Chapter 3

Wallpaper patterns

Wallpaper patterns are patterns that repeat infinitely in two independent directions. They have been used over many centuries by artists creating mosaics, for example in the Alhambra, the famous Islamic palace in Granada (Spain).

However, mosaics are not the only occurrence of such patterns: they occur in many physical systems, including stripes on tigers and zebras, spots on leopards, cells in beehives, and patterns in sand dunes and in chemical reactions.

The aim of this chapter is to discuss and describe the symmetries of these patterns. The fact that the pattern repeats in 2 independent directions means that the set of translational symmetries of a wallpaper pattern forms what's called a *lattice*, so we begin by studying these.

3.1 Lattices and their symmetries

Definition 3.1. A ***lattice*** in $\mathbb{R}^n$ is a subset of the form

$$L = \{m_1\mathbf{a}_1 + m_2\mathbf{a}_2 + \cdots + m_n\mathbf{a}_n \mid m_i \in \mathbb{Z} \text{ for all } i = 1, \ldots, n\}$$

where $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is a basis of $\mathbb{R}^n$. One writes $\mathbb{Z}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ for this set, and we say it is the lattice ***generated by*** $\mathbf{a}_1, \ldots, \mathbf{a}_n$. ☆

It is easy to check that a lattice is a subgroup of $\mathbb{R}^n$; in particular putting all $m_i = 0$ shows $0 \in L$ (and 0 is the identity element of the group $\mathbb{R}^n$).

In dimension 1 ($n = 1$), a lattice is just $\mathbb{Z}\{a\} = \{m\,a \mid m \in \mathbb{Z}\}$ for some $a \neq 0$ and looks like this: (one of these points is 0, then $\pm a$, $\pm 2a$ etc).

From now on we just consider lattices in the plane. One 2-dimensional example is shown in Fig. 3.1.

Question: What symmetries can a lattice have? The easy observation is that the translations of a lattice L are precisely the elements of L themselves. Also a rotation

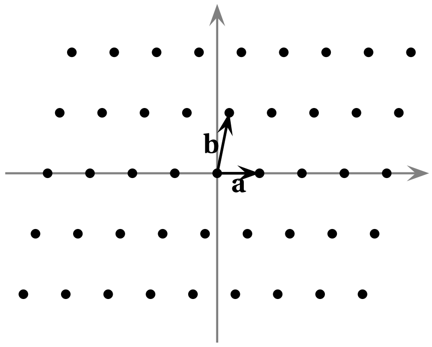


FIGURE 3.1: The lattice in the plane generated by vectors **a** and **b**

by π about the origin is always a symmetry of any lattice. On the other hand, which other rotations and (glide) reflections are symmetries of L varies from one lattice to another. To begin with, there is one important restriction:

Theorem 3.2 (Crystallographic restriction theorem). *Let $L \subset \mathbb{R}^2$ be a lattice and suppose R is a rotation preserving L. Then R has order $2, 3, 4$ or 6.*

In other words, R is a rotation through an integer multiple of $\frac{\pi}{2}$ or $\frac{\pi}{3}$.

Proof: Choose $\mathbf{a}$ to be a shortest non-zero vector in L, and suppose first that R has order $k \geq 7$. Then R (or some power of it) is $R_{2\pi/k}$. Then $R_{2\pi/k}\mathbf{a} \in L$ as L is preserved by R. Consequently, $\mathbf{u} := R_{2\pi/k}\mathbf{a} - \mathbf{a} \in L$ and satisfies

$$\begin{aligned} |\mathbf{u}|^2 &= |R_{2\pi/k}\mathbf{a} - \mathbf{a}|^2 \\ &= |R_{2\pi/k}\mathbf{a}|^2 - 2\,(R_{2\pi/k}\mathbf{a}) \cdot \mathbf{a} + |\mathbf{a}|^2 \\ &= 2|\mathbf{a}|^2 - 2|\mathbf{a}|^2 \cos(2\pi/k) \\ &= 2|\mathbf{a}|^2\,(1 - \cos(2\pi/k)). \end{aligned}$$

If $k > 6$, then $\cos(2\pi/k) > 1/2$ and thus $|\mathbf{u}|^2 < |\mathbf{a}|^2$. It follows that $|\mathbf{u}| < |\mathbf{a}|$ which contradicts the minimal length property of $\mathbf{a}$.

If $k = 5$, we repeat this argument using instead $R_{4\pi/5} = R^2_{2\pi/5}$. Setting $\mathbf{v} := R_{4\pi/5}\mathbf{a} + \mathbf{a}$ which must also be in L, we find $|\mathbf{v}| < |\mathbf{a}|$ leading to a similar contradiction. ❐

We are now ready to state the 5 possible symmetry groups of lattices. In order to distinguish them it is useful to choose $\mathbf{a}$ and $\mathbf{b}$ according to the following convention:

- choose $\mathbf{a}$ so that no other non-zero vector in L is shorter than $\mathbf{a}$ (as before), and
- choose $\mathbf{b}$ to be the (or a) smallest vector in L which is linearly independent of $\mathbf{a}$; note that if $\mathbf{b}$ satsifes this, then so does $-\mathbf{b}$, and out of these two we select the one that makes $|\mathbf{a} - \mathbf{b}| \leq |\mathbf{a} + \mathbf{b}|$.

