Homework 7. Solutions

1. *Find coordinate basis vectors, first quadratic form and unit normal vector field for the following surfaces:*

a) sphere of the radius R:

$$\mathbf{r}(\varphi,\theta) \qquad \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases} \tag{1}$$

$(0 \le \varphi < 2\pi,\ 0 \le \theta \le \pi)$,

b) cylinder

$$\mathbf{r}(\varphi,h) \qquad \begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \\ z = h \end{cases} \qquad (0 \le \varphi < 2\pi, -\infty < h < \infty) \tag{2}$$

c) graph of the function $z = F(x,y)$

$$\mathbf{r}(u,,v) \qquad \begin{cases} x = u \\ y = v \\ z = F(u,v) \end{cases} \qquad (-\infty < u < \infty, -\infty < v < \infty) \tag{3}$$

in the case if $F(u,v) = F = Au + 2Buv + Cv^2$.

a) *sphere* $x^2+y^2+z^2=R^2$ *(of the radius R):*

$$\mathbf{r}(\theta,\varphi) \qquad \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases} \tag{1}$$

$(0 \le \varphi < 2\pi,\ 0 \le \theta \le \pi)$,

$$\mathbf{r}_\theta\big|_{\theta,\varphi} = \frac{\partial\mathbf{r}(\varphi,\theta)}{\partial\theta} = \begin{pmatrix} R\cos\theta\cos\varphi \\ R\cos\theta\sin\varphi \\ -R\sin\theta \end{pmatrix},\ \mathbf{r}_\varphi\big|_{\theta,\varphi} = \frac{\partial\mathbf{r}(\varphi,\theta)}{\partial\varphi} = \begin{pmatrix} -R\sin\theta\sin\varphi \\ R\sin\theta\cos\varphi \\ 0 \end{pmatrix},\ \mathbf{n}(\theta,\varphi) = \frac{\mathbf{r}(\theta,\varphi)}{R} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

(Sometimes we denote by $\mathbf{r}_\theta$ by ∂_θ and $\mathbf{r}_\varphi$ by ∂_φ.)

Check that $\mathbf{n}(\theta,\varphi)$ is indeed unit normal vector:

$$(\mathbf{n},\mathbf{n}) = \sin^2\theta(\cos^2\varphi+\sin^2\varphi)+\cos^2\theta = 1\,,$$

$$(\mathbf{n},\mathbf{r}_\theta) = R\sin\theta\cos\theta(\cos^2\varphi+\sin^2\varphi) - R\sin\theta\cos\theta = 0,\ (\mathbf{n},\mathbf{r}_\varphi) = R\sin^2\theta(-\cos\varphi\sin\varphi+\cos\varphi\sin\varphi) = 0\,.$$

Unit normal vector is defined up to a sign; $-\mathbf{n}$ is unit normal vector too.

Calculate now first quadratic form. $(\mathbf{r}_\theta,\mathbf{r}_\theta) = R^2\cos^2\theta(\cos^2\varphi+\sin^2\varphi)+R^2\sin^2\theta = R^2$, $(\mathbf{r}_\theta,\mathbf{r}_\varphi)=0$, $(\mathbf{r}_\varphi,\mathbf{r}_\varphi) = R^2\sin^2\theta(\sin^2\varphi+\cos^2\varphi) = R^2\sin^2\theta$. Thus

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_\theta,\mathbf{r}_\theta) & (\mathbf{r}_\theta,\mathbf{r}_\varphi) \\ (\mathbf{r}_\varphi,\mathbf{r}_\theta) & (\mathbf{r}_\varphi,\mathbf{r}_\varphi) \end{pmatrix} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix}$$

$$dl^2 = G_{11}d\theta^2 + 2G_{12}d\theta d\varphi + G_{22}d\varphi^2 = R^2d\theta^2 + R^2\sin^2\theta d\varphi^2$$

The length of the curve $\mathbf{r}(t) = \mathbf{r}(\theta(t),\varphi(t))$ with $\theta=\theta(t),\varphi=\varphi(t)$, $t_1 \le t \le t_2$ is given by the integral:

$$\int_{t_1}^{t_2}\sqrt{G_{11}\dot\theta^2 + 2G_{12}\dot\theta\dot\varphi + G_{22}\dot\varphi^2}dt = \int_{t_1}^{t_2}\sqrt{R^2\dot\theta^2 + R^2\sin^2\theta\dot\varphi^2}dt \tag{1a}$$

b) cylinder $x^2 + y^2 = R^2$

$$\mathbf{r}(\varphi, h) \qquad \begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \\ z = h \end{cases} \qquad (0 \le \varphi < 2\pi, -\infty < h < \infty) \tag{2}$$

$$\mathbf{r}_\varphi\big|_{\varphi,h} = \frac{\partial \mathbf{r}(\varphi,h)}{\partial \varphi} = \begin{pmatrix} -R\sin\varphi \\ R\cos\varphi \\ 0 \end{pmatrix}, \ \mathbf{r}_h\big|_{\varphi,h} = \frac{\partial \mathbf{r}(\varphi,h)}{\partial h} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ \mathbf{n}(\varphi,h) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$$

Sometimes we denote $\mathbf{r}_\varphi$ by ∂_φ and $\mathbf{r}_h$ by ∂_h.

