30 July 2017

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: Lioville Theorem prohibits conformal bijection of plane on the disc, whilst there is no problem of conformal bikjection half-plane to disc

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31 July 2017

Study conformal mapings, Dirichle problem (Lavrentiev book)

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Let C be a curve with pieces (arcs) C_α. (e.g. polygon)

Let $f = f(t)$ be a function on C, which has jumps. Let D_i be internal points of arcs C_α, where f has a jump. Let A_i be vertices A_α: the arc C_α goes from the vertex A_α to the vertex $A_{\alpha+1}$. Note that at vertices we have a jump of angles: at the vertex A_α the jump

$$\delta\varphi_\alpha = \varphi_\alpha^{(+)} - \varphi_\alpha^{(-)} \,.$$

Let $F(z)$ be function with jumpes h_k at the points D_k, where curve is smooth and with jumps H_α at the vertices A_α

Consider (with Lavrentiev-Shabad) the new function

$$\tilde{F}(z) = F(z) + \frac{1}{\pi}\sum_k \arg(z - D_k) - \sum_\alpha \frac{H_\alpha}{\delta\varphi_a}\arg(z - A_\alpha)$$

This function has no jumps!!!

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1 August 2017

How look linear factional maps which send disc $x^2 + y^2 = 1$ onto itself: this is representation of the group $SL(2,R)/??$ in Poincare disc.

(Linear fractional bijections of upper-half plane onto itself are $SL(2,R)/Z_2$?)

If

$$w = \frac{Az + B}{Cz + D}$$

and $|w| = 1$ if $|z| = 1$, then

$$z\bar{z} = 1 \to w\bar{w} = 1, i.e. (Az + B)(\bar{A}\bar{z} + \bar{B}) = (Cz + D)(\bar{C}\bar{z} + \bar{D})$$

we come to

$$|A|^2 + |B|^2 + (A\bar{B}z + c.c.) = |C|^2 + |D|^2 + (C\bar{D}z + c.c.)\,.$$

We see that for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$,

$$|A|^2 + |B|^2 = |C|^2 + |D|^2\,, \quad \text{and}\,, \quad A\bar{B} - C\bar{D} = 0 \tag{$*$}$$

This means that g up to dilation preserves hermitian scalar product with signature (1.1), we come to the group $SU(1,1)/\{1.-1\}$.

Indeed Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ Then conditions (*) means that

$$g^+ \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot g = \begin{pmatrix} \bar{A} & \bar{C} \\ \bar{B} & \bar{D} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

i.e. removing scaling factor λ and putting $\det g = 1$ we come to

$$g^+ = \begin{pmatrix} \bar{A} & \bar{C} \\ \bar{B} & \bar{D} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} D & -B \\ -C & A \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} D \\ C \end{pmatrix}$$

i.e. $D = \bar{A}, C = \bar{B}$ Thus symmetries of Poincare disc is $SU(1,1)/\{1,-1\}$, where

$$SU(1.1) = \left\{ g = \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} : |A|^2 - |B|^2 = 1 \right\}\,.$$

How look linear factional maps which send disc $x^2 + y^2 = 1$ onto itself: this is representation of the group $SL(2,R)/??$ in Poincare disc.

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$$SU(1.1) = \left\{ g = \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} : |A|^2 - |B|^2 = 1 \right\} .$$

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2 August 2017

Let ξ be a point in the unit disc, then the trasformation

$$w = w(z) = \frac{z = \xi}{1 - \bar{\xi} z}$$

trasforms disc onto disc and the point ξ on the centre.

M (), :

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3 August 2017

Let $u = u(x,y)$ be harmonic function in the disc, and let z_0 be its singulr point. Consider conjugaate function v:

$$v_x = -u_y \, , v_y = -u_x$$

it is antiderivative of 1-form $dv = u_y dx - u_x dy$. It defines multivalued function $V = \int (u_y dx - u_x dy)$ whic is defined up to a period

$$\Pi = \int_C (u_y dx - u_x dy) \, .$$

We come to multivalued holomorphic function $u + iv$ with period Π. On the other hand the function

$$\frac{\Pi}{2\pi} Log(z - z_0)$$

is multivalued, with the same period. Hence the function:

$$F(z) = u(z) + iv(z) - \frac{\Pi}{2\pi} \log(z - z_0)$$

has disconnected leaves. This is multivalued function $f(z) \to f(z) + i\Pi$. Consider the function

$$G(z) = \exp\left(-\frac{2\pi F(z)}{\Pi}\right) = \exp\left(u(z) + iv(z) - \frac{\Pi}{2\pi}\log(z - z_0)\right) = (z-z_0)\exp\left(-\frac{2\pi F(z)}{\Pi}\right)$$

This is ONE-VALUED! holomorphic function:

$$f(z) \to f(z) + i\Pi\,, G(z) \to \exp\frac{2\pi}{\Pi}\left(F(z) + i\Pi\right) = G(z)\,.$$

(here ther are slight inconveniences.....)

This is holomorphic function in the disc. Consider function

$$G$$

, , !

