

## Solutions of Homework 7

**1** Let  $C$  be an ellipse in the plane  $\mathbf{E}^2$  such that its foci are at points  $F_1 = (-1, 0)$  and  $F_2 = (1, 0)$  and it passes through the point  $K = (0, 2)$ .

Write down the analytical formula which defines this ellipse.

Find the area of this ellipse.

The foci of this ellipse are on the  $OX$  axis, and the centre of this ellipse is at the point midpoint of the segment  $[F_1F_2]$ , it is the origin  $L(0, 0)$ . Hence the analytical equation is  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , where  $a$  is semi-major axis, and  $b$  is semi-minor axis,  $a \geq b$ . This ellipse intersects the axis  $OY$  at the point  $x = 0, y = \pm b$ . Hence the point  $K = (0, 2) = (0, b)$ , i.e. semi-minor axis  $b = 2$ . Calculating the distances between the point  $K$  and foci we calculate the length of the semi-major axis

$$|KF_1| + |KF_2| = \sqrt{2^2 + 1} + \sqrt{2^2 + 1} = 2\sqrt{5} = 2a \Rightarrow a = \sqrt{5}$$

We see that  $a = \sqrt{5}$  and  $b = 2$ , thus analytical formula for the ellipse is

$$\frac{x^2}{5} + \frac{y^2}{4} = 1.$$

Area of the ellipse is equal to  $S = \pi ab = 2\sqrt{5}\pi$ .

**2** Let  $C$  be an ellipse in the plane  $\mathbf{E}^2$  such that its foci are at the points  $F_1 = (-5, 0)$ ,  $F_2 = (16, 0)$ . It is known that the point  $K = (0, 12)$  belongs to the ellipse.

Find intersections of the ellipse with  $OX$  and  $OY$  axis.

Find the area of the ellipse.

We know that  $K = (0, 12)$ , We have that the point  $K = (0, 12)$  belongs to the ellipse. Hence

$$|KF_1| + |KF_2| = \sqrt{12^2 + (-5)^2} + \sqrt{12^2 + 16^2} = \sqrt{169} + 4\sqrt{25} = 33.$$

Hence for an arbitrary  $D$  point on the ellipse  $|DF_1| + |DF_2| = 33$ . If the ellipse intersects  $OX$  axis at the point with coordinate  $(x, 0)$  then

$$|KF_1| + |KF_2| = 33 = |x - (-5)| + |x - 16| = |x + 5| + |x - 16|.$$

1-st case:  $x > 16$ ,  $|x + 5| + |x - 16| = 2x - 11 = 33 \Rightarrow x = 22$ .

2-nd case:  $x < -5$ ,  $-x - 5 - x + 16 = -2x + 11 = 33 \Rightarrow x = -11$ .

The point of intersection the  $OY$  axis is the point  $K' = (0, -12)$  symmetrical to the point  $K$ .

Hence

The ellipse intersects axis at the points  $(22, 0), (-11, 0), (0, \pm 12)$ .

Now find the area of this ellipse.

To calculate area we calculate length of axis of the ellipse, the minor and the major. Foci of this ellipse are on the  $OX$  axis. Ellipse intersects  $OX$  axis at the points  $(-11, 0)$  and  $(22, 0)$ . Hence major axis is equal to  $2a = 33$ . The distance between foci  $2c = 16 - (-5) = 21$ . Hence the minor axis  $b$  is equal to

$$b = \sqrt{a^2 - c^2} = \sqrt{\left(\frac{33}{2}\right)^2 - \left(\frac{21}{2}\right)^2} = \frac{1}{2}\sqrt{33^2 - 21^2} = \frac{1}{2}\sqrt{(33-21)(33+21)} =$$

$$\frac{1}{2}\sqrt{12 \cdot 54} = \frac{1}{2}\sqrt{3 \cdot 4 \cdot 3 \cdot 9 \cdot 2} = 9\sqrt{2}.$$

One may calculate semi-minor axis  $b$  also in another way: The centre is at the midpoint of the segment  $(F_1, F_2) = (-5, 16)$ : the point  $(\frac{11}{2}, 0)$ . If semi-minor axis is equal to  $b$  then consider the point  $P = (\frac{11}{2}, b)$ . This point is at the equal distances from foci thus  $|PF_1| + |PF_2| = 2|PF_1| = 2\sqrt{(\frac{11}{2} - (-5))^2 + b^2} = 33$ . We come to

$$\left(\frac{11}{2} - (-5)\right)^2 + b^2 = \left(\frac{21}{2}\right)^2 + b^2 \Rightarrow b = \frac{1}{2}\sqrt{33^2 - 21^2} = 9\sqrt{2}.$$

