On Kac Peterson Formula; calculation of volumes of groups corresponding to algebras $sl(n)$

Let $\mathcal{G}$ be simple Lie algebra. Then Kac Peterson formula tells that volume $V(G)$ of compact connected simple connected group $G(\mathcal{G})$ is defined by the formula,

$$V^2(G) = (8\pi)^2 J(4\pi i\rho)\,,$$

where $\rho = \frac{1}{2}\sum_{\alpha\in\triangle_+}\alpha$ is Weyl vector,and J Jacobian of the map $\mathcal{G}\to G$:

$$J(x) = \det\left(\frac{1-e^{-ad_x}}{ad_x}\right)$$

(We suppose that covectors and vectors are identified via Killing Cartan metric

$$\phi(x,y) = -\mathrm{Tr}\,(ad_x ad_y)$$

$\phi = \langle\,,\,\rangle$)

This implies that

$$V = (2\pi\sqrt{2})^{dim\,\mathcal{G}}\prod_{\alpha\in\triangle_+} f(2\pi\phi(\rho,\alpha))\,,\qquad \text{where}\quad f(x) = \frac{\sin x}{x}\,.$$

(See equation (4.32.1) in V.Kac, D.Peterson "Infinite-dimensioanal Lie algebras, Theta functions and modular forms", Advances in Math,. **13**, pp.125—264 (1984))

Let us apply this formula to the most simple case $su(2)$, then to $sl(n,C)$ (we mean calculate volume of corresponding simple simply connected Lie groups.

volume of $SU(2)$

)

Consider generators $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$:

$$[\mathbf{e}_i,\mathbf{e}_k] = \varepsilon_{ikm}\mathbf{e}_m\,.$$

One can see that

$$\mathrm{Tr}(ad_{e_i}ad_{e_k}) = -2\delta_{ik}\,,\text{this means that } \phi(\mathbf{e}_i,\mathbf{e}_k) = 2\delta_{ik}$$

Consider Cartan algebra h spanned by vector $\mathbf{e}_3$. We have that

$$[i\mathbf{e}_3,\mathbf{e}_\pm] = \pm\mathbf{e}_\pm\,,\quad \text{where } \mathbf{e}_\pm = \mathbf{e}_1 \pm i\mathbf{e}_2\,.$$

Thus we have two roots α_+,α_-:

$$\alpha_\pm\in h^*: \alpha_\pm(\mathbf{e}_1) = \pm\frac{1}{i} \text{ i.e. } \begin{cases}\alpha_+\in h^*\,, & \text{such that } \alpha_+(\mathbf{e}_1) = -i\\ \alpha_-\in h^*\,, & \text{such that } \alpha_+(\mathbf{e}_1) = i\end{cases}\,.$$

We see vectors $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ have length $\sqrt{2}$ and covectors α_+,α_- have length $\frac{1}{\sqrt{2}}$ (matrix of the Cartan-Killing metric in the basis $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ is $\begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix}$,) Weyl vector $\rho = \frac{\alpha}{2}$.

Hence

$$2\pi\phi(\rho,\alpha_+) = \pi|\alpha_+|^2 = \frac{\pi}{2}\,.$$

We come to

$$Vol(su(2)) = (2\pi\sqrt{2})^{dim\,\mathcal{G}} \prod_{\alpha\in\triangle_+} \frac{\sin(2\pi\phi(\rho,\alpha))}{2\pi\phi(\rho,\alpha)} =$$

$$= (2\pi\sqrt{2})^3 \frac{\sin(2\pi\phi(\rho,\alpha))}{2\pi\phi(\rho,\alpha)} = (2\pi\sqrt{2})^3 \frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} = 32\sqrt{2}\pi^2$$

Notice that $su(2)$ is three-dimensional sphere and volume of three-dimensional sphere is proportional to π^2: volume of sphere of radius R is equal to $2\pi R^3$: volume of 3-dimensional sphere of radius 1 is equal to

$$Vol(S^3) = \frac{2\pi^{\frac{k+1}{2}}}{\Gamma\left(\frac{k+1}{2}\right)}\Big|_{k=3} = 2\pi^2$$

One can say that volume of $su(2)$ is equal to the volume of S^3 with radius $R = 2\sqrt{2}$. (it has to be clarified.)

First we will study roots of algebra $sl(n, C)$.