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4 August 2017

Why the function multivalued function $\arctan\frac{y}{x}$ appeared often in calculations? Because this function is conjugate to $\log\sqrt{x^2+y^2}$:

$$\log z = \log(x + iy)\log\sqrt{x^2+y^2} + \arctan\frac{y}{x}$$

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5 August 2017

Why the function multivalued function $\arctan\frac{y}{x}$ appeared often in calculations? Because this function is conjugate to $\log\sqrt{x^2+y^2}$:

$$\log z = \log(x + iy)\log\sqrt{x^2+y^2} + \arctan\frac{y}{x}$$

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6 August 2017

I know well angle function and its relation with Green function:

In fact Green function generalises this concept.

recall sketchly : Let $G_U == G(z_0, z)$ be a Green function of the domain U, i.e. the function such that

1) $G \approx \log(z - z_0)$ in a vicnity of the point z_0,

2) it is harmonic elswhere in U except a point z_0

3) it vanishes at boundary ∂U

Using the identity:

$$\int_U u\Delta v - \int_U v\Delta u = \int_{\partial U} u * dv - \int_{\partial V} v * du = \qquad (identity)$$

we solve the boundary problem

$$\begin{cases} \Delta F = f \\ F\big|_{\partial U} = \mu \end{cases}, \qquad F = \int_U G \circ f + \int_{\partial U} *dG\mu$$

where $\circ$ is convolution. (this is called Dirichle problem if $f \equiv 0$.)

On the other hand one can come to the Green function G_U trhough

1) Green function $G_\infty(z_0, z) = -\frac{1}{2\pi}\log(z - z_0)$

2) and the "angle function" defined on the boundary solving (with Arsenin) the integral equation. Recall that angle function defines for every curve the function on plane which is equal to the angle that we look at this curve *.

Remark *We wrote identity (ident.) in a way to emphasize as much as possible the relation between 'angle' function and Green function. This identity is written in standard way as following:*

$$\int_U u\Delta v - \int_U v\Delta u = \oint_{\partial U} u\frac{\partial v}{\partial n}d\mathbf{S} - \oint_{\partial V} v\frac{\partial u}{\partial n}d\mathbf{S} = \qquad (identity')$$

(see any book). In fact for every vector field $\mathbf{A}$

$$\underbrace{\int}_{C} \mathbf{A}d\mathbf{s}_{\textit{flux of } \mathbf{A} \textit{ trhough the surface } C} = \int \Omega\rfloor\mathbf{A}\,,$$

where Ω *is volume* 2*-form, and*

$$\frac{\partial f}{\partial n} = \operatorname{grad} f$$

and $*$ *is Hodge operation:*

$$*df = \Omega\rfloor\operatorname{grad} f$$

Notice that for harmonic function u, $d^{-1}(*du)$ *is conjugate function:*

$$\Delta u(x,y) = 0 \Leftrightarrow u(x,y) + id^{-1}(*du(x,y)) = F(z) \textit{ is holomorphoc function}$$

---

* another name: double layer potential

, *e.g.*

$$d^{-1}(*d\log\sqrt{x^2+y^2}) = \arctan\frac{y}{x}\,, \log x^2+y^2+i\arctan\frac{y}{x} = \log(x+iy)$$

it is how angle function relates with normal derivative of Green function.

Thus on the space $C(\partial U)$ of functions on boundary we have to linear operators:
First operator:

$$C(\partial U)\ni\nu\mapsto\mu\in C(\partial U):\quad s(t)=\int_{\partial U}L(t,t')\nu(t')dt'$$

Second operator

$$C(\partial U)\ni\nu\mapsto W\in C(U):\quad W(\mathbf{r})=\int_{U}L(\mathbf{r},t')\nu(t')dt'$$

On the other hand we know that for the angle function L the difference of values of these operators on the boundary ∂U is equal to $\pi\nu(t)$:

$$C(\partial U)\ni\nu\mapsto\left(\lim_{\mathbf{r}\to t} int_U L(\mathbf{r},t')\nu(t')dt'\right)-\int_{\partial U}L(t,t')\nu(t')dt'=\pi\nu(t)$$

Thus to solve the Dirichle problem, i.e. to reconstruct harmonic function W by its value ν at the boundary $\partial U_{0\_}$ we first solving integral equation find a function ν such that

$$\pi\nu+\int_{\partial U}L(t,t')\nu(t')dt'=\mu(t)$$

then we reconstruct W in terms of ν.

Example
Tro to recontruct harmonic function W in the disc $x^2+y^2<1$ using different methods.
First: Green function:

$$W=\int_{x^2+y^2=1}*dG$$

but here we need to know $G=G(z_0,0)$ We already know the conformal map whoch transforms circle to circle, and we know the Green fucntion for $z_0=0$, hence

$$G(z_0,z)=-\frac{1}{2\pi}\log\left|\frac{z-z_0}{1-\bar{z_0}z}\right|$$

Another way to calculate:

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7 August 2017

Calculation for Green function of disc and half-plane.

