Uniformisation of bounded domain

This etude contains three topics:

In the first topic we will discuss the following formula: If M is an hypersurface surface in $\mathbf{E}^n$, then the function

$$F_M(\mathbf{r}) = \text{the angle that a surface } M \text{ is seen from the point } \mathbf{r}$$

obeys the equation

$$\Delta F_M = 0\,, F_M\big|_M = \pi????????$$

(in the case if M is regular surface*)

In the second topic we will discuss There are two well know constructions: Dirichle integral, which produces the harmonic function $h\colon U \to \mathbf{R}$ function on the domain U of $\mathbf{E}^2$ obeying boundary conditions

$$h\big|_{\partial U} = u\,.$$

In the third topic we will consider holomorphic map $F\colon U \to D$ which maps domain U to the disc $|w| < 1$ in $\mathbf{C}$.

There is a deep relation between these two constructions. and the conformlal maps. This we will study in the third topic.

§1

Let $C\colon \mathbf{r} = \mathbf{r}(t)$ be a curve in $\mathbf{E}^2$. Consider the function

$$W(\mathbf{R}) = \text{the angle at which one sees the curve } C \text{ from the point } \mathbf{R} = \int_C F^* \omega_{\text{angle}} \mathbf{R}\,. \tag{1.1}$$

One can see that the function $W(\mathbf{R})$ tends to zero at infinity

it obeys the boundary condition????

$$W\big|_C = \pi\,. \tag{1.2}$$

no and it is harmonic function:

$$\Delta W = \frac{\partial^2 W(x,y)}{\partial x^2} + \frac{\partial^2 W(x,y)}{\partial y^2} = 0\,. \tag{1.3}$$

Condition (1.2) is almost evident AND WRONG!, We will prove (1.3) later.

To deal with function (1.1) we will express it as an intergral.

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* some technical conditions: Lyapunov surfaces????

Consider 1-form $\omega = \omega_{\text{angle}}$

$$\omega = \omega_{\text{angle}} = \frac{xdy - ydx}{x^2 + y^2} = d\varphi \text{ in polar coordinates}\,, \tag{1.4}$$

then one can see that

$$W(\mathbf{R}) = \int_{F_*(\mathbf{R})C} \omega = \int_C F^*(R)\omega\,, \tag{1.5}$$

where F: is a map

$$F\colon \quad F(\mathbf{r}) = \mathbf{r} - \mathbf{R}$$

Examples

1. Let AB be a segment of straight line between points $A = (a, b)$ and $B = (1, 1)$ on the axis $Ox = X$, $A = (a, 0)$, $B = (b, 0)$.

Consider an arbitrary point $P = (X, Y)$ on $\mathbf{E}^2$. Let $N = (0, X)$ be a projection of the point P on the axis OX. Then it is obvious that

$$W(P) = W(X, Y) = \angle PAN - \angle PBN = \arctan\frac{X - a}{Y} - \arctan\frac{X - b}{Y}\,. \tag{1.Ex1.1}$$

We come to the same formula using (1.4):

$$F^*(P)\omega = F^*(X, Y)\left(\frac{xdy - ydx}{x^2 + y^2}\right) = \frac{(x - X)dy - (y - Y)dx}{(x - X)^2 + (y - Y)^2}\,,$$

a curve C: $\begin{cases} x = t \\ y = 0 \end{cases}$, $a \le t \le b$, and

$$W(P) = W(X, Y) = \int_C F^*(P)\omega = \int_{\begin{cases} x = t \\ y = 0 \end{cases} a \le t \le b} \frac{(x - X)dy - (y - Y)dx}{(x - X)^2 + (y - Y)^2} = \int_a^b \frac{Y dt}{(X - t) + Y^2} \tag{1, Ex.1.2}$$

Integrating we will come to formula (1, Ex 1.1).

