Locally Euclidean Geometries

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Following the book of Schafarevitch—Nikulin we will classify here locally Euclidean 2-dimensional Riemannian manifold. The answer is:

0) 1) Cylindre,

2) Twisted cylindre,

2) Lobachevsky plane of tori

3)

1 Locally Eucldean Geometry

Let (M, G) be a Riemannian manifold.

We say that it is locally Euclidean if in a vicinity of every point there are local coordinates u_i such that

$$G(u, v) = du_1^2 + \cdots + du_n^2 \,,$$

where n is dimension of manifold. We mostly consider here the case $n = 2$.

The condition above means that for every point $\mathcal{D} \in M$ there exists small neighboorhoud $O_\varepsilon(\mathcal{D})$ such that $O_\varepsilon(D)$ is isometric to the interior of the disc $O_\varepsilon(D)$, where D is an arbitrary point of Euclidean plane $\mathbf{E}^2$.

The radius ε of the neighborhood depends on a point. Suppose that the following additional condition is fixed: there is $r > 0$ such that for every point $\mathcal{D} \in M$ the neighboorhoud $O_r(D)$ is isometric to the interior of the disc $O_\varepsilon(D)$, where D is an arbitrary point of Euclidean plane $\mathbf{E}^2$. This means that for every point $\mathcal{D}$ there exist local coordinates $(u_\mathcal{D}, v_\mathcal{D})$[1] such that

$$G(u, v) = du_\mathcal{D}^2 + dv_\mathcal{D}^2 \,, \quad u_\mathcal{D}(\mathcal{D}) = v_\mathcal{D}(\mathcal{D}) = 0 \,, \tag{1.1}$$

[1] now and later we will consider only 2-dimensional case. Sure these and many considerations can be easily generalised for an arbitrary n.

and the following additional condition is obeyed:

$$\text{local coordinates } u_{\mathcal{D}}, v_{\mathcal{D}} \text{ run in the disc } 0 \le u_{\mathcal{D}}^2 + v_{\mathcal{D}}^2 < r \ , \tag{1.2}$$

where radius r does not depend on a choice of a point D.

Riemannian manifold is locally Euclidean condition (1.1) is obeyed. If both conditions (1.1) and (1.2) are obeyed, then Riemannian manifold is uniforly locally Eucldean

In the case if M is compact manifold, then evidenlty locally Euclidean Riemannian manifold is uniformly locally Euclidean. In general this is not true. E.g. the sheet $-1 < x < 1$ of $\mathbf{E}^2$ is evidently locally Euclidean in the sense but it is not uniformly locally Euclidean since the radius r becomes smaller and smaller when $x \to 1$).

We will call coordinates $(u_{\mathcal{D}}, v_{\mathcal{D}})$ Euclidean coordinates on M adjusted to the point $\mathcal{D}$.

2 Equivalence on E^2 and discontinuous action of group

Let r be an equivalence relation of points on $\mathbf{E}^2$, and there exists $\delta > 0$ such that for every two r-equivalen distinct points A, B the distance between these points is greater than δ: $d(A, B) > \delta$ if ArB but $A \neq B$.

We say that an isometry g is an r- isometry, if it preserves r-equivalence:

$$ARB \Leftrightarrow A^g R B^g \, .$$

Lemma 1. *Let $\Gamma = \Gamma_r$ be a set of isometries of $\mathbf{E}^2$ such that*

$$ArB \Leftrightarrow \textit{there exist } g \in \Gamma \textit{ such that } B = A^g \, ,$$

WLOG choose an arbitrary point $A \in \mathbf{E}^2$, since ArA then there exists $F \in \Gamma$ such that $F(A) = A$. Hence F is identity. This follows from the lemma:

Lemma 2. *If F is an isometry with a fixed point such that sends any point to the equivalent point, then $F =$ identity.*

Let F be an arbitrary isometry such that

$$\text{for an arbitrary point } D \ DrF(D) \, .$$

One can prove that $F \in \Gamma_r$. Indeed choose an arbitrary point D. Since $DrF(D)$, thus there exists an isometry $g \in \Gamma$ such that $F(D) = g(D)$. Due to the lemma $g^{-1} \circ Fi = identity$, i.e. $F = g$.

Thus we see that the set Γ is a group.

