Geometry of differential operators on R

Let

$$A = t^\delta \underbrace{\left(s\frac{\partial^{n+1}}{\partial x^{n+1}} + a\hat{w}\frac{\partial^n}{\partial x^n} + b\hat{w}^2\frac{\partial^{n-1}}{\partial x^{n-1}}\right)}_{\text{terms of the order } n+1} +$$

$$+\underbrace{\left(p\frac{\partial^n}{\partial x^n} + c\hat{w}\frac{\partial^{n-1}}{\partial x^{n-1}} + \dots\right)}_{\text{terms of the order } n} + \underbrace{\left(q\frac{\partial^n}{\partial x^{n-1}} + \dots\right)}_{\text{terms of the order } n-1} + \dots$$

be differential operator of the order $n+1$ and of the weight δ on the algebra $\mathcal{F}$ of densities on **R**. Here $\hat{w} = t\frac{\partial}{\partial t}$, $a = a(x), \dots$

E.g. if $\Psi(x,t) = \varphi(x)t^5$ is a density $\varphi(x)|dx|^5$ of the weight 5, and $A = t^{-3}\left(\frac{\partial^3}{\partial x^3} + p(x)\frac{\partial^2}{\partial x^2} + b(x)\hat{w}\frac{\partial}{\partial x}\right)$ then

$$A\Psi = t^{-3}\left(\frac{\partial^3\Psi}{\partial x^3} + p(t)\frac{\partial^2\Psi}{\partial x^2} + b(x)t\frac{\partial}{\partial t}\frac{\partial}{\partial x}\right) = \left(\frac{d^3\varphi}{dx^3} + p(t)\frac{d^2\varphi}{dx^2} + 4b(x)\frac{d\varphi}{dx}\right)|dx|^2 .$$

The canonical scalar product in the space $\mathcal{F}$ is defined by condition that for densities $\Psi = \Psi(x)t^\lambda$ and $\Phi = \Phi(x)t^\mu$

$$\langle \Psi, \Phi\rangle = \begin{cases} 0 \text{ if } \lambda+\mu \neq 1 \\ \int \Psi(x)\Phi(x)dx \ \text{ if } \lambda+\mu \neq 1 \end{cases} .$$

Thus we have the conjugation of derivatives:

$$x^\dagger = x, \left(\frac{\partial}{\partial x}\right)^\dagger = -\frac{\partial}{\partial x}, t^\dagger = t, \left(\frac{\partial}{\partial t}\right)^\dagger = \frac{2}{t} - \frac{\partial}{\partial t},$$

In particular:

$$\hat{w}^\dagger = \left(t\frac{}{d\partial}\right)^\dagger = 1 - \left(t\frac{\partial}{\partial t}\right)^\dagger = 1 - \hat{w}^+$$

and

$$\hat{w}^\dagger(t^\sigma\Psi) = t^\sigma(1-\sigma-\hat{w})\Psi .$$

§1. *Subprincipal symbol*

Now we find the restrictions on operator A posed by the condition that it is self conjugate operator (up to sign), i.e.

$$A^\dagger = (-1)^{n+1}A$$

, Then we discuss the geometrical nature of coefficients.

We check the condition of self-conjugality (*is it right word?*) step by step for lower and lower derivatives [1].

First consider the terms which are proportional to derivatives of the order $\geq n$ with respect to x:

$$A = t^\delta \, \left(s\frac{\partial^{n+1}}{px^{n+1}} + a\hat{w}\frac{\partial^n}{\partial x^n} + p\frac{\partial^n}{\partial x^n} + \dots \right) ,$$

$$A^\dagger = \left[t^\delta \, \left(s\frac{\partial^{n+1}}{\partial x^{n+1}} + a\hat{w}\frac{\partial^n}{\partial x^n} + p\frac{\partial^n}{\partial x^n} + \dots \right) \right]^\dagger =$$

