HEISENBERG GROUP

In this lecture we introduce the Heisenberg algebra (over the real and complex numbers) and the simply connected and reduced Heisenberg groups. We largely follow Folland [F].

1. Real Heisenberg Lie Algebra

A symplectic vector space can be built from a real vector space V by $V \oplus V^*$ with $\sigma((x \oplus f, x' \oplus f') = f(x') - f'(x)$.

Notation of Takhtajan: p. 82.

Definition: The Hesienberg algebra $\mathfrak{h}_n$ is a Lie algebra with generators $e^1, \dots, e^n, f_1, \dots f_n, c$ and relations

$$[e^k, f_\ell] = c\delta^k_\ell, \quad [e^k, c] = [f_\ell, c] = 0.$$

There is an exact sequence

$$0 \to \mathbb{R} \to \mathfrak{g}_n \to V \to 0$$

and

$$[x, y] = \omega(\bar{x}, \bar{y})c.$$

A choice of symplectic basis gives the generators and relations above. In coordinates

$$[(p, q, t), (p', q', t')] = (0, 0, (pq' - qp')).$$

1.1. **Finite matrix representation of the Heisenberg algebra.** Folland (p. 18): Let

$$m(p, q, t) = \begin{pmatrix} 0 & \vec{p} & t \\ 0 & \vec{0} & q_1 \\ 0 & \vec{0} & q_2 \\ & \dots & \\ 0 & \vec{0} & q_n \\ 0 & \vec{0} & 0 \end{pmatrix}.$$

Then

$$m(p, q, t)m(p, q, t') = m(0, 0, \langle p, q' \rangle)$$

and

$$[m(p, q, t), m(p, q, t')] = m(0, 0, \langle p, q' \rangle - \langle p', q \rangle).$$

Note that the center is $m(0, 0, t)$ so c is the matrix with a 1 in the upper right corner entry.

2. Heisenberg group

There is a unique simply connected group with this Lie algbra: The Heisenberg group is $\mathbb{R}^{2n} \times \mathbb{R}$ with group law

$$(x, \xi, t) \cdot (x', \xi', t') = (x + x', \xi + \xi', t + t' + \frac{1}{2}\sigma((x, \xi), (x', \xi')))$$

with $\sigma((x, \xi), (x', \xi'))) = \langle \xi, x' \rangle - \langle \xi', x \rangle$.

Then

$$\begin{aligned} (x', \xi', t') \cdot (x, \xi, t) &= (x + x', \xi + \xi', t + t' - \tfrac{1}{2}\sigma((x, \xi), (x', \xi'))) \\ &= (0, 0, -\sigma((x, \xi), (x', \xi')))((x, \xi, t) \cdot (x', \xi', t'). \end{aligned}$$

In complex notation, we have

$$(z, t) \cdot (z', t') = (z + z', t + t' + \Im(z\bar{z'}).$$

Note that the only difference if we interchange the order of multiplication is the sign of $\Im z\bar{z'}$.

2.1. Finite matrix representation.

Let

$$M(p, q, t) = I + m(p, q, t) = \begin{pmatrix} 1 & \vec{p} & t & & & \\ 0 & 1 & \vec{0} & q_1 & & \\ 0 & 0 & 1 & \vec{0} & q_2 & \\ & \cdots & & & & \\ 0 & 0 & & \vec{0} & 1 & q_n \\ 0 & \vec{0} & 1 & & & \end{pmatrix}.$$

Then

$$M(p, q, t)M(p', q', t') = M(p + p', q + q', t + t' + \langle p, q \rangle).$$

One has

$$e^{m(p,q,t)}e^{m(p',q',t')} = e^{m(p+p',q+q',t+t'+\frac{1}{2}(pq'-p'q))}.$$

This follows because

$$e^{m(p,q,t)} = I + m(p, q, t) + \frac{1}{2}m(0, 0, pq) = M(p, q, t + \frac{1}{2}pq).$$

We thus have,

$$\exp : \mathfrak{h}_n \to \mathcal{H}_n, \qquad \log : \mathcal{H}_n \to \mathfrak{h}_n,$$

and we use it to endow $\mathbb{R}^{2n+1}$ with a group law by $X \iff e^{m(X)}$ so that $(q, p, t) \equiv e^{m(p,q,t)}$.

