iOn Duistermaat-Heckman localisation Theorem

Here we will give a formulation (with supermathematics flavor), the proof and concrete calculations for DH (Dustermaat-Heckman) localisation formula. This etude is based on the paper of Zaboronsky and Schwarz [1] and my etude [2] (see the previous etude on this topic) which was based on calculations of A.Belavin.)

If a form, is invariant with respect to odd vector field $Q = d \circ \iota_{\mathbf{K}} + \iota_{\mathbf{K}} \circ d = \sqrt{\mathcal{L}_{\mathbf{K}}}$ where $\mathcal{L}_{\mathbf{K}}$ is Lie derivative with respect to $U(1)$-vector field $\mathbf{K}$, then integral of this form over manifold M is localised at the zero locus of vector fied K. This is the meaning. of Dustermaat-Heckman localisation formula.

During this text it will always be assumed that M is compact manifold and $\mathbf{K}$ is compact vector field on it, i.e. vector field which generates $U(1)$ action. We denote by

$$Q_{\mathbf{K}} = d + \iota_{\mathbf{K}}\,, \quad \text{in “supernotations” } Q_{\mathbf{K}} = \xi^i \frac{\partial}{\partial x^i} + K^i(x)\frac{\partial}{\partial \xi^i}\,,$$

where $x^i, \xi^i = dx^i$ are local coordiantes on ΠTM.

Odd vector field $Q_{\mathbf{K}}$ is a “square root” of a Lie derivative $\mathcal{L}_K = \iota_{\mathbf{K}} \circ d + d \circ \iota_{\mathbf{K}}$:

$$\mathcal{L}_{\mathbf{K}} = Q_{\mathbf{K}}^2 = \left(\xi^i \frac{\partial}{\partial x^i} + K^i(x)\frac{\partial}{\partial \xi^i}\right)^2 = K^i(x)\frac{\partial}{\partial x^i} + \xi^r \frac{\partial K^i}{\partial \xi^r}\frac{\partial}{\partial \xi^i}\,, \tag{1}$$

or in classical notations

$$\mathcal{L}_{\mathbf{K}} = Q_{\mathbf{K}}^2 = (d + \iota_k)^2 = d \circ \iota_{\mathbf{K}} + \iota_{\mathbf{K}} \circ d\,.$$

We formulate the following version of DH localisation theorem:

Theorem *Let $H = H(x, dx)$ be a $Q_{\mathbf{K}}$-invatriant form on M, i.e.*

$$dH + \iota_{\mathbf{K}} H = 0\,. \tag{2}$$

Then the integral $\int_M H(x, dx)$ is localised at locus of K. This means follows: let U_K be an arbitrary $U(1)$-invariant tubular neighborhood of locus of K and let G_U be a form such that it is equal to 1 at the locus of vector field $\mathbf{K}$ and the form vanishes out of neighborhood $U_{\mathbf{K}}$:*

$$G_U\big|_{locus\ of\ \mathbf{K}} = 1, \quad G_U\big|_{M\setminus U_K} = 0\,. \tag{3}$$

(We will prove the existence of such a form)

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* *the condition to be $U(1)$-invariant may be is not necessary. We will use it for constructing $U(1)$-ivariant partition of unity. This condition is absent in the paper [1].*

Then

$$\int_M H = \int_M H G_U \,. \tag{4}$$

Example Let M be a symplectic manifold, i.e. non-degenerate closed two-form Ω is defined on M (M is even-dimensional). Let $h = h(x)$ be a Hamiltonian such that its Hamiltonian vector field D_h (D_h: $\iota_{D_h}\Omega = -dh$) is compact, i.e. it defines $U(1)$ action. Consider the form

$$H(x, dx) = \exp i\left(\Omega + h\right) \,. \tag{5}$$

This form is $Q_{\mathbf{K}}$-invariant. Indeed since K is hamiltonian vector field D_h hence

$$\iota_{\mathbf{K}}\Omega + dh = 0 \text{.i.e.}\, Q_{\mathbf{K}}(h+\Omega) = 0 \Rightarrow Q_{\mathbf{K}} H = 0 \,.$$

Then

$$\int H(x, dx) = \int \exp i\left(\Omega + h\right) = \frac{i^n}{n!}\int \exp ih \underbrace{\Omega \wedge \ldots \wedge \Omega}_{n \text{ times}}$$

is localised.

