Conic sections and Kepler law. Newton—Lagrange—-A.Giventale

This etude is written on the base of my conversations with Alexander Giventale in May 2014, and his letters to me, in April 2017

§1 **Introduction**

Conic section is an intersection of plane with surface of cone. Let $M\colon k^2x^2+k^2y^2=z^2$ be conic surface, and let $\pi\colon Ax+By+Cz=1$ be a plane[3]

Conic section

$$C\colon \begin{cases} k^2x^2+k^2y^2=z^2\\ Ax+By+Cz=1\end{cases} \tag{1.0}$$

$$\text{1. \textit{Conic sections are ellipses or parabolas or hyperbolas}}\,, \tag{1.1}$$

Kepler Law:

$$\begin{gathered}\text{trajectories of particle in gravitational field are conic section}\\ \text{In particular oribits of planets are ellipses}\end{gathered} \tag{1.2}$$

These two statements are common place for everybody.

Almost everybody who has learnt a bit of higher mathematics knows how to prove statement (1.1) (exercise in Analytical Geometry). Everybody who learnt little bit calculus knows that using the gravitational law, that

$$F=\gamma\frac{mM_{\text{sun}}}{R^2} \tag{1.3}$$

one can prove statement (1.2), showing that the solution of differential equation

$$\frac{d^2}{dt^2}\mathbf{R}=\frac{\mathbf{F}}{m}=-\gamma\frac{M_{\text{sun}}\mathbf{R}}{R^3}\,, \tag{1.3a}$$

is a conic section(m is a mass of planet, and M_{sun} is Solar mass). Standard proofs are based on calculus: solving equation (1.3a) in polar coordinates and calculating the integral we come to conic section [1)] (see any book in Theoretical Mechanics or Appendix 1 in the end of this text).

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[3] we do not conisder degenerate case when plane intersects origin.

[1)] Usually equation (1.3) is called Newton Graviational Law, and equation (3a) Newton Second law. There is an alternative point of view that Robert Hook formulated statement (1.3) in the letter to Newton, and Newton who invented Calculus, deduced the Kepler law, statement (2), solving differential equation (1.3a).

Here we discuss how to come to the statement (1.2) avoiding caclulations of integras.

Instead conic section (1.0) one can consider its orthogonal projection on the plane $z = 0$. For example if

$$C\colon \begin{cases} k^2x^2 + k^2y^2 = z^2 \\ Ax + By + z = 1 \end{cases} \text{ then its orthogonal projection } C_{proj}\colon \begin{cases} k^2x^2 + k^2y^2 = (1 - Ax - By)^2 \\ z = 0 \end{cases} \tag{1.0}$$

The following statement which is almost evident plays cruciar role in this etude:

$$\textit{orthogonal projections of conic sections on the plane is also a conic sections.} \tag{4}$$

In fact orbits of planets are these projections....

§2

This etude has the following history. Few months ago I was preparing the new lecture for Geometry students about conic sections. I had a task to explain that sections of conic surface with plane are conic sections (ellipses, parabolas or hyperbolas). The problem was that I could not find a beautiful and short explanation of the statement (1) without using extracurricular material. Another trouble was that at that time I had serious problems with eyes, and my access to books was very restricted. Finally I prepared the following explanation: Instead statement (1) I proved to students the statement (4), then I deduced statement (1) as the corollary of the statement (4). Honeslty considerations looked very bulky, I did not like them, but I had no choice.

I never forget early morning 30 March. In five hours the lecture will begin, where I have to explain to students the statement (1) on conic sections. I feel me very unhappy and unsatisfied with the way how I want to deliver the lecture, since the geometry of my considerations on the lecture look very vague. Suddenly I ask me a question: where are the foci of the projected conic section? (The foci of initial conic sections are at the points where Dandellen spheres touch the plane which sects the conic surface.) When the question was put, the answer come almost immediately: It is the vertex of the cone that is one of the foci of projected conic section. This was crucial: I immediately remembered, the May 2014.

`I am in the Davis University on 80 years celebration of my teacher Albert S. Schwarz. I meet there Alexander Givental. Alexander`

Givental is famous mathematician, but he is also very much engaged in
teaching mathematics to keeds (see his homepage in Berkeley University).
During conference dinner we are sharing the same table, and Alexander
is explaining me some beautiful properties of conic section.
In particular he is telling me the sentence:
if you consider a projection of ellipse on the horisontal plane,
you come naturally to Kepler law...

Recalling this phraze in the morning 30 March 2017, I immediately realised the geometrical meaning of my construction (1.4), which I made preparing the lecture, and I understood what Alexander Givental wanted to tell me three years ago.

