Thick morphism in the third order

This is an attempt of straightforward calculations for Voronov's thick morphism $M_1 \to M_2$.

We consider manifolds M_1 and M_2. Let (x^i) be local coordinates on M_1 and let (y^α) be local coordinates on M_2. Respectively let (p_i, x^i) be local coordinates on T^*M_1 and let (q_α, y^β) be local coordinates on T^*M_2

Let

$$S(x,q) = S(x) + S^\alpha(x) q_\alpha + \frac{1}{2} S^{\alpha\beta} q_\beta q_\alpha + \frac{1}{6} S^{\alpha\beta\gamma} q_\gamma q_\beta q_a + \frac{1}{24} S^{\alpha\beta\gamma\pi} q_\pi q_\gamma q_\beta q_\alpha + \dots$$

be a function which defines Lagrangian surface Λ_S in $T^*M_1 \times (-T^*M_2)$:

$$\Lambda_S = \left\{ (p_i, x^j, q_\alpha, y^\beta) \colon p_i = \frac{\partial S(x,q)}{\partial x^i}, y^\alpha = \frac{\partial S(x,q)}{\partial q_\alpha} \right\}$$

(Λ_S is Lagrangian with respect to canonical symplectic form $dp_i \wedge dx^i - dq_\alpha dy^\alpha$ on $T^*M_1 \times T^*M_2$.)

This Lagrangian surface define V-thick morphism φ_S $M_1 \overset{\to}{\to} M_2$ in the following way:

$$C(M_2) \ni g = g(y) \to f = f(x)$$

such that Lagrangian surfaces defined by graphs of functions f and g are related with Lagrangian surface Λ_S, i.e.

$$p_i = \frac{\partial f(x)}{\partial x^i} = \frac{\partial S(x,q)}{\partial x^i}, \quad q_\alpha = \frac{\partial g(y)}{\partial y^a}, \quad y^\alpha = \frac{\partial S(x,q)}{\partial q^\alpha}.$$

One can see that in local coordinates

$$f(x) = S(x,q) + g(y) - y^\alpha q_\alpha$$

(see T.Voronov [1].)

We suppose that $g \to \varepsilon g$ is infinitesimal function. Thick morphism defines non-linear map of infinitesimal functions εg on M_2 to infinitesimal functions on M_1.

Solve these equations indictively up to an arbitrary order. Since $y^\alpha = \frac{\partial S(x,q)}{\partial q_\alpha}$ and $q_\alpha = \frac{\partial g(y)}{\partial y^\alpha}$ hence we see that $q_\alpha = q_\alpha(x)$ and $y^a = y^\alpha(x)$ can be recurrently expressed due to the relation:

$$q_\alpha(x) = \frac{\partial g(y)}{\partial y^\alpha}\Big|_{y^\beta = S^{\beta\pi} q_\pi + \dots}$$

We have

$$y^\alpha = \frac{\partial S(x,q)}{\partial q_\alpha} = S^\alpha(x) + S^{\alpha\beta} q_\beta + \frac{1}{2} S^{\alpha\beta\gamma} q_\gamma q_\beta + \frac{1}{6} S^{\alpha\beta\gamma\pi} q_\pi q_\gamma q_\beta + \dots$$

and expanding in Taylor series we come to

$$q_\alpha = \varepsilon \frac{\partial g(y)}{\partial y^\alpha}\Big|_{y^a = S^\alpha + S^{\alpha\beta} q_\beta + \frac{1}{2} S^{\alpha\beta\gamma} q_\gamma q_\beta + \dots} =$$

$$\varepsilon \frac{\partial g(y)}{\partial y^\alpha} \left(S^\alpha(x) + S^{\alpha\beta}(x) q_\beta + \frac{1}{2} S^{\alpha\beta\gamma}(x) q_\gamma q_\beta + \dots \right) =$$

$$\varepsilon \sum \frac{1}{n!} \frac{\partial^{n+1} g(y)}{\partial y^\alpha \partial y^{\sigma_1} \dots \partial y^{\sigma_n}} \left(S^\alpha(x)\right) T^{\sigma_1} \dots T^{\sigma_n} \,,$$

where

$$T^\alpha = S^{\alpha\beta} q_\beta + \frac{1}{2} S^{\alpha\beta\gamma} q_\gamma q_\beta + \frac{1}{6} S^{\alpha\beta\gamma\pi} q_\pi q_\gamma q_\beta + \dots$$

