Differential Galois theory and non-regular linear differential equations

Our goals are

• to provide a quite explicit description (in terms of a proper quiver with relations) of polynomial differential equations in one variable,

• to provide some basics on differential Galois theory.

Then we will show how the second one applies to the first one. The description of such equations would be given in three steps:

(1) a description of linear differential equations in Laurent power series (requires some algebra - but not much of it),

(2) a description of linear differential equations on a punctured disc $D^{\times}$ (requires some sheaf background in complex analysis to understand all proofs),

(3) a description of the most general case (the result would be explained in a closed, and [hopefully] understandable form; a [comprehensible] reference is supposed to be provided on the proofs).

On all steps (1), (2), (3) one can apply the Differential Galois Theory to have some insight into the picture.

To proceed we need to define the objects with which we wish to work with. Our main field would be $\mathbb{C}$, even if in some cases it is easy to see how to replace it by a more advanced field. We denote

(1) by $\mathbb{C}((x))$ the Laurent power series of x, i.e. the sums of the form $\sum\limits_{n\geq n_0\in\mathbb{Z}} a_n x^n$,

(2) by $\mathrm{Hol}_0(D^{\times})$ the holomorphic functions on $D^{\times} := \{x \in \mathbb{C} \mid 0 < |x| < 1\}$ which are meromorphic at 0,

(3) by $\mathbb{C}[x]$ the polynomials of x, by $\mathbb{C}(x)$ the rational functions of x.

On all these algebras/fields we define (complex) derivation ∂ such that $\partial(x) = 1$. We also consider the corresponding rings of differential operators

$$\mathbb{C}((x))[\partial], \quad \mathrm{Hol}_0(D^{\times})[\partial], \quad \mathbb{C}(x)[\partial]. \tag{1}$$

One can think about these algebras as of (skew-)polynomials in ∂ with an appropriate field of coefficients.

We wish to offer the following theorems as the starters to our course.

Theorem 1 (An analogue of the fundamental theorem of algebra)**.** *For any*

$$a_{n-1}, ..., a_0 \in \mathbb{C}((x))$$

there exists $d \in \mathbb{Z}_{>0}$ *and* $\lambda_1, ..., \lambda_n \in \mathbb{C}((x^{\frac{1}{d}}))$ *such that*

$$\partial^n + a_{n-1}\partial^{n-1} + ... + a_0 = (\partial - \lambda_1)(\partial - \lambda_2)...(\partial - \lambda_n).$$

Theorem 2. *For any "reasonable" differential equation over* $\mathbb{C}((x))$ *there exists* $d \in \mathbb{Z}_{>0}$ *such that this equation over* $\mathbb{C}((x^{\frac{1}{d}}))$ *is equivalent to the system of completely independent equations of the form*

$$(x\partial - q(x))^m f = 0, \quad q(x) \in \mathbb{C}((x^{\frac{1}{d}})), m \in \mathbb{Z}_{\geq 1}.$$

Proof. No proof for now, but is similar to [M, p. 25, Proposition 1.5]. □

Together with the above results, we will also review the classical Picard-Vessiot theory for (homogeneous) linear differential equations over a differential field (k, ∂) of characteristic zero (following [vdPS]). In particular, we will present the basic properties of Picard-Vessiot (PV) extensions (the analogue of the splitting field), the Galois group associated to the linear differential equation, and the Galois correspondence between the algebraic subgroups of the Galois group and differential fields between k and the PV extension.

Given a linear differential equation in vector form

$$(\dagger) \qquad \partial Y = AY$$

over the differential field k. One can naturally associate to (†) a $k[\partial]$-module M (analogous to how one associates a $k[x]$-module to a linear transformation of a k-vector space). Suppose $\partial Y = BY$ is another linear differential equation over k and N is the associated $k[\partial]$-module. We will see that M and N are isomorphic (as $k[\partial]$-modules) iff there is an invertible matrix Z over k such that

$$B = Z^{-1}AZ - Z^{-1}\partial(Z)$$

In this case we say that the two equations are equivalent over k (or that the second equation is obtained from the first by a gauge transformation via Z). Time permitting we will further look at properties of $k[\partial]$-modules and their relation to linear differential equations.

Going back to the case when $k = \mathbb{C}((x))$, the differential field of Laurent series, we will see that the differential equation (†) can be put in a normal form over an algebraic extension of k. From this we will see (the classical fact) that in this case the solution of the equation is of the form Hx^Le^Q. We will show that in this case the Galois group of the equation is the smallest linear algebraic group containing a certain commutative group of diagonalizable matrices (the exponential torus) and one more element (the formal monodromy), and both of these can be computed from the normal form.

References

[M] B. Malgrange, Équations différentielles à coefficients polynomiaux. (French) [Differential equations with polynomial coefficients] Progress in Mathematics, 96. Birkhäuser Boston, Inc., Boston, MA, 1991.

[vdPS] M. van der Put and M. Singer. Galois theory of linear differential equations.