Remark 1 Note that the example is basic example in classical background. Compact vector field $\mathbf{K}$ appears naturally in this example as hamiltonian vector field of Hamiltonin h. In Schwarz-Zaboronsky approach the vector field $\mathbf{K}$ appears independetly without symplectic structure and Hamiltonian. In this approach the localisation formula is stated for a function $H(x, dx)$ on ΠTM (sum of differential forms of different orders). The classical condition that sum of differential forms is invariant with respect to equivariant differential $d_{\mathbf{K}} = d + \iota_{\mathbf{K}}$ becomes the condition that that "function" $H(x, dx)$ is invariant with respect to odd vector field $Q_{\mathbf{K}} = \sqrt{\text{Lie derivative with respect to vector field } \mathbf{K}}$.

Remark 2 partiion of unity for form

Proof of Theorem First we prove the existence of a form $G_U = G_U(x, dx)$ which obeys the condition (3), then we will show that an arbitrary $Q_{\mathbf{K}}$-invariant "function" (form) which obeys conditions (3) yields the localisation formula (4).

Using partiion of unity arguments consider a function $F = F(x)$ such that

$$F(x)\big|_{\text{locus of } \mathbf{K}} = 0, \quad F(x)\big|_{M \setminus U_K} = 1 \,. \tag{6}$$

(We may consider partiion of unity subordiante to covering $V_1 \cup V_2$, where $V_1 = U_{\mathbf{K}}$ and $V_2 = M \setminus$locus of K.

We may assume that $F(x)$ is $\mathbf{K}$-invariant function. (Here we use the $U(1)$-ivariance of neighborhood of locus (see the footnote.)).

It is useful to consider the differential 1-form

$$\omega_{\mathbf{K}} \colon \omega_{\mathbf{K}}(\mathbf{x}) \langle \mathbf{K}, , \mathbf{x} \rangle \,, \omega_i = g_{im} K^m dx^i \,, \tag{7}$$

where $\langle \mathbf{K}, , \mathbf{x} \rangle$ is $U(1)$-invariant Riemannian metric on M. Now we are ready to define form G_U which obeys the condition (3):

$$G_U(x, dx) = 1 - Q_{\mathbf{K}}\left(\frac{\omega_{\mathbf{K}}(x, dx)}{Q_{\mathbf{K}}\omega_{\mathbf{K}}} F(x)\right) \tag{8}$$

Straightforward calculations show that this function obeys conditions (3). Indeed $F(x) = 0$ if x belongs to locus of K (and in a vicinity of the locus), hence the right hand side of equation (8) is well-defined on the locus of $\mathbf{K}$, where the form $\omega_{\mathbf{K}}$ is not defined. Using the fact that $Q_{\mathbf{K}}\left(\frac{\omega_{\mathbf{K}}(x,dx)}{Q_{\mathbf{K}}\omega_{\mathbf{K}}}\right) = 1$ (if $\mathbf{K}(x) \neq 0$) we immediately come to the condition (3).

Let $\tilde{G}_U = \tilde{G}_U(x, dx)$ be an arbitrary $Q_{\mathbf{K}}$-invariant form which obeys the condition (3). Then consider the difference $L(x, dx) = \tilde{G}_U - G_U$. The form $L(x, dx)$ is $Q_{\mathbf{K}}$-invariant and it is equal to 0 at the locus of K, Hence

$$L(x, dx) = Q_{\mathbf{K}}\left(\frac{\omega_{\mathbf{K}}(x, dx)}{Q_{\mathbf{K}}\omega_{\mathbf{K}}} L(x, dx)\right) . \tag{9}$$

