\theta r_\alpha = r_{\alpha+\theta/2}, \\ r_\alpha R_\theta = r_{\alpha-\theta/2}, & r_\beta r_\alpha = R_{2(\beta-\alpha)}. \end{cases} \tag{2.8}$$

These can be readily checked by matrix multiplication and some trigonometric identities. (You can also show this by drawing some diagrams in the plane, and working in polar coordinates.)

Proposition 2.8. *Every element of* $\mathsf{O}(2)$ *is either a rotation or a reflection, as described above.*

Proof: Lat $A \in \mathsf{O}(2)$. Since orthogonal transformations preserve the length of vectors, the unit vector $\mathbf{e}_1 = (1,0)$ is sent to a unit vector, so is of the form $(\cos\theta, \sin\theta)$ for some θ. This will be the first column of the matrix A. Now consider the second column $A\mathbf{e}_2$. Since A is orthogonal, we have $\mathbf{e}_2 \cdot \mathbf{e}_1 = 0 \Longrightarrow A\mathbf{e}_2 \cdot A\mathbf{e}_1 = 0$. Thus $A\mathbf{e}_2$ is a unit vector orthogonal to $(\cos\theta, \sin\theta)$. There are two such possible vectors subtending an angle of $\theta + \pi/2$ and $\theta - \pi/2$ with the positive x-axis. In the two cases the matrices of A are

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \text{and} \quad A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix},$$

respectively. The first of these is the rotation R_θ and the second is the reflection $r_{\theta/2}$. ❒

2.2 Finite subgroups of $\mathsf{O}(2)$.

First we describe two classes of finite subgroup of $\mathsf{O}(2)$, and below (Theorem 2.9) we show these are the only possibilities.

(1). C_n is a subgroup of $\mathsf{SO}(2)$ and is defined to be the set of all rotations about the origin through multiples of $2\pi/n$. Thus,

$$\mathsf{C}_n := \{I, R_{2\pi/n}, R_{4\pi/n}, R_{6\pi/n} \dots, R_{2(n-1)\pi/n}\}.$$

Here I is the 2×2 identity matrix. By setting $R := R_{2\pi/n}$, one can also write it as

$$\mathsf{C}_n = \{I, R, R^2, R^3, \dots, R^{n-1}\}.$$

It is therefore isomorphic to the cyclic group of order n. This is the symmetry group of shapes like those in Fig. 2.2.

FIGURE 2.2: Two figures with cyclic symmetry

(2). D_n, called the *dihedral group of order* $2n$, is the symmetry group of the regular n-gon, at least for $n \geq 3$. Contrary to C_n, the dihedral group D_n is not a subgroup of $\mathsf{SO}(2)$. In fact, it contains C_n and, in addition, n reflections. For example, the dihedral group order 6 is given by

$$\mathsf{D}_3 = \{I, R_{2\pi/3}, R_{4\pi/3}, r_0, r_{\pi/3}, r_{2\pi/3}\},$$

as we saw in Chapter 1. In general,

$$\mathsf{D}_n = \mathsf{C}_n \cup r_0\mathsf{C}_n = \left\{I, R, R^2, R^3, \dots, R^{n-1}, r_0, r_{\pi/n}, r_{2\pi/n}, \dots, r_{(n-1)\pi/n}\right\}$$

where $R = R_{2\pi/n}$. Notice that $r_{j\pi/n} = r_0 R_{-2\pi j/n} \in r_0\mathsf{C}_n$. Notice also that C_n and $r_0\mathsf{C}_n$ are two (left) cosets of D_n.

There are two special cases: $\mathsf{D}_1 = \{I, r_0\}$, which is just a single reflectional symmetry, and $\mathsf{D}_2 = \{I, R_\pi, r_0, r_{\pi/2}\}$ which is the symmetry of a rectangle (see Fig. 2.3 below)

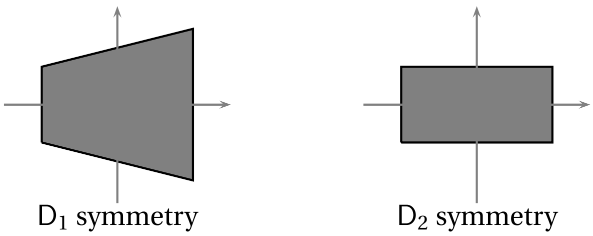


FIGURE 2.3: Figures with D_1 and D_2 symmetry

The dihedral group D_n is isomorphic to the abstract dihedral group $\mathrm{Dih}(2n)$ (see the Appendix).

