

Gamma function

We know very well the integral representation of Gamma function

$$\Gamma(x) = (x-1)! = \int_0^\infty t^{x-1} e^{-t} dt \quad (1)$$

We also know very well that

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad (2)$$

where

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt.$$

In fact Euler worked with a different definition of Gamma-function, which was natural continuation on all x of factorial. Using this definition he came to formulae (1) and (2).

I cannot avoid temptation to recall here this topic.

In fact Euler observed gamma-function in the following way: He noted that

$$k! = \frac{(N+k)!}{(k+1)_N} = \frac{N!(N+1)_k}{(k+1)_N} = \lim_{N \rightarrow \infty} \frac{N!N^k}{(k+1)_N},$$

where $(A)_r = A(A+1) \dots (A+r-1)$. Hence one can define

$$x! = \lim_{N \rightarrow \infty} \frac{N!N^x}{(x+1)_N}$$

and respectively

$$\Gamma(x) = (x-1)! = \lim_{N \rightarrow \infty} \frac{N!N^{x-1}}{(x)_N}. \quad (3)$$

This definition looks not very beautiful, but one can easily imply equations (1), (2) and another identities.

E.g. one can easily see that equation (3) implies that

$$\frac{1}{\Gamma(x)} = x e^{\gamma x} \prod_{k=1}^{\infty} \left(\left(1 + \frac{x}{k}\right) e^{-\frac{x}{k}} \right),$$

where $\gamma = \lim_{N \rightarrow \infty} (1 + \frac{1}{2} + \dots + \frac{1}{N} - \log N)$ is Euler constant.

Indeed

$$\frac{1}{\Gamma(x)} =$$