Irreducible components of third rank tensors

Let T^i_{km} be a tensor in E^3 such that $T^i_{km} = T^i_{mk}$, i.e. $T^i(x) = T^i_{km}x^k x^m$. What are its irreducible components[1](with respect to $SO(3)$)?

$$T_{ikm} = T^n_{i(km)} + T^s_{(ikm)}\,, \qquad (18 = 10 + 8)$$

where T^s is totally symmetric and T^n is normal:

$$T^s_{ikm} = \frac{1}{3}(T_{ikm} + T_{kim} + T_{mki})\,, T^n_{ikm} = \frac{2}{3}T_{ikm} - \frac{1}{3}T_{kim} - \frac{1}{3}T_{mki})\,,$$

$$T^n_{ikm}x^i x^k x^m = 0\,, \text{i.e. } T^n_{ikm} + T^n_{kim} + T^n_{mki} = 0,\ (T^n_{ikm} = T^n_{imk})\,. \qquad (1.1c)$$

Now consider restriction on the group $SO(3)$.

Space of tensors $T^s_{(ikm)}$ is irreducible with respect to the group $GL(3)$, but it is reducible with respect to $SO(3)$. Tensor T^s_{ikm} is a sum of $l = 3$ (symmetric traceless tensor) and of $l = 1$:

$$T^s_{imk} = T^{l=2}_{(imk)} + (a_i\delta_{mk} + a_m\delta_{ik} + a_k\delta_{mi})\ , \quad (10 = 7 + 3)\,, \qquad (1.2)$$

where $T^{l=2}_{imk}$ is symmetric traceless tensor and $a_i = \frac{1}{5}T^s_{irr}$,

Little bit more care about the 8-dimensional space of tensors $T^n_{i(mk)}$, obeying conditions (1.1c) It is one of equivalent irreducible spaces with respect to $GL(3)$ ($27 = 10 + 8 + 8 + 1$), and with respect to $SO(3)$ it is a sum of $l = 2$ and $l = 1$ subspaces ($8 = 5 + 3$): Indeed tensor $T^n_{i(km)}$ is the following sum

$$T^n_{i(mk)} = T^{l=2}_{i(mk)} + (2b_i\delta_{mk} - b_m\delta_{ik} - b_k\delta_{mi})\ , (1.3a)$$

where $b_i = \frac{1}{4}T_{irr}$ and tensor $T^{l=2}_{ikm}$ obeys conditions:

$$T^{l=2}_{irr} = 0\,, \quad T^{l=2}_{ikm} = T^{l=2}_{imk}\,, \quad T^{l=2}_{ikm}x^i x^k x^m = 0\,, \quad (1.3b)$$

One can see that it is pseudotensor of angular momentum $l = 2$

$$T^{l=2}_{ikm} = \varepsilon_{ikr}t_{rm} + \varepsilon_{imr}t_{rk}\,, \quad t_{mn} = \frac{1}{3}\varepsilon_{ikm}T^{l=2}_{ikn}$$

where t_{rk} is traceless symmetric tensor ($l = 2$).

Exrcise Analyze these maps in detail. Show that map above is one-one correspondence.

[1] (We need it considering analogues of quadropole expansion for current $J^i(x)$)

Finally collecting these formulae we see that tensor T_{imk} which is symmetric with second and third index possesses one field of $l = 3$, one field of $l = 2$ and two vector fields:

$$T_{ikm} = \underbrace{t_{ikm}}_{l=3} + \underbrace{\varepsilon_{ikr}t_{rm} + \varepsilon_{imr}t_{rk}}_{l=2} + \underbrace{a_i\delta_{mk} + a_m\delta_{ik} + a_k\delta_{mi}}_{l=1} + \underbrace{2b_i\delta_{mk} - b_m\delta_{ik} - b_k\delta_{mi}}_{l=1} \tag{1.4}$$

Remark Notice that one can consider instead decomposition (1.3) the decomposition:

$$T^n_{i(mk)} = T^{l=2}_{i(mk)} + (2b_i\delta_{mk} - b_m\delta_{ik} - b_k\delta_{mi}) \ , (1.3'a)$$

where instead conditions (1.3b) we have for the tensor $T^{l=2}_{ikm}$ the conditions:

$$T^{l=2}_{rri} = 0 \, , \quad T^{l=2}_{ikm} = T^{l=2}_{imk} \, , \quad T^{l=2}_{ikm}x^ix^kx^m = 0 \, , \quad (1.3'b)$$

In this case condition $b_i = \frac{1}{4}T_{irr}$ for (1.3a) transforms to condition '$b_i = -\frac{1}{2}T_{rri}$.

iBoth conditions define the same subsapces. This follows from the uniqueness of decomposition of space on irreducible components. Onwe can see it by brute force: $T^{rri} = 0 \leftrightarrow T_{iir} = 0$ for tensors obeu=ying condition (1.1c).

