Zorn lemma application

We apply Zorn lemma to show that every ideal belongs to maximal ideal. Can we apply Zorn lemma to show that every family of ideals has maximal ideal? Answer is: "No" since positive answer means that ring is noetherian. Let us see how conditions of applying Zorn lemma are failed.

Let I be an arbitrary ideal in an arbitrary ring $\mathbf{R}$.

Consider two sets of ideals:

$$M_1 = \{ \text{ set of all finitely generated ideals of ring } \mathbf{R} \text{ which belong to } I\}$$

and

$$M_2 = \{ \text{ set of all ideals of ring } \mathbf{R} \text{ which possesses the ideal } I\}\,,$$

The first set for not-Noetherian rings is not inductive: it may happen that M_1 possesses well-ordered subset which has not upper bound in M_1. E.g. consider ring $\mathbf{R} = \mathbf{R}[x_i]$, ring of polynomials with real coefficients which depend on variables $\{x_i\}, i = 1, 2, 3, \dots$ (every polynomial depend on finite number of variablles). Consider ideals

$$\alpha_k = \text{polynomials which vanish if } x_1 = x_2 = \ldots = x_k = 0$$

and

$$I = \text{polynomials which vanish if at least on of variables is equal to zero}$$

The ideal I is not finitely generated. Well-ordered set of finitely generated ideals α_k which belong to I has not upper bound in the set M_1 of finitely generated ideals. We cannot use Zorn lemma to prove that the family M_1 possesses maximal ideal: the fact that M_1 possesses maximal ideal means that I is finitely generated. Indeed let β be maximal element in M_1, then $\beta = I$, since if $\beta \neq I$, we can expand β. Recall that the fact that every family of ideals possesses maximal ideal is nothing but Noether condition.

The second set is inductive: every well-ordered subset $N \subseteq M_2$ has upper bound in M_2. Indeed consider subset N of ideals $N = \{\alpha_\iota\}, N \subseteq M$ such that for arbitrary two ideals $\alpha_{\iota_1}, \alpha_{\iota_2} \in N$

$$\alpha_{\iota_1} \subseteq \alpha_{\iota_2}\,, \text{or } \alpha_{\iota_2} \subseteq \alpha_{\iota_2}\,.$$

(This is well-ordered subset in poset[1] M of ideals.)

One can see that N has upper bound in M (not obligatory in N itself!). Indeed consider union of all ideals in N:

$$T_N = \sum_\iota \alpha$$

All ideals α do not possess 1,hence $T \neq \mathbf{R}$ since it does not possess 1 and $T\mathbf{R} \in T$. Hence T is ideal. We see that every well-ordered subset in M_2 has upper bond. Hence by Zorn lemma we come to the existence of maximal element in M_2. In particular we proved very imprortant

[1] 'poset' means *partially ordered set*

Proposition Every ideal is contained in maximal ideal.

Resumé. Statement about existence of maximal ideals (in a given family) sometimes follows just from Zorn lemma, sometimes follows from noetherian property.

Consider another good example.

Proposition If x is not nilpotent element of ring $\mathbf{R}$, then there exist prime ideal which does not possess this element.

Now present the proof based on Zorn lemma.

Proof. Consider the set

$$M = \text{set of all ideals which do not contain any power of } x$$

M is not empty, since $\{0\} \in M$.

For noetherian ring everything is done: every set of ideals in noetherian ring possesses maximal ideal [3)] and maximal ideal is automatically simple.

Now forget about the condition of being noetherian ring.

The set M due to Zorn lemma possess maximal element (this maximal element is not necessary maximal ideal!) since every well-ordered subset in M has upper bond

Hence we come to the ideal β such that

$$\forall N\,, \quad x^N \notin \beta_x$$

Prove that β is a prime ideal.

The proof follows from the lemma

Lemma For an arbitrary $a \not\beta$ there exist N such that

$$x^N \in (a) + \beta$$

Indeed let $ab \in \beta$. Suppose that $a, b \notin \beta$. The lemma implies that there exist $M.N$ such that $x^M - pa \in \beta, x^N - qb \in \beta$ for some elements $p, q \in \mathbf{R}$. Hence $x^{N+M} \in \beta$ since $ab \in \beta$, but $x^{N+M} \notin \beta$. Contradiction.

It remains to prove the lemma. Suppose $a \notin \beta$. Consider subring generated by β and a. It is all the ring or the ideal. The condition that β is maximal ideal which does not possess all the powers of x implies the statement of lemma.

