It is well-known that Laplacian is

$$\Delta + c_n R \tag{1}$$

invariant with respect to conformal transformations

$$\tilde{g}_{ik} = e^{2\sigma} g_{ik} \tag{2}$$

Here c_n is conastand depending on dimension of the space, R is a scalar curvature.

What is the exact statement?

Why R appears?

Let $\mathbf{s} = s|Dx|^\lambda$ be a density of weight λ. Consider our operator $\hat{\Delta}$ acting on this density:

$$\hat{\Delta}_{g,\Gamma} = \hat{\Delta}\mathbf{s} = \left(\partial_m (g^{mn}\partial_n s) + (2\lambda - 1)\Gamma^i \partial_i s\right) |Dx|^\lambda + \lambda \partial_i \Gamma^i \mathbf{s} + \lambda(\lambda - 1)\Gamma^i \Gamma_i \mathbf{s} \tag{3}$$

here Γ_i is a connection on densities. We know that

$$\hat{\Delta}\mathbf{s} = \rho^\lambda \Delta(\rho^{-\lambda}\mathbf{s}) \,, \tag{4}$$

where $\rho = \sqrt{\det g}|Dx|$ is volume form on Riemannian manifold, $\Gamma_i = -\partial_i \log \rho$ and Δ is Laplace-Beltrami Laplacian on functions:

$$\Delta F = \frac{1}{\rho}\partial_m \left(\rho g^{mn}\partial_n F\right) \tag{5}$$

Now study how operator $\hat{\Delta}$ changes under conformal transformations:

$$g \to \tilde{g} = e^{2\sigma} g \,, \rho \to \tilde{\rho} = \rho(\sqrt{\det g})^{\frac{n}{2}} = \rho e^{n\sigma} \,, \Gamma_i \to \tilde{\Gamma}_i = -\partial_i \log \tilde{\rho} = \Gamma_i - n\partial_i \sigma \,. \tag{6}$$

Then we come to operator

$$\hat{\Delta}' = \hat{\Delta}'_{\tilde{g},\tilde{\Gamma}} = \partial_m (\tilde{g}^{mn}\partial_n) + (2\lambda - 1)\tilde{\Gamma}^i \partial_i + \lambda \partial_i \tilde{\Gamma}^i + \lambda(\lambda - 1)\tilde{\Gamma}^i \tilde{\Gamma}_i \tag{7}$$

Study the relation between operators $\hat{\Delta}'$ and $\hat{\Delta}$.

Using equations (6) we come to

$$\hat{\Delta}' = \hat{\Delta}_{\tilde{g},\tilde{\Gamma}} = \partial_m \left(e^{-2\sigma} g^{mn}\partial_n\right) + (2\lambda - 1)\tilde{\Gamma}^i \partial_i + \lambda \partial_i \tilde{\Gamma}^i + \lambda(\lambda - 1)\tilde{\Gamma}^i \tilde{\Gamma}_i =$$

$$\partial_m \left(e^{-2\sigma} g^{mn}\partial_n\right) + (2\lambda - 1)e^{-2\sigma} g^{ij}(\Gamma_j - n\partial_j \sigma)\partial_i + \lambda \partial_i \left(e^{-2\sigma} g^{ij}\left(\Gamma_j - n\partial_j \sigma\right)\right) +$$

$$\lambda(\lambda - 1)e^{-2\sigma} g^{ij}(\Gamma_i - n\partial_i \sigma)(\Gamma_j - n\partial_j \sigma) =$$

$$e^{-2\sigma}\left[\partial_m \left(g^{mn}\partial_n\right) - 2\partial_m \sigma g^{mn}\partial_n\right] +$$

$$+e^{-2\sigma}(2\lambda-1)g^{ij}(\Gamma_j - n\partial_j\sigma)\partial_i +$$

$$+e^{-2\sigma}\lambda\left[-2\partial_i\sigma g^{ij}(\Gamma_i - n\partial_j\sigma) + \partial_i\Gamma^i - n\partial_i\left(g^{ij}\partial_j\sigma\right)\right] +$$

$$e^{-2\sigma}\lambda(\lambda-1)g^{ij}(\Gamma_i - n\partial_i\sigma)(\Gamma_j - n\partial_j\sigma) =$$

$$e^{-2\sigma}\left[\partial_m(g^{mn}\partial_n\ ) + (2\lambda-1)\Gamma^i\partial_i + \lambda\partial_i\Gamma^i + \lambda(\lambda-1)\Gamma^i\Gamma_i\right]$$

