Thick morphisms of vector spaces.

Let U, W be a vector spaces, and $L: U \to W$ linear map from U to W. One can consider its adjont $L^*: V^* \to U^*$:

$$L^*: W^* \to U^* \text{ such that for arbitr. } \mathbf{Y} \in U, \langle \mathbf{Y}, L^*(\omega)\rangle = \langle L(\mathbf{Y}), \omega\rangle \,. \tag{0.1}$$

In the case if $W = U^*$ both L and L^* map $U \to U^*$. Map L is self-adjoint (anti-self adjoint if $L = L^*$. This is standard text-book stuff. Using Vornov's thick mrophisms in fact generalise this statement for non-linear maps.

All the constructions will be based on Voronov's thick morphisms.

For linear spaces U, W consider linear space

$$T = U \oplus U^* \oplus W \oplus W^* \tag{1.1}$$

If V is an arbitrary vector space then denote by ω_V 2-form on vector space $V \oplus V^*$ such that

$$\begin{cases} \omega_V(\mathbf{X}, \mathbf{Y}) = 0 \text{ if } \mathbf{X}\,, \mathbf{Y} \in V\,, \\ \omega_V(t, \mathbf{X}) = t(\mathbf{X}) \text{ if } \mathbf{X} \in V \in, t \in V^* \\ \omega_V(t, s) = 0 \text{ if } t\,, s \in V^* \end{cases} , \text{in coordinates}\, \omega_U = dp_i \wedge du^i \tag{1.1a}$$

if p_i coordinates on V^* dual to coordinates to u^j.

Spaces $U \oplus U^*$ and $W \oplus W^*$can be provided with canonical sympletic structures ω_U, ω_W, and the space $T = U \otimes U^* \otimes W \otimes W^*$ with sympelctic structure

$$\Omega = \omega_U - \omega_V = dp_i du^i - dq_a dw^a \tag{1.1b}$$

where we denote by u^i some coordinates on U, p_i coordinates on U^* dual to u^i, and w^a some coordinates on W, q_a coordinates on W^* dual to w^a,

Definition Let $S = S(u, q)$ be a smooth function on $V \times W^*$. This function defines Lagrangian surface

$$\mathcal{L}_S = \left\{ (u^i, p_j, w^a, q_b) \colon\ p_j = \frac{\partial S(u,q)}{\partial u^i}, w^a = \frac{\partial S(u,q)}{\partial q_a} \right\} \tag{1.2a}$$

Definition The function S and Lagrangian surface $\mathcal{L}_S$ define thich morhism

$$\Phi_S \colon U \to W\,. \tag{1.2b}$$

Thick morphism Φ_S defines pull-back of functions

$$\Phi_S^*\colon U \leftarrow W\,. \tag{1.2c}$$

which is the following non-linear map:

$$C(W) \ni g \to f = f(u) = g(w) + S(u,q) = q(w)\,, \tag{1.3}$$

Geometrical meaning of this map: Function g defines Lagrangian surface

$$\mathcal{L}_g = \left\{ (w^a, q_b)\colon\ q_a = \frac{\partial g(w)}{\partial w^a} \right\} \tag{1.4a}$$

This is -graph of the function dg in the symplectic space $W \oplus W^*$. Respectively, function f defines Lagrangian surface $\mathcal{L}_f = \left\{ (x^i, p_j)\colon\ p_j = \frac{\partial f(x)}{\partial x^i} \right\}$. This is -graph of the function dg in the symplectic space $W \oplus W^*$. The ansatz (1.3) means that Lagrangian surfaces $\mathcal{L}_F, \mathcal{L}_g$ are related via the Lagrangian surface $\mathcal{L}_S$:

$$\mathcal{L}_f = \mathcal{L}_S \circ \mathcal{L}_g \tag{1.5}$$

Note that the relations (1.5) define pullback up to a constant. (ecrire mieux)

In the special case if

$$S = \Psi^a(x) q_a$$

then thick morphism Φ_S is nothing but the morphism $w^a = \Psi^a(x)$.