Check that $\mathbf{n}(\varphi,h)$ is indeed unit normal vector:

$$(\mathbf{n},\mathbf{n}) = \cos^2\varphi + \sin^2\varphi = 1, \ (\mathbf{n},\mathbf{r}_\varphi) = R\cos\varphi\sin\varphi(-1+1) = 0, \ (\mathbf{n},\mathbf{r}_h) = 0$$

Unit normal vector is defined up to a sign; $-\mathbf{n}$ is unit normal vector too.

Calculate now first quadratic form. $(\mathbf{r}_\varphi,\mathbf{r}_\varphi) = R^2(\sin^2\varphi + \cos^2\varphi) = R^2$, $(\mathbf{r}_\varphi,\mathbf{r}_h) = 0$, $(\mathbf{r}_h,\mathbf{r}_h) = 1$. Thus

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_\varphi,\mathbf{r}_\varphi) & (\mathbf{r}_\varphi,\mathbf{r}_h) \\ (\mathbf{r}_h,\mathbf{r}_\varphi) & (\mathbf{r}_h,\mathbf{r}_h) \end{pmatrix} = \begin{pmatrix} R^2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$dl^2 = G_{11}d\varphi^2 + 2G_{12}d\varphi dh + G_{22}dh^2 = R^2 d\varphi^2 + dh^2$$

The length of the curve $\mathbf{r}(t) = \mathbf{r}(\varphi(t), h(t))$ with $\varphi = \varphi(t), h = h(t)$, $t_1 \le t \le t_2$ is given by the integral:

$$\int_{t_1}^{t_2} \sqrt{G_{11}\dot\varphi^2 + 2G_{12}\dot\varphi\dot h + G_{22}\dot h^2}\,dt = \int_{t_1}^{t_2} \sqrt{R^2\dot\varphi^2 + \dot h^2}\,dt \tag{2a}$$

c) graph of the function $z = F(x,y)$

$$\mathbf{r}(u,v) \qquad \begin{cases} x = u \\ y = v \\ z = F(u,v) \end{cases} \qquad (-\infty < u < \infty, -\infty < v < \infty) \tag{3}$$

in the case if $F(u,v) = Au^2 + 2Buv + Cv^2 + \dots$.

$$\mathbf{r}_u\big|_{u,v} = \frac{\partial \mathbf{r}(u,v)}{\partial u} = \begin{pmatrix} 1 \\ 0 \\ F_u \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2Au + 2Bv + \dots \end{pmatrix}, \ \mathbf{r}_u\big|_{u=v=0} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbf{r}_v\big|_{u,v} = \frac{\partial \mathbf{r}(u,v)}{\partial v} = \begin{pmatrix} 0 \\ 1 \\ F_v \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2Bu + 2Cv + \dots \end{pmatrix}, \ \mathbf{r}_v\big|_{u=v=0} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

$$\mathbf{n}(u,v) = \frac{1}{\sqrt{1 + F_u^2 + F_v^2}} \begin{pmatrix} -F_u \\ -F_v \\ 1 \end{pmatrix}, \ \mathbf{n}(u,v)\big|_{u=v=0} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Sometimes we denote $\mathbf{r}_u$ by ∂_u and $\mathbf{r}_v$ by $\mathbf{r}_v$ and ∂_v.

Check that $\mathbf{n}(u,v)$ is indeed unit normal vector: $(\mathbf{n},\mathbf{n}) = \frac{1}{1+F_u^2+F_v^2}(F_u^2 + F_v^2 + 1) = 1$, $(\mathbf{n},\mathbf{r}_u) = \frac{1}{\sqrt{1+F_u^2+F_v^2}}(F_u - F_u) = 0$, $(\mathbf{n},\mathbf{r}_v) = \frac{1}{\sqrt{1+F_u^2+F_v^2}}(F_v - F_v) = 0$. Calculate now first quadratic form. $(\mathbf{r}_u,\mathbf{r}_u) = 1 + F_u^2$, $(\mathbf{r}_u,\mathbf{r}_v) = F_uF_v$, $(\mathbf{r}_v,\mathbf{r}_v) = 1 + F_v^2$. Thus

