L_∞ algeborids and L_∞ morphisms

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Contents

[1]The content of these notes is based on my discussions with Ted Voronov about algebroids mainly under the influence of his notes and remarks.

Let (M, P) be a Poisson manifold. It defines Lie algebroid $T^*M \to M$ of 1-forms. It will be our most inspiring example. This is fundamental object. During these lectures we often return to it,

1 Lie algebroids. Definition of all its manifestations.

1.1 Lie algebroid

We give a definition of Lie algebroid. As usual in these lectures we will consider four different manifestations of Lie algebroid. m

I-st manifestation.

Let $E \to M$ be a vector bundle, $[[\,,\,]]$ commutator of sections and $\mathbf{a}$– anchor map: linear map of $E \to M$ to tangent bundle TM:

$$[[\mathbf{s}_1, \mathbf{s}_2]] = -[[\mathbf{s}_2, \mathbf{s}_1]]\,, [[\mathbf{s}_1, f\mathbf{s}_2]] = (-1)^{\cdots} f[[\mathbf{s}_1, f\mathbf{s}_2]] + \mathbf{a}(\mathbf{s}_1)f\mathbf{s}_2 \,\text{(Leibnitz rule)}$$

and Jacobi identity is obeyed

$$[[\,[[\mathbf{s}_1, \mathbf{s}_2]], \mathbf{s}_3]] + \text{cyclic permutation} = 0$$

It is useful to write local formulae for commutator and anchor.

Let x^μ be local coordinates on the base M. Then set of linearly independent local sections $\{\mathbf{e}_a(x)\}$ ($a = 1, \ldots, n$, where n is a dimension of fibre) define local coordinates $(x^\mu . s^a)$ on $\mathbf{E}$: $(x^\mu, y^a) \mapsto y^a \mathbf{e}_a(x)$. We denote by

$$c^a_{bc}\colon \quad c^a_{bc}(x)\mathbf{e}_a(x) = [[\mathbf{e}_b(x), \mathbf{e}_c(x) \text{ and } \alpha^\mu_a\colon \quad \alpha^\mu_a \partial_\mu = \mathbf{a}(\mathbf{e}_a) \tag{1.1}$$

Then we have that for two arbitrary sections $\mathbf{s}_1(x) = y^a_1(x)\mathbf{e}_a(x)$,$\mathbf{s}_2(x) = y^a_2(x)\mathbf{e}_a(x)$

$$[[\mathbf{s}_1, \mathbf{s}_2]] = [[y^a_1(x)\mathbf{e}_a(x), y^b_2(x)\mathbf{e}_b(x)]] =$$
$$y^a_1(x)y^b_2(x)c^d_{ba}(x) + y^a_1(x)\alpha^\mu_a(x)\partial_\mu y^b_2(x) - y^a_2(x)\alpha^\mu_a(x)\partial_\mu y^b_1(x)$$

Exercise Show that

$$\mathbf{a}\left([[\mathbf{s}_1, \mathbf{s}_2]]\right) = [\mathbf{a}(\mathbf{s}_1), \mathbf{a}(\mathbf{s}_2)]$$

This is the condition of morphisms of algebroid to the tangent bundle algeborid (see alter). Later we also will give a conceptual proof of this statement.

Let $E \to M$ be an algebroid with commutator $[[\,,\,]]$ and anchor map a.

Now we consider other three manifestations of an alfgebroid.

II-nd manifestation of algebroid For an algebroid $E \to M$ consider fibbre bundle $\Pi E \to M$ with opposite parities of fibres (Π parity reversing functor). One can define an homological vector field Q on ΠE of weight $\sigma = 1$ defined by commutator relations and anchor in the following way: In local coordinates (x^μ, y^a) (see (1.1))

$$Q = \xi^a \xi^b c^d_{ba}(x) \frac{\partial}{\partial \xi^d} + \xi^a \alpha^\mu_a(x) \frac{\partial}{\partial x^\mu} \,. \tag{1.2}$$

Here (x^μ, ξ^a) are local coordinates on ΠE corresponding to local coordinates (x^μ, y^a) on E.

If $\mathbf{s}_1(x)$, $\mathbf{s}_2(x)$ are two sections, then

$$[[\mathbf{s}_1, \mathbf{s}_2]] = \Pi\left([[Q, \Pi\mathbf{s}_1], \Pi\mathbf{s}_2]\right) \,.$$

The condition that Q defines the algebroid is equivalent to the condition that $Q^2 = 0$.

1.2 Examples of algebroid

1.2.1 Tangent bundle algebroid in all its manifestations

Let M be a manifold. Consider tangent bundle algebroid $TM \to M$ with $[[\ ,\ ]]$ be equal to usual commutator $[\ ,\ ]$ of vector and anchore–identity map.

Consider all other manifestations of this algebroid.

II-nd manifestation: tangent bundle $\Pi TM \to M$ with homological vector field, de Rham differential

$$Q = dx^m \frac{\partial}{x^m}$$

(x^m-local coordinates on M.) If $\mathbf{s}_1, \mathbf{s}_2$ two sections of tangent bundle $TM \to M$

$$[[]]$$

III-rd manifestation: contangent bundle $T*M$ with canonical symplectic structure. IY-th manifestation: contangent bundle $\Pi T*M$ with canonical odd symplectic structure, Schouten commutator.

1.2.2 Algebroid $T^*M \to M$ in all its manifestations for Poisson manifold (M, P)

We define commutator and anchor by relations:

$$[[df, dg]] = d\{f, g\}\,, \mathbf{a}(df) = D_f$$

II-nd manifestation: Fibre bundle ΠT^*M with homological vector field

$$Q = \theta_i\theta_j\partial_r P^{ij}\frac{\partial}{\partial\theta_{\mathbf{r}}} \pm \theta_i P^{ij}\frac{\partial}{\partial x^j}$$

III-nd manifestation: Linear (in fibres) even Poisson bracket on TM, i.e. Poisson bracket defined by relations

$$\{v^i, v^j\}_0 = v^r\partial_r P^{ij}\,,\ \{v^i, x^j\} = P^{ij}$$

II-nd manifestation: Linear (in fibres) odd Poisson bracket on ΠTM, Koszul bracket, odd Poisson bracket defined by relations

$$\{\theta^i, \theta^j\}_0 = \theta^r\partial_r P^{ij}\,,\ \{\theta^i, x^j\} = P^{ij}$$

1.3 Morphisms of algebroids

Let $E_1 \to M$, $E_2 \to M$ be two algebroids on the same base. We define morphisms of these algebroids in all manifestations.

Note that it is very inmprotant the special case: the morphism of aarbitrary algebroid $E \to M$ on the tangent algebroid $TM \to M$. I-st manifestation

2 L_∞ algeborids. Definition in all its manifestations.

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