# Riemannian Geometry

it is a draft of Lecture Notes of H.M. Khudaverdian.
Manchester, 19 March, 2020

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# 1 Riemannian manifolds

## 1.1 Manifolds. Tensors. (Recollection)

### 1.1.1 Manifolds

I recall briefly basics of manifolds and tensor fields on manifolds.

An $n$-dimensional manifold $M = M^n$ is a space[1]

such that in a vicinity of an arbitrary point one can consider local coordinates $\{x^1, \dots, x^n\}$. (We say that in a vicinity of this point a manifold $M$ is covered by local coordinates $\{x^1, \dots, x^n\}$). One can consider different local coordinates. If coordinates $\{x^1, \dots, x^n\}$ and $\{x^{1'}, \dots, x^{n'}\}$ both are defined in a vicinity of the given point then they are related by *bijective transition functions* which are defined on domains in $\mathbf{R}^n$ and taking values also in $\mathbf{R}^n$:

$$\begin{cases} x^{1'} = x^{1'}(x^1, \dots, x^n) \\ x^{2'} = x^{2'}(x^1, \dots, x^n) \\ \dots \\ x^{n-1'} = x^{n-1'}(x^1, \dots, x^n) \\ x^{n'} = x^{n'}(x^1, \dots, x^n) \end{cases} \tag{1.1}$$

We say that $n$-dimensional manifold is *differentiable* or *smooth* if all transition functions are diffeomorphisms, i.e. they are smooth. Invertability implies that Jacobian matrix is non-degenerate:

$$\det \begin{pmatrix} \frac{\partial x^{1'}}{\partial x^1} & \frac{\partial x^{1'}}{\partial x^2} \dots & \frac{\partial x^{1'}}{\partial x^n} \\ \frac{\partial x^{2'}}{\partial x^1} & \frac{\partial x^{2'}}{\partial x^2} \dots & \frac{\partial x^{2'}}{\partial x^n} \\ \dots & & \\ \frac{\partial x^{n'}}{\partial x^1} & \frac{\partial x^{n'}}{\partial x^2} \dots & \frac{\partial x^{n'}}{\partial x^n} \end{pmatrix} \neq 0 \,. \tag{1.2}$$

(If bijective function $x^{i'} = x^{i'}(x^i)$ is smooth function, and its inverse, the transition function $x^i = x^i(x^i)$ is also smooth function, then matrices $||\frac{\partial x^{i'}}{\partial x^i}||$ and $||\frac{\partial x^i}{\partial x^{i'}}||$ are both well defined, hence condition (1.2) is obeyed.

[1] A space $M$ is a topological space, i.e. it is covered by a collection $\mathcal{F}$ of sets, which are called *open* sets. This collection obeys the following axioms

i) the union of an arbitrary set of open sets is an open set

ii) the intersection of finite number of open sets is an open set

iii) the whole space $M$ and the empty set $\emptyset$ are open sets

**Example**

open domain in $\mathbf{E}^n$

A good example of manifold is an open domain $D$ in $n$-dimensional vector space $\mathbf{R}^n$. Cartesian coordinates on $\mathbf{R}^n$ define global coordinates on $D$. On the other hand one can consider an arbitrary local coordinates in different domains in $\mathbf{R}^n$. E.g. one can consider polar coordinates $\{r, \varphi\}$ in a domain $D = \{x, y\colon\, y > 0\}$ of $\mathbf{R}^2$ defined by standard formulae:

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}, \tag{1.3}$$

$$\det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{pmatrix} = \det\begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix} = r \tag{1.4}$$

or one can consider spherical coordinates $\{r, \theta, \varphi\}$ in a domain $D = \{x, y, z\colon\, x > 0, y > 0, z > 0\}$ of $\mathbf{R}^3$ (or in other domain of $\mathbf{R}^3$) defined by standard formulae

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases},$$

,

$$\det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{pmatrix} = \det\begin{pmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} = r^2\sin\theta \tag{1.5}$$

Choosing domain where polar (spherical) coordinates are well-defined we have to be aware that coordinates have to be well-defined and transition functions (1.1) have to obey condition (1.2), i.e. they have to be diffeomorphisms. E.g. for domain $D$ in example (1.3) Jacobian (1.4) does not vanish if and only if $r > 0$ in $D$.

Consider another examples of manifolds, and local coordinates on manifolds.

**Example**

*Circle $S^1$ in* $\mathbf{E}^2$

Consider circle $x^2 + y^2 = R^2$ of radius $R$ in $\mathbf{E}^2$.

One can consider on the circle different local coordinates

i) *polar coordinate* $\varphi$:

$$\begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \end{cases} \quad , \qquad 0 < \varphi < 2\pi$$

(this coordinate is defined on all the circle except a point $(R, 0)$),

ii) *another polar coordinate* $\varphi'$:

$$\begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \end{cases} \quad , \qquad -\pi < \varphi < \pi\,,$$

this coordinate is defined on all the circle except a point $(-R, 0)$,

iii) stereographic coordinate $t$ with respect to north pole of the circle

$$\begin{cases} x = \frac{2R^2 t}{t^2+R^2} \\ y = R\frac{t^2-R^2}{t^2+R^2} \end{cases} \quad , \quad t = \frac{Rx}{R-y}\,, \tag{1.6}$$

this coordinate is defined at all the circle except the north pole,

iiii) stereographic coordinate $t'$ with respect to south pole of the circle

$$\begin{cases} x = \frac{2R^2 t'}{t'^2+R^2} \\ z = R\frac{R^2-t'^2}{t'^2+R^2} \end{cases} \quad , \quad t' = \frac{Rx}{R+y}\,,$$

this coordinate is defined at all the points except the south pole.

We considered four different local coordinates on the circle $S^1$. Write down some transition functions (1.1) between these coordinates

- polar coordinate $\varphi$ coincide with polar coordinate $\varphi'$ in the domain $x^2 + y^2 > 0$, and in the domain $x^2 + y^2 < 0$ $\varphi' = \varphi - 2\pi$.
- Transition function from polar coordinate $\varphi$ to stereographic coordinates $t$ is $t = \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)$,
- transition function from stereographic coordinate $t$ to stereographic coordinate $t'$ is

$$t' = \frac{R^2}{t}\,,$$

(see Homework 0.)

**Example**

*Sphere $S^2$ in $\mathbf{E}^3$*

Consider sphere $x^2+y^2+z^2=R^2$ of radius $a$ in $\mathbf{E}^3$.
One can consider on the sphere different local coordinates
i) *spherical coordinates on domain of sphere $\theta,\varphi$:*

$$\begin{cases} x = a\sin\theta\cos\varphi \\ y = a\sin\theta\sin\varphi \\ z = a\cos\theta \end{cases}, \quad 0<\theta<\pi\,, -\pi<\varphi<\pi$$

ii) stereographic coordinates $u,v$ with respect to north pole of the sphere

$$\begin{cases} x = \frac{2a^2u}{a^2+u^2+v^2} \\ y = \frac{2a^2v}{a^2+u^2+v^2} \\ z = a\frac{u^2+v^2-a^2}{a^2+u^2+v^2} \end{cases}, \quad \frac{x}{u}=\frac{y}{v}=\frac{a-z}{a}, \quad \begin{cases} u = \frac{ax}{a-z} \\ v = \frac{ay}{a-z} \end{cases}.$$

iii) stereographic coordinates $u',v'$ with respect to south pole of the sphere

$$\begin{cases} x = \frac{2a^2u'}{a^2+u'^2+v'^2} \\ y = \frac{2a^2v}{a^2+u'^2+v'^2} \\ z = a\frac{a^2-u'^2-v'^2}{a^2+u^2+v^2} \end{cases}, \quad \frac{x}{u'}=\frac{y}{v'}=\frac{a+z}{a}, \quad \begin{cases} u' = \frac{ax}{a+z} \\ v' = \frac{ay}{a+z} \end{cases}.$$

(see also Homework 0)

Spherical coordinates are defined elsewhere except poles and the meridians $y=0, x\leq 0$.

Stereographical coordinates $(u,v)$ are defined elsewhere except north pole;
stereographic coordinates $(u',v')$ are defined elsewhere except south pole.

One can consider transition function between these different coordinates.E.g. transition functions from spherical coordinates i) to stereographic coordinates $(u,v)$ are

$$\begin{cases} u = \frac{ax}{a-z} = \frac{a\sin\theta\cos\varphi}{1-\cos\theta} = a\mathrm{cotan}\,\frac{\theta}{2}\cos\varphi \\ v = \frac{ay}{a-z} = \frac{a\sin\theta\sin\varphi}{1-\cos\theta} = a\mathrm{cotan}\,\frac{\theta}{2}\sin\varphi \end{cases},$$

and transition function from stereographic coordinates $u,v$ to stereographic coordinates $(u',v')$ are

$$\begin{cases} u' = \frac{a^2u}{u^2+v^2} \\ v' = \frac{a^2v}{u^2+v^2} \end{cases},$$

(see Homework 0.)

**Remark**

† One very important property of stereographic projetion which we do not use in this course but it is too beautiful not to mention it: under stereographic projection all points of the circle of radius $R = 1$ with rational coordinates $x$ and $y$ and only these points transform to rational points on line. Thus we come to Pythagorean triples $a^2 + b^2 = c^2$. The same is for unit sphere: the stereographic projection establishes one-one correspondence between points on the unit sphere with rational coordinates and rational points on the plane.

### 1.1.2 Tensors on Manifold

*tangent vector and tangent vector space*

Tangent vector at the given point can be considered as a derivation of function at this point.

For an arbitrary (smooth) function $f$ defined in a vicinity of a given point $\mathbf{p}$ a tangent vector $\mathbf{A}(x) = A^i(x)\frac{\partial}{\partial x^i}$ defines the directional derivative of this function

$$\mathbf{A}: \quad f \mapsto \partial_{\mathbf{A}} f\big|_{\mathbf{p}} = A^i(x)\frac{\partial f}{\partial x^i}\big|_{\mathbf{p}}\,.$$

Using the chain rule one can see that under changing of coordinates it transforms as follows:

$$\mathbf{A} = A^i(x)\frac{\partial}{\partial x^i} = A^i(x)\frac{\partial x^{i'}(x)}{\partial x^i}\frac{\partial}{\partial x^{i'}} = A^{i'}(x'(x))\frac{\partial}{\partial x^{i'}}\,,$$

i.e.

$$A^{i'}(x') = \frac{\partial x^{i'}}{\partial x^i}A^i(x)\,. \tag{1.7}$$

This leads as to the following equivalent definition of the tangent vector.

**Definition** Let $M = M^n$ be $n$-dimensional manifold, and $\mathbf{p}$ the point on it. To define a vector $\mathbf{A}$ tangent to the manifold at the point $\mathbf{p}$ we assign to an arbitrary given local coordinates $\{x^i\}$ the array $\{A^i\}$ $(i = 1, \ldots, n)$ of numbers (components) such that under changing of local coordinates this array transforms according to equation (1.7):

| `coordinates` | | `components of vector` | |
|---|---|---|---|
| $\{x^i\}$ | $\to$ | $\{A^i\}$ | such that $A^{i'} = \frac{\partial x^{i'}(x)}{\partial x^i}\big|_{\mathbf{p}} A^i\,.$ (1.8) |
| $\{x^{i'}\}$ | $\to$ | $\{A^{i'}\}$ | |

**Definition** Tangent vector space $T_{\mathbf{p}}M$ at the point $\mathbf{p}$ is the space of vectors tangent to the manifold at the point $M$.

1 *-form (covector) in a given point*

We defined above vectors of tangent space $T_{\mathbf{p}}M$. Now we consider dual obejcts: we consider cotangent space $T^*_{\mathbf{p}}M$ (for every point $\mathbf{p}$ on manifold $M$)— space of linear functions on tangent vectors, i.e. space of 1-forms which sometimes are called *covectors.*:

Linear function, 1-form $\omega = \omega_i dx^i$ is a function on tangent vectors:

$$T_{\mathbf{p}}M \ni \mathbf{A} = A^i \frac{\partial}{\partial x^i}\,, \omega\left(\mathbf{A}\right) = \omega_m dx^m \left(A^i \frac{\partial}{\partial x^i}\right) = \omega_m A^i \underbrace{dx^i \left(\frac{\partial}{\partial x^m}\right)}_{\delta_i^m} = \omega_m A^m\,.$$

If we consider new coordinates $x^{i'} = x^{i'}(x)$, then

$$\omega = \omega_i dx^i = \omega_i \left(\frac{\partial x^i}{\partial x^{i'}} dx^{i'}\right) = \underbrace{\omega_i \frac{\partial x^i}{\partial x^{i'}}}_{\omega_{i'}} dx^{i'}$$

i.e., nne-form (covector) $\omega = \omega_i(x) dx^i$ transforms as follows

$$\omega_{m'}(x') = \frac{\partial x^m(x')}{\partial x^{m'}} \omega_m(x)\,. \tag{1.9}$$

Differential form sometimes is called *covector*.

In the same way as for vectors we may give definition of covectors in the following way:

**Definition** Let $M = M^n$ be $n$-dimensional manifold, and $\mathbf{p}$ the point on it. To define a *covector* $\mathbf{A}$ at the point $\mathbf{p}$, (the linear function on tangent vectors at $\mathbf{p}$) we assign to an arbitrary given local coordinates $\{x^i\}$ the collection $\{\omega_i\}$ $(i = 1, \ldots, n)$ of numbers (components) such that under changing of local coordinates this collection transforms according to equation (1.9):

$$\begin{array}{ccc} \texttt{coordinates} & & \texttt{components of covector} \\ \{x^i\} & \to & \{\omega_i\} \\ & & \\ \{x^{i'}\} & \to & \{\omega_{i'}\} \end{array} \quad \text{such that}\, \omega_{i'} = \frac{\partial x^i(x')}{\partial x^{i'}}\Big|_{\mathbf{p}} \omega_i\,. \tag{1.10}$$

**Remark** Notice the difference between formulae (1.7) and (1.9). In formulae (1.7), (1.8) transformation is performed by matrix of derivatives

$\partial x^{i'} \partial x^i$ from coordinates $x^i$ to the new coordinates $x^{i'}$, and in formula (1.9) transformation is performed by the *inverse* matrix, matrix of derivatives $\partial x^i \partial x^{i'}$ from new coordinates $x^{i'}$ to the initial coordinates $x^i$.

*Tensors*:

**Definition** Consider geometrical object such that in arbitrary local coordinates $(x^i)$ it is given by components

$$Q = \left\{ Q^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q} \right\} \quad , i_1, \dots, i_p; j_1, \dots, j_q = 1, 2, \dots, n \,,$$

and under changing of coordinates this object is transformed in the following way:

$$Q^{i'_1 i'_2 \dots i'_p}_{j'_1 j'_2 \dots j'_q}(x') = \frac{\partial x^{i'_1}}{\partial x^{i_1}} \frac{\partial x^{i'_2}}{\partial x^{i_2}} \dots \frac{\partial x^{i'_p}}{\partial x^{i_p}} \frac{\partial x^{j_1}}{\partial x^{j'_1}} \frac{\partial x^{j_2}}{\partial x^{j'_2}} \dots \frac{\partial x^{j_q}}{\partial x^{j'_q}} Q^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q}(x) \,. \tag{1.11}$$

We say that this is *p-times contravariant, q-times covariant tensor of valence* $\begin{pmatrix} p \\ q \end{pmatrix}$*, or shorter, tensors of the type* $\begin{pmatrix} p \\ q \end{pmatrix}$.

`Caution: this tensor possess` $n^{p+q}$ `components.`

Sometimes it is useful to view $\begin{pmatrix} p \\ q \end{pmatrix}$-tensor as

$$Q = Q^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q} \frac{\partial}{\partial x^{i_1}} \otimes \frac{\partial}{\partial x^{i_2}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_p}} dx^{j_1} \otimes dx^{j_2} \otimes \dots \otimes dx^{j_q}$$

(Compare with definition of vector: $\mathbf{A} = A^i \frac{\partial}{\partial x^i}$ and covector (1-form) $\omega = \omega_i dx^i$).

**Examples**

Note that vector field (1.7) is nothing but tensor field of valency $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and 1-form (1.9) is nothing but tensor field of valency $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

One can consider *contravariant* tensors of the rank $p$

$$T = T^{i_1 i_2 \dots i_p}(x) \frac{\partial}{\partial x^{i_1}} \otimes \frac{\partial}{\partial x^{i_2}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_k}}$$

with components $\{T^{i_1 i_2 \dots i_k}\}(x)$. Under changing of coordinates $(x^1, \dots, x^n) \to (x^{1'}, \dots, x^{n'})$ (see (1.1)) they transform as follows:

$$T^{i'_1 i'_2 \dots i'_p}(x') = \frac{\partial x^{i'_1}}{\partial x^{i_1}} \frac{\partial x^{i'_2}}{\partial x^{i_2}} \dots \frac{\partial x^{i'_p}}{\partial x^{i_p}} T^{i_1 i_2 \dots i_p}(x) \,. \tag{1.12}$$

One can consider *covariant* tensors of the rank $q$

$$S = S_{j_1 j_2 \ldots j_q} dx^{j_1} \otimes dx^{j_2} \otimes \ldots dx^{j_q}$$

with components $\{S_{j_1 j_2 \ldots j_q}\}$. Under changing of coordinates $(x^1, \ldots, x^n) \to (x^{1'}, \ldots, x^{n'})$ they transform as follows:

$$S_{j'_1 j'_2 \ldots j'_q}(x') = \frac{\partial x^{i_1}}{\partial x^{i'_1}} \frac{\partial x^{i_2}}{\partial x^{i'_2}} \ldots \frac{\partial x^{i_p}}{\partial x^{i'_p}} S_{j_1 j_2 \ldots j_q}(x) \,.$$

E.g. if $S_{ik}$ is a covariant tensor of rank 2 then

$$S_{i'k'}(x') = \frac{\partial x^i(x')}{\partial x^{i'}} \frac{\partial x^k(x')}{\partial x^{k'}} S_{ik}(x) \,. \tag{1.13}$$

[**Example**] *linear operator and bilinear form*
Consider in Linear space $V$ the following two geometrical objects

- Bilinear form $B(\mathbf{x}, \mathbf{y})$

$$B(\lambda \mathbf{x} + \mu \mathbf{x}', \mathbf{y}) = \lambda B(\mathbf{x}, \mathbf{y}) + \mu B(\mathbf{x}', \mathbf{y}) \,.$$

  In an arbitrary basis $\{\mathbf{e}_i\}$ the bilinear form is presented by the matrrix

$$B_{ik} = B(e_i, \mathbf{e}_k)$$

- Linear operator
  $A \colon V \to V$,

$$A(\lambda \mathbf{x} + \mu \mathbf{x}') = \lambda A(\mathbf{x}) + \mu A(\mathbf{x}') \,.$$

  In an arbitrary basis $\{\mathbf{e}_i\}$ the linear operator $A$ is presented by the matrrix

$$P_i^k P(\mathbf{e}_i) = P_i^k \mathbf{e}_k$$

  These both objexts, bilinear form and linear operator, are both presented in an arbitrary basis by $2 \times 2$ marices. However they are `different` objects!

Consider an arbitrary linear changing of coordinates from coordinates $\{x^i\}$ to new linear coordinates $\{x^{i'}\}$:

$$x^{i'} = Q^{i'}_k x^k\,, \text{respectively} x^i = P^i_{i'} x^{i'}\,, \tag{1.14}$$

(matrix $Q^{i'}_i$ is inverse to the matrix $P^i_{i'}$: $Q^{i'}_k P^k_{j'} = \delta^{i'}_{j'}$).

Bilinear form $B$ transforms as

$$B_{i'k'} = \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^k}{\partial x^{k'}} = P^i_{i'} P^k_{k'} B_{ik}$$

Bilinear form is the covariant tensor of the valency $\binom{0}{2}$. Under the same linear changing of coordinates (1.14) linear operator $A$ transforms as

$$A^{i'}_{k'} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^k}{\partial x^{k'}} A^i_k = Q^{i'}_i P^k_{k'} A^i_k\,.$$

Linear operator is the tensor of the of the valency $\binom{1}{1}$.

**Remark** One can calulate determinant of the linear operator, however the determinant of bilinear form is not well-defined. In its turn one can consider symmetric bilinear form, which is represented by symmetrical matrix, however one cannot consider symmetric linear operator in linear space: it is ill-defined notion

**Remark** *Einstein summation rules*

In our lectures we always use so called *Einstein summation convention.* it implies that when an index occurs twice in the same expression in upper and in lower postitions, then the expression is implicitly summed over all possible values for that index. Sometimes it is called dummy indices summation rule.

Using Einstein summation rules we avoid to write bulky expressions. Later we will see that these notations are really very effective. E.g. equation (1.7) in 'standard' notations will appear as

$$\text{for every } i' = 1, \dots, n \;\; A^{i'}(x') = \sum_{i=1}^{n} \frac{\partial x^{i'}}{\partial x^i} A^i(x)\,.$$

## 1.2 Riemannian manifold

### 1.2.1 Riemannian manifold— manifold equipped with Riemannian metric

**Definition** The Riemannian manifold $(M, G)$ is a manifold equipped with a Riemannian metric.

The Riemannian metric $G$ on the manifold $M$ defines the length of the tangent vectors and the length of the curves.

**Definition** Riemannian metric $G$ on n-dimensional manifold $M^n$ defines for every point $\mathbf{p} \in M$ the scalar product of tangent vectors in the tangent space $T_{\mathbf{p}}M$ smoothly depending on the point $\mathbf{p}$.

