On Barnes sinus-functions

We define Barnes sinus-function (its logarithm) by relation

$$Sn(z|a_1,\dots,a_n) = \Psi(z|a_1,\dots,a_n) - (-1)^n \Psi\left(\sum a_i - z|a_1,\dots,a_n\right) , \tag{1}$$

where

$$\Psi(z|a_1,\dots,a_n) = \frac{d}{ds}\zeta(s,z|a_1,\dots,a_n)\big|_{s=0} = \left[\frac{d}{ds}\sum_{r_1,\dots r_n=1}^{\infty}\frac{1}{(z+a_1r_1+\dots a_nr_n)^s}\right]_{\text{regularised}}$$

$$= \int_0^\infty \left(A(t) - A_-(t) - A_0e^{-t}\right)\frac{dt}{t} ,$$

where

$$A(t) = \frac{e^{-zt}}{\prod_i (1-e^{-a_it})} \tag{2}$$

Respectively for the function Sn we have that

$$Sn(z|a_1,\dots,a_n) = \int_0^\infty \left(A^{(Sn)}(t) - A_-^{(Sn)}(t) - A_0^{(Sn)}e^{-t}\right)\frac{dt}{t}$$

with

$$A^{(Sn)}(z,t) = \frac{e^{-zt}-(-1)^n e^{(\sum a_i - z)t}}{\prod_i (1-e^{-a_it})} . \tag{3}$$

Dividing the numerator and denominator of this fraction on $\prod_i e^{\frac{a_it}{2}}$ we come to

$$A_n^{(Sn)}(z,t) = \begin{cases} \dfrac{\mathrm{sh}\left(\frac{\sum_i a_i}{2}-z\right)t}{\prod_i \mathrm{sh}\,\frac{a_it}{2}} & \text{for even } n \\ \dfrac{\mathrm{ch}\left(\frac{\sum_i a_i}{2}-z\right)t}{\prod_i \mathrm{sh}\,\frac{a_it}{2}} & \text{for odd } n \end{cases} \tag{4}$$

Study this function. First of all this is an odd function, and in particular $A_0^{(Sn)} = 0$. Thus one can see that

$$Sn(z|a_1,\dots,a_n) = \int_0^\infty \left(A^{(Sn)}(t) - A_-^{(Sn)}(t) - A_0^{(Sn)}e^{-t}\right)\frac{dt}{t} = \frac{1}{2}\int_{-\infty}^{\infty} A^{(Sn)}(t)\frac{dt}{t+i0} \tag{5}$$

(where function $A^{(Sn)}(t)$ is defined for negative t by its odness: $A^{(Sn)}(t) = A^{(Sn)}(-t)$.

The function Sn is not physical but it does not possesses zero term.

One can see that

$$Sn_{2k}(z|a_1,a_2,\dots,a_{2k}) = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\mathrm{sh}\left(\frac{\sum_i a_i}{2}-z\right)t}{\prod_i \mathrm{sh}\,\frac{a_it}{2}}\frac{dt}{t+i0}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\mathrm{sh}\,\left(\frac{\frac{a_1}{2}+\sum_{i>1} a_i}{2} - z\right)t}{\mathrm{sh}\,\frac{a_1 t}{2}\prod_{i>1}\mathrm{sh}\,\frac{a_i t}{2}}\frac{dt}{t+i0} =$$

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{\mathrm{sh}\,\frac{a_1 t}{2}\mathrm{ch}\,\left(\sum_{i>1}\frac{a_i}{2} - z\right)t + \mathrm{ch}\,\frac{a_1 t}{2}\mathrm{sh}\,\left(\frac{\sum_{i>1} a_i}{2} - z\right)t}{\mathrm{sh}\,\frac{a_1 t}{2}\prod_{i>1}\mathrm{sh}\,\frac{a_i t}{2}}\frac{dt}{t+i0} =$$

$$Sn_{2k-1}()z|a_2,\dots,a_{2k}) + \frac{1}{2}\mathrm{coth}\,\frac{a_1 t}{2}\int_{-\infty}^{\infty} \frac{\mathrm{sh}\,\left(\frac{\sum_{i=2}^{2k} a_i}{2} - z\right)t}{\prod_{i=2}^{2k}\mathrm{sh}\,\frac{a_i t}{2}}\frac{dt}{t+i0} \tag{6}$$

