Dear Tian, yesterday we have so interesting talk, I absolutelyu forgot to speak little bit about Shur lemma.

Shur Lemma

It is the powerful tool in representation theory. The statement and proof is very short and beautiful.

Consider an arbitrary group G. We say that T is linear representation of the group G in a vector space V if

$$g \in G \mapsto T_g\colon \quad V \to V \text{ is linear map and } T_{g_1} \circ T_{g_2} = T_{g_1 g_2}\,.$$

We say that U is invariant subspace of V if $T_g U \to U$ for every g. Sure, zero subspace and space V itself are invariant subspaces. We call them trivial subspaces.

We say that representation $[G, V]$ is *irreducible* if vector space V does not possess non-trivial invariant subspaces.

Let $[G, M]$ and $[G, N]$ are two linear representations of a given group G in vector spaces M and N.

We say that linear operator $F\colon M \to N$ is an interwinning operator if it commutes with an action of group G:

$$\forall g \in G\,, T_g \circ F = F \circ \tilde{T}_g\,.$$

We say that two representations are equivalent if there exist interwinning operator $F: M \to N$.

Lemma (Shur)

Let G, M be $[G, N]$ two *irreducible* linear representations of a given group G in finite-dimensional vector spaces M and N. Then

1. These representations are not equivalent if $\dim M \neq \dim N$.

2. If $M = N$ and basic field is algebraically closed then interwinning operator is identity operator (up to a scalar multiplier).

Shur lemma says that equivalent representations have the same dimension, and intewinning operator is defined up to a scalar operator.

Proof. Let $[G, M]$ be $[G, N]$ two linear representations of a given group G in vector spaces M and N, and let $F\colon M \to N$ be an interwinning operator between these spaces. Then evidently kernel of F and image of F are invariant subspaces. This means that kernel of F is zero subspace in M and image of F is N If these representations are irreducible, i.e. F is isomorphism of M on N

Let $M = N$ and basic field be closed. Suppose F is interwinning operator. If $F\ \emptyset$ then It has at least one non-zero eigenvector (with eigenvalue λ which is a root of polynomial $\det(F - \lambda I)$). Consider invariant subspace of M which is a kernel of interwinning operator

$F - \lambda I$. It not zero, since it contains eigenvector. Hence it has to coincide with M, since representation is irreducible. Hence $F = \lambda I$.

This lemma is a basis of representation theory.

I just would like to give you one problem related with this lemma.

Question Show that for an arbitrary N there exist two $N \times N$ matrices A, B such that for an arbitrary $N \times N$ matrix C, the condition

$$[C, A] = CA - AC = [C, B] = CB - BC = 0$$

implies that C is equal to scalar matrix λI.