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Here I will try to caclulate the continual integral for free aprticle using some linear algebra stuff

To calculate the continual integral for free function we deal with exponent of the polynomial

$$F = (x_0 - x_1)^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_2)^2 + \ldots + (x_{N-1} - x_N)^2 \,, \qquad (0.1)$$

where initial and final points are fixed

$$x_0 = a, x_N = b$$

One has to calculate the gaussian integral e^{iCF}:

$$\int \exp(icF) dx_1 dx_2 \ldots dx_{N-1}$$

In the classical book of Feynman the author caclculates the integral, step by step performing the trick: every next integration over dx_{i+1} gives the same answer as the previous up to the coefficient depending on i.

We will present here the straightforward calculation which is based on the linear algebra.

F in equation (0.1) is quadratic polynomial over $N-1$ variables $x_1, \ldots, x_{N-1}$. We perform affine transformation to the new cooridnates such that in these coordinates F will have only quadratic and zero order terms.

Moreover we will try to consider transfromations with unity Jacobian.

Consider first affine transformation

$$\begin{cases} \xi_1 = x_1 - x_0 = x_1 - a \\ \xi_2 = x_2 - x_1 \\ \xi_3 = x_3 - x_2 \\ \ldots \\ \xi_{N-1} = x_{N-1} - x_N \end{cases} \Leftrightarrow \begin{cases} x_1 = \xi_1 + a \\ x_2 = \xi_2 + \xi_1 + a \\ x_3 = \xi_3 + \xi_2 + \xi_1 + a \\ \ldots \\ x_{N-1} = \xi_{N-1} + \ldots + \xi_1 + a \end{cases}$$

The "linear" part iof this transfromation is not orthogonal transformation, but it is unimodular transformation. In the new coordinates

$$F = \sum_{i,k=1}^{N-1} M_{ik} x^i x^k + \sum_{i=1}^{N-1} L_i x^i + N = \xi_1^2 + \xi_2^2 + \xi_3^2 + \ldots + \xi_{N-1}^2 + (b - a - \xi_1 - \xi_2 - \ldots - \xi_{N-1})^2 \,. \qquad (1.1)$$

Now consider new coordinates

$$\xi_i = \eta_i + \frac{b-a}{N} \,, \qquad i = 1, \ldots, N-1$$

One can see that in these coordinates linear terms in (1.1) will be killed:

$$F = \sum_{i,k=1}^{N-1} M_{ik}x^i x^k + \sum_{i=1}^{N-1} L_i x^i + N = \xi_1^2 + \xi_2^2 + \xi_3^2 + \ldots + \xi_{N-1}^2 + (a - b - \xi_1 - \xi_2 - \ldots - \xi_{N-1})^2 =$$

$$\left(\eta_1^2 + \eta_2^2 + + \ldots + \eta_{N-1}^2\right) + 2\left(\eta_1 + \ldots + \eta_{N-1}\right)\frac{b-a}{N} + \frac{(N-1)(b-a)^2}{N^2} +$$

$$+ \left(\frac{a-b}{N} - \eta_1 - \eta_2 - \ldots - \eta_{N-1}\right)^2 =$$

$$= 2\left(\eta_1^2 + \eta_2^2 + \ldots + \eta_{N-1}^2\right) + 2\sum_{i<j}\eta_i\eta_j + \frac{(b-a)^2}{N}\,.$$

The linear temrs are cancelled, and we have that

$$F = \tilde{M}_{ik}u^i u^k + \frac{(b-a)^2}{N}\,,$$

where

$$\tilde{M}_{ik} = \delta_{ik} + t_i t_k\,, \qquad (t_i = (1, \ldots, 1))\,.$$

Now we can calculate the integral:

$$I = \int e^{-cF} dx^1 dx^2 \ldots dx^{N-1} = \int e^{-c\left(M_{ik}x^i x^k + L_i x^i + N\right)} dx^1 dx^2 \ldots dx^{N-1} =$$

$$\int e^{-c\left(\sum_{k=1}^{N-1}\xi_k^2 + \left(b-a-\sum_{m=1}^{N-1}\xi_m^2\right)^2\right)} \underbrace{\left(\frac{\partial(x_1, \ldots, x_{N-1})}{\partial\xi_1, \ldots, \xi_{N-1})}\right)}_{\text{equals to 1}} d\xi_1 d\xi_2 \ldots d\xi_{N-1} =$$

$$\int e^{-c\left(2\sum_{k=1}^{N-1}\xi_k^2 + \left(b-a-\sum_{m=1}^{N-1}\xi_m^2\right)^2\right)} d\xi_1 d\xi_2 \ldots d\xi_{N-1} =$$

$$\int e^{-c\left(2\sum_{i,k=1}^{N-1}\eta_i\eta_k + \frac{(b-a)^2}{N}\right)} \underbrace{\left(\frac{\partial(\xi_1, \ldots, \xi_{N-1})}{\partial\eta_1, \ldots, \eta_{N-1})}\right)}_{\text{equals to 1}} d\eta_1 d\eta_2 \ldots d\eta_{N-1} =$$

$$\int e^{-c\left(N_{ik}\eta^i\eta^k + \frac{(b-a)^2}{N}\right)} d\eta_1 d\eta_2 \ldots d\eta_{N-1} =$$

$$\left(\frac{\pi}{c}\right)^{\frac{N-1}{2}} \sqrt{\frac{1}{\det(\tilde{M})}}\, e^{-c\left(\frac{(b-a)^2}{N}\right)}\,.$$

Notice that the matrix $\tilde{M}$ has eigenvector $\mathbf{t}$ with eigenvalue N, and all other $N-2$ eigenvectors wchih are orthogonal to the vector $\mathbf{t}$ with the eigenvalue 1. Hence

$$\det \tilde{M} = N\,,$$

and we have for integral:

$$I = \int e^{-c\left(N_{ik}\eta^i\eta^k + \frac{(b-a)^2}{N}\right)} d\eta_1 d\eta_2 \dots d\eta_{N-1} =$$

$$\sqrt{\frac{\pi}{\det(c\tilde{M})}} e^{-c\left(\frac{(b-a)^2}{N}\right)} = \left(\frac{\pi}{c}\right)^{\frac{N-1}{2}} \frac{1}{\sqrt{N}} e^{-c\left(\frac{(b-a)^2}{N}\right)}\,.$$