Remarks (1) Note that $\mathsf{D}_1 = \{I, r_0\}$ and $\mathsf{C}_2 = \{I, R_\pi\}$ are isomorphic as abstract groups, since all groups of order 2 are isomorphic. Although D_1 and C_2 are isomorphic, they are not conjugate subgroups of $\mathsf{O}(2)$.

(2) Consider the restriction of the homomorphism $\det : \mathsf{O}(2) \to \mathbb{Z}_2$ to D_n; that is, $\det|_{\mathsf{D}_n} : \mathsf{D}_n \to \mathbb{Z}_2$. The kernel consists of the $A \in \mathsf{D}_n$ for which $\det(A) = 1$. Therefore, $\ker(\det) = \mathsf{C}_n$. In particular, C_n is a normal subgroup of D_n (ie, $\mathsf{C}_n \lhd \mathsf{D}_n$).

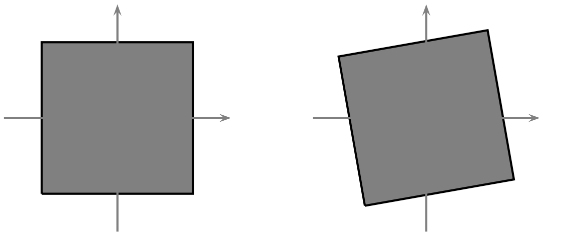


FIGURE 2.4

Reminder Let G be a group and $H \le G$. Then, for any $g \in G$,

$$gHg^{-1} = \{ghg^{-1} \mid h \in H\}$$

is a subgroup ***conjugate*** to H. For each $g \in G$, conjugation by g is an isomorphism of G with itself (this is the action denoted μ in Chapter 1 (Sec. 1.3)).

What does this conjugacy mean in $O(2)$? First we conjugate R_θ by R_φ:

$$R_\varphi R_\theta R_\varphi^{-1} = R_{\varphi+\theta-\varphi} = R_\theta.$$

The conjugation does nothing, which is due to the fact that $SO(2)$ is Abelian. Now conjugate r_0 by R_ϕ:

$$R_\phi r_0 R_\phi^{-1} = r_{(0+\phi/2)+\phi/2} = r_\phi.$$

This tells us that r_0 and r_ϕ are conjugate in $O(2)$, for any ϕ, and thus in $O(2)$, all reflections are conjugate.

Taking this further, one finds that different 'instances' of the dihedral group are conjugate, by which one means that $D_n = C_n \cup r_0 C_n$ and $D'_n = C_n \cup r_\phi C_n$ are conjugate subgroups of $O(2)$. Indeed, since C_n is Abelian,

$$\begin{aligned} R_\phi D_n R_\phi^{-1} &= R_\phi C_n R_\phi^{-1} \cup R_\phi r_0 C_n R_\phi^{-1} \\ &= C_n \cup R_\phi r_0 R_\phi^{-1} C_n \\ &= C_n \cup r_\phi C_n \\ &= D'_n. \end{aligned}$$

In the context of group actions, the main importance of conjugacy is in Proposition 1.6: points in the same orbit have conjugate stabilizers.

We now classify *all* finite subgroups of $O(2)$.

Theorem 2.9. *Let G be a finite subgroup of $O(2)$. Then, either $G = C_n$ for some $n \ge 1$, or G is conjugate to D_n for some $n \ge 1$.*

Proof: We consider two cases: $G \le SO(2)$ and $G \not\le SO(2)$.

First suppose G is a subgroup of $\mathsf{SO}(2)$ of order n. We want to show that $G = \mathsf{C}_n$. Since $G \leq \mathsf{SO}(2)$, $G = \{\mathbf{I}, R_{\theta_1}, R_{\theta_2}, \ldots, R_{\theta_{n-1}}\}$ with $\theta_1, \theta_2, \ldots, \theta_{n-1} \in (0, 2\pi)$. Let $\theta_0 := \min\{\theta_1, \theta_2, \ldots, \theta_{n-1}\}$. We claim that each θ_j is an integer multiple of θ_0. To see this, suppose to the contrary that, say, θ_j is not an integer multiple of θ_0. Then there is an integer m such that $\theta_j \in (m\theta_0, (m+1)\theta_0)$. Then $|\theta_j - m\theta_0| < \theta_0$. It follows that $R_{\theta_j}(R_{\theta_0})^{-m} = R_{\theta_j - m\theta_0}$ which is a rotation through a positive angle strictly less than θ_0 despite being an element of G. This is a contradiction, so proving the claim.

