Toy example of thick morphism

Let V be a vector space.

Let u^i be coordinates on V.

Let w_i be arbitrary coordinates on V^*.

Consider on vector space $N = V \oplus V^*$ canonical symplectic structure

$$dp_i \wedge du^i = dw_i \wedge dq^i \,. \tag{1a}$$

where p_i, q^j are coordinates on V^*, V dual to coordinates u^i, w_j.

Consider on $N \times N$ the symplectic structure

$$\Omega = \omega_1 - \omega_2 \tag{1b}$$

If (u, v) are coordinates on the first examplaire of N and (q, w) are coordinates on the second exemplaire of N then

$$\Omega = dp_i \wedge du^i - dw_i \wedge dq^i \,. \tag{1c}$$

Let $S(u, q)$ be an arbitrary (smooth) function. It defines Lagrangian surface

$$\Lambda_S = \left\{ (u, p, q, w) \colon p_i = \frac{\partial S(u,q)}{\partial u^i} \,,\ w_i = \frac{\partial S(u,q)}{\partial q^i} \right\} \tag{2a}$$

This is Lagrangian since its dimension is equal to n ($n = \dim V$) and due to the construction

$$\left(dS = p_i du^i + w_m dq^m\right)\big|_{\Lambda_S} \Rightarrow \left(d^2 S = 0 = d\left(p_i du^i + w_m dq^m\right) = dp_i \wedge du^i + dw_m \wedge dq^m = \Omega\right)\big|_{\Lambda_S} \,. \tag{2b}$$

This Lagrangian surface defines canonical relation $\sim_S$: on sympletic vector space $N = V \oplus V^*$:

$$(u, p) \sim_S (q, u) \text{ if } (u, p, q, w) \in \Lambda_S \,, \text{i.e. } p_i = \partial S(u, q\partial u^i) \,, w_i = \partial S(u, q\partial q^i) \tag{3a}$$

For example bilinear form $S = u^i S_{ik} q^k$ defines canonical relation

$$(u, p) \sim_S (q, u) \text{ if } \begin{cases} p_i = S_{ik} q^k \\ w_i = u^m S_{mi} \end{cases} \tag{3b}$$

This canonical relation may define canonical transformation if it defines bijection of N on N. E,g, if bikinear form S is non-degenerate, then canonical relation (3b) defines canonical transformation

$$\begin{cases} u^m = w_k S^{km} \\ p_i = S_{ik} q^k \end{cases} \tag{3c}$$

Now define thick morphisms