Hilbert Theorem of non-embedding of hyperbolic space, sine-Gorodon, e.t.c.

Here I try to give some ideas of proof of the theorem, that Lobachevsky plane as a whole cannot be immersed in $\mathbf{E}^3$*. I knew this statement long time, considering it as very important statement, which just has to be known.*

I did not realise how much it is related with such "working" questions as assymptotic directions, Chebyshev net and ... sine-Gordon equation:

$$\frac{\partial^2 F}{\partial x \partial y} = \sin F \tag{01}$$

Suppose that M: $\mathbf{r} = \mathbf{r}(u, v)$ is immersion of Lobachevsky plane in $\mathbf{E}^3$. Let $\Pi(\mathbf{x}, \mathbf{y})$ be second quadratic form. Since Gaussian curvture is equal to -1, hence in arbitrary point $\mathbf{p} \in M$ there exist two directions $\mathbf{l}_1, \mathbf{l}_2$ such that the second quadratic form vanishes on every vector which is in one of these directions.

These directions define a net at least locally. We may choose local coordinates u, v such that second quadratic form is equal to

$$\Pi = \Pi(u, v) du dv \tag{1.a}$$

in these local coordinates.

Proposition One may choose local coordinates u, v such that second quadratic form has appearance (1a) and the first quadratic form has appearnace

$$G = du^2 + 2\cos\Theta(u, v) du dv + dv^2\,. \tag{1b}$$

To prove it consider basic coordinate vectors $\mathbf{e}_\alpha = \frac{\partial \mathbf{r}}{\partial u^\alpha}$.

We have that

$$\frac{\partial \mathbf{e}_\pi}{\partial u^a} = (\partial_\alpha \mathbf{e}_\pi)_{||} + (\partial_\alpha \mathbf{e}_\pi)_\perp = \nabla_\alpha \mathbf{e}_\pi + \Pi(\mathbf{e}_\pi, \mathbf{e}_\alpha)\mathbf{n} = \Gamma^\rho_{\alpha\pi}\mathbf{e}_\rho + \Pi_{\alpha\pi}\mathbf{n}\,.$$

(Here as usual $\mathbf{n}$ is normal unit vector).

We have that

$$\frac{\partial^2 \mathbf{e}_\pi}{\partial u^a \partial u^\beta} = \frac{\partial}{\partial u^\beta}\left(\Gamma^\rho_{\alpha\pi}\mathbf{e}_\rho + \Pi_{\alpha\pi}\mathbf{n}\right) = \partial_\beta \Gamma^\rho_{\alpha\pi}\mathbf{e}_\rho + \Gamma^\rho_{\alpha\pi}\partial_\beta \mathbf{e}_\rho + \partial_\beta \Pi_{\alpha\pi}\mathbf{n} + \Pi_{\alpha\pi}\partial_\beta \mathbf{n} =$$

$$\partial_\beta \Gamma^\rho_{\alpha\pi}\mathbf{e}_\rho + \Gamma^\rho_{\alpha\pi}\left(\Gamma^\sigma_{\beta\rho}\mathbf{e}_\sigma + \Pi_{\beta\rho}\mathbf{n}\right) + \partial_\beta \Pi_{\alpha\pi}\mathbf{n} - \Pi_{\alpha\pi}\Pi^\rho_\beta \mathbf{e}_\rho\,.$$

(We use equation for Shape operator: $\partial_\beta \mathbf{n} = -S^\rho_\beta \mathbf{e}_\rho$, $\Pi^\rho_\beta = S^\rho_\beta = \Pi_{\beta\alpha} g^{\alpha\rho}$.)

We have that

$$\frac{\partial^2 \mathbf{e}_\pi}{\partial u^a \partial u^\beta} = \frac{\partial^2 \mathbf{e}_\pi}{\partial u^\beta \partial u^\alpha} =$$

(so called Peterson-Codazzi integrability conditions) give as that

$$\partial_\beta \Gamma^\rho_{\alpha\pi}\mathbf{e}_\rho + \Gamma^\rho_{\alpha\pi}\left(\Gamma^\sigma_{\beta\rho}\mathbf{e}_\sigma + \Pi_{\beta\rho}\mathbf{n}\right) + \partial_\beta \Pi_{\alpha\pi}\mathbf{n} - \Pi_{\alpha\pi}\Pi^\rho_\beta \mathbf{e}_\rho =$$

$$\partial_\alpha \Gamma^\rho_{\beta\pi}\mathbf{e}_\rho + \Gamma^\rho_{\beta\pi}\left(\Gamma^\sigma_{\alpha\rho}\mathbf{e}_\sigma + \Pi_{\alpha\rho}\mathbf{n}\right) + \partial_\alpha \Pi_{\beta\pi}\mathbf{n} - \Pi_{\beta\pi}\Pi^\rho_\alpha \mathbf{e}_\rho$$

