Geometry of differential operators on R. II

Example of operator

(All notations from the previous file)

We already know the canonical pencil of $n+1$-th order operators on $\mathbf{R}$.

Now we will write down an example of $n+1$-th order operator on $\mathbf{R}$ provided with volume form $\rho \in \mathcal{F}_1$.

Consider the operator:

$$D\colon \qquad \Psi |dx|^\sigma \mapsto D_\sigma \Psi = |dx|^{\sigma+1} \rho^\sigma \frac{d}{dx}\left(\frac{\Psi}{\rho^\sigma}\right) = |dx|^{\sigma+1}\left(\frac{d\Psi}{dx} + \Gamma_\bullet \Psi\right)$$

which sends $\mathcal{F}_\sigma$ to $\mathcal{F}_{\sigma+1}$.

(Here $\Gamma_\bullet = -\frac{d}{dx}\log\rho$ is a flat connection corresponding to the volume form ρ.)

One can assign to this operator the operator

$$\hat{D} = t\left(\frac{\partial}{\partial x} + \Gamma_\bullet \hat{w}\right)$$

on the whole space $\mathcal{F}$ of all the densities.

Operator $\hat{D}$ is self-adjoint operator.

Then we come to the following operator which sends $\mathcal{F}_\lambda$ to $\mathcal{F}_{\lambda+n+1}$:

Consider n-th order operator

$$L_n = \hat{D}^n = \left[t\left(\frac{d}{dx} + \Gamma_\bullet \hat{w}\right)\right]^n = t^n\left(\frac{\partial^n}{\partial x^n} + A_n \frac{\partial^{n-1}}{\partial x^{n-1}} + B_n \frac{\partial^{n-2}}{\partial x^{n-2}} + \ldots\right)$$

One can see that

$$L_{n+1} = \hat{D} L_n = t\left(\frac{\partial}{\partial x} + \Gamma_\bullet \hat{w}\right) t^n \left(\frac{\partial^n}{\partial x^n} + A_n \frac{\partial^{n-1}}{\partial x^{n-1}} + B_n \frac{\partial^{n-2}}{\partial x^{n-2}} + \ldots\right) =$$

$$t^{n+1}\left(\frac{\partial}{\partial x} + \Gamma_\bullet (n+\hat{w})\right)\left(\frac{\partial^n}{\partial x^n} + A_n \frac{\partial^{n-1}}{\partial x^{n-1}} + B_n \frac{\partial^{n-2}}{\partial x^{n-2}} + \ldots\right) =$$

$$t^{n+1}\left(\frac{\partial^{n+1}}{\partial x^{n+1}} + A_{n+1} \frac{\partial^n}{\partial x^n} + B_{n+1} \frac{\partial^{n-1}}{\partial x^{n-1}} + \ldots\right)$$

Thus we come to recurrent relations. For A_n, $A_1 = \Gamma_\bullet \hat{w}$, $A_{n+1} = A_n + \Gamma_\bullet(n+\hat{w})$, i.e.

$$A_n = n\Gamma_\bullet \hat{w} + (1+2+\ldots+n-1)\Gamma_\bullet = \frac{n}{2}(2\hat{w}+n-1)\Gamma_\bullet$$

For B_n: $B_1 = 0$,

$$B_{n+1} = B_n + (\partial_x + (n+\hat{w})\Gamma_\bullet) A_n, = B_n + (\partial_x + (n+\hat{w})\Gamma_\bullet)\frac{n+1}{2}(2\hat{w}+n)\Gamma_\bullet, \text{ i.e.}$$

,
$$B_{n+1} = B_n + \frac{(n+1)(n+2\hat{w})}{2}\partial_x\Gamma_\bullet + \frac{(n+1)(n+\hat{w})(n+2\hat{w})}{2}\Gamma_\bullet^2 .$$
One can see (this is long calculations) that

$$B_n = \frac{n(n-1)}{6}\left[(3\hat{w}+n-2)\,\Gamma'_\bullet + \left(3\hat{w}^2 + 3(n-1)\hat{w} + \frac{(n-2)(3n-1)}{4}\right)\Gamma_\bullet^2\right]$$

(*Tu a calculer ca...*)

We see

$$L_{n+1} = t^{n+1}\left[\frac{\partial^{n+1}}{\partial x^{n+1}} + \frac{n+1}{2}\,(2\hat{w}+n)\,\Gamma_\bullet\frac{\partial^n}{\partial x^n} + B_{n+1}\frac{\partial^{n-1}}{\partial x^{n-1}} + \dots\right]$$

(*Coefficient A_n is right here? ...*)

This operator defines the pencil: It transforms the density $\Psi|dx|^\lambda$ of the weight λ into the density $\Phi(x)|dx|^{\lambda+n-1}$ of the weight $\lambda+n-1$, where

$$\Phi(x) = \frac{\partial^{n+1}\Psi}{\partial x^{n+1}} + \frac{n+1}{2}\,(2\lambda+n)\,\Gamma_\bullet\frac{\partial^n\Psi}{\partial x^n} + B_{n+1}\frac{\partial^{n-1}\Psi}{\partial x^{n-1}} + \dots$$

