Homework 1. Solutions

1 *Show that* $(\mathbf{x},\mathbf{y}) = x^1y^1+x^2y^2+x^3y^3$ *is a scalar product in* $\mathbf{R}^3$. *Show that* $(\mathbf{x},\mathbf{y}) = x_1y_1+x_2y_2-x_3y_3$ *does not define scalar product in* $\mathbf{R}^3$.

You need to check the conditions of symmericity, linearity and non-degeneracy for scalar product. First two conditions obviously are obeyed. Condition of non-degeneracy is obeyed for first scalar product $(\mathbf{x},\mathbf{x}) = (x^1)^2+(x^2)^2+(x^3)^2 \geq 0$. Contrary for $(\mathbf{x},\mathbf{y}) = x_1y_1+x_2y_2-x_3y_3$ it *is not obeyed*: e.g. for $\mathbf{x} = (1,1,2)$ $(\mathbf{x},\mathbf{x}) = 0$; hence it is not scalar product.

2* *Prove Cauchy–Bunyakovsky–Schwarz inequality* $(\mathbf{x},\mathbf{y})^2 \leq (\mathbf{x},\mathbf{x})(\mathbf{y},\mathbf{y})$ *where* $\mathbf{x},\mathbf{y}$ *are arbitrary two vectors and* $(\ ,\ )$ *is a scalar product in Euclidean space and* $\mathbf{E}^n$.

Consider quadratic polynomial $P(t) = \sum_{i=1}^n (tx^i+y^i)^2 = At^2+2Bt+C$, where $A = \sum_{i=1}^n (x^i)^2 = (\mathbf{x},\mathbf{x})$, $B = \sum_{i=1}^n (x^iy^i) = (\mathbf{x},\mathbf{y})$, $C = \sum_{i=1}^n (y^i)^2 = (\mathbf{y},\mathbf{y})$. We see that equation $P(t) = 0$ has at most one root (and this is the case if only vector $\mathbf{x}$ is collinear to the vector $\mathbf{y}$). This means that discriminant of this equation is less or equal to zero. But discriminant of this equation is equal to $4B^2-4AC$. Hence $B^2 \leq AC$. It is just CBS inequality. ($(\mathbf{x},\mathbf{y})^2 = (\mathbf{x},\mathbf{x})(\mathbf{y},\mathbf{y})$, i.e. discriminant is equal to zero $\Leftrightarrow$ vectors $\mathbf{x}$, $\mathbf{y}$ are colinear.

3 *Denote by* $T_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$ *Calculate matrices* T_φ^{-1}, $T_\varphi T_\psi$.

Straightforward calculations show that $T_\varphi^{-1} = T_{-\varphi}$ and $T_\varphi T_\psi = T_{\varphi+\psi}$.

4 *Show that under transformation* $\begin{pmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \end{pmatrix} = T_\varphi \circ \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}$ *orthonormal basis transforms to orthonormal one. How coordinates of vectors change if we rotate the orthonormal basis* $(\mathbf{e}_1,\mathbf{e}_2)$ *on the angle* $\varphi = \frac{\pi}{3}$ *anticlockwise?*

We have to check that $(\mathbf{e}'_1,\mathbf{e}'_1) = (\mathbf{e}'_2,\mathbf{e}'_2) = 1$ and $(\mathbf{e}'_1,\mathbf{e}'_2) = 0$. Calculations show that $(\mathbf{e}'_1,\mathbf{e}'_1) = (\cos\varphi\mathbf{e}_1+\sin\varphi\mathbf{e}_2, \cos\varphi\mathbf{e}_1+\sin\varphi\mathbf{e}_2) = \cos^2\varphi(\mathbf{e}_1,\mathbf{e}_1)+2\cos\varphi\sin\varphi(\mathbf{e}_1,\mathbf{e}_2)+\sin^2\varphi(\mathbf{e}_2,\mathbf{e}_2) = 1$, $(\mathbf{e}'_2,\mathbf{e}'_2) = (-\sin\varphi\mathbf{e}_1+\cos\varphi\mathbf{e}_2, -\sin\varphi\mathbf{e}_1+\cos\varphi\mathbf{e}_2) = 1$, $(\mathbf{e}'_1,\mathbf{e}'_2) = (\cos\varphi\mathbf{e}_1+\sin\varphi\mathbf{e}_2, -\sin\varphi\mathbf{e}_1+\cos\varphi\mathbf{e}_2) = 0$.

Now answer the second question.

