Riemannian Geometry (31082, 41082)

2011

COURSEWORK

Starred questions are for the 15 credit version

This assignment counts for 20% of your marks.

Solutions are due by 1-st April

Write solutions in the provided spaces.

STUDENTS'S NAME:

1

(a) Explain why the positive-definiteness of a Riemannian metric implies its non degeneracy.

(b) Consider $\mathbf{R}^2$ with the metric $G = \frac{\lambda(dx^2+dy^2)}{(1+x^2+y^2)^2}$, where λ is a constant ($\lambda > 0$).

Show that this Riemannian manifold is isometric to a sphere with a point removed, in $\mathbf{E}^3$. Find the radius $R = R(\lambda)$ of this sphere.

(*You may use the standard stereographic coordinates on the sphere of radius R.*)

Give an example of non-identity transformation of coordinates x, y such that it preserves the metric G.

* *Give an example of non-linear transformation of coordinates x, y such that it preserves the metric G.*

(c) Find the length of the shortest curve on the cone $x^2 + y^2 - k^2z^2 = 0$, joining the points $A = (kh, 0, h)$ and $B = (-kh, 0, h)$.

2

(a) Write down the formula for the volume element on a Riemannian manifold M with metric $G = g_{ik}dx^i dx^k$.

* Show explicitly that the volume element is invariant under changing of coordinates.

(b) Calculate the total volume of n-dimensional space $\mathbf{R}^n$ equipped with the Riemannian metric

$$G = e^{-u_1^2-u_2^2-\ldots-u_n^2}\left((du_1)^2 + (du_2)^2 + \ldots + (du_n)^2\right) .$$

(c) Evaluate the area of the part of the sphere of radius $R = 1$ between the planes given by equations $2x + 2y + z = 1$ and $2x + 2y + z = 2$.

3

(a) Explain what is meant by Christoffel symbols and write down their transformation law.

Let ∇ be an affine connection on the 2-dimensional manifold M such that in local coordinates (u, v), $\nabla_{\frac{\partial}{\partial u}} \left(u^2 \frac{\partial}{\partial v}\right) = 3u\frac{\partial}{\partial v} + u\frac{\partial}{\partial u}$.

Calculate the Christoffel symbols Γ^u_{uv} and Γ^v_{uv} of this connection.

* Let $\Gamma^{i(1)}_{km}$ be the Christoffel symbols of a connection $\nabla^{(1)}$ and $\Gamma^{i(2)}_{km}$ be the Christoffel symbols of a connection $\nabla^{(2)}$. Show that the linear combinations $f\Gamma^{i(1)}_{km} + g\Gamma^{i(2)}_{km}$, (where f and g are some functions) are Christoffel symbols for some connection if and only if $f + g \equiv 1$.

(b) Consider a cone $\mathbf{r}(h, \varphi)$: $\begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases}$ in $\mathbf{E}^3$.

Calculate the induced connection on the cone (the connection induced by canonical flat connection in the ambient Euclidean space: $\nabla_{\mathbf{X}}\mathbf{Y} = \left(\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}\right)_{\text{tangent}}$.)

Calculate the Riemannian metric on the cone induced by the canonical metric in ambient Euclidean space $\mathbf{E}^3$ and calculate explicitly the Levi-Civita connection of this metric using the Levi-Civita Theorem.

Calculate the Christoffel symbols of the cone using equations of motions of Lagrangian of the "free" particle on the cone.

Compare the results of these three calculations for the connection on the cone.

(c) Calculate Levi-Civita connection of the Riemannian metric on the sphere in stereographic coordinates:

$$G = \frac{4R^4(du^2 + dv^2)}{(R^2 + u^2 + v^2)^2}$$

at the point $u = v = 0$

* at an arbitrary point.

Consider a surface M in $\mathbf{E}^3$ defined by the equation $\begin{cases} x = u \\ y = v \\ z = F(u, v) \end{cases}$.

Consider a point $\mathbf{p}$ on M with coordinates $u = x_0, v = y_0$ such that (x_0, y_0) is a point of local extremum for the function F.

Calculate Christoffel symbols of Levi-Civita connection at the point $\mathbf{p}$.

(a) * Suppose a curve C: $x^i = x^i(t)$ is a geodesic on a Riemannian manifold in an arbitrary parameterisation. Show that the velocity vector of this curve remains tangent during parallel transport.

Consider a vertical ray C: $x(t) = x_0, y(t) = y_0 + t$, $0 \le t < \infty$, $(y_0 > 0)$ on the Lobachevsky plane. Find the parallel transport $\mathbf{X}(t)$ of the vector $\mathbf{X}_0 = \partial_y$ attached at the initial point (x_0, y_0) to an arbitrary point of the ray C. (You may use the fact that the vertical ray C is a geodesic in the Lobachevsky plane.)

(**b**) Find the length of the shortest curve on the sphere of the radius R which joins points A and B on the sphere with spherical coordinates in the case if $\theta_A = \theta_B = \theta_0, \varphi_A = 0, \varphi_B = \pi$ and in the case if $\theta_A = \theta_B = \theta_0, \varphi_A = 0, \varphi_B = \frac{\pi}{2}$.

(**c**) On the sphere $x^2 + y^2 + z^2 = R^2$ in $\mathbf{E}^3$ consider points $A = (0, 0, R)$, $B = (R, 0, 0)$ and $C = (R\cos\varphi, R\sin\varphi, 0)$. Consider the isosceles triangle ABC. (Sides of this triangle are the shortest curves joining these points.) Show that:

$$KS(\triangle ABC) = \alpha + \beta + \gamma - \pi, \tag{1}$$

where K is the Gaussian curvature of the sphere, $S(\triangle ABC)$ is the area of the triangle $\triangle ABC$ and α, β, γ are angles of this triangle.

(c) Consider the points $B = (a, \sqrt{1 - a^2})$, $C = (-a, \sqrt{1 - a^2})$ on the Lobachevsky plane realised as half-plane. Consider vertical lines $x = \pm a$ and the half-circle $x^2 + y^2 = 1$ which pass through the points B and C.

Find the angles β, γ between these lines and the circle.

* *Show that the distance between points $B'_h = (a, h)$ and $C'_h = (-a, h)$ on vertical lines tends to zero if $h \to \infty$.*

* *Find the area $S(D)$ of the domain D delimited by the vertical lines and above half-circle and show that $KS(D) = \beta + \gamma - \pi$, where $K = -1$ is Gaussian curvature of Lobachevsky plane. (Compare with the formula (1).)*

(*You may use the fact that Lobachevsky metric of half-plane is $G = \frac{dx^2 + dy^2}{y^2}$.)*