Characteristics and bicharacteristics. Solution of equation $\Phi(x,u,p)=0$

Introduction: Characteristic

Let us recall simple things.

Characteristic is vector field (direction) tangent to the surface. It appears naturally when we solve linear differential equation

$$A^a\partial_a u = C \tag{1}$$

Solution $u=u(x)$ of this equation defines hypersurface M_u in $\mathbf{R}^{n+1}$ (with coordinates $(x^1,\dots,x^n,u)$). Equation (1) can be rewritten as

$$M\colon\quad \mathcal{L}_{\mathbf{X}} M_u = 0,\quad \text{where } \mathbf{X} = A^a\partial_a - C\partial_u \text{ is } \textit{characteristic}\,.$$

Let $N=N^{n-1}$ be $n-1$-dimensional surface in $\mathbf{R}^{n+1}$. which is transversal to vector field $\mathbf{Y}$. Then taking exponents of vector field $\mathbf{X}$ we come to the surface M defined in a vicinity of the surface N.

Typical Cauchy problem: Let ϕ be a function defined on the $n-1$-dimensional subspace $L=L^{n-1}\subset\mathbf{R}^n$. Find function u on domain in $\mathbf{R}^{n+1}$, solution of equation (1) such that $u=\varphi$ on this subspace. *Solution* Take a $n-1$-dimensional surface Y defined by the function φ on L. If this surface is transversal to characterisic

Now we go to more delicate matter–bicharacteristic.

First two very important definition.

Let $M=M^n$ be an arbitrary n-dimensional manifold. Consider $2n+1$ dimensional space of first jets $J_1=J_1(M^n,\mathbf{R})$ with local coordinates (x^a,p_b,u). Usually we consider $M^n=\mathbf{R}^n$, n-dimensional affine space,

One can define on $J^1(M,\mathbf{R})$ the contact 1-form α:

$$\alpha = du - p_a dx^a$$

It is is easy to see the invariant meaning of this form: value of α at every vector ξ attached at the point $\mathbf{p}$ is equal to the value of the form $\mathbf{p}$ at the projection of this vector on M^n.

Let $\Phi=\Phi(x^a,p_a,u)$ be an arbitrary (smooth function) in the $2n+1$ dimensional space of first jets $J_1=J_1(\mathbf{R}^n,\mathbf{R})$ with coordinates (x^a,p_b,s). We assign to this function the following ector field on J^1:

$$D_\Phi = \frac{\partial\Phi(x,p,u)}{\partial p^a}\frac{\partial}{\partial x^a} - \left(\frac{\partial\Phi(x,p,u)}{\partial x^a} + p_a\frac{\partial\Phi(x,p,u)}{\partial u}\right)\frac{\partial}{\partial u} + p_a\frac{\partial\Phi(x,p,u)}{\partial p^a}\frac{\partial}{\partial u}\,. \tag{2.0}$$

We explain the geometrical meaning of this formula. At every point $\mathbf{p} = (x, p, u)$ of Jets space the function Φ defines the $2n$-dimensional tangent space $D_{\mathbf{p}}$ of vectors which annihilate the form $d\Phi$ and the $2n$-dimensional tangent space $K_{\mathbf{p}}$ of vectors which annihilate the contact form α. The intersection of these vector spaces $K_{\mathbf{p}}$ and $D_{\mathbf{p}}$ is $2n-1$-dimensional vector space $P_{\mathbf{p}}$ of vectors which annihilate 1-form $d\Phi$ and contact 1-form α:

$$P_{\mathbf{p}} = D_{\mathbf{p}} \cap K_{\mathbf{p}}\,.$$

Now notice that $P_{\mathbf{p}}$ is hyperspace of symplectic space $K_{\mathbf{p}}$. Vector $\mathbf{X}$ is vector in $K_{\mathbf{p}}$ which is symplectoorthogonal to $K_{\mathbf{p}}$. It is defined up to multiplier. In the special case if $2n$-dimensional planes are not transversal, then $\mathbf{X} = 0$.

Consider non-linear first order partial differential equation

$$\Phi\left(x^a, u, \frac{\partial u}{\partial x^b}\right) = 0 \tag{2.1}$$

on function $u(\mathbf{x})$ $(\mathbf{x} \in \mathbf{R}^n)$.

Solution of this equation $u = S(x)$ defines n-dimensional surface, integral of two distributions—-distribution $\mathcal{D}$ of $2n$-dimensional planes which annihilate $d\Phi_u$ and distribution $\mathbf{K}$ of planes which annihilate contact form α.

Thus at every point (except special points) it is defined vector field D_Φ tangent to manifold M_Φ.

This vector field is bicharacteristics.

It is useful write down bicharacterstic in the special case when function $\Phi(x,,p,u) = H(x,p)$ does not depend explicitly on u:

Equation $\Phi(x, \frac{\partial S}{\partial x^i}, S) = 0$	Equation $H(x, \frac{\partial S}{\partial x^i}) = 0$
hypersurface $\Phi(x, p, u) = 0$	Equation $H(x, p) = 0$
in $2n+1$-dim. jets space $J^1(\mathbf{R}^n, R)$	in $2n$-dimensional space
contact 1-form $\alpha = du - p_a dx^a$	symplectic 2-form $\omega = dp_a \wedge dq^a$
for an arbitr. gen. point $\mathbf{p}$ on a surface $\Phi = 0$	for an arbitr. gen. point $\mathbf{p}$ on a surface $H = 0$
D_p is $2n$-dim. plane orthogonal to $d\Phi$	D_p is $2n-1$-dim. plane orthog. to dH
$K_{\mathbf{p}}$ is $2n$-dim.plane orthog. to contact form $d\alpha$	all the tangent plane to $H = 0$
bicharacteristic $\mathbf{X}_\Phi$	bicharacteristic D_H
symplectoorthog. to $K_{\mathbf{p}}$ and $D_{\mathbf{p}}$	symplectoortog. to $D_{\mathbf{p}}$

Properties of bicharacteristic field.

$$\mathcal{L}_{D_\Phi}\alpha = d\Phi + \Phi_u \alpha\,. \tag{$***$}$$

Let Y^{n-1} be a surface in V^n and φ function on Y^{n-1}

We have to find function u such that

$$\Phi(x,u,p) = 0\,,\,,\, u\big|_Y = \varphi$$

'*Solution*

Characteristic vector $\mathbf{X} = D_\Phi$ field,

$$\mathbf{X} = -\frac{\partial \Phi}{\partial p_m}\frac{\partial}{\partial_m} + \left(\frac{\partial \Phi}{\partial x^m} + p_m \frac{\partial \Phi}{\partial y}\right)\frac{\partial}{\partial} + p_r \frac{\partial \Phi}{\partial p_r}\frac{\partial}{\partial u}$$

is

i) tangent to the surface M

ii) it vanishes contact form

iii) it is symplectoorthogonal to all vectors which are contact and tangent

Due to relation (***) the flux of this vector field preserves all this stuff.