Geometry of differential operators on $\mathbf{R}$

Let

$$A = t^\delta \underbrace{\left(s\frac{\partial^{n+1}}{\partial x^{n+1}} + a\hat{w}\frac{\partial^n}{\partial x^n} + b\hat{w}^2\frac{\partial^{n-1}}{\partial x^{n-1}}\right)}_{\text{terms of the order } n+1} +$$

$$+ \underbrace{\left(p\frac{\partial^n}{\partial x^n} + c\hat{w}\frac{\partial^{n-1}}{\partial x^{n-1}} + \dots\right)}_{\text{terms of the order } n} + \quad \underbrace{\left(q\frac{\partial^n}{\partial x^{n-1}} + \dots\right)}_{\text{terms of the order } n-1} \quad + \dots$$

be differential operator of the order $n+1$ and of the weight δ on the algebra $\mathcal{F}$ of densities on $\mathbf{R}$. Here $\hat{w} = t\frac{\partial}{\partial t}$, $a = a(x), \dots$

E.g. if $\Psi(x,t) = \varphi(x)t^5$ is a density $\varphi(x)|dx|^5$ of the weight 5, and $A = t^{-3}\left(\frac{\partial^3}{\partial x^3} + p(x)\frac{\partial^2}{\partial x^2} + b(x)\hat{w}\frac{\partial}{\partial x}\right)$ then

$$A\Psi = t^{-3}\left(\frac{\partial^3 \Psi}{\partial x^3} + p(t)\frac{\partial^2 \Psi}{\partial x^2} + b(x)t\frac{\partial}{\partial t}\frac{\partial}{\partial x}\right) = \left(\frac{d^3\varphi}{dx^3} + p(t)\frac{d^2\varphi}{dx^2} + 4b(x)\frac{d\varphi}{dx}\right)|dx|^2 .$$

The canonical scalar product in the space $\mathcal{F}$ is defined by condition that for densities $\Psi = \Psi(x)t^\lambda$ and $\Phi = \Phi(x)t^\mu$

$$\langle \Psi, \Phi \rangle = \begin{cases} 0 \text{ if } \lambda + \mu \neq 1 \\ \int \Psi(x)\Phi(x)dx \quad \text{if } \lambda + \mu \neq 1 \end{cases} .$$

Thus we have the conjugation of derivatives:

$$x^\dagger = x, \left(\frac{\partial}{\partial x}\right)^\dagger = -\frac{\partial}{\partial x}, t^\dagger = t, \left(\frac{\partial}{\partial t}\right)^\dagger = \frac{2}{t} - \frac{\partial}{\partial t},$$

In particular:

$$\hat{w}^\dagger = \left(t\frac{}{d\partial}\right)^\dagger = 1 - \left(t\frac{\partial}{\partial t}\right)^\dagger = 1 - \hat{w}^+$$

and

$$\hat{w}^\dagger(t^\sigma \Psi) = t^\sigma(1 - \sigma - \hat{w})\Psi .$$

§1. *Subprincipal symbol*

Now we find the restrictions on operator A posed by the condition that it is self conjugate operator (up to sign), i.e.

$$A^\dagger = (-1)^{n+1}A$$

, Then we discuss the geometrical nature of coefficients.

We check the condition of self-conjugality step by step for lower and lower derivatives [1].

In this paragraph we consider only the terms which are proportional to derivatives of the order $n+1$ and n with respect to x:

$$A = t^{\delta} \left(s\frac{\partial^{n+1}}{px^{n+1}} + a\hat{w}\frac{\partial^n}{\partial x^n} + p\frac{\partial^n}{\partial x^n} + \dots \right),$$

$$A^{\dagger} = \left[t^{\delta} \left(s\frac{\partial^{n+1}}{\partial x^{n+1}} + a\hat{w}\frac{\partial^n}{\partial x^n} + p\frac{\partial^n}{\partial x^n} + \dots \right) \right]^{\dagger} =$$

$$(-1)^{n+1} t^{\delta} \frac{\partial^{n+1}}{\partial x^{n+1}} (s(x)\cdot) + \hat{w}^{\dagger} \left[t^{\delta}(-1)^n \frac{\partial^n}{\partial x^n} (a(x)\cdot) \right] + t^{\delta}(-1)^n \frac{\partial^n}{\partial x^n} (p(x)\cdot) + \dots =$$

$$= (-1)^{n+1} t^{\delta} s \frac{\partial^{n+1}}{\partial x^{n+1}} + (-1)^n t^{\delta} \left(-a\hat{w} - (n+1)\frac{ds}{dx} + (1-\delta)a + p \right) \frac{\partial^n}{\partial x^n} + \dots$$

