Integration of vector-valued forms

§1 Archimedes principle and Gauss- Ostrogradsky law

In mathematical physics we often have to use Stokes formula for integrals which take vector values. Very good example is the deriving of Archimedes principle. If $p = p(\mathbf{r})$ is a presure of liquid then the force acting on the body is equal to

$$\mathbf{F} = \oint_{\partial D} p(\mathbf{r}) d\mathbf{s} = \int_D \nabla p(\mathbf{r}) dV \,. \tag{1a}$$

In the case if $p(\mathbf{r}) = consta$ then this integral is equal to zero. In the case if $p(\mathbf{r}) = mgz$, then we come to

$$\mathbf{F} = \oint_{\partial D} p(\mathbf{r}) d\mathbf{s} = \int_D \nabla p(\mathbf{r}) dV = mgV = \text{weight of the liquid that the body displaces}\,. \tag{1a}$$

This is standard Archimedes principle.

There is a simple truck to derive the formula (1). It is the following: take the scalar product of an arbitrary *constant* vector $\mathbf{a}$ with integrand in (1) in surface integral in (1a). Then we come to the flux of vector field $p \cdot \mathbf{a}$ through surface ∂D:

$$\mathbf{F} \cdot \mathbf{a} = \oint_{\partial D} (p(\mathbf{r}) \cdot \mathbf{a})\, d\mathbf{s}\,. \tag{2a}$$

Apply Gauss-Ostogradsky law to this flux we come to

$$\mathbf{F} \cdot \mathbf{a} = \oint_{\partial D} (p(\mathbf{r}) \cdot \mathbf{a})\, d\mathbf{s} = \int \text{div}\ (p(\mathbf{r})\mathbf{a}) = \mathbf{a} \cdot \int_D \nabla p(\mathbf{r}) dV\,. \tag{1a}$$

This relation implies (1a) since it is true for an arbitrary constant vector $\mathbf{a}$.

Two words about essence of Gauss-Ostogradsky law for students For student who know differential forms.

Let $\mathbf{K}$ is an arbitrary vector field and respectively M be an arbitrary surface in $\mathbf{E}^3$. Then the flux of vector field $\mathbf{K}$ throug the surface M is equal to integral of 2-form $\omega_{\mathbf{K}} = \Omega \rfloor \mathbf{K}$ over surface M:

$$\text{F'lux of } K \text{ via } M = \int_M \mathbf{K} d\mathbf{s} = \int \Omega \rfloor \mathbf{K}\,, \tag{3a}$$

where Ω is a volume form. Due to Stokes Theorem ($\int_{\partial D} \omega = \int_D d\omega$ we have that in the case if surface M is a boundary, $M = \partial D$ then the integral (3a) is equal to the integral of 3-form $\Omega \rfloor \mathbf{K}$ over body D:

$$\text{F'lux of } K \text{ via } \partial D = \int_{\partial D} \mathbf{K} d\mathbf{s} = \int_{\partial D} \Omega \rfloor \mathbf{K} = \int_D d\,(\Omega \rfloor \mathbf{K})\ . \tag{3b}$$

Now the Cartan formula gives that

$$d\,(\Omega \rfloor \mathbf{K}) = \mathcal{L}_{\mathbf{K}} \Omega = (\text{div}_\Omega\, \mathbf{K})\, \Omega$$

and

$$\int_{\partial D} \mathbf{K} d\mathbf{s} = \int_{\partial D} \Omega \rfloor \mathbf{K} = \int_D d\left(\Omega \rfloor \mathbf{K}\right) = \int \left(\operatorname{div}_\Omega \mathbf{K}\right) \Omega \qquad (3b)$$

In coordinates: if volume form $\Omega = \rho dx \wedge dy \wedge dz$ then

$$\operatorname{div}_\Omega \mathbf{K} = \frac{d\left(\Omega \rfloor \mathbf{K}\right)}{\Omega} = \frac{d\left(\rho dx \wedge dy \wedge dz \rfloor \left(K_x \partial_x + K_y \partial_y + K_z \partial_z\right)\right)}{\rho dx \wedge dy \wedge dz} =$$

$$\frac{d\left(\rho\left(K_x dy \wedge dz - K_y dx \wedge dz + K_z dx \wedge dy\right)\right)}{\rho dx \wedge dy \wedge dz} = \frac{1}{\rho}\left(\frac{\partial(\rho K_x)}{\partial x} + \frac{\partial(\rho K_y)}{\partial y} + \frac{\partial(\rho K_z)}{\partial z}\right) =$$

$$\frac{\partial K_x}{\partial x} + \frac{\partial K_y}{\partial y} + \frac{\partial \rho K_z)}{\partial z} + K_x \partial \log \rho \partial x + K_y \partial \log \rho \partial y + K_z \partial \log \rho \partial z\,.$$

Now return to the integral (1). The derivation above is alright but it looks little bit artificial. We may avoid it introducind *vector-valued forms*.

