On Kac Peterson Formula

Let $\mathcal{G}$ be simple Lie algebra. Then Kac Peterson formula tells that volume $V(G)$ of compact connected simple connected group $G(\mathcal{G})$ is defined by the formula,

$$V^2(G) = (8\pi)^2 J(4\pi i\rho)\,,$$

where $\rho = \frac{1}{2}\sum_{\alpha\in\triangle_+}\alpha$ is Weyl vector,and J Jacobian of the map $\mathcal{G}\to G$:

$$J(x) = \det\left(\frac{1-e^{-ad_x}}{ad_x}\right)$$

(We suppose that covectors and vectors are identified via Killing Cartan metric

$$\phi(x,y) = -\mathrm{Tr}\,(ad_x ad_y)$$

$\phi = \langle\,,\,\rangle$)

This implies that

$$V = (2\pi\sqrt{2})^{dim\,\mathcal{G}}\prod_{\alpha\in\triangle_+} f(2\pi\phi(\rho,\alpha))\,,\qquad \text{where}\quad f(x) = \frac{\sin x}{x}\,.$$

(See equation (4.32.1) in V.Kac, D.Peterson "Infinite-dimensioanal Lie algebras, Theta functions and modular forms", Advances in Math,. **13**, pp.125—264 (1984))

Let us apply this formula to the most simple case $su(2)$, then to $sl(n,C)$ (we mean calculate volume of corresponding simple simply connected Lie groups.

volume of $SU(2)$

)

Consider generators $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$:

$$[\mathbf{e}_i,\mathbf{e}_k] = \varepsilon_{ikm}\mathbf{e}_m\,.$$

One can see that

$$\mathrm{Tr}(ad_{e_i}ad_{e_k}) = -2\delta_{ik}\,,\text{this means that } \phi(\mathbf{e}_i,\mathbf{e}_k) = 2\delta_{ik}$$

Consider Cartan algebra h spanned by vector $\mathbf{e}_3$. We have that

$$[i\mathbf{e}_3,\mathbf{e}_\pm] = \pm\mathbf{e}_\pm\,,\quad \text{where } \mathbf{e}_\pm = \mathbf{e}_1 \pm i\mathbf{e}_2\,.$$

Thus we have two roots α_+,α_-:

$$\alpha_\pm\in h^*: \alpha_\pm(\mathbf{e}_1) = \pm\frac{1}{i}\ \text{i.e.}\ \begin{cases}\alpha_+\in h^*\,, & \text{such that } \alpha_+(\mathbf{e}_1) = -i\\ \alpha_-\in h^*\,, & \text{such that } \alpha_+(\mathbf{e}_1) = i\end{cases}.$$

We see vectors $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ have length $\sqrt{2}$ and covectors α_+,α_- have length $\frac{1}{\sqrt{2}}$ (matrix of the Cartan-Killing metric in the basis $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ is $\begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix}$,) Weyl vector $\rho = \frac{\alpha}{2}$.

Hence

$$2\pi\phi(\rho,\alpha_+) = \pi|\alpha_+|^2 = \frac{\pi}{2}\,.$$

We come to

$$Vol(su(2)) = (2\pi\sqrt{2})^{dim\,\mathcal{G}} \prod_{\alpha\in\triangle_+} \frac{\sin(2\pi\phi(\rho,\alpha))}{2\pi\phi(\rho,\alpha)} =$$

$$= (2\pi\sqrt{2})^3 \frac{\sin(2\pi\phi(\rho,\alpha))}{2\pi\phi(\rho,\alpha)} = (2\pi\sqrt{2})^3 \frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} = 32\sqrt{2}\pi^2$$

Notice that $su(2)$ is three-dimensional sphere and volume of three-dimensional sphere is proportional to π^2: volume of sphere of radius R is equal to $2\pi R^3$: volume of 3-dimensional sphere of radius 1 is equal to

$$Vol(S^3) = \frac{2\pi^{\frac{k+1}{2}}}{\Gamma\left(\frac{k+1}{2}\right)}\Big|_{k=3} = 2\pi^2$$

One can say that volume of $su(2)$ is equal to the volume of S^3 with radius $R = 2\sqrt{2}$. (it has to be clarified.)

Before we study roots system of algebra $sl(n,C)$.

