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Here I will try to caclulate the continual integral for free aprticle using some linear algebra stuff

To calculate the continual integral for free function we deal with exponent of the polynomial

$$F = (x_0 - x_1)^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_2)^2 + \ldots + (x_{N-1} - x_N)^2 \,,$$

where initial and final points are fixed

$$x_0 = a, x_N = b$$

*.

Instead doind standard calculations (see the book of Feynman) we will present here two calcualtions based on the linear algebra.

§1 **Straightforward calculations** The function F is the polynomial of order 2 on the $N-1$ variables $x_1, x_2, \ldots, x_N$ it is equal to

$$F = M_{ik} x^i x^k + L_i x^i + N \,, (i, k = 1, \ldots, N-1)$$

where the $(N-1) \times (N-1)$ matrix M_{ik} is the symmetrical matrix which is equal to

$$M = \begin{pmatrix} 2 & -1 & 0 & 0 & 0\ldots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0\ldots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0\ldots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1\ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2\ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots\ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & 1\ldots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1\ldots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1\ldots & 1 & -1 & 2 \end{pmatrix},$$

the vector L is equal to

$$L = (-2a, 0, 0, 0, 0, \ldots, 0, 0, -2b)$$

and the scalar N is equal to

$$N = a^2 - b^2$$

The orthogonal transformations make the matrix M the diagonal matrix,

We see that

$$\int e^{-cF} dx^1 dx^2 \ldots dx^{N-1} = \int e^{-c\left(M_{ik} x^i x^k + L_i x^i + N\right))} dx^1 dx^2 \ldots dx^{N-1} \,.$$

* One have to calculate the gaussian integral e^{iCF}

Now choose the vector $\mathbf{a}$ such that $2M\mathbf{a} = -\mathbf{L}$, i.e.

$$\mathbf{a} = -\frac{1}{2}M^{-1}\mathbf{L}\,.$$

and perform the translation $x^i = a^i + y^i$: we come to

$$\int e^{-cF} dx^1 dx^2 \dots dx^{N-1} = \int e^{-c\left(M_{ik}x^i x^k + L_i x^i + N\right)} dx^1 dx^2 \dots dx^{N-1} =$$

$$\int e^{-c\left(M_{ik}(y^i+a^i)(y^k+a^k) + L_i (y^i+a^i)^i + N\right)} dx^1 dx^2 \dots dx^{N-1} =$$

$$\int e^{-c\left(M_{ik}y^i y^k + M_{ik}a^i a^k + L_i a^i + N\right)} dx^1 dx^2 \dots dx^{N-1} =$$

$$\left(\frac{\pi}{c}\right)^{N-1} \frac{1}{\sqrt{\det M}} e^{-c\left(M_{ik}a^i a^k + L_i a^i + N\,,\right)} \quad \text{where } \mathbf{a} = -\tfrac{1}{2}M^{-1}\mathbf{L} \text{ and } N = b^2 - a^2\,.$$

§2 **More sophisticated linear algebra calculations**

Consider linear transformation

$$\begin{cases} \xi_1 = x_1 - x_0 = x_1 - a \\ \xi_2 = x_2 - x_1 \\ \xi_3 = x_3 - x_2 \\ \dots \\ \xi_{N-1} = x_{N-1} - x_N \end{cases} \Leftrightarrow \begin{cases} x_1 = \xi_1 + a \\ x_2 = \xi_2 + \xi_1 + a \\ x_3 = \xi_3 + \xi_2 + \xi_1 + a \\ \dots \\ x_{N-1} = \xi_{N-1} + \dots + \xi_1 + a \end{cases}$$

This is not linear transformation, and its "linear" part is not orthogonal transformation, but it is unimodular transformation. In the new coordinates

$$F = \sum_{i,k=1}^{N-1} M_{ik}x^i x^k + \sum_{i=1}^{N-1} L_i x^i + N = \xi_1^2 + \xi_2^2 + \xi_3^2 + \dots + \xi_{N-1}^2 + (b - a - \xi_1 - \xi_2 - \dots - \xi_{N-1})^2\,.$$

Find translation which kills linear terms: $\xi_i = \eta_i + z_i$ such that only sconstants and quadratic terms will survive. One can see that this is

$$\xi_i = \eta_i + \frac{b-a}{N}\,.$$

Check it:

$$F = \sum_{i,k=1}^{N-1} M_{ik}x^i x^k + \sum_{i=1}^{N-1} L_i x^i + N = \xi_1^2 + \xi_2^2 + \xi_3^2 + \dots + \xi_{N-1}^2 + (a - b - \xi_1 - \xi_2 - \dots - \xi_{N-1})^2 =$$

$$\left(\eta_1^2 + \eta_2^2 + + \dots + \eta_{N-1}^2\right) + 2\left(\eta_1 + \dots + \eta_{N-1}\right)\frac{b-a}{N} + \frac{(N-1)(b-a)^2}{N^2} +$$

$$+\left(\frac{a-b}{N}-\eta_1-\eta_2-\ldots-\eta_{N-1}\right)^2=$$

$$=2\left(\eta_1^2+\eta_2^2+\ldots+\eta_{N-1}^2\right)+2\sum_{i<j}\eta_i\eta_j+\frac{(b-a)^2}{N}\,.$$