L

ie algebras $sl(n, C)$

Denote by E_{ij} $n \times n$ matrix such that its all entries vanish except the intry in i-th column amd j-th row which is equal to 1. These matrices span over C $gl(n, \mathbf{C})$ Lie algebra. Traceless matrices span Lie algebra $sl(n, \mathbf{C})$. Denote by $\mathbf{t}H$ Cartan algebra of diagonal matrices in $gl(n, C)$ and respectively by H Cartan algebra of traceless diagonal matrices in $sl(n, C)$. Denote by $\alpha^{ij}, (i \neq j)$ linear functions on Lie algebra $\mathbf{t}H$ (i.e. elements of $\mathbf{t}H^*$) such that

$$\alpha^{ij}(t^m E_{mm} a = t^i - t^j\,.$$

One can see $\{\alpha^{ij}\}$ are roots — they are equal to values of observables (Cartan subalgebra $\mathbf{t}H$) on root vectors $\{E_{ij}\}$:

$$\forall \mathbf{t} \in H, \mathbf{t} = t^m E_{mm},\ \hat{\mathbf{t}} E_{ik} = [\mathbf{t}, E_{ik}] = (t^i - t^k) E_{ik}$$

$$\begin{pmatrix} \text{matrices } E_{ij}, (i > j) \text{ are positive weight vectors} \\ \text{matrices } E_{ij}, (i < j) \text{ are negative weight vectors} \end{pmatrix}$$

Teperj poschitajem metriku Cartana Killinga na algebre $gl(n, \mathbf{C})$ ad its subalgebra $sl(n, \mathbf{C})$.

$$\phi(X, Y) = \mathrm{Tr}\,(\hat{X} \circ \hat{Y})$$

where $\hat{X} = ad_X$: $\hat{X} Y = [X, Y]$. Notice that for every matrix X for coefficients of expansion we have:

$$X = X^{\pi\rho} E_{\pi\rho} \Rightarrow X^{\pi\rho} = \mathrm{Tr}\,(X E_{\rho\pi})\,.$$

Hence for metric coefficients we have

$$g_{ik|pq} = \mathrm{Tr}\,(\hat{E}_{ik} \circ \hat{E}_{pq}) = \mathrm{Tr}\,(([E_{ik}, [E_{pq}, E_{\alpha\beta}]])\, E_{\beta\alpha}) =$$

$$Tr\,(E_{ik} E_{pq} E_{\alpha\beta} E_{\beta_\alpha} - E_{ik} E_{\alpha\beta} E_{pq} E_{\beta_\alpha} - E_{pq} E_{\alpha\beta} E_{ik} E_{\beta_\alpha} + E_{\alpha\beta} E_{pq} E_{ik} E_{\beta_\alpha}) =$$

$$2N\delta_{iq}\delta_{kp} - 2\delta_{ik}\delta_{pq}\,.$$

We see that this metric is degenerate: identity matrix is zeroeigenvector, in other words algebra $gl(n,\mathbf{C})$ is not semisimple, it possesses the centre. The corank of the metric is just one—the algebra $sl(n,\mathbf{C})$ is semisimple. Calculate Kartan-Killing on $sl(n,\mathbf{C})$. For every $X \in sl(n,\mathbf{C})$

$$\operatorname{Tr} X\big|_{gl(N,\mathbf{C})} = \operatorname{Tr} X\big|_{sl(N,\mathbf{C})}$$

since $\hat{X} I = [X, I] = 0$. Hence metric is defined by the same formula.

Choose the basis in $gl(n,\mathbf{C})$:

$$E_{pq}, p \neq q \; T_i = E_{ii} - E_{nn}, (i = 1, \dots, n-1)\,.$$

Our next step to calculate scalar products of roots. We see from previous calculations that non-zero metric entries are only the following:

$$\text{for every two indices } p, q \text{ such that } p \neq q,\; \phi(E_{pq}, E_{qp}) = 2n$$

$$\text{for every two indices } i, j \begin{cases} \phi(T_i, T_j) = 2n \text{ if } i \neq j \\ \phi(T_i, T_j) = 4n \text{ if } i = j \end{cases}$$

and all other entries vanish.

