Monge cone and characteristic for equation $F(x,u,p,q)=0$

Consider partial differential equation

$$F(x,y,u,p,q)=0$$

on function $u=u(x,y)$, where $p=u_x, q=u_y$.

Function u, solution of this equation defines surface u_F.

Differential equation is a function on space of 1-forms

Pick an arbitrary point $\mathbf{p}$ at the surface M_u.

The set of all $1-forms$ at the space $T^*_{\mathbf{p}}R^3$ which obey the equation $F\big|_{\mathbf{p}}=0$ defines one-parametric curve in projective space P^{*2}. This curve is Monge cone.

The dula of this curve, the curve in P^2 defined by vectors, directions, which annihilate these forms.

This is *dual Monge cone.*

Calculations:

Consider projective curve of 1-forms $[a(s):b(s):c(s)]=[p(s):q(s):-1]$ such that $F(p,q)=0$ (at the point $\mathbf{p}$, i.e.)

It annihilates tangent vectors. A curve $[\alpha(s):\beta(s):\gamma(s)]$ dual to the curve $[a(s):b(s):c(s)]$ can be defined by relation

$$\det\begin{pmatrix} a(s) & \dot a(s) & \alpha(s)\\ b(s) & \dot b(s) & \beta(s)\\ c(s) & \dot c(s) & \gamma(s)\end{pmatrix}=0$$

For the Monge curve $[p(s):q(s):-1]$ we have

$$\det\begin{pmatrix} p(s) & \dot p(s) & \alpha(s)\\ q(s) & \dot q(s) & \beta(s)\\ -1 & 0 & \gamma(s)\end{pmatrix}=0\Rightarrow\begin{bmatrix}\alpha(s)\\ \beta(s)\\ \gamma(s)\end{bmatrix}=\left[\begin{pmatrix}p(s)\\ q(s)\\ -1\end{pmatrix}\times\begin{pmatrix}\dot p(s)\\ \dot q(s)\\ 0\end{pmatrix}\right]=\begin{bmatrix}\dot q(s)\\ -\dot p(s)\\ p(s)\dot q(s)-q(s)\dot p(s)\end{bmatrix}.$$

Recalling that

$$F_p\dot p+F_q\dot q=0$$

we come to

$$[\alpha(s):\beta(s):\gamma(s)]=[\dot q(s):-\dot p(s):p(s)\dot q(s)-q(s)\dot p(s)]=[F_p:F_q:pF_p+qF_q]$$

This is characteristic, more precisely the projection of bicharacteristic on the points space.

For every point $\mathbf{p} = (x, y, u)$ and for every point $[p : q : -1]$ on the Monge cone over point $\mathbf{p}$ (Monge cone is one-parametric curve in projective space) there is a direction

$$[dx : dy : du] = [F_p : F_q : pF_p + qF_q] ,$$

which is a point on the dula curve corresponding to the point $[p : q : -1]$.

This direction is nothing is defined by the projection of bicharacteristic

$$\mathbf{X} = K_F = F_p$$

(see the file on bicharacteristics) on the points space.

We defined in the previous file $\mathbf{X}$ as a field sympelctorthogonal to tnagent vector fields which annihilate the contact strucutre.

Here we will calculate the bicharacteristics $\mathbf{X}$ direction (i.e. vectgor field (up to multiplier) just using Monge cone considerations.

For every point (x, y, u, p, q) on the manifold M_f we will define

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{u} \\ \dot{p} \\ \dot{q} \end{pmatrix}$$

We consider curve

$$\begin{pmatrix} x(s) \\ y(s) \\ u(s) \\ p(s) \\ q(s) \end{pmatrix}$$

We have already that

$$[\dot{x} : \dot{y} : \dot{u}] = [F_p : F_q : pF_p + qF_q]$$

It is convenient to choose $\dot{x} = F_p, \dot{y} = F_q$ (we calculate up to a multiplier.) Now caclulate $\dot{p}, \dot{q}$.

$$\frac{dp(s)}{ds} = \frac{d}{ds} u_x(x(s, y(s))) = p_x x_s + p_y y_s$$

Use the fact that

$$p_y = q_x \,, \quad \text{Lagrangian surface}$$

We have

$$\frac{dp(s)}{ds} = \frac{d}{ds} u_x(x(s,y(s))) = p_x x_s + p_y y_s = p_x F_p + p_y F_q = p_x F_p + q_x F_q =$$

$$(F_x + F_u u_x + F_p p_x + F_q q_x) - F_x - F_u u_x \, .$$

Now note that

$$(F_x + F_u u_x + F_p p_x + F_q q_x) = \frac{d}{dx} \left(F(x,y,u,p,q) \right)_{u=u(x,y)\,,p=p(x,y)\,,q=q(x,y)} = 0 \, .$$

Hence we come to

$$\frac{dp(s)}{ds} = -F_x + F_u u_x \, .$$

In the same way we calculate

$$\frac{dq(s)}{ds} = -F_y + F_u u_y \, .$$

nd we come to bicharacteristic direction

$$dx : dy : du : dp : dq = F_p : F_q : pF_p + qF_q : -(F_x + u_x F_u) : -(F_x + u_x F_u)$$

This is in accordance with the fact that bicharacteristic vector field is equal to

$$\mathbf{X} = K_F = F_p \frac{\partial}{\partial x} + F_q \frac{\partial}{\partial y} - (F_x + u_x F_u) \frac{\partial}{\partial p} + (F_y + u_y F_u) \frac{\partial}{\partial q} + (pF_p + qF_q) \frac{\partial}{\partial u}$$

If we ingnore u we come to Hamiltonian vector field.