A number such that its square finishes on it and Veselov's comment on it

Find a number x such that its square , a number x^2 finishes with it. More precisely this means the following: We say that a number x *has m-digits and it is finished with it* if

$$10^m \leq x < 10^{m+1}\,, and \quad x^2 - x \text{divides } 10^m\,. \tag{1}$$

Remark E.g. a number $x = 0625$ has 4 digits and it is finished with it:

$$625^2 - 625 = 390000\,.$$

This is very old problem for me It is one of the first problems which I solved.

Theorem There are two exactly two sequences

$$a_1, a_2, \dots, a_n, \dots = 5, 25, 625,$$

and

$$b_1, a_2, \dots, a_n, \dots = 6, 76, 376,$$

such that

1) all the numbers in these sequences obey the cohdition, that their squares finishes by them.

2) Any number a_n in the first sequence possesses not more than n digits: $a_n < 10^n$.

2) Any number such that its square finish by it belongs to the first or to the second sequence

This follows from the following induction statement

Lemma

Suppose by induction that a number a_n contains n digits (zeroes are permitted (see the Remark)). and it is finished by it. Then there is exists a number a_{n+1} which contains $n+1$ digits (zeroes are permitted (see the Remark)) and it is finished by it.

$$a_n < 10^n, a_n^2 - a_n = 0(mod 10^n) \Rightarrow \text{there exists a number } a_{n+1} \text{ such that}$$

$$a_{n+1}\colon a_{n+1} < 10^n, a_{n+1}^2 - a_{n+1} = 0(mod 10^{n+1})\,, \tag{4}$$

where $a_{n+1} = 10^n x + a_n$ ($x = 0, 1, \dots, 9$) (if $x = 0$ then a number a_{n+1} may "slip" a digit. (see the remark after equation (1).)) First note that

$$a_n^2 - a_n = 10^n s_n$$

with s_n integer. Hence

$$a_{n+1}^2 - a_{n+1} = (10^n x + a_n)^2 - (10^n x + a_n) = 10^n \left(10^n x^2 + 2x a_n + s_n - x\right)\,.$$

This expression has to be divisible on 10^{n+1}. Hence one has to choose x such that

$$x(2a_n - 1) + s_n = 0(mod 10) \tag{5}$$

Consider examples.

Take $N = 1$ and $a_1 = 5$, $(10^{n_1} = 10)$,

$a_1^2 - a_1 = 20$, $s_1 = \frac{a_1^2 - a_1}{10} = 2$. Choose x in (5). We have

$$9x + 2 = 0 \Rightarrow x = 2(mod 10)\,, a_2 = 10 \cdot 2 + 5 = 25\,.$$

Take $N = 2$ and $a_2 = 25$,$(10^{n_2} = 100)$,

$a_2^2 - a_2 = 600$, $s_2 = \frac{a_2^2 - a_2}{100} = 6$. Choose x in (5). We have

$$49x + 6 = 0(mod 10)\,, 9x + 6 = 0(mod 10)\,, x = 6(mod 10)\,, a_3 = 100 \cdot 6 + 25 = 625\,.$$

Take $N = 3$ and $a_3 = 625$,$(10^{n_3} = 1000)$,

$a_3^2 - a_3 = 390.000$, $s_3 = \frac{a_3^2 - a_3}{1000} = 390$. Choose x in (5). We have

$$1249x + 390 = 0(mod 10)\,, 9x = 0(mod 10)\,, x = 0(mod 10)\,, a_4 = 100 \cdot 0 + 25 = 0625\,.$$

Take $N = 4$ and $a_4 = 0625$, $(10^{n_4} = 10.000)$,

$a_4^2 - a_4 = 390.000$, $s_4 = \frac{a_4^2 - a_4}{10.000} = 39$. Choose x in (5). We have

$$1249x + 39 = 0(mod 10)\,, 9x + 9 = 0(mod 10), x = 9.\,, a_5 = 10.000 \cdot 9 + 0625 = 90625$$

Take $N = 5$ and $a_5 = 90625$, $(10^{n_5} = 100.000)$,

$a_5^2 - a_5 = 8212800000$, $s_5 = \frac{a_5^2 - a_5}{100.000} = 82128$. Choose x in (5). We have

$$181249x + 82128 = 0(mod 10)\,, 9x + 8 = 0(mod 10)\,, x = 8.\,, a_6 = 100.000 \cdot 8 + 90625 = 890.625$$

Take $N = 6$ and $a_6 = 890625$, $(10^{n_6} = 1.000.000)$, . $a_6^2 - a_6 = 793212.000.000$ $s_5 = \frac{a_5^2 - a_5}{100.000} = 793212$. Choose x in (5). We have

$$1781249x + 793212 = 0(mod 10)\,, 9x + 2 = 0(mod 10)\,, x = 2\,, a_7 = 1.000.000 \cdot 2 + 890625 = 2.890.m625$$

We come to the answer:

there is the infinite sequence $\{5, 25, 625, 0625, 90625, 890625, 2890625, \ldots\}$ such that

$$5^2 = 25, 25^2 = 625, 625^2 = 390.625, 90625^2 = 8212$$