Locally Euclidean Geometries

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Locally Euclidean geometries and hyperbolic geometry

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`This talks is based on the book of S.Nikuin, R.Shafarevich`

Abstract

Geometry on the surface of the cylinder is locally Euclidean. An "ant-mathematician" who lives on the cylinder will not distinguish the geometry of the surface at small distances from the Euclidean geometry; the Pythagorean Theorem will be almost the same, and for "not too large" triangles the sum of the angles will be π. In the first part of talk, we will study locally Euclidean two-dimensional geometries. We will study these geometries by using discrete subgroups of the isometry group of the Euclidean plane E^2. The list of locally Euclidean geometries is exhausted by the geometries on the surface of the cylinder, on the surface of the torus, on the surface of the "twisted cylinder" (the Moebius band), and on the so-called Klein bottle. In the second part of the talk, we will consider the set of locally Euclidean geometries, and will show that this set can be naturally parametrized by the points of the Lobachevsky (hyperbolic) plane.

Contents

1 (Uniformly) locally Eucldean surfaces

We consider locally Euclidean 2-dimensional geometries. An arbitrary 2-dimensional geometry can be considered as 2-dimensional Riemannian surface—(M, G). M is a surface, and G defines scalar product of tangent vectors, i.e. length of an arbitrary curve. For arbitrary curve $\mathbf{x} = \mathbf{x}(t), t_1 \le t \le t_2$

$$\text{length of the curve} = \int_{t_1}^{t_2} \sqrt{(\mathbf{v}(t), \mathbf{v}(t))}dt\,, \text{ scalar product } (\mathbf{v}(t), \mathbf{v}(t)) = G(\mathbf{v}(t), \mathbf{v}(t))\,,$$

where $\mathbf{v}(t)$ is velocity vector. In local coordinates x^i, the curve has appearance $\mathbf{x}(t) = x^i(t)$, $G = g_{ik}(x)dx^i dx^k$, $\mathbf{v}(t) = v^i(x(t))\partial_i = \frac{dx^i(t)}{dt}\frac{\partial}{\partial x^i}$, and the scalar product of velocity vector on itself is equal to

$$(\mathbf{v}, \mathbf{v}) = v^i(x(t))g_{ik}(x(t))v^k(x(t)) = \frac{dx^i(x(t))}{dt} g_{ik}(x(t)) \frac{dx^k(x(t))}{dt}\,,$$

i.e.

$$\text{length of the curve} = \int_{t_1}^{t_2} \sqrt{(\mathbf{v}(t), \mathbf{v}(t))}dt = \int_{t_1}^{t_2} \sqrt{\frac{dx^i(x(t))}{dt} g_{ik}(x(t)) \frac{dx^k(x(t))}{dt}}\,.dt$$

We say that M is (uniformly) locally Eucldean if

1) in a vicinity of arbitrary points there exist Eucldean coordinates, i.e. the coordinates u, v such that $G = du^2 + dv^2$ in these coordinates.

2) This neighborhood is enough large: there exists $r > 0$ such that all local cEucldiean coordinates u, v are defined in the circle of radius $\ge r$.

Remark Surface is called locally Euclidean if the first condition is obeyed. In the case if the second condition is obeyed also, the surface is called *uniformly locally Euclidean.*

Exercise Show that the surface of sphere is not locally Eucldean.

Exercise Show that domain $a < x < b$ of $\mathbf{E}^2$ is locally Eucldean but it is not uniformly locally Eucldean.

(We suppose that metric on the surface M in $\mathbf{E}^3$ is the metric induced from $\mathbf{E}^3$.)

In this talk we will consider only uniformly locally Euclidean surface, and we will call them sometimes just locally Eucldean.

2 Examples of locally Euclidean surfaces and subgroups of $\mathbf{E}(2)$.

Exercise Consider surface of cylinder.

2.1 Subgroups of group $\mathbf{E}(2)$ and surfaces

Let Γ be an arbitrary subgroup of group of isometries of $\mathbf{E}^2$.

Assign to the group Γ a space M_Γ of orbits of G-group action on $\mathbf{E}^2$, $M_\Gamma = \mathbf{E}^2 \backslash \Gamma$, i.e.

