Bernoulli numbers, Bernoulli polynomials...

We know that Bernoulli numbers appear everywhere in mathematics. I will consider here two topics: basic formulae for integrals leading to Euler Maclaurin type formulae and Fourier transformations formulae for calculations of ζ function at even points. In both cases naturally appear the same series of polynomials...

Shortly: Bernoulli Polynomials are polynomials $\{B_n\}$ of the degree n which are convinient fro integration by part as well as polynomials $\{x^n\}$. They also are orthogonal to constant function. These properties Bernoulli numbers are values of Bernoulli polynomials in boundary points.

§1 Integral and area of trapezium

Everybody who heard about integral knows that $\int_a^b f(t)dt$ equals approximately to the area of trapezoid with altitude $(b-a)$ and parallel sides equal to the values of the function f at the points a, b:

$$\int_a^b f(t)dt \approx (b-a)\cdot\frac{f(a)+f(b)}{2} \tag{1}$$

Very simple question: How this formula follows from the formula of integration by parts ($\int f(x)dx = xf(x) - \ldots$)? (I was surprirsed realising that I never asked myself this simple question before.)

Answer: Instead $\int f(x)dx = xf(x) - \int xf'(x)dx$ take $\int f(x)dx = (x+c)f(x) - \int (x+c)f'(x)dx$ putting $x+c$ instead x, where c is an arbitrary constant. Thus we come to

$$\int_a^b f(t)dt = (x+c)f(x)\big|_a^b - \int_a^b (t+c)f'(t)dt \tag{2}$$

Now if we choose $c = -\frac{a+b}{2}$ we come to (1).

$$\int_a^b f(t)dt = \left(x - \frac{a+b}{2}\right) f(x)\big|_a^b - \int_a^b \left(x - \frac{a+b}{2}\right) f'(t)dt = \frac{b-a}{2}(f(a)+f(b)) + \ldots \tag{2a}$$

One can go further performing integration by part. Keeping in mind formula (2a) instead an expansion

$$\int f(x)dx = xf(x) - \frac{x^2}{2}f'(x) + \frac{x^3}{3!}f''(x) - \frac{x^4}{4!}f'''(x) + \ldots \tag{3}$$

we consider an expansion

$$\int f(x)dx = B_1(x)f(x) - \frac{B_2(x)}{2}f'(x) + \frac{B_3(x)}{3!}f''(x) - \frac{B_4(x)}{4!}f'''(x) + \ldots, \tag{3a}$$

where polynomials $\{B_1(x), B_2(x), B_3(x), \ldots\}$ are defined by the relations $\frac{dB_{k+1}(x)}{dx} = kB_k(x)$:

$$B_1(x) = x + c_1,\ B_2(x) = 2\left(\frac{x^2}{2} + c_1x + c_2\right),\ B_3(x) = 6\left(\frac{x^3}{6} + c_1\frac{x^2}{2} + c_2x + c_3\right),$$

$$B_4(x) = 24\left(\frac{x^4}{24} + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3x + c_4\right),\ \text{and so on}, \tag{3b}$$

where $c_1, c_2, c_3, \ldots$ are an arbitrary constants. We have for an interval (a, b) that

$$\int_a^b f(t)dt = \sum_{n=1}^N \frac{B_n(x)}{n!} f^{(n-1)}(x)|_a^b + (-1)^N \int_a^b B_N(t)f^{(N)}(t)dt =$$

$$B_1(b)f(b) - B_1(a)f(a) + \frac{B_2(b)f'(b) - B_2(a)f'(a)}{2} + \frac{B_3(b)f''(b) - B_3(a)f''(a)}{2} + \ldots \tag{4}$$

Now encouraged by the trapezoid formula choose $c_1 = -\frac{a+b}{2}$. Then

$$B_2(a) = B_2(b)\,. \tag{5}$$

since $B_1(a) = B_1(b)$ if $c_1 = -\frac{a+b}{2}$. We want to keep the relation (5) for all $B_k(x)$ for $k \geq 2$:

$$B_k(a) = B_k(b) \quad \text{for all } k \geq 2 \tag{5a}$$

In this case the formula (4) becomes:

$$\int_a^b f(t)dt = (b-a)\cdot\frac{f(a)+f(b)}{2} + \sum_n B_n(a)\left(f^{(n-1)}(b) - f^{(n-1)}(a)\right) \tag{6}$$

