## On canonical isomorphisms $T^*E = T^*E^*$ for vector bundle $E$

Kirill Mackenzie told and explained many many times the construction of remarkable isomorphism between cotangent bundles of vector bundle and its dual: for an arbitrary vector bundle $E$ $T^*E = T^*E^*$, where $E^*$ is a bundle dula to $E$.

The first thing that you try to apply this construction to it is to consider tangent bundle: $E = TM$ or cotangent bundle $E = T^*M$ and consider isomorphism $T^*TM = T^*T^*M$. On the other hand in this special case the canonical symplectic structure on the cotangent bundle $T^*M$ implies the canonical isomorphism between tangent and cotangent bundles of the manifold $T^*M$: $T^*T^*M = TT^*M$. Hence in the special case of $E = TM$ the "Mackenzie" isomorphism combined with isomorphism induced by cnonical symplectic structure implies the canonical isomorphisms $T^*TM = TT^*M$.

In this etude we would like to reconstruct the "Mackenzie" isomorphism $T^*E = T^*E^*$ and its special case, the isomorphism $TT^*M = T^*TM$ using pedestrian's arguments. In the first section we conisder the special case $E = TM$ and establish the isomorphism$TT^*M = T^*TM$. In the second question we will establish the "Mackenzie" isomorphism $T^*E = T^*E^*$.

Our notations are little bit inconsistent: in the first paragraph we denote coordintes by $x^i, ....$ and new ones by $\tilde{x}^\mu, ....$ In the second paragraph our notations are much more traditional: indices of new coordinates are denoted by the same letters with "prime" indices$(x^i \to x^{i'})$.

### Canonical isomorphism $TT^*M = T^*TM$

Let $M$ be manifold. Establish and study canonical isomorphisms $TT^*M = T^*TM = T^*T^*M$.

Perform calculations in local coordinates. It may sounds surprising but calculations in local coordinates are transparent and illuminating.

First consider local coordinates on $TM$ and $T^*M$ corresponding to local coordinates $(x^i)$ on $M$. Local coordinates for $TM$ are $(x^i, t^j)$: every vector $\mathbf{r} \in TM$ is a vector $t^i \frac{\partial}{\partial x^i}$, $t^i(\mathbf{r}) = dx^i(\mathbf{r})$. If $\tilde{x}^\mu = \tilde{x}^\mu(x^i)$ are new local coordinates on $M$ then

$$d\tilde{x}^\mu \left( t^i \frac{\partial}{\partial x^i} \right) = \frac{\partial \tilde{x}^\mu(x^i)}{\partial x^i} dx^i \left( t^i \frac{\partial}{\partial x^i} \right) = \frac{\partial \tilde{x}^\mu(x^i)}{\partial x^i} t^i \,. \tag{1.1}$$

Hence changing of local coordinates in $TM$ is

$$(x^i, t^j) \mapsto (\tilde{x}^\mu, \tilde{t}^\nu), \quad \tilde{x}^\mu = \tilde{x}^\mu(x^i),\ \tilde{t}^\mu = \begin{pmatrix} \mu \\ i \end{pmatrix} t^i, \tag{1.2}$$

where we denote $\frac{\partial \tilde{x}^\mu(x^i)}{\partial x^i}$ by $\begin{pmatrix} \mu \\ i \end{pmatrix}$.

Respectively local coordinates for $T^*M$ are $(x^i, p_j)$. For every 1-form $\omega \in T^*M$ $p_i = \omega\left(\frac{\partial}{\partial x^i}\right)$. Under changing of local coordinates on $M$ $\tilde{x}^\mu = \tilde{x}^\mu(x^i)$, coordinates $(p_i)$ change to new coordinates $(p_\mu)$:

$$p_\mu = w\left(\frac{\partial}{\partial \tilde{x}^\mu}\right) = \omega\left(\frac{\partial x^i(\tilde{x}^\mu)}{\partial \tilde{x}^\mu}\frac{\partial}{\partial x^i}\right) = \frac{\partial x^i(\tilde{x}^\mu)}{\partial \tilde{x}^\mu} p_i$$

Hence changing of local coordinates in $T^*M$ is

$$(x^i, p_k) \mapsto (\tilde{x}^\mu, \tilde{p}_\nu), \quad \tilde{x}^\mu = \tilde{x}^\mu(x^i),\ p_\mu = \begin{pmatrix} i \\ \mu \end{pmatrix} p_i, \tag{1.3}$$

where we denote $\frac{\partial x^i(\tilde{x}^\mu)}{\partial \tilde{x}^\mu}$ by $\begin{pmatrix} i \\ \mu \end{pmatrix}$