It means that in every coordinate system $(x^1, \dots, x^n)$ a metric $G = g_{ik} dx^i dx^k$ is defined by a matrix valued smooth function $g_{ik}(x)$ $(i = 1, \dots, n; k = 1, \dots n)$ such that for any two vectors

$$\mathbf{A} = A^i(x)\frac{\partial}{\partial x^i},\ \mathbf{B} = B^i(x)\frac{\partial}{\partial x^i},$$

tangent to the manifold $M$ at the point $\mathbf{p}$ with coordinates $x = (x^1, x^2, \dots, x^n)$ $(\mathbf{A}, \mathbf{B} \in T_{\mathbf{p}}M)$ the scalar product is equal to:

$$\langle \mathbf{A}, \mathbf{B} \rangle_G \big|_{\mathbf{p}} = G(\mathbf{A}, \mathbf{B})\big|_{\mathbf{p}} = A^i(x) g_{ik}(x) B^k(x) =$$

$$\begin{pmatrix} A^1 \dots A^n \end{pmatrix} \begin{pmatrix} g_{11}(x) & \dots & g_{1n}(x) \\ \dots & \dots & \dots \\ g_{n1}(x) & \dots & g_{nn}(x) \end{pmatrix} \begin{pmatrix} B^1 \\ \cdot \\ \cdot \\ \cdot \\ B^n \end{pmatrix} \tag{1.15}$$

where

- $G(\mathbf{A}, \mathbf{B}) = G(\mathbf{B}, \mathbf{A})$, i.e. $g_{ik}(x) = g_{ki}(x)$ (symmetricity condition)
- $G(\mathbf{A}, \mathbf{A}) > 0$ if $\mathbf{A} \neq \mathbf{0}$, i.e.

  $g_{ik}(x) u^i u^k \geq 0$, $g_{ik}(x) u^i u^k = 0$ iff $u^1 = \dots = u^n = 0$ (positive-definiteness)
- $G(\mathbf{A}, \mathbf{B})\big|_{\mathbf{p}=x}$, i.e. $g_{ik}(x)$ are smooth functions.

The matrix $||g_{ik}||$ of components of the metric $G$ we also sometimes denote by $G$.

Now we establish rule of transformation for entries of matrix $g_{ik}(x)$, of metric $G$.

Notice that an arbitrary matrix entry $g_{ik}$ is nothing but scalar product of vectors $\partial_i, \partial_k$ at the given point:

$$g_{ik}(x) = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k} \right\rangle , \quad \text{in coordinates } (x^1, \dots, x^n) \tag{1.16}$$

Use this formula for establishing rule of transformations of $g_{ik}(x)$. In the new coordinates $x^{i'} = (x^{1'}, \dots, x^{n'})$ according this formula we have that

$$g_{i'k'}(x') = \left\langle \frac{\partial}{\partial x^{i'}}, \frac{\partial}{\partial x^{k'}} \right\rangle , \quad \text{in coordinates } (x^1, \dots, x^n) \,.$$

Now using chain rule, linearity of scalar product and formula (1.16) we see that

$$g_{i'k'}(x') = \left\langle \frac{\partial}{\partial x^{i'}}, \frac{\partial}{\partial x^{k'}} \right\rangle = \left\langle \frac{\partial x^i}{\partial x^{i'}} \frac{\partial}{\partial x^i}, \frac{\partial x^k}{\partial x^{k'}} \frac{\partial}{\partial x^k} \right\rangle$$
$$= \frac{\partial x^i}{\partial x^{i'}} \underbrace{\left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k} \right\rangle}_{g_{ik}(x)} \frac{\partial x^k}{\partial x^{k'}} = \frac{\partial x^i}{\partial x^{i'}} g_{ik}(x) \frac{\partial x^k}{\partial x^{k'}} \tag{1.17}$$

This transformation law implies that $g_{ik}$ entries of matrix $||g_{ik}||$ are components of *covariant tensor field $G = g_{ik} dx^i dx^k$ of rank 2*(see equation (1.13)).

*One can say that Riemannian metric is defined by symmetric covariant smooth tensor field $G$ of the rank 2 which defines scalar product in the tangent spaces $T_{\mathbf{p}}M$ smoothly depending on the point $\mathbf{p}$. Components of tensor field $G$ in coordinate system are functions $g_{ik}(x)$:*

$$G = g_{ik}(x) dx^i \otimes dx^k \,,$$
$$\langle \mathbf{A}, \mathbf{B} \rangle = G(\mathbf{A}, \mathbf{B}) = g_{ik}(x) dx^i \otimes dx^k \, (\mathbf{A}, \mathbf{B}) \,. \tag{1.18}$$

In practice it is more convenient to perform transformation of metric $G$ under changing of coordinates in the following way:

$$G = g_{ik} dx^i \otimes dx^k = g_{ik} \left( \frac{\partial x^i}{\partial x^{i'}} dx^{i'} \right) \otimes \left( \frac{\partial x^k}{\partial x^{k'}} dx^{k'} \right) =$$

$$\frac{\partial x^i}{\partial x^{i'}} g_{ik} \frac{\partial x^k}{\partial x^{k'}} dx^{i'} \otimes dx^{k'} = g_{i'k'} dx^{i'} \otimes dx^{k'}\,, \text{ hence } g_{i'k'} = \frac{\partial x^i}{\partial x^{i'}} g_{ik} \frac{\partial x^k}{\partial x^{k'}}\,. \quad (1.19)$$

We come to transfomation rule (1.17).

Later by some abuse of notations we sometimes omit the sign of tensor product and write a metric just as

$$G = g_{ik}(x) dx^i dx^k\,.$$

### 1.2.2 Examples

- $\mathbf{R}^n$ with canonical coordinates $\{x^i\}$ and with metric

  $$G = (dx^1)^2 + (dx^2)^2 + \cdots + (dx^n)^2$$

  $G = ||g_{ik}|| = \mathrm{diag}\ [1, 1, \ldots, 1]$

  Recall that this is a basis example of $n$-dimensional Euclidean space $\mathbf{E}^n$, where scalar product is defined by the formula:

  $$G(\mathbf{X}, \mathbf{Y}) = \langle \mathbf{X}, \mathbf{Y} \rangle = g_{ik} X^i Y^k = X^1 Y^1 + X^2 Y^2 + \cdots + X^n Y^n\,.$$

  In the general case if $G = ||g_{ik}||$ is an arbitrary symmetric positive-definite metric then $G(\mathbf{X}, \mathbf{Y}) = \langle \mathbf{X}, \mathbf{Y} \rangle = g_{ik} X^i Y^k$. One can show that there exists a new basis $\{\mathbf{e}_i\}$ such that in this basis $G(\mathbf{e}_i, \mathbf{e}_k) = \delta_{ik}$. This basis is called orthonormal basis. (See the Lecture notes in Geometry)

  Scalar product in vector space defines the *same* scalar product at all the points. In general case for Riemannian manifold scalar product depends on a point. In Riemannian manifold we consider arbitrary transformations from local coordinates to new local coordinates.

- Euclidean space $\mathbf{E}^2$ with polar coordinates in the domain $y > 0$ ($x = r\cos\varphi, y = r\sin\varphi$):

  $dx = \cos\varphi dr - r\sin\varphi d\varphi, dy = \sin\varphi dr + r\cos\varphi d\varphi$. In new coordinates the Riemannian metric $G = dx^2 + dy^2$ will have the following appearance:

  $G = (dx)^2 + (dy)^2 = (\cos\varphi dr - r\sin\varphi d\varphi)^2 + (\sin\varphi dr + r\cos\varphi d\varphi)^2 = dr^2 + r^2 (d\varphi)^2$

  We see that for matrix $G = ||g_{ik}||$

  $$\underbrace{G = \begin{pmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{in Cartesian coordinates}}, \qquad \underbrace{G = \begin{pmatrix} g_{rr} & g_{r\varphi} \\ g_{\varphi r} & g_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}}_{\text{in polar coordinates}}$$

- `Circle (without a point)`

  Interval $[0, 2\pi)$ in the line $0 \leq x < 2\pi$ with Riemannian metric

$$G = a^2 dx^2 \tag{1.20}$$

  Renaming $x \mapsto \varphi$ we come to habitual formula for metric for circle without a point of the radius $a$: $x^2+y^2 = a^2$ embedded in the Euclidean space $\mathbf{E}^2$:

$$G = a^2 d\varphi^2 \qquad \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \end{cases} , 0 < \varphi < 2\pi, \quad \text{or } -\pi < \varphi < \pi\,. \tag{1.21}$$

  Rewrite this metric in stereographic coordinate $t$:

$$G = a^2 d\varphi^2 = 4a^4 dt^2 (a^2 + t^2)^2\,, \quad \text{where } t = \frac{ax}{a-y} = \frac{a^2\cos\varphi}{a - a\sin\varphi} = \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)\,. \tag{1.22}$$

  (See (1.6) and Homeworks 0 and 2.)

- `Cylinder surface (without a line)`

  Consider domain in $\mathbf{R}^2$, $D = \{(x, y)\colon\,,\ 0 \leq x < 2\pi$ with Riemannian metric

$$G = a^2 dx^2 + dy^2 \tag{1.23}$$

  We see that renaming variables $x \mapsto \varphi$, $y \mapsto h$ we come to habitual, familiar formulae for metric in standard polar coordinates for cylinder surface of the radius $a$ without a line embedded in the Euclidean space $\mathbf{E}^3$:

$$G = a^2 d\varphi^2 + dh^2 \qquad \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases} , 0 < \varphi < 2\pi, -\infty < h < \infty \tag{1.24}$$

  (Coordinate $\varphi$ is well defined for $-\pi < \varphi < \pi$ also.)

- `Sphere without...`

  Consider domain in $\mathbf{R}^2$, $0 < x < 2\pi$, $0 < y < \pi$ with metric $G = dy^2 + \sin^2 y dx^2$ We see that renaming variables $x \mapsto \varphi$, $y \mapsto h$ we come to habitual, familiar formulae for metric in standard spherical coordinates for sphere $x^2+y^2+z^2=a^2$ of the radius $a$ embedded in the Euclidean space $\mathbf{E}^3$:

$$G = a^2 d\theta^2 + a^2\sin^2\theta d\varphi^2 \qquad \begin{cases} x = a\sin\theta\cos\varphi \\ y = a\sin\theta\sin\varphi \\ z = a\cos\theta \end{cases} \quad , \quad 0<\theta<\pi\,,\, 0<\varphi<2\pi\,. \tag{1.25}$$

(See examples also in the Homeworks.)

If we omit the condition of positive-definiteness for Riemannian metric we come to so called *Pseudoriemannian metric.* Manifold equipped with pseudoriemannian metric is called pseudoriemannian manifold. Pseudoriemannian manifolds appear in applications in the special and general relativity theory.

In pseudoriemanninan space scalar product $(\mathbf{X},\mathbf{X})$ may take an arbitrary real values: it can be positive, negative, it can be equal to zero. Vectors $\mathbf{X}$ such that $(\mathbf{X},\mathbf{X})=0$ are called null-vectors.

For example consider 4-dimensional linear space $\mathbf{R}^4$ with pseudometric

$$G = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2\,.$$

For an arbitrary vector $\mathbf{X}=(a^0,a^1,a^2,a^3)$ scalar product $(\mathbf{X},\mathbf{X})$ is positive if $(a^0)^2 > (a_1)^2+(a_2)^2+(a_3)^2$, and it is negative if $(a^0)^2 < (a_1)^2+(a_2)^2+(a_3)^2$, and $\mathbf{X}$ is null-vector if $(a^0)^2 = (a_1)^2+(a_2)^2+(a_3)^2$. It is so called Minkovski space. The coordinate $x^0$ plays a role of the time: $x^0=ct$, where $c$ is the value of the speed of the light. Vectors $\mathbf{X}$ such that $(\mathbf{X},\mathbf{X})>0$ are called time-like vectors and they called space-like vectors if $(\mathbf{X},\mathbf{X})<0$.

### 1.2.3 Scalar product $\rightarrow$ Length of tangent vectors and angle between them

The Riemannian metric defines scalar product of tangent vectors attached at the given point. Hence it defines the length of tangent vectors and angle between them. If $\mathbf{X}=X^m\frac{\partial}{\partial x^m}$, $\mathbf{Y}=Y^m\frac{\partial}{\partial x^m}$ are two tangent vectors at the given point $\mathbf{p}$ of Riemannian manifold with coordinates $x^1,\dots,x^n$, then we have that lengths of these vectors equal to

$$|\mathbf{X}| = \sqrt{\langle\mathbf{X},\mathbf{X}\rangle} = \sqrt{g_{ik}(x)X^iX^k}, \quad |\mathbf{Y}| = \sqrt{\langle\mathbf{Y},\mathbf{Y}\rangle} = \sqrt{g_{ik}(x)Y^iY^k}, \tag{1.26}$$

and an 'angle' $\theta$ between these vectors is defined by the relation

$$\cos\theta = \frac{\langle\mathbf{X},\mathbf{Y}\rangle}{|\mathbf{X}|\cdot|\mathbf{Y}|} = \frac{g_{ik}X^iY^k}{\sqrt{g_{ik}(x)X^iX^k}\sqrt{g_{ik}(x)Y^iY^k}} \tag{1.27}$$

**Remark** We say 'angle' but we calculate just cosinus of angle.

**Example** Let $M$ be 3-dimensional Riemannian manifold, and $\mathbf{p} \in M$ a point in it. Suppose that the manifold $M$ is equipped with local coordinates $x, y, z$ in a vicinity of this point, and the expression of Riemannian metric in these local coordinates is

$$G = \frac{dx^2 + dy^2 + dz^2}{(1 + x^2 + y^2 + z^2)^2} \,. \tag{1.28}$$

Consider the vectors $\mathbf{X} = a\partial_x + b\partial_y + c\partial_z$ and $\mathbf{Y} = p\partial_x + q\partial_y + r\partial_z$, attached at the point $\mathbf{p}$, with coordinates $x = 2, y = 2, z = 1$. Find the lengths of vectors $\mathbf{X}$ and $\mathbf{Y}$ and find cosinus of the angle between these vectors.

We see that matrix of Riemannian metric is

$$||g_{ik(x)}|| = \begin{pmatrix} \frac{1}{(1+x^2+y^2+z^2)^2} & 0 & 0 \\ 0 & \frac{1}{(1+x^2+y^2+z^2)^2} & 0 \\ 0 & 0 & \frac{1}{(1+x^2+y^2+z^2)^2} \end{pmatrix} \text{ i. e. } g_{ik}(x, y, z) = \frac{\delta_{ik}}{(1 + x^2 + y^2 + z^2)^2} \,,$$

where $g_{ik}(x)$ are entries of matrix: $G = g_{ik}(x) dx^i dx^k$, ($\delta_{ik}$ is Kronecker symbol: $\delta_{ik} = 1$ if $i = k$ and it vanishes otherwise).

According to formulae above

$$|\mathbf{X}| = \sqrt{\langle \mathbf{X}, \mathbf{X} \rangle} = \sqrt{g_{ik}(x, y, z) X^i X^k}\Big|_{\mathbf{p}} = \sqrt{\frac{\delta_{ik} X^i X^k}{(1 + x^2 + y^2 + z^2)^2}}\Big|_{x=2,y=2,z=1} =$$

$$\sqrt{\frac{a^2 + b^2 + c^2}{(1 + 2^2 + 2^2 + 1^2)^2}} = \frac{\sqrt{a^2 + b^2 + c^2}}{10} \,,$$

$$|\mathbf{Y}| = \sqrt{\langle \mathbf{Y}, \mathbf{Y} \rangle} = \sqrt{g_{ik}(x, y, z) Y^i Y^k}\Big|_{\mathbf{p}} = \sqrt{\frac{\delta_{ik} Y^i Y^k}{(1 + x^2 + y^2 + z^2)^2}}\Big|_{x=2,y=2,z=1} =$$

$$\sqrt{\frac{p^2 + q^2 + r^2}{(1 + 2^2 + 2^2 + 1^2)^2}} = \frac{\sqrt{p^2 + q^2 + r^2}}{10} \,,$$

and

$$\cos\theta = \frac{\langle \mathbf{X}, \mathbf{Y} \rangle}{|\mathbf{X}||\mathbf{Y}|} = \frac{g_{ik}(x, y, z) X^i Y^k\big|_{\mathbf{p}}}{\sqrt{g_{pq}(x, y, z) X^p X^q}\sqrt{g_{rs}(x, y, z) Y^r Y^s}} = \frac{\frac{\delta_{ik} X^i Y^k}{(1+x^2+y^2+z^2)^2}}{|\mathbf{X}||\mathbf{Y}|}$$

$$= \frac{\frac{ap+bq+cr}{(1+2^2+2^2+1)^2}}{\frac{\sqrt{a^2+b^2+c^2}}{10}\frac{\sqrt{p^2+q^2+r^2}}{10}} == \frac{ap+bq+cr}{\sqrt{a^2+b^2+c^2}\sqrt{p^2+q^2+r^2}} \,.$$

This example is related with the notion of so called *conformally euclidean metric* (see the next paragraph, 1.2.4).

### 1.2.4 Conformally Euclidean metric

Let $(M, G)$ be a Riemannian manifold.

**Definition** We say that metric $G$ is locally conformally Euclidean in a vicinity of the point $\mathbf{p}$ if in a vicinity of this point there exist local coordinates $\{x^i\}$ such that in these coordinates metric has an appearance

$$G = \sigma(x)\delta_{ik}dx^i dx^k = \sigma(x)\left((dx^1)^2 + \cdots + (dx^n)^2\right) \,, \tag{1.29}$$

i.e. it is proportional to 'Euclidean metric'. We call coordinates $\{x^i\}$ *conformall* coordinates or *isothermic* coordinates if condition (1.29) holds.

We say that metric is conformally Euclidean if it is locally conformally Euclidean in the vicinity of every point. We say that Riemannian manifold $(M.G)$ is conformally Euclidean if the metric $G$ on it is conformally Euclidean

One can see that if metric is conformally Euclidean in a vicinity of some point $\mathbf{p}$, then the angle between vectors, more precisely the cosinus of the angle between vectors attached at this point (see equation (1.27)) is the same as for Eucldean metric. Indeed, let $G$ be conformally Euclidean metric and let $x^i$ be local coordinates such that the metric has an appearance (1.29) in these coordinates. Let $\mathbf{X}, \mathbf{Y}$ be two non-vanishing vectors $\mathbf{X} = X^m(x)\frac{\partial}{\partial x^m}, \mathbf{Y} = Y^m(x)\frac{\partial}{\partial x^m}$ ($|\mathbf{X}| \neq 0, |\mathbf{Y}| \neq 0$) attached at a same point. Then

$$\cos\theta = \frac{\langle \mathbf{X}, \mathbf{Y}\rangle}{|\mathbf{X}|\cdot|\mathbf{Y}|} = \frac{g_{ik}X^iY^k}{\sqrt{g_{ik}(x)X^iX^k}\sqrt{g_{ik}(x)Y^iY^k}} =$$

$$\frac{\sigma(x)\delta_{ik}X^iY^k}{\sqrt{\sigma(x)\delta_{ik}(x)X^iX^k}\sqrt{\sigma(x)\delta_{ik}(x)Y^iY^k}} = \frac{\sum_k X^kY^k}{\sqrt{\sum_k(x)X^kX^k}\sqrt{\sum_k Y^kY^k}} \,. \tag{1.30}$$

(Note that coefficient $\sigma$ in equation (1.29) has to be positive.)

**Remark** One can show that the condition of 'preserving the angles' is not only necessary condition but it is also sufficient condition for metric to be conformally Euclidean (see the problem 1 in Homework 2).

Now consider examples.

First It is instructive to recall the example considered in previous subsection 1.2.3), where Riemannian metric in a vicinity of a point had an appearance (1.28) This is example of Riemannian manifold which is locally conformally Euclidean in a vicinity of a point $\mathbf{p}$.

Another

**Example** Consider the surface of cylinder with the metric

$$G = a^2 d\varphi^2 + dh^2 \tag{1.31}$$

(see equation (1.25) ). In a vicinity of every point one can conisder coordinates $\begin{cases} u = a\varphi \\ v = h \end{cases}$. It is evident that in these coordinates $G = du^2 + dv^2$, i.e. this Riemannian manifold is conformally Euclidean.

**Remark**. In fact we proved more: for metric of cylinder in coordinates $u, v$, the coefficient $\sigma(x) \equiv 1$, i,e. in these coordinates metric is not only *locally conformally Euclidean*, but also it is *locally Euclidean.* We will study this question later in details. (see paragraph "Locally Euclidean Riemannian manifold" later).

Later we consider also another important examples.

It is important, that the following Theorem takes a place:

**Theorem** (Gauss) *Every 2-dimensional Riemannian manifold is locally conformally Euclidean, i.e. for arbitrary 2-dimensional Riemannian manifold, in a vicinity of arbitrary point, there exist coordinates $u', v'$ such that in these coordinates Riemannian metric*

$$G = A(u,v)du^2 + 2B(u,v)dudv + C(u,v)dv^2 = \sigma(u',v')\left(du'^2\right) \tag{1.32}$$

We will not prove this theorem footnotethe proof is easy and almost evident for anaylitical manifolds, and it is hard for smooth manifolds, but consider many examples of 2-dimensional Riemannian manifolds, with suitable conformall coordinates.