Recall that if $G_U(z_0, z)$ is Green function of domain U then the identity

$$\int_U (u\Delta v - v\Delta v) = \int_{\partial U} (u * dv - v * du) \tag{0.1}$$

implies that the function

$$u(\zeta) = \mathcal{G}_U(\zeta, z) f(z)\Omega + \int_{\partial U} *d_{(z)} G(\zeta, z)\mu\,, \quad (\Omega \text{is area form on } z\text{-plane}) \tag{0.2}$$

is the solution of boundary problem

$$\begin{cases} \Delta u = f \\ u\big|_{\partial U} = \mu \end{cases} \tag{0.3}$$

Green function for disc

Having a map

$$z \to w = \frac{z - \zeta}{1 - \bar{\zeta} z}$$

which maps unic disc onto unit disc and boundary on boundary, and the point ζ on the centre we define

$$G(z_0, z) = -\frac{1}{2\pi} \log \left| \frac{z - z_0}{1 - \bar{z}_0 z} \right|$$

One can see that

$$*dG_z = \Omega \rfloor \operatorname{grad} G_z = r dr \wedge d\varphi \rfloor \left(\frac{\partial G}{\partial r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial G}{\partial \varphi} \frac{\partial}{\partial \varphi} \right) = r \frac{\partial G}{\partial r} \frac{\partial}{\partial r} d\varphi - \frac{1}{r} \frac{\partial G}{\partial \varphi} dr$$

($z = x + iy = r\cos\varphi_i r \sin\varphi$, $z_0 = R\cos\theta + iR\sin\theta$) We need derivatives of Green function on boundary $r = 1$:

$$*dG_z \big| |z| = 1 = -\Omega \rfloor \operatorname{grad} G_z \big| |z| = 1 = -r \frac{\partial G}{\partial r} \frac{\partial}{\partial r} \big|_{r=1} d\varphi =$$

$$-r \frac{\partial}{\partial r} \left(-\frac{1}{2\pi} \log \left| r^{i\varphi} - Re^{i\theta} \right| + \frac{1}{2\pi} \log \left| 1 - Re^{-i\theta} r^{e^{i\varphi}} \right| \right) \big|_{r=1} d\varphi =$$

$$-r \frac{\partial}{\partial r} \left(-\frac{1}{2\pi} \log \sqrt{R^2 - 2Rr\cos(\theta - \varphi) r + r^2} + \frac{1}{2\pi} \log \sqrt{1 - 2Rr\cos(\theta - \varphi) r + R^2 r^2} \right) \big|_{r=1} d\varphi =$$

$$-\frac{1}{2\pi} \frac{1 - R^2}{1 - 2R\cos(\theta - \varphi) + R^2} d\varphi$$

Remark (Peut etre un problem de sign?)

This we have the solution of Dirichle problme for the circle: if W harmonic function in the disc D: $x^2+y^2<1$ such that $W_{\partial D}=\mu(\varphi)$ $(r\to 1????)$ then

$$W(R,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\mu(\varphi)\frac{1-R^2}{1-2R\cos(\theta-\varphi)+R^2}d\varphi\,.$$

Look at this formula. It is useful sometimes to write it:

$$W(R,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\mu(\varphi)\frac{1-R^2}{(1-Re^{i(\theta-\varphi)})(1-Re^{-i(\theta-\varphi)})}d\varphi\,.$$

Notice also that If $\mu(\varphi)=C$ then obviously

$$W(R,\theta)=\begin{cases} C \text{ for } R<1\\ \frac{C}{2} \text{ for } R=1\end{cases}$$

if $\mu(\varphi)=e^{in\varphi}$, $n\geq 0$ then

$$W(R,\theta)=\begin{cases} z^n=R^ne^{in\varphi} \text{ for } R<1\\ \text{for } R=1\end{cases}$$

then we come to harmonic polynomials.....

Remark *In the formula (*) there is a jump, so to be more precise we have to write:*

$$W(R,\theta)=\lim_{R\to R_-}\frac{1}{2\pi}\int_0^{2\pi}\mu(\varphi)\frac{1-R^2}{(1-Re^{i(\theta-\varphi)})(1-Re^{-i(\theta-\varphi)})}d\varphi\,.$$

sure this is important only for $R=1$:

$$W(1,\theta)=\lim_{R\to 1_-}\frac{1}{2\pi}\int_0^{2\pi}\mu(\varphi)\frac{1-R^2}{(1-Re^{i(\theta-\varphi)})(1-Re^{-i(\theta-\varphi)})}d\varphi\,.$$

Remark 2 *All this can be done much more easilly using Fourier transform : Indeed if $\mu(\varphi=e^{ik\theta})$ then $W=r^ke^{ik\varphi}$, hence*

$$W(R,\theta)=\sum\mu_kR^ke^{ik\theta}=\frac{1}{2\pi}\sum_k\left(\int_0^{2\pi}\mu(\varphi)e^{-ik\varphi}d\varphi\right)R^{|k|}e^{ik\theta}=$$

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{k=-\infty}^{k=\infty}R^{|k|}e^{ik(\theta-\varphi)}\right)d\varphi=\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{1-Re^{i(\theta-\varphi)}}+\frac{1}{1-Re^{-i(\theta-\varphi)}}-1\right)d\varphi$$

