Stirling+Fedoruk

Famous Stirling formula is a good problem which explains staionary phase method including its non-trivial part. I deduce standard Stirling formula using Ted Voronov's comments on Fedoruk paper

`I returned to this text again in June 2020...(see another version in math Blog on October 2019)`

Method of stationary phase... Almost everybody knows that it is about how to calculate the leading term in in the integral

$$\int \exp \frac{iS(x)}{\hbar} u(x)dx$$

if $\hbar \to 0$. If x_0 is unique stationary point of function S,

$$S(x) = S(x_0) + A_{ab}(x^a - x_0^a)(x^b - x_0^b) + S_+(x)$$

and Hessian at this point is not degenerate, then

$$\int \exp \frac{iS(x)}{\hbar} u(x)dx = C \exp \frac{iS(x_0)}{\hbar} \frac{(\pi\hbar)^{\frac{n}{2}}}{\sqrt{\det A}} (1 + O(\hbar)) \ . \qquad (01)$$

People say that if $\hbar \to 0$ then the main contibution to the integral is given by vicinity of the stationary point, and this is just gaussian integral.

What about other terms of expansion?

There is deep and powerfull statement: All other terms are polynomial in $\hbar$! Namely

$$\int \exp \frac{iS(x)}{\hbar} u(x)dx = C \exp \frac{iS(x_0)}{\hbar} \frac{(\pi\hbar)^{\frac{n}{2}}}{\sqrt{\det A}} \left(\exp\left(-\frac{\hbar}{2i} A^{ab} \frac{\partial}{\partial x^a}\frac{\partial}{\partial x^b}\right)\right)\left(\exp \frac{iS_+(x)}{\hbar}\right) \qquad (02)$$

whereas the expression in big brackest is an expansion over positive powers of $\hbar$., i.e. $O(\hbar)$ in equation (01) is polynomial on $\hbar$,

$$\text{Only positive powers of } \hbar \text{ have imput in } O(\hbar) \qquad (03)$$

On one hand everybody knows the statement (01), on the other way the statement (02) is not simple, and it is highly non-trivial.

The best way to see this statement it is to study famous Stirling formula.

Recall that Striling formula says that

$$N! = \left(\frac{N}{e}\right)^N \sqrt{2\pi N}\left(1 + \frac{1}{12N} + \frac{1}{288N^2} + \dots\right) \qquad \text{(Stirling)}$$

The "easy part" of Stirling formula, i.e. the statement

$$N! \approx \left(\frac{N}{e}\right)^N \sqrt{2\pi N}\,, \quad N! = \left(\frac{N}{e}\right)^N \sqrt{2\pi N}\left(1 + O\left(\frac{1}{N}\right)\right)\,,$$

is related with (01). Equation (Stirling) is the 'difficult part'. of Stirling formula. In particular studying this equation it is impossible to avoid the question: why in formal series appear only negative powers of N (compare with (02,03)).

Calculations of Stirling formula

This is standard that

$$N! = \int_0^\infty t^N e^{-t} dt = \int_0^\infty e^{-t+N\log t} dt\,.$$

Calculate it.

To use stationary phase method consider substitution $t = N(1+x)$ (The point $t = -N$ is stationary point, we consider $t = N(1+x)$ not just $t = 1+x$ to have 'big coefficient' in a vicinity of stationary point)

Thus we have:

$$N! = \int_0^\infty t^N e^{-t} dt = \int_0^\infty e^{-t+N\log t} dt = \int_{-1}^\infty e^{-N(1+x)+N\log[N(1+x)]} N dx =$$

$$\int_{-1}^\infty e^{-N(1+x)+N\log(N(1+x))} N dx = e^{-N} N^{N+1} \int_{-1}^\infty e^{N(\log(1+x)-x)} dx =$$

$$N\left(\frac{N}{e}\right)^N \int_{-1}^\infty e^{-N\left(\frac{x^2}{2}-\frac{x^3}{3}+\frac{x^4}{4}+\dots\right)} dx = N\left(\frac{N}{e}\right)^N \int e^{-N\frac{x^2}{2}} e^{N\left(\frac{x^3}{3}-\frac{x^4}{4}+\dots\right)} dx\,. \quad (2)$$

To go further use identity:

$$\sqrt{2\pi N} e^{\frac{-Nx^2}{2}} = \int e^{\frac{-k^2}{2N}} e^{ikx} dk \quad (***)$$

The meaning of why we use use this identity we will see later.

