The regular representation of group contains all irreducible representations

The well-known (Burnside?) lemma states that for finite group G

$$\sum n_i^2 = |G|\,,$$

where n_i is a dimension of irreducible representation. E.g. the group S_3 has two 1-dimensional irreducible representations: (identical and alternating) and one 2-dimensional representation (symmetry of the triangle in the plane) and $1^2 + 2^2 + 1^2 = 6$.

There are many different proofs of this lemma. I will present here the proof which makes me to remember the story of the commode which is full of the dishes (this proof) but these dishes never breake.....

The proof is founded on the following construction:

Let ρ be a representation of a group G in finite-dimensional space V. Then arbitrary element ω in the dual space V^* defines the map $\iota_{\rho,\omega}$ of V in the space $\mathcal{R}_G$ of of functions on G:

$$\iota_{\rho,\omega}(\mathbf{x})\colon = \iota_{\rho,\omega}(\mathbf{x}|g) = \langle \rho_g(\mathbf{x}), \omega\rangle\,. \tag{1}$$

The kernel of this map, the subspace $V' \subseteq V$ is invariant with respect to the action of the group G. Hence $V' = 0$ if $\omega \neq 0$ and the representation ρ is irreducible representation. Hence map (1) of V into regular representation $\mathcal{R}$ is injection map in the case if representation ρ is irreducible and $\omega \neq 0$.

Observation. Let (ρ_1, V_1) and (ρ_2, V_2) be two irreducible representations of group G, and ω_1, ω_2 be two non-zero covectors, $\omega_1 \in V_1^*$, $\omega_2 \in V_2^*$. Consider the reciprocal position of images of spaces V_1 and V_2 under injection (1) under the injection (1). The following statement is valid:

if representations ρ_1, ρ_2 are not equivalent and then the images intersect only by zero element. If these representations are equivalent, $\rho_1 \approx \rho_2$ i.e. there exists interwining bijection $T\colon V_1 \to V_2, \rho_2 \circ T = T \circ \rho_1$, then the images intersect only by zero element if and only if covectors ω_1 and $T^*\omega_2$ are linear independent. In the case if these covectors are proportional then the images coicide.

Indeed suppose that there exist $\mathbf{x} \in V_1, \mathbf{y} \in V_2$ such that

$$\forall g\ \langle \rho_1(g)(\mathbf{x}), \omega_1\rangle = \langle \rho_2(g)(\mathbf{y}), \omega_2\rangle\,, \quad (\omega_1 \neq 0, \mathbf{x} \neq 0)\,. \tag{2}$$

This condition defines invariant subspaces in V_1 and V_2. Since representations are irreducible and $\mathbf{x} \neq 0$ we see that this condition holds for all $\mathbf{x} \in V_1$. Since the injectivity of maps (1) we see that condition (2) defines interwinning operator T on V_1 such that $T(\mathbf{x}) = \mathbf{y}$. Hence by Schur lemma these representations are equvalent. We may identify spaces V_1 and V_2 by the interwinning operator. In this case $T(\mathbf{x}) = \lambda\mathbf{x}$, i.e. $\omega_1 = \lambda\omega_1$.

Using this observations perform the following operation:

Let $\{\rho_a, V_\alpha\}$ be a set of *all* irreducible representations of the group G. They all are finite-dimnesional. For every representation $(\rho_\alpha, V_\alpha$ consider an arbitrary basis $\{\omega_{i_\alpha}\} = \{\omega_1, \ldots, \omega_{n_\alpha}\}$ in V_α (n_α is dimension of the space V_α). Then an arbitrary irreducible representation (ρ_α, V_α) defines n_α embeddings $\iota_{\omega_{i_\alpha}, \rho_\alpha}$ such that all images of these embeddings intersect by zero element in the "universal commode", the regular representation $\mathcal{R}$. Two non-equivalent representations $(\rho_\alpha, V_\alpha), (\rho_\beta, V_\beta)$, $\rho_\alpha \not\approx \rho_\beta$ define two collections of subspaces in $\mathcal{R}$, the collection of n_α subspaces, and the collection of second n_β subspaces. All these subspaces will intersect by zero element. Consider the union of the embeddings in the "universal commode" $\mathcal{R}$ of all pairwise non-equivalent representation; every representation ρ_α is embedded n_α times:

$$\oplus_{\rho_\alpha : \rho_\alpha \not\approx \rho_{\alpha'}} \oplus_{i_\alpha} \iota_{\omega_{i_\alpha}, \rho_\alpha}(V_\alpha) \subseteq \mathcal{R}, \tag{3}$$

i.e.

$$\sum_\alpha (\dim V_\alpha)^2 = \sum n_\alpha^2 \le \dim \mathcal{R} = |G| \,. \tag{3a}$$

(The number relation 3a is the corollary of the relation (3)) It remains to prove that order of G is not bigger than $\sum_\alpha n_\alpha^2$.

Consider in $\mathcal{R}$ the subspace W which is orthogonal to all subspaces dual to images of subspaces V_α. This subspace contains some irreducible subspace of functions (ρ_0, V_0). This subspace of functions is equivalent to one of irreducible representation $(\rho_{\alpha_0}, V_{\alpha_0})$ and hence it is equivalent to its image (1) to the subspace in the "universal commode" $\mathcal{R}$. Two euivalent irreducible subspaces in the same space have to coincide by Schur Lemme. Hence $W = 0$. We proved that

$$\oplus_{\rho_\alpha : \rho_\alpha \not\approx \rho_{\alpha'}} \oplus_{i_\alpha} \iota_{\omega_{i_\alpha}, \rho_\alpha}(V_\alpha) = \mathcal{R}, \tag{3}$$

i.e.

$$\sum_\alpha (\dim V_\alpha)^2 = \sum n_\alpha^2 = \dim \mathcal{R} = |G| \,. \tag{3a}$$

The proof is finished.

Many years ago giving this proof to students in Yerevan University I told them the story about "universal commode": you buy after marriage the commode in two three months it becomes full ofthings, there is not free space, but everything is put there safely. This is exactly what happens here: all irreducible representations are packed in the regular representation.