Let $A(x)$ be a linear operator on tangent vectors: $A(x)\colon \quad T_xM \to T_x(M)$. Then one can define $[A, A]$ which is linear operator from $T_xM \wedge T_xM \to T_xM$. This is a special case of Nevenhuisen bracket. We do it in straightforwar way, then comne to this formula using general formalism.

Let $A(x)$ be an operator-valued function on manifold M. Consider the following function on vector fields:

$$\mathcal{N}(\mathbf{X}, \mathbf{Y}) = [L(\mathbf{X}),\, L(\mathbf{Y})] - L\left([\mathbf{X},\, L(\mathbf{Y})] - [\mathbf{Y},\, L(\mathbf{X})]\right) + L\left(L\left([\mathbf{X},\, \mathbf{Y}]\right)\right) .$$

where $[\ ,\ ]$ is commutator of vector fields. $\mathcal{N}(\mathbf{X}, \mathbf{Y})$ is vector field on M which is antisymmetric:

$$\mathcal{N}(\mathbf{X}, \mathbf{Y}) = -\mathcal{N}(\mathbf{Y}, \mathbf{X})$$

Fact $\mathcal{N}(\mathbf{X}, \mathbf{Y})$ is not only linear over vector fields, it is linear over algebra of functions on M: in particular for arbitrary function f

$$\mathcal{N}(f\mathbf{X}, \mathbf{Y}) = f\mathcal{N}(\mathbf{X}, \mathbf{Y})\,,$$

(this implies linearity over functions),

Show it, Note that $[f\mathbf{X}, Y] = f[\mathbf{X}, \mathbf{Y}] - (\mathbf{Y}f)\mathbf{X}$. Hence

$$\mathcal{N}(f\mathbf{X}, \mathbf{Y}) = [L(f\mathbf{X}),\, L(\mathbf{Y})] - L\left([f\mathbf{X},\, L(\mathbf{Y})] - [\mathbf{Y},\, L(f\mathbf{X})]\right) + L\left(L\left([f\mathbf{X},\, \mathbf{Y}]\right)\right) =$$

$$f\,[L(\mathbf{X}),\, L(\mathbf{Y})] - (L(\mathbf{Y})f)\, L(\mathbf{X}) - fL\left([\mathbf{X},\, L(\mathbf{Y})]\right) + (L(\mathbf{Y})f)\, L(\mathbf{X}) +$$

$$+ fL\left([\mathbf{Y},\, L(f\mathbf{X})]\right) + (\mathbf{Y}f)\, L\left(L\left(\mathbf{X}\right)\right) + fL\left(L\left([\mathbf{X},\, \mathbf{Y}]\right)\right) - (\mathbf{Y}f)\, L\left(L\left(\mathbf{X}\right)\right) = f\mathcal{N}(\mathbf{X}, \mathbf{Y})\,.$$

In components

$$\mathcal{N}(\mathbf{X}, \mathbf{Y}) = N^i_{kp} X^k Y^p,$$

where

$$N^m_{kp}\partial_m = \mathcal{N}(\partial_k, \partial_p) = [L(\partial_k),\, L(\partial_p)] - L\left([\partial_k,\, L(\partial_p)] - [\partial_p,\, L(\partial_k)]\right) + L\left(L\left([\partial_k,\, \partial_p]\right)\right) =$$

$$L^i_k \partial_i L^m_p - L^i_p \partial_i L^m_k - L^m_r \left(\partial_k L^r_p - \partial_p L^r_k\right)$$

Theorem (Neveinhuisen) Operator valued function $L(x)$ = vector valued differential 1-form defines vector valued differential 2-form:

$$L\colon L = dx^k L^i_k \partial_i \to [L, L] = dx^p \wedge dx^k \left(L^i_k \partial_i L^m_p - L^m_r \partial_k L^r_p\right) .$$

This is bracket of L with itself. In fact Nevenhuisen defines bracket for all vector fields valued differential forms. We will describe them using supermathematics.

General approach

For manifold M of dimension n consider $n|n$-dimensional supermanifold ΠTM, i.e. nothing that tangent bundle TM with changing parity of fibers.

Note that usual k-form on M $dx^{i_1}...dx^{i_k} w_{i_1...i_k}$ defines function which is even if k is even and odd if k is odd.

Vector-valued differential form is nothing but vector field on ΠTM such that its vetical components vanish. Sure the vanishing of vertical components is not covariant condition. One has to define canonical lifting on whole ΠTM

Let $A^i(x,\xi)\partial_i, B^i(x,\xi)\partial_i$ be vector valued differential forms

To define he Dutch bracket we consider canonical lifting:

$$\mathbf{A} = A^i \partial_i \mapsto \hat{\mathbf{A}} = \mathbf{A}^i(x,\xi) + (-1)^{p(\mathbf{A})} \xi^k \partial_k A^i(x,\xi) \frac{\partial}{\partial \xi i}$$

One can see that this lifting is well-defined by the condition that it is commute with de Rham differential $\mathbf{x}^i \partial_i$

Then

$$[\mathbf{A},\mathbf{B}]_{\text{Nevenhuisen}}\colon\ \widehat{[\mathbf{A},\mathbf{B}]}_{\text{Nevenhuisen}} = [\hat{A},\hat{B}]$$

This is all ■