Wronskians and Differential Operators

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Let us recall the Wronskian of n functions, $\Phi = (\phi_1, \dots, \phi_n)$ where $\phi_i \in C^\infty(\mathbb{R})$, is defined as,

$$W(\Phi) = det \begin{pmatrix} \phi_1 & \partial_x \phi_1 & \cdots & \partial_x^{n-1} \phi_1 \\ \vdots & \vdots & \ddots & \vdots \\ \phi_n & \partial_x \phi_n & \cdots & \partial_x^{n-1} \phi_n \end{pmatrix}$$

There is the following fundamental theorem relating Wronskians to linear differential operators (the theorem and its proof may be found in a books discussing the basic properties of differential equations, for example [Hurewicz]):

Theorem

Let $\Phi = (\phi_1, \dots, \phi_n)$ be n linearly independant smooth functions and suppose that $W(\Phi)(x) \neq 0$. Then there exists a unique linear operator of order n, L_Φ, defined in a neighbourhood of x, such that

i) The top term of L_Φ, i.e. the coefficient of ∂_x^n, is 1.

ii) The set of solutions of the differential equation, $L_\Phi(f) = 0$, is spanned by $\{\phi_1, \dots, \phi_n\}$.

An explicit expression for L_Φ is given by,

$$L_\Phi(f) = \frac{W(\Phi, f)}{W(\Phi)}$$

where we have written Φ, f for $(\phi_1, \dots, \phi_n, f)$. This theorem reveals that there is a very close relation between the coefficients of linear operators and so called "generalized Wronskians" to be defined below. However this theorem is not coordinate independant and our first task is to remedy this situation.

Proposition

Let $\phi_i \in D^\lambda(\mathbb{R})$, $i = 1, \dots, n$ and $\Phi = (\phi_1, \dots, \phi_n)$, then $W(\Phi)$ is a density of weight $n(\lambda + \frac{1}{2}(n-1))$.

Proof.

$$W(\Phi)_\alpha = |\Phi_\alpha \partial_x \Phi_\alpha \cdots \partial_x^{n-1}\Phi_\alpha| = |\Phi_\beta \partial_x y_x^\lambda \Phi_\beta \cdots \partial_x^{n-1} y_x^\lambda \Phi_\beta|$$

We have that $\partial_x^k (y_x^\lambda \Phi_\beta = \sum_{j=0}^k \binom{k}{j} \partial_x^j (y_x^\lambda) \partial_x^{k-j}(\Phi_\beta)$. However using the bilinearity and anticommutativity of the determinant we have that only the top term in this expansion isn't killed, as the lower derivatives are proportianal to earliar columns. We therefore have that,

$$W(\Phi)_\alpha = y_x^{n\lambda} |\Phi_\beta \partial_x \Phi_\beta \cdots \partial_x^{n-1}\Phi_\beta|$$

Using the fact that $\partial_x^k = y_x^k \partial_y^k + O(\partial_y^{k-1})$, and the same argument above via the properties of the determinant we have thatwe may simply replace ∂_x^k by $y_x^k \partial_y^k$ in the expression. This finally gives,

$$W(\Phi)_\alpha = y_x^{n\lambda} |\Phi_\beta y_x \partial_y \Phi_\beta \cdots y_x^{n-1} \partial_y^{n-1} \Phi_\beta| = y_x^{n(\lambda + \frac{1}{2}(n-1))} W(\Phi)_\beta$$

□

In particular note that if the weight of the ϕ_i's is $\frac{1-n}{2}$ then the Wronskian is a function. Moreover as the Wronskian is a density, in general, we can make the assumption that $W(\Phi) \neq 0$ and this makes sense. We can now give a coordinate independant description of the above theorem.

Proposition

Let $\Phi = (\phi_1, \ldots, \phi_n)$ where $\phi_i \in D^\lambda(\mathbb{R})$. Assume that $W(\Phi)(x) \neq 0$, then we have a unique linear operator, $L_\Phi : D^\lambda(\mathbb{R}) \to D^{\lambda+n}(\mathbb{R})$, defined in a neighbourhood of x of order n such that

i) The top term of L_Φ is 1

ii) The set of solutions to the differential equation $L_\Phi(f) = 0$ is spanned by $\{\phi_1, \ldots, \phi_n$.

