Homework 4. Solutions

1 *Calculate the Christoffel symbols of the canonical flat connection in* $\mathbf{E}^3$ *in*

a) cylindrical coordinates $(x = r\cos\varphi, y = r\sin\varphi, z = h)$,

b) spherical coordinates.

(For the case of sphere try to make calculations at least for components $\Gamma^r_{rr}, \Gamma^r_{r\theta}, \Gamma^r_{r\varphi}, \Gamma^r_{\theta\theta}, \ldots, \Gamma^r_{\varphi\varphi}$*)*

Remark One can calculate Christoffel symbols using Levi-Civita Theorem (Homework 5). There is a third way to calculate Christoffel symbols: It is using approach of Lagrangian. This is may be the easiest and most elegant way. (see the Homework 6)

In cylindrical coordinates (r, φ, h) we have

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = h \end{cases} \quad \text{and} \begin{cases} r = \sqrt{x^2+y^2} \\ \varphi = arctan\frac{y}{x} \\ h = z \end{cases}$$

We know that in Cartesian coordinates all Christoffel symbols vanish. Hence in cylindrical coordinates (see in detail lecture notes):

$$\Gamma^r_{rr} = \frac{\partial^2 x}{\partial^2 r}\frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial^2 r}\frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial^2 r}\frac{\partial r}{\partial z} = 0\,,$$

$$\Gamma^r_{r\varphi} = \Gamma^r_{\varphi r} = \frac{\partial^2 x}{\partial r\partial\varphi}\frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial r\partial\varphi}\frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial r\partial\varphi}\frac{\partial r}{\partial z} = -\sin\varphi\cos\varphi + \sin\varphi\cos\varphi = 0\,.$$

$$\Gamma^r_{\varphi\varphi} = \frac{\partial^2 x}{\partial^2\varphi}\frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial^2\varphi}\frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial^2\varphi}\frac{\partial r}{\partial z} = -x\frac{x}{r} - y\frac{y}{r} = -r\,.$$

$$\Gamma^\varphi_{rr} = \frac{\partial^2 x}{\partial^2 r}\frac{\partial\varphi}{\partial x} + \frac{\partial^2 y}{\partial^2 r}\frac{\partial\varphi}{\partial y} + \frac{\partial^2 z}{\partial^2 r}\frac{\partial\varphi}{\partial z} = 0\,.$$

$$\Gamma^\varphi_{\varphi r} = \Gamma^\varphi_{r\varphi} = \frac{\partial^2 x}{\partial r\partial\varphi}\frac{\partial\varphi}{\partial x} + \frac{\partial^2 y}{\partial r\partial\varphi}\frac{\partial\varphi}{\partial y} + \frac{\partial^2 z}{\partial r\partial\varphi}\frac{\partial\varphi}{\partial z} = -\sin\varphi\frac{-y}{r^2} + \cos\varphi\frac{x}{r^2} = \frac{1}{r}$$

$$\Gamma^\varphi_{\varphi\varphi} = \frac{\partial^2 x}{\partial^2\varphi}\frac{\partial\varphi}{\partial x} + \frac{\partial^2 y}{\partial^2\varphi}\frac{\partial\varphi}{\partial y} + \frac{\partial^2 z}{\partial^2\varphi}\frac{\partial\varphi}{\partial z} = -x\frac{-x}{r^2} - y\frac{y}{r^2} = 0\,.$$

All symbols $\Gamma^{\cdot}_{\cdot h}, \Gamma^{\cdot}_{h\cdot}$ vanish

$$\Gamma^r_{rh} = \Gamma^r_{hr} = \Gamma^r_{hh} = \Gamma^r_{\varphi h} = \Gamma^r_{h\varphi} = \Gamma^\varphi_{hr} = dots = 00$$

since $\frac{\partial^2 x}{\partial h\partial\ldots} = \frac{\partial^2 y}{\partial h\partial\ldots} = \frac{\partial^2 z}{\partial h\partial\ldots} = 0$

For all symbols $\Gamma^h_{\cdot\cdot}$ $\Gamma^h_{\cdot\cdot} = \frac{\partial^2 z}{\partial\cdot\partial\cdot}$ since $\frac{\partial h}{\partial x} = \frac{\partial h}{\partial y} = 0$ and $\frac{\partial h}{\partial y} = 1$. On the other hand all $\frac{\partial^2 z}{\partial\cdot\partial\cdot}$ vanish. Hence all symbols $\Gamma^h_{\cdot\cdot}$ vanish. ∎

b) spherical coordinates

$$\begin{cases} x = r\sin\cos\varphi \\ y = r\sin\sin\varphi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \arccos\frac{z}{\sqrt{x^2+y^2+z^2}} \\ \varphi = \arctan\frac{y}{x} \end{cases}$$

We already know the fast way to calculate Christoffel symbol using Lagrangian of free particle and this method work for a flat connection since flat connection is a Levi-Civita connection for Euclidean metric

So perform now brute force calculations only for some components. (Then later (in homework 6) we will calculate using very quickly Lagrangian of free particle.)