If $\lambda = -\frac{n}{2}$ then

$$\Phi(x) = \frac{\partial^{n+1}\Psi}{\partial x^{n+1}} + B_{n+1}\frac{\partial^{n-1}\Psi}{\partial x^{n-1}} + \dots$$

One can see that it is $\sim$ to Schwarzian...

$$B_{n+1} = c_{n+1}B_2$$

(*Nous avons calculer hier*)

Second example

Go further. One can see that for $D = t(\partial_x + \hat{w}\Gamma_\bullet)$

$$\hat{w}D - D\hat{w} = D$$

This commutation relation implies that $n+1$-th order operator

$$K_n = D^n\hat{w} + (-1)^{n+1}\hat{w}^\dagger\left(D^\dagger\right)^n = D^n\hat{w} + (-1)^{n+1}(1-\hat{w})\left((-1)^nD\right)^=$$

$$D^n(2\hat{w}+n-1) = t^n(2\hat{w}+n-1)\left(\partial_x^n + \dots\right)$$

is self-conjugate operator.

Let $r = r(x)t$ be a density of the weight 1. it is useful to consider also the operator which is anticommutator:

$$\mathcal{K}_n = \frac{1}{2}\left(K_n \circ tr(x) + tr(x)\circ K_n\right) = \frac{1}{2}\left(D^n(2\hat{w}+n-1)\circ tr(x) + tr(x)\circ D^n(2\hat{w}+n-1)\right)$$

$$= t^{n+1}(2\hat{w}+n)r(x)\left(\partial_x^n+\dots\right)$$

Proposition Let $\Delta_{n+1} = t^{n+1}\partial_x^{n+1}+\dots$ be an arbitrary self-adjoint operator. It has the following appearance:

$$\Delta_{n+1} = t^{n+1}\left(\partial_x^{n+1}+\frac{n+1}{2}(2\hat{w}+n)\Gamma_\bullet\partial_x^n+\beta_{n+1}\partial_x^{n-1}+\dots\dots\right),$$

where

$$\beta_{n+1} = \frac{1}{2}\left[\theta\hat{w}^2+\left(n(n+1)\Gamma'_\bullet+n\theta\right)\hat{w}+q\right]$$

(see the previous file. *I am not sure about the term β_n, recalculate it, please.*)

We come to decomposition:

$$\Delta_{n+1} = D^{n+1}+t^{n+1}\frac{n+1}{2}(2\hat{w}+n)r\partial_x^n+\dots$$

where $r = \Gamma_\bullet - \Gamma_\bullet$. On the other hand

$$t^{n+1}\frac{n+1}{2}(2\hat{w}+n)r\partial_x^n+\dots = \dots\mathcal{K}_n+\dots\text{operator of the order} \leq n-1 \text{ w.r.s. to } x$$

where $\mathcal{K}$ is self-adjoint operator defined above.

Another useful formulae

It is useful to rewrite self-conjugality conditions in terms of the corresponding pencil. Let A be an operator on $\mathcal{F}$ of the weight δ and A_λ the corresponding pencil:

$$A_\lambda\colon \Psi(x)|dx|^\lambda \mapsto \left(A(x,\partial_x,\hat{w})\Psi(x)t^\lambda\right)\big|_{t=|dx|^\lambda+\delta}.$$

or : $A_\lambda = A(x,\partial_x,\hat{w})\big|_{\hat{w}=\lambda}$

One can see that

$$A_\lambda^\dagger = (A^\dagger)_{1-\delta-\lambda}$$

and in particular

$$A_\lambda^\dagger = A_{1-\delta-\lambda}$$

if A is self-adjoint operator.

It is useful to write down the "test-operator" which has the form $t^n\partial_x\hat{w}^k$ Note that

$$\hat{w}^m D^n = D^n(\hat{w}+n)^m.$$

Consider the following (anti)self-adjoint operator produced via the operator $\hat{w}^r D^n$ of the order $r+n$:

$$K_n^m = \hat{w}^m D^n + (-1)^{n+m}\left(\hat{w}^m D^n\right) = \hat{w}^m D^n + (-1)^m\left(D^n\hat{w}^{\dagger m}\right) = \hat{w}^m D^n + (-1)^m\left(D^n\left(1-\hat{w}\right)^m\right) =$$
$$\hat{w}^m D^n + \left(D^n\left(\hat{w}-1\right)^m\right) = D^n\left(\left(\hat{w}+n\right)^m+\left(\hat{w}-1\right)^m\right)$$

We come to self-adjoint operator of the weight n

$$K_n = D^n\left(\left(\hat{w}+n\right)^m+\left(\hat{w}-1\right)^m\right) = t^n\partial_x^n\left(2\hat{w}^m+(n-2)\hat{w}^{m-1}+\dots\right)+\dots$$

If r is a density of the weight l then calculating anticommutator we come to the self-adjoint operator

$$\mathcal{K}_n^m = \frac{1}{2}\left[K_n^m, s\right] = t^{n+l}\partial_x^n\left(\hat{w}^r+\dots\right)$$

it is a basic operator.