Huigens principle

(Here I will present my calculations based on memories and textbooks...) Consider differential

Consider in $\mathbf{E}^n$ differential equation

$$\begin{cases} u_{tt} - \Delta u = 0 \\ u(t,\mathbf{x})\big|_{t=0} = \varphi(\mathbf{x}) \\ \frac{\partial u(t,\mathbf{x})}{\partial t}\big|_{t=0} = \psi(xx) \end{cases}$$

One can see that formal solution in Fourrier series will be

$$u(\mathbf{x},t) = \int e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}\left(\varphi(y)\cos kt + \psi(y)\frac{\sin kt}{k}\right)d^n\mathbf{k}d^n\mathbf{y}\,, \qquad (*)$$

$\mathbf{k}, \mathbf{x}$ are vectors, k is modulus of vector $\mathbf{k}$ $k = |\mathbf{k}|$ (Here and later we often omit coefficients: e.g. in the formula above we have omitted the coefficient $(2\pi)^{???}$). (All integrals ar assumed to be generalised functions.)

We calculate this integral and show that for odd n it implies Huigens.

Consider Green function

$$G^{(0)}(\mathbf{x}-\mathbf{y},t) = \int e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}\cos ktd^n\mathbf{k}\,.$$

(One can rewrite (*) in the following way: $u = G * \varphi + \partial_t G * \psi$).

Now calculate it. Ne can perform integration over sphere and radius. Volume form in $\mathbf{E}^n$ in spherical coordiantes will be $d^n k = d\Omega_n k^{n-1}dk$ and

$$G^{(0)}(\mathbf{x}-\mathbf{y},t) = \int e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}\cos ktd^n\mathbf{k} = \int e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}\cos ktd\Omega_n k^{n-1}dk =$$

$$\int_0^\infty \cos ktk^{n-1}\left(\left(\int_0^\pi e^{ik|\mathbf{x}-\mathbf{y}|\cos\theta}\sin^{n-2}\theta d\theta\right)d\Omega_{n-1}\right)dk =$$

$$w_{n-1}\int_0^\infty \cos ktk^{n-1}\left(\int_{-1}^1 e^{ik|\mathbf{x}-\mathbf{y}|u}(1-u^2)^{\frac{n-3}{2}}du\right)dk =$$

Here w_n is volume of $n-1$-dimensional unit sphere (unit sphere in $\mathbf{E}^n$),

$$w_0 = 2, w_1 = 2\pi\,, w_2 = 4\pi\ \ldots, w_n = 2\pi^{\frac{n+1}{2}}\Gamma\left(\frac{n+1}{2}\right).$$

(It is funny to note that volume of 0-dimensional sphere $\sigma_0 = 2$ is given by the general formula.)

Now we spcialize calculations for odd n. In the case if n is odd, then $(1-u^2)^{\frac{n-3}{2}}$ is just polynomial on u. We have that for odd n

$$G^{(0)}(\mathbf{x}-\mathbf{y},t) = w_{n-1}\int_0^\infty \cos kt k^{n-1}\left(\int_{-1}^1 e^{ik|\mathbf{x}-\mathbf{y}|u}\underbrace{(1-u^2)^{\frac{n-3}{2}}}_{\text{polynomial}\, P_n(u)}du\right)dk =$$

$$w_{n-1}\int_0^\infty \cos kt k^{n-1}\left(\left(P\left(\frac{d}{dz}\right)\int_{-1}^1 e^{zu}du\right)\Big|_{z=ik|\mathbf{x}-\mathbf{y}|}\right)dk =$$

$$(-1)^{n-1}w_{n-1}\left(\frac{d}{dt}\right)^{n-1}\int_0^\infty \cos kt\left(\left(P\left(\frac{d}{dz}\right)\int_{-1}^1 e^{zu}du\right)\Big|_{z=ik|\mathbf{x}-\mathbf{y}|}\right)dk\,.$$

One can say it in another way:

