Projetive class of connections

§1. Reparameterising of geodesics and changing of connection.

We try to expose here some common facts of projective geometry about geodesics and projective class of connections

Let M be a manifold. We consider only symmetric connections.

We say that two symmetric connections ∇ and $\tilde\nabla$ belong to the same *projective* class if there exists 1-form ω such that

$$\left(\tilde\nabla - \nabla\right)(\mathbf{X},\mathbf{Y}) = \mathbf{X}\omega(\mathbf{Y}) + \mathbf{Y}\omega(\mathbf{X})\,, \tag{1.1a}$$

i.e. for Christoffel symbols

$$\tilde\Gamma^i_{km} - \Gamma^i_{km} = \delta^i_k\omega_m + \delta^i_m\omega_k\,, \tag{1.1b}$$

Proposition Two symmetric connections belong to the same projective class if and only if their all geodesics coincide up to reparameterisation.

Prove it. Let $\mathbf{x}(t)$ be a geodesic of connection ∇, i.e.

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{km}(\mathbf{x}(t))\frac{dx^k}{dt}\frac{dx^m}{dt} = 0\,. \tag{1.2a}$$

Let a connection $\tilde\nabla$ belong to the projective class of the connection ∇. Show that there exist $\tau=\tau(t)$ such that

$$\frac{d^2\tilde x^i}{d\tau^2} + \tilde\Gamma^i_{km}(\tilde{\mathbf{x}}(\tau))\frac{d\tilde x^k}{d\tau}\frac{d\tilde x^m}{d\tau} = 0\,. \tag{1.2b}$$

for $\tilde{\mathbf{x}}(\tau) = \mathbf{x}(t)$ for $\tau=\tau(t)$.

We rewrite (1.2a) using that $\mathbf{x}_t = \mathbf{x}_\tau\tau_t$:

$$\frac{d^2\tilde x^i}{d\tau^2}\tau_t^2 + \frac{d\tilde x^i}{d\tau}\tau_{tt} + \Gamma^i_{km}(\tilde{\mathbf{x}}(\tau))\frac{d\tilde x^k}{d\tau}\frac{d\tilde x^m}{d\tau}\tau_t^2 = 0\,. \tag{1.3}$$

Dividing this equation on τ_t^2 and substracting from the equation (1.2b) we come to

$$-\frac{d\tilde x^i}{d\tau}\frac{\tau_{tt}}{\tau_t^2} + \left(\tilde\Gamma^i_{km} - \Gamma^i_{km}\right)\frac{d\tilde x^k}{d\tau}\frac{d\tilde x^m}{d\tau} = -\frac{d\tilde x^i}{d\tau}\frac{\tau_{tt}}{\tau_t^2} + 2\frac{d\tilde x^i}{d\tau}\omega_m\left(\tilde{\mathbf{x}}(\tau)\right)\frac{d\tilde x^m}{d\tau} =$$

$$-\frac{d\tilde x^i}{d\tau}\left[\frac{\tau_{tt}}{\tau_t^2} - 2\omega_m(\tilde{\mathbf{x}}(t))\frac{d\tilde x^m(t)}{d\tau}\right] = -\frac{dx^i(t)}{dt}\left[\frac{d\log\tau_t}{dt} - 2\omega_m(\mathbf{x}(t))\frac{dx^m(t)}{d}\right]t_\tau^2 = 0\,. \tag{1.4}$$

Thus we see that if $\mathbf{x}(t)$ is a geodesic of the connection ∇ then $\tilde{\mathbf{x}}(\tau)$ is a geodesic of the connection $\tilde\nabla$ if $\tau=\tau(t)$ obeys the differential equation:

$$\frac{d}{dt}\log\tau_t - 2\omega_m(\mathbf{x}(t))\frac{dx^m(t)}{dt}\,. \tag{1.4b}$$

i.e. if

$$\tau(t) = \exp\left(\int^t F(t')dt'\right) \text{ where } F(t) = \int^t \left[2\omega_m(\mathbf{x}(t'))\frac{dx^m(t')}{dt'}\right] dt' \qquad (1.5)$$

Now prove the converse implication.

Suppose that an arbitrary geodesic $\mathbf{x}(t)$ of the connection ∇ becomes the geodesic of the connection ∇ after suitable reparameterisation $\tau(t)$: $\tilde{\mathbf{x}}(\tau) = \mathbf{x}(t)$. Then comparing the equations (1.2a) and (1.2b) we come instead the equation (1.4) to the equation

$$-\frac{d\tilde{x}^i}{d\tau}\frac{\tau_{tt}}{\tau_t^2} + S^i_{km}\frac{d\tilde{x}^k}{d\tau}\frac{d\tilde{x}^m}{d\tau} = 0, \text{ where } S^i_{km} = \tilde{\Gamma}^i_{km} - \Gamma^i_{km}$$

We have to prove that in this case there exists 1-form ω such that $S^i_{km} = \delta^i_k\omega_m + \delta^i_m\omega_k$.

