On canonical isomorphisms $T^*E = T^*E^*$ for vector bundle E

§0

Kirill Mackenzie told and explained many many times the construction of remarkable isomorphism between cotangent bundles of vector bundle and its dual: for an arbitrary vector bundle E

$$T^*E = T^*E^*\,, \tag{0.1}$$

where E^* is a bundle dula to E.

The first thing that you try to apply this constrution to it is to consider tangent bundle: $E = TM$ or cotangent bundle $E = T^*M$ and consider isomorphism

$$T^*TM = T^*T^*M \tag{0.2}$$

On the other hand in this special case the canonical symplecctic structure on the cotangent bundle T^*M implies the canonical isomorphism between tangent and cotangent bundles of the manifold T^*M:

$$T^*T^*M = TT^*M \tag{0.3}$$

Hence in the special case of $E = TM$ the "Mackenzie" isomorphism (0.1) induces the canonical isomorphisms:

$$T^*T^*M = T^*TM = TT^*M\,. \tag{0.4}$$

In particual we see that in this case there is a canonical isomorphism

$$T^*TM = TT^*M\,. \tag{0.5}$$

In this etude we would like to reconstruct the "Mackenzie" isomorphism (0.1) and its special case the isomorphisms (0.4) and (0.5) using pedestrian's arguments. In the first section we conisder the special case $E = TM$ and establish the isomorphisms (0.5) and (0.4)

Our notations are little bit inconsistent: in the first paragraph we denote indices of coordinates by latin letters i, j, k and new ones by greek letters. In the second paragraph our notations are much more traditional: indices of new coordinates are denoted by the same letters with "prime" indices ($x^i \to x^{i'}$).

Canonical isomorphism $TT^*M = T^*TM$

Let M be manifold. Establish and study canonical isomorphisms $TT^*M = T^*TM = T^*T^*M$.

Perform calculations in local coordinates. It may sounds surprising but calculations in local coordinates are transparent and illuminating.

First consider local coordinates on TM and T^*M corresponding to local coordinates (x^i) on M. Local coordinates for TM are (x^i, t^j): every vector $\mathbf{r} \in TM$ is a vector $t^i \frac{\partial}{\partial x^i}$, $t^i(\mathbf{r}) = dx^i(\mathbf{r})$. If $\tilde{x}^\mu = \tilde{x}^\mu(x^i)$ are new local coordinates on M then

$$d\tilde{x}^\mu \left(t^i \frac{\partial}{\partial x^i}\right) = \frac{\partial \tilde{x}^\mu(x^i)}{\partial x^i} dx^i \left(t^i \frac{\partial}{\partial x^i}\right) = \frac{\partial \tilde{x}^\mu(x^i)}{\partial x^i} t^i \,. \tag{1.1}$$

Hence changing of local coordinates in TM is

$$(x^i, t^j) \mapsto (\tilde{x}^\mu, \tilde{t}^\nu), \quad \tilde{x}^\mu = \tilde{x}^\mu(x^i),\ \tilde{t}^\mu = \begin{pmatrix} \mu \\ i \end{pmatrix} t^i, \tag{1.2}$$

where we denote $\frac{\partial \tilde{x}^\mu(x^i)}{\partial x^i}$ by $\begin{pmatrix} \mu \\ i \end{pmatrix}$.

Respectively local coordinates for T^*M are (x^i, p_j). For every 1-form $\omega \in T^*M$ $p_i = \omega\left(\frac{\partial}{\partial x^i}\right)$. Under changing of local coordinates on M $\tilde{x}^\mu = \tilde{x}^\mu(x^i)$, coordinates (p_i) change to new coordinates (p_μ):

$$p_\mu = w\left(\frac{\partial}{\partial \tilde{x}^\mu}\right) = \omega\left(\frac{\partial x^i(\tilde{x}^\mu)}{\partial \tilde{x}^\mu} \frac{\partial}{\partial x^i}\right) = \frac{\partial x^i(\tilde{x}^\mu)}{\partial \tilde{x}^\mu} p_i$$

Hence changing of local coordinates in T^*M is

$$(x^i, p_k) \mapsto (\tilde{x}^\mu, \tilde{p}_\nu), \quad \tilde{x}^\mu = \tilde{x}^\mu(x^i),\ p_\mu = \begin{pmatrix} i \\ \mu \end{pmatrix} p_i, \tag{2}$$

where we denote $\frac{\partial x^i(\tilde{x}^\mu)}{\partial \tilde{x}^\mu}$ by $\begin{pmatrix} i \\ \mu \end{pmatrix}$

Now using (1),(2) we define coordinates on the spaces TT^*M, T^*TM and T^*T^*M.