Analogous for $n = 2k-1$. Sure the last integral possesses non-pleasant term at the power $t = 0$

Note that one can find another representation for the function Sn. To see it return to general formulae: Let $A(t)$ be an arbitrary rapidly decreasing at infinity function which is smooth at all the points excluding origin, and has finite Laurent series at origin. The function $\zeta_A(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}A(t)dt$ is well-defined analytical function for $\Re s > C$, and one can consider analytical continuation of this function for all s: for enough big N

$$\zeta_A^{\mathrm{reg.}}(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}A(t)dt = \frac{1}{\Gamma(s)}\int_0^1 t^{s-1}A(t)dt + \frac{1}{\Gamma(s)}\int_1^\infty t^{s-1}A(t)dt =$$

$$\sum_{k\leq N}\frac{A_k}{\Gamma(s)(k+s)} + \frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\left(A(t)dt - \sum_{k\leq N}A_k t^k\right)dt + \frac{1}{\Gamma(s)}\int_1^\infty t^{s-1}A(t)dt\,,$$

where $A(t) = \sum A_k t^k$ in a vicinity of origin. For example $\zeta_A^{\mathrm{reg.}}(0) =$

$$\left[\sum_{k\leq 0}\frac{A_k}{\Gamma(s)(k+s)} + \frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\left(A(t)dt - \sum_{k\leq 0}A_k t^k\right)dt + \frac{1}{\Gamma(s)}\int_1^\infty t^{s-1}A(t)dt`\right]_{s=0} = A_0$$

since $\Gamma(s) \sim \frac{1}{s}$ in a vicinity of origin and

$$\Gamma_A = \left(\frac{d}{ds}\zeta_A^{\mathrm{reg.}}(s)\right)|_{s=0} =$$

$$\sum_{k<0}\frac{A_k}{k} + \frac{d}{ds}\left(\frac{1}{\Gamma(s+1)}\right)_{s=0} A_0 + \int_0^1\left(A(t)dt - \sum_{k\leq 0}A_k t^k\right)\frac{dt}{dt} + \int_1^\infty A(t)\frac{dt}{t} =$$

$$\int_1^\infty\left(-\sum_{k<0}A_k t^k - e^{-t}A_0\right)\frac{dt}{dt} + \int_0^1\left(A(t)dt - \sum_{k<0}A_k t^k - e^{-t}A_0\right)\frac{dt}{dt} + \int_1^\infty A(t)\frac{dt}{t} +$$

$$\left(-\Gamma'(1) + \int_0^1\frac{e^{-t}-1}{t}dt) + \int_1^\infty\frac{e^{-t}}{t}dt\right)A_0 = \tag{$***$}$$

$$\int_0^\infty \left(A(t) - \sum_{k<0} A_k t^k - e^{-t} A_0 \right) \frac{dt}{t} \,. \tag{7}$$

(The term (***) vanishes. This can be checked straightforwardly or to prove it in the following way: Coefficient in front of A_0 does not depend on a function A_0. Take $A(t) = e^{-t}$, then it is evidently vanishes $\blacksquare$)

Revenons á nos moutons. We return to formula (2). It is evident from the formulae above that odd part of function $A(t)$ (2) is a function $A^{(Sn)}$. Using this fact calculate

$$\int_{-\infty}^{\infty} A(t) \frac{dt}{t - i0} \tag{8}$$

Fact Here we use that $A(t)$ is meromorphic function in a vicinity of origin. Exact calculations show that

$$\int_{-\infty}^{\infty} A(t) \frac{dt}{t - i0} = \int_{|t| \geq \varepsilon} \left(A^{\mathrm{even}}(t) + A^{\mathrm{odd}}(t) \right) \frac{dt}{t - i0} = 2 \int_{\varepsilon}^{\infty} A^{\mathrm{odd}}(t) \frac{dt}{t} + + \int_{|z| = \varepsilon, \mathrm{Im}\, z < 0} A^{\mathrm{even}}(z) \frac{dz}{z} =$$