Introduction to Geometry

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1 Preliminary notes. Lines and planes in Euclidean space

1.1 n-dimensional Euclidean space.

Recalling of vector space. Vector space V on real numbers is a set of vectors with operations "$+$" (addition of vectors) and "$\cdot$" (multiplication of vector on real number, sometimes called coefficients, scalars). These operations obey the axioms For an arbitrary vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$, $\mathbf{a}+(\mathbf{b}+\mathbf{c})=(\mathbf{a}+\mathbf{b})+\mathbf{c}$, existence of zero: O: $0+\alpha=\alpha$, If $\lambda, \mu \in \mathbf{R}$ then $(\lambda+\mu)\mathbf{a}=\lambda\mathbf{a}+\mu\mathbf{a}$, $(\lambda\mu)\mathbf{a}=\lambda(\mu\mathbf{a})$ and $\lambda(\mathbf{a}+\mathbf{b})=\lambda\mathbf{a}+\lambda\mathbf{b}$.

Basis in vector space. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be an ordered set of vectors in the vector space V such that for any vector $\mathbf{a} \in V$ there exists an unique ordered n-tuple $(a^1, \dots, a^n)$ of coefficients such that $\mathbf{a} = a^1\mathbf{e}_1 + \dots + a^n\mathbf{e}_n$. Then vector space V is finite-dimensional, n is called *dimension* of vector space V, the ordered set $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ is called a *basis* of vector space V. A basis of V is a set of linear independent vectors which spans the vector space V.

A basic example of n-dimensional vector space over real numbers is a space of ordered n-tuples of real numbers. If $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ are two vectors, $\mathbf{x} = (x^1, \dots, x^n)$, $\mathbf{y} = (y^1, \dots, y^n)$ then $\mathbf{x} + \mathbf{y} = (x_1 + y_1, \dots, x_n + y_n)$ and multiplication on scalars is defined as $\lambda \cdot (x^1, \dots, x^n) = (\lambda x^1, \dots, \lambda x^n)$ $(\lambda \in \mathbf{R})$.

One can consider set of ordered n-tuples in $\mathbf{R}^n$ as the set of points. Two points $\mathbf{a}, \mathbf{b} \in \mathbf{R}^n$ define a vector: if $\mathbf{a} = (a^1, \dots, a^n)$, $\mathbf{b} = (b^1, \dots, b^n)$, then the vector $\mathbf{ab}$ attached to the point $\mathbf{a}$ has coordinates $= (b^1 - a^1, b^2 - a^2 \dots, b^n - a^n)$ [1].

Consider vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n \in \mathbf{R}^n$:

$$\begin{array}{rl} \mathbf{e}_1 = & (1,0,0\dots,0,0) \\ \mathbf{e}_2 = & (0,1,0\dots,0,0) \\ \dots & \quad\quad \dots \\ \mathbf{e}_n = & (0,0,0\dots,0,1) \end{array} \tag{1.1}$$

One can see that the ordered set $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a basis in $\mathbf{R}^n$. For every vector $\mathbf{x}$ we have unique expansion

$$\mathbf{x} = \sum_{i=1}^{n} x^i \mathbf{e}_i, \qquad \text{(we will use sometimes condensed notations } \mathbf{x} = x^i\mathbf{e}_i)$$

Euclidean space

In the Euclidean space one have additional structure: *scalar product of vectors.* Scalar product in a vector space V is a function $(\mathbf{x}, \mathbf{y})$ on a pair of vectors which takes real values and satisfies the the following conditions:

$$\begin{array}{c} (\mathbf{x}, \mathbf{y}) = (\mathbf{y}, \mathbf{x}) \quad \text{(symmetricity condition)} \\ (\lambda\mathbf{x} + \mu\mathbf{x}', \mathbf{y}) = \lambda(\mathbf{x}, \mathbf{y}) + \mu(\mathbf{x}', \mathbf{y}) \quad \text{(linearity condition)} \\ (\mathbf{x}, \mathbf{x}) \geq 0 \ , (\mathbf{x}, \mathbf{x}) = 0 \Leftrightarrow \mathbf{x} = 0 \quad \text{(non-degeneracy condition)} \end{array} \tag{1.2}$$