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_u,\mathbf{r}_u) & (\mathbf{r}_u,\mathbf{r}_v) \\ (\mathbf{r}_v,\mathbf{r}_u) & (\mathbf{r}_v,\mathbf{r}_v) \end{pmatrix} = \begin{pmatrix} 1 + F_u^2 & F_uF_v \\ F_uF_v & 1 + F_v^2 \end{pmatrix}$$

$$dl^2 = G_{11}d\varphi^2 + 2G_{12}dd\varphi dh + G_{22}dh^2 = (1+F_u^2)du^2 + 2F_uF_vdudv + (1+F_v^2)dv^2$$

At the point $u = v = 0$, $F_u = F_v = 0$ and

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_u, \mathbf{r}_u) & (\mathbf{r}_u, \mathbf{r}_v) \\ (\mathbf{r}_v, \mathbf{r}_u) & (\mathbf{r}_v, \mathbf{r}_v) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ dl^2 = du^2 + dv^2$$

The length of the curve $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ with $u = u(t), v = v(t)$ can be calculated by the integral:

$$\int_{t_1}^{t_2} \sqrt{G_{11}\dot{u}^2 + 2G_{12}\dot{u}\dot{v} + G_{22}\dot{v}^2}dt = \int_{t_1}^{t_2} \sqrt{(1+F_u^2)\dot{u}^2 + 2F_uF_v\dot{u}\dot{v} + (1+F_v^2)\dot{v}^2}dt \tag{3a}$$

2 *Show that there are two straight lines which pass through the point* $(3, 4, 12)$ *on the saddle* $z = xy$ *and lie on this saddle.*

Show that this is true for an arbitrary point of the saddle.

Let $\mathbf{r}_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ be an arbitrary point on the saddle: $z_0 = x_0y_0$.

Consider the following two lines: the line

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{a}, \text{ where } \mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ y_0 \end{pmatrix}, \text{ i.e. } \begin{cases} x = x_0 + t \\ y = y_0 \\ z = z_0 + y_0 t \end{cases}$$

and the line

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{b}, \text{ where } \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ x_0 \end{pmatrix}, \text{ i.e. } \begin{cases} x = x_0 + t \\ y = y_0 \\ z = z_0 + y_0 t \end{cases}$$

It is easy to check that these both lines belong to the saddle: $xy = (x_0 + t)y_0 = z_0 + ty_0 = z$ and $xy = x_0(y_0 + t) = z_0 + x_0 t$.

One the other hand it is easy to see that it is all: If $\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$ is an arbitrary straight line on the saddle passing through the point (x_0, y_0, z_0), $x_0y_0 = z_0$ then

$$xy = (x_0 + at)(y_0 + bt) = z = z_0 + ct \text{ for all } t$$

Hence $ab = 0$. Thus $a = 0$ or $b = 0$. We see that through the an arbitrary point on the saddle pass exactly two straight lines.

3. *Consider on the sphere (1) the following curves:*

$$C_1\colon \ \mathbf{r} = \mathbf{r}(\theta(t), \varphi(t)), \ 0 \le t \le 2\pi, \text{ where } \theta(t) = \theta_0, \varphi(t) = t, \ \textit{(circle)}$$

$$C_2\colon \ \mathbf{r} = \mathbf{r}(\theta(t), \varphi(t)), \ 0 \le t \le \pi, \text{ where } \theta(t) = t, \varphi(t) = \varphi_0, \ \textit{(semicircle)}$$

Sketch these curves.

Calculate length of these curves considering them in the ambient Euclidean space. Calculate length of these curves using first quadratic form.

Calculate the length of C_1 in ambient Euclidean space, i.e. from the point of view of the External Observer:

$$\mathbf{r}(t)\colon \quad \begin{cases} x(t) = R\sin\theta_0\cos\varphi(t) = R\sin\theta_0\cos t \\ y(t) = R\sin\theta_0\sin\varphi(t) = R\sin\theta_0\sin t \\ z(t) = R\cos\theta_0 \end{cases}$$

It is the circle (latitude) of the radius $R\sin\theta_0$. Its length is equal to $L = 2\pi R\sin\theta_0$.

Now calculate the length of this curve from the point of view of internal observer: $\begin{cases} \theta(t) = \theta_0 \\ \varphi(t) = t \end{cases}$. Using quadratic form (see the equation (1a) and the equation before it) we come to the same answer:

$$\int_0^{2\pi} \sqrt{G_{11}\dot\theta^2 + 2G_{12}\dot\theta\dot\varphi + G_{22}\dot\varphi^2}dt = \int_0^{2\pi} \sqrt{R^2\dot\theta^2 + R^2\sin^2\theta(t)\dot\varphi^2}dt = \int_0^{2\pi} \sqrt{R^2\sin^2\theta_0}dt = 2\pi R\sin\theta_0$$

Now calculate the length of the curve C_2.

In the ambient Euclidean space it is the semicircle (meridian) of the radius R. Its length is equal to πR.