Example 2

Let C be an arc of unit circle $x^2 + y^2 = 1$:

$$C\colon \begin{cases} x = \cos t \\ y = \sin t \end{cases} \alpha \le t \le \beta$$

then equation (1.5) implies that

$W_C(P) =$: angle such that one can see the arc C from the point P $=$

$$\int_{\begin{cases} x=\cos t \\ y=\sin t\end{cases}, \alpha\le t\le\beta} \frac{(x-X)dy-(y-Y)dx}{(x-X)^2+(y-Y)^2} = \int_\alpha^\beta \frac{(x(t)-X)dy(t)-(y(t)-Y)dx(t)}{(x(t)-X)^2+(y(t)-Y)^2} =$$

$$\int_\alpha^\beta \frac{(\cos t - r\sin\theta)\cos t-(\sin t - r\sin\theta)(-\cos t)}{(\cos t - r\cos\theta)^2+(\sin t-\sin\theta)^2}dt = \int_\alpha^\beta \frac{1-r\cos(\theta-\varphi)}{1+r^2-2r\cos(\theta-\varphi)}dt\,.$$

Here we denote $X=r\cos\theta, Y=r\sin\theta$ coordinates of point P.

Integrating we come to

$$W_{S^1}(P) = \int_\alpha^\beta \frac{1-r\cos(\theta-\varphi)}{1+r^2-2r\cos(\theta-\varphi)}dt = \arctan\left(\frac{\sin(t-\theta)}{\cos(t-\theta)-r}\right)\Big|_\alpha^\beta\,.$$

(Sure one can guess antiderivative knowing that this is the angle (1.1).)

The expressions calculated above are potentials of the double layer. We will use them later.

Consider the kernel $A_{S^1}(\mathbf{r},\mathbf{R})$ ($\mathbf{r}=(\cos\varphi,\sin\varphi)$) is the point on the circle, $\mathbf{R}=(R\cos\theta, R\sin\theta)$ is the observation point, and $A_{S^1}(\mathbf{r},\mathbf{R})d\varphi$ is equal to the differential of the angle at which one sees the arc $d\varphi$ from the point $\mathbf{R}$. We have

$$A_{S^1}(\mathbf{r},\mathbf{R})d\varphi = \frac{\cos\angle(\mathbf{n},\mathbf{R})d\varphi}{|\mathbf{r}-\mathbf{R}|} = \frac{1-R\cos(\theta-\varphi)}{1+R^2-2R\cos(\theta-\varphi)}d\varphi\,.$$

Compare with expressions above.

Useful exercies Show straightforwardly that the integral over circle is equal to 2π or π or 0 depending on the position of the point $\mathbf{R}$.

Write down also the angle function for unit sphere:

$$A_{S^2} = (\mathbf{R},\mathbf{r}) = \frac{\cos\angle(\mathbf{n},\mathbf{R})}{|\mathbf{r}-\mathbf{R}|} = \frac{(\mathbf{r}-\mathbf{R},\mathbf{R})}{|\mathbf{r}-\mathbf{R}|^2}$$

here $\mathbf{r}=(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta)$ is a point on the unit sphere, $\mathbf{R}=(R\Theta\cos\Phi, Rsin\Theta\sin\Phi, R\cos\Theta$ is the observation point We come to

$$A_{S^2} = (\mathbf{R},\mathbf{r}) = \frac{(\mathbf{r}-\mathbf{R},\mathbf{R})}{|\mathbf{r}-\mathbf{R}|^2}\frac{1-R\cos(\Theta-\theta)-R\sin\Theta\sin\theta\,(\cos(\Phi-\varphi)-1)}{1+^2-2R\cos\Theta-\theta)-2R\sin\Theta\sin\theta\,(\cos(\Phi-\varphi)-1)}$$

Remark Sure one can write the formula for 1-form like in Example 2, this is the form

$$\frac{dx\wedge dy\wedge dz}{r^2dr}\frac{xdy\wedge dz+ydz\wedge dx+zdx\wedge dy}{r^3} = \sin\theta d\theta d\varphi$$

(see Appendix)

Dirichle problem

Recall Dirichle problem Let U be ball-like domain, ρ a functionon U and μ a function on the boundary. We have to find a function F such that

$$\begin{cases} \Delta F = \rho \\ F\big|_{\partial U} = \mu \end{cases} \tag{2.1}$$

To solve this problem we have to find Green function $G_U(\mathbf{r}, \mathbf{R})$, the function such that

$$\begin{cases} \Delta_{(\mathbf{r})} G(\mathbf{r}, \mathbf{R}) = \delta(\mathbf{r} - \mathbf{R}) \\ G(\mathbf{r}, \mathbf{R})\big|_{\mathbf{R} \in \partial U} = 0 \end{cases} \tag{2.2}$$

If we find Green function then one can see that

$$f(R) = \int f(\mathbf{r}) G(\mathbf{r}, \mathbf{R}) + c \oint \mu(\mathbf{r}) \mathrm{grad}_{(\mathbf{r})} G$$

is a solution of (2.1)

We can go another scenario.

