Comments on the Coursework (Discussion of solutions)

April 2017

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Question **a)** —no problem.

Question **b)** —almost no problem.

The question about A^{2017} is a simple question. Every year it changes little bit ($A^{2015} \to A^{2016} \to A^{2017} \to \ldots$). This year is a "prime" year (2017 is prime number), but on the other hand $2017 = 1 + 63 \cdot 32$. Hence $A^{2017} = A$.

Question **c)**. Students did it, On the other hand many students did not find all three solutions: We have that $A^3 = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$ for $\varphi = \frac{\pi}{3}$. Hence

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{where } 3\theta = \varphi$$

We have to divide the angle φ on 3. It is evident that $\theta = \frac{\varphi}{3} = \frac{\pi}{9}$, i.e. $A = \begin{pmatrix} \cos\frac{\pi}{9} & -\sin\frac{\pi}{9} \\ \sin\frac{\pi}{9} & \cos\frac{\pi}{9} \end{pmatrix}$ is a solution. Almost all students did it. On the other hand there are also two other solutions:

$$\theta_2 = \frac{\varphi + 2\pi}{3} = \frac{7\pi}{9}\,, A = \begin{pmatrix} \cos\frac{7\pi}{9} & -\sin\frac{7\pi}{9} \\ \sin\frac{7\pi}{9} & \cos\frac{7\pi}{9} \end{pmatrix}\,,$$

and

$$\theta_3 = \frac{\varphi + 4\pi}{3} = \frac{13\pi}{9}\,, A = \begin{pmatrix} \cos\frac{13\pi}{9} & -\sin\frac{13\pi}{9} \\ \sin\frac{13\pi}{9} & \cos\frac{13\pi}{9} \end{pmatrix}\,.$$

Question **d)**. Many students did it, almost 70%, but they did it using just brute force, doing straightforward calculations. Sure calculations were not very easy, and some students did mistakes in these calculations. On the other hand this problem has short and elegant solution based on the following observation: the vector $\mathbf{a}$ and an arbitrary vector which is orthogonal to the vector $\mathbf{a}$ are eigenvectors of the operator A.

Indeed it is evident that $A(\mathbf{a}) = \mathbf{a}$ since $\mathbf{a} \times \mathbf{a} = 0$. Hence $\mathbf{a}$ is eigenvector with eigenvalue 1. Now the next observation: one can see that *if vector* $\mathbf{b}$ *is orthogonal to the vector* $\mathbf{a}$ *then* $\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) = -|\mathbf{a}|^2\mathbf{b}$. Hence

$$A(\mathbf{b}) = \mathbf{b} - \mathbf{a} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{b} + |\mathbf{a}|^2\mathbf{b} = (1 + |\mathbf{a}|^2)\mathbf{b}\,, \text{if } \mathbf{b} \text{ is orthog. to } \mathbf{a}.$$

We see that every vector which is proportional to the vector $\mathbf{a}$ is eigenvector with eigenvalue 1, and all vectors which are orthogonal to the vector $\mathbf{a}$ are eigenvectors with eigenvalue $1 + |\mathbf{a}|^2$.

One can choose a basis such that the first vector of this basis is proprotional to the vector $\mathbf{a}$ and two other vectors are orthogonal to the vector $\mathbf{a}$. Matrix of the operator A in this basis is $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+|\mathbf{a}|^2 & 0 \\ 0 & 0 & 1+|\mathbf{a}|^2 \end{pmatrix}$. Now it is evident that

$$\operatorname{Tr} A = 1 + 2(1+|\mathbf{a}|^2) = 1 + 2\cdot 2 = 5\,, \det A = (1+|\mathbf{a}|^2)^2 = 4\,,$$

since $|\mathbf{a}| = \sqrt{\frac{9}{169} + \frac{16}{169} + \frac{144}{169}} = 1$.

This is really nice solution, is not it? ¡I am happy that a few students did it in this way! (In fact much more much more than "a few" students realised that vector $\mathbf{a}$ is eigenvector, but they still were looking for other eigenvector using brute force caclulations.)

