Proof that the three altitudes of a triangle meet at a point

Does not use the Jacobi identity explicitly

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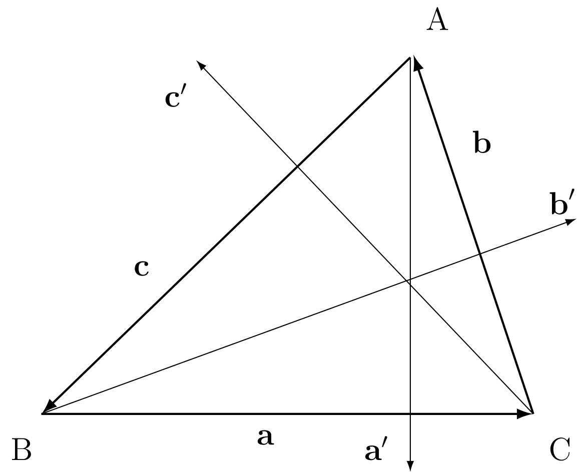

Sides of the triangle are $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ as shown,
with $\mathbf{a}+\mathbf{b}+\mathbf{c} \quad = \quad 0.$

Define

$$\mathbf{a}' = \mathbf{a}\times(\mathbf{b}\times\mathbf{c})$$
$$\mathbf{b}' = \mathbf{b}\times(\mathbf{c}\times\mathbf{a})$$
$$\mathbf{c}' = \mathbf{c}\times(\mathbf{a}\times\mathbf{b})$$

Clearly these are in the correct directions for the altitudes.
Hence equations of the altitudes are:

$$\mathbf{r} = \mathbf{r}_A + \lambda_A\,\mathbf{a}' \qquad (\text{line } l_1)$$
$$\mathbf{r} = \mathbf{r}_B + \lambda_B\,\mathbf{b}' \qquad (\text{line } l_2)$$
$$\mathbf{r} = \mathbf{r}_C + \lambda_C\,\mathbf{c}' \qquad (\text{line } l_3)$$

where $\mathbf{r}_A$ is the position vector of A and
λ_A is the parameter of the altitude through A *etc.*

l_1 and l_2 meet when

$$\mathbf{r}_A + \lambda_A^{12}\,\mathbf{a}' \quad = \quad \mathbf{r}_B + \lambda_B^{12}\,\mathbf{b}'$$

where λ_A^{12}, λ_B^{12} are the values of λ_A, λ_B at the meeting point of l_1 and l_2.
Since $\mathbf{r}_B - \mathbf{r}_A \quad = \quad \mathbf{c}$ we have

$$\lambda_A^{12}\,\mathbf{a}' \quad = \quad \mathbf{c} + \lambda_B^{12}\,\mathbf{b}' \qquad (3)$$

Similarly when l_2 and l_3 meet we have

$$\mathbf{r}_B + \lambda_B^{23}\,\mathbf{b}' \quad = \quad \mathbf{r}_C + \lambda_C^{23}\,\mathbf{c}'$$

$$\lambda_B^{23}\,\mathbf{b}' \quad = \quad \mathbf{a} + \lambda_C^{23}\,\mathbf{c}' \qquad (4)$$

Now need to show that $\lambda_B^{12} \quad = \quad \lambda_B^{23}$. From (3) and (4)

$$\lambda_B^{12}\,\mathbf{b}' \quad = \quad \lambda_A^{12}\,\mathbf{a}' - \mathbf{c}$$

$$\text{and} \qquad \lambda_B^{23}\,\mathbf{b}' \quad = \quad \lambda_C^{23}\,\mathbf{c}' + \mathbf{a}$$

Since $\mathbf{a}.\mathbf{a}' = \mathbf{b}.\mathbf{b}' = \mathbf{c}.\mathbf{c}' = 0$

$$\begin{aligned} \lambda_B^{12}\,\mathbf{b}'.\mathbf{a} \quad &= \quad 0 - \mathbf{c}.\mathbf{a} \\ \lambda_B^{23}\,\mathbf{b}'.\mathbf{c} \quad &= \quad 0 + \mathbf{a}.\mathbf{c} \end{aligned}$$

$$\therefore \quad \lambda_B^{12}\,\mathbf{b}'.\mathbf{a} \quad = \quad -\lambda_B^{23}\,\mathbf{b}'.\mathbf{c} \qquad (5)$$

Since $\mathbf{c} \quad = \quad -\mathbf{a} - \mathbf{b}$ we have

$$\mathbf{b}'.\mathbf{c} \quad = \quad -\mathbf{b}'.(\mathbf{a} + \mathbf{b}) \quad = \quad -\mathbf{b}'.\mathbf{a}.$$

Substitution in (5) proves the result.