We denote the points of the space M_Γ, by handwriting letters $\mathcal{A}, \mathcal{B}, C, \mathcal{D}, \ldots$. These points are orbits of group Γ action. Every point $A \in \mathbf{E}^2$ produces the point $\mathcal{A} \in \mathbf{E}^2 \backslash \Gamma$, the equivalence class of a point A with respect to the gropu Γ:

$$\mathcal{A} = [A]_\Gamma \,, \qquad g \in \Gamma \,, A' = g(A) \in [A] \,.$$

To establish the geometry on M we define the distance between points $\mathcal{A}, \mathcal{B}$ as the minimal distance between the orbit $\{A^g\}$ and $\{B^g\}$:

$$\text{if } \mathcal{A} = [A] \text{ and } \mathcal{B} = [B] \text{ then } d(\mathcal{A}, \mathcal{B}) = \min_{g, g' \in \Gamma} d(A^g, B^{g'})$$

Exercise 1 Let $\mathbf{a} \neq 0$ be an arbitrary vector in $\mathbf{E}^2$ and $\Gamma = T_{n\mathbf{a}}$ be group of translations generated by the translation on vector $\mathbf{a}$:

$$T_{n\mathbf{a}} \colon \mathbf{r} \to \mathbf{r} + n\mathbf{a} \,.$$

Describe the geometry M_Γ and show that this is geometry of cylinder.

Exercise 2 Let $\Gamma = C_2$ be group of reflection with respect to the line l.

Describe the geometry M_Γ and show that this is not uniformly locally Euclidean manifold.

2.2 Uniformly discontinuous subgroups of $\mathbf{E}(2)$

We say that the subgroup Γ acts properly discontinuous on $\mathbf{E}^2$ if there exists δ such that for an arbitrary point $A \in \mathbf{E}^2$, and for an arbitrary non-identity element $g \in \Gamma$

$$d(A, g(A)) \geq \delta \,.$$

In other words it means that the distance between distinct points of an arbitrary orbit exceeds the δ.

Why these groups are interesting? Because every such group defines locally Eucldean manifold.

Proposition 1. *If action of group Γ has a fixed point (there exists $\mathbf{r}_0 \in \mathbf{E}^2$ such that for all $g \in \Gamma$, $g(\mathbf{r}_0) = \mathbf{r}_0$) (il faut dire mieux), then M_Γ is not uniformly locally Euclidean.*

Proof. If $A = \mathbf{r}_0$ is a fixed point, then for arbitrary $g \neq 1$ $d(A, A^g) \geq \delta$, and on the other hand $d(A, A^g) = 0$, if Γ acts uniformly discontinuou. Contradiction. $\square$

Proposition 2. *If $\Gamma \in \mathbf{E}(2)$ is uniformly discontinuous group, then the group Γ is uniformly discontinuous.*

`Sketch of the proof.`

Let Γ be uniformly discontinuous:

$$\exists \delta \geq 0\,,\,\text{such that for an arbitrary } A \in \mathbf{E}^2\,, g \in \Gamma\,, d(A^g, A) < \delta \Rightarrow g = 1\,.$$

and let B be an arbitrary point which belong to the disc $D_{\frac{\delta}{r3}}(A)$ Consider orbits $\mathcal{A}$ and $\mathcal{B}$ of these points. It is easy to see from triangle inequality that for arbitrary points $A' \in \mathcal{A}$ and $B' \in \mathcal{B}$ the distance between these points is bigger or equal to $r = \frac{\delta}{2}$. Indeed let $A" = A^g$, and $B' = B^h$. Denote $\tilde{B} = B^{hg^{-1}}$. Then $d(A', B') = d\left(A, \left(B^h\right)^{g^{-1}}\right) = d\left(A, \tilde{B}\right)$, and by triangle inequality

$$d(A', B') = d(A, \tilde{B}) \geq \left| d(B, \tilde{B}) - d(A, B) \right| > \delta$$

if $B \neq \tilde{B}$.