Question **e)**. The last part of this question is really difficult question.

Some students did it using not not easy analysis of inequalities. Only one studnet gave clear geometrical solution. I will present here two nice geometrical solutions.

First solution Consider the parallelogram $\Pi_{\mathbf{e},\mathbf{f}}$ formed by basis vectors. This parallelogram is a unit square. It obviously does not possess integer points except vertices. The linear operator P with matrix transforms the square $\Pi_{\mathbf{e},\mathbf{f}}$ onto the paralellogram $\Pi_{\mathbf{a},\mathbf{b}}$. The inverse operator P^{-1} transforms the parallelogram $\Pi_{\mathbf{a},b}$ onto the square $\Pi_{\mathbf{e}_x,\mathbf{e}_y}$. The idea of the proof is the following: the matrix of the operator P and the matrix of the inverse operator P^{-1} in the basis $\mathbf{e},\mathbf{f}$ have integer entries. This implies that all integer points of the unit square are in one-one correspondence with integer points of the parallelogram $\Pi_{\mathbf{a},\mathbf{b}}$, and the unit square has no integer points in it except vertices.

Second solution The proof follows from

Lemma Any triangle with vertices in integer points has area equal or bigger than $\frac{1}{2}$.

Proof of the lemma: Let triangle be formed by two vectors $\mathbf{c},\mathbf{d}$. Take the parallelogram $\Pi_{\mathbf{c},\mathbf{d}}$ corresponding to this triangle. This parallelogram has vertices in integer points. Hence by determinant formula its area is bigger or equal to 1. Hence the area of triangle is bigger or equal to 1/2.

Now based on the lemma prove that the parallelogram $\Pi_{\mathbf{a},\mathbf{b}}$ has no a point with integer coordinates except vertices. Take any point A in this parallelogram. In the case if A does not coincide with one of vertices, then one can form at least three triangles in this parallelogram which do not interesect (and even 4 triangles if this point is an interior point). Suppose that A has integer coefficients. Then by lemma we see that area of parallelogram is bigger or equal than $3 \cdot \frac{1}{2} = \frac{3}{2} > 1$. Contradiction with the fact that parallelogram has an area 1[1)]

There are another beautiful proofs of this fact. It has to be mentioned that all this stuff is related with continuous fractions. and the Pick formula that states that any convex polygon with vertices in integer points has the area

$$S = \frac{E}{2} + I - 1\,,$$

where E is a number of points which belong to edges (including vertices), and I the number of points which belong to interior of the polygon. (In fact we are on the way to prove the Pick formula).

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Almost nobody had problems to answer the question 2a)

Answering question 2b) almost everybody calculated right the matrix P of the operator $P_1 \cdot P_2$

Many students when proving the fact that $P_1 \circ P_2$ is also orthogonal operator preserving orientation did it using brute force: they just calculated straightforwardly that $P^T \cdot P = \mathbf{id}$ and $\det P = 1$ for matrix P of operator $P_1 \circ P_2$. Instead doing these calculations one can deduce it from the properties of operators P_1, P_2 or corresponidng matrices. E.g. an operator $P = P_1 \circ P_2$ is orthogonal because it is product of two orthogonal operators:

$$P^T \cdot P = (P_1 \cdot P_2)^T \cdot (P_1 \cdot P_2) = P_2^T \cdot (P_1^T \cdot P_1) \cdot P_2 = P_2^T \cdot P_2 = \mathbf{id}$$

and its determionant is equal to 1 because

$$\det P = \det(P_1 \cdot P_2) = \det P_1 \det P_2 = 1 \cdot 1 = 1\,.$$

One does not need to do straightworward calculations of determinant of the matrix P.

[1)] This solution was suggested by 2-nd year student, Ruth Reinolds. Now she is PhD student in Edinburgh

Some students came to the conclusion that $P = P_1 \cdot P_2$ is rotation operator just on the base that this operator preserves orientation, This is not enough: operator P is rotation operator since

1) *it preserves orientation*

2) *and it is orthogonal operator.*

Both conditions have to be checked.

