Davidik, here is

On a interpretation of Poisson formula

§1 Poisson Formula

Let $F(x)$ be a good function on real numbers which tends to zero at infinity. Let $G(k)$ be the component of its Fourier expansion:

$$F(x) = \int_{-\infty}^{\infty} G(k) e^{ikx} dk \tag{1}$$

Note that*

$$G(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(x) e^{-ikx} dx \tag{2}$$

and in particularly

$$G(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(x) dx \tag{2'}$$

One can prove the following very beautiful identity.

$$\sum_{n\in\mathbf{Z}} F(na) = \frac{2\pi}{a} \sum_{n\in\mathbf{Z}} G\left(\frac{2\pi n}{a}\right), \tag{3}$$

Here a is an arbitrary parameter. Summation goes over all integers. This is famous Poisson identity. It says that *sum of the values of function over a lattice coincides with sum of the values of its Fourier image over reciprocal lattice.*

Example 1. Consider $F(x) = e^{-|x|}$. One can see that

$$F(x) = e^{-|x|} = \int_{-\infty}^{\infty} \frac{e^{ikx} dk}{\pi(1+k^2)}, \quad G(k) = \frac{1}{\pi(1+k^2)} \tag{4}$$

and respectively

$$G(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|x|} e^{-ikx} dx = \frac{1}{2\pi} \left(\int_0^{\infty} e^{-x} e^{-ikx} dx + \int_0^{\infty} e^{-x} e^{ikx} dx \right) = \frac{1}{\pi(1+k^2)} \tag{5}$$

Then Poisson formula gives that for $a > 0$

$$\sum_{n\in\mathbf{Z}} e^{-|n|a} = 1 + 2\sum_{n=1}^{\infty} e^{-na} = \frac{2\pi}{a} \sum_{n\mathbf{Z}} \frac{1}{\pi\left(1 + \frac{4\pi^2 n^2}{a^2}\right)} = \sum_{n\in\mathbf{Z}} \frac{2a}{a^2 + 4\pi^2 n^2} \tag{6}$$

---

* We do not need formulae (2),(3) for obtaining Poisson formula. We need them for interpretations

Example 2 Consider $F(x) = e^{-x^2}$. One can see that

$$F(x) = e^{-x^2} = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{\frac{-k^2}{4}} e^{ikx} dk, \quad G(k) = \frac{e^{-\frac{k^2}{4}}}{2\sqrt{\pi}} \tag{4}$$

and respectively

$$G(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-x^2} e^{-ikx} dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(x+\frac{ik}{2})^2 - \frac{k^2}{4}} dx = \frac{1}{2\sqrt{\pi}} e^{-\frac{k^2}{4}} \tag{5}$$

Then Poisson formula gives that for $a > 0$

$$\sum_{n \in \mathbf{Z}} e^{-n^2 a^2} = 1 + 2 \sum_{n=1}^{\infty} e^{-n^2 a^2} = \frac{\sqrt{\pi}}{a} \sum_{n \in \mathbf{Z}} e^{\frac{-\pi^2 n^2}{a^2}} \tag{6}$$

§2 Proof of the Poisson Formula

Before analysing it meaning prove it:

Proof of Poisson formula

Consider a function

$$H(x) = \sum_{n \in \mathbf{Z}} F(x + na)$$

This is periodic function: $H(x) = H(x + a)$. Consider its Fourier expansion in series:

$$H(x) = \sum_{n \in \mathbf{Z}} c_n e^{\frac{2i\pi nx}{a}}, \quad \text{with} \quad c_n = \frac{1}{a} \int_0^a H(x) e^{\frac{-2i\pi nx}{a}} dx$$

(To calculate c_k we multiply the relation above on $e^{\frac{-2i\pi kx}{a}}$ integrate it over x using the relation $\int_0^a e^{\frac{2i\pi(n-k)x}{a}} dx = a\delta_{kn}$).

