Homework 1

1 Show that the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2 \ldots, \mathbf{a}_m\}$ in vector space V is linearly dependent if at least one of these vectors is equal to zero.

2 Show that any three vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ in $\mathbf{R}^2$ are linearly dependent.

3 Let 3 vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ in vector space V belong to the span of 2 vectors $\{\mathbf{a}, \mathbf{b}\}$ of this vector space, i.e. vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are expressible as linear combinations of vectors $\mathbf{a}$ and $\mathbf{b}$. Prove that vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ are linearly dependent.

4 Let $\{\mathbf{a}, \mathbf{b}\}$ be two vectors in the vector space V such that

i) these vectors are linearly independent

ii) for an arbitrary vector $\mathbf{x} \in V$ vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{x}\}$ are linearly dependent.

What is a dimension of the vector space V?

Is an ordered set $\{\mathbf{a}, \mathbf{b}\}$ a basis in the vector space V?

5 Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a basis in 3-dimensional vector space V. Show that

a) all vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are not equal to zero.

b) an arbitrary vector $\mathbf{a} \in V$ can be expressed as a linearly combination of the basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in a unique way, i.e.$\mathbf{a} = a^1\mathbf{e}_1 + a^2\mathbf{e}_2 + a^3\mathbf{e}_3$ and if

$$\mathbf{a} = a^1\mathbf{e}_1 + a^2\mathbf{e}_2 + a^3\mathbf{e}_3 = a^{1'}\mathbf{e}_1 + a^{2'}\mathbf{e}_2 + a^{3'}\mathbf{e}_3 \text{ then } a^1 = a^{1'}, a^2 = a^{2'}, a^3 = a^{3'} .$$

Remark The following statement is very useful: *Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be an ordered set of vectors in the vector space V such that an arbitrary vector $\mathbf{x} \in V$ can be expressed as a linear combination of the vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ in a unique way. Then one can show that V is an n-dimensional vector space and an ordered set $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis in V.*

6† Show that the ordered set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \ldots, \mathbf{e}_n\}$ of vectors is a basis in $\mathbf{R}^n$ in the case if

$$\begin{array}{llllll}
\mathbf{e}_1 & = (1, & 2, & 3, & 4, \ldots, & n) \\
\mathbf{e}_2 & = (0, & 1, & 2, & 3, \ldots, & n-1) \\
\mathbf{e}_3 & = (0, & 0, & 1, & 2, \ldots, & n-2) \\
\ldots & & & & & \\
\mathbf{e}_n & = (0, & 0, & 0, & 0, \ldots, & 1)
\end{array}$$

7 Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a basis of 3-dimensional vector space V. Is a set of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ a basis of V in the case if

a) $\mathbf{e}'_1 = \mathbf{e}_2$, $\mathbf{e}'_2 = \mathbf{e}_1$, $\mathbf{e}'_3 = \mathbf{e}_3$;

b) $\mathbf{e}'_1 = \mathbf{e}_1$, $\mathbf{e}'_2 = \mathbf{e}_1 + 3\mathbf{e}_3$, $\mathbf{e}'_3 = \mathbf{e}_3$;

c) $\mathbf{e}'_1 = \mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{e}'_2 = 3\mathbf{e}_1 - 3\mathbf{e}_2$, $\mathbf{e}'_3 = \mathbf{e}_3$;

d) $\mathbf{e}'_1 = \mathbf{e}_2$, $\mathbf{e}'_2 = \mathbf{e}_1$, $\mathbf{e}'_3 = \mathbf{e}_1 + \mathbf{e}_2 + \lambda\mathbf{e}_3$ (where λ is an arbitrary coefficient)?