Thus we see that in the case if two points A and B are closer than $\frac{\delta}{2}$, then the distance between orbits $\mathcal{A}$ and $\mathcal{B}$ is equal to the distance $d(A, B)$. This proves that M_Γ is uniformly locally Euclidean if Γ is uniformly discontinous.

the orbit of the point B and the orbit of the point A the disc of radius

2.3 Uniformly discontinuous subgroups of $\mathbf{E}(2)$

First of all recall the classic Theorem:

Theorem 1. *(Schalle+?) Any isometry of* $\mathbf{E}^2$ *is rotation, or translation or glided reflection.*

This Theorem possesses two statements. First that an arbitrary (even non-linear map) which is isometry has linear appearance

$$F(\mathbf{r}) = A\mathbf{r} + \mathbf{b}\,,$$

where A is linear operator, and the second statement that this linear map is rotation (with respect to some centre) or translation or glided reflection.

We can prove the first statement under the assumption that $F(\mathbf{r})$ is smooth map of $\mathbf{E}^2$ in $\mathbf{E}^2$.

Suppose now that $F(\mathbf{r}) = A(\mathbf{r}) + \mathbf{b}$.

I-st case) orientation is preserved, i.e. $\det A = 1$. If $A = 1$ then it is translation, if $A \neq 1$ then operator $A - 1$ is invertible, and

$$\mathbf{r} = A(\mathbf{r}) + \mathbf{b} = \mathbf{a} + (A\,(\mathbf{r} - \mathbf{a}))\ \text{, where }, \mathbf{a} = (1 - A)^{-1}(\mathbf{b})\,,$$

i.e. this affine transformation is a rotation around the point $O - \mathbf{a}$.

II-nd case) orientation is not preserved, i.e. $\det A = -1$. This operator has eigenvector $\mathbf{n}$

One can see that we come to reflection with respect to the line along the vector $\mathbf{n}$ and translation along $\mathbf{n}$, i.e. glided reflection.

Now using this Theorem classify uniformly discontinuous subgroups of isometry group.

From no on we exclude the trivial case when $G = e$.

Let Γ be a subgroup of $\mathbf{E}(2)$ which acts on $\mathbf{E}^2$ uniformly discontinuous, i.e. there exists $\delta > 0$ such that for an arbitrary point $A \in \mathbf{E}^2$ and an arbitrary $g \in \Gamma$

$$d(A, A^g) \leq \delta \Rightarrow g = 1\,.$$

It follows from proposition 1 that the subgroup Γ contains only translations and non-trivial glided reflections.

If group $\Gamma = e$ this is trivial: $M_\Gamma = \mathbf{E}^2$.

Denote by Γ_0 the subgroup of Γ which preserve orientation, i.e. subgroup of translations.

Proposition 3. *The subgroup* Γ_0 *of uniformly discontinuous grroup* Γ *of orientation preserving transformations is*

- *the group of translations generated by arbitrary non-zero vector* $\mathbf{a}$

$$\Gamma_0 = \Gamma_0^{\mathbf{a}} = \{T_{n\mathbf{a}}\colon T_{n\mathbf{a}}(\mathbf{r}) = \mathbf{r} + n\mathbf{a},\ , \text{where } n = 0, \pm 1, \pm 2, \dots\}$$

- *the group of translations generated by arbitrary two non-zero linearly independent vectors* $\mathbf{a}, \mathbf{b}$

$$\Gamma_0 = \Gamma_0^{\mathbf{a},\mathbf{b}} = \{T_{m\mathbf{a}+n\mathbf{b}}\colon T_{m\mathbf{a}+n\mathbf{b}}(\mathbf{r}) = \mathbf{r}+m\mathbf{a}+n\mathbf{b},\ , \text{where } m, n = 0, \pm 1, \pm 2, \dots\}$$

On the base of this proposition study the group Γ.

I-st case

Let $\Gamma_0 = \{T_{n\mathbf{a}}\}$. There are two possibilites. First if $\Gamma = \Gamma_0 = \{T_{n\mathbf{a}}\}$, and $M = \mathbf{E}^2 \backslash \Gamma$ is cylindre.