It follows then that

$$|\mathbf{a}| \le |\mathbf{b}| \le |\mathbf{a}-\mathbf{b}| \le |\mathbf{a}+\mathbf{b}|$$

and the different symmetry types correspond to which of these inequalities are equalities (it is easy to see they can't all four be equal):

Theorem 3.3. *There are 5 types of lattice, classified according to their symmetry group. These are, with* $\mathbf{a}$ *and* $\mathbf{b}$ *chosen as described,*

Oblique	$\lvert\mathbf{a}\rvert < \lvert\mathbf{b}\rvert < \lvert\mathbf{a}-\mathbf{b}\rvert < \lvert\mathbf{a}+\mathbf{b}\rvert$,
Rectangular	$\lvert\mathbf{a}\rvert < \lvert\mathbf{b}\rvert < \lvert\mathbf{a}-\mathbf{b}\rvert = \lvert\mathbf{a}+\mathbf{b}\rvert$,
Centred rectangular	$\lvert\mathbf{a}\rvert < \lvert\mathbf{b}\rvert = \lvert\mathbf{a}-\mathbf{b}\rvert < \lvert\mathbf{a}+\mathbf{b}\rvert$,
"	$\lvert\mathbf{a}\rvert = \lvert\mathbf{b}\rvert < \lvert\mathbf{a}-\mathbf{b}\rvert < \lvert\mathbf{a}+\mathbf{b}\rvert$,
Square	$\lvert\mathbf{a}\rvert = \lvert\mathbf{b}\rvert < \lvert\mathbf{a}-\mathbf{b}\rvert = \lvert\mathbf{a}+\mathbf{b}\rvert$,
Hexagonal	$\lvert\mathbf{a}\rvert = \lvert\mathbf{b}\rvert = \lvert\mathbf{a}-\mathbf{b}\rvert < \lvert\mathbf{a}+\mathbf{b}\rvert$.

Notice that the centred rectangular lattice has two possibilities: this is explained in Remark 3.5 below. The hexagonal lattice is sometimes called a triangular lattice, and the centred rectangular is sometimes called a rhombic lattice. See Figure 3.2 for illustrations.

Continuing with the convention for choosing $\mathbf{a}$ and $\mathbf{b}$ described above, we now describe the symmetries of each lattice. For any lattice L, write $\mathcal{W}_L \le \mathsf{E}(2)$ for the symmetry group of L.

For the following definition, consider the homomorphism $\pi : \mathsf{E}(2) \to \mathsf{O}(2)$ defined by

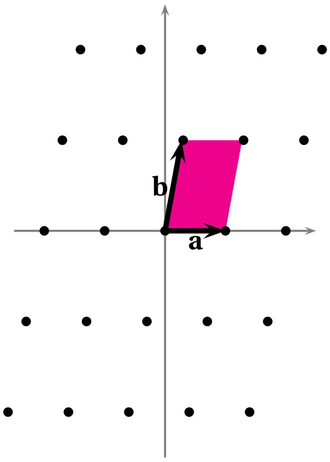
$$\pi(A \mid \mathbf{v}) = A.$$

Note that $\ker\pi = \mathbb{R}^2$, the subgroup of translations.

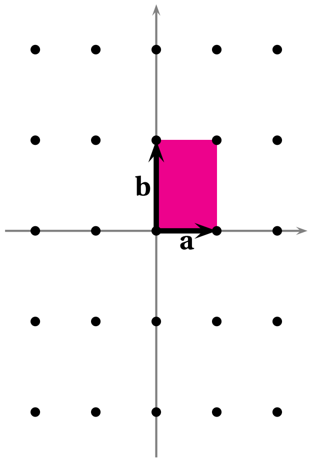
Definition 3.4. Given any subgroup $G \le \mathsf{E}(2)$, then $\ker(\pi|_G)$ (equivalently $G \cap \mathbb{R}^2$) is called the ***translation subgroup*** of G, and the image $\pi(G) \subset \mathsf{O}(2)$ is called the ***point group*** of G, and usually denoted J or J_G. ☆

In symbols, $(A|\mathbf{u}) \in G$ implies $A \in J_G$. The point group thus involves ignoring any translational components of a group; for example, a glide reflection becomes just the corresponding reflection. As we have pointed out before, the translation subgroup of $\mathcal{W}_L$ is just L. We therefore discuss the other symmetries below.

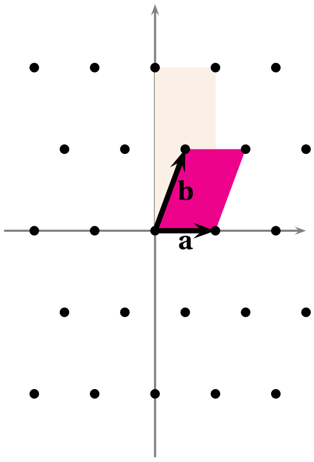
- For the *oblique* lattice (see Figure 3.2), the only symmetries are the half-turns (rotations by π) about different points. And there are 4 different centres of rotation: the points of the lattice $m\mathbf{a} + n\mathbf{b}$, the centres of the parallelograms formed from $\mathbf{a}$ and $\mathbf{b}$, namely $(m+1/2)\mathbf{a}+(n+1/2)\mathbf{b}$; the centres of the horizontal lines $(m+1/2)\mathbf{a}+n\mathbf{b}$ and finally the centres of the skew lines $m\mathbf{a}+(n+1/2)\mathbf{b}$. These are all rotations by π (they all differ by translations, that is, are of the form $(R_\pi \mid \mathbf{v}) \in \mathsf{E}(2)$), and therefore the point group is $J = \langle R_\pi\rangle = \mathsf{C}_2$. Note that the 4