Lemma 1. *The induced group law on $\mathcal{H}_n$ is*

$$(p, q, t)(p', q', t') = (p + p', q + q', t + t' + \frac{1}{2}(pq' - qp')).$$

Proof: By definition,

$$\begin{aligned}(p,q,t)(p',q',t') = \log e^{m(p,q,t)}e^{m(p','q,t')} &= \log M(p,q,t+\tfrac{1}{2}pq)M(p',q,t'+\tfrac{1}{2}p'q') \\ &= \log M(p+p',q+q',t+\tfrac{1}{2}pq+\tfrac{1}{2}p'q'+pq')).\end{aligned}$$

We need (ξ, x, s) so that

$$\exp m(\xi,x,s) = M(\xi,x,s+\frac{1}{2}x\xi) = M(p+p',q+q',t+\frac{1}{2}pq+\frac{1}{2}p'q'+pq')).$$

Obviously, $\xi = p+p', x = q+q'$ and

$$s+\frac{1}{2}(p+p')(q+q') = \frac{1}{2}pq+\frac{1}{2}p'q'+pq')) \implies s = \frac{1}{2}pq+\frac{1}{2}p'q'+pq'))-\frac{1}{2}(p+p')(q+q') = \frac{1}{2}(pq'-qp')).$$

What is the identity element of $\mathcal{H}_n$? It is $(0,0,0)$. Also, the inverse of (p,q,t) is $(-p,-q,-t)$:

$$(p,q,t)(-p,-q,-t) = (0,0,0+\frac{1}{2}(p(-q)-q(-p))).$$

Also the center is

$$\mathcal{Z} = \{(0,0,t) : t \in \mathbb{R}\}.$$

2.2. **Reduced Heisenberg group.** The *reduced Heisenberg group* $\mathbb{H}_n^{\mathrm{red}}$ is $\mathbb{C}^n \times S^1$ with group law:

$$\begin{aligned}&(x,\xi,e^{it})\cdot(x',\xi',e^{it'}) \\ &= (x+x',\xi+\xi',e^{i(t+t'+\frac{1}{2}\sigma((x,\xi),(x',\xi'))}).\end{aligned}$$

That is

$$\mathcal{H}_n^{red} = \mathcal{H}_n/\{0,0,k) : k \in \mathbb{Z}\}.$$

3. Schrödinger representation

The Schrödinger representation ρ is a unitary representation of $\mathcal{H}_n$ on $L^2(\mathbb{R}^n, dx)$. Its derived representation, the infinitesimal Schrödinger representation $d\rho$, is a Lie algebra representation of $\mathfrak{h}_n$ by skew-Hermitian operators on $L^2(\mathbb{R}^n, dx)$.

Notation: x_j as an operator is multiplication by x_j. We write $X = (x_1, \dots, x_n)$ and $qX = \sum_j q_j x_j$. These are all understood to be multiplication operators,

$$qXf(x) = \langle q, x\rangle f(x).$$

Let $P_j = \frac{h}{2\pi i}\frac{\partial}{\partial x_j}$. Also let $D_j = \frac{1}{2\pi i}\frac{\partial}{\partial x_j}$. Then

$$pP = \frac{h}{2\pi i}\sum_j p_j \frac{\partial}{\partial x_j}.$$

3.1. **Infinitesimal Schrödinger representation.** See Folland (1.8). The infinitesimal Schrödinger representation is the homorphism

$$d\rho_h(p,q,t) = 2\pi i(hpD + xX + tI).$$

Lemma 2. *$d\rho_h$ is a homomorphism from $\mathfrak{h}_n$ to the skew Hermitian operators on $\mathcal{S}(\mathbb{R}^n)$.*

Proof: we must check that

$$[d\rho_h(p,q,t), d\rho_h(p',q',t')] = d\rho_h(0,0,pq'-qp')$$

which is equivalent to checking that

$$[2\pi i(hpD + qX + tI), 2\pi i(hp'D + q'X + t'I) = 2\pi i(pq'-qp').$$

The left side is

$$[2\pi ihpD, q'X] + [qX, 2\pi ihp'D] = h[p\cdot \frac{\partial}{\partial x_j}, q'X] + h[qX, p'\frac{\partial}{\partial x_j}] = h(pq'-qp').$$

3.2. **Schrödinger representation. Notation**

- Takhtajan (page 89) writes $P = \frac{\hbar}{i}\frac{d}{dx}$ and $Qf(x) = xf(x)$. Then $[P,Q] = -iI$.
- von Neumann:
$$[P,Q] = \frac{h}{2\pi i}I, \qquad P = \frac{h}{2\pi i}\frac{d}{dx}.$$
-