The space TT^*M is tangent space to the space T^*M. The local coordinates on TT^*M corresponding to local coordinates (x^i, p_j) on T^*M are coordinates $(x^i, p_j; \xi^k, \rho_m)$; $\xi^k = dx^i(\mathbf{r}), \rho_m = dp_m(\mathbf{r})$. Under changing of local coordinates (x^i) to coordinates $\tilde{x}^\mu = \tilde{x}^\mu(x^i)$ coordinates (ξ^i) and (ρ_m) transform to new coordinates $(\tilde{\xi}^\mu)$ and $(\tilde{\rho}_\nu)$ respectively. It follows from (1) that

$$\tilde{\xi}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^i} \xi^i + \frac{\partial \tilde{x}^\mu}{\partial p_i} \rho_i = \begin{pmatrix} \mu \\ i \end{pmatrix} \xi^i \tag{3}$$

because $\frac{\partial \tilde{x}^\mu}{\partial p_i} = 0$. For transformation of coordinates (ρ_m) calculations are longer:

$$\tilde{\rho}_\mu = \frac{\partial \tilde{p}_\mu}{\partial x^i} \xi^i + \frac{\partial \tilde{p}_\mu}{\partial p_i} \rho_i$$

We see that $\frac{\partial \tilde{p}_\mu}{\partial p_i} = \frac{\partial}{\partial p_i}\left(\tilde{p}_\mu = \begin{pmatrix} k \\ \mu \end{pmatrix} p_k\right) = \begin{pmatrix} i \\ \mu \end{pmatrix}$ and

$$\frac{\partial \tilde{p}_\mu}{\partial x^i} = \frac{\partial}{\partial x^i}\left(\tilde{p}_\mu = \begin{pmatrix} k \\ \mu \end{pmatrix} p_k\right) = \begin{pmatrix} \nu \\ i \end{pmatrix} \begin{pmatrix} k \\ \nu\mu \end{pmatrix} p_k,$$

where we denote as always by $\begin{pmatrix} \nu \\ i \end{pmatrix}$ the partial derivative $\frac{\partial \tilde{x}^\mu}{\partial x^i}$ and by $\begin{pmatrix} k \\ \nu\mu \end{pmatrix}$ the partial derivative $\frac{\partial^2 x^k}{\partial \tilde{x}^\nu \partial \tilde{x}^\mu}$. The summation over repeated indices is assumed. Finally we come to

$$\tilde{\rho}_\mu = \frac{\partial \tilde{p}_\mu}{\partial x^i}\xi^i + \frac{\partial \tilde{p}_\mu}{\partial p_i}\rho_i = \xi^i \begin{pmatrix} \nu \\ i \end{pmatrix} \begin{pmatrix} k \\ \nu\mu \end{pmatrix} p_k + \begin{pmatrix} i \\ \mu \end{pmatrix} \rho_i \tag{4}$$

Summarising:

Proposition 1 *To local coordinates (x^i) on M one can naturally assign local coordinates on TT^*M $(x^i, p_j; \xi^k, \rho_m)$ such that under changing of coordinates $(x^i) \mapsto (\tilde{x}^\mu)$ on M these coordinates transform in the following way*

$$\tilde{p}_\mu = \begin{pmatrix} j \\ \mu \end{pmatrix} p_j, \quad \tilde{\xi}^\mu = \begin{pmatrix} \mu \\ i \end{pmatrix} \xi^i, \quad \tilde{\rho}_\mu = \xi^i \begin{pmatrix} \nu \\ i \end{pmatrix} \begin{pmatrix} k \\ \nu\mu \end{pmatrix} p_k + \begin{pmatrix} i \\ \mu \end{pmatrix} \rho_i \tag{$*$}$$

Now consider coordinates on T^*TM and their transformation rules. If (x, t) coordinates on TM (see (1)) and (x, t, π, τ) corresponding coordinates on T^*TM ($\pi_k = \omega\left(\frac{\partial}{\partial x^k}\right)$, $\tau_m = \omega\left(\frac{\partial}{\partial t^m}\right)$) then according to (2) under changing of coordinates $(x^i) \mapsto (\tilde{x}^\mu)$, the coordinates (π_m) transform to coordinates $(\tilde{\pi}_\mu)$, the coordinates (τ_k) transform to coordinates $(\tilde{\tau}_\nu)$ such that

$$\tilde{\pi}_\mu = \frac{\partial x^i}{\partial \tilde{x}^\mu}\pi_i + \frac{\partial t^k}{\partial \tilde{x}^\mu}\tau_k, \quad \tilde{\tau}_\nu = \frac{\partial x^i}{\partial \tilde{t}^\nu}\pi_i + \frac{\partial t^k}{\partial \tilde{t}^\nu}\tau_k$$