Sure we can solve Dirichle problem just using angle function. Recall shortly the calculation of angle function

$$Angle_C(\mathbf{R}) = \int_C (x-X)dy - (y-Y)dx$$

(compare with $*d\log|z| = \frac{xdy-ydx}{(x-X)^2+(y-Y)^2}$). Put $x=\cos\varphi, y=\sin\varphi$ we come to

$$L(R,\theta,\varphi)d\varphi = \frac{(x-X)dy-(y-Y)dx}{(x-X)^2+(y-Y)^2}|_{X=R\cos\theta, Y=R\sin\theta, x=\cos\varphi, y=\sin\varphi} =$$

$$\frac{1-R\cos(\theta-\varphi)}{1-2R\cos(\theta-\varphi)+R^2}d\varphi$$

Now we solve Dirichle problem via integral equation.

Consider the distribution $\nu(\varphi)$ on the circle. It defines thefunction:

$$U(R,\theta) = \int_0^{2\pi} \nu(\varphi)L(R,\theta,\varphi)d\varphi = \int_0^{2\pi} \nu(\varphi)\frac{1-R\cos(\theta-\varphi)}{1-2R\cos(\theta-\varphi)+R^2}d\varphi\,.$$

This function is harmonic in the interior of the disc , in the exterior of the dis, but it has the jump:

$$U(R,\theta)\big|_{R=1-0_+} = U(R,\theta)\big|_{R=1} + \pi\nu(\theta)\,,\quad U(R,\theta)\big|_{R=1+0_+} = U(R,\theta)\big|_{R=1} - \pi\nu(\theta)\,.$$

In particular we come to the harmonic function in the disc such that its values at the boundary are equal to

$$\mu(\varphi) = U(R,\theta)\big|_{R=1-0_+} = U(R,\theta)\big|_{R=1} + \pi\nu(\theta) =$$

$$\int_0^{2\pi} \nu(\varphi)\frac{1-R\cos(\theta-\varphi)}{1-2R\cos(\theta-\varphi)+R^2}|_{R=1}d\varphi + \pi\nu(\theta) = \frac{1}{2}\int_0^{2\pi}\nu(\varphi)d\varphi + \pi\nu(\theta)$$

This is linear integral equation on the function $\nu(\theta)$:

$$\mu(\theta) = \frac{1}{2}\int_0^{2\pi}\nu(\varphi)d\varphi + \pi\nu(\theta)\,.$$

Solving it we come to

$$\nu(\theta) = \frac{\mu(\theta)}{\pi} - \frac{1}{4\pi^2}\int_0^{2\pi}\mu(\varphi)d\varphi\,.$$

Hence

$$U(R,\theta) = \int_0^{2\pi}\nu(\varphi)L(R,\theta,\varphi)d\varphi =$$

$$\int_0^{2\pi}\left[\frac{\mu(\varphi)}{\pi}-\frac{1}{4\pi^2}\int_0^{2\pi}\mu(\tau)d\tau\right]L(R,\theta,\varphi)d\varphi=$$

$$\int_0^{2\pi}\mu(\varphi)\frac{L(R,\theta,\varphi)}{\pi}-\int_0^{2\pi}\frac{L(R,\theta,\varphi}{\pi}d\varphi\cdot\frac{1}{4\pi^2}\int_0^{2\pi}\mu(\tau)d\tau=$$

$$\int_0^{2\pi}\mu(\varphi)\frac{L(R,\theta,\varphi)}{\pi}-2\pi\cdot\frac{1}{4\pi^2}\int_0^{2\pi}\mu(\varphi)d\varphi=\frac{1}{\pi}\int_0^{2\pi}\mu(\varphi)\left(L(R,\theta,\varphi)-\frac{1}{2}\right)d\varphi=$$

$$\frac{1}{\pi}\int_0^{2\pi}\mu(\varphi)\left(\frac{1-R\cos(\theta-\varphi)}{1-2R\cos(\theta-\varphi)+R^2}-\frac{1}{2}\right)d\varphi=\frac{1}{2\pi}\int_0^{2\pi}\mu(\varphi)\left(\frac{1-R^2}{1-2R\cos(\theta-\varphi)+R^2}-\frac{1}{2}\right)d\varphi$$

Thus we come to the solutioj of Dirichle problem.

Remark *Sure using angle function we can reconstruct not only* $*dG$ *but the Green fucntion also. We take the ordinary Green function,* $G_\infty(z_0,z)$ *and using Dirichle problem solution we will find harmonic function* W *such that* $G_\infty+W$ *becomes the Green function: this harmonic function is the soluton of Dirichle problem* $W\big|_{boundary}=-G_\infty\big|_{boundary}$

Upper-half plane: Green function, Dirichle problem

G We know that $w=\frac{iz+1}{z+i}$ maps $\mathbf{H}\leftrightarrow D$ (in fact this is not arbitrary: see the blog on 10-th August)

taking composition we come to the map:

$$w=w(\xi,z)=\frac{\frac{iz+1}{z+i}-\frac{i\zeta+1}{\zeta+i}}{1-\frac{-i\bar\zeta+1}{\zeta-i}\frac{iz+1}{z+i}}$$

maps upper half plane H on the disc $x^2+y^2<1$ and a point ζ on the centre.

