Orthogonal matrices with rational coefficients...

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The best is to use Cayley transfromation

$$g = (1-X)^{-1}(1+X)$$

where

$$X = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

is an element in $so(3)$. We have

$$(1+X)^{-1} = \begin{pmatrix} 1 & a & b \\ -a & 1 & c \\ -b & -c & 1 \end{pmatrix} = \frac{1}{\det(1+X)} \begin{pmatrix} 1+c^2 & -a-bc & ac-b \\ a-bc & 1+b^2 & -c-ab \\ ac+b & c-ab & 1+a^2 \end{pmatrix} =$$

$$\frac{1}{1+a^2+b^2+c^2} \begin{pmatrix} 1+c^2 & -a-bc & ac-b \\ a-bc & 1+b^2 & -c-ab \\ ac+b & c-ab & 1+a^2 \end{pmatrix}.$$

Hence

$$SO(3) \ni g = \frac{1}{1+a^2+b^2+c^2} \begin{pmatrix} 1+c^2 & -a-bc & ac-b \\ a-bc & 1+b^2 & -c-ab \\ ac+b & c-ab & 1+a^2 \end{pmatrix} \begin{pmatrix} 1 & -a & -b \\ a & 1 & -c \\ b & c & 1 \end{pmatrix} =$$

$$\frac{1}{1+a^2+b^2+c^2} \begin{pmatrix} 1+c^2-a^2-b^2 & -2a-2bc & 2ac-2b \\ 2a-2bc & 1+b^2-a^2-c^2 & -2c-2ab \\ 2ac+2b & 2c-2ab & 1+a^2-b^2-c^2 \end{pmatrix}$$

The funny fact (Sasha Karabegov told me about it ‘hundred years’ ago): If $||a_{ik}||$ $i,k = 1,2,3$ is orthogonal 3×3 matrix, and all entries are not equal to zero then for matrix $||b_{ik||}$ such that $b_{ik} = \frac{1}{a_{ik}}$ then

$$\det ||b_{ik}|| = 0$$

In particular

$$\det \begin{pmatrix} \frac{1}{1+c^2-a^2-b^2} & \frac{1}{-2a-2bc} & \frac{1}{2ac-2b} \\ \frac{1}{2a-2bc} & \frac{1}{1+b^2-a^2-c^2} & \frac{1}{-2c-2ab} \\ \frac{1}{2ac+2b} & \frac{1}{2c-2ab} & \frac{1}{1+a^2-b^2-c^2} \end{pmatrix} = 0\,.???!!!$$

(if all denominators do not vanish)