It follows that every element of G is of the form $R^m = R_{m\theta_0}$ where $R = R_{\theta_0}$. Since G has order n, it follows that

$$G = \{I, R, R^2, \ldots, R^{n-1}\}$$

whence $R^n = I$ and $\theta_0 = 2\pi/n$. Thus $G = \mathsf{C}_n$.

Now suppose $G \not\leq \mathsf{SO}(2)$. Let $G_0 = G \cap \mathsf{SO}(2)$ (necessarily a subgroup) and let $n = |G_0|$. Then by part (1), there is an integer n such that $G_0 = C_n$.

Consider the determinant homomorphism again, but now just on G, so $\det : G \to \mathbb{Z}_2$. Its kernel is G_0. It follows from the first isomorphism theorem that $G/\mathsf{C}_n \simeq \mathbb{Z}_2$. Consequently, G has just two (left) cosets of $G_0 = \mathsf{C}_n$. Let $r \in G \setminus G_0$ (so $\det r = -1$ meaning that r is a reflection). Then

$$G = \mathsf{C}_n \cup r\mathsf{C}_n,$$

which means that G is conjugate to D_n (by the discussion above). ❐

2.3 General Euclidean transformations of the plane.

As we have seen, general Euclidean transformations in the plane are obtained by combining orthogonal transformations and translations. In particular, as stated in Definition 2.2, the translation by $\mathbf{v} \in \mathbb{R}^2$ is the map

$$\begin{aligned} T_{\mathbf{v}} : \mathbb{R}^2 &\to \mathbb{R}^2 \\ \mathbf{x} &\mapsto \mathbf{x} + \mathbf{v}. \end{aligned}$$

There is of course one of these translations for each $\mathbf{v} \in \mathbb{R}^2$, and the set of all translations is therefore $\mathbb{R}^2$; it forms a group whose binary operator is simply vector addition (because the composition of two translations is another translation: $T_{\mathbf{u}} \circ T_{\mathbf{v}} = T_{\mathbf{u}+\mathbf{v}}$; see Problem 2.3).

Recall that a Euclidean transformation of the plane can be written using the Seitz symbol $(A \,|\, \mathbf{v})$, see Equation (2.2). We want to understand what sort of transformation is represented by a given Seitz symbol.

FIGURE 2.5: Glide reflection (along a horizontal line)

Particular cases of Euclidean transformation:

• Let $(I \mid \mathbf{v}) \in \mathsf{E}(2)$ where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity matrix in $\mathbb{R}^2$. Then,

$$(I \mid \mathbf{v}) \cdot \mathbf{x} = I\mathbf{x} + \mathbf{v} = \mathbf{x} + \mathbf{v} = T_{\mathbf{v}}(\mathbf{x}).$$

So $(I \mid \mathbf{v}) = T_{\mathbf{v}}$ is just the translation by $\mathbf{v}$.

• Let $(R_\theta \mid \mathbf{v}) \in \mathsf{E}(2)$ with $\theta \neq 0$. It turns out that this is a rotation, but about some point other than the origin, called the ***centre of rotation***. Where is that centre of rotation? Call it $\mathbf{c}$. Now the centre of rotation is the only point not moved by the rotation, so is the unique point satisfying $(R_\theta \mid \mathbf{v}) \cdot \mathbf{c} = \mathbf{c}$. That is, $R_\theta \mathbf{c} + \mathbf{v} = \mathbf{c}$. This implies

$$\mathbf{v} = \mathbf{c} - R_\theta \mathbf{c} = (I - R_\theta)\mathbf{c}. \tag{2.9}$$

Now, the matrix

$$I - R_\theta = \begin{pmatrix} 1 - \cos\theta & \sin\theta \\ -\sin\theta & 1 - \cos\theta \end{pmatrix}$$

is invertible since $\det(I - R_\theta) = 2 - 2\cos\theta > 0$ as $\theta \neq 0$. Therefore, we can rewrite (2.9) as $\mathbf{c} = (I - R_\theta)^{-1}\mathbf{v}$. It follows that,

the Seitz symbol $(R_\theta \mid \mathbf{v})$ corresponds to a rotation through an angle θ about the point $\mathbf{c} = (I - R_\theta)^{-1}\mathbf{v}$.