Hence Green function for half-plane is

$$G(z_0,z)=\log|w(z_0,z)| \tag{3.1}$$

Remark this is very stupid way to calculate Green function. Much easier another way: see the blog on 10 August!

It seems that here it is much easier way to calculate Dirichle problem solution straightforwardly, using angle function. We will do it, but in tomorrow file you will see that it is much easier to calculate straightforwardly Green function using reciprocity method.

Indeed due to

$$A_C(\mathbf{R})=\int_C\frac{(x-X)dy-(y-Y)dx}{(x-X)^2+(y-Y)^2}$$

Put $x=t,y=0$ we come to

$$L(\mathbf{R},t)dt=\frac{Y dt}{(t-X)^2+Y^2}=d\left(\arctan\frac{t-X}{y}\right)$$

The potential of double layer with density $\nu(t)$ (compare with disc) is equal to

$$U(X,Y) = \int_{-\infty}^{\infty} \nu(t) \frac{Y dt}{(t-X)^2+Y^2} eqno(2.1)$$

in the same way as for the circle

$$U(X,0_+) = \pi\nu(x) + U(X,0),$$

but for the boundary of half plane this function vanishes: the calculations are simpler, we do not need to solve the integral equation. The function (2.1) is the solution of Dirichle problem:

$$\begin{cases} \Delta W = 0 \\ W\big| = \mu(x) \end{cases} \Rightarrow W(X,Y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \mu(t) \frac{Y dt}{(t-X)^2+Y^2} \qquad (2.2)$$

Remark *One can see straightforwardly that the function*

$$F(X) = \lim_{Y\to 0} \frac{Y dt}{(t-X)^2+Y^2} = \pi\delta(t-x)\,.$$

Indeed $F(X) = 0$ *for all* $X \neq t$ *and* $\int F(X)dX = \pi$

This implies the boundary condition

Calculate Green function using solution of Dirichle problem: Let

$$G_{\mathbf{H}}(z_0,z) = G_{\text{classic}}(z_0,z) + W =$$

$$-\frac{1}{2\pi}\log|z-z_0| + W = -\frac{1}{2\pi}\log\sqrt{(x-X)^2+(y-Y)^2} + W(x,y)$$

The condition that $G_{\mathbf{H}}$ vanishes at absolute implies that

$$W(x,y)\big|_{y=0} = \frac{1}{2\pi}\log\sqrt{(x-X)^2+(y-Y)^2}\big|_{y=0} = \frac{1}{2\pi}\log\sqrt{(x-X)^2+Y^2}$$

thus we have that in Dirichle problem (2.2) $\mu(t) = \frac{1}{2\pi}\log\sqrt{(t-X)^2+Y^2}$, thus

$$W(x,y) = \frac{1}{2\pi^2}\int_{-\infty}^{\infty} \log\sqrt{(t-X)^2+Y^2}\frac{y dt}{(t-x)^2+y^2}$$

and

$$G_{\mathbf{H}}(z_0,z) = -\frac{1}{2\pi}\log\sqrt{(x-X)^2+(y-Y)^2} + \frac{1}{2\pi^2}\int_{-\infty}^{\infty} \log\sqrt{(t-X)^2+Y^2}\frac{y dt}{(t-x)^2+y^2}\,.$$

It seems that calculation of the integral here are not an easy task. But instead, note that the Green function can be immediately obtained using symmetry arguments:

$$G_{\mathbf{H}}(z_0,z) = -\frac{1}{2\pi}\left(\log|z-z_0| - \log|z-\bar{z}_0|\right) = -\frac{1}{2\pi}\log\sqrt{(x-X)^2+(y-Y)^2} + \frac{1}{2\pi}\log\sqrt{(x-X)^2+$$

Thus:

$$\int_{-\infty}^{\infty} \log\sqrt{(t-X)^2+Y^2}\frac{ydt}{(t-x)^2+y^2} = \ldots \log|z-\bar z_0| = \ldots \log\sqrt{(x-X)^2+(y+Y)^2} +$$

Beautiful, is not it???

7 August 2017

Calculation for Green function of disc and half-plane.