We calculated lengths of semi-minor and semi-major axes:

$$a = \frac{33}{2}, b = 9\sqrt{2}.$$

Hence the area of the ellipse is equal to

$$S = \pi \cdot \text{vertical half-axis} \cdot \text{horizontal half-axis} = \pi ab = \frac{297\sqrt{2}}{2}.$$

**3** Let  $H$  be hyperbola in the plane  $\mathbf{E}^2$  such that it passes through the point  $P = (3, 2)$ , and its foci are at the points  $F_1 = (0, 2)$ ,  $F_2 = (0, -2)$ . Find the intersection points of the hyperbola with  $OY$  axis.

By geometrical definition of hyperbola we have that for arbitrary point  $K$  of hyperbola

$$||KF_1| - |KF_2|| = ||PF_1| - |PF_2|| = |\sqrt{(3-0)^2 + (2-2)^2} - \sqrt{(3-0)^2 + (2-(-2))^2}| = |3-5| = 2. \blacksquare$$

If the hyperbola intersects  $OY$  axis at the point  $(0, y)$  then

$$||KF_1| - |KF_2|| = 2 = ||y-2| - |y-(-2)|| = ||y-2| - |y+2||.$$

This means that  $|y-2| - |y+2| = \pm 2$ .

$$1\text{-st case } |y-2| - |y+2| = 2 \Rightarrow y = -1$$

2-nd case  $|y - 2| - |y + 2| = -2 \Rightarrow y = 1$

Hyperbola intersects  $OY$  axis at the points  $(0, \pm 1)$ .

This hyperbola does not intersect the axis  $OX$  because the points of this axis are on the equal distance from foci of the hyperbola.

4 Consider in the plane the curves  $C_1$ ,  $C_2$  and  $C_3$  which are given in some Cartesian coordinates  $(x, y)$  by equations  $C_1: 4x^2 + 4x + y^2 = 0$ ,  $C_2: 4x^2 + 4x - y^2 = 0$ ,

$C_3: 4x^2 + 4x + y = 0$ .

Show that  $C_1$  is ellipse,  $C_2$  is hyperbola, and  $C_3$  is parabola

We have

$$C_1: 4x^2 = 4x + y^2 = 0 \Leftrightarrow 4x^2 = 4x + 1 + y^2 = 1 \Leftrightarrow 4\left(x + \frac{1}{2}\right)^2 + y^2 = 1.$$

Choose new Cartesian coordinates  $\begin{cases} x + \frac{1}{2} = x' \\ y = y' \end{cases}$  we come to

$$C_1: 4x^2 = 4x + y^2 = 0 \Leftrightarrow 4x^2 = 4x + 1 + y^2 = 1 \Leftrightarrow 4\left(x + \frac{1}{2}\right)^2 + y^2 = 1 \Leftrightarrow y'^2 + 4x'^2 = 1.$$

This is canonical equation of ellipse. ( $x'$  is the second coordinate, and  $y'$  is the first)

Now consider  $C_2$ :

$$C_2: 4x^2 = 4x - y^2 = 0 \Leftrightarrow 4x^2 = 4x + 1 - y^2 = 1 \Leftrightarrow 4\left(x + \frac{1}{2}\right)^2 - y^2 = 1.$$

Choose new Cartesian coordinates  $\begin{cases} x + \frac{1}{2} = x' \\ y = y' \end{cases}$  we come to

$$C_2: 4x^2 = 4x - y^2 = 0 \Leftrightarrow 4x^2 = 4x + 1 - y^2 = 1 \Leftrightarrow 4\left(x + \frac{1}{2}\right)^2 - y^2 = 1 \Leftrightarrow 4x'^2 - y'^2 = 1.$$

We see that this is canonical equation of hyperbola

Now consider  $C_3$ :

$$C_3: 4x^2 + 4x + y = 0.$$

This is equation of parabola. To make it canonical we have to choose Cartesian coordinates  $\tilde{x}, \tilde{y}$  such that in these coordinates the curve  $C_3$  have the appearance

$$C_3: \tilde{y}^2 = 2p\tilde{x}.$$

We have

$$\begin{aligned} \mathbf{C}_3: 4x^2 + 4x + y = 0 &\Leftrightarrow (4x^2 + 4x + 1) + (y - 1) = 0 \Leftrightarrow 4\left(x + \frac{1}{2}\right)^2 + (y - 1) = 0 \\ &\Leftrightarrow \left(x + \frac{1}{2}\right)^2 = \frac{1}{4}(1 - y) = 0 \end{aligned}$$