The linear temrs are cancelled, and we have that

$$F=\tilde{M}_{ik}u^iu^k+\frac{(b-a)^2}{N}\,,$$

where

$$\tilde{M}_{ik}=\delta_{ik}+t_it_k\,,\qquad (t_i=(1,\ldots,1))\,.$$

Now we can calculate the integral:

$$I=\int e^{-cF}dx^1dx^2\ldots dx^{N-1}=\int e^{-c\left(M_{ik}x^ix^k+L_ix^i+N\right)}dx^1dx^2\ldots dx^{N-1}=$$

$$\int e^{-c\left(\sum_{k=1}^{N-1}\xi_k^2+\left(b-a-\sum_{m=1}^{N-1}\xi_m^2\right)^2\right)}\underbrace{\left(\frac{\partial(x_1,\ldots,x_{N-1}}{\partial\xi_1,\ldots,\xi_{N-1})}\right)}_{\text{equals to 1}}d\xi_1d\xi_2\ldots d\xi_{N-1}=$$

$$\int e^{-c\left(2\sum_{k=1}^{N-1}\xi_k^2+\left(b-a-\sum_{m=1}^{N-1}\xi_m^2\right)^2\right)}d\xi_1d\xi_2\ldots d\xi_{N-1}=$$

$$\int e^{-c\left(2\sum_{i,k=1}^{N-1}\eta_i\eta_k+\frac{(b-a)^2}{N}\right)}\underbrace{\left(\frac{\partial(\xi_1,\ldots,\xi_{N-1}}{\partial\eta_1,\ldots,\eta_{N-1})}\right)}_{\text{equals to 1}}d\eta_1d\eta_2\ldots d\eta_{N-1}=$$

$$\int e^{-c\left(N_{ik}\eta^i\eta^k+\frac{(b-a)^2}{N}\right)}d\eta_1d\eta_2\ldots d\eta_{N-1}=$$

$$\left(\frac{\pi}{c}\right)^{\frac{N-1}{2}}\sqrt{\frac{1}{\det(\tilde{M})}}e^{-c\left(\frac{(b-a)^2}{N}\right)}\,.$$

Notice that the matrix $\tilde{M}$ has eigenvector $\mathbf{t}$ with eigenvalue N, and all other $N-2$ eigenvectors wchih are orthogonal to the vector $\mathbf{t}$ with the eigenvalue 1. Hence

$$\det\tilde{M}=N\,,$$

and we have for integral:

$$I=\int e^{-c\left(N_{ik}\eta^i\eta^k+\frac{(b-a)^2}{N}\right)}d\eta_1d\eta_2\ldots d\eta_{N-1}=$$

$$\sqrt{\frac{\pi}{\det(c\tilde{M})}} e^{-c\left(\frac{(b-a)^2}{N}\right)} = \left(\frac{\pi}{c}\right)^{\frac{N-1}{2}} \frac{1}{\sqrt{N}} e^{-c\left(\frac{(b-a)^2}{N}\right)}.$$

§3 Returning to Quantum mechanics

Now return to Quantum Mechanics:

We calculate now the wave function of free particle using continual integral and the classial action.

Let particle starts at the point x_0, t_0 and ends at the point x_1, t_1 Divide $[t_0, t_1]$ on

$$[t_0, t_1, t_2, \ldots, t_{N-1}, t_N]\,, \qquad (t_N \mapsto t_1)$$

and consider the integral

$$\int\int\int \ldots \int e^{\frac{i}{\hbar} S_{\text{free}}(x_0,t_0;x_1,t_1)} e^{\frac{i}{\hbar} S_{\text{free}}(x_1,t_1;x_2,t_2)} e^{\frac{i}{\hbar} S_{\text{free}}(x_2,t_2;x_3,t_3)} \ldots$$

$$e^{\frac{i}{\hbar} S_{\text{free}}(x_{N-2},t_{N-2};x_{N-1},t_{N-1})} e^{\frac{i}{\hbar} S_{\text{free}}(x_{N-1},t_{N-1};x_N,t_N)} dx^1 dx^2 dx^3 \ldots dx^{N-2} dx^{N-1} =$$

$$\int \prod_{i=1}^{N} \exp\left[\frac{i}{\hbar} S_{\text{free}}(x_{i-1}, t_{i-1}; x_i, t_i)\right] \prod_{j=1}^{N-1} dx^j\,,$$

where

$$S_{\text{free}}(x_0, t_0; x_1, t_1) = \frac{m(x_1 - x_0)^2}{2t}$$

is the classical action.

