Orthocentre of triangle and related problems

Three heights of triangle intersect at the point. This is well-known statement *

I remember how I was surprised when I realised that this happens since orthocentre of $\triangle ABC$ coincides with orthocentre of 'double' triangle $\triangle A'B'C' = 2\times\triangle ABC$!

Remark We will use notation $\triangle A'B'C' = 2\times\triangle ABC$ for triangle such that

$$\begin{array}{c} \text{side } A'B' \text{ passes via the point } C \text{ and } A'B'||AB \\ \text{side } B'C' \text{ passes via the point } A \text{ and } B'C'||BC \\ \text{side } A'C' \text{ passes via the point } B \text{ and } A'C'||AC \end{array} \tag{1}$$

We consider another remarkable point of $\triangle ABC$:

$$\text{intersection of heights of double triangle} \triangle A'B'C' = 2\times\triangle ABC \tag{2}$$

or in other words the orthocentre of the 'quatre' triangle $\triangle\tilde{A}\tilde{B}\tilde{C} = 2\times\triangle A'B'C'$, where $\triangle A'B'C' = 2\times\triangle ABC$.

Many years ago I was solving the following problem : Let $ABCD$ be thetraedron such that all its fsces are four euqal triangles with sides a, b, c. Calculate its volume** At that time I came to the following very beautiful solution of this problem:

Consider right parallelipiped $ABCDA'B'C'D'$ with sides x, y, z such that this tetrahedron is inscribed in this parallelepiped:

$$\begin{cases} AD = BC = A'D' = B'C' = x \\ AB = CD = A'B' = C'D' = y \\ AA" = BB' = CC" = DD" = z \end{cases}, \quad \text{where} \begin{cases} x^2+y^2 = a^2 \\ y^2+z^2 = b^2 \\ z^2+x^2 = c^2 \end{cases} \; i.e. \begin{cases} x = \sqrt{\frac{a^2+c^2-b^2}{2}} \\ y = \sqrt{\frac{b^2+a^2-c^2}{2}} \\ z = \sqrt{\frac{b^2+c^2-a^2}{2}} \end{cases}$$

We see that triangle which forms tetrahedron has to be acute (not obtuse), since x, y, z have to be positive (or non zero)

Now we see that volume of our tetrahedron is equal to

$$\mathrm{Vol}\,(AB'CD') = \mathrm{Vol}\,(ABCDA'B'C'D') - \mathrm{Vol}\,(BACB') - \mathrm{Vol}\,(C'B'D'C) -$$

$$\mathrm{Vol}\,(DACD') - \mathrm{Vol}\,(A'B'D'A) = xyz - \frac{xyz}{6}\cdot 4 = \frac{xyz}{3} =$$

$$\frac{\sqrt{2}}{12}\sqrt{(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)} \tag{$*$}$$

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* Arnold makes it famous claiming that this happens due to Jacobi identity. (see my etude:) However we will speak here about other topic.

** THis problem comes from 1984 when I was tutoring Vahagn Minasian...

Yes, this is beautiful. However there is also another solution. Thirty years ago trying to construct this tetrahedron, I came the equations

$$\begin{cases} {a'}^2 + h^2 = a^2 \\ {b'}^2 + h^2 = b^2 \\ {c'}^2 + h^2 = c^2 \end{cases} . \tag{**}$$

Here a, b, c are edges of the $\triangle ABC$, $AB = c, BC = a$ and $AC = b$. For an arbitrary point P on the plane denote by $a' = PA$, $b' = PB$ and $c' = PC$. If you find a point P such that these equations are fullfilled then,

$$\mathrm{Vol}\,(tetrahedron) = \frac{h \cdot \mathrm{Areaofthe}\,\triangle ABC}{3} =$$

I could not solve these equations. A week ago i told this problem to my friend, Hovik Nersessian. He suggested that a point P is related with orthocentre... Due to hom I realised that the following statements are obeyed:

Theorem There is unique point P such that

$$\begin{cases} {a'}^2 - {b'}^2 = b^2 - a^2 \\ {b'}^2 - {c'}^2 = c^2 - b^2 \\ {c'}^2 - {a'}^2 = a^2 - c^2 \end{cases}$$

and this point is *the orthocentre of triangle ABC.*

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$$\begin{cases} {a'}^2 - {b'}^2 = a^2 - b^2 \\ {b'}^2 - {c'}^2 = b^2 - c^2 \\ {c'}^2 - {a'}^2 = c^2 - a^2 \end{cases}$$

and this point is *the orthocentre of 'double' triangle ABC.*