$$(-1)^{n+1} t^\delta \frac{\partial^{n+1}}{\partial x^{n+1}} \left(s(x) \cdot \right) + \hat{w}^\dagger \left[t^\delta (-1)^n \frac{\partial^n}{\partial x^n} \left(a(x) \cdot \right) \right] + t^\delta (-1)^n \frac{\partial^n}{\partial x^n} \left(p(x) \cdot \right) + \dots =$$

$$(-1)^{n+1} t^\delta \left(s\frac{\partial^{n+1}}{\partial x^{n+1}} + (n+1)\frac{\partial s}{\partial x}\frac{\partial^n}{\partial x^n} \right) + (-1)^n t^\delta (1-\delta-\hat{w}) \left(a\frac{d^n}{\partial x^n} \right) + t^\delta (-1)^n p\frac{\partial^n}{\partial x^n} + \dots =$$

$$= (-1)^{n+1} t^\delta s\frac{\partial^{n+1}}{\partial x^{n+1}} + (-1)^n t^\delta \left(-a\hat{w} - (n+1)\frac{ds}{dx} + (1-\delta)a + p \right) \frac{\partial^n}{\partial x^n} + \dots$$

Hence

$$0 = (-1)^{n+1} A^\dagger - A =$$

$$t^\delta s\frac{\partial^{n+1}}{\partial x^{n+1}} + t^\delta \left(a\hat{w} + (n+1)\frac{ds}{dx} - (1-\delta)a - p \right) \frac{d^n}{dx^n} - t^\delta \, \left(s\frac{\partial^{n+1}}{\partial x^{n+1}} + a\hat{w}\frac{\partial^n}{\partial x^n} + p\frac{\partial^n}{\partial x^n} + \dots \right)$$

$$= t^\delta \left((n+1)\frac{ds}{dx} + (\delta - 1)a - 2p \right) \frac{\partial^n}{\partial x^n} + \dots$$

We come to the condition

$$\frac{n+1}{2}\frac{ds}{dx} + (\delta-1)a - 2p = 0 \, , , \text{ i.e. } p = \frac{n+1}{2}\left(\frac{ds}{dx} \right) + \frac{1}{2}(1-\delta)$$

and

$$A = t^\delta \left(s\frac{\partial^{n+1}}{\partial x^{n+1}} + a\hat{w}\frac{\partial^n}{\partial x^n} + p\frac{\partial^n}{\partial x^n} + \dots \right) =$$

$$A = t^\delta s\frac{\partial^{n+1}}{\partial x^{n+1}} + \frac{1}{2}\left((n+1)\frac{ds}{dx} + (2\hat{w} + 1 - \delta)a(x) \right) \frac{\partial^n}{\partial x^n} \dots$$

Now we study how to transform s and a under coordinate transformation, then we will go to the next terms.

§2. *Geometric meaning of Subprincipal symbol*

[1] All that we will do here must be compared with our calculations with T.Voronov for second order operators. The motto is that we descend from higher derivatives to lower: for n-th order operators the terms of n-th order behave like the term of second order of Laplacian, the terms of $n-1$-th order behave like the term of first order of Laplacian, the terms of $n-2$-th order behave like the term of zeroth order of Laplacian.

We consider how a transforms with respect to an arbitrary coordinate transformation. Consider arbitrary coordinate transformation: $(x,t) \mapsto (y,\tau)$[1]. :

$$\begin{cases} x = x(y) \\ t = x_y \tau \end{cases}, \begin{cases} y = y(x) \\ \tau = y_x t \end{cases}$$

Then to calculate how operator A will transform we note that $t = x_y \tau$,

$$\hat{w}_{(t)} = t\frac{\partial}{\partial t} = t = x_y \tau \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} = x_y \tau y_x \frac{\partial}{\partial \tau} = \tau \frac{\partial}{\partial \tau} = \hat{w}_{(\tau)}$$

and

$$\frac{\partial}{\partial x} = y_x \frac{\partial}{\partial y} + \frac{\partial \tau}{\partial x}\frac{\partial}{\partial \tau} = y_x \frac{\partial}{\partial y} + t y_{xx}\frac{\partial}{\partial \tau} = y_x \frac{\partial}{\partial y} + \frac{y_{xx}}{y_x} w_{(\tau)}$$

