Riemannian Geometry

COURSEWORK 2018

Discussions

`Here we discuss the solutions of the coursework. (The printed solutions of coursework problems and marked courseworks will be distributed on 8-th may before revision lecture and tutorial`

1

Consider a surface M, the upper sheet of the cone

$$\mathbf{r}(h,\varphi)\colon \begin{cases} x = h\cos\varphi \\ y = h\sin\varphi \\ z = kh \end{cases}, \qquad 0 \le \varphi < 2\pi\,,\ h > 0\,.$$

Find the length of the shortest curve C which belongs to the surface M, starts at the point $(h_0, 0, kh_0)$ and ends at the point $(-h_0, 0, kh_0)$.

This exercise was made by many students. Students calculated right the parameters of the unfolded surface: this is a sector of the circle with radius $R = h_0\sqrt{k^2+1}$ and the angle

$$\theta = \frac{\text{circumference of the base circle}}{R} = \frac{2\pi}{\sqrt{1+k^2}}\,.$$

However If you take the images of points $A = (h, 0, kh)$ and $B = (-h, 0, kh)$ on the unfolded surface, then the angle between these points will be $\theta/2$, not θ. (A few students considered wrong angle.) To calculate the length of the segment AB (this is the shortest distance) one may consider the height $OD \perp AB$ of the isocseles $\triangle AOB$, then $\angle DOA = \frac{\theta}{4}$ and

$$|AB| = 2|AO|\sin\frac{\theta}{4}\,, \tag{1.1a}$$

or instead one may consider cosine rule for this this isocseles triangle:

$$|AB| = \sqrt{2|AO|^2 - 2|AO|^2\cos\frac{\theta}{2}}\,. \tag{1.1b}$$

Many students prefered to write the solutions in the form (1.1b). The expression (1.1a) seems to be better if you want to compare the length of the shortest curve with the circumference of the based circle (see the Remark above).

A few students instead unfolding the conical surface, considered new coordinates

$$u, v: \quad \begin{cases} u = \sqrt{1+k^2}h\cos\frac{\varphi}{\sqrt{k^2+1}} \\ v = \sqrt{1+k^2}h\sin\frac{\varphi}{\sqrt{k^2+1}} \end{cases},$$

such that in these coordinates surface is locally Euclidean: $du^2+dv^2 = (1+k^2)dh^2+h^2d\varphi^2$, and they calculated the distance in these coordinates. This is approach is rigorous and nice.

Few students, instead finding the closest curve calculated the length of the arc of the base circle: $L = \pi h_0$. Obviously this is not the shortest (see also Remark below).

Remark The length of the shortest curve: $L = 2h_0\sqrt{1+k^2}\sin\frac{\pi}{2\sqrt{1+k^2}}$ (see equation (1.1a) or the solutions). One can see that L is less than curcumference of the based circle, and in the limit $k \to \infty$ it tends to this answer:

$$L = 2h\sqrt{1+k^2}\sin\frac{\pi}{2\sqrt{1+k^2}} < \pi h\,, \text{since } \frac{\sin x}{x} \leq 1 \text{ for small } x$$

and

$$\lim_{k\to\infty}\left(2h\sqrt{1+k^2}\sin\frac{\pi}{2\sqrt{1+k^2}}\right) = \pi h\,, \text{ since } \lim_{x\to 0}\frac{\sin x}{x} = 1\,.$$

2

You know that the Riemannian metric on the sphere of radius R in the stereographic coordinates is expressed by the formula

$$G_{\text{stereogr.}} = \frac{4R^4(du^2+dv^2)}{(R^2+u^2+v^2)^2}\,.$$

a) Give an example of a non-identity transformation of coordinates u, v such that it preserves this metric.

b) Give an example of a non-linear transformation of coordinates u, v such that it preserves this metric.

(Hint: You may find this transformation considering transformations of the sphere.)

c) Find the length of the line $v = au$ in $\mathbf{R}^2$ with respect to this metric.

Why the length of this curve does not depend on the parameter a?

a) Almost all students have give the example of non-dentical linear transformation of coordinates u, v. This was a simple question.

b) For an example of non-linear transformatin many students have chosen the inversion of stereographic coordinates u, v: $u = \frac{R^2 u}{u^2+v^2}$, respectively $v = \frac{R^2 v}{u^2+v^2}$, but only few of them showed explicitly that this transformation in fact preserves the metric.

One can show an example of non-linear transformation without performing straightforward calculations: take an arbitrary orthogonal thransformation of points of sphere, which is not the rotation around Oz axis, e,g, the transformation $\mathbf{r} \to -\mathbf{r}$ or you may take rotation on arbitrary non-zero angle around an arbitrary axis which does not coincide with Oz axis (see the solutions), and you will come to non-linear transformation of stereographic coordiantes preserving the metric[1)].

c) I am very happy that almost all students did this exercise succesfully. Many students did straightforward brute force calculations and came to the right answer $L = 2\pi R$. Usually these students noted that it is the length of the great circle. About four-five students made msitakes doing these straightforward calculations and as the consequence, they missed the geometrical interpretation of the answer.

3.

a) Evaluate the area of the part of the sphere of radius $R = 1$ between the planes given by equations $2x + 2y + z = 1$ and $2x + 2y + z = 2$.

b) Consider the plane $\mathbf{R}^2$ with standard coordinates (x, y) equipped with Riemannian metric

$$G = (1 + x^2 + y^2) e^{-a^2x^2 - a^2y^2} \left(dx^2 + dy^2\right) .$$

Calculate the total area of this plane.

No problem arised in this question.

Performing the exercise a) students had to recall the standard formula on the distance between the origin and the plane, and almost all students (except two or three) did it succesfully.

Few students were confused during caclulations of integral for area in the exercise b).

4

1) Helas, it seems to me that nobody did this in spite of the hint to the question.

a) Consider the points $A = (0, 0, R)$, $B = (R, 0, 0)$ and $C = (R\cos\varphi, R\sin\varphi, 0)$ on the sphere $x^2 + y^2 + z^2 = R^2$ in $\mathbf{E}^3$ $(0 < \varphi < \pi)$. Consider the isosceles triangle ABC on this sphere. (Sides of this triangle are the arcs of great circles joining these points.) Show that:

$$\frac{Area\ (\triangle ABC)}{R^2} = \alpha + \beta + \gamma - \pi\,, \tag{4.1}$$

where α, β, γ are angles of this triangle.

b) Consider the upper half-plane $y > 0$ with the Riemannian metric

$$G = \frac{dx^2 + dy^2}{y^2}$$

(the Lobachevsky plane).

In the Lobachevsky plane consider the domain D defined by

$$D = \{x, y\colon \quad x^2 + y^2 \geq 1\,,\ -a \leq x \leq a\}\,, \tag{4.2}$$

where a is a parameter such that $0 < a < 1$.

Find the area of the domain D (with respect to the metric G).

Show that

$$\textit{Area of the domain } D = \pi - \beta - \gamma\,,$$

where β, γ are angles between the vertical lines $x = \pm a$ and arc of the circle delimiting the domain D.

Consider the points $A_t = (-a, t)$ and $B_t = (a, t)$ on the vertical rays $x = \pm a$ delimiting the domain D. Show that the distance between these points tends to 0 if $t \to \infty$.

Explain why the domain D can be considered as an isosceles triangle.

Why it can be said that the third angle of this triangle vanishes.

The part a) of this question is extremely simple, elementary question, which was given just to compare the answer (4.1) with answer (4.2) to the question b) which was indeed a good question. Two angles of the isosceles triange ABC are evidently equal to $\frac{\pi}{2}$, and the angle between two meridians is equal to φ, hence $\alpha + \beta + \gamma - \pi = \varphi$ On the other hand comparing this triangle with the north hemisphere we see that

$$\text{Area of } \triangle ABC = \text{Area of North hemisphere} \cdot \frac{\varphi}{2\pi} = 2\pi R^2 \cdot \frac{\varphi}{2\pi} = R^2\varphi\,.$$

Hence

$$\frac{\text{Area }(\triangle ABC)}{R^2} = \varphi = \alpha + \beta + \gamma - \pi = \frac{\pi}{2} + \frac{\pi}{2} + \varphi - \pi \,.$$

That is all. About half of students did it using just this elementary considerations. Other students instead solving this very easy question decided `to use a sledgehammer to crack a nut` and they were trying to present the solution of much more difficult problem: to prove equation (4.1) *for arbitrary spherical triangle.*

b) Solving this problem almost all students have noted correctly that the length of the horizontal segment A_tB_t tends to zero if $t \to \infty$, but their estimations of the length of this segment were wrong: $|A_tB_t| = \frac{2a}{t}$, not $\frac{2a}{t^2}$.

Another comment: the following two statements almost nobody stated exactly[2)]

1) the distance (the lengtth of the shortest arc between the points A_t and B_t) is $\leq$ than the length of the segment A_tB_t.

2) Angles in Lobachevsky plane and in Euclidean are the same, since the metric is conformally Eucldiean (however it is not locally Euclidean: curvature is not equal to zero!)

5

a) Let ∇ be an affine connection on the 2-dimensional manifold M such that in local coordinates (u, v), $\nabla_{\frac{\partial}{\partial u}} \left(u^2 \frac{\partial}{\partial v}\right) = 3u\frac{\partial}{\partial v} + u\frac{\partial}{\partial u}$.