$$-e^{-2\sigma}\left[2+n(2\lambda-1)\right](\nabla\sigma)^m\partial_m +$$

$$\lambda e^{-2\sigma}\left[-2\partial_i\sigma\Gamma^i + 2n(\nabla\sigma)^m\partial_m\sigma - n\partial_i(\nabla^i\sigma)\right] +$$

$$-2e^{-2\sigma}\lambda(\lambda-1)n\Gamma^i\partial_i\sigma + +e^{-2\sigma}\lambda(\lambda-1)n^2\nabla^i\sigma\partial_i\sigma\,, \tag{8}$$

where $\nabla^i\sigma = \mathrm{grad}_{\,g}\sigma = g^{ij}\partial_j\sigma$. Use also that

$$\mathcal{L}_{\mathbf{A}}\mathbf{s} = (A^m\partial_m s + \lambda\partial_m A^m)|Dx|^\lambda \tag{9}$$

and the fact that:

$$\Delta\sigma = \partial_n\nabla^m\sigma = \partial_n(g^{mn}\partial_m\sigma) - \Gamma^m\partial_m\sigma = (\hat\Delta)_{\lambda=0}\sigma\,. \tag{10}$$

Hence we have

$$\hat\Delta'\mathbf{s} = \hat\Delta_{\tilde g = e^{2\sigma}g}\mathbf{s} = e^{-2\sigma}\hat\Delta_g\mathbf{s}$$

$$-\left[2+n(2\lambda-1)\right]\mathcal{L}_{\nabla\sigma}\mathbf{s} + \lambda\left[2+n(2\lambda-1)\right]\partial_m\nabla^m\sigma\mathbf{s} +$$

$$\lambda e^{-2\sigma}\left[-2\partial_i\sigma\Gamma^i + 2n(\nabla\sigma)^m\partial_m\sigma - n\partial_i(\nabla^i\sigma)\right]\mathbf{s} +$$

$$-2e^{-2\sigma}\lambda(\lambda-1)n\Gamma^i\partial_i\sigma + e^{-2\sigma}\lambda(\lambda-1)n^2\nabla^i\sigma\partial_i\sigma\mathbf{s} =$$

$$e^{-2\sigma}\left(\Delta\mathbf{s} - \left[2+n(2\lambda-1)\right]\mathcal{L}_{\nabla\sigma}\mathbf{s} + 2\lambda[1+n(\lambda-1)]\left(\nabla^i\sigma\partial_i\sigma - \Gamma_i\sigma\right)\mathbf{s} + n\lambda\left[2+n(\lambda-1)\right]\nabla^i\sigma\partial_i\sigma\mathbf{s}\right)$$

$$= e^{-2\sigma}\left(\Delta\mathbf{s} - \left[2+n(2\lambda-1)\right]\mathcal{L}_{\nabla\sigma}\mathbf{s} + 2\lambda[1+n(\lambda-1)]\left(\Delta\sigma\right)\mathbf{s} + n\lambda\left[2+n(\lambda-1)\right]\nabla^i\sigma\partial_i\sigma\mathbf{s}\right)$$

or in other way:

$$e^{2\sigma}\hat\Delta' = \hat\Delta - \left(2+n(2\lambda-1)\right)\mathcal{L}_{\nabla\sigma} + 2\lambda\left(1+n(\lambda-1)\right)\Delta\sigma + n\lambda\left(2+n(\lambda-1)\right)\nabla^i\sigma\partial_i\sigma\,. \tag{11}$$

or in other way:

$$\hat\Delta' = e^{-2\sigma}\left(\hat\Delta + a(n,\lambda)\mathcal{L}_{\nabla\sigma} + \Phi\right)\,,$$

where

$$a(n,\lambda) = \left(2+n(2\lambda-1)\right)\,,\ \Phi = 2\lambda\left(1+n(\lambda-1)\right)\Delta\sigma + n\lambda\left(2+n(\lambda-1)\right)\nabla^i\sigma\partial_i\sigma\,. \tag{11a}$$

We see that symbols of operator pencils $\hat{\Delta}'$ and $\hat{\Delta}$ coincide up to multiplier $e^{2\sigma}$. The pencil $\hat{\Delta}$ defines self-adjoint operator in algebra of densities. The operator $e^{2\sigma}\hat{\Delta}'$ differs from operator Δ on antiself-adjoint operator $\mathcal{L}$ and scalar function.

