Angle function

In this etude we will speak about one function which can be defined in a very simple way, and it has relations with such beautiful problem of mathematics as Dirichle problem for domain, conformal map. e.t.c

Let $C\colon \mathbf{r} = \mathbf{r}(t)$ be a curve in $\mathbf{E}^2$. Consider the function

$$W(\mathbf{R}) = \text{the angle at which one sees the curve } C \text{ from the point } \mathbf{R} \tag{1.1}$$

What can we say about this function?

1) This function is harmonic function:

$$\Delta W = \frac{\partial^2 W(x,y)}{\partial x^2} + \frac{\partial^2 W(x,y)}{\partial y^2} = 0\,, \quad \text{for all the points } \mathbf{r} \notin C$$

(x, y are standard Cartesian coordinates in $\mathbf{E}^2$)

on the surface C this function has jump of values

Example 1 1. Let AB be a segment of straight line between points $A = (a, b)$ and $B = (1, 1)$ on the axis $Ox = X$, $A = (a, 0)$, $B = (b, 0)$.

Consider an arbitrary point $P = (X, Y)$ on $\mathbf{E}^2$. Let $N = (0, X)$ be a projection of the point P on the axis OX. Then it is obvious that

$$W(P) = W(X,Y) = \angle PAN - \angle PBN = \arctan\frac{X-a}{Y} - \arctan\frac{X-b}{Y}\,. \tag{1.Ex1.1}$$

Example 2 Let $u(\varphi)$, $\nu(\varphi)$ be functions on unit circle $x^2 + y^2 = 1$ then

$$W(r,\theta) = \int_0^{2\pi} \frac{1 - r\cos(\theta-\varphi)}{1 + r^2 - 2r\cos(\theta-\varphi)}\nu(\varphi)d\varphi = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1 + r^2 - 2r\cos(\theta-\varphi)}f(\varphi)d\varphi$$

is a harmonic function in circle with "wall" on the circle. the jump is equal to $\pi\nu(\varphi)$.

Example 3 Let $u(x)$ be a function on axis OX. Then

$$W(x,y) = \int \frac{yu(t)}{(x-t)^2 + y^2}dt$$

is harmonic function with wall on the axis OX

One can easy to generalise this example considering instead C hypersurface pf dimension $n-1$ in $\mathbf{E}^n$, and instead angle cohomology (see later) For example for (1.1)

$$W(X,Y) = \int F^*_{\mathbf{R}}(xdy - ydx) = \int \frac{(x-X)dy - (y-Y)dx}{(x-X)^2 + (y-Y)^2}$$

for the case $n > 1$ one have to consider instead 1-form -cohomology $\frac{xdy - ydx}{x^2+y^2}$. the $n-1$-form

$$\omega\colon dx^1 \wedge \ldots \wedge dx^n = r^{n-1} dr \wedge \omega$$

3)This function is related wih the potential of double layer (see later)

4) This function is related with Dirichle problem: find a function W which is harmonic in a domain U if its value on the boundary ∂U is equal to the given function

5) Conformal mapping: Let U be a function which is conjugate to the function W, i.e.

$$F(x, y) = W + iU$$

is meromorphic function