Choose the ordering

$$\{E_{21}, E_{31}, E_{32}, \dots, E_{n1}, \dots, E_{n\,n-1}, |T_1, T_2, \dots, T_{n-1}, E_{12}, \dots, E_{1n}, E_{23}, \dots, E_{2n}, \dots, E_{n-1\,n}\}$$

of basic vectors. Then we see that $(n^2-1) \times (n^2-1)$ matrix of Cartan-Killing metric has the following appearance

$$||G|||_{sl(n,\mathbf{C})} = 2n \begin{pmatrix} 0 & 0 & I \\ 0 & K & 0 \\ I & 0 & 0 \end{pmatrix}\,,$$

where I is $\frac{n^2-n}{2} \times \frac{n^2-n}{2}$ unity matrix,a and K is $n-1 \times n-1$ matrix such that

$$K = \begin{pmatrix} 2 & 1 & 1\dots & 1 & 1 \\ 1 & 2 & 1\dots & 1 & 1 \\ \dots & & & & \\ 1 & 1 & 1\dots & 1 & 2 \end{pmatrix}$$

The inverse matrix (we need it to calculate the scalar product of covectors (roots)) has the appearance

$$||G^{-1}|||_{sl(n,\mathbf{C})} = \frac{1}{2n} \begin{pmatrix} 0 & 0 & I \\ 0 & K^{-1} & 0 \\ I & 0 & 0 \end{pmatrix}\,,$$

where $n-1 \times n-1$ matrix K^{-1} has the appearance:

$$K^{-1} = \frac{1}{n} \begin{pmatrix} n-1 & -1 & -1\dots & -1 & -1 \\ -1 & n-1 & -1\dots & -1 & -1 \\ \dots & & & & \\ -1 & -1 & -1\dots & -1 & n-1 \end{pmatrix}$$

Thus we will be able to calculate scalar producst of roots.

Roots, coroots

Calculate the components of roots α^{ik} in the basis T_i (Recall that $T_i = E_{ii} - E_{nn}$). Calculating $\alpha^{ik}(T_i)$ we come to components of roots, covectors:

$$\alpha^{12} = \begin{pmatrix} \alpha^{ik}(T_1) \\ \alpha^{ik}(T_2) \\ \alpha^{ik}(T_3 \\ \alpha^{ik}(T_4) \\ \dots \\ \alpha^{ik}(T_{n-1}) \\ \alpha^{ik}(T_n) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ \dots \\ 0 \\ 0 \end{pmatrix},$$

For example for simple roots $\alpha^{i,i+1}$ we have:

$$\alpha^{23} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ \dots \\ 0 \\ 0 \end{pmatrix}, \dots, \alpha^{n-2,n-1} = \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \alpha^{n-1,n} = \begin{pmatrix} 1 \\ 1 \\ \dots \\ 1 \\ 1 \\ 1 \\ 2 \end{pmatrix},$$

Every positive (negative) root is combination of simple roots with positive (negative)integers, e.g.

$$\alpha^{36} = \alpha^{34} + \alpha^{45} + \alpha^{56}.$$

Cartan-Killing metric defines isomorphism between vectors and covectrs: To every covector $f \in \mathcal{G}^*$ corresponds the vector $\xi_f \in \mathcal{G}$ such that

$$\phi(\xi_f, x) = f(x), \quad \text{for every vector } x.$$

In particular define coroots α_{pq} dula to roots. From the explicit expression for Cartan-Killing metric in the basis $\{E_{21}, \dots, E_{n,n-1}T_1 \dots T_{n-1}, E_{12}, \dots E_{n-1,n}\}$ we have that

$$\alpha_{pq} = \frac{1}{2n} K^{-1} \alpha^{pq}.$$

In particular for simple roots we have

$$\alpha_{12} = \frac{1}{2n^2} \begin{pmatrix} n-1 & -1 & -1\dots & -1 & -1 \\ -1 & n-1 & -1\dots & -1 & -1 \\ \dots \\ -1 & -1 & -1\dots & -1 & n-1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ \dots \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2n} (1, -1, 0, \dots, 0)..$$

$$\alpha_{23} = \frac{1}{2n^2}\begin{pmatrix} n-1 & -1 & -1\dots & -1 & -1 \\ -1 & n-1 & -1\dots & -1 & -1 \\ \dots \\ -1 & -1 & -1\dots & -1 & n-1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ \dots \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2n}\left(\, 0, 1, -1, 0, \dots, 0 \,\right) ..$$