• Finally, consider $(r_\alpha \mid \mathbf{v}) \in \mathsf{E}(2)$, where r_α is a reflection. First a definition:

Definition 2.10. A ***glide reflection*** is a transformation of the plane consisting of a reflection followed by a translation parallel to the line of reflection (see Fig. 2.5). ✩

Given the transformation $(r_\alpha \mid \mathbf{v})$, let ℓ be the line of reflection of r_α, and write

$$\mathbf{v} = \mathbf{v}_\perp + \mathbf{v}_\parallel$$

where $\mathbf{v}_\perp$ is perpendicular to ℓ and $\mathbf{v}_\parallel$ is parallel to (lies in) ℓ. It is easy to check that

$$(r_\alpha \mid \mathbf{v}) = (I \mid \mathbf{v}_\parallel)\,(r_\alpha \mid \mathbf{v}_\perp).$$

Now I claim that $(r_\alpha \mid \mathbf{v}_\perp)$ is the reflection in the line parallel to ℓ shifted (translated) by $\frac{1}{2}\mathbf{v}_\perp$. Let ℓ_1 be that line. Then it is parametrized by $\frac{1}{2}\mathbf{v}_\perp + t\mathbf{u}$, where $\mathbf{u}$ is a unit vector parallel to ℓ. Consider the effect of $(r_\alpha \mid \mathbf{v}_\perp)$ on points of ℓ_1:

$$\begin{aligned} (r_\alpha \mid \mathbf{v}_\perp) \cdot (\tfrac{1}{2}\mathbf{v}_\perp + t\mathbf{u}) &= r_\alpha(\tfrac{1}{2}\mathbf{v}_\perp + t\mathbf{u}) + \mathbf{v}_\perp \\ &= (-\tfrac{1}{2}\mathbf{v}_\perp + t\mathbf{u}) + \mathbf{v}_\perp \\ &= \tfrac{1}{2}\mathbf{v}_\perp + t\mathbf{u}. \end{aligned}$$

Here we used the fact that since $\mathbf{v}_\perp$ is perpendicular to ℓ, the reflection maps it to its opposite: $r_\alpha \mathbf{v}_\perp = -\mathbf{v}_\perp$, and since $\mathbf{u}$ is parallel to ℓ, the reflection r_α leaves it unchanged. Consequently, the points on the line ℓ_1 are fixed by $(r_\alpha \mid \mathbf{v}_\perp)$, and so it must be the reflection in that line.

Finally, $(I \mid \mathbf{v}_\perp)$ is a translation parallel to ℓ and hence parallel to ℓ_1, showing that $(r_\alpha \mid \mathbf{v})$ is in general a glide reflection, and it is a simple reflection if the glide part $\mathbf{v}_\parallel = \mathbf{0}$.

We have thus proved the following:

Proposition 2.11. *The geometric effect of* $(A \mid \mathbf{v})$ *depends on* A *as follows:*

(1). $(I \mid \mathbf{v}) = T_\mathbf{v}$*, translation by* $\mathbf{v}$*,*

(2). $(R_\theta \mid \mathbf{v})$ *with* $\theta \neq 0$ *is the rotation through* θ *about* $\mathbf{c} = (I - R_\theta)^{-1}\mathbf{v}$*, and*

(3). $(r_\alpha \mid \mathbf{v})$ *is a glide reflection as described above (with* $\mathbf{v} = \mathbf{v}_\perp + \mathbf{v}_\parallel$*), and is a pure reflection if* $\mathbf{v}_\parallel = \mathbf{0}$*.*

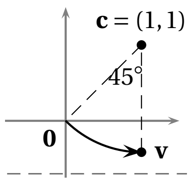
So far we have discussed how a given $(A \mid \mathbf{v})$ acts on the plane (rotation, glide-reflection etc). It is also useful to ask the converse question: if we are given a Euclidean transformation, how do we write it as $(A \mid \mathbf{v})$? This is in fact easier:
• Firstly, $\mathbf{v}$ is found simply as the image of $\mathbf{0}$ under the transformation, because

$$(A \mid \mathbf{v}) \cdot \mathbf{0} = A\mathbf{0} + \mathbf{v} = \mathbf{v}.$$

• If the Euclidean transformation is a rotation about some point $\mathbf{c}$ through an angle θ, then $A = R_\theta$, and
• if it is a (glide) reflection in a line ℓ, then A is the reflection in the line through the origin parallel to ℓ.

