Lie Theorem of existence and uniqueness of highest vector

Important lemma

Let $L = H + X_+$ be Lie algebra of upper triangular $n \times n$ matrices. (H is Lie algebra of diagonal matrices, X_+ is sublagebra if strictly upper trinagular matirces.)

Lemma Irreducible representation of L in complex space V is one-dimensional.

Proof.

Let V is irreducible representation of Lie algebra $L = H + X_+$.

Let V_0 be a subspace of V such that it is irreducible with respect to subalgebra X_+. Then it is easy to see that V_0 is one-dimensional.

The proof easy follows from Shur Lemma and commutation relations. Indeed first note that operators $\{E_{1i}\}$ belong to the centre of the algebra X_+. Hence all these operators are proprotional to identity operator on V_0. On the other hand all the operators $\{E_{1i}\}$ except the operator E_{12} are traceless on V_0 since they belong to subalgera $[X_+, X_+]$ (they are expressed via commutators). Hence we proved that for $i = 3, \dots, n$ all the operators $E_{1i} = 0$ on V_0 and E_{12} is a scalar operator. using this fact and Shur lemma we immediately come to the result that

Now take an arbitrary such a subspace V_0 in V which is irreducible one-dimensional representation of X_+. For every $x \in X_+$ denote by $\lambda(x)$ the eigenvalue of x on V_0:

$$x(\mathbf{e}_0) = \lambda(x)\mathbf{e}_0$$

where $\mathbf{e}_0 \in V_0$.

Consider the flag of subspaces

$$V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_k$$

such that all the factor spaces $V_{r+1} \backslash V_r$ have weightfunction $\lambda(x)$ for elements of the algebra X_+: if $\{\mathbf{e}_0, e_1, \dots, \mathbf{e}_k\}$ s a basis in V_k adjusted to the flag ($\mathbf{e}_s \in V_s$) then

$$\text{for every } x \in X_+ \qquad x(\mathbf{e}_s) = \lambda(x)\mathbf{e}_s + \ldots$$

where dots are vectors belonging to V_{s-1}.

Suppose now that V_k is maximal flag with weight function $\lambda(x)$ in the vector space V, i.e. for every non-zero vector $\mathbf{f} \in V$ such that $\mathbf{f} \not\in V_k$ there exists an element $x \in X_+$ such that $x(\mathbf{f}) - \lambda(x)x$ does not belong to V_k also.

By definition V_k is invariant with respect to subalgebra X_+. Show that this is invariant with respect to the whole algebra L also.

Indeed consider a vector $\mathbf{y} = E_{pp}\mathbf{e}$, where $\mathbf{e} \in V_k$ ($E_{pp} \in H$) Suppose that $\mathbf{y} \not\in V_k$. From commutation relations

$$[E_{pp}, E_{ij}] = \delta_{pi}E_{pj} - \delta_{jp}E_{ip} \tag{1}$$

it follows that

$$x(\mathbf{y}) = \lambda(x)\mathbf{y} + \ldots$$

where we denote by dots vector in V_k. Hence the flag V_k is not maximal. Contradiction. So we proved that the flag space V_k is invariant with respect to the whole alge bra L.

Hence the flag space V_k coincides with the space V, $V = V_k$. Consider again an arbitrary element $x \in X_+$. The fact that $\lambda(x)$ is the weight function of the flag, i.e.

$$x(\mathbf{e}_k) = \lambda(x) + \text{vector in } V_{k-1}$$

means that trace of an arbitrary operator x is equal to $N\lambda(x)$, where N is dimension of the space $V = V_k$. On the other hand due to commutation relations (*) trace of an arbitrary operator $x \in X_+$ is equal to 0. Hence we have proved that $\lambda(x) = 0$ (for $x \in X_+$). Now we see that the action of Lie algebra $L = H + X_+$ on the space V is reduced to the action of abelian Lie algebra H. Hence V is 1-dimensional. Lemma is proved.

Now we are ready to prove the Theorem.

A finite dimensional representation of the algebra $gl(n, C)$ or $sl(n, C)$ has a (up to a coefficient) highest vector. It is a unique (up to a multiplier) if representation is irreducible.

Indeed consider in W a subspace V which is irreducible representation of the subalgebra $L = H + X_+$ (for $sl(n, C)$ we consider traceless diagoanl matrices). Due to the lemma this is one-dimensional vector space. It is spanned by a vector $\mathbf{e}$. This vector is highest vector.

Let W be finite-dimensional