Little bit work and we come to the following formula:

$$\frac{\partial^k}{\partial x^k} = \left(y_x \frac{\partial}{\partial y} + \frac{y_{xx}}{y_x}\hat{w}_{(\tau)} \right)^k =$$

$$y_x^k \frac{\partial^k}{\partial y^k} + \left((1 + 2 + \ldots + (k-1)) y_x^{k-2} y_{xx} + k\left(\frac{y_{xx}}{y_x}\right) y_x^{k-2} y_{xx}\hat{w} \right) \frac{\partial^{k-1}}{\partial y^{k-1}} =$$

$$y_x^k \frac{\partial^k}{\partial y^k} + \left(\frac{k(k-1)}{2} + k\hat{w} \right) y_x^{k-2} y_{xx} \frac{\partial^{k-1}}{\partial y^{k-1}} .$$

(This can be easily calculated by induction.)

Now we are ready to calculate transformation of coefficients:

$$A = t^\delta s \frac{\partial^{n+1}}{\partial x^{n+1}} + t^\delta \frac{1}{2}\left((n+1)\frac{ds}{dx} + (2\hat{w} + \delta - 1)a(x) \right) \frac{\partial^n}{\partial x^n} + \ldots =$$

$$\tau^\delta x_y^\delta s \left(y_x \frac{\partial}{\partial y} + \frac{y_{xx}}{y_x}\hat{w} \right)^{n+1} + \frac{1}{2}\tau^\delta x_y^\delta \left((n+1)\frac{ds}{dx} + (2\hat{w} + \delta - 1)a(x) \right) \left(y_x \frac{\partial}{\partial y} + \frac{y_{xx}}{y_x}\hat{w} \right)^n + \ldots =$$

$$\tau^\delta y_x^{-\delta} s \left(y_x^{n+1}\frac{\partial^{n+1}}{\partial y^{n+1}} + \left(\frac{n(n+1)}{2} + (n+1)\hat{w} \right) y_{xx} y_x^{n-1} \frac{\partial^n}{\partial y^n} \right) +$$

$$+ \frac{1}{2}\tau^\delta y_x^{-\delta} \left((n+1)\frac{ds}{dx} + (2\hat{w} + \delta - 1)a(x) \right) y_x^n \frac{\partial^n}{\partial y^n} \ldots$$

Dots means terms which are of the order $\leq n-1$

Denote $\tilde{s} = y_x^{n+1-\delta} s$ (principal symbol in new coordinates).

[1] it would be enough to fix the weight of the densities, i.e. consider the action of the operator on the subspace $\mathcal{F}_\lambda$ of the densitites of the fixed weight λ and consider only coordinate transformations $y = y(x)$ but we prefer to consider the general case.

Calculate terms separately. For the first term we have:

$$\tau^\delta y_x^{-\delta} s \left(y_x^{n+1} \frac{\partial^{n+1}}{\partial y^{n+1}} + \left(\frac{n(n+1)}{2} + (n+1)\hat{w} \right) y_{xx} y_x^{n-1} \frac{\partial^n}{\partial y^n} \right) =$$

$$\tau^\delta \left(\tilde{s} \frac{\partial^{n+1}}{\partial y^{n+1}} + \left(\frac{n(n+1)}{2} + (n+1)\hat{w} \right) \tilde{s} y_{xx} y_x^{-2} \frac{\partial^n}{\partial y^n} \right) ,$$

