The problem of the coins

Along the paper of Mario Marteli and Gerald Ganon:
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It is the very old problem how to find the odd coin from 12 coins by 4 measurements...

DIVIDE ET IMPERA

For finding the strategy we first solve three preliminary problems:

$$1) \to 2) \to 3) \to 4)$$

Proposition 1 Number of coins is equal to 3^n and **odd coin is heavier**. One can find it by n measurements.

The proof (trivial): The case $n = 1$ is evident. Let $n = k + 1$. During the first measurement we put 3^k coins on left and 3^k on the right. If "$left > right$" by induction we choose odd coin in the left set by k measurements. If "left=right" we do it with 3^k remaining coins.

Proposition 2 Number of coins is 3^n. Coins are marked: they are red or black and red coins are heavier or equal by weight to black, i.e. odd coin is heavier if it is red and it is lighter if it is black. One can find it by n measurements.

The proof. The case $n = 1$ is trivial again. Let the statement is right for $n = k$.

Let $n = k + 1$. We always can choose $2a$ red and $2b$ black coins such that $a + b = 3^n$ $(a = 0, 1, 2, \ldots)$ and put on the left pan a red coins and b black coins, the same on the right pan. If "$left > right$" hence the odd coin is one of the a red or one of the b black. We take $3^n = a + b$ coins (a red from the left and b right from the right and by n measurements find the odd one.

If "$left = right$" the we find odd one in the remaining 3^n coins.

Proposition 3 Number of coins is equal to $s_n = \frac{3^n-1}{2}$, $(s_1 = 1, s_2 = 4, s_3 = 13, s_4 = 40, s_5 = 121, \ldots$ $(s_{n+1} = 3s_n + 1)$.

The coins are not marked and we do not know is the odd coin heavier or odd but we have **additionally the coin which sure is not odd.** One can find it by n measurements!?

Proof The case $n = 1$ is evident. We compare the coin with test one and see is it odd or not.

Let we can find the odd one for $n = k$ and consider the case $n = k + 1$.

$$s_{k+1} = s_k + s_k + s_k + 1$$

We put aside s_k coins. We put s_k coins and additional test coin on the left pan and $s_k + 1$ coins on the right pan. If "left¿right" we throw out the test coin, mark the s_k coins on the left pan by red paint, mark the $s_k + 1$ coins on the right pan by black pain. We have $s_k + s_k + 1 = 3^k$ marked coins. Using Proposition 2) we find the odd coin by k measurements.

If "left=right" we know that odd coin is in the rest s_k coins and we find it with test coins by k measurements.

Finally we come to

Theorem Number of coins is equal to $p_n = \frac{3^n-3}{2}$, $(p_1 = 0, p_2 = 3, p_3 = 12, p_4 = 39, p_5 = 120, \ldots$ $(p_{n+1} = 3p_n + 3)$.

We do not know is the odd coin heavier or odd!

One can find it by n measurements!!!

Proof

Let $n = k + 1$.

One can see that

$$p_{k+1} = (p_k + 1) + (p_k + 1) + (p_k + 1)$$

We put $p_k + 1$ coins on the left side and $p_k + 1$ on the right.

a) If "$left > right$" It means that odd coin is "red" if it is on the left pan and it is right if it is on the right pan (See Proposition 2). We mark coins on the left pan by red and coins on the right pan by black, add one coin from the remaining coins. (This additional coin is sure not odd, so it can be marked as we want red or black) $(p_k + 1) + (p_k + 1) + 1 = 3^k$. Hence we have 3^k marked coins and we can find the odd coin amongst them, according to Proposition 2)

b) "left=right". In this case we know that odd coin is in the remaining $p_k + 1$ coins. But $p_k + 1 = s_k$ (See Proposition 3)

We take remainig s_k coins and the test coins from the coins which we measured also and due to Proposition 3) we find it by k measurements.

It is all!

Example Let we have 12 $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ coins.

First meauserement: $\{1, 2, 3, 4\}$ on the left and $\{5, 6, 7, 8\}$ on the black.

a)If $\{1, 2, 3, 4\} > \{5, 6, 7, 8\}$ we mark $\{1, 2, 3, 4\}$ by red, $\{5, 6, 7, 8\}$ by black and consider $\{1_r, 2_r, 3_r, 4_r, 5_b, 6_b, 7_b, 8_b, 10_b\}$.

We do second measurement: $\{1_r, 2_r, 5_b\}$ on the left and $\{3_r, 4_r, 6_b\}$ on the right. If $\{1_r, 2_r, 5_b\} > \{3_r, 4_r, 6_b\}$ it means that odd is or on the left and red or on the right and black. We take the coins $\{1_r, 2_r, 5_b\}$ and find by the third measurement the odd one, putting 1_r on the left and $_b$ on the right.

If $\{1_r, 2_r, 4_b\} = \{3_r, 4_r, 5_b\}$ it means that the odd is in the set $\{7_b, 8_b, 10_b\}$ and we find it by one measurement, putting 7_b on the left and 8_b on the right.

Now consider the second case:

a)If $\{1, 2, 3, 4\} = \{5, 6, 7, 8\}$. Then the odd is in the set $\{9, 10, 11, 12\}$. We take the coin 1 as the test coin and add it to the set $\{9, 10, 11, 12\}$. Consider the set $\{1, 9, 10, 11, 12\}$ where 1 is the test coin (sure not odd).

We do the second measurement according to the proof of Proposition 3): putting the coins $\{1,9\}$ on the left pan and $\{10,11\}$ on the right pan. Now if $\{1,9\} > \{10,11\}$ we mark (coin by red and 10 and 11 by black and during the third measurement we put the coin 9 on the left pan and 10 on the right pan and find the odd coin in the set $\{9,10,11\}$. If $\{1,9\} = \{10,11\}$ then the odd coin is the coin 12 and we do not need the third measurement.