Next day I contacted Alexander iby e-mail. He immediately sent me the detailed answer

First, Alexander sent me the article [3]. In this article he gives detailed geometrical explanation why trajectories of planets are conic section, and he does not use the caclulus. You can find this article on the homepage of A.Givental. Thirty years ago, an abstract of this article was included by V. I. Arnold into his joint with V. V. Kozlov and A. I. Neishtadt survey of classical mechanics [2].

In this letter A. Givental also told me that Alain Chensiner noted him that his (A.G.) considerations are very close to Lagrange's proof (see the paper [1]), and Alexander sent me the Lagrange article [1], and the letter where he explained this.

I do not want here to retell the work [3] of A.Givental. I just try based on the letter of A.Givental to retell the simple proof of statement (1.2) avoiding calculus. This construction can be traced to the work [1] of Lagrange. Of course the work [3] contains complete picture of these considerations.

My modest contribution to this etude is related with the fact that trying to find a simple proof of classical statement (1.1), I realised the importance of the statement (4),a nd on the base of it I try to retell the constuctions of the papers [1] and [3].

§3 **Simple exlanation of Kepler Law.**

Consider differential equation (1.3a) of particle in gravitational field

$$\ddot{\mathbf{R}} = -\frac{k\mathbf{R}}{R^3}\,. \tag{3.1}$$

let $\mathbf{R} = \mathbf{R}(t)$ be a solution of this equation. In Appendix we explicitly solved the equation and showed that it is conic section. Now we will do something much simpler. Without calculating explicitly integrals (see (App1.9), (App1.9a)) we just will show that $\mathbf{R}(t)$ is a conic section, that is it obeys equation (App1.10). Further we will use considerations from Appendix which has nothing to do with calculus (equations (App1.1)—(App1.8))

The preservation of angular momentum implies that the solution $\mathbf{R} = \mathbf{R}(t)$ is a curve in the plane. Choose Cartesian coordinates x, y, z such that

$$\mathbf{R}(t) = x(t)\mathbf{e}_x + y(t)\mathbf{e}_z \tag{3.2}$$

, and angular momentum

$$\mathbf{M} = (x(t)\dot{y}(t) - y(t)\dot{x}(t))\mathbf{e}_z = \mu\mathbf{e}_z\,. \tag{3.3}$$

(see for details Appendix).

Choose an arbitrary solution of equation (3.1) Let $x_0(t)$, $y = y_0(t)$ be components (3.2) of this solution of equation (3.1):

$$\begin{pmatrix} \ddot{x}_0\ (t) \\ \ddot{y}_0\ (t) \end{pmatrix} = -\frac{k}{\left(x_0^2(t) + y_0^2(t)\right)^{\frac{3}{2}}} \begin{pmatrix} x_0(t) \\ y_0(t) \end{pmatrix}\,. \tag{3.4}$$

Now consider the pair of differential equationts on function $f = f(t)$

1) homogeneous linear differential equation

$$\frac{d^2 f}{dt^2} = -k\frac{f}{r_0^3(t)}\,, \tag{3.5}$$

and non-homogeneous linear differential equation

$$\frac{d^2 f}{dt^2} = -k\frac{f}{r_0^3(t)} + \frac{\mu^2}{r_0^3(t)}\,, \tag{3.5a}$$

where function $\mathbf{r}_0(t)$ is defined by solutions $x_0(t)$, $y_0(t)$ (3.4):

$$r_0(t) = \sqrt{x_0^2(t) + y_0^2(t)}\,.$$

We see that $x_0(t)$, $y_0(t)$ are two independent solutions of second order homogeneous equation (we suppose that angular momentum $\mu = x\dot{y} - y\dot{x}$ does not vanish).

Let $g = g_0(t)$ be an arbiitrary solution of linear not homogeneous equation (3.5a). Then the space of all solutions of equation (3.5a) is

$$f(t) = Ax_{{}_0}(t) + By_{{}_0}(t) + g_0(t) \tag{3.6}$$

also that differential equation (3.1) implies that the functio

$$r_0(t) = \sqrt{x_0^2(t) + y_0^2(t)}$$

also obeys the differential equation

$$\frac{d^2 r}{dt^2} = -\frac{k}{\,} + \frac{\mu}{r^3}\,. \tag{3.5}$$

(see formulae) We consider the solution $x_0(t), y_0(t)$ of the equation (1) Consider the function

$$F\colon\, , F(t) = \left(x_0^2(t) + y_0^2(t)\right)^{-\frac{3}{2}}\,.$$

This function is defined by the solution of the equation (1). $r(t) = \sqrt{x^2(t) + y^2(t)}$.