From the point of view of internal observer using (1a) we come to the same answer

$$\int_0^{\pi} \sqrt{G_{11}\dot\theta^2 + 2G_{12}\dot\theta\dot\varphi + G_{22}\dot\varphi^2}dt = \int_0^{\pi} \sqrt{R^2\dot\theta^2 + R^2\sin^2\theta(t)\dot\varphi^2}dt = \int_0^{\pi} \sqrt{R^2}dt = \pi R$$

3a* *Take arbitrary two points A, B on the curve C_2. Show that the arc of the curve C_2 is the shortest curve on the sphere between these points, i.e. for an arbitrary curve C on the sphere which starts at the point A and ends at the point B the length of C is greater or equal than the length of this arc of C_2.*

How to find the shortest curve between two arbitrary points on the sphere?

Let point A has coordinates θ_1, φ_0 and the point B has coordinates θ_2, φ_0. (They are both on the curve C_2, $\varphi = \varphi_0$.) Consider an arbitrary curve $\theta = \theta(t)$, $\varphi = \varphi(t)$, $t_1 \le t \le t_2$ starting at the point A and ending at the point B: $\theta(t_1) = \theta_1, \varphi(t_1) = \varphi_0$ and $\theta(t_2) = \theta_2, \varphi(t_2) = \varphi_0$. Then the length of this curve is given by the formula (2a):

$$L = \int_{t_1}^{t_2} \sqrt{R^2\dot\theta^2 + R^2\sin^2\theta(t)\dot\varphi^2}dt \ge \int_{t_1}^{t_2} \sqrt{R^2\dot\theta^2}dt = \int_{t_1}^{t_2} R\dot\theta dt = R\int_0^{\pi} d\theta = |t_2 - t_1|R \quad \blacksquare$$

We see that the shortest distance is the length of the shortest arc of the great circle which passes trough these points. This is true for any two points on the sphere.

4. *Consider on the sphere (1) the following circles:*

C_1: $x = R\cos t,\ y = R\sin t,\ z = 0$ *(Equator),*

C_2: $x = R\cos t,\ y = 0,\ z = R\sin t$ *("Greenwich" Meridian),*

C_3: $x = R\sin\theta_0\cos t,\ y = R\sin\theta_0\sin t,\ z = \cos\theta_0$ *(Circle of constant latitude)*

($0 \le t < 2\pi$)

Calculate normal curvatures at points of these circles.

Let C be an arbitrary curve on the sphere. What values can take the normal curvature at points of this curve?

For the curve C_1 at the point $\mathbf{r}(t) = \begin{pmatrix} R\cos t \\ R\sin t \\ 0 \end{pmatrix}$ the velocity vector is equal to $\mathbf{v}(t) = \begin{pmatrix} -R\sin t \\ R\cos t \\ 0 \end{pmatrix}$.

Speed is equal to R. Acceleration vector at this point is equal to $\mathbf{a}(t) = \begin{pmatrix} -R\cos t \\ -R\sin t \\ 0 \end{pmatrix}$. Acceleration vector is orthogonal to the sphere, i.e. it is collinear to the unit normal vector: $\mathbf{a}(t) = -\mathbf{r}(t) = -R\mathbf{n}$. Hence normal curvature is equal to $\kappa_n = \frac{(\mathbf{a},\mathbf{n})}{(\mathbf{v},\mathbf{v})} = \frac{-R}{R^2} = -\frac{1}{R}$. The same solution without calculations: The circle of equator is great circle. Vector of centripetal acceleration is orthogonal to the surface. Hence normal curvature has to coincide (up to a sign) with a usual curvature. It is equal to $\frac{1}{R}$. Hence $\kappa_n = \pm\frac{1}{R}$ (sign depends on the direction of normal vector $\mathbf{n}$).

For the curve C_2, "Greenwich" meridian answer is the same: speed is equal to R acceleration is equal to R and it is orthogonal to the surface. Hence normal curvature is equal to $-\frac{1}{R}$. It is equal to usual curvature since Greenwich meridian as Equator is an arc of the great circle.

Now consider the circle C_3. The centre of this circle is the point on z-axis with coordinates $(0, 0, R\cos\theta_0)$. The radius of this circle is equal to $R\sin\theta_0$. The usual curvature of this curve is equal to $1/R\sin\theta_0$, but it is not normal curvature because normal (centripetal) acceleration vector is not orthogonal to the surface. The angle between $\mathbf{n}$ and the vector of normal acceleration is equal to $\pi/2 - \theta_0$. Hence normal curvature (up to a sign) is equal to

$$\kappa_n = \frac{\underbrace{\frac{v^2}{R\sin\theta_0}}_{centripet.\ accel} \cdot \sin\theta_0}{v^2} = \frac{1}{R}$$