Hence

$$0 = (-1)^{n+1} A^{\dagger} - A =$$

$$t^{\delta} s \frac{\partial^{n+1}}{\partial x^{n+1}} + t^{\delta} \left(a\hat{w} + (n+1)\frac{ds}{dx} - (1-\delta)a - p \right) \frac{d^n}{dx^n} - t^{\delta} \left(s\frac{\partial^{n+1}}{\partial x^{n+1}} + a\hat{w}\frac{\partial^n}{\partial x^n} + p\frac{\partial^n}{\partial x^n} + \dots \right)$$

$$= t^{\delta} \left((n+1)\frac{ds}{dx} + (\delta - 1)a - 2p \right) \frac{\partial^n}{\partial x^n} + \dots .$$

Hence we come to the condition:

$$\frac{n+1}{2}\frac{ds}{dx} + (\delta-1)a - 2p = 0\,,\,, \text{ i.e. } p = \frac{n+1}{2}\left(\frac{ds}{dx}\right) + \frac{1}{2}(1-\delta)\,.$$

We see that self-conjugate operator A in the terms proportional to the order $n+1$ and n looks like

$$A = t^{\delta} \left(s\frac{\partial^{n+1}}{\partial x^{n+1}} + a\hat{w}\frac{\partial^n}{\partial x^n} + p\frac{\partial^n}{\partial x^n} + \dots \right) =$$

$$A = t^{\delta} s \frac{\partial^{n+1}}{\partial x^{n+1}} + \frac{1}{2} \left((n+1)\frac{ds}{dx} + (2\hat{w} + 1 - \delta)a(x) \right) \frac{\partial^n}{\partial x^n} \dots$$

Now we study how transform s and a under coordinate transformation.

§2. *Geometric meaning of Subprincipal symbol*

We consider how a transforms with respect to an arbitrary coordinate transformation.

[1] All that we will do here could be compared with our calculations with T.Voronov for second order operators. The motto is that we descend from higher derivatives to lower: for n-th order operators the terms of n-th order behave like the term of second order of Laplacian, the terms of $n-1$-th order behave like the term of first order of Laplacian, the terms of $n-2$-th order behave like the term of zeroth order of Laplacian.

Consider arbitrary coordinate transformation: $(x,t) \mapsto (y,\tau)$[1]. :

$$\begin{cases} x = x(y) \\ t = x_y \tau \end{cases}, \begin{cases} y = y(x) \\ \tau = y_x t \end{cases}$$

Then to calculate how operator A will transform we note that $t = x_y \tau$,

$$\hat{w}_{(t)} = t\frac{\partial}{\partial t} = t = x_y \tau \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} = x_y \tau y_x \frac{\partial}{\partial \tau} = \tau \frac{\partial}{\partial \tau} = \hat{w}_{(\tau)}$$

and

$$\frac{\partial}{\partial x} = y_x \frac{\partial}{\partial y} + \frac{\partial \tau}{\partial x}\frac{\partial}{\partial \tau} = y_x \frac{\partial}{\partial y} + t y_{xx} \frac{\partial}{\partial \tau} = y_x \frac{\partial}{\partial y} + \frac{y_{xx}}{y_x} w_{(\tau)}$$

Little bit work and we come to the following formula:

$$\frac{\partial^k}{\partial x^k} = \left(y_x \frac{\partial}{\partial y} + \frac{y_{xx}}{y_x} \hat{w}_{(\tau)} \right)^k =$$

$$y_x^k \frac{\partial^k}{\partial y^k} + \left((1 + 2 + \ldots + (k-1)) y_x^{k-2} y_{xx} + k \left(\frac{y_{xx}}{y_x} \right) y_x^{k-2} y_{xx} \hat{w} \right) \frac{\partial^{k-1}}{\partial y^{k-1}} + \ldots =$$

$$y_x^k \frac{\partial^k}{\partial y^k} + \left(\frac{k(k-1)}{2} + k\hat{w} \right) y_x^{k-2} y_{xx} \frac{\partial^{k-1}}{\partial y^{k-1}} + \ldots .$$

(This can be easily calculated by induction.)

