*In the coursework in Riemannian geometry appeared an integral. Its straighforward calculations is interesting...*

**Appendix**

Straightforward calculations of the lenght of the curve $C'$ lead to the following integral:

$$I(z =) \int_0^{2\pi} \frac{d\varphi}{1 - z\cos\varphi}\,,\,(|z| < 1)$$

Here I present two different ways to calcualte this integral.

*First way*

This integral can be calculated explicitly, the answer is beautiful:

$$I(z) = \int_0^{2\pi} \frac{d\varphi}{1 - z\cos\varphi} = \frac{2\pi}{\sqrt{1-z^2}}\,.$$

Do it. One can see that for $|z| < 1$,

$$I(z) = \int_0^{2\pi} \frac{d\varphi}{1 - z\cos\varphi} = \int_0^{2\pi} \left(1 + z\cos\varphi + z^2\cos^2\varphi + \ldots\right) d\varphi =$$

$$\int_0^{2\pi} \left(\sum z^n \cos^n\varphi\right) d\varphi = \sum_{n=0}^{\infty} c_n z^n\,, \text{where } c_n = \int_0^{2\pi} \cos^n \varphi d\varphi\,.$$

Calculate $c_n$:

$$c_n = \int_0^{2\pi} \cos^n \varphi d\varphi = \int_0^{2\pi} \left(\frac{e^{i\varphi} + e^{-i\varphi}}{2}\right)^n d\varphi = \begin{cases} 2\pi \frac{C_{2k}^k}{2^k} \text{for } n = 2k \\ 0 \;\; \text{for } n = 2k+1 \end{cases}.$$

since $(a+b)^n = \sum_j C_n^j a^j b^{n-j}$, and $\int_0^{2\pi} e^{ik\varphi} d\varphi = 0$ if $k \neq 0$. Hence we have that

$$I(z) = \int_0^{2\pi} \frac{d\varphi}{1 - z\cos\varphi} = 2\pi \sum_{n=0}^{\infty} c_n z^n = \sum_{n=0}^{\infty} z^n = \sum \frac{C_{2n}^n z^{2n}}{2^n} = \frac{2\pi}{\sqrt{1-z^2}}\,.$$

**Remark** One can see that

$$P(z) = \sum C_{2n}^n z^n = \frac{2\pi}{\sqrt{1-4z}}\$$$

Looking at this funtion it is difficult to avoid temptation to write something like:

$$\sum C_{2n}^n = P(1) = .... = \frac{2\partial}{\sqrt{-3}} ???!$$

*Second way*

$$I(z) = \int_0^{2\pi} \frac{d\varphi}{1 - z\cos\varphi} = \int_0^{2\pi} \frac{\sin\varphi d\varphi}{\sin\varphi(1 - z\cos\varphi)} = \int_0^{2\pi} \frac{d\cos\varphi}{(\sqrt{1-\cos^2\varphi})(1 - z\cos\varphi)} =$$

$$\int_0^1 \frac{dw}{(\sqrt{1-w^2})(1-zw)} = \frac{1}{2}\int_0^1 \frac{dw}{(\sqrt{1-w^2})(1-zw)}\,.$$

Now consider an integrand, the function $F(w) = \frac{1}{\sqrt{1-w^2}(1-zw)}$ in the plane excluding the neigborhood of the interval $[-1,1]$ which connects the branching points of this function. We take the following branch $F'(w)$ of this function such that it is holomorphic function in the plane without neighborhood of the segment [1] Now

[1] If $P = w = u + iv$ is an arbitrary point of complex plane, $A = -1$ and $B = 1$, and $\varphi$ is an angle between $AB$ and $AP$ (anti-clock wise), and $\psi$ is an angle between $BA$ and $BP$ (anti-clock wise), then $F' = \sqrt{|AP||BP|} e^{i\frac{\phi+\psi}{2}}$. In particular $F(w) = i\sqrt{w^2-1}$ if $w$ is a real number which is greater than 1.

we note that the integral of function over the great circle tends to zero. The function $F'(w)$ has a pole at the point $w = \frac{1}{z}$. Hence if $|z| < 1$ $F(z')$ is holomorphic function in plane without noiborhood of interval $AB$. We have:

$$0 = \int_{C_1} F'(w)dw + \int_{C_2} F(w)dw = I(z) - \frac{1}{z}\int_{C_2} \frac{1}{i\sqrt{w^2-1}\left(w-\frac{1}{z}\right)} =$$

$$I(z) - \frac{2\pi i}{z}\left(\frac{1}{i\sqrt{w^2-1}}\right)\Big|_{w=\frac{1}{z}} = I(z) - \frac{2\pi}{\sqrt{1-z^2}} \Rightarrow I(z) = \frac{2\pi}{\sqrt{1-z^2}}\,.$$

where we denote by $C_1$ the closed curve around the interval $AB$, and $C_2$ the circle of small radius around