The space $TT^*M$ is tangent space to the space $T^*M$. The local coordinates on $TT^*M$ corresponding to local coordinates $(x^i, p_j)$ on $T^*M$ are coordinates $(x^i, p_j; \xi^k, \rho_m)$; $\xi^k = dx^i(\mathbf{r}), \rho_m = dp_m(\mathbf{r})$. Under changing of local coordinates $(x^i)$ to coordinates $\tilde{x}^\mu = \tilde{x}^\mu(x^i)$ coordinates $(\xi^i)$ and $(\rho_m)$ transform to new coordinates $(\tilde{\xi}^\mu)$ and $(\tilde{\rho}_\nu)$ respectively. It follows from (1.1) that

$$\tilde{\xi}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^i}\xi^i + \frac{\partial \tilde{x}^\mu}{\partial p_i}\rho_i = \begin{pmatrix} \mu \\ i \end{pmatrix} \xi^i .$$

because $\frac{\partial \tilde{x}^\mu}{\partial p_i} = 0$. For transformation of coordinates $(\rho_m)$ calculations are longer:

$$\tilde{\rho}_\mu = \frac{\partial \tilde{p}_\mu}{\partial x^i}\xi^i + \frac{\partial \tilde{p}_\mu}{\partial p_i}\rho_i .$$

We see that $\frac{\partial \tilde{p}_\mu}{\partial p_i} = \frac{\partial}{\partial p_i}\left(\tilde{p}_\mu = \begin{pmatrix} k \\ \mu \end{pmatrix} p_k\right) = \begin{pmatrix} i \\ \mu \end{pmatrix}$ and

$$\frac{\partial \tilde{p}_\mu}{\partial x^i} = \frac{\partial}{\partial x^i}\left(\tilde{p}_\mu = \begin{pmatrix} k \\ \mu \end{pmatrix} p_k\right) = \begin{pmatrix} \nu \\ i \end{pmatrix}\begin{pmatrix} k \\ \nu\mu \end{pmatrix} p_k,$$

where we denote as always by $\begin{pmatrix} \nu \\ i \end{pmatrix}$ the partial derivative $\frac{\partial \tilde{x}^\mu}{\partial x^i}$ and by $\begin{pmatrix} k \\ \nu\mu \end{pmatrix}$ the partial derivative $\frac{\partial^2 x^k}{\partial \tilde{x}^\nu \partial \tilde{x}^\mu}$. The summation over repeated indices is assumed. Finally we come to

$$\tilde{\rho}_\mu = \frac{\partial \tilde{p}_\mu}{\partial x^i}\xi^i + \frac{\partial \tilde{p}_\mu}{\partial p_i}\rho_i = \xi^i \begin{pmatrix} \nu \\ i \end{pmatrix}\begin{pmatrix} k \\ \nu\mu \end{pmatrix} p_k + \begin{pmatrix} i \\ \mu \end{pmatrix}\rho_i .$$

Summarising:

**Proposition 1** *Let $(x^i, p_j; \xi^k, \rho_m)$ be local coordiantes on $TT^*M$ described above. Under changing of coordinates $(x^i) \mapsto (\tilde{x}^\mu)$ on $M$ these coordinates transform in the following way*

$$\tilde{p}_\mu = \begin{pmatrix} j \\ \mu \end{pmatrix} p_j, \quad \tilde{\xi}^\mu = \begin{pmatrix} \mu \\ i \end{pmatrix}\xi^i, \quad \tilde{\rho}_\mu = \xi^i \begin{pmatrix} \nu \\ i \end{pmatrix}\begin{pmatrix} k \\ \nu\mu \end{pmatrix} p_k + \begin{pmatrix} i \\ \mu \end{pmatrix}\rho_i . \tag{$*$}$$

Now consider analogously coordinates on $T^*TM$ and their transformation rules. If $(x,t)$ coordinates on $TM$ (see (1)) and $(x,t,\pi,\tau)$ corresponding coordinates on $T^*TM$ $(\pi_k = \omega\left(\frac{\partial}{\partial x^k}\right), \tau_m = \omega\left(\frac{\partial}{\partial t^m}\right))$ then according to (2) under changing of coordinates $(x^i) \mapsto (\tilde{x}^\mu)$, the coordinates $(\pi_m)$ transform to coordinates $(\tilde{\pi}_\mu)$, the coordinates $(\tau_k)$ transform to coordinates $(\tilde{\tau}_\nu)$ such that

$$\tilde{\pi}_\mu = \frac{\partial x^i}{\partial \tilde{x}^\mu}\pi_i + \frac{\partial t^k}{\partial \tilde{x}^\mu}\tau_k, \quad \tilde{\tau}_\nu = \frac{\partial x^i}{\partial \tilde{t}^\nu}\pi_i + \frac{\partial t^k}{\partial \tilde{t}^\nu}\tau_k$$

