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The following lemma is useful.

Let M be n-dimensional Riemannian manifold. Consider the spray of the geodesics $x^i = x^i(t, \mathbf{n})$, (differential equation:

$$x^i(t,\tau)\colon \quad \begin{cases} x^i tt + \Gamma^i_{km} x^k_t x^m_t = 0 \\ x^i(t)\big|_{t=0} = 0 \\ x^i(t,\mathbf{n})\big| t = 0 = \mathbf{n}\,. \end{cases} \qquad \mathbf{n} \text{ is unit vector}$$

Thus we have the exponential map which assigns to every unit vector $\mathbf{n} \in T_{\mathbf{pt}}M$, the geodesics.

The natural question can be asked. Consider the disc of the radius R, i.e. the set of the points $x^i(t, \mathbf{n})\colon t \in [0, R]$. Does the boundary of thedisc is perpendicular to the geodesics, i.e. is it true that

$$\langle x_t(t, \mathbf{n}), \delta x\rangle = 0$$

Let τ_α ($\alpha = 1, \dots, n-1$) be cooridnates on unit sphere $\mathbf{n} = \mathbf{n}(\tau_\alpha)$.

Denote

$$L = \langle \mathbf{x}_t, \mathbf{x}_\tau\rangle = g_{ij} x^i_t x^j_\tau\,.$$

It is evident that $L(t,\tau)\big|_{t=0} = 0$ since this is scalar product of unit vector on the tangent vector to the unit sphere. Prove that it vanishes for all t.

Let $E = \frac{1}{2}|x^i_t|^2$. This is equal to constant. Hence

$$0 = \left(\frac{1}{2} g_{mn} x^m_t x^n_t\right)_\tau = \frac{1}{2} x^p_\tau \partial_p g_{mn} x^m_t x^n_t + g_{mn} x^m_{t\tau} x^n_t = \frac{1}{2} x^p_\tau \partial_p g_{mn} x^m_t x^n_t + (g_{mn} x^m_\tau x^n_t)_t =$$

$$= \frac{1}{2} x^p_\tau \partial_p g_{mn} x^m_t x^n_t + L_t - x^r_t \partial_r g_{mn} x^m_\tau x^n_t - g_{mn} x^m_\tau x^n_{tt} =$$

$$L_t - \frac{1}{2}\left(\partial_m g_{np} + \partial_n g_{mp} - \partial_p g_{mn}\right) x^p_\tau x^m_t x^n_t - g_{mn} x^m_\tau x^n_{tt} = L_t - g_{mn} x^m_\tau \left(x^n_{tt} + \Gamma^i_{mn} x^m_t x^n_t\right) = L_t = 0\,. \blacksquare$$

Invariant proof.

Consider two vector fields $\mathbf{v}, \mathbf{Y}$ which commute Then We have

$$0 = \partial_{\mathbf{Y}}\left(\frac{1}{2}\langle \mathbf{v}, \mathbf{v}\rangle\right) = \langle \nabla \mathbf{Y} \mathbf{v}], \mathbf{v}\rangle = \langle \nabla_{\mathbf{v}} \mathbf{Y}, \mathbf{v}\rangle = \partial_{\mathbf{v}}\langle \mathbf{Y}, \mathbf{v}\rangle - \langle \mathbf{Y}, \nabla_{\mathbf{v}} \mathbf{v}\rangle\,.$$