Now considering the following chain of identities:

$$\sum_{n \in \mathbf{Z}} F(na) = H(0) = \sum_{n \in \mathbf{Z}} c_n = \frac{1}{a} \sum_{n \in \mathbf{Z}} \int_0^a H(x) e^{\frac{-2i\pi nx}{a}} dx = \frac{1}{a} \sum_{n,m \in \mathbf{Z}} \int_0^a F(x+ma) e^{\frac{-2i\pi nx}{a}} dx =$$

$$\frac{1}{a} \sum_{n,m \in \mathbf{Z}} \int_{ma}^{(m+1)a} F(t) e^{\frac{-2i\pi n(t-ma)}{a}} dt = \frac{1}{a} \sum_{n \in \mathbf{Z}} \int_{-\infty}^{\infty} F(t) e^{\frac{-2i\pi nt}{a}} dt =$$

$$\frac{2\pi}{a} \sum_{n \in \mathbf{Z}} G\left(\frac{2\pi n}{a}\right)$$

§3 *Poisson Formula and approximation of integral by series*

It is commonplace the relation between Darboux series and integrals:

$$\int_{\infty}^{\infty} F(x)dx \approx a \sum_{n\in\mathbf{Z}} F(na), \quad \text{if } a \text{ is small, i.e.} \quad \int F(x)dx = \lim_{n\to\infty}\left(a\sum_{n\in\mathbf{Z}} F(na)\right)$$

Poisson formula gives exact meaning to the asymptotic of integral by series. Note that Poisson formula (3) can be rewritten in the way:

$$a\sum_{n\in\mathbf{Z}} F(na) = 2\pi \sum_{n\in\mathbf{Z}} G\left(\frac{2\pi n}{a}\right) = 2\pi G(0) + 2\pi \sum_{n\neq 0} G\left(\frac{2\pi n}{a}\right)$$

Now using $(2')$ we come to:

$$a\sum_{n\in\mathbf{Z}} F(na) = \int_{-\infty}^{\infty} F(x)dx + 2\pi \sum_{n\neq 0} G\left(\frac{2\pi n}{a}\right)$$

or

$$\int_{-\infty}^{\infty} F(x)dx = \frac{1}{a}\sum_{n\in\mathbf{Z}} F(na) - 2\pi \sum_{n\neq 0} G\left(\frac{2\pi n}{a}\right) \qquad \text{(approximation of integral)}$$

This formula gives approximation of integral by series.

Consider again examples 1 and 2.

Example 3

Function $f = e^{-x}$

Apply the approximation formula to the formulae in the example 1. We come to $(a > 0)$:

$$a\sum_{n\in\mathbf{Z}} e^{-|n|a} = 2\pi G(0) + 2\pi \sum_{n\neq 0} \frac{1}{\pi\left(1+\frac{4\pi^2 n^2}{a^2}\right)} = \int_{-\infty}^{\infty} e^{-|x|}dx + \sum_{n=1}^{\infty} \frac{4}{\left(1+\frac{4\pi^2 n^2}{a^2}\right)},$$

It can be rewritten with boundary term:

$$\int_0^{\infty} e^{-x}dx = \frac{a}{2} + a\sum_{n=1}^{\infty} e^{-na} - \sum_{n=1}^{\infty} \frac{4}{\left(1+\frac{4\pi^2 n^2}{a^2}\right)}$$

Example 4

Function $f = e^{-x^2}$

It follows from the Example 2 and the last approximation formula that

$$a\sum_{n\in\mathbf{Z}} e^{-n^2a^2} = \sqrt{\pi} + 2\sqrt{\pi}\sum_{n=1}^{\infty}, e^{\frac{-\pi^2 n^2}{a^2}}$$

i.e.

$$\int_{-\infty}^{\infty} e^{-x^2} dx = a \sum_{n \in \mathbf{Z}} e^{-n^2 a^2} - 2\sqrt{\pi} \sum_{n=1}^{\infty}{}' e^{\frac{-\pi^2 n^2}{a^2}}$$

N.B. The last formula is related with the preancestor of Seeley formula: It is about the following: Let λ_n be eigenvalues of the Laplace operator, then the asymptotic of the following function $Z(t) = \sum_n e^{-\lambda_n t}$ can be expressed in terms of basic terms. E.g. if λ_n are eigenvalues of operator ∂^2 on the closed interval $[0.L]$ then

$$Z(t) = \sum_n e^{-\lambda_n t} = \sum_n e^{\frac{-n^2 t}{l^2}} \approx \frac{l}{\sqrt{t}}$$

In the general case:

One can prove the following: Let λ_i be frequencies for d-dimensional drum, i.e. eigenvalues of the Laplacian acting on this drum: Then

$$Z(t) = \frac{V}{t^{d/2}} + \dots$$

i.e. one can estimate dimension of the drum and its volume just hearing it!!!!