Nice fractions and Platonic bodies

(reflections on the Klein book and Lueroth Theorem...)

This text is the continuation of the text on Lueroth theorem.

We consider fractions—rational functions on indeterminate t or if you prefer elements of simple transcendent extensions.

Consider an arbitrary irreducible fraction $R(x) = f(x)/g(x)$. Suppose its degree is equal N. Consider the equation

$$\frac{f(x)}{g(x)} = \frac{f(t)}{g(t)} \tag{1}$$

If this equation has exactly N solutions which are fractions on t (i.e. elements in $\mathbf{C}(\mathbf{t})$ then we call this fraction *nice fraction*

Suppose fractions $\{\psi_0(t), \psi(t_1), \ldots, \psi_N(t)\}$ $(\psi_0(t) = t)$ are solutions of equation (1) for the nice fractions $F = f/g$, i.e. $f(\psi(t))/g(\psi(t)) \equiv f(t)/g(t)$. Their degrees coincide and degree of ψ must be equal to 1 (Is it vraie?) The N substitutions $\{\psi_i(t)\}$ form invariance group of the fraction $f(x)/g(x)$.

Every nice fraction R of degree N defines the finite subgroup H_R of fractional-linear transformations containing N elements

The inverse is true too:

Let H be the finite subgroup of the group $PGL(2, C)$ of fractional linear transformations. Consider the polynomial

$$P_H(x) = \prod_{g_i \in H} (x - t^{g_i}) = x^N + b_{N-1}(t)x^{N_1} + \ldots + b_1(t)x + b_0$$

where t^g is the action of fraction-linear transformation on t. This polynomial has N roots $\psi_i(t) = t^{g_i}$ Here N is number of elements in H and $b_k(t)$ are fractions on t. At least one of these fractions is not trivial (because polynomial has root $x = t$). Take one of the fractions $b_k = f(t)/g(t)$. This coefficient is some symmetric polynomial on the roots $\{\psi_i(t)\}$ hence *this is a nice fraction which has just the same roots that polynomial P_H*. Of course we could consider another coefficient, come to another nice fraction with the same roots. It is just related with Lueroth theorem: all coefficients have to be expressed via one of non-trivial coefficient. Moreover if R is a nice fraction then $\frac{aR+b}{cR+d}$ is a nice fraction which defines the same group.

If $F = f(z)/g(z)$ is the nice fraction then the fractions $F'(z) = aF(z) + b/cF(z) + d$ $(a, b, c, d \in \mathbf{C})$ and the fraction $\tilde{F}(z) = F\,(az + b/cz + d)$ are nice fraction too. The nice fractions $F, F' = aF + b/cF + d$ have the same ramification points. They define the same invariance group.

The nice fractions $F(z), \tilde{F}(z) = F\,(az + b/cz + d)$ define conjugate invariance subgroups.

Example Consider the fractions x^2 and $\frac{x+1}{x}$. They both are nice fractions with different ramification points (see the previous text). The transformation $x \to \frac{x+1}{x+1}$ transforms ramification points $[0, \infty]$ of x^2 to $[-1, 1]$. $x^2 \to (x + 1/x + 1)^2$. One can see that this function is just fractional-linear transformation of $x + 1/x$:

$$\frac{\left(x + \frac{1}{x}\right) + 2}{\left(x + \frac{1}{x}\right) - 2} = \left(\frac{x+1}{x-1}\right)$$

More tricky to show the relation between nice fractions x^3 and $\frac{x^3-3x-1}{x(x+1)}$ considered in the previous text.

But e.g. fraction $x^3/overx^3+(1-x)^3$ is the nice fraction produced from x^3 by changing ramif. points ($[0, \infty] \to [0, 1]$)

Now we go to very fundamental result belonging to Klein:

To every Platonic body corresponds finite subgroup of rational transformation and a class of nice fraction. It is obvious. One can prove that it is all!

This fact is dual to the kindergarten result: there is a finite number of Platonic bodies. There are five Platonic bodies: tetrahedra, hexahedra, octahedra, dodecahedral and icosahedra. Plus we consider also degenerated bodies: diedres (They have to sides which are regular n-gones.) The point is that not only the statement about the finitness of Platonic bodies is analogous to the statement about the finite number of groups (up to conjugancy)

Consider arbitrary Platonic body. Let every vertex is incident to r edges and every side is n-gon. If E is number of edges then Γ (number of sides) is equal to $En/2$ and V(number of vertices) is equal to $Er/2$, According to Euler identity

$$V - E + \Gamma = 2 \qquad (E1)$$

. Hence

$$\frac{1}{r} + \frac{1}{n} = \frac{1}{2} + \frac{1}{E} \qquad (E2)$$

It is very easy to solve this diophantine equation. If $r \geq 6$ then $1/r + 1/n \leq 1/2$ because $n \geq 3$. Hence $r \leq 5$. Consider cases $r = 2, 3, 4, 5$

1. $r = 5$

If $n \geq 4$ then $1/r + 1/n \leq 1/5 + 1/4 < 1/2$. Hence only $n = 3$ is allowed. If $n = 3$ then $E = 30$ we come to *icosahedra*:

$$V = 12, E = 30, \Gamma = 10$$

2. $r = 4$. If $n \geq 4$ then $1/r + 1/n \leq 1/4 + 1/4 \leq 1/2$. Hence again only $n = 3$ is allowed. If $n = 3$ then $E = 12$. We come to octahedra:

$$V = 6, E = 12, \Gamma = 8$$

3. $r = 3$. If $n \geq 6$ then $1/r + 1/n \leq 1/3 + 1/6 \leq 1/2$. Hence only $n = 3, 4, 5$ are allowed.

a) If $n = 5$ we come to *dodecahedral* (dual to icosahedra): :

$$V = 10, E = 30, \Gamma = 12$$

b) if $n = 4$ we come to *hexahedral* (dual to octahedra): :

$$V = 8, E = 12, \Gamma = 6$$

c) if $n = 3$ we come to *tetrahedra*: :

$$V = 4, E = 6, \Gamma = 4$$

4. $r = 2$. It is degenerate case; $E = n$ We come to diedre.

