Linear algebra and volume element of $G_{k,n}$

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Let $V_{k,n}$ be a space of $k \times n$ real marices.

We consider the Euclidean metric in $V_{k,n}$ induced by the norm

$$||M|| = \mathrm{Tr}\,(MM^+)\,, \qquad \text{the scalar product is}\,, \langle M, N\rangle = \mathrm{Tr}\,(MN^+)\,.$$

Denote by $[M]$ the subspace in $V_{k,n}$ spanned by the left action of $GL(k,\mathbf{R})$ on matrix M:

$$[M] = \{gM,\, g \in GL(k,\mathbf{R})\}\,.$$

In components $[M]$ is the set of matrices $M_{ia} = \lambda_{ik} M_{ka}$.

We denote by $\mathcal{V}_{k,n}$ the open set of non-degenerate matrices in V.

Every matrix M in $\mathcal{V}$ defines k^2-dimensional subspace $[M]$ in $V_{k,n}$.

Consider an arbitrary matrix $M \in \mathcal{V}_{k,n}$.

For an arbitrary matrix N consider the matrix

$$N'_{(N.M)} = N - \lambda M$$

such that the distance between N' and M is minimal:

$$N'_{(N.M)} = N - NM^+(MM^+)^{-1}M\,.$$

We see that

$$d(N,[M]) = \min_{\lambda \in GL(k)} ||N - \lambda M|| = ||N - NM^+(MM^+)^{-1}M||\,,$$

where M is an arbitrary matrix in $[M]$. Note that the matrix $N' = N - NM^+(MM^+)^{-1}M$ is orthogonal to the plane $[M]$, more:

$$N'M = 0\,.$$

(This is more than $\langle N'M\rangle = 0$.)

One can see that for arbitrary matrix N,

$$d(N,[M]) = ||N'|| = \min_{\lambda \in GL(k)} ||N - \lambda M|| = ||N - NM^+(MM^+)^{-1}M||\,,$$

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One can see that

$$d(N,[\lambda M]) = d(N,[M])\,,$$

and

$$N'_{(\lambda N,M)} = \lambda N'_{(N,M)}\,.$$

Using these relations we are ready to define the distance between two planes:

$$d\left([N],[M]\right) = \sqrt{\mathrm{Tr}\,\left(\left(N'_{(N,M)}N'^{+}_{(N,M)}\right)(NN^{+})^{-1}\right)}$$

Is it good???

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$$\frac{d\left(N,[M]\right)}{\sqrt{\det(MM^{+})}} = \frac{||N - NM^{+}(MM^{+})^{-1}M||}{\sqrt{\det(MM^{+})}}$$

Using the fact tat $N - NM^{+}(MM^{+})^{-1}M$ is orthogonal to the plane $[M]$ we come to the answer

$$d\left([N],[M]\right) = \frac{||N - NM^{+}(MM^{+})^{-1}M||}{\sqrt{\det(MM^{+})}} = \frac{\sqrt{\mathrm{Tr}\,\left(NN^{+} - NM^{+}(MM^{+})^{-1}MN^{+}\right)}}{\sqrt{\det(MM^{+})}}$$

In particular for infinitesimal tangent vectors $N = M + m$ we have

$$ds^2 = \frac{\mathrm{Tr}\,\left(mm^{+} - mM^{+}(MM^{+})^{-1}Mm^{+}\right)}{\det(MM^{+})} =$$

$$\frac{m_{ia}\left(\delta_{ij}\left(\delta_{ab} - \left(M^{+}(MM^{+})^{-1}M\right)_{ab}\right)\right)m_{jb}}{\det(MM^{+})}\,.$$

Now go to non-homogeneous (affine) coordinates in the space of planes

$$M_{ia} = \left(\delta_{ij}, W_{i\alpha}\right)\,, m_{ia} = \left(\mathbf{0}\,, m_{i\alpha}\right)\,,\quad i,j = 1,\ldots,k,\; a = 1.\ldots,N, \alpha = k+1,\ldots,N,$$

We see that in these affine coordinates

$$ds^2 = \frac{\mathrm{Tr}\,\left(mm^{+} - mW^{+}(\mathbf{1} + WW^{+})^{-1}Wm^{+}\right)}{\det(\mathbf{1} + WW^{+})} =$$

$$\frac{m_{i\alpha}\left(\delta_{ij}\left(\delta_{\alpha\beta} - \left(W^{+}(\mathbf{1} + WW^{+})^{-1}W\right)_{\alpha\beta}\right)\right)m_{j\beta}}{\det(\mathbf{1} + WW^{+})}\,.$$

Calculate the determinant of the metric. First calculate the determinant of operator

$$L\colon \quad L_{\alpha\beta} = \delta_{\alpha\beta} - \left(W^{+}(\mathbf{1} + WW^{+})^{-1}W\right)_{\alpha\beta}$$

Matrix W defines the operator which maps $\mathbf{R}^k$ to $\mathbf{R}^{n-k}$ Notice that arbitrary vector which is orthogonal to the image of this operator:

$$\mathbf{t}\colon W_{i\alpha} t_\alpha = 0\,,$$

$L(\mathbf{t}) = \mathbf{t}$, i.e. it is the eigenvector of operator L with eigenvalue 1.

On the other hand for arbitrary vector which belongs to the image of this operator

$$\mathbf{l}\colon \quad l_\alpha = l_k W_{k\alpha}\,, \text{(linear compbination of rows)}$$

we have

$$L\mathbf{l}_\alpha = \left(\delta_{\alpha\beta} - \left(W^{+}\left(\mathbf{1} + WW^{+}\right)^{-1} W\right)_{\alpha\beta}\right) l_k W_{k\beta} = l_k W_{k\alpha} -$$

$$-W_{i\alpha}\left(\mathbf{1} + WW^{+}\right)^{-1}_{ij} W_{j\beta} l_k W_{k\beta} l_k W_{k\alpha} = -W_{i\alpha}\left(\left(\mathbf{1} + WW^{+}\right)^{-1} WW^{+}\right)_{ik} l_k$$

i.e.

$$(L\mathbf{l})_\alpha = \tilde{l}_k W_{k\alpha}\,, \text{ where } \tilde{l}_k = l_k - \left(\left(\mathbf{1} + WW^{+}\right)^{-1}\left(WW^{+}\right)\right)_{kn} l_n\,.$$

This means that $\det L$ is equal to the product of 1 (the determinant of this operator restricted on vectors orthogonal to the image of W) on the determinant of the operator $\mathbf{1} - \left(\left(\mathbf{1} + WW^{+}\right)^{-1}\left(WW^{+}\right)\right)$:

$$\det L = 1 \cdot \det\left(\mathbf{1} - \left(\left(\mathbf{1} + WW^{+}\right)^{-1}\left(WW^{+}\right)\right)\right) = \frac{1}{\det((\mathbf{1} + WW^{+})}$$

Hence

$$\det G = (\det L)^k \left(\frac{1}{\sqrt{\det(\mathbf{1} + WW^{+})}}\right)^{n(n-k)} = \left(\frac{1}{\det(\mathbf{1} + WW^{+})}\right)^{\frac{n(n-k)}{2} + k???}$$