For the second term we have:

$$\frac{1}{2} \tau^\delta y_x^{-\delta} \left((n+1) \frac{ds}{dx} + (2\hat{w} + \delta - 1) a(x) \right) y_x^n \frac{\partial^n}{\partial y^n} =$$

$$\frac{1}{2} \tau^\delta \left((n+1) y_x^{n-\delta} \frac{d}{dx} (\tilde{s} y_x^{\delta-n-1}) + (2\hat{w} + \delta - 1) a(x) y_x^{n-\delta} \right) \frac{\partial^n}{\partial y^n} =$$

$$\frac{1}{2} \tau^\delta \left((n+1) \frac{ds}{dy} + (n+1)(\delta - n - 1) y_{xx} y_x^{-2} \tilde{s} + (2\hat{w} + 1 - \delta) a(x) y_x^{n-\delta} \right) \frac{\partial^n}{\partial y^n}$$

Now add the first and the second terms:

$$A = \tau^\delta \left(\tilde{s} \frac{\partial^{n+1}}{\partial y^{n+1}} + \left(\frac{n(n+1)}{2} + (n+1)\hat{w} \right) \tilde{s} y_{xx} y_x^{-2} \frac{\partial^n}{\partial y^n} \right) +$$

$$\frac{1}{2} \tau^\delta \left((n+1) \frac{ds}{dy} + (n+1)(\delta - n - 1) y_{xx} y_x^{-2} \tilde{s} + (2\hat{w} + \delta - 1) a(x) y_x^{n-\delta} \right) \frac{\partial^n}{\partial y^n} + \ldots =$$

$$\tau^\delta \tilde{s} \frac{\partial^{n+1}}{\partial y^{n+1}} + \tau^\delta \frac{n+1}{2} \frac{ds}{dy} \frac{\partial^n}{\partial y^n} +$$

$$\frac{1}{2} \tau^\delta \left(n(n+1) + 2(n+1)\hat{w} + (n+1)(\delta - n - 1) \right) \tilde{s} y_{xx} y_x^{-2} \frac{\partial^n}{\partial y^n} +$$

$$\frac{1}{2} \tau^\delta (2\hat{w} + \delta - 1) a(x) y_x^{n-\delta} \frac{\partial^n}{\partial y^n} + \ldots =$$

$$\tau^\delta \tilde{s} \frac{\partial^{n+1}}{\partial y^{n+1}} + \tau^\delta \frac{n+1}{2} \frac{ds}{dy} \frac{\partial^n}{\partial y^n} +$$

$$\frac{1}{2} \tau^\delta (2\hat{w} + \delta - 1) \left(a(x) y_x^{n-\delta} + (n+1) \tilde{s} \frac{y_{xx}}{y_x^2} \right) \frac{\partial^n}{\partial y^n} + \ldots =$$

Claim We see that if in old coordinates

$$A = t^\delta s \frac{d^{n+1}}{dx^{n+1}} + t^\delta \frac{1}{2} \left((n+1) \frac{ds}{dx} + (2\hat{w} + \delta - 1) a(x) \right) \frac{d^n}{dx^n} + \ldots =$$

then in new coordinates

$$A = \tau^\delta \tilde{s} \frac{d^{n+1}}{dy^{n+1}} + \tau^\delta \frac{1}{2} \left((n+1) \frac{ds}{dy} + (2\hat{w} + \delta - 1) \tilde{a}(x) \right) \frac{d^n}{dy^n} + \ldots =$$

where

$$\tilde{s} = s y_x^{n+1-\delta}$$

and

$$\tilde{a} = a(x) y_x^{n-\delta} + (n+1)\tilde{s}\frac{y_{xx}}{y_x^2} = y_x^{-\delta}\left(a + (n+1)s\frac{\partial \log y_x}{\partial x}\right) y_x^n$$

Remark In the case $n = 2$ this is just the connection on the volume forms. (Principal symbol equals to $2s$)

