On canonical isomorphisms $TT^*M = T^*TM = T^*T^*M$

I wrote this file two-three years ago. Now I just added the pencil of isomorphisms.

Let M be manifold. Establish and study canonical isomorphisms $TT^*M = T^*TM = T^*T^*M$

Calculations in coordinates

It may sounds surprising but calculations in coordinates are transparent and illuminating.

First consider local coordinates on TM and T^*M corresponding to local coordinates (x^i) on M.

Local coordinates for TM are (x^i, t^j): every vector $\mathbf{r} \in TM$ is a vector $t^i \frac{\partial}{\partial x^i}$, $t^i(\mathbf{r}) = dx^i(\mathbf{r})$. If $\tilde{x}^\mu = \tilde{x}^\mu(x^i)$ are new local coordinates on M then

$$d\tilde{x}^\mu \left(t^i \frac{\partial}{\partial x^i} \right) = \frac{\partial \tilde{x}^\mu(x^i)}{\partial x^i} dx^i \left(t^i \frac{\partial}{\partial x^i} \right) = \frac{\partial \tilde{x}^\mu(x^i)}{\partial x^i} t^i \,.$$

Hence changing of local coordinates in TM is

$$(x^i, t^j) \mapsto (\tilde{x}^\mu, \tilde{t}^\nu), \quad \tilde{x}^\mu = \tilde{x}^\mu(x^i), \ \tilde{t}^\mu = \begin{pmatrix} \mu \\ i \end{pmatrix} t^i, \tag{1}$$

where we denote $\frac{\partial \tilde{x}^\mu(x^i)}{\partial x^i}$ by $\begin{pmatrix} \mu \\ i \end{pmatrix}$.

Respectively local coordinates for T^*M are (x^i, p_j). For every 1-form $w \in T^*M$ $p_i = w\left(\frac{\partial}{\partial x^i}\right)$. Under changing of local coordinates on M $\tilde{x}^\mu = \tilde{x}^\mu(x^i)$, coordinates (p_i) change to new coordinates (p_μ):

$$p_\mu = w\left(\frac{\partial}{\partial \tilde{x}^\mu}\right) = w\left(\frac{\partial x^i(\tilde{x}^\mu)}{\partial \tilde{x}^\mu} \frac{\partial}{\partial x^i}\right) = \frac{\partial x^i(\tilde{x}^\mu)}{\partial \tilde{x}^\mu} p_i$$

Hence changing of local coordinates in T^*M is

$$(x^i, p_k) \mapsto (\tilde{x}^\mu, \tilde{p}_\nu), \quad \tilde{x}^\mu = \tilde{x}^\mu(x^i), \ p_\mu = \begin{pmatrix} i \\ \mu \end{pmatrix} p_i, \tag{2}$$

where we denote $\frac{\partial x^i(\tilde{x}^\mu)}{\partial \tilde{x}^\mu}$ by $\begin{pmatrix} i \\ \mu \end{pmatrix}$

Now using (1),(2) we define coordinates on the spaces TT^*M, T^*TM and T^*T^*M.

The space TT^*M is tangent space to the space T^*M. The local coordinates on TT^*M corresponding to local coordinates (x^i, p_j) on T^*M are coordinates $(x^i, p_j; \xi^k, \rho_m)$; $\xi^k =$

$dx^i(\mathbf{r}), \rho_m = dp_m(\mathbf{r})$. Under changing of local coordinates (x^i) to coordinates $\tilde{x}^\mu = \tilde{x}^\mu(x^i)$ coordinates (ξ^i) and (ρ_m) transform to new coordinates $(\tilde{\xi}^\mu)$ and $(\tilde{\rho}_\nu)$ respectively. It follows from (1) that

$$\tilde{\xi}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^i}\xi^i + \frac{\partial \tilde{x}^\mu}{\partial p_i}\rho_i = \begin{pmatrix}\mu\\ i\end{pmatrix}\xi^i \tag{3}$$

because $\frac{\partial \tilde{x}^\mu}{\partial p_i} = 0$. For transformation of coordinates (ρ_m) calculations are longer:

$$\tilde{\rho}_\mu = \frac{\partial \tilde{p}_\mu}{\partial x^i}\xi^i + \frac{\partial \tilde{p}_\mu}{\partial p_i}\rho_i$$

We see that $\frac{\partial \tilde{p}_\mu}{\partial p_i} = \frac{\partial}{\partial p_i}\left(\tilde{p}_\mu = \begin{pmatrix}k\\ \mu\end{pmatrix} p_k\right) = \begin{pmatrix}i\\ \mu\end{pmatrix}$ and

$$\frac{\partial \tilde{p}_\mu}{\partial x^i} = \frac{\partial}{\partial x^i}\left(\tilde{p}_\mu = \begin{pmatrix}k\\ \mu\end{pmatrix} p_k\right) = \begin{pmatrix}\nu\\ i\end{pmatrix}\begin{pmatrix}k\\ \nu\mu\end{pmatrix} p_k,$$

