Hidden hyperbolicity

In this étude we consider one geometrical problem, which have two manifestations. It seems to be the standard Euclidean problem, but it possesses the hidden hyperbolicity. We first explain how the example is arised, then consider its solution in terms of Euclidean and Hyperbolic Geometry

Source

Consider realisation of hyperbolic (Lobachevsky) plane as upper half-plane. Let C be a circle in H,

The circle in H is the circle Euclidean circle also. Let A be a centre of hyperbolic circle and O be a centre of C considered as the Eucldean circle, and let A be a centre of C considered as the hyperbolic circle: This means that if two curves γ, γ' ar arbitrary hyperbolic geodesics passing through the point A, and $L_\gamma, L_{\gamma'}$ are points of the intersections of these geodesics with circle C, $L_\gamma = \gamma \times C$, $L_{\gamma'} = \gamma' \times C$ then the lengths (hyperbolic) of these geodesics coincide. In particular this implies that all these geodesics intersect the circle C under the angle $\frac{\pi}{2}$.

Now look at this picture from the point of view of Euclidean geometry. All gedesics are half-circles with centres on the absolute, the line $x = 0$, (except the geodesicis which are the vertical lines) Angles are the same. If geodesic γ is represented by the half-circle with the centre at the point K, then the segment KL_γ, radius of this half-circle is tangent to the circle C. Thus we come to conclusion that for the arbitrary point K on the absolute the (Euclidean) length of the tangent from the point K to the circle C is equal to the (Euclidean) length of the segment KA.

The considerations above make us to formulate the following problem of Eucldean geometry.

Let C be a circle in the Euclidean plane and let A, be an arbitrary point on this plane.

Consider the locus $M_{C,A}$ of the points K such that the length of the tangent from K to the circle C is equal to the length of the segment KA:

$$M_{C,A} = \{K\colon \quad KA = \text{lenght of the tangent from } K \text{ to the circle.}\}\,. \tag{1}$$

Find the locus $M_{C,A}$.

This problem looks as standard geometrical question (in Euclidean gepmetry). Temporary forgetting where this problem comes from, we will discuss first solution of this problem just in terms of Eucldean geometry

Solution

Let A' be a point which is inverse to the point A with respect to the circle C: Points O (centre of the circle), A and A' belong to the same ray r_{OA} and

$$|OA| \cdot |OA'| = 1\,. \tag{2}$$

(We suppose that the circle C has unit length.)

Consider the set of circles which are passing throug the points A and A'. One can see that for every such a circle,

$$\text{it intersects the circle } C \text{ under the right angle} \tag{3}$$

This means that for centre K of such circle, the tangent to the circle C is the radius of this circle. We see that the locus $M_{C,A}$ = the locus of the centres of circles passing through the points A and A' = the locus of the points which are on the same (Euclidean) distance from the points A and A'. This is the line l which is ortogonal to the AA' and passes through th emiddle point P of the segment AA' ($P \in AA', |AP| = |PA'|$.

So we proved that

$$M_{C,A} = l\colon \quad d(l, A) = d(l, A')\,. \tag{4}$$

Without loss of generality suppose that point A belongs to the interior of the circle, then the point A' is out of the circle.

Let M be a point of intersection of the ray r_{OA} with the circle M, and N be a point on the circle Cm which is on the continuation of the ray r_{OA}. Denote by a the length of the segment OA, then due to (2) $|OA'| = \frac{1}{a}$, and $|OP| = \frac{1}{2}\left(a + \frac{1}{a}\right)$. One can see that

$$|MA| \cdot |MA'| = (1+a)\left(1 + \frac{1}{a}\right) = 2 + a + \frac{1}{a} = 2\left(1 + \frac{1}{2}\left(a + \frac{1}{a}\right)\right) = |MN| \cdot |MP|\,. \tag{3}$$

The relation (2) means that inversion with centre at the point O with radius $r = 1$ transforms point A to the point A' and vice versa. We call this inversion the first inversion, the inversion I_I. The relation (3) means that the inversion with centre at the point N and with radius $r' = \sqrt{(1+a)\left(1 + \frac{1}{a}\right)}$ also ransforms point A to the point A' and vice versa. This inversion also transforms the circle C to the line l and vice versa, since a point M of the circle is the centre of the inversion. We call this inversion the second inversion, the inversion I_{II}:

$$I_I\colon \quad A \leftrightarrow A'\,, \quad I_{II}\colon \quad A \leftrightarrow A'\,, \text{and } C \leftrightarrow l\,. \tag{4}$$

We are ready to prove the Claim. Let K be an arbitrary point on the line l.

Consider the circle C_K with the centre at the point K and with the radius $|KA|$. This circle passes trhough the points A, A'm hence according to equation (4) its inverse $I_{II}(C_K)$, is passinig through these points also. Since the curve $I_{II}(C_K)$ is the circle (or line) passing via the points A, A' hence it is orthogonal to the line l. Hence the circle C_K is orthogonal to the curve $C = I_{II}(l)$. This means that tangent from the point K to the circle C is the radius, thus length of the tangent is equal to the $|KA|$.

Meaning in hyperbolic geometry

The considerations of the first paragraph show that the fact that $l = M_{C,A}$ has explanations in 'hyperbolic world'.

Indeed the line l divides the plane on two half-planes. Consider the half-plane the circle C belongs to, as a model of hyperbolic plane *.

One of he points A or A' is in the circle C. Consider the Lobachevsky plane formed by this half plane, with l is absolute. The circle C will be the circle in the hyperbolic plane also. One can see that the point A is the centre of this hyperbolic circle.

Indeed due to lemma all the geodesics interesect the circle C under the right angle, i.e. the points of l belong to this locus.

Remark What happens in the case if the point A belongs to the circle. The absolute l is tangent to the circle, and all the half-circles with centre on the absolute are orthogonal to the cyrcle. Our circle is nothing but `horocycle`.

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* Recall shrotly what is it. One can consider Cartesian coordinates (x, y) such that the line l is $y = 0$ the half-plane is $y \geq 0$. Then hyperbolic plane H can be defined as as this half-plane with Riemannian metric $G = \frac{dx^2+dy^2}{y^2}$. The geodesics of this metric (lines of hyperbolic plane) are vertical lines $x = a$ and upper half-circles with centre on the absolute l: $\begin{cases} (x-a)^2 + y^2 = R^2 \\ y > 0 \end{cases}$. The distance between two points $A_1 = (x_1, y_1)$ and $A_2 = (x_2, y_2)$, the length of the geodesic passing via points A_1, A_2 can be defined alternatively by cross-ratio of the points

$$d(A_1, A_2) = |\log\left(A_1, A_2, A_0, A_\infty\right))| = \left|\log\left(\frac{z_1 - z_0}{z_1 - z_\infty} : \frac{z_2 - z_0}{z_1 - z_\infty}\right)\right|,$$

where points A_0, A_∞ are points of intersection of the half-circle with absolute, $z_1 = x_1 + iy_1$,$z_2 = x_2 + iy_1$. E.g. for two points $A_1 = (0, a_1)$, $A_2 = (o, a_2)$ on the vertical line (this is geodesic)

$$d(A_1, A_2) = |\log|(A_1, A_2, 0, \infty))| = \left|\log\left(\frac{ia_1 - 0}{ia_1 - \infty} : \frac{ia_2 - 0}{ia_1 - \infty}\right)\right| = \left|\log\left(\frac{a_1}{a_2}\right)\right|.$$