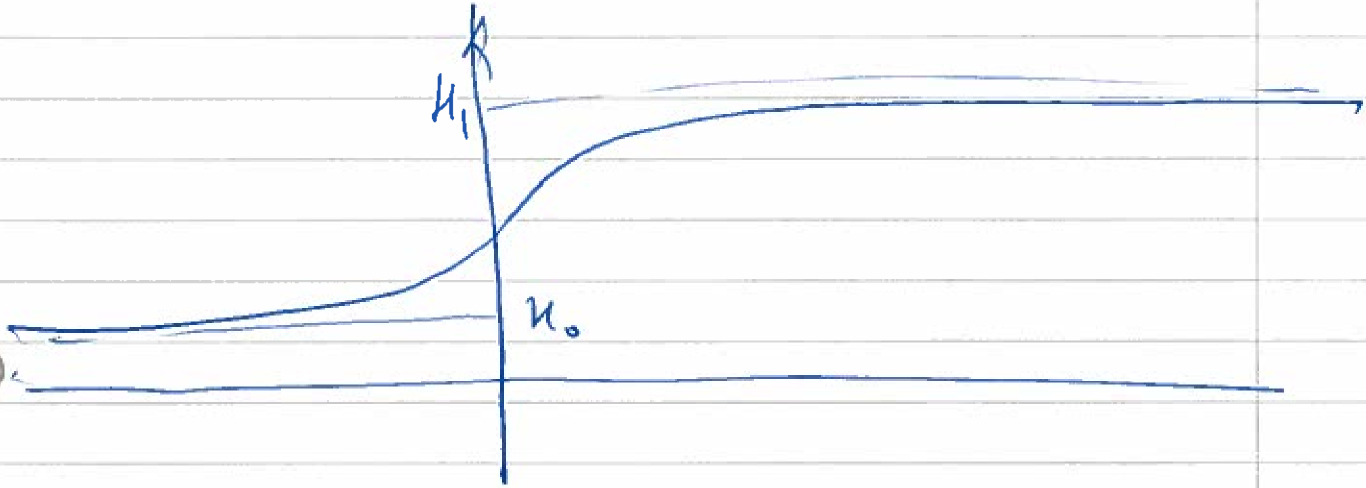
12 December
X-th lecture

# Adiabatic Perturb. theory and Adiabatic Invariants.




Let Hamiltonian $H_0$ 'slowly' transforms to Hamiltonian $H_1$

('slowly' - what it means
$t \gg \frac{1}{\Delta\omega}$ )

$$\Psi(x,t) = \sum C_n(t)\varphi_n(x)$$

Here: $\varphi_n(x)$ - eigen functions of Hamiltonian $H(t)$ such that

$$\langle \varphi_n, \dot{\varphi}_n \rangle = 0$$

Lemma:

$$\Psi(x,t) = \exp\left[-\frac{i}{\hbar}\int_0^T E(z)\,dt\right]\left[\varphi_n(t) + \ldots\right]$$

Hierarchy of functions

is not changed!

1-st function remains first
2-nd " ——————— " 2-nd

Number (numero) is preserved

Soft inflation does not change
ranking

What is the number (numero) of state?

Study this question in Quasiclassics

$$\hat{H} = -\frac{\hbar^2}{2m}\Delta + U(\vec{r})$$

$$\Psi = e^{\frac{i}{\hbar}S_\hbar(x,E)}$$

$$S_\hbar(x,E) = S(x,E) + \frac{\hbar}{i}\sigma(x,E) + \ldots$$

$$\hat{H}\Psi = E\Psi$$

$$(\hat{H} - E)\Psi = (\hat{H} - E)e^{\frac{i}{\hbar}S_\hbar(x,E)} =$$

$$= \frac{1}{2m}\frac{\hbar}{i}\frac{\partial}{\partial x^a}\left(\left(\frac{\partial S}{\partial x^a} + \frac{\hbar}{i}\frac{\partial \sigma}{\partial x^a}\right)e^{\frac{i}{\hbar}S_\hbar}\right) + (\hat{U} - E)e^{\frac{i}{\hbar}S} = 0$$

