One combinatorial problem

Autumn, 2012

It was long long long ago when I solved the following exercise: Denote by S_k *a number of sequences of* n *natural numbers* $\{1,2,3,\dots,k\}$ *such that all the numbers are on the wrong places, i.e. the first number is not* 1*, the second number is not* 2*, e.t.c.*

I forget the calculations. I just rememeber that they were not nice, but the answer was beautiful, something like $S_k \approx k!/e$*. Two months ago I found a following beautiful solution. Here it is*:

$$\text{One can see that} \quad \sum_{k=1}^{n} C_n^k S_k = n!\,, \tag{1}$$

where $C_n^k = \begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!}$. (The right hand sight of equation (1) is the number of all permutations of the set with n elements. The summand $C_n^k S_k$ in the left hand side is the number of permutations such that exactly $n-k$ elements are fixed.)

Recall that the n-th derivative of the product FG of two functions F and G is given by the formula

$$\left(\frac{d^n}{dx^n}\right)(F(x)G(x)) = \sum_{k=1}^{n} C_n^k \left(\frac{d}{dx}\right)^k (F(x)) \left(\frac{d}{dx}\right)^{n-k} (G(x))\,.$$

Comparing this formula with relation (1) we see that if we choose

$$F = \sum \frac{S_k}{k!} x^k\,, \qquad \text{and } G = e^x\,,$$

then

$$\left(\frac{d^n}{dx^n}\right)(F(x)G(x))_{x=0} = \left(\frac{d^n}{dx^n}\right)(F(x)e^x)_{x=0} = \sum_{k=1}^{n} C_n^k S_k = n!\,.$$

Hence

$$F(x)e^x = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

in a vicinity of $x=0$. We come to the answer: the sequence S_k is such that

$$F = \sum_{k=0}^{\infty} \frac{S_k}{k!} = \frac{e^{-x}}{1-x}\,.$$

Using this formula we write down the explicit formula for S_k. Denote by $s_k = \frac{S_k}{k!}$. We have that

$$\sum_{k=0}^{\infty} \frac{S_k}{k!} x^k = \sum_{k=0}^{\infty} s_k x^k = \frac{e^{-x}}{1-x} = \left(1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \dots\right)\left(1 + x + x^2 + x^3 + \dots\right)$$

$$= 1+(1-1)\,x+\left(1-1+\frac{1}{2!}\right)x^2+\left(1-1+\frac{1}{2!}-\frac{1}{3!}\right)x^3+\left(1-1+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right)x^4+\dots$$

i.e.

$$s_k = \frac{S_k}{k!} = \sum_{p=0}^{k} \frac{(-1)^p}{p!}, \text{ and } S_k = k!\sum_{p=0}^{k} \frac{(-1)^p}{p!}\,.$$

In particular

$$s_\infty = \lim_{k\to\infty} = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} = \frac{1}{e}\,,$$

i.e. the probability that all terms of the sequence $\{1, 2, 3, \dots, N\}$ are on the wrong places equals to $\approx \frac{1}{e}$ when $N \to \infty$.