Choose new Cartesian coordinates

$$\begin{cases} \tilde{x} = 1 - y \\ \tilde{y} = x + \frac{1}{2} \end{cases}$$

We see that in these coordinates

$$C_3: \tilde{y}^2 = 2p\tilde{x}, \quad \text{with } p = \frac{1}{8}.$$

**5** Let  $H$  be hyperbola considered in the exercise **3**.

Consider in the plane  $\mathbf{E}^2$  the ellipse such that it passes through the foci of the hyperbola  $H$ , and its foci are at the points where hyperbola  $H$  intersects axis  $OY$ . Write down equation of this ellipse.

The foci of this ellipse are on the axis  $OY$  and its centre is at the origin, hence the equation of this ellipse is

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1, (a > b)$$

The ellipse passes through the foci of the hyperbola the points  $(0, \pm 2)$ . Hence  $\frac{y^2}{a^2} = 1$  for  $x = 0$ , i.e.  $a = 2$ . Consider the point  $(b, 0)$  on this ellipse. The sum of distances from this point to foci is equal to

$$\sqrt{b^2 + 1^2} + \sqrt{b^2 + (-1)^2} = 2\sqrt{b^2 + 1} = 2a = 4. \Rightarrow b = 3.$$

Hence the equation of the ellipse is

$$\frac{y^2}{4} + \frac{x^2}{3} = 1.$$

**6** The ellipse  $C$  on the plane  $\mathbf{E}^2$  has foci at the vertices  $A = (-1, -1)$  and  $C = (1, 1)$  of the square  $ABCD$ , and it passes through the other two vertices  $B = (-1, 1)$  and  $D = (1, -1)$  of this square.

Find new Cartesian coordinates  $(u, v)$  (express them via initial coordinates  $(x, y)$ ) such that the ellipse  $C$  has canonical form  $C$ :  $\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1$  in these coordinates.

Write down the equation of ellipse  $C$  in initial Cartesian coordinates  $(x, y)$

Calculate the area of this ellipse.

Consider new Cartesian coordinates  $OX'Y'$  such that axis  $OX'$  goes from focus  $A = (-1, -1)$  to the focus  $C = (1, 1)$ . In these new Cartesian coordinates equation of the ellipse is

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1$$

The foci are on the axis  $AC$ , the semi-minor axis of the ellipse:  $b = |OB| = \sqrt{2}$ ,  $b = \sqrt{2}$ . To calculate  $a$  use the fact that  $2a$  is equal to the sum of the distances from the arbitrary point to the foci:  $2a = |BC| + |BA| = 2|BA| = 4$ , and  $a = 2$ . We see that equation of the ellipse in new Cartesian coordinates is

$$\frac{x'^2}{4} + \frac{y'^2}{2} = 1$$

Now find the relation between new coordinates  $(x', y')$  and the genuine coordinates  $(x, y)$ .

The line  $AC$  is the bisectrix of the angle  $XOY$ . Hence we come from Cartesian coordinates  $OXY$  to Cartesian coordinates  $OX'Y'$  by rotation on the angle  $\frac{\pi}{4}$ : Having in mind that  $\cos \frac{\pi}{4} = \frac{\sin \pi}{4} = \frac{\sqrt{2}}{2}$ , and that the new coordinates  $(x', y')$  of the point  $A$  are  $(-\sqrt{2}, 0)$ , and the new coordinates  $(x', y')$  of the point  $C$  are  $(0, \sqrt{2})$  we see that

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{cases} x' = \frac{x+y}{\sqrt{2}} \\ y' = \frac{-x+y}{\sqrt{2}} \end{cases}$$

Thus equation of ellipse is

$$\frac{x'^2}{4} + \frac{y'^2}{2} = 1 \Leftrightarrow \frac{1}{4} \left( \frac{x+y}{\sqrt{2}} \right)^2 + \frac{1}{2} \left( \frac{-x+y}{\sqrt{2}} \right)^2 = 1 \Leftrightarrow 3x^2 + 3y^2 - 2xy = 8.$$

Area of the ellipse is equal to the product of semi-major axis on semi-minor axis and on  $\pi$ :

$$S = \pi\sqrt{2} \cdot 2 = 2\pi\sqrt{2}.$$

**7** Consider a curve defined in Cartesian coordinates  $(x, y)$  by the equation

$$C: \quad px^2 + py^2 + 2xy + \sqrt{2}(x + y) = 0,$$

where  $p$  is a parameter.

