In mathematical physics if one says 'conformal structure of Riemannian manifold M' it means or

A Transformations of manifold M which preserve metric g.

B It is considered Equivalence class, conformal class $[g]$ of Riemannian metric. Metric g' belongs to conformal class $[g]$ if

$$g' = e^{\sigma(x)} g$$

If dimension of M is greater than 2 then conformal structure **A** possesses at most finite parametric family of transofrmations For example all conformal transformations of Euclidean space V^n are exhausted by orthogonal transformations, homothety, translations and inversion if $n \geq 3$ (Lioville Theorem). The algebra of conformal transofrmations $c0(n)$ is nothing but $so(n+1.1)$. Infinitesimal conformal transformations are:

$$\text{infinitesimal rotations}: x^i \mapsto x^i + \varepsilon B^i_k x^k, \quad (B^i_k = -B^k_i)$$

$$\text{infinitesimal translations}: x^i \mapsto x^i + \varepsilon t^i,$$

dilations (homothety):

$$x^i \mapsto x^i + \varepsilon x^i\,,$$

and special conformal transformations generated by inversion and translations:

$$K^i\colon \qquad x^i \mapsto x^i + \varepsilon K^i = O\left(Ox^i + \varepsilon t^i\right)\,,$$

where O is inversion with respect to origin: $Ox^i = \frac{x^i}{|x|^2}$,

$$x^i \mapsto x^i + \varepsilon K^i = O\left(Ox^i + \varepsilon t^i\right) = \frac{\frac{x^i}{|x|^2} + \varepsilon t^i}{\left|\frac{x^i}{|x|^2} + \varepsilon t^i\right|^2} =$$

$$\frac{\frac{x^i}{|x|^2} + \varepsilon t^i}{\frac{1}{|x|^2} + \frac{2\varepsilon t^i x^i}{|x|^2}} = \frac{x^i + \varepsilon t^i |x|^2}{1 + 2\varepsilon t^i x^i} = x^i + \varepsilon t^i (x,x) - 2\varepsilon x^i (t,x)$$

Thus we see that generators of algebra $co(n)$ are

$$\underbrace{x^i\partial_j - x^j\partial_i}_{\text{inf.rotations}}, \quad \underbrace{\partial_i}_{\text{inf.translations}}, \quad \underbrace{x^m\partial_m}_{\text{inf.homothety}}, \quad \underbrace{|x|^2\partial_i - 2x^i x^m \partial_m}_{\text{inf.special conf.transform.}}.$$

We have

$$|co(n)| = \frac{n(n-1)}{2} + n + 1 + n = \frac{(n+2)(n+1)}{2} = |so(n+1.1)|$$

One can show that conformal transformations of R^n are orthogonal of conic in $R^{n+1.1}$

Now consider second group.

It is related with the group $CO_0(n) = O(n) \otimes \mathbf{R}^*$ (i.e. orthogonal transformations and homothety)

Its algebra is $co_0(n) \oplus \mathbf{R}$.

Lemma The algebra $co(n) = so(n+1.1)$ is Cartan prolongaion of the algbera $co_0(n)$.

Proof of the Lemma

First recall Cartan prolongation

Let $\mathcal{G} = \mathcal{G}_0$ be an arbitrary Lie algebra.

Consider graded Lie algebra

$$\mathcal{G} = \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_1 \oplus \dots ,$$

where

$$\mathcal{G}_0 \quad \text{is Lie algebra } \mathcal{G}_0 ,$$

$$\mathcal{G}_{-1} = V \quad \text{is the vector space, such that } \mathcal{G}_0(n) \text{ faithfully acts on } V ,$$

We assume:

$$[v_1, v_2] = 0 \quad \text{for every two elements } v_1, v_2 \in \mathcal{G}_{-1}$$

and

$$[h, v] = -[v, h] = h(v) \quad \text{for arbitrary } h \in \mathcal{G}_0 \text{ and } v \in \mathcal{G}_{-1}$$

($h(v)$ is the actioh of element h on vector v.) Now we construct the spaces $\mathcal{G}_i$ $(i = 1, 2, 3, \dots)$ *i-th Cartan prolongations of Lie algebra $\mathcal{G}_0$* in the following way:

Consider the space

$$T_i(V) = Hom\left(\underbrace{V \times V \times \dots \times V}_{i+1 \text{ times}}, V\right) \underbrace{V \otimes V \otimes \dots \otimes V}_{i+1 \text{ times}} \otimes V$$

of tensors of rank $i+2$ of valency $\begin{pmatrix} 1 \\ i+1 \end{pmatrix}$.

Now we define vector space $\mathcal{G}_i$ as the subspace of tensors in $T_i(V)$ which are symmetric:

$$T_i(V) \ni t\left(\dots \underbrace{u}_{m\text{-th place}} \dots \underbrace{v}_{n\text{-th place}} \dots\right) = t\left(\dots \underbrace{v}_{m\text{-th place}} \dots \underbrace{u}_{n\text{-th place}} \dots\right)$$

and obey the following condition:

for arbitrary i vectors $v_1, \dots, v_i \in V$ there exists an element $h \in \mathcal{G}_0$ such that

$$h(u) = t\,(u, v_1, \dots, v_i)$$

for arbitrary vector $u \in \mathbf{R}^n = \mathcal{G}_{-1}$.

The Lie commutator in spaces $\mathcal{G}_i$ (i=1,2,3,...) is the following: for every $t \in \mathcal{G}_i$, $v \in \mathcal{G}_{-1}$

$$\mathcal{G}_{i-1} \ni [t, v]: \quad [t, v](u_1, \dots, u_i) = t(v, u_1, \dots, u_i)\,,$$

for every $t \in \mathcal{G}_i$, $h \in \mathcal{G}_0$

$$\mathcal{G}_i \ni [h, t]: \quad [h, t](u_1, \dots, u_{i+1}) =$$

$t\,([h, u_1], u_2, \dots, u_{i+1}) + t\,(u_1, [h, u_2], u_3, \dots, u_{i+1}) + \dots + t\,(u_1, u-2, \dots, u_i, [h, u_{i+1}])$ (up to a constant

(up to a constant) of symmetric tensors in the space $Hom(V \times V, V) = V^* \times V^* \times V$ such that for every vectors $v_1, v_2 \in V$

$$t(v)$$

$$\mathcal{G}_0 \quad \text{is Lie algebra } co_0(n)\,,$$