A number such that its square finishes on it and Veselov's comment on it

Find a number x such that its square , a number x^2 finishes with it. More precisely this means the following: We say that a number x *has m-digits and it is finished with it* if

$$10^m \leq x < 10^{m+1}\,, and \quad x^2 - x \,\text{divides}\, 10^m\,. \qquad (1)$$

Remark E.g. a number $x = 0625$ has 4 digits and it is finished with it:

$$625^2 - 625 = 390000\,, 390000\,\text{divides}\, 10.000\,.$$

This is very old problem for me [1]) About five years ago Sasha Veselov made very beatuiful comment on this. It follows from the **Theorem** below that 10 is not a prime number!

Indeed, suppose that 10 is prime. Then 10-adic is a field and in the field *there is no* solutions of equation

$$x^2 = x$$

This contradicts the statement of the theorem.

One can say that Veselov's statement is very sophisticated way to prove that a number 10 is not a prime.

Theorem *There are two exactly two sequences*

$$a_1, a_2, \ldots, a_n, \ldots = 5, 25, 625, 0625, 90625, 890625, 2890625, \ldots$$

and

$$b_1, b_2, \ldots, b_n, \ldots = 6, 76, 376, \ldots$$

such that

1) all the numbers in these sequences obey the condition, that their squares finishes by them.

2) Any number a_n in the first sequence possesses not more than n digits: $a_n < 10^n$, respectively any number b_n in the second sequence possesses not more than n digits: $a_n < 10^n$.

3) Any number such that its square finish by it belongs to the first or to the second sequence

This follows from the following inductive statement

Lemma

Suppose by induction that a number a_n contains n digits (zeroes are permitted (see the Remark)), *and it is finished by it.* Then there is exists a number a_{n+1} which contains $n+1$ digits (zeroes are permitted (see the Remark)) *and it is finished by it.*

$$a_n < 10^n, a_n^2 - a_n = 0 (mod 10^n) \Rightarrow \text{there exists a number } a_{n+1} \text{ such that}$$

[1]) It is one of the first problems which I solved when I was a kid

$$a_{n+1}\colon a_{n+1} < 10^n, a_{n+1}^2 - a_{n+1} = 0(mod 10^{n+1}), \tag{4}$$

where $a_{n+1} = 10^n x_n + a_n$ $(x = 0, 1, \ldots, 9)$ (if $x = 0$ then a number a_{n+1} may "slip" a digit. (see the remark after equation (1).)) First note that

$$a_n^2 - a_n = 10^n s_n ,$$

and

$$b_m^2 - b_m = 10^m t_m ,$$

with s_n and t_m which are integers. Hence

$$a_{n+1}^2 - a_{n+1} = (10^n x_n + a_n)^2 - (10^n x_n + a_n) = 10^n \left(10^n x^2 + 2x_n a_n + s_n - x\right) .$$

and respectively

$$b_{m+1}^2 - b_{m+1} = (10^m x + b_m)^2 - (10^m x + b_m) = 10^m \left(10^m x^2 + 2x_m b_m + t_m - x\right) .$$

We see that expressions $10^n \left(10^n x^2 + 2xa_n + s_n - x\right)$ and $\left(10^m (10^m x^2 + 2x_m b_m + t_m - x\right)$ has to be divisible on 10^{n+1}. Hence one has to choose x_n such that

$$x_n(2a_n - 1) + s_n = 0(mod 10), \tag{5}$$

where

$$a_{n+1} = 10^n x_n + a_n \tag{6}$$

or respectively

$$y_m(2b_m - 1) + t_m = 0(mod 10), \tag{5'}$$

where

$$b_{n+1} = 10^n x_n + b_n \tag{6'}$$

Solve equation (5) or equation (5')

$$\begin{array}{lll}
a = 1, & x + s = 0, & x = 9s(mod 10) \\
a = 2, & 3x + s = 0, & x = 3s(mod 10) \\
a = 3, & 5x + s = 0, & \text{no solutions} \\
a = 4, & 7x + s = 0, & x = 7s(mod 10) \\
a = 5, & 9x + s = 0, & x = s(mod 10) \\
a = 6, & x + s = 0, & x = 9s(mod 10) \\
a = 7, & 3x + s = 0, & x = 3s(mod 10) \\
a = 8, & 5x + s = 0, & \text{no solutions} \\
a = 9, & 7x + s = 0, & x = 7s(mod 10)
\end{array} \tag{7}$$