$\Gamma^r_{rr} = 0$ since $\frac{\partial^2 x^i}{\partial^2 r} = 0$.

$$\Gamma^r_{r\theta} = \Gamma^r_{\theta r} = \frac{\partial^2 x}{\partial r \partial\theta}\frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial r\partial\theta}\frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial r\partial\theta}\frac{\partial r}{\partial z} = \cos\theta\cos\varphi\frac{x}{r} + \cos\theta\sin\varphi\frac{y}{r} - \sin\theta\frac{z}{r} = 0\,,$$

$$\Gamma^r_{\theta\theta} = \frac{\partial^2 x}{\partial^2\theta}\frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial^2\theta}\frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial^2\theta}\frac{\partial r}{\partial z} = -r\sin\theta\cos\varphi\frac{x}{r} - r\sin\theta\sin\varphi\frac{y}{r} - r\cos\theta\frac{z}{r} = -r$$

$$\Gamma^r_{r\varphi} = \Gamma^r_{\varphi r} = \frac{\partial^2 x}{\partial r\partial\varphi}\frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial r\partial\varphi}\frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial r\partial\varphi}\frac{\partial r}{\partial z} = -\sin\theta\sin\varphi\frac{x}{r} + \sin\theta\cos\varphi\frac{y}{r} = 0$$

and so on....

2 *Let ∇ be an affine connection on a 2-dimensional manifold M such that in local coordinates (u, v) it is given that $\Gamma^u_{uv} = v$, $\Gamma^v_{uv} = 0$.*

Calculate the vector field $\nabla_{\frac{\partial}{\partial u}}\left(u\frac{\partial}{\partial v}\right)$.

Using the properties of connection and definition of Christoffel symbols have

$$\nabla_{\frac{\partial}{\partial u}}\left(u\frac{\partial}{\partial v}\right) = \partial_{\frac{\partial}{\partial u}}(u)\frac{\partial}{\partial v} + u\nabla_{\frac{\partial}{\partial u}}\left(\frac{\partial}{\partial v}\right) =$$

$$\frac{\partial}{\partial v} + u\left(\Gamma^u_{uv}\frac{\partial}{\partial u} + \Gamma^v_{uv}\frac{\partial}{\partial v}\right) = \frac{\partial}{\partial v} + u\left(v\frac{\partial}{\partial u} + 0\right) = \frac{\partial}{\partial v} + uv\frac{\partial}{\partial u}\,.$$

3 *Let ∇ be an affine connection on the 2-dimensional manifold M such that in local coordinates (u, v)*

$$\nabla_{\frac{\partial}{\partial u}}\left(u\frac{\partial}{\partial v}\right) = (1+u^2)\frac{\partial}{\partial v} + u\frac{\partial}{\partial u}\,.$$

Calculate the Christoffel symbols Γ^u_{uv} and Γ^v_{uv} of this connection.

Using the properties of connection we have $\nabla_{\frac{\partial}{\partial u}}\left(u\frac{\partial}{\partial v}\right) = u\nabla_{\frac{\partial}{\partial u}}\left(\frac{\partial}{\partial v}\right) +$

$$\partial_{\frac{\partial}{\partial u}}(u)\frac{\partial}{\partial v} = u\left(\Gamma^u_{uv}\frac{\partial}{\partial u} + \Gamma^v_{uv}\frac{\partial}{\partial v}\right) + 1\cdot\frac{\partial}{\partial v} = (1 + u\Gamma^v_{uv})\frac{\partial}{\partial v} + u\Gamma^u_{uv}\frac{\partial}{\partial u} = \left(1+u^2\right)\frac{\partial}{\partial v} + u\frac{\partial}{\partial u}.$$

Hence $1 + u^2 = 1 + u\Gamma^v_{uv}$ and $u\Gamma^v_{uv} = u$, i.e. $\Gamma^v_{uv} = u$ and $\Gamma^u_{uv} = 1$. ■

4 a) *Consider a connection such that its Christoffel symbols are symmetric in a given coordinate system: $\Gamma^i_{km} = \Gamma^i_{mk}$.*

Show that they are symmetric in an arbitrary coordinate system.

b*) Show that the Christoffel symbols of connection ∇ are symmetric (in any coordinate system) if and only if

$$\nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X}, \mathbf{Y}] = 0\,,$$

for arbitrary vector fields $\mathbf{X}, \mathbf{Y}$.

c)* Consider for an arbitrary connection the following operation on the vector fields:

$$S(\mathbf{X}, \mathbf{Y}) = \nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X}, \mathbf{Y}]$$

and find its properties.

Solution

a) Let $\Gamma^i_{km} = \Gamma^i_{mk}$. We have to prove that $\Gamma^{i'}_{k'm'} = \Gamma^{i'}_{m'k'}$

We have

$$\Gamma^{i'}_{k'm'} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\Gamma^i_{km} + \frac{\partial x^r}{\partial x^{k'}\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^r}\,. \tag{1}$$

Hence

$$\Gamma^{i'}_{m'k'} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^m}{\partial x^{m'}}\frac{\partial x^k}{\partial x^{k'}}\Gamma^i_{mk} + \frac{\partial x^r}{\partial x^{m'}\partial x^{k'}}\frac{\partial x^{i'}}{\partial x^r}$$

But $\Gamma^i_{km} = \Gamma^i_{mk}$ and $\frac{\partial x^r}{\partial x^{m'}\partial x^{k'}} = \frac{\partial x^r}{\partial x^{k'}\partial x^{m'}}$. Hence

$$\Gamma^{i'}_{m'k'} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^m}{\partial x^{m'}}\frac{\partial x^k}{\partial x^{k'}}\Gamma^i_{mk} + \frac{\partial x^r}{\partial x^{m'}\partial x^{k'}}\frac{\partial x^{i'}}{\partial x^r} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^m}{\partial x^{m'}}\frac{\partial x^k}{\partial x^{k'}}\Gamma^i_{km} + \frac{\partial x^r}{\partial x^{k'}\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^r} = \Gamma^{i'}_{k'm'}\,.$$

b) The relation

$$\nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X},\mathbf{Y}] = 0$$

holds for all fields if and only if it holds for all basic fields. One can easy check it using axioms of connection (see the next part). Consider $\mathbf{X} = \frac{\partial}{\partial x^i}, \mathbf{Y} = \frac{\partial}{\partial x^j}$ then since $[\partial_i, \partial_j] = 0$ we have that

$$\nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X},\mathbf{Y}] = \nabla_i\partial_j - \nabla_j\partial_i = \Gamma^k_{ij}\partial_k - \Gamma^k_{ji}\partial_k = (\Gamma^k_{ij} - \Gamma^k_{ji})\partial_k = 0$$

