Homework 5. Solutions.

Christoffel symbols and Lagrangians

1 *Consider the Lagrangian of "free" particle* $L = \frac{1}{2} g_{ik}\dot{x}^i\dot{x}^k$ *for Riemannian manifold with a metric* $G = g_{ik}dx^i dx^k$.
Write down Euler-Lagrange equations of motion for this Lagrangian and compare them with differential equations for geodesics on this Riemannian manifold.

In fact show that

$$\underbrace{\frac{\partial L}{\partial x^i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}}_{\text{Euler-Lagrange equations}} \quad \Leftrightarrow \quad \underbrace{\frac{d^2x^i}{dt^2} = \Gamma^i_{km}\dot{x}^k\dot{x}^m}_{\text{Equations for geodesics}} \quad , \tag{1}$$

where

$$\Gamma^i_{km} = \frac{1}{2}g^{ij}\left(\frac{\partial g_{jk}}{\partial x^m} + \frac{\partial g_{jm}}{\partial x^k} - \frac{\partial g_{km}}{\partial x^j}\right) . \tag{2}$$

Solution: see the lecture notes.

2 *a) Write down the Lagrangian of of "free" particle* $L = \frac{1}{2} g_{ik}\dot{x}^i\dot{x}^k$ *for Euclidean plane in polar coordinates. Calculate Christoffel symbols for canonical flat connection in polar coordinates using Euler-Lagrange equations for this Lagrangian. Compare with answers which you obtained by the direct use of the formula (2). b) Do the same for cylindrical coordinates in* $\mathbf{E}^3$.

Solution. Canonical flat connection is Levi-Civita connection of Euclidean metric $G = dx^2 + dy^2$. Hence we can calculate Christoffel symbols using Lagrangian method.

Euclidean metric in polar coordinates is $dr^2 + r^2 d\varphi^2$. Hence the Lagrangian of the free particle is

$$L = \frac{\dot{r}^2 + r^2\dot{\varphi}^2}{2}$$

Euler-Lagrange equations:

1) for r:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \ddot{r} = \frac{\partial L}{\partial \varphi} = r\dot{\varphi}^2$$

i.e.

$$\ddot{r} - r\dot{\varphi}^2 = 0 \Rightarrow \Gamma^r_{rr} = \Gamma^r_{\varphi r} = \Gamma^r_{r\varphi} = 0, = \Gamma^r_{\varphi\varphi} = -r.$$

2) for φ:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = \frac{d}{dt}\left(r^2\dot{\varphi}\right) = r^2\ddot{\varphi} + 2r\dot{r}\dot{\varphi} = \frac{\partial L}{\partial \varphi} = 0,$$

i.e.

$$\ddot{\varphi} + \frac{2}{r}\dot{r}\dot{\varphi} = 0 \Rightarrow \Gamma^\varphi_{rr} = \Gamma^\varphi_{\varphi\varphi} = 0, \Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = \frac{1}{r}.$$

b) cylindrical coordinates in $\mathbf{E}^3$. Calculations almost the same as for polar coordinates in $\mathbf{E}^2$. $G = dr^2 + r^2 d\varphi^2 + dh^2$,

$$L = \frac{\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{h}^2}{2}$$

for r: $\ddot{r} - r\dot{\varphi}^2 = 0 \Rightarrow$

$$\Gamma^r_{rr} = \Gamma^r_{\varphi r} = \Gamma^r_{r\varphi} = \Gamma^r_{rh} = \Gamma^r_{hr} = \Gamma^r_{h\varphi} = \Gamma^r_{\varphi h} = \Gamma^r_{hh} = 0\,, \Gamma^r_{\varphi\varphi} = -r.$$

for φ, $r^2\ddot{\varphi} + 2r\dot{r}\dot{\varphi} = \frac{\partial L}{\partial \varphi} = 0$, i.e. $\ddot{\varphi} + \frac{2}{r}\dot{r}\dot{\varphi} = 0 \Rightarrow$

$$\Gamma^{\varphi}_{rr} = \Gamma^{\varphi}_{rh} = \Gamma^{\varphi}_{hr} = \Gamma^{\varphi}_{\varphi\varphi} = \Gamma^{\varphi}_{\varphi h} = \Gamma^{\varphi}_{h\varphi} = \Gamma^{\varphi}_{hh} = 0,\ \Gamma^{\varphi}_{r\varphi} = \Gamma^{\varphi}_{\varphi r} = \frac{1}{r}$$