This is a mistake to think that $\det P = 1$ implies that P is orthogonal operator. Please avoid it!

Many students just ignored to calculate the axis of rotation operator P.

The answer on the last question: that $\Phi \approx \sqrt{2}\theta$ can be done by study of the formula

$$\cos\Phi = \cos\theta + \frac{1}{2}(\cos^2\theta - 1)\,. \qquad (*)$$

Some students ca,e the answer, but only few analyzed it properly.

There is of course another very beautiful geometrical interpretation of this result. If angle θ is very small, then infinitesimally, action of rotation operator is $P(\mathbf{x}) = \mathbf{x} + \theta\mathbf{w} \times \mathbf{x}$, where $\mathbf{w}$ is the vector of angular velocity. Hence the result of infinitesimal rotations around axis $\mathbf{w}_1$ and $\mathbf{w}_2$ is the rotation around axis $\mathbf{w}_1 + \mathbf{w}_2$, and the length of the vector $\mathbf{e}_x + \mathbf{e}_z$ is equal to $\sqrt{2}$.

Angular velocity is the vector, two infinitesimal rotations are described by the sum of two vectors of angular velocity!

Unfortunately nobody even tried to do iit in this way.

Few students tried to use L'Hopital's rule. Using this rule one has to be carreful with the fact that derivative of the function arccosx ($\cos^{-1} x$) at the point $x = 1$ is not well-defined.

The function $\cos\theta$ is almost equal to 1 at small θ and

$$\cos\theta = 1 - \frac{\theta^2}{2} + o(\theta^2)$$

for small θ. Using this formula one can easy come to the answer.

Finally I will show not the most elegant, but the simple and clear solution, which does not use much calculus: We have to solve equation

$$\Phi(\theta): \quad \cos\Phi = \cos\theta - \frac{1}{2}\sin^2\theta\,.$$

Transform it:

$$\cos\Phi = 1 - 2\sin^2\frac{\Phi}{2} = \cos\theta - \frac{1}{2}\sin^2\theta = 1 - 2\sin^2\frac{\theta}{2} - 2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}\,.$$

i.e.

$$\sin^2\frac{\Phi}{2} = \left(1 + \cos^2\frac{\theta}{2}\right)\sin^2\frac{\theta}{2} = \left(2 - \sin^2\frac{\theta}{2}\right)\sin^2\frac{\theta}{2}\,,$$

and

$$\Phi = 2\mathrm{arcsin}\left(\sqrt{2 - \frac{\sin^2\frac{\theta}{2}}{2}}\sin\frac{\theta}{2}\right) = 2\cdot\sqrt{2}\frac{1}{2}\theta + 0(\theta) = \sqrt{2}\theta + o(\theta)\,.$$

3

This question was relatively much simpler than previous ones.

Question **a** —no problem

Question **b** —no problem

Question **c** —essentially it was alright.

Many students have guessed the right answer for the function G, $G = e^{-x}$.

The curious student may ask the question: how to describe the all solutions? One can do it solving the equation

$$G_x : G_y = y : x\,, \tag{**}$$

$G = e^x f\left(e^x y\right)$, where f is an arbitrary smooth function.

On the other hand some students were trying to find G as a function obeying differential equation (**), but many of them were confused in calculations.

Here I cannot avoid temptation to note the following statement: for an arbitrary 1-form $\omega = a(x,y)dx + b(x,y)dy$ in $\mathbf{E}^2$ there exists a function G such that a form $\sigma = G\omega$ is an exact form[1)]. This statement has the following very nice and non-expected corollary in Physics: let ω be 1-form in the Thermodymnamics first law ($\omega = PdV + dU$, where P is pressure, V-volume, and U-internal energy) then one may choose a function G to be equal to $\frac{1}{T}$, where T is temperature:

$$G\omega = \frac{1}{T}\left(PdV + dU\right) = dS\,,$$

where S is entropy. Using 1-forms one can define temperature!

[1)] This is true only in $\mathbf{E}^2$. E.g. the form $xdy + dz$ in $\mathbf{E}^3$ cannot be transformed to the exact form by multiplication on a function