# Thick morphisms and spinors

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*The talk is based on the work with Ted Voronov*

# Contents

# Papers that talk is based on are

[1]. H.M.Khudaverdian, Th.Voronov “Thick morphisms, higher Koszul brackets, and $L_\infty$-algebroids”, math-arXiv:1808.10049

[2]. H.M.Khudaverdian, Th.Voronov “Thick morphisms of supermanifolds, quantum mechanics and spinor representation”, math-arXiv:1909.00290

[3] Th. Voronov, *Nonlinear pullback on functions and a formal category extending the category of supermanifolds]*, arXiv: 1409.6475

[4] Th. Voronov, *Microformal geometry*, arXiv: 1411.6720

# Abstract...

For an arbitrary morphism $\varphi\colon\ M \to N$ of (super)manifolds, the pull-back $\phi^* C^\infty(N) \to C^\infty(M)$ is a linear map of space of functions. Moreover it is homomorphism of algebra $C^\infty(N)$ into algebra $C^\infty(M)$.
In 2014 Voronov have introduced *thick morphisms* of (super)manifolds which define generally non-linear pull-back of functions. This construction was introduced as an adequate tool to describe $L_\infty$ morphisms of algebras of functions provided with the structure of homotopy Poisson algebra.

Thick morphism $\Phi = \Phi_S\colon \quad M \Rrightarrow N$ can be defined by the "action" $S(x, q)$, where $x$ are local coordinates on $M$ and $q$ are coordinates of momenta in $T^*N$. The pull-back of thick morphism $\Phi_S^*\colon\ C^\infty(N) \to C^\infty(M)$, is non-linear map in the case if the action $S(x, q)$ is not linear over $q$.
In this approach we come to fundamental concepts of Quantum Mechanics.
In particular thick morphisms with quadratic action give naturally the spinor representation.

Let $M_1, M_2$ be two (super)manifolds.

$$\underbrace{M_1}_{x^i\text{- loc.coord.}}\,, \qquad \underbrace{M_2}_{y^a\text{- loc.coord.}}\,,$$

Consider also cotangent bundles $T^*M_1$ and $T^*M_2$.

$$\underbrace{T^*M_1}_{x^i, p_j\text{- loc.coord.}}\,, \qquad \underbrace{T^*M_2}_{y^a, q_b\text{- loc.coord.}}\,,$$

$p_i$ are components of momenta which are conjugate to $x^i$, respectively $q_a$ are components of momenta which are conjugate to $y^a$.

**Remark** We consider even and odd coordinates, i.e. $M_1, M_2$ are supermanifolds; parity of any coordinate coincide with the parity of corresponding component of momenta:

$$p(p_i) = p(x^i)\,, \quad p(q_b) = p(y^b)\,.$$

# Definition of thick morphism. (T.Voronov)

$M_1, M_2$–two (super)manifolds
Consider symplectic manifold $T^*M_1 \times (-T^*M_2)$
equipped with canonical symplectic structure

$$\omega = \omega_1 - \omega_2 = \underbrace{dp_i \wedge dx^i}_{\text{coord. on } T^*M_1} - \underbrace{dq_a \wedge dy^a}_{\text{coord. on } T^*M_2}$$

Function $S = S(x, q)$—action
It defines Lagrangian surface $\Lambda_S \subset T^*M_1 \times (-T^*M_2)$:

$$\Lambda_S = \left\{ (x, p, y, q) : \quad p_i = \frac{\partial S(x,q)}{\partial x^i}, y^b = \frac{\partial S(x,q)}{\partial q_b} \right\}$$

# Lagrangian surface—canonical relation—thick morphism

Lagr. surf. $\Lambda_S$ is canon. relation $\Phi_s$ in $T^*M_1 \times (-T^*M_2)$

$$(x^i, p_j) \sim_S (y^a, q_b) \leftrightarrow (x^i, p_j, y^a, q_b) \in \Lambda_S\,, (\Phi_S = \sim_S)\,.$$

$$\Phi = \Phi_s \text{ is a thick morphism } M_1 \Rrightarrow M_2$$

It defines pull-back $\Phi_S^*$ of functions

$$\Phi_S^*\colon\ \mathfrak{M}_2 = C(M_2) \to \mathfrak{M}_1 = C(M_1)\,,$$

such that for every function $g = g(y) \in \mathfrak{M}_2$,

$$f = f(x) = (\Phi_S^* g)(x)\colon \Lambda_f = \Phi_S \circ \Lambda_g\,,$$

where $\Lambda_f, \Lambda_g$ are Lagrangian surfaces, graphs of $df, dg$ in $T^*M_1, T^*M_2$.