Consider also the following two linear differential equations (homogeneous and non-homogeneous)

$$\ddot{u} = -kF_0 u\,, \qquad \ddot{u} = -kF_0 u + CF_0\,.$$

Solutions of the first equation form 2-dimensional linear space, solutions of the second equation form 2-dimensional affine space. The improtant observation is that

One can see that equation (1) implies that for the function $r(t)$

$$\frac{d^2 r(t)}{dt^2} = \frac{d^2}{dt^2}\left(\sqrt{x^2(t) + y^2(t)}\right) = \frac{d}{dt}\left(\frac{x(t)\frac{dx(t)}{dt} + y(t)\frac{dy(t)}{dt}}{\sqrt{x^2(t) + y^2(t)}}\right) =$$

$$\frac{x(t)\frac{d^2x(t)}{dt^2} + y(t)\frac{d^2y(t)}{dt^2}}{\sqrt{x^2(t) + y^2(t)}} + \left(\frac{\left(\frac{dx(t)}{dt}\right)^2 + \left(\frac{dy(t)}{dt}\right)^2}{\sqrt{x^2(t) + y^2(t)}}\right) - \left(\frac{x(t)\frac{dx(t)}{dt} + y(t)\frac{dy(t)}{dt}}{\sqrt{x^2(t) + y^2(t)}}\right)$$

Appenidix 1

In this Appnedix we will give the standard proof based on the calculus, that trajectory of particle in gravitational filed is a conic section

Let $\mathbf{R}=\mathbf{R}(t)$ is a solution of differential equation (1.3a):

$$\frac{d^2}{dt^2}\mathbf{R}=\frac{\mathbf{F}}{m}=-\gamma\frac{M_{\rm sun}\mathbf{R}}{R^3}\,, \qquad (Ap1.1)$$

This implies that angular momentum

$$\mathbf{M}=m\dot{\mathbf{R}}\times\mathbf{R} \qquad (App1.2)$$

is conserved: $\dot{M}=m\dot{\mathbf{R}}\times\dot{\mathbf{R}}+m\ddot{\mathbf{R}}\times\mathbf{R}=0+\left(-\gamma\frac{mM_{\rm sun}\mathbf{R}}{R^3}\right)\times\mathbf{R}=0$. Thus vector $\mathbf{R}(t)$ is orthogonal to the vector $\mathbf{M}$. Consider Cartesian coordinates x,y,z such that $\mathbf{M}$ is directed along axis OZ, and $\mathbf{R}(t)=x(t)\mathbf{e}_x+y(t)\mathbf{e}_y$ belongs to the plane OXY:

$$\mathbf{R}(t)=x(t)\mathbf{e}_x+y(t)\mathbf{e}_y\,,\quad \mathbf{M}(t)=\mu\mathbf{e}_z\,,(\mu\text{ is constant})\,. \qquad (App1.5)$$

Let φ,r be polar coordinates in the plane OXY, $x=r\cos\varphi, y=r\sin\varphi$. Then equation (1.3a) and (1.5) imply that

$$\mu=m(x\dot{y}-y\dot{x})=r\cos\varphi(\dot{r}\sin\varphi+r\dot{\varphi}\cos\varphi)-r\sin\varphi(\dot{r}\cos\varphi-r\dot{\varphi}\sin\varphi)=mr^2\dot{\varphi}\,, (App1.6a)$$

and

$$\ddot{r}=\frac{d}{dt}\dot{r}=\frac{d}{dt}\left(\frac{x\dot{x}+y\dot{y}}{r}\right)=\frac{x\ddot{x}+y\ddot{y}}{r}+\frac{\dot{x}^2+\dot{y}^2}{r}-\frac{(x\dot{x}+y\dot{y})^2}{r^3}=$$

$$\frac{x\left(-\gamma\frac{M_{\rm sun}x}{r^3}\right)+y\left(-\gamma\frac{M_{\rm sun}y}{r^3}\right)}{r}+\frac{(\dot{x}^2+\dot{y}^2)(x^2+y^2)-(x\dot{x}+y\dot{y})^2}{r^3}=$$

$$-\gamma\frac{M_{\rm sun}}{r^2}+\frac{(x\dot{y}-y\dot{x})^2}{r^3}=-\gamma\frac{M_{sun}}{r^2}+\frac{\mu^2}{m^2r^3}\,, \qquad (App1.7a)$$

i.e.

