Second lecture

$\mathcal{H}$ — Hilbert space of states.

F — physical magnitude

$\Downarrow$

assign to F self-adjoint operator $\hat{F}$ on $\mathcal{H}$

$$\langle \psi, \hat{F}\psi \rangle = \langle \hat{F}\psi, \psi \rangle.$$

Translation TABLE

Let system 'be prepared' at state Ψ, $\Psi \in H$. We perform $\boxed{N}$ identical experiments

Result of measurement of magnitude F

$\hat{F}$ — is self-adjoint $\Downarrow$ f_i is real number

$\Psi = \varphi_i$ $\hat{F}\varphi_i = f_i \varphi_i$ φ_i is eigenvector of observable $\hat{F}$ with eigenvalue f_i	F is equal to f_i in all experiments. (f_i is real: $f_i = \langle \varphi, F\varphi \rangle = \overline{\langle \varphi, F\varphi \rangle}$
$\Psi = C_m \varphi_m + C_k \varphi_k$ (superposition of states φ_m, φ_n) (Suppose $\lvert\varphi_m\rvert = \lvert\varphi_n\rvert = 1$	F is equal to $\begin{cases} f_m \text{ in } n_m \\ f_k \text{ in } n_k \end{cases}$ experiments $\varphi_m \perp \varphi_k$ if $f_m \neq f_k$ $n_m : n_k = \lvert C_m\rvert^2 : \lvert C_k\rvert^2$ $n_m + n_k = N$
$\Psi = \sum C_m \varphi_m$ $(\forall m \ \lvert\varphi_m\rvert = 1)$	F is equal to f_m in n_m experiments $n_m \sim \lvert C_m\rvert^2$

$$\overline{F} = \frac{\sum n_m f_m}{N} = \frac{\sum f_m \lvert C_m\rvert^2}{N} = \frac{\langle \Psi, \hat{F}\Psi \rangle}{\langle \Psi, \Psi \rangle}$$

($\frac{n_m}{N}$ — probability)

Math. Appendix

$\hat{F}$ - self-adjoint operator in $\mathcal{H}$, $\boxed{\dim \mathcal{H} < \infty}$!
then there exists orthonormal basis
$\{\vec{f}_i\}$ of eigen-vectors.

Proof

Consider

$$S(\Psi) = \langle \Psi, \hat{F}\Psi \rangle \qquad \delta S = \langle \delta\Psi, \hat{F}\Psi \rangle$$

on $|\Psi| = 1$.

this is real function on compact ($\dim \mathcal{H} < \infty$)

$$M_0 = \{\Psi : \ S|_M - \text{minimum}\}$$

~~On~~ M_0 defines subspace of vectors with minimum eigenvalues.

Then by induction.

If $\lambda_i \neq \lambda_j \Rightarrow \langle f_i, f_j \rangle = 0$
if $\lambda_i = \lambda_j$ we can make them orthogonal.

$\langle \delta\Psi, F\Psi \rangle + \langle \Psi, F\delta\Psi \rangle = 0$

$\mathrm{Re} \ \langle \delta\Psi, F\Psi \rangle = 0 \qquad z_1 a_1 + z_2 a_2$

a_1, a_n

$\mathcal{H} = \mathbb{C}^2 \qquad \Psi = \begin{pmatrix} a \\ b \end{pmatrix} = a\uparrow + b\downarrow$

$$\underbrace{\hat{S}_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{S}_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \hat{S}_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

Pauli matrices [$\{i\hat{S}_x, i\hat{S}_y, i\hat{S}_z\}$ - generators of $su(2)$]

Observables $\hat{S}_x$ measures x component of spin of electron

" —— " $\hat{S}_y$ " —— " y " —— "

$\hat{S}_z$ " —— " z " —— "

$\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$ measures spin of electron

($\hat{S}^2$ belong to universal enveloping algebra of $su(2)$)

$\hat{S}_z(\uparrow) = \hat{S}_z\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\uparrow$

state $\uparrow$ has $S_z = \frac{1}{2}$

$\hat{S}_z(\downarrow) = \hat{S}_z\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2}\downarrow$

state $\downarrow$ has $S_z = -\frac{1}{2}$

$\Psi = (\uparrow + \downarrow) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For state $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ spin $S_z = \frac{1}{2}$ with probability $\frac{1}{2}$ and
$S_z = -\frac{1}{2}$ with the same probability

$\hat{S}_x\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \hat{S}_x\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

For state $\Psi = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ S_z can take two values $\pm\frac{1}{2}$,
S_x takes value $+\frac{1}{2}$.

