On Jordan normal form

you may find in any good textbook of linear algebra what is it Jordan normal form. I discussed with my students appearance of Jordan normal form in many other places, in particular in decomposition of finite Abelian groups. Here is the exposition of standard linear algebra subject from this point of view.

Consider a pair (A, V) where A is a linear operator on vector space V. If V is vector field over algebraically closed field (e.g. $\mathbf{C}$, we consider only this case) then A has an eigenvector, i.e. V possesses one-dimensional subspace L_A invariant with respect to operator A.

Definition Let A be a non-zero linear operator acting on finite-dimensional vector space V. We say that the pair (A, V) is Jordan cell if the vector space V possesses unique one-dimensional invariant subspace.

Example Let $J^{\lambda,n} = (A, \mathbf{R}^n)$ be a pair of n-dimensional arithmetic vector space $\mathbf{R}^n$ and an operator A such that in the canonical basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$

$$A\mathbf{e}_1 = \lambda\mathbf{e}_1, \quad \text{and } A_{\mathbf{e}_i} = \lambda\mathbf{e}_i + \mathbf{e}_{i-1} \quad \text{for } i = 2, \ldots, n$$

It is easy to see that this is Jordan cell. Indeed let $\mathbf{f} = x^i\mathbf{e}_i$ be a vector which defines invariant subspace: $\mathbf{f} \neq 0$ and $A\mathbf{f} = \mu\mathbf{f}$, i.e.

$$A(x^i\mathbf{e}_i) = x^1\lambda\mathbf{e}_1 + x^2(\lambda\mathbf{e}_2 + \mathbf{e}_1) + \ldots + x^n(\lambda\mathbf{e}_n + \mathbf{e}_{n-1}) = \mu x^i\mathbf{e}_i$$

This implies that $x_i = 0$ if $i = 2, \ldots, n$, i.e. $\mathbf{f}$ is proportional to $\mathbf{e}_1$. We see that there is a unique invariant subspace.

Note that if (A, V) is a Jordan cell and L is invariant subspace then the sequence

$$0 \to L \to V \to [V] \to 0$$

is not splitted with respect to operator A, i.e. every $n-1$-dimensional subspace in V is not invariant with respect to operator A.

Statement A If V is Jordan cell with respect to linear operator A acting on it, then (V, A) is isomorphic to $J^{\lambda,n}$ for some number λ and natural n.

This statement can be formulated in the following equivalent way:

Statement A′ Let n-dimensional vector space V be a Jordan cell with respect to operator A. Then there exists a number λ and a flag of invariant subspaces $\{V_k\}$, $k = 1, \ldots, n$ such that every V_k provided with action of A on it is isomorphic to $J^{\lambda,k}$.

This flag is unique.

It is easy to see that this statement implies

Proposition A″ Let A be N-dimensional space where acts an operator A. Then V is a direct sum of Jordan cells.

Prove the statement by induction on dimension.

For $n = 1$ it is alright. Suppose it is true if $\dim V = k$. Prove it for $\dim V = k + 1$.

It is illuminating first to consider the case $k = 1$.

Let V be two-dimensional Jordan cell with respect to operator A. Let $\mathbf{e}_1 \neq 0$ be an eigenvector of operator A: $A\mathbf{e} = \lambda\mathbf{e}$. By definition of Jordan cell it is unique (up to a coefficient).

Consider an arbitrary non-zero vector $\mathbf{f}$ which is not colinear to the vector $\mathbf{e}$. Let $Af = \mu\mathbf{e} + a\mathbf{f}$. Vector $\mathbf{f}$ is not eigenvector since V is Jordan cell. Hence $a \neq 0$. By changing $\mathbf{f} \to \frac{1}{a}\mathbf{f}$ we come to $A\mathbf{f} = \mu\mathbf{f} + \mathbf{e}$. Show that $\mu = \lambda$. If this is not the case then the vector $\mathbf{f}' = \mathbf{f} + \frac{\mathbf{e}}{\mu-\lambda}$ is second eigen-vector of operator A:

$$A\mathbf{f}' = A\left(\mathbf{f} + \frac{\mathbf{e}}{\mu - \lambda}\right) = \mu\mathbf{f} + \frac{\lambda\mathbf{e}}{\mu - \lambda} = \mu\left(\mathbf{f} + \frac{\mathbf{e}}{\mu - \lambda}\right) = \mu\mathbf{f}'\,.$$

Contradiction. Hence $\mu = \lambda$ and $A\mathbf{f} = \lambda\mathbf{e} + \mathbf{f}$. Thus pair (V, A) is Jordanian cell $J^{\lambda,2}$.

Now consider general case. Let V ne $n+1$-dimensional Jordanian cell, $n \geq 1$. Let $\mathbf{e}_1 \neq 0$ be an eigenvector of operator A: $A\mathbf{e} = \lambda\mathbf{e}$. By definition of Jordan cell this is unique (up to a coefficient) eigenvector.

