**Double complex and its spectral sequences 1**

Now we give a brief sketch on the topic how to apply spectral sequences technique for calculations of cohomology of double complexes. (See for the details for example [16].)

Let $E^{**} = \{E^{p.q}\}$ $(p, q = 0, 1, 2, ...)$ be a family of abelian groups (modules, vector spaces) on which are defined two differentials $\partial_1$ and $\partial_2$ which define complexes in rows and in columns of $E^{*.*}$ and which commute with each other:

$$\partial_1 \colon E^{p.q} \to E^{p.q+1}\,, \partial_1^2 = 0,\ \partial_2 \colon E^{p.q} \to E^{p+1.q}\,, \partial_2^2 = 0,\ \partial_1\partial_2 = \partial_2\partial_1\,. \qquad (A2.1)$$

$\{E^{**}, \partial_1, \partial_2\}$ is called double complex.

( It is convenient to consider $E^{p.q}$ for all integers $p$ and $q$ fixing that $E^{p.q} = 0$ if $p < 0$ or $q < 0$.)

One can consider "antidiagonals": $\mathcal{D}^m = \{E^{p.m-p}\}$ $(p = 0, 1, ..., m)$ which form complex with differential

$$Q = (-1)^q \partial_2 + \partial_1 \qquad (A2.2)$$

which evidently obeys to condition $Q^2 = 0$.

$$0 \to \mathcal{D}^0 \xrightarrow{Q} \mathcal{D}^1 \xrightarrow{Q} \mathcal{D}^2 \to \dots\,. \qquad (A2.3)$$

The cohomologies $H^m(Q)$ of this complex are called the cohomologies of double complex $(E^{**}, \partial_1, \partial_2)$.

The rows and the columns complexes define the cohomologies $H(\partial_1)$ and $H(\partial_2)$ of $E^{**}$.

One can consider the filtration corresponding to the double complex $\{E^{*.*}, \partial_1, \partial_2\}$

$$\ldots \subseteq X^m \subseteq X^{m+1} \subseteq \ldots \subseteq X^1 \subseteq X^0 \qquad (A2.4)$$

$$\text{where} \qquad X^k = \bigoplus_{q \geq 0, p \geq k} E^{p.q} \qquad (A2.5)$$

and sequence of the spaces $\{E_r^{p.q}\}$ $(r = 0, 1, 2, \ldots$ corresponding to this filtration

$$E_r^{p.q} = Z_r^{p.q} / B_r^{p.q} \quad (E_0^{p.q} = E^{p.q})\,. \qquad (A2.6)$$

In (A2.6) $Z_r^{p.q}$ ("$r$-th order cocycles") is the space of the elements in $E^{p.q}$ which are leader terms of cocycles of the differential $Q$ up to $r$–th order w.r.t. the filtration (A2.4), i.e.

$$\{Z_r^{p.q}\} = \{E_r^{p.q} \ni c\colon \quad \exists \tilde{c} = c (mod\, X_{p+1})\, \text{such that}\, Q\tilde{c} = 0 (mod\, X_{p+r})\}\,. \qquad (A2.7)$$

It means that there exists $\tilde{c} = (c, c_1, c_2, \ldots, c_{r-1})$ where $c_i \in E^{p+i.q-i}$ such that
$Q(c, c_1, c_2, \ldots, c_{r-1}) \subseteq X_{p+r}$ :

$$\partial_1 c = 0, \partial_2 c = \partial_1 c_1, \partial_2 c_1 = \partial_1 c_2, \ldots, \partial_2 c_{r-2} = \partial_1 c_{r-1},\, \text{so}\, Q\tilde{c} = \partial_2 c_{r-1} \in X_{p+r}\,.$$

Correspondingly $B_r^{p.q}$ is the space of up to $r$–th order borders:

$$\{B_r^{p.q}\} = \{E_r^{p.q} \ni c\colon \quad \exists \tilde{b} \in X_{p-r+1}\, \text{such} \quad \text{that}\, Q\tilde{b} = c\,. \qquad (A2.8)$$

It means that there exist $\tilde{c} = (b_0, b_1, b_2, \ldots, b_{r-1})$ where $b_i \in E^{p-i.q+i}$ and
$Q(b_0, b_1, b_2, \ldots, b_{r-1}) = c$:

$$\partial_1 b_0 + \partial_2 b_1 = c, \partial_1 b_1 + \partial_2 b_2 = 0, \partial_1 b_2 + \partial_2 b_3 = 0, \ldots, \partial_1 b_{r-1} = 0\,. \qquad (A2.9)$$

For example $E_1^{p.q} = H(\partial_1, E^{p.q})$.