Now we are ready to calculate transformation of coefficients:

$$A = t^\delta s \frac{\partial^{n+1}}{\partial x^{n+1}} + t^\delta \frac{1}{2} \left((n+1)\frac{ds}{dx} + (2\hat{w} + \delta - 1) a(x) \right) \frac{\partial^n}{\partial x^n} + \ldots =$$

$$\tau^\delta x_y^\delta s \left(y_x \frac{\partial}{\partial y} + \frac{y_{xx}}{y_x} \hat{w} \right)^{n+1} + \frac{1}{2} \tau^\delta x_y^\delta \left((n+1)\frac{ds}{dx} + (2\hat{w} + \delta - 1) a(x) \right) \left(y_x \frac{\partial}{\partial y} + \frac{y_{xx}}{y_x} \hat{w} \right)^n + \ldots =$$

$$\tau^\delta y_x^{-\delta} s \left(y_x^{n+1} \frac{\partial^{n+1}}{\partial y^{n+1}} + \left(\frac{n(n+1)}{2} + (n+1)\hat{w} \right) y_{xx} y_x^{n-1} \frac{\partial^n}{\partial y^n} \right) +$$

$$+ \frac{1}{2} \tau^\delta y_x^{-\delta} \left((n+1)\frac{ds}{dx} + (2\hat{w} + \delta - 1) a(x) \right) y_x^n \frac{\partial^n}{\partial y^n} + \ldots$$

Dots means terms which are of the order $\leq n-1$ with respect to variables x, y

Denote by $\tilde{s} = y_x^{n+1-\delta} s$ (principal symbol in new coordinates). We come to

$$A = \tau^\delta \left(\tilde{s} \frac{\partial^{n+1}}{\partial y^{n+1}} + \left(\frac{n(n+1)}{2} + (n+1)\hat{w} \right) \tilde{s} y_{xx} y_x^{-2} \frac{\partial^n}{\partial y^n} \right) +$$

[1] it would be enough to fix the weight of the densities, i.e. consider the action of the operator on the subspace $\mathcal{F}_\lambda$ of the densitites of the fixed weight λ and consider only coordinate transformations $y = y(x)$ but we prefer to consider the general case.

$$\frac{1}{2}\tau^\delta\left((n+1)\frac{ds}{dy}+(n+1)(\delta-n-1)y_{xx}y_x^{-2}\tilde{s}+(2\hat{w}+\delta-1)a(x)y_x^{n-\delta}\right)\frac{\partial^n}{\partial y^n}+\ldots=$$

$$\tau^\delta\tilde{s}\frac{\partial^{n+1}}{\partial y^{n+1}}+\tau^\delta\frac{n+1}{2}\frac{ds}{dy}\frac{\partial^n}{\partial y^n}+$$

$$\frac{1}{2}\tau^\delta(2\hat{w}+\delta-1)\left(a(x)y_x^{n-\delta}+(n+1)\tilde{s}\frac{y_{xx}}{y_x^2}\right)\frac{\partial^n}{\partial y^n}+\ldots=$$

Claim We see that if in coordinates x,t

$$A=t^\delta s\frac{d^{n+1}}{dx^{n+1}}+t^\delta\frac{1}{2}\left((n+1)\frac{ds}{dx}+(2\hat{w}+\delta-1)a(x)\right)\frac{d^n}{dx^n}+\ldots=$$

then in new coordinates $y=y(x)$, $\tau=y_xt$ ($t\sim|dx|,\tau\sim|dy|$)

$$A=\tau^\delta\tilde{s}\frac{d^{n+1}}{dy^{n+1}}+\tau^\delta\frac{1}{2}\left((n+1)\frac{ds}{dy}+(2\hat{w}+\delta-1)\tilde{a}(x)\right)\frac{d^n}{dy^n}+\ldots=$$

where

$$\tilde{s}=sy_x^{n+1-\delta}\,,\ \tilde{a}=a(x)y_x^{n-\delta}+(n+1)\tilde{s}\frac{y_{xx}}{y_x^2}=y_x^{-\delta}\left(a+(n+1)s\frac{\partial\log y_x}{\partial x}\right)y_x^n\,.$$