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implies that the function

$$u(\zeta) = \mathcal{G}_U(\zeta, z) f(z)\Omega + \int_{\partial U} *d_{(z)} G(\zeta, z)\mu\,, \quad (\Omega \text{is area form on } z\text{-plane}) \tag{0.2}$$

is the solution of boundary problem

$$\begin{cases} \Delta u = f \\ u\big|_{\partial U} = \mu \end{cases} \tag{0.3}$$

Green function for disc

Having a map

$$z \to w = \frac{z-\zeta}{1-\bar\zeta z}$$

which maps unic disc onto unit disc and boundary on boundary, and the point ζ on the centre we define

$$G(z_0, z) = -\frac{1}{2\pi}\log\left|\frac{z-z_0}{1-\bar z_0 z}\right|$$

One can see that

$$*dG_z = \Omega \rfloor \mathrm{grad}\, G_z = r dr\wedge d\varphi \rfloor \left(\frac{\partial G}{\partial r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial G}{\partial \varphi}\frac{\partial}{\partial \varphi}\right) = r\frac{\partial G}{\partial r}\frac{\partial}{\partial r}d\varphi - \frac{1}{r}\frac{\partial G}{\partial \varphi}dr$$

($z = x + iy = r\cos\varphi_i r\sin\varphi$, $z_0 = R\cos\theta + iR\sin\theta$) We need derivatives of Green function on boundary $r = 1$:

$$*dG_z\big||z| = 1 = -\Omega\rfloor \mathrm{grad}\, G_z\big||z| = 1 = -r\frac{\partial G}{\partial r}\frac{\partial}{\partial r}\big|_{r=1} d\varphi =$$

$$-r\frac{\partial}{\partial r}\left(-\frac{1}{2\pi}\log\left|r^{i\varphi} - Re^{i\theta}\right| + \frac{1}{2\pi}\log\left|1 - Re^{-i\theta} r^{e^{i\varphi}}\right|\right)\big|_{r=1} d\varphi =$$

$$-r\frac{\partial}{\partial r}\left(-\frac{1}{2\pi}\log\sqrt{R^2-2Rr\cos(\theta-\varphi)r+r^2}+\frac{1}{2\pi}\log\sqrt{1-2Rr\cos(\theta-\varphi)r+R^2r^2}\right)\Big|_{r=1}d\varphi =$$

$$-\frac{1}{2\pi}\frac{1-R^2}{1-2R\cos(\theta-\varphi)+R^2}d\varphi$$

Remark (Peut etre un problem de sign?)

This we have the solution of Dirichle problme for the circle: if W harmonic function in the disc D: $x^2+y^2<1$ such that $W_{\partial D}=\mu(\varphi)$ $(r\to 1????)$ then

$$W(R,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\mu(\varphi)\frac{1-R^2}{1-2R\cos(\theta-\varphi)+R^2}d\varphi\,.$$

Look at this formula. It is useful sometimes to write it:

$$W(R,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\mu(\varphi)\frac{1-R^2}{(1-Re^{i(\theta-\varphi)})(1-Re^{-i(\theta-\varphi)})}d\varphi\,.$$

Notice also that If $\mu(\varphi)=C$ then obviously

$$W(R,\theta)=\begin{cases} C \text{ for } R<1\\ \frac{C}{2} \text{ for } R=1\end{cases}$$

if $\mu(\varphi)=e^{in\varphi}$, $n\geq 0$ then

$$W(R,\theta)=\begin{cases} z^n=R^ne^{in\varphi} \text{ for } R<1\\ \text{ for } R=1\end{cases}$$

then we come to harmonic polynomials.....

Remark *In the formula (*) there is a jump, so to be more precise we have to write:*

$$W(R,\theta)=\lim_{R\to R_-}\frac{1}{2\pi}\int_0^{2\pi}\mu(\varphi)\frac{1-R^2}{(1-Re^{i(\theta-\varphi)})(1-Re^{-i(\theta-\varphi)})}d\varphi\,.$$

sure this is important only for $R=1$:

$$W(1,\theta)=\lim_{R\to 1_-}\frac{1}{2\pi}\int_0^{2\pi}\mu(\varphi)\frac{1-R^2}{(1-Re^{i(\theta-\varphi)})(1-Re^{-i(\theta-\varphi)})}d\varphi\,.$$

Remark 2 *All this can be done much more easilly using Fourier transform : Indeed if $\mu(\varphi=e^{ik\theta})$ then $W=r^ke^{ik\varphi}$, hence*

$$W(R,\theta)=\sum\mu_kR^ke^{ik\theta}=\frac{1}{2\pi}\sum_k\left(\int_0^{2\pi}\mu(\varphi)e^{-ik\varphi}d\varphi\right)R^{|k|}e^{ik\theta}=$$

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{k=-\infty}^{k=\infty} R^{|k|}e^{ik(\theta-\varphi)}\right)d\varphi = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{1-Re^{i(\theta-\varphi)}}+\frac{1}{1-Re^{-i(\theta-\varphi)}}-1\right)d\varphi$$

Sure we can solve Dirichle problem just using angle function. Recall shortly the calculation of angle function

$$Angle_C(\mathbf{R}) = \int_C (x-X)dy-(y-Y)dx$$

(compare with $*d\log|z| = \frac{xdy-ydx}{(x-X)^2+(y-Y)^2}$). Put $x=\cos\varphi, y=\sin\varphi$ we come to

$$L(R,\theta,\varphi)d\varphi = \frac{(x-X)dy-(y-Y)dx}{(x-X)^2+(y-Y)^2}\Big|_{X=R\cos\theta, Y=R\sin\theta, x=\cos\varphi, y=\sin\varphi} =$$

$$\frac{1-R\cos(\theta-\varphi)}{1-2R\cos(\theta-\varphi)+R^2}d\varphi$$

Now we solve Dirichle problem via integral equation.

