§1 **Conformally flat manifolds, and conformal class of metric**

We say that manifold M is provided with conformal flat structure if it is provided by atlas $\{x^i_{(\alpha)}\}$ (which is compatible with smooth structure) such that transition functions $\Psi_{\alpha\beta}$ are conformal transformations.

Conformal transformations are:

i) translation

$$x^i_{(\beta)} = x^i_{(\alpha)} + a^i \qquad \text{(Conf.trans.1)}$$

ii) Orthogonal transformations

$$x^i_{(\beta)} = P^i_k x^i_{(\alpha)}, \ \text{ where } P^i_k \text{ is an orhogonal matrix}\,, \qquad \text{(Conf.trans.2)}$$

iii) Inversion

$$x^i_{(\beta)} = \frac{x^i_{(\alpha)}}{\sum_i x^i_{(\alpha)} x^i_{(\alpha)}} \qquad \text{(Conf.trans.3)}$$

and any composition of these transformations.

We say that Riemannian metric is conformally flat if in a vicinity of an arbitrary point there exist coordinates x^i such that in these coordinates $g_{ik} = e^\sigma(x)\delta_{ik}$.

Very important exercise:

Proposition

Let metric be flat in coordinates x^i, i.e. it has an appearance $g_{ik} = \delta_{ik}$ in this coordinates. Then

1) This metric has an an appearance $g_{ik} = e^\sigma(x)\delta_{i'k'}$ in new coordinates $x^{i'}$ if and only if changing of coordinates is conformal, i.e. it is a composition of transformations (Conf.trans.).

In other words transformations of local cartesian coordinates (Conf.trans.1—Conf.trans.3) and only these transformations and their compositions preserve the angles.

2) The following statement *is not true*:

Let metric has an appearance $g_{ik} = e^\sigma(x)\delta_{ik}$ in coordinates x^i where $\sigma(x)$ is an arbitrary smooth function. Then there exist new coordinates such that in these new coordinates metric is flat, i.e. it has an appearance $g_{ik} = e^\sigma(x)\delta_{ik}$

Show counterexample.

Theorem A manifold M can be endowed with flat conformal structure if and only if there exist conformally flat metric on this manifold.

Sketch of the proof.

Let atlas $\{x^i_{(\alpha)}\}$ provides M with flat conformal structure, i.e. all transition functions are conformal (See equations (Conf.trans.)). Consider locally defined Riemannian metrics $g^{(\alpha)}_{ik} = \delta_{ik}$ and 0-cochain $\{\sigma_{(\alpha)}\}$ such that

$$\sigma_{(\alpha)} - \sigma_{(\beta)} = t_{\alpha\beta}\,,$$

where $t_{\alpha\beta}$ is logarifm of Jacobian of conformal transformations from coordinates $x^i_{(\alpha)}$ to coordinates $x^i_{(\beta)}$. One can see that the right hand side defined closed 1-cochain, hence 0-cochain $\sigma_{(\alpha)}$ is a coboundary. Using partition we see that

$$\sigma_{(\alpha)} = \ldots ?? \sum_\gamma t_{\alpha\gamma}\varphi_\gamma$$

and $g_{ik} = \mathbf{e}^{\sigma(\alpha)}\delta_{ik}$ defines globally conformally flat Riemannian metric on M.

Exercise: Reconstruct exactly the right hand side of the right hand side of the lasst expression.

Now prove the converse implication

Let $\{x^i_\alpha\}$ be an atlas such that in this atlas Riemannian metric has an appearance

$$g^{(\alpha)}_{ik} = \mathbf{e}^{\sigma_{(\alpha)}}\delta_{ik}$$

Under change of coordinates $x^i_{(\beta)} = x^i_{(\beta)}(x^i_{(\alpha)})$ metric is changed on the scalar function, i.e. angles do not change. Thus this follows from Proposition 1 that transition functions are conformal ■

§2 **Conformally flat metrics , and Weil tensor**

1Exercise Calculate variation of curvature tensor R^i_{klm} under conformal variation of metrics.

1Exercise Write down the tensor (linear combination of R^i_{klm}, R^i_k and R with coefficients formed by the tensor g_{ik} such that this tensor is invariant with respect to conformal transformations.

Thus we will write Weyl tensor:

$$C^i_{klm} = R^i_{klm} + \ldots$$

Theorem: Weyl curvature vanishes if and only if the metric is conformally flat, i.e. flat conformal structure exists.