How looks this curve

if  $p > 1$ ? if  $p = 1$ ? if  $-1 < p < 1$ ? if  $p = -1$ ? if  $p < -1$ ?

Find an affine transformation

$$\begin{cases} x = au + bv + e \\ y = cu + dv + f \end{cases}, \quad \left( \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0 \right) \quad (1)$$

which transforms this curve to the circle  $u^2 + v^2 = 1$  in the case if  $p > 1$

We have

$$C: px^2 + py^2 + 2xy + \sqrt{2}(x + y) = 0.$$

Rotate coordinates on the angle  $\frac{\pi}{4}$ :

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos \frac{\pi}{4} & \sin \frac{\pi}{4} \\ -\sin \frac{\pi}{4} & \cos \frac{\pi}{4} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \quad \begin{cases} x = \frac{u+v}{\sqrt{2}} \\ y = \frac{-u+v}{\sqrt{2}} \end{cases}$$

(Compare with changing of coordinates in the previous exercise).

We have in new Cartesian coordinates  $(u, v)$

$$\begin{aligned} C: px^2 + py^2 + 2xy + \sqrt{2}(x + y) &= p \left( \frac{u+v}{\sqrt{2}} \right)^2 + p \left( \frac{-u+v}{\sqrt{2}} \right)^2 + 2 \left( \frac{u+v}{\sqrt{2}} \right) \left( \frac{-u+v}{\sqrt{2}} \right) + 2u = \\ &= (p-1)u^2 + (p+1)v^2 + 2u = 0. \end{aligned}$$

1)  $p > 1$

$$C: (p-1)u^2 + (p+1)v^2 + 2u = (p-1) \left( u + \frac{1}{p+1} \right)^2 + (p-1)v^2 = \frac{1}{p-1}$$

This is an ellipse

2)  $p = 1$

$$(p-1)u^2 + (p+1)v^2 + 2u = 0 = 2V^2 + 2U = 0.$$

This is a parabola.

3)  $-1 < p < 1$

$$C: (p-1)u^2 + (p+1)v^2 + 2u = (p+1)v^2 - (1-p) \left( u + \frac{1}{p+1} \right)^2 = \frac{1}{p-1}$$

This is hyperbola.

4)  $p = -1$

$$(p-1)u^2 + (p+1)v^2 + 2u = 2(u - u^2) = 2u(u-1).$$

two parallel lines

5)  $p < -1$  it is again ellipse:

$$C: (p-1)u^2 + (p+1)v^2 + 2u = (p-1) \left( u + \frac{1}{p+1} \right)^2 + (p-1)v^2 = \frac{1}{p-1}$$

it is again an ellipse.

Consider the following transformation from Cartesian coordinates  $u, v$  to new coordinates  $u', v'$ :

$$\begin{cases} u' = \sqrt{p-1}u \\ v' = \sqrt{p+1}v \end{cases}.$$

In these coordinates (they are not Cartesian coordinates!) the curve  $C$  for will become a circle.

8\* (pursuit problem) Consider two point in the plane  $\mathbf{E}^2$ ,  $A$ , and  $B$ . Let point  $A$  starts moving at the origin, and moves along  $OY$  with constant velocity  $v$ :  $\begin{cases} x = 0 \\ y = vt \end{cases}$ .

Let point  $B$  starts moving at the point  $(L, 0)$ , its speed is equal to  $v$ , and velocity vector is directed on the particle  $A$ , i.e. at any moment of time the particle  $B$  moves in the direction of the segment  $BA$  with the same speed  $v$ .

Of course the particle  $B$  never will reach the particle  $A$  because their speeds are the same. On the other hand the particle  $B$  asymptotically will be tended to vertical axis. What is the distance between these particles at  $t \rightarrow \infty$ ?

Hint: Consider the reference frame in which particle  $A$  is not moved, i.e. consider coordinates  $\begin{cases} x' = x \\ y' = y + vt \end{cases}$ .

Show that in these coordinates the trajectory of particle  $B$  will be a parabola.

See my homepage the very last etude on Geometry.