A number such that its square finishes on it and Veselov's comment on it

Find a number x such that its square , a number x^2 finishes with it. More precisely this means the following: We say that a number x *has m-digits and it is finished with it* if

$$10^m \leq x < 10^{m+1}\,, and \quad x^2 - x \text{divides } 10^m\,. \tag{1}$$

Remark E.g. a number $x = 0625$ has 4 digits and it is finished with it:

$$625^2 - 625 = 390000\,.$$

This is very old problem for me It is one of the first problems which I solved.

Theorem There are two exactly two sequences

$$a_1, a_2, \ldots, a_n, \ldots = 5, 25, 625,$$

and

$$b_1, a_2, \ldots, a_n, \ldots = 6, 76, 376,$$

such that

1) all the numbers in these sequences obey the cohdition, that their squares finishes by them.

2) Any number a_n in the first sequence possesses not more than n digits: $a_n < 10^n$.

2) Any number such that its square finish by it belongs to the first or to the second sequence

This follows from the following induction statement

Lemma

Suppose by induction that a number a_n contains n digits (zeroes are permitted (see the Remark)) and *it is finished by it*:

$$a_n < 10^n\,, and \quad a_n^2 - a_n \quad \text{divides } 10^n\,. \tag{3}$$

Then there exists a number $x = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ such that a number $a_{n+1} = 10^n x + a_n$ contains $n + 1$ digits and *it is finished by it*:

$$a_{n+1} < 10^{n+1}\,, a_{n+1}^2 - a_{n+1} \quad \text{divides } 10^{n+1}\,. \tag{4}$$

(if $x = 0$ then a number a_{n+1} may "slip" a digit. (see the remark after equation (1).))

Prove the lemma, i.e. prove that (3) implies (4). First note that by inductive hypothesis (3)

$$a_n^2 - a_n = 10^n s_n$$

with s_n integer. Hence

$$a_{n+1}^2 - a_{n+1} = (10^n x + a_n)^2 - (10^n x + a_n) = 10^n \left(10^n x^2 + 2xa_n + s_n - x\right)\,.$$

This expression has to be divisible on 10^{n+1}. Hence one has to choose x such that

$$2xa_n + s_n - x \quad \text{has to be divisible on } 10 \tag{5}$$

Consider examples.
Take $N=1$ and $a_1=5$. $a_1^2-a_1=20$, $s_1=\frac{a_1^2-a_1}{10}=2$. Choose x in (5). We havd

$$2xa_n+s_n-x=2\cdot x\cdot 5+2-x \quad \text{is divisible on } 10 \Rightarrow x=2, a_2=2, a_2=25\,.$$

take $N=2$ and $a_2=25$. $a_2^2-a_2=600$, $x=s_2=\frac{a_2^2-a_2}{100}=6$. $a_3=10^n x+a_2=625$
take $N=3$ and $a_3=625$. $a_2^2-a_2=390.000$, $x=s_2=\frac{a_2^2-a_2}{1000}=390$. $a_4=a_3=625$
take $N=4$ and $a_4=0625$. , $x=s_4=\frac{a_2^2-a_2}{10000}=39$. $a_5=90625$
$N=5$ and $a_5=90625$. , $x=s_5=\frac{a_5^2-a_5}{100000}=39$. $a_5=90625$