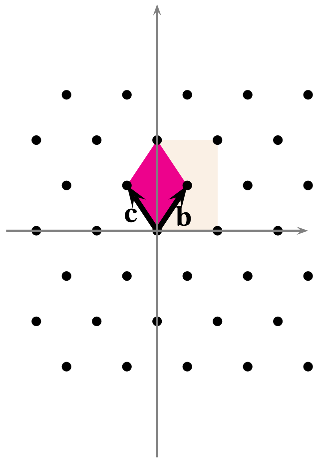
(a) Oblique



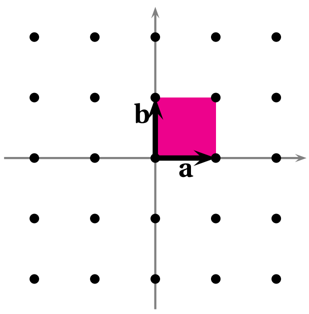
(b) Rectangular



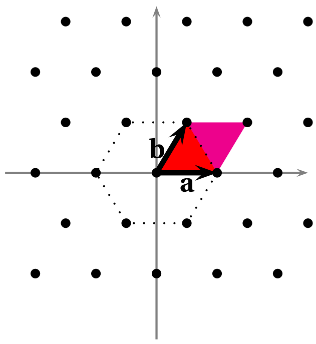
(c) Centred rectangular



(c′) Centred rectangular as a rhombus



(d) Square



(e) Hexagonal

FIGURE 3.2: The 5 types of planar lattice

types of centre of rotation are of distinct type, where by *distinct types* we mean that each cannot be transformed into another by the lattice symmetries.

- ▹ For the *rectangular* lattice there are the same rotations as for the oblique lattice, and in addition reflections in horizontal and vertical mirrors. Thus $J = \{I, R_\pi, r_0, r_{\pi/2}\} = \mathsf{D}_2$.
- ▹ For the *centred rectangular* lattice, the point group is also $J = \mathsf{D}_2$.
- ▹ For a *square* lattice, $J = \mathsf{D}_4$.
- ▹ For a *hexagonal* lattice, $J = \mathsf{D}_6$.

Remarks 3.5 (On centered rectangular lattices). (1) Centred rectangular lattices are often called *rhombic lattices*. Which you call it depends on your 'point of view', or more precisely which generators you choose. For example, in Fig. 3.2(c), the lattice is generated by $\{\mathbf{a},\mathbf{b}\}$ as shown, but also by $\{\mathbf{b},\mathbf{c}\}$ where $\mathbf{c} = \mathbf{b}-\mathbf{a}$. The vectors $\mathbf{a}$ and $2\mathbf{b}-\mathbf{a}$ span a rectangle (shaded in the figure), with the vector $\mathbf{b}$ in the centre (hence 'centred rectangular lattice'). On the other hand, $\mathbf{b}$ and $\mathbf{c}$ define two sides of a rhombus (as illustrated in Fig. 3.2(c′)). Note that there are some cases where $\mathbf{a},\mathbf{b}$ are the shortest vectors, and some where $\mathbf{b},\mathbf{c}$ are, depending on the angle between $\mathbf{a}$ and $\mathbf{b}$, but this does not effect the symmetry of the lattice.

(2) Since the point groups of the rectangular and centred rectangular lattice are both D_2, some explanation of why they are distinguished is needed. Denote by L_1 a rectangular lattice and by L_2 a centred rectangular lattice. In spite of having isomorphic translation group and the same point group, $\mathcal{W}_{L_1}$ and $\mathcal{W}_{L_2}$ are *not* isomorphic, because the point group is acting differently on the lattice in the two cases, as shown in the following table (compare with Fig. 3.2):

Rectangular		Centered rectangular	
r_0	$r_{\pi/2}$	r_0	$r_{\pi/2}$
$\mathbf{a}\mapsto\mathbf{a}$	$\mathbf{a}\mapsto-\mathbf{a}$	$\mathbf{a}\mapsto\mathbf{a}$	$\mathbf{a}\mapsto-\mathbf{a}$
$\mathbf{b}\mapsto-\mathbf{b}$	$\mathbf{b}\mapsto\mathbf{b}$	$\mathbf{b}\mapsto\mathbf{a}-\mathbf{b}$	$\mathbf{b}\mapsto\mathbf{b}-\mathbf{a}$

❞

Notice that the elements of J in these cases act on the lattice L (permute its elements): this is a general fact we see for any subgroup of $\mathsf{E}(2)$ (see Proposition 3.6 below).

One immediate property of the symmetry group of a lattice is the following:
if $(A\mid\mathbf{v})\in\mathcal{W}_L$ *then* $\mathbf{v}\in L$ *and* $(A\mid\mathbf{0})\in\mathcal{W}_L$.
This is because, firstly $\mathbf{v} = (A\mid\mathbf{v})\cdot\mathbf{0}$, and $\mathbf{0}\in L$ and hence $\mathbf{v}\in L$. Secondly, since $\mathbf{v}\in L$ implies $(I\mid-\mathbf{v})\in\mathcal{W}_L$, it follows that $(A\mid\mathbf{0}) = (I\mid-\mathbf{v})(A\mid\mathbf{v})\in L$ (this product is readily checked).