We denote by $[c]_r$ the equivalence class of the element $c$ in the $E_r^{p.q}$ if $c \in Z_r^{p.q}$.

It is easy to see that the sequence $\{E_r^{p.q}\}$ $r = 0, 1, 2, \ldots$ is stabilized after finite number of the steps: $(E_{r_0}^{p.q} = E_{r_0+1}^{p.q} = \ldots = E_\infty^{p.q}$, where $r_0 = max\{p+1, q+1\}$.

Let $H^m(Q, X_p)$ be cohomologies groups of double complex truncated by filtration (A2.4) (we come to $H^m(Q, X_p)$ considering $\{\mathcal{D} \cap X^p, Q\}$ as subcomplex of (A2.3), $H^m(Q) = H^m(Q, X^0)$. We denote by ${}_{(p)}H^m(Q)$ the image of $H^m(Q, X_p)$ in $H(Q)$ under the homomorphism induced by the embedding $\mathcal{D} \cup X_p \to \mathcal{D}$. The spaces ${}_{(p)}H^m(Q)$ are embedded in each other

$$0 \subseteq {}_{(m)}H^m(Q) \subseteq {}_{(m-1)}H^m(Q) \subseteq \ldots {}_{(1)}H^m(Q) \subseteq {}_{(0)}H^m(Q) = H^m(Q)\,. \tag{A2.10}$$

The spaces $E_\infty^{p.q.}$ considered above are related with (A2.10) by the following relations:

$$E_\infty^{p.m-p} =_{(p)} H^m(Q) \big/ {}_{(p+1)}H^m(Q)\,. \tag{A2.11}$$

In particular $E_\infty^{0.m}$ is canonically embedded in $H^m(Q)$.

The formula (A2.11) is the basic formula which expresses the cohomology $H(Q)$ of the double complex $\{E^{p.q}, \partial_1, \partial_2\}$ in terms of $\{E_\infty^{p.q}\}$. From (A2.10, A2.11) it follows that

$$H^m(Q) \simeq \bigoplus_{i=0}^m E^{p-i.i}\,. \tag{A2.12}$$

The essential difference of (A2.12) from (A2.11) is that in (A2.12) the isomorphism of l.h.s. and of r.h.s. *is not canonical.*

The importance of the sequence $\{E_r^{*.*}\}$ $(r = 0, 1, 2, \ldots)$ is explained by the fact that its terms (and so $\{E_\infty^{*.*}\}$) can be calculated in a recurrent way. Namely one can consider differentials (See for details [16.]) $d_r$: $E_r^{p.q} \to E_r^{p+r.q+1-r}$ such that $\{E_r^{*.*}, d_r\}$ form spectral sequence, i.e.

$$E_{r+1}^{*.*} = H(d_r, E_r^{*.*}). \tag{A2.13}$$

The differentials $d_r$ are constructed in the following way: $d_0 = \partial_1$: $E^{p.q} = E_0^{p.q} \to E^{p.q+1} = E_0^{p.q+1}$.
If $c \in E^{p.q}$ and $\partial_1 c = 0 \leftrightarrow [c]_1 \in E_1^{p.q}$ then $d_1[c] = [\partial_2 c]$, $d_1$: $E_1^{p.q} \to E_1^{p+1.q}$.
In general case for $[c]_r \in E_r^{p.q}$ $d_r[c]_r = [Q\tilde{c}]_r$ $d_r$: $E_r^{p.q} \to E_1^{p+r.q+1-r}$,

where $\tilde{c}$: $c - \tilde{c} \in X^{p+r}$ (see the definition (A2.7) of $Z_r^{p.q}$).

One can show that definition of $d_r$ is correct, $d_r^2 = 0$ and (A2.13) is obeyed [16].

Using (A2.13) one come after finite number of steps to $E_\infty^{p.q}$ calculating each $E_r^{p.q}$ as the cohomology group of the $E_{r-1}^{p.q}$: $E_1^{p.q} = H(d_0, E^{p.q})$, $E_2^{p.q} = H(d_1, E_1^{p.q}$ and so on.

The spaces $E_r^{p.q}$ can be considered intuitively as $r$–th order (with respect to differential $\partial_2$) cohomologies of differential $Q$ . The operator $\partial_1$ is zeroth order approximation for differential $Q$. The calculations of $E_\infty^{p.q}$ via (A2.13) can be considered as perturbational calculations.

