Textbook approach to quantum mechanics based on Hilbert spaces and self-adjoint operators in these spaces is not sufficient.
Algebraic approach= starting point is a unital associative algebra with involution. Self-adjoint elements of this algebra are physical observables
Geometric approach= starting point is a convex set (the set of states)
Particles and quasiparticles are elementary excitations of translation-invariant stationary state. Scattering matrix, inclusive scattering matrix, Green functions, generalized Green functions.

New results (applications to string theory).
Lectures for physicists(on Tuesday at 8 pm)
BRST, conformal field theories, string theory.
Weekly homework assignments.

Convex set $\mathcal{C}$ in vector space contains together with every two points an interval connecting these points.
If $e_i \in \mathcal{C}$ and $p_i \geq 0, \sum p_i = 1$ then $\sum p_i e_i \in \mathcal{C}$.
This point is called a mixture of e_i's with probabilities (weights) p_i.
Extreme points of $\mathcal{C}$ = points that cannot be represented in non-trivial way as mixtures of other points of $\mathcal{C}$
We always assume that convex sets we consider are closed. Then for any probability distribution on a subset of $\mathcal{C}$ we can consider a mixture of points of this subset as the integral $\int \omega(\lambda)\rho(\lambda)d\lambda$ where $\rho(\lambda)$ is the density of the distribution.
If $\mathcal{C}$ is compact then every point of $\mathcal{C}$ is a mixture of extreme points.

Quantum mechanics in Hilbert space
State= density matrix= positive-definite self-adjoint operator K with unit trace: $trK = 1$.
The set of density matrices is convex, its extreme points (pure states) correspond to vectors of Hilbert space having unit norm:
$K_\Psi(x) = \langle x, \Psi \rangle \Psi$.
If $\Psi' = \lambda\Psi$ then $K_{\Psi'} = K_\Psi$.
Density matrix K can be diagonalized, there exist a basis of eigenvectors e_i with non-negative eigenvalues p_i and $\sum p_i = 1$.
$K = \sum p_i K_{e_i}$, hence K is a mixture of pure states K_{e_i} with probabilities (weights) p_i

Evolution of pure state $i\hbar\frac{d\Psi}{dt} = \hat{H}\Psi$ (Schrödinger equation), $\hat{H}$ -self-adjoint operator (Hamiltonian)
Evolution operator:
$\Psi(t) = \hat{U}(t)\Psi(0), i\hbar\frac{d\hat{U}}{dt} = \hat{H}\hat{U}(t), \hat{U}(t) = e^{\frac{-it\hat{H}}{\hbar}}$
$\hat{H}$ is a generator of the evolution operator $\hat{U}(t)$
Evolution of density matrix:

$$i\hbar\frac{dK}{dt} = -\hat{H}K + K\hat{H} = [K, \hat{H}], K(t) = \hat{U}(t)K\hat{U}^{-1}(t)$$

Observables = self-adjoint operators
$\langle A\rangle_K = trAK$ -expectation value of A in state K

Generalizations. Geometric approach-set of states. Algebraic approach-algebra of observables. We define the algebra of observables $\mathcal{A}$ as unital (having unit element) associative algebra over complex numbers with involution denoted by *. (An involution is an antilinear map $A \to A^*$ obeying $A^{**} = A, (AB)^* = B^*A^*$.) We assume that the algebra is a topological space and all operations are continuous, but for many of our statements these requirements are not sufficient (one should assume that we have a Banach algebra ($||AB|| \leq ||A|| \cdot ||B||$) or C^*-algebra ($||A^*A|| = ||A||^2$) or impose some other conditions). In our terminology an automorphism preserves not only operations in the algebra, but also the involution.

Physical observables= self-adjoint elements $(A = A^*)$. Jordan algebra of self-adjoint elements with respect to the operation $A \circ B = AB + BA$.
States = linear functionals ω on A obeying $\omega(A^*A) \geq 0$ (positive linear functionals)
Normalized state $\omega(1) = 1$.
Expectation value $\langle f(A) \rangle_\omega = \omega(f(A)$.
For C^*-algebras one can define $f(A)$ for any continuous function f; this allows us to define probability distribution of physical observable A in normalized state ω.

