

# One method of Calculating Integral $\int_1^\infty \frac{\exp(-at)}{t} dt$ for small $a$ and Seeley Formulae

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We consider calculating of the integral  $\int_1^\infty \frac{\exp(-at)}{t} dt$  using ideas inspired by Seeley formulae. First the calculation. For estimating the behaviour of  $\int_1^\infty \frac{\exp(-at)}{t} dt$  for large  $a$  one can consider integration by parts

$$\int_1^\infty \frac{\exp(-at)}{t} dt = \frac{e^{-a}}{a} \sum_{n=0}^N \frac{n!(-1)^n}{a^n} + \frac{(-1)^N N!}{a^{N+1}} \int_1^\infty e^{-at} t^{N+1} dt \quad (1)$$

**Remark** One can see that this formula does not give convergent power series. On the other hand for every fixed  $N$  the reminder term is  $\frac{e^{-a} o(1)}{a^N}$ .

For small  $a$  the Eq.(1) is helpless.

We consider the trivial identity

$$a^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \exp(-at) dt \quad \text{for } s > 0 \quad (2)$$

One can rewrite the l.h.s. of (2) in the way to consider its analytical continuation for neighbourhood of  $s = 0$

$$\begin{aligned} a^{-s} &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \exp(-at) dt = \\ &= \frac{1}{\Gamma(s)} \left( \int_0^1 t^{s-1} (\exp(-at) - 1) dt + \int_0^1 t^{s-1} dt + \int_1^\infty t^{s-1} \exp(-at) dt \right) \\ &= \frac{1}{\Gamma(s)} \left( B(s) + \frac{1}{s} \right) \end{aligned} \quad (3)$$

where we denote

$$B(s) = \left( \int_0^1 t^{s-1} (\exp(-at) - 1) dt + \int_1^\infty t^{s-1} \exp(-at) dt \right)$$

Hence

$$a^{-s} = \frac{1}{\Gamma(s)} \left( B(s) + \frac{1}{s} \right) \quad (4)$$

The both parts of (3) are defined and can be differentiated in the neighbourhood of  $s = 0$ . Using that

$$\frac{1}{\Gamma(s)} \Big|_{s=0} = 0 \quad \text{and} \quad \left( \frac{d}{ds} \frac{1}{\Gamma(s)} \right) \Big|_{s=0} = \left( \frac{d}{ds} \left( \frac{s}{\Gamma(s+1)} \right) \right) \Big|_{s=0} = 1 \quad (5)$$

we come from (4) to

$$\frac{d}{ds} a^{-s} \Big|_{s=0} = -\log a = B(0) - \Gamma'(1) \quad (6)$$

the derivative  $\Gamma'(1)$  is the famous Euler constant

$$\Gamma'(1) = \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} - \log n \right) \quad (7)$$

We see from (2—6) that

$$B(0) = \int_0^1 \frac{\exp(-at) - 1}{t} dt + \int_1^\infty \frac{\exp(-at)}{t} dt = \Gamma'(1) - \log a \quad (8)$$

This gives representation for  $\int_1^\infty \frac{\exp(-at)}{t} dt$  which is convenient for small  $a$

$$\begin{aligned} \int_1^\infty \frac{\exp(-at)}{t} dt &= \Gamma'(1) - \log a - \int_0^1 \frac{\exp(-at) - 1}{t} dt = \\ &= \Gamma'(1) - \log a - \sum_{n=1}^{\infty} \frac{(-1)^n a^n}{n!n} = \\ &= -\log a + \Gamma'(1) + a - \frac{a^2}{2 \cdot 2!} + \frac{a^3}{3 \cdot 3!} - \frac{a^4}{4 \cdot 4!} + \dots \end{aligned} \quad (9)$$

(where  $\Gamma'(1)$  is nothing but Euler constant  $C$ :

$\Gamma'(1) = C = \lim_{n \rightarrow \infty} (1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log n)$ ) Empirically  $B(0)$  in (8) can be considered as renormalized value of the integral

$$\int_1^\infty t^{s-1} \exp(-at) dt = \dots \quad (10)$$

And naively (which happens often in Quantum Field Theory) one can calculate it using Frullany formula:

$$\int \frac{f(at) - f(bt)}{t} dt = f(0) \cdot \log \frac{b}{a}$$

The exact result differs from naive expectation on the Euler constant !!!.

In fact all this stuff is the reduction of the Seeley formulae

I think this very trivial exercise is useful

shtoby vvesti populjarno ideologiju perenormirovok v formulakh Sili (Seleey)

**Remark 2**

Denote  $I(a) = \int_1^\infty \frac{\exp(-at)}{t} dt$  and  $K(a) = \sum_{n=1}^{\infty} \frac{(-1)^n a^n}{n!n}$

We note that

$$\Gamma'(1) = \int_0^\infty \log t e^{-t} dt = \int_0^1 \log t e^{-1} dt + \int_1^\infty \log t e^{-1} dt = K(1) + I(1)$$

hence we come to the identity at  $a = 1$ . Helas our formula does not give series for  $a = 1...$

**Remark 3** Tut vsjo zatsepleno: Naprimer

$$\int_0^1 \frac{e^{-at} - 1}{at} dt = \int_0^1 \log t e^{-at} dt$$

The second integral is related with  $\Gamma'(1)$ .