Remark In the case $n=2$ this is just the connection on the volume forms. (Principal symbol equals to $2s$)

Resumé
In the general case $\frac{2a}{n+1}$ is a "connection"[3]. We denote

$$\gamma=\frac{2a}{n+1},a=\frac{(n+1)\gamma}{2}$$

Then we can rewrite the operator

$$A=t^\delta s\frac{d^{n+1}}{dx^{n+1}}+t^\delta\frac{1}{2}\left((n+1)\frac{ds}{dx}+(2\hat{w}+\delta-1)a(x)\right)\frac{d^n}{dx^n}+\ldots=$$

$$t^\delta\frac{d^{n+1}}{dx^{n+1}}+t^\delta\frac{n+1}{4}\left(2\frac{ds}{dx}+(2\hat{w}+\delta-1)\gamma(x)\right)\frac{d^n}{dx^n}+\ldots=$$

One can consider the canonical pencil of n-th order operators of the degree δ which send the density of the weight λ to the density of the weight $\lambda+\delta$

$$\Psi(x)|dx|^\lambda\mapsto\left(s\frac{d^{n+1}\Psi(x)}{dx^{n+1}}+\frac{n+1}{4}\left(2\frac{ds}{dx}+(2\lambda+\delta-1)\gamma(x)\right)\frac{d^n\Psi}{dx^n}+\ldots\right)|dx|^{\lambda+\delta}=$$

[3] It transforms as upper connection. Here may be a coefficient $\frac{n+1}{2}$ is chosen wrong

where γ is a connection

Exercise Consider the previous construction for the case $n = 1$ ("laplacian") and $n = 0$ (vector field)

The next step is to consider the subsubprincipal symbol, which is of highly interest[4]

Before going to the next step note that for the canonical pencil above if we put:

$$\begin{cases} \delta = n+1 \\ 2\gamma + \delta = 1 \end{cases} \leftrightarrow \begin{cases} \lambda = -\frac{n}{2} \\ \delta = 2 \end{cases}$$

then we come to the constructions of the book Ovsienko=Tabachnikov.

§3. *Subsubprincipal symbol*

Now we check the condition of self-conjugancy ($A = (-1)^{n+1}A^\dagger$) up to the order $n-1$ with respect to x:

From previous considerations it follows that the operator

$$A = t^\delta \underbrace{\left(s\frac{\partial^{n+1}}{\partial x^{n+1}} + a\hat{w}\frac{\partial^n}{\partial x^n} + b\hat{w}^2\frac{\partial^{n-1}}{\partial x^{n-1}}\right)}_{\text{terms of the order } n+1} + \underbrace{\left(p\frac{\partial^n}{\partial x^n} + c\hat{w}\frac{\partial^{n-1}}{\partial x^{n-1}} + \dots\right)}_{\text{terms of the order } n} + \underbrace{\left(q\frac{\partial^n}{\partial x^{n-1}} + \dots\right)}_{\text{terms of the order } n-1} +$$

$$= t^\delta \left(s\frac{\partial^{n+1}}{\partial x^{n+1}} + a\hat{w}\frac{\partial^n}{\partial x^n} + p\frac{\partial^n}{\partial x^n} + \dots\right) =$$

$$A = t^\delta s\frac{\partial^{n+1}}{\partial x^{n+1}} + \frac{1}{2}\left((n+1)\frac{ds}{dx} + (2\hat{w} + 1 - \delta)a(x)\right)\frac{\partial^n}{\partial x^n} + (b\hat{w}^2 + c\hat{w} + q)\frac{\partial^{n-1}}{\partial x^{n-1}} + \dots\dots$$

Now find the restrictions which are imposed by the condition $A = (-1)^{n+1}A^\dagger$: We have that

$$(-1)^{n+1}A^\dagger\Psi = \frac{\partial^{n+1}}{\partial x^{n+1}}(t^\delta s\Psi) - \frac{n+1}{2}\frac{\partial^n}{\partial x^n}\left(t^\delta\frac{ds}{dx}\Psi\right) -$$

$$\frac{n+1}{2}\frac{\partial^n}{\partial x^n}\left(t^\delta\left(\hat{w} + \frac{\delta-1}{2}\right)\gamma\Psi\right) + \frac{\partial^{n-1}}{\partial x^{n-1}}\left(t^\delta(\hat{w}^2 b + \hat{w}c + q)\psi\right) =$$