Consider the distribution $\nu(\varphi)$ on the circle. It defines thefunction:

$$U(R,\theta) = \int_0^{2\pi}\nu(\varphi)L(R,\theta,\varphi)d\varphi = \int_0^{2\pi}\nu(\varphi)\frac{1-R\cos(\theta-\varphi)}{1-2R\cos(\theta-\varphi)+R^2}d\varphi\,.$$

This function is harmonic in the interior of the disc , in the exterior of the dis, but it has the jump:

$$U(R,\theta)\big|_{R=1-0_+} = U(R,\theta)\big|_{R=1}+\pi\nu(\theta)\,,\quad U(R,\theta)\big|_{R=1+0_+} = U(R,\theta)\big|_{R=1}-\pi\nu(\theta)\,.$$

In particular we come to the harmonic function in the disc such that its values at the boundary are equal to

$$\mu(\varphi) = U(R,\theta)\big|_{R=1-0_+} = U(R,\theta)\big|_{R=1}+\pi\nu(\theta) =$$

$$\int_0^{2\pi}\nu(\varphi)\frac{1-R\cos(\theta-\varphi)}{1-2R\cos(\theta-\varphi)+R^2}\Big|_{R=1}d\varphi+\pi\nu(\theta) = \frac{1}{2}\int_0^{2\pi}\nu(\varphi)d\varphi+\pi\nu(\theta)$$

This is linear integral equation on the function $\nu(\theta)$:

$$\mu(\theta) = \frac{1}{2}\int_0^{2\pi}\nu(\varphi)d\varphi+\pi\nu(\theta)\,.$$

Solving it we come to

$$\nu(\theta) = \frac{\mu(\theta)}{\pi}-\frac{1}{4\pi^2}\int_0^{2\pi}\mu(\varphi)d\varphi\,.$$

Hence

$$U(R,\theta)=\int_0^{2\pi}\nu(\varphi)L(R,\theta,\varphi)d\varphi=$$

$$\int_0^{2\pi}\left[\frac{\mu(\varphi)}{\pi}-\frac{1}{4\pi^2}\int_0^{2\pi}\mu(\tau)d\tau\right]L(R,\theta,\varphi)d\varphi=$$

$$\int_0^{2\pi}\mu(\varphi)\frac{L(R,\theta,\varphi)}{\pi}-\int_0^{2\pi}\frac{L(R,\theta,\varphi}{\pi}d\varphi\cdot\frac{1}{4\pi^2}\int_0^{2\pi}\mu(\tau)d\tau=$$

$$\int_0^{2\pi}\mu(\varphi)\frac{L(R,\theta,\varphi)}{\pi}-2\pi\cdot\frac{1}{4\pi^2}\int_0^{2\pi}\mu(\varphi)d\varphi=\frac{1}{\pi}\int_0^{2\pi}\mu(\varphi)\left(L(R,\theta,\varphi)-\frac{1}{2}\right)d\varphi=$$

$$\frac{1}{\pi}\int_0^{2\pi}\mu(\varphi)\left(\frac{1-R\cos(\theta-\varphi)}{1-2R\cos(\theta-\varphi)+R^2}-\frac{1}{2}\right)d\varphi=\frac{1}{2\pi}\int_0^{2\pi}\mu(\varphi)\left(\frac{1-R^2}{1-2R\cos(\theta-\varphi)+R^2}-\frac{1}{2}\right)d\varphi$$

Thus we come to the solutioj of Dirichle problem.

Remark *Sure using angle function we can reconstruct not only* $*dG$ *but the Green fucntion also. We take the ordinary Green function,* $G_\infty(z_0,z)$ *and using Dirichle problem solution we will find harmonic function* W *such that* $G_\infty+W$ *becomes the Green function: this harmonic function is the soluton of Dirichle problem* $W\big|_{boundary}=-G_\infty\big|_{boundary}$

Upper-half plane: Green function, Dirichle problem

G We know that $w=\frac{iz+1}{z+i}$ maps $\mathbf{H}\leftrightarrow D$

taking composition we come to the map:

$$w=w(\xi,z)=\frac{\frac{iz+1}{z+i}-\frac{i\zeta+1}{\zeta+i}}{1-\frac{-i\bar\zeta+1}{\bar\zeta-i}\frac{iz+1}{z+i}}$$

maps upper half plane H on the disc $x^2+y^2<1$ and a point ζ on the centre.

Hence Green function for half-plane is

$$G(z_0,z)=\log|w(z_0,z)| \tag{3.1}$$

It seems that here it is much easier to calculate Dirichle problem solution straightforwardly, using angle function. We will do it, but in tomorrow file you will see that it is much easier to calculate straightforwardly Green function using reciprocity method.