### 1.2.5 Length of curves

Let $\gamma$: $x^i = x^i(t), (i = 1, \dots, n))$ $(a \le t \le b)$ be a curve on the Riemannian manifold $(M, G)$.

At the every point of the curve the velocity vector (tangent vector) is defined:

$$\mathbf{v}(t) = \begin{pmatrix} \dot{x}^1(t) \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}^n(t) \end{pmatrix} = \frac{dx^i(t)}{dt}\frac{\partial}{\partial x^i}$$

**Remark** Note that $\mathbf{v}(t)$ is a vector; check transformation rules:

$$\frac{dx^i(t)}{dt}\frac{\partial}{\partial x^i} = \frac{dx^i(t)}{dt}\frac{\partial x^{i'}}{\partial x^i}\frac{\partial}{\partial x^{i'}} = \frac{\partial x^{i'}}{\partial x^i}\frac{dx^i(t)}{dt}\frac{\partial}{\partial x^{i'}} = \frac{dx^{i'}(t)}{dt}\frac{\partial}{\partial x^{i'}}\,.$$

The length of velocity vector $\mathbf{v} \in T_xM$ (vector $\mathbf{v}$ is tangent to the manifold $M$ at the point $x$) equals to

$$|\mathbf{v}|_x = \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle_G\big|_x} = \sqrt{g_{ik}v^iv^k}\big|_x = \sqrt{g_{ik}\frac{dx^i(t)}{dt}\frac{dx^k(t)}{dt}}\big|_x\,.$$

For an arbitrary curve its length is equal to the integral of the length of velocity vector:

$$L_\gamma = \int_a^b \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle_G\big|_{x(t)}}dt = \int_a^b \sqrt{g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t)}dt\,. \tag{1.33}$$

Bearing in mind that metric (1.18) defines the length we often write metric in the following form

$$G = ds^2 = g_{ik}dx^idx^k$$

**Example 1** Consider 2-dimensional Riemannian manifold with metric

$$||g_{ik}(u,v)|| = \begin{pmatrix} g_{11}(u,v) & g_{12}(u,v) \\ g_{21}(u,v) & g_{22}(u,v) \end{pmatrix}\,.$$

Then

$$G = ds^2 = g_{ik}du^idv^k = g_{11}(u,v)du^2 + 2g_{12}(u,v)dudv + g_{22}(u,v)dv^2\,.$$

The length of the curve $\gamma$: $u = u(t), v = v(t)$, where $t_0 \le t \le t_1$ according to (1.33) is equal to $L_\gamma = \int_{t_0}^{t_1}\sqrt{\langle \mathbf{v}, \mathbf{v}\rangle} = \int_{t_0}^{t_1}\sqrt{g_{ik}(x)\dot{x}^i\dot{x}^k} =$

$$\int_{t_0}^{t_1}\sqrt{g_{11}\left(u\left(t\right), v\left(t\right)\right)u_t^2 + 2g_{12}\left(u\left(t\right), v\left(t\right)\right)u_tv_t + g_{22}\left(u\left(t\right), v\left(t\right)\right)v_t^2}dt\,. \tag{1.34}$$

**Example** Consider Lobachevsky (hyperbolic) plane. We consider upper-half model of Lobachevsky (hyperbolic) plane:

$$G = \frac{dx^2 + dy^2}{y^2}\,, \quad (y > 0)$$

Consider in Lobachevsky plane the curve $C$: $\begin{cases} x = x_0 \\ y = t \end{cases}$, $a < t < b$ and claculate its length:

$$L_C = \int_a^b \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \int_a^b \sqrt{g_{ik}(x)\dot{x}^i \dot{x}^k} =$$

$$\int_a^b \sqrt{g_{11}\left(x\left(t\right), y\left(t\right)\right) x_t^2 + 2g_{12}\left(x\left(t\right), y\left(t\right)\right) x_t y_t + g_{22}\left(x\left(t\right), y\left(t\right)\right) y_t^2}dt =$$

$$\int_a^b \sqrt{\frac{1}{y^2}\left(x_t^2 + y_t^2\right)}dt = \int_a^b \sqrt{\frac{1}{t^2}\left(0+1\right)}dt = \int_a^b \frac{dt}{t} = \left|\log \frac{a}{b}\right|\,.$$

The length of curves defined by the formula(1.33) obeys the following natural conditions

- It coincides with the usual length in the Euclidean space $\mathbf{E}^n$ ($\mathbf{R}^n$ with standard metric $G = (dx^1)^2 + \dots + (dx^n)^2$ in Cartesian coordinates). E.g. for 3-dimensional Euclidean space

$$L_\gamma = \int_a^b \sqrt{g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t)}dt = \int_a^b \sqrt{(\dot{x}^1(t))^2 + (\dot{x}^2(t))^2 + (\dot{x}^3(t))^2}dt$$

- It does not depend on parameterisation of the curve

$$L_\gamma = \int_a^b \sqrt{g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t)}dt = \int_{a'}^{b'} \sqrt{g_{ik}(x(\tau))\dot{x}^i(\tau)\dot{x}^k(\tau)}d\tau\,,$$

($x^i(\tau) = x^i(t(\tau))$, $a' \le \tau \le b'$ while $a \le t \le b$) since under changing of parameterisation

$$\dot{x}^i(\tau) = \frac{dx(t(\tau))}{d\tau} = \frac{dx(t(\tau))}{dt}\frac{dt}{d\tau} = \dot{x}^i(t)\frac{dt}{d\tau}\,.$$

- It *does not depend on coordinates on Riemannian manifold $M$*

$$L_\gamma = \int_a^b \sqrt{g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t)}dt = \int_a^b \sqrt{g_{i'k'}(x'(t))\dot{x}^{i'}(t)\dot{x}^{k'}(t)}dt\,.$$

This immediately follows from transformation rule (1.72) for Riemannian metric:

$$g_{i'k'}\dot{x}^{i'}(t)\dot{x}^{k'}(t) = g_{ik}\left(\frac{\partial x^i}{\partial x^{i'}(t)}\dot{x}^{i'}(t)\right)\left(\frac{\partial x^k}{\partial x^{k'}}\dot{x}^{k'}(t)\right) g_{ik}\,\dot{x}^i(t)\dot{x}^k(t)\,.$$

- It is additive: length of the sum of two curves is equal to the sum of their lengths. If a curve $\gamma = \gamma_1 + \gamma$, i.e. $\gamma\colon x^i(t), a \le t \le b$, $\gamma_1\colon x^i(t), a \le t \le c$ and $\gamma_2\colon x^i(t), c \le t \le b$ where a point $c$ belongs to the interval $(a, b)$ then $L_\gamma = L_{\gamma^1} + L_{\gamma^2}$.

One can show that formula (1.33) for length is defined uniquely (up to a constant multiplier) by these conditions. More precisely one can show under some technical conditions one may show that any local additive functional on curves which does not depend on coordinates and parameterisation, and depends on derivatives of curves of order $\le 1$ is equal to (1.33) up to a constant multiplier. To feel the taste of this statement you may do the following exercise:

**Exercise** Let $A = A\left(x(t), y(t), \frac{dx(t)}{dt}, \frac{dy(t)}{dt}\right)$ be a function such that an integral $L = \int A\left(x(t), y(t), \frac{dx(t)}{dt}, \frac{dy(t)}{dt}\right) dt$ over an arbitrary curve $\gamma$ in $\mathbf{E}^2$ does not change under reparameterisation of this curve and under an arbitrary isometry, i.e. translation and rotation of the curve. Then one can easy show (show it!) that

$$A\left(x(t), y(t), \frac{dx(t)}{dt}, \frac{dy(t)}{dt}\right) = c\sqrt{\left(\frac{dx(t)}{dt}\right)^2 + \left(\frac{dy(t)}{dt}\right)^2},$$

where $c$ is a constant, i.e. it is a usual length up to a multiplier

## 1.3 Riemannian structure on the surfaces embedded in Euclidean space

Let $M$ be a surface embedded in Euclidean space. Let $G$ be Riemannian structure on the manifold $M$.

Let $\mathbf{X}, \mathbf{Y}$ be two vectors tangent to the surface $M$ at a point $\mathbf{p} \in M$. An External Observer calculate this scalar product viewing these two vectors as vectors in $\mathbf{E}^3$ attached at the point $\mathbf{p} \in \mathbf{E}^3$ using scalar product in $\mathbf{E}^3$. An Internal Observer will calculate the scalar product viewing these two vectors as vectors tangent to the surface $M$ using the Riemannian metric $G$ (see the formula (1.39)). Respectively

If $L$ is a curve in $M$ then an External Observer consider this curve as a curve in $\mathbf{E}^3$, calculate the modulus of velocity vector (speed) and the length of the curve using Euclidean scalar product of ambient space. An Internal Observer ("an ant") will define the modulus of the velocity vector and the length of the curve using Riemannian metric.

**Definition** Let $M$ be a surface embedded in the Euclidean space. We say that metric $G_M$ on the surface is induced by the Euclidean metric if the scalar product of arbitrary two vectors $\mathbf{A}, \mathbf{B} \in T_\mathbf{p}M$ calculated in terms of the metric $G$ equals to Euclidean scalar product of these two vectors:

$$\langle \mathbf{A}, \mathbf{B} \rangle_{G_M} = \langle \mathbf{A}, \mathbf{B} \rangle_{G_{\text{Euclidean}}} \tag{1.35}$$

In other words we say that Riemannian metric on the embedded surface is induced by the Euclidean structure of the ambient space if External and Internal Observers come to the same results calculating scalar product of vectors tangent to the surface.

In this case modulus of velocity vector (speed) and the length of the curve is the same for External and Internal Observer.

### 1.3.1 Internal and external observers

*Tangent vectors, coordinate tangent vectors*

Here we recall basic notions from the course of Geometry which we will need here.

Let $\mathbf{r} = \mathbf{r}(u, v)$ be parameterisation of the surface $M$ embedded in the Euclidean space:

$$\mathbf{r}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}$$

Here as always $x, y, z$ are Cartesian coordinates in $\mathbf{E}^3$.

Let $\mathbf{p}$ be an arbitrary point on the surface $M$. Consider the plane formed by the vectors which are adjusted to the point $\mathbf{p}$ and tangent to the surface $M$. We call this plane *plane tangent to $M$ at the point $\mathbf{p}$* and denote it by $T_{\mathbf{p}}M$.

For a point $\mathbf{p} \in M$ one can consider a basis in the tangent plane $T_pM$ adjusted to the parameters $u, v$.

Tangent basis vectors at any point $(u, v)$ are

$$\mathbf{r}_u = \frac{\partial \mathbf{r}(u, v)}{\partial u} = \begin{pmatrix} \frac{\partial x(u,v)}{\partial u} \\ \frac{\partial y(u,v)}{\partial u} \\ \frac{\partial z(u,v)}{\partial u} \end{pmatrix} = \frac{\partial x(u, v)}{\partial u} \frac{\partial}{\partial x} + \frac{\partial y(u, v)}{\partial u} \frac{\partial}{\partial y} + \frac{\partial z(u, v)}{\partial u} \frac{\partial}{\partial z}$$

and

$$\mathbf{r}_v = \frac{\partial \mathbf{r}(u, v)}{\partial v} = \begin{pmatrix} \frac{\partial x(u,v)}{\partial v} \\ \frac{\partial y(u,v)}{\partial v} \\ \frac{\partial z(u,v)}{\partial v} \end{pmatrix} = \frac{\partial x(u, v)}{\partial v} \frac{\partial}{\partial x} + \frac{\partial y(u, v)}{\partial v} \frac{\partial}{\partial y} + \frac{\partial z(u, v)}{\partial v} \frac{\partial}{\partial z}$$

**Definition** We call basis vectors $\mathbf{r}_u, \mathbf{r}_v$ adjusted to parameters (coordinates) $u, v$ *coordinate basis vectors*

Every vector $\mathbf{X} \in T_pM$ can be expanded over the basis of coordinate basis vectors:

$$\mathbf{X} = X_u \mathbf{r}_u + X_v \mathbf{r}_v,$$

where $X_u, X_v$ are coefficients, components of the vector $\mathbf{X}$.

Internal Observer views the basis vector $\mathbf{r}_u \in T_pM$ as the vector $\partial_u$. Why? The vector $\mathbf{r}_u$ attached at the point $\mathbf{p}$ is a velocity vector for the curve $\gamma_{\mathbf{r}_u}(t)\colon \begin{cases} u = u_0 + t \\ v = v_0 \end{cases}$ starting at the point $\mathbf{p}$ ($(u_0, v_0)$ are coordinates of the point $\mathbf{p}$). If $f = f(u, v)$ is a function on the surface $M$, then one can see that directional derivative of this function along a vector $\mathbf{r}_u$ is defined by $\frac{\partial}{\partial u}$:

$$\partial_u f(u, v)\big|_{\mathbf{p}} = \frac{d}{dt} f\left(\gamma_{\mathbf{r}_u}(t)\right) = \frac{d}{dt} f\left(u_0 + t, v_0\right) .$$

Respectively the basis vector $\mathbf{r}_v \in T_pM$ for an Internal Observer, is velocity vector for the curve $\gamma_{\mathbf{r}_v}(t)\colon \begin{cases} u = u_0 \\ v = v_0 + t \end{cases}$ starting at the point $\mathbf{p}$ and Internal Observer denotes this vector $\partial_v$:

$$\partial_v f(u, v)\big|_{\mathbf{p}} = \frac{d}{dt} f\left(\gamma_{\mathbf{r}_v}(t)\right) = \frac{d}{dt} f\left(u_0, v_0 + t\right) .$$

For an arbitrary vector $\mathbf{X}$ which is tangent to surface $M$ at the point $\mathbf{p}$, ($\mathbf{X} \in T_pM$)

| `External observer` | `Internal observer` |
|---|---|
| $\mathbf{X} = a\mathbf{r}_u + b\mathbf{r}_v$ | $\mathbf{X} = a\partial_u + b\partial_v$ |

**Example** Consider sphere of radius $R$ in $\mathbf{E}^3$, $x^2 + y^2 + z^2 = R^2$. In spherical coordinates

$$\mathbf{E}^3 \ni \mathbf{r} = \mathbf{r}(\theta, \varphi) \begin{cases} x = R \sin\theta \cos\varphi \\ y = R \sin\theta \sin\varphi \\ x = R \cos\theta \end{cases} ,$$

these coordinates are well-defined for $0 < \theta < \frac{\pi}{2}$ and $0 < \varphi < 2\pi$. For coordinate basis vectors $\mathbf{r}_\theta$ and $\mathbf{r}_\varphi$ we have:

$$\mathbf{r}_\theta = \frac{\partial \mathbf{r}(\theta, \varphi)}{\partial \theta} = \frac{\partial}{\partial \theta} \begin{pmatrix} x(\theta, \varphi) \\ y(\theta, \varphi) \\ z(\theta, \varphi) \end{pmatrix} = \frac{\partial}{\partial \theta} \begin{pmatrix} R \sin\theta \cos\varphi \\ R \sin\theta \sin\varphi \\ R \cos\theta \end{pmatrix} =$$

$$\begin{pmatrix} R\cos\theta\cos\varphi \\ R\cos\theta\sin\varphi \\ -R\sin\theta \end{pmatrix} = R\cos\theta\cos\varphi\frac{\partial}{\partial x} + R\cos\theta\sin\varphi\frac{\partial}{\partial y} - R\sin\theta\frac{\partial}{\partial z}\,,$$

and respectively

$$\mathbf{r}_\varphi = \frac{\partial \mathbf{r}(\theta,\varphi)}{\partial\theta} = \frac{\partial}{\partial\varphi}\begin{pmatrix} x(\theta,\varphi) \\ y(\theta,\varphi) \\ z(\theta,\varphi) \end{pmatrix} = \frac{\partial}{\partial\varphi}\begin{pmatrix} R\sin\theta\cos\varphi \\ R\sin\theta\sin\varphi \\ R\cos\theta \end{pmatrix} =$$

$$\begin{pmatrix} -R\sin\theta\sin\varphi \\ R\sin\theta\cos\varphi \\ 0 \end{pmatrix} = -R\sin\theta\sin\varphi\frac{\partial}{\partial x} + R\sin\theta\cos\varphi\frac{\partial}{\partial y}\,. \tag{1.36}$$

Here is a table how observers look at the objects on sphere:

| | INTERNAL OBSERVER | EXTERNAL OBSERVER |
|---|---|---|
| point on $S^2$ | 2 coordinates $\theta, \varphi$ | 3 coordinates $\mathbf{r} = \mathbf{r}(\theta,\varphi)$ |
| curve on $S^2$ | $\theta(t), \varphi(t)$ | $\mathbf{r}(t) = \mathbf{r}(\theta(t,\varphi(t)))$ |
| coordinate tangent vectors to $S^2$ | $\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\varphi}$ | $\mathbf{r}_\theta\,, \quad \mathbf{r}_\varphi$ |
| tangent vector to $S^2$ | $a\frac{\partial}{\partial\theta} + b\frac{\partial}{\partial\varphi}$ | $A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} + C\frac{\partial}{\partial z} = a\mathbf{r}_\theta + b\mathbf{r}_\varphi$ |

*Explicit formulae for induced Riemannian metric (First Quadratic form)*

Now we are ready to write down the explicit formulae for the Riemannian metric on the surface induced by metric (scalar product) in ambient Euclidean space (see the Definition (1.35). We will return to induced metric again in next paragraph1.3.2.

Let $M\colon \mathbf{r} = \mathbf{r}(u,v)$ be a surface embedded in $\mathbf{E}^3$.

The formula (1.35) means that scalar products of basic vectors $\mathbf{r}_u = \partial_u, \mathbf{r}_v = \partial_v$ has to be the same calculated on the surface or in the ambient space, i.e. calculated by Internal observer, or by External observer. For example scalar product $\langle\partial_u,\partial_v\rangle_M = g_{uv}$ calculated by the Internal Observer is the same as a scalar product $\langle\mathbf{r}_u,\mathbf{r}_v\rangle_{\mathbf{E}^3}$ calculated by the External Observer, scalar product $\langle\partial_v,\partial_v\rangle_M = g_{uv}$ calculated by the Internal Observer is the same as a scalar product $\langle\mathbf{r}_v,\mathbf{r}_v\rangle_{\mathbf{E}^3}$ calculated by the External Observer and so on:

$$G = \begin{pmatrix} g_{uu} & g_{uv} \\ g_{vu} & g_{vv} \end{pmatrix} = \begin{pmatrix} \langle\partial_u,\partial_u\rangle & \langle\partial_u,\partial_v\rangle \\ \langle\partial_v,\partial_u\rangle & \langle\partial_v,\partial_v\rangle \end{pmatrix} = \begin{pmatrix} \langle\mathbf{r}_u,\mathbf{r}_u\rangle_{\mathbf{E}^3} & \langle\mathbf{r}_u,\mathbf{r}_v\rangle_{\mathbf{E}^3} \\ \langle\mathbf{r}_v,\mathbf{r}_u\rangle_{\mathbf{E}^3} & \langle\mathbf{r}_v,\mathbf{r}_v\rangle_{\mathbf{E}^3} \end{pmatrix} \tag{1.37}$$

where as usual we denote by $\langle\ ,\ \rangle_{\mathbf{E}^3}$ the scalar product in the ambient Euclidean space.

**Remark** It is convenient sometimes to denote parameters $(u, v)$ as $(u^1, u^2)$ or $u^\alpha$ ($\alpha = 1, 2$) and to write $\mathbf{r} = \mathbf{r}(u^1, u^2)$ or $\mathbf{r} = \mathbf{r}(u^\alpha)$ ($\alpha = 1, 2$) instead $\mathbf{r} = \mathbf{r}(u, v)$

In these notations:

$$G_M = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} = \begin{pmatrix} \langle \mathbf{r}_u, \mathbf{r}_u \rangle_{\mathbf{E}^3} & \langle \mathbf{r}_u, \mathbf{r}_v \rangle_{\mathbf{E}^3} \\ \langle \mathbf{r}_u, \mathbf{r}_v \rangle_{\mathbf{E}^3} & \langle \mathbf{r}_v, \mathbf{r}_v \rangle_{\mathbf{E}^3} \end{pmatrix}, \quad g_{\alpha\beta} = \langle \mathbf{r}_\alpha, \mathbf{r}_\beta \rangle\,,$$

$$G_M = g_{\alpha\beta} du^\alpha du^\beta = g_{11} du^2 + 2 g_{12} du dv + g_{22} dv^2 \tag{1.38}$$

where $(\ ,\ )$ is a scalar product in Euclidean space.

The formula (1.38) is the formula for induced Riemannian metric on the surface $\mathbf{r} = \mathbf{r}(u, v)$ [2].

If $\mathbf{X}, \mathbf{Y}$ are two tangent vectors in the tangent plane $T_pC$ then $G(\mathbf{X}, \mathbf{Y})$ at the point $p$ is equal to scalar product of vectors $\mathbf{X}, \mathbf{Y}$:

$$(\mathbf{X}, \mathbf{Y}) = (X^1\mathbf{r}_1 + X^2\mathbf{r}_2, Y^1\mathbf{r}_1 + Y^2\mathbf{r}_2) = \tag{1.39}$$

$$X^1(\mathbf{r}_1, \mathbf{r}_1)Y^1 + X^1(\mathbf{r}_1, \mathbf{r}_2)Y^2 + X^2(\mathbf{r}_2, \mathbf{r}_1)Y^1 + X^2(\mathbf{r}_2, \mathbf{r}_2)Y^2 =$$
$$X^\alpha(\mathbf{r}_\alpha, \mathbf{r}_\beta)Y^\beta = X^\alpha g_{\alpha\beta} Y^\beta = G(\mathbf{X}, \mathbf{Y})$$

### 1.3.2 Formulae for induced metric

We obtained (1.38) from equation (1.35).