We see that system (5) *has no solutions* if $a = 3$ or if $a = 8$. On the other hand in the equation (5) when we change a_n on a_{n+1} then according equation (6) the multiplier $2a_n - 1$ is changed on the $2a_{n+1} - 1 = 2\left(10^n x + a_n\right) - 1$. We see that the multiplier$(2a - 1)$ remains the same modulo 10 This means that the multiplier $(2a - 1)$ never will be equal to 5.

Lemma is proved.
Consider examples.

1 Take $N = 1$ and $a_1 = 5$, $(10^{n_1} = 10)$,
$a_1^2 - a_1 = 20$, $s_1 = \frac{a_1^2 - a_1}{10} = 2$. Choose x_2 in (5). We have

$$(2a_1 - 1)x_2 + 2 = 0(mod 10)\,, 9x_2 + 2 = 0 \Rightarrow x_2 = 2(mod 10)\,, a_2 = 10 \cdot 2 + 5 = 25\,.$$

Respectively take $M = 1$ and $b_1 = 6$, $(10^{m_1} = 10)$,
$b_1^2 - b_1 = 30$, $t_1 = \frac{b_1^2 - b_1}{10} = 3$. Choose y_2 in (5').
We have

$$(2b_1 - 1)y_2 + 3 = 0 \Rightarrow y_2 = 7(mod 10)\,, b_2 = 10 \cdot 7 + 6 = 76\,.$$

Thus we have two sequences

$$\{5, 25, \ldots\}; \{6, 76, \ldots\}$$

2 Take $N = 2$ and $a_2 = 25$,$(10^{n_2} = 100)$,
$a_2^2 - a_2 = 600$, $s_2 = \frac{a_2^2 - a_2}{100} = 6$. Choose x in (5). We have

$$(2a_2 - 1)x_2 + s_2 = 0(mod 10)\,, 49x_2 + 6 = (mod 10)\,, x_2 = 6(mod 10)\,, a_3 = 100 \cdot 6 + 25 = 625\,.$$

Respectively take $M = 2$ and $b_2 = 76$,$(10^{n_2} = 100)$,
$b_2^2 - b_2 = 5700$, $t_2 = \frac{b_2^2 - b_2}{100} = 57$. Choose x in (5). We have

$$151x + 57 = 0(mod 10)\,, x + 7 = 0(mod 10)\,, x = 3(mod 10)\,, b_3 = 100 \cdot 3 + 76 = 376\,.$$

Thus we have two sequences

$$\{5, 25, 625 \ldots\}; \{6, 76, 376 \ldots\}$$

3 Take $N = 3$ and $a_3 = 625$,$(10^{n_3} = 1000)$,
$a_3^2 - a_3 = 390.000$, $s_3 = \frac{a_3^2 - a_3}{1000} = 390$. Choose x in (5). We have

$$(2a_3 - 1)x_3 + s_3 = 0(mod 10)\,, 1249x + 390 = 0(mod 10)\,, 9x = 0(mod 10)\,, x = 0(mod 10)\,,$$

$$a_4 = 100 \cdot 0 + 25 = 0625\,.$$

Respectively take $M = 3$ and $b_3 = 376$,$(10^{n_3} = 1000)$,
$b_3^2 - b_3 = 141000$, $t_3 = \frac{a_3^2 - a_3}{1000} = 141$. Choose x in (5). We have