§2 Bernoulli polynomials and numbers

The condition (5a) fixes uniquely all constants $c_2, c_3, \ldots, c_4, \ldots$ in (3). We come to recurrent formula for polynomials $B_n(x)$:

$$B_0(x) \equiv 1, \quad \begin{cases} B_k(x)\colon \ \frac{dB_k(x)}{dx} = kB_{k-1}(x) \\ \int_a^b B_k(x)dx = 0, \text{ i.e. } B_{k+1}(a) = B_{k+1}(b) \end{cases} \quad (k = 1,2,3,\ldots) \tag{2.1}$$

One can say roughly that polynomials $B_n(x) = x^n + \ldots$ are "deformations" of polynomials x^n suitable for integration by part.

These polynomials are:

$$\begin{array}{c} B_0(x) = 1 \\ B_1(x) = x - \frac{a+b}{2} \\ B_2(x) = (x-a)(x-b) + \frac{1}{6}(b-a)^2 \\ B_3(x) = (x-a)^3 - \frac{3}{2}(x-a)^2(b-a) + \frac{1}{2}(x-a)(b-a)^2 \\ B_4(x) = (x-a)^4 - 2(x-a)^3(b-a) + (x-a)^2(b-a)^2 - \frac{1}{30}(b-a)^4 \\ \ldots. \end{array} \tag{2.1a}$$

Consider *normalised* polynomials choosing $a = 0, b = 1$:

$$B_0(x) = 1,\ B'_n(x) = nB_{n-1}, \int_0^1 B_n(x)dx = 0, \text{ i.e. } B_{n+1}(0) = B_{n+1}(0), n = 1,2,\ldots:$$

$$\begin{array}{c} B_0(x) = 1 \\ B_1(x) = x - \frac{1}{2} \\ B_2(x) = x^2 - x + \frac{1}{6} \\ B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x \\ B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30} \\ B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{x}{6} \\ B_6(x) = x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{3}x^2 + \frac{1}{42} \\ B_7(x) = x^7 - \frac{7}{2}x^6 + \frac{7}{2}x^5 - \frac{7}{6}x^3 + \frac{1}{6}x \\ \ldots \end{array} \tag{2.2}$$

Exercise 1 Show that relation between normalised polynomials $B_n^{[0,1]}$ in (7) and polynomials $B_n^{[a,b]}$ is

$$B_n^{[a,b]}(x) = (a-b)^n B_n^{[0,1]}\left(\frac{x-a}{b-a}\right) \tag{2.3}$$

This formula controls the behaviour of Bernoulli polynomials under changing of a, b.

We define *Bernoulli number* b_n as a value of polynomial (2.2) at the points 0 or 1 (or polynomial (2.1a) divided by a coefficient $(b-a)^n$)

$$b_n = B_n(0) = B_n(1).$$

We have

$$b_0 = 1, b_1 = -\frac{1}{2}, b_2 = \frac{1}{6}, b_3 = 0, b_4 = -\frac{1}{30}, b_5 = 0, b_6 = \frac{1}{42}, b_7 = 0, \dots$$

Interesting observation:

Proposition 1 *Bernoulli numbers b_n are equal to zero if n is an odd number bigger than 1.*

This proposition follows from the following very beautiful property of Bernoulli polynomials:

Proposition 2 *Let $\{B_n(x)\}$ be a set of Bernoulli polynomials corresponding to the interval (a,b) (see eq. (7)). Let P be a reflection with respect to the middle point $\frac{a+b}{2}$ of the interval (a,b):*

$$P\colon\ x \mapsto a + b - x \tag{2.4}$$

Then all Bernoulli polynomials (except B_1) are eigenvectors of this transformation:

$$B_n(Px) = B_n(x) \text{ for all even } n,\ n = 0, 2, 4, \dots \tag{2.5a}$$

and

$$B_n(Px) = -B_n(x) \text{ for all odd } n \geq 3,\ n = 3, 5, 7, \dots \tag{2.5b}$$

Indeed it follows from (2.5b) that $b_n = B_n(a) = -B_n(b) = -b_n$ for odd $n \geq 3$. Thus $b_n = 0$ for $n = 3, 5, 7, \dots$.

