Valya Ovsienko likes to say that Schwarzian has more than 600 different manifestations. Yesterday we discussed with my student Adam Biggs one of them. Here is the result of our discussions.

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Schwarzian and ... normal gauging conditions

Normal gauging is tremendously powerfull tool in geometry. E.g. the quickest way to define invariants of gauge field (connection) is to consider connection $A_\mu(x)$ in so called "normal gauge":

$$A'_\mu(x): \qquad A_\mu(x)(x^\mu - x_{_0}^\mu) = 0\,, \quad (A'_\mu = gA_\mu g^{-1} + g^{-1}\partial_\mu g)\,. \tag{1}$$

(Here A_μ takes values in the Lie algebra Lie group $\mathcal{G}$, $g(x)$ is the function in G.) Coefficients of Taylor series expansion are curvature of connection and its covariant derivatives at the point $\mathbf{p}$. (Here x^μ are local coordiantes in the vicinity of the point $\mathbf{p}$ with coordinates $x_{_0}^\mu$.) E.g. if $A_\mu(x)$ is electromagnetic field given in a vicinity of the point $x_{_0}^\mu = 0$ in normal gauge then $A_\mu(x) = F_{\mu\nu}x^\nu + \dots$ where $F_{\mu\nu}$ is the value of electromagnetic field tensor. Another example: if Riemannian metric is given in normal coordinates: $g_{\mu\nu}x^\nu = \delta_{\mu_\nu}x^\nu$ then

$$g_{\mu\nu}(x) = \delta_{\mu\nu} + \dots R_{\mu\alpha\nu\beta}(x^\alpha - x_{_0}^\alpha)(x^\beta - x_{_0}^\beta) + \dots$$

where $R_{\mu\alpha\nu\beta}$ is curvature tensor at the point $\mathbf{p}$. The normal gauging condition $g_{\mu\nu}x^\nu = \delta_{\mu_\nu}x^\nu$ is strictly related with condition $\Gamma^\alpha_{\mu\nu}x^\mu x^\nu = 0$ for geodesic coordinates. You can read about normal gauge in different textbooks *.

Now *revenons a nos moutons.* Let x be a local coordinate on projective line $P\mathbf{R}^1$ which fixes projective structure in a vicinity of the point $\mathbf{p}$: one admits the changing of coordinates $x \mapsto \frac{ax+b}{cx+d}$. Let $F(x)$ be a local expression for diffeomorphism of $P\mathbf{R}^1$.

Recall that Shwarzian equals to the following density of the weight 2:

$$\mathcal{S}^F = \left(\frac{F_{xxx}}{F_x} - \frac{3}{2}\frac{F_{xx}^2}{F_x^2}\right)|Dx|^2\,. \tag{2}$$

This is non-trivial 1-cocycle of diffeomorphisms which vanishes on projective transformations. Projective transformations have three degrees of freedom. Consider "normal" gauging of the diffeomorphism: the new diffeomorphism F' such that it difffers form F on projective transformation and F' is identity in a vicinity of $\mathbf{p}$ up to the third order terms: $F'\colon F = G \circ F'$ where $G = \frac{ax+b}{cx+d}$ is projective transformation such that

$$F'(x) = (x - x_{_0}) + O((x - x_{_0})^3), : \qquad F'\big|_{x=\mathbf{p}} = \frac{dF'(x)}{dx}\big|_{x=\mathbf{p}} = \frac{d^2F'(x)}{dx^2}\big|_{x=\mathbf{p}} = 0\,.$$

The value of gauged diffeomorphsim F' at the point $\mathbf{p}$ is the Schwarzian of F at the point $\mathbf{p}$. Calculate it.

* One can find the excellent exposition of the geometry of normal gauging in one of Appendices of the famous article: "On the heat equation and the index theorem" of M. Atiyah, R. Bott and V.K. Patodi)

If $F(x)=a+b(x-x_0)+c(x-x_0)^2+d(x-x_0)^3+o((x-x_0)^3)$ then consider composition of projective tansformations such that they "kill" derivatives: $G_1\colon x\mapsto x-a$ (translation), $G_2\colon x\mapsto x\mapsto \frac{x}{b}$ and special projective transformation $G_3\colon x\mapsto \frac{x}{1+px}$ with $p=\frac{c}{b}$. Then we come to

$$F'(x)=G_3\circ G_2\circ G_1\circ F=\frac{(x-x_0)+\frac{c}{b}(x-x_0)^2+\frac{d}{b}(x-x_0)^3+o((x-x_0)^3)}{1+p((x-x_0)+\frac{c}{b}(x-x_0)^2+o((x-x_0)^2))}=$$

$$(x-x_0)^2+d'(x-x_0)^3+o((x-x_0)^3,$$

where

$$d'=\frac{d}{b}-\frac{c^2}{b^2}=\left(\frac{6F_{xxx}}{F_x}-\frac{(2F_{xx})^2}{F_x^2}\right)\Big|_{x=x_0}=6\left(\frac{F_{xxx}}{F_x}-\frac{3}{2}\frac{F_{xx}^2}{F_x^2}\right)\Big|_{x=x_0}.$$

We come (up to a multiplier) to the Schwarzian (2). Schwarzian appears in the way as curvature of gauge field....