and so on

$$\alpha_{n-2n-1} = \frac{1}{2n^2}\begin{pmatrix} n-1 & -1 & -1\dots & -1 & -1 \\ -1 & n-1 & -1\dots & -1 & -1 \\ \dots \\ -1 & -1 & -1\dots & -1 & n-1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{2n}\left(\, 0, \dots, 0, 1, -1 \,\right) ..$$

and

$$\alpha_{n-1n2} = \frac{1}{2n^2}\begin{pmatrix} n-1 & -1 & -1\dots & -1 & -1 \\ -1 & n-1 & -1\dots & -1 & -1 \\ \dots \\ -1 & -1 & -1\dots & -1 & n-1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \dots \\ 1 \\ 2 \end{pmatrix} = \frac{1}{2n}\left(\, 0, 0, \dots, 0, 1 \,\right) ..$$

Now we can calulate all scalar products and all lengths.

For example:

It is useful to calculate coroots α_{pq} which are Now we can perfoirm calculations, for example

$$|\alpha^{12}|^2 == \langle\alpha^{12}, \alpha^{12}\rangle_{CK} = \phi(\alpha^{12}, \alpha^{12}) = \alpha^{12}(\alpha_{12}) = \frac{1}{2n}\left(\, 1, -1, 0, 0\dots, 0, 0 \,\right)\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ \dots \\ 0 \\ 0 \end{pmatrix} = \frac{1}{n}\,,$$

$$\langle\alpha^{12}, \alpha^{23}\rangle_{CK} = \phi(\alpha^{12}, \alpha^{23}) = \frac{1}{2n}\left(\, 1, -1, 0, 0\dots, 0, 0 \,\right)\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \dots \\ 0 \\ 0 \end{pmatrix} = -\frac{1}{2n}$$

and angle between roots is eqaul to $\frac{2\pi}{3}$.

One can see that all the simple roots have the same length $\frac{1}{\sqrt{n}}$.

Now caculate the Weyl covector ρ which is equal to halph of the sum of positive roots:

$$\rho = \frac{1}{2}\sum_{p<q}\alpha^{pq} = \frac{1}{2}\sum_{p<q}\left(\sum_{i=p}^{q-1}\alpha^{i,i+1}\right) = \sum_{k=1}^{n-1}k(n-k)\alpha_{k,k+1}\,.$$

The vector ρ' which is dual to covector ρ is given by the formula

$$\rho' = \frac{1}{2}\left((n-1)\alpha_{12} + 2(n-2)\alpha_{23} + 3(n-3)\alpha_{34} + \ldots + (n-2)2\alpha_{n-2,n-1} + (n-1)\alpha_{n-1,n}\right) =$$

$$\frac{1}{4n}\,(\,n-1, n-3, n-5, \ldots, -n+3\,)$$

Now we are able to caclulate the volume.

Later Cartan-Killing scalar product we will denote just by $\langle\ ,\ \rangle$.

We have

$$\langle\rho,\alpha^{12}\rangle = \alpha^{12}(\rho') = \begin{pmatrix} 1 \\ -1 \\ 0 \\ \cdots \\ 0 \end{pmatrix}\frac{1}{4n}\,(\,n-1, n-3, n-5, \ldots, -n+3\,) = \frac{1}{2n}$$

The same is for all other simple roots including the root $\alpha^{n-1,n}$:

$$\langle\rho,\alpha^{12}\rangle = \langle\rho,\alpha^{23}\rangle = \ldots\langle\rho,\alpha^{n-1,n}\rangle = \frac{1}{n}$$

Hence we have that for an arbitrary root α^{pq}

$$\langle\rho,\alpha_{pq}\rangle = \langle\rho,\alpha^{p,p+1} + \ldots + \alpha^{q-1,q} = \frac{q-p}{2n}\,.$$

Now return to the formula for volume of the group G_n which is compact connected simply connected Lie group such that complexification of its Lie algebra is equal to $gl(n,\mathbf{C})$.

Kac-Peterson formula and our caculationn give

$$Vol(G_n) = (2\pi\sqrt{2})^{dimG_n}\prod_{\alpha^{pq}:p<q}\frac{\sin\left(2\pi\langle\rho,\alpha^{pq}\rangle\right)}{2\pi\langle\rho,\alpha^{pq}\rangle}$$

$$VolG = (2\pi\sqrt{2})^{n^2-1}\prod_{k=1}^{n-1}\left(\frac{\sin\left(\frac{k\pi}{n}\right)}{\frac{k\pi}{n}}\right)^{n-k}\,.$$