Homework 8

1 Let M be a surface embedded in Euclidean space $\mathbf{E}^3$. We say that the triple of vector fields $\{\mathbf{e},\mathbf{f},\mathbf{n}\}$ is adjusted to the surface M if $\mathbf{e},\mathbf{f},\mathbf{n}$ be three vector fields defined on the points of this surface such that they form an orthonormal basis at any point, so that the vectors $\mathbf{e},\mathbf{f}$ are tangent to the surface and the vector $\mathbf{n}$ is orthogonal to the surface.

Consider the derivation formulae for adjusted vector fields $\{\mathbf{e},\mathbf{f},\mathbf{n}\}$:

$$d\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\\ \mathbf{n}\end{pmatrix} = \begin{pmatrix}0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{pmatrix}\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\\ \mathbf{n}\end{pmatrix}, \tag{1}$$

where a,b,c are 1-forms on the surface M.

Write down the explicit expression for connection, Weingarten operator, (shape operator), the mean curvature and the Gaussian curvature of M in terms of 1-forms a,b,c and vector fields $\{\mathbf{e},\mathbf{f},\mathbf{n}\}$.

2 Show that in derivation formulae $\begin{cases} da + b\wedge c = 0\\ db + c\wedge a = 0\\ dc + a\wedge b = 0\end{cases}$.

3 Find explicitly a triple of vector fields $\{\mathbf{e},\mathbf{f},\mathbf{n}\}$ adjusted to the surface M if M is a) cylinder, b) cone c) sphere.

4 Using results of the previous exercise find explicit expression for derivation formulae (1) in the case if the surface M is a) cylinder, xb) cone, c) sphere, and deduce from these results the formulae for Gaussian and mean curvature for cylinder, cone and sphere

5 Consider surface M which is given by equation

$$\mathbf{r}(u,v)\colon\quad \begin{cases} x = u\\ y = v\\ z = F(u,v)\end{cases}$$

Find explicitly a triple of vector fields $\mathbf{e},\mathbf{f},\mathbf{n}$ adjusted to the surface M.

Suppose the origin (point $u=v=0$) is a stationary point of the function $F(u,v)$, i.e. $F_u = F_v = 0$ at $u=v=0$.

Calculate in this case vector 1-forms $d\mathbf{e}, d\mathbf{f}, d\mathbf{n}$, 1-forms a,b,c at origin, and calculate Gaussian and mean curvature at origin.

6 a) Find a triple of vector fields $\{\mathbf{e},\mathbf{f},\mathbf{n}\}$ adjusted to the surface M if a Riemannian metric on a surface M is given by the formula $G = a(u,v)du^2 + b(u,v)dv^2$.
b*) Calculate 1-form a in derivation formulae in the special case if $a = b = \sigma(u,v)$, i.e. u,v are conformal coordinates on the surface. calculate Gaussian curvature. Show that it is expressed by the formula:

$$K = -\frac{1}{2\sigma}\frac{\partial^2\sigma(u,v)}{\partial u^2} + \frac{\partial^2\sigma(u,v)}{\partial v^2}\,.$$

7 Calculate Gaussian curvature for surface M if induced Riemannian metric is equal to

$$G = (u^2 + v^2)(du^2 + dv^2)$$

* Show that this surface is locally Euclidean: find coordinates p, q, $p = p(u, v), q = q(u, v)$ such that $G = dp^2 + dq^2$ in these coordinates.

8 Choose conformal coordinates on sphere of radius R and calculate the curvature of sphere.

Deduce that sphere *is not locally Euclidean surface*, i.e. there are no local coordinates on sphere such that induced metric in these coordinates is equal to $G = du^2 + dv^2$.

9* Let u, v be coordinates on the locally Euclidean surface M such that $G = du^2 + dv^2$. Let p, q be another coordinates such that

$$w = p + iq = F(z)\big|_{z=x+iy},$$

where $F = F(z)$ is a holomorphic function. Show that p, q are conformal coordinates also in the case if

a) $F = e^z$.

b)* F is an arbitrary (non-zero) holomorphic function.