Homework 5. Solutions (part a)

Remark *Here are the solutions of exercises 1,2,3,4 and 5 of homework 5. The problems 6 and 7 and 8 of homework 5 we will consider during tutorial on Tuesday 13 March; respectively the solutions of these exercises will appear on the web after 13 March.*

1 *Calculate the Christoffel symbols of the canonical flat connection in* $\mathbf{E}^3$ *in*

a) cylindrical coordinates ($x = r\cos\varphi, y = r\sin\varphi, z = h$),

b) spherical coordinates.

(For the case of sphere try to make calculations at least for components $\Gamma^r_{rr}, \Gamma^r_{r\theta}, \Gamma^r_{r\varphi}, \Gamma^r_{\theta\theta}, \ldots, \Gamma^r_{\varphi\varphi}$*)*

Remark *One can calculate Christoffel symbols using Levi-Civita Theorem (see the Homework6). There is a third way to calculate Christoffel symbols: It is using approach of Lagrangian. This is may be the easiest and most elegant way. (see one of the next Homeworks). Now we perform brute force calculations which just use the fact that Christoffel symbols of flat connection vanish in Cartesian coordinates.*

In cylindrical coordinates (r, φ, h) we have

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = h \end{cases} \quad \text{and} \begin{cases} r = \sqrt{x^2 + y^2} \\ \varphi = arctan\frac{y}{x} \\ h = z \end{cases}$$

'We know that in Cartesian coordinates all Christoffel symbols vanish. Hence in cylindrical coordinates (see in detail lecture notes):

$$\Gamma^r_{rr} = \frac{\partial^2 x}{\partial^2 r}\frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial^2 r}\frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial^2 r}\frac{\partial r}{\partial z} = 0\,,$$

$$\Gamma^r_{r\varphi} = \Gamma^r_{\varphi r} = \frac{\partial^2 x}{\partial r\partial\varphi}\frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial r\partial\varphi}\frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial r\partial\varphi}\frac{\partial r}{\partial z} = -\sin\varphi\cos\varphi + \sin\varphi\cos\varphi = 0\,.$$

$$\Gamma^r_{\varphi\varphi} = \frac{\partial^2 x}{\partial^2 \varphi}\frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial^2 \varphi}\frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial^2 \varphi}\frac{\partial r}{\partial z} = -x\frac{x}{r} - y\frac{y}{r} = -r\,.$$

$$\Gamma^\varphi_{rr} = \frac{\partial^2 x}{\partial^2 r}\frac{\partial \varphi}{\partial x} + \frac{\partial^2 y}{\partial^2 r}\frac{\partial \varphi}{\partial y} + \frac{\partial^2 z}{\partial^2 r}\frac{\partial \varphi}{\partial z} = 0\,.$$

$$\Gamma^\varphi_{\varphi r} = \Gamma^\varphi_{r\varphi} = \frac{\partial^2 x}{\partial r\partial\varphi}\frac{\partial \varphi}{\partial x} + \frac{\partial^2 y}{\partial r\partial\varphi}\frac{\partial \varphi}{\partial y} + \frac{\partial^2 z}{\partial r\partial\varphi}\frac{\partial \varphi}{\partial z} = -\sin\varphi\frac{-y}{r^2} + \cos\varphi\frac{x}{r^2} = \frac{1}{r}$$

$$\Gamma^\varphi_{\varphi\varphi} = \frac{\partial^2 x}{\partial^2 \varphi}\frac{\partial \varphi}{\partial x} + \frac{\partial^2 y}{\partial^2 \varphi}\frac{\partial \varphi}{\partial y} + \frac{\partial^2 z}{\partial^2 \varphi}\frac{\partial \varphi}{\partial z} = -x\frac{-x}{r^2} - y\frac{y}{r^2} = 0\,.$$

All symbols $\Gamma^{\cdot}_{\cdot h}, \Gamma^{\cdot}_{h\cdot}$ vanish

$$\Gamma^r_{rh} = \Gamma^r_{hr} = \Gamma^r_{hh} = \Gamma^r_{\varphi h} = \Gamma^r_{h\varphi} = \Gamma^\varphi_{hr} = dots = 00$$

since $\frac{\partial^2 x}{\partial h \partial \dots} = \frac{\partial^2 y}{\partial h \partial \dots} = \frac{\partial^2 z}{\partial h \partial \dots} = 0$