Proposition 3.6. *Let G be any subgroup of $\mathsf{E}(2)$. The point group J_G acts on its translation subgroup L_G.*

Since the point group is the image in $\mathsf{O}(2)$ of G, if $A \in J_G$ then there is a $\mathbf{v} \in \mathbb{R}^2$ for which $(A \mid \mathbf{v}) \in G$. The very concise statement of the proposition is therefore saying that, if $(A \mid \mathbf{v}) \in G$, then $A\mathbf{u} \in L_G$ for all $\mathbf{u} \in L_G$. This follows from properties of groups and normal subgroups (see Problem 3.18), but we give a direct proof. Recall first that a vector $\mathbf{u} \in L_G$ if and only if $(I \mid \mathbf{u}) \in G$ (property of the kernel of π).

Proof: Suppose $(A \mid \mathbf{v}) \in G$ and let $\mathbf{u} \in L$. Then $(I \mid \mathbf{u}) \in G$. Now use conjugation on G:

$$\begin{aligned}
(A \mid \mathbf{v})\,(I \mid \mathbf{u})\,(A \mid \mathbf{v})^{-1} &= (A \mid \mathbf{v})\,(I \mid \mathbf{u})\,(A^{-1} \mid -A^{-1}\mathbf{v}) \\
&= (A \mid \mathbf{v} + A\mathbf{u})(A^{-1} \mid -A^{-1}\mathbf{v}) \\
&= (AA^{-1} \mid \mathbf{v} + A\mathbf{u} + A(-A^{-1}\mathbf{v})) \\
&= (I \mid \mathbf{v} + A\mathbf{u} - \mathbf{v}) \\
&= (I \mid A\mathbf{u}).
\end{aligned}$$

We conclude that indeed $(I \mid A\mathbf{u}) \in G$, and hence $A\mathbf{u} \in L_G$ as required. ❒

3.2 Wallpaper groups

Definition 3.7. A ***wallpaper group*** $\mathcal{W}$ is a subgroup of $\mathsf{E}(2)$ with the following properties:

(1). its translation subgroup $L = \mathcal{W} \cap \mathbb{R}^2$ is a lattice.

(2). its point group $J = J_{\mathcal{W}}$ is finite.

☆

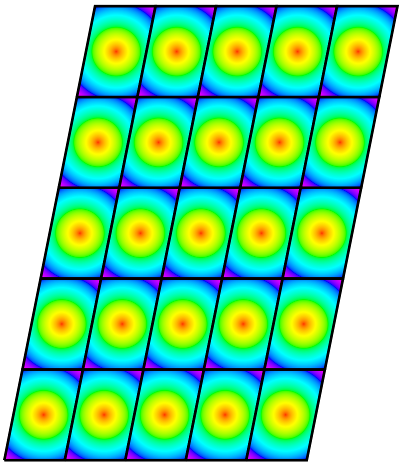
It follows that the point group of a wallpaper group is either C_n or D_n (for some $n \geq 1$ — see Theorem 2.9). It then follows from the Crystallographic Restriction Theorem (Theorem 3.2) that the only possibilities are $n = 1, 2, 3, 4$ or 6.

Theorem 3.8. *There are 17 distinct wallpaper groups.*

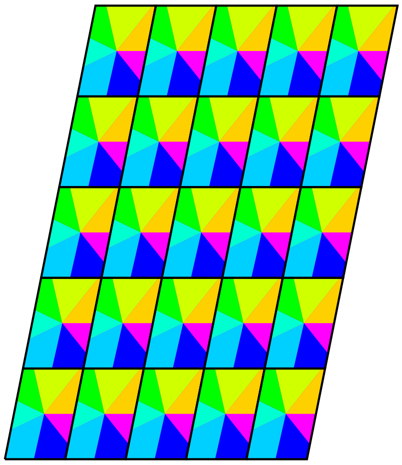
Here 'distinct' means that no two in the list are isomorphic as abstract groups.

Outline proof: Let $\mathcal{W}$ be a wallpaper group with lattice L and point group J. Then L is one of the 5 types of lattices of Theorem 3.3. Now, from Proposition 3.6, J acts on L. Let J_L be the point group of $\mathcal{W}_L$. It follows that $J \leq J_L$. Now, go through each type of lattice and enumerate all possible subgroups of J_L. This gives the classification. (There are one or two subtleties to take into account, such as glide-reflections.)

As an example, suppose $L = \mathbb{Z}\{\mathbf{a}, \mathbf{b}\}$ is an oblique lattice. Then, $J_L = \mathsf{C}_2 = \{I, R_\pi\}$. If $\mathcal{W}$ is such that its associated lattice L is oblique, then $J \leq J_L$ implies that



(a) $J = J_L = C_2$
(p2—with rotational symmetry)



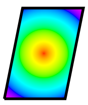
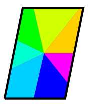
(b) $J = \mathbb{1}$
(p1—with no rotational symmetry)

FIGURE 3.3: The two wallpaper patterns with an oblique lattice

$J = C_2$ or $J = \mathbb{1}$ (see Fig. 3.3, where the colouring of (b) has broken the rotational symmetry).