Straightforward calculations are the following: For the curve C_3 velocity vector at the point $\mathbf{r} = \begin{pmatrix} R\sin\theta_0\cos t \\ R\sin\theta_0\sin t \\ R\cos\theta_0 \end{pmatrix}$ is equal to $\mathbf{v}(t) = \begin{pmatrix} -R\sin\theta_0\sin t \\ R\sin\theta_0\cos t \\ 0 \end{pmatrix}$. Speed is equal to $R\sin\theta_0$. Acceleration vector at this point is equal to $\mathbf{a}(t) = \begin{pmatrix} -R\sin\theta_0\cos t \\ -R\sin\theta_0\sin t \\ 0 \end{pmatrix}$. Normal vector is equal to $\mathbf{n}(t) = \begin{pmatrix} \sin\theta_0\cos t \\ \sin\theta_0\sin t \\ \cos\theta_0 \end{pmatrix}$. We come to the same answer:

$$\kappa_n = \frac{(\mathbf{a}, \mathbf{n})}{\mathbf{v}^2} = \frac{-R\sin^2\theta_0\cos^2 t - R\sin^2\theta_0\cos^2 t}{R^2\sin^2\theta_0(\sin^2 t + \cos^2 t)} = -\frac{1}{R}$$

A unit normal vector is not defined uniquely. If $\mathbf{n}$ is unit normal vector then $-\mathbf{n}$ is normal unit vector too. Changing of $\mathbf{n} \to -\mathbf{n}$ changes the sign of the normal acceleration component $(\mathbf{n}, \mathbf{a})$. Therefore the sign of normal curvature is changed too.

We see that normal curvature for the circle of latitude is equal to $1/R$ as for great circles (up to a sign)

Now show that for an arbitrary (smooth) curve on the sphere of the radius R the normal curvature is equal to $1/R$ (up to a sign) at all points of the curve. Let $\mathbf{v}(t)$ be a velocity vector, $\mathbf{a}(t)$ be an acceleration vector and $\mathbf{n}(t) = \frac{\mathbf{r}(t)}{R}$ be a unit normal vector at the point $\theta(t), \varphi(t)$ on the sphere.

Note that $\frac{d\mathbf{n}(t)}{dt} = \frac{\mathbf{v}(t)}{R}$ because $\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt}$ and $\mathbf{n}(t) = \frac{\mathbf{r}(t)}{R}$. Then normal acceleration is equal to $\frac{-v^2}{R}$ since $\mathbf{a}_n = a_n\mathbf{n}$ where

$$\alpha_n = (\mathbf{n}(t), \mathbf{a}(t)) = \left(\mathbf{n}(t), \frac{d}{dt}\mathbf{v}(t)\right) = \frac{d}{dt}(\mathbf{n}(t), \mathbf{v}(t)) - \left(\frac{d\mathbf{n}(t)}{dt}, \mathbf{v}(t)\right) = \frac{d}{dt}(0) - \left(\frac{\mathbf{v}(t)}{R}, \mathbf{v}(t)\right) = -\frac{v^2}{R}$$

and normal curvature is equal to $\kappa_n = \frac{(\mathbf{v},\mathbf{n})}{v^2} = \frac{a_n}{v^2} = -\frac{1}{R}$

Normal curvature for an arbitrary curve on the sphere is always equal to $1/R$

5. *Consider on the cylinder (2) the following curves:*
C_1: $x = R\cos t,\ y = R\sin t,\ z = h_0$ *(circle),*
C_2: $x = R\cos t,\ y = R\sin t,\ z = vt$ *(helix),*
C_3: $x = R\cos\varphi_0,\ y = R\sin\varphi_0,\ z = t$ *(straight line).*
Calculate normal curvatures at points of these curves.

Let C be an arbitrary curve on the cylinder. What values can take the normal curvature at points of this curve?

For the points of the circle C_1 normal curvature is equal (up to a sign) to $1/R$. Indeed consider point moving around C_1 with constant speed ($x = R\cos\omega t, y = R\sin\omega t, z = h_0$). Speed is equal to $v = \omega R$ and the acceleration vector $\mathbf{a} = -R\omega^2\cos\omega t\partial_x - R\omega^2\sin\omega t\partial_y$ is orthogonal to the surface: $\mathbf{a} = \omega^2 R\mathbf{n}$, where we choose unit normal vector to be $\mathbf{n} = -(x/R, y/R, 0)$ at the points (x, y, z) of the cylinder ($\mathbf{n} = (-\cos\omega t, -\sin\omega t, 0)$). Normal curvature is equal to $\kappa_n = (\mathbf{a}, \mathbf{n})/(\mathbf{v}, \mathbf{v}) = \omega^2 R/\omega^2 R^2 = \frac{1}{R}$. If we choose $\mathbf{n} = +(x/R, y/R, 0) = (\cos\omega t, \sin\omega t, 0)$ then normal curvature would change a sign: $\mathbf{n} \to -\mathbf{n}$, $(\mathbf{a}, \mathbf{n}) \to -(\mathbf{a}, \mathbf{n})$ and $\kappa_{normal} \to -\kappa_{normal} = -\frac{1}{R}$.