For all symbols $\Gamma^h_{\cdot\cdot}$ $\Gamma^h_{\cdot\cdot} = \frac{\partial^2 z}{\partial \cdot \partial \cdot}$ since $\frac{\partial h}{\partial x} = \frac{\partial h}{\partial y} = 0$ and $\frac{\partial h}{\partial y} = 1$. On the other hand all $\frac{\partial^2 z}{\partial \cdot \partial \cdot}$ vanish. Hence all symbols $\Gamma^h_{\cdot\cdot}$ vanish. ∎

b) spherical coordinates

$$\begin{cases} x = r \sin \cos \varphi \\ y = r \sin \sin \varphi \\ z = r \cos \theta \end{cases} \qquad \begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \arccos \frac{z}{\sqrt{x^2+y^2+z^2}} \\ \varphi = \arctan \frac{y}{x} \end{cases}$$

The fast way to calculate Christoffel symbol is to use Levi-Civita Theorem, or it is even faster to use Lagrangian of free particle. Perform now brute force calculations only for some components. (Then later we will calculate using Levi-Civita Theorem or Lagrangian of free particle.)

$\Gamma^r_{rr} = 0$ since $\frac{\partial^2 x^i}{\partial^2 r} = 0$.

$$\Gamma^r_{r\theta} = \Gamma^r_{\theta r} = \frac{\partial^2 x}{\partial r \partial \theta} \frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial r \partial \theta} \frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial r \partial \theta} \frac{\partial r}{\partial z} = \cos\theta \cos\varphi \frac{x}{r} + \cos\theta \sin\varphi \frac{y}{r} - \sin\theta \frac{z}{r} = 0\,,$$

$$\Gamma^r_{\theta\theta} = \frac{\partial^2 x}{\partial^2 \theta} \frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial^2 \theta} \frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial^2 \theta} \frac{\partial r}{\partial z} = -r \sin\theta \cos\varphi \frac{x}{r} - r \sin\theta \sin\varphi \frac{y}{r} - r\cos\theta \frac{z}{r} = -r$$

$$\Gamma^r_{r\varphi} = \Gamma^r_{\varphi r} = \frac{\partial^2 x}{\partial r \partial \varphi} \frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial r \partial \varphi} \frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial r \partial \varphi} \frac{\partial r}{\partial z} = -\sin\theta \sin\varphi \frac{x}{r} + \sin\theta \cos\varphi \frac{y}{r} = 0$$

and so on....

2 *Let ∇ be an affine connection on a 2-dimensional manifold M such that in local coordinates (u, v) it is given that $\Gamma^u_{uv} = v$, $\Gamma^v_{uv} = 0$.*

Calculate the vector field $\nabla_{\frac{\partial}{\partial u}} \left(u \frac{\partial}{\partial v} \right)$.

Using the properties of connection and definition of Christoffel symbols have

$$\nabla_{\frac{\partial}{\partial u}} \left(u \frac{\partial}{\partial v} \right) = \partial_{\frac{\partial}{\partial u}} (u) \frac{\partial}{\partial v} + u \nabla_{\frac{\partial}{\partial u}} \left(\frac{\partial}{\partial v} \right) =$$

$$\frac{\partial}{\partial v} + u \left(\Gamma^u_{uv} \frac{\partial}{\partial u} + \Gamma^v_{uv} \frac{\partial}{\partial v} \right) = \frac{\partial}{\partial v} + u \left(v \frac{\partial}{\partial u} + 0 \right) = \frac{\partial}{\partial v} + uv \frac{\partial}{\partial u}\,.$$

3 *Let ∇ be an affine connection on the 2-dimensional manifold M such that in local coordinates (u, v)*

$$\nabla_{\frac{\partial}{\partial u}} \left(u \frac{\partial}{\partial v} \right) = (1 + u^2) \frac{\partial}{\partial v} + u \frac{\partial}{\partial u}\,.$$

Calculate the Christoffel symbols Γ^u_{uv} and Γ^v_{uv} of this connection.

