Huigens principle

(Here I will present my calculations based on memories and textbooks...)

Consider in $\mathbf{E}^n$ differential equation

$$\begin{cases} u_{tt} - \Delta u = 0 \\ u(t, \mathbf{x})\big|_{t=0} = \varphi(\mathbf{x}) \\ \frac{\partial u(t,\mathbf{x})}{\partial t}\big|_{t=0} = \psi(x) \end{cases} \tag{1}$$

Using Fourrier transformation one can see that formal solution in Fourrier series will be

$$C_n \int e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} \left(\varphi(y) \cos kt + \psi(y) \frac{\sin kt}{k}\right) d^n\mathbf{k} d^n\mathbf{y}\,, (), \tag{1a}$$

where $C_n = 2\pi^{-\frac{n}{2}}$, $\mathbf{k}, \mathbf{x}$ are vectors, k is modulus of vector $\mathbf{k}$, $k = |\mathbf{k}|$, where coefficients $A(\mathbf{k}), B(\mathbf{k})$

Symmetry of solutions with respect to initial data

Let $u = u^{\{\varphi,\psi\}}$ be the solution (1a) of wave equation with boundary conditions (1). Then it is easy to see that for function $v = u_t$

$$v = u^{\{\varphi',\psi'\}} \text{ with } \varphi' = \psi\,, \text{and } \psi' = \Delta\varphi\,. \tag{2a}$$

since

$$v\big|_{t=0} = u_t\big|_{t=0}\,, \text{and } v_t\big|_{t=0} = u_{tt}\big|_{t=0} = \Delta u\big|_{t=0} = \Delta\varphi\,. \tag{2b}$$

In particular if the function u is the solution of wave equation with boundary conditions $u(t, \mathbf{x})\big|_{t=0} = 0$ and $\frac{\partial u(t,\mathbf{x})}{\partial t}\big|_{t=0} = \varphi(x)$, then the function $v = u_t$ is the solution of wave equation with boundary conditions $v(t, \mathbf{x})\big|_{t=0} = \varphi$ and $\frac{\partial v(t,\mathbf{x})}{\partial t}\big|_{t=0} = 0$, and

$$u^{\{\varphi,\psi\}} = u^{\{0,\psi\}} + u_t^{\{0,\varphi\}}. \tag{2c}$$

We calculate this integral, show that for odd n it implies Huigens and try to reveal some geometrical reasons of this fact based on the article of Roger Howe *.

Preliminary calculation: Calculate preliminary the average of the function $e^{i\mathbf{k}(\mathbf{x}-y)}$ over unit $n-1$-dimensional sphere of the radius k.

Denote by σ_n area of n-dimensional unit sphere:

$$\sigma_0 = 2, \sigma_1 = 2\pi\,, \sigma_2 = 4\pi\ \ldots, \sigma_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}\,. \tag{3}$$

* Roger Howe. On the role of the Heisenberg group in harmonic analysis. Bulletin (New Series) OF THE AMERICAN MATHEMATICAL SOCIETY, Volume 3, Number 2, September 1980

(It is funny to note that volume of 0-dimensional sphere $\sigma_0 = 2$ is given by the general formula.)

Function $\mathbf{kx}$ is not constant on $n-1$ dimensional sphere $kx = 1$, but it is constant on $n-2$ dimensional spheres $\mathbf{kx}\cos\theta = c$ (θ is angle between $\mathbf{k}$ and $\mathbf{x}$ and $|c| \leq 1$). Using this fact we calculate average of function $f(\mathbf{kx}) = e^{i\mathbf{kx}}$ over the sphere of radius k:

$$< f\mathbf{kx} >_k = \frac{1}{k^{n-1}\sigma_{n-1}} \int_0^\pi f(kx\cos\theta)\sigma_{n-2}(k\sin\theta)^{n-2}kd\theta =$$

$$= \frac{\sigma_{n-2}}{\sigma_{n-1}} \int_0^\pi f(kx\cos\theta)\sin\theta)^{n-3}d\sin\theta = \frac{\sigma_{n-2}}{\sigma_{n-1}} \int_{-1}^1 f(kxu)(1-u^2)^{\frac{n-3}{2}}du \qquad (3a)$$

