Let $L = L(q, \dot{q})$ be a Lagrangian on $\mathbf{E}^3$ which is invariant with respect to rotations up to derivatives:

$$\delta_i L = d\alpha_i, \quad \text{i.e.}\,, \mathcal{L}_{\hat{M}_i} L = \frac{d}{dt}\alpha_i(q) = \frac{\partial \alpha_i(q)}{\partial q^k}\dot{q}^k\,, \tag{1}$$

where

$$\hat{M}_i = \varepsilon_{imk} q^m \partial_k, \quad \begin{cases} \hat{M}_1 = y\partial_z - z\partial_y \\ \hat{M}_2 = z\partial_x - x\partial_z \\ \hat{M}_1 = x\partial_y - y\partial_x \end{cases}, \quad [\hat{M}_i, \hat{M}_k] = \varepsilon_{ikm}\hat{M}_m\,. \tag{2}$$

Show that this generalised symmetry is not essentially generalised, i.e. one can redefine Lagrangian $\tilde{L} = L - F$ such that for new Lagrangian $\delta_i \tilde{L} = 0$, i,e, we can find a function $F = F(q)$ such that

$$\delta_i F = \alpha_i, \; i.e. \; \mathcal{L}_{M_i}\left(\frac{dF}{dt}\right) = \mathcal{L}_{M_i}\left(\frac{\partial F(q)}{\partial q^k}\dot{q}^k\right) = \frac{d}{dt}\alpha_i(q) = \frac{\partial \alpha_i(q)}{\partial q^k}\dot{q}^k\,, \tag{3}$$

i.e. for the new Lagrangian $\tilde{L} = L - F$,

$$\delta_i \tilde{L} = \delta_i (L - F) = 0\,. \tag{4}$$

To show the existence of 'coboundary' F which transform generalised symmetry to usual one, recall that it follows from equation (1) that

$$d\left((\delta\alpha)_{ij}\right) = (\delta\,(d\alpha))_{ij} = (\delta^2 L)_{ij} = 0 \Rightarrow (\delta\alpha)_{ij} = \omega_{ij} \text{ are constants} \tag{5}$$

i.e.

$$(\delta\alpha)_{ij} = \delta_i\alpha_j - \delta_j\alpha_i - \varepsilon_{ijm}\alpha_m = \hat{M}_i(\alpha_j) - \hat{M}_j(\alpha_i) - \varepsilon_{ijm}\alpha_m = \omega_{ij} \tag{5a}$$

is a cocycle in constants. On the other hand

$$\omega_{ij} = \varepsilon_{ijm} t_m\,, \quad (t_m = \frac{1}{2}\varepsilon_{mpq}\omega_{pq}). \tag{6}$$

(This simple relation means that $H^2(so(3), \mathbf{R}) = 0$)

If we redefine $\alpha_i \mapsto \alpha_i + t_i$ then for new α_i we have

$$(\delta\alpha)_{ij} = \omega_{ij} \mapsto (\delta\alpha)_{ij} = 0\,.$$

So from now on we will consider that cocyclw in equation (5) vanishes: $\omega_{ij} = 0$:

$$\delta_i\alpha_j - \delta_j\alpha_i - \varepsilon_{ijm}\alpha_m = \hat{M}_i(\alpha_j) - \hat{M}_j(\alpha_i) - \varepsilon_{ijm}\alpha_m = 0\,. \tag{5c}$$

Now we solve equation (3) using condition (5c). Applying operator $\hat{M}_i$ to (5c) we come to

$$0 = \hat{M}_i\left(\hat{M}_i(\alpha_j) - \hat{M}_j(\alpha_i) - \varepsilon_{ijm}\alpha_m\right) = \hat{M}^2\alpha_j - \left[\hat{M}_i\hat{M}_j\right]\alpha_i - \hat{M}_j\left(\hat{M}_i\alpha_i\right) - \varepsilon_{ijm}\hat{M}_i\alpha_m =$$

$$\hat{M}^2\alpha_j - \varepsilon_{ijm}\hat{M}_m\alpha_i - \hat{M}_j\left(\hat{M}_i\alpha_i\right) - \varepsilon_{ijm}\hat{M}_i\alpha_m = \hat{M}^2\alpha_j - \hat{M}_j\left(\hat{M}_i\alpha_i\right) = 0$$

Thus we come to the equation

$$\hat{M}^2\alpha_j = \hat{M}_j\left(\hat{M}_i\alpha_i\right) . \tag{7}$$

Consider the expansion of $\alpha_i(q)$ over harmonics:

$$\alpha_i(q) = \sum_l \alpha_i^{(l)}(q) \, , \tag{8}$$

where we denote by $F^{(l)}$ the function which is eigenfunction of operator M^2 with eigenvalue $l(l+1)$*

$$\hat{M}^2 F^{(l)} = l(l+1)F^{(l)} .$$

Observation Condition (5c) implies that zeorth harmonic in (8) vanishes:

$$\alpha_i(q) = \sum_{l\geq 1} \alpha_i^{(l)}(q)$$

Indeed $l=0$ harmonics does not depend on θ, φ, $\alpha_i^{(0)}(q) = \alpha_i^{(0)}(r)$, hence $\delta_i\alpha_j^{(0)} = 0$. This implies that $\varepsilon_{ijm}\alpha_m^{(0)} = 0$, i.e. $\alpha_m^{(0)} = 0$.

Now using this Observation, put expansion (8) in (7). We come to

$$\hat{M}^2\alpha_i^{(l)} = l(l+1)\alpha_i^{(l)} = \hat{M}_i(\hat{M}_k\alpha_k^{(l)}) \, , \quad l = 1, 2, 3, \ldots$$

i.e.

$$\alpha_i = \sum_{l\geq 1}\alpha_i^{(l)} = \hat{M}_i\left(\sum_{l\geq 1}\frac{\hat{M}_k\alpha_k^{(l)}}{l(l+1)}\right)$$

We see that

$$\alpha_i = \delta_i F \text{ where} F = \sum_{l\geq 1}\frac{\hat{M}_k\alpha_k^{(l)}}{l(l+1)}$$

Thus we solved equation (3).

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* In spherical coordinates

$$F^{(l)}(q) = \sum_{m=-l}^{l} c_m(r)\mathbf{Y}_{lm}(\theta,\varphi) \, , \quad \mathbf{Y}_{lm}(\theta,\varphi) = P_{lm}(\theta)e^{im\varphi}$$

where P_{lm} are adjoint Legendre polynomials. In Cartesian coordinates $\mathbf{Y}_{lm}$ is restriction on the sphere of harmonic polynomial ($\Delta P(x,y,z) = 0$) of the weight l