$$\frac{1}{2m}\left(\frac{\partial S}{\partial x^a} + \frac{\hbar}{i}\frac{\partial \sigma}{\partial x^a} + \ldots\right)\left(\frac{\partial S}{\partial x^a} + \frac{\hbar}{i}\frac{\partial \sigma}{\partial x^a} + \ldots\right) +$$

$$+ \frac{1}{2m}\frac{\hbar}{i}\Delta S + U - E = 0$$

$$\left\{\frac{1}{2m}\left(\frac{\partial S}{\partial x^a}\right)^2 + U - E\right\} + \frac{1}{2m}\frac{\hbar}{i}\left[\Delta S + 2\frac{\partial S}{\partial x^a}\frac{\partial \sigma}{\partial x^a}\right] =$$

$$= 0.$$

$$\frac{1}{2m}\left(\frac{\partial S}{\partial x^a}\right)^2 + U - E = 0, \quad \text{(Hamilton-Jacobi)}$$

$$\underbrace{\Delta S + 2\,\frac{\partial S}{\partial x^a}\,\frac{\partial \sigma}{\partial x^a} = 0}_{\text{Transport equation,}}$$

$n = 1.$

$$S = \pm\int^x \sqrt{2m(E - U)}\,dt = \pm\int^x p(t)\,dt.$$

transport:

$$S_{xx} + 2S_x\sigma_x = 0 \qquad (S_x = p)$$

$$\sigma_x = -\frac{S_{xx}}{2S_x}$$

$$\sigma = \frac{C}{\sqrt{p}}$$

$$\Psi = \frac{A}{\sqrt{p}}\,e^{-\frac{1}{i\hbar}\int_0^x p(t)\,dt} + \frac{B}{\sqrt{p}}\,e^{+\frac{1}{i\hbar}\int_0^x p(t)\,dt}$$

two waves.

How to <u>stitch</u> these waves?

# Regression Stationary phase method

$$\int_a^b e^{i\lambda f(t)} \varphi(t)\,dt =$$

suppose

1) $t_0$ is stationary point for $f(t)$: $f'(t_0)=0$
2) and $t_0 \in (a, b)$

~~3) $\varphi''(t_0)$~~ <

$$\approx e^{i\lambda f(t_0)} \varphi(t_0) \cdot \frac{1}{\sqrt{|f'(t_0)|}} e^{\pm \frac{i\pi}{4}}$$

