Duistermaat-Heckman localisation formula and locus of vector fields.

12. October 2013

§0

About ~~two~~ two years ago (summer 2012) Sasha Belavin explained how to calculate an integral

$$Z(t) = \int e^{t\, d_K \omega} \qquad (0.1)$$

(ω - 1-form, $d_K = d + L_K$). He ~~exp~~ showed first that this integral does not depend on t, then showed that it is localised at zeros of vector field K:

$$Z(t) \sim \frac{1}{\sqrt{\det \frac{\partial K}{\partial x}}}\Bigg|_{K=0}. \qquad (0.2)$$

It is typical localisation formula.

I tried to revive these calculations. ~~that~~ On one hand they are leading to Duistermaat-Heckman formula in more less general case.
On the other hand ~~we may discuss~~ it is interesting to analyze geometrical meaning of answer.

§1.

Two words about Duistermaat-Heckman formula (localisation) (DHL)-formula.

Let ~~M be compact manifold~~

(M^{2n}, Ω) be compact symplecticic manifold

Let H be an Hamiltonian, such that the vector field

$$K = D_H : \ \Omega \lrcorner D_H = -dH$$

is a compact vector field
(i.e. it generates compact subgroup e^{tK} in the group of diffeomorphisms)

Then

$\int \underbrace{\Omega^n}_{\substack{\Omega \wedge \dots \wedge \Omega \\ n\text{-times}}} e^{iH}$ is localised at zero locus of vector field K

and

$$\int \Omega^n e^{iH} = \sum_{x_i : K(x_i) = 0} e^{\frac{i\pi}{4} \operatorname{sgn} H(x_i)} \frac{e^{iH(x_i)}}{\sqrt{\det \operatorname{Hess} H(x_i)}}$$

(we suppose that $K(x_i)$ are not-degenerate).

This is famous Duistermaat-Heckman formula.

We will consider here a special but very illuminating case of this formula.

We consider now the following set up:

Let w be 1-form on M ($\dim M = 2n$)

such that $\Omega = dw$ defines symplectic structure.

(of course condition $\Omega = dw$ is in contradiction with compactness of M: $\int \Omega^n \neq 0$, but we ignore now this. E.g. we suppose that M is not compact)

Let K be a vector field such that

$$\mathcal{L}_K w = dw \lrcorner K + d(w \lrcorner K) = 0$$

Then it is evident that K is Hamiltonian vector field of $H = w \lrcorner K$

$$\Omega \lrcorner K = dw \lrcorner K = -d(w \lrcorner K) = -dH.$$

$$\begin{tikzcd} & w \arrow[dl, dash, "\Omega = d\Omega"'] \arrow[dr, dash, "H = w\lrcorner K"] & \\ \Omega & & H \end{tikzcd}$$

$$d_K w = (d + l_K) w = \Omega + H;$$

We see that

$$\int \Omega^n e^{iH} = \int e^{iH + i\Omega} = \int e^{i d_K w}$$

We come to integral (0).

§ 2

Calculation of $\int e^{id_K\omega}$.

Consider

$$Z(t) = \int_M e^{itd_K\omega} \qquad \boxed{d_K^2 = \mathcal{L}_K}$$

Show that $Z(t)$ does not depend on t.

$$\frac{dZ(t)}{dt} = i\int d_K\omega\, e^{itd_K\omega} =$$

$$= i\int_M d_K\left(\omega e^{itd_K\omega}\right) = i\int_M d\left(\omega e^{itd_K\omega}\right) = 0 \qquad (2.1)$$

(under some technical conditions),

[$\int \mathcal{L}_K\omega = 0$ since form $\mathcal{L}_K\omega$ has rank $\leq 2n-1$)

We see that $Z(t)$ does not depend on t.
Hence we can calculate $Z(t)$ at $t \to \infty$.

$$\int e^{itd_K\omega} = \int e^{it(\Omega + H)} =$$

$$d\omega = \Omega\,,\quad \omega \lrcorner K = H. \quad (\mathcal{L}_K\omega = 0)$$

$$= \sum \frac{i^n t^n}{n!}\int \Omega^n e^{itH} = \frac{i^m t^m}{m!}\int \Omega^m e^{itH} \qquad (2.2)$$

$$(\dim M = 2m)$$

Calculate using stationary phase method:

$$dH = d(\omega \lrcorner K) = -\,d\omega \lrcorner K \qquad (2.3)$$

Locus of dH = locus of K

We see that at stationary point $dH=0$
Hessian is:

$$\frac{\partial^2}{\partial x^i \partial x^k} H\Big|_{K=0} = \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^k}(w_r K^r)\Big|_{K=0} = \frac{\partial}{\partial x^i}(\Omega_{ip} K^p) =$$

$$(d(w \lrcorner K) = -dw \lrcorner K)$$

$$= \underline{\Omega_{ip}\frac{\partial K^p}{\partial x^r}}$$

$$\underline{(H(x_i)=0 \text{ for } K(x_i)=0)}$$

Hence.

$$\int \Omega^n e^{itH} \sim \sum \frac{\sqrt{\det \Omega}\, e^{itH(x_i)}}{\sqrt{\det\left(\Omega \cdot \frac{\partial K}{\partial x}\right)}} \sim$$

where $e^{itH(x_i)} \to 0$

$$\sim \sum_{x_i} \frac{\cancel{e^{itH(x_i)}}\, 1}{\sqrt{\det \frac{\partial K}{\partial x}}}$$

Note: $\frac{\partial K}{\partial x}$ is linear operator at points where $K(x)=0$;

$$L_K = \frac{\partial K}{\partial x}: \quad L_K u = -[K, u]\big|.$$

We see that answer does not depend on choise of $\underline{w}$.

$$\int \sim \frac{1}{\sqrt{\det \frac{\partial K}{\partial x}}}$$

Our formula is a special case of DHL formula. (In particular $H(x_i)=0$).
On the other hand this formula emphasizes the role of vector field K: It states that

$$\int e^{it(d\omega + i_K \omega)} = \sum_{x_i : K(x_i)=0} \frac{c}{\sqrt{\det \frac{\partial K}{\partial x}}}$$

depends only on K at locus in the case if ω is an "arbitrary" K-invariant 1-form (of course $d\omega$ is not-degenerate).

It is useful to study DHL formula in its supersymmetric manifestation.

(See A. Schwarz, O. Zaboronsky "Supersymmetry and localization" CMP (1995 or 1996))

12 X 2013