Hence one can find a value of λ such that operator is almost the same up to scalar:

$$2 + (2\lambda - 1)n = 0 \text{ i.e. } \lambda_0 = \frac{1}{2} - \frac{1}{n}$$

In this case we have that for equation (11)

$$2\lambda(1 + n(\lambda - 1))\big|_{\lambda=\lambda_0} = \frac{n-2}{2}\,,\ n\lambda(2 + n(\lambda - 1))\big|_{\lambda=\lambda_0} = -\frac{(n-2)^2}{4}\,. \qquad (12)$$

We see that on the densities of weight $\lambda_0 = \frac{1}{2} - \frac{1}{n}$ the following realtion holds:

$$\hat{\Delta}' = e^{-2\sigma}\left(\hat{\Delta} - \frac{n-2}{2}\Delta\sigma - \frac{(n-2)^2}{4}\partial_m\sigma\nabla^m\sigma\right)\,. \qquad (13)$$

Thus we come to construction of the operator:

$$L = \hat{\Delta} - \frac{n-2}{4(n-1)}R \text{ acting on densities of weight } \lambda = \tfrac{1}{2} - \tfrac{1}{n}.$$

Now we want to go further. Return to the formula (11)

$$\hat{\Delta}' = e^{-2\sigma}\left(\hat{\Delta} + a(n,\lambda)\mathcal{L}_{\nabla\sigma} + \Phi\right)\,,$$

where

$$a(n,\lambda) = (2 + n(2\lambda - 1))\ ,\ \Phi = 2\lambda\left(1 + n(\lambda - 1)\right)\Delta\sigma + n\lambda\left(2 + n(\lambda - 1)\right)\nabla^i\sigma\partial_i\sigma\,. \quad (11a)$$

ecall that operator pencil $\hat{\Delta}$ is constructed vial Beltrami-Laplace, and we know that $\hat{\Delta}$ is self-adjoint operator. Consider operator

$$\hat{\Delta}_\delta = \rho^\delta\hat{\Delta} = (\det g)^{\frac{n}{2}}|Dx|^\delta\hat{\Delta}$$

This operator sends λ-densities to $\lambda + \delta$ dentisites. This is easy exercise to check that this is alos self-adjoint operator! We denote by $\hat{\Delta}$ the Beltrami-Laplac pencil defined by Riemannian ,metric g_{ik} $\hat{\Delta}'$ the pencil cooresponding to the metric $\tilde{g_{ik}} = e^{2\sigma}g_{ik}$, and we denote by $\hat{\Delta}_{(\delta)}$ the weighted pencil $\hat{\Delta}_{(\delta)} = \rho^\delta\hat{\Delta} = \rho^\delta\hat{\Delta}_{(0)}$.

The relation (11a) will take the form:

$$\hat{\Delta}'_{(\delta)} = \tilde{\rho}^\delta\hat{\Delta}' = (\det\tilde{g})^{\frac{n\delta}{2}}|Dx|^\delta\hat{\Delta}' = e^{n\sigma\delta}\rho^\delta\hat{\Delta}' = e^{n\sigma\delta}e^{-2\sigma}\rho^\delta\left(\hat{\Delta} + a(n,\lambda)\mathcal{L}_{\nabla\sigma} + \Phi\right) =$$

$$e^{(n\delta-2)\sigma}\rho^\delta\left(\hat{\Delta}+a(n,\lambda)\mathcal{L}_{\nabla\sigma}+\Phi\right)=e^{(n\delta-2)\sigma}\left(\hat{\Delta}_{(\delta)}+a(n,\lambda)\rho^\delta\mathcal{L}_{\nabla\sigma}+\rho^\delta\Phi\right).$$

Now we put

$$\delta=\frac{2}{n}\,,\quad \text{then } \rho^\delta=\det g\,.$$

We come to the statement:

If weight $\delta=\frac{2}{n}$ then the weighted pencil obeys the transformation:

$$\hat{\Delta}'_{\frac{2}{n}}=\hat{\Delta}_{\frac{2}{n}}+\det g\left(a(n,\lambda)\mathcal{L}\nabla\sigma+\Phi\right)$$

Note that

$$a(n,\hat{\lambda})^*=-a(n,\lambda)$$

We come to

Proposition I Weighted Operators $\hat{\Delta},\hat{\Delta}'$ both are self-adjoint weighted operators with the same principal symbol in the case if $\delta=\frac{1}{2n}$.

Using Vornov-Khudaverdian Theorem we come to the conclusion that

$$\hat{\Delta}=t^\delta\left(S^{ab}\partial_b\partial_a+\partial_b S^{ba}\partial_a++\left(2\hat{\lambda}+\delta-1\right)\Gamma^a\partial_a+\hat{\lambda}\partial_a\mathcal{G}^a+\hat{\lambda}(\hat{\lambda}+\delta-1)\Gamma^a\Gamma_a\right)$$