[1] $\mathbf{R}^n$ considered as a set of points is called affine space

A function $B(\mathbf{x}, \mathbf{y})$ on pair of vectors which satisfies first condition is called *bilinear form* on vector space. Bilinear form $B(\mathbf{x}, \mathbf{y})$ which satisfies the second condition is called *symmetric bilinear form.* So scalar product is nothing but symmetric bilinear form on vectors which is positively deifined ($(\mathbf{x}, \mathbf{x}) \geq 0$) and is non-degenerate ($(\mathbf{x}, \mathbf{x}) = 0 \Rightarrow \mathbf{x} = 0$).

Definition Euclidean space is vector space equipped with scalar product.

For example one can consider $\mathbf{R}^n$ as Euclidean space provided by the scalar product

$$(\mathbf{x}, \mathbf{y}) = x^1 y^1 + \cdots + x^n y^n \tag{1.3}$$

Exercise a) Check that it is indeed scalar product. b) Show that operation $(\mathbf{x}, \mathbf{y}) = x_1 y_1 + x_2 y_2 - x_3 y_3$ *does not define scalar product* in $\mathbf{R}^3$.

One can see that for scalar product (1.3) and for the basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ defined by the relation (1.1) the following relations hold:

$$(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \tag{1.4}$$

Definition The basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ in n-dimensional Euclidean space which obeys the conditions (1.4) is called *orthonormal basis.*

One can prove that every (finite-dimensional) Euclidean space possesses orthonormal basis. Later by default we consider only orthonormal bases in Euclidean spaces; respectively scalar product will be defined by the formula (1.3).

The scalar product of vector on itself defines the *length of the vector*:

$$\text{Length of the vector } \mathbf{x} = ||\mathbf{x}|| = \sqrt{(\mathbf{x}, \mathbf{x})} = \sqrt{(x^1)^2 + \cdots + (x^n)^2} \tag{1.5}$$

If we consider Euclidean space $\mathbf{E^n}$ as the set of points then the distance between two points $\mathbf{x}, \mathbf{y}$ is the length of corresponding vector:

$$\text{distance between points } \mathbf{x}, \mathbf{y} = |\mathbf{x} - \mathbf{y}| = \sqrt{(y^1 - x^1)^2 + \cdots + (y^n - x^n)^2} \tag{1.6}$$

Geometrical properties of scalar product

We recall very important formula how scalar (inner) product is related with the angle between vectors:

$$(\mathbf{x}, \mathbf{y}) = x^1 y^1 + x^2 y^2 = |\mathbf{x}||\mathbf{y}| \cos \varphi \tag{1.7}$$

where φ is an angle between vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbf{E}^2$.

This formula is valid also in the three-dimensional case and any n-dimensional case for $n \geq 1$. It gives as a tool to calculate angle between two vectors:

$$(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^2 + \cdots + x^ny^n = |\mathbf{x}||\mathbf{y}| \cos\varphi \tag{1.8}$$

In particulary it follows from this formula that

$$\begin{gathered}\textit{angle between vectors } \mathbf{x}, \mathbf{y} \textit{ is acute if scalar product } (\mathbf{x}, \mathbf{y}) \textit{ is positive} \\ \textit{angle between vectors } \mathbf{x}, \mathbf{y} \textit{ is obtuse if scalar product } (\mathbf{x}, \mathbf{y}) \textit{ is negative} \\ \textit{vectors } \mathbf{x}, \mathbf{y} \textit{ are perpendicular if scalar product } (\mathbf{x}, \mathbf{y}) \textit{ is equal to zero}\end{gathered} \tag{1.9}$$