3) for h, $\ddot{h} = 0$,

$$\Gamma^{h}_{rr} = \Gamma^{h}_{r\varphi} = \Gamma^{h}_{\varphi r} = \Gamma^{h}_{rh} = \Gamma^{h}_{hr} = \Gamma^{h}_{\varphi\varphi} = \Gamma^{h}_{\varphi h} = \Gamma^{h}_{h\varphi} = \Gamma^{h}_{hh} = 0,$$

3 *Write down the Lagrangian of "free" particle $L = \frac{1}{2}g_{ik}\dot{x}^i\dot{x}^k$ for the sphere of radius R in $\mathbf{E}^3$ in spherical coordinates. Calculate Christoffel symbols of Levi-Civita connection in spherical coordinates using Euler-Lagrange equations for this Lagrangian. (The induced Riemannian metric on the sphere equals $G = R^2d\theta^2 + R^2\sin^2\theta d\varphi^2$.)*

Solution.

Riemannian metric on sphere in spherical coordinates is $R^2d\theta^2 + R^2\sin^2\theta d\varphi^2$. Hence the Lagrangian of the free particle is

$$L = \frac{R^2\dot{\theta}^2 + R^2\sin^2\theta\dot{\varphi}^2}{2}$$

Euler-Lagrange equations:

for θ: $\ddot{\theta} = \sin\theta\cos\theta\dot{\varphi}^2$, hence

$$\ddot{\theta} = \sin\theta\cos\theta\dot{\varphi}^2 \Rightarrow \Gamma^{\theta}_{\theta\theta} = \Gamma^{\theta}_{\theta\varphi} = \Gamma^{\theta}_{\varphi\theta} = 0\,, \Gamma^{\theta}_{\varphi\varphi} = -\sin\theta\cos\theta\,.$$

for φ, $\frac{d}{dt}\left(R^2\sin^2\theta\dot{\varphi}\right) = 0$, i.e. $\sin^2\theta\ddot{\varphi} + 2\sin\theta\cos\theta\dot{\theta}\dot{\varphi} = 0$, hence

$$\ddot{\varphi} + 2\mathrm{cotan}\,\theta\ \dot{\theta}\dot{\varphi} = 0 \Rightarrow \Gamma^{\varphi}_{\theta\theta} = \Gamma^{\varphi}_{\varphi\varphi} = 0\,, \Gamma^{\varphi}_{\varphi\theta} = \Gamma^{\varphi}_{\theta\varphi} = \mathrm{cotan}\,\theta\,.$$

4 Calculate Christoffel symbols of Levi-Civita connection for Riemannian metric $G = adu^2 + bdv^2$. Lagrangian

$$L = \frac{a\dot{u}^2 + b\dot{v}^2}{2}$$

Euler-lagrange equations

for u:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}}\right) = \frac{d}{dt}(a\dot{u}) = a_u\dot{u}^2 + a_v\dot{v}\dot{u} + a\ddot{u} = \frac{\partial L}{\partial u} = \frac{a_u\dot{u}^2 + b_u\dot{v}^2}{2}$$

hence

$$\ddot{u} + \frac{1}{2}\frac{a_u}{a}\dot{u}^2 + \frac{a_v}{a}\dot{v}\dot{u} - \frac{1}{2}\frac{b_u}{a}\dot{u}^2 \Rightarrow \Gamma^u_{uu} = \frac{1}{2}\frac{a_u}{a}\,, \Gamma^u_{uv} = \Gamma^u_{vu} = \frac{1}{2}\frac{a_v}{a}, \Gamma^u_{vv} = -\frac{1}{2}\frac{b_u}{a}\,,$$

for v:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{v}}\right) = \frac{d}{dt}(b\dot{v}) = b_v\dot{v}^2 + b_u\dot{u}\dot{v} + b\ddot{b} = \frac{\partial L}{\partial v} = \frac{a_v\dot{u}^2 + b_v\dot{v}^2}{2}$$

hence

$$\ddot{v} + \frac{1}{2}\frac{b_v}{b}\dot{b}^2 + \frac{b_u}{b}\dot{u}\dot{v} - \frac{1}{2}\frac{a_v}{b}\dot{v}^2 \Rightarrow \Gamma^v_{vv} = \frac{1}{2}\frac{b_v}{b}\,, \Gamma^v_{vu} = \Gamma^v_{uv} = \frac{1}{2}\frac{b_u}{b}, \Gamma^v_{uu} = -\frac{1}{2}\frac{a_v}{b}\,.$$