Consider now a point moving around the curve C_2 (helix) with constant speed: $x = R\cos\omega t$, $y = R\sin\omega t$, $z = vt$. Then velocity vector is equal to $\mathbf{v} = -R\omega\sin\omega t\partial_x + R\omega\cos\omega t\partial_y + v\partial_z$ and $\mathbf{a} = -R\omega^2\cos\omega t\partial_x - R\omega^2\sin\omega t\partial_y$ (it is normal (centripetal) acceleration). We see that $v = \sqrt{\omega^2 R^2 + v^2}$, $\mathbf{a}_n = (\mathbf{a}, \mathbf{n}) = \omega^2 R$ (we choose $\mathbf{n} = -(x/R, y/R, 0)$ at the point (x, y, z) of the cylinder) and normal curvature is equal to

$$\kappa_n = \frac{\omega^2 R}{\omega^2 R^2 + v^2}$$

(In the case if we change $\mathbf{n} \to -\mathbf{n}$ then normal curvature will change a sigh too) One can see that

$$0 < \kappa_{normal} = \frac{\omega^2 R}{\omega^2 R^2 + v^2} \leq \frac{\omega^2 R}{\omega^2 R^2} = \frac{1}{R}$$

On the cylinder normal curvature varies from zero (for straight line) to $1/R$: $0 \leq \kappa_{normal} \leq \frac{1}{R}$

For the straight line C_3 normal curvature obviously is equal to 0: e.g. if particle moves on C_3 with constant speed, then its acceleration is equal to zero.

Show now that for an arbitrary curve on the cylinder (2) normal curvature takes values in the interval $\left[0, \frac{1}{R}\right]$. For convenience we choose normal unit vector attached at the point (x, y, z) of the cylinder $\mathbf{n} = -(x/R, y/R, 0)$. (In this case normal curvature of circle and helix were positive).

Consider an arbitrary smooth curve $\mathbf{r} = \mathbf{r}(\varphi(t), h(t))$ on the cylinder (2) and take any point $P = \mathbf{r} = (x, y, z)$ on it ($x^2 + y^2 + z^2 = R^2$). Let $\mathbf{v}$ be velocity vector at this point, $\mathbf{v} = (v_x, v_y, v_z)$. Consider expansion of the vector $\mathbf{v}$ on horizontal and vertical components. $\mathbf{v} = \mathbf{v}_{horisont} + \mathbf{v}_{vertical}$, $\mathbf{v}_{horisontal} = v_x\partial_x + v_y\partial_y$, $\mathbf{v}_{vertical} = v_z\partial_z$. Normal (centripetal) acceleration $\mathbf{a}_n$ is equal to $\mathbf{a}_n = -a_n\mathbf{n}$, where $a_n = \frac{|\mathbf{v}_{horisontal}|^2}{R} = \frac{v_x^2 + v_y^2}{R}$. We have for normal curvature:

$$\kappa_{normal} = \frac{\frac{v^2_{horisontlal}}{R}}{(\mathbf{v}, \mathbf{v})} = \frac{v^2_{horisontlal}}{R(v^2_{horisontlal} + v^2_{vertical})} = \frac{v_x^2 + v_y^2}{R(v_x^2 + v_y^2 + v_z^2)} \leq \frac{1}{R}\,.$$

We see that normal curvature could be less or equal than $1/R$ (normal curvature of the circle) and bigger or equal than 0 (normal curvature for straight line) (See also the example in Lecture notes.)

Remark Note that one can consider on the cylinder $x^2 + y^2 = R^2$ a circle of very small radius r. The curvature (usual curvature which we studied before) of this circle will be equal to $1/r$. We see that usual curvature of curve can be very big, but normal curvature cannot be bigger than $1/R$.

6. *Calculate shape operator for an arbitrary point of the sphere (1).*

We use results of calculations of vectors $\mathbf{r}_\theta, \mathbf{r}_\varphi$ and of unit normal vector $\mathbf{n}(\theta, \varphi)$ from the exercise 1a. By the definition (see lecture notes) the action of shape operator on any tangent vector $\mathbf{v}$ is given by the formula $S\mathbf{v} = -\partial_\mathbf{v} S$. Hence for basis vectors $\mathbf{r}_\theta = \partial_\theta, \mathbf{r}_\varphi = \partial_\varphi$

$$S\mathbf{r}_\theta = -\partial_\theta \mathbf{n}(\theta, \varphi) = -\partial_\theta\left(\frac{\mathbf{r}(\theta, \varphi)}{R}\right) = -\left(\frac{\partial_\theta \mathbf{r}(\theta, \varphi)}{R}\right) = -\frac{\mathbf{r}_\theta}{R}$$

and

$$S\mathbf{r}_\varphi = -\partial_\varphi \mathbf{n}(\theta,\varphi) = -\partial_\varphi \left(\frac{\mathbf{r}(\theta,\varphi)}{R}\right) = -\left(\frac{\partial_\varphi \mathbf{r}(\theta,\varphi)}{R}\right) = -\frac{\mathbf{r}_\varphi}{R}$$