$$t^\delta s\frac{\partial^{n+1}}{\partial x^{n+1}} + \frac{1}{2}\left((n+1)\frac{ds}{dx} + (2\hat{w} + 1 - \delta)a(x)\right)\frac{\partial^n}{\partial x^n} +$$

$$\frac{n(n+1)}{2}t^\delta\left(\hat{w} + \frac{\delta-1}{2}\right)\frac{d\gamma}{dx}\frac{\partial^{n-1}\Psi}{\partial x^{n-1}} + t^\delta\left[b(1-\delta-\hat{w})^2 + c(1-\delta-\hat{w}) + q\right]\frac{\partial^{n-1}\Psi}{\partial x^{n-1}} + \dots\dots$$

Comparing with operator A we see that the condition $A = (-1)^{n+1}A^\dagger$ implies that

$$\frac{n(n+1)}{2}\left(\hat{w} + \frac{\delta-1}{2}\right)\frac{d\gamma}{dx} + b(1-\delta-\hat{w})^2 + c(1-\delta-\hat{w}) + q = b\hat{w}^2 + c\hat{w} + q$$

[4] We have also to construct the operator globally, but this we can do defining a connection.

Thus

$$c = \frac{n(n+1)}{2}\frac{d\gamma}{dx} + b(\delta - 1)$$

We denote $b = \frac{\theta}{2}$. (To compare with second order operators.)

We come to the following statement:

Theorem The self-conjugate operator of the order $n+1$ on the algebra of densitites has the following appearance:

$$A = t^\delta s \frac{\partial^{n+1}}{\partial x^{n+1}} + \frac{1}{2}\left((n+1)\frac{ds}{dx} + (2\hat{w} + \delta - 1)a(x)\right)\frac{\partial^n}{\partial x^n} +$$

$$\frac{1}{2}\left[\theta\hat{w}^2 + \left(n(n+1)\frac{d\gamma}{dx} + \theta(\delta - 1)\right)\hat{w} + q\right]\frac{\partial^{n-1}}{\partial x^{n-1}} + \ldots\ldots$$

Here s is the density of the weight $\delta - n - 1$, γ is connection and Brans-Dicke scalar is related to Schwarzian.

Considering the restriction of this operator on the space $\mathcal{F}_\lambda$ we come to the following pencil of operators. Any density of the weight λ $\Psi|dx|^\lambda$ is transformed to the density of the weight $\mu\lambda + \delta$:

$$\Psi|dx|^\lambda \mapsto \Phi(x)|dx|^{\lambda+\delta}$$

where

$$\Phi = s\frac{d^{n+1}\Psi}{dx^{n+1}} + \frac{1}{2}\left((n+1)\frac{ds}{dx} + (2\lambda + \delta - 1)a(x)\right)\frac{d^n\Psi}{dx^n} +$$

$$\frac{1}{2}\left[\theta\lambda^2 + \left(n(n+1)\frac{d\gamma}{dx} + \theta(\delta - 1)\right)\lambda + q\right]\frac{d^{n-1}}{dx^{n-1}} + \ldots$$

§4. *Special case*

Consider operator of the weight δ on the densitites o the weight λ such that

$$\begin{cases} \delta - n - 1 = 0 \\ 2\lambda + \delta - 1 = 0\,. \end{cases}$$

i.e.

$$\begin{cases} \delta = 1 + n \\ \lambda = -\frac{n}{2} \end{cases}$$

The principal symbol s becomes the scalar we put it $s = 1$, subprincipal symbol vanishes. We come to the operator

$$\Psi|dx|^{-\frac{n}{2}} \mapsto \Phi(x)|dx|^{1+\frac{n}{2}}$$

where

$$\Phi(x) = \frac{d^{n+1}\Psi}{dx^{n+1}} + \frac{1}{2}\left[\theta\frac{n^2}{4} + \left(n(n+1)\frac{d\gamma}{dx} + \theta n\right)\left(\frac{-n}{2}\right) + q\right]\frac{d^{n-1}}{dx^{n-1}} + \ldots =$$

$$\frac{d^{n+1}\Psi}{dx^{n+1}} + \ldots\theta\frac{d^{n-1}\Psi}{dx^{n+1}} + \ldots\frac{d\gamma}{dx}$$

The next step is to consider how θ transform under coordinate transformations.,,