Symmetries of Clairaut equation

It is well-known that the Clairaut equation

$$y - xy' = f(y') \tag{1}$$

has the one-parametric family of solutions:

$$y = kx + f(k) \tag{2a}$$

and the special soluction, their envelope: $\varphi(x)$,

$$\varphi(x) = k(x)x + f(k(x)) \colon \quad \frac{\partial\varphi}{k} = x + f'(k) = 0\,. \tag{2b}$$

This is standard.

On the other hand the Clairaut equation (1) can be approached in the following way: Consider in 3-dimensional space $E = (p, x, y)$ $(p = y')$ the following transformation

$$(x, p, y) \to (\tilde{x}, \tilde{p}, \tilde{y}) \quad \begin{cases} \tilde{x} = p \\ \tilde{p} = -x \\ \tilde{y} = y - px \end{cases}$$

This is so called Legendre transformation of contact space E. Under this transformation the characteristic 1-form $pdx - dy$ is not changed , and the Clairaut equation transforms to the algebraic equation

$$\tilde{y} = f(\tilde{x})$$

, every solution (2a) is transformed to the generalised solution, the line $\begin{cases} \tilde{x} = k \\ \tilde{p} = \text{arbitrary number} \\ \tilde{y} = f(k) \end{cases}$,

and the special solution (2b), $y = \varphi(x)$ is transformed to the solution, the curve $\begin{cases} \tilde{x} = t \\ \tilde{p} = f'(t) \\ \tilde{y} = f(t) \end{cases}$,

Our aim to find the *one-parametric famiily of contact trasnformations which include the Legendre transformation and see how look the image of solutions (2a), (2b) under this transformation.*

Consider the Hamiltnian $H = \frac{p^2}{2} + \frac{x^2}{2}$ of harmonic oscillator. This Hamiltonian defines the contact vector field

$$\mathbf{X}_H = \frac{\partial H}{\partial p}\frac{\partial}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial}{\partial p} + \left(p\frac{\partial H}{\partial p} - H\right)\frac{\partial}{\partial y}\,,$$

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the transformation (2a) preserves the distribution of planes defined by 1-form $pdx - dy$, thus it can be applied to an arbitrary differential equation $F(x, p, y) = 0$ $(p = y')$.

and it induces the contact transformation

$$\begin{pmatrix} x \\ p \\ y \end{pmatrix} \to \begin{pmatrix} \tilde{x}_t \\ \tilde{p}_t \\ \tilde{y}_t \end{pmatrix} ,$$

such that

$$\begin{cases} \frac{d\tilde{x}}{dt} = \frac{\partial H}{\partial \tilde{p}} = \tilde{p} \\ \frac{d\tilde{p}}{dt} = -\frac{\partial H}{\partial \tilde{x}} = -\tilde{x} \\ \frac{d\tilde{y}}{dt} = \left(\frac{\tilde{p}^2}{2} - \frac{\tilde{x}^2}{2} \right) \end{cases}$$

Solving this equation and taking in account the boudnary conditions

$$\begin{pmatrix} \tilde{x}_t \\ \tilde{p}_t \\ \tilde{y}_t \end{pmatrix} \Big|_{t=0} = \begin{pmatrix} x \\ p \\ y \end{pmatrix}$$

we come to

$$\begin{cases} x_t = x \cos t + p \sin t \\ p_t = -x \sin t + p \cos t \\ y_t = y + \frac{1}{4} p^2 \sin 2t - \frac{1}{4} x^2 \sin 2t + \frac{1}{2} px \left(\cos 2t - 1 \right) \end{cases}$$

We see that for $t = 0$ this is identity contact transformation, and for $t = \frac{\pi}{2}$ this the trasnformation

$$\begin{cases} x_{\frac{\pi}{2}} = p \\ p_{\frac{\pi}{2}} = -x \\ y_{\frac{\pi}{2}} = y - px \end{cases}$$