We see that shape operator is equal to $S = -\frac{I}{R}$, where I is an identity operator. Its matrix in the basis $\partial_\theta, \partial_\varphi$ is equal to

$$-\begin{pmatrix} \frac{1}{R} & 0 \\ 0 & \frac{1}{R} \end{pmatrix}.$$

In the case if we choose the opposite direction for unit normal vector then we will come to the same answer just with changing the signs: if $\mathbf{n} \to -\mathbf{n}$, $S \to -S$.

7. *Calculate shape operator for an arbitrary point of the cylinder (2).*

We use results of calculations vectors $\mathbf{r}_\varphi, \mathbf{r}_h$ and for unit normal vector $\mathbf{n}(\varphi, h)$ from the exercise 1b. By the definition the action of shape operator on any tangent vector $\mathbf{v}$ is given by the formula $S\mathbf{v} = -\partial_\mathbf{v} S$. Hence for basis vectors $\mathbf{r}_\varphi = \partial_\varphi, \mathbf{r}_h = \partial_h$

$$S\mathbf{r}_\varphi = -\partial_\varphi \mathbf{n}(\varphi, h) = -\partial_\varphi \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} \sin\varphi \\ -\cos\varphi \\ 0 \end{pmatrix} = -\frac{\mathbf{r}_\varphi}{R}$$

and

$$S\mathbf{r}_h = -\partial_h \mathbf{n}(\varphi, h) = -\partial_h \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

For an arbitrary tangent vector $\mathbf{X} = a\mathbf{r}_\varphi + b\mathbf{r}_h$, $S\mathbf{X} = -a/R\mathbf{r}_\varphi$. We see that shape operator transforms tangent vectors to tangent vectors. Its matrix in the basis $\mathbf{r}_\varphi, \mathbf{r}_h$ is equal to

$$-\begin{pmatrix} \frac{1}{R} & 0 \\ 0 & 0 \end{pmatrix}$$

In the case if we choose the opposite direction for unit normal vector then we will come to the same answer just with changing the signs: if $\mathbf{n} \to -\mathbf{n}$, $S \to -S$.

8. *Calculate shape operator for the surface (3) at the point $u = v = 0$.*

We use results of calculations vectors $\mathbf{r}_u, \mathbf{r}_v$ and for unit normal vector $\mathbf{n}(u, v)$ from the exercise 1c For basic vectors $\mathbf{r}_u = \partial_u, \mathbf{r}_v = \partial_v$ we have $S\mathbf{r}_u = -\partial_u \left(\mathbf{n}(u,v)\right)_{|u=v=0} =$

$$-\partial_u \left(\frac{1}{\sqrt{1+F_u^2+F_v^2}} \begin{pmatrix} -F_u \\ -F_v \\ 1 \end{pmatrix}\right)_{|u=v=0} = \left(\frac{1}{\sqrt{1+F_u^2+F_v^2}}\right)_{|u=v=0} \begin{pmatrix} F_{uu} \\ F_{uv} \\ 1 \end{pmatrix}_{|u=v=0} = \begin{pmatrix} 2A \\ 2B \\ 0 \end{pmatrix} = 2A\mathbf{r}_u + 2B\mathbf{r}_v$$

and $S\mathbf{r}_v = -\partial_v \left(\mathbf{n}(u,v)\right)_{|u=v=0} =$

$$-\partial_v \left(\frac{1}{\sqrt{1+F_u^2+F_v^2}} \begin{pmatrix} -F_u \\ -F_v \\ 1 \end{pmatrix}\right)_{|u=v=0} = \left(\frac{1}{\sqrt{1+F_u^2+F_v^2}}\right)_{|u=v=0} \begin{pmatrix} F_{vu} \\ F_{vv} \\ 1 \end{pmatrix}_{|u=v=0} = \begin{pmatrix} 2B \\ 2C \\ 0 \end{pmatrix} = 2B\mathbf{r}_u + 2C\mathbf{r}_v$$

The matrix of the shape operator in the basis $\mathbf{r}_u, \mathbf{r}_v$ is $\begin{pmatrix} 2A & 2B \\ 2B & 2C \end{pmatrix}$.

