Consider the angle φ with the vertex is at the origin. Consider the rod of the length $l = 1$. What is the equilibrium position of the rod in this angle under the gravitational force. This problem was sugested by John Parkinson on tutorials. I was really surprised realising that this problem is not trivial.

For simplicity consider the case if bisectrix of the angle is vertical. Thus we have to study the stationary points of the function

$$F = x + y\,, \qquad \text{under the constrain } x^2 + y^2 - 2xy\cos\varphi = 1\,.$$

Solving this constrain we see that there are two triangles if $x < 1$, one triangle if $1 < x \leq \frac{1}{\sin\varphi}$, and no triagnle if $x > \frac{1}{\sin\varphi}$:

$$\begin{cases} y_+ = x\cos\varphi + \sqrt{1 - x^2\sin^2\varphi} & \text{if } x < \frac{1}{\sin\varphi} \\ y_- = x\cos\varphi - \sqrt{1 - x^2\sin^2\varphi} & \text{if } x \leq 1 \end{cases}$$

Symmetry arguments:

$$F(x, y) = F(y, x)$$

give us that one of the stationary points of the function F is the isocseles triangle:

$$x_0 = \frac{1}{2\sin\frac{\varphi}{2}}\,, (x_0 > 11)\,.$$

(isocseles triangle). In this case $x > 1$, and this point is a stationary point of the function F_+.

Is this point equilibrium point? Consider a function F_+ in the vicinity of this point:

$$x = \frac{1}{2\sin\frac{\varphi}{2}} + u\,,$$

We come to

$$\Phi_+(u) = x + y = x + x\cos\varphi + \sqrt{1 - x^2\sin^2\varphi} =$$

$$(1 + \cos\varphi)\left(\frac{1}{2\sin\frac{\varphi}{2}} + u\right) + \sqrt{1 - \left(\frac{1}{2\sin\frac{\varphi}{2}} + u\right)^2 \sin^2\varphi} = \left(\frac{\cos^2\frac{\varphi}{2}}{\sin\frac{\varphi}{2}} + 2\cos^2\frac{\varphi}{2}u\right) +$$

$$\sqrt{1 - \cos^2\frac{\varphi}{2} - 4\sin\frac{\varphi}{2}\cos^2\frac{\varphi}{2}u - 4\sin^2\frac{\varphi}{2}\cos^2\frac{\varphi}{2}u^2} = \left(\frac{\cos^2\frac{\varphi}{2}}{\sin\frac{\varphi}{2}} + 2\cos^2\frac{\varphi}{2}u\right) +$$

$$\sin\frac{\varphi}{2}\sqrt{1 - 4\frac{\cos^2\frac{\varphi}{2}}{\sin\frac{\varphi}{2}}u - 4\cos^2\frac{\varphi}{2}u^2} = \left(\frac{\cos^2\frac{\varphi}{2}}{\sin\frac{\varphi}{2}} + 2\cos^2\frac{\varphi}{2}u\right) +$$

$$\sin\frac{\varphi}{2}\left(1 - 2\frac{\cos^2\frac{\varphi}{2}}{\sin\frac{\varphi}{2}}u - 2\cos^2\frac{\varphi}{2}u^2 - \frac{1}{8}\left(4\cos^2\varphi\left(\frac{u}{\sin\frac{\varphi}{2}} + u^2\right)\right)^2 + \frac{1}{16}\left(\frac{4\cos^2\frac{\varphi}{2}}{\sin\frac{\varphi}{2}}\right)^3 + \dots\right) =$$

(Here we use the binom formula:

$$\sqrt{1+x} = 1 + \frac{1}{2}x + \frac{1}{2}\cdot\frac{1}{2}\left(-\frac{1}{2}\right)x^2 + \frac{1}{6}\cdot\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)x^3 + \dots = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + + \dots =$$