$$(2b_3 - 1)y_3 + t_3 = 0(mod 10)\,, 751y_3 + 141 = 0(mod 10)\,, y_3 + 1 = 0(mod 10)\,, y_3 = 9\,,$$

$$b_4 = 1000y_3 + b_3 = 9376\,.$$

Thus we have two sequences

$$\{5, 25, 625, 0625 \ldots\}; \{6, 76, 376, 9376 \ldots\}$$

4 Take $N = 4$ and $a_4 = 0625$, $(10^{n_4} = 10.000)$,

$a_4^2 - a_4 = 390.000$, $s_4 = \frac{a_4^2 - a_4}{10.000} = 39$. Choose x in (5). We have

$$(2a_4 - 1)x_4 + 39 = 0(mod10)\,, 1249x_4 + 39 = 0(mod10) 9x_4 + 9 = 0(mod10), x_4 = 9,\ ,$$

$$a_5 = 10.000 \cdot 9 + 0625 = 90625$$

Respectively take $M = 4$ and $b_4 = 9376$, $(10^{n_4} = 10.000)$,

$b_4^2 - b_4 = 8.790.000$, $t_4 = \frac{b_4^2 - b_4}{10.000} = 8790$. Choose y_4 in (5'). We have

$$(2b_4 - 1)y_4 + 8790 = 0(mod10)\,, 18751y_4 + 0 = 0(mod10)\,, y_4 = 0$$

$$b_5 = 09376$$

Thus we have two sequences

$$\{5, 25, 625, 0625, 90625 \ldots\}; \{6, 76, 376, 9376, 09376 \ldots\}$$

5 Take $N = 5$ and $a_5 = 90625$, $(10^{n_5} = 100.000)$,

$a_5^2 - a_5 = 8212800000$, $s_5 = \frac{a_5^2 - a_5}{100.000} = 82128$. Choose x in (5). We have

$181249x + 82128 = 0(mod10)\,, 9x + 8 = 0(mod10)\,, x = 8.\,, a_6 = 100.000 \cdot 8 + 90625 = 890.625$

Respectively take $M = 5$ and $b_5 = 09376$, $(10^{n_5} = 100.000)$,

$b_5^2 - b_5 = 87900000$, $s_5 = \frac{b_5^2 - b_5}{100.000} = 879$. Choose y_5 in (5). We have

$18751y_5 + 879 = 0(mod10)\,, y_5 + 9 = 0(mod10)\,, y_5 = 1.\,, b_6 = 100.000 \cdot 1 + 09376 = 109.376$

Thus we have two sequences

$$\{5, 25, 625, 0625, 90625, 890625, \ldots\}; \{6, 76, 376, 9376, 09376 \ldots\}$$

6 Take $N = 6$ and $a_5 = 890625$, $(10^{n_5} = 100.000)$,

$a_5^2 - a_5 = 793212000000$, $s_6 = \frac{a_6^2 - a_6}{100.000} = 793212$. Choose x in (5). We have

$$1781249x_6 + 793212 = 0(mod10)\,, 9x_6 + 2 = 0(mod10)\,, x_6 = 2.\,,$$

$$a_7 = 100.000 \cdot 2 + 890625 = 2.890.625$$

Respectively take $M = 6$ and $b_6 = 109.376$, $(10^{m_6} = 100.000)$,

$b_6^2 - b_6 = 11.96.300.000$, $s_5 = \frac{b_5^2 - b_5}{100.000} = 11.963$. Choose y_6 in (5). We have

$218,751y_6 + 11.963 = 0(mod10)\,, y_6 + 3 = 0(mod10)\,, y_6 = 7.\,, b_7 = 100.000 \cdot 1 + 109376 = 7.109.376$

Thus we have two sequences

$$\{5, 25, 625, 0625, 90625, 890625, 2890625, \ldots\};$$
$$\{6, 76, 376, 9376, 09376, 109376, 7109376 \ldots\}$$

which obey the Theorem[2)] unfortunately this is the edge of capacities of my calculatrice