# Explicit expression

Thick morphism $M \Rrightarrow N$ defines the pull-back $\Phi_g^*: \; C^\infty(N) \to C^\infty(M)$, such that

$$\Phi_S^* g = f(x) = g(y) + S(x,q) - y^a q_a\,,$$

where $y^a$ and $q_a$ are defined from the equations

$$y^a = \frac{\partial S(x,q)}{\partial q_a}\,, \quad q_a = \frac{\partial g(y)}{\partial y^a}$$

We see that $\Lambda_f = \Phi_S \circ \Lambda_g$ since

$$p_i = \frac{\partial f}{\partial x^i} = \frac{\partial}{\partial x^i}\left(g(y) + S(x,q) - y^a q_a\right) = \frac{\partial S(x,q)}{\partial x^i}\,.$$

# Thick morphism is usual map if $S(x,q) = S^a(x)q_q$

## Example

Generating function $S = S^a(x)q_a$

$$(\Phi_S^* g)(x) = g(y) + S(x,q) - y^a q_a = g(y) + \underbrace{(S^a(x) - y^a)}_{\text{vanishes}} q_a = g(S^a(x))$$

Thick morphism $M_1 \overset{\Phi_s}{\Rrightarrow} M_2$ is usual morphism $M_1 \overset{y^a = S^a(x)}{\to} M_2$.

# Thick morphism in general case

In general case the pull-back is non-linear:

$$f(x) = (\Phi_S^* g)(x) = \left( S(x,q) + g(y) - y^i q_i \right) \Big|_{y = \frac{\partial S(x,q)}{\partial q}, q = \frac{\partial g}{\partial y}} ,$$

## Example

$S(x,q) = xq + \frac{1}{2}aq^2$, $g(y) = \frac{1}{2}ky^2$ then $q = ky$,
$y = y(x)$ is defined by relation

$$y = \frac{\partial S(x,q)}{\partial q} = x + aq = x + ky \Rightarrow y = \frac{x}{1-k} ,$$

and

$$f(x) = \Phi_S^*(g)(x) = S(x,q) + g(y) - yq = \frac{kx^2}{2(1-k)} .$$

# Application of thick morphisms: $L_\infty$ morphisms

Consider two homotopy Poisson algebras defined on space of functions $C^\infty(M_i)$ by Hamiltonian $Q_i$ $(i=1,2)$
We say that Hamiltonians $Q_1, Q_2$ are connected by the action $S(x,q)$ if

$$Q_1\left(x^i, p_j = \frac{\partial S(x,q)}{\partial x^j}\right) \equiv Q_2\left(y^a = \frac{\partial S(x,q)}{\partial q_a}, q_b\right)$$

($x^i$-coordinates on $M_1$ and $q_a$ momenta on fibers of $T^*M_2$)

**Theorem**

*The pull-back $\Phi_S^*$ of the thick morphism $\Phi_S$ is $L_\infty$ morphism of homotopy Poisson algebra $(C^\infty(M_2), Q_2)$ on homotopy Poisson algebra $(C^\infty(M_1), Q_1)$. (Th.Voronov, 2014)*

# What is it ‘Homotopy Poisson algebras’. Recalling

Let $M$ be a supermanifold, and let $Q = Q(x,p)$ be an odd Hamiltonian, odd function defined on cotangent bundle $T^*M$ This Hamiltonian defines hmotopy Poisson bracket on algebra of functions $C^\infty(M)$. The chain of brackets ican be defined by

$$\{f_1\} = (Q, f_1)\big|_M, \quad \{f_1, f_2\} = ((Q, f_1), f_2)\big|_M,$$

$$\{f_1, f_2, f_3\} = (((Q, f_1), f_2) f_3)\big|_M, \quad \text{and so on:}$$

$$\{f_1, \ldots, f_n\} = \underbrace{(\ldots(}_{n-\text{times}} Q, f_1), \ldots, f_n)\big|_M,$$

$( \, , \, )$— canonical even Poisson bracket on $T^*M$
$Q$ obeys condition $(Q,Q) = 0$—Jacobi identity.
The chain of brackets $\{f_1, \ldots, f_n\}$ becomes an usual odd Poisson) bracket if Hamiltonian $Q$ is quadratic on momenta.