depending on sign of $\lambda$

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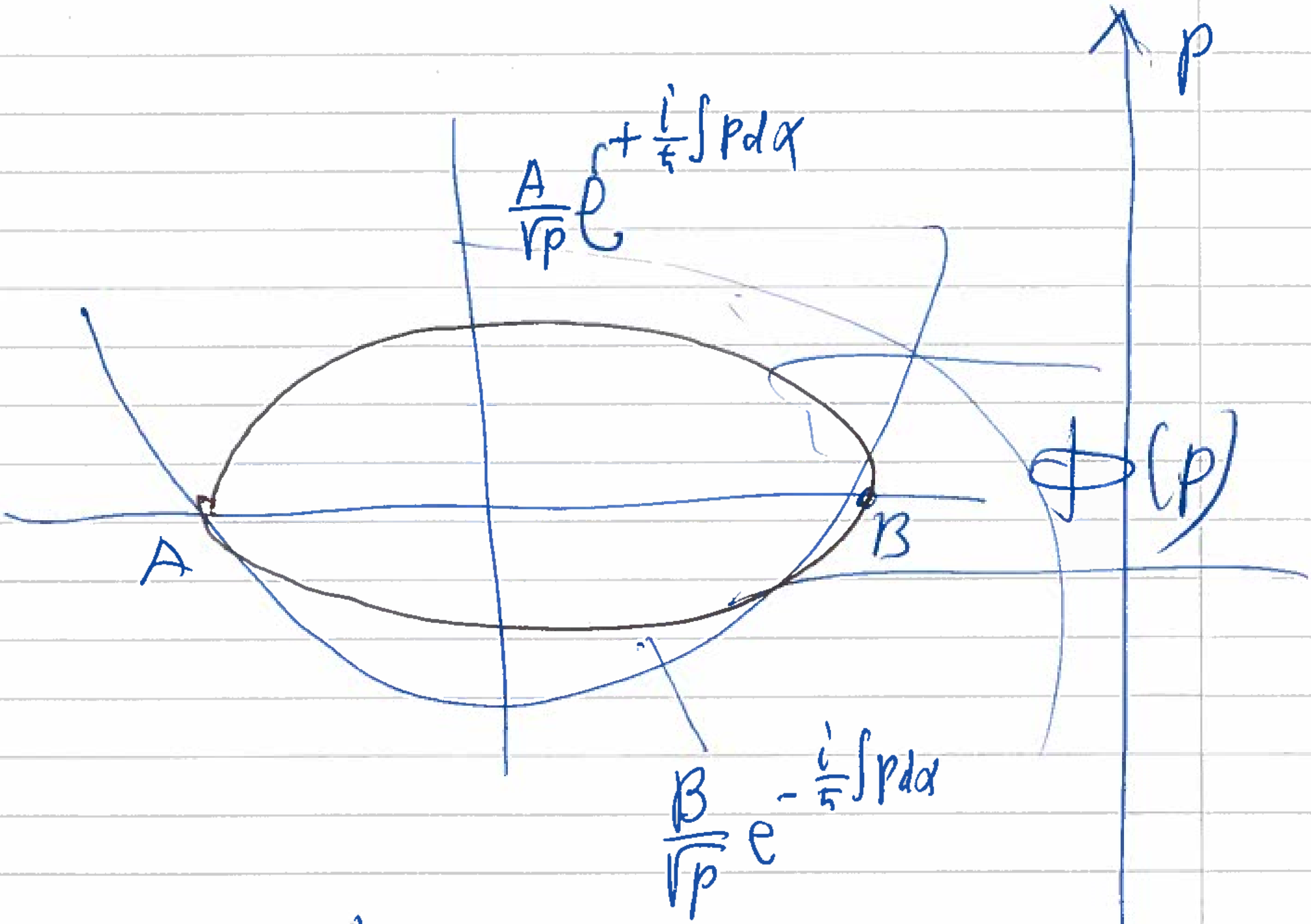
for real integral:

$$\int_a^b e^{\lambda f(t)} \varphi(t)\,dt = \frac{1}{\sqrt{f''(t_0)}} e^{\lambda f(t_0)} \varphi(t_0)$$

$f''(t_0) \cancel{>} 0$

Exercise: $n! = \int^{\infty} e^{-t} t^n\,dt = \ldots$ $(t = nz!)$

Now return to stitching (p. 4)




In a vicinity of points A, B quasiclassic works in momentum represent.

