Afiine and linear connections

Linear connection are known to everybody in theoretical physics under the name of so called "Christofells symbols". Here I will try to explain what is it affine connection and its relations to linear connections. Respectively we will try to consider relations between Euclidean connection and Levi-Chivita connection.

§0 Some standard stuff

Let (P, π, M, G) be a principal bundle over manifold M with structure group G. M is a base, π is a projection on the base M, Lie group G is a structure group. Structure group G acts on the right transitively and free on all fibres $\pi^{-1}(x)$. The base can be covered by open sets $\{V_\alpha\}$ $M = \cup_\alpha V_\alpha$ such that all $\pi^{-1}V_\alpha$ are homeomorphic to $V_\alpha \times G$:

$$\pi^{-1}(V_\alpha) \longleftrightarrow V_\alpha \times G \tag{0.1}$$

Let $\{p_\alpha(x)\}$ ($p_\alpha(x)\colon V_\alpha \to P$ such that $\pi \circ p_\alpha = \mathbf{id}|_{V_\alpha}$) be the collection of local sections *. Thus we come to local coordinates $\{(x^\mu_{(\alpha)}, g_{(\alpha)})\}$ on the domains $\{\pi^{-1}(V_\alpha)\}$ of the bundle:

$$(x^\mu_{(\alpha)}, g_{(\alpha)}) \xmapsto{\varphi_\alpha} p_\alpha(x) \circ g \tag{0.2}$$

where $x^\mu_{(\alpha)}$ are local coordinates of points in V_α (in some atlas on M), $\mu = 1, 2, \ldots, n$, where n is a dimension of M and $g_{(\alpha)} \in G$, (index α later will be often omitted)**. With some abuse of notations we often identify points $\mathbf{x}$ in V_α with there local coordinates $x^\mu_{(\alpha)}$.

Consider local transition functions $\Psi_{\alpha\beta}$: $(V_\alpha \cap V_\beta) \times G \to (V_\alpha \cap V_\beta) \times G$: $\Psi_{\alpha\beta} = \varphi_\alpha^{-1} \circ \varphi_\beta$. One can see that they following:

$$\Psi_{\alpha\beta}(x, g) = (x, h_{\alpha\beta}(x) \circ g), \quad \text{where } h_{\alpha\beta}(x)\colon\ p_\beta(x) = p_\alpha(x) \circ h_{\alpha\beta} \tag{0.3}$$

Indeed

$$\Psi_{\alpha\beta}(x, g) = \varphi_\alpha^{-1} \circ \varphi_\beta(x, g) = \varphi_\alpha^{-1}(p_\beta(x)g) = \varphi_\alpha^{-1}(p_\alpha(x)h_{\alpha\beta}(x)g) = (x, h_{\alpha\beta}(x)g) \quad (0,3)$$

Remark Please note that in the equation (0.3) the action of group on fibre coordinates is left multiplication.— It is passive action of changing coordinates of the fibre. In the definition of the fibre bundle there is right action of the group: it is an active action– the point is changed.

Connection in the principal bundle

One can define a connection—one-form Ω on P with values in the Lie algebra $\mathcal{G}(G)$ of the group G such that

1. $\Omega(\tilde{\xi}) = \xi$ if the vertical vector $\tilde{\xi}$ is tangent to the curve $p \circ (1 + t\xi + \ldots)$ at the point $t = 0$

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* One can define local section over V_α by formulae $p_\alpha(x) = \psi_\alpha(x, g_\alpha(x))$ where ψ_α establish homeomorphism (0.1) and $g_\alpha(x)$ are arbitrary (smooth) functions, e.g. $g_\alpha(x) \equiv 1$.

** the homeomorphisms $\{\psi_\alpha\}$ in (0.1) provide coordinatisation for the choice of local sections $p_\alpha(x) = \psi(x, e)$ (see the first footnote)

2. It is invariant with respect to the action of group G: $R_g^*\Omega = Ad_{g^{-1}}\Omega$

We say that the vector $\mathbf{X} \in TP$ is *horisontal* vector if $\Omega(\mathbf{X}) = 0$.

The tangent space T_pP for an arbitrary point p in the bundle P is a direct sum of vertical subspace and horisontal subspace:

$$T_pP = T_p^{\rfloor}P \oplus T_p^{||}P \qquad \text{and } \ R_gT_p^{||} = T_{pg}^{||}$$

In local coordinates

$$\Omega = g^{-1}dg - g^{-1}A_\mu^{(\alpha)}dx^u g$$

where $A_\mu^{(\alpha)}dx^u$ is local one-form with values in the Lie algebra $\mathcal{G}(G)$. ($A_\mu^{(\alpha)}(x)$ is sometimes called Yang Mills field.)

§1 **Linear connection on the linear frames bundle**

Let M be a manifold and $P(M)$ the principal bundle of linear frames, i.e.