The function $S = S(x, q)$ which defines thick morphism $\Phi_S \colon M \Rrightarrow N$ we call in this paper'action'[1]. Why?

Let $H = H(x, p)$ be Hamiltonian defined in cotangent bundle $T^*M$ and let $\mathfrak{S} = \mathfrak{S}(t, x, y)$ be the action of classical mechanics for the path $x(\tau)$, $0 \leq \tau \leq t$ which obeys equations of motion, starts at the point $x$ at $\tau = 0$, and ends at the point $y$ at time $\tau = t$.

[1] In the pioneer works of T.Voronov, where thick morphism was suggested, this function was called just "generating function"

Its Legendre transform

$$S(t,x,q) = yq - \mathfrak{S}(t,x,y)\,,\text{where } y = \frac{\partial \mathfrak{S}}{\partial q}\,.$$

It obeys Hamilton Jacobi equation

$$S(x,q,t)\colon \begin{cases} \frac{\partial S}{\partial t} = H\left(\frac{\partial S}{\partial q}, q\right) \\ S(x,q)\big|_{t=0} = xq \end{cases}$$

Denote $S(t,x,q) = \exp tH$.

## Example

free particle

$$H_{\text{free}} = \frac{p^2}{2m}$$

$$\exp tH_{\text{free}}:$$

$$\mathfrak{S}(t,x,y) = \frac{m(y-x)^2}{2t}\,, \quad S(t,x,q) = xq + \frac{q^2 t}{2m}\,,$$

## Example

harmonic oscillator

$$H=\frac{p^2+x^2}{2}\,,$$

$$\exp tH_{\text{oscillator}}:$$

$$\mathfrak{S}(t,x,y)=\frac{x^2+y^2}{2}\operatorname{cotan} t-\frac{yx}{\sin t}\,,\ S(t,x,q)=xq\cos t+\frac{x^2+q^2}{2}\tan t\,,$$

### Theorem

*Let action $S(t,x,q)$ is an exponent of Hamiltonian $H$:*

$$S(t,x,q) = \exp tH$$

*Consider the one-parametric group of thick morphism $\Phi_t \colon M \Rrightarrow M$ generated by $S(t,x,q)$. For an arbitrary function $g = g(x)$ consider*

$$f_t(x) = \Phi_t^*(g)$$

*The function $f_t(x)$ obeys the Hamilton-Jacobi equation:*

$$\frac{\partial f_t(x)}{\partial t} = H\left(x, \frac{\partial f}{\partial x}\right), \quad f_t(x)\big|_{t=0} = g(x).$$

# Quantum thick morphisms

$S_\hbar(x,q)$-quantum action, power series in $q$ and $\hbar$

The corresponding `quantum thick morphism` performs the pull-back:

$$\Phi^{*\text{quant.}}_{S_\hbar}(w)(x) = \int_{T^*N} e^{\frac{i}{\hbar}\left(S_\hbar(x,q)-y^i q_i\right)} w(y) DqDy\,.$$

$DqDp$ is invariant Lioville measure on $T^*M$

# Quantum thick morphisms → classical thick morphisms

One can see this using stationary phase method:
For $w_\hbar = e^{\frac{i}{\hbar}g(y)}$

$$\lim_{\hbar\to 0}\left[\frac{\hbar}{i}\left(\log\left(\Phi^{*\text{quant.}}_{S_\hbar}(w_\hbar)\right)\right)\right] = \lim_{\hbar\to 0}\left[\frac{\hbar}{i}\left(\log\left(e^{\frac{i}{\hbar}\left(g(y)+S_\hbar(x,q)-y^i q_i\right)}\right)\right)\right] =$$