Denote by Aut the group of automorphisms of the algebra $\mathcal{A}$ commuting with involution. This group naturally acts on states. In quantum system the state depends on time and this dependence can be described by the evolution operator $U(t)$ transforming $\omega(0)$ into $\omega(t)$. This is so called Schrödinger picture; it is equivalent to Heisenberg picture where the elements of $\mathcal{A}$ depend of time, but the states do not depend: $\omega(t)(A) = \omega(A(t))$.

We say that A is an infinitesimal automorphism if $e^{At} \in \mathcal{A}$ (if A generates a one-parameter subgroup of $\mathcal{A}$). It is easy to check that A is a derivation (obeys Leibniz rule $A(xy) = (Ax)y + x(Ay)$), this follows from the relation

$$(1 + tA + ..)x \cdot (1 + tA + ..)y = (1 + tA + ...)(xy).$$

Conversely, a derivation A compatible with involution (obeying $(Ax)^* = A(x^*)$) specifies an infinitesimal automorphism if the equation $dV/dt = AV$ has a solution with initial value $V(0) = 1$. Derivations compatible with involution constitute a Lie algebra; one can say that this is a Lie algebra of $Aut(\mathcal{A})$.

The equation of motion has the form $\frac{d\omega}{dt} = H(t)\omega(t)$. Here $H(t)$ stands for infinitesimal automorphism, or, more generally, for a derivation. It plays the role of the Hamiltonian of the quantum system. If H does not depend on t the evolution operators obey $U(t+\tau) = U(t)U(\tau)$. In textbook quantum mechanics ω corresponds to a density matrix K (namely $\omega(A) = tr(KA)$) and $H(K) = [K, \hat{H}]/i\hbar$. Stone theorem: Self-adjoint operators in Hilbert space are in one-to-one correspondence with strongly continuous one-parameter groups of unitary operators: $\hat{H} \to \hat{U}(t) = e^{it\hat{H}}$.

To specify a quantum system we should fix an algebra with involution $\mathcal{A}$ and an infinitesimal automorphism H (or a family of infinitesimal automorphisms $H(t)$). Then the evolution is defined by a family of of automorphisms $U(t)$ that obeys the equation $dU/dt = H(t)U(t)$. The automorphisms $U(t)$ specify the evolution of states (positive linear functionals, i.e. linear functionals obeying $\omega(A^*A) \geq 0$).

If $A \to \hat{A}$ is a representation of $\mathcal{A}$ (i.e. an involution preserving homomorphism of algebra $\mathcal{A}$ into the algebra of bounded linear operators in Hilbert space $\mathcal{E}$) then every normalized vector $\Phi \in \mathcal{E}$ specifies a normalized state ω by the formula $\omega(A) = \langle \hat{A}\Phi, \Phi \rangle$ (more generally a density matrix K specifies a state $\omega(A) = tr(K\hat{A})$). Conversely, for any algebra $\mathcal{A}$ and any state ω we can construct a pre-Hilbert space $\mathcal{E}$ and a representation of $\mathcal{A}$ by operators in $\mathcal{E}$ such that for some cyclic vector $\Phi \in \mathcal{E}$ we have $\omega(A) = \langle \hat{A}\Phi, \Phi \rangle$. (The vector Φ is cyclic in the following sense: every other vector can be represented in the form $\hat{A}\Phi$ where $A \in \mathcal{A}$.) This construction (GNS-construction) is essentially unique (up to equivalence).

Let us sketch the proof of this theorem. Let us assume that the representation we need is constructed. Let us introduce in $\mathcal{A}$ a scalar product by the formula $\langle A, B\rangle = \omega(B^*A)$. The map $\nu : A \to \mathcal{E}$ sending A to $\hat{A}\Phi$ preserves this scalar product: $\langle \hat{A}\Phi, \hat{B}\Phi\rangle = \langle \hat{B}^*\hat{A}\Phi, \Phi\rangle = \widehat{\langle (B^*A)\Phi, \Phi\rangle} = \omega(B^*A)$. Cyclicity of Φ implies that this map is surjective; this allows us to identify $\mathcal{E}$ with the quotient of $\mathcal{A}$ with respect to null vectors. (A null vector is a vector that is orthogonal to all other vectors.) The relation $\nu(BA) = \hat{B}\nu(A)$ allows us to describe our representation in terms of the algebra $\mathcal{A}$ and state ω. Namely we construct $\mathcal{E}$ as $\mathcal{A}$ factorized with respect to null vectors of the scalar product.