Resumé

In the general case $\frac{2a}{n+1}$ is a connection. We denote

$$\gamma = \frac{2a}{n+1}, a = \frac{(n+1)\gamma}{2}$$

Then we can rewrite the operator

$$A = t^\delta s\frac{d^{n+1}}{dx^{n+1}} + t^\delta \frac{1}{2}\left((n+1)\frac{ds}{dx} + (2\hat{w} + \delta - 1)a(x)\right)\frac{d^n}{dx^n} + \ldots =$$

$$t^\delta \frac{d^{n+1}}{dx^{n+1}} + t^\delta \frac{n+1}{4}\left(2\frac{ds}{dx} + (2\hat{w} + \delta - 1)\gamma(x)\right)\frac{d^n}{dx^n} + \ldots =$$

One can consider the canonical pencil of n-th order operators of the degree δ which send the density of the weight λ to the density of the weight $\lambda + \delta$

$$\Psi(x)|dx|^\lambda \mapsto \left(s\frac{d^{n+1}\Psi(x)}{dx^{n+1}} + \frac{n+1}{4}\left(2\frac{ds}{dx} + (2\lambda + \delta - 1)\gamma(x)\right)\frac{d^n\Psi}{dx^n} + \ldots\right)|dx|^{\lambda+\delta} =$$

where γ is a connection

Exercise Consider the previous construction for the case $n = 1$ ("laplacian") and $n = 0$ (vector field)

The next step is to consider the subsubprincipal symbol, which is of highly interest for the case

We have also to construct the operator globally, but this we can do defining a connection.

§3. *Subsubprincipal symbol*

Now we check the condition of self-conjugancy ($A = (-1)^{n+1}A^\dagger$) up to the order $n-1$ with respect to x:

From previous considerations it follows that the operator

$$A = t^\delta \underbrace{\left(s\frac{\partial^{n+1}}{\partial x^{n+1}} + a\hat{w}\frac{\partial^n}{\partial x^n} + b\hat{w}^2\frac{\partial^{n-1}}{\partial x^{n-1}}\right)}_{\text{terms of the order } n+1} + \underbrace{\left(p\frac{\partial^n}{\partial x^n} + c\hat{w}\frac{\partial^{n-1}}{\partial x^{n-1}} + \ldots\right)}_{\text{terms of the order } n} + \underbrace{\left(q\frac{\partial^n}{\partial x^{n-1}} + \ldots\right)}_{\text{terms of the order } n-1} +$$

$$= t^\delta \left(s\frac{\partial^{n+1}}{\partial x^{n+1}} + a\hat{w}\frac{\partial^n}{\partial x^n} + p\frac{\partial^n}{\partial x^n} + \dots \right) =$$

$$A = t^\delta s\frac{\partial^{n+1}}{\partial x^{n+1}} + \frac{1}{2}\left((n+1)\frac{ds}{dx} + (2\hat{w}+1-\delta)a(x)\right)\frac{\partial^n}{\partial x^n} + (b\hat{w}^2 + c\hat{w} + q)\frac{\partial^{n-1}}{\partial x^{n-1}} + \dots\dots$$

Now find the restrictions which are imposed by the condition $A = (-1)^{n+1}A^\dagger$: We have that

$$(-1)^{n+1}A^\dagger\Psi = \frac{\partial^{n+1}}{\partial x^{n+1}}(t^\delta s\Psi) - \frac{n+1}{2}\frac{\partial^n}{\partial x^n}\left(t^\delta\frac{ds}{dx}\Psi\right) -$$

$$\frac{n+1}{2}\frac{\partial^n}{\partial x^n}\left(t^\delta\left(\hat{w} + \frac{\delta-1}{2}\right)\gamma\Psi\right) + \frac{\partial^{n-1}}{\partial x^{n-1}}\left(t^\delta(\hat{w}^2 b + \hat{w}c + q)\psi\right) =$$

$$t^\delta s\frac{\partial^{n+1}}{\partial x^{n+1}} + \frac{1}{2}\left((n+1)\frac{ds}{dx} + (2\hat{w}+1-\delta)a(x)\right)\frac{\partial^n}{\partial x^n} +$$

$$\frac{n(n+1)}{2}t^\delta\left(\hat{w} + \frac{\delta-1}{2}\right)\frac{d\gamma}{dx}\frac{\partial^{n-1}\Psi}{\partial x^{n-1}} + t^\delta\left[b(1-\delta-\hat{w})^2 + c(1-\delta-\hat{w}) + q\right]\frac{\partial^{n-1}\Psi}{\partial x^{n-1}} + \dots\dots$$