We see that commutator for basic fields $\nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X},\mathbf{Y}] = 0$ if and only if $\Gamma^k_{ij} - \Gamma^k_{ji} = 0$.

c) One can easy check it by straightforward calculations or using axioms for connection that $S(\mathbf{X},\mathbf{Y})$ is a vector-valued bilinear form on vectors. In particularly $S(f\mathbf{X}, Y) = fS(\mathbf{X},\mathbf{Y})$ for an arbitrary (smooth) function. Show this just using axioms defining connection:

$$S(f\mathbf{X}, Y) = \nabla_{f\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}(f\mathbf{X}) - [f\mathbf{X},\mathbf{Y}] = f\nabla_X\mathbf{Y} - f\nabla_{\mathbf{Y}}\mathbf{X} - \partial_{\mathbf{Y}}f\mathbf{X} + [\mathbf{Y}, f\mathbf{X}] =$$

$$f\nabla_X\mathbf{Y} - f\nabla_{\mathbf{Y}}\mathbf{X} - (\partial_{\mathbf{Y}}f)\mathbf{X} + \partial_{\mathbf{Y}}f\mathbf{X} + f\,[\mathbf{Y},\mathbf{X}] = f(\nabla_X\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X},\mathbf{Y}]) = fS(\mathbf{X},\mathbf{Y})$$

5 *Consider the surface M in the Euclidean space $\mathbf{E}^n$. Calculate the induced connection in the following cases*

a) $M = S^1$ in $\mathbf{E}^2$,

b) M— parabola $y = x^2$ in $\mathbf{E}^2$,

c) cylinder in $\mathbf{E}^3$.

d) cone in $\mathbf{E}^3$.

e) sphere in $\mathbf{E}^3$.

f) saddle $z = xy$ in $\mathbf{E}^3$

Solution.

a) Consider polar coordinate on S^1, $x = R\cos\varphi, y = R\sin\varphi$. We have to define the connection on S^1 induced by the canonical flat connection on $\mathbf{E}^2$. It suffices to define $\nabla_{\frac{\partial}{\partial\varphi}}\frac{\partial}{\partial\varphi} = \Gamma^\varphi_{\varphi\varphi}\partial_\varphi$.

Recall the general rule. Let $\mathbf{r}(u^\alpha)$: $\quad x^i = x^i(u^\alpha)$ is embedded surface in Euclidean space $\mathbf{E}^n$. The basic vectors $\frac{\partial}{\partial u^\alpha} = \frac{\partial \mathbf{r}(u)}{\partial u^\alpha}$. To take the induced covariant derivative $\nabla_{\mathbf{X}}\mathbf{Y}$ for two tangent vectors $\mathbf{X},\mathbf{Y}$ we take a usual derivative of vector $\mathbf{Y}$ along vector $\mathbf{X}$ (the derivative with respect to canonical flat connection: in Cartesian coordiantes is just usual derivatives of components) then we take the tangent component of the answer, since in general derivative of vector $\mathbf{Y}$ along vector $\mathbf{X}$ is not tangent to surface:

$$\nabla_{\frac{\partial}{\partial u^\alpha}}\frac{\partial}{\partial u^\beta} = \Gamma^\gamma_{\alpha\beta}\frac{\partial}{\partial u^\gamma} = \left(\nabla^{(\text{canonical})}_{\partial_\alpha}\frac{\partial}{\partial u^\beta}\right)_{\text{tangent}} = \left(\frac{\partial^2\mathbf{r}(u)}{\partial u^\alpha\partial u^\beta}\right)_{\text{tangent}}$$

$(\nabla_{\text{canonical } \partial_\alpha} \frac{\partial}{\partial u^\beta})$ is just usual derivative in Euclidean space since for canonical connection all Christoffel symbols vanish.)

In the case of 1-dimensional manifold, curve it is just tangential acceleration!:

$$\nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial u} = \Gamma^u_{uu} \frac{\partial}{\partial u} = \left(\nabla^{(\text{canonical})}_{\partial_u} \frac{\partial}{\partial u} \right)_{\text{tangent}} = \left(\frac{d^2\mathbf{r}(u)}{du^2} \right)_{\text{tangent}} = \mathbf{a}_{\text{tangent}}$$

For the circle S^1, $(x = R\cos\varphi, y = R\sin\varphi)$, in $\mathbf{E}^2$. We have

$$\mathbf{r}_\varphi = \frac{\partial}{\partial\varphi} = \frac{\partial x}{\partial\varphi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\varphi}\frac{\partial}{\partial y} = -R\sin\varphi\frac{\partial}{\partial x} + R\cos\varphi\frac{\partial}{\partial y}\,,$$

$$\nabla_{\frac{\partial}{\partial\varphi}} \frac{\partial}{\partial\varphi} = \Gamma^\varphi_{\varphi\varphi}\partial_\varphi = \left(\nabla^{(\text{canonic.})}_{\partial_\varphi} \partial_\varphi \right)_{\text{tangent}} = \left(\frac{\partial}{\partial\varphi}\mathbf{r}_\varphi \right)_{\text{tangent}} =$$

$$\left(\frac{\partial}{\partial\varphi}\left(-R\sin\varphi\right)\frac{\partial}{\partial x} + \frac{\partial}{\partial\varphi}\left(R\cos\varphi\right)\frac{\partial}{\partial y} \right)_{\text{tangent}} = \left(-R\cos\varphi\frac{\partial}{\partial x} - R\sin\varphi\frac{\partial}{\partial y} \right)_{\text{tangent}} = 0,$$

since the vector $-R\cos\varphi\frac{\partial}{\partial x} - R\sin\varphi\frac{\partial}{\partial y}$ is orthogonal to the tangent vector $\mathbf{r}_\varphi$. In other words it means that acceleration is centripetal: tangential acceleration equals to zero.

