

Two formulae for determinants

Let A be $m \times n$ matrix and B be $n \times m$ matrix. Then

$$\det(1 + MN) = \det(1 + NM). \quad (1)$$

Proof. $\text{Tr}(MN)^k = \text{Tr}(NM)^k$. Hence characteristic polynomials $\det(1 + zMN)$ and $\det(1 + zNM)$ coincide. Thus we come to (1). How to prove it in another way?

It follows from (1) that if $M = (x^1, x^2, \dots, x^n)$ is $1 \times n$ matrix then

$$\det(\delta^{ik} + x^i x^k) = \det(1 + M^+ M) = \det(1 + M M^+) = \det(1 + x^i x^i) = 1 + (x^1)^2 = \dots + (x^n)^2$$

The relation (1) has very interesting "generalisation":

Let \mathcal{B}, \mathcal{D} be $p \times p$ matrices such that their entries are... odd "numbers", i.e. anticommuting elements of some \mathbf{Z}_2 -algebra (e.g. odd elements of Grassmann algebra):

$$\mathcal{B} = ||\mathcal{B}_{ik}|| \text{ such that } \mathcal{B}_{ik}\mathcal{B}_{rs} = -\mathcal{B}_{rs}\mathcal{B}_{ik}$$

the same for \mathcal{D} :

$$\mathcal{D} = ||\mathcal{D}_{ik}|| \text{ such that } \mathcal{D}_{ik}\mathcal{D}_{rs} = -\mathcal{D}_{rs}\mathcal{D}_{ik}$$

Then the following identity holds:

$$\det(1 + \mathcal{B}\mathcal{D}) = 1, \quad (2)$$

if \mathcal{B} is symmetrical matrix and \mathcal{D} is antisymmetrical matrix: $\mathcal{B}_{ik} = -\mathcal{B}_{ki}$, $\mathcal{D}_{ik} = \mathcal{D}_{ki}$. This is very important identity*. This identity follows from the fact that for an arbitrary $k = 1, 2, 3, \dots$

$$\text{Tr}(\mathcal{B}\mathcal{D})^k = 0. \quad (3)$$

Indeed in the same way as for identity (1) the relation (3) implies that characteristic polynomial $\det(1 + z\mathcal{B}\mathcal{D})$ equals to 1. Proof of (3) immediately follow from the facts that

$$\text{Tr} A^+ = \text{Tr} A, \quad \text{Tr}(AB) = (-1)^{p(B)p(A)} \text{Tr}(BA), \quad \text{and } \mathcal{B}^+ = \mathcal{B}, \quad \mathcal{D}^+ = -\mathcal{D}.$$

E.g.

$$(\mathcal{B}\mathcal{D}\mathcal{B}\mathcal{D})^+ = \mathcal{D}^+ \mathcal{B}^+ \mathcal{D}^+ \mathcal{B}^+ = \mathcal{D}\mathcal{B}\mathcal{D}\mathcal{B}.$$

Hence

$$\text{Tr}(\mathcal{B}\mathcal{D})^2 = \text{Tr}(\mathcal{B}\mathcal{D}\mathcal{B}\mathcal{D}) = \text{Tr}(\mathcal{B}\mathcal{D}\mathcal{B}\mathcal{D})^+ = \text{Tr}(\mathcal{D}\mathcal{B}\mathcal{D}\mathcal{B}) = -\text{Tr}(\mathcal{B}\mathcal{D}\mathcal{B}\mathcal{D}) = -\text{Tr}(\mathcal{B}\mathcal{D})^2,$$

i.e. $\text{Tr}(\mathcal{B}\mathcal{D})^2 = 0$.

* In particular it follows from this identity that square root of Berezinian (superdeterminant) of linear canonical transformation is equal to the determinant of its boson-boson sector