We can do these calculations in a little bit other way.

The Riemannian structure of Euclidean space— standard Euclidean metric in Euclidean coordinates is given by

$$G_{\mathbf{E}^3} = (dx)^2 + (dy)^2 + (dz)^2\,. \tag{1.40}$$

Then the induced metric (1.35) on the surface $M$ defined by equation $\mathbf{r} = \mathbf{r}(u, v)$ is equal to

$$G_M = G_{\mathbf{E}^3}\big|_{\mathbf{r}=\mathbf{r}(u,v)} = \left((dx)^2 + (dy)^2 + (dz)^2\right)\big|_{\mathbf{r}=\mathbf{r}(u,v)} = G_M = g_{\alpha\beta} du^\alpha du^\beta \tag{1.41}$$

[2] it is called sometimes First Quadratic Form of this surface.

i.e. $((dx)^2+(dy)^2+(dz)^2)\big|_{\mathbf{r}=\mathbf{r}(u,v)} =$

$$\left(\frac{\partial x(u,v)}{\partial u}du+\frac{\partial x(u,v)}{\partial v}dv\right)^2+\left(\frac{\partial y(u,v)}{\partial u}du+\frac{\partial y(u,v)}{\partial v}dv\right)^2+\left(\frac{\partial z(u,v)}{\partial u}du+\frac{\partial z(u,v)}{\partial v}dv\right)^2=$$

$$(x_u^2+y_u^2+z_u^2)du^2+2(x_ux_v+y_uy_v+z_uz_v)dudv+(x_v^2+y_v^2+z_v^2)dv^2$$

We see that

$$G_M=g_{\alpha\beta}du^\alpha du^\beta=g_{11}du^2+2g_{12}dudv+g_{22}dv^2\,, \tag{1.42}$$

where for matrix $||g_{\alpha\beta}||$, $(\alpha,\beta=1,2)$,

$$||g_{\alpha\beta}||=\begin{pmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix}=\begin{pmatrix} g_{uu} & g_{uv}\\ g_{vu} & g_{vv}\end{pmatrix}=$$

$$\begin{pmatrix} (x_u^2+y_u^2+z_u^2) & (x_ux_v+y_uy_v+z_uz_v)\\ (x_ux_v+y_uy_v+z_uz_v) & (x_v^2+y_v^2+z_v^2)\end{pmatrix}=\begin{pmatrix} \langle\mathbf{r}_u,\mathbf{r}_u\rangle_{\mathbf{E}^3} & \langle\mathbf{r}_u,\mathbf{r}_v\rangle_{\mathbf{E}^3}\\ \langle\mathbf{r}_v,\mathbf{r}_u\rangle_{\mathbf{E}^3} & \langle\mathbf{r}_v,\mathbf{r}_v\rangle_{\mathbf{E}^3}\end{pmatrix}. \tag{1.43}$$

We come to same formula (1.38).

**Example** Consider again sphere of radius $R$ in $\mathbf{E}^3$, $x^2+y^2+z^2=R^2$ in stereographic coordinates. We calculated coordinate tangent vectors to this sphere in (1.36). Now calculate induced Riemannian metric:

$$G_{S^2}=\left(dx^2+dy^2+dz^2\right)\big|_{x=R\sin\theta\cos\varphi,y=R\sin\theta\sin\varphi,x=R\cos\theta\cos\theta}=$$

$$[d(R\sin\theta\cos\varphi)]^2+[d(R\sin\theta\sin\varphi)]^2+[d(R\cos\theta)]^2=$$

$$[R\cos\theta\cos\varphi d\theta-R\sin\theta\sin\varphi d\varphi]^2+[R\cos\theta\sin\varphi d\theta+R\sin\theta\cos\varphi d\varphi]^2+[-R\sin\theta d\theta]^2=$$

$$(R^2\sin^2\theta\sin^2\varphi+R^2\sin^2\theta\cos^2\varphi)d\varphi^2+(R^2\cos^2\theta\cos^2\varphi+R^2\sin^2\theta)d\theta^2=R^2d\theta^2+R^2\sin^2\theta d\varphi^2\,.$$

We see that

$$G_{S^2}=R^2d\theta^2+R^2\sin^2\theta d\varphi^2\,, ||g_{\alpha\beta}||=\begin{pmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix}=\begin{pmatrix} g_{\theta\theta} & g_{\theta\varphi}\\ g_{\varphi\theta} & g_{\varphi\varphi}\end{pmatrix}=\begin{pmatrix} R^2 & 0\\ 0 & R^2\sin^2\theta\end{pmatrix}. \tag{1.44}$$

**Remark** Sometimes it is useful to use the following "condensed" notations. We denote Cartesian coordinates $(x,y,z)$ of Euclidean space by $x^i$, $(i=1,2,3)$. Let surface $M$ be given in local parameterisation $x^i=x^i(u^\alpha)$. Riemannian metric of Euclidean space (1.40) has appearance

$$G_{\mathbf{E}}=dx^i\delta_{ik}dx^k\,. \tag{1.45}$$

and calculations (1.41) —(1.43) for induced metric (1.41) has appearance

$$G_M = dx^i \delta_{ik} dx^k \big|_{x^i = x^i(u^\alpha)} = \frac{\partial x^i(u)}{\partial u^\alpha} \delta_{ik} \frac{\partial x^k(u)}{\partial u^\beta} du^\alpha du^\beta = g_{\alpha\beta}(u) du^\alpha du^\beta \tag{1.46}$$

(See also remark above before equation (1.38)). One can rewrite (1.46) in the following way:

$$g_{\alpha\beta} = \frac{\partial x^i(u)}{\partial u^\alpha} \delta_{ij} \frac{\partial x^j(u)}{\partial u^\beta} . \quad (\alpha, \beta = 1, 2, ) .$$

It is instructive to come to this equation straightforwardly from equation (1.35) and definition (1.18). We have that due to (1.35)

$$g_{\alpha\beta} = g_{\pi\rho} dx^\pi dx^\rho \left( \frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial u^\beta} \right) = G_M \left( \frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial u^\beta} \right) = G_{\mathbf{E}^3} \left( \mathbf{r}_\alpha, \mathbf{r}_\beta \right) =$$

$$\delta_{pq} dx^p dx^q \left( \frac{\partial x^i}{\partial u^\alpha} \frac{\partial}{\partial x^i}, \frac{\partial x^j}{\partial u^\beta} \frac{\partial}{\partial x^j} \right) = \frac{\partial x^i}{\partial u^\alpha} \left[ \delta_{pq} dx^p dx^q \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \right] \frac{\partial x^j}{\partial u^\beta} = \frac{\partial x^i}{\partial u^\alpha} \delta_{ij} \frac{\partial x^j}{\partial u^\beta} .$$

Representation (1.46) in condenced notations is very useful. It is easy to see that this formula works for arbitrary dimensions, i.e. if we have $m$-dimensional manifold embedded in $n$-dimensional Euclidean space. We just have to suppose that in this case $i = 1, \dots, n$ and $\alpha = 1, \dots, m$; manifold is given by parameterisation $x^i = x^i(u^\alpha)$ $(\alpha = 1, \dots, m)$. Moreover in the case if manifold is embedded not in Euclidean space but in an arbitrary Riemannian space then one can see that we come to the induced metric

$$G_M = dx^i g_{ik} \left( (x(u)) \right) dx^k \big|_{x^i = x^i(u^\alpha)} = \frac{\partial x^i(u)}{\partial u^\alpha} g_{ik} \left( (x(u)) \right) \frac{\partial x^k(u)}{\partial u^\beta} du^\alpha du^\beta = g_{\alpha\beta}(x(u)) du^\alpha du^\beta$$

Check explicitly again that length of the tangent vectors and curves on the surface calculating by External observer (i.e. using Euclidean metric (1.40)) *is the same* as calculating by Internal Observer, ant (i.e. using the induced Riemannian metric (1.38), (1.42)). Let $\mathbf{X} = X^\alpha \mathbf{r}_\alpha = a\mathbf{r}_u + b\mathbf{r}_v$ be a vector tangent to the surface $M$. The square of the length $|\mathbf{X}|$ of this vector calculated by External observer (he calculates using the scalar product in $\mathbf{E}^3$) equals to

$$|\mathbf{X}|^2 = \langle \mathbf{X}, \mathbf{X} \rangle = \langle a\mathbf{r}_u + b\mathbf{r}_v, a\mathbf{r}_u + b\mathbf{r}_v \rangle = a^2 \langle \mathbf{r}_u, \mathbf{r}_u \rangle + 2ab \langle \mathbf{r}_u, \mathbf{r}_v \rangle + b^2 \langle \mathbf{r}_v, \mathbf{r}_v \rangle \tag{1.47}$$

where $\langle\, ,\, \rangle$ is a scalar product in $\mathbf{E}^3$. The internal observer will calculate the length using Riemannian metric (1.38) (1.42):

$$G(\mathbf{X}, \mathbf{X}) = \begin{pmatrix} a, & b \end{pmatrix} \cdot \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & G_{22} \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = g_{11} a^2 + 2 g_{12} ab + g_{22} b^2 \tag{1.48}$$

*External observer (person living in ambient space* $\mathbf{E}^3$*) calculates the length of the tangent vector using formula* (1.47)*. An ant living on the surface calculates length of this vector in internal coordinates using formula* (1.48)*. External observer deals with external coordinates of the vector, ant on the surface with internal coordinates. They come to the same answer.*

Let $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ $a \leq t \leq b$ be a curve on the surface.

Velocity of this curve at the point $\mathbf{r}(u(t), v(t))$ is equal to

$$\mathbf{v} = \mathbf{X} = \xi\mathbf{r}_u + \eta\mathbf{r}_v \text{where } \xi = u_t, \eta = v_t \colon \quad \mathbf{v} = \frac{d\mathbf{r}(t)}{dt} = u_t\mathbf{r}_u + v_t\mathbf{r}_v\,.$$

The length of the curve is equal to

$$L = \int_a^b |\mathbf{v}(t)|dt = \int_a^b \sqrt{\langle \mathbf{v}(t), \mathbf{v}(t)\rangle_{\mathbf{E}^3}}dt = \int_a^b \sqrt{\langle u_t\mathbf{r}_u + v_t\mathbf{r}_v, u_t\mathbf{r}_u + v_t\mathbf{r}_v\rangle_{\mathbf{E}^3}}dt = \tag{1.49}$$

$$\int_a^b \sqrt{\langle \mathbf{r}_u, \mathbf{r}_u\rangle_{\mathbf{E}^3} u_t^2 + 2\langle \mathbf{r}_u, \mathbf{r}_v\rangle_{\mathbf{E}^3} u_t v_t + \langle \mathbf{r}_v, \mathbf{r}_v\rangle_{\mathbf{E}^3} v_t^2}d\tau =$$

$$\int_a^b \sqrt{g_{11}u_t^2 + 2g_{12}u_t v_t + g_{22}v_t^2}dt \tag{1.50}$$

*An external observer will calculate the length of the curve using* (1.49)*. An ant living on the surface calculate length of the curve using* (1.50) *using Riemannian metric on the surface. They will come to the same answer.*

### 1.3.3 Induced Riemannian metrics. Examples.

We consider already an example of induced Riemannian metric on sphere in spherical coordinates. Now we consider here other examples of induced Riemannian metric on some surfaces in $\mathbf{E}^3$. using calculations for tangent vectors (see (1.38)) or explicitly in terms of differentials (see (1.41) and (1.42)).

First of all consider the general case when a surface $M$ is defined by the equation $z - F(x, y) = 0$. One can consider the following parameterisation of this surface:

$$\mathbf{r}(u, v)\colon \quad \begin{cases} x = u \\ y = v \\ z = F(u, v) \end{cases} \tag{1.51}$$

Then coordinate tangent vectors $\mathbf{r}_u, \mathbf{r}_v$ are

$$\mathbf{r}_u = \begin{pmatrix} 1 \\ 0 \\ F_u \end{pmatrix} \quad \mathbf{r}_v = \begin{pmatrix} 0 \\ 1 \\ F_v \end{pmatrix} \tag{1.52}$$

,

$$(\mathbf{r}_u, \mathbf{r}_u) = 1 + F_u^2, \quad (\mathbf{r}_u, \mathbf{r}_v) = F_u F_v, \quad (\mathbf{r}_v, \mathbf{r}_v) = 1 + F_v^2$$

and induced Riemannian metric (first quadratic form) (1.38) is equal to

$$||g_{\alpha\beta}|| = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_u, \mathbf{r}_u) & (\mathbf{r}_u, \mathbf{r}_v) \\ (\mathbf{r}_u, \mathbf{r}_v) & (\mathbf{r}_v, \mathbf{r}_v) \end{pmatrix} = \begin{pmatrix} 1 + F_u^2 & F_u F_v \\ F_u F_v & 1 + F_v^2 \end{pmatrix} \tag{1.53}$$

$$G_M = ds^2 = (1 + F_u^2) du^2 + 2 F_u F_v du dv + (1 + F_v^2) dv^2 \tag{1.54}$$

and the length of the curve $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ on $C$ $(a \leq t \leq b)$ can be calculated by the formula:

$$L = \int \int_a^b \sqrt{(1 + F_u^2) u_t^2 + 2 F_u F_v u_t v_t + (1 + F_v)^2 v_t^2} dt$$

One can calculate (1.54) explicitly using (1.41):

$$G_M = \left(dx^2 + dy^2 + dz^2\right)\big|_{x=u, y=v, z=F(u,v)} = (du)^2 + (dv)^2 + (F_u du + F_v dv)^2 =$$

$$= (1 + F_u^2) du^2 + 2 F_u F_v du dv + (1 + F_v^2) dv^2 \,. \tag{1.55}$$

*Cylinder*

Cylinder is given by the equation $x^2 + y^2 = a^2$. One can consider the following parameterisation of this surface:

$$\mathbf{r}(h, \varphi) \colon \quad \begin{cases} x = a \cos \varphi \\ y = a \sin \varphi \\ z = h \end{cases} \tag{1.56}$$

We have $G_{cylinder} = \iota^* G_{\mathbf{E}^3} = (dx^2 + dy^2 + dz^2)\big|_{x = a\cos\varphi, y = a \sin\varphi, z = h} =$

$$= (-a \sin \varphi d\varphi)^2 + (a \cos \varphi d\varphi)^2 + dh^2 = a^2 d\varphi^2 + dh^2 \tag{1.57}$$

The same formula in terms of scalar product of tangent vectors:

$$\text{coordinate basis vectors } \mathbf{r}_h = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{r}_\varphi = \begin{pmatrix} -a\sin\varphi \\ a\cos\varphi \\ 0 \end{pmatrix} \tag{1.58}$$

,

$$(\mathbf{r}_h, \mathbf{r}_h) = 1, \quad (\mathbf{r}_h, \mathbf{r}_\varphi) = 0, \quad (\mathbf{r}_\varphi, \mathbf{r}_\varphi) = a^2$$

and

$$||g_{\alpha\beta}|| = \begin{pmatrix} (\mathbf{r}_u, \mathbf{r}_u) & (\mathbf{r}_u, \mathbf{r}_v) \\ (\mathbf{r}_u, \mathbf{r}_v) & (\mathbf{r}_v, \mathbf{r}_v) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a^2 \end{pmatrix},$$

$$G = dh^2 + a^2 d\varphi^2 \tag{1.59}$$

and the length of the curve $\mathbf{r}(t) = \mathbf{r}(h(t), \varphi(t))$ on the cylinder $(a \leq t \leq b)$ can be calculated by the formula:

$$L = \int_a^b \sqrt{h_t^2 + a^2\varphi_t} dt \tag{1.60}$$

*Cone*

Cone is given by the equation $x^2 + y^2 - k^2z^2 = 0$. One can consider the following parameterisation of this surface:

$$\mathbf{r}(h, \varphi)\colon \quad \begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases} \tag{1.61}$$

Calculate induced Riemannian metric:
We have

$$G_{conus} = \iota^* G_{\mathbf{E}^3} = \left(dx^2 + dy^2 + dz^2\right)\big|_{x=kh\cos\varphi, y=kh\sin\varphi, z=h} =$$

$$(k\cos\varphi dh - kh\sin\varphi d\varphi)^2 + (k\sin\varphi dh + kh\cos\varphi d\varphi)^2 + dh^2$$

$$G_{conus} = k^2h^2d\varphi^2 + (1+k^2)dh^2, \ ||g_{\alpha\beta}|| = \begin{pmatrix} 1+k^2 & 0 \\ 0 & k^2h^2 \end{pmatrix} \tag{1.62}$$

The length of the curve $\mathbf{r}(t) = \mathbf{r}(h(t), \varphi(t))$ on the cone $(a \leq t \leq b)$ can be calculated by the formula:

$$L = \int_a^b \sqrt{(1+k^2)h_t^2 + k^2h^2\varphi_t^2} dt \tag{1.63}$$

*Circle* `(again)`

Circle of radius $R$ is given by the equation $x^2 + y^2 = R^2$. Consider standard parameterisation $\varphi$ of this surface:

$$\mathbf{r}(\varphi)\colon \quad \begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \end{cases}$$

Calculate induced Riemannian metric (first quadratic form)

$$G_{S^1} = \iota^* G_{\mathbf{E}^3} = \left(dx^2 + dy^2\right)\big|_{x=R\cos\varphi, y=R\sin\varphi} =$$

$$(-R\sin\varphi d\varphi)^2 + (a\cos\varphi d\varphi)^2 = (R^2\cos^2\varphi + R^2\sin^2\varphi)d\varphi^2 = R^2 d\varphi^2\,.$$

One can consider stereographic coordinates on the circle (see Example in the subsection 1.1) A point $x, y\colon x^2 + y^2 = R^2$ has stereographic coordinate $t$ if points $(0,1)$ (north pole), the point $(x,y)$ and the point $(t,0)$ belong to the same line, i.e. $\frac{x}{t} = \frac{R-y}{R}$, i.e.

$$t = \frac{Rx}{R-y}, \qquad \begin{cases} x = \frac{2tR^2}{R^2+t^2} \\ y = \frac{t^2-R^2}{t^2+R^2}R \end{cases} \quad . \quad \text{since } x^2 + y^2 = R^2\,.$$

Induced metric in coordinate $t$ is

$$G = (dx^2 + dy^2)\big|_{x=x(t), y=y(t)} = \left(d\left(\frac{2tR^2}{R^2+t^2}\right)\right)^2 + \left(d\left(\frac{t^2-R^2}{R^2+t^2}R\right)\right)^2 =$$

$$\left(\frac{2R^2 dt}{R^2+t^2} - \frac{4t^2R^2dt}{(R^2+t^2)^2}\right)^2 + \left(-\frac{4R^2tdt}{(t^2+R^2)^2}\right)^2 = \frac{4R^4dt^2}{(R^2+t^2)^2}\,. \tag{1.64}$$

(See for detail Homework 2[3].

**Remark** Stereographic coordinates very often are preferable since they define birational equivalence between circle and line.

*Sphere* `(again...)`

Sphere of radius $R$ is given by the equation $x^2 + y^2 + z^2 = R^2$. Consider first stereographic coordinates

$$\mathbf{r}(\theta,\varphi)\colon \quad \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases} \tag{1.65}$$

[3]One can also obtain this formula in a very beautiful way using inversion (see Appendices)

We already calculated the coordinate basis in (1.36) and we calculated induced Riemannian metric in (1.44):

$$, \qquad G_{S^2} = R^2 d\theta^2 + R^2 \sin^2\theta d\varphi^2 \,, \quad ||g_{\alpha\beta}|| = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix} \tag{1.66}$$

One comes to the same answer calculating scalar product of coordinate tangent vectors:

coordinate tangent vectors are $\mathbf{r}_\theta = \begin{pmatrix} R\cos\theta\cos\varphi \\ R\cos\theta\sin\varphi \\ -R\sin\theta \end{pmatrix}$ , $\mathbf{r}_\varphi = \begin{pmatrix} -R\sin\theta\sin\varphi \\ R\sin\theta\cos\varphi \\ 0 \end{pmatrix}$

,

$$(\mathbf{r}_\theta, \mathbf{r}_\theta) = R^2, \quad (\mathbf{r}_h, \mathbf{r}_\varphi) = 0, \quad (\mathbf{r}_\varphi, \mathbf{r}_\varphi) = R^2\sin^2\theta$$

and

$$||g|| = \begin{pmatrix} (\mathbf{r}_u, \mathbf{r}_u) & (\mathbf{r}_u, \mathbf{r}_v) \\ (\mathbf{r}_u, \mathbf{r}_v) & (\mathbf{r}_v, \mathbf{r}_v) \end{pmatrix} =$$

$$\begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix}, \quad G_{S^2} = ds^2 = R^2 d\theta^2 + R^2\sin^2\theta d\varphi^2$$

The length of the curve $\mathbf{r}(t) = \mathbf{r}(\theta(t), \varphi(t))$ on the sphere of the radius $a$ ($a \le t \le b$) can be calculated by the formula:

$$L = \int_a^b R\sqrt{\theta_t^2 + \sin^2\theta \cdot \varphi_t^2} dt \tag{1.67}$$

One can consider on sphere as well as on a circle stereographic coordinates:

$$\begin{cases} u = \frac{Rx}{R-z} \\ v = \frac{Ry}{R-z} \end{cases}, \qquad \begin{cases} x = \frac{2uR^2}{R^2+u^2+v^2} \\ y = \frac{2vR^2}{R^2+u^2+v^2} \\ z = \frac{u^2+v^2-R^2}{u^2+v^2+R^2} R \end{cases} \tag{1.68}$$

In these coordinates Riemannian metric is

$$G = (dx^2 + dy^2 + dz^2)\big|_{x=x(u,v), y=y(u,v), z=z(u,v)} =$$

$$\left(d\left(\frac{2uR^2}{R^2+u^2+v^2}\right)\right)^2 + \left(d\left(\frac{2vR^2}{R^2+u^2+v^2}\right)\right)^2 + \left(d\left(1 - \frac{2R^2}{R^2+u^2+v^2}\right) R\right)^2 =$$

$$= \frac{4R^4(du^2+dv^2)}{(R^2+u^2+v^2)^2}\,. \tag{1.69}$$

(See for detail Homework 2[4].

