volume of sphere

Denote by w_{n+1} or σ_n volume of n-dimensional sphere. (Notation w_{n+1} is sign of respect to old books) We know very well that

$$w_n = \sigma_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \tag{1}$$

The quickest way to show it is to use identity

$$\int e^{-\mathbf{x}^2} d^n x = \pi^{\frac{n}{2}} = w_n \int e^{-r^2} r^{n-1} dr.$$

Using this formula one can calculate the volume d_n of unit ball:

$$d_n = \int_0^1 r^{n-1} \omega_n dr = \frac{w_n}{n}\,.$$

Nothice that inductive formula via B-function relates $w_n, w_n + 1$:

$$w_{n+1} = \int_0^\pi \sin^{n-1} w_n \theta d\theta = 2w_n \int_0^1 (1-u^2)^{\frac{n-2}{2}} du = \int (1-t)^{\frac{n-2}{2}} t^{\frac{1}{2}} dt = w_n B\left(\frac{\frac{n}{2}, 1}{2}\right) =$$

$$\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} w_n = \frac{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} w_n\,.$$

This is in accordance with equation (1).

Note that it is reasonable to say that "volume" of "unit sphere" on the line, the set of two points ± 1 is equal to 2. This is in accordance with equation (1).