§2 Oriented area—vector valued form

Let $\mathbf{r} = \mathbf{r}(\xi, \eta)$ be a local parameterisation of surface M in $\mathbf{E}^3$.

Then surface element of M is equal to

$$d\sigma = |\mathbf{r}_\xi \times \mathbf{r}_\eta| d\xi \wedge d\eta = \sqrt{\mathbf{r}_\xi^2 \mathbf{r}_\eta^2 - \left(\mathbf{r}_\xi \cdot \mathbf{r}_\eta\right)^2} d\xi \wedge d\eta =$$

$$\sqrt{\left(x_\xi y_\eta - x_\eta y_\xi\right)^2 + \left(x_\xi z_\eta - x_\eta z_\xi\right)^2 + \left(z_\xi y_\eta - z_\eta y_\xi\right)^2} d\xi \wedge d\eta$$

A normal unit vector to the surface is equal to $\mathbf{n} = \frac{\mathbf{r}_\xi \times \mathbf{r}_\eta}{|\mathbf{r}_\xi \times \mathbf{r}_\eta|}$ and vector surface element is equal to

$$d\mathbf{s} = \mathbf{n} d\sigma = \left(\mathbf{r}_\xi \times \mathbf{r}_\eta\right) d\xi \wedge d\eta$$

We see that vector surface element is expressed trough vector valued 2-form:

$$\vec{\omega}: \quad \vec{\omega}(\mathbf{r}_\xi, \mathbf{r}_\eta) = \mathbf{r}_\eta \times \mathbf{r}_\eta\,.$$

In Cartesian coordinates

$$\vec{\omega} = dx \wedge dy \partial_z - dx \wedge dz \partial_y + dy \wedge dz \partial_x$$

In arbitrary coordinates $u^i = (u^1, u^2, u^3)$

$$\vec{\omega} = \sqrt{\det g} \epsilon_{ikm} du^i \wedge du^k g^{mn} \partial_m\,,$$

where g_{ik} is Riemannina metric in coordinates u^i.

We have that

$$\int_M d\mathbf{s} = \int_M \mathbf{n} d\sigma = \int_M \vec{\omega}\,.$$

In these notations we have immediately that

01.10.2013

Yesterday Grisha Vekstein showed me the surface integral:

$$\int_M (ds \times \nabla\varphi) \tag{1}$$

He realises well that this integral is equal to zero. How to show it properly?

First of all naive approach. Use the formulae of "naive" vector calculus:
Take an arbitrary constant vector $\mathbf{a}$. Then we have

$$\mathbf{a} \cdot \int_M (d\mathbf{s} \times \nabla\varphi) = \int_M d\mathbf{s} \cdot (\nabla\varphi \times \mathbf{a}) \tag{1a}$$

If $M = \partial D$ then due to Ostogradsky-Gauss Theorem we have that

$$\mathbf{a} \cdot \int_M (d\mathbf{s} \times \nabla\varphi) = \int_M d\mathbf{s} \cdot (\nabla\varphi \times \mathbf{a}) = \int_D \text{div } (\nabla\varphi \times \mathbf{a}) = 0\,,$$

since $\mathbf{a}$ is constant vector and

$$\text{div } (\nabla\varphi \times \mathbf{a}) = \text{rot}\, \nabla\varphi \times \mathbf{a} = 0$$

Hence the integral (1) is equal to zero too if $M = \partial D$.

Alright, the asnwer is equal to zero. Ifg this is the case in arbitrary coordinates, then it has to be a nice formula, On the other formulae for gradient of function, vector product, e.t.c. in general heavily depend on additional structures such that Riemannian metric, volume form, e.t.c.

E.g. if g is Riemannian metric in a given coordinates, then for gradient of a function $\nabla\varphi = g^{ik} \frac{\partial\varphi}{\partial x^k} \frac{\partial}{\partial x^i}$, and for vector product of two vectors $\mathbf{a}, \mathbf{b}$ $(\mathbf{a} \times \mathbf{b})_i = \sqrt{\det g} \epsilon_{ikm} a^k b^m$, and respectively

$$(\mathbf{a} \times \mathbf{b})^i = \sqrt{\det g} g^{ij} \epsilon_{jkm} a^k b^m,$$

Now what object have to be considered instead constant vector $\mathbf{a}$, which obviously is meaningless in covariant approach. Consider an arbitary covector ω_i (one-form $\omega = \omega_i dx^i$)*.

We consider now the value of 1-form ω on the integrand $d\mathbf{s} \times \nabla\phi$ instead (1a) the integral

$$\int_M (d\mathbf{s} \times \nabla\varphi) = \int_M d\mathbf{s} \cdot (\nabla\varphi \times \mathbf{a}) \tag{1a}$$

The fact that the integral is equal to zero, means that the integral in fact *does not depend* on metric.

One can see that the integral (1) can be viewed in the following way:

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* Why covector, not vector? Later we will see that closed covector palys the role of constant vector.