Geometrical meaning of Crammer rule.

The Crammer rule which you can find in any handbooks for mathematical calcualtions for engineers may be seems to be little bit annoying for mathematicians. But it has a very simple and beautiful geometrical meaning

We know Crammer rule. It states the following:
Consider n simultaneous linear equations for n unknowns:

$$A\mathbf{x} = \mathbf{c}\,, \tag{0.1}$$

where A is $n \times n$ matrix, $\mathbf{c}$ is $n \times 1$ matrix with real entries , $\mathbf{x}$ is $n \times 1$ of unknowns. (We can view $\mathbf{x}, \mathbf{c}$ as vectors $\mathbf{x} = x^i \mathbf{e}_i$ in $\mathbf{R}^n$ and A as a linear operator).

The solution of this system, the vector $\mathbf{x} = A^{-1}\mathbf{c}$ can be calculated in many different ways. The following receipt of calculations is practical:

If we remove i-th row from the matrix A and put instead it the vector $\mathbf{c}$ we come to the matrix which we denote by A_i: If matrix A can be considered as the ordered set of n vectors:

$$A = (\mathbf{a}_1, \ldots, \mathbf{a}_n) \tag{0.2}$$

then

$$A_1 = (\mathbf{c}, \mathbf{a}_2, \ldots, \mathbf{a}_n)\,, A_2 = (\mathbf{a}_1, \mathbf{c}, \mathbf{a}_3, \ldots, \mathbf{a}_n)\,, A_{n-1} = (\mathbf{a}_1, \ldots, \mathbf{a}_{n-2}, \mathbf{c}, \mathbf{a}_n)\ , A_n = (\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}, \mathbf{c})$$

Crammer rule tells that in the case if $\det A \neq 0$ then the solution of the system (0.1) is

$$x^i = \frac{\det A_i}{\det A}\,, (i = 1, 2, \ldots, n) \tag{0.3}$$

This rule is may be the best known formula in Linear Algebra for the wide community of non-mathematicians. (For example you can find it in any mathematical manual for engineers.)

There are million proofs of this elementary formula. I would like to expose here just one which looks nice (and which can be generalised for graded spaces).

Crammer identity and Crammer rule

We use exterior n-form on $\mathbf{R}^n$. Exterior n-form $\omega(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ is bilinear n-form (function on n vectors which is linear with respect to all vectors) and which is *antysymmetrical* with respect to any two vectors:

$$\omega(\ldots, \mathbf{x}_i, \ldots, \mathbf{x}_j, \ldots) = -\omega(\ldots, \mathbf{x}_j, \ldots, \mathbf{x}_i, \ldots)\,. \tag{0.4}$$

In particular this means that for any vector $\mathbf{x}$

$$\omega(\ldots, \mathbf{x}, \ldots, \mathbf{x}, \ldots) = 0\,. \tag{0,4a}$$

(In fact conditions (0.4) and (0,4a) for bilinear forms are equivalent: show it.)

An example of exterior n-form is determinant: Choose a basis and consider

$$\omega(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n) = \det(\mathbf{x}_1, \dots, \mathbf{x}_n)\,, \tag{0.5}$$

where $(\mathbf{x}_1, \dots, \mathbf{x}_n)$ in the right hand side is $n \times n$ matrix composed of vectors $\mathbf{x}_1, \dots, \mathbf{x}_n$ in the chosen basis.

Exercise Any exterior form is proportional to (0.5).

Exterior n-form in $\mathbf{R}^n$ defines the volume of n-parallelepiped: $\omega(\mathbf{x}_1, \dots, \mathbf{x}_n)$ can be considered as a volume of parallelepiped formed by vectors $\mathbf{x}_1, \dots, \mathbf{x}_n$.

