18.01.2012

On calculation of one determinant

Let $\|a_{ik}\|$ be $n\times n$ matrix.
Calculate determinant of $(n+1)\times(n+1)$ matrix:

$$\det\left(\begin{array}{c|c} a_{ik} & u_i \\ \hline v_k & 0 \end{array}\right)$$

$$\det\left(\begin{array}{c|c} a_{ik} & u_i \\ v_k & 0 \end{array}\right) = \lim_{\varepsilon\to 0}\det\left(\begin{array}{c|c} a_{ik} & u_i \\ v_k & \varepsilon \end{array}\right) =$$

$$= \lim_{\varepsilon\to 0}\det\left(a_{ik} - \frac{u_i v_k}{\varepsilon}\right)\cdot\varepsilon =$$

$$= \lim_{\varepsilon\to 0}\left[\det a_{ij}\,\det\left(\delta - \frac{a^{im}u_m v_k}{\varepsilon}\right)\varepsilon\right] =$$

$$= \lim_{\varepsilon\to 0}\det a\left[1 + \mathrm{Tr}\left(\frac{a^{im}v_m u_k}{\varepsilon}\right) + O(\varepsilon)\right]\varepsilon =$$

$$\lim_{\varepsilon\to 0}\det a\left[1 + \frac{a^{im}v_m u_i}{\varepsilon} + O(\varepsilon)\right]\varepsilon =$$

We do not need to calculate higher traces!

$$= (\det a)\, a^{im} v_m u_i \qquad (a_{ir}a^{rj} = \delta_i^j)$$

Geom. meaning: If $a_{ik}x^i x^k = 0$ $(i = 1,\dots,n)$ quadric in $\mathbb{P}^{n-1}$
then $\det\left(\begin{array}{c|c} a_{im} & u_i \\ u_m & 0 \end{array}\right) = 0$ defines pencil of lines tangent to this quadric.