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ie algebras $sl(n,C)$

Denote by E_{ij} $n\times n$ matrix such that its all entries vanish except the intry in i-th column amd j-th row which is equal to 1. These matrices span over C $gl(n,\mathbf{C})$ Lie algebra. Traceless matrices span Lie algebra $sl(n,\mathbf{C})$. Denote by $\mathbf{t}H$ Cartan algebra of diagonal matrices in $gl(n,C)$ and respectively by H Cartan algebra of traceless diagonal matrices in $sl(n,C)$. Denote by $\alpha^{ij}, (i\neq j)$ linear functions on Lie algebra $\mathbf{t}H$ (i.e. elements of $\mathbf{t}H^*$) such that

$$\alpha^{ij}(t^m E_{mm} a = t^i - t^j\,.$$

One can see $\{\alpha^{ij}\}$ are roots — they are equal to values of observables (Cartan subalgebra $\mathbf{t}H$) on root vectors $\{E_{ij}\}$:

$$\forall \mathbf{t}\in H, \mathbf{t} = t^m E_{mm},\ \hat{\mathbf{t}}E_{ik} = [\mathbf{t}, E_{ik}] = (t^i - t^k)E_{ik}$$

$$\begin{pmatrix} \text{matrices } E_{ij}, (i>j) \text{ are positive weight vectors} \\ \text{matrices } E_{ij}, (i<j) \text{ are negative weight vectors} \end{pmatrix}$$

Teperj poschitajem metriku Cartana Killinga na algebre $gl(n,\mathbf{C})$ ad its subalgebra $sl(n,\mathbf{C})$.

$$\phi(X,Y) = \mathrm{Tr}\,(\hat{X}\circ\hat{Y})$$

where $\hat{X} = ad_X$: $\hat{X}Y = [X,Y]$. Notice that for every matrix X for coefficients of expansion we have:

$$X = X^{\pi\rho}E_{\pi\rho} \Rightarrow X^{\pi\rho} = \mathrm{Tr}\,(XE_{\rho\pi})\,.$$

Hence for metric coefficients we have

$$g_{ik|pq} = \mathrm{Tr}\,(\hat{E}_{ik}\circ\hat{E}_{pq}) = \mathrm{Tr}\,(([E_{ik},[E_{pq},E_{\alpha\beta}]])\,E_{\beta\alpha}) =$$

$$Tr\,(E_{ik}E_{pq}E_{\alpha\beta}E_{\beta_\alpha} - E_{ik}E_{\alpha\beta}E_{pq}E_{\beta_\alpha} - E_{pq}E_{\alpha\beta}E_{ik}E_{\beta_\alpha} + E_{\alpha\beta}E_{pq}E_{ik}E_{\beta_\alpha}) =$$

$$2N\delta_{iq}\delta_{kp} - 2\delta_{ik}\delta_{pq}\,.$$

We see that this metric is degenerate: identity matrix is zeroeigenvector, in other words algebra $gl(n,\mathbf{C})$ is not semisimple, it possesses the centre. The corank of the metric is just one—the algebra $sl(n,\mathbf{C})$ is semisimple. Calculate Kartan-Killing on $sl(n,\mathbf{C})$. For every $X \in sl(n,\mathbf{C})$

$$\mathrm{Tr}\, X\big|_{gl(N,\mathbf{C})} = \mathrm{Tr}\, X\big|_{sl(N,\mathbf{C})}$$

since $\hat{X}I = [X, I] = 0$. Hence metric is defined by the same formula.

Choose the basis in $gl(n,\mathbf{C})$:

$$E_{pq}, p \neq q\ T_i = E_{ii} - E_{nn}, (i = 1, \ldots, n-1)\,.$$

Our next step to calculate scalar products of roots. We see from previous calculations that non-zero metric entries are only

for every $p, q = 1, \ldots, n$ such that $p \neq q\,, \phi(E_{pq}, E_{qp}) = 2n\,,$ and $\begin{cases} = \phi(T_i T_j) = 2 \text{ if } i \neq j \\ \phi(T_1, T_1) = \ldots = \phi(T_{n-1}, T_{n-1}) = 4 \end{cases}$

and all other entries vanish.

Choose the ordering

$$\{E_{21}, E_{31}, E_{32}, \ldots, E_{n1}, \ldots, E_{n\,n-1}, | T_1, T_2, \ldots, T_{n-1}, E_{12}, \ldots, E_{1n}, E_{23}, \ldots, E_{2n}, \ldots, E_{n-1\,n}\}$$

of basic vectors. Then we see that $(n^2-1)\times(n^2-1)$ matrix of Cartan-Killing metric has the following appearance

$$||G|||_{sl(n,\mathbf{C})} = \begin{pmatrix} 0 & 0 & 2nI \\ 0 & K & 0 \\ 2nI & 0 & 0 \end{pmatrix}\,,$$

where I is $\frac{n^2-n}{2} \times \frac{n^2-n}{2}$ unity matrix,a and K is $n-1 \times n-1$ matrix such that

$$K = \begin{pmatrix} 4 & 2 & 2\ldots & 2 & 2 \\ 2 & 4 & 2\ldots & 2 & 2 \\ \ldots & & & & \\ 2 & 2 & 2\ldots & 2 & 4 \end{pmatrix}$$

