We know that the standard way to calculate metric in stereographic projection is annuyingly long exercise.

Consider another way to do it.

We use the following fact:

If function $f$ vanihes at the given point, then derivative of function $fg$ at this point is equal to $f'(x_0)g(x_0)$.

This trivial statement very often facilitates calculations.

Let $S^2$ be a sphere of radius $x^2+y^2+z^2=1$ with stereographic coordinates $u, v$

$$\begin{cases} u = \frac{2x}{1-z} \\ v = \frac{2y}{1-z} \end{cases}, \qquad \begin{cases} x = \frac{2u}{1+u^2+v^2} \\ y = \frac{2v}{1+u^2+v^2} \\ z = \frac{u^2+v^2-1}{1+u^2+v^2} \end{cases}$$

To calculate metric in stereographic coordinates we have to calculate:

$$\left(dx^2+dy^2+dz^2\right)_{x=x(u,v),y=y(u,v),z=z(u,v)}$$

If you did these calculations you remember that straightforward caclulations are not difficult, but annuying, moreover the final answer appears something as as a "rabbit of the hat"

We consider here much more illuminating calculations.

First of all calculate (1c) at the point $u=v=0$ $(x=y=0, z=-1)$. We see that due to the vanishing of functions $u$ and $v$ at this point we have:

$$\left(dx^2+dy^2+dz^2\right)_{x=0,y=0,z=-1} =$$

$$d\left(\frac{2u}{1+u^2+v^2}\right)^2_{u=v=0} + d\left(\frac{2v}{1+u^2+v^2}\right)^2_{u=v=0} + d\left(\frac{u^2+v^2-1}{1+u^2+v^2}\right)^2_{u=v=0} = 4du^2+4dv^2\,.$$

Now use the fact that metric on the sphere is invariant with respect to rotations.

Metric of sphere in stereographic coordinates is invariant with respect to rotations—-i.e. rotations of sphere around axis $OZ$. To calculate metric at arbitrary point of the plae $(u,v)$ we use the invariance of the metric with respect to rotations around other axis. Let $\mathbf{pt}--(u_0,0)$ be an arbitrary point of the line $(u,0)$. Consider the rotation of the sphere around axis $0Y$: $(x,y,z)\to(\tilde{x},y,\tilde{z})$ which transforms the point $\mathbf{pt}$ to the origin (respectively the point $(x_0,0,z_0)$) corresponding to the point $\mathbf{pt}$ to the point $(\tilde{x},\tilde{y},\tilde{z}) = (0,0,-1)$.

$$\tilde{u} = \frac{\tilde{x}}{1-\tilde{z}} = \frac{x\cos\varphi + z\sin\varphi_0}{1-(-x\sin\varphi_0+z\cos\varphi_0)} = \frac{\frac{2u}{1+u^2}\cos\varphi) + \frac{u^2r-1}{u^2+1}\sin\varphi_0}{1-\left(-\frac{2u}{1+u^2}\sin\varphi_0+\frac{u^2-1}{u^2+1}\cos\varphi_0\right)}\,,$$

$$\tilde v=\frac{y}{1-\tilde z}=\frac{y}{1-(-x\sin\varphi_0+z\cos\varphi_0)}=\frac{\frac{2v}{1+u^2}}{1-\left(-\frac{2u}{1+u^2}\sin\varphi_0+\frac{u^2-1}{u^2+1}\cos\varphi_0\right)}\,,$$

where angle $\varphi_0$ is chosen such that $\tilde x=0,\tilde z=-1$:

$$\operatorname{ctg}\varphi_0=\frac{1-u_0^2}{2u_0}$$

Now one can easy calculate differentials $d\tilde u$ and $d\tilde v$ at the origin—i.e. differentials $du,dv$ at the point **pt**: Since for the point **pt** $x=0,u=0$, and $\tilde z=-1$, $\tilde x=0$ we can very quickly calculate $d\tilde v$ at give point:

$$d\tilde v\big|_{(u,v)=(u_0,0)}=d\left(\frac{\frac{2v}{1+u^2}}{1-\left(-\frac{2u}{1+u^2}\sin\varphi+\frac{u^2-1}{u^2+1}\cos\varphi\right)}\right)=\frac{d\left(\frac{2v}{1+u^2}\right)}{2}\big|_{v=0}==\frac{dv}{1+u^2}\,,$$

and little bit more difficult (since we need to calculate $\sin\varphi$) but still quick we can caluclate the differential $d\tilde u$ (at the given point):

$$d\tilde u\big|_{(u,v)=(u_0,v)}=d\left(\frac{\frac{2u}{1+u^2}\cos\varphi_0+\frac{u^2-1}{u^2+1}\sin\varphi_0}{1-\left(-\frac{2u}{1+u^2}\sin\varphi_0+\frac{u^2-1}{u^2+1}\cos\varphi_0\right)}\right)=$$

$$\frac{d\overbrace{\left(2u\cos\varphi_0+(u^2+1)\sin\varphi_0\right)}^{\text{vanishes at } u=u_0}}{2(1+u^2)}=\frac{d\left(\overbrace{\left(2u\left(\frac{1-u_0^2}{2u_0}\right)+(u^2-1)\right)}^{\text{vanishes at } u=u_0}\sin\varphi_0\right)}{2(1+u^2)}=$$

$$\frac{d\left(\overbrace{\left(2u\left(\frac{1-u_0^2}{2u_0}\right)+(u^2+1)\right)}^{\text{vanishes at } u=u_0}\sin\varphi_0\right)}{2(1+u^2)}=$$

$$\frac{\sin\varphi_0 du}{(1+u^2)^2}\left(\frac{1-u^2}{2u}+u\right)\big|_{u=u_0}=\frac{du}{1+u^2}\big|_{u=u_0}$$

The result of calculation: we see that at the origin metric is equal to $4du^2+4dv^2$. It follows from the previous calculations that at the point **pt**: $(u,v)=(u_0,v)$

$$G=4(d\tilde u)^2+4d\frac{\tilde v)^2=4du^2+4dv^2}{(1+u^2)^2}\,.$$

Metric is invariant with rotation (corresponding to rotation along axis $OZ$). Hence we come to the answer:

$$G=\frac{4du^2+4dv^2}{(1+u^2+v^2)^2)}\,.$$

This answer can be easily generalised for $n$-dimensional sphere: