One way to define thick morphisms

Let $M=\mathbf{R}^m$, $N=\mathbf{R}^n$ be two linear spaces. Consider a function $S(x,q)$ where $x\in M$ and $q\in T^*N$ such that

$$S(x,q)=S_0(x)+\varphi^i(x)q_i+S_+(x,q)$$

where S_+ possesses terms of order by q higher or equal than 2:

$$S_+(x,q)=\frac{1}{2}\Pi^{ij}(x)q_iq_j+\frac{1}{6}\Pi^{ijk}(x)q_iq_jq_k+\dots$$

Define the action of (quantum) thick morphism We have

$$\Phi^*_S(w(y))=\Psi(x)=\int e^{\frac{i}{\hbar}\left(S(x,q)-y^iq_i\right)}w(y)dydq=\int e^{\frac{i}{\hbar}S(x,q)}\bar{w}(q)dq\,,$$

where $\bar{w}(q)$ is Fourrier image of function $w(y)$:

$$\bar{w}(q)=\int w(y)e^{-\frac{i}{\hbar}qy}dy\quad\text{and}\quad w(y)=\int\bar{w}(q)e^{\frac{i}{\hbar}qy}dq,\text{, (up to multiplier)}\,.$$

Now use the standard identity that up to multiplier

$$f(q)\bar{w}(q)=\int\left[f\left(\frac{\hbar}{i}\frac{\partial}{\partial y}\right)w(y)\right]e^{-\frac{i}{\hbar}qy}dy\,.$$

Indeed

$$f(q)\bar{w}(q)=\int f(k)\bar{w}(k)\delta(k-q)dk=\int f(k)\bar{w}(k)e^{\frac{i}{\hbar}(k-q)y}dkdy=$$

$$\int\left[f\left(\frac{\hbar}{i}\frac{\partial}{\partial y}\right)e^{\frac{i}{\hbar}ky}\bar{w}(k)\right]e^{-\frac{i}{\hbar}qy}dkdy=\int\left[f\left(\frac{\hbar}{i}\frac{\partial}{\partial y}\right)w(y)\right]e^{-\frac{i}{\hbar}qy}dy\,.$$

Hence

$$\Phi^*_S(w(y))=\Psi(x)=\int e^{\frac{i}{\hbar}\left(S(x,q)-y^iq_i\right)}w(y)dydq=\int e^{\frac{i}{\hbar}S(x,q)}\bar{w}(q)dq=$$

$$\int\left(e^{\frac{i}{\hbar}S(\left(x,\frac{\hbar}{i}\frac{\partial}{\partial y}\right)}w(y)\right)e^{-iqy}dydq=$$

$$\left(e^{\frac{i}{\hbar}S(\left(x,\frac{\hbar}{i}\frac{\partial}{\partial y}\right)}w(y)\right)\delta(y)dy=\left(e^{\frac{i}{\hbar}S(\left(x,\frac{\hbar}{i}\frac{\partial}{\partial y}\right)}w(y)\right)\big|_{y=0}$$

$$\left(e^{\frac{i}{\hbar}\left(S_0(x)+\varphi(x)\frac{\hbar}{i}\frac{\partial}{\partial y}+S_+\left(x,\frac{\hbar}{i}\frac{\partial}{\partial y}\right)\right)}w(y)\right)\big|_{y=0}=e^{\frac{i}{\hbar}S_0(x)}\left(e^{\varphi(x)\frac{\partial}{\partial y}}e^{\frac{i}{\hbar}S_+\left(x,\frac{\hbar}{i}\frac{\partial}{\partial y}\right)}w(y)\right)\big|_{y=0}=$$

$$\left(e^{\frac{i}{\hbar}S_+\left(x,\frac{\hbar}{i}\frac{\partial}{\partial y}\right)}w(y)\right)\big|_{y=\varphi(x)}\,.\tag{1}$$

We suppose here that $S_0(x)\equiv0$.