The inverse matrix (we need it to calculate the scalar product of covectors (roots)) has the appearance

$$||G^{-1}|||_{sl(n,\mathbf{C})} = \begin{pmatrix} 0 & 0 & \frac{1}{2n}I \\ 0 & K^{-1} & 0 \\ \frac{1}{2n}I & 0 & 0 \end{pmatrix}\,,$$

where

$$K^{-1} = \frac{1}{2}\begin{pmatrix} 1-\frac{1}{n} & -\frac{1}{n} & -\frac{1}{n}\ldots & -\frac{1}{n} & -\frac{1}{n} \\ -\frac{1}{n} & 1-\frac{1}{n} & -\frac{1}{n}\ldots & -\frac{1}{n} & -\frac{1}{n} \\ \ldots & & & & \\ -\frac{1}{n} & -\frac{1}{n} & -\frac{1}{n}\ldots & -\frac{1}{n} & 1-\frac{1}{n} \end{pmatrix}$$

Thus we will be able to calculate scalar producst of roots.

Roots

Calculate the components of roots α^{ik} in the basis T_i (Recall that $T_i = E_{ii} - E_{nn}$). Calculating $\alpha^{ik}(T_i)$ we come to components of roots, covectors:

$$\alpha^{12} = \begin{pmatrix} \alpha^{ik}(T_1) \\ \alpha^{ik}(T_2) \\ \alpha^{ik}(T_3 \\ \alpha^{ik}(T_4) \\ \dots \\ \alpha^{ik}(T_{n-1}) \\ \alpha^{ik}(T_n) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ \dots \\ 0 \\ 0 \end{pmatrix}, \alpha^{23} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ \dots \\ 0 \\ 0 \end{pmatrix}, \dots, \alpha^{n-2,n-1} = \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \alpha^{n-1,n} = \begin{pmatrix} 1 \\ 1 \\ \dots \\ 1 \\ 1 \\ 1 \\ 2 \end{pmatrix},$$

We wrote components of simple roots $\alpha^{s\,s+1}$. (Every positive (negative) root is combination of simple roots with positive (negative)integers, e.g.

$$\alpha^{36} = \alpha^{34} + \alpha^{45} + \alpha^{56}\,.$$

Now we can perfoirm calculations, for example

$$|\alpha^{12}|^2 = \phi(\alpha^{12}, \alpha^{12}) = \alpha^{12*} K^{-1} \alpha^{12} = \frac{1}{2}\,(\,1, -1, 0, 0 \dots, 0, 0\,) \begin{pmatrix} 1-\frac{1}{n} & -\frac{1}{n} & -\frac{1}{n} \dots & -\frac{1}{n} & -\frac{1}{n} \\ -\frac{1}{n} & 1-\frac{1}{n} & -\frac{1}{n} \dots & -\frac{1}{n} & -\frac{1}{n} \\ \dots \\ -\frac{1}{n} & -\frac{1}{n} & -\frac{1}{n} \dots & -\frac{1}{n} & 1-\frac{1}{n} \end{pmatrix} \Bigg($$

$$\frac{1}{2}\,(\,1, -1, 0, 0 \dots, 0, 0\,) \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ \dots \\ 0 \\ 0 \end{pmatrix} = 1\,,$$

$$\phi(\alpha^{12}, \alpha^{23}) = \alpha^{12*} K : wq^{-1} \alpha^{23} = \frac{1}{2}\,(\,1, -1, 0, 0 \dots, 0, 0\,) \begin{pmatrix} 1-\frac{1}{n} & -\frac{1}{n} & -\frac{1}{n} \dots & -\frac{1}{n} & -\frac{1}{n} \\ -\frac{1}{n} & 1-\frac{1}{n} & -\frac{1}{n} \dots & -\frac{1}{n} & -\frac{1}{n} \\ \dots \\ -\frac{1}{n} & -\frac{1}{n} & -\frac{1}{n} \dots & -\frac{1}{n} & 1-\frac{1}{n} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ \dots \\ 0 \\ 0 \end{pmatrix}$$

$$\frac{1}{2}\,(\,1, -1, 0, 0 \dots, 0, 0\,) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \dots \\ 0 \\ 0 \end{pmatrix} = -\frac{1}{2}$$