Locally Euclidean geometries and hyperbolic geometry

H.M.Khudaverdian

`This talks is based on the book of S.Nikuin, R.Shafarevich`

Abstract: *Geometry on the surface of the cylinder is locally Euclidean. An "ant-mathematician" who lives on the cylinder will not distinguish the geometry of the surface at small distances from the Euclidean geometry; the Pythagorean Theorem will be almost the same, and for "not too large" triangles the sum of the angles will be π. In the first part of talk, we will study locally Euclidean two-dimensional geometries. We will study these geometries by using discrete subgroups of the isometry group of the Euclidean plane E^2. The list of locally Euclidean geometries is exhausted by the geometries on the surface of the cylinder, on the surface of the torus, on the surface of the "twisted cylinder" (the Moebius band), and on the so-called Klein bottle. In the second part of the talk, we will consider the set of locally Euclidean geometries, and will show that this set can be naturally parametrized by the points of the Lobachevsky (hyperbolic) plane.*

We consider locally Euclidean 2-dimensional geometries. An arbitrary 2-dimensional geometry can be considered ass 2-dimensional Riemannian surface—(M, G). M is a surface, and G defines scalar product of tangent vectors, i.e. length of an arbitrary curve. For arbitrary curve $\mathbf{x} = \mathbf{x}(t), t_1 \leq t \leq t_2$

$$\text{length of the curve} = \int_{t_1}^{t_2} \sqrt{(\mathbf{v}(t), \mathbf{v}(t))} dt\,, \text{ scalar product } (\mathbf{v}(t), \mathbf{v}(t)) = G(\mathbf{v}(t), \mathbf{v}(t))\,,$$

where $\mathbf{v}(t)$ is velocity vector. In local coordinates x^i, the curve has appearance $\mathbf{x}(t) = x^i(t)$, $G = g_{ik}(x)dx^i dx^k$, $\mathbf{v}(t) = v^i(x(t))\partial_i = \frac{dx^i(t)}{dt}\frac{\partial}{\partial x^i}$, and the scalar product of velocity vector on itself is equal to

$$(\mathbf{v}, \mathbf{v}) = v^i(x(t))g_{ik}(x(t))v^k(x(t)) = \frac{dx^i(x(t))}{dt} g_{ik}(x(t)) \frac{dx^k(x(t))}{dt}\,,$$

i.e.

$$\text{length of the curve} = \int_{t_1}^{t_2} \sqrt{(\mathbf{v}(t), \mathbf{v}(t))} dt = \int_{t_1}^{t_2} \sqrt{\frac{dx^i(x(t))}{dt} g_{ik}(x(t)) \frac{dx^k(x(t))}{dt}}\, .dt$$

We say that M is (uniformly) locally Eucldean if

1) in a vicinity of arbitrary points there are Eucldean coordiantes, i.e. the coordinates u, v such that $G = du^2 + dv^2$ in these coordinates

2) This neighborhood is enough large: there exists $\delta > 0$ i.e. these coordinates u, v are defined in the circle of radius δ

Remark Surface is called locally Euclidean if the first condition is obeyed. In the case if the second condition is obeyed also, the surface is called *uniformly locally Euclidean.*

Exercise Show that the surface of sphere is not locally Eucldean.

Exercise Show that domain $a < x < b$ of $\mathbf{E}^2$ is locally Eucldean but it is not uniformly locally Eucldean.

(We suppose that metric on the surface M in $\mathbf{E}^3$ is the metric induced from $\mathbf{E}^3$.)

In this talk we will consider only uniformly locally Euclidean surface, and we will call them just locally Eucldean.

First of all examples of locally Euclidean surfaces.

We will consider the following set of examples.

Let Γ be subgroup of isometries of $\mathbf{E}^2$. We say that the subgroup Γ acts properly discontinuous on $\mathbf{E}^2$ if there exists δ such that for an arbitrary point $A \in \mathbf{E}^2$, and for an arbitrary non-identity element $g \in \Gamma$

$$d(A, g(A)) \geq \delta$$

In other words it means that the distance between distinct points of an arbitrary orbit exceeds the δ.

Why these groups are interesting? Because every such group defines locally Eucldean manifold.

Indeed, let Γ be an arbitrary subgroup of group of isometries of $\mathbf{E}^2$.

Assign to the group K a space M_Γ of orbits of G-group action on $\mathbf{E}^2$, $M_\Gamma = \mathbf{E}^2 \backslash \Gamma$, i.e.

The points of the space M_Γ, the orbits, we denote by handwriting letters $\mathcal{A}, \mathcal{B}, C, \mathcal{D}, \ldots$

Every point $A \in \mathbf{E}^2$ produces the point $\mathcal{A} \in \mathbf{E}^2 \backslash \Gamma$, the equivalence class of a point A with respect to the gropu Γ:

$$\mathcal{A} = [A]_\Gamma \,, \qquad g \in \Gamma \,, A' = g(A) \in [A] \,.$$

To establish the geometry on M we define the distance between points $\mathcal{A}, \mathcal{B}$ as the minimal distance between the orbit $\{A^g\}$ and $\{B^g\}$:

$$\text{if } \mathcal{A} = [A] \text{ and } \mathcal{B} = [B] \text{ then } d(\mathcal{A}, \mathcal{B}) = \min_{g, g' \in \Gamma} d(A^g, B^{g'})$$

Exercise 1 Let Γ be group of rweflection with resepct to the line l.

Describe the geometry M_Γ and show that this is not uniformly locally Euclidean manifold.

Theorem M_Γ is uniformly locally Eucldean, if and only if the group Γ is uniformly discontinuous.

`Sketch of the proof.`

Let Γ be uniformly discontinuous:

$$\exists \delta \geq 0\,,\,,\text{such that for an arbitrary } A \in \mathbf{E}^2\,, g \in \Gamma\,, d(A^g, A) < \delta \Rightarrow g = 1\,.$$

and let B be an arbitrary point which belong to the disc $D_{\frac{\delta}{r3}}(A)$ Consider orbits $\mathcal{A}$ and $\mathcal{B}$ of these points. It is easy to see from triangle inequality that for arbitrary points $A' \in \mathcal{A}$ and $B' \in \mathcal{B}$ the distance between these points is bigger or equal to $r = \frac{\delta}{2}$. Indeed let $A" = A^g$, and $B' = B^h$. Denote $\tilde{B} = B^{hg^{-1}}$. Then $d(A', B') = d\left(A, \left(B^h\right)^{g^{-1}}\right) = d\left(A, \tilde{B}\right)$, and by triangle inequality

$$d(A', B') = d(A, \tilde{B}) \geq \left| d(B, \tilde{B}) - d(A, B) \right| > \delta$$

if $B \neq \tilde{B}$.

Thus we see that in the case if two points A and B are closer than $\frac{\delta}{2}$, then the distance between orbits $\mathcal{A}$ and $\mathcal{B}$ is equal to the distance $d(A, B)$. This proves that M_Γ is uniformly locally Euclidean if Γ is uniformly discontinous.

the orbit of the point B and the orbit of the point A the disc of radius