Notice that we showed that metric on sphere is conformally Euclidean.

*Saddle (paraboloid)*

Consider paraboloid $z = x^2 - y^2$. It can be rewritten as $z = axy$ and it is called sometimes "saddle" (rotation on the angle $\varphi = \pi/4$ transforms $z = x^2 - y^2$ onto $z = 2xy$.) We considered this example in homework 3 Paraboloid and saddle they are ruled surfaces which are formed by lines.

Examples of other quadratic surfaces see in in Appendix.

## 1.4 Isometries of Riemannian manifolds.

### 1.4.1 Riemannian metric induced by map

Let $M$ be a manifold, and let $(N, G(N))$ be a Riemannian manifold. Let $F$ be a map from $M$ to $N$, $F\colon \quad M \longrightarrow N$. We do not suppose that $F$ is diffeomorphism, we even do not suppose that manifolds $M$ and $N$ have the same dimension. We just suppose that $F$ is differentiable map, i.e. local expressions for $F$ are smooth functions.

We can define $F^*G$, a *"Riemannian metric" on manifold $M$ induced by the map $F\colon M \to N$.*

**Remark** It is better to say that $F^*G$ is $\begin{pmatrix}0\\2\end{pmatrix}$ valency symmetric tensor field, which defines Riemannian metric on manifold $M$ in the case if this tensor field is positive definite (see counterexample below). It is why we put the expression `Riemannian metric` in quotes.

Describe the object $F^*G$ on $M$ in local coordinates.

Let $x^a$, $(a = 1, \dots, m)$ be local coordinates on $m$-dimensional manifold $M$ in a vicinity of some point $\mathbf{p}_M$ on $M$. Consider a point $\mathbf{p}_N = F(\mathbf{p}_M)$ on manifold $N$ and let $y^i$, $(i = 1, \dots, n)$ be local coordinates on $n$-dimensional manifold $N$ in a vicinity of point $\mathbf{p}_N$. If in local coordinates $y^i$, Riemannian metric $G^{(N)}$ on $N$ has appearance

$$G^{(N)} = g^{(N)}_{ij}(y)dy^i dy^j\,,$$

[4]Another beautiful deduction of this formula see in Appendices (Inversion)

then in local coordinates $x^a$, Riemannian metric $F^*(G^{(N)})$ on $M$ has appearance

$$F^*(G^{(N)}) = g^{(M)}_{ab}(x)dx^a dx^b = dx^a \frac{\partial y^i(x)}{\partial x^a} g^{(N)}_{ij}(y(x)) \frac{\partial y^j(x)}{\partial x^b} dx^b \,, \quad `$$

i.e.

$$g^{(M)}_{ab}(x) = \frac{\partial y^i(x)}{\partial x^a} g^{(N)}_{ij}(y(x)) \frac{\partial y^j(x)}{\partial x^b} \,, \tag{1.70}$$

where $y^i = y^i(x^a)$ is expression of map $F$ in local coordinates $x^a$ and $y^i$,

**Remark** The object $F^*(G^{(N)})$ on $M$ is *pull-back* of metric $G^{(N)}$ under the map $F\colon M \to N$. This is $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ valency tensor field. In the case if axioms of Riemannian metric are obeyed, i.e. $F^*G$ defines on $M$ Riemannian metric (see paragraph 1.2.1) then pull-back of Riemannian metric is Riemannian metric (see also example and counterexample below)

**Example** The induced metric on surfaces in $\mathbf{E}^3$ is a special example of this general construction. Indeed embedding $\iota\colon M \to \mathbf{E}^3$ is a map from points of 2-dimensional manifold $M$ to points of 3-dimensional Euclidean space $\mathbf{E}^3$. Applying (1.70) to this map we come to formulae (1.42) for induced metric:

$$G_M = \iota^* G_{\mathbf{E}^3} = \iota^*(dx^2 + dy^2 + dz^2) =$$

$$G_{\mathbf{E}^3}\big|_{\mathbf{r}=\mathbf{r}(u,v)} = \left((dx)^2 + (dy)^2 + (dz)^2\right)\big|_{\mathbf{r}=\mathbf{r}(u,v)} = G_M = g_{\alpha\beta} du^\alpha du^\beta$$

(or another manifestation of this formula, equation (1.46)).

**CounterExample** Consider projection of 3-dimensional Euclidean space $\mathbf{E}^3$ with Cartesian coordinates $(x, y, z)$ ($G_{\mathbf{E}^3} = dx^2 + dy^2 + dz^2$) on Euclidean spaces $\mathbf{E}^2$ with Cartesian coordinates $(u, v)$ ($G_{\mathbf{E}^2} = du^2 + dv^2$)

$$F\colon M = \mathbf{E}^3 \ni (x, y, z) \to (u, v) \in \mathbf{E}^2 = N \text{ such that } \begin{cases} u = x \\ v = y \end{cases}$$

Then pull-back

$$F^* G_{\mathbf{E}_2} = F^*\left(du^2 + dv^2\right) = dx^2 + dy^2$$

does not define Euclidean (Riemannian) metric on $\mathbf{E}^3$ since the condition of positive definitness (see paragraph 1.2.1) is not satisfied.

### 1.4.2 Diffeomorphism, which is an isometry

Let $(M_1, G_{(1)})$, $(M_2, G_{(2)})$ be two Riemannian manifolds— manifolds equipped with Riemannian metric $G_{(1)}$ and $G_{(2)}$ respectively.

Loosely speaking isometry is the diffeomorphism of Riemannian manifolds which preserves the distance.

**Definition** Let $F$ be a diffeomorphims (one-one smooth map with smooth inverse) of manifold $M_1$ on manifold $M_2$.

We say that diffeomorphism $F$ is an isometry of Riemannian manifolds $(M_1, G_{(1)})$ and $(M_2, G_{(2)})$ if it preserves the metrics, i.e. $G_{(1)}$ is pull-back of $G_{(2)}$:

$$F^* G_{(2)} = G_{(1)} \,. \tag{1.71}$$

According (1.70) this means that

$$F^*\left(g_{(2)ab}(y) dy^a dy^b\right) = g_{(2)ab}(y) dy^a dy^b\big|_{y=y(x)} =$$

$$g_{(2)ab}(y(x)) \frac{\partial y^a(x)}{\partial x^i} dx^i \frac{\partial y^b(x)}{\partial x^k} dx^k = g_{(1)ik}(x) dx^i dx^k \,,$$

i.e.

$$g_{(1)ik}(x) = \frac{\partial y^a(x)}{\partial x^i} g_{(2)ab}(y(x)) \frac{\partial y^b(x)}{\partial x^k} \,, \tag{1.72}$$

where $y^a = y^a(x)$ is local expression for diffeomorphism $F$. We say that diffeomorphism $F$ is *isometry* of Riemanian manifolds $(M_1, G_{(1)})$ and $M_2, G_{(2)}$. The difference of this equation with equation (1.70) is that $F$ in (1.70) was just a differentiable map, which is not a diffeomorphism. In (1.72) diffeomorphism $F$ establishes one-one correspondence between local coordinates on manifolds $M_1$ and $M_2$. The left hand side of equation (1.72) can be considered as a local expression of metric $G_{(2)}$ in coordinates $x^i$ on $M_2$ and the right hand side of this equation is local expression of metric $G_{(1)}$ in coordinates $x^i$ on $M_1$. Diffeomorphism $F$ identifies manifolds $M_1$ and $M_2$ and it can be considered as changing of coordinates.

**Example** Consider surface of cylinder $C$, $x^2 + y^2 = a^2$ in $\mathbf{E}^3$ with induced Riemannian metric $G_C = a^2 d\varphi^2 + dh^2$ (see equations (1.56) and (1.57)). If we remove the line $l\colon\ x = a, y = 0$ from the cylinder surface $C$ we come to surface $C' = C \backslash l$. Consider a map $F$ of this surface in Euclidean space $E^2$ with Cartesian coordinates $u, v$ (with standard Euclidean metric $G_{Eucl} = du^2 + dv^2$):

$$F\colon \qquad \begin{cases} u = a\varphi \\ v = h \end{cases} \qquad 0 < \varphi < 2\pi \,. \tag{1.73}$$

One can see that $F$ is the diffeomorphism of $C'$ on the domain $0 < u < 2\pi a$ in $\mathbf{E}^2$ and this diffeomorphism is an isometry: it transforms the metric $G_{Eucl}$

on Euclidean space in metric $G_C$ on cylinder, i.e. pull-back condition (1.71) is obeyed:

$$F^* G_{Eucl} = F^* \left(du^2 + dv^2\right) = \left(du^2 + dv^2\right)\big|_{u=a\varphi, v=h} = a^2 d\varphi^2 + dh^2 = G_1 \,.$$

We see that cylinder surface with removed line is isometric to domain in $\mathbf{E}^2$ and the map $F$ establishes this isometry.

**Remark** Notice that if $F$ is diffeomorphism of manifold $M_1$ on a Riemannian manifold $(M_2, G_{(2)}$, then it defines Riemannian structure, the pull-back $G_1 = F^*(G(2))$ on $M_1$, and $F$ is isometry of Riemannian manifold $(M_1, G_{(1)})$ on Riemannian manifold $(M_2, G_{(2)})$.

### 1.4.3 Isometries of Riemannian manifold on itself

**Definition** Let $(M, G)$ be a Riemannian manifold. We say that a diffeomorphism $F$ is an isometry of Riemannian manifold on itself if it preserves the metric, i.e. $F^*G = G$. In local coordinates this means that

$$g_{ik}(x) = g_{pq}(x'(x)) \frac{\partial x^p(x')}{\partial x^i} \frac{\partial x^q(x')}{\partial x^k} \,, \tag{1.74}$$

where $x' = x'(x)$ is a local expression for diffeomorphism $F$. **Example** Let $\mathbf{E}^2$ be Euclidean plane with metric $dx^2 + dy^2$ in Cartesian coordinates $x, y$. Consider the transformation

$$\begin{cases} x' = p + ax + by \\ y' = q + cx + dy \end{cases}$$

is isometry if and only if the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an orthogonal matrix, i.e. if the trasformation above is combination of translation, rotation and reflection.

Another example **Example** Consider Lobachevsky (hyperbolic) plane: an upper half-plain $(y > 0)$ in $\mathbf{R}^2$ equipped with Riemannian metric

$$G = \frac{dx^2 + dy^2}{y^2} \,,$$

One can see that the map

$$\begin{cases} x = \lambda x' \\ y = \lambda y' \end{cases} \,, (\lambda > 0)$$

($\lambda > 0$) is an isometry of the Lobachevsky plane on itself. Are there other isometries? Yes there are (See the disucssion of these questions in Homeworks.)

### 1.4.4 Locally Euclidean Riemannian manifolds

It is useful to formulate the local isometry condition between Riemannian manifold and Euclidean space. A neighbourhood of every point of $n$-dimensional manifold is diffeomorphic to $\mathbf{R}^n$. Let as usual $\mathbf{E}^n$ be $n$-dimensional Euclidean space, i.e. $\mathbf{R}^n$ with standard Riemannian metric $G = dx^i \delta_{ik} dx^k = (dx^1)^2 + \cdots + (dx^n)^2$ in Cartesian coordinates $(x^1, \ldots, x^n)$.

**Definition** We say that $n$-dimensional Riemannian manifold $(M, G)$ is locally isometric to Euclidean space $\mathbf{E}^n$, i.e. it is locally Euclidean Riemannian manifold, if for every point $\mathbf{p} \in M$ there exists an open neighboorhood $D$ (domain) containing this point, $\mathbf{p} \in D$ such that $D$ is isometric to a domain in Euclidean space. In other words in a vicinity of every point $\mathbf{p}$ there exist local coordinates $u^1, \ldots, u^n$ such that Riemannian metric $G$ in these coordinates has an appearance

$$G = du^i \delta_{ik} du^k = (du^1)^2 + \cdots + (du^n)^2 \,. \tag{1.75}$$

The coordinates $(u^1, \ldots, u^n)$ are called *locally Euclidean coordinates.*

Consider examples.

**Example** Consider again cylinder surface..

We know that cylinder is not diffeomorphic to plane (cylinder surface is $S^1 \times \mathbf{R}$, $\mathbf{E}^2 = \mathbf{R} \times \mathbf{R}$, and circle is not diffeomorphic to line ). In the previous subsection we cutted the line from cylindre. Thus we came to surface diffeomorphic to plane. We established that this surface is isometric to Euclidean plane. (See equation (1.73) and considerations above.) Local isometry of cylinder to the Euclidean plane, i.e. the fact that it is locally Euclidean Riemannian surface immediately follows from the fact that under changing of local coordinates $u = a\varphi, v = h$ in equation (1.73), the standard Euclidean metric $du^2 + dv^2$ transforms to the metric $G_{cylinder} = a^2 d\varphi^2 + dh^2$ on cylinder.

**Remark** Strictly speaking we consider all the points except the points on the cutted line (with coordinate $\varphi - 0$). On the other hand for the points on cuttng line we can consider instead coordinate $\varphi$ another coordinate $\varphi' = \varphi - \pi$, $-\pi < \varphi' < \pi$, and we will come to the same answer. In this case the cutted line will be the line $\varphi/ = \pi$.

**Example** Now show that cone is locally Euclidean Riemannian surface, i.e, it is locally isometric to the Euclidean plane. This means that we have to find local coordinates $u, v$ on the cone such that in these coordinates induced metric $G|_c$ on cone would have the appearance $G|_c = du^2 + dv^2$. Recall calculations of the metric on cone in coordinates $h, \varphi$ where

$$\mathbf{r}(h,\varphi)\colon \begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases},$$

$x^2+y^2-k^2z^2 = k^2h^2\cos^2\varphi + k^2h^2\sin^2\varphi - k^2h^2 = k^2h^2 - k^2h^2 = 0$. We have that metric $G_c$ on the cone in coordinates $h, \varphi$ induced with the Euclidean metric $G = dx^2 + dy^2 + dz^2$ is equal to

$$G_c = \left(dx^2 + dy^2 + dz^2\right)\big|_{x=kh\cos\varphi, y=kh\sin\varphi, z=h} = (k\cos\varphi dh - kh\sin\varphi d\varphi)^2 +$$

$$(k\sin\varphi dh + kh\cos\varphi d\varphi)^2 + dh^2 = (k^2+1)dh^2 + k^2h^2d\varphi^2\,.$$

In analogy with polar coordinates try to find new local coordinates $u, v$ such that $\begin{cases} u = \alpha h\cos\beta\varphi \\ v = \alpha h\sin\beta\varphi \end{cases}$, where $\alpha, \beta$ are parameters. We come to $du^2 + dv^2 =$

$$(\alpha\cos\beta\varphi dh - \alpha\beta h\sin\beta\varphi d\varphi)^2 + (\alpha\sin\beta\varphi dh + \alpha\beta h\cos\beta\varphi d\varphi)^2 = \alpha^2dh^2 + \alpha^2\beta^2h^2d\varphi^2.$$

Comparing with the metric on the cone $G_c = (1+k^2)dh^2 + k^2h^2d\varphi^2$ we see that if we put $\alpha = \sqrt{k^2+1}$ and $\beta = \frac{k}{\sqrt{1+k^2}}$ then $du^2 + dv^2 = \alpha^2dh^2 + \alpha^2\beta^2h^2d\varphi^2 = (1+k^2)dh^2 + k^2h^2d\varphi^2$.

Thus in new local coordinates

$$\begin{cases} u = \sqrt{k^2+1}h\cos\frac{k}{\sqrt{k^2+1}}\varphi \\ v = \sqrt{k^2+1}h\sin\frac{k}{\sqrt{k^2+1}}\varphi \end{cases}$$

induced metric on the cone becomes $G|_c = du^2 + dv^2$, i.e. cone locally is isometric to the Euclidean plane ∎

Of course these coordinates are local.— Cone and plane are not homeomorphic, thus they are not globally isometric.

**Example and counterexample**

Consider domain $D$ in Euclidean plane with two metrics:

$$G_{(1)} = du^2 + \sin^2 v dv^2\,, \quad \text{and} \quad G_{(2)} = du^2 + \sin^2 u dv^2 \tag{1.76}$$

Thus we have two different Rimeannian manifolds $(D, G_{(1)})$ and $(D, G_{(2)})$. Metrics in (1.76) look similar. But.... It is easy to see that the first one is locally isometric to Euclidean plane, i.e. it is locally Euclidean Riemannian manifold since $\sin^2 v dv^2 = d(-\cos v)^2$: in new coordinates $u' = u, v' = \cos v$ Riemannian metric $G_{(1)}$ has appearance of standard Euclidean metric:

$$(du')^2 + (dv')^2 = (du)^2 + (d(\cos v))^2 = du^2 + \sin^2 v dv^2 = G_{(1)}\,.$$

This is not the case for second metric $G_{(2)}$. If we change notations $u \mapsto \theta$, $v \mapsto \varphi$ then $G_{(2)} = d\theta^2 + sin\theta^2 d\varphi^2$. This is local expression for Riemannian metric induced on the sphere of radius $R = 1$. Suppose that there exist coordinates $u' = u'(\theta, \varphi)$ $v' = v'(\theta, \varphi)$ such that in these coordinates metric has Eucldean appearance. This means that locally geometry of sphere is as a geometry of Euclidean plane. On the other hand we know from the course of Geometry that this is not the case: sum of angles of triangels on the sphere is not equal to $\pi$, sphere cannot be bended without shrinking (creasing). Later in this course we will return to this question....

There are plenty other examples:

2) Plane with metric $\frac{4R^4(dx^2+dy^2)}{(R^2+x^2+y^2)^2}$ is isometric to the sphere with radius $R$.

3) Disc with metric $\frac{du^2+dv^2}{(1-u^2-v^2)^2}$ is isometric to half plane with metric $\frac{dx^2+dy^2}{4y^2}$.

(see also exercises in Homeworks and Coursework.)

## 1.5 Volume element in Riemannian manifold

The volume element in $n$-dimensional Riemannian manifold with metric $G = g_{ik}dx^i dx^k$ is defined by the formula

$$\sqrt{\det g}\, dx^1 dx^2 \dots dx^n\,. \tag{1.77}$$

If $D$ is a domain in the $n$-dimensional Riemannian manifold with metric $G = g_{ik}dx^i$ then its volume is equal to to the integral of volume element over this domain.

$$V(D) = \int_D \sqrt{\det g}\, dx^1 dx^2 \dots dx^n\,. \tag{1.78}$$

Note that in the case of $n = 1$ volume is just the length, in the case if $n = 2$ it is area.

Why this formula for volume form? One can see that volume form (1.77) is invariant with respect to changing of coordinates i.e. if $y^1, \dots, y^n$ are new coordinates: $x^1 = x^1(y^1, \dots, y^n), x^2 = x^2(y^1, \dots, y^n)$...,

$$x^i = x^i(y^p), i = 1, \dots, n \quad , p = 1, \dots, n$$

and $\tilde{g}_{pq}(y)$ matrix of the metric in new coordinates:

$$\tilde{g}_{pq}(y) = \frac{\partial x^i}{\partial y^p} g_{ik}(x(y)) \frac{\partial x^k}{\partial y^q} \,. \tag{1.79}$$

Then

$$\sqrt{\det g_{ik}(x)}\, dx^1 dx^2 \dots dx^n = \sqrt{\det \tilde{g}_{pq}(y)}\, dy^1 dy^2 \dots dy^n \tag{1.80}$$

This follows from (1.79). Namely

$$\sqrt{\det g_{ik}(y)}\, dy^1 dy^2 \dots dy^n = \sqrt{\det \left( \frac{\partial x^i}{\partial y^p} g_{ik}(x(y)) \frac{\partial x^k}{\partial y^q} \right)}\, dy^1 dy^2 \dots dy^n$$

Using the fact that $\det(ABC) = \det A \cdot \det B \cdot \det C$ and $\det \left( \frac{\partial x^i}{\partial y^p} \right) = \det \left( \frac{\partial x^k}{\partial y^q} \right)$[5] we see that from the formula above follows:

$$\sqrt{\det g_{ik}(y)}\, dy^1 dy^2 \dots dy^n = \sqrt{\det \left( \frac{\partial x^i}{\partial y^p} g_{ik}(x(y)) \frac{\partial x^k}{\partial y^q} \right)} dy^1 dy^2 \dots dy^n =$$

$$\sqrt{\left( \det \left( \frac{\partial x^i}{\partial y^p} \right) \right)^2} \sqrt{\det g_{ik}(x(y))} dy^1 dy^2 \dots dy^n =$$

$$\sqrt{\det g_{ik}(x(y))} \det \left( \frac{\partial x^i}{\partial y^p} \right) dy^1 dy^2 \dots dy^n = \tag{1.81}$$

Now note that

$$\det \left( \frac{\partial x^i}{\partial y^p} \right) dy^1 dy^2 \dots dy^n = dx^1 \dots dx^n$$

according to the formula for changing coordinates in $n$-dimensional integral [6]. Hence

$$\sqrt{\det g_{ik}(x(y))} \det \left( \frac{\partial x^i}{\partial y^p} \right) dy^1 dy^2 \dots dy^n = \sqrt{\det g_{ik}(x(y))} dx^1 dx^2 \dots dx^n \tag{1.82}$$

---

[5] determinant of matrix does not change if we change the matrix on the adjoint, i.e. change columns on rows.