Since $\frac{\partial x^i}{\partial \tilde{t}^\nu} = 0$ and $\frac{\partial t^k}{\partial \tilde{t}^\nu} = \frac{\partial x^k}{\partial \tilde{x}^\mu}$ then

$$\tilde{\tau}_\nu = \begin{pmatrix} k \\ \nu \end{pmatrix} \tau_k$$

. For $\tilde{\pi}_\mu$ we have

$$\tilde{\pi}_\mu = \begin{pmatrix} i \\ \mu \end{pmatrix} \pi_i + \frac{\partial}{\partial \tilde{x}^\mu}\left(\frac{\partial x^k}{\partial \tilde{x}^\nu} t^\nu\right)\tau_k = \begin{pmatrix} i \\ \mu \end{pmatrix} \pi_i + \begin{pmatrix} k \\ \mu\nu \end{pmatrix}\begin{pmatrix} \nu \\ i \end{pmatrix} t^i \tau_k\,.$$

Summarising:

**Proposition 2** *Let $(x^i, t^j; \pi^k, \tau^m)$ be local coordinates on $T^*TM$ described above. Under changing of coordinates $(x^i) \mapsto (\tilde{x}^\mu)$ on $M$ these coordinates transform in the following way*

$$\tilde{\tau}_\mu = \begin{pmatrix} j \\ \mu \end{pmatrix} \tau_j, \quad \tilde{t}^\mu = \begin{pmatrix} \mu \\ i \end{pmatrix} t^i, \quad \tilde{\pi}_\mu = t^i \begin{pmatrix} \nu \\ i \end{pmatrix}\begin{pmatrix} k \\ \nu\mu \end{pmatrix} \tau_k + \begin{pmatrix} i \\ \mu \end{pmatrix} \pi_i \qquad (**)$$

Comparing Propositions 1 and 2 we come to

**Observation**

As it was described above assign to local coordiantes $x^i$ on a manifold $M$ local coordinates $(x^i, t^j; \pi_k, \tau_j)$ on $T^*TM$ and local coordiantes $(x^i, p_i; \xi^k, \rho_m)$ on $T^*TM$ which are described above. The map

$$t^i = \xi^i, \;\; \tau_j = p_j, \;\; \pi_k = \rho_k \qquad (1.***)$$

establishes the isomorphism between the spaces $T^*TM$ and $TT^*M$ which does not depend on the choice of local coordinates $x^i$ on $M^*$.

Note that canonical symplectic structure $\Omega = dp_i \wedge dx^i$ establishes canonical isomorphism between spaces $TT^*M$ and $T^*T^*M$:

$$dx^i \leftrightarrow \frac{\partial}{\partial p_i}\,, \quad dp_i \leftrightarrow -\frac{\partial}{\partial x^i}$$

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* In fact one can consider the *pencil* of maps $t^i = \mathbf{a}\xi^i, \;\tau_j = \mathbf{b}p_j, \;\pi_k = \mathbf{ab}\rho_k$ where $\mathbf{a}, \mathbf{b} \neq 0$.

Combining this isomorphism with isomorphism (1.$***$) we come to the isomoprphism $T^*TM = T^*T^*M$ which looks like

$$(x^i, t^i, \pi_k, -p_i) \leftrightarrow (x^i, p_i; \pi_k, t^i)$$

Of course there is two-parametric freedom (see the footnote).

## §2 Canoncial isomorphism in general case: $T^*E = T^*E^*$

In the previous paragraph we constructed canonical isomorphism $T^*E = T^*E^*$ in the case where vector bundle $E$ is tangent (cotangent bundle).

Now consdier general case. Let $(x^\mu, s^i)$ be local coordiantes on bundle $E$. Let under changing of coordinates $x^{\mu'} = x^{\mu'}(x)$ fibre coordiantes $s^i$ transform in the following way: $s^{i'} = \Psi^{i'}_k(x)s^k$. Then Respectively coordinates $s_k$ of dual fibre will transform as $s_{k'} = \Phi^i_{k'}s_i$, where transition matrices $\Psi$ and $\Phi$ are inverse to each other: $\Psi^{i'}_k\Phi^k_{j'} = \delta^{i'}_{j'}$.