Now suppose $\Gamma \neq \Gamma_0$, i.e. Γ possesses glided reflections. Let $S = S_{l,\mathbf{b}} \in \Gamma$ (reflection with respect to the line l directed along the vector $\mathbf{b}$ and translation on the vector $\mathbf{b}$). Then $S^2 = T_{2\mathbf{b}}$. Hence $\mathbf{b} = \frac{k\mathbf{a}}{2}$ for some integer k. This integer k has to be odd, since if $p = 2p$ then the transformation $T_{-p\mathbf{a}}S$ is the reflection, and it possesses fixed point (see the porposition 1). We see that in this case G is generated by translation $T_{\mathbf{a}}$ and glided reflection $S_{l,\frac{\mathbf{a}}{2}}$

In this case $M = \mathbf{E}^2 \backslash \Gamma$ is twisted cylindre, (Mobius strip with infinite sides)

II-nd case

Let $\Gamma_0 = \{T_{n\mathbf{a}}\}$. Study again two possibilites.

First: $\Gamma = \Gamma_0 = \{T_{n\mathbf{a}}\}$. In this case

$M = \mathbf{E}^2 \backslash \Gamma$ is a torus.

Suppose now that $\Gamma \neq \Gamma_0$ and it possesses glided reflections. One can see that this can happen only if vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal to each other. We will come to $M = \mathbf{E}^2 \backslash \Gamma$ is a Klein bottle.

Theorem 2. *Let Γ be uniformly discontinuous subgroup of isometries group. Then the following possibilities may occur (ca il faut dire mieux!!!)*

- *I-st case (trivial)*

  $\Gamma = e$ *has only identity element. Then* $M = M \backslash \Gamma = \mathbf{E}^2$

- *II-nd case*

  Group $\Gamma = \{T_{m\mathbf{a}}\}$ *is generated by translation on vector* $\mathbf{a}$*, where* $\mathbf{a}$ *is an arbitrary non-zero vector. Then* $M = M \backslash \Gamma$ *is cylindre*

- *III-rd case*

  Group is generated by translation on vector $\mathbf{a}$*, and glided reflection* $S_{l,\frac{\mathbf{a}}{2}}$*, where the line* l *goes along the vector* $\mathbf{a}$ *Then* $M = M\backslash\Gamma$ *is twisted cylindre*

- *IV-th case*

  Group $\Gamma = \{T_{m\mathbf{a}+n\mathbf{b}}\}$ *is generated by translation on vectors* $\mathbf{a}$ *and* $\mathbf{b}$*, where* $\mathbf{a}, \mathbf{b}$ *are arbitrary linearly independent vectors. Then* $M = M\backslash\Gamma$ *is a torus*

- *V-th case*

  Group is generated by translation on vectors $\mathbf{a}$ *and* $\mathbf{b}$*, and glided reflection* $S_{l,\frac{\mathbf{a}}{2}}$*, where the line* l *goes along the vector* $\mathbf{a}$ *where* $\mathbf{a}, \mathbf{b}$ *are arbitrary non-zero vectors which are orthogonal! to each other. Then* $M = M\backslash\Gamma$ *is a Klein bottle.*

3 The final classification

We classified in the Theorem above all locally Euclidean surfaces which are generated by uniformly discontinuous groups.

Is an arbitrary uniformly locally Euclidean surface M generated by uniformluy discontinuous group? Yes!

To prove this statement we consider the following covering

Take an arbitrary point $A \in \mathbf{E}^2$ and $\mathcal{A} \in M$, and consider the map

$$\begin{array}{c} \mathbf{E}^2 \\ p \downarrow \\ M \end{array}$$

such that

- $p(A) = \mathcal{A}$.
- Let B be an arbitrary point in $\mathbf{E}^2$.

  If d

We come to

Theorem 3. *Let M be an arbitrary uniformly locally Euclidean surface. Then there exists uniformly discontinuous group Γ such that*

$$M = M_\Gamma$$

It follows from this Theorem and Proposition 3 the following corollary

Corollary 1. *Let M be an arbitrary uniformly locally Eucldean surface. Then the following cases occur:*

- *M is $\mathbf{E}^2$*
- *M is cylindre*
- *M is twisted cylindre*
- *M is torus*
- *M is Klein bottle*
-

4 Space of locally Eucldean geometries

Geometries on cylindres are similar to each other, the same about geometries of twisted cylindres and Klein bottles.

