Cubic and quadric equations; Galois theory for pedestrians.

This text is written on the base of the book of A. Khovansky

We suppose that Galois group exists. THis is not trivial....

We suppose that the main field is a field of characteristic 0 which possesses all roots of unity. which possesses all roots of unity.

Proposition Let A be an algebra over field K, and let G be a finite commutative group acting on this algebra.

Then every element x of the algebra A can be represented as a sum of elements $w_1, \dots, w_k$ such that for every w

Then there exist basis

Calculations for cubic equation

It is conveniuent to consider the equation $x^3 + px + q = 0$ with

$$\begin{gathered} x_1 + x_2 + x_3 = 0 \\ x_1x_2 + x_2x_3 + x_3x_1 = p \\ x_1x_2x_3 = -q \end{gathered}$$

We introduce Lagrange variables

$$\begin{aligned} w_0 &= x_1 + x_2 + x_3 \\ w_I &= x_1 + \varepsilon x_2 + \varepsilon^2 x_3 \qquad (\varepsilon = e^{\frac{2\pi i}{3}}) . \\ w_{II} &= x_2 + \varepsilon x_1 + \varepsilon^2 x_3 \end{aligned}$$

$w_0 = 0$. We have

$$\begin{gathered} w_0^3 = 0 \\ u = w_I^3 = x_1^3 + x_2^3 + x_3^3 + 3\varepsilon(x_1^2x_2 + x_2^2x_3 + x_3^2x_1) + 3\varepsilon^2(x_1x_2^2 + x_2x_3^2 + x_3x_1^2) + 6x_1x_2x_3 \\ v = w_{II} = x_1^3 + x_2^3 + x_3^3 + 3\varepsilon^2(x_1^2x_2 + x_2^2x_3 + x_3^2x_1) + 3\varepsilon(x_1x_2^2 + x_2x_3^2 + x_3x_1^2) + 6x_1x_2x_3 \end{gathered}$$

Hence

$$u + v = 2(x_1^3 + x_2^3 + x_3^3) + 3\varepsilon(1+\varepsilon)(x_1^2x_2 + \dots) + 12x_1x_2x_3$$

$\dots$ means all permutations.Since

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - ac - bc) , \tag{$*$}$$

$w_0 = 0$ and $\varepsilon(1+\varepsilon) = -1$, hence

$$u+v = 6x_1x_2x_3 - 3\left(x_1x_2(-x_3) + x_1x_3(-x_2) + x_2x_3(-x_1)\right) + 12x_1x_2x_3 = 27x_1x_2x_3 = -27q ,$$

$$u - v = 3(\varepsilon - \varepsilon^2)(x_1^2x_2 + x_2^2x_3 + x_3^2x_2) + 3(\varepsilon^2 - \varepsilon)(x_1x_2^2 + x_2x_3^2 + x_3x_2^2) =$$

$$3(\varepsilon - \varepsilon)^2 \left(x_1x_2(x_1 - x_2) + x_2x_3(x_2 - x_3) + x_3x_1(x_3 - x_1) .\right)$$

and

$$(u-v)^2 = 9(\varepsilon-\varepsilon^2)^2\left[x_1x_2(x_1-x_2)+x_2x_3(x_2-x_3)+x_3x_1(x_3-x_1)\right]^2 = -27\left[K+L\right]\ ,$$

where

$$K = x_1^2x_2^2(x_1-x_2)^2 + x_2^2x_3^2(x_2-x_3)^2 + x_3^2x_1^2(x_3-x_1)^2\,,$$

$$L = 2x_1x_2x_3\left[x_1(x_1-x_2)(x_3-x_1)+x_2(x_1-x_2)(x_2-x_3)+x_3(x_2-x_3)(x_3-x_1)\right]\ .$$

using equation (*) and fact that $x_1+x_2+x_3=0$ we see that

$$K = x_1^2x_2^2(x_1+x_2)^2 + x_2^2x_3^2(x_2+x_3)^2 + x_3^2x_1^2(x_3+x_1)^2 - 4(x_1^3x_2^3+x_2^3x_3^3+x_3^3x_1^3) = ,$$

$$3x_1^2x_2^2x_3^2 - 4(x_1x_2+x_2x_3+x_3x_1)(x_1^2x_2^2+x_2^2x_3^2+x_3^2x_1^2-x_1^2x_2x_3-x_1x_2^2x_3-x_1x_2x_3^2) - 12x_1^2x_2^2x_3^2$$