We already know the fundamental solutions of equation $\Delta\Phi_n = \delta(\mathbf{r})$ (see Appendix), and we know functions potentials of double layer...

We may try to express solutions and Green function in this way.

W

Study relation between these approaches Let G_∞ be fundamental solution of Laplacian in $\mathbf{E}^n$

$$G_\infty^{(2)} = 2\pi \log r, G_\infty^{(3)} = -4\pi \log r, \ldots, G_\infty^{(n)} = \frac{\sigma_{n-1}}{n-2} \log r, \ldots,$$

(see Appendix)

Later we will show how to deduce G knowing G_∞ and using potential of double layer. Now we just will see the relation between these functions. We consider fundamental Green second identity: for arbitrary (generalised) functions B, C

$$\int_U (U\Delta V - V\Delta U)\Omega = \int_{\partial U} \Omega \rfloor \left(B \mathrm{grad}\, C - C \mathrm{grad}\, B\right)$$

where Ω is volume form and

$$\Delta F \Omega = \mathcal{L}_{\mathrm{grad}\, F} \Omega = d\left(\Omega \rfloor \mathrm{grad}\, F\right)$$

in a more convenitional notations

$$\int_U (U\Delta V - V\Delta U) d^n x = \oint_{\partial U} \left(B \mathrm{grad}\, C - C \mathrm{grad}\, B\right) d\mathbf{S}$$

Let G be a Green function of Dirichle

We perform considerations for $n = 2$ and $n = 3$.

n=2

Conisder unit circle $U\colon x^2 + y^2 < 1$. Find function harmonic in U such that its value on the boundary is the function $F(\varphi)$

$$\Delta u = 0\,,\, u\big|$$

Later we will discuss in detail what happens at boundary.

Instead using Green function method we will use functions considered in the first paragraph.

Table of functions:

Double layer Potential of unit circle:

$$w(r,\theta = \int \frac{1 - r\cos(\theta - \varphi)}{1 + r^2 - 2r\cos(\theta - \varphi)} \nu(\varphi) d\varphi$$

1. If $\nu(\varphi) = 1$ then w is angle (see the first paragraph).
2. For arbitrary (continuous) $\nu(\varphi)$

$$w(r,\varphi)\big|_{r\to 1_-} = \pi\nu(\varphi) + w(r,\varphi)\big|_{r=1}\,, \quad w(r,\varphi)\big|_{r\to 1_+} = -\pi\nu(\varphi) + w(r,\varphi)\big|_{r=1}\,.$$

If w is harmonic function

In the previous paragraph we described the solution of Dirichle problem. It seemed to be not so difficult. Sure we were save because we knew that this problem has solution and it is unique. In fact the Dirichle problem for domain U in some sence is equivalent to biholomorphme map

Appendix. Green function

We consider n-dimensional Euclidean space $\mathbf{E}^n$.

Let $\Omega = dx^1 \wedge \ldots \wedge dx^n$ be a volume form on $\mathbf{E}^n$.

Note that in 'polar' coordinates

$$\Omega = \sigma_n r^{n-1} dr \sigma_{\mathrm{sph}}\,,$$

where $r = \sqrt{(x^1)^2 + \ldots + (x^n)^2}$ and

$$\sigma_{\mathrm{sph}} = (-1)^k x^k \frac{\Omega}{r^n dx^k} = \sum_{k=1}^{n} (-1)^k \frac{x^k dx^1 \ldots dx^{k-1} \wedge dx^{k+1} \wedge \ldots \wedge dx^n}{r^n}\,.$$

Namely:

$$r^{n-1}dr\sigma_{\rm sph}=r^{n-2}\sum_{i=1}^n\left(x^idx^i\right)\sum_{j=1}^n\left((-1)^k\frac{x^kdx^1\dots dx^{k-1}\wedge dx^{k+1}\wedge\dots\wedge dx^n}{r^n}\right)=\Omega\,. \qquad (App1.1)$$

(Note that area σ_n of n-dimensional unit sphere S^n is equal to

$$\sigma_n=\int_{S^n\subset\mathbf{E}^{n+1}}\sigma_{\rm sph}=\frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} \qquad (App1.1b)$$

*)

Consider on $\mathbf{E}^n\backslash 0$ spherically invariant function $\Phi=\Phi(r)$ such that $n-1$-form

$$\omega_\Phi=\Omega\rfloor\operatorname{grad}\Phi-\mathcal{L}_{\operatorname{grad}\Phi}$$

is closed form in $\mathbf{E}^n/0$.