9 *Calculate principal curvatures, Gaussian and mean curvature at the points of the sphere (1) using results of exercise 4 or using results of the exercise 6.*

a) using results of exercise 4: We obtained in the exercise 4 that for any point of an arbitrary curve normal curvature is equal to $-1/R$ (or $1/R$ if we change the direction of unit normal vector). We know that $k_- \leq \kappa_{normal} \leq k_+$. Hence we come to conclusion that $k_- = k_+ = \kappa_{normal} = -\frac{1}{R}$. Hence Gaussian curvature $K = k_- \cdot k_+ = \frac{1}{R^2}$ and mean curvature $H = k_- + k_+ = -\frac{2}{R}$.

Principal curvatures and mean curvature as well as unit normal vector and shape operator are defined up to a sign. If we change $\mathbf{n} \to -\mathbf{n}$ principal curvatures and mean curvature will change the sign but Gaussian curvature remains intact.

b) using results of the exercise 6): we obtained in the exercise 6 shape operator $S = -\begin{pmatrix} \frac{1}{R} & 0 \\ 0 & \frac{1}{R} \end{pmatrix}$.

Gaussian curvature $K = k_- \cdot k_- = \det S = \frac{1}{R^2}$ and mean curvature $H = k_- + k_+ = \operatorname{Tr} S = -\frac{2}{R}$. We have that $k_- \cdot k_+ = \frac{1}{R^2}$ and $k_- + k_+ = -\frac{2}{R}$. Hence $k_- = k_+ = -\frac{1}{R}$. If we change $\mathbf{n} \to -\mathbf{n}$ principal curvatures and mean curvature will change the sign but Gaussian curvature remains intact.

10 *Calculate principal curvatures, Gaussian and mean curvature at the points of the cylinder (2) using results of exercise 5 or using results of the exercise 7.*

a) using results of exercise 5: We obtained in the exercise 5 that for any point of an arbitrary curve normal curvature takes values in the interval $[0, 1/R] : 0 \leq \kappa_{normal} \leq \frac{1}{R}$. We know that $k_- \leq \kappa_{normal} \leq k_+$. Hence we come to conclusion that $k_- = 0$, $k_+ = \frac{1}{R}$. Hence Gaussian curvature $K = k_- \cdot k_+ = 0$ and mean curvature $H = k_- + k_+ = \frac{1}{R}$. Principal curvatures and mean curvature as well as unit normal vector and shape operator are defined up to a sign. If we change $\mathbf{n} \to -\mathbf{n}$ principal curvatures and mean curvature will change the sign but Gaussian curvature remains intact.

b) using results of the exercise 7): we obtained in the exercise 7 shape operator $S = -\begin{pmatrix} \frac{1}{R} & 0 \\ 0 & 0 \end{pmatrix}$. Gaussian curvature $K = k_- \cdot k_+ = \det S = 0$ and mean curvature $H = k_- + k_+ = \operatorname{Tr} S = -\frac{1}{R}$. We have that $k_- \cdot k_+ = 0$ and $k_- + k_+ = \frac{1}{R}$. Hence $k_- = 0$, $k_+ = -\frac{1}{R}$.

Answers are the same up to a sign: we choose different directions for $\mathbf{n}$ in Exercises 5 and 7: If we change $\mathbf{n} \to -\mathbf{n}$ principal curvatures and mean curvature will change the sign but Gaussian curvature remains intact.

11 *Calculate Gaussian and mean curvature of the surface (3) at the point $u = v = 0$.*

We already calculated shape operator at the point $u = v = 0$: its matrix an the basis ∂_u, ∂_v is equal to $S = \begin{pmatrix} 2A & 2B \\ 2B & 2C \end{pmatrix}$. Gaussian curvature $K = k_- \cdot k_+ = \det S = 4AC - 4B^2$ and mean curvature $H = k_- + k_+ = \operatorname{Tr} S = 2A + 2C$.

12 *Assume that the action of shape operator at the tangent coordinate vectors ∂_u, ∂_v at the given point $\mathbf{p}$ of the surface $\mathbf{r} = \mathbf{r}(u, v)$ is defined by the relations: $S(\partial_u) = 2\partial_u + 2\partial_v$ and $S(\partial_v) = -\partial_u + 5\partial_v$. Calculate principal curvatures, Gaussian and mean curvatures of the surface at this point.*

We see that the matrix of the shape operator in the basis ∂_u, ∂_v is equal to

$$S = \begin{pmatrix} 2 & -1 \\ 2 & 5 \end{pmatrix}$$

Hence Gaussian curvature $K = \det S = 12$ and mean curvature $H = \operatorname{Tr} S = 7$ To calculate principal curvatures k_-, k_+ note that

$$\begin{cases} k_- + k_+ = H = 7 \\ k_- \cdot k+ = K = 12 \end{cases}$$

Hence $k_- = 3, k_+ = 4$; κ_-, k_+ are eigenvalues of the shape operator.