Example 2.12. Find the Seitz symbol for the rotation through $\pi/4$ about the point $(1, 1)$.

Solution: first find where the origin ends up under the rotation. It is the point a distance $\sqrt{2}$ from $(1, 1)$ in the vertical downwards direction, which is $(1, 1-\sqrt{2})$. Thus $\mathbf{v} = (1, 1-\sqrt{2})^T$, and the Seitz symbol for the transformation is $(R_{\pi/4} \mid \mathbf{v})$.



2.4 Classification of triangles using symmetry

These ideas can be applied to classification of any type of geometric figure in the plane, and we illustrate how this works with the simplest case: triangles.

A triangle is determined by its three vertices, which we assume to be 3 distinct points in the plane. Note that this allows the three points to be collinear, where the triangle has zero area, giving a rather degenerate triangle! We call the triangles with non-zero area *proper triangles*. In Euclidean geometry, two triangles are said to be ***congruent*** if there is a Euclidean transformation taking one to the other, or in other words, taking the vertices of one to the vertices of the other. You may remember theorems such as, two triangles are congruent if and only if the three sides of one triangle have the same lengths as the three sides of the other triangle.

We can rephrase congruence in terms of group actions as follows. Let $\mathcal{T}$ denote the set of all triangles in the plane:

$$\mathcal{T} = \left\{ \{A, B, C\} \subset \mathbb{R}^2 \mid A, B, C \text{ are distinct} \right\}.$$

The Euclidean group $\mathsf{E}(2)$ acts in a natural way on $\mathcal{T}$ by

$$g \cdot \{A, B, C\} = \{g(A),\ g(B),\ g(C)\}.$$

The orbit $\mathsf{E}(2)\cdot\{A, B, C\}$ is then, by definition, the set of all triangles congruent to ABC.

Question: which triangles have non-trivial symmetry? The symmetry of a triangle corresponds to the set of Euclidean transformations preserving it. That is, it is the stabilizer of the triangle. Our question therefore becomes, which triangles have non-trivial stabilizer?

Consider a triangle ABC (or $\{A, B, C\}$), and let $g \in \mathsf{E}(2)$. There are several ways in which one can have $g\cdot\{A, B, C\} = \{A, B, C\}$, depending on how g permutes the vertices (recall that there are 3 types of permutation of 3 objects: the identity, a transposition and a 3-cycle):

(1). g fixes all three: $g(A) = A$, $g(B) = B$ and $g(C) = C$

(2). g swaps two: for example $g(A) = A$, $g(B) = C$ and $g(C) = B$, or

(3). g acts as a 3-cycle, say $g(A) = B$, $g(B) = C$ and $g(C) = A$.

Case (1): if g fixes all three points, then assuming g is not the identity, g must be a reflection and the three points must lie on the line of reflection (recall that a rotation only fixes one point, while a glide-reflection fixes none, and we are assuming the three vertices are distinct). That is, it is a degenerate triangle (as described above).

Case (2): g fixes the point A and swaps B and C. It follows that the lengths of AB and AC are equal, so the triangle is isosceles (including equilateral as a special case of isosceles). Since g only fixes one of the vertices, there are two possibilities aside from the identity, namely a reflection in a line through A, and a rotation about A. If g is a reflection, then this would be the traditional picture of a line of reflection of an isosceles triangle (though possibly a degenerate, flat, triangle). On the other hand, if g is a rotation with centre at A, and exchanges the two points B and C, it must be a rotation by π and the vertices are collinear, with A at the mid-point of the other two.

Case (3): g cycles the 3 vertices. It follows that the three sides of the triangle are all equal (since for example, $g(AB) = BC$), and so the triangle is equilateral, and g is a rotation by $\pm 2\pi/3$ about its centre.