Thus we come to the statement

We see that

$$\omega_\Phi=r^{n-1}dr\rfloor\left(\frac{d\Phi(r)}{dr}\frac{\partial}{\partial r}\right)=r^{n-1}\frac{d\Phi(r)}{dr}\sigma_{\rm sph}$$

Hence we have that for $\Phi=r^{2-n}$ (and $\Phi=\log r$ for $\mathbf{E}^2$) the form ω_Φ is closed non-exact form.

The function

$$\Phi_n=\begin{cases}r^{2-n}\text{if } n\neq 2\\ \log|r|\text{if } n=2\end{cases}$$

defines closed non exact form, non-vanishing charge (cohomology): if C is a sphere such that origin belong to it, then

$$\int_C\Omega\rfloor\operatorname{grad}\Phi_n=\oint_C\operatorname{grad}\Phi=\begin{cases}\frac{\sigma_{n-1}}{2-n}\text{if } n\neq 2\\ 2\pi\text{ if } n=2\end{cases} \qquad (1.2a)$$

Notice that condition that form $\Omega\rfloor\operatorname{grad}\Phi_n$ is closed, means that

$$\text{for } \mathbf{r}\neq 0, \Delta\Phi=0\,.$$

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* Indeed in

$$\int_{\mathbf{E}^{n+1}}e^{-r^2}\Omega=\left(\int e^{-x^2}dx\right)^{n+1}=\pi^{\frac{n+1}{2}}\,,$$

and on the other hand in 'polar' coordinates

$$\int_{\mathbf{E}^{n+1}}e^{-r^2}\Omega=\int_{\mathbf{E}^{n+1}}e^{-r^2}r^ndr\sigma_{\rm sph}=\int_{r=1}\sigma_{\rm sph}\cdot\int_{\mathbf{E}^{n+1}}e^{-r^2}r^{n-1}dr\sigma_{\rm sph}=\sigma_n\Gamma\left(\frac{n+1}{2}\right)$$

This implies (App1.1b).

Indeed by definition

$$\Delta\Phi_n = \mathrm{div}\,\mathrm{grad}\,\Phi_n = \frac{\mathcal{L}_{\mathrm{grad}}\,\Phi_n}{\Omega} = \frac{d\,(\Omega\rfloor \mathrm{grad}\,\Phi_n)}{\Omega} = 0\,.$$

We come to **Proposition** Functions Φ_n are harmonic functions out of the origin:

Theorem Laplacian of function Φ_n, considered in the sence of generalised functions is eproportional to δ-function:

$$\Delta\Phi_n = \frac{\sigma_{n-1}}{2-n}\delta(\mathbf{r})\,, \quad (2\pi\delta(r)\,, \quad \text{in the case if } n=2 \qquad (App1.3)$$

One can say that closedness of the form $\Omega\rfloor\mathrm{grad}\,\Phi_n$ means that function Φ_n is harmonic function, and non-exactness of this form means that Φ_n is proportional to δ- function**

We will try to make the proof of this Theorem 'almost' proper.

Let $\rho(\mathbf{r})$ (density function) vanishes for big R. We calculate derivatives in the sense of generalised functions

$$(\Delta\Phi_n, \rho) = (\Phi_n, \Delta\rho)\,.$$

Calculate the right hand side explicitly:

$$(\Phi_n, \Delta\rho) = \int_{\mathbf{E}^n} \Phi_n(\mathbf{r})\Delta\rho(\mathbf{r})d^n x = \lim_{\varepsilon\to 0+} \int_{r>\varepsilon} \Phi_n(\mathbf{r})\Delta\rho(\mathbf{r})d^n x = \lim_{\varepsilon\to 0_+} I_\varepsilon$$