Let $(x^\mu, s^i; \rho_\mu, \pi_i)$ be local coordinates in $T^*E$ and respectively let $(x^\mu, s_i; \zeta_\mu, t^i)$ be local coordinates in $T^*E^*$. Reccall that as usual we suppose that 1-form $\omega \in T^*E$ has local coordinates $(x^\mu, s^i; \rho_\mu, \pi_i)$ if it is the function on vectors tangent to the manifold $E$ at the point $(x^\mu, s^i)$ such that

$$\omega\left(\frac{\partial}{\partial x^\mu}\right) = \rho_\mu, \quad \omega\left(\frac{\partial}{\partial s^i}\right) = \pi_i$$

Respectively we suppose that 1-form $\omega \in T^*E^*$ has local coordinates $(x^\mu, s_i; \zeta_\mu, t_i)$ if it is the function on vectors tangent to the manifold $E^*$ at the point $(x^\mu, s_i)$ such that

$$\omega\left(\frac{\partial}{\partial x^\mu}\right) = \zeta_\mu, \quad \omega\left(\frac{\partial}{\partial s_i}\right) = t_i\,.$$

Write down transformation for fields under coordiante trasnformation $x^{\mu'} = x^{\mu'}(x^\mu)$. We have

$$\begin{cases} s^{i'} = \Psi^{i'}_k(x)s^k \\ \rho_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\rho_\mu + \frac{\partial s^i}{\partial x^{\mu'}}\pi_i\,, \\ \pi_{i'} = \frac{\partial x^\mu}{\partial s^{i'}}\rho_\mu + \frac{\partial s^i}{\partial s^{i'}}\pi_i \end{cases} \qquad \begin{cases} s_{i'} = \Phi^k_{i'}(x)s_k \\ \zeta_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\zeta_\mu + \frac{\partial s_i}{\partial x^{\mu'}}t^i \\ t^{i'} = \frac{\partial x^\mu}{\partial s_{i'}}\zeta_\mu + \frac{\partial s_i}{\partial s_{i'}}t^i \end{cases}$$

Note that

$$\frac{\partial x^\mu}{\partial s^{i'}} = \frac{\partial x^\mu}{\partial s_{i'}} = 0,\ \frac{\partial s^i}{\partial s^{i'}} = \Phi^i_{i'},\ \frac{\partial s_i}{\partial s_{i'}} = \Psi^{i'}_i\,,$$

and

$$\frac{\partial s^i}{\partial x^{\mu'}}\pi_i = \frac{\partial \Phi^i_{i'}}{\partial x^{\mu'}}s^{i'}\pi_i = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial \Phi^i_{i'}}{\partial x^\mu}s^{i'}\pi_i = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial \Phi^i_{i'}}{\partial x^\mu}\Psi^{i'}_k s^k\pi_i\,, \tag{2.2$a$}$$

$$\frac{\partial s_i}{\partial x^{\mu'}}t^i = \frac{\partial \Psi^{i'}_i}{\partial x^{\mu'}}s_{i'}t^i = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial \Psi^{i'}_i}{\partial x^\mu}s_{i'}t^i = \frac{\partial x^\mu}{\partial x^\mu}\frac{\partial \Psi^{i'}_i}{\partial x^\mu}\Phi^k_{i'}s_kt^i\,. \tag{2.2$b$}$$

Introducing $L^i_{\mu k}$ such that

$$L^i_{\mu k} = \frac{\partial \Psi^{i'}_k}{\partial x^\mu} \Phi^i_{i'} = -\frac{\partial \Phi^i_{i'}}{\partial x^\mu} \Psi^{i'}_k, \ (\Psi \circ \Phi = 1)$$

we see that the transformations (2.1) have the following nice appearance:

$$\begin{cases} s^{i'} = \Psi^{i'}_k(x) s^k \\ \rho_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \left(\rho_\mu - L^i_{\mu k} s^k \pi_i\right) \\ \pi_{i'} = \Phi^i_{i'} \pi_i \end{cases}, \qquad \begin{cases} s_{i'} = \Phi^k_{i'}(x) s_k \\ \zeta_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \left(\zeta_\mu + L^i_{\mu k} t^k s_i\right) \\ t^{i'} = \Psi^{i'}_i t^i \end{cases} \tag{2.3}$$

Put

$$\begin{cases} \pi_i = \alpha s_i \\ t^i = \beta s^i \\ \zeta_\mu = \gamma \rho_\mu \end{cases}$$

We see that this map is invariant with respect to changing of coordinates if $\beta = -\gamma\alpha$. In particular we can put $\alpha = -1$, $\beta = \gamma = 1$ We come to isomorphism $T^*E = T^*E^*$ defined in local coordinates by condition that

$$(x^\mu, s^i; \rho_\mu, \pi_i) \leftrightarrow (x^\mu, s_i; \zeta_\mu, t^i), \quad \text{such that } \rho_\mu = \zeta_\mu, \pi_i = -s_i, t^i = s^i$$

(Compare with (1.***))

We constructed the isomorphism ∎