Proof. The majority of the content of this proposition is contained in the above theorem by working in some fixed coordinate frame. We must just check that $L_\Phi(f)_\alpha = y_x^{\lambda+n} L_\Phi(f)_\beta$. We have,

$$L_\Phi(f)_\alpha = \frac{W(\Phi, f)_\alpha}{W(\Phi)_\alpha} = \frac{y_x^{(n+1)(\lambda + \frac{n}{2})} W(\Phi, f)_\beta}{y_x^{n(\lambda + \frac{n-1}{2})} W(\Phi)_\beta} = y_x^{\lambda+n} L(\Phi)_\beta$$

□

If we fix some coordinate frame we may expand L_Φ as, $L_\Phi = \sum_{i=0}^{n} a_i \partial_x^i$. We then see that we have,

$$a_i = (-1)^{n-i} \frac{W^{0,\ldots\widehat{i}\ldots,n}(\Phi)}{W^{0,\ldots,n-1}(\Phi)}$$

where for $0 \leq j_1 < \cdots < j_n$, the *generalised Wronskian*, $W^{j_1,\ldots,j_n}$, is defined by,

$$W^{j_1,\ldots,j_n}(\Phi) = |\partial_x^{j_1}\Phi \cdots \partial_x^{j_n}\Phi|$$

this is in general not a tensor. We thus in principle know what all terms of such linear operators look like. We shall now give a proof of the classical appearance of Schwarzian like objects appearing as coefficients of ∂_x^{n-2} term of special weights. Recall that if the weights of the ϕ_i's is $\frac{1-n}{2}$ then the statement $W^{0,\ldots,n-2,n}(\Phi) = 0$ is well defined as $W^{0,\ldots,n-2,n}(\Phi) = \partial_x W^{0,\ldots,n-1}(\Phi)$ and is thus the derivative of a function. It therefore makes sense to say that the term $a_{n-1} = 0$ in the above expansion, for this weight.

Lemma

Let $L : D^{\frac{1-n}{2}}(\mathbb{R}) \to D^{\frac{1+n}{2}}(\mathbb{R})$ be a linear differential operator of order n and suppose that $a_n = 1$ and $a_{n-1} = 0$, where $L = \sum_{i=0}^{n} a_i \partial_x^i$. Then a_{n-2} transforms as a multiple of a Schwarziand and this multiple is $\frac{(n+1)n(n-1)}{12}$.

Proof. By working in a local coordinate frame we may use classical results on the existence of solutions to linear differential equations to find n solutions, ϕ_i, such that $L(\phi_i) = 0$. We then have by the uniqueness property that $L = L_\Phi$, at least locally, for $\Phi = (\phi_1, \ldots, \phi_n)$. The condition that $a_{n-1} = 0$ is then just $\partial_x(W(\Phi)) = W^{0,\ldots,n-2,n}(\Phi) = 0$.

Let us calculate how $W^{0,\ldots\widehat{n-2}\ldots,n}(\Phi)$ transforms. We will use the following expansion of ∂_x^k;

$$\partial_x^k = y_x^k \partial_y^k + \frac{k(k-1)}{2} y_x^{k-2} y_{xx} \partial_y^{k-1}$$

$$+\frac{k(k-1)(k-2)}{2}\left(\frac{1}{3} y_x^{k-3} y_{xxx} + \frac{k-3}{4} y_x^{k-4} y_{xx}^2\right)\partial_y^{k-2} + O(\partial_y^{k-3})$$

Using this we know compute,

$$\partial_x^{n-1}\Phi_\alpha = y_x^{\frac{1-n}{2}}\left(y_x^{n-1}\partial_y^{n-1}\Phi_\beta + \frac{1-n}{2} y_x^{n-3} y_{xx} \partial_y^{n-2}\Phi_\beta\right) + O(\partial_y^{n-3}\Phi_\beta)$$

$$\partial_x^n \Phi_\alpha = y_x^{\frac{1-n}{2}}(y_x^n \partial_y^n \Phi_\beta - \frac{(n+1)n(n-1)}{12}(y_x^{n-3} y_{xxx} - \frac{3}{2} y_x^{n-4} y_{xx}^2)\partial_y^{n-2}\Phi_\beta) + O(\partial_y^{n-3}\Phi_\beta)$$

Combinig these and using the properties of the determinant as well as the condition that $W^{0,\ldots,n-2,n}(\Phi) = 0$, we then get,

$$W^{0,\ldots\widehat{n-2}\ldots,n}(\Phi)_\alpha = y_x^2 W^{0,\ldots\widehat{n-2}\ldots,n}(\Phi)_\beta$$

$$+\frac{(n+1)n(n-1)}{12}(y_x^{-1} y_{xxx} - \frac{3}{2} y_x^{-2} y_{xx}^2) W^{0,\ldots,n-1}(\Phi)_\beta$$

and this therefore gives us the required statement, recall that $W^{0,\ldots,n-1}(\Phi)$ is a function,

$$a_{(\alpha)n-2} = y_x^2 a_{(\beta)n-2} + \frac{(n+1)n(n-1)}{12}(y_x^{-1} y_{xxx} - \frac{3}{2} y_x^{-2} y_{xx}^2)$$

□