Projective transformations: pedestrian's point of view

§0. Projective transformations

The group of projective transformations of n-dimensional projective space $\mathbf{R}P^n$ is $PGL(n,\mathbf{R})$. It is is a factor of $GL(n+1,R)$ by matrices which do not move any line, i.e. it is the factor of $SL(n+1,R)$ by the centre of this group. This is immediate consequence of definition of projective space as set of lines in $\mathbf{R}^{n+1}$ passing through the origin— the linear transformations in $GL(n+1,\mathbf{R})$ induces projective transformations. If $[x^0 : x^1 : \ldots : x^n]$— homogeneous coordinates in $\mathbf{R}P^n$ and $u^1, u^2, \ldots, u^n$ not-homogeneous coordinates in $\mathbf{R}P^n$ such that

$$u^i = \frac{x^n}{x^0}, \qquad i = 1, \ldots, n, \; x^0 \neq 0$$

(in a chart of $\mathbf{R}P^n$) then projective transformation in homogeneous coordinates is

$$x^\mu \mapsto A^\mu_\nu x^\nu \quad (\mu, \nu = 0, 1, 2, \ldots, n) \tag{0.2}$$

and in nothomogeneous coordinates it will be

$$u^i \mapsto \frac{A^i_k u^k + t^i}{1 + \omega_r u^r} \tag{0.3}$$

The definitions of projective group follows from (1).

Let us stand on naive point of view,. Projective transformations has to be understood in the following way:

Projective transformations are transformations which transform lines to lines

The transformation (3) transforms an arbitrary line to the line*. This is because line in $\mathbf{R}P^n$ it is a plane in $\mathbf{R}^{n+1}$, and the transformation (0.2) transforms planes to planes.

Hence it is projective transformation in the naive sense.

Question: *Is the converse true? Is it true that any transformations transforming lines to lines are all described by (3)?*

Yes, it is

This statement and its proof sure was known centuries ago , but not so easy to find it in modern textbooks. I show that this is true, on infinitesimal level. All calculations are elementary.

§1. Infinitesimal transformations of lines to lines

Let F be a transformation of $\mathbf{R}P^n$ which transforms straight lines to straight lines. (In other words a vector field $\mathbf{F} = F^m(u)\frac{\partial}{\partial u^m}$ is considered).

Take an arbitrary line: $s_r u^r = 1$. Write down the condition that this line transforms to a line:

$$s'_m u^m - 1 = 0 \text{ if and only if } s_m u'^m - 1 = 0 \tag{1.1}$$

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* I do not want to make this statement exact, involving "infinities"

where $s'_m = s'_m(s_1, \ldots, s_n)$ are coefficients of the transformed line, u'^m are coordinates of transformed points:

$$s'_m(s) = s_m + \varepsilon L_m(s), \;\; u'^m = u^m + \varepsilon F^m(u), \quad (\varepsilon^2 = 0) \tag{1.2}$$

(The condition $\varepsilon^2 = 0$ encodes the fact that we consider infinitesimal small magnitudes of the first order only) Taking care about proportionality coefficient in (1.2) we come to the equation:

$$(s_m + \varepsilon L_m(s))u^m - 1 = (1 + \varepsilon\Lambda(u, s))\,(s_m(u^m + \varepsilon F^m(u) - 1)), \qquad (\varepsilon^2 = 0)$$

Due to $\varepsilon^2 = 0$ this equation is equivalent to the equation:

$$L_m(s)u^m - s_m F^m(u) = \Lambda(s, u)(s_m u^m - 1) \tag{1.3}$$

Solve this equation in polynomials (Why in polynomials?).

§2. Description of vector fields transforming lines to lines

The equation (1.3) means that $(s_m u^m - 1)$ divides $L_m(s)u^m - s_m F^m(u)$.

We have to find polynomials $F^m(u)$ which obey this equation, tio.e. F such that there exist polynomials (Why polynomials?) L_m, Λ such that the equation (2.1) is obeyed.

It is linear equation. Solve it in steps

I case. 0-degree polynomials: $F^m(u) = F^m$:

$$L_m(s)u^m - s_m F^m = \Lambda(s, u)(s_m u^m - 1)$$

We see that arbitrary F^m obeys the equation. (We can put $L_m(s) = s_m s_r F^r$, $\Lambda = s_r F^r$)

II case. First degree polynomials: $F^m(u) = F^m_p u^p$. Then

$$L_m(s)u^m - s_m F^m_p u^p = \Lambda(s, u)(s_m u^m - 1)$$

We see that arbitrary F^m_p obeys the equation. (We can put $L_m(s) = F^p_m s_p$, $\Lambda = s_r F^r$)

III case. Second degree polynomials: $F^m(u) = F^m_{pq} u^p u^q$. Then

$$L_m(s)u^m - s_m F^m_{pq} u^p u^q = \Lambda(s, u)(s_m u^m - 1)$$

Solve this equation. Let $\Lambda(s, u) = Ł(s) + Ł_k(s)s^k + \ldots$. Comparing zeroth order terms with respect to u we come to condition $\Lambda(s) = 0$. Comparing terms of the first order we come to the equation:

$$L_m(s)u^m = -\Lambda_k(s)s_k, \text{ i.e. } \Lambda_m(s) = -L_m(s)$$

Now looking on second order terms we come to

$$-s_m F^m_{pq} u^p u^q = Ł_k(s)s^k s_m u^m = -L_k(s)u^k s_m u^m, i.e.$$

$$2F^m_{pq} = L_p\delta^m_q + L_q\delta^m_p$$

We come to solution:

$$F^m_{pq} = \frac{1}{2}\left(t_p\delta^m_q + t_q\delta^m_p\right), \text{ where } t_i \text{ are constants}$$

where we can put $L_r(s) = t_r, Ł(s,r) = -t_r$:

IV and the last case. Show that in the higher degrees there are not solutions. Indeed let F is of the order k over u with $k \geq 3$. Then left hand side posseses temrs of first degree and third degree over u. Hence Λ cannot be a polynomial. If degree of Ł is less or equal to n then the right hand side possesses the terms pf the order n and $n+1$.

Collecting all the cases we come to the answer

The algebra of transformations (vector fields) which transform line to the line is the following:

$$F^m(u) = r^m + a^m_p u^p + \frac{1}{2}\left(t_p\delta^m_q + t_q\delta^m_p\right)u^p u^q\,.$$

One can easy see that this algebra coincides with algebra of infinitesimal transformations from (0.3) ∎