where we denote as always by $\begin{pmatrix}\nu\\ i\end{pmatrix}$ the partial derivative $\frac{\partial \tilde{x}^\mu}{\partial x^i}$ and by $\begin{pmatrix}k\\ \nu\mu\end{pmatrix}$ the partial derivative $\frac{\partial^2 x^k}{\partial \tilde{x}^\nu \partial \tilde{x}^\mu}$. The summation over repeated indices is assumed. Finally we come to

$$\tilde{\rho}_\mu = \frac{\partial \tilde{p}_\mu}{\partial x^i}\xi^i + \frac{\partial \tilde{p}_\mu}{\partial p_i}\rho_i = \xi^i\begin{pmatrix}\nu\\ i\end{pmatrix}\begin{pmatrix}k\\ \nu\mu\end{pmatrix} p_k + \begin{pmatrix}i\\ \mu\end{pmatrix}\rho_i \tag{4}$$

Summarising:

Proposition 1 *To local coordinates (x^i) on M one can naturally assign local coordinates on TT^*M $(x^i, p_j; \xi^k, \rho_m)$ such that under changing of coordinates $(x^i) \mapsto (\tilde{x}^\mu)$ on M these coordinates transform in the following way*

$$\tilde{p}_\mu = \begin{pmatrix}j\\ \mu\end{pmatrix} p_j, \quad \tilde{\xi}^\mu = \begin{pmatrix}\mu\\ i\end{pmatrix}\xi^i, \quad \tilde{\rho}_\mu = \xi^i\begin{pmatrix}\nu\\ i\end{pmatrix}\begin{pmatrix}k\\ \nu\mu\end{pmatrix} p_k + \begin{pmatrix}i\\ \mu\end{pmatrix}\rho_i \tag{$*$}$$

Now consider coordinates on T^*TM and their transformation rules. If (x,t) coordinates on TM (see (1)) and (x,t,π,τ) corresponding coordinates on T^*TM ($\pi_k = w\left(\frac{\partial}{\partial x^k}\right)$, $\tau_m = w\left(\frac{\partial}{\partial t^m}\right)$) then according to (2) under changing of coordinates $(x^i) \mapsto (\tilde{x}^\mu)$, the coordinates (π_m) transform to coordinates $(\tilde{\pi}_\mu)$, the coordinates (τ_k) transform to coordinates $(\tilde{\tau}_\nu)$ such that

$$\tilde{\pi}_\mu = \frac{\partial x^i}{\partial \tilde{x}^\mu}\pi_i + \frac{\partial t^k}{\partial \tilde{x}^\mu}\tau_k, \quad \tilde{\tau}_\nu = \frac{\partial x^i}{\partial \tilde{t}^\nu}\pi_i + \frac{\partial t^k}{\partial \tilde{t}^\nu}\tau_k$$

Since $\frac{\partial x^i}{\partial \tilde{t}^\nu} = 0$ and $\frac{\partial t^k}{\partial \tilde{t}^\nu} = \frac{\partial x^k}{\partial \tilde{x}^\mu}$ then

$$\tilde{\tau}_\nu = \begin{pmatrix} k \\ \nu \end{pmatrix} \tau_k$$

. For $\tilde{\pi}_\mu$ we have

$$\tilde{\pi}_\mu = \begin{pmatrix} i \\ \mu \end{pmatrix} \pi_i + \frac{\partial}{\partial \tilde{x}^\mu} \left(\frac{\partial x^k}{\partial \tilde{x}^\nu} t^\nu \right) \tau_k = \begin{pmatrix} i \\ \mu \end{pmatrix} \pi_i + \begin{pmatrix} k \\ \mu\nu \end{pmatrix} \begin{pmatrix} \nu \\ i \end{pmatrix} t^i \tau_k \,.$$

Summarising:

Proposition 2 *To local coordinates (x^i) on M one can naturally assign local coordinates on T^*TM $(x^i, t^j; \pi_k, \tau_j)$ such that under changing of coordinates $(x^i) \mapsto (\tilde{x}^\mu)$ on M these coordinates transform in the following way*

$$\tilde{\tau}_\mu = \begin{pmatrix} j \\ \mu \end{pmatrix} \tau_j, \quad \tilde{t}^\mu = \begin{pmatrix} \mu \\ i \end{pmatrix} t^i, \quad \tilde{\pi}_\mu = t^i \begin{pmatrix} \nu \\ i \end{pmatrix} \begin{pmatrix} k \\ \nu\mu \end{pmatrix} \tau_k + \begin{pmatrix} i \\ \mu \end{pmatrix} \pi_i \qquad (**)$$

Comparing Propositions 1 and 2 we see that the map

$$t^i = \xi^i, \;\; \tau_j = p_j, \;\; \pi_k = \rho_k$$

establishes isomorphism between the spaces T^*TM and TT^*M which does not depend on the choice of local coordinates. In fact one can consider the *pencil* of maps

$$t^i = \mathbf{a}\xi^i, \;\; \tau_j = \mathbf{b}p_j, \;\; \pi_k = \mathbf{ab}\rho_k$$

where $\mathbf{a}, \mathbf{b} \neq 0$.