On one integral

Let $A(t)$ be a function on $\mathbf{R}$ such that

$$A(t) = \sum_{k \geq l} \Psi_k t^k \tag{1}$$

in a vicinity of 0 and $\int t^n A(t) dt$ converges at infinity for any n.

Consider a function

$$\zeta_A(s) = \frac{1}{G(s)} \int_0^\infty t^{s-1} A(t) dt,$$

where $\Gamma(s)$ is the usual Gamma-function. This integral is well-defined for enough big s.

Perform analytical prolongation and calculate the derivative of the function ζ_A at the point $s = 0$:

$$\zeta'_A(0) = \frac{d}{ds} \zeta_A(s)\big|_{s=0} = \sum_{k \neq 0} \frac{\Psi_k}{k} + C\Psi_0 + \int_1^\infty A(t) \frac{dt}{t}\,, \text{where } C = -\Gamma'(1)\,.$$

Statement:

$$\zeta'_A(0) = \int_0^\infty \left(A(t) - \sum_{k<0} \Psi_k t^k - e^{-t} \Psi_0 \right) \frac{dt}{t}\,. \tag{2}$$

Proof. Calculate straightforwardly this integral as a sum of the integral over $[0, 1]$ and the integral over $[1, \infty$: Using equation 1 we come to

$$\int_0^1 \left(A(t) - \sum_{k<0} \Psi_k t^k - e^{-t} \Psi_0 \right) \frac{dt}{t} = \int_0^1 \left(\sum_{k \geq 0} \Psi_k t^k - e^{-t} \Psi_0 \right) \frac{dt}{t} = \sum_{k>0} \frac{\Psi_k}{k} + C_1 \Psi_0\,, \text{where } C_1 = \int_0^1 (1 - e^{-t}) \frac{dt}{t}\,,$$

and

$$\int_1^\infty \left(A(t) - \sum_{k<0} \Psi_k t^k - e^{-t} \Psi_0 \right) \frac{dt}{t} = \int_1^\infty A(t) \frac{dt}{t} - \sum_{k<0} \Psi_k \int_1^\infty t^{k-1} dt - \Psi_0 \int_1^\infty e^{-t} \frac{dt}{t} =$$

$$\int_1^\infty A(t) \frac{dt}{t} + \sum_{k<0} \frac{\Psi_k}{k} + C_2 \Psi_0\,, \text{where } C_2 = -\int_1^\infty e^{-t} \frac{dt}{t}\,.$$

Adding these two integrals and subtracting integral (2) we come to:

$$\int_0^\infty \left(A(t) - \sum_{k<0} \Psi_k t^k - e^{-t} \Psi_0 \right) \frac{dt}{t} - \zeta'_A(0) = (C_1 + C_2 - C)\, \Psi_0\,. \tag{3}$$

It remains to prove that $C_1 + C_2 = C$. The nice way to show it is the following: Consider the function $A(t) = e^{-t}$. Then $\Psi_0 = 1$ and in equation (4) the left hand side is obviously vanished. Hence

$$C_1 + C_2 - C = 0\,, \text{i.e.} \int \frac{(1 - e^{-t}) dt}{t} - \int \frac{e^{-t} dt}{t} = \Gamma'(1)\,.$$

We come to this result without calculations.

One can formulate the result of the statement as the following identity: For an arbitrary function $A(t)$

$$\frac{d}{ds} \left(\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} A(t) dt \right) \Big|_{s=0} = \int_0^\infty \frac{A(t) - A_-(t) - A_0 e^{-t}}{t} dt\,.$$