Consider the space of geometries on tori.

Every lattice $T_{\mathbf{a},\mathbf{b}}$ defines geometry $M_{\mathbf{a},b} = M\backslash\{T_{m\mathbf{a}+n\mathbf{b}}\}$.

Two geometries coincide if lattices are isometric.

We say that two geometries $M_{\mathbf{a},b}$ and $M_{\mathbf{a}',b'}$ are similar if $\mathbf{a}' = \lambda\mathbf{a}$ and $\mathbf{b}' = \lambda\mathbf{b}$.

Proposition 4. *Two geometries $M_{\mathbf{a},\mathbf{b}}$ and $M_{\mathbf{a}',\mathbf{b}'}$ are similar if and only if*

$$\begin{pmatrix} \mathbf{a}' \\ \mathbf{b}' \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}$$

such that

$$ps - qr = \pm 1$$

and

$$p, q, r, s \quad \text{are integers}$$

in other words if these lattices are related by unimodular transformation in integers.

Now identify all similar tori and tori which are related with each other by unimodular transfromation (arbitrary?)

Find distance between geometries.

Let $T_{\mathbf{a},\mathbf{b}}$ be a lattice.

Assign to this lattice the complex number

$$z = \frac{a_x + ia_y}{b_x + ib_y}$$

in the case if $\mathbf{a} \times \mathbf{b}$ is poisitve, and the inverse complex number in the case if $\mathbf{a} \times \mathbf{b}$ is negative.

We see that a class of similar geometries define the point in upper-half plane.

One can say that set of points on the upper half plane is the set of all geometries on tori.

Define distance on this set!

Recall that under transformation transformation

$$\frac{\mathbf{a}}{\mathbf{b}} = \frac{a_x + ia_y}{b_x + ib_y} \leftrightarrow \frac{p\mathbf{a} + q\mathbf{b}}{r\mathbf{a} + s\mathbf{b}}$$

geometry is no changed if $ps - rq = \pm 1$, and p, q, r, s are integeres. Now we find distance function which is invariant with respect to the action of this group. Here we consider instead integeres arbitrary real numbers.

In terms of complex coordinates $z = \frac{a_x+ia_y}{b_x+ib_y}$ we come to transformations

$$w = \frac{az + b}{cz + d}, \quad \text{with } ad - bc = 1\,, w = \frac{a\bar{z} + b}{c\bar{z} + d}, \quad \text{with } ad - bc = -1\,, \quad (4.1)$$

This is so called extended $SL(2, \mathbf{R})$ group

Find a geometry which is invariant with respect to this action.

First find geodesics, "straight lines" of this geometry.

Notice that reflection with respect to vertical axis, and inversion with a centre at real axis, $z = a$ (a is real) are symmetry transformation:

$$w = 2a - \bar{z}\,, \qquad w = a + \frac{1}{\bar{z} - a}$$

The set of fixed points of this transformations are vertical lines and half-circles with centre at real axis. These are geodesics due to the following

Lemma 1. *Let C be a locus of fixed points of isometry. If it is a curve, then this curve is geodesic.*

Consider points on the vertical line $z = it$, and define the distance between these points.

Transformation $z \to \lambda z$, and $z \to -\frac{\lambda}{z}$ are symmetry transfromations, hence for arbitrary points $z = ia, ib$ and for arbitrary real λ

$$d\left(ia, ib\right) = d\left(i\lambda, i\lambda b\right) = d\left(i\frac{\lambda}{a}, i\frac{\lambda}{b}\right) =$$

Moreover since vertical line is geodesic hence for arbitrary three points ia, ib, ic

$$d(ia, ic) = d(ia, ib) + d(ib, ic)\,, \qquad \text{if } a < b < c$$

It follows from these equations that $d(ia, ib) = \log\frac{a}{b}$

Theorem 4. *Riemannian metric which is invariant with resepct to the action of the group* (4.1) *is the Lobachevsky metric which up to multiplier is defined by equation*

$$G = \frac{dzd\bar{z}}{(z-\bar{z})(\bar{z}-z)}$$