**Homework 4. Solutions**

**1** *Calculate the area of parallelograms formed by the vectors* $\mathbf{a}, \mathbf{b}$ *if*

a) $\mathbf{a} = (1, 2, 3), \mathbf{b} = (1, 0, 1)$;
b) $\mathbf{a} = (2, 2, 3), \mathbf{b} = (1, 1, 1)$;
c) $\mathbf{a} = (5, 8, 4), \mathbf{b} = (10, 16, 8)$.

*Solution*
Area of parallelogram formed by the vectors $\mathbf{a}, \mathbf{b}$ is equal to the length of the vector $\mathbf{c} = \mathbf{a} \times \mathbf{b}$.

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = (a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z) \times (b_x\mathbf{e}_x + b_y\mathbf{e}_y + b_z\mathbf{e}_z) = a_xb_x\mathbf{e}_x \times \mathbf{e}_x + a_xb_y\mathbf{e}_x \times \mathbf{e}_y + a_xb_z\mathbf{e}_x \times \mathbf{e}_z +$$

$$a_yb_x\mathbf{e}_y \times \mathbf{e}_x + a_yb_y\mathbf{e}_y \times \mathbf{e}_y + a_yb_z\mathbf{e}_y \times \mathbf{e}_z + a_zb_x\mathbf{e}_z \times \mathbf{e}_x + a_zb_y\mathbf{e}_z \times \mathbf{e}_y + a_zb_z\mathbf{e}_z \times \mathbf{e}_z =$$

$$(a_xb_y - a_yb_x)\mathbf{e}_z + (a_yb_z - a_zb_y)\mathbf{e}_x + (a_zb_x - a_xb_z)\mathbf{e}_y$$

a) $S = |\mathbf{a} \times \mathbf{b}| = |-2\mathbf{e}_z + 2\mathbf{e}_x + 2\mathbf{e}_y|$, $S = \sqrt{4+4+4} = 2\sqrt{3}$.
b) $S = |\mathbf{a} \times \mathbf{b}|$. $\mathbf{a} \times \mathbf{b} = -\mathbf{e}_x + \mathbf{e}_y$, $S = \sqrt{1+1} = \sqrt{2}$
c) Vectors $\mathbf{a} = (5, 8, 4), \mathbf{b} = (10, 16, 8)$ are collinear, hence $\mathbf{a} \times \mathbf{b} = 0$, $S = 0$.

**2** *Prove the inequality* $(ad - bc)^2 \leq (a^2 + b^2)(c^2 + d^2)$

*a) by a direct calculation*
*b) considering vector product of vectors* $\mathbf{x} = a\mathbf{e}_x + b\mathbf{e}_y$ *and vectors* $\mathbf{y} = c\mathbf{e}_x + d\mathbf{e}_y$

*Solution*
a) $-2adbc \leq a^2c^2 + b^2d^2$ because $(ac+bd)^2 \geq 0$. Hence $(ad-bc)^2 = a^2d^2 + b^2c^2 - 2adbc \leq a^2d^2 + b^2c^2 + a^2c^2 + b^2d^2 = (a^2 + b^2)(c^2 + d^2)$.

b) $\mathbf{x} \times \mathbf{y} = (ad - bc)\mathbf{e}_z$. Length of this vector is the area of the parallelogram spanned by the vectors $\mathbf{x}, \mathbf{y}$. On the other hand area of parallelogram is less or equal than area of the rectangle with the same sides: $S = |ad - bc| = |\mathbf{x}||\mathbf{y}|\sin\varphi \leq |\mathbf{x}||\mathbf{y}| = \sqrt{(a^2 + b^2)(c^2 + d^2)}$

**3** *Show that for any two vectors* $\mathbf{a}, \mathbf{b} \in \mathbf{E}^3$ *the following identity is satisfied*

$$(\mathbf{a}, \mathbf{a})(\mathbf{b}, \mathbf{b}) = (\mathbf{a}, \mathbf{b})^2 + (\mathbf{a} \times \mathbf{b}, \mathbf{a} \times \mathbf{b})\,.$$

*Write down this identity in components.*
*Compare this identity with CBS inequality from the previous homework.*

*Solution*: It is exactly the problem in Coursework. The solution will be put later.

Notice that for $n = 2, 3$ this identity is more strong statement than CBS inequality: $(a_x^2 + a_y^2 + a_z^2)(b_x^2 + b_y^2 + b_z^2) \geq (a_xb_x + a_yb_y + a_zb_z)^2$. CBS inequality $(\mathbf{a}, \mathbf{a})(\mathbf{b}, \mathbf{b}) \geq |\mathbf{a}|^2|\mathbf{b}|^2$ follows from the identity (1.10).