$$-9x_1^2x_2^2x_3^2 - 4(x_1x_2+x_2x_3+x_3x_1)\left[(x_1x_2+x_2x_3+x_3x_1)^2 - 3(x_1^2x_2x_3+x_1x_2^2x_3+x_1x_2x_3^2)\right] =$$

$$= -9x_1^2x_2^2x_3^2 - 4(x_1x_2+x_2x_3+x_3x_1)^3 = -9q^2-4p^3\,.$$

and

$$L = 2x_1x_2x_3\left[x_1(x_1-x_2)(x_3-x_1)+x_2(x_1-x_2)(x_2-x_3)+x_3(x_2-x_3)(x_3-x_1)\right] = 2x_1x_2x_3M\,.$$

with

$$M = x_1(x_1-x_2)(x_3-x_1)+x_2(x_1-x_2)(x_2-x_3)+x_3(x_2-x_3)(x_3-x_1) =$$

$$\big(-x_1^3+x_1(x_2+x_3)-x_1x_2x_3\big)+\big(-x_2^3+x_2(x_1+x_3)-x_1x_2x_3\big)+\big(-x_3^3+x_3(x_1+x_2)-x_1x_2x_3\big) =$$

$$-2(x_1^3+x_2^3+x_3^3) - 3x_1x_2x_3 = -9x_1x_2x_3$$

since $x_1+x_2+x_3=0$. Hence

$$L = 2x_1x_2x_3M = -18x_1^2x_2^2x_3^2 = -18q^2\,.$$

We come to

$$(u-v)^2 = -27(K+L) = -27(-9q^2-4p^3-18q^2) = 27(27q^2+4p^3)\,.$$

Finally we have

$$\begin{cases} u+v = w_I^3 + w_{II}^3 = -27q \\ u-v = w_I^3 - w_{II}^3 = \pm 27\sqrt{q^2+\frac{4p^3}{27}} \end{cases}$$

or:

$$\begin{cases} w_0 = x_1+x_2+x_3 \\ w_I = x_1+\varepsilon x_2+\varepsilon^2 x_3 \\ w_{II} = x_2+\varepsilon x_1+\varepsilon^2 x_3 \end{cases} \qquad (\varepsilon = e^{\frac{2\pi i}{3}})\,, \qquad \text{with} \begin{cases} w_I^3 + w_{II}^3 = -27q \\ w_I^3 - w_{II}^3 = \pm 27\sqrt{q^2+\frac{4p^3}{27}} \end{cases}$$

One can easy to solve the system of linear equations. E.g. adding first equation mulitplied by ε^2 with second and third we come to the answer for x_3 and so on:

$$w_I + \varepsilon^2 w_{II} + w_0 = 3x_1$$
$$w_{II} + \varepsilon^2 w_I + w_0 = 3x_2 \varepsilon^2 w_0 + w_1 + w_2 = 3x_3$$

Hence

$$\begin{cases} x_1 = \frac{w_I + \varepsilon^2 w_{II}}{3} \\ x_2 = \frac{w_{II} + \varepsilon^2 w_I}{3} \\ x_3 = \frac{(w_I + w_{II})\varepsilon}{3} \end{cases}, \quad \text{with} \quad \begin{cases} w_I^3 + w_{II}^3 = -27q \\ w_I^3 - w_{II}^3 = \pm 27\sqrt{q^2 + \frac{4p^3}{27}} \end{cases}$$

i.e.

$$\begin{cases} w_I^3 = 27\left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right) \\ w_{II}^3 = 27\left(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right) \end{cases} \text{or} \begin{cases} w_I^3 = 27\left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right) \\ w_{II}^3 = 27\left(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right) \end{cases}$$

(We suppose that the branch of function $\sqrt{}$ is chosen) These equations define w_I, w_{II} not uniquely. There are six solutions:

$$w_I = 3\left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right), w_{II} = 3\left(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)$$
$$w_I = 3\varepsilon\left(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right), w_{II} = 3\left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)$$
$$w_I = 3\left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right), w_{II} = 3\left(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)$$
$$w_I = 3\left(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right), w_{II} = 3\left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)$$
$$w_I = 3\left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right), w_{II} = 3\left(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)$$
$$w_I = 3\left(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right), w_{II} = 3\left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)$$