Proposition

Let ω be an arbitrary exterior n-form on $\mathbf{R}^n$ and vector $\mathbf{c}$ belongs to the span of the vectors $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$, i.e.

$$\mathbf{c} = c^k \mathbf{a}_k = c^1 \mathbf{a}_1 + c^2 \mathbf{a}_2 + \dots + c^n \mathbf{a}_n$$

Then the following identity takes place

$$\omega(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n)\mathbf{c} =$$

$$\omega(\mathbf{c}, \mathbf{a}_2, \mathbf{a}_3, \dots, \mathbf{a}_n)\mathbf{a}_1 + \omega(\mathbf{a}_1, \mathbf{c}, \mathbf{a}_3, \dots, \mathbf{a}_n)\mathbf{a}_2 + \dots + \omega(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \dots, \mathbf{a}_{n-1}, \mathbf{c})\mathbf{a}_n \tag{1.1}$$

We call this identity *Crammer* identity

Remark Here and everywhere c^k is k-th component of the vector $\mathbf{c}$, not the k-th power of the c!!!)

Crammer rule immediately follows from the Crammer identity. Indeed let ω be a non-degenerate exterior n form. Then the equation (0.1) means that

$$\mathbf{c} = x^i \mathbf{a}_i = c^1 \mathbf{a}_1 + c^2 \mathbf{a}_2 + \dots + c^n \mathbf{a}_n\,, \tag{1.2}$$

where $\mathbf{a}_i$ are rows of the matrix A (see (0.2)). On the other hand due to Crammer identity (1.1)

$$\mathbf{c} = \frac{\omega(\mathbf{c}, \mathbf{a}_2, \mathbf{a}_3, \dots, \mathbf{a}_n)}{\omega(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n)}\mathbf{a}_1 + \frac{\omega(\mathbf{a}_1, \mathbf{c}, \mathbf{a}_3, \dots, \mathbf{a}_n)}{\omega(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n)}\mathbf{a}_2 + \dots + \frac{\omega(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \dots, \mathbf{a}_{n-1}, \mathbf{c})}{\omega(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n)}\mathbf{a}_n =$$

$$c^1 \frac{\det(\mathbf{c}, \mathbf{a}_2, \mathbf{a}_3, \dots, \mathbf{a}_n)}{\det(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n)} + c^2 \frac{\det(\mathbf{a}_1, \mathbf{c}, \mathbf{a}_3, \dots, \mathbf{a}_n)}{\det(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n)} + \dots c^n \frac{\det(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \dots, \mathbf{a}_{n-1}, \mathbf{c})}{\det(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n)} \tag{1.3}$$

Comparing (1.2)and (1.3) we come to (0.3).

It remains to prove Crammer identity (1.1):

Proof of Crammer identity.

It is just one enough long line: Let $c = c^1\mathbf{a}_1 + c^2\mathbf{a}_2 + \ldots + c^n\mathbf{a}_n$. Then using linearity and anitsymmetricity (0.4), (0.4a) we come to

$$\omega(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)\mathbf{c} = \omega(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)\left(c^1\mathbf{a}_1 + c^2\mathbf{a}_2 + \ldots + c^n\mathbf{a}_n\right) =$$

$$c_1\omega(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)\mathbf{a}_1 + c_2\omega(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)\mathbf{a}_2 + \ldots + c_n\omega(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)\mathbf{a}_n =$$

$$\omega(c_1\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)\mathbf{a}_1 + \omega(\mathbf{a}_1, c_2\mathbf{a}_2, \ldots, \mathbf{a}_n)\mathbf{a}_2 + \ldots + \omega(\mathbf{a}_1, \mathbf{a}_2, \ldots, c_n\mathbf{a}_n)\mathbf{a}_n =$$

$$\omega(c_1\mathbf{a}_1+c_2\mathbf{a}_2+\ldots+c_n\mathbf{a}_n, \mathbf{a}_2, \ldots, \mathbf{a}_n)\mathbf{a}_1+\omega(\mathbf{a}_1, c_1\mathbf{a}_1+c_2\mathbf{a}_2+\ldots+c_n\mathbf{a}_n, \ldots, \mathbf{a}_n)\mathbf{a}_2+\ldots+\ldots =$$

$$\omega(\mathbf{c}, \mathbf{a}_2, \ldots, \mathbf{a}_n)\mathbf{a}_1 + \omega(\mathbf{a}_1, \mathbf{c}, \ldots, \mathbf{a}_n)\mathbf{a}_2 + \ldots + \omega(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{n-1}, \mathbf{c})\mathbf{a}_n.$$

It is worth to note that these considerations can be generalised for linear operators on Z_2-spaces (superspaces) (Here very interesting mathematics begins (see the works on Berezinians of T. Voronov and mine.))

All the best
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