Nijenhuis

Nijenhuis bracket of differential forms valued vector fields

Let $A(x)$ be a linear operator on tangent vectors: $A(x)$: $T_xM \to T_x(M)$. Then one can define $[A,A]$ which is linear operator from $T_xM \wedge T_xM \to T_xM$. This is a special case of Nijenhuis bracket. We first do it in straightforward way, then comne to this formula using general formalism.

Let $A(x)$ be an operator-valued function on manifold M. Consider the following function on vector fields:

$$\mathbf{X},\mathbf{Y} \mapsto \mathcal{N}(\mathbf{X},\mathbf{Y}) = [L(\mathbf{X}),\, L(\mathbf{Y})] - L\left([\mathbf{X},\, L(\mathbf{Y})]\right) - L\left([L(\mathbf{X}),\, \mathbf{Y}]\right) + L\left(L\left([\mathbf{X},\, \mathbf{Y}]\right)\right)\,.$$

where $[\ ,\ ]$ is commutator of vector fields: $[\mathbf{A},\mathbf{B}] = [\mathbf{A},\mathbf{B}]^i\partial_i = (A^r\partial_r B^i - B^r\partial_r A^i)\partial_i$.

$\mathcal{N}(\mathbf{X},\mathbf{Y})$ is vector field on M. One can see which that:

$$\mathcal{N}(\mathbf{X},\mathbf{Y}) = -\mathcal{N}(\mathbf{Y},\mathbf{X})$$

Fact $\mathcal{N}(\mathbf{X},\mathbf{Y})$ is not only linear over vector fields, it is linear over algebra of functions on M: in particular for arbitrary function f

$$\mathcal{N}(f\mathbf{X},\mathbf{Y}) = f\mathcal{N}(\mathbf{X},\mathbf{Y})\,,$$

(this implies linearity over functions),

This statement means that at every point x_0, $\mathcal{N}$ is linear function on vectors $\mathbf{X},\mathbf{Y}$ tangent to M at this point.

Show it, Note that $[f\mathbf{X},Y] = f[\mathbf{X},\mathbf{Y}] - (\mathbf{Y}f)\mathbf{X}$. Hence

$$\mathcal{N}(f\mathbf{X},\mathbf{Y}) = [L(f\mathbf{X}),\, L(\mathbf{Y})] - L\left([f\mathbf{X},\, L(\mathbf{Y})]\right) - L\left([L(f\mathbf{X}),\, \mathbf{Y}]\right) + L\left(L\left([f\mathbf{X},\, \mathbf{Y}]\right)\right) =$$

$$f\,[L(\mathbf{X}),\, L(\mathbf{Y})] - (L(\mathbf{Y})f)\, L(\mathbf{X}) - fL\left([\mathbf{X},\, L(\mathbf{Y})]\right) + (L(\mathbf{Y})f)\, L(\mathbf{X}) +$$

$$-fL\left([L(\mathbf{X}),\mathbf{Y}]\right) + (\mathbf{Y}f)\, L\left(L\left(\mathbf{X}\right)\right) + fL\left(L\left([\mathbf{X},\, \mathbf{Y}]\right)\right) - (\mathbf{Y}f)\, L\left(L\left(\mathbf{X}\right)\right) = f\mathcal{N}(\mathbf{X},\mathbf{Y})\,.$$

In components

$$\mathcal{N}(\mathbf{X},\mathbf{Y}) = N^i_{kp}X^kY^p,$$

where

$$N^m_{kp}\partial_m = \mathcal{N}(\partial_k,\partial_p) = [L(\partial_k),\, L(\partial_p)] - L\left([\partial_k,\, L(\partial_p)] - [\partial_p,\, L(\partial_k)]\right) + L\left(L\left([\partial_k,\, \partial_p]\right)\right) =$$

$$L^i_k\partial_i L^m_p - L^i_p\partial_i L^m_k - L^m_r\left(\partial_k L^r_p - \partial_p L^r_k\right)$$

Theorem (Neveinhuisen) Operator valued function $L(x)$ = vector valued differential 1-form defines vector valued differential 2-form:

$$L\colon L = dx^k L^i_k \partial_i \to [L, L] = dx^p \wedge dx^k \left(L^i_k \partial_i L^m_p - L^m_r \partial_k L^r_p \right) .$$

This is bracket of L with itself. In fact Nijenhuis defines bracket for vector fields valued in differential forms of arbitrary rank. We will describe them using supermathematics.

General approach

For manifold M of dimension n consider $n|n$-dimensional supermanifold ΠTM, i.e. nothing that tangent bundle TM with changing parity of fibers.

Note that usual k-form on M $dx^{i_1} \wedge \ldots \wedge dx^{i_k} \omega_{i_1 \ldots i_k}$ defines a function

$$\omega(x, dx) = dx^{i_1} \ldots dx^{i_k} \omega_{i_1 \ldots i_k}(x)$$

which is even if k is even and odd if k is odd (dx^i are odd variables: $dx^i dx^j = -dx^j dx^i$.)

Differential form-valued vector field $\mathbf{X}(x, dx) = X^i(x, dx)\partial_i$ is nothing but vector field on ΠTM such that its vertical components vanishes. Sure the vanishing of vertical components is not covariant condition. Better to say, that form-valued vector field defines differentiation of functions on M with values in differential forms. One can define canonical lifting of this object on vector field on whole ΠTM:

$$\mathbf{X}(x, dx) = X^i(x, dx)\partial_i \mapsto \mathbf{X}\widehat{(x, dx)} = X^i(x, dx)\partial_i + (-1)^{p(\mathbf{X})} dx^k \partial_k X^i(x, dx) \frac{\partial}{\partial dx^i}$$

$p(\mathbf{X}) = 0$ if it values in forms of even rank and it is equal to 1 if it values in forms of an odd rank.

This canonical lifting is uniquely defined by the condition that veector field $\widehat{\mathbf{X}}$ commutes wioth de Rham differential $d = dx^k \partial_k$.

Definition of Nijenuis bracket Let $\mathbf{A} = A^i(x, dx)\partial_i, \mathbf{B} = B^i(x, dx)\partial_i$ be vector valued differential forms Then

$$[\mathbf{A}, \mathbf{B}]_{\mathrm{N}}\colon\ \widehat{[\mathbf{A}, \mathbf{B}]}_{\mathrm{N}} = [\hat{\mathbf{A}}, \hat{\mathbf{B}}] ,$$

where on the right hand side is usual commutato of vector fields. This is all ■