Solving diophantine equation is key step to prove that there are few Platonic bodies.

Now we will show that the fact that there are few finite groups in group of rational transformations is reduced to simple diophantine equation too. (Of course this can be treated pure geometrically: Instead the group of fractional transformations we can consider the group $S0(3)$ and to prove that this group has finite number of finite subgroups (so called cristollagr.....))

Let H be finite subgroup. Let $f(x)/g(x)$ be its corresponding fraction and characteristic equation:

$$\frac{f(x)}{g(x)} = \frac{f(t)}{g(t)}$$

Here I repeat up to some modifications Klein considerations:

Let $F(z)$ be a rational function. It maps **CP** on **CP**. We say that the point $z = z_0$ has an index $n = \mu(z_0|F)$ if z_0 is the zero of the order n for the function $F(z) - F(z_0)$. Evidently the index is invariant under rational fraction-linear transformations of z and $F(z)$. (In the case $z_0 = \infty$, or $F(z) = \infty$ index can be easily calculated with the help of transformation $z \to 1/z$). If the point z_0 has the index n then the derivative has zero of the order $n-1$ at the point z_0. On the other hand one can show that if a degree * $\deg F$ of the rational function F is equal to N then the degree of the derivative is equal to $2N-2$. (It is easy to see for the special gauging when for rational function $F(z)$ all points with index ≥ 2 are regular points of **CP** and its image is regular too. E.g. one can always by suitable fraction-linear transformation) Hence we come to the following relation:

for arbitrary rational function $F(z)$

$$\sum_{z\in\mathbf{CP}} [\mu(z|F(z))-1] = \sum_{z:\ F'(z)=0} [\mu(z|F'(z))] = 2\deg F - 2$$

The relation

$$\sum_{z\in\mathbf{CP}} [\mu(z|F(z))-1] = 2\deg F - 2 \qquad (E^*1)$$

can be considered as an analogue of Euler identity (E1) *Proof* Suppose that for the rational function $F(z)$ the points $z=0,\infty$ have an order 1. (You come by this condition by suitable fraction-linear transformation which does not affect the relation (E^*1)). In this case let $F=f(z)/g(z)$ where polynomials f,g and $g(z)$ has not degenerate zeros. All zeros of of the function F' are the same that for the function $fg'-gf'$. This function has the degree $N-2$.

Making suitable fraction-linear transformation of z and of F we can restrict the function F by the condition

The relation (E^81) has the following geometrical meaning:If $F(z)$: $\mathbf{CP}\to\mathbf{CP}$ is N-sheaves covering ($N=\deg F$) with ramification points $z_1,\dots,z_k$ such that ramification index at the point z_i is equal to ν_i ($\nu_i\geq 2$) then the relation $\sum_k(\nu_k-1)=2N-2$ holds. This is the special case of the so called Riemann Hourvitz formula: Let F be a complex map from complex surface $\bar{R}$ of the genre $\bar{g}$ onto complex surface R of the genre g. If F is N-sheaves covering with ramification points $z_1,\dots,z_k$ such that ramification index at the point z_i is equal to ν_i ($\nu_i\geq 2$) then the relation $\sum_k(\nu_k-1)=2N-2+g+\dots$ holds.

Now *revenons a nos moutons*

Let H be finite subgroup of the group of rational transformations of **CP**.

Consider nice fraction $F_H(z)$ corresponding to this group. Let z_0 be a point of an index $n\geq 2$, then the point $w=F(z_0)$ is ramification point. To the point z_0 corresponds stationary subgroup of the index n. Hence there are N/n points with conjugated stationary subgroups. (N is an order of group H). Note that the degree of nice fraction F_H is equal to N. We see that for nice fraction there are numbers $\{n_1,n_2,\dots\}$ such that there are for

* As always we define $\deg F=\max f,g$ if F is a ration of two polynomials f,g and these polynomials are coprime

every n_k there are N/n_k points of the index $n_k \geq 2$. Then using previous identity we come to the relation:

$$\sum_k \frac{N}{n_k}(n_k - 1) = 2N - 2$$

(Compare with relation (E2)). Dividing by N we come to

$$\sum_k \left(1 - \frac{1}{n_k}\right) = 2 - \frac{2}{N} \qquad (E^*2)$$

This is remarkable identity from Klein book which can be considered as analogue (E2) for Kleinian subgroups.

Solve this diophantine equation. Firs of all note that it follows immediately from (E^*2) that there could be only two or three ramification points! Indeed right hand side is greater than one and smaller than 2. Hence there can be only two possibilities.

I. two ramification points:

$$\frac{1}{n_1} + \frac{1}{n_2} = \frac{2}{N}$$

II. Three ramification points:

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = 1 + \frac{2}{N}$$

One can see that the first equation has only solution $n_1 = n_2 = N$. For every N nice fraction has two ramification points of degree N. Gauge this points to be $[0, \infty]$. We come to the function z^N to diedre.