Calculate the Christoffel symbols Γ^u_{uv} and Γ^v_{uv} of this connection.

b) Let ∇ be an arbitrary connection on a manifold M. Show that

$$\cos F \nabla_{\mathbf{A}}(\sin F\mathbf{B}) - \sin F \nabla_{\mathbf{A}}(\cos F\mathbf{B}) = (\partial_{\mathbf{A}} F)\mathbf{B}\,,$$

where $\mathbf{A}, \mathbf{B}$ are arbitrary vector fields and F is an arbitrary function.

c) Let $\Gamma^{i(1)}_{km}$ be the Christoffel symbols of a connection $\nabla^{(1)}$ and $\Gamma^{i(2)}_{km}$ be the Christoffel symbols of a connection $\nabla^{(2)}$. Show, that the linear combinations $\frac{2}{3}\Gamma^{i(1)}_{km} + \frac{1}{3}\Gamma^{i(2)}_{km}$, are Christoffel symbols for some connection.

Explain, why $\frac{1}{2}\Gamma^{i(1)}_{km} + \frac{1}{3}\Gamma^{i(2)}_{km}$ are not Christoffel symbols for any connection.

Almost all students answered questions a) and b).

Qiestion c) was not easy question.

[2)] These drawbacks did not affect the mark, however I would like to note them.

Many students stated that the linear combinations $\frac{2}{3}\Gamma^{i(1)}_{km} + \frac{1}{3}\Gamma^{i(2)}_{km}$, are Christoffel symbols for some connection, and on the other hand that the linear combinations $\frac{1}{2}\Gamma^{i(1)}_{km} + \frac{1}{3}\Gamma^{i(2)}_{km}$, are *not* the Christoffel symbols for any connection, because $1/3 + 2/3 = 1$ and $1/2 + 1/3 \neq 1$; a part of these students were trying to explain why.

One solution of this question is to check how transform under changing of coordinates the linear combinations $\gamma\Gamma^{i(1)}_{km} + \mu\Gamma^{i(2)}_{km}$. It is easy to see that this combination transforms as Christoffel symbol of some connection if and only if $\lambda + \mu = 1$. Three or four students did this exerise in this way.

About five or six students were trying to do it in the following way: they consider symbols

$$\mathcal{G}^i_{km} = \lambda\Gamma^{i(1)}_{km} + \mu\Gamma^{i(2)}_{km}$$

and the operation:

$$S_{\mathbf{X}}(\mathbf{Y}) = \lambda\nabla^{(1)}_{\mathbf{X}}\mathbf{Y} + \mu\nabla^{(2)}_{\mathbf{X}}\mathbf{Y}$$

One can see that this operation obeys axioms of connection: if and only if $\lambda + \mu = 1$. In particular for Leibnitz rule

$$S_{\mathbf{X}}(f\mathbf{Y}) = \lambda\nabla^{(1)}_{\mathbf{X}}(f\mathbf{Y}) + \mu\nabla^{(2)}_{\mathbf{X}}(f\mathbf{Y}) = \lambda f\nabla^{(1)}_{\mathbf{X}}(\mathbf{Y}) + \mu f\nabla^{(2)}_{\mathbf{X}}(\mathbf{Y}) + (\lambda + \mu)\partial_{\mathbf{X}} f\mathbf{Y} =$$

$$fS_{\mathbf{X}}\mathbf{Y} + \partial_{\mathbf{X}} f\mathbf{Y}\,, \quad \text{if and only if } \lambda + \mu = 1\,.$$

This is nice approach. All the students who tried this method received credits. Two of them did it properly.

6

Let M be a surface considered in question **1** *(the upper sheet of a cone),*

a) Calculate the induced connection on this surface (the connection induced by the canonical flat connection in the ambient Euclidean space: $\nabla_{\mathbf{X}}\mathbf{Y} = \left(\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}\right)_{\text{tangent}}$).

b) Calculate the Riemannian metric on the cone induced by the canonical metric in ambient Euclidean space $\mathbf{E}^3$ and calculate explicitly the Levi-Civita connection of this metric using the Levi-Civita Theorem.

c) Calculate the Christoffel symbols of Levi–Civita connection on the cone using Lagrangian of free particle.

In this exercise students have to calculate Christoffel symbols on the surface of cone in $\mathbf{E}^3$. of induced connection, and of Levi-Civita connection with use of Levi-Civita standard formula or Lagrangin of free particle.

This exercise is almost bookwork, on the other hand some calculations (e.g. calculations for components of $\Gamma^h_{\varphi\varphi}$ of inudced connection need a time.)

Typical mistakes

a) wrong calculation of normal unit vector to the surface of cone

b) Few students received different answers for Christoffel symbols calculating induced connection and Levi-Civita connection. It is easy to make a mistake in calculations, but you have know firmly that

$$\text{Levi-Civita connection of induced Riemannian metric} = \text{induced connection}$$

In this exercise you have calculated the same Christoffel symbols using three different methods.

This note helps to avoid mistakes in calculations.