This we have for the integral

$$I = \int \prod_{i=1}^{N} \exp\left[\frac{i}{\hbar} S_{\text{free}}(x_{i-1}, t_{i-1}; x_i, t_i)\right] \prod_{j=1}^{N-1} dx^j = \int \prod_{i=1}^{N} \exp\left[\frac{im}{2\varepsilon\hbar}(x_i - x_{i-1})^2\right] \prod_{j=1}^{N-1} dx^j\,,$$

where

$$\varepsilon = \frac{t_N - t_0}{N+1}\,,$$

We have that according the results of calculations in the previous paragraph

$$G(x_s, t_s, x_f, t_f) = \int \exp\left[\frac{im}{2\varepsilon\hbar} \sum (x_i - x_{i-1})^2\right] dx^1 dx^2 \ldots dx^n =$$

$$\left(\frac{\pi}{2c}\right)^{\frac{N}{2}} e^{-c\left(\frac{(b-a)^2}{N}\right)}, \text{with } c = \tfrac{m}{2\varepsilon\hbar}\,,$$

i.e.

$$G(x_s, t_s, x_f, t_f) = \left(\frac{\pi}{2c}\right)^{\frac{N}{2}} e^{-c\left(\frac{(b-a)^2}{N}\right)} =$$

§4 One linear problem question

Making this calculation we in fact calculated the matrix

$$M=\begin{pmatrix} 2 & -1 & 0 & 0 & 0\dots & 0 & 0 & 0\\ -1 & 2 & -1 & 0 & 0\dots & 0 & 0 & 0\\ 0 & -1 & 2 & -1 & 0\dots & 0 & 0 & 0\\ 0 & 0 & -1 & 2 & -1\dots & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 2\dots & 0 & 0 & 0\\ \dots & \dots & \dots & \dots & \dots\dots & \dots & \dots & \dots\\ 0 & 0 & 0 & 0 & 1\dots & 2 & -1 & 0\\ 0 & 0 & 0 & 0 & 1\dots & -1 & 2 & -1\\ 0 & 0 & 0 & 0 & 1\dots & 1 & -1 & 2 \end{pmatrix},$$

Indeed the quadratic form of this matrix is

$$Q_M=\sum_{i,k=1}^{N-1}M_k x^i x^k=(x_1-a)^2+\sum_{i=2}^{N-1}(x_k-x_{k-1})^2+(b-x_{N-1})^2+2ax_1-a^2+2bx_{N-1}-b^2\,.$$

In the new coordinates

$$\begin{cases}\xi_1=x_1-x_0=x_1\\ \xi_2=x_2-x_1\\ \xi_3=x_3-x_2\\ \dots\\ \xi_{N-1}=x_{N-1}-x_N\end{cases}\Leftrightarrow\begin{cases}x_1=\xi_1\\ x_2=\xi_2+\xi_1\\ x_3=\xi_3+\xi_2+\xi_1\\ \dots\\ x_{N-1}=\xi_{N-1}+\dots+\xi_1\end{cases}$$

(this is linear unimodular transformation, and it is different from the trasforation considered in §2.) we have

$$Q_M=\sum_{i,k=1}^{N-1}M_k x^i x^k=(x_1-a)^2+\sum_{i=2}^{N-1}(x_k-x_{k-1})^2+(b-x_{N-1})^2+2ax_1-a^2+2bx_{N-1}-b^2=$$

$$(\xi_1-a)^2+\sum_{i=2}^{N-1}\xi_k^2+(b-\xi_1-\dots-\xi_{N_1})^2+2a\xi_1-a^2+2b(\xi_1+\dots+\xi_{N-1})-b^2=$$

$$\sum_{i=1}^{N-1}\xi_i^2+\sum_{i,k=1}^{N-1}\xi_i\xi_k=\xi^i\left(\delta_{ik}+t_it_k\right)\xi^k\,,$$

with

$$\mathbf{t}=(1,1,\dots,1)$$

We have

$$M = P^* M_\xi P$$

In particular since the transformation is unimodular we can calculate the the characteristic polynomial of M. It is equal to

$$P_M(\lambda) = \det(M - \lambda) = \det\left((1-\lambda)\,\delta_{ik} + t_i t_k\right) \,.$$

To calculate the determinant of this matrix notice the vector $\mathbf{t}$ is the eigenvector with the eigenvalue $(1-\lambda+(N-1) = \lambda+N$, and all the vectors which are orthogonal to the vector $\mathbf{t}$ has eigenvalue $1-\lambda$. Hence

$$P_M(\lambda) = \det(M - \lambda) = \det\left((1-\lambda)\,\delta_{ik} + t_i t_k\right) = (\lambda + N)(1-\lambda)^{N-2}$$