Thus we see that $Q_{\mathbf{K}}$-invariatn form $G_U(x, dx)$ in (8) which obeys the condition (3) as well as an arbitrary $Q_{\mathbf{K}}$-invariatn form $\tilde{G}_U(x, dx)$ which obeys the condtion (3) obey the condition that

$$\begin{aligned} G_U(x, dx) &= 1 + Q_{\mathbf{K}}(...) \\ \tilde{G}_U(x, dx) &= 1 + Q_{\mathbf{K}}(...) \end{aligned}$$

This immediately implies the relation (4):

$$\int_M H(x, dx) G_U(x, dx) = \int_M H(x, dx)(1 + Q_{\mathbf{K}}(\ldots)) = \int_M H(x, dx)$$

since $\int_M Q_{\mathbf{K}}(\ldots) = 0^{**}$ ■

Concrete calculations

Now based on the Theorem we present concrete calculations.

Let $H = H(x, dx)$ be $Q_{\mathbf{K}}$ invariant form and locus (zero locus) of $U(1)$-invariant vector field $\mathbf{K}$ is a set $\{x_i\}$ of isolated points.

Using bump-form G_U, the form which vanishes out vicinites of points $\{x_i\}$ (see the considerations above) we calculate $\int_M H(x, dx)$.

Lemma For an arbitrary $Q_{\mathbf{K}}$-invariant form $H(x, dx)$ the integral

$$Z(t) = \int H(x, dx) e^{itQ_{\mathbf{K}}(\omega_{\mathbf{K}})} ,$$

where $\omega_{\mathbf{K}}$ is $U(1)$-invariant form (7) does not depend on t.

Proof:

$$\frac{dZ(t)}{dt} = i \int_M H(x, dx) Q_{\mathbf{K}}\omega_{\mathbf{K}} e^{itQ_{\mathbf{K}}(\omega_{\mathbf{K}})} = i \int_M Q_{\mathbf{K}}\left(H(x, dx) e^{itQ_{\mathbf{K}}(\omega_{\mathbf{K}})}\right) = 0 .$$

Now using lemma and bump-form which localises integrand in vicinity of points $\{x_i\}$ we come to

$$\int_M H(x, dx) = \int_M H(x, dx) G_U(x, dx) = \left(\int_M H(x, dx) G_U(x, dx) e^{itQ_{\mathbf{K}}(\omega_{\mathbf{K}})}\right)\Big|_{t=0}$$

** since $Q_K = d + \iota_K$, and $\iota_K \omega$ 'does not contain' top form. This follows also from the vanishing of divergence of odd vector field $Q_{\mathbf{K}}$ with respect to canonical volume form in ΠTM

$$= \left(\int_M H(x,dx) G_U(x,dx) e^{itQ_{\mathbf{K}}(\omega_{\mathbf{K}})} \right) \big|_{t\to\infty}$$

Using method of stationary phase and assuming that $d\omega$ is non-degenerate at locus of $\mathbf{K}$ we calculate the last integral (see [4]) and come to the answer

$$\int_M H(x,dx) == \left(\int_M H(x,dx) G_U(x,dx) e^{itQ_{\mathbf{K}}(\omega_{\mathbf{K}})} \right) \big|_{t\to\infty} = \sum_{x_i} \frac{i^n}{n!} \frac{H(x,dx)\big|_{x_i}}{\sqrt{\frac{\partial K}{\partial x}\big|_{x_i}}}$$

If $H(x,dx)\big|_{x_i} = H_0(x_i)$, where $H(x,dx) = H_0(x) + H_1(x,dx) + \ldots$ is a sum of differential forms.

References

[1] Albert Schwarz and Oleg Zaboronsky. *Supersymmetry and localisation.* arXiv: hep-th/951112v1

[2] A. Nersessian *Antibrackets and non-Abelian equivariant cohomology* arXix: hep-th/951081

[3] *On the Duistermaat-Heckman localisation formula and Integrable systems* arXiv: hep-th/9402041v1

[4] homepage: maths.manchester.ac.uk/khudian/Etudes/Geometry/Dustermaat-Heckman localisation formula. *Etude based on the fragment of the lecture of A.Belavin in Bialoveza, summer 2012.*