**4** *Find a vector* $\mathbf{n}$ *such that the following conditions hold:*
*1) It has a unit length*
*2) it is orthogonal to the vectors* $\mathbf{a} = (1, 2, 3)$ *and* $\mathbf{b} = (1, 3, 2)$.
*3) An ordered triple* $\{\mathbf{a}, \mathbf{b}, \mathbf{n}\}$ *has an orientation opposite to the orientation of the basis of Euclidean space.*

Consider a vector $\mathbf{N} = \mathbf{a} \times \mathbf{b}$ and a vector $\frac{\mathbf{N}}{|\mathbf{N}|}$. The vector $\mathbf{N}$ is orthogonal to vectors $\mathbf{a}, \mathbf{b}$ (vector product) and a vector $\frac{\mathbf{N}}{|\mathbf{N}|}$ is a unit vector. I tremains to solve the problem of orientation. Both vectors $\pm\frac{\mathbf{N}}{|\mathbf{N}|}$ are unit vectors which are orthogonal to vectors $\mathbf{a}, \mathbf{b}$. On the other hand the ordered triple $\{\mathbf{a}, \mathbf{b}\mathbf{N}\}$ is a basis and thgis basis has the same orientation as a basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. This can be checked by straightforward calculations, or in other way this follows from the definition of vector product and the fact that vectors $\mathbf{N} \neq 0$ (i.e. the triple is a basis). Hence the vector

$$\mathbf{n} = -\frac{\mathbf{N}}{|\mathbf{N}|} = -\frac{(\mathbf{e}_x + 2\mathbf{e}_y + 3\mathbf{e}_z) \times (\mathbf{e}_x + 3\mathbf{e}_y + 2\mathbf{e}_z)}{||\mathbf{N}|} = \frac{5\mathbf{e}_x - \mathbf{e}_y - \mathbf{e}_z}{3\sqrt{3}}$$

**5** *Consider system of simultaneous equations*

$$\begin{cases} ax + by + cz = d \\ x + 2y + 3z = 1 \end{cases}$$

*Find conditions on parameters $a, b, c$ such that this system has no solutions.*

*Could this system have exactly one solution?*

*Algebraic solution*: $x = 1 - 2y - 3z$. Put it in the first equation we come to the equation $(b-2a)y + (c-3a)z = d - a$. This equation has no solution if and only if $b - 2a = c - 3a = 0$ and $d - a \neq 0$.

*Geometric solution*: Consider planes $\alpha$ defined by the equation $ax + by + cz = d$ and plane $\alpha'$ defined by the equation $x + 2y + 3z = 1$. Vector $\mathbf{N}_1 = (a, b, c)$ is orthogonal to the plane $\alpha$ and vector $\mathbf{N}_2 = (1, 2, 3)$ is orthogonal to the plane $\alpha'$.

The system above has no solutions if planes are parallel and they do not coincide. Planes are parallel, or coincide iff vectors $\mathbf{N}_1$ and $\mathbf{N}_2$ are colinear: $a = \lambda, b = 2\lambda, c = 3\lambda$ $(\lambda \neq 0)$. If $d = \lambda$ $(\lambda \neq 0)$ then these planes coincide. If $d \neq \lambda$ $(\lambda \neq 0)$ these planes are parallel and they do not coincide. System above has no solution if and only $a = \lambda, b = 2\lambda = 2a, c = 3\lambda$ and $d \neq \lambda$. (If $\lambda = 0$ then first equation defines empty set (if $d \neq 0$) or all $\mathbf{E}^3$ if $d = 0$).

Two planes if they are not parallel coincide or intersect by the line. Hence the system cannot have exactly one solution.

We come to porism: or no solution or infinitely many solutions.

**6** *Write down an equation of the plane $\alpha$ such that $\alpha$ is orthogonal to the vector $\mathbf{N} = (1, 2, 3)$ and the point $A = (2, 3, 5)$ belongs to this plane.*

*Find the distance between this plane and the point $B = (1, 0, 0)$.*

If $\alpha$ is orthogonal to the vector $\mathbf{N} = (1, 2, 3)$ then it can be defined by the equation $x + 2y + 3z = D$. On the other hand the point $A = (2, 3, 5) \in \alpha$. Hence $2 + 2 \cdot 3 + 3 \cdot 5 = 23 = D$. Hence equation of the plane $\alpha$ is $x + 2y + 3z = D = 23$.