Using the properties of connection we have

$$\nabla_{\frac{\partial}{\partial u}}\left(u\frac{\partial}{\partial v}\right) = u\nabla_{\frac{\partial}{\partial u}}\left(\frac{\partial}{\partial v}\right) + \partial_{\frac{\partial}{\partial u}}(u)\frac{\partial}{\partial v} =$$

$$= u\left(\Gamma^u_{uv}\frac{\partial}{\partial u} + \Gamma^v_{uv}\frac{\partial}{\partial v}\right) + 1\cdot\frac{\partial}{\partial v} = (1+u\Gamma^v_{uv})\frac{\partial}{\partial v} + u\Gamma^u_{uv}\frac{\partial}{\partial u} = \left(1+u^2\right)\frac{\partial}{\partial v} + u\frac{\partial}{\partial u}.$$

Hence $1+u^2 = 1+u\Gamma^v_{uv}$ and $u\Gamma^v_{uv} = u$, i.e. $\Gamma^v_{uv} = u$ and $\Gamma^u_{uv} = 1$.

4 *Let ∇ be an affine connection on a 2-dimensional manifold M such that, in local coordinates (x,y), all Christoffel symbols vanish except $\Gamma^x_{xx} = xy$, $\Gamma^y_{xx} = -1$ and $\Gamma^y_{yy} = y$. Show that for the vector field $\mathbf{X} = \partial_x + x\partial_y$, $\nabla_{\mathbf{X}}\mathbf{X} = xy\mathbf{X}$.*

$$\nabla_{\mathbf{X}}(\mathbf{X}) = \nabla_{\partial_x + x\partial_y}(\partial_x + x\partial_y) = \nabla_{\partial_x}(\partial_x) + \nabla_{\partial_x}(x\partial_y) + x\nabla_{\partial_y}(\partial_x) + x\nabla_{\partial_y}(x\partial_y) =$$

$$\nabla_{\partial_x}(\partial_x) + (\partial_{\partial_x}x)\partial_y + x\nabla_{\partial_x}(\partial_y) + x\nabla_{\partial_y}(\partial_x) + x(\partial_y x)\partial_y + x^2\nabla_{\partial_y}(\partial_y) =$$

$$\Gamma^x_{xx}\partial_x + \Gamma^y_{xx}\partial_y + \partial_y + x(\Gamma^x_{xy}\partial_x + \Gamma^y_{xy}\partial_y) + x(\Gamma^x_{yx}\partial_x + \Gamma^y_{yx}\partial_y) + 0 + x^2(\Gamma^x_{yy}\partial_x + \Gamma^y_{yy}\partial_y) =$$

$$= xy\partial_x - \partial_y + \partial_y + x\cdot 0 + x\cdot 0 + x^2 y\partial_y = xy(\partial_x + x\partial_y) = xy\mathbf{X}\,.$$

5 a) *Consider a connection such that its Christoffel symbols are symmetric in a given coordinate system: $\Gamma^i_{km} = \Gamma^i_{mk}$.*

Show that they are symmetric in an arbitrary coordinate system.

b) Show that the Christoffel symbols of connection ∇ are symmetric (in any coordinate system) if and only if*

$$\nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X},\mathbf{Y}] = 0\,, \tag{5.1}$$

for arbitrary vector fields $\mathbf{X},\mathbf{Y}$.

c) Consider for an arbitrary connection the following operation on the vector fields:*

$$S(\mathbf{X},\mathbf{Y}) = \nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X},\mathbf{Y}]$$

and find its properties.

Solution

a) Let $\Gamma^i_{km} = \Gamma^i_{mk}$. We have to prove that $\Gamma^{i'}_{k'm'} = \Gamma^{i'}_{m'k'}$

We have

$$\Gamma^{i'}_{k'm'} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\Gamma^i_{km} + \frac{\partial x^r}{\partial x^{k'}\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^r}\,. \tag{1}$$

Hence

$$\Gamma^{i'}_{m'k'} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^m}{\partial x^{m'}}\frac{\partial x^k}{\partial x^{k'}}\Gamma^i_{mk} + \frac{\partial x^r}{\partial x^{m'}\partial x^{k'}}\frac{\partial x^{i'}}{\partial x^r}$$