What happens in the case of general tensor. In this case first two words about decomposition on irreducible subspaces with respect to group $GL(n)$ (We temporarlly consider arbitrary dimension)

Consider first decompostion on tensors symmetric and antisymmetric with respect to k, m:

$$T_{ikm} = T_{i(km)} + T_{i[km]}$$

We decompose $T_{i(km)}$ on the sum of symmetric and normal tensor as in (1.1) and $T_{i[mk]}$ on a sum of antisymmetric and normal

Then for general symmetric group we will have

$$T_{ikm} = \underbrace{t_{ikm}}_{\text{symmetric}} + \underbrace{u^n_{i(km)} + v^n_{i[km]}}_{\text{two equival. irreduc. repres.}} + \underbrace{s_{[ikm]}}_{\text{antisymmetric}} \, ,$$

where $u^n_{(ikm)} = v^n_{[ikm]} = 0$:

$$t_{ikm} = \frac{1}{6}(T_{ikm} + T_{imk} + T_{kmi} + T_{kim} + T_{mik} + T_{mki}) \, ,$$

$$u^n_{ikm} = \frac{1}{6}(2T_{ikm} + 2T_{imk} - T_{kmi} - T_{kim} - T_{mik} - T_{mki}) \, ,$$

$(u^n_{ikm} = u^n_{imk}, u^n_{ikm}x^ix^mx^k = 0).)$

$$s_{ikm} = \frac{1}{6}(T_{ikm} - T_{imk} + T_{kmi} - T_{kim} + T_{mik} - T_{mki})\,,$$

$$v^n_{ikm} = \frac{1}{6}(2T_{ikm} + 2T_{imk} - T_{kmi} + T_{kim} - T_{mik} + T_{mki})\,,$$

($v^n_{ikm} = -v^n_{imk}, v^n_{ikm}\xi^i\xi^m\xi^k = 0$), ξ is odd variable)).

For dimensions we have

$$n^3 = \underbrace{\frac{n(n+1)(n+2)}{6}}_{\text{symmetric}} + \underbrace{2 \times \frac{n(n+1)(n-1)}{3}}_{\text{two equival. irreduc. repres.}} + \underbrace{\frac{n(n-1)(n-2)}{6}}_{\text{antisymmetric}}\,,$$

In particular for $n = 3$

$$27 = 10 + 8 + 8 + 1$$

Restrict on the group $SO(3)$. We already did for symmetric part $T_{i(mk)} = t_{imk} + u^n_{imk}$ (see (1.4)) Analogously we do for for antisymmetric part. For $n = 3$ $s_{imk} = s\varepsilon_{imk}$ defines pseudoscalar. For tensor v^n_{imk}:

$$v^n_{imk} = -v^n_{ikm}, \quad v^n_{imk}\xi^i\xi^m\xi^k = 0), \qquad \text{i.e.}\, \varepsilon_{imk}v_{imk} = 0$$

we come to decomposition like in (1.3'a):

$$v^n_{imk} = \tau_{ij}\varepsilon_{imk} + (\delta_{im}\beta_k - \delta_{ik}\beta_m)\,.$$

where τ_{ij} is symmetric traceless tensor ($l = 2$) and β_i is a vector ($l = 1$) We come to decomposition:

$$T_{ikm} = \underbrace{t_{ikm}}_{l=3} + \underbrace{a_i\delta_{mk} + \alpha_m\delta_{ik} + \alpha_k\delta_{mi}}_{l=1} + \underbrace{\varepsilon_{ikr}t_{rm} + \varepsilon_{imr}t_{rk}}_{l=2} + \underbrace{2b_i\delta_{mk} - b_m\delta_{ik} - b_k\delta_{mi}}_{l=1} +$$

$$\underbrace{\tau_{ij}\varepsilon_{jkm}}_{l=2} + \underbrace{2\beta_i\delta_{mk} - \beta_m\delta_{ik} - \beta_k\delta_{mi}}_{l=1} + \underbrace{v\varepsilon_{ikm}}_{l=0}$$

We will have one particle with $l = 3$, two with $l = 2$, three vector fields, $l = 1$ and one pseudoscalar:

$$27 = 10 + 8 + 8 + 1 = (7 + 3) + (5 + 3) + (5 + 3) + 1$$