

## Homework 4

**1** Calculate the Christoffel symbols of the canonical flat connection in  $\mathbf{E}^3$  in

a) cylindrical coordinates  $(x = r \cos \varphi, y = r \sin \varphi, z = h)$ ,

b) spherical coordinates.

(For the case b) try to make calculations at least for components  $\Gamma_{rr}^r, \Gamma_{r\theta}^r, \Gamma_{r\varphi}^r, \Gamma_{\theta\theta}^r, \dots, \Gamma_{\varphi\varphi}^r$ )

**2** Let  $\nabla$  be an affine connection on a 2-dimensional manifold  $M$  such that in local coordinates  $(u, v)$  it is given that  $\Gamma_{uv}^u = v, \Gamma_{uv}^v = 0$ .

Calculate the vector field  $\nabla_{\frac{\partial}{\partial u}} \left( u \frac{\partial}{\partial v} \right)$ .

**3** Let  $\nabla$  be an affine connection on the 2-dimensional manifold  $M$  such that in local coordinates  $(u, v)$

$$\nabla_{\frac{\partial}{\partial u}} \left( u \frac{\partial}{\partial v} \right) = (1 + u^2) \frac{\partial}{\partial v} + u \frac{\partial}{\partial u}.$$

Calculate the Christoffel symbols  $\Gamma_{uv}^u$  and  $\Gamma_{uv}^v$  of this connection.

**4 a)** Consider a connection such that its Christoffel symbols are symmetric in a given coordinate system:  $\Gamma_{km}^i = \Gamma_{mk}^i$ .

Show that they are symmetric in an arbitrary coordinate system.

b\*) Show that the Christoffel symbols of connection  $\nabla$  are symmetric (in any coordinate system) if and only if

$$\nabla_{\mathbf{X}} \mathbf{Y} - \nabla_{\mathbf{Y}} \mathbf{X} - [\mathbf{X}, \mathbf{Y}] = 0,$$

for arbitrary vector fields  $\mathbf{X}, \mathbf{Y}$ .

c)\* Consider for an arbitrary connection the following operation on the vector fields:

$$S(\mathbf{X}, \mathbf{Y}) = \nabla_{\mathbf{X}} \mathbf{Y} - \nabla_{\mathbf{Y}} \mathbf{X} - [\mathbf{X}, \mathbf{Y}]$$

and find its properties.

**5** Let  $\nabla_1, \nabla_2$  be two different connections. Let  ${}^{(1)}\Gamma_{km}^i$  and  ${}^{(2)}\Gamma_{km}^i$  be the Christoffel symbols of connections  $\nabla_1$  and  $\nabla_2$  respectively.

a) Find the transformation law for the object:  $T_{km}^i = {}^{(1)}\Gamma_{km}^i - {}^{(2)}\Gamma_{km}^i$  under a change of coordinates. Show that it is  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$  tensor.

b)\*? Consider an operation  $\nabla_1 - \nabla_2$  on vector fields and find its properties.

**6 \*** a) Consider  $t_m = \Gamma_{im}^i$ . Show that the transformation law for  $t_m$  is

$$t_{m'} = \frac{\partial x^m}{\partial x^{m'}} t_m + \frac{\partial^2 x^r}{\partial x^{m'} \partial x^{k'}} \frac{\partial x^{k'}}{\partial x^r}.$$

b) <sup>†</sup> Show that this law can be written as

$$t_{m'} = \frac{\partial x^m}{\partial x^{m'}} t_m + \frac{\partial}{\partial x^{m'}} \left( \log \det \left( \frac{\partial x}{\partial x'} \right) \right) .$$

**7** Calculate Christoffel symbols of the connection induced on the surface  $M$  in  $\mathbf{E}^n$  equipped with canonical flat connection.

- a)  $M = S^1$  in  $\mathbf{E}^2$
- b)  $M$ — parabola  $y = x^2$  in  $\mathbf{E}^2$
- c) cylinder, cone and sphere in  $\mathbf{E}^3$ .
- d) saddle  $z = xy$  in  $\mathbf{E}^3$