This identity implies that

$$\int e^{-n\frac{x^2}{2}} \varphi(x) dx = \frac{1}{\sqrt{2\pi n}} \int e^{\frac{-k^2}{2n}} e^{ikx} dk dx \int \varphi(p) e^{ipx} dp =$$

$$\frac{1}{\sqrt{2\pi n}} \int e^{\frac{-k^2}{2n}} e^{ikx} \varphi(p) e^{ipx} dp dk dx =$$

$$\frac{2\pi}{\sqrt{2\pi n}} \int e^{\frac{-k^2}{2n}} \delta(k+p) \varphi(p) dp dk = \int e^{\frac{-k^2}{2n}} \varphi(k) dk = \frac{2\pi}{\sqrt{2\pi n}} e^{-\frac{1}{2n}\left(\frac{d}{dx}\right)^2} \varphi(x)\big|_{x=0}$$

We see that identity (***) allows us to rewrite (2) in terms of differential operator. [In fact doing this trick we formally were using identity (***) we used the following identity (Fedoruk)

$$\int f(x_0-x)u(x)dx = \int \tilde{f}(k)e^{ik(x_0-x)}\tilde{u}(p)e^{ipx}dxdkdp =$$

$$\int \tilde{f}(k)e^{ikx_0}\tilde{u}(p)e^{i(p-k)x}dxdkdp \int \tilde{f}(k)e^{ikx_0}\tilde{u}(p)\delta(p-k)dkdp =$$

$$\int \tilde{f}(k)e^{ikx_0}\tilde{u}(k)dk = f\left(\frac{1}{i}\frac{d}{dx}\right)u(x)\big|_{x=0}.$$

]

Namely apply identity (***) to equation (2):

$$N! = \left(\frac{N}{e}\right)^N \sqrt{2\pi N}\int_{-1}^{\infty} e^{-N\left(\frac{x^2}{2}-\frac{x^3}{3}+\frac{x^4}{4}+\dots\right)}dx =$$

$$\left(\frac{N}{e}\right)^N \sqrt{2\pi N}\int e^{-N\frac{x^2}{2}}e^{N\left(\frac{x^3}{3}-\frac{x^4}{4}+\dots\right)}dx =$$

$$\left(\frac{N}{e}\right)^N \sqrt{2\pi N}\int e^{-N\frac{x^2}{2}} \underbrace{e^{N\left(\frac{x^3}{3}-\frac{x^4}{4}+\dots\right)}}_{\text{terms of order } \geq 3 \text{ in } x} dx =$$

$$\left(\frac{N}{e}\right)^N \sqrt{2\pi N}\left\{e^{-\frac{1}{2N}\left(\frac{1}{i}\frac{d}{dx}\right)^2}\left[e^{N\left(\frac{x^3}{3}-\frac{x^4}{4}+\dots\right)}\right]\right\}_{x=0}.$$

... and here the mystery began: since the expression $\left[e^{N\left(\frac{x^3}{3}-\frac{x^4}{4}+\dots\right)}\right]$ possesses only the terms of the order ≥ 3 in exponent the last integral possess only terms which are proportional to $\frac{1}{N}$:

$$\frac{1}{N}\approx\hbar$$

$$\int e^{-\frac{1}{N}\left(\frac{d}{dx}\right)^2}\left[e^{N\left(\frac{x^3}{3}-\frac{x^4}{4}+\dots\right)}\right]dx = 1+\frac{1}{12N}+\frac{1}{288N^2}+\dots$$

Try to show it at least partially:

$$N! = \left(\frac{N}{e}\right)^N \sqrt{2\pi N}\left\{e^{-\frac{1}{2N}\left(\frac{1}{i}\frac{d}{dx}\right)^2}\left[e^{N\left(-\frac{x^3}{6}+\frac{x^4}{24}+\dots\right)}\right]\right\}_{x=0} =$$

$$\left(\frac{N}{e}\right)^N \sqrt{2\pi N}\left[1-\frac{1}{2N}\left(\frac{1}{i}\frac{d}{dx}\right)^2+\frac{1}{8N^2}\left(\frac{1}{i}\frac{d}{dx}\right)^4-\frac{1}{48N^3}\left(\frac{1}{i}\frac{d}{dx}\right)^6+\dots\right]$$

acting on

$$\left[1+N\left(\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\frac{x^6}{6}+\frac{x^7}{7}-\frac{x^8}{8}+\dots\right)+\frac{1}{2}N^2\left(\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}+\dots\right)^2+\dots\right]=$$

$$\left(\frac{n}{e}\right)^n\sqrt{2\pi n}\left[1+\frac{1}{2n}\frac{d^2}{dx^2}+\frac{1}{8n^2}\frac{d^4}{dx^4}+\frac{1}{48n^3}\frac{d^6}{dx^6}+\frac{1}{384n^4}\frac{d^8}{dx^8}+\dots\right]\text{ acting on}$$

$$\left[1+n\left(\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\frac{x^6}{6}+\frac{x^7}{7}-\frac{x^8}{8}+\dots\right)+\frac{1}{2}n^2\left(\frac{x^6}{9}-\frac{x^7}{6}+\frac{x^8}{16}+\frac{2x^8}{15}+\dots\right)^2+\dots\right]=$$

