1. Fundamental theorem of finite Abelian groups. Constructive approach.

Assume that A is an additive Abelian group of order n. First we represent n as a product of prime powers, $n = p_1^{k_1} \dots p_s^{k_s}$, where p_i are distinct primes and $k_i > 0$. For each $i, 1 \leq i \leq s$, denote by A_i the subgroup of A comprised of the elements of A whose order is a power of p_i:

$$A_i = \{x \in A | p_i^k x = 0 \text{ for some } k\}.$$

Then

$$A = \bigoplus_{i=1}^{s} A_i$$

is an internal direct sum and the order of A_i is $p_i^{k_i}$. Then we decompose each subgroup A_i into an internal direct sum of cyclic subgroups according to the following procedure.

Assume that A is an additive Abelian group of order p^N, where p is a prime and $N > 0$. Denote by A_i the subgroup of A comprised of the elements of A whose order divides p^i,

$$A_i = \{x \in A | p^i x = 0\}.$$

Let $k \geq 0$ be such that p^{k+1} is the largest order of an element of A. Then

$$\{0\} = A_0 \subset A_1 \subset \ldots \subset A_k \subset A_{k+1} = A.$$

Lemma 1. *For $0 \leq j \leq k - i + 1$ we have the inclusion $p^j A_{i+j} \subset A_i$.*

Proof. Let $x \in p^j A_{i+j}$. Then there exists $y \in A_{i+j}$ such that $x = p^j y$ and $p^{i+j} y = 0$. Now, $p^i x = p^i(p^j y) = p^{i+j} y = 0$, which means that $x \in A_i$. $\square$

For each i satisfying $1 \leq i \leq k + 1$, consider the factor group $B_i = A_i/A_{i-1}$. The order of each nonzero element of B_i is p. Therefore, B_i is a vector space over the field $\mathbb{Z}/p\mathbb{Z}$. Consider the following inductive construction. Choose a set $\lambda_{k+1} \subset A_{k+1} = A$ such that the set of cosets $\lambda_{k+1} + A_k$ is a basis for B_{k+1}. Then the set $p\lambda_{k+1} + A_{k-1} \subset B_k$ is linearly independent in B_k. Complete the set $p\lambda_{k+1} \subset A_k$ by a set $\lambda_k \subset A_k$ so that $\lambda_k \cup (p\lambda_{k+1}) + A_{k-1}$ be a basis in B_k. Then the set $(p\lambda_k) \cup (p^2\lambda_{k+1}) + A_{k-2} \subset B_{k-1}$ is linearly independent in B_{k-1}. Complete the set $(p\lambda_k) \cup (p^2\lambda_{k+1}) \subset A_{k-1}$ by a set $\lambda_{k-1} \subset A_{k-1}$ so that $\lambda_{k-1} \cup (p\lambda_k) \cup (p^2\lambda_{k+1}) + A_{k-2} \subset B_{k-1}$ be a basis in B_{k-1}. By the last step we will have selected sets $\lambda_i \subset A_i, 2 \leq i \leq k+1$, and the set

$$(p\lambda_2) \cup (p^2\lambda_3) \cup \ldots \cup (p^k\lambda_{k+1}) \subset A_1 = B_1$$

will be linearly independent. We complete it by a set λ_1 to a basis for $A_1 = B_1$. Then the elements of the set

$$\lambda = \bigcup_{i=1}^{k+1} \lambda_i$$

generate cyclic subgroups of A which form an internal direct sum,

$$A = \bigoplus_{x \in \lambda} \langle x \rangle.$$