One can develop this scheme considering in perturbative calculations not the operator $\partial_1$, but $\partial_2$ as zeroth order approximation.

Instead filtration (A2.4) one has consider the "transposed" filtration

$$\ldots \subseteq {}^tX^m \subseteq {}^tX^{m+1} \subseteq \ldots \subseteq {}^tX^1 \subseteq X^0 \tag{A2.14}$$

$$\text{where} \qquad {}^tX^k = \bigoplus_{p\geq 0, q\geq k} E^{p.q}$$

and corresponding transposed spaces $\{{}^tE_r^{p.q}\}$. For example

$$E_1^{p.q} = H(\partial_1, E^{p.q}), \quad {}^tE_r^{p.q} = H(\partial_2, E^{p.q})\,.$$

Instead spectral sequence $\{E_r^{*.*}, d_r\}$ one has to consider transposed spectral sequence $\{{}^tE_r^{*.*}, {}^td_r\}$:

$$d_0 = \partial_1, \to {}^td_0 = \partial_2\,; d_1[c]_1 = [\partial_2 c]_1, \to {}^td_1[c]_1 = [\partial_1 c]_1\,,$$

and so on.

The relations between spaces $\{E_\infty^{p.q}\}$ and $\{{}^tE_\infty^{p.q}\}$ which express in different ways the cohomology $H(Q)$ is one of the applications of the method described here.

**Example.** Let $\mathbf{c} = (c_0, c_1.c_2)$ where $c_0 \in E^{0.2}, c_1 \in E^{1.1}, c_2 \in E^{2.0}$ be cocycle of the differential $Q$: $Q(c_0, c_1.c_2) = 0$ i.e. $\partial_1 c_0 = 0, \partial_2 c_0 = -\partial_1 c_1, \partial_2 c_1 = \partial_1 c_2$. To the leading term $c_0$ of this cocycle w.r.t. the filtration (A2.4) corresponds the element $[c_0]_\infty$ in $E_\infty^{0.2}$ which represents the cohomology class of the cocycle $\mathbf{c}$ in $E_\infty^{0.2}$.

In the case if the equation $(c_0, c_1.c_2) + Q(b_0, b_1) = (0, c_1', c_2')$ has a solution, i.e. the leading term $c_0$ of the cocycle $\mathbf{c}$ can be cancelled by changing of this cocycle on a coboundary, then the element $[c_1']_\infty \in E_\infty^{1.1}$ represents the cohomology class of the cocycle $\mathbf{c}$ in $E_\infty^{1.1}$.

In the case if the equation $(c_0, c_1.c_2)+Q(b_0, b_1) = (0, 0, \tilde{c}_2)$ have a solution, i.e. the leading term and next one both can be cancelled, by redefinition on a coboundary, then $[\tilde{c}_2]_\infty \in E_\infty^{2.0}$ represents the cohomology class of the cocycle $\mathbf{c}$ in $E_\infty^{2.0}$.

To put correspondences between the cohomology class of the cocycle $\mathbf{c}$ and corresponding elements from transposed spaces ${}^tE_\infty^{0.2}$, ${}^tE_\infty^{1.1}$ ${}^tE_\infty^{1.1}$ we have to do the same, changing only the definition of leading terms, which we have to consider now w.r.t. the filtration (A2.14).

To the leading term $c_2$ of this cocycle w.r.t. the filtration (A2.14) corresponds the element $[c_2]_\infty$ in ${}^tE_\infty^{2.0}$ which represents the cohomology class of the cocycle $\mathbf{c}$ in ${}^tE_\infty^{2.0}$. In the case if the equation $(c_0, c_1.c_2) + Q(b_0, b_1) = (c_0', c_1', 0)$ has a solution, i.e. the leading term $c_0$ of the cocycle $\mathbf{c}$ can be cancelled by changing of on a coboundary, then the element $[c_1']_\infty$ represents the cohomology class of the cocycle $\mathbf{c}$ in ${}^tE_\infty^{1.1}$. In the case if the equation $(c_0, c_1.c_2) + Q(b_0, b_1) = (\tilde{c}_0, 0, 0)$ has a solution, then $[\tilde{c}_0]$ represents the cohomology class of the cocycle $\mathbf{c}$ in ${}^tE_\infty^{0.2}$.

[16] Postnikov,M.M.: Lectures on Geometry, *Semestre III, Lecture #19, Semestre V, Lecture #23* Moscow, Nauka, (1987).