(If we put $f \equiv 1$ we come to the formula (3) for volumes of sphere* One can see that answers for even and odd will be different. For odd n it is just elementary function, and for even n they are expressed via special function

$$J(x) = \int_0^\pi e^{ix\cos\theta}d\theta \qquad (4)$$

In more details. First consider special cases (we often omit later all the coefficients....)

$$n = 2, J(x) = F_2(x) = \sigma_0 \int_0^\pi e^{ix\cos\varphi}d\varphi = \int_0^{2\pi} e^{ix\cos\varphi}d\varphi\,, \qquad (5a)$$

$$n = 3, F_3(x) = \sigma_1 \int_0^\pi e^{ix\cos\varphi}\sin\varphi d\varphi = 2\pi \int_{-1}^1 e^{ixu}du = 2i\frac{\sin x}{x}\,. \qquad (5b)$$

Now one can see that the answer for $n = 2$ produces all the answers for even n and the answer for $n = 3$ produces all the answers for odd n: all fnctions $F_n(x)$ can be produced from function $J(a)$ for even n and function $f(a) = \frac{\sin a}{a}$ by differentiation, e,g,

$$F_6(x) = \int e^{ix\cos\theta}\sin^4\theta d\theta = \int e^{ix\cos\theta}(1-\cos^2\theta)^2 d\theta = \left(1 + \frac{d^2}{dx^2}\right)^2 J(x) \qquad (6a)$$

$$F_7(x) = \int_0^\pi e^{ix\cos\theta}\sin^5\theta d\theta = \int_0^\pi e^{ix\cos\theta}\sin^4\theta d\cos\theta = \int_0^\pi e^{ixu}(1-u^2)^2 du = \qquad (6b)$$

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* Indeed we have $1 =< 1 >= \frac{\sigma_{n-2}}{\sigma_{n-1}} \int_{-1}^1 (1-u^2)^{\frac{n-3}{2}}du$.. Hence $\frac{\sigma_{n-2}}{\sigma_{n-1}} =$

$$\int_{-1}^1 (1-u^2)^{\frac{n-3}{2}}du = \int_0^1 (1-x)^{\frac{n-3}{2}}x^{-\frac{1}{2}}dx = B\left(\frac{n-1}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} = \frac{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}$$

$$\left(1-\left(i\frac{d}{dx}\right)^2\right)^2\int_0^\pi e^{ixu}du=\left(1+\frac{d^2}{dx^2}\right)^2\frac{\sin x}{x}\,. \tag{6b}$$

(We often omit coefficients proportional to volumes of spheres.)

One can see that in general case **PROPOSITION**

$F_n(x)=P\left(\frac{d}{dx}\right)J(x)$ for even n and $F_n(x)=P\left(\frac{d}{dx}\right)\frac{\sin x}{x}$.

Now we return to the integral (*). Using functions $F_n(a)$ which are averaging of exponent over spere we come to

$$u(t,\mathbf{x})=C_n\int e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}\left(\varphi(y)\cos kt+\psi(y)\frac{\sin kt}{k}\right)d^n\mathbf{k}d^n\mathbf{y}\,.=$$

$$=\int F_n(k|\mathbf{x}-\mathbf{y}|)\left(\varphi(y)\cos kt+\psi(y)\frac{\sin kt}{k}\right)k^{n-1}dkd^n\mathbf{y}= \tag{7}$$

We denote

$$G_n^{(0)}(\mathbf{x},\mathbf{y},t)=\int e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}\cos kt\,d^n\mathbf{k}=\int F_n(k|\mathbf{x}-\mathbf{y}|)\cos kt\,k^{n-1}dk= \tag{8a}$$

and

$$G_n^{(1)}(\mathbf{x},\mathbf{y},t)=\int e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}\frac{\sin kt}{k}\,d^n\mathbf{k}\,.\int F_n(k|\mathbf{x}-\mathbf{y}|)\frac{\sin kt}{k}k^{n-1}dk\,. \tag{8b}$$