It remains to prove the lemma. Let A be a fixed point of the isometry F, and $BrF(B)$ for every point B. Consider the domain $O_{\frac{\delta}{2}}(A)$. For an arbitrary point X in this domain $F(X)$ belongs to the domain too, since F is isometry, and $F(A) = A$. Let $Y = F(X)$. Then YrX, but the distance between these points is less than δ. Hence $F(X) = X$. We proved that F is identity in the interior of the disc. Show that this is true for an arbotrary point X. Conisder on the ray AX an arbitrary point B which is insider the disc $O_{\frac{\delta}{2}}(A)$. Isometry sends line to the line, and rays to the rays, and points A and B remain intact under the action of isometry F. Hence $F(X) = X$.

2.1 Properly discontinuous and uniformly discontinuous action

We say that group Γ acts on manifold M *properly discontinuous* if for arbitrary compact $K \subset M$, if the equation

$$g\colon K \cap g(K) \neq$$

has only finite number of solutions.

In the case if M is provided with metric, we say that group Γ acts on manifold M *uniformly discontinuous* if there exists δ such that $\delta > 0$ and for every point $A \in M$,

$$d(A, g(A)) \leq \delta \Rightarrow g = \text{identity}$$

Theorem 1. *For manifold with metric these two definitions are equivalent. Metric is iniformly discontinuos if and only if it is properly discontinous*

Proof. Let group Γ acts on manifold M with metric properly discontinous. Show that its action is iniformly discontinuous. Pick an arbitrary point $A \in M$ □

Lemma 3. *Let Γ be uniformly discontinuous subgroup of isometries, then an isometry $F \in \Gamma$ is identity if it has at least one fixed point*

Lemma 4. *(Chasles' lemma) Let F be an isometry of $\mathbf{E}^2$. Then the following dichotomy is obeyed:*

F preserves orientation, and F is rotation, i.e. there exists a point O such that for an arbitrary point B

$$F(B) = A + \mathrm{Rot}_\varphi(AB)$$

or

F changes the orientation, and there exist a line $\mathbf{l}$, and a vector $\mathbf{N}$ directed along this line such that for arbitrary point B

$$F(B) = \mathrm{Reflect}_{\mathbf{l}}(B) + \mathbf{N}$$

These two lemmas imply the Theorem:

Theorem 2.

We say that the subgroup Γ of isometries is *properly discontinuous* if for an arbitrary compact set K the equation

$$F\colon\ F \in \Gamma, K \cup F(K) \neq$$

there exist only finite number of isometries in Γ such that

Lemma 5. *If F belongs to*

These definitions are equivalent. Indeed let Γ be properly discontinuous.

lee d $g \in \Gamma$ such that $g(F(D)) = D$, i.e. $F = g^{-1}$. Due to the lemma $g \circ F =$ identity, i.e.

it follows from this lemma that the set Γ contains all isometries which send isometries which preserve relation r Indeed let H be an arbitrary r-isometry,

One can see that Γ_r is subroup of isometries.

We define the action of group $\Gamma = \Gamma_R$ on $\mathbf{E}^2$ in the following way.

Let Γ be a set of isometries of $\mathbf{E}^2$ such that for arbitrary two points $A, B \in \mathbf{E}^2$

$$ARB \Leftrightarrow \text{there exist } g \in \Gamma \text{ such that } B = A^g\,,$$

i.e.

2.2 Uniformly discontinuous groups on plane

Let Γ be properly discontinbuous group acting on $\mathbf{E}^2$.

First suppose that Γ does not possess transformations changing orientation.

If Γ possesses at least one non-identity element, hence this element is a translation $T = T_{\mathbf{a}}$ since rotations (presence of fixed points) and Chasles' transformation (it changes orinetation) are forbidden. Choose a translation such that vector $\mathbf{a}$ has minimal length. It can be maximum two elements of minimal length[2]. Thus we come to groups

- $\mathbf{Z}$ group of translation $\{T_{m\mathbf{a}}\}$:

$$\Gamma \ni A\colon A(\mathbf{x}) = \mathbf{x} + m\mathbf{a}\,, m = 0, 1, 2, 3, \dots$$

- $\mathbf{Z} \times \mathbf{Z}$ group of translation $\{T_{m\mathbf{a}+n\mathbf{b}}\}$:

$$\Gamma \ni A\colon A(\mathbf{x}) = \mathbf{x} + m\mathbf{a} + n\mathbf{b}\,,\ m, n = 0, 1, 2, 3, \dots$$