$$g(y_0) + S(x,q^0) - y_0^i q_i^0\,,$$

where $y_0 = y_0(x)$ and $q^0 = q^0(x)$ are defined (depending on $x$) by the sationary point condition: $y_0^i = \frac{\partial S(x,q)}{\partial q_i}|q_i = q_i^0$ and $q_i^0 = \frac{\partial g(y)}{\partial y^i}\big|_{y^i = y_0^i}$, and $S(x,q) = \lim_{\hbar\to 0} S_\hbar(x,q)$.
We come to the classical thick morphism:

$$\lim_{\hbar\to 0}\left[\frac{\hbar}{i}\left(\log\left(\Phi^{*\text{quant.}}_{S_\hbar}\left(e^{\frac{i}{\hbar g(y)}}\right)\right)\right)\right] = \Phi^{\text{class}}_{S_0}(g)(x)\,.$$

# Legendre transform $\to$ Fourier transform

Legendre transform is quasiclassics of Fourier transform:
Legendre: $g(p) = G(x) = px$ such that $G'(x) = p$

$$e^{\frac{i}{\hbar}(G(x)-px)}dx \approx e^{\frac{i}{\hbar}g(p)}.$$

Classical thick morphisms — Hamilton Jacobi equation
Quantum thick morphisms— Shrodinger equation:

# What is a spinor

Thick morphism acts on functions on $n$ variables. On the other hand it is defined by an action $S(x, q)$ which depends on $2n$ variables.
This strongly resembles spinor representation if one recalls that the spinor representation (in the orthogonal or symplectic settings) can be seen as action of transformations of a large space on objects such as functions or half-forms that live on a (half-dimensional) maximally isotropic subspace.

Symplectic (orthogonal) spinor is a function on space of half-dimensions, which transforms under the action of spinor group

$V$-vector space, $\dim V = N$, $X = V \oplus V^*$, $\dim X = 2N$.

$$X \ni \mathbf{A} = \begin{pmatrix} a^i \\ \alpha_j \end{pmatrix} \longrightarrow h_{\mathbf{A}} = a^i \hat{p}_i + \alpha_i \hat{q}^i = \frac{\hbar}{i} a^i \frac{\partial}{\partial x^i} + \alpha_j x^j \,,$$

$$X \ni \mathbf{B} = \begin{pmatrix} b^i \\ \beta_j \end{pmatrix} \longrightarrow h_{\mathbf{B}} = b^i \hat{p}_i + \beta_j \hat{q}^j = \frac{\hbar}{i} b^i \frac{\partial}{\partial x^i} + \beta_j x^j \,,$$

$\mathbf{A}, \mathbf{B}$-vectors in $2N$-dimensional space $\to h_{\mathbf{A}}, h_{\mathbf{B}}$ operators on space of functions on $N$ variables.
Symplectic scalar product $\to$ commutators

$$\langle \mathbf{A}, \mathbf{B} \rangle = a^i \beta_i - \alpha_j b^j = \frac{i}{\hbar} [h_{\mathbf{A}}, h_{\mathbf{B}}]$$

$$\left( [\hat{p}_i, \hat{p}_j] = [\hat{q}^i, \hat{q}^j] = 0 \,, [\hat{p}_i, \hat{q}^j] = \frac{\hbar}{i} \right)$$

$$h_{\mathbf{A}} \to \mathfrak{O}^{-1} h_{\mathbf{A}} \mathfrak{O} = h_{\mathbf{A}'}\,,\ h_{\mathbf{B}} \to \mathfrak{O}^{-1} h_{\mathbf{B}} \mathfrak{O} = h_{\mathbf{B}'}\,,$$

$$\mathbf{A} \longrightarrow \mathbf{A}' = g(\mathbf{A})\,,\, \mathbf{B} \longrightarrow \mathbf{B}' = g(\mathbf{B})\,,$$

$$\langle \mathbf{A}, \mathbf{B} \rangle = \frac{i}{\hbar}[h_{\mathbf{A}}, h_{\mathbf{B}}] = \frac{i}{\hbar}\left[\mathfrak{O}^{-1} h_{\mathbf{A}} \mathfrak{O}, \mathfrak{O}^{-1} h_{\mathbf{B}} \mathfrak{O}\right] = \frac{i}{\hbar}[h_{\mathbf{A}'}, h_{\mathbf{B}'}] = \langle \mathbf{A}', \mathbf{B}' \rangle\,.$$