If ω is a stationary state then the group $U(t)$ descends to a group $\hat{U}(t)$ of unitary transformations of corresponding space $\mathcal{E}$ (the scalar product $\omega(B^*A)$ is $U(t)$-invariant).
The generator $\hat{H}$ of $\hat{U}(t)$ = Hamiltonian.
We say that the stationary state ω is a ground state if the spectrum of $\hat{H}$ is non-negative.
This definition agrees with the standard definition of the ground state. Let us apply the GNS construction to the algebra of bounded operators in Hilbert space $\mathcal{E}$ and to a state represented by a vector $\Phi \in \mathcal{E}$ where Φ is an eigenvector of the Hamiltonian: $\hat{H}\Phi = E\Phi$. Then the space given by GNS construction can be identified with $\mathcal{E}$ and the generator of $\hat{U}(t)$ is equal to $\hat{H} - E$.

In Hamiltonian formalism a (pure) state of classical mechanical system (at the time t) is characterized by $2n$ numbers: $p = (p_1, ..., p_n)$ (generalized momenta) and $q = (q^1, ..., q^n)$ (generalized coordinates). (Together these numbers specify a point of $2n$-dimensional space called the phase space of the system).
More generally we can define a state as a probability distribution on the phase space. The set $\mathcal{D}$ of probability distributions is a convex set, the pure states can be identified with extreme points of this set. Every state can be represented in unique way as a mixture of pure states.

The evolution of a pure state is governed by Hamiltonian equations
$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \frac{dp}{dt} = -\frac{\partial H}{\partial q},$
where the function $H(p, q, t)$ is called the Hamiltonian. The evolution of general state is governed by Liouville equation
$\frac{d}{dt}\rho(p, q, t) = \{H, \rho(p, q, t)\}.$
If $U(t)$ denotes the evolution operator (the operator transforming the state at the moment 0 into the state at the moment t) we can write Liouville equation in the form $\frac{d}{dt}U(t) = LU(t)$ where $L\rho = \{H, \rho\}$. Notice that ρ is in general a generalized function on phase space. To verify the Liouville equation it is sufficient to check that for pure states (represented by δ-functions) it is equivalent to Hamiltonian equations.

A physical quantity (an observable) can be considered as a real function $f(p,q)$ on the phase space. It follows from the chain rule that
$\frac{d}{dt}f(p(t),q(t)) = -\frac{\partial f}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial f}{\partial q}\frac{\partial H}{\partial p}$
or, equivalently,
$\frac{d}{dt}f(p(t),q(t)) = \{f,H\}$.
Let us denote by $\mathcal{A}$ the set of all complex continuous functions on the phase space considered as an algebra with respect to the conventional addition and multiplication of functions.
Notice that every state ω (considered as a probability distribution on the phase space) specifies a linear functional on $\mathcal{A}$ by the formula $\omega(f) = \int f\omega$. This functional obeys the positivity condition: $\omega(f) > 0$ if $f > 0$.

QM as a deformation of classical mechanics. Consider family of algebras $\mathcal{A}_\hbar$ depending smoothly on the parameter $\hbar$ assuming that for $\hbar = 0$ we have a commutative algebra with the product that will be denoted $A \cdot B$. All algebras are defined on the same vector space (i.e. the addition and multiplication by a number do not depend on $\hbar$, but the multiplication of elements A, B of the algebra, denoted by $A \cdot_\hbar B$) smoothly depends on $\hbar$.) The commutator $[A, B]_\hbar = A \cdot_\hbar B - B \cdot_\hbar A$ vanishes for $\hbar = 0$, therefore we can introduce a new operation $\{A, B\}$ (Poisson bracket) using the formula

$$[A, B]_\hbar = i\{A, B\}\hbar + O(\hbar^2).$$

It is easy to verify that $\mathcal{A}_0$ (the algebra with commutative multiplication $A \cdot B$ that we have for $\hbar = 0$) with the new operation is Poisson algebra, i.e. the new operation satisfies the axioms of Lie algebra and

$$\{A \cdot B, C\} = \{A, C\} \cdot B + A \cdot \{B, C\}.$$

The proof: take identities $[A, B] = -[B, A]$,
$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$,
$[AB, C] = [A, C]B + A[B, C]$.
Decompose with respect to $\hbar$.