Now suppose that the group Γ possesses at least one element which does not preserve orientation, i.e. Chales' e lement

$$\Gamma \ni S\colon S(\mathbf{x}) = \mathbf{x}_{||} + \mathbf{x}_{\perp} + \mathbf{c} \tag{2.1}$$

where vector $\mathbf{c} = \mathbf{c}_{||}$. We have $S^2 = T_{2\mathbf{c}}$, i.e. our grop possess subgroup $\{T_{n\mathbf{a}}\}$ or subgroup $\{T_{m\mathbf{a}+m\mathbf{b}}\}$, where $\mathbf{a}, \mathbf{b}$ are linearly independent.

I-case. Group Γ is generated by trasnformations S in (2.1) and translation $T_{\mathbf{a}}$, i.e.

$$S^2 = T_{2\mathbf{c}} = T_{m\mathbf{a}}, m = 0, 1, 2, 3, 4, \dots$$

If m is even, $m = 2k$, then $ST_{-k\mathbf{a}}$ is reflection, and it possess fixed points, Hence m is odd, and we see that $\mathbf{c} = \frac{2k+1}{2}\mathbf{a}$. One can consider $\mathbf{c} = \frac{\mathbf{a}}{2}$ changing $S \mapsto ST_{k\mathbf{a}}$.

If we choose the basis $\mathbf{e}, \mathbf{f}$ such that $\mathbf{e}$ is parallel to the axis of Chasles' reflection, and $\mathbf{f}$ is orthogonal to the axis of Chasles' reflection, then we come to

[2] if there are three vectors in general position, then equivalent points will be very close to each other

Thus we see that group G is equal to

$$\Gamma \ni A\colon\ A(x\mathbf{e}+y\mathbf{f}) = \left(x+\frac{m}{2}\right)\mathbf{e}+(-1)^m y\mathbf{f}$$

Now we consider the second case when group Γ possesses subgroup $\{T_{m\mathbf{a}+n\mathbf{b}}\}$.

Show that in this case $\mathbf{a}$ and $\mathbf{b}$ have to be orthogonal to each other. In the same way like above we conclude that Chasles' element is reflection with respecto to axis directed along $\mathbf{a}$ and translation on the vector $\frac{\mathbf{a}}{2}$.

Now notice that

$$S \circ T_{m\mathbf{b}} = T_{m\mathbf{b}'} \circ S\,, \quad \text{where } \mathbf{b}' = \mathbf{b}_{||} - \mathbf{b}_{\perp}$$

and

$$S \circ T_{m\mathbf{b}} \circ S = T_{m\mathbf{b}'} \circ T_{\mathbf{a}}$$

This implies that vector $\mathbf{b}$ has to be parallel to $\mathbf{a}$ or orthogonal (not to priduce too dense lattice of equivalent points.)

3 Main statement

Theorem 3. *Let (M,G) be locally Euclidean 2-dimensional manifold in stronger sense, i.e. both conditions* (1.1) *and* (1.2) *are obeyed.*

Then

We prove this Theorem in three steps.

I-st step. We will construct surjection $\pi\colon \mathbf{E}^2 \to M$ which is isometry for small distances (less than r)

II-step On the base of this surjection we will consider the subroup of isometries which preserve π

$$\Gamma_\pi = \{F\colon\ F \text{ is isometry and } \ \pi \circ F = \pi\}$$

and will prove that this group is uniformly discontinuous.

Proof. Then we will prove that the Riemannian manifold M is isometric to the factor of $\mathbf{E}^2$ with respect to the gropu Γ_π, i.e. M is plane $\mathbf{E}^2$ or cylindre, or twisted cylinder, or torus, or Klein bottle.

Choose two arbitrary points D on $\mathbf{E}^2$ and $\mathcal{D}$ on M and conisder the covering $\pi\colon \mathbf{E}^2 \to M$ such that $\pi(D) = \mathcal{D}$ and for every point $X \in \mathbf{E}^2$ $\pi(X)$ is defined in the following way

Consider the vector $\mathbf{X} = OX$.

Choose points $D = X_0, X_1, X_2, dots X_n = X$ on the segment DX such that the distance between these points is less than r, recurrently define $F(X_i)$ for $i = 0, 1, 2, \dots, n$.