$$\Phi(p) \longrightarrow \Psi(x) = \int \Phi(p') e^{\frac{ip'x}{\hbar}} dp' =$$

$$= \int e^{\frac{i}{\hbar}\Delta(p') + \frac{ip'x}{\hbar}}$$

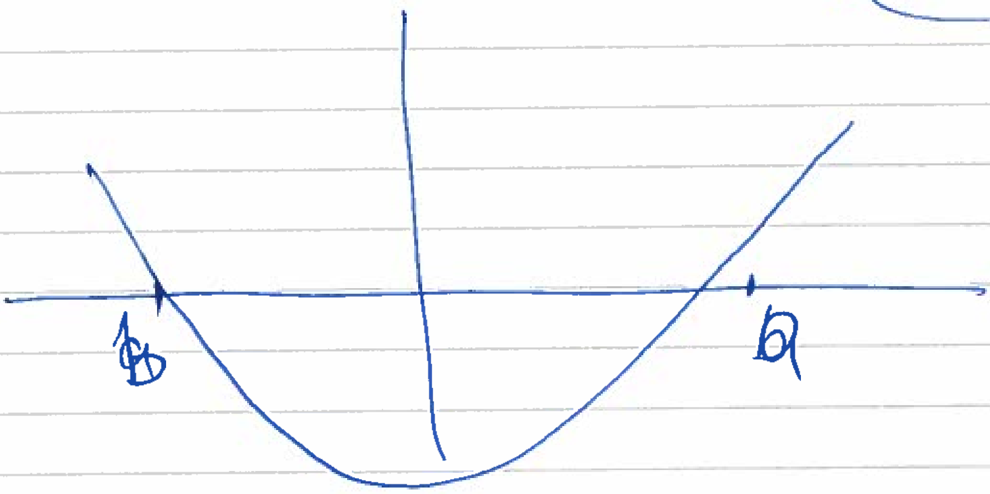
Maslov index $\pm \frac{i\pi}{4}$

$$\Downarrow$$

stitching:

$$\Psi = \frac{C}{\sqrt{p}} \sin\left(\frac{1}{\hbar}\int_b^x p\,dx + \frac{\pi}{4}\right)$$

Maslov index




$$\frac{1}{\hbar}\int_b^a p\,dx + \frac{\pi}{2} = \pi(n+1)$$

$$\underbrace{\oint p\,dx}_{} = 2\pi\hbar\left(n+\frac{1}{2}\right) = \int_S dp \wedge dq$$

action takes 'integer' values!

$\Downarrow$

Adiabatic invariant in classical mechanics

Adiabatic invariant

$$H(\lambda) = \frac{p^2}{2m} + U(q, \lambda)$$

$\lambda = \lambda(t)$ "slowly" changes in time

$I(p, q, \lambda)$ - adiabatic invariant

$$I(p(t_0+t), q(t_0+t), t_0+\varepsilon t) =$$
$$= I(p(t_0), q(t_0), t_0) + O(\varepsilon t)$$

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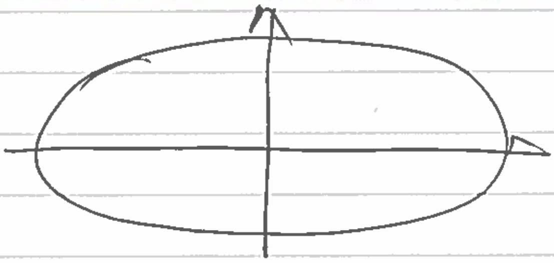
Fix $\lambda$



$$I = S = \int p\,dq$$

Adiabatic invariant $= 2\pi\hbar(n+1)$
$\quad\quad\quad\quad \parallel$
Numero of state

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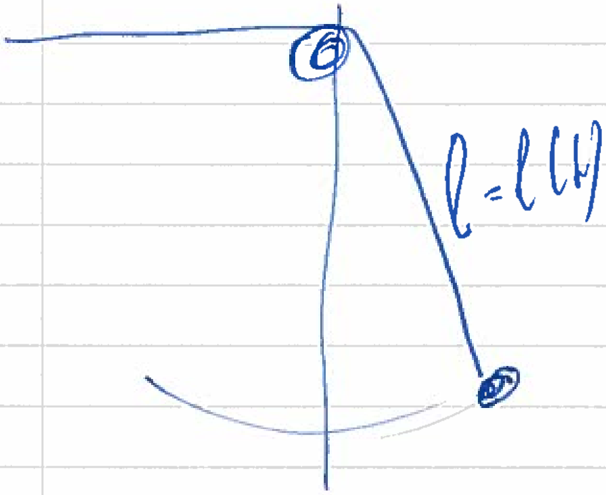
Exampler

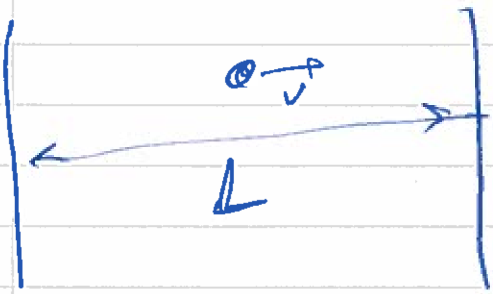



$$T = 2\pi\sqrt{\frac{l}{g}}$$

$$\omega = \sqrt{\frac{g}{l}}$$

$$I = \frac{E}{\omega} \sim$$ number of state

action




$$L \cdot V = I$$

$N \sim I$ numero of state

END