The Schrödinger representation ρ_h is defined by:

- (Folland)
$$\rho_h(p,q,t) = \exp 2\pi i(pD + qX + tI).$$
- Von Neumann:
$$U(\alpha) = e^{\frac{2\pi i}{\hbar}\alpha P}, \qquad V(\beta) = e^{\frac{2\pi i}{\hbar}\beta Q}.$$

Definition: Schrödinger representation ([?], footnote)

$$U(\alpha)f(q) = f(q+\alpha), \quad V(\beta)f(q) = e^{\frac{2\pi i}{\hbar}}f(q).$$
$$S(\alpha,\beta) = e^{-\frac{i}{2}\langle\alpha,\beta\rangle}U(\alpha)V(\beta) = e^{\frac{i}{2}\langle\alpha,\beta\rangle}V(\beta)U(\alpha).$$

Lemma 3. *Then*

$$e^{iqX}f(x) = e^{i\langle q,x\rangle}f(x), \quad e^{2\pi ipP}f(x) = f(x+\hbar p).$$

The first statement is obvous. The second follows because

$$e^{\sum_j p_jD_j}e^{i\langle x,\xi\rangle} = e^{\langle p,\xi_j\rangle}e^{i\langle x,\xi\rangle} = e^{i\langle (x+p),\xi\rangle}.$$

Let us show that the definitions in Folland and von Neumann are consistent:

Lemma 4. *We have*

$$\rho_h(\alpha,\beta,0) = \exp 2\pi i(\alpha D + \beta X) = S(\alpha,\beta).$$

For the exponential map from a Lie algebra to the Lie group, we have:

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\cdots},$$

where $\cdots$ denote higher order commutators. Since the Heisenberg Lie algebra is 2-step nilpotent, we have exactly:

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}.$$

Since $[A, B]$ lies in the center, we also get

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} = e^B e^A e^{-\frac{1}{2}[B,A]}.$$

Lemma 5.

$$\rho_h(p, q, t)f(x) = e^{2\pi it} e^{i\pi pq} e^{2\pi iqx} f(x + p)$$

Proof. We may set $t = 0$ with no loss of generality. With a different definition of t we consider

$$g(t, x) = e^{2\pi it(pD+qX)} f(x).$$

By definition it solves

$$\frac{d}{dt} g(t, x) = (2\pi i)(pD + qX)g(t, x), \quad g(0, x) = f(x).$$

That is,

$$\frac{\partial g}{\partial t} - \sum_{j=1}^{n} p_j \frac{\partial}{\partial x_j} g(t, x) = 2\pi i\langle q, \vec{x}\rangle g(t, x).$$

Let $\vec{x}(t) = x - t\vec{p}, x = x(t) + t\vec{p}$. Then the left side is

$$\frac{\partial}{\partial t} g(t, x - t\vec{p}).$$

We rewrite the equation as

$$\frac{\partial}{\partial t} g(t, x - t\vec{p}) = 2\pi i\langle q, \vec{x} - t\vec{p}\rangle g(t, x - t\vec{p}).$$

We then integrate to get

$$\tfrac{d}{dt} \log g(t, x - tp) = 2\pi i\langle q, x\rangle - 2\pi it\langle q, p\rangle \implies \log g(t, x - tp) = \log g(0, x) + t2\pi i\langle q, x\rangle - i\pi qpt^2$$

$$\implies g(t, x - tp) = f(x)e^{2\pi it\langle q,x\rangle} e^{-i\pi t^2\langle p,q\rangle}$$

$$\implies g(t, y) = e^{2\pi it\langle q,y\rangle} e^{2\pi it^2\langle q,p\rangle} e^{-i\pi t^2\langle p,q\rangle} f(y + tp),$$

with $y = x - tp$. We then set $t = 1$ to conclude the proof. $\square$

Lemma 6. *ρ_h is a representation. That is,* $\rho_h(p, q)\rho_h(r, s) = \rho_h(p + r, q + s, \frac{1}{2}(ps - qr))$.

$$e^{2\pi i(pD+qX)} e^{2\pi i(rD+sX)} = e^{i\pi(ps-qr)} e^{2\pi i[(p+r)D+(q+s)X)}.$$

Proof.

$$\begin{aligned}\rho_h(p,q)\rho_h(r,s)f(x) &= e^{i\pi pq}e^{2\pi ixq}(\rho_h(r,s)f)(x+p)\\ &= e^{i\pi pq}e^{2\pi ixq}e^{i\pi rs}e^{2\pi i\langle s,x+p\rangle}f(x+p+r).\end{aligned}$$

