Huigens principle

(Here I will present my calculations based on memories and textbooks...) Consider differential

Consider in $\mathbf{E}^n$ differential equation

$$\begin{cases} u_{tt} - \Delta u = 0 \\ u(t,\mathbf{x})\big|_{t=0} = \varphi(\mathbf{x}) \\ \frac{\partial u(t,\mathbf{x})}{\partial t}\big|_{t=0} = \psi(xx) \end{cases}$$

One can see that formal solution in Fourrier series will be

$$C_n \int e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} \left(\varphi(y)\cos kt + \psi(y)\frac{\sin kt}{k}\right) d^n\mathbf{k} d^n\mathbf{y}\,, (), \tag{$*$}$$

where $C_n = 2\pi^{-\frac{n}{2}}$, $\mathbf{k}, \mathbf{x}$ are vectors, k is modulus of vector $\mathbf{k}\, k = |\mathbf{k}|$.

We calculate this integral and show that for odd n it implies Huigens.

Preliminary calculation: Calculate preliminary the average of the function $e^{i\mathbf{k}(\mathbf{x}-y)}$ over unit $n-1$-dimensional sphere (in $\mathbf{k}$ space.

Denote by σ_n area of n-dimensional unit sphere:

$$\sigma_0 = 2, \sigma_1 = 2\pi\,, \sigma_2 = 4\pi\ \ldots, \sigma_n = 2\pi^{\frac{n+1}{2}}\Gamma\left(\frac{n+1}{2}\right).$$

(It is funny to note that volume of 0-dimensional sphere $\sigma_0 = 2$ is given by the general formula.)

Function $\mathbf{kx}$ is not constant on $n-1$ dimensional sphere $kx = 1$, but it is constant on $n-2$ dimensional spheres $\mathbf{kx}\cos\theta = c$ (θ is angle between $\mathbf{k}$ and $\mathbf{x}$ and $|c| \leq 1$). We have

$$F_n(kx) = < e^{i\mathbf{kx}} >_{k=1} = \frac{1}{\sigma_{n-1}} \int_{k=1} e^{i\mathbf{kx}} d\Omega_{n-1} = \frac{\sigma_{n-2}}{\sigma_{n-1}} \int_0^\pi e^{ix\cos\theta} \sin^{n-2}\theta d\theta$$

One can see that answers for even and odd will be different. For odd n it is just elementary function, and for even n they are expressed via special function $j(x) = \int_0^\pi e^{ix\cos\theta} d\theta$.

In moe details. First consider special cases (we ofen omit later all the coefficients....)

$$n = 2, J(x) = F_2(x) = \sigma_0 \int_0^\pi e^{ix\cos\varphi} d\varphi = \int_0^{2\pi} e^{ix\cos\varphi} d\varphi\,,$$

$$n = 3, F_3(x) = \sigma_1 \int_0^\pi e^{ix\cos\varphi} \sin\varphi d\varphi = 2\pi \int_{-1}^1 e^{ixu} du = 2i\frac{\sin x}{x}\,.$$

It is easy to see that the answer for $n = 0$ produces all the answers for even n and the answer for $n = 3$ produces all the answers for odd n:

One can see that all fnctions $F_(x)$ can be produced from function $J(a)$ and $f(a) = \frac{\sin a}{a}$ by differentiation, e,g,

$$F_7(x) = \sigma_5 \int_0^\pi e^{ix\cos\theta} \sin^5\theta d\theta = s_5 \int_0^\pi e^{ix\cos\theta} \sin^4\theta d\cos\theta = s_5 \int_0^\pi e^{ixu}(1-2u^2+u^4)du =$$

$$\left(1 + 2\frac{d^2}{du^2} + 4\frac{d^4}{du^4}\right) \int_0^\pi e^{ixu} du = 2i\sigma_5 \left(1 + 2\frac{d^2}{dx^2} + 4\frac{d^4}{dx^4}\right) \frac{\sin x}{x}$$

Now we return to the integral (*). Calculate it for odd n. Using functions $F_n(a)$ which are averaging of exponent over spere we come to

$$u(t, \mathbf{x}) = C_n \int e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} \left(\varphi(y)\cos kt + \psi(y)\frac{\sin kt}{k}\right) d^n\mathbf{k} d^n\mathbf{y} \, . = \int F_n(k|\mathbf{x}-\mathbf{y}|) \left(\varphi(y)\cos kt + \psi(y)\frac{\sin}{k}\right.$$

We denote

$$G_n^{(0)}(\mathbf{x}, \mathbf{y}, t) = \int F_n(k|\mathbf{x}-\mathbf{y}|) \left(\varphi(y)\cos kt + \psi(y)\frac{\sin kt}{k}\right) k^{n-1} dk =$$

We see that

$$u(\mathbf{x}, t) = \int G(\mathbf{x}, \mathbf{y}, t) \left(\varphi(y)\cos kt + \psi(y)\frac{\sin kt}{k}\right) d^n\mathbf{k} d^n\mathbf{y}$$