One then has to eliminate repeated cases. For example, whatever the type of lattice (oblique, hexagonal etc) if the point group is trivial $J = \mathbb{1}$ then the wallpaper group is just the translation group so $\mathcal{W} \simeq \mathbb{Z}^2$, and information on the type of lattice is irrelevant to the symmetry. ❐

To make a pattern with a given wallpaper group $\mathcal{W}$ of symmetries, first draw its lattice. Then decorate one tile (parallelogram, rectangle, square, or hexagon) so that the decoration has the symmetries required and no more. This must then be repeated in each tile. In Figure 3.3(a) the pattern in the basic parallelogram must have 180° symmetry while in (b) (where $J = \mathbb{1}$) it must not. Examples are shown below, but of course many other patterns work too.

(a)  (b) 

Notations There are two standard notations for the 17 wallpaper groups, the (classical) crystallographic notation and the (newer) orbifold notation. The symbol in the orbifold notation is called the ***signature*** of the pattern.

TABLE 3.1: The 17 different wallpaper groups

Lattice type:	**oblique**		**rectangular**					**cent. rect.**	
Crystallographic	p1	p2	pm	pg	pmm	pmg	pgg	cm	cmm
Orbifold signature	o	2222	$**$	$\times\times$	$*2222$	$22*$	$22\times$	$*\times$	$2*22$
Point group	$\mathbb{1}$	C_2	D_1	D_1	D_2	D_2	D_2	D_1	D_2

Lattice type:	**square**			**hexagonal**				
Crystallographic	p4	p4m	p4g	p3	p3m1	p31m	p6	p6m
Orbifold signature	442	$*442$	$4*2$	333	$*333$	$3*3$	632	$*632$
Point group	C_4	D_4	D_4	C_3	D_3	D_3	C_6	D_6

TABLE 3.2: Generators for each of the 17 wallpaper groups.

o	$\{T_{\mathbf{a}}, T_{\mathbf{b}}\}$	442	$\{T_{\mathbf{e}_1}, R_{\pi/2}\}$
2222	$\{T_{\mathbf{a}}, T_{\mathbf{b}}, R_\pi\}$	$*442$	$\{T_{\mathbf{e}_1}, R_{\pi/2}, r_0\}$
$**$	$\{T_{a\mathbf{e}_1}, T_{b\mathbf{e}_2}, r_0\}$	$4*2$	$\{(R_{\pi/2} \mid \mathbf{e}_1), r_0\}$
$\times\times$	$\{T_{a\mathbf{e}_1}, T_{b\mathbf{e}_2}, (r_0 \mid \frac{a}{2}\mathbf{e}_1)\}$	333	$\{T_{\mathbf{e}_1}, R_{2\pi/3}\}$
$*2222$	$\{T_{a\mathbf{e}_1}, T_{b\mathbf{e}_2}, R_\pi, r_0\}$	$*333$	$\{T_{\mathbf{e}_2}, R_{2\pi/3}, r_0\}$
$22*$	$\{T_{a\mathbf{e}_1}, T_{b\mathbf{e}_2}, R_\pi, (r_0 \mid b\mathbf{e}_2)\}$	$3*3$	$\{T_{\mathbf{e}_1}, R_{2\pi/3}, r_0\}$
$22\times$	$\{T_{a\mathbf{e}_1}, T_{b\mathbf{e}_2}, R_\pi, , (r_0 \mid \mathbf{u}), (r_{\pi/2} \mid \mathbf{u})\}$	632	$\{T_{\mathbf{e}_1}, R_{\pi/3}\}$
$*\times$	$\{T_{2a\mathbf{e}_1}, T_{a\mathbf{e}_1+b\mathbf{e}_2}, r_0\}$	$*632$	$\{T_{\mathbf{e}_1}, R_{\pi/3}, r_0\}$
$2*22$	$\{T_{2a\mathbf{e}_1}, T_{a\mathbf{e}_1+b\mathbf{e}_2}, R_\pi, r_0\}$		

Here we write translations and elements of $O(2)$ when possible, otherwise we use the Seitz notation. The translation part in $22\times$ is $\mathbf{u} = \frac{1}{2}(a\mathbf{e}_1 + b\mathbf{e}_2)$. We do not claim that these are a minimal set of generators (and often they are not). For $4*2$, see Example 3.9 and Problem 3.15.

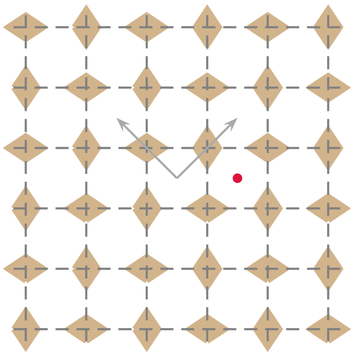
To understand these notations, I recommend reading the wikipedia page on Wallpaper Groups. (The orbifold notation is sometimes called the Thurston-Conway notation, after William Thurston and John Conway.)

Briefly, in the orbifold notation, the integers refer to centres of rotation; thus in 442, each square contains two inequivalent centres of 4-fold rotation and one of 2-fold rotation. The $*$ refers to there being a reflection , and the $\times$ to a glide reflection. The difference between, say, $*3$ and $3*$ is that $3*$ means there is a centre of 3-fold rotation and a line of reflection but the centre of rotation does not lie on the line of reflection, while in $*3$ the centre of rotation does lie on the line of reflection.