$$|\mathbf{x}| = \sqrt{(\mathbf{x}, \mathbf{x})} \tag{1.10}$$

Remark Geometrical intuition tells us that cosinus of the angle between two vectors has to be less or equal to one and it is equal to one if and only if vectors $\mathbf{x}, \mathbf{y}$ are collinear. Comparing with (1.8) we come to the inequality:

$$\begin{gathered}(\mathbf{x}, \mathbf{y})^2 = \left(x^1y^1 + \cdots + x^ny^n\right)^2 \leq ((x^1)^2 + \cdots + (x^n)^2)\,((y^1)^2 + (\cdots + (y^n)^2) = (\mathbf{x}, \mathbf{x})(\mathbf{y}, \mathbf{y}) \\ \text{and}(\mathbf{x}, \mathbf{y})^2 = (\mathbf{x}, \mathbf{x})(\mathbf{y}, \mathbf{y}) \quad \text{if vectors are colienar, i.e. } x^i = \lambda y^i\end{gathered} \tag{1.11}$$

This is famous Cauchy–Buniakovsky–Schwarz inequality, one of most important inequalities in mathematics. (See for more details Homework 1)

One can consider different orthonormal bases in $\mathbf{E}^n$.

If $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\}$ are two orthonormal bases then $(\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n) = (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)T$ where T is $n \times n$ transition matrix:

$$(\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n) = (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n) \begin{pmatrix} a_{11} & a_{12} \ldots & a_{1n} \\ a_{21} & a_{22} \ldots & a_{2n} \\ a_{21} & a_{22} \ldots & a_{2n} \\ \ldots & \ldots\ldots & \ldots \\ a_{(n-1)1} & a_{(n-1)2} \ldots & a_{(n-1)n} \\ a_{n1} & a_{n2} \ldots & a_{nn} \end{pmatrix} \tag{1.12}$$

The condition that new basis is orthonormal basis too implies that $T^tT = I$ (unity matrix), i.e. T is *orthogonal* matrix.

Example Consider 2-dimensional Euclidean space $\mathbf{E}^2$ with orthonormal basis $\{\mathbf{e}, \mathbf{f}\}$: $(\mathbf{e}, \mathbf{e}) = (\mathbf{f}, \mathbf{f}) = 1$ (i.e. $|\mathbf{e}| = |\mathbf{f}| = 1$) and $(\mathbf{e}, \mathbf{f}) = 0$ (i.e. vectors

$\mathbf{e}, \mathbf{f}$ are orthogonal). Let $\{\mathbf{e}', \mathbf{f}'\}$ be a new basis:

$$(\mathbf{e}', \mathbf{f}') = (\mathbf{e}, \mathbf{f})T = (\mathbf{e}, \mathbf{f}) \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \text{ i.e. } \mathbf{e}' = \alpha\mathbf{e} + \gamma\mathbf{f},\ \mathbf{f}' = \beta\mathbf{e} + \delta\mathbf{f}$$

new basis is orthonormal basis also,$(\mathbf{e}', \mathbf{e}') = (\mathbf{f}', \mathbf{f}') = 1$ and $(\mathbf{e}', \mathbf{f}') = 0$,i.e. transition matrix is an orthogonal matrix:

$$\begin{aligned} 1 = (\mathbf{e}', \mathbf{e}') &= (\alpha\mathbf{e} + \gamma\mathbf{f}, \alpha\mathbf{e} + \gamma\mathbf{f}) = \alpha^2 + \gamma^2 = 1 \\ 0 = (\mathbf{e}', \mathbf{f}') &= (\alpha\mathbf{e} + \gamma\mathbf{f}, \beta\mathbf{e} + \delta\mathbf{f}) = \alpha\beta + \gamma\delta = 0 \\ 0 = (\mathbf{f}', \mathbf{e}') &= (\beta\mathbf{e} + \delta\mathbf{f}, \alpha\mathbf{e} + \gamma\mathbf{f}) = \alpha\beta + \gamma\delta = 0 \\ 1 = (\mathbf{f}', \mathbf{f}') &= (\beta\mathbf{e} + \delta\mathbf{f}, \beta\mathbf{e} + \delta\mathbf{f}) = \beta^2 + \delta^2 = 1 \end{aligned}$$