This transformation preserves symplectic scalar product.

spinor group $\ni \mathfrak{O} \to g\colon \mathbf{A} \to \mathbf{A}'$, $g \in$ symplectic group $Sp(n)$

# Where are usual spinors?

$$SO(N) = Sp(-N)$$

We come to usual (orthogonal spinors) changing a parity

symplectic group $Sp(n)$ – – – – – – Orthogonal group $O(n)$

| | | |
|---|---|---|
| $X = V \oplus V^*$ | – – – | $\Pi X = \Pi V \oplus \Pi V^*$ |
| symplectic space | – – – | Euclidean space |
| linear operator $h_{\mathbf{A}}$ | – – – | linear operator $\gamma_{\mathbf{A}}$ |
| symplectic group $Sp(N)$ | – – – | Orthogonal group $O(N)$ |
| acting on space of functions | – – – | acting on space of functions |
| of commuting coordinates | – – – | of anticommuting coordinates |
| $x_1, \ldots, x_N$ | – – – | $\xi_1, \ldots, \xi_N$ |
| Spinor representation | – – – | Spinor representation |
| is infinite dimensional | – – – | is finite dimensional |

In the symplectic case spinors called metaplectic spinors.

# Spinor group and thick morphisms

Return to thick morphisms
Spinor group $\{\mathfrak{O}\}$ can be defined as subgroup of qiantum thick morphisms corresponding to quadratic Hamiltonians.

Let $M, N$ be two (super)manifolds. Recall that the classical action $S = S(x, q)$ connects Hamiltonian $H_M$ on $T^*M$ with Hamiltonian $H_N$ on $T^*N$ if

$$H_M\left(x, \frac{\partial S(x,q)}{\partial x}\right) \equiv H_N\left(\frac{\partial S(x,q)}{\partial q}, q\right).$$

Let $\Delta_M = H_M(\hat{x}, \hat{p})$ be a linear operator on $M$ =the quantum Hamiltonian (operator depending on $\hat{x} = x$ and $\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$, and respectively let $\Delta_N = H_N(\hat{y}, \hat{y})$ be a linear operator on $N$ =the quantum Hamiltonian (operator depending on $\hat{y} = y$ and $\hat{q} = \frac{\hbar}{i}\frac{\partial}{\partial y}$,

## Definition

We say that the quantum thick morphism $\Phi_{S_\hbar}$ $M \Rrightarrow N$ connects operators $\Delta_M$ and $\Delta_N$ if the pull-back $\Phi_{S_h}$ of quantum thick morphism commutes with these operators. i.e.

$$\Delta_M \circ \Phi^*_{S_\hbar} = \Phi^*_{S_\hbar} \circ \Delta_N\,, \quad \left(\Delta_N = \left(\Phi^*_{S_\hbar}\right)^{-1} \circ \Delta_M \Phi^*_{S_\hbar}\right)$$

# Quantum morpshims $\to$ classical morphisms

## Theorem

*Let $S_\hbar(x,q)$ be a quantum action such that quantum thick morphism $\Phi_{S_\hbar}$ connects quantum Hamiltonians $\Delta_M$ and $\Delta_N$. then*

- *classical thick morphism $\Phi_{S_0}$ defined by classical action $S_0(x,q) = \lim_{\hbar\to 0} S_\hbar$ connects classical Hamiltonians $H_M$ and $H_N$ (symbols of operators $\Delta_M$ and $\Delta_N$).*
- *If $\Delta_M$ and $\Delta_N$ are operators, such that Hamiltonians (their symbols) $H_M, H_N$ are linear then the condition that quantum thick morphism $\Phi_{S_\hbar}$ connects quantum Hamiltonians $\hat{H}_M$ and $\hat{H}_N$ does not depend on $\hbar$; in particular the condition that classical action connects two linear classical Hamiltonians is equivalent to the condition quantum version.*

# Definition of spinor group in terms of thick morphisms

To define spinor group we have to consider thick morphisms corresponding to quadratic Hamiltonians/

## Definition

Spinor group is the group of thick diffeomorphisms $\Phi_S$ corresponding to quadratic Hamiltonians.