[6] Determinant of the matrix $\left( \frac{\partial x^i}{\partial y^p} \right)$ of changing of coordinates is called sometimes Jacobian. Here we consider the case if Jacobian is positive. If Jacobian is negative then formulae above remain valid just the symbol of modulus appears.

Thus we come to (1.80).

**Remark** Students who know the concept of exterior forms can read the volume element as $n$-form $\sqrt{\det g}\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$

In the next paragraph we will give another motivation of this formula from linear algebra.

### 1.5.1 Motivation: Gram formula for volume of parallelepiped

In this short paragraph we consider formulae for volume of $n$-dimensional parallelepiped, and we explain how formulae (1.77), (1.78) are related with basic formulae in geometry. For simplicity one can consider just the case if $n = 2, 3$.

Let $\mathbf{E}^n$ be Euclidean vector space equipped with orthonormal basis $\{\mathbf{e}_i\}$.

Let $\{\mathbf{a}_i\} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ be an arbitrary row of $n$-vectors in $\mathbf{E}^n$. Consider $n$-dimensional parallelepiped $\Pi_{\{\mathbf{a}_i\}}$ formed by these vectors: $\Pi_{\{\mathbf{a}_i\}} : \mathbf{r} = t^i \mathbf{v}_i, 0 \leq t^i \leq 1$. The volume of this parallelepiped is equal to

$$Vol(\Pi_{\{\mathbf{a}_i\}}) = \det A\,, \tag{1.83}$$

where the matrix $A = ||\mathbf{a}_i^m||$ is defined by expansion of vectors $\{\mathbf{a}_i\}$ over orthonormal basis $\{\mathbf{e}_i\}$: $\mathbf{a}_i = \mathbf{e}_m a_m^i$ (Volume vanishes ($Vol(\Pi_{\mathbf{a}_i}) = \det A = 0$) $\Leftrightarrow$ if $\{\mathbf{a}_i\}$ is not a basis.)

Now consider the scalar product (Riemannian metric) in $\mathbf{E}^n$ in the basis $\mathbf{a}_1, \dots, \mathbf{a}_n$:

$$g_{ik} = \langle \mathbf{a}_i, \mathbf{a}_k \rangle\,, \tag{1.84}$$

where $\langle\ ,\ \rangle$ is scalar product in $\mathbf{E}^n$: $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}$. We see that in (1.84)

$$g_{ij} = \langle \mathbf{a}_i, \mathbf{a}_k \rangle = \langle \sum_m a_i^m \mathbf{e}_m, \sum_n a_j^n \mathbf{e}_n \rangle = a_i^m \delta_{mn} a_i^n = (A^T \cdot A)_{ij} \Rightarrow \det G = (\det A)^2\,,$$

where $G = ||g_{ij}||$. Comparing with formula (1.83) we come to formula:

$$Vol(\Pi_{\mathbf{a}_i}) = \sqrt{\det g_{ik}} \tag{1.85}$$

This formula is called Gram formula, and the matrix $G = ||g_{ik}||$ is called Gram matrix for the vectors $\{\mathbf{a}_i\}$. Gram formula justifies equations (1.77) and (1.78) [7].

[7] We see that $n$-dimensional parallelepiped $\Pi_{\{\mathbf{a}_i\}}$ in new coordinates $t^i$ corresponding

**Remark** One can easy see that formula (1.85) works for arbitrary $n$-dimensional parallelogramm in $m$-dimensional space. Indeed if $\alpha_1, \dots, \alpha_n$ are just arbitrary $n$ vectors in $m$-dimensional Euclidean space then if $n < m$, the formula (1.83) s failed (matrix $A$ is $m \times n$ matrix), but formula (1.85) works. For example the area of parallelogram formed by arbitrary vectors $\mathbf{a}_1, \mathbf{a}_2$ in $\mathbf{E}^n$ is equal to

$$\sqrt{\det \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}} = \sqrt{\det \begin{pmatrix} \langle \mathbf{a}_1, \mathbf{a}_1 \rangle & \langle \mathbf{a}_1, \mathbf{a}_1 \rangle \\ \langle \mathbf{a}_1, \mathbf{a}_1 \rangle & \langle \mathbf{a}_1, \mathbf{a}_1 \rangle \end{pmatrix}}.$$

### 1.5.2 Examples of calculating volume element

Consider first very simple example: Volume element of plane in Cartesian coordinates, metric $g = dx^2 + dy^2$. Volume element is equal to

$$\sqrt{\det g} dxdy = \sqrt{\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} dxdy = dxdy$$

Volume of the domain $D$ is equal to

$$V(D) = \int_D \sqrt{\det g} dxdy = \int_D dxdy$$

If we go to polar coordinates:

$$x = r\cos\varphi, y = r\sin\varphi \tag{1.86}$$

Then we have for metric:

$$G = dr^2 + r^2 d\varphi^2$$

because

$$dx^2 + dy^2 = (dr\cos\varphi - r\sin\varphi d\varphi)^2 + (dr\sin\varphi + r\cos\varphi d\varphi)^2 = dr^2 + r^2 d\varphi^2 \tag{1.87}$$

---

to the basis $\{\mathbf{a}_i\}$ becomes $n$-dimensional cube, Standard Euclidean metric $G = dx^i \delta_{ik} dx^k$ (in orthonormal basis $\{e_i\}$) transforms to $G = dx^i \delta_{ik} dx^k = (a^i_m dt^m)\delta_{ik}(a^k_n dt^n) = (A^T A)_{mn} dt^m dt^n$ and

$$\mathrm{Volume}\Pi_{\{\mathbf{a}_i\}} = \int_{\mathbf{x}\in\Pi} dx^1 \dots dx^n = \int_{0\le t_i \le 1} \sqrt{G} dt^1 \dots dt^n = \sqrt{\det G} = \sqrt{\det A^T A}\,.$$

Volume element in polar coordinates is equal to

$$\sqrt{\det g} dr d\varphi = \sqrt{\det \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}} dr d\varphi = r dr d\varphi\,.$$

*Lobachesvky plane.*

In coordinates $x, y$ $(y > 0)$ metric $G = \frac{dx^2+dy^2}{y^2}$, the corresponding matrix $G = \begin{pmatrix} 1/y^2 & 0 \\ 0 & 1/y^2 \end{pmatrix}$. Volume element is equal to $\sqrt{\det g} dx dy = \frac{dxdy}{y^2}$ .

*Sphere in stereographic coordinates* In stereographic coordinates

$$G = \frac{4R^4(du^2 + dv^2)}{(R + u^2 + v^2)^2} \tag{1.88}$$

(It is isometric to the sphere of the radius $R$ without North pole in stereographic coordinates (see the Homeworks.))

Calculate its volume element and volume. It is easy to see that:

$$G = \begin{pmatrix} \frac{4R^4}{(R^2+u^2+v^2)^2} & 0 \\ 0 & \frac{4R^4}{(R^2+u^2+v^2)^2} \end{pmatrix} \qquad \det g = \frac{16R^8}{(R^2 + u^2 + v^2)^4} \tag{1.89}$$

and volume element is equal to $\sqrt{\det g} du dv = \frac{4R^4 du dv}{(R^2+u^2+v^2)^2}$

One can calculate volume in coordinates $u, v$ but it is better to consider homothety $u \to Ru, v \to Rv$ and polar coordinates: $u = Rr\cos\varphi, v = Rr\sin\varphi$. Then volume form is equal to $\sqrt{\det g} du dv = \frac{4R^4 du dv}{(R^2+u^2+v^2)^2} = \frac{4R^2 r dr d\varphi}{(1+r^2)^2}$.

Now calculation of integral becomes easy:

$$V = \int \frac{4R^2 r dr d\varphi}{(1 + r^2)^2} = 8\pi R^2 \int_0^\infty \frac{r dr}{(1 + r^2)^2} = 4\pi R^2 \int_0^\infty \frac{du}{(1 + u)^2} = 4\pi R^2\,.$$

Domain in Lobachevsky plane.

Consider in Lobachevsky plane the domain $D_a$ such that

$$D_a = \{x, y\colon \quad x^2 + y^2 \geq 1, |x| \leq a\}\,, (|a| \leq 1)\,. \tag{1.90}$$

**Remark** Note that vertical lines and half-circle are geodesics. One can see that the distance between these lines tends to zero[8]see the coursework). If

[8] (

we denote by $A$ a point $(a, \sqrt{1-a^2})$ and by $B$ the point $(-a, \sqrt{1-a^2})$, then the domain $D_a$ can be cinsidered as a 'triangle' with vertices at the point $A, B, C$ where $C$ is a point at infinity. The meaning of this remark we will study later.

One can calculate the area of this domain, using area form on Lobachevsky plane

$$V(D_a) = \int_{-a \leq y \leq a\,, x^2+y^2 \geq 1} \frac{dxdy}{y^2} = 2 \arcsin a \tag{1.91}$$

(See in detail Homework and coursework) We will discuss later the geometrical meaning of this formula.

*Segment of the sphere.*

Consider sphere of the radius $a$ in Euclidean space with standard Riemannian metric

$$a^2 d\theta^2 + a^2 \sin^2 \theta d\varphi^2$$

This metric is nothing but first quadratic form on the sphere (see (1.3.3)). The volume element is

$$\sqrt{\det g} d\theta d\varphi = \sqrt{\det \begin{pmatrix} a^2 & 0 \\ 0 & a^2 \sin\theta \end{pmatrix}} d\theta d\varphi = a^2 \sin\theta d\theta d\varphi$$

Now calculate the volume of the segment of the sphere between two parallel planes, i.e. domain restricted by parallels $\theta_1 \leq \theta \leq \theta_0$: Denote by $h$ be the height of this segment. One can see that

$$h = a\cos\theta_0 - a\cos\theta_1 = a(\cos\theta_0 - a\cos\theta_1)$$

There is remarkable formula which express the area of segment via the height $h$:

$$V = \int_{\theta_1 \leq \theta \leq \theta_0} \left(a^2 \sin\theta\right) d\theta d\varphi = \int_{\theta^0}^{\theta^1} \left(\int_0^{2\pi} \left(a^2 \sin\theta\right) d\varphi\right) d\theta =$$

$$\int_{\theta^1}^{\theta^0} 2\pi a^2 \sin\theta d\theta = 2\pi a^2 (\cos\theta_0 - \cos\theta_1) = 2\pi a(a\cos\theta_0 - acos\theta_1) = 2\pi a h \tag{1.92}$$

E.g. for all the sphere $h = 2a$. We come to $S = 4\pi a^2$. It is remarkable formula: area of the segment is a polynomial function of radius of the sphere and height (Compare with formula for length of the arc of the circle)

# 2 Covariant differentiaion. Connection. Levi Civita Connection on Riemannian manifold

## 2.1 Differentiation of vector field along the vector field.— Affine connection

How to differentiate vector fields on a (smooth )manifold $M$?

Recall the differentiation of functions on a (smooth )manifold $M$.

Let $\mathbf{X} = \mathbf{X^i}(\mathbf{x})\mathbf{e_i}(\mathbf{x}) = \frac{\partial}{\partial \mathbf{x^i}}$ be a vector field on $M$. Recall that vector field [9] $\mathbf{X} = \mathbf{X^i e_i}$ defines at the every point $x_0$ an infinitesimal curve: $x^i(t) = x_0^i + tX^i$ (More exactly the equivalence class $[\gamma(t)]_{\mathbf{X}}$ of curves $x^i(t) = x_0^i + tX^i + \dots$).

Let $f$ be an arbitrary (smooth) function on $M$ and $\mathbf{X} = X^i \frac{\partial}{\partial x^i}$. Then derivative of function $f$ along vector field $\mathbf{X} = X^i \frac{\partial}{\partial x^i}$ is equal to

$$\partial_{\mathbf{X}} f = \nabla_{\mathbf{X}} f = X^i \frac{\partial f}{\partial x^i}$$

The geometrical meaning of this definition is following: If $\mathbf{X}$ is a velocity vector of the curve $x^i(t)$ at the point $x_0^i = x^i(t)$ at the "time" $t = 0$ then the value of the derivative $\nabla_{\mathbf{X}} f$ at the point $x_0^i = x^i(0)$ is equal just to the derivative by $t$ of the function $f(x^i(t))$ at the "time" $t = 0$:

$$\text{if} \quad X^i(x)\big|_{x_0 = x(0)} = \frac{dx^i(t)}{dt}\Big|_{t=0}, \quad \text{then} \quad \nabla_{\mathbf{X}} f\big|_{x^i = x^i(0)} = \frac{d}{dt} f\left(x^i\left(t\right)\right)\Big|_{t=0} \tag{2.1}$$

**Remark** In the course of Geometry and Differentiable Manifolds the operator of taking derivation of function along the vector field was denoted by "$\partial_{\mathbf{X}} f$". In this course we prefer to denote it by "$\nabla_{\mathbf{X}} f$" to have the uniform notation for both operators of taking derivation of functions and vector fields along the vector field.

One can see that the operation $\nabla_X$ on the space $C^\infty(M)$ (space of smooth functions on the manifold) satisfies the following conditions:

- $\nabla_{\mathbf{X}} (\lambda f + \mu g) = \lambda \nabla_{\mathbf{X}} f + \mu \nabla_{\mathbf{X}} g$ where $\lambda, \mu \in \mathbf{R}$ (linearity over numbers )

[9] here like always we suppose by default the summation over repeated indices. E.g.$\mathbf{X} = X^i \mathbf{e}_i$ is nothing but $\mathbf{X} = \sum_{i=1}^n X^i \mathbf{e}_i$

- $\nabla_{h\mathbf{X}+g\mathbf{Y}}(f) = h\nabla_{\mathbf{X}}(f) + g\nabla_{\mathbf{Y}}(f)$ (linearity over the space of functions)
- $\nabla_{\mathbf{X}}(\lambda fg) = f\nabla_{\mathbf{X}}(\lambda g) + g\nabla_{\mathbf{X}}(\lambda f)$ (Leibnitz rule)

$$(2.2)$$

**Remark** One can prove that these properties characterize vector fields:operator on smooth functions obeying the conditions above is a vector field. (You will have a detailed analysis of this statement in the course of Differentiable Manifolds.)

How to define differentiation of vector fields along vector fields.

The formula (2.1) cannot be generalised straightforwardly because vectors at the point $x_0$ and $x_0 + tX$ are vectors from different vector spaces. (We cannot substract the vector from one vector space from the vector from the another vector space, because *apriori* we cannot compare vectors from different vector space. One have to define an operation of transport of vectors from the space $T_{x_0}M$ to the point $T_{x_0+tX}M$ defining the transport from the point $T_{x_0}M$ to the point $T_{x_0+tX}M$).

Try to define the operation $\nabla$ on vector fields such that conditions (2.2) above be satisfied.

### 2.1.1 Definition of connection. Christoffel symbols of connection

**Definition** Affine connection on $M$ is the *operation* $\nabla$ which assigns to every vector field $\mathbf{X}$ a linear map, (but not necessarily $C(M)$-linear map!) (i.e. a map which is linear over numbers not necessarily over functions) $\nabla_{\mathbf{X}}$ on the space of vector fields on $M$:

$$\nabla_{\mathbf{X}}(\lambda\mathbf{Y} + \mu\mathbf{Z}) = \lambda\nabla_{\mathbf{X}}\mathbf{Y} + \mu\nabla_{\mathbf{X}}\mathbf{Z}, \qquad \text{for every } \lambda, \mu \in \mathbf{R} \tag{2.3}$$

(Compare the first condition in (2.2)).
which satisfies the following conditions:

- for arbitrary (smooth) functions $f, g$ on $M$

  $$\nabla_{f\mathbf{X}+\mathbf{g}\mathbf{Y}}(\mathbf{Z}) = f\nabla_{\mathbf{X}}(\mathbf{Z}) + g\nabla_{\mathbf{Y}}(\mathbf{Z}) \qquad (C^\infty(M)\text{-linearity}) \tag{2.4}$$

  (compare with second condition in (2.2))

- for arbitrary function $f$

$$\nabla_{\mathbf{X}}(f\mathbf{Y}) = (\nabla_{\mathbf{X}} f)\,\mathbf{Y} + f\nabla_{\mathbf{X}}(\mathbf{Y}) \qquad \text{(Leibnitz rule)} \tag{2.5}$$

  Recall that $\nabla_{\mathbf{X}} f$ is just usual derivative of a function $f$ along vector field: $\nabla_{\mathbf{X}} f = \partial_{\mathbf{X}} f$.

  (Compare with Leibnitz rule in (2.2)).

  *The operation $\nabla_{\mathbf{X}}\mathbf{Y}$ is called covariant derivative of vector field $\mathbf{Y}$ along the vector field $\mathbf{X}$.*

Write down explicit formulae in a given local coordinates $\{x^i\}$ ($i = 1, 2, \ldots, n$) on manifold $M$.

Let

$$\mathbf{X} = X^i \mathbf{e}_i = X^i \frac{\partial}{\partial x^i} \qquad \mathbf{Y} = Y^i \mathbf{e}_i = Y^i \frac{\partial}{\partial x^i}$$

The basis vector fields $\frac{\partial}{\partial x^i}$ we denote sometimes by $\partial_i$ sometimes by $\mathbf{e}_i$

Using properties above one can see that

$$\nabla_{\mathbf{X}}\mathbf{Y} = \nabla_{X^i \partial_i} Y^k \partial_k = X^i \left(\nabla_i \left(Y^k \partial_k\right)\right), \qquad \text{where } \nabla_i = \nabla_{\partial_i} \tag{2.6}$$

Then according to (2.4)

$$\nabla_i \left(Y^k \partial_k\right) = \nabla_i \left(Y^k\right) \partial_k + Y^k \nabla_i \partial_k$$

Decompose the vector field $\nabla_i \partial_k$ over the basis $\partial_i$:

$$\nabla_i \partial_k = \Gamma^m_{ik} \partial_m \tag{2.7}$$

and

$$\nabla_i \left(Y^k \partial_k\right) = \frac{\partial Y^k(x)}{\partial x^i} \partial_k + Y^k \Gamma^m_{ik} \partial_m, \tag{2.8}$$

$$\nabla_{\mathbf{X}}\mathbf{Y} = X^i \frac{\partial Y^m(x)}{\partial x^i} \partial_m + X^i Y^k \Gamma^m_{ik} \partial_m, \tag{2.9}$$

In components

$$(\nabla_{\mathbf{X}}\mathbf{Y})^m = X^i \left(\frac{\partial Y^m(x)}{\partial x^i} + Y^k \Gamma^m_{ik}\right) \tag{2.10}$$

Coefficients $\{\Gamma^m_{ik}\}$ are called *Christoffel symbols* in coordinates $\{x^i\}$. These coefficients define covariant derivative—**connection**.

If operation of taking covariant derivative is given we say that the connection is given on the manifold. Later it will be explained why we us the word "connection"

We see from the formula above that to define covariant derivative of vector fields, connection, we have to define Christoffel symbols in local coordinates.

