Killing vectors and Levi-Ciovita conenction

This lecture is the addition to the subsection 1.6 "Infinitesimal isometries of Riemannian manifold"

Let M be Riemannian manifold with Riemannian metric G. Recall that a vector field $\mathbf{K}$ is Killing vector field, i.e. it defines infinitesimal isometry, (given in local coordinates by the formula $x^i \mapsto x^i + \varepsilon K^i$, $\varepsilon^2 = 0$) if Lie derivative of metric with respect to vector field vanishes:

$$\mathcal{L}_{\mathbf{K}} G = K^i \frac{\partial g_{mn}(x)}{\partial x^i} + \frac{\partial K^i(x)}{\partial \partial x^m} g_{in}(x) + \frac{\partial K^i(x)}{\partial \partial x^n} g_{im}(x) = 0\,. \tag{1}$$

(see subsection "Infinitesimal isometries of Riemannian manifold" of Lecture notes.)

Let ∇ be Levi-Civita connection of Riemannian metric: in local coordinates Christoffel symbols of this connection are

$$\Gamma^i_{km} = \frac{1}{2} g^{ij}(x) \left(\frac{\partial g_{jk}}{\partial x^m} + \frac{\partial g_{jm}}{\partial x^k} - \frac{\partial g_{km}}{\partial x^j} \right)\,, \tag{2}$$

where g^{ij} is tensor inverse to g_{ik}.

Consider the following construction.

Let $\mathbf{K}$ be an arbitrary vector fields (not necessarily Killing vector field) on manifold M, Consider the following operation on vector fields: to every vector field $\mathbf{X}$ we assign the vector field

$$\left(\nabla_{\mathbf{K}} - \mathcal{L}_{\mathbf{K}}\right) \mathbf{X}, \tag{3}$$

where ∇ is Levi-Civita connection (2) and w$\mathcal{L}_{\mathbf{K}}$ is Lie derivative of $\mathbf{X}$ with respect to $\mathbf{K}$:

$$\mathcal{L}_{\mathbf{K}} \mathbf{X} = [\mathbf{K}, \mathbf{X}], \quad [\mathbf{K}, \mathbf{X}]^i = K^m x \frac{\partial X^i(x)}{\partial x^m} - X^m x \frac{\partial K^i(x)}{\partial x^m},$$

Hence we have that equation (3) can be written as

$$\left(\nabla_{\mathbf{K}} - \mathcal{L}_{\mathbf{K}}\right) \mathbf{X} =$$

$$\left(\left(K^m \left(\frac{\partial X^i}{\partial x^m} + \Gamma^i_{mn} X^n \right) \right) - \left(K^m \frac{\partial X^i}{\partial x^m} - X^m \frac{\partial K^i}{\partial x^m} \right) \right) \frac{\partial}{\partial x^i} =$$

$$X^m \frac{\partial K^i}{\partial x^m} + K^m \Gamma^i_{mn} X^n = X^m \frac{\partial K^i}{\partial x^m} + X^n \Gamma^i_{mn} K^M = X^m \frac{\partial K^i}{\partial x^m} + X^n \Gamma^i_{nm} K^m = \nabla_{\mathbf{X}} \mathbf{K} \tag{4}$$

since Levi-Civita connection is symmetric: $\Gamma^i mn = \Gamma^i_{nm}$. We see that operation (3) defines linear operator $A_{\mathbf{K}}$ in the following way: to every vector $\mathbf{X}$ tangent to manifold at the point $\mathbf{p}$, $\mathbf{X} \in T_{\mathbf{p}} M$ operator $A_{\mathbf{K}}$ assigns vector $\nabla_{\mathbf{X}} \mathbf{K}$:

$$T_{\mathbf{p}} M \ni \mathbf{X} \mapsto A_{\mathbf{K}}(\mathbf{X}) = \nabla_{\mathbf{X}} \mathbf{K} = \left(\left(\frac{\partial K^i}{\partial x^m} + \Gamma^i_{mn} K^n \right) X^m \right) \frac{\partial}{\partial x^i} \tag{4a}$$