We see that in coordinate φ, $\Gamma^\varphi_{\varphi\varphi} = 0$. ■

Additional work: Perform calculation of Christoffel symbol in stereographic coordinate t:

$$x = \frac{2tR^2}{R^2 + t^2}, y = \frac{R(t^2 - R^2)}{t^2 + R^2}\,.$$

In this case

$$\mathbf{r}_t = \frac{\partial}{\partial t} = \frac{\partial x}{\partial t}\frac{\partial}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial}{\partial y} = \frac{2R^2}{(R^2+t^2)^2}\left((R^2 - t^2)\frac{\partial}{\partial x} + 2tR\frac{\partial}{\partial x} \right)\,,$$

$$\nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} = \Gamma^t_{tt}\partial_t = \left(\nabla^{(\text{canonic.})}_{\partial_t} \partial_t \right)_{\text{tangent}} = \left(\frac{\partial}{\partial t}\mathbf{r}_t \right)_{\text{tangent}} = (\mathbf{r}_{tt})_{\text{tangent}} =$$

$$\left(-\frac{4t}{t^2+R^2}\mathbf{r}_t + \frac{2R^2}{(R^2+t^2)^2}\left(-2t\frac{\partial}{\partial x} + 2R\frac{\partial}{\partial y} \right) \right)_{\text{tangent}}$$

In this case $\mathbf{r}_{tt}$ is not orthogonal to velocity: to calculate $(\mathbf{r}_{tt})_{\text{tangent}}$ we need to extract its orthogonal component:

$$(\mathbf{r}_{tt})_{\text{tangent}} = \mathbf{r}_{tt} - \langle \mathbf{r}_{tt}, \mathbf{n}_t \rangle \mathbf{n}$$

We have

$$\mathbf{n}_t = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{1}{R^2 + t^2}\left(2tR\partial_x + (t^2 - R^2)\partial_y \right)\,,$$

where $\langle \mathbf{r}_t, \mathbf{n} \rangle = 0$. Hence $\langle \mathbf{r}_{tt}, \mathbf{n}_t \rangle = \frac{-4R^3}{(t^2+R^2)^2}$ and

$$(\mathbf{r}_{tt})_{\text{tangent}} = \mathbf{r}_{tt} - \langle \mathbf{r}_{tt}, \mathbf{n}_t \rangle \mathbf{n} =$$

$$\left(-\frac{4t}{t^2+R^2}\mathbf{r}_t + \frac{2R^2}{(R^2+t^2)^2}\left(-2t\frac{\partial}{\partial x} + 2R\frac{\partial}{\partial y} \right) \right) + \frac{4R^3}{(t^2+R^2)^2}\cdot\frac{1}{R^2+t^2}\left(2tR\partial_x + (t^2 - R^2)\partial_y \right) = \frac{-2t}{t^2+R^2}\mathbf{r}_t$$

We come to the answer:

$$\nabla_{\partial_t}\partial_t = \frac{-2t}{t^2+R^2}\partial_t, \quad \mathrm{i.e.}\Gamma^{\mathrm{t}}_{\mathrm{tt}} = \frac{-2\mathrm{t}}{\mathrm{t}^2 + \mathrm{R}^2}$$

Of course we could calculate the Christoffel symbol in stereographic coordinates just using the fact that we already know the Christoffel symbol in polar coordinates: $\Gamma^{\varphi}_{\varphi\varphi} = 0$, hence

$$\Gamma^t_{tt} = \frac{dt}{d\varphi}\frac{d\varphi}{dx}\frac{d\varphi}{dx}\Gamma^{\varphi}_{\varphi\varphi} + \frac{d^2\varphi}{dt^2}\frac{dt}{d\varphi} = \frac{d^2\varphi}{dt^2}\frac{dt}{d\varphi}$$

It is easy to see that $t = R\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)$, i.e. $\varphi = 2\arctan\frac{t}{R} - \frac{\pi}{2}$ and

$$\Gamma^t_{tt} = \frac{d^2\varphi}{dt^2}\frac{dt}{d\varphi} = \frac{\frac{d^2\varphi}{dt^2}}{\frac{d\varphi}{dt}} = -\frac{2t}{t^2+R^2}\,.$$

b) For parabola $x = t, y = t^2$

$$\mathbf{r}_t = \frac{\partial}{\partial t} = \frac{\partial x}{\partial t}\frac{\partial}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial}{\partial y} = \frac{\partial}{\partial x} + 2t\frac{\partial}{\partial y}\,,$$

$$\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t} = \Gamma^t_{tt}\partial_t = \left(\nabla^{(\text{canonic.})}_{\partial_t}\partial_t\right)_{\text{tangent}} = \left(\frac{\partial}{\partial t}\mathbf{r}_t\right)_{\text{tangent}} = (\mathbf{r}_{tt})_{\text{tangent}} = \left(2\frac{\partial}{\partial y}\right)_{\text{tangent}}$$