If $\mathbf{a} = x\mathbf{e}_x+y\mathbf{e}_2 = x'\mathbf{e}'_x+y'\mathbf{e}'_2$ and $T_\varphi = T_{\frac{\pi}{3}} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ is the matrix of bases transformation then we have:

$$\mathbf{a} = (\mathbf{e}_x,\mathbf{e}_y)\begin{pmatrix} x \\ y \end{pmatrix} = (\mathbf{e}'_x,\mathbf{e}'_y)\begin{pmatrix} x' \\ y' \end{pmatrix} = (\mathbf{e}_x,\mathbf{e}_y)T_{\frac{\pi}{3}}\begin{pmatrix} x' \\ y' \end{pmatrix} = (\mathbf{e}_x,\mathbf{e}_y)\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

Hence

$$\begin{pmatrix} x \\ y \end{pmatrix} = T_{\frac{\pi}{3}}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

and

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = T_{\frac{\pi}{3}}^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = T_{-\frac{\pi}{3}}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

5 *Find normal and parametric equations of the line which passes through the point* $(2,3)$ *and making an angle* $\varphi = 30^\circ$ *with* x*-axis.*

Direction vector of the line is $\mathbf{N} = (1, \tan 30^\circ)$. Hence parametric equation is

$$\begin{cases} x = 2+t, \\ y = 3+\frac{\sqrt{3}}{3}t, \end{cases} \qquad -\infty < t < \infty$$

Excluding t we come to $y = 3+\frac{\sqrt{3}}{3}(x-2)$, or $y-\frac{\sqrt{3}}{3}x = 3-\frac{2\sqrt{3}}{3}$. In this equation coefficients are defined up to a multiplier: $1 \mapsto \frac{1}{\lambda}$, $\frac{-\sqrt{3}}{3} \mapsto \frac{-\sqrt{3}}{3\lambda}$. To come to the normal form we choose coefficient λ such that $\lambda = \sqrt{a^2+b^2} = \frac{2}{\sqrt{3}}$ and we come finally to normal equation:

$$\frac{\sqrt{3}}{2}y-\frac{1}{2}x = \frac{3\sqrt{3}}{2}-1, \text{ or } \frac{\sqrt{3}}{2}(y-3)-\frac{1}{2}(x-2) = 0\,.$$

6 *Find an equation of the line passing through the point* $(0, 1)$ *and which is orthogonal to the line* $y - 2x = 0$.

The vector $(-2, 1)$ is orthogonal to the line $y - 2x = 0$. Vector $(1, 2)$ is orthogonal to the vector $(-2, 1)$. Hence the vector $(1, 2)$ is orthogonal to the line which we have to find. The equation of this line will be $x + 2y = c$ or $(x - 0) + 2(y - 1) = 0$ because the point $(0, 1)$ belongs to the line. We see that equation of the line is $x + 2y = 2$.

7 *Calculate a distance between the point* (x_0, y_0) *and the line* $y - kx = b$ *using*

a) geometrical methods

b) "brute force": just a minimum of distance between a point (x_0, y_0) *and an arbitrary point on the line, i.e. minimum of the function:* $\sqrt{(x - x_0)^2 + (y - y_0)^2}$ *with* $y = kx + b$

To use geometrical methods we write normal equation of the line:

$$\frac{y}{\sqrt{1+k^2}} - \frac{kx}{\sqrt{1+k^2}} = \frac{b}{\sqrt{1+k^2}}$$

Hence the distance will be:

$$d = \pm\left(\frac{y_0}{\sqrt{1+k^2}} - \frac{kx_0}{\sqrt{1+k^2}} - \frac{b}{\sqrt{1+k^2}}\right)$$

Now calculate the distance using "brute" force: Consider function: distance between point (x_0, y_0) and arbitrary point of the line:

$$f(x) = \sqrt{(x - x_0)^2 + (y - y_0)^2}\Big|_{y=kx+b} = \sqrt{(x - x_0)^2 + (kx + b - y_0)^2}$$

The stationary point of this function $x = x_1$ is given by the condition:

$$\frac{df(x)}{dx}\Big|_{x=x_1} = \frac{x_1 - x_0 + k(kx_1 + b - y_0)}{\sqrt{(x_1 - x_0)^2 + (kx_1 + b - y_0)^2}} = 0,\ x_1 = \frac{x_0 + ky_0 - kb}{1+k^2},\ y_1 = kx_1 + b = \frac{k^2y_0 + kx_0 + b}{1+k^2}$$