Comparing with operator A we see that the condition $A = (-1)^{n+1}A^\dagger$ implies that

$$\frac{n(n+1)}{2}\left(\hat{w} + \frac{\delta-1}{2}\right)\frac{d\gamma}{dx} + b(1-\delta-\hat{w})^2 + c(1-\delta-\hat{w}) + q = b\hat{w}^2 + c\hat{w} + q$$

Thus

$$c = \frac{n(n+1)}{2}\frac{d\gamma}{dx} + b(\delta-1)$$

We denote $b = \frac{\theta}{2}$. (To compare with second order operators.)

We come to the following statement:

Theorem The self-conjugate operator of the order $n+1$ on the algebra of densitites has the following appearance:

$$A = t^\delta s\frac{\partial^{n+1}}{\partial x^{n+1}} + \frac{1}{2}\left((n+1)\frac{ds}{dx} + (2\hat{w}+\delta-1)a(x)\right)\frac{\partial^n}{\partial x^n} +$$

$$\frac{1}{2}\left[\theta\hat{w}^2 + \left(n(n+1)\frac{d\gamma}{dx} + \theta(\delta-1)\right)\hat{w} + q\right]\frac{\partial^{n-1}}{\partial x^{n-1}} + \dots\dots$$

Here s is the density of the weight $\delta - n - 1$, γ is connection and Brans-Dicke scalar is related to Schwarzian.

Considering the restriction of this operator on the space $\mathcal{F}_\lambda$ we come to the following pencil of operators. Any density of the weight λ $\Psi|dx|^\lambda$ is transformed to the density of the weight $\mu\lambda + \delta$:

$$\Psi|dx|^\lambda \mapsto \Phi(x)|dx|^{\lambda+\delta}$$

where

$$\Phi = s\frac{d^{n+1}\Psi}{dx^{n+1}} + \frac{1}{2}\left((n+1)\frac{ds}{dx} + (2\lambda+\delta-1)a(x)\right)\frac{d^n\Psi}{dx^n} +$$

$$\frac{1}{2}\left[\theta\lambda^2 + \left(n(n+1)\frac{d\gamma}{dx} + \theta(\delta-1)\right)\lambda + q\right]\frac{d^{n-1}}{dx^{n-1}} + \dots$$

§4. *Special case*

Consider operator of the weight δ on the densitites o the weight λ such that

$$\begin{cases} \delta - n - 1 = 0 \\ 2\lambda + \delta - 1 = 0\,. \end{cases}$$

i.e.

$$\begin{cases} \delta = 1 + n \\ \lambda = -\frac{n}{2} \end{cases}$$

The principal symbol s becomes the scalar we put it $s = 1$, subprincipal symbol vanishes. We come to the operator

$$\Psi|dx|^{-\frac{n}{2}} \mapsto \Phi(x)|dx|^{1+\frac{n}{2}}$$

where

$$\Phi(x) = \frac{d^{n+1}\Psi}{dx^{n+1}} + \frac{1}{2}\left[\theta\frac{n^2}{4} + \left(n(n+1)\frac{d\gamma}{dx} + \theta n\right)\left(\frac{-n}{2}\right) + q\right]\frac{d^{n-1}}{dx^{n-1}} + \dots =$$

$$\frac{d^{n+1}\Psi}{dx^{n+1}} + \dots\theta\frac{d^{n-1}\Psi}{dx^{n+1}} + \dots\frac{d\gamma}{dx}$$