We have

$$I_\varepsilon = \int_{r>\varepsilon} \Phi_n(\mathbf{r})\Delta\rho(\mathbf{r})d^n x = \int_{r<\varepsilon<R} \left(\Phi_n(\mathbf{r})\Delta\rho(\mathbf{r}) - \rho(\mathbf{r})\Delta\Phi_n(\mathbf{r})\right) d^n x\,.$$

since $\Delta\Phi_n(r) \equiv 0$ for $\mathbf{r} \neq 0$ then

$$I_\varepsilon = \int_{r<\varepsilon<R} \left(\Phi_n(\mathbf{r})\mathrm{div}\,\mathrm{grad}\,\rho(\mathbf{r}) - \rho(\mathbf{r})\mathrm{div}\,\mathrm{grad}\,\Phi_n(\mathbf{r})\right) d^n x =$$

$$\int_{r<\varepsilon<R} \left(\Phi_n(\mathbf{r})d\left(\Omega\rfloor\mathrm{grad}\,\rho(\mathbf{r})\right) - \rho(\mathbf{r})d\left(\Omega\rfloor\mathrm{grad}\,\Phi_n(\mathbf{r})\right)\right)\,.$$

Now using Stokes Theorem we rewrite this integral as integral over the boundary of the integration domain of $n-1$-form:

$$I_\varepsilon = \int_{\partial[r<\varepsilon<R]=[r=R]-[r=\varepsilon]} \left(\Phi_n(\mathbf{r})\left(\Omega\rfloor\mathrm{grad}\,\rho(\mathbf{r})\right) - \rho(\mathbf{r})\left(\Omega\rfloor\mathrm{grad}\,\Phi_n(\mathbf{r})\right)\right)\,. \qquad (App1.4a)$$

This integral can be in other notations:

$$I_\varepsilon = \oint_{\partial[r<\varepsilon<R]=[r=R]-[r=\varepsilon]} \left(\Phi_n(\mathbf{r})\mathrm{grad}\,\rho(\mathbf{r}) - \rho(\mathbf{r})\mathrm{grad}\,\Phi_n(\mathbf{r})\right) d\mathbf{S}\,. \qquad (App1.4b)$$

** Cohomology— generalsed functions

Now using the fact that function ρ vanishes at big R we rewrite (1.4a) as

$$I_\varepsilon = -\left[\int_{r=\varepsilon} \left(\Phi_n(\mathbf{r})\,(\Omega\rfloor \operatorname{grad}\rho(\mathbf{r})) - \rho(\mathbf{r})\,(\Omega\rfloor \operatorname{grad}\Phi_n(\mathbf{r}))\right)\right] .$$

Look carefully on two integrals in this last expression. The first integral

$$-\int_r \left(\Phi_n(\mathbf{r})\,(\Omega\rfloor \operatorname{grad}\rho(\mathbf{r}))\right)$$

tends to zero if $\varepsilon \to 0$ since are of the sphere of radius r is equal to $\sigma_{n-1}\varepsilon^{n-1}$.

For the second integral using the cohomology formula (1.2a) we come to

$$\lim_{\varepsilon\to 0_+} I_\varepsilon = -\left[-\int_{r=\varepsilon} \rho(\mathbf{r})\,(\Omega\rfloor \operatorname{grad}\Phi_n(\mathbf{r}))\right] = \rho(0)\int_r \Omega\rfloor \operatorname{grad}\Phi_n = \begin{cases} \rho(0)\frac{\sigma_{n-1}}{2-n} \text{ if } n \neq 2 \\ 2\pi\rho(0) \text{ if } n = 2 \end{cases}$$

Thus we calculated $(\Delta\Phi_n, \rho)$ and Theorem is proved.

This Theorem may be used to construct Green functions of Δ on $\mathbf{E}^n$. (see details in the second paragraph) Consider the function

$$G_n(\mathbf{r}, \mathbf{R}) = \begin{cases} \frac{2-n}{\sigma_{n-1}}\Phi_n(\mathbf{r}-\mathbf{R}) \text{ if } n \neq 2 \\ \frac{1}{2\pi}\log(\mathbf{r}-\mathbf{R}) \text{ if } n = 2 \end{cases}$$

Due to Theorem for every ρ with compact support the function

$$U(\mathbf{R}) = \int \rho(\mathbf{r})G(\mathbf{r}, \mathbf{R})d^x$$