Conic sections and Kepler law. Newton—Lagrange—-A.Giventale

Conic section is intersection of plnae with surface of cone.

$$\textit{1. Ellipses are conic sections} \tag{1}$$

$$\text{Kepler Law: } \textit{2. The orbit of Planets are ellipses.} \tag{2}$$

These two statements are common place for everybody.

Almost everybody who have learnt a bit of higher mathematics knows how to prove statement (1) (exercise in Analytical Geometry). Everybody who learnt little bit calculus knows that using the fact that

$$F = \gamma \frac{mM}{R^2} \tag{3}$$

one can prove statement (2), showing that the solution of differential equation

$$\frac{d^2}{dt^2}\mathbf{R} = \frac{\mathbf{F}}{m} = -\gamma \frac{M\mathbf{R}}{R^3}\,, \tag{3a}$$

is a conic section(m is a mass of planet, and M is Solar mass). On the other hand all standard proofs are based on calculus: solving equation (3a) in polar coordinates come to and calculating the integral we come to conic section (see any book in Theoretical Mechanics) [1)]

The following statement which is almost evident plays cruciar role in this etude:

$$\textit{orthogonal projections of conic sections on the plane which are conic sections also.} \tag{4}$$

In fact orbits of planets are these projections....

This etude has the following history. In May 2014 I was in Davis University on 80 years celebration of my Teacher Albert S. Schwarz. I met there Alexander Givental. Alexander is famous mathematician, but he is also very much engaged in teaching mathematics (see his homepage in Berkeley University) During conference dinner we were sharing the same table,

[1)] Usually equation (3) is called Newton Graviational Law, and equation (3a) Newton Second law. There is an alternative point of view that Robert Hook formulated statement (3) in the letter to Newton, and Newton who invented Calculus, based on (3) and (3a) deduced Kepler law, statement (2).

and Givental was explaining me some beautiful properties of conic section. In particular he told me the sentence: if you consider a projection of ellipse on the horisontal plane, you come naturally to Kepler law. At that moment I did not understand the meaning of this phraze... About three years passed. Few months ago I preparing the lecture for Geometry students about conic sections, realised that I cannot find a beautiful and short explanation of the statement (1) without using extracurricular material. Another trouble was that because of problems of my eyes my access to books was very restricted. Finally I prepared the lecture where to prove the statement (1) I was forced to use the statement (4). Honeslty I did not like this way of considerations, but I had no choice. 30 March, morning, in four hours will begin the lecture, I am very unsatisfied, since I do not understand the menaing of the statement (4). Suddenly I decided just to check, where are the foci of the projected cone. (The foci of initial conic sections are related with Dandellen spheres.) In a moment I realised that the vertex of the cone is one of the foci!. The next second I recalled the the converstaion with Alexander Givental, his phraze: this is projected conic secion which is related with Kepler law. Sure I was very happy, I immediately contacted Alexander and he sent me his article all the material that this etude is based on. My modest contribution to this etude is related with the fact that I was desperately trying to find as much as possible easy proof of classical statement (1).

This etude is written on the base of my conversations with Alexander Giventale in may 2014, his paper [3], and his letter to me, where he explained me that main idea of his paper can be traced in to the manuscript [1] of Lagrange. My modest contribution is that few months ago preparing lectures for students I noted that explaining conic sections, this is useful to consider projection of conics on the plane. Thinking about the geometrical meaning of the projection of conic, I realised that, the vertex of the cone will be one of the foci of the projected ellipse. Immediately I remembered that three years ago when I was in US Davis Univerrsity (celebretation of 80 years of my teacher A.S.Schwarz) I met A.Givental who told me many beautiful stories about ellipses, and I recalled his sentence: To see Kepler's Law you have to consider the projection of ellipse. The rest was easy.

Considee differential equation of particle in gravitational field

$$\ddot{\mathbf{r}} = k\frac{\mathbf{r}}{r^3}\,. \tag{1}$$

The solution $\mathbf{r} = \mathbf{r}(t)$ is a curve in the plane (preservation of angular momentum), and we know that this is conic section (ellipse, hyperbola, parabola) with focus at the origin (Kepler law) The standard proof of this fact (see any book in Mechanics) contains calulation of integrals. A. Giventale had a beatiful point of view on the geometrical origin of the Kepler first law. (This was anounced in [2] and published in [3] after 30 years). On the other hand A.Givental told me that Alain Chensiner noted him that this idea can be traced in the work [1] of Lagrange ' Here is the exposition of this result. (This is due to results of A.Givental in his paper, and in his letter to me.)

Let particle moves in the plane, and let $x(t), y(t)$ be components of $\mathbf{r}(t)$ (x, y are Cartesian coordinates of the plane)

$$\begin{pmatrix} \ddot{x}\ (t) \\ y\ (t) \end{pmatrix} = -\frac{k}{(x^2(t)+y^2(t))^{\frac{3}{2}}} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} . \tag{1a}$$

Note also that differential equation (1) implies that the function

$$r_0(t) = \sqrt{x_0^2(t) + y_0^2(t)}$$

obeys the differential equation

$$\frac{d^2 r}{dt^2} = -\frac{k}{r} + \frac{C}{r^3} . \tag{2}$$

We consider the solution $x_0(t), y_0(t)$ of the equation (1) Consider the function

$$F\colon , F(t) = \left(x_0^2(t) + y_0^2(t)\right)^{-\frac{3}{2}} .$$

This function is defined by the solution of the equation (1). $r(t) = \sqrt{x^2(t) + y^2(t)}$.

Consider also the following two linear differential equations (homogeneous and non-homogeneous)

$$\ddot{u} = -kF_0 u , \qquad \ddot{u} = -kF_0 u + CF_0 .$$

Solutions of the first equation form 2-dimensional linear space, solutions of the second equation form 2-dimensional affine space. The improtant observation is that

One can see that equation (1) implies that for the function $r(t)$

$$\frac{d^2 r(t)}{dt^2} = \frac{d^2}{dt^2}\left(\sqrt{x^2(t)+y^2(t)}\right) = \frac{d}{dt}\left(\frac{x(t)\frac{dx(t)}{dt} + y(t)\frac{dy(t)}{dt}}{\sqrt{x^2(t)+y^2(t)}}\right) =$$

$$\frac{x(t)\frac{d^2x(t)}{dt^2} + y(t)\frac{d^2y(t)}{dt^2}}{\sqrt{x^2(t)+y^2(t)}} + \left(\frac{\left(\frac{dx(t)}{dt}\right)^2 + \left(\frac{dy(t)}{dt}\right)^2}{\sqrt{x^2(t)+y^2(t)}}\right) - \left(\frac{x(t)\frac{dx(t)}{dt} + y(t)\frac{dy(t)}{dt}}{\sqrt{x^2(t)+y^2(t)}}\right)$$

[1] Lagrange *DES PERTURBATIONS DES COMETES.*— SECTION DDEUXIEME. Integrations des equations differentielles de l'orbite non-altere. pp.419—430, 1785

[2] V. I. Arnold, V. V. Kozlov, A. I. Neishtadt *Mathematical aspects of classical and celestial mechanics.* Dynamical systems III (Encyclopaedia of Mathematical Sciences), Springer, 1987.

[3] Alexander Givental *Keplers Laws and Conic Sections* Arnold Mathematical Journal March 2016, Volume 2, Issue 1, pp 139148