Example 3.9 (p4g $= 4{*}2$).

Here we consider in detail the pattern of lozenge shapes shown on the right which we will see has wallpaper group p4g or orbifold symbol $4{*}2$. In this orbifold notation, the number 4 is used to say that a pattern has 4-fold rotational symmetry. The $*$ tells us that a pattern has lines of reflection, and its appearance after the 4 means that the centre of rotation does not lie on a line of reflection. Finally, the 2 means there are centres of 2-fold rotation, and these do lie on the lines of reflection.

Let us look in detail at the symmetries of the pattern.



$\mathcal{W}$ is p4g, with $J = \mathrm{D}_4$

In the figure on the left, the vertical and horizontal dashed lines are clearly lines of reflection. Let us say 1 unit is the side of the squares formed by the reflection lines.

The lattice of translations is generated by the two vectors shown in the figure, which are equal to $(1,1)$ and $(1,-1)$ (they can of course be placed anywhere in the diagram).

The centres of 4-fold rotation do not lie on the lines of reflection (they lie mid-way between, at the centres of the squares, such as at the red dot), and hence in the orbifold notation, the $*$ is after the 4, not before. There are also centres of 2-fold symmetry (half-turns) at the centre of each lozenge, where the lines of reflection intersect. This justifies the orbifold symbol.

However there are other symmetries which follow from the ones we have described. Note that the point group J contains the horizontal and vertical reflections r_0 and $r_{\pi/2}$, and the rotation $R_{\pi/2}$ (and its powers). The group generated by these is D_4, and so there must also be 'diagonal' reflections $r_{\pm\pi/4}$ in the point group, and these could arise either from reflections or from glide-reflections. However, from the figure one can see there are no pure reflections of this form, but there are glide reflections (whence the g in p4g). In particular, if we take any diagonal line through the centre of a lozenge, the reflection makes every vertical lozenge horizontal and *vice versa*, so composing this with a horizontal or vertical translation by 1 unit restores the orientation of the lozenges and hence is a symmetry.

There are other glide reflections in the symmetry group. They are given by reflecting in any line mid-way between two lines of reflection followed by a translation by 1 unit parallel to the line. The Seitz symbol is, for example, $(r_0 \mid (1,1)^T)$, where the origin is chosen at the centre of a lozenge.

3.3 Problems

3.1 Suppose $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors. Show that they are orthogonal if and only if $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}|$.

3.2 Consider the lattice $L = \mathbb{Z}^2$. Show that L can be generated by the vectors $\mathbf{a} = (7,3)$ and $\mathbf{b} = (9,4)$.

3.3 Extending the previous problem, show that $L = \mathbb{Z}^2$ is generated by integer vectors (a,b) and (c,d) whenever $ad-bc = \pm 1$. [Hint: Consider the matrix $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and show the inverse matrix has integer entries iff $\det A = \pm 1$.]

3.4 Let $L = \{(x,y) \in \mathbb{R}^2 \mid y \in \mathbb{Z}, \sqrt{2}(x-y) \in \mathbb{Z}\}$. First show L is a subgroup of $\mathbb{R}^2$ (under vector addition). Second show that

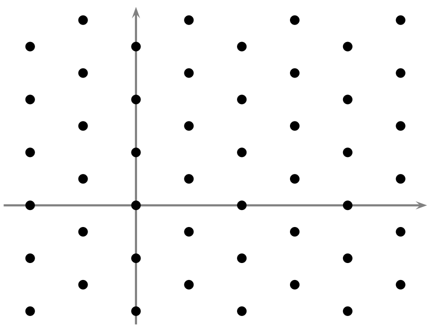
$$L = \left\{ \begin{pmatrix} a + b\sqrt{2} \\ a \end{pmatrix} \in \mathbb{R}^2 \mid a, b \in \mathbb{Z} \right\},$$

and hence find two generators of L and deduce that it is a lattice.

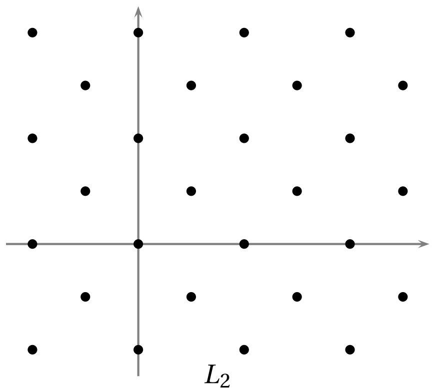
3.5 Consider the two lattices in $\mathbb{R}^2$ defined by,

$$L_1 = \{(2m+n, \tfrac{1}{2}n) \mid m,n \in \mathbb{Z}\} \quad \text{and} \quad L_2 = \{(2m+n, n) \mid m,n \in \mathbb{Z}\}$$

shown in the figures below. In each case, determine vectors $\mathbf{a}, \mathbf{b}$ according to the conventions, and find the point group. Describe how the point group acts on the lattice. Which of the 5 types of lattice is each of these?



