On one properties of discriminants

Let

$$P(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0 \qquad (1)$$

be a polynomial where a_i are indeterminants over $\mathbf{C}$.

Consider its derivative a polynomial

$$Q(x) = nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots + a_1 \qquad (2)$$

. It is convenient to consider fields

$$Ł = \mathbf{C}(a_1, a_2, \ldots, a_{n-1}) \quad \text{and} \qquad K = Ł(a_0) = \mathbf{C}(a_1, a_2, \ldots, a_{n-1}, a_0) \qquad (2)$$

Consider also *discriminant* of the polynomial $P \in K[x]$ (1) i.e. resultant of the polynomials $P(x)$ and $Q(x)$ over the field K (polynomial $Q \in L[x] \subseteq K[x]$). If $\mu_i, (i = 1, \ldots, n)$—roots of the polynomial $P(x)$ then

$$D = \prod_{i \neq j} (\mu_i - \mu_j)$$

Discriminant is a resultant of the polynomial and its derivative Q: if δ_α $(\alpha = 1, \ldots, n-1)$ are roots of derivative Q then up to a coefficient

$$D(P) = R(P, Q) = \prod_{i,\alpha} (\mu_i - \delta_\alpha) = \prod_\alpha P(\delta_\alpha) = \prod_\alpha Q(\mu_i)$$

One can see that discriminant is a polynomial of degree $n-1$ with respect to indeterminant a_0. It is convenient later to denote the indeterminant a_0 by the letter y and consider *discriminant* as a polynomial with respect to $y = a_0$ over the field L:

$$D = d(y) = \prod_\alpha P(\delta_\alpha) = \left(y + a_1\delta_1 + \ldots + a_{n-1}\delta_1^{n-1} + \delta_1^n\right) \cdot \ldots \cdot \left(y + a_1\delta_1 + \ldots + a_{n-1}\delta_1^{n-1} + \delta_1^n\right) =$$

$$y^{n-1} + \ldots + (-1)^? a_1^n$$

Polynomial $d(y)$ is an irreducible polynomial over field L as well as derivative polynomial $Q(x)$

Consider polynomial

$$G(x) = -\int_0^x Q(u)du = y - P(x)$$

If

$$y = y_i = F(\delta_i)$$

then δ_i is a root of the polynomial $P(x)$. Hence polynomial P and its derivative has joint root, i.e.

$$d(y_i) = 0 \quad \text{if} \quad y_i = F(\delta_i)$$

We see that y_i belongs to the field $L(\delta_i)$. $D(y)$ is irreducible, hence there exist a polynomial $F(y)$ such that

$$\delta_i = F(y_i)$$

Proposition.

Fields $Ł_D = Ł[x]/(Q(x))$ and $Ł_Q = Ł_D = Ł[y]/(Q(y))$ are isomorphic, i.e. adding to the field Ł one of the roots δ_i of derivative polynomial $\mathbf{Q}(x)$ leads to the same field as adding to the Ł a number $y_i = -P_0(\delta_i)$.

It follows from the Proposition that

$$P(x, y) = (x - F(y))^2 P_{n-2} \quad \text{projected on the field } Ł_D, i.e. \qquad (main)$$

or in the other words

Let $I =< P(x, y), d(y) >$ be an ideal generated by polynomials P, d.

Polynomial $F(y))$ considered above is defined modulo $D(y)$, respectively polynomial $G(x)$ is defined modulo $Q(x)$. We can take their degree $\leq n-1$.