Contribution is gven by the terms of order $4,6$ and 8. More in detail: consider the action of the terms $\left[\frac{1}{2n}\left(\frac{1}{i}\frac{d}{dx}\right)^2\right]^\lambda$ which act on the monoms

$$F_{p_1}F_{p_2}\dots F_{p_r},$$

where every F_{p_k} is a monom of the order p_k, $(p_i\geq 3)$ which belong to the term

$$n\left(\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\frac{x^6}{6}+\dots\right).$$

The condition that the action of the operator $\left[\frac{1}{2n}\left(\frac{1}{i}\frac{d}{dx}\right)^2\right]^\lambda$ on the monom $F_{p_1}F_{p_2}\dots F_{p_r}$, is not zero is the following:

$$2\lambda=p_1+p_2+\dots+p_r\,,$$

and the power of n corresponding to this monom is equal to

$$\left(\frac{1}{n}\right)^\lambda\cdot n^r=n^{r-\lambda}\,.$$

Use this formula.

First of all note that all $p_i\geq 3$, hence $2\lambda\geq 3r$, hence

$$r-\lambda\leq r-\frac{3r}{2}<0\,.$$

Thus we have proved that in the expansion of $n!$ the contribution is given by negative powers of n.

Now calculate the contribution of power $\frac{1}{n^k}$ for $k=1,2,3\dots$.

We have that

$$\begin{cases}2\lambda=p_1+\dots+p_r\\ r-\lambda=-k\end{cases},\quad(p_i\geq 3)\,.$$

Hence

$$\begin{cases}2\lambda=p_1+\dots+p_r\geq 3r\\ \lambda=r+k\end{cases}\Rightarrow 2r+2k\geq 3r,\text{ i.e. } r\leq 2k$$

Thus we see that `contribution to terms of order` $\frac{1}{n^k}$ `is given by action of` $\exp -\frac{1}{n}\left(\frac{1}{i}\frac{d}{dx}\right)^2$ `on terms which possess not more than` $2k$ `monoms`

`I Calculate contribution to` $\frac{1}{n}$, $k=1$:

$$\begin{cases} 2\lambda = p_1 + \ldots + p_r \geq 3r \\ \lambda = r+1 \end{cases} \Rightarrow r = 1, 2\,.$$

$$a) r = 1\,, \begin{cases} 2\lambda = p_1 \\ \lambda = r+1 = 2 \end{cases}, p_1 = 4\,, \quad b) r = 2\,, \begin{cases} 2\lambda = p_1 + p_2 \geq 3r \\ \lambda = r+1 = 3 \end{cases}, p_1 = p_2 = 3\,.$$

`I Calculate contribution to` $\frac{1}{n^2}$, $k=2$:

$$\begin{cases} 2\lambda = p_1 + \ldots + p_r \geq 3r \\ \lambda = r+2 \end{cases} \Rightarrow r = 1, 2, 3, 4\,.$$

$$a) r = 1\,, \begin{cases} 2\lambda = p_1 \\ \lambda = r+2 = 3 \end{cases}, p_1 = 6\,,$$

$$b) r = 2\,, \begin{cases} 2\lambda = p_1 + p_2 \geq 3r \\ \lambda = r+2 = 4 \end{cases}, 2\lambda = 8,\ p_1 = 3, p_2 = 5 \text{ or } p_1 = p_2 = 4\,,$$

$$c) r = 3\,, \begin{cases} 2\lambda = p_1 + p_2 + p_3 \geq 3r \\ \lambda = r+2 = 5 \end{cases}, 2\lambda = 10\,, p_1 = p_2 = 3, p_3 = 4\,,$$

$$d) r = 4\,, \begin{cases} 2\lambda = p_1 + p_2 + p_3 + p_4 \\ \lambda = r+2 = 6 \end{cases}, 2\lambda = 12\,, p_1 = p_2 = p_3 = p_4 = 3\,,$$

`On the base of these considerations calculate` $n!$ `up to the terms` $\frac{1}{n}$. `We have according previous considerations that`

$$n! = \exp\left(-\frac{1}{2n}\left(\frac{1}{i}\frac{d}{dx}\right)^2\right)\exp\left(n\left(\frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \ldots\right)\right)_{x=0} =$$

$$\left(\frac{n}{e}\right)^n \sqrt{2\pi n}\left[1 + \frac{1}{8n^2}\frac{d^4}{dx^4}\left(-n\frac{x^4}{4}\right) + \frac{1}{48n^3}\frac{d^6}{dx^6}\left(+\frac{1}{2}n^2\frac{x^6}{9}\right) + \ldots\right] =$$