First define $\pi(X_1)$ as a point on manifold M with coordinates $u_{\mathcal{D}} = (X_1)_1, v_{\mathcal{D}} = (X_1)_2$ where $(u_{\mathcal{D}}, v_{\mathcal{D}})$ are Euclidean coordinates adjusted to the point $\pi(D) = \pi(X_0) = \mathcal{D}$ (see section) , and $((X_1)_1, (X_1)_2)$ are components of the vector DX_1, then recurrently: if we already have defined $\pi(X_i)$ for point X_i $(i = 1, 2, \dots, n-1)$ we define $\pi(X_{i+1})$ as a point on manifold M with coordinates

$$u_{X_i} = (X_{i+1})_1 - (X_i)_1, v_{X_i} = (X_{i+1})_2 - (X_i)_2\,,$$

where (u_{X_i}, v_{X_i}) are Euclidean coordinates adjusted to the point X_i and $((X_{i+1})_1 - (X_i)_1, (X_{i+1})_2 - (X_i)_2)$ are components of the vector X_iX_{i+1}.

One can see that function π is well defined on all the plane, its value does not depend on on a choice of partition $O = X_0, X_1, X_2, dots X_n = X$. □

One can show that the map $\pi\colon \mathbf{E}^2 \to M$ preserves small distances:

$$d(\varphi(A), \varphi(B)) = d(A, B)\,, \quad \text{if } d(A, B) < r$$

(here as always r is parameter which defines locality of geometry on M (see (1.2)))

This is evident for points which are at the distance $< r$ from initial pointi D, and one can prove this recurrently for arbitrary close point A, B considering partitions $D = A_0, A_1, \dots, A_n = A$ and $D = B_0, B_1, \dots, B_n = B$ and 'small' trapecies $A_iB_iB_{i+1}A_{i+1}$.

(see details in Shaf) One can see that the map π is surjection.

Prove it. Let $\mathcal{B}$ be an arbitrary point on M. Consider the set of points $\mathcal{B}_0\mathcal{B}_1\mathcal{B}_2\dots\mathcal{B}_n$ such that $\mathcal{B}_n = \mathcal{B}$ and the distance between points is less tham r. Thus we will see recurrently that all these points are covered by π.

Choose an arbitrary point $\mathcal{B} \in M$.

Now we will conisder a group Γ and will show that

$$M = \mathbf{E}^2 \backslash \Gamma$$

We define this group as a group of isometries which preserve the covering map π, i.e. Consider an arbitrary isometry F of $\mathbf{E}^2$. We say that isometry F belongs to group Γ for arbotrary point $D \in \mathbf{E}^2$

$$\pi\left(F\left(D\right)\right)=\pi\left(D\right)\ ,$$

i.e.

$$\Gamma=\{\text{isometries } F \text{ of } \mathbf{E}^2 \text{ such that } \pi\circ F\equiv\pi\}$$

One can see that this is a group. We still now nothing about the group Γ, except that it possesses identity element. Hovever we can easy to prove that this group is uniformly discontinuous.

Let $F\in\Gamma$ be an arbitrry non-identical element in Γ [3], i.e there exists $B\in\mathbf{E}^2$ such that $B'=F(B)\neq B$.

Suppose that $d(B,B')<r$. Consider on manifold M points $\mathcal{B}=\pi(B)$ and $\mathcal{B}'=\pi(B')$. Then due to the properties of surjection π, $d(\mathcal{B},\mathcal{B}')<r$ also, i.e. the point $\mathcal{B}'$ belongs to the chart $(u_{\mathcal{B}},v_{\mathcal{B}})$. π is local bijection, hence $\mathcal{B}\neq\mathcal{B}'$. On the other hand $\mathcal{B}=\mathcal{B}'$ since $F\in\Gamma$. Contradiction.

Theorem 4. *The surjection π is a covering. The group Γ acts freely on the preimages $\pi^{-1}(\mathcal{D})$.*

Consider preimages of the point $\mathcal{B}\in M$.

Let $\{B_i\}$ be a set of the points such that

$$\text{for every } B_i\ \pi(B_i)=\mathcal{B}$$

This set possesses at least one point.

Exercise

Consider

[3] notice that we still know nothing does this element exist or no.