Or in matrix notations: The matrix $T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is orthogonal matrix if and only if

$$T^t T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^t \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha^2 + \gamma^2 & \alpha\beta + \gamma\delta \\ \alpha\beta + \gamma\delta & \beta^2 + \delta^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.13}$$

Find orthogonal transition matrices. Consider the matrix $T_\varphi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$. In this case the new basis is:

$$(\mathbf{e}', \mathbf{f}') = (\mathbf{e}, \mathbf{f})T_\varphi = (\mathbf{e}, \mathbf{f}) \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = \begin{pmatrix} \cos\varphi\,\mathbf{e} + \sin\varphi\,\mathbf{f} \\ -\sin\varphi\,\mathbf{e} + \cos\varphi\,\mathbf{f} \end{pmatrix} \tag{1.14}$$

One can see that that new basis $\{\mathbf{e}', \mathbf{f}'\}$ is orthogonormal basis too and T rotates the basis $(\mathbf{e}, \mathbf{f})$ on the angle φ (see Homework 1). One can also explicitly check that the matrix T_φ satisfies conditions (1.13).

Not every orthogonal matrix is the rotation matrix. E.g. the $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ reflects the basis: $(\mathbf{e}_1, \mathbf{e}_2) \mapsto (-\mathbf{e}_1, \mathbf{e}_2)$. This is not rotation. On the other hand one can prove that an arbitrary 2×2 orthogonal matrix T is rotation if and only if $\det T = 1$. In other words orthogonal transformation which preserves orientation is rotation. This implies that an arbitrary orthogonal transition matrix T is either rotation (if $\det T = 1$) or reflection (if $\det T = -1$).

Let (x, y) be components of the vector $\mathbf{a}$ in the basis $(\mathbf{e}, \mathbf{f})$, and (x', y') be components of the vector $\mathbf{a}$ in the rotated basis $\{\mathbf{e}', \mathbf{f}'\}$.

Then it follows from (1.14) that

$$\mathbf{a} = x'\mathbf{e}' + y'\mathbf{f}' = (\mathbf{e}', \mathbf{f}') \begin{pmatrix} x' \\ y' \end{pmatrix} = (\mathbf{e}, \mathbf{f}) T_\varphi \begin{pmatrix} x' \\ y' \end{pmatrix} = (\mathbf{e}, \mathbf{f}) \begin{pmatrix} x \\ y \end{pmatrix} .$$

Hence

$$\begin{pmatrix} x \\ y \end{pmatrix} = T_\varphi \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x'\cos\varphi - y'\sin\varphi \\ x'\sin\varphi + y'\cos\varphi \end{pmatrix} \tag{1.15}$$

and respectively

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = T_{-\varphi} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\varphi + y\sin\varphi \\ -x\sin\varphi + y\cos\varphi \end{pmatrix} \tag{1.16}$$

because $T_\varphi^{-1} = T_{-\varphi}$.

1.2 Lines in $\mathbf{E}^2$

Equation of the line

$$ax + by = c \tag{1.17}$$

It intersects with abscissa (x-axis) at the point $(c/a, 0)$ and with ordinate (y-axis) at the point $(0, c/b)$. It is parallel to x- line if $a = 0$ and it is parallel to y-axis if $b = 0$. If point x_0, y_0 belongs to the line (1.17) then $ax_0 + by_0 = c$. Equation above can be rewritten in the follwoing way:

$$a(x - x_0) + b(y - y_0) = 0\,. \tag{1.18}$$

The line l_{AB} which passes points A, B, $A = (x_0, y_0)$ and $B = (x_1, y_1)$ obeys the equation

$$\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0} \tag{1.19}$$

It can be rewritten in the way:

$$y = k(x - x_0) + y_0, \text{ where coefficient } k = \tfrac{y_1 - y_0}{x_1 - x_0}, \quad (x_1 \neq x_0)\,. \tag{1.20}$$