### 2.1.2 Transformation of Christoffel symbols for an arbitrary connection

Let $\nabla$ be a connection on manifold $M$. Let $\{\Gamma^i_{km}\}$ be Christoffel symbols of this connection in given local coordinates $\{x^i\}$. Then according (2.7) and (2.8) we have

$$\nabla_{\mathbf{X}}\mathbf{Y} = X^m \frac{\partial Y^i}{\partial x^m}\frac{\partial}{\partial x^i} + X^m \Gamma^i_{mk} Y^k \frac{\partial}{\partial x^i},$$

and in particularly

$$\Gamma^i_{mk}\partial_i = \nabla_{\partial_m}\partial_k$$

Use this relation to calculate Christoffel symbols in new coordinates $x^{i'}$

$$\Gamma^{i'}_{m'k'}\partial_{i'} = \nabla_{\partial'_m}\partial_{k'}$$

We have that $\partial_{m'} = \frac{\partial}{\partial x^{m'}} = \frac{\partial x^m}{\partial x^{m'}}\frac{\partial}{\partial x^m} = \frac{\partial x^m}{\partial x^{m'}}\partial_m$. Hence due to properties (2.4), (2.5) we have

$$\Gamma^{i'}_{m'k'}\partial_{i'} = \nabla_{\partial_{m'}}\partial_{k'} = \nabla_{\partial'_m}\left(\frac{\partial x^k}{\partial x^{k'}}\partial_k\right) = \left(\frac{\partial x^k}{\partial x^{k'}}\right)\nabla_{\partial'_m}\partial_k + \frac{\partial}{\partial x^{m'}}\left(\frac{\partial x^k}{\partial x^{k'}}\right)\partial_k =$$

$$\left(\frac{\partial x^k}{\partial x^{k'}}\right)\nabla_{\frac{\partial x^m}{\partial x^{m'}}\partial_m}\partial_k + \frac{\partial^2 x^k}{\partial x^{m'}\partial x^{k'}}\partial_k = \frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\nabla_{\partial_m}\partial_k + \frac{\partial^2 x^k}{\partial x^{m'}\partial x^{k'}}\partial_k$$

$$\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\Gamma^i_{mk}\partial_i + \frac{\partial^2 x^k}{\partial x^{m'}\partial x^{k'}}\partial_k = \frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\Gamma^i_{mk}\frac{\partial x^{i'}}{\partial x^i}\partial_{i'} + \frac{\partial^2 x^k}{\partial x^{m'}\partial x^{k'}}\frac{\partial x^{i'}}{\partial x^k}\partial_{i'}$$

Comparing the first and the last term in this formula we come to the transformation law:

If $\{\Gamma^i_{km}\}$ are Christoffel symbols of the connection $\nabla$ in local coordinates $\{x^i\}$ and $\{\Gamma^{i'}_{k'm'}\}$ are Christoffel symbols of this connection in new local coordinates $\{x^{i'}\}$ then

$$\Gamma^{i'}_{k'm'} = \frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^i}\Gamma^i_{km} + \frac{\partial^2 x^r}{\partial x^{k'}\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^r} \tag{2.11}$$

**Remark** Christoffel symbols do not transform as tensor. If the second term is equal to zero, i.e. transformation of coordinates are linear (see the Proposition on flat connections) then the transformation rule above is the the same as a transformation rule for tensors of the type $\begin{pmatrix}1\\2\end{pmatrix}$ (see the formula (1.11)). In general case this is not true. Christoffel symbols do not transform as tensor under arbitrary non-linear coordinate transformation: see the second term in the formula above.

**Remark** On the other hand note that *difference of two arbitrary connections is a tensor.* If $\Gamma^i_{km}$ and $\tilde{\Gamma}^i_{km}$ are corresponding Chrstoffel symbols then it follows from (1.11)) that their difference $T^i_{km} = \Gamma^i_{km} - \tilde{\Gamma}^i_{km}$ transforms as a tensor:

$$T^{i'}_{k'm'} = \Gamma^{i'}_{k'm'} - \tilde{\Gamma}^{i'}_{k'm'} = \frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^i}\left(\Gamma^i_{km} - \tilde{\Gamma}^i_{km}\right) = \frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^i}T^i_{km}$$

(See for detail the Homework 5.)

### 2.1.3 Canonical flat affine connection

It follows from the properties of connection that it is suffice to define connection at vector fields which form basis at the every point using (2.7), i.e. to define Christoffel symbols of this connection.

**Example** Consider $n$-dimensional Euclidean space $\mathbf{E}^n$ with Cartesian coordinates $\{x^1, \dots, x^n\}$.

Define connection such that all Christoffel symbols are equal to zero in these Cartesian coordinates $\{x^i\}$.

$$\nabla_{\mathbf{e}_i}\mathbf{e}_k = \Gamma^m_{ik}\mathbf{e}_m = 0, \quad \Gamma^m_{ik} = 0 \tag{2.12}$$

Does this mean that Christoffel symbols are equal to zero in an arbitrary Cartesian coordinates if they equal to zero in given Cartesian coordinates?

Does this mean that Christoffel symbols of this connection equal to zero in arbitrary coordinates system?

it follows from transformation rules (2.11) for Christoffel symbols that Christofel symbols vanish also in new coordinates $x^{i'}$ if and only if

$$\frac{\partial^2 x^i}{\partial x^{m'}\partial x^{i'}} = 0, \text{ i.e. } x^i = b^i + a^i_k x^k \tag{2.13}$$

i.e. the relations between new and old coordinates are linear. We come to simple but very important

**Proposition** *Let all Christoffel symbols of a given connection be equal to zero in a given coordinate system $\{x^i\}$. Then all Christoffel symbols of this connection are equal to zero in an arbitrary coordinate system $\{x^{i'}\}$ such that the relations between new and old coordinates are linear:*

$$x^{i'} = b^i + a^i_k x^k \tag{2.14}$$

*If transformation to new coordinate system is not linear, i.e.* $\frac{\partial^2 x^i}{\partial x^{m'} \partial x^{i'}} \neq 0$ *then Christoffel symbols of this connection in general are not equal to zero in new coordinate system $\{x^{i'}\}$.*

**Definition** We call connection $\nabla$ flat if there exists coordinate system such that all Christoffel symbols of this connection are equal to zero in a given coordinate system.

In particular connection (2.12) has zero Christoffel symbols in arbitrary Cartesian coordinates.

**Corollary** Connection has zero Christoffel symbols in arbitrary Cartesian coordinates if it has zero Christoffel symbols in a given Cartesian coordinates.

Hence the following definition is correct:

**Definition** A connection on the Euclidean space $\mathbf{E}^n$ which Christoffel symbols vanish in Cartesian coordinates is called *canonical flat connection.*

**Remark** Canonical flat connection in Euclidean space is uniquely defined, since Cartesian coordinates are defined globally. On the other hand on arbitrary manifold one can define flat connection locally just choosing any arbitrary *local* coordinates and define *locally flat connection* by condition that Christoffel symbols vanish in these local coordinates. This does not mean that one can define flat connection *globally.* We will study this question after learning transformation law for Christoffel symbols.

**Remark** One can see that flat connection is symmetric connection.

**Example** Consider a connection (2.12) in $\mathbf{E}^2$. It is a flat connection. Calculate Christoffel symbols of this connection in polar coordinates

$$\begin{cases} x = r\cos\varphi \\ y = y\sin\varphi \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2} \\ \varphi = \arctan\frac{y}{x} \end{cases} \tag{2.15}$$

Write down Jacobians of transformations—matrices of partial derivatives:

$$\begin{pmatrix} x_r & y_r \\ x_\varphi & y_\varphi \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -r\sin\varphi & r\cos\varphi \end{pmatrix}, \qquad \begin{pmatrix} r_x & \varphi_x \\ r_y & \varphi_y \end{pmatrix} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & -\frac{y}{x^2+y^2} \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{x}{x^2+y^2} \end{pmatrix} \tag{2.16}$$

According (2.11) and since Chrsitoffel symbols are equal to zero in Cartesian coordinates $(x, y)$ we have

$$\Gamma^{i'}_{k'm'} = \frac{\partial^2 x^r}{\partial x^{k'} \partial x^{m'}} \frac{\partial x^{i'}}{\partial x^r}, \tag{2.17}$$

where $(x^1, x^2) = (x, y)$ and $(x^{1'}, x^{2'}) = (r, \varphi)$. Now using (2.16) we have

$$\Gamma^r_{rr} = \frac{\partial^2 x}{\partial r \partial r} \frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial r \partial r} \frac{\partial r}{\partial y} = 0$$

$$\Gamma^r_{r\varphi} = \Gamma^r_{\varphi r} = \frac{\partial^2 x}{\partial r \partial \varphi} \frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial r \partial \varphi} \frac{\partial r}{\partial y} = -\sin\varphi\cos\varphi + \sin\varphi\cos\varphi = 0\,.$$

$$\Gamma^r_{\varphi\varphi} = \frac{\partial^2 x}{\partial \varphi \partial \varphi} \frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial r \partial \varphi} \frac{\partial r}{\partial y} = -x\frac{x}{r} - y\frac{y}{r} = -r\,.$$

$$\Gamma^\varphi_{rr} = \frac{\partial^2 x}{\partial r \partial r} \frac{\partial \varphi}{\partial x} + \frac{\partial^2 y}{\partial r \partial r} \frac{\partial \varphi}{\partial y} = 0\,.$$

$$\Gamma^\varphi_{\varphi r} = \Gamma^\varphi_{r\varphi} = \frac{\partial^2 x}{\partial r \partial \varphi} \frac{\partial \varphi}{\partial x} + \frac{\partial^2 y}{\partial r \partial \varphi} \frac{\partial \varphi}{\partial y} = -\sin\varphi\frac{-y}{r^2} + \cos\varphi\frac{x}{r^2} = \frac{1}{r}$$

$$\Gamma^\varphi_{\varphi\varphi} = \frac{\partial^2 x}{\partial \varphi \partial \varphi} \frac{\partial \varphi}{\partial x} + \frac{\partial^2 y}{\partial \varphi \partial \varphi} \frac{\partial \varphi}{\partial y} = -x\frac{-y}{r^2} - y\frac{x}{r^2} = 0\,. \tag{2.18}$$

Hence we have that the covariant derivative (2.12) in polar coordinates has the following appearance

$$\nabla_r \partial_r = \Gamma^r_{rr}\partial_r + \Gamma^\varphi_{rr}\partial_\varphi = 0\,,, \qquad \nabla_r \partial_\varphi = \Gamma^r_{r\varphi}\partial_r + \Gamma^\varphi_{r\varphi}\partial_\varphi = \frac{\partial_\varphi}{r}$$

$$\nabla_\varphi \partial_r = \Gamma^r_{\varphi r}\partial_r + \Gamma^\varphi_{\varphi r}\partial_\varphi = \frac{\partial_\varphi}{r}, \qquad \nabla_\varphi \partial_\varphi = \Gamma^r_{\varphi\varphi}\partial_r + \Gamma^\varphi_{\varphi\varphi}\partial_\varphi = -r\partial_r \tag{2.19}$$

**Remark** Later when we study geodesics we will learn a very quick method to calculate Christoffel symbols.

### 2.1.4 Space of connections

.

We defined axiomatically connection. How look aspace of all conneccions? Denote by $\mathcal{A}_M$ the space of all connecctions on manifold $M$. We still have only one example of connection: canonical flat connection in $\mathbf{E}^n$.

In other words we proved that space $\mathcal{A}_{\mathbf{E}^n}$ is not empty. Are there another connections?

**Proposition** Let $\nabla$ be an arbitrary connection on manifold $M$. Then operation $\widetilde{\nabla}$ is connection if and only if the difference

$$\widetilde{\Gamma}^i_{km} - \Gamma^i_{km} = T^i_{km} \tag{2.20}$$

is tensor of valency $\begin{pmatrix}1\\2\end{pmatrix}$. Here as usual $\Gamma^i_{km}$ are Christoffel symbols of connection $\Delta$ in local coordinates $\{x^i\}$ and $\widetilde{\Gamma}^i_{km}$ are Christoffel symbols of connection $\widetilde{\Delta}$ in same local coordinates $\{x^i\}$.

In other words space of connections is affine space associated with vector space of tensor fileds of valency $\begin{pmatrix}1\\2\end{pmatrix}$.

Roughly speaking one can say that there are as many connections as many tensor fields of valency $\begin{pmatrix}1\\2\end{pmatrix}$.

`Proof`

Recall that under change $\{x^i\} \to \{x^{i'}\}$ of local coordinates Christofel symbols of connection transform according equation (2.11), i.e.

$$\Gamma^{i'}_{k'm'} = \frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^i}\Gamma^i_{km} + \frac{\partial^2 x^r}{\partial x^{k'}\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^r}$$

and

$$\widetilde{\Gamma}^{i'}_{k'm'} = \frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^i}\widetilde{\Gamma}^i_{km} + \frac{\partial^2 x^r}{\partial x^{k'}\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^r}\,. \tag{2.21}$$

Substracting these equations we come to

$$T^{i'}_{k'm'} = \widetilde{\Gamma}^{i'}_{k'm'} - \Gamma^{i'}_{k'm'} = \frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^i}\left(\widetilde{\Gamma}^i_{km} - \Gamma^i_{km}\right) = \frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^i}T^i_{km}\,,$$

i.e. $T^i_{km}$ are components of tensor of valency $\begin{pmatrix}1\\2\end{pmatrix}$ (see paragraph 1.1.2).

`Another proof`

Let operations $\nabla$ and $\tilde{\nabla}$ be two connections. Consider operation

$$T(\mathbf{X},\mathbf{Y}) = \tilde{\nabla}_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{X}}\mathbf{Y}\,.$$

Using axioms for connection (see paragraph 2.1.1) we show that $T(\mathbf{X},\mathbf{Y})$ is linear operation and moreover for arbitrary functions $f,g$

$$T(f\mathbf{X},g\mathbf{Y}) = fgT(\mathbf{X},\mathbf{Y})\,. \tag{2.22}$$

This implies that it is tensor. It follows from linearity and equation (2.22) that

$$T\left(\mathbf{X},\mathbf{Y}\right) = T\left(X^k(x)\partial_k, Y^m(x)\partial_m\right) = X^k(x)Y^m(x)T(\partial_k,\partial_m)T^i_{km}\partial_i\,, \text{where } T(\partial_k,\partial_m) = T^i_{km}\partial_i\,.$$

.

**Example** Let $\nabla$ be an affine connection on a 2-dimensional manifold $M$ such that in local coordinates $(x,y)$

$$\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} = x\frac{\partial}{\partial y}\,.$$

Calculate the Christoffel symbols $\Gamma^x_{xx}$ and $\Gamma^y_{xx}$.

Let $\omega = a(x,y)dx + b(x,y)dy$ be the differential 1-form on $M$. Consider the operation

$$\tilde{\nabla}_{\mathbf{X}}\mathbf{Y} = \nabla_{\mathbf{X}}\mathbf{Y} + \omega(\mathbf{Y})\mathbf{X}$$

Show that this operation is also an affine connection, and calculate the Christoffel symbols $\tilde{\Gamma}^x_{xx}$, and $\tilde{\Gamma}^y_{xx}$ of this new connection.

First calculate Christoffel symbols symbols $\Gamma^x_{xx}$ and $\Gamma^y_{xx}$ of initial connection. We have:

$$\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} = \Gamma^x_{xx}\frac{\partial}{\partial x} + \Gamma^y_{xx}\frac{\partial}{\partial y} = x\frac{\partial}{\partial y} \Rightarrow\ \Gamma^x_{xx} = 0\,, +\Gamma^y_{xx} = x\,.$$

Check straightforwardly that the operation

$$\tilde{\nabla}_{\mathbf{X}}\mathbf{Y} = \nabla_{\mathbf{X}}\mathbf{Y} + \omega(\mathbf{Y})\mathbf{X}$$

is also connection, i.e. it obeys obeys conditions (2.3), (2.4) and (2.5) (see paragraph 2.1.1) [10].

[10] One can see that the difference between connection $\nabla$ and new operation $\tilde{\nabla}$ is defined by tensor: $T(\mathbf{X},\mathbf{Y}) = \tilde{\nabla}_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{X}}\mathbf{Y} = \mathbf{X}\omega(\mathbf{Y})$ Due to Proposition this implies that $\tilde{\nabla}$ is a connection too.

- $$\widetilde{\nabla}_{\mathbf{X}}(\lambda\mathbf{Y}+\mu\mathbf{Z}) = \nabla_{\mathbf{X}}(\lambda\mathbf{Y}+\mu\mathbf{Z}) + \omega(\lambda\mathbf{Y}+\mu\mathbf{Z})\,\mathbf{X} =$$
$$\lambda\left(\nabla_{\mathbf{X}}\mathbf{Y}+\omega(\mathbf{Y})\mathbf{X}\right)\mu\left(\nabla_{\mathbf{X}}\mathbf{Z}+\omega(\mathbf{Z})\mathbf{Y}\right) = \lambda\widetilde{\nabla}_{\mathbf{X}}\mathbf{Y}+\mu\widetilde{\nabla}_{\mathbf{X}}\mathbf{Z}\,.$$
  Hence condition (2.3) is obeyed.

- $$\widetilde{\nabla}_{f\mathbf{X}+\mathbf{g}\mathbf{Y}}\mathbf{Z} = \nabla_{f\mathbf{X}+\mathbf{g}\mathbf{Y}}\mathbf{Z}+\omega(\mathbf{Z})(f\mathbf{X}+\mathbf{g}\mathbf{Y}) =$$
$$f\nabla_{\mathbf{X}}\mathbf{Z}+g\nabla_{\mathbf{Y}}\mathbf{Z}+f\omega(\mathbf{Z})\mathbf{X}+g\omega(\mathbf{Z})\mathbf{Y} = f\widetilde{\nabla}_{\mathbf{X}}\mathbf{Z}+g\widetilde{\nabla}_{\mathbf{Y}}\mathbf{Z}$$
  Hence condition (2.4) is obeyed.

- $$\widetilde{\nabla}_{\mathbf{X}}(f\mathbf{Y}) = \nabla_{\mathbf{X}}(f\mathbf{Y})+\omega(f\mathbf{Y})\mathbf{X} = f\nabla_{\mathbf{X}}\mathbf{Y}+\partial_{\mathbf{X}}f\mathbf{Y}+f\omega(\mathbf{Y})\mathbf{X} = f\widetilde{\nabla}_{\mathbf{X}}\mathbf{Y}+\partial_{\mathbf{X}}f\mathbf{Y}$$
  Hence condition (2.5) is also obeyed.

Hence we proved that $\widetilde{\nabla}$ is also the connection.

Now express the Christoffel symbols of new connection through the Christoffel symbols of connection $\nabla$

We have that

$$\omega\left(\frac{\partial}{\partial x}\right) = (adx+bdy)\left(\frac{\partial}{\partial x}\right) = a\,,\text{and }\omega\left(\frac{\partial}{\partial y}\right) = (adx+bdy)\left(\frac{\partial}{\partial y}\right) = b$$

hence

$$\widetilde{\nabla}_{\partial_x}\partial_x = \widetilde{\Gamma}^x_{xx}\partial_x+\widetilde{\Gamma}^y_{xx}\partial_y = \nabla_{\partial_x}\partial_x+\omega(\partial_x)\partial_x = \Gamma^x_{xx}\partial_x+\Gamma^y_{xx}\partial_y+a(x,y)\partial_x\,,$$

hence

$$\widetilde{\Gamma}^x_{xx} = \Gamma^x_{xx}+a\,,\widetilde{\Gamma}^y_{xx} = \Gamma^y_{xx}\,.$$

In the same way fpr other symbols. E.g.

$$\widetilde{\nabla}_{\partial_x}\partial_y = \widetilde{\Gamma}^x_{xy}\partial_x+\widetilde{\Gamma}^y_{xy}\partial_y = \nabla_{\partial_x}\partial_y+\omega(\partial_y)\partial_x = \Gamma^x_{xy}\partial_x+\Gamma^y_{xy}\partial_y+b(x,y)\partial_x\,,$$

hence

$$\widetilde{\Gamma}^x_{xy} = \Gamma^x_{xy}+b\,,\widetilde{\Gamma}^y_{xy} = \Gamma^y_{xy}\,.$$

## 2.2 Connection induced on the surfaces

Let $M$ be a manifold embedded in Euclidean spaceCanonical flat connection on $\mathbf{E}^N$ induces the connection on surface in the following way.

Let $\mathbf{X}, \mathbf{Y}$ be tangent vector fields to the surface $M$ and $\nabla^{\text{can.flat}}$ a canonical flat connection in $\mathbf{E}^N$. In general

$$\mathbf{Z} = \nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y} \quad \text{is not tangent to manifold } M \tag{2.23}$$

Consider its decomposition on two vector fields:

$$\mathbf{Z} = \mathbf{Z}_{tangent} + \mathbf{Z}_{\perp}, \nabla^{\text{can.flat}}_{\mathbf{X}}, \mathbf{Y} = \left(\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}\right)_{tangent} + \left(\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}\right)_{\perp}, \tag{2.24}$$

where $\mathbf{Z}_{\perp}$ is a component of vector which is orthogonal to the surface $M$ and $\mathbf{Z}_{||}$ is a component which is tangent to the surface. Define an induced connection $\nabla^M$ on the surface $M$ by the following formula

$$\nabla^M: \qquad \nabla^M_{\mathbf{X}}\mathbf{Y} := \left(\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}\right)_{tangent} \tag{2.25}$$

One can see that this formula really defines the connection on surface $M$, i.e. the operation defined by this relation obeys all axioms of connection. Indeed it is easy to see that for arbitrary vector fields $\mathbf{X}$ and $\mathbf{Y}$, the vector field $\nabla^M_{\mathbf{X}}\mathbf{Y}$ is tangent vector field, and this operation obeys relations (2.3), (2.4) and (2.5). For example check Leibnitz rule:

$$\nabla^M_{\mathbf{X}}(f\mathbf{Y}) = \left(\nabla^{\text{can.flat}}_{\mathbf{X}}(f\mathbf{Y})\right)_{tangent} = \left(\partial_{\mathbf{X}}f\mathbf{Y} + f\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}\right)_{tangent} =$$

$$\partial_{\mathbf{X}}f\mathbf{Y} + f\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}_{tangent} = \partial_{\mathbf{X}}f\mathbf{Y} + f\nabla^M_{\mathbf{X}}\mathbf{Y}\,.$$

We mainly apply this construction for 2-dimensional manifolds (surfaces) in $\mathbf{E}^3$.