Since $\frac{\partial x^i}{\partial \tilde{t}^\nu} = 0$ and $\frac{\partial t^k}{\partial \tilde{t}^\nu} = \frac{\partial x^k}{\partial \tilde{x}^\mu}$ then

$$\tilde{\tau}_\nu = \begin{pmatrix} k \\ \nu \end{pmatrix} \tau_k$$

. For $\tilde{\pi}_\mu$ we have

$$\tilde{\pi}_\mu = \begin{pmatrix} i \\ \mu \end{pmatrix} \pi_i + \frac{\partial}{\partial \tilde{x}^\mu}\left(\frac{\partial x^k}{\partial \tilde{x}^\nu} t^\nu\right)\tau_k = \begin{pmatrix} i \\ \mu \end{pmatrix} \pi_i + \begin{pmatrix} k \\ \mu\nu \end{pmatrix} \begin{pmatrix} \nu \\ i \end{pmatrix} t^i \tau_k \,.$$

Summarising:

Proposition 2 *To local coordinates (x^i) on M one can naturally assign local coordinates on T^*TM $(x^i, t^j; \pi_k, \tau_j)$ such that under changing of coordinates $(x^i) \mapsto (\tilde{x}^\mu)$ on M these coordinates transform in the following way*

$$\tilde{\tau}_\mu = \begin{pmatrix} j \\ \mu \end{pmatrix} \tau_j, \quad \tilde{t}^\mu = \begin{pmatrix} \mu \\ i \end{pmatrix} t^i, \quad \tilde{\pi}_\mu = t^i \begin{pmatrix} \nu \\ i \end{pmatrix} \begin{pmatrix} k \\ \nu\mu \end{pmatrix} \tau_k + \begin{pmatrix} i \\ \mu \end{pmatrix} \pi_i \tag{$**$}$$

Comparing Propositions 1 and 2 we come to

Observation 1

Let (x^i, t^i) be local coordinates on TM and respectively (x^i, p_i) be local coordinates on T^*M. Respectively one can consider local coordinates $(x^i, t^i; \pi_k, \tau_j)$ on T^*TM and $(x^i, p_i; \xi^k, \rho_m)$ on T^*TM. The map

$$t^i = \xi^i, \quad \tau_j = p_j, \quad \pi_k = \rho_k \tag{1.*}$$

establishes isomorphism between the spaces T^*TM and TT^*M which does not depend on the choice of local coordinates*.

Note canonical symplectic structure $\Omega dp_i \wedge dx^i$ establishes isomorphism between spaces TT^*M and T^*T^*M: if $(x^i, p_i; r_m, \kappa^r))$ are local coordiantes on T^*TM then this isomorphism looks like

$$\rho_m = r_m, \xi^i = -\kappa_i, \tag{1.**}$$

Combining with isomorphism (1.*) we see that isomoprphism $T^*TM = T^*T^*M$ looks like

$$(x^i, t^i, \pi_k, -p_i) \leftrightarrow (x^i, p_i; \pi_k, t^i) \tag{1.***}$$

Of course there is two-parametric freedom related with (footnote).

§2 **Canoncial isomorphism in general case:** $T^*E = T^*E$

In the previous paragraph we constructed canonical isomorphism $T^*E = T^*E$ in the case where vector bundle E is tangent (cotangent bundle).

Now cosndier general case. Let (x^μ, s^i) be local coordiantes on bundle E. Under changing of coordinates $x^{\mu'} = x^{\mu'}(x)$ fibre coordiantes s^i transform in the following way:

$$s^{i'} = \Psi^{i'}_k(x) s^k \,.$$

Respectively dual fibre coordinates s_k transform as:

$$s_{k'} = \Phi^i_{k'} s_i \,,$$

where matrices Ψ and Φ are inverse to each other: $\Psi^{i'}_k \Phi^k_{j'} = \delta^{i'}_{j'}$.