L_1



L_2

3.6 Let S be the subset of $\mathbb{R}^2$ consisting of points $(3n+1, 4m-2)$ (with $m, n \in \mathbb{Z}$). Find the set (group) of translations of $\mathbb{R}^2$ preserving the set S; that is, find

$$L = \left\{\mathbf{v} \in \mathbb{R}^2 \mid \mathbf{x}+\mathbf{v} \in S \ \ \forall \mathbf{x} \in S\right\}.$$

[Hint: Let $T_{\mathbf{v}}(\mathbf{x}) = \mathbf{y}$. Then $\mathbf{y} = \mathbf{x}+\mathbf{v} \in S$, or $\mathbf{v} = \mathbf{y}-\mathbf{x}$ (with $\mathbf{x}, \mathbf{y} \in S$).]

3.7 [From 2016 exam] Consider the following lattice in the plane:

$$L = \left\{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{Z},\ y \in 2\mathbb{Z},\ x + \tfrac{1}{2}y \in 2\mathbb{Z}\right\}.$$

(i). Show that the two vectors $\mathbf{u}_1 = (2,0)$ and $\mathbf{u}_2 = (0,4)$ both belong to L, and sketch a diagram showing all the points of L that lie in the rectangle $0 \le x \le 6$ and $0 \le y \le 8$. Deduce that $L \neq \mathbb{Z}\{\mathbf{u}_1, \mathbf{u}_2\}$.

(ii). Find two vectors $\mathbf{a}$ and $\mathbf{b}$ such that $L = \mathbb{Z}\{\mathbf{a}, \mathbf{b}\}$, proving carefully that this is the case.

(iii). Show that the point group of the lattice L has order 4 by finding appropriate elements of its symmetry group $\mathcal{W}_L < \mathsf{E}(2)$, expressed in the form $(A \mid \mathbf{v})$, and state why it is Abelian.

3.8 Let $u > 0$ and consider the 1-dimensional lattice $\mathbb{Z}\{u\}$. Show that the infinite dihedral group $\mathsf{Dih}(\infty)$ (see appendix) acts on this lattice, via

$$a \cdot x = -x, \quad \text{and} \quad b \cdot x = u - x.$$

(You need to show that these two transformations do indeed preserve L and that they satisfy the relations defining $\mathsf{Dih}(\infty)$).)

3.9 Consider the planar lattice $L = \mathbb{Z}\left\{\begin{pmatrix}3\\0\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}\right\}$. Show this is an oblique lattice, and by choosing two appropriate elements, show that it contains a subset which is a rectangular lattice.

3.10 Which of the 5 types of lattice is $L = \mathbb{Z}\left\{\begin{pmatrix}2\\0\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}\right\}$?

3.11 Let $\mathbf{a}, \mathbf{b}$ be two perpendicular vectors of different lengths in $\mathbb{R}^2$, say $|\mathbf{b}| > |\mathbf{a}| > 0$, and let $L = \mathbb{Z}\{\mathbf{a}, \mathbf{b}\}$ be the resulting 'rectangular' lattice. Let $\mathbf{r} \in \mathsf{E}(2)$ be any of the reflections that preserve L. Show that there is $\mathbf{v} \in L$ such that either $r = (r_0 \mid \mathbf{v})$ or $r = (r_{\pi/2} \mid \mathbf{v})$. Deduce that the group $\mathcal{W}_L$ of all symmetries of this lattice is generated by $\{\mathbf{a}, \mathbf{b}, R_\pi, r_0\}$ (why is $r_{\pi/2}$ not needed?).

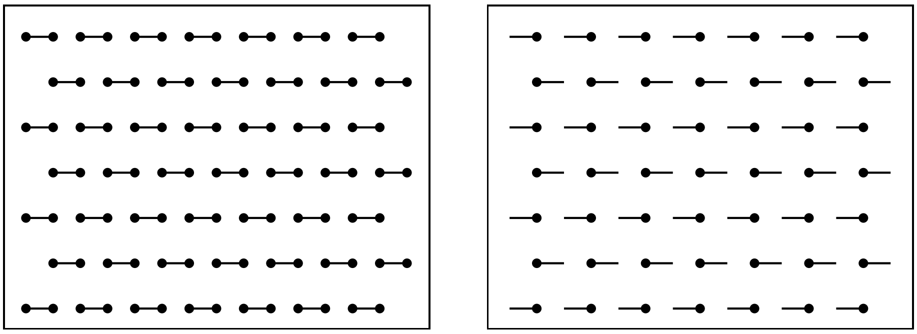
3.12 Let $L = \mathbb{Z}\{\mathbf{a}, \mathbf{b}\}$ be any lattice in the plane. There are many possible centres of symmetry: points $\mathbf{c}$ for which a rotation by π about $\mathbf{c}$ (denoted $h(\mathbf{c})$ in Problem 2.14) is a symmetry of the lattice.

(i). Show that $\mathbf{c}_1 = \frac{1}{2}\mathbf{a}$ and $\mathbf{c}_2 = \frac{1}{2}(\mathbf{a}+\mathbf{b})$ are two such points.

(ii). Show that $h(\mathbf{c})$ preserves the lattice if and only if $\mathbf{c} \in \frac{1}{2}L$, where $\frac{1}{2}L$ is the lattice

$$\tfrac{1}{2}L = \{\mathbf{u} \in \mathbb{R}^2 \mid 2\mathbf{u} \in L\}.$$

3.13 For each of the following wallpaper patterns, draw generators of the translation lattice and find the point group. Finally determine which of the 17 wallpaper groups it is.