Triangle type	Symmetry
scalene proper	$\mathbb{1}$
isosceles proper	D_1
equilateral	D_3
scalene degenerate	D_1
isosceles degenerate	$\mathsf{D}_2 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$

TABLE 2.1: Symmetry types of triangles

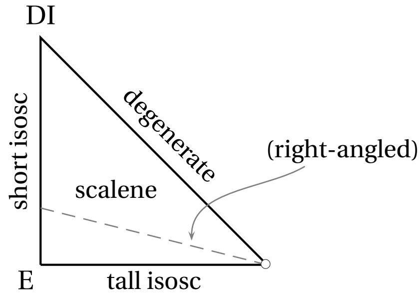


FIGURE 2.6: A representation of the classification of triangles

The remaining possibility is that the triangle is a proper scalene triangle (scalene means having three sides of different lengths). These triangles have trivial stabilizer, so 'trivial symmetry'.

See Table 2.1 for a complete summary. (Note that an equilateral triangle cannot be degenerate without having all three points coinciding, which is not allowed, so every equilateral triangle is a proper triangle. On the other hand, an isosceles triangle can be degenerate (marked DI on the diagram) by having one vertex at the mid-point of the other two). See also Fig. 2.6, where the point E represents equilateral triangles, the dashed line represents right-angled triangles (which in general are scalene, so from the symmetry standpoint are not special). An isosceles triangle is short △ or tall Δ depending on whether the equal angles are less than or greater than the third angle, respectively.

2.5 Problems

2.1 Deduce from Definition 2.1 that any Euclidean transformation of $\mathbb{R}^n$ is injective.

2.2 Show the set $\mathsf{O}(n)$ of orthogonal $n \times n$ matrices forms a subgroup of $\mathsf{GL}(n)$. (Recall (see Appendix) that $\mathsf{GL}(n)$ is the group of all invertible $n \times n$ matrices.)

2.3 Let V be a vector space, which we know is a group under vector addition. For each $\mathbf{v} \in V$ there is the translation $T_{\mathbf{v}}(\mathbf{x}) = \mathbf{x} + \mathbf{v}$. Show that this defines an action T of V on itself, which coincides with the left translation defined in Section 1.3.

2.4 Find the eigenvalues of R_θ and r_α. How are the eigenvectors of r_α related to the line of reflection?

2.5 Verify the identities in Eq. (2.8). Let r_α be a reflection, and find the angle of rotation of $r_\alpha R_\theta r_\alpha^{-1}$. Deduce that, for each n, C_n is a normal subgroup of $\mathsf{O}(2)$.

2.6 Show that the map $p : \mathsf{E}(n) \to \mathsf{O}(n)$ given by $p(A \mid \mathbf{v}) = A$ is a homomorphism, and deduce that the set of translations in $\mathsf{E}(n)$ is a normal subgroup.

2.7 In contrast to problem 2.6, show that the map $p' : \mathsf{E}(n) \to \mathbb{R}^n$ defined by $p'(A \mid \mathbf{v}) = \mathbf{v}$ is *not* a homomorphism.

2.8 Describe the transformation with each of the following Seitz symbols:

$$\text{(i) } (I \mid \mathbf{v}), \quad \text{(ii) } (R_\pi \mid \mathbf{v}), \quad \text{(iii) } (r_{\pi/4} \mid \mathbf{v}), \quad \text{(iv) } (r_0 \mid \mathbf{v}).$$

where $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathbb{R}^2$.

2.9 Let $(A \mid \mathbf{v}) \in \mathsf{E}(2)$. Show that the inverse transformation is given by

$$(A \mid \mathbf{v})^{-1} = (A^{-1} \mid -A^{-1}\mathbf{v}).$$

2.10 Write the Seitz symbol for each of the following Euclidean transformations:
(i) the rotation through $\pi/2$ about the point (1,1);
(ii) the reflection in the line $y = x + 1$;
(iii) the glide reflection consisting of the reflection in (ii) followed by a translation by (1,1) (which is parallel to the line of reflection).

2.11 For $\mathbf{v} \in \mathbb{R}^n$, as usual let $T_\mathbf{v}$ denote the translation $\mathbf{x} \mapsto \mathbf{x} + \mathbf{v}$. Let $A \in \mathsf{O}(n)$. Show that conjugation of a translation by A is another translation, and in particular,

$$A T_\mathbf{v} A^{-1} = T_{A\mathbf{v}}.$$

2.12 If r_L is the reflection in the line L, and $g \in \mathsf{E}(2)$ is a Euclidean transformation, show that $g\, r_L\, g^{-1}$ is the reflection in the line $g(L)$. [Hint: use Problem 2.15.]

