On Barnes functions

Consider ring B with generators $\{\sigma_a\}$, where a is an arbitrary complex number. We consider also functions s_a such that

$$s_a(t) = \frac{1}{e^{-at}}$$

One can consider homomorphism ι of the ring B to the algebra of functions defined by relations

$$\iota(\sigma_a) = s_a\,.$$

Denote by N the kernel of this isomorphism.

Fact

For arbitrary complex numbers $a_1, a_2, \dots a_n$, the elements

$$\sigma_{a_1+\ldots+a_n} \prod (\sigma_{a_i} - 1) - (\sigma_{a_1+\ldots+a_n} - 1) \prod (\sigma_{a_i})$$

belong to the subring N, that is the functions

$$s_{a_1+\ldots+a_n}(t) \prod (s_{a_i}(t) - 1) - (s_{a_1+\ldots+a_n}(t) - 1) \prod (s_{a_i}(t))$$

vanish

For example elements

$$\sigma_{a+b} - \sigma_{a+b}(\sigma_a + \sigma_b) + \sigma_a\sigma_b\,, \quad \sigma_{a+b+c} - \sigma_{a+b+c}\left(\sigma_a + \sigma_b + \sigma_c - \sigma_a\sigma_b - \sigma_a\sigma_c - \sigma_b\sigma_c\right) - \sigma_a\sigma_b\sigma_c$$

belong to the subring N and

$$s_{a+b}(t) - s_{a+b}(t)(s_a(t) + s_b(t)) + s_a(t)s_b(t) \equiv 0$$
$$s_{a+b+c}(t) - s_{a+b+c}(t)\left(s_a(t) + s_b(t) + s_c(t) - s_a(t)s_b(t) - s_a(t)s_c(t) - s_b(t)s_c(t)\right) - s_a(t)s_b(t)s_c(t) \equiv 0$$

Question: How looks ring N???

Let r be an arbitrary element of the ring B,

Now we consider on the space of functions the linear map R:

$$R\colon \quad A(t) \mapsto R(A)(t) = A_-(t) - A_0 e^{-t}$$

(see the étude in)

One can assing to every monom $r = \sigma_{a_1} \dots \sigma_{a_n} \in B$ the following Barnes function

$$B_r(z|a_1, \dots, a_n) = \exp\left(\int_0^\infty \left(e^{-zt}\iota(r)(t) - R\left[e^{-zt}\iota(r)\right](t)\right)\frac{dt}{t}\right) =$$
$$\exp\left(\int_0^\infty \left(e^{-zt}s_{a_1}(t)\dots s_{a_n}(t) - R\left[e^{-zt}s_{a_1}\dots s_{a_n}\right](t)\right)\frac{dt}{t}\right).$$

For example to the monom $\sigma_a\sigma_b\sigma_c$ corresponds the barnes function

$$B(z|a,b,c) = \exp\left(\int_0^\infty \left(\frac{e^{-zt}}{(1-e^{-at})(1-e^{-bt})(1-e^{-ct})} - R\left[\frac{e^{-zt}}{(1-e^{-at})(1-e^{-bt})(1-e^{-ct})}\right]\right)\frac{dt}{t}\right).$$