The statement of this Proposition 2 is irrelevant to the choice of a, b. To prove the Proposition it is suffice to consider the special case $a = -b$. In this case the transformation P in (2.4) is just $x \mapsto -x$. Thus in this case the statement of Proposition is that Bernoulli polynomials $B_n(x)$ are even polynomials ($B_n(x) = B_n(-x)$) if n is even, and they are odd polynomials if n is an odd number greater than 1 ($B_n(x) = -B_n(-x)$).

Prove it by induction. Suppose that for $n \leq 2N$ this is true. Then consider polynomial $B_{2N}(x)$. We have that $\int_{-a}^{a} B_{2N}(x)dx = 0$, hence $\int_0^a B_{2N}(x)dx = 0$ since by induction hypothesis this is an even polynomial. Hence

$$B_{2N+1}(x) = (2N+1)\int_0^x B_{2N}(t)dt\,.$$

Indeed this polynomial obeys the differential equation $B'_{2N+1}(x) = (2N+1)B_{2N}(x)$ This polynomial is also an odd polynomial. Hence it obeys the boundary condition $\int_{-a}^{a} B_{2N+1}(x)dx = 0$. It remains to prove that B_{2N+2} is an even polynomial. We have that $B_{2N+2}(x) = \int_0^x B_{2N+1}(t)dt + c_{2N+2}$, where c_{2N+2} is a constant chosen by the boundary condition $\int_a^a B_{2N+2}(x)dx = 0$. We see that B_{2N+2} is even since B_{2N+1} is an odd polynomial and constant is an even polynomial.

§3 Integral and area of trapezium (revisited)

Now equipped by the knowledge of formulae return to the last formula from the first paragraph:

$$\int_a^b f(t)dt = (b-a)\cdot\frac{f(a)+f(b)}{2} + \sum_{n\geq 2} B_n(a)\left(f^{(n-1)}(b) - f^{(n-1)}(a)\right) = \tag{3.1}$$

$$(b-a)\cdot\frac{f(a)+f(b)}{2} + \sum_{n\geq 2} b_n(b-a)^n\left(f^{(n-1)}(b) - f^{(n-1)}(a)\right) = \tag{3.1a}$$

$$(b-a)\cdot\frac{f(a)+f(b)}{2} + \sum_{k\geq 1} b_{2k}(b-a)^{2k}\left(f^{(2k-1)}(b) - f^{(2k-1)}(a)\right) = \tag{3.1a}$$

$$(b-a)\cdot\frac{f(a)+f(b)}{2} + \frac{1}{6}(b-a)^2\left(f'(b) - f'(a)\right) - \frac{1}{30}(b-a)^4\left(f'''(b) - f'''(a)\right) + \dots$$

We are ready to write down assymptotic formula for series: Dividing the interval $[0,1]$ on $N+1$ parts consider the formula above for any interval $\left[\frac{k}{N}, \frac{k+1}{N}\right]$, then making summation we come to:

$$\int_0^1 f(x)dx = \frac{1}{2}f(0)+\left(f\left(\frac{1}{N}\right)+f\left(\frac{2}{N}\right)+\ldots+f\left(\frac{N-1}{N}\right)\right)+\frac{1}{2}f(0)+\sum_{k\geq 1}\frac{b_{2k}}{N^{2k}}\left(f^{(2k-1)}(1)-f^{(2k-1)}(0)\right)$$

Exercise Use this formula for the functions $f=x^r$ to express sums $\sum_{i=1}^{N} i^r$ via Bernoulli numbers.

§4 **Fourier image of Bernoulli polynomials and ζ-function**

Bernoulli polynomials are deformations of x^n which are convenient for integration by part. Function e^x is eigenvalue of derivation operator. This means that Bernoulli polynomials have "good" Fourier transform. Do calculations. Consider Forier polynomials for the interval $[0,1]$ (see 2.2) and an orthonormal basis $c_k\{e^{2\pi ikx}\}$ where .... Indeed

$$\langle B_n(x), e^{2\pi ik}\rangle = \int_0^1 B_n(x), e^{2\pi ikx} \sim \frac{1}{k^n}$$

Hence

$$\langle B_n(x), e^{2\pi ik}\rangle \sim \sum \frac{1}{k^{2n}} = \zeta(2n)$$

Notice that square of the norms of Bernoulli polynomials can be expressed via Bernoulli numbers due to their properties...