It follows from formulae (3) and (4) that for every vector field $\mathbf{X}$ on M

$$A_{\mathbf{K}}(\mathbf{X}) = (\nabla\mathbf{X} - \mathcal{L}_{\mathbf{K}})\,\mathbf{X}\,. \tag{4b}$$

In the left hand side of this formula $\mathbf{X}$ is the value of vector field at the given point $\mathbf{p}$ and on the right hand side covarinat derivative and Lie derivative act on vector field. More accurate we have to write this formula in the following way:

$$A_{\mathbf{K}}\left(\mathbf{X}\big|_{\mathbf{p}}\right) = \left((\nabla_{\mathbf{K}} - \mathcal{L}_{\mathbf{K}})\,\mathbf{X}\right)\big|_{\mathbf{p}} \tag{4c}$$

Now we formulate and prove the Proposition

Proposition Let M be Riemannian manifold and ∇ be its Levi-Civita connection. Vector field $\mathbf{K}$ is Killing vector field of Riemannina manifold M if and only if the linear operator $A_{\mathbf{K}}$ defined by (4) is antisymmetric operator, i.e. if for arbitrary vector fields $\mathbf{X}, \mathbf{Y}$

$$\langle A_{\mathbf{K}}(\mathbf{X}), \mathbf{Y}\rangle = -\langle \mathbf{X}, A_{\mathbf{K}}(\mathbf{Y})\rangle\,. \tag{5}$$

where $\langle\ ,\ \rangle$ is Riemannian scalar product:

$$\langle \mathbf{X}, \mathbf{Y}\rangle = X^i(x) g_{ik}(x) Y^k(x)\,.$$

Proof

Let $\mathbf{X}, \mathbf{Y}$ be two arbitrary vector fields.

The condition that ∇ is Levi-Civita connection means that

$$\mathcal{L}_{\mathbf{K}}\langle \mathbf{X}, \mathbf{Y}\rangle = \partial_{\mathbf{K}}\langle \mathbf{X}, \mathbf{Y}\rangle = \langle \nabla_{\mathbf{K}}\mathbf{X}, \mathbf{Y}\rangle + \langle \mathbf{X}, \nabla\mathbf{Y}\rangle\,. \tag{6}$$

(See the subsection in Lecture notes about Levi-Civita connection.) (Lie derivative of function is just directional derivative: $\mathcal{L}_{\mathbf{K}}F = \partial_{\mathbf{K}}F$).

The condition that $\mathbf{K}$ is Killing vector field is the condition that $\mathbf{K}$ preserves Riemannian metric, scalar product (see (1)). Hence

$$\mathcal{L}_{\mathbf{K}}\langle \mathbf{X}, \mathbf{Y}\rangle = \partial_{\mathbf{K}}\langle \mathbf{X}, \mathbf{Y}\rangle = \langle \mathcal{L}_{K}\mathbf{X}, \mathbf{Y}\rangle + \langle \mathbf{X}, \mathcal{L}_{\mathbf{X}}\mathbf{Y}\rangle \tag{6a}$$

Now substract an equation (6a) from the equation (6). Using relations (4a,4b,4c) we come to

$$\langle(\nabla_{\mathbf{K}} - \mathcal{L}_{\mathbf{K}})\,\mathbf{X}, \mathbf{Y}\rangle + \langle \mathbf{X}, (\nabla_{\mathbf{K}} - \mathcal{L}_{\mathbf{K}})\,\mathbf{Y}\rangle = \langle \nabla_{\mathbf{X}}\mathbf{K}, \mathbf{Y}\rangle + \langle \mathbf{X}, \nabla_{\mathbf{Y}}\mathbf{K}\rangle = \langle A_{\mathbf{K}}(\mathbf{X}), \mathbf{Y}\rangle + \langle \mathbf{X}, A_{\mathbf{K}}(\mathbf{Y})\rangle = 0\,.\blacksquare$$

Thus relation (5) is proved $\blacksquare$