Introduction to Geometry (20222)

2010

COURSEWORK

This assignment counts for 20% of your marks.

Solutions are due by 26 April

Write solutions in the provided spaces.

STUDENTS'S NAME:

a) Let (x^1, x^2, x^3) be coordinates of the vector $\mathbf{x}$, and (y^1, y^2, y^3) be coordinates of the vector $\mathbf{y}$ in $\mathbf{R}^3$.

Does the formula $(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^2 + x^2y^3 + x^3y^2 + x^3y^3$ define a scalar product on $\mathbf{R}^3$? Justify your answer.

b) Vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in Euclidean space are orthogonal to each other and all of them have non-zero length.

Prove that these vectors are linearly independent.

c) Let $\mathbf{x}, \mathbf{y}$ be two vectors in the Euclidean space $\mathbf{E}^2$ such that the length of the vector $\mathbf{x}$ is equal to 1, the length of the vector $\mathbf{y}$ is equal to 13 and scalar product of these vectors is equal to 12.

Find a vector $\mathbf{e}$ in $\mathbf{E}^2$ (express it through the vectors $\mathbf{x}$ and $\mathbf{y}$) such that the following conditions hold

i) an ordered pair $\{\mathbf{e}, \mathbf{x}\}$ is an orthonormal basis in $\mathbf{E}^2$,

ii) the vector $\mathbf{e}$ has an obtuse angle with the vector $\mathbf{y}$.

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a) Consider the matrix

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Calculate the matrix A^2 in the case if $\theta = \frac{\pi}{4}$.
Calculate the matrix A^{18} in the case if $\theta = \frac{\pi}{6}$.

b) Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal basis in 3-dimensional Euclidean space $\mathbf{E}^3$. Let P be a linear operator acting on 3-dimensional Euclidean space $\mathbf{E}^3$, such that for arbitrary two vectors $\mathbf{x}, \mathbf{y} \in \mathbf{E}^3$

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle P\mathbf{x}, P\mathbf{y} \rangle ,$$

where $\langle \mathbf{x}, \mathbf{y} \rangle$ is a scalar product in $\mathbf{E}^3$.

Given that $P\mathbf{e}_1 = \mathbf{e}_2$, $P\mathbf{e}_2 = \mathbf{e}_3$ and $\det P > 0$, calculate $P\mathbf{e}_3$.

Find a vector $\mathbf{f} \neq 0$ such that $P\mathbf{f} = \mathbf{f}$.

What is a geometrical meaning of this vector?

c) On the plane OXY find the horizontal line l: $y = c$ and the point $F = (0, f)$ on the OY axis such that for all the points M on the parabola $y = x^2$ the distance $|MF|$ equals to the distance between the point M and the line l.

3

Throughout this question, $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ is an orthonormal basis for Euclidean space $\mathbf{E}^3$.

a) Consider vector $\mathbf{a} = 2\mathbf{e}_x + 3\mathbf{e}_y + 6\mathbf{e}_z$ in $\mathbf{E}^3$.

Show that the angle θ between vectors $\mathbf{a}$ and $\mathbf{e}_z$ belongs to the interval $\left(\frac{\pi}{6}, \frac{\pi}{4}\right)$.

Find a unit vector $\mathbf{b}$ such that it is orthogonal to vectors $\mathbf{a}$ and $\mathbf{e}_z$, and the angle between vectors $\mathbf{b}$ and $\mathbf{e}_x$ is acute.

b) Denote by $\Pi(\mathbf{a}, \mathbf{b}, \mathbf{c})$ a parallelepiped formed by the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathbf{E}^3$ attached at the fixed point. Denote by $V\left(\Pi(\mathbf{a}, \mathbf{b}, \mathbf{c})\right)$ a volume of this parallelepiped.