3.14 Consider the functions of two variables,

$$f(x,y) = \sin(x) + \sin(y) \quad \text{and} \quad g(x,y) = \sin(x) - 2\sin\left(\tfrac{1}{2}x\right)\cos\left(\tfrac{\sqrt{3}}{2}y\right).$$

The contours of f and g are shown in Figure 3.4: the lighter, or green, regions are where the function takes positive values and the darker (violet) ones are where the function is negative. Let $\mathcal{W}_f$ and $\mathcal{W}_g$ be their symmetry groups (wallpaper groups).

(a) In each case, find the translation subgroup of $\mathcal{W}$. Which of the 5 types of lattice is this translation subgroup?

(b) Find the point groups J_f and J_g (first just by looking at the diagrams, and then check that these transformations do indeed preserve the function in question).

(c) How is this changed if we allow transformations that change f to $-f$ and g to $-g$? More formally, find the stabilizer of each function under the action of $G = \mathsf{E}(2) \times \mathbb{Z}_2$, where $\mathbb{Z}_2 = \{1, -1\}$ and $(T, s) \cdot f = s f \circ T^{-1}$, for $T \in \mathsf{E}(2)$ and $s \in \{\pm 1\}$.

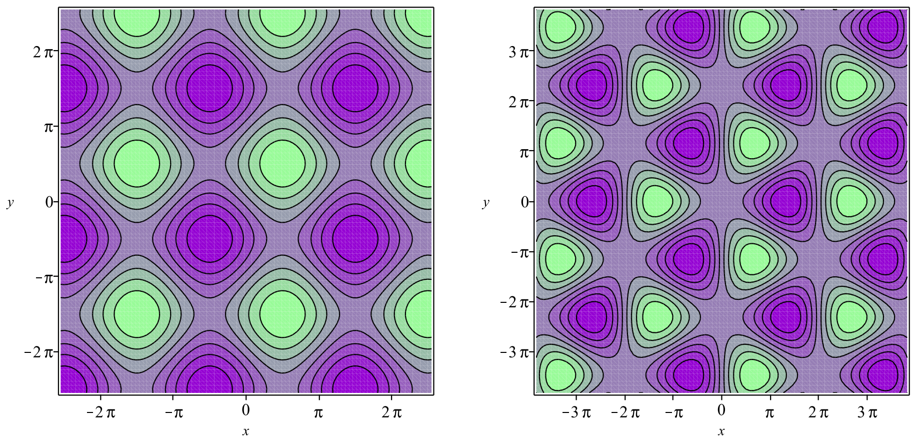


FIGURE 3.4: See Problem 3.14. The left-hand figure shows the contours of the function f, the right-hand one the contours of g

3.15 Refer to Example 3.9, and choose the origin to be at the centre of one of the lozenges. Here we discuss how the group of symmetries is generated. Show that each of the following Euclidean transformations are in the symmetry group:

$$(R_{\pi/2} \mid \mathbf{e}_1), \quad (r_0 \mid \mathbf{0})$$

where $\mathbf{e}_1 = (1,0)^T$.
(a) Show that the product (composite) $g = (R_{\pi/2} \mid \mathbf{e}_1)(r_0 \mid \mathbf{0})$ is a glide-reflection, and find the line of reflection.
(b) Show that g^2 is one of the vectors that generate the lattice of translations.
(c) Show the other generator of that lattice is the square of the 'reverse' product

$$k = (r_0 \mid \mathbf{0})(R_{\pi/2} \mid \mathbf{e}_1).$$

(d) Conclude that the wallpaper group for this pattern is generated by $(R_{\pi/2} \mid \mathbf{e}_1)$ and $(r_0 \mid \mathbf{0})$.

3.16 If we identify $\mathbb{R}^2$ with the complex numbers $\mathbb{C}$, then there are two famous lattices defined using the complex numbers:

- Gaussian integers $G = \{a + b\mathrm{i} \mid a, b, \in \mathbb{Z}\} = \mathbb{Z}\{1, \mathrm{i}\}$.
- Eisenstein integers: $E = \{a + b\omega \mid a, b, \in \mathbb{Z}\} = \mathbb{Z}\{1, \omega\}$, where $\omega = \mathrm{e}^{2\pi\mathrm{i}/3} = \frac{1}{2}(-1 + \sqrt{3}\,\mathrm{i})$.

Determine which of the 5 types each of these lattices is.
[The interesting thing about these lattices is that they are not only groups, but rings as well, as you can check. For the Eisenstein case, one uses the fact that $\omega^2 + \omega + 1 = 0$.]

3.17 Find all homomorphisms
(a) from $\mathbb{Z}_2$ to $\mathbb{Z}_2 \times \mathbb{Z}_2$, and
(b) from $\mathbb{Z}_4$ to $\mathbb{Z}_6$. [Hint: If H is a cyclic group generated by a, and $\phi : H \to G$ a homomorphism, then ϕ is entirely determined by knowing $\phi(a)$, because $\phi(a^2) = \phi(a)^2$ etc.]

3.18† (a) Prove the following lemma:

Lemma 3.10. *Let G be a group and $H \lhd G$ a normal subgroup. Then the action of G on itself by conjugation restricts to an action of G on H. Moreover, if H is abelian, this defines an action of the quotient group G/H on H.*

(b) Let $G = \mathsf{D}_n$ and $H = \mathsf{C}_n$ (which is a normal subgroup). Determine the resulting action of G/H on H.

(c) Deduce Proposition 3.6 from this lemma.