Homework 9

1 Let ∇ be a connection on n-dimensional manifold M and $\{R^i{}_{rmn}\}$ be the components of the curvature tensor of a connection ∇ in local coordinates $(x^1, x^2, \ldots, x^n)$.

a) For arbitrary vector fields $\mathbf{A}, \mathbf{B}$ and $\mathbf{D}$ calculate the vector field

$$(\nabla_{\mathbf{A}}\nabla_{\mathbf{B}} - \nabla_{\mathbf{B}}\nabla_{\mathbf{A}})\,\mathbf{D} - \nabla_{\mathbf{C}}\mathbf{D}\,,$$

where the vector field $\mathbf{C}$ is a commutator of vector fields $\mathbf{A}$ and $\mathbf{B}$:

$$\mathbf{C} = C^i\frac{\partial}{\partial x^i} = [\mathbf{A}, \mathbf{B}] = \left(A^m\frac{\partial B^i(x)}{\partial x^m} - B^m\frac{\partial A^i(x)}{\partial x^m}\right)\frac{\partial}{\partial x^i}\,.$$

b) Calculate the vector field

$$(\nabla_{\mathbf{A}}\nabla_{\mathbf{B}} - \nabla_{\mathbf{B}}\nabla_{\mathbf{A}})\,\mathbf{D}$$

in the case if for vector fields $\mathbf{A}$ and $\mathbf{B}$ components A^i and B^m are constants (in the local coordinates $(x^1, \ldots, x^n)$)

c) Calculate the vector field

$$(\nabla_{\mathbf{A}}\nabla_{\mathbf{B}} - \nabla_{\mathbf{B}}\nabla_{\mathbf{A}})\,\mathbf{A} - \nabla_{\mathbf{A}}\mathbf{A}$$

in the case if $\mathbf{A} = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}$, $\mathbf{B} = x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2}$.

(*You have to express the answers in terms of components of the vector fields and components of the curvature tensor* $R^i{}_{rmn}$.)

Consider a surface M in $\mathbf{E}^3$ defined by the equation

$$\begin{cases} x = u \\ y = v \\ z = F(u, v) \end{cases} \qquad (1)\ .$$

2* Calculate explicitly the component R_{1212} of the Riemannian curvature tensor at the point with coordinates $u = v = 0$ in the case if $F(u, v) = \frac{1}{2}(au^2 + 2buv + bv^2)$, where a, b, c are parameters.

3 * Consider a point $\mathbf{p}$ on the surface M with coordinates $u = x_0, v = y_0$ such that (x_0, y_0) is a point of local extremum for the function F.

Using the results of previous exercise calculate the component R_{1212} of the Riemannian curvature tensor at the point $\mathbf{p}$.

4 † Using the results of the calculations in the previous exercise calculate the Riemannian curvature tensor at the arbitrary point of the surface (1).