Spinor group

Let $\texttt{Cliff}(V,Q)$ be Clifford algebra of vector space V equipped with bilinear form Q. According the lemma that

$$\texttt{Cliff}\left((Q_1,V_1)\right)\oplus(Q_2,V_2)=\texttt{Cliff}(Q_1,V_1)\hat{\otimes}\texttt{Cliff}(Q_2)\,,$$

we come to theorem:

Clifford algebra $\texttt{Cliff}(Q,V)$ *is isomorphic to wedge product of* p *algberas of doube numbers* $(a+b\varepsilon,\ \varepsilon^2=1)$*,* $r-p$ *algberas of complex numbers* $(a+b\varepsilon,\ \varepsilon^2=-1)$*, and* $n-r$ *algberas of dual numbers* $(a+b\varepsilon,\ \varepsilon^2=0)$*, where* r *is the rank of the form* Q*,* $(p,r-p)$ *is the signature of the form,* n *is the dimension of vector space* V*, i.e.* p,q,r *are integers such that quadratic form* Q *in* V *can be reduced to the form*

$$Q=x_1^2+\ldots+x_p^2-x_{p+1}^2-\ldots x_r^2$$

in some linear coordinates

Example $\mathbf{E}^2$ with $Q(\mathbf{x})=-x_1^2-x_2^2$, then

$$\texttt{Cliff}(\mathbf{E}^2,Q=(-,-))=\mathbf{C}\hat{\otimes}\mathbf{C}=\mathbf{H}\ \text{(quaternions)}$$

One can view the algebra $\texttt{Cliff}(V,Q)$ as unital algebra such that vector space V is subpsace in this algebra, and all elements are generated by vectors of V such that $\mathbf{x}^2=Q(\mathbf{x})\cdot 1$.

We denote vectors and their image in Clifford algebra by the same letter

Now solve the following excises:

Exercise

$$\mathbf{x}^{-1}=\frac{\mathbf{x}}{Q(\mathbf{x})}\,.$$

Exercise

$$\mathbf{v}^{-1}\mathbf{x}\mathbf{v}=L_{\mathbf{v}}(\mathbf{x})=2\frac{(\mathbf{v},\mathbf{x})}{Q(\mathbf{v})}\mathbf{v}-\mathbf{x}$$

and it is orthogonal operator (if $Q(\mathbf{v})\neq 0$).

Exercise

Definition Consider a set of all elements

$$\mathbf{v}_1\cdot\mathbf{v}_2\ldots\mathbf{v}_k$$

$k=0,1,2,\ldots$, if $k=0$ then the element is a number $c\neq 0$, where all $\mathbf{v}_i$ are vectors of non-zero length:

$$Q(\mathbf{v}_1)\neq 0\ \ Q(\mathbf{v}_2)\neq 0\ \ \ldots Q(\mathbf{v}_k)\neq 0\,.$$

In other words we consider Clifford algebra forgetting the operation of $+$.

This set is a group. This group projects on the group $SO(n, Q)$, the kernel of projection is the group $\mathbf{R}^*$ of non-zero constants.

The group which we construct is generated by non-zero constants and vectors of non-zero length