! means that the problem is used in other problems or in lectures

* - a problem for extra credit

** - I do not know a solution of this problem (this does not mean that it is difficult, I did not try).

The solutions should be submitted to schwarz@math.ucdavis.edu

Please, indicate the numbers of problems you solved in the subject line.

0!. A complex vector space E is equipped with non-negative scalar product. Prove that we can obtain pre Hilbert space factorizing E with respect to vectors with$\langle x, x\rangle = 0$.

Hint. Check that these vectors constitute a linear subspace of E. Use $|\langle x, y\rangle \leq \sqrt{\langle x, x\rangle}\sqrt{\langle y, y\rangle}$.

Density matrices are defined as positive-definite self-adjoint operators having unit trace and acting in complex Hilbert space .

1*. Prove that the set of density matrices is convex. Check that extreme points of this set are one-dimensional projectors $K_\Psi(x) = \langle x, \Psi\rangle \Psi$ where $||\Psi|| = 1$ (they are in one-to-one correspondence with non-zero vectors of Hilbert space with identification $\Psi \sim \lambda\Psi$).

2. Prove that the set of density matrices in two-dimensional Hilbert space is a three-dimensional ball and set of of its extreme points is a two-dimensional sphere.

3**.Prove that automorphisms of the set of density matrices are in one-to-one correspondence with unitary operators.

The linear envelope $\mathcal{T}$ of the set of density matrices is the space of all self-adjoint operators belonging to trace class. (A self-adjoint operator belongs to trace class if it has discrete spectrum and the the series of its eigenvalues is absolutely convergent.) We consider $\mathcal{T}$ as a normed space with the norm $||T|| = \sum |\lambda_k|$ where λ_k are eigenvalues of T. By definition an automorphism of the set of density matrices is a bicontinuous linear operator in $\mathcal{T}$ generating a bicontinuous map of the set density matrices. (One says that a map is bicontinuous if it is continuous and has continuous inverse.). It is obvious that a unitary operator specifies an automorphism of the set of density matrices by the formula $T \to UTU^{-1}$. One should prove that every automorphism has this form.