First gamma-function Γ_1, Euler gamma function Γ_E and Froullani integral

We know that for first gamma-function $\Gamma_1 = \Gamma_1(z|a)$,

$$\log\Gamma_1(z|a) = \frac{p}{ps}\zeta(s,z|a) = \int_0^\infty \left(\frac{e^{-zt}}{1-e^{-at}} - \dots\right)\frac{dt}{t} = \int_0^\infty \left(\frac{e^{-zt}}{1-e^{-at}} - \frac{1}{at} - \left(\frac{1}{2-z}\right)e^{-t}\right)\frac{dt}{t} =$$

Now chaning $t \mapsto at$ we come to the integral:

$$\log\Gamma_1(z|a) = \int_0^\infty \left(\frac{e^{-\frac{z}{a}t}}{1-e^{-t}} - \frac{1}{t} - \left(\frac{1}{2} - \frac{z}{a}\right)e^{-\frac{t}{a}}\right)\frac{dt}{t} =$$

$$\underbrace{\int_0^\infty \left(\frac{e^{-\frac{z}{a}t}}{1-e^{-t}} - \frac{1}{t} - \left(\frac{1}{2} - \frac{z}{a}\right)e^{-t}\right)\frac{dt}{t}}_{\text{I-st}} + \left(\frac{1}{2} - \frac{z}{a}\right)\underbrace{\int_0^\infty \left(e^{-t} - e^{-\frac{t}{a}}\right)\frac{dt}{t}}_{\text{II-nd integral}}$$

First integral is just integral for usual Gamma-function with argument $\frac{z}{a}$. The Second integral is just usual Froullani integral *:

$$\int_0^\infty \left(e^{-t} - e^{-\frac{t}{a}}\right)\frac{dt}{t} = -\log a\,.$$

Using Froullani integral we can deduce the fact that the first integral is the usual Euler gamma function. Indeed one can show that the first integral obeys the standard relation for logarithm of Euler gamma function $\log\Gamma(w+1) = \log\Gamma(w) + \log w$, i.e. it is just Euler Gamma function (up to scaling of argument). Show it: Denote temporarly the first inetgral by $\Psi(w)$, where $w = \frac{z}{a}$. Then one can see that

$$\Psi(w+1) - \Psi(w) =$$

$$\int_0^\infty \left(\frac{e^{-(w+1)t}}{1-e^{-t}} - \frac{1}{t} - \left(\frac{1}{2} - w + 1\right)e^{-t}\right)\frac{dt}{t} - \int_0^\infty \left(\frac{e^{-wt}}{1-e^{-t}} - \frac{1}{t} - \left(\frac{1}{2} - w\right)e^{-t}\right)\frac{dt}{t} =$$

$$\int_0^\infty \left(-e^{-wt} - e^{-t}\right)\frac{dt}{t} = \log w = \log\frac{z}{a}$$

Thus we see that

$$\log\Gamma_1(z|a) = \log\Gamma_E\left(\frac{z}{a}\right) + \left(\frac{z}{a} - \frac{1}{2}\right)\log a$$

Resumé First Barnes function differs from usual gamma function on the Froullani untegral factor.

It is interestijng to see how to calculate analytical continuation for Froullani untegral....

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* Froullani integral:

$$\frac{f(at) - f(bt)}{t} = f(0)\log\frac{b}{a}$$

To calculate this integral you have just to calculate limits at 0 and at the infinity!!!