[If $\Psi = \sum c_m \varphi_m$ and $\hat{F}$ is measured f_m then
after measurement system is at the
state φ_m]

$$\Psi = \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

$$\begin{pmatrix} c_+ \\ c_- \end{pmatrix} = c_+ \uparrow + c_- \downarrow$$

$$\bar{S}_z = \frac{\langle \Psi, \hat{S}_z \Psi \rangle}{\langle \Psi, \Psi \rangle} = \frac{\langle \begin{pmatrix} c_+ \\ c_- \end{pmatrix}, \frac{1}{2} c_+ \uparrow - \frac{1}{2} c_- \downarrow \rangle}{\langle c_+ \uparrow + c_- \downarrow, c_+ \uparrow + c_- \downarrow \rangle} =$$

$$= \frac{\frac{1}{2}|c_+|^2 - \frac{1}{2}|c_-|^2}{\cancel{2}|c_+|^2 + |c_-|^2}$$

Spin S_z is measured to be $\frac{1}{2}$ with probability

$$p_+ = \frac{|c_+|^2}{|c_+|^2 + |c_-|^2} \text{ and}$$

it is equal to $-\frac{1}{2}$ with probability

$$p_- = \frac{|c_-|^2}{|c_+|^2 + |c_-|^2}.$$

$$\bar{S}_x = \frac{\langle \Psi, \hat{S}_x \Psi \rangle}{\langle \Psi, \Psi \rangle} = \frac{\langle \begin{pmatrix} c_+ \\ c_- \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_+ \\ c_- \end{pmatrix} \rangle}{|c_+|^2 + |c_-|^2} =$$

$$= \frac{\frac{1}{2}(c_+ \bar{c}_- + c_- \bar{c}_+)}{|c_+|^2 + |c_-|^2},$$

Here we put $\hbar = 1$

Note: S_z, S_x cannot be SIMULTANEOUSLY MEASURED

$$[S_x, S_z] = -i S_y \neq 0.$$

For $\Psi = \begin{pmatrix}\psi_+ \\ \psi_-\end{pmatrix} = \psi_+ \uparrow + \psi_- \downarrow$

$$\langle S_i \rangle = \frac{\langle \Psi, \hat{S}_i \Psi \rangle}{\langle \Psi, \Psi \rangle}$$

Thus we define a map

$$\mathbb{C}P^1 \longrightarrow S^2$$

$$[\Psi] = \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} \longrightarrow \vec{S} = \frac{\langle \Psi, \hat{S} \Psi \rangle}{\langle \Psi, \Psi \rangle}$$

$\Psi \to g\Psi$ $\quad \langle \Psi g^{-1}, S g \Psi \rangle = \langle \Psi, g^{-1} S g \Psi \rangle$

$$[SU(2), \mathbb{C}P^1] \longrightarrow [SO(3), S^2].$$

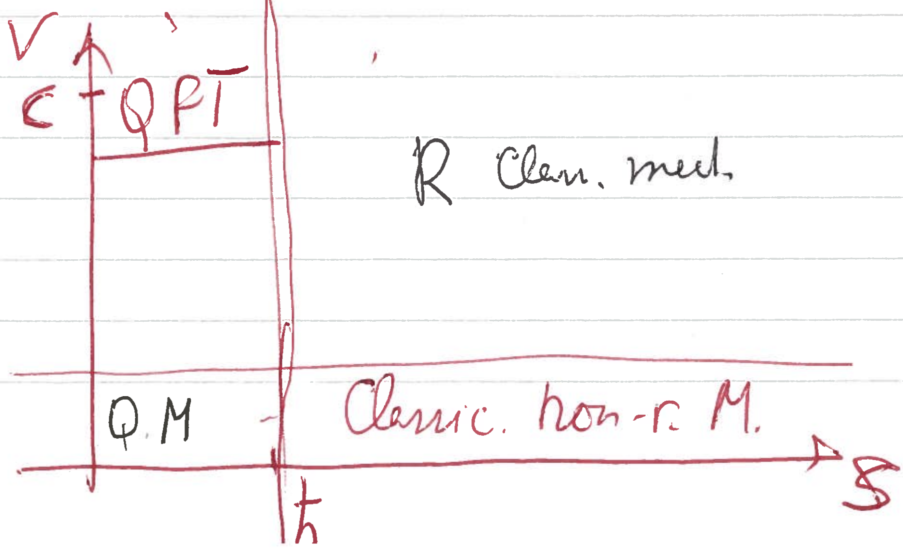
$$\mathbb{C}P^1 \simeq S^2 \xrightarrow{\text{stereogr. proj}} S^2.$$

Two words about $\hbar$

$\hbar \approx 6 \cdot 10^{-34}$ j·sec

$S \gg \hbar$ — Classical mech.