On the other hand,

$$\rho_h(p+r,q+s,\tfrac{1}{2}(ps-qr))f(x) = e^{2\pi i\frac{1}{2}(ps-qr))}e^{i\pi(\langle p+r,q+s\rangle}e^{2\pi i\langle q+s,x\rangle}f(x+p+r).$$

They are equal since

$$e^{i\pi pq}e^{i\pi rs}e^{2\pi i\langle s,p\rangle} = e^{2\pi i\frac{1}{2}(ps-qr))}e^{i\pi(\langle p+r,q+s\rangle}.$$

□

Another version: If we conjugate by the Fourier transform, we get

$$\rho'(p,q,t) = \mathcal{F}\rho(p,q,t)\mathcal{F}^{-1} = \rho(-q,p,t)$$

which acts by

$$\rho'(p,q,t) = e^{2\pi it}e^{2\pi i\langle pX-qD\rangle}.$$

Thus,

$$\rho'(p,q,t)f(x) = e^{-i\pi pq}e^{2\pi i\langle p,x\rangle}f(x-q).$$

3.3. **Weyl commutation relations.**

Lemma 7.

$$U(q)V(p) = e^{-iqp}V(p)U(q).$$

Proof. Let us check directly:

$$U(q)V(p)f(x) = (V(p)f)(x-q) = e^{i\langle p,x-q\rangle}f(x-q).$$

On the other hand,

$$V(p)U(q)f(x) = e^{i\langle p,x\rangle}(U(q)f)(x) = e^{ipx}f(x-q).$$

□

Note that

$$U(q) = \rho'(0,q,0) = \rho(-q,0,0), \quad V(p) = \rho'(p,0,0) = \rho(0,p,0).$$

4. Fock space representation

In dimension 3, the Heisenberg is the Lie algebra generated by the position operator Q= multiplication by x, the momentum operator $P = D = \frac{1}{i}\frac{d}{dx}$ and 1 with only one non-trivial commutation relation:

$$[Q,P] = i.$$

It is convenient to complexify the algebra and choose the generators

$$a = \frac{1}{\sqrt{2}}(Q+iP), \;\; a^* = \frac{1}{\sqrt{2}}(Q-iP)$$

which are known as the annihilation and creation operators. As differential operators, $a = \frac{1}{\sqrt{2}}(x + \frac{d}{dx}), a^* = \frac{1}{\sqrt{2}}(x - \frac{d}{dx})$. They satisfy:

$$[a, a^*] = \frac{1}{2}[-i[Q, P] + i[P, Q]] = 1.$$

We include $\hbar$ in the definition of quantization by putting $P = \hbar D = \frac{\hbar}{i}\frac{d}{dx}$. Then the commutation relations become

$$[Q, P] = i\hbar, \quad [a, a^*] = \hbar.$$

Let

$$a_j^* = \frac{1}{\sqrt{2}}(x_j - \frac{\partial}{\partial x_j}), \ \ a_j = \frac{1}{\sqrt{2}}(x_j + \frac{\partial}{\partial x_j}).$$

Also define the 'number operator' by

$$\mathbf{N} = \sum_j a_j^* a_j.$$

In the Bargmann representation,

$$a_j^* \to z_j, \ \ a_j^* \to \frac{\partial}{\partial z_j}.$$

The Number operator is

$$N = \sum_j z_j \frac{\partial}{\partial z_j}.$$

The unique vacuum state is the constant function 1. The other eigenfunctions are the monomials z^n, and the eigenvalue is the degree.

Definition: Let (L, σ) be a symplectic vector space. A Weyl system over (L, σ) is a pair (K, W) where K is a complex Hilbert space and $W : L \to U(K)$ such that

$$W(z)W(z') = e^{\frac{1}{2}i\sigma(z,z')}W(z + z').$$

As an example, a complex Hilbert space defines (L, σ) where L is the underlying real vector space and $\sigma(z, z') = \Im(z, z')$.