One can consider the one-parametric family $L^*_\hbar$

$$L^*_\hbar(g(y)) = f(x)\,,\text{such that } \Psi(x) = e^{\frac{i}{\hbar}f(x)} = \Phi^*_S\left(e^{\frac{i}{\hbar}g(y)}\right)\,, \tag{2}$$

and its limit

$$L^*_{S\text{classic.}}(g(y)) = \lim_{\hbar\to 0} L^*_\hbar(g(y))\,. \tag{2a}$$

One can see that this is a la 'Legendre':

$$L^*_{\text{classic.}}(g(y)) = S(x,q) + g(y) - y^i q_i\,,$$

where

$$y^i = \frac{\partial S(x,q)}{\partial q_i} = \varphi^i(x) + \Pi^{ij}(x)q_j + \Pi^{ijk}(x)q_j + \dots \text{ such that } q_i = \frac{\partial g(y)}{\partial y^i}\,,.$$

Here we have to be carefull taking limits.

Calculate (2):

$$L^*_\hbar(g(y)) = f(x) = \frac{\hbar}{i}\log\left(\Psi(x)\right) = \frac{\hbar}{i}\log\left(\Phi^*_S\left(e^{\frac{i}{\hbar}g(y)}\right)\right) =$$

$$\frac{\hbar}{i}\log\left(e^{\frac{i}{\hbar}S_+\left(x,\frac{\hbar}{i}\frac{\partial}{\partial y}\right)}\left(e^{\frac{i}{\hbar}g(y)}\right)\right)\Big|_{y=\varphi(x)} =$$

$$\frac{\hbar}{i}\log\left[e^{\frac{i}{\hbar}g(y)}\left\{e^{-\frac{i}{\hbar}g(y)}\left(e^{\frac{i}{\hbar}S_+\left(x,\frac{\hbar}{i}\frac{\partial}{\partial y}\right)}\left(e^{\frac{i}{\hbar}g(y)}\right)\right)\right\}\right]\Big|_{y=\varphi(x)} =$$

$$\frac{\hbar}{i}\left[\frac{i}{\hbar}g(y) + \log\left\{e^{-\frac{i}{\hbar}g(y)}\left(e^{\frac{i}{\hbar}S_+\left(x,\frac{\hbar}{i}\frac{\partial}{\partial y}\right)}\left(e^{\frac{i}{\hbar}g(y)}\right)\right)\right\}\right]\Big|_{y=\varphi(x)} =$$

$$\left[g(y) + \frac{\hbar}{i}\log\left\{e^{-\frac{i}{\hbar}g(y)}\left(e^{\frac{i}{\hbar}S_+\left(x,\frac{\hbar}{i}\frac{\partial}{\partial y}\right)}\left(e^{\frac{i}{\hbar}g(y)}\right)\right)\right\}\right]\Big|_{y=\varphi(x)} =$$

$$\left[g(y) + \frac{\hbar}{i}\log\left\{e^{-\frac{i}{\hbar}g(y)}\left(e^{\frac{i}{\hbar}S_+\left(x,\frac{\hbar}{i}\frac{\partial}{\partial y}\right)}\left(e^{\frac{i}{\hbar}g(y)}\right)\right)\right\}\right]\Big|_{y=\varphi(x)} =$$

$$\left[g(y) + \frac{\hbar}{i}\log\left\{e^{\frac{i}{\hbar}S_+\left(x,\frac{\partial g}{\partial y}\right)} + \dots\right\}\right]\Big|_{y=\varphi(x)}\,, \tag{3}$$

where we denote by dots the terms of order equal or higher than zero by $\hbar$.

Ochenj khochetsia zakljuchitj iz etoj formuly shto for classical thick morphisms

$$\Phi_S(g(y)) = f(x) = \lim_{\hbar\to 0} f_\hbar(x) = \lim_{\hbar\to 0}\left[g(y) + \frac{\hbar}{i}\log\left\{e^{\frac{i}{\hbar}S_+\left(x,\frac{\partial g}{\partial y}\right)} + O(1)\right\}\right]\Big|_{y=\varphi(x)} = \tag{3a}$$

$$\left(g(y) + S_+\left(x,\frac{\partial g}{\partial y}\right)\right)\Big|_{y=\varphi(x)}\,.$$

No eto zhe ne tak!

We have to look on (3) more carefully