Homework 5. Solutions

1.

Calculate the differentials of the following 1*-forms:*

a) xdx ,

b) xdy

c) $xdx + ydy$,

d)$xdy + ydx$,

e) $xdy - ydx$

f) $x^4dy + 4x^3ydx$,

g) $xdy + ydx + dz$,

h) $xdy - ydx + dz$.

For each 1-forms listed above find a function f *(0-form) such that* $df = \omega$*, if possible. If it is not possible, explain why.*

General remark: if $d\omega \neq 0$ then the equation $\omega = df$ has no solution, because if $\omega = df$ then $dw = d(df) = 0$. In other words exact form necessarily has to be closed. The inverse implication is true if the form is defined on the whole Euclidean space.

a) $d(xdx) = dx \wedge dx = 0.\, xdx = df$ where $f = \frac{x^2}{2} + c$, where c is a constant.

b) $d(xdy) = dx \wedge dy \neq 0$. Hence this form is not exact.

c) $d(xdx + ydy) = dx \wedge dx + dy \wedge dy = 0$. $xdx + ydy = d\left(\frac{x^2+y^2}{2} + c\right)$, ($c$ is a constant).

d) $d(xdy + ydx) = dx \wedge dy + dy \wedge dx = dx \wedge dy - dx \wedge dy = 0$. $xdy + ydx = d(xy + c)$,where c is a constant.

e) $d(xdy - ydx) = dx \wedge dy - dy \wedge dx = 2dx \wedge dy \neq 0$. Hence this form is not exact.

f) $d(x^4dy+4x^3ydx) = 4x^3dx\wedge dy+4x^3dy\wedge dx = 4x^3(dx\wedge dy+dy\wedge dx) = 0$. $x^4dy+4x^3ydx = d(x^4y+c)$, where c is a constant.

g) $d(xdy + ydx + dz) = dx \wedge dy + dy \wedge dx + ddz = 0$. $xdy + ydx + dz = d(xy + z + c)$, where c is a constant.

h) $d(xdy - ydx + dz) = dx \wedge dy - dy \wedge dx = 2dx \wedge dy \neq 0$. The form is not exact.

2

Consider one-form

$$\omega = \frac{xdy - ydx}{x^2 + y^2} \tag{1}$$

This form is defined in $\mathbf{E}^2 \setminus 0$.

Calculate differential of this form.

Write down this form in polar coordinates

Find a function f *such that* $\omega = df$.

Is this function defined in the same domain as ω*?*

First calculate differential in cartesian coordinates with "brute force"

$$d\omega = d\left(\frac{xdy - ydx}{x^2 + y^2}\right) = \frac{d(xdy - ydx)}{x^2 + y^2} - (xdy - ydx) \wedge d\left(\frac{1}{x^2 + y^2}\right) = \frac{2dx \wedge dy}{x^2 + y^2} +$$

$$\frac{(xdy - ydx) \wedge d(x^2 + y^2)}{(x^2 + y^2)^2} = \frac{2dx \wedge dy}{x^2 + y^2} + \frac{(xdy - ydx) \wedge (2xdx + 2ydy)}{(x^2 + y^2)^2} =$$

$$\frac{2dx \wedge dy}{x^2 + y^2} + \frac{2x^2dy \wedge dx + 2y^2dy \wedge dx}{(x^2 + y^2)^2} = 0\,.$$

Much more illuminating to write down this form in polar coordinates then calculate its differential. We know already that $xdy - ydx = r^2 d\varphi$. Indeed

$dx = d(r\cos\varphi) = \cos\varphi dr - r\sin\varphi d\varphi = \frac{x}{r}dr - yd\varphi$ and $dy = d(r\sin\varphi) = \sin\varphi dr + r\cos\varphi d\varphi = \frac{y}{r}dr + xd\varphi$. Hence

$$xdy - ydx = x\left(\frac{y}{r}dr + xd\varphi\right) - y\left(\frac{x}{r}dr - yd\varphi\right) = (x^2+y^2)d\varphi \text{ and } \frac{xdy - ydx}{x^2+y^2} = d\varphi$$

Hence the form is closed.

For the form $\omega = \frac{xdy - ydx}{x^2+y^2}$ one can consider the function $f = \varphi = \arctan\frac{y}{x}$, such that $\omega = df$, but the function f is not well-defined on whole $\mathbf{E}^2$. It is well-defined e.g. we remove the ray $(-\infty, 0]$.

Note that ω is defined in $\mathbf{E}^2 \setminus 0$, but f is defined on $\mathbf{E}^2 \setminus (-\infty, 0]$.

On the other hand it is well defined in any domain where we can define one-valued continuous function $f = \varphi$, i.e. the domain does not contain a loop which rotates around origin. (The function $f = \varphi$ is multi-valued function in the domain $\mathbf{R}^2 \setminus 0$ which contains loops rotating around origin). E.g. one can see that for an arbitrary convex domain which does not contain the origin, or for an arbitrary domain which does not contain a ray $[-\infty, 0]$ a function $f = \varphi$ is well defined one-valued function.

3*

Let $\omega = a(x,y)dx + b(x,y)dy$ be a closed form in $\mathbf{E}^2$, $d\omega = 0$.

Consider the function

$$f(x,y) = x\int_0^1 a(tx,ty)dt + y\int_0^1 b(tx,ty)dt \qquad (2)$$

Show that

$$\omega = df\,.$$

This proves that an arbitrary closed form in $\mathbf{E}^2$ is an exact form.

Why we cannot apply the formula (2) to the form ω defined by the expression (1)?

Perform the calculations: $df = f_x d + f_y dy$.

$$f_x = \int_0^1 a(tx,ty)dt + x\int_0^1 a_x(tx,ty)tdt + y\int_0^1 b_x(tx,ty)tdt\,.$$

and

$$f_y = \int_0^1 b(tx,ty)dt + x\int_0^1 a_y(tx,ty)tdt + y\int_0^1 b_y(tx,ty)tdt\,.$$

On the other hand $d\omega = d(adx + bdy) = (b_x - a_y)dx \wedge dy = 0$. Hence $b_x = a_y$ and

$$f_x = \int_0^1 a(tx,ty)dt + x\int_0^1 a_x(tx,ty)tdt + y\int_0^1 a_y(tx,ty)tdt = \int_0^1 \left(\frac{d}{dt}\left(ta(tx,ty)\right)\right) = ta(tx,ty)\big|_0^1 = a(x,y)\,,$$

because

$$\frac{d}{dt}\left(ta(tx,ty)\right) = a(tx,ty) + xta_x(tx,ty) + yta_y(tx,ty).$$

Analogously

$$f_y = \int_0^1 b(tx,ty)dt + x\int_0^1 b_x(tx,ty)tdt + y\int_0^1 b_y(tx,ty)tdt = \int_0^1 \left(\frac{d}{dt}\left(tb(tx,ty)\right)\right) = tb(tx,ty)\big|_0^1 = b(x,y)\,,$$

We see that $f_x = a(x,y)$ and $f_y = b(x,y)$, i.e. $df = adx + bdy$ ■