Two formulae for determinants

Let A be $m \times n$ matrix and B be $n \times m$ matrix. Then

$$\det(1 + MN) = \det(1 + NM)\,. \tag{1}$$

Proof. $\mathrm{Tr}\,(MN)^k = \mathrm{Tr}\,(NM)^k$. Hence characteristic polynomials $\det(1+zMN)$ and $\det(1+zNM)$ coincide. Thus we come to (1). How to prove it in another way?

It follows from (1) that if $M = (x^1, x^2, \ldots, x^n)$ is $1 \times n$ matrix then

$$\det(\delta^{ik} + x^i x^k) = \det(1 + M^+ M) = \det(1 + MM^+) = \det(1 + x^i x^i) = 1 + (x^1)^2 = \ldots + (x^n)^2$$

The relation (1) has very interesting "generalisation":

Let $\mathcal{B}, \mathcal{D}$ be $p \times p$ matrices such that there entries are... odd "numbers", i.e. anticommuting elements of some $\mathbf{Z_2}$-algebra (e.g. odd elements of Grassmann algebra):

$$\mathcal{B} = ||\mathcal{B}_{ik}|| \text{ such that } \mathcal{B}_{ik}\mathcal{B}_{rs} = -\mathcal{B}_{rs}\mathcal{B}_{ik}$$

the same for $\mathcal{D}$:

$$\mathcal{D} = ||\mathcal{D}_{ik}|| \text{ such that } \mathcal{D}_{ik}\mathcal{D}_{rs} = -\mathcal{D}_{rs}\mathcal{D}_{ik}$$

Then the following identity holds:

$$\det(1 + \mathcal{B}\mathcal{D}) = 1\ , \tag{2}$$

if $\mathcal{B}$ is symmetrical matrix and $\mathcal{D}$ is antisymmetrical matrix: $\mathcal{B}_{ik} = -\mathcal{B}_{ik}$, $\mathcal{D}_{ik} = \mathcal{D}_{ik}$. This is very important identity*. This identity follows from the fact that for an arbitrary $k = 1, 2, 3, \ldots$

$$\mathrm{Tr}\ (\mathcal{B}\mathcal{D})^k = 0\,. \tag{3}$$

Indeed in the same way as for identity (1) the relation (3) implies that characteristic polynomial $\det(1+z\mathcal{B}\mathcal{D})$ equals to 1. Proof of (3) immediately follow from the facts that

$$\mathrm{Tr}\, A^+ = \mathrm{Tr}\, A,\ \ \mathrm{Tr}\,(AB) = (-1)^{p(B)p(A)}\mathrm{Tr}\,(BA),\ \text{ and }\ \mathcal{B}^+ = \mathcal{B},\ \mathcal{D}^+ = -\mathcal{D}\,.$$

E.g.

$$(\mathcal{B}\mathcal{D}\mathcal{B}\mathcal{D})^+ = \mathcal{D}^+\mathcal{B}^+\mathcal{D}^+\mathcal{B}^+ = \mathcal{D}\mathcal{B}\mathcal{D}\mathcal{B}\,.$$

Hence

$$\mathrm{Tr}\,(\mathcal{B}\mathcal{D})^2 = \mathrm{Tr}\ (\mathcal{B}\mathcal{D}\mathcal{B}\mathcal{D}) = \mathrm{Tr}\ (\mathcal{B}\mathcal{D}\mathcal{B}\mathcal{D})^+ = \mathrm{Tr}\ (\mathcal{D}\mathcal{B}\mathcal{D}\mathcal{B}) = -\mathrm{Tr}\ (\mathcal{B}\mathcal{D}\mathcal{B}\mathcal{D}) = -\mathrm{Tr}\,(\mathcal{B}\mathcal{D})^2\,,$$

i.e. $\mathrm{Tr}\,(\mathcal{B}\mathcal{D})^2 = 0$.

* In particular it follows form this identity that square root of Berezinian (superdeterminant) of linear canonical transformation is equal to the determinant of its boson-boson sector