To calculate $(\mathbf{r}_{tt})_{\text{tangent}}$ we need to extract its orthogonal component: $(\mathbf{r}_{tt})_{\text{tangent}} = \mathbf{r}_{tt} - \langle\mathbf{r}_{tt},\mathbf{n}_t\rangle\mathbf{n}$, where $\mathbf{n}$ is an orthogonal unit vector: $\langle\mathbf{n},\mathbf{r}_t\rangle = 0, \langle\mathbf{n},\mathbf{n}\rangle = 1$:

$$\mathbf{n}_t = \frac{1}{\sqrt{1+4t^2}}\left(-2t\partial_x + \partial_y\right)\,.$$

We have

$$(\mathbf{r}_{tt})_{\text{tangent}} = \mathbf{r}_{tt} - \langle\mathbf{r}_{tt},\mathbf{n}_t\rangle\mathbf{n} = 2\partial_y - \left\langle 2\partial_y, \frac{1}{\sqrt{1+4t^2}}\left(-2t\partial_x+\partial_y\right)\right\rangle \frac{1}{\sqrt{1+4t^2}}\left(-2t\partial_x+\partial_y\right) =$$

$$\frac{4t}{1+4t^2}\partial_x + \frac{8t^2}{1+4t^2}\partial_y = \frac{4t}{1+4t^2}\left(\partial_x + 2t\partial_y\right) = \frac{4t}{1+4t^2}\partial_t$$

We come to the answer:

$$\nabla_{\partial_t}\partial_t = \frac{4t}{1+4t^2}\partial_t, \quad \text{i.e.}\Gamma^{\text{t}}_{\text{tt}} = \frac{4\text{t}}{1+4\text{t}^2}$$

Remark Do not be surprised by resemblance of the answer to the answer for circle in stereographic coordinates.

c) *Cylinder*

$\mathbf{r}(h,\varphi)$: $\begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases}$.

$\partial_h = \mathbf{r}_h = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\partial_\varphi = \mathbf{r}_\varphi = \begin{pmatrix} -a\sin\varphi \\ a\cos\varphi \\ 0 \end{pmatrix}$

Calculate

$$\nabla_{\partial_h}\partial_h = \Gamma^h_{hh}\partial_h + \Gamma^{\varphi}_{hh}\partial_\varphi = \left(\frac{\partial^2\mathbf{r}}{\partial h^2}\right)_{\text{tangent}} = 0 \,\text{since}\; \mathbf{r}_{hh} = 0.$$

Hence $\Gamma^h_{hh} = \Gamma^{\varphi}_{hh} = 0$

$$\nabla_{\partial_h}\partial_\varphi = \nabla_{\partial_\varphi}\partial_h = \Gamma^h_{h\varphi}\partial_h + \Gamma^{\varphi}_{h\varphi}\partial_\varphi = \left(\frac{\partial^2\mathbf{r}}{\partial h\partial\varphi}\right)_{\text{tangent}} = 0 \,\text{since}\; \mathbf{r}_{h\varphi} = 0$$

Hence $\Gamma^h_{h\varphi} = \Gamma^h_{\varphi h} = \Gamma^\varphi_{h\varphi} = \Gamma^\varphi_{\varphi h} = 0$.

$$\nabla_{\partial_\varphi}\partial_\varphi = \Gamma^h_{\varphi\varphi}\partial_h + \Gamma^\varphi_{\varphi\varphi}\partial_\varphi = \left(\frac{\partial^2 \mathbf{r}}{\partial\varphi\partial\varphi}\right)_{\text{tangent}} = \left(\begin{pmatrix} -a\cos\varphi \\ -a\sin\varphi \\ 0 \end{pmatrix}\right)_{\text{tangent}} = 0$$

since the vector $\mathbf{r}_{\varphi\varphi} = \begin{pmatrix} -a\cos\varphi \\ -a\sin\varphi \\ 0 \end{pmatrix}$ is orthogonal to the surface of cylinder. Hence $\Gamma^h_{h\varphi} = \Gamma^h_{\varphi h} = \Gamma^\varphi_{h\varphi} = \Gamma^\varphi_{\varphi h} = 0$

We see that for cylinder all Christoffel symbols in cylindrical coordinates vanish. This is not big surprise: in cylindrical coordinates metric equals $dh^2 = a^2 d\varphi^2$. This due to Levi-Civita theorem one can see that Levi-Civita connection which is equal to induced connection vanishes since all coefficients are constants.

d) *Cone*

For cone: $x^2 + y^2 = k^2 z^2$ we have $\mathbf{r}(h,\varphi) = \begin{cases} x = kh\cos\ \varphi \\ y = kh\sin\varphi \\ z = h \end{cases}$

$$\frac{\partial}{\partial h} = \mathbf{r}_h = \begin{pmatrix} k\cos\varphi \\ k\sin\varphi \\ 1 \end{pmatrix},\ \frac{\partial}{\partial\varphi} = \mathbf{r}_\varphi = \begin{pmatrix} -kh\sin\varphi \\ kh\cos\varphi \\ 0 \end{pmatrix},\ \mathbf{n} = \frac{1}{\sqrt{1+k^2}}\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix}$$

We have $\mathbf{r}_{hh} = 0$, hence $\nabla_{\partial_h}\partial_h = 0$. i.e. $\Gamma^h_{hh} = \Gamma^\varphi_{hh} = 0$.