Hence we come to

$$q_\alpha = \varepsilon q_\alpha^{(1)} + \varepsilon^2 q_\alpha^{(2)} + \varepsilon^3 q_\alpha^{(3)} + \dots \qquad (*)$$

where recurrently:

$$q_\alpha^{(1)} = q_\alpha^{(1)}(x) = \frac{\partial g(y)}{\partial y^\alpha} \left(S^\alpha(x)\right) = l_\alpha \,,$$

$$q_\alpha^{(2)} = q_\alpha^{(2)}(x) = l_{\alpha\sigma} S^{\sigma\beta} l_\beta \,, \quad l_{\alpha\sigma} = \frac{\partial^2 g(y)}{\partial y^\alpha \partial y^\sigma}(y)\Big|_{y^\alpha = S^\alpha(x)}$$

$$q_\alpha^{(3)} = q_\alpha^{(3)}(x) = l_{\alpha\sigma} S^{\sigma\beta} l_{\beta\omega} S^{\omega\pi} l_\pi + \frac{1}{2} l_{\alpha\sigma} S^{\sigma\beta\gamma} l_\beta l_\gamma + \frac{1}{2} l_{\alpha\sigma} S^{\alpha\beta} S^{\sigma\omega} l_\beta l_\omega \,, \quad l_{\alpha\sigma} = \frac{\partial^2 g(y)}{\partial y^\alpha \partial y^\sigma}(y)\Big|_{y^\alpha = S^\alpha(x)}$$

One can express all answers in terms of series (*). We use notation:

$$S(x,q) = S(x) + S^\alpha(x) q_\alpha + \frac{1}{2} S^{\alpha\beta} q_\beta q_\alpha + \frac{1}{6} S^{\alpha\beta\gamma} q_\gamma q_\beta q_a + \frac{1}{24} S^{\alpha\beta\gamma\pi} q_\pi q_\gamma q_\beta q_\alpha + + \frac{1}{120} S^{\alpha\beta\gamma\pi\rho} q_\rho q_\pi q_\gamma q_\beta q_\alpha + \dots =$$

$$= \sum_{r \geq 0} S_r(x,q) = S_0 + \sum_{r \geq 0} V_r^\alpha(x,q) q_\alpha$$

We have

$$f(x) = S(x,q) + g(y) - y^\alpha q_\alpha =$$

$$\sum_r S_r(x,q) + g\left(\frac{\partial S(x,q)}{\partial q^\alpha} = V^\alpha(x) + \sum_{r \geq 1} V(x,q) q_\alpha \right) - \left(\frac{\partial S(x,q)}{\partial q^\alpha} q_\alpha = \sum_r r S_r(x,q) \right) .$$

$$S_0(x) + \sum_{r\geq 2} S_r\left(x, l_\alpha + \sum_{k\geq 2} q_\alpha^{(k)}\right) + g\left(S^\alpha + \sum_{r\geq 2} S_r\left(x, l_\alpha + \sum_{k\geq 2} q_\alpha^{(k)}\right)\right)$$

Example

$$g = c + \varepsilon y_\alpha^r \Rightarrow f = f(x) = c + S(x,r)(\varepsilon?????)$$

$$g = c + \varepsilon y_\alpha^r + \frac{1}{2}t_{\alpha\beta}y^\alpha y^\beta \Rightarrow f = f(x) = ?????$$

Example

Let $M_1 = M_2 = \mathbf{R}$ and

$$S(x,q) = a(x) + \varphi(x)q + \frac{1}{2}bq^2$$

Then

$$\begin{cases} y = \varphi(x) + b(x)q \\ q = g'(y) \end{cases} \Rightarrow \begin{cases} y = \varphi(x) + b(x)g'(y) \\ q = g'(\varphi(x) + b(x)q) \end{cases} \Rightarrow$$

$$\begin{cases} y = \varphi + bg'(\varphi + bg') = \varphi + bl' + bl''b(l' + bl''bl') + \frac{b}{2}l'''bl'bl' + \dots \\ q = g'(\varphi(x) + b(x)q) = l' + l''b(l' + l''bl') + \frac{1}{2}l'''bl'bl' + \end{cases}$$

where $l = g(\varphi)$, $l' = g'(y)|big|_{y=\varphi(x)}$, $l'' = g''(y)|big|_{y=\varphi(x)}$,... We have

$$g(y) = g(\varphi + bq) = \dots$$

and

$$f(x) = a+\varphi q+\frac{1}{2}bq^2+g(y)-yq = a+\varphi q+\frac{1}{2}bq^2+g(y)-(\varphi+bq)q = a+\frac{1}{2}bq^2+g(\varphi+bq)-bq^2$$

$$a + \frac{1}{2}b(l' + bl'l'' + b^2l''l''l' + \frac{1}{2}b^2l'l'l''')^2$$