Let $(x^\mu, s^i; \rho_\mu, \pi_i)$ be local coordinates in T^*E and respectively let $(x^\mu, s_i; \zeta_\mu, t^i)$ be local coordinates in T^*E^*. Reccall that as usual we say that if $\omega \in T^*E$ is 1-form with

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* In fact one can consider the *pencil* of maps

$$t^i = \mathbf{a}\xi^i, \quad \tau_j = \mathbf{b} p_j, \quad \pi_k = \mathbf{ab}\rho_k \tag{footnote}$$

where $\mathbf{a}, \mathbf{b} \neq 0$.

local coordinates $(x^\mu, s^i; \rho_\mu, \pi_i)$ if it is the function on vectors tangent to the manifold E at the point (x^μ, s^i) such that

$$\omega\left(\frac{\partial}{\partial x^\mu}\right) = \rho_\mu, \quad \omega\left(\frac{\partial}{\partial s^i}\right) = \pi_i$$

Respectively we say that $\omega \in T^*E^*$ is 1-form with coordinates $(x^\mu, s_i; \zeta_\mu, t_i)$ if it is the function on vectors tangent to the manifold E^* at the point (x^μ, s_i) such that

$$\omega\left(\frac{\partial}{\partial x^\mu}\right) = \zeta_\mu, \quad \omega\left(\frac{\partial}{\partial s_i}\right) = t_i\,.$$

Write down transformation for fields under coordiante trasnformation $x^{\mu'} = x^{\mu'}(x^\mu)$. We have

$$\begin{cases} s^{i'} = \Psi^{i'}_k(x) s^k \\ \rho_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\rho_\mu + \frac{\partial s^i}{\partial x^{\mu'}}\pi_i \\ \pi_{i'} = \frac{\partial x^\mu}{\partial s^{i'}}\rho_\mu + \frac{\partial s^i}{\partial s^{i'}}\pi_i \end{cases}, \qquad \begin{cases} s_{i'} = \Phi^k_{i'}(x) s_k \\ \zeta_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\zeta_\mu + \frac{\partial s_i}{\partial x^{\mu'}}t^i \\ t^{i'} = \frac{\partial x^\mu}{\partial s_{i'}}\zeta_\mu + \frac{\partial s_i}{\partial s_{i'}}t^i \end{cases} \tag{2.1}$$

Note that

$$\frac{\partial x^\mu}{\partial s^{i'}} = \frac{\partial x^\mu}{\partial x^{\mu'}} = 0, \ \frac{\partial s^i}{\partial s^{i'}} = \Phi^i_{i'}, \ \frac{\partial s_i}{\partial s_{i'}} = \Psi^{i'}_i\,,$$

and

$$\frac{\partial s^i}{\partial x^{\mu'}}\pi_i = \frac{\partial \Phi^i_{i'}}{\partial x^{\mu'}} s^{i'}\pi_i = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial \Phi^i_{i'}}{\partial x^\mu} s^{i'}\pi_i = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial \Phi^i_{i'}}{\partial x^\mu}\Psi^{i'}_k s^k \pi_i\,, \tag{2.2a}$$

$$\frac{\partial s_i}{\partial x^{\mu'}}t^i = \frac{\partial \Psi^{i'}_i}{\partial x^{\mu'}} s_{i'} t^i = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial \Psi^{i'}_i}{\partial x^\mu} s_{i'} t^i = \frac{\partial x^\mu}{\partial x^\mu}\frac{\partial \Psi^{i'}_i}{\partial x^\mu}\Phi^k_{i'} s_k t^i\,, \tag{2.2b}$$

Introducing $L^i_{\mu k}$ such that

$$L^i_{\mu k} = \frac{\partial \Psi^{i'}_k}{\partial x^\mu}\Phi^i_{i'} = -\frac{\partial \Phi^i_{i'}}{\partial x^\mu}\Psi^{i'}_k =, \text{ since } \Psi \circ \Phi = 1$$

we see that the transformations (2.1) have the following appearance:

$$\begin{cases} s^{i'} = \Psi^{i'}_k(x) s^k \\ \rho_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\left(\rho_\mu - L^i_{\mu k} s^k \pi_i\right) \\ \pi_{i'} = \Phi^i_{i'}\pi_i \end{cases}, \qquad \begin{cases} s_{i'} = \Phi^k_{i'}(x) s_k \\ \zeta_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\left(\zeta_\mu + L^i_{\mu k} t^k s_i\right) \\ t^{i'} = \Psi^{i'}_i t^i \end{cases} \tag{2.3}$$

Put

$$\begin{cases} \pi_i = \alpha s_i \\ t^i = \beta s^i \\ \zeta_\mu = \gamma \rho_\mu \end{cases}$$

We see that this map is invariant with respect to changing of coordinates if $\beta = -\gamma\alpha$. In particular we can put $\alpha = -1$, $\beta = \gamma = 1$ We come to isomorphism $T^*E = T^*E^*$ defined in local coordinates by condition that

$$(x^\mu, s^i; \rho_\mu, \pi_i) \leftrightarrow (x^\mu, s_i; \zeta_\mu, t^i), \quad \text{such that } \rho_\mu = \zeta_\mu, \pi_i = -s_i, t^i = s^i$$

(Compare with (1.***))