We have that $\mathbf{r}_{h\varphi} = \mathbf{r}_{\varphi h} = \begin{pmatrix} -k\sin\varphi \\ k\cos\varphi \\ 0 \end{pmatrix} = \frac{\mathbf{r}_\varphi}{h}$, i.e. $\nabla_{\partial_h}\partial_\varphi = \nabla_{\partial_\varphi}\partial_h = \frac{\mathbf{r}_\varphi}{h}$:

$$\Gamma^\varphi_{h\varphi} = \Gamma^\varphi_{\varphi,h} = \frac{1}{h}, \quad \Gamma^h_{h\varphi} = \Gamma^h_{\varphi h}.$$

Now calculate $\mathbf{r}_{\varphi\varphi}$: $\mathbf{r}_{\varphi\varphi} = \begin{pmatrix} -kh\cos\varphi \\ -kh\sin\varphi \\ 0 \end{pmatrix}$. This vector is neither tangent to the cone nor orthogonal to the cone: $0 \neq \langle \mathbf{r}_{\varphi\varphi}, \mathbf{n}\rangle = -\frac{kh}{\sqrt{1+k^2}}$. Hence we have consider its decomposition:

$$\mathbf{r}_{\varphi\varphi} = \underbrace{\mathbf{r}_{\varphi\varphi} - \langle \mathbf{r}_{\varphi\varphi}, \mathbf{n}\rangle\mathbf{n}}_{\text{tangent component}} + \underbrace{\langle \mathbf{r}_{\varphi\varphi}, \mathbf{n}\rangle\mathbf{n}}_{\text{orthogonal component}}$$

Hence we have

$$\nabla_\varphi\partial_\varphi = (\mathbf{r}_{\varphi\varphi})_{tangent} = \mathbf{r}_{\varphi\varphi} - \langle \mathbf{r}_{\varphi\varphi}, \mathbf{n}\rangle\mathbf{n} = \mathbf{r}_{\varphi\varphi} + \frac{kh}{\sqrt{1+k^2}}\mathbf{n} =$$

$$\begin{pmatrix} -kh\cos\varphi \\ -kh\sin\varphi \\ 0 \end{pmatrix} + \frac{kh}{1+k^2}\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix} = -\frac{hk^2}{1+k^2}\begin{pmatrix} k\cos\varphi \\ k\sin\varphi \\ 1 \end{pmatrix} = -\frac{hk^2}{1+k^2}\mathbf{r}_h,$$

i.e.

$$\Gamma^h_{\varphi\varphi} = -\frac{hk^2}{1+k^2},\ \Gamma^\varphi_{\varphi\varphi} = 0\,.$$

e) *Sphere*

For the sphere $\mathbf{r}(\theta,\varphi)$: $\begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases}$, we have

$$\frac{\partial}{\partial\theta} = \mathbf{r}_\theta = \begin{pmatrix} R\cos\theta\cos\varphi \\ R\cos\theta\sin\varphi \\ -R\sin\theta \end{pmatrix}, \ \frac{\partial}{\partial\varphi} = \mathbf{r}_\varphi = \begin{pmatrix} -R\sin\theta\sin\varphi \\ R\sin\theta\cos\varphi \\ 0 \end{pmatrix}, \ \mathbf{n} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

Calculate

$$\nabla_{\partial_\theta}\partial_\theta = \Gamma^\theta_{\theta\theta}\partial_\theta + \Gamma^\varphi_{\theta\theta}\partial_\varphi = \left(\frac{\partial^2\mathbf{r}}{\partial\theta^2}\right)_{\text{tangent}} = 0$$

since $\frac{\partial^2\mathbf{r}}{\partial\theta^2} = -R\mathbf{n}$ is orthogonal to the sphere. Hence $\Gamma^\theta_{\theta\theta} = \Gamma^\varphi_{\theta\theta} = 0$.

Now calculate

$$\nabla_{\partial_\theta}\partial_\varphi = \Gamma^\theta_{\theta\varphi}\partial_\theta + \Gamma^\varphi_{\theta\varphi}\partial_\varphi = \left(\frac{\partial^2\mathbf{r}}{\partial\theta\partial\varphi}\right)_{\text{tangent}}.$$

We have

$$\frac{\partial^2\mathbf{r}}{\partial\theta\partial\varphi} = \operatorname{cotan}\theta\,\mathbf{r}_\varphi,$$

hence

$$\nabla_{\partial_\theta}\partial_\varphi = \Gamma^\theta_{\theta\varphi}\partial_\theta + \Gamma^\varphi_{\theta\varphi}\partial_\varphi = \left(\frac{\partial^2\mathbf{r}}{\partial\theta\partial\varphi}\right)_{\text{tangent}} = \operatorname{cotan}\theta\,\mathbf{r}_\varphi, i.e.$$

$\Gamma^\theta_{\theta\varphi} = 0, \Gamma^\varphi_{\theta\varphi} = \operatorname{cotan}\theta$

Now calculate

$$\nabla_{\partial_\varphi}\partial_\theta = \Gamma^\theta_{\varphi\theta}\partial_\theta + \Gamma^\varphi_{\varphi\theta}\partial_\varphi = \left(\frac{\partial^2\mathbf{r}}{\partial\varphi\partial\theta}\right)_{\text{tangent}}.$$

We have

$$\frac{\partial^2\mathbf{r}}{\partial\varphi\partial\theta} = \operatorname{cotan}\theta\,\mathbf{r}_\varphi,$$

hence

$$\nabla_{\partial_\theta}\partial_\varphi = \Gamma^\theta_{\theta\varphi}\partial_\theta + \Gamma^\varphi_{\theta\varphi}\partial_\varphi = \left(\frac{\partial^2\mathbf{r}}{\partial\theta\partial\varphi}\right)_{\text{tangent}} = \operatorname{cotan}\theta\,\mathbf{r}_\varphi, i.e.$$

$\Gamma^\theta_{\varphi\theta} = 0, \Gamma^\varphi_{\varphi\theta} = \operatorname{cotan}\theta$. Of course we did not need to perform these calculations: since ∇ is symmetric connection and $\nabla_{\partial_\varphi}\partial_\theta = \nabla_{\partial_\theta}\partial_\varphi$, i.e.