2.13 Given any transformation $(A \mid \mathbf{v}) \in \mathsf{E}(2)$, define the 3×3 invertible matrix, written in block form,

$$\psi((A \mid \mathbf{v})) = \left(\begin{array}{c|c} A & \mathbf{v} \\ \hline 0 & 1 \end{array}\right) \in \mathsf{GL}_3(\mathbb{R}).$$

(i) Let $g = (r_{\pi/4} \mid \mathbf{u}) \in \mathsf{E}(2)$, where $\mathbf{u} = (1,1)^T$. Write down $\psi(g)$ and calculate $\psi(g)^2$ and compare with $\psi(g^2)$.
(ii) Show that the map $\psi : \mathsf{E}(2) \to \mathsf{GL}_3(\mathbb{R})$ is a homomorphism.

2.14 (i) A rotation in the plane through an angle π is called a ***half-turn***. By using the homomorphism $\mathsf{E}(2) \to \mathsf{O}(2)$ taking $(A \mid \mathbf{v})$ to A, show that the composite of two half-turns is a translation.

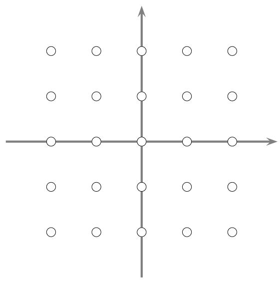
(ii) For $\mathbf{c} \in \mathbb{R}^2$, denote by $h(\mathbf{c})$ the half-turn with centre $\mathbf{c}$. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$. Express the translation $h(\mathbf{b})h(\mathbf{a})$ explicitly in terms of $\mathbf{a}$ and $\mathbf{b}$.

2.15 Complete the following table, showing the geometric type (point/line ...) of the set of points fixed under each type of Euclidean transformation:

Type	fixed point set
Identity	
Translation	
Rotation	
Reflection	
Glide-reflection	

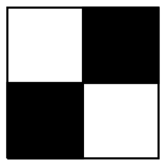
Note that the different geometric types of fixed point set almost distinguishes between the 5 types of transformation.

2.16 Consider the subset $L \subset \mathbb{R}^2$ shown below consisting of all points in the plane both of whose coordinates are integers (that is, $L = \mathbb{Z}^2$). Which rotations and reflections in $\mathsf{O}(2)$ preserve L (map L to itself)?



2.17† Describe all rotations (centre + angle) and lines of reflection in the plane which preserve the set L defined in the previous problem.

2.18 (a) Consider a 2×2 chessboard with 2 black squares and 2 white as shown. Which reflections and rotations preserve the chessboard (with its colouring: i.e. sending black squares to black and white to white). You should use the notation introduced on the sheet on symmetries of the square.



(b) How do we know in advance (i.e., without finding them) that these transformations will form a subgroup of D_4? [Hint: Consider the action of D_4 on the set $\{C, C'\}$ where C is the chessboard shown, and C' the same board but with the colours swapped.]

2.19 Continuing the previous question, now consider the action of $\mathbb{Z}_2 = \{e, c\}$ (with $c^2 = e$) on the chessboard where $c \in \mathbb{Z}_2$ acts by changing the colour of every square: black to white and white to black. Combine this with the action of D_4 by rotations and reflections to form an action of the product $G = \mathsf{D}_4 \times \mathbb{Z}_2$ (eg $(R_{\pi/2}, c)$ acts by rotating by $\pi/2$ and then changing the colours). What is the symmetry group of the chessboard, as a subgroup of G? (In other words, which elements of G 'preserve' the chessboard with its colouring?)

2.20 Find all homomorphisms from $\mathbb{Z}_2$ to each of $\mathbb{Z}_3$ and $\mathbb{Z}_4$.

2.21 † Find the possible symmetry types of sets of 4 distinct points in the plane.

2.22 † Find the possible symmetry types of quadrilaterals in the plane. (A quadrilateral consists of 4 points *in order*: that is, in $ABCD$ there is no edge joining A to C nor B to D. (There should be a different answer than in the previous question.))