Comparing the terms at $\mathbf{e}_\rho$ and $\mathbf{n}$ we come to:

$$\partial_\beta \Gamma^\rho_{\alpha\pi} - \partial_\alpha \Gamma^\rho_{\beta\pi} + \Gamma^\rho_{\beta\sigma}\Gamma^\sigma_{\alpha\pi} - \Gamma^\rho_{\alpha\sigma}\Gamma^\sigma_{\beta\pi} = \Pi^\rho_\beta \Pi_{\alpha\pi} - \Pi^\rho_\alpha \Pi_{\beta\pi}\,,$$

i.e.

$$R^\rho_{\pi\beta\alpha} = \Pi^\rho_\beta \Pi_{\alpha\pi} - \Pi^\rho_\alpha \Pi_{\beta\pi}\,, \text{(Gaus\'s conditions)}$$

and

$$\partial_\alpha \Pi_{\beta\pi} - \Gamma^\rho_{\alpha\pi}\Pi_{\rho\beta} = \partial_\beta \Pi_{\alpha\pi} - \Gamma^\rho_{\beta\pi}\Pi_{\rho\alpha}\,, \text{(Peterson Kodazzi conditions)}$$

Taking traces of Gauss conditions we come to the stadnard realtion between Gaussian and scalar curvature:

$$R = R^\beta_{\pi\beta\alpha} g^{\pi\alpha} = [\mathrm{Tr}\,\Pi]^2 - \mathrm{Tr}\,\Pi^2 = 2\det S = 2K = 2\frac{\det \Pi}{\det g}$$

(Here we are abusing little bit lower and upper indices, e.g. $S^\alpha_\rho = -\Pi_{\alpha\beta}g^{\beta\rho}$).

Consider Peterson Codazzi relations:
for indices $\alpha\beta\pi = 1\,1\,1$ empty conditions (as well as for$\alpha\beta\pi = 2\,2\,2$)
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for indices $\alpha\beta\pi = 1\,2\,2$ we have:

$$\partial_1 P = P(\Gamma^1_{11} - \Gamma^2_{21})$$

as well as for indices $\alpha\beta\pi = 2\,1\,1$ we have:

$$\partial_2 P = P(\Gamma^2_{22} - \Gamma^1_{12})$$

Here $\Pi = \begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix}$.
These conditions do not look nice, but we are in dimension 2.
Note that Gaussian curvature is constant. Choose it 1: We see that

$$\det \Pi = -P^2 \cdot \det g = -1 \Rightarrow P = \sqrt{\det g}\,.$$

Note that

$$\delta\sqrt{\det g} = \frac{1}{2}\sqrt{\det g}g^{ik}\delta g_{ik}\,,$$

i.e.

$$\frac{\partial P}{\partial u^\alpha} = \sqrt{\det g}\frac{\partial\sqrt{\det g}}{\partial u^\alpha} = P\Gamma^\pi_{\alpha\pi}\,.$$

We see that Petrson Codazzi conditions mean that

$$\Gamma^1_{11} + \Gamma^2_{12} = \Gamma^1_{11} - \Gamma^2_{12} \Rightarrow \Gamma^2_{12} = 0\,, \quad \Gamma^1_{21} + \Gamma^2_{22} = \Gamma^2_{22} - \Gamma^2_{12} \Rightarrow \Gamma^1_{12} = 0\,,$$

Thus we see that Peterson-Codazzi conditions with constant curvature imply that

$$\Gamma^1_{12} = \Gamma^2_{12} = 0 \Rightarrow \Gamma_{12;1} = \Gamma_{12;2} = 0 \Rightarrow \frac{\partial g_{11}}{\partial u^2} = \frac{\partial g_{22}}{\partial u^1} = 0\,.$$

We see that first quadratic form is

$$G = A(u)du^2 + 2B(u,v)dudv + C(v)dv^2$$

Taking antiderivatives $\tilde u = \int \sqrt{A(u)}du$ and $\tilde v = \int \sqrt{C(v)}dv$

We come to coordinates $u = \tilde u, v = \tilde v$ such that in these coordinates

$$G = du^2 + 2\cos\Theta(u,v)dudv + dv^2$$

and second quadratic form in these coordinates is equal to

$$\Pi = 2\sin\Theta(u,v)dudv$$