$S \sim \hbar$ — Quantum Mech



~~Another ex~~ $\mathcal{H} = \overline{C(\mathbb{R}^3)} = L^2(\mathbb{R}^3)$

$$\Psi = \Psi(x, y, z)$$

Measure coordinates x, y, z, momenta p_x, p_y, p_z.

$$\hat{x}\Psi = x\Psi, \quad \hat{y}\Psi = y\Psi, \quad \hat{z}\Psi = z\Psi$$

$$\hat{p}_x\Psi = \frac{\hbar}{i}\frac{\partial\Psi}{\partial x}, \quad \hat{p}_y\Psi = \frac{\hbar}{i}\frac{\partial\Psi}{\partial y}, \quad \hat{p}_z\Psi = \frac{\hbar}{i}\frac{\partial\Psi}{\partial z}$$

$$[\hat{p}_i, \hat{q}_k] = \frac{\hbar}{i}\mathbb{1}.$$

Math. Appendix

[We have presentation of Weyl ~~group~~ algebra (Heisenberg algebra) in $\mathcal{H}$)

There is a problem to define eigenvectors eigenvalues.

$$\cancel{\delta(x - x_0} \quad \underbrace{\delta(\vec{r} - \vec{r}_0), \quad e^{\frac{i}{\hbar}\vec{p}\cdot\vec{x}}}$$

these generalised function do not belong to $\mathcal{H}$

To deal with these objects we need ~~ex~~ a short .... to realm of ~~geor~~ general. function

$$\hat{X}\,\delta(\vec{r} - \vec{r}_0) = \sigma_x\,\delta(\vec{r} - \vec{r}_0) \qquad \frac{\hbar}{i}\frac{\partial}{\partial x} e^{\frac{i}{\hbar}\vec{p}\vec{x}} = p_x X$$

Yes, but ~~a~~ both functions DOES NOT belong to $\mathcal{H}$

Moreover $\delta(\vec{r} - \vec{r}_0)$ is not even a function (in a common sense)

Generalised eigen vectors

Let $\mathcal{H}$ - be Hilbert space, (e.g. $\mathcal{H} = L^2(\mathbb{R}^3) = C(\mathbb{R}^3)$
[$\mathcal{H} = \{\Psi(\vec{r}): \int \bar{\Psi}\Psi < \infty$)

M - space (of parameters)

$$\underbrace{R}_{\text{test functions}} \subset L^2(M) \subset \underbrace{R'}_{\text{general. functions.}}$$

We say that $f = f_a(x)$ $(a \in M)$
is generalised eigen vector ~~if~~ of operator A
if
f_a is generalised eigen function on M of
with values in Hilbert space $\mathcal{H}$
$\hat{A} f_a = \lambda(a) f_a$
[e.e. $f = f_a(x)$ if $\mathcal{H} = \{\Psi(\vec{r}) \ldots$]
such that
for every test function $\varphi(a) \in R$

$$\hat{A} \underbrace{\int f_a \varphi(a)\, d\mu_M}_{\text{vector in } \mathcal{H}} = \underbrace{\int \lambda(a) f_a \varphi(a)\, d\mu_M}_{\text{Vector in } \mathcal{H}},$$

Example $\mathcal{H} = L^2(\mathbb{R}^3) = \overline{C(\mathbb{R}^3)}$
$M = \mathbb{R}^3$
$\hat{A} = \hat{x}$
$f = \delta(x-a)$ is general. eigenf. of $\hat{x}$
$\hat{x}\delta(x-a) = a\,\delta(x-a)$

$$\hat{x} \int \delta(x-a) \varphi(a)\, da = \int a\,\delta(x-a) \varphi(a)\, da$$

$$\hat{x}\varphi(x) \qquad = \qquad x\varphi(x)$$