The vector $\mathbf{r}_1 - \mathbf{r}_0$, where $\mathbf{r}_1 = (x_1, y_1)$ is perpendicular to the line and distance between point $\mathbf{r}_0 = (x_0, y_0)$ and $\mathbf{r}_1 = (x_1, y_1)$ is the distance d between the point $\mathbf{r}_0 = (x_0, y_0)$ and the line. The distance is:

$$d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} = \sqrt{\left(\frac{x_0 + ky_0 - kb}{1+k^2} - x_0\right)^2 + \left(\frac{k^2y_0 + kx_0 + b}{1+k^2} - y_0\right)^2} =$$

$$\sqrt{\frac{(y_0 - b)^2 + k^2x_0^2 - 2kx_0(y_0 - b)}{1+k^2}} = \pm\left(\frac{y_0 - kx_0 - b}{\sqrt{1+k^2}}\right)$$

We come to the same answers, but calculations are little bit boring.

8 *Calculate a distance between the point* $A = (1, 1)$ *and the line* $x + 2y = 1$

Normal equation of the line $x + 2y = 1$ will be

$$\frac{x}{\sqrt{1+4}} + \frac{2y}{\sqrt{1+4}} = \frac{1}{\sqrt{1+4}},\quad \frac{\sqrt{5}}{5}x + \frac{2\sqrt{5}}{5}y = \frac{\sqrt{5}}{5}$$

So the distance between point $(1, 1)$ and this line will be (up to a sign):

$$\frac{\sqrt{5}}{5}\cdot 1 + \frac{2\sqrt{5}}{5}\cdot 1 - \frac{\sqrt{5}}{5} = \frac{2\sqrt{5}}{5}$$

9 *Write down an equation of the line passing via point* $A = (x_0, y_0)$ *and which is tangent to the circle* $(x - a)^2 + (y - b)^2 = R^2$

How many solutions does this problem have?

We consider three different solutions of this problem.

First solution

Note that:

$$\text{there are no solution if } (x_0-a)^2+(y_0-b)^2 < R^2, \text{ i.e. point } A \text{ is in the interior of the circle,} \tag{8.1a}$$

$$\text{there is one solutions if } (x_0-a)^2+(y_0-b)^2 = R^2, \text{ i.e.point } A \text{ is on the circle} \tag{8.1b}$$

$$\text{there are two solutions if } (x_0-a)^2+(y_0-b)^2 > R^2, \text{ i.e. point } A \text{ is in the exterior of the circle.} \tag{8.1c}$$

Now perform "blind" calculations. Let $\alpha x+\beta y=c$ with $\alpha^2+\beta^2=1$ be a normal equation of the line. It is convenient to consider φ such that $\alpha=\cos\varphi, \beta=\sin\varphi$. The normal equation of the line will be rewritten as

$$x\cos\varphi+y\sin\varphi=c$$

This line has to be on the distance $h=R$ from the point (a,b)— centre of the circle and it has to pass through the point A. Hence

$$\begin{cases} x_0\cos\varphi+y_0\sin\varphi=c, & \text{i.e. point } A \text{ belongs to the line} \\ a\cos\varphi+b\sin\varphi-c=\pm R, & \text{line is tangent to the circle} \end{cases} \tag{8.2}$$

We see that normal equation of the line is $(x-x_0)\cos\varphi+(y-y_0)\sin\varphi=0$ and the condition $(a-x_0)\cos\varphi+(b-y_0)\sin\varphi=\pm R$ is obeyed, i.e.

$$[(x_0-a)\cos\varphi+(y_0-b)\sin\varphi]^2=R^2 \tag{8.3}$$

We have to solve this equation with respect to coefficients $\cos\varphi, \sin\varphi$

Use standard trigonometry. Choose δ such that $\tan\delta=\frac{y_0-b}{x_0-a}$. Then

$$x_0-a=L\cos\delta, \quad y_0-a=L\sin\delta \quad \text{with } L=\sqrt{(x_0-a)^2+(y_0-b)^2} \tag{8.4}$$

Equation (8.3) can be rewritten as:

$$(L\cos\delta\cos\varphi+L\sin\delta\sin\varphi)^2=L^2\cos^2(\varphi-\delta)=R^2, \qquad \cos^2(\varphi-\delta)=\frac{R^2}{L^2} \tag{8.5}$$

We see that in full accordance with geometry (see beginning of the solution) we have following three cases:

I If $R^2>L^2$, i.e. point A is in the interior of the circle, the equation (8.5) has no solution and there is no such line (as we see geometrically above (see 8.1a)).