$$\Gamma^\theta_{\varphi\theta} = \Gamma^\theta_{\theta\varphi} = 0 \ \Gamma^\varphi_{\varphi\theta} = \Gamma^\varphi_{\theta\varphi} = \operatorname{cotan}\theta\,.$$

and finally

$$\nabla_{\partial_\varphi}\partial_\varphi = \Gamma^\theta_{\varphi\varphi}\partial_\theta + \Gamma^\varphi_{\varphi\varphi}\partial_\varphi = \left(\frac{\partial^2\mathbf{r}}{\partial\varphi^2}\right)_{\text{tangent}}.$$

We have

$$\frac{\partial^2\mathbf{r}}{\partial\varphi^2} = \mathbf{r}_{\varphi\varphi} = \begin{pmatrix} -R\sin\theta\cos\varphi \\ -R\sin\theta\sin\varphi \\ 0 \end{pmatrix}.$$

The vector $\mathbf{r}_{\varphi\varphi}$ is not proportional to normal vector $\mathbf{n}$, i.e. it is not orthogonal to the sphere; the vector $\mathbf{r}_{\varphi\varphi}$ is not tangent to sphere, i.e. it is not orthogonal to vector $\mathbf{n}$: $0 \neq \langle \mathbf{r}_{\varphi\varphi}, \mathbf{n}\rangle = -R\sin^2\theta$. We decompose the vector $\mathbf{r}_{\varphi\varphi}$ on the sum of tangent vector and orthogonal vector:

$$\mathbf{r}_{\varphi\varphi} = \underbrace{\mathbf{r}_{\varphi\varphi} - \mathbf{n}\langle\mathbf{r}_{\varphi\varphi}, \mathbf{n}\rangle}_{\text{tangent vector}} + \mathbf{n}\langle\mathbf{r}_{\varphi\varphi}, \mathbf{n}\rangle,$$

We see that

$$\left(\frac{\partial^2 \mathbf{r}}{\partial \varphi^2}\right)_{\text{tangent}} = \mathbf{r}_{\varphi\varphi} - \mathbf{n}\langle \mathbf{r}_{\varphi\varphi}, \mathbf{n}\rangle = \mathbf{r}_{\varphi\varphi} + R\sin^2\theta\mathbf{n} = \begin{pmatrix} -R\sin\theta\cos\varphi \\ -R\sin\theta\sin\varphi \\ 0 \end{pmatrix} + R\sin^2\theta \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} =$$

$$\begin{pmatrix} -R\cos^2\theta\sin\theta\cos\varphi \\ -R\cos^2\theta\sin\theta\sin\varphi \\ R\sin^2\theta\cos\theta \end{pmatrix} = -\sin\theta\cos\theta \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} = -\sin\theta\cos\theta\mathbf{r}_\theta\,.$$

We have

$$\nabla_{\partial_\varphi}\partial_\varphi = \Gamma^\theta_{\varphi\varphi}\partial_\theta + \Gamma^\varphi_{\varphi\varphi}\partial_\varphi = \left(\frac{\partial^2 \mathbf{r}}{\partial\varphi\partial\varphi}\right)_{\text{tangent}} = -\sin\theta\cos\theta\mathbf{r}_\theta,\ \textit{i.e.}$$

$\Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta, \Gamma^\varphi_{\varphi\varphi} = 0$.

f) *Saddle*

For saddle $z = xy$: We have $\mathbf{r}(u,v)$: $\begin{cases} x = u \\ y = v \\ z = uv \end{cases}$, $\partial_u = \mathbf{r}_u = \begin{pmatrix} 1 \\ 0 \\ v \end{pmatrix}$, $\partial_v = \mathbf{r}_v = \begin{pmatrix} 0 \\ 1 \\ u \end{pmatrix}$ It will be useful also to use the normal unit vector $\mathbf{n} = \frac{1}{\sqrt{1+u^2+v^2}}\begin{pmatrix} -v \\ -u \\ 1 \end{pmatrix}$.

Calculate:

$$\nabla_{\partial_u}\partial_u = \Gamma^u_{uu}\partial_u + \Gamma^v_{uu}\partial_v = \left(\frac{\partial^2 \mathbf{r}}{\partial u^2}\right)_{\text{tangent}} = (\mathbf{r}_{uu})_{\text{tangent}} = 0\,\text{since}\ \mathbf{r}_{uu} = 0.$$

Hence $\Gamma^u_{uu} = \Gamma^v_{uu} = 0$.

Analogously $\Gamma^u_{vv} = \Gamma^v_{vv} = 0$ since $\mathbf{r}_{vv} = 0$.

Now calculate $\Gamma^u_{uv}, \Gamma^v_{uv}, \Gamma^u_{vu}, \Gamma^v_{vu}$:

$$\nabla_{\partial_u}\partial_v = \nabla_{\partial_v}\partial_u = \Gamma^u_{uv}\partial_u + \Gamma^v_{uv}\partial_v = (\mathbf{r}_{uv})_{\text{tangent}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{\text{tangent}}$$

Using normal unit vector $\mathbf{n}$ we have: $(\mathbf{r}_{uv})_{\text{tangent}} = \mathbf{r}_{uv} - \langle \mathbf{r}_{uv}, \mathbf{n}\rangle\mathbf{n} = \Gamma^u_{uv}\partial_u + \Gamma^v_{uv}\partial_v =$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{\text{tangent}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{1+u^2+v^2}}\begin{pmatrix} -v \\ -u \\ 1 \end{pmatrix}\right\rangle \frac{1}{\sqrt{1+u^2+v^2}}\begin{pmatrix} -v \\ -u \\ 1 \end{pmatrix} =$$

$$\frac{1}{1+u^2+v^2}\begin{pmatrix} v \\ u \\ u^2+v^2 \end{pmatrix} = \frac{v}{1+u^2+v^2}\begin{pmatrix} 1 \\ 0 \\ v \end{pmatrix} + \frac{u}{1+u^2+v^2}\begin{pmatrix} 0 \\ u \\ u \end{pmatrix} = \frac{v\mathbf{r}_u + u\mathbf{r}_v}{1+u^2+v^2}.$$