Coefficient k is equal to the tangent of the angle between x-axis and the line l_{AB}. (If $x_1 = x_0$ then line l_{AB} is parallel to the y-axis. Its equation is $x = x_1$.) (Sometimes it is called "slop" or "gradient")

Parametric equation of the line

Write down an equation of the line which passes via point $\mathbf{r}_0 = (x_0, y_0)$ and has direction vector $\mathbf{v} = (v^1, v^2)$, i.e. vector $\mathbf{v}$ is attached at the point $\mathbf{r}_0$ and goes along the line. It is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} \text{ or in components } \begin{cases} x = x_0 + tv_x \\ y = y_0 + tv_y \end{cases} \tag{1.21}$$

Here t is a parameter. It takes an arbitrary real values. The equation above defines the line spanned by the direction vector $\mathbf{v}$ attached at the point $\mathbf{r}_0$. We say that the equation (1.21) gives *parametric equation of the line.* If we interpret the parameter t as a "time" then direction vector $\mathbf{v}$ can be interpreted as a "velocity" vector. If we exclude parameter t from these equations we come to standard (1.17)-type equation of the line. Note that sometimes it is convenient to choose direction vector $\mathbf{v}$ having unit length.

Normal equation of the line and distance between point and the line

Consider line l given by the equation (1.17) (or equation(1.18)). It is easy to see that vector $\mathbf{N}_{\text{perpend}} = (a, b)$ is orthogonal to this line. Indeed taking two arbitrary points $\mathbf{r} = (x, y)$ and $\mathbf{r}_0 = (x_0, y_0)$ on the line l we see that equation(1.18) $(a(x - x_0) + b(y - y_0) = 0)$, means that scalar product of the vector $\mathbf{N}_{\text{perpend}}$ with the vector $\mathbf{r} - \mathbf{r}_0$ which is directed along the line is equal to zero: $(\mathbf{N}_{\text{perpend}}, \mathbf{r} - \mathbf{r}_0) = 0$. Hence vector $\mathbf{N}_{\text{perpend}} = (a, b)$ is perpendicular (normal) to the line l. Consider unit vector $\mathbf{n}$ collinear to $\mathbf{N}$:

$$\mathbf{n} = \frac{\mathbf{N}}{|\mathbf{N}|} = (n_x, n_y) = \left(\frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}}\right) \tag{1.22}$$

and rewrite equation for the line in the way: $a'x + b'y = c'$ with $a'^2 + b'^2 = 1$, $(a' = \frac{a}{\sqrt{a^2+b^2}}, b' = \frac{b}{\sqrt{a^2+b^2}}, c' = \frac{c}{\sqrt{a^2+b^2}})$. This equation can be rewritten in a more geometric way: a point $\mathbf{r} = (x, y)$ belongs to the line l if and only if $(\mathbf{n}, \mathbf{r}) = c'$. On the other hand If $\mathbf{r}_0$ is an arbitrary point on the line then $(\mathbf{n}, \mathbf{r}_0) = c'$. Hence we can rewrite the equation in the following way:

$$(\mathbf{n}, \mathbf{r}) = (\mathbf{n}, \mathbf{r}_0), \text{ i.e. } (\mathbf{n}, \mathbf{r} - \mathbf{r}_0) = 0, \tag{1.23}$$

i.e.all vectors starting and ending on the line are orthogonal to the vector $\mathbf{n}$.

Definition We say that an equation $ax+by = c$ or $a(x-x_0)+b(y-y_0) = 0$ is *normal equation* of the line if $a^2 + b^2 = 1$.

The formula (1.23) for $\mathbf{n} = (a, b)$ gives geometric meaning of normal equation of the line.