$$\frac{d}{dt}\left(\frac{\dot{r}^2}{2}\right)=\dot{r}\ddot{r}=-\gamma\frac{M_{\rm sun}}{r^2}\dot{r}+\frac{\mu^2}{m^2r^3}\dot{r}=\frac{d}{dt}\left(\gamma\frac{M}{r}-\frac{\mu^2}{2m^2r^2}\right)\,.$$

Thus we come to integral of motion:

$$\frac{d}{dt}\left(\frac{m\dot{r}^2}{2}-\gamma\frac{mM_{\rm sun}}{r}+\frac{\mu^2}{2mr^2}\right)=0\Rightarrow\frac{m\dot{r}^2}{2}-\gamma\frac{mM_{\rm sun}}{r}+\frac{\mu^2}{2mr^2}=E\,. \qquad (App1.8)$$

We come to first order differential equations [2]:

$$\begin{cases}\frac{dr(t)}{dt}=\sqrt{\frac{2}{m}}\sqrt{E+\gamma\frac{mM}{r}-\frac{\mu^2}{2mr^2}}\\ \frac{d\varphi(t)}{dt}=\frac{\mu}{mr^2}\end{cases}\Rightarrow\frac{d\varphi}{dr}=\frac{\frac{\mu}{r^2}}{\sqrt{2mE+\gamma\frac{2mM_{\rm sun}}{r}-\frac{\mu^2}{r^2}}} \qquad (App1.8a)$$

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[2] $E=\frac{m\dot{r}^2}{2}-\gamma\frac{mM_{\rm sun}}{r}+\frac{\mu^2}{2mr^2}$ is energy, $\frac{m\dot{r}^2}{2}$ is kinetik enery, and $U_{\rm eff}=-\gamma\frac{mM_{\rm sun}}{r}+\frac{\mu^2}{2mr^2}$ is effective potential energy

thus

$$\varphi = \int \frac{\frac{\mu dr}{r^2}}{\sqrt{2mE + \gamma \frac{2m^2 M_{\text{sun}}}{r} - \frac{\mu^2}{r^2}}} \tag{$App1.9$}$$

Integrating we come to

$$\int \frac{\frac{\mu dr}{r^2}}{\sqrt{2mE + \gamma \frac{2m^2 M_{\text{sun}}}{r} - \frac{\mu^2}{r^2}}}, = -\int \frac{\mu d\left(\frac{1}{r}\right)}{\sqrt{\gamma^2 \frac{m^2 M_{\text{sun}}^2}{\mu^2} + 2mE - \mu^2 \left(\frac{1}{r} - \gamma \frac{m^2 M_{\text{sun}}}{\mu^2}\right)^2}} =$$

$$\arcsin\left(\frac{\mu}{\sqrt{\gamma^2 \frac{m^2 M_{\text{sun}}^2}{\mu^2} + 2mE}}\right)\left(\frac{1}{r} - \gamma \frac{m^2 M_{\text{sun}}}{\mu^2}\right) \tag{$App1.9a$}$$

i.e.

$$\sin\varphi = \frac{\mu}{\sqrt{\gamma^2 \frac{m^2 M_{\text{sun}}^2}{\mu^2} + 2mE}}\left(\frac{1}{r} - \gamma \frac{m^2 M_{\text{sun}}}{\mu^2}\right) = \frac{1}{\sqrt{1 + \frac{2E\mu^2}{\gamma^2 M_{\text{sun}}^2 m^3}}}\left(\frac{\mu^2}{\gamma m^2 M_{\text{sun}}}\frac{1}{r} - 1\right) \tag{$App1.9b$}$$

Introduce notations:

$$e = \sqrt{1 + \frac{2E\mu^2}{\gamma^2 M_{\text{sun}}^2 m^3}}, k = \frac{\mu^2}{\gamma m^2 M_{\text{sun}}},$$

then (App1.9b) has appearance:

$$e \sin\varphi = \frac{k}{r} - 1 \tag{$App1.10$}$$

This is conic section with excentricet ε, and one of the foci at the origin.

Remark `in fact it has to be` $\cos\varphi$, `not` $\sin\varphi$

[1] Lagrange *DES PERTURBATIONS DES COMETES.*— SECTION DDEUXIEME. Integrations des equations differentielles de l'orbite non-altere. pp.419—430, 1785

[2] V. I. Arnold, V. V. Kozlov, A. I. Neishtadt *Mathematical aspects of classical and celestial mechanics.* Dynamical systems III (Encyclopaedia of Mathematical Sciences), Springer, 1987.

[3] Alexander Givental *Keplers Laws and Conic Sections* Arnold Mathematical Journal March 2016, Volume 2, Issue 1, pp 139148

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