Example Consider algebra of functions on $\mathbf{R}$, how looks ideal β_x. $\beta_x = (x - a)$, where $a \neq 0$ maixmal ideals are points of $\mathbf{R}$

specA

Let A be a ring. What can we say about $SpecA$?

If ring A is an integer domain and in aprticular it does not possess nilpotents then $\{0\}$ is a prime ideal. In general it is not the case, but we can consider the set of all ideals containing the ideal $\{0\}$, and the maximal element of this set wiil be the maximal ideal.

[3)] Indeed suppose this is wrong. Then we come to infintie ascending series of ideals

The maximal ideal is prime (opposite is not true!). We come to point of $SpecA$. Moreover we see that every point of $SpecA$ is contained in the point of $SpecmA$– every prime ideal is contained in some maximal ideal. Now recall that factor of A over maximal ideal is a field.

We call system uncompatible if there exist polynomials P_i such that $\sum P_i F_i \equiv 1$, i.e. $< F_1, \dots, F_n >=< F_i >$. span all the ring.

Corollalry (Hilbert's weak Theorem) Let $F_i(T_\alpha) = 0$ be a system of compatible equations. Then there exist a field L such that a system have solution in this field.

Theorem above implies that one can take as such a field, a factor of A over some maximal ideal. In this field factors of T_i are roots,

Geometrical points

Let A be an algebra over ring K, a ring induced by system of equations

$$F_i(T_\alpha) = 0 \, i = 1, \dots, n, \alpha = 1, \dots, m$$

where $F_i \in K[T - \alpha]$.

Let L be an arbitrary extension of the ring K, and $X(L)$ set of solutions of this system in L. Simple but important statement

$$X(L) = Hom_K(A, L)$$

Indeed let τ be a homomorphism of A in L:

$$\tau\colon \quad A = K[T_1, \dots, T_m] \backslash I_{\text{equations}}$$

where we denote by $I_{\text{equations}}$ the ideal generated by polynomials F_i:

$$I_{\text{equations}} =< F_1(T_1, \dots T_m), \dots, F_n(T_1, \dots T_m) > \ .$$

Denote by l_α the value of φ on equivalence classes of elements $[T_i]$:

$$l_\alpha = \tau([T_a]), \quad \text{where}[T_a] = \{x \in K\colon \quad x - T_a \in I_{\text{equations}}\}$$

One can see that for arbitrary polynomial F_i in equations

$$F_i(l_1, \dots, l_m) = F_i\left(\tau\left([T_1]\right), \dots, \tau\left([T_1]\right)\right) = \tau\left(F_i\left([T_1]\right), \dots, \left([T_m]\right)\right) = \tau\left(\left[F_i\left(T_1\right], \dots, T_m\right)\right]\right) = 0, . \blacksquare$$

Now prove the converse implication.

Let elements $l_1, \dots, l_m \in L$ of algebra L be a solution of equations

$$F_i(l_1), \dots . l_m) = 0 \, .$$

Consider the homomorphism φ of the ring $KT_1, \dots, T_m$ in L such that $\varphi(T_\alpha) = l_\alpha$:

$$\varphi\colon \quad \varphi(P(T_1, \dots, T_m)) = P(l_1, \dots, l_m)$$

for an arbitrary polynomial $P \in K[T_1, \ldots, T_m]$.

This homomorphism is well-defined on factor-algebra A since it vanishes on polynomials F_i.

$$|SpecA| = 1$$

What can we say about a ring if it has only one prime ideal. If ring is not-empty then one can consider maximal ideal which possesses zero ideal. Hence there is at least one point– a maximal ideal α.

If there is exactlone point, it means that zero ideal is not prime, i.e. ring possesses at least one nilpotent element.

Show that maximal ideal possesses all nilpotents.

Let $\theta \in \alpha$. The value of θ on a point α is equal to zero. If θ is not nilpotent then there exist a point x such that $\theta(x) \neq 0$, i.e. there is a prime ideal, which does not possess θ, that it is there is another point of A. We come to

Proposition If ring A possesses just one point, then it is maximum ideal and all its elements are nilpotents.

Counterexample Consider for $\mathbf{Z}$ a local ring at the point (3):

$$O_3 = \{\frac{p}{q}: \quad 3 \not| q\}$$

This ring possesses just one maximum ideal (3), but it has two points: point (3) and the point (0).

Let A be a ring with just one point