Indeed due to

$$A_C(\mathbf{R})=\int_C\frac{(x-X)dy-(y-Y)dx}{(x-X)^2+(y-Y)^2}$$

Put $x=t, y=0$ we come to

$$L(\mathbf{R},t)dt=\frac{Y dt}{(t-X)^2+Y^2}=d\left(\arctan\frac{t-X}{y}\right)$$

The potential of double layer with density $\nu(t)$ (compare with disc) is equal to

$$U(X,Y) = \int_{-\infty}^{\infty} \nu(t) \frac{Y dt}{(t-X)^2 + Y^2} eqno(2.1)$$

in the same way as for the circle

$$U(X, 0_+) = \pi\nu(x) + U(X, 0),$$

but for the boundary of half plane this function vanishes: the calculations are simpler, we do not need to solve the integral equation. The function (2.1) is the solution of Dirichle problem:

$$\begin{cases} \Delta W = 0 \\ W\big| = \mu(x) \end{cases} \Rightarrow W(X,Y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \mu(t) \frac{Y dt}{(t-X)^2 + Y^2} \tag{2.2}$$

Remark *One can see straightforwardly that the function*

$$F(X) = \lim_{Y \to 0} \frac{Y dt}{(t-X)^2 + Y^2} = \pi\delta(t-x)\,.$$

Indeed $F(X) = 0$ *for all* $X \neq t$ *and* $\int F(X)dX = \pi$

This implies the boundary condition

Calculate Green function using solution of Dirichle problem: Let

$$G_{\mathbf{H}}(z_0, z) = G_{\mathrm{classic}}(z_0, z) + W =$$

$$-\frac{1}{2\pi} \log|z - z_0| + W = -\frac{1}{2\pi} \log\sqrt{(x-X)^2 + (y-Y)^2} + W(x,y)$$

The condition that $G_{\mathbf{H}}$ vanishes at absolute implies that

$$W(x,y)\big|_{y=0} = \frac{1}{2\pi} \log\sqrt{(x-X)^2 + (y-Y)^2}\big|_{y=0} = \frac{1}{2\pi} \log\sqrt{(x-X)^2 + Y^2}$$

thus we have that in Dirichle problem (2.2) $\mu(t) = \frac{1}{2\pi} \log\sqrt{(t-X)^2 + Y^2}$, thus

$$W(x,y) = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} \log\sqrt{(t-X)^2 + Y^2} \frac{y dt}{(t-x)^2 + y^2}$$

and

$$G_{\mathbf{H}}(z_0, z) = -\frac{1}{2\pi} \log\sqrt{(x-X)^2 + (y-Y)^2} + \frac{1}{2\pi^2} \int_{-\infty}^{\infty} \log\sqrt{(t-X)^2 + Y^2} \frac{y dt}{(t-x)^2 + y^2}\,.$$

It seems that calculation of the integral here are not an easy task. But instead, note that the Green function can be immediately obtained using symmetry arguments:

$$G_{\mathbf{H}}(z_0, z) = -\frac{1}{2\pi} \left(\log|z - z_0| - \log|z - \bar{z}_0|\right) = -\frac{1}{2\pi} \log\sqrt{(x-X)^2 + (y-Y)^2} + \frac{1}{2\pi} \log\sqrt{(x-X)^2 +$$

Thus:

$$\int_{-\infty}^{\infty} \log\sqrt{(t-X)^2+Y^2}\frac{ydt}{(t-x)^2+y^2} = \ldots\log|z-\bar{z}_0| = \ldots\log\sqrt{(x-X)^2+(y+Y)^2}+$$

Beautiful, is not it???

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8 August 2017

Calculation for half-plane, encore

Yesterday we did difficult caclulations for half-plane. On the other hand it is obvious that

$$-\frac{1}{2\pi}\left(\log|z-z_0|-\log|z+\bar{z}_0|\right)$$

is a Green function! This imeediately gives answer for $*dG$ on the absolute and implies many identities! inncluding the relation for the integral (see the end of yesterday file.)

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10 August 2017

Conformal map and in terms of Green function

This is the topic why I began to read the Lavrentiev inspired by French book. Now it is alright. If we know the Green function $G_U(z_0, z)$ then it defines in $U\backslash\{z_0\}$ holomorphic function $\hat{G} = G + iU$ where U is defined up to period $\Pi = \oint \ldots$

Thus we come to the map $e^{\frac{\ldots G}{\Pi}}$.

(comment ca marche: the question of bijection????)

Example How to construct $\mathbf{H} \to D$ We know that

$$G_{\mathbf{H}} = \ldots\left(\log|z-z_0| - \log|z-\bar{z}_0|\right)$$

(coefficient $-\frac{1}{2\pi}$)

Hence $G(z_0, z) =\approx \frac{z-z_0}{z-\bar{z}_0}$

is it unique? Yes up to the coefficient $e^{i\varphi}$.

In the blog on 7 August we considered function

$$\frac{G(z_0,z)-G(z_0,\zeta)}{1-G(\bar{z_0}.\zeta)G(z_0,z)}$$

Thus we see that this function is equal to $\frac{z-\zeta}{z-\zeta}$ up to a coefficient.

Remark *We used here that every F holomorphic map of Disc $|z| < 1$ onto itself such that $F(\zeta) = 0$ is nothing but* $\frac{z-\zeta}{1-\bar{\zeta}z}e^{i\varphi}$