But $\Gamma^i_{km} = \Gamma^i_{mk}$ and $\frac{\partial x^r}{\partial x^{m'} \partial x^{k'}} = \frac{\partial x^r}{\partial x^{k'} \partial x^{m'}}$. Hence

$$\Gamma^{i'}_{m'k'} = \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^m}{\partial x^{m'}} \frac{\partial x^k}{\partial x^{k'}} \Gamma^i_{mk} + \frac{\partial x^r}{\partial x^{m'} \partial x^{k'}} \frac{\partial x^{i'}}{\partial x^r} = \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^m}{\partial x^{m'}} \frac{\partial x^k}{\partial x^{k'}} \Gamma^i_{km} + \frac{\partial x^r}{\partial x^{k'} \partial x^{m'}} \frac{\partial x^{i'}}{\partial x^r} = \Gamma^{i'}_{k'm'} .$$

b*) One can check it by striaghtforward calculations, using definition of commutator:

$$\nabla_{\mathbf{X}} \mathbf{Y} - \nabla_{\mathbf{Y}} \mathbf{X} - [\mathbf{X}, \mathbf{Y}] =$$

$$X^i \left(\partial_i Y^k + \Gamma^k_{ir} Y^r \right) \partial_k - Y^i \left(\partial_i X^k + \Gamma^k_{ir} X^r \right) \partial_k - \left(X^i \partial_i Y^k - Y^i \partial_i X^k \right) = (X^i Y^r - Y^i X^r) \Gamma^k_{ir} \partial_k =$$

$$X^i Y^r \Gamma^k_{ir} \partial_k = -Y^r X^i) \Gamma^k_{ri} \partial_k = X^i Y^r (\Gamma^k_{ir} - \Gamma^k_{ri}) \partial_k = 0 \Leftrightarrow \Gamma^k_{ir} = \Gamma^k_{ri} .$$

Another solution Check using properties of connection and commutator that the expression

$$S(\mathbf{X}, \mathbf{Y}) = \nabla_{\mathbf{X}} \mathbf{Y} - \nabla_{\mathbf{Y}} \mathbf{X} - [\mathbf{X}, \mathbf{Y}] , \tag{5.2}$$

is linear with respect to $C^\infty(M)$:

$$S(f\mathbf{X}, \mathbf{Y}) = S(\mathbf{X}, f\mathbf{Y}) = f S(\mathbf{X}, \mathbf{Y}) . \tag{5.3}$$

Indeed

$$S(f\mathbf{X}, Y) = \nabla_{f\mathbf{X}} \mathbf{Y} - \nabla_{\mathbf{Y}} (f\mathbf{X}) - [f\mathbf{X}, \mathbf{Y}] = f \nabla_X \mathbf{Y} - f \nabla_{\mathbf{Y}} \mathbf{X} - \partial_{\mathbf{Y}} f \mathbf{X} + [\mathbf{Y}, f\mathbf{X}] =$$

$$f \nabla_X \mathbf{Y} - f \nabla_{\mathbf{Y}} \mathbf{X} - (\partial_{\mathbf{Y}} f) \mathbf{X} + \partial_{\mathbf{Y}} f \mathbf{X} + f [\mathbf{Y}, \mathbf{X}] = f (\nabla_X \mathbf{Y} - \nabla_{\mathbf{Y}} \mathbf{X} - [\mathbf{X}, \mathbf{Y}]) = f S(\mathbf{X}, \mathbf{Y}) .$$

The the condition $S(\mathbf{X}, \mathbf{Y}) = -S(\mathbf{Y}, \mathbf{X})$ implies equation (5.1).

Hence it follows from equation (5.2) that it is enoug to prove identity (5.1) only for basic fields: Consider $\mathbf{X} = \frac{\partial}{\partial x^i}, \mathbf{Y} = \frac{\partial}{\partial x^j}$ then since $[\partial_i, \partial_j] = 0$ we have that

$$\nabla_{\mathbf{X}} \mathbf{Y} - \nabla_{\mathbf{Y}} \mathbf{X} - [\mathbf{X}, \mathbf{Y}] = \nabla_i \partial_j - \nabla_j \partial_i = \Gamma^k_{ij} \partial_k - \Gamma^k_{ji} \partial_k = (\Gamma^k_{ij} - \Gamma^k_{ji}) \partial_k = 0$$

We see that commutator for basic fields $\nabla_{\mathbf{X}} \mathbf{Y} - \nabla_{\mathbf{Y}} \mathbf{X} - [\mathbf{X}, \mathbf{Y}] = 0$ if and only if $\Gamma^k_{ij} - \Gamma^k_{ji} = 0$.

c†) We proved it above: see equations (5.2), and equation (5.3)