Statement For odd n the Green function belongs is generating by differential operator from the function

$$f = \int_0^\infty \cos kt\frac{\sin k|\mathbf{x}-\mathbf{y}|}{k|\mathbf{x}-\mathbf{y}|}dk = \frac{\pi}{2}\mathrm{sgn}\,\left(|\mathbf{x}-\mathbf{y}|-t\right)+\frac{\pi}{2}\mathrm{sgn}\,\left(|\mathbf{x}-\mathbf{y}|+t\right)$$

First perform calculations for $n=3$:

$$\text{for } n=3\ G^{(0)}(\mathbf{x}-\mathbf{y},t) = \int e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}\cos kt d^3\mathbf{k}\,.$$

Preliminary calculation: Calculate preliminary the average of the function $e^{i\mathbf{k}(\mathbf{x}-y)}$ over unit $n-1$-dimensional sphere (in $\mathbf{k}$ space.

Function $\mathbf{kx}$ is not constant on $n-1$ dimensional sphere $kx=1$, but it is constant on $n-2$ dimensional spheres $\mathbf{kx}\cos\theta = c$ (θ is angle between $\mathbf{k}$ and $\mathbf{x}$ and $|c|\le 1$). We have

$$F_n(kx) = < e^{i\mathbf{kx}} >_{k=1} = \frac{1}{\sigma_{n-1}}\int_{k=1} e^{i\mathbf{kx}}d\Omega_{n-1} = \frac{\sigma_{n-2}}{\sigma_{n-1}}\int_0^\pi e^{ix\cos\theta}\sin^{n-2}\theta d\theta$$

One can see that answers for even and odd will be different. For odd n it is just elementary function, and for even n they are expressed via special function $j(x) = \int_0^\pi e^{ix\cos\theta}d\theta$.

In moe details. First consider special cases (we ofen omit later all the coefficients....)

$$n=2, J(x) = F_2(x) = \sigma_0\int_0^\pi e^{ix\cos\varphi}d\varphi = \int_0^{2\pi} e^{ix\cos\varphi}d\varphi\,,$$

$$n=3, F_3(x) = \sigma_1\int_0^\pi e^{ix\cos\varphi}\sin\varphi d\varphi = 2\pi\int_{-1}^1 e^{ixu}du = 2i\frac{\sin x}{x}\,.$$

It is easy to see that the answer for $n = 0$ produces all the answers for even n and the answer for $n = 3$ produces all the answers for odd n:

One can see that all fnctions $F_(x)$ can be produced from function $J(a)$ and $f(a) = \frac{\sin a}{a}$ by differentiation, e,g,

$$F_7(x) = \sigma_5 \int_0^\pi e^{ix\cos\theta} \sin^5\theta d\theta = s_5 \int_0^\pi e^{ix\cos\theta} \sin^4\theta d\cos\theta = s_5 \int_0^\pi e^{ixu}(1-2u^2+u^4)du =$$

$$\left(1 + 2\frac{d^2}{du^2} + 4\frac{d^4}{du^4}\right) \int_0^\pi e^{ixu} du = 2i\sigma_5 \left(1 + 2\frac{d^2}{dx^2} + 4\frac{d^4}{dx^4}\right) \frac{\sin x}{x}$$

Now we return to the integral (*). Calculate it for odd n. Using functions $F_n(a)$ which are averaging of exponent over spere we come to

$$u(t, \mathbf{x}) = C_n \int e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} \left(\varphi(y)\cos kt + \psi(y)\frac{\sin kt}{k}\right) d^n\mathbf{k} d^n\mathbf{y}\,. = \int F_n(k|\mathbf{x}-\mathbf{y}|) \left(\varphi(y)\cos kt + \psi(y)\frac{\sin kt}{k}\right) k^{n-1} dk d^n\mathbf{y} =$$

We denote

$$G_n^{(0)}(\mathbf{x}, \mathbf{y}, t) = \int F_n(k|\mathbf{x}-\mathbf{y}|) \left(\varphi(y)\cos kt + \psi(y)\frac{\sin kt}{k}\right) k^{n-1} dk =$$

We see that

$$u(\mathbf{x}, t) = \int G(\mathbf{x}, \mathbf{y}, t) \left(\varphi(y)\cos kt + \psi(y)\frac{\sin kt}{k}\right) d^n\mathbf{k} d^n\mathbf{y}$$