We see that

$$u(\mathbf{x},t)=\int G^{(0)}(\mathbf{x},\mathbf{y},t)\varphi(\mathbf{y})d^n\mathbf{y}+\int G^{(1)}(\mathbf{x},\mathbf{y},t)\psi(\mathbf{y})d^n\mathbf{y} \tag{8c}$$

Sometimes we use functions $P^{(0)}(\mathbf{x},t)$ and $P^{(1)}(\mathbf{x},t)$:

$$G^{(0)}(\mathbf{x},\mathbf{y},t)=P^{(0)}(\mathbf{x}-\mathbf{y},t)\,,G^{(1)}(\mathbf{x},\mathbf{y},t)=P^{(1)}(\mathbf{x}-\mathbf{y},t)\,. \tag{8d}$$

We may rewrite (8c) as

$$u=P^{(0)}\circ\varphi+P^{(1)}\circ\psi\,,$$

where "$\circ''$ is convolution.

Now one can see literally that for odd n integral is localised on the light cone. This is easy to check for propagators P_1,P_2: Let $n=2m+1$ be odd then according to previous calculations for for function $F_n(u)$ and elementary formulae of differentiation of trigonometric functions we have:

$$P_0(\mathbf{x},t)=\int e^{i\mathbf{k}\mathbf{x}}\cos kt\,d^{2m+1}\mathbf{k}=\int F_{2m+1}(kx)\cos kt\,k^{2m}dk=$$

$$\left(\int_0^{\frac{\pi}{2}}e^{iu\cos\theta}\sin^{2m-1}\theta d\theta\right)\Big|_{u=kx}\cos kt\,k^{2m}dk=\left(\int_0^{\frac{\pi}{2}}e^{iu\cos\theta}\sin^{2m-2}\theta d\cos\theta\right)\Big|_{u=kx}\cos kt\,k^{2m}dk$$

$$= \left(\int_{-1}^{1} e^{iuz}(1-z^2)^{\frac{m-1}{2}} dz\right)|_{u=kx} \cos kt\, k^{2m} dk == \left(\left(1-\left(i\frac{d}{du}\right)^2\right)^{m-1} \int_{-1}^{1} e^{iuz} dz\right)|_{u=kx} \cos kt\, k^{2m} dk$$

$$= \left(\left(1+\frac{d^2}{du^2}\right)^{m-1} \frac{\sin u}{u}\right)|_{u=kx} \cos kt\, k^{2m} dk =$$

Comparing with paper of Howe

Try to compare propagators P_1, P_2 (see equations 8d) with those in paper of Howe. Denote as in Howe Fourrier image of F by $\hat{F}$. We have

$$\hat{P}_1(\mathbf{k}, w) = \int e^{i\mathbf{k}'\mathbf{x}} \cos k't e^{-i(\mathbf{kx}-wt)} d^n\mathbf{k}' d^n\mathbf{x} dt =$$

$$\left(\delta(k-w)+\delta(k-w)\right) d^n k dw = k^2 \delta(k^2-w^2) d\mu$$

where

$$d\mu = \frac{d^n k}{k}\,, \text{is an invariant measure in } \mathbf{E}^{n+1} \text{ with Minkovsku metric}$$

and

$$\hat{P}_2(\mathbf{k}, w) = \int e^{i\mathbf{k}'\mathbf{x}} \frac{\sin k't}{k'} e^{-i(\mathbf{kx}-wt)} d^n\mathbf{k}' d^n\mathbf{x} dt =$$

$$\left(\delta(k-w)+\delta(k-w)\right) d^n k dw = k^2 \delta(k^2-w^2) d\mu$$

Action of group $SL(2)$

Consider on $\mathbf{E}^n$ operators

$$\Delta = g^{ik}\partial_i\partial_k, \rho^2 = g_{ik}x^i x^k\,, E = \frac{n+1}{2} + D$$

here $g_{ik} = \mathrm{diag}\,(1,-1,\dots,-1)$ We see that

$$[\Delta, \rho^2] = [g^{ik}\partial_i\partial_k, g_{pq}x^p x^q] = 2(n+1) + 4x^i\partial_i = 4E$$

$$[E, \rho^2] = 2E\,, \quad [E, \Delta] = 2\Delta$$

Now notice that