II If $R^2=L^2$, i.e. according to (8.6) point A is at the circle. The equation (8.5) will have a form $\cos(\varphi-\delta)=\pm 1$, i,e, $\cos\varphi:\sin\varphi=\cos\delta:\sin\delta=x_0-a:y_0-b$. Equation of the line (8.2) can be written in the form

$$(x_0-a)(x-x_0)+(y_0-b)(y-y_0)=0$$

There is only one line obeying conditions (8.2) (see (8.1b)).

III If $R^2>L^2$, i.e. point A is in exterior of the circle. To solve equation (8.5) in this case it is convenient to consider an angle ψ such that $\cos\psi=\frac{R}{L}$. Then solution of equation (8.5) will be

$$\cos\varphi:\sin\varphi=\cos(\delta+\psi):\sin(\delta+\psi) \text{ or } \cos\varphi:\sin\varphi=\cos(\delta-\psi):\sin(\delta-\psi)$$

We have two lines obeying conditions (8.2) (see (8.1a))

$$(x-x_0)\cos(\delta+\psi)+(y-y_0)\sin(\delta+\psi)=0 \quad \text{or} \quad (x-x_0)\cos(\delta-\psi)+(y-y_0)\sin(\delta-\psi)=0\,.$$

Coefficients $\cos(\delta \pm \psi), \sin(\delta \pm \psi)$ can be expressed via $\tan\delta = \frac{y_0-b}{x_0-a}$, and $\cos\psi = \frac{R}{R}$ with standard trigonometric formulae.

Second solution

One can avoid these calculations using the fact that if we consider translation $x \mapsto x-a, y \mapsto y-b$ and a suitable rotation then initial data can be reduced to much simpler ones: Circle will be given by the equation $x^2+y^2=R^2$ and a point A will have coordinates $(x_0, 0)$. Then all previous calculations becomes very short:

I $x_0^2 < R^2$, no solutions.

II $x_0^2 = R^2$, unique solution: tangent line is $x = x_0$.

III $x_0^2 > R^2$. Consider normal equation of the line $(x-x_0)\cos\varphi + y\sin\varphi = 0$. Calculate distance between centre of the circle and line. It is equal to to $x_0^2\cos^2\varphi = R^2$. Hence $\cos\varphi = \pm\frac{R}{x_0}$. Hence there are two solutions:

$$\frac{R}{x_0}(x-x_0) \pm \sqrt{1-\frac{R^2}{x_0^2}}\, y = 0 \tag{8.6}$$

Third solution

This problem can be solved in many other ways, e.g. just considering triangle $\triangle AOB$ where O is centre of the circle, B is touching point, or one can consider the following very beatiful solution based on the idea that line is tangent to the circle if and only if corresponding simultaneous equations have only one solution: Do it. Solve it just for the case where $a = b = y_0 = 0$. Then equation of the line will be $y = k(x-x_0)$. Line is tangent to the circle $x^2+y^2=R^2$ if and only if the simultaneous equations

$$\begin{cases} x^2+y^2=R^2 \\ y = k(x-x_0) \end{cases}$$

have exactly one solution: it will be coordinates of tangent point. Put $y = k(x-x_0)$ we come to quadratic equation:

$$x^2 + k^2(x-x_0)^2 = R^2, (k^2+1)x^2 = 2k^2x_0x + (x_0^2 - R^2) = 0$$

This quadratic equation has one solution if and only if $D = (2k^2x_0)^2 - 4(k^2+1)(x_0^2-R^2) = 0$, i.e.

$$k^2x_0^2 - k^2R^2 - R^2 = 0, \quad \text{hence} \quad k = \pm\frac{R}{\sqrt{x_0^2-R^2}}$$

Hence equation of the line will be

$$y \pm \frac{(x-x_0)R}{\sqrt{x_0^2-R^2}} = 0\,.$$

(Compare it with (8.6)).

10 *Find the locus formed by centres of segments of the length 1, such that their endpoints lie on the axes OX, OY.*

Consider the segment AB of the length 1 such that the point $A = (x, 0)$ is on the x-axis and the point $B = (0, y)$ is on the y-axis. Then $x^2+y^2=1$. The centre of this segment has coordinates $\left(\frac{x}{2}, \frac{y}{2}\right)$.

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = \frac{x^2+y^2}{4} = \frac{1}{4}$$

The centre is on the circle of the radius $\frac{1}{2}$. On the other hand if a point $M = (a, b)$ belongs to this circle then $a^2+b^2=\frac{1}{4}$. Hence the segment AB with $A = (2a), B = (0, 2b)$ has the length 1. Thus we showed that the locus of the centres of these segments is the set of points of the circle $x62 + y^2 = \frac{1}{4}$