Let $\mathcal{F}$ be the space of entire holomorphic functions f on $\mathbb{C}$ with inner product

$$\frac{1}{\pi}\int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} dL(z).$$

An orthonormal basis is

$$e_n(z) = \frac{z^n}{\sqrt{n!}}.$$

The evaluation functional is

$$f(a) = \langle f, E_a \rangle, \quad E_a(z) = e^{z\bar{a}},$$

i.e.

$$E_a(z) = \sum_{n=0}^{\infty} \langle E_a, e_n \rangle \frac{z^n}{\sqrt{n!}} = \sum_{n \geq 0} \frac{z^n \bar{a}^n}{n!}.$$

Define

$$W_{\mathcal{F}}(z)f(w) = e^{-|z|^2} e^{w\bar{z}} f(w - z).$$

Thus,

$$W_{\mathcal{F}}(z) : \mathcal{F} \to \mathcal{F}.$$

Also,

$$W_{\mathcal{F}}(z_1)W_{\mathcal{F}}(z_2) = e^{-\Im(z_1\bar{z_2}}W_{\mathcal{F}}(z_1 + z_2).$$

Note that

$$(z_1, z_2) \to \Im z_1\overline{z_2}$$

is a symplectic inner product. If $z_1 = x_1 + i\xi_1, z_2 = x_2 + i\xi_2$ then

$$\Im z_1\overline{z_2} = \Im(x_1 + i\xi_1)(x_2 - i\xi_2) = x_2\xi_1 - x_1\xi_2.$$

Thus,

$$z \to W_{\mathcal{F}}(z)$$

is a unitary representation of the Weyl commutation relations. Unitarity: with $v = w - z$,

$$\begin{aligned} ||W_{\mathcal{F}}(z)f||^2_{\mathcal{F}} &= \tfrac{1}{\pi}\int_{\mathbb{C}} |W_{\mathcal{F}}(z)f(w)|^2 e^{-|w|^2} dL(w) \\ &= \tfrac{1}{\pi}\int_{\mathbb{C}} |e^{-|z|^2}e^{w\bar{z}}f(w-z)|^2 e^{-|w|^2} dL(w) \\ &= \tfrac{1}{\pi}\int_{\mathbb{C}} e^{-2|z|^2}e^{w\bar{z}+z\bar{w}}|f(w-z)|^2 e^{-|w|^2} dL(w) \\ &= \tfrac{1}{\pi}\int_{\mathbb{C}} e^{-2|z|^2}e^{(v+z)\bar{z}+z\overline{(v+z)}}|f(v)|^2 e^{-|v+z|^2} dL(v) \\ &= \tfrac{1}{\pi}\int_{\mathbb{C}} |f(v)|^2 e^{-|v|^2} dL(v), \end{aligned}$$

where in the last step we used that

$$-2|z|^2 + (v+z)\bar{z} + z\overline{(v+z)} - |v+z|^2 = -|v|^2.$$

Further,

$$W_{\mathcal{F}}E_0 = e^{-|z|^2}E_a,$$

so that $W_{\mathcal{F}}$ is irreducible. Indeed, any irrep must contain a vacuum vector, which is $E_0 = 1$. But if $f \perp W_{\mathcal{F}}(a)E_0$ for all a then $f \equiv 0$. Alternatively, if P is an orthogonal projection commuting with all $W_{\mathcal{F}}(a)$ and $f = PE_0$ then

$$f(z) = \langle PE_0, E_z\rangle = e^{|z|^2}\langle PE_0, W_{\mathcal{F}}(z)E_0\rangle = \langle PE_{-z}, E_0\rangle = \overline{f(-z)} \implies f = C.$$

Bargmann-Fock transform: By the Stone-von Neumann theorem, there must exist a unitary $U : L^2(\mathbb{R}) \to \mathcal{F}$ intertwining the Schrödinger and Bargmann-Fock representations, i..e. $W_{\mathcal{F}}(a)U = UW(a)$ and $UH_0 = E_0$. Then

$$Uf(z) = \langle Uf, E_z\rangle = \langle f, U^*E_z\rangle = e^{-|z|^2}\langle f, U^*W_{\mathcal{F}}(z)E_0\rangle = e^{-|z|^2}\langle W(-z)f, H_0\rangle.$$

Then,

$$Uf(z) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-(x^2+y^2)}e^{-2ixy}f(t+x)e^{-t^2/2}dt = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} B(z,t)f(t)dt.$$

Here,

$$B(z,t) = \exp[-z^2 - t^2/2 + zt].$$

Also: Takhtajan: pages 112-113 discusses Fock space. Pages 118-139 cover Weyl relations.

4.1. **Quantum field theory.** In §12 Weyl defines operators $q(x), p(x')$ and has the commutation rule

$$[q(x), p(x')] = i\delta(x - x').$$

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