Points on Integer Distances

Yulii Rudyak gave me the following problem:

X is the set of the points on the plane, such that not all the points are on the same line and all the distances between the points are integers. Prove that in this case X has to be a finite set.

Remark One can easily to construct for every N a set of N points which are not on the same line and all the distances are integer. One can also construct the infinite set X of points not on the same line with rational distances between them.

E.g. consider the points $\{A_0, A_1, \ldots . A_p, \ldots\}$ which have following cartesian coordinates: A_0 has coordinates (0.2) and the point A_k has coordinates $(k - \frac{1}{k})$ $(k = 0, 1, 2, 3, \ldots)$. It is easy to see that all the distances between these points are rational. On the other hand for arbitrary N under dilatation $(x, y) \to (N!x, N!Y)$ the subset of first $N+1$ points transforms to the set with integer distances.

Now we go to the proof of the Rudyak,s statement.

Let A, B, C be three points from the set X which are not on the same line and there lengths are equal to $|AB| = c$, $|BC| = a$

We consider the subset X_{pq} which is defined in the following way:

$$X_{pq}\colon \quad \mathbf{r} \in X_{pq} \quad \text{iff } \mathbf{r} \in X \quad \text{and} \begin{cases} |\rho(\mathbf{r}, A) - \rho(\mathbf{r}, B)| = p\,, \\ |\rho(\mathbf{r}, A) - \rho(\mathbf{r}, C| = q \end{cases} \tag{1}$$

In other words X_{pq} are the points of X whose distances between A and B differs on p and between A and C differs on q.

It is evident that the integers p and q are restricted by integers c and b respectively: $0 \le p \le c$,$0 \le q \le b$,

Hence it remains to prove that all X_{pq} are finite sets, because

$$X = \bigcup_{p \le c, q \le b} X_{pq} \tag{2}$$

The fact that every X_{pq} is the finite sets immediately follows from the fact that belongs simultaneously to hyperbole Γ^1_p $\mathbf{r}$ $\mathbf{r}$ with focuses in the points A and B, defined by the first equation in (1) and to the hyperbole Γ^2_q with focuses in the points A and C defined by the second equation in (1). Hence these hyperboles have not more than 4 general points, because their focuses do not belong to the same line. (The degenerated hyperboles: Γ^1_0, Γ^2_0—lines and Γ^1_c and Γ^2_b—rayons are considered also like hyperboles. "Hyperboles" Γ^1_c and Γ^2_b — rayons have infinite set of general points if A, B, C would be on the same line)

We came to the conclusion that the number of ponts in X_{pq} is not more that 4. From (3) and (2) it follows that number of points in X is not more than $4bc$. The statement is proved.

Compare this problem with the problem discussed in "n-points on the plane"