Fix in $\mathbf{E}^3$ the following two vectors

$$\mathbf{a} = \frac{1}{3}(\mathbf{e}_x + 2\mathbf{e}_y + 2\mathbf{e}_z)\ ,\ \mathbf{b} = \frac{1}{3}(2\mathbf{e}_x - 2\mathbf{e}_y + \mathbf{e}_z)\,.$$

Show that for an arbitrary vector $\mathbf{c}$, $V\left(\Pi(\mathbf{a}, \mathbf{b}, \mathbf{c})\right) \leq |\mathbf{c}|$.

Find a unit vector $\mathbf{c}$ such that $V\left(\Pi(\mathbf{a}, \mathbf{b}, \mathbf{c})\right) = 1$ and the basis $\{\mathbf{c}, \mathbf{b}, \mathbf{a}\}$ has an orientation opposite to the orientation of the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$.

c) Consider a triangle $\triangle ABC$ in $\mathbf{E}^3$, formed by the vectors $\mathbf{a} = (2, 10, 25)$ and $\mathbf{b} = (5, 4, -2)$ attached at the point A.

Calculate the area of the triangle $\triangle ABC$.

Calculate the length of the height AM of this triangle. ($AM \perp BC$ and the point M belongs to the line BC.)

Consider the vector $\mathbf{h} = AM$ and the vector $\mathbf{d} = \mathbf{c} \times (\mathbf{a} \times \mathbf{b})$, where $\mathbf{c} = \mathbf{b} - \mathbf{a}$.

Are the vectors $\mathbf{h}$ and $\mathbf{d}$ linearly independent or not? Justify your answer.

4

a) Given a vector field $\mathbf{G} = r\frac{\partial}{\partial r} + \frac{\partial}{\partial \varphi}$ in polar coordinates express it in Cartesian coordinates ($x = r\cos\varphi$, $y = r\sin\varphi$).

b) Consider the function $f = r^2 \sin 2\varphi$ and the vector fields $\mathbf{A} = x\partial_x + y\partial_y$, $\mathbf{B} = x\partial_y - y\partial_x$.

Calculate $\partial_{\mathbf{A}} f$ and $\partial_{\mathbf{B}} f$.

Calculate the value of the 1-form $\omega = (x^2 + y^2)de^{-x^2-y^2}$ on the vector fields $\mathbf{A}$, $\mathbf{B}$.

(c) Show that the 1-form $\omega = 3x^2y^2dx + 2x^3ydy$ is an exact form.

Show that the 1-form $\omega = dx - xydy$ is not an exact form.

Find a function $f(x, y)$ such that for an arbitrary closed curved C in $\mathbf{E}^2$

$$\int_C f(x,y)\,(dx - xydy) = 0\,.$$

5

(**a**) Consider in $\mathbf{E}^2$ the ellipse $\mathbf{r}(t)$: $x = a\cos t, y = b\sin t$, $0 \le t < 2\pi$, $a > b > 0$.

Find the velocity $\mathbf{v} = \frac{d\mathbf{r}(t)}{dt}$ and acceleration $\mathbf{a}(t) = \frac{d^2\mathbf{r}(t)}{dt^2}$ vectors.

Find the points of this curve where speed is increasing.

Find the points of this curve where speed takes maximum value.

(**b**) Consider in $\mathbf{E}^2$ the curve $\mathbf{r}(t)$: $x = t^2 - t, y = 2t$, $0 < t < 1$.

Sketch the image of this curve.

Calculate the integral of the differential form $\omega = xdy + y^2dx$ over this curve.

How does this integral change under the reparameterisation $t = \sin\tau$, $(0 < \tau < \frac{\pi}{2})$?

How does this integral change under the reparameterisation $t = \cos\tau$, $(0 < \tau < \frac{\pi}{2})$?

c) Let C be an ellipse in $\mathbf{E}^2$ such that the sum of the distances between an arbitrary point of C and its foci $F_1 = (0,0)$ and $F_2 = (1,0)$ equals to 3:

$$C = \{P\colon\ |P - F_1| + |P - F_2| = 3\}\,.$$

Write down the equation of this ellipse in polar coordinates.

Sketch this ellipse and write down its equation in Cartesian coordinates.

Find the integral of the 1-form $\omega = ydx$ over this ellipse.