Consider $n-1$ vector space $[V] = V \backslash L_{\mathbf{e}}$, where $L_{\mathbf{e}}$ is one-dimensional invariant subspace spanned by the vector $\mathbf{e}$. $[V]$ is Jordan cell too with respcet to the action of operator $[A]$. and dimension of this Jordanian cell is equal to n. Hence by inductive hypothesis, it is isomorphic to $J^{\mu,n}$ for some eigenvalue μ, i.e. there exists a basis $\{[\mathbf{f}_1], [\mathbf{f}_2], \ldots [\mathbf{f}_n]\}$ in the factor-space $[V]$ such that $A[\mathbf{f}_1] = \mu[\mathbf{f}_1]$ and $A[\mathbf{f}_{i+1}] = \mu[\mathbf{f}_{i+1}] + [\mathbf{f}_i]$. This means that for vectors $\{\mathbf{e}, \mathbf{f}_1, \ldots, \mathbf{f}_n\}$

$$\begin{cases} A\mathbf{e} = \lambda\mathbf{e} \\ A\mathbf{f}_1 = \mu\mathbf{f}_1 + a_1\mathbf{e} \\ A\mathbf{f}_2 = \mu\mathbf{f}_2 + \mathbf{f}_1 + a_2\mathbf{e} \\ A\mathbf{f}_3 = \mu\mathbf{f}_3 + \mathbf{f}_2 + a_3\mathbf{e} \\ A\mathbf{f}_4 = \mu\mathbf{f}_4 + \mathbf{f}_3 + a_4\mathbf{e} \\ \ldots\ldots\ldots\ldots \\ A\mathbf{f}_n = \mu\mathbf{f}_n + \mathbf{f}_{n-1} + a_n\mathbf{e} \end{cases}$$

Consider the first two equations in the system above. We already showed that $a_1 \neq 0$ and $\mu = \lambda$ since if $a_1 = 0$ or $\lambda \neq \mu$ then the operator A possesses second eigen vector in the span of vectors $\mathbf{e}$ and $\mathbf{f}$ (see the case $k = 2$ above).

Hence by multiplying all the vectors $\mathbf{f}_i$ by $\frac{1}{a_1}$ we come to the equations

$$\begin{cases} A\mathbf{e} = \lambda\mathbf{e} \\ A\mathbf{f}_1 = \lambda\mathbf{f}_1 + \mathbf{e} \\ A\mathbf{f}_2 = \lambda\mathbf{f}_2 + \mathbf{f}_1 + a_2\mathbf{e} \\ A\mathbf{f}_3 = \lambda\mathbf{f}_3 + \mathbf{f}_2 + a_3\mathbf{e} \\ A\mathbf{f}_4 = \lambda\mathbf{f}_4 + \mathbf{f}_3 + a_4\mathbf{e} \\ \ldots\ldots\ldots\ldots \\ A\mathbf{f}_n = \lambda\mathbf{f}_n + \mathbf{f}_{n-1} + a_n\mathbf{e} \end{cases}$$

Now one can see (again) by induction that that we have the flag of spaces $\{V_k\}$ such of these spaces is $J^{\lambda,k}$.

Indeed consider spaces V_k which are spans of vectors $\mathbf{e}, \mathbf{f}_1, \ldots, \mathbf{f}_{k-1}$. All these spaces are Jordanian cells with same eigenvalue λ. By induction hypothesis there exists a basis $\mathbf{e}, \mathbf{f}'_1, \mathbf{f}'_2, \ldots, \mathbf{f}'_{n-1}$ in V_{n-1} such that the

$$\begin{cases} A\mathbf{e} = \lambda\mathbf{e} \\ A\mathbf{f}'_1 = \lambda\mathbf{f}'_1 + \mathbf{e} \\ A\mathbf{f}'_2 = \lambda\mathbf{f}'_2 + \mathbf{f}'_1 \\ A\mathbf{f}'_3 = \lambda\mathbf{f}'_3 + \mathbf{f}'_2 \\ A\mathbf{f}'_4 = \lambda\mathbf{f}'_4 + \mathbf{f}'_3 \\ \ldots\ldots\ldots\ldots \\ A\mathbf{f}'_{n-1} = \lambda\mathbf{f}'_{n-1} + \mathbf{f}_{n-2} \end{cases} .$$

Then for vector $\mathbf{f}_n$

$$A\mathbf{f}_n = \lambda\mathbf{f}_n + b_0\mathbf{e} + b_1\mathbf{f}'_1 + b_2\mathbf{f}'_2 + b_3\mathbf{f}'_3 + \ldots + b_{n-1}\mathbf{f}'_{n-1}$$

If we choose

$$\mathbf{f}'_n = \mathbf{f}_n - b_0\mathbf{f}_1 + -b_1\mathbf{f}_2 - \ldots$$

we come to the answer ■

Jordanclells in Abelian group classification

Consider the abelian finite group

$$C_k = \mathbf{Z} \backslash k\mathbf{Z}$$

Then if $k = p^2$ (p is prime number) then C_k is Jordanian cell. The group C_{p^n} possesses subgroup C_p and the sequence

$$0 \to C_p \to C_p^2 \to C_p \to 0$$

is not splitted.

Standard classfication theorem says that C_N is direct sum of the groups $C_{p_k^{n_k}}$ where $N = \prod_i p_i^{n_i}$ is expansion over primes This is expansion on Jordanian cells.

C_{p^k} is "analog" of k-dimensional Jordanian cell.