This holds for the arbitrary velocity vector $\mathbf{v} = \frac{d\mathbf{x}}{d\tau}$. Hence this follows from the following algebraic statement:

Let $\mathbf{S} = \mathbf{S}(\mathbf{X}, \mathbf{Y})$ be symmetric bilinear form on the vector space V with values in this space such that for arbitrary vector $\mathbf{X}$ the quadratic form $\mathbf{S}(\mathbf{X}, \mathbf{X})$ is collinear to $\mathbf{X}$. Then there exists 1-form (covector) $\mathbf{t}$ such that

$$\mathbf{S}(\mathbf{X}, \mathbf{Y}) = \mathbf{X}\mathbf{t}(\mathbf{Y}) + \mathbf{Y}\mathbf{t}(\mathbf{x}), \qquad (t_i = \frac{1}{n+1}S^k_{ki}).$$

(See the proof in the Appendix.) The next statement is much more intriguing.

§2. Thomas bundle.

Let M be a manifold equipped with projective class of connections, i.e. a class $[\nabla]$ of connections is defined on it. (We say that two connections belong to the same class if the condition (1.1) is obeyed.) E.g. if M is $\mathbf{R}^n$ then consider connection which Christoffel symbols vanish in cartesian coordinates The projective class of connections have Christoffel symbols $\Gamma^i_{km} = \delta^i_k\omega_m + \delta^i_m\omega_k$ where ω_k is an arbitrary covector. Geodesics of these connections are straight lines (in Cartesian coordinates.)

Add a new coordinate t, i.e. consider expanded manifold $\hat{M}$ with coordinates $x^\mu = (t, x^i), t = x^0$. We come to so called so called Thomas one-dimensional bundle $\hat{M}$.

Under changing of coordinates $x^{i'} = x^{i'}(x^i)$ the coordinate x^0 changes in the following way:

$$x^{0'} = \log\left(\det\left(\frac{\partial x^i}{\partial x^{i'}}\right)\right) + x^0$$

Theorem. Let $[\nabla]$ be a projective class of connections on M. One can assign to this class the connection $\hat{\nabla}$ in $\hat{M}$ with the following Christoffel symbols: $\Pi^\mu_{\nu\rho}$ $(\mu, \nu, \rho = 0, 1, 2, \dots, n)$ are such that

$$\Pi^i_{km} = \Gamma^i_{km} + \frac{1}{n+1}\left(\delta^i_k\gamma_m + \delta^i_k\gamma_m\right)$$

for $i = 1, 2, 3, \dots, n$ where $\gamma_i = -\Gamma^k_{ik}$,

$$\Gamma^i_{k0} = \Gamma^i_{0k} = -\frac{\delta^i_k}{n+1}$$

and so on.

Remark Please pay attention that in the formulae above the right hand side is defined up to

$$\Gamma^i_{km} \to \Gamma^i_{km} + \delta^i_k \omega_m + \delta^i_m \omega_k$$

Appendix

Here we prove the following simple statement:

Let $\mathbf{S} = \mathbf{S}(\mathbf{X}, \mathbf{Y})$ *be symmetric bilinear form on the vector space* V *with values in this space such that*

$$\mathbf{S}(\mathbf{X}, \mathbf{X}) \ \ \textit{is collinear to } \mathbf{X} \textit{ for arbitrary vector } \mathbf{X} \ . \qquad (App1)$$

Then there exists 1*-form (covector)* $\mathbf{t}$ *such that*

$$\mathbf{S}(\mathbf{X}, \mathbf{Y}) = \mathbf{X}\mathbf{t}(\mathbf{Y}) + \mathbf{Y}\mathbf{t}(\mathbf{x}), \qquad (t_i = \frac{1}{n+1} S^k_{ki}) \, . \qquad (App2)$$

Proof. We have that $S^i_{km} = S^i_{mk}$ and $S^i_{km} a^k a^m = a^i \lambda(\mathbf{a})$ for arbitrary vector $\mathbf{a} = a^i$. This means that for arbitrary i, j $a^j S^i_{km} a^k a^m = a^i S^j_{km} a^k a^m$.

Consider the set vectors $\mathbf{a}$ with first component $a^1 = 0$. Then for $i = 2, 3, \ldots, n$ we come

$$a^1 S^i_{km} a^k a^m = S^d_{bc} a^b a^c + 2 S^d_{1c} a^c + S^d_{11} = a^i S^1_{km} a^k a^m = a^d \left(S^1_{bc} a^b a^c + 2 S^1_{1c} a^c + S^1_{11} \right) \qquad (App3)$$

where indices b, c, d run over $\{2, 3, \ldots, n\}$. Comparing these quadratic polynomials we see that

$$S^d_{bc} = \delta^d_b S^1_{1c} + \delta^d_c S^1_{1b}, \quad S^d_{1c} = S^d_{c1} = \delta^d_c S^1_{11}, \quad S^d_{11} = 0 \, . \qquad (App4)$$

Consider the object $(t_1, t_2, \ldots, t_n)$ such that $t_1 = \frac{1}{2} S^1_{11}, \ t_a = S^1_{1a} \, (a = 2, \ldots, n)$. (We still do not know is it a vector!) It is easy to see from equations (App 4) that

$$S^i_{km} = \delta^i_k t_m + \delta^i_m t_k$$

We come to App 2: $t_m = \frac{1}{n+1} \mathrm{Tr} S = \frac{1}{n+1} S^k_{km}$ is a covector.