This is not normal equation of the plane $\alpha$: $1^2 + 2^2 + 3^2 = 14 \neq 1$. Dividing by $\sqrt{14}$ we come to normal equation:

$$x + 2y + 3z - 23 = 0 \Leftrightarrow \frac{x + 2y + 3z - 23}{\sqrt{14}} = 0, \ \frac{1}{\sqrt{14}}x + \frac{2}{\sqrt{14}}y + \frac{3}{\sqrt{14}}z - \frac{23}{\sqrt{14}} = 0.$$

Having normal equation we calculate the distance between the point $B = (1, 0, 0)$ and the plane $\alpha$:

$$d(B, \alpha) = \left| \frac{x + 2y + 3z - 23}{\sqrt{14}} \Big|_{x=1, y=0, z=0} \right| = \frac{22}{\sqrt{14}} = \frac{11\sqrt{14}}{7}$$

**3** *Write down an equation of the plane (standard and parametric) passing through the points $A = (x_1, y_1, z_1) = (1, 1, 1)$, $B = (x_2, y_2, z_2) = (1, 2, 3)$, $C = (x_3, y_3, z_3) = (2, 2, 0)$.*

Consider vector $\mathbf{N} = AB \times AC = (\mathbf{r}_2 - \mathbf{r}_1) \times (\mathbf{r}_3 - \mathbf{r}_1)$. It is orthogonal to the sides $AB, AC$ of the triangle $\triangle ABC$. Hence it is orthogonal to the plane. The point $\mathbf{r} = (x, y, z)$ belongs to the plane if and only if vector $\mathbf{r} - \mathbf{r}_1$ is orthogonal to the vector $\mathbf{N}$. We have $\mathbf{r} \in \alpha \iff 0 = (\mathbf{r} - \mathbf{r}_1, \mathbf{N}) = (\mathbf{r} - \mathbf{r}_1, (\mathbf{r}_2 - \mathbf{r}_1) \times (\mathbf{r}_3 - \mathbf{r}_1)] =$

$$= \det \begin{pmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{pmatrix} = 0$$

This is an equation of the plane.

E.g. if $A = \mathbf{r}_1 = (1,1,1)$, $B = \mathbf{r}_2 = (1,2,3), C = \mathbf{r}_3 = (2,2,0)$ then equation of the plane will be

$$\det \begin{pmatrix} x-1 & y-1 & z-1 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix} = 2y - 3x - z + 2 = 0$$

Another solution (using parametric equation) *Write down an equation of the plane (standard and parametric) passing through the points* $A = (x_1, y_1, z_1) = (1,1,1)$, $B = (x_2, y_2, z_2) = (1,2,3)$, $C = (x_3, y_3, z_3) = (2,2,0)$.

It is easy to write parametric equation of this plane The plane $\alpha$ which we have to define is spanned by the vectors $\mathbf{a} = \mathbf{r}_2 - \mathbf{r}_1$, $\mathbf{b} = \mathbf{r}_3 - \mathbf{r}_1$ attached at the point $\mathbf{r}_1$. Hence we have parametric equation:

$$\mathbf{r}(u,v) = \mathbf{r}_1 + u(\mathbf{r}_1 - \mathbf{r}_1) + v(\mathbf{r}_3 - \mathbf{r}_1); \quad \begin{cases} x = x_1 + u(x_2 - x_1) + v(x_3 - x_1) \\ y = y_1 + u(y_2 - y_1) + v(y_3 - y_1) \\ z = z_1 + u(z_2 - z_1) + v(z_3 - z_1) \end{cases}, i.e. \begin{cases} x = 1 + v \\ y = 1 + u + v \\ z = 1 + 2u - v \end{cases}$$

Using "brute force" we can exclude parameters $u, v$ from these equations. But it is not beautiful. Do it in another way. (see for details lecture notes §1.4) (Note hat sometimes the method of "brute force" is much more effective than the method above).

One can compare these two solutions:

To double check that the solution is right, compare it with parametric formula:

$$2y - 3x - z + 2 = 2(1 + u + v) - 3(1 + v) - (1 + 2u - v) + 2 = 0\,.$$

**8**$^\dagger$ *Find a line $l$ passing through the point* $(1,0,0)$ *such that all points of this line belong to the one-sheeted hyperboloid* $x^2 + y^2 - z^2 = 1$.

Let $x = 1 + at, y = bt, z = t$ be a parametric equation of this line. ($t$ is a parameter $-\infty < t < \infty$).

The condition that all points of the line belong to the hyperboloid means that $(1+at)^2+b^2t^2-c^2t^2-1 \equiv 0$. On the other hand $(1 + at)^2 + b^2t^2 - c^2t^2 = (a^2 + b^2 - c^2)t^2 + 2at$. Hence $a = 0, b = \pm c$. There are exactly two lines: $x = 1, y = z$ and $x = 1, y = -z$. One can show that via every point of this hyperboloid pass two lines. One-sheeted hyperboloid is *ruled* surface.