### 2.2.1 Calculation of induced connection on surfaces in $\mathbf{E}^3$.

Let $\mathbf{r} = \mathbf{r}(u, v)$ be a surface in $\mathbf{E}^3$. Let $\nabla^{\text{can.flat}}$ be a flat connection in $\mathbf{E}^3$. Then

$$\nabla^M: \qquad \nabla^M_{\mathbf{X}}\mathbf{Y} := \left(\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}\right)_{||} = \nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y} - \mathbf{n}(\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}, \mathbf{n}), \tag{2.26}$$

where $\mathbf{n}$ is normal unit vector field to $M$. Consider a special example

**Example** (Induced connection on sphere) Consider a sphere of the radius $R$ in $\mathbf{E}^3$:

$$\mathbf{r}(\theta,\varphi)\colon \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases}$$

then

$$\mathbf{r}_\theta = \begin{pmatrix} R\cos\theta\cos\varphi \\ R\cos\theta\sin\varphi \\ -R\sin\theta \end{pmatrix}, \mathbf{r}_\varphi = \begin{pmatrix} -R\sin\theta\sin\varphi \\ R\sin\theta\cos\varphi \\ 0 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} sin\theta\cos\varphi \\ sin\theta\sin\varphi \\ \cos\theta \end{pmatrix},$$

where $\mathbf{r}_\theta = \frac{\partial \mathbf{r}(\theta,\varphi)}{\partial\theta}$, $\mathbf{r}_\varphi = \frac{\partial \mathbf{r}(\theta,\varphi)}{\partial\varphi}$ are basic tangent vectors and $\mathbf{n}$ is normal unit vector.

Calculate an induced connection $\nabla$ on the sphere.

First calculate $\nabla_{\partial_\theta}\partial_\theta$.

$$\nabla_{\partial_\theta}\partial_\theta = \left(\frac{\partial \mathbf{r}_\theta}{\partial\theta}\right)_{tangent} = (\mathbf{r}_{\theta\theta})_{tangent}.$$

On the other hand one can see that $\mathbf{r}_{\theta\theta} = \begin{pmatrix} -Rsin\theta\cos\varphi \\ -Rsin\theta\sin\varphi \\ -R\cos\theta \end{pmatrix} = -R\mathbf{n}$ is proportional to normal vector, i.e. $(\mathbf{r}_{\theta\theta})_{tangent} = 0$. We come to

$$\nabla_{\partial_\theta}\partial_\theta = (\mathbf{r}_{\theta\theta})_{tangent} = 0 \Rightarrow \Gamma^\theta_{\theta\theta} = \Gamma^\varphi_{\theta\theta} = 0\,. \qquad (2.27)$$

**Remark** Notice that equation (2.27) follows from teh fact that $\mathbf{r}_{\theta\theta}$ is centripetal acceleration which is directed along $\mathbf{r} \sim \mathbf{n}$.

Now calculate $\nabla_{\partial_\theta}\partial_\varphi$ and $\nabla_{\partial_\varphi}\partial_\theta$.

$$\nabla_{\partial_\theta}\partial_\varphi = \left(\frac{\partial \mathbf{r}_\varphi}{\partial\theta}\right)_{tangent} = (\mathbf{r}_{\theta\varphi})_{tangent}\,, \nabla_{\partial_\varphi}\partial_\theta = \left(\frac{\partial \mathbf{r}_\theta}{\partial\varphi}\right)_{tangent} = (\mathbf{r}_{\varphi\theta})_{tangent}$$

We have

$$\nabla_{\partial_\theta}\partial_\varphi = \nabla_{\partial_\varphi}\partial_\theta = (\mathbf{r}_{\varphi\theta})_{tangent} = \begin{pmatrix} -R\cos\theta\sin\varphi \\ R\cos\theta\cos\varphi \\ 0 \end{pmatrix}_{tangent}.$$

We see that the vector $\mathbf{r}_{\varphi\theta}$ is orthogonal to $\mathbf{n}$:

$$\langle \mathbf{r}_{\varphi\theta}, \mathbf{n}\rangle = -R\cos\theta\sin\varphi\sin\theta\cos\varphi + R\cos\theta\cos\varphi\sin\theta\sin\varphi = 0.$$

Hence

$$\nabla_{\partial_\theta}\partial_\varphi = \nabla_{\partial_\varphi}\partial_\theta = (\mathbf{r}_{\varphi\theta})_{tangent} = \mathbf{r}_{\varphi\theta} = \begin{pmatrix} -R\cos\theta\sin\varphi \\ R\cos\theta\cos\varphi \\ 0 \end{pmatrix} = \text{cotan}\,\theta\mathbf{r}_\varphi\,.$$

We come to

$$\nabla_{\partial_\theta}\partial_\varphi = \nabla_{\partial_\varphi}\partial_\theta = \text{cotan}\,\theta\partial_\varphi \Rightarrow \Gamma^\theta_{\theta\varphi} = \Gamma^\theta_{\varphi\theta} = 0,\ \Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = \text{cotan}\,\theta \quad (2.28)$$

Finally calculate $\nabla_{\partial_\varphi}\partial_\varphi$

$$\nabla_{\partial_\varphi}\partial_\varphi = (\mathbf{r}_{\varphi\varphi})_{tangent} = \left(\begin{pmatrix} -R\sin\theta\cos\varphi \\ -R\sin\theta\sin\varphi \\ 0 \end{pmatrix}\right)_{tangent}$$

Projecting on the tangent vectors to the sphere (see (2.26)) we have

$$\nabla_{\partial_\varphi}\partial_\varphi = (\mathbf{r}_{\varphi\varphi})_{tangent} = \mathbf{r}_{\varphi\varphi} - \mathbf{n}\langle\mathbf{n}, \mathbf{r}_{\varphi\varphi}\rangle =$$

$$\begin{pmatrix} -R\sin\theta\cos\varphi \\ -R\sin\theta\sin\varphi \\ 0 \end{pmatrix} - \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} (-R\sin\theta\cos\varphi\sin\theta\cos\varphi - R\sin\theta\sin\varphi\sin\theta\sin\varphi) =$$

$$-\sin\theta\cos\theta \begin{pmatrix} R\cos\theta\cos\varphi \\ R\cos\theta\sin\varphi \\ -R\sin\theta \end{pmatrix} = -\sin\theta\cos\theta\mathbf{r}_\theta,$$

i.e.

$$\nabla_{\partial_\varphi}\partial_\varphi = -\sin\theta\cos\theta\mathbf{r}_\theta \Rightarrow \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta,\ \Gamma^\varphi_{\varphi\varphi} = \Gamma^\varphi_{\varphi\varphi} = 0\,. \quad (2.29)$$

## 2.3 Levi-Civita connection

We learned that one can consider different connections on manifold $M$. It turns out that on Riemannian manifold there is one distinguished connection. This is Levi-Civita connection.

### 2.3.1 Symmetric connection

**Definition**. We say that connection is symmetric if its Christoffel symbols $\Gamma^i_{km}$ are symmetric with respect to lower indices

$$\Gamma^i_{km} = \Gamma^i_{mk} \tag{2.30}$$

The canonical flat connection and induced connections considered above are symmetric connections.

*Invariant definition of symmetric connection*

A connection $\nabla$ is symmetric if for an arbitrary vector fields $\mathbf{X}, \mathbf{Y}$

$$\nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X}, \mathbf{Y}] = 0 \tag{2.31}$$

If we apply this definition to basic fields $\partial_k, \partial_m$ which commute: $[\partial_k, \partial_m] = 0$ we come to the condition

$$\nabla_{\partial_k}\partial_m - \nabla_{\partial_m}\partial_k = \Gamma^i_{mk}\partial_i - \Gamma^i_{km}\partial_i = 0$$

and this is the condition (2.30).

### 2.3.2 Levi-Civita connection. Theorem and Explicit formulae

Let $(M, G)$ be a Riemannian manifold.

**Definition-Theorem**

*A symmetric connection $\nabla$ is called Levi-Civita connection if it is compatible with metric, i.e. if it preserves the scalar product:*

$$\partial_{\mathbf{X}}\langle \mathbf{Y}, \mathbf{Z}\rangle = \langle \nabla_{\mathbf{X}}\mathbf{Y}, \mathbf{Z}\rangle + \langle \mathbf{Y}, \nabla_{\mathbf{X}}\mathbf{Z}\rangle \tag{2.32}$$

*for arbitrary vector fields $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$.*

*There exists unique levi-Civita connection on the Riemannian manifold.*

*In local coordinates Christoffel symbols of Levi-Civita connection are given by the following formulae:*

$$\Gamma^i_{mk} = \frac{1}{2} g^{ij}\left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^m} - \frac{\partial g_{mk}}{\partial x^j}\right). \tag{2.33}$$

*where $G = g_{ik}dx^i dx^k$ is Riemannian metric in local coordinates and $||g^{ik}||$ is the matrix inverse to the matrix $||g_{ik}||$.*

*Proof*

Suppose that this connection exists and $\Gamma^i_{mk}$ are its Christoffel symbols. Consider vector fields $\mathbf{X} = \partial_m, \mathbf{Y} = \partial_i$ and $\mathbf{Z} = \partial_k$ in (2.32). We have that

$$\partial_m g_{ik} = \langle \Gamma^r_{mi}\partial_r, \partial_k\rangle + \langle \partial_i, \Gamma^r_{mk}\partial_r\rangle = \Gamma^r_{mi} g_{rk} + g_{ir}\Gamma^r_{mk}\,. \tag{2.34}$$

for arbitrary indices $m, i, k$.

Denote by $\Gamma_{mik} = \Gamma^r_{mi} g_{rk}$ we come to

$$\partial_m g_{ik} = \Gamma_{mik} + \Gamma_{mki}, \text{ i.e.}$$

Now using the symmetricity $\Gamma_{mik} = \Gamma_{imk}$ since $\Gamma^k_{mi} = \Gamma^k_{im}$ we have

$$\Gamma_{mik} = \partial_m g_{ik} - \Gamma_{mki} = \partial_m g_{ik} - \Gamma_{kmi} = \partial_m g_{ik} - (\partial_k g_{mi} - \Gamma_{kim}) =$$

$$\partial_m g_{ik} - \partial_k g_{mi} + \Gamma_{kim} = \partial_m g_{ik} - \partial_k g_{mi} + \Gamma_{ikm} = \partial_m g_{ik} - \partial_k g_{mi} + (\partial_i g_{km} - \Gamma_{imk}) =$$
$$\partial_m g_{ik} - \partial_k g_{mi} + \partial_i g_{km} - \Gamma_{mik}\,.$$

Hence

$$\Gamma_{mik} = \frac{1}{2}(\partial_m g_{ik} + \partial_i g_{mk} - \partial_k g_{mi}) \Rightarrow \Gamma^k_{im} = \frac{1}{2} g^{kr} \left(\partial_m g_{ir} + \partial_i g_{mr} - \partial_r g_{mi}\right) \qquad (2.35)$$

We see that if this connection exists then it is given by the formula(2.33).

On the other hand one can see that (2.33) obeys the condition (2.34). We prove the uniqueness and existence.

since $\nabla_{\partial_i} \partial_k = \Gamma^m_{ik} \partial_m$.

Consider examples.

### 2.3.3 Levi-Civita connection of $\mathbf{E}^n$

For Euclidean space $\mathbf{E}^n$ in standard Cartesian coordinates

$$G_{\text{Eucl}} = (dx^1)^2 + \cdots + (dx^n)^2 = \delta_{ik} dx^i dx^k$$

Components of metric are constants (they are equal to 0 or 1). Hence obviously Christoffel symbols of Levi-Civita connection in Cartesian coordinates according formula (2.33) vanish:

$$\Gamma^I_{km} = 0 \text{ in Cartesian coordiantes}$$

Recalling canonical flat connection (see 2.1.3) we come to simple but important observation:

**Observation** Levi-Civita connection coincides with canonical flat connection in Euclidean space $\mathbf{E}^n$. They have vanishing Cristoffel symbols in Cartesian coordinates.

### 2.3.4 Levi-Civita connection on 2-dimensional Riemannian manifold with metric $G = adu^2 + bdv^2$.

**Example** Consider 2-dimensional manifold with Riemannian metrics

$$G = a(u,v)du^2 + b(u,v)dv^2, \qquad G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} a(u,v) & 0 \\ 0 & b(u,v) \end{pmatrix}$$

Calculate Christoffel symbols of Levi Civita connection.

Using (2.35) we see that:

$$\begin{aligned}
\Gamma_{111} &= \tfrac{1}{2}\left(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}\right) = \tfrac{1}{2}\partial_1 g_{11} = \tfrac{1}{2}a_u \\
\Gamma_{211} = \Gamma_{121} &= \tfrac{1}{2}\left(\partial_1 g_{12} + \partial_2 g_{11} - \partial_1 g_{12}\right) = \tfrac{1}{2}\partial_2 g_{11} = \tfrac{1}{2}a_v \\
\Gamma_{221} &= \tfrac{1}{2}\left(\partial_2 g_{12} + \partial_2 g_{12} - \partial_1 g_{22}\right) = -\tfrac{1}{2}\partial_1 g_{22} = -\tfrac{1}{2}b_u \\
\Gamma_{112} &= \tfrac{1}{2}\left(\partial_1 g_{12} + \partial_1 g_{12} - \partial_2 g_{11}\right) = -\tfrac{1}{2}\partial_2 g_{11} = -\tfrac{1}{2}a_v \\
\Gamma_{122} = \Gamma_{212} &= \tfrac{1}{2}\left(\partial_2 g_{21} + \partial_1 g_{22} - \partial_2 g_{21}\right) = \tfrac{1}{2}\partial_1 g_{22} = \tfrac{1}{2}b_u \\
\Gamma_{222} &= \tfrac{1}{2}\left(\partial_2 g_{22} + \partial_2 g_{22} - \partial_2 g_{22}\right) = \tfrac{1}{2}\partial_2 g_{22} = \tfrac{1}{2}b_v
\end{aligned} \tag{2.36}$$

To calculate $\Gamma^i_{km} = g^{ir}\Gamma_{kmr}$ note that for the metric $a(u,v)du^2 + b(u,v)dv^2$

$$G^{-1} = \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{a(u,v)} & 0 \\ 0 & \frac{1}{b(u,v)} \end{pmatrix}$$

Hence

$$\begin{aligned}
&\Gamma^1_{11} = g^{11}\Gamma_{111} = \tfrac{a_u}{2a}, \quad \Gamma^1_{21} = \Gamma^1_{12} = g^{11}\Gamma_{121} = \tfrac{a_v}{2a}, \quad \Gamma^1_{22} = g^{11}\Gamma_{221} = \tfrac{-b_u}{2a} \\
&\Gamma^2_{11} = g^{22}\Gamma_{112} = \tfrac{-a_v}{2b}, \quad \Gamma^2_{21} = \Gamma^2_{12} = g^{22}\Gamma_{122} = \tfrac{b_u}{2b}, \quad \Gamma^2_{22} = g^{22}\Gamma_{222} = \tfrac{b_v}{2b}
\end{aligned} \tag{2.37}$$

### 2.3.5 Example of the sphere again

Calculate Levi-Civita connection on the sphere.

On the sphere first quadratic form (Riemannian metric) $G = R^2 d\theta^2 + R^2 \sin^2\theta d\varphi^2$. Hence we use calculations from the previous example with

$a(\theta,\varphi) = R^2, b(\theta,\varphi) = R^2\sin^2\theta$ $(u=\theta, v=\varphi)$. Note that $a_\theta = a_\varphi = b_\varphi = 0$. Hence only non-trivial components of $\Gamma$ will be:

$$\Gamma^\theta_{\varphi\varphi} = \frac{-b_\theta}{2a} = \frac{-\sin 2\theta}{2}, \qquad \left(\Gamma_{\varphi\varphi\theta} = \frac{-R^2\sin 2\theta}{2}\right), \tag{2.38}$$

$$\Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = \frac{b_\theta}{2b} = \frac{\cos\theta}{\sin\theta} \qquad \left(\Gamma_{\theta\varphi\varphi} = \frac{R^2\sin 2\theta}{2}\right) \tag{2.39}$$

All other components are equal to zero:

$$\Gamma^\theta_{\theta\theta} = \Gamma^\theta_{\theta\varphi} = \Gamma^\theta_{\varphi\theta} = \Gamma^\varphi_{\theta\theta} = \Gamma^\varphi_{\varphi\varphi} = 0$$

**Remark** Note that Christoffel symbols of Levi-Civita connection on the sphere coincide with Christoffel symbols of induced connection calculated in the subsection "Connection induced on surfaces". later we will understand the geometrical meaning of this fact.

## 2.4 Levi-Civita connection = induced connection on surfaces in $\mathbf{E}^3$

We know already that *canonical flat connection of Euclidean space is the Levi-Civita connection of the standard metric on Euclidean space.* (see section 2.3.3.) Now we show that Levi-Civita connection on surfaces in Euclidean space coincides with the connection induced on the surfaces by canonical flat connection. We perform our analysis for surfaces in $\mathbf{E}^3$.

Let $M\colon \mathbf{r} = \mathbf{r}(u,v)$ be a surface in $\mathbf{E}^3$. Let $G$ be induced Riemannian metric on $M$ and $\nabla$ Levi–Civita connection of this metric.

We know that the induced connection $\nabla^{(M)}$ is defined in the following way: for arbitrary vector fields $\mathbf{X}, \mathbf{Y}$ tangent to the surface $M$, $\nabla^M_{\mathbf{X}}\mathbf{Y}$ equals to the projection on the tangent space of the vector field $\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}$:

$$\nabla^M_{\mathbf{X}}\mathbf{Y} = \left(\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}\right)_{\text{tangent}},$$

where $\nabla^{\text{can.flat}}$ is canonical flat connection in $\mathbf{E}^3$ (its Christoffel symbols vanish in Cartesian coordinates). We denote by $\mathcal{A}_{\text{tangent}}$ a projection of the vector $A$ attached at the point of the surface on the tangent space: $\mathcal{A}_\perp = \mathcal{A} - \mathbf{n}(\mathcal{A},\mathbf{n})$, ($\mathbf{n}$ is normal unit vector field to the surface.)

**Theorem** *Induced connection on the surface $\mathbf{r} = \mathbf{r}(u, v)$ in $\mathbf{E}^3$ coincides with Levi-Civita connection of Riemannian metric induced by the canonical metric on Euclidean space $\mathbf{E}^3$.*

*Proof*

Let $\nabla^M$ be induced connection on a surface $M$ in $\mathbf{E}^3$ given by equations $\mathbf{r} = \mathbf{r}(u, v)$. Considering this connection on the basic vectors $\mathbf{r}_h, \mathbf{r}_v$ we see that it is symmetric connection. Indeed

$$\nabla^M_{\partial_u}\partial_v = (\mathbf{r}_{uv})_{\text{tangent}} = (\mathbf{r}_{vu})_{\text{tangent}} = \nabla^M_{\partial_v}\partial_u \,. \Rightarrow \Gamma^u_{uv} = \Gamma^u_{vu}, \Gamma^v_{uv} = \Gamma^v_{vu} \,.$$

Prove that this connection preserves scalar product on $M$. For arbitrary tangent vector fields $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ we have

$$\partial_{\mathbf{X}}\langle \mathbf{Y}, \mathbf{Z}\rangle_{\mathbf{E}^3} = \langle \nabla^{\text{can. flat}}_{\mathbf{X}}\mathbf{Y}, \mathbf{Z}\rangle_{\mathbf{E}^3} + \langle \mathbf{Y}, \nabla^{\text{can. flat}}_{\mathbf{X}}\mathbf{Z}\rangle_{\mathbf{E}^3} \,.$$

since canonical flat connection in $\mathbf{E}^3$ preserves Euclidean metric in $\mathbf{E}^3$ (it is evident in Cartesian coordinates). Now project the equation above on the surface $M$. If $\mathcal{A}$ is an arbitrary vector attached to the surface and $\mathcal{A}_{\text{tangent}}$ is its projection on the tangent space to the surface, then for every tangent vector $\mathbf{B}$ scalar product $\langle \mathcal{A}, \mathbf{B}\rangle_{\mathbf{E}^3}$ equals to the scalar product $\langle \mathcal{A}_{\text{tangent}}, \mathbf{B}\rangle_{\mathbf{E}^3} = \langle \mathcal{A}_{\text{tangent}}, \mathbf{B}\rangle_M$ since vector $\mathcal{A} - \mathcal{A}_{\text{tangent}}$ is orthogonal to the surface. Hence we deduce from (2) that $\partial_{\mathbf{X}}\langle \mathbf{Y}, \mathbf{Z}\rangle_M =$

$$\langle \left(\nabla^{\text{can. flat}}_{\mathbf{X}}\mathbf{Y}\right)_{\text{tangent}}, \mathbf{Z}\rangle_{\mathbf{E}^3} + \langle \mathbf{Y}, \left(\nabla^{\text{can. flat}}_{\mathbf{X}}\mathbf{Z}\right)_{\text{tangent}}\rangle_{\mathbf{E}^3} = \langle \nabla^M_{\mathbf{X}}\mathbf{Y}, \mathbf{Z}\rangle_M + \langle \mathbf{Y}, \nabla^M_{\mathbf{X}}\mathbf{Z}\rangle_M \,.$$

We see that induced connection is symmetric connection which preserves the induced metric. Hence due to Levi-Civita Theorem it is unique and is expressed as in the formula (2.33).

**Remark** One can easy to reformulate and prove more general statement: Let $M$ be a submanifold in Riemannian manifold $(E, G)$. Then Levi-Civita connection of the metric induced on this submanifold coincides with the connection induced on the manifold by Levi-Civita connection of the metric $G$.

# 3 Parallel transport and geodesics

## 3.1 Parallel transport