Hence $\Gamma^u_{uv} = \Gamma^u_{vu} = \frac{v}{1+u^2+v^2}$ and $\Gamma^v_{uv} = \Gamma^v_{vu} = \frac{u}{1+u^2+v^2}$. ■

Sure one may calculate this connection as Levi-Civita connction of the induced Riemannian metric using explicit Levi-Civita formula or using method of Lagrangian of free particle.

6 *Let ∇_1, ∇_2 be two different connections. Let ${}^{(1)}\Gamma^i_{km}$ and ${}^{(2)}\Gamma^i_{km}$ be the Christoffel symbols of connections ∇_1 and ∇_2 respectively.*

a) Find the transformation law for the object : $T^i_{km} = {}^{(1)}\Gamma^i_{km} - {}^{(2)}\Gamma^i_{km}$ under a change of coordinates. Show that it is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ tensor.

b)$^{*?}$ Consider an operation $\nabla_1 - \nabla_2$ on vector fields and find its properties.

Christoffel symbols of both connections transform according the law (1). The second term is the same. Hence it vanishes for their difference:

$$T^{i'}_{k'm'} = {}^{(1)}\Gamma^{i'}_{k'm'} - {}^{(2)}\Gamma^{i'}_{k'm'} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\left({}^{(1)}\Gamma^i_{km} - {}^{(2)}\Gamma^i_{km}\right) = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}T^i_{km}$$

We see that $T^{i'}_{'km'}$ transforms as a tensor of the type $\begin{pmatrix}1\\2\end{pmatrix}$.

b) One can do it in invariant way. Using axioms of connection study $T = \nabla_1 - \nabla_2$ is a vector field. Consider

$$T(\mathbf{X}, \mathbf{Y}) = \nabla_{1\mathbf{X}}\mathbf{Y} - \nabla_{2\mathbf{X}}\mathbf{Y}$$

Show that $T(f\mathbf{X}, \mathbf{Y}) = fT(\mathbf{X}, \mathbf{Y})$ for an arbitrary (smooth) function, i.e. it does not possesses derivatives:

$$T(f\mathbf{X}, \mathbf{Y}) = \nabla_{1fX}\mathbf{Y} - \nabla_{2fX}\mathbf{Y} = (\partial_{\mathbf{X}}f)\mathbf{Y} + f\nabla_{1\mathbf{X}}\mathbf{Y} - (\partial_{\mathbf{X}}f)\mathbf{Y} - f\nabla_{2\mathbf{X}}\mathbf{Y} = fT(\mathbf{X}, \mathbf{Y}).$$

7 * a) *Consider $t_m = \Gamma^i_{im}$. Show that the transformation law for t_m is*

$$t_{m'} = \frac{\partial\ x^m}{\partial x^{m'}}t_m + \frac{\partial^2 x^r}{\partial x^{m'}\partial x^{k'}}\frac{\partial x^{k'}}{\partial x^r}\,.$$

b) † *Show that this law can be written as*

$$t_{m'} = \frac{\partial\ x^m}{\partial x^{m'}}t_m + \frac{\partial}{\partial x^{m'}}\left(\log\det\left(\frac{\partial x}{\partial x'}\right)\right)\,.$$

Solution. Using transformation law (1) we have

$$t_{m'} = \Gamma^{i'}_{i'm''} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^k}{\partial x^{i'}}\frac{\partial x^m}{\partial x^{m'}}\Gamma^i_{km} + \frac{\partial x^r}{\partial x^{i'}\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^r}$$

We have that $\frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^k}{\partial x^{i'}} = \delta^k_i$. Hence

$$t_{m'} = \Gamma^{i'}_{i'm''} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^k}{\partial x^{i'}}\frac{\partial x^m}{\partial x^{m'}}\Gamma^i_{km} + \frac{\partial x^r}{\partial x^{i'}\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^r} = \delta^k_i\frac{\partial x^m}{\partial x^{m'}}\Gamma^i_{km} + \frac{\partial x^r}{\partial x^{i'}\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^r} = \frac{\partial x^m}{\partial x^{m'}}t_m + \frac{\partial x^r}{\partial x^{i'}\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^r}\,. \blacksquare$$

b) † When calculating $\frac{\partial}{\partial x^{m'}}\left(\log\det\left(\frac{\partial x}{\partial x'}\right)\right)$ use very important formula:

$$\delta\det A = \det A\,\mathrm{Tr}\,(A^{-1}\delta A) \to \delta\log\det A = \mathrm{Tr}\,(A^{-1}\delta A)\,.$$

Hence

$$\frac{\partial}{\partial x^{m'}}\left(\log\det\left(\frac{\partial x}{\partial x'}\right)\right) = \frac{\partial x^{i'}}{\partial x^r}\frac{\partial^2 x^r}{\partial x^{i'}\partial x^{m'}}$$

and we come to transformation law for (1).

To deduce the formula for $\delta\det A$ notice that

$$\det(A + \delta A) = \det A\det(1 + A^{-1}\delta A)$$

and use the relation: $\det(1 + \delta A) = 1 + \mathrm{Tr}\,\delta A + O(\delta^2 A)$