Let line l be given by normal equation $ax + by = c$, $(a^2 + b^2 = 1)$ and $\mathbf{r}_1 = (x_1, y_1)$ be an arbitrary point on the plane $\mathbf{E}^2$. Our aim is to calculate the distance $\rho(\mathbf{r}_1, l)$ between the line l and the point $\mathbf{r}_1$.

Let $\mathbf{N}_{\mathbf{r}_1, l}$ be a vector which is perpendicular to the line l and which starts at the point $\mathbf{r}_1$ and ends at the line l. Length of this vector is just the distance required. Let $\mathbf{r}$ be an arbitrary point on the line l. Then $|\mathbf{N}_{\mathbf{r}_1, l}| = \pm(\mathbf{n}, \mathbf{r} - \mathbf{r}_1)$. On the other hand according (1.23) $(\mathbf{n}, \mathbf{r} - \mathbf{r}_1) = ax + by - ax_1 - by_1 = c - ax_1 - by_1$. We come to the

Proposition. *If $ax + by = c$ is a normal equation of the line l, $(a^2 + b^2 = 1)$ then distance between an arbitrary point $\mathbf{r} = (x, y)$ and a line l up to a sign is equal to $ax + by - c$:*

$$d(\mathbf{r}, l) = |ax + by - c| \qquad \text{if} \quad \mathbf{r} = (x, y) \tag{1.24}$$

1.3 Orientation in vector space

In the three-dimensional Euclidean space except scalar (inner) product, one can consider another important operation: vector product. For defining this operation we need additional structure:*orientation*.

We say that a basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ have the same orientation as the basis $\{\mathbf{a}', \mathbf{b}', \mathbf{c}'\}$ if they both obey right hand rule or if they both obey left hand rule. In the other case we say that these bases have opposite orientation.

How to make this conception more mathematical?

Consider the set of *all* bases in the given vector space V. (We can assume $V = \mathbf{R}^n$.)

If $(\mathbf{e}_1, \ldots \mathbf{e}_n)$, $(\mathbf{e}'_1, \ldots \mathbf{e}'_n)$ are two bases then one can consider the matrix T—transition matrix which transforms the old basis to the new one (see (1.12)). The transition matrix is not degenerate, i.e. determinant of this matrix is not equal to zero.

Definition Let $(\mathbf{e}_1, \ldots \mathbf{e}_n), (\mathbf{e}'_1, \ldots \mathbf{e}'_n) \in \mathbf{R}^n$ be two bases in $\mathbf{R}^n$ and T be transition matrix. We say that these two bases have the same orientation if $\det T > 0$. We say that the basis $(\mathbf{e}_1, \ldots \mathbf{e}_n), (\mathbf{e}'_1, \ldots \mathbf{e}'_n)$ has an orientation opposite to the orientation of the basis $(\mathbf{e}'_1, \ldots \mathbf{e}'_n)$ if $\det T < 0$.

It is easy to see that we come to the equivalence relation in the set of all bases: The set of all bases is a sum of two distinct subsets.

Any two bases which belong to the same subset have the same orientation. Any two bases which belong to different subsets have opposite orientation.

Definition *An orientation of a vector space is an equivalence class of bases in this vector space.*

Note that fixing any basis we fix orientation, considering the subset of all bases which have the same orientation that the given basis.

There are two orientations. Every basis has the same orientation as a given basis or opposite orientation.

Definition *An oriented vector space is a vector space equipped with orientation.*

Example Let $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ be any basis in $\mathbf{E}^3$ and $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are arbitrary three vectors in $\mathbf{R}^3$:

$$\mathbf{a} = a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z \ \mathbf{b} = b_x\mathbf{e}_x + b_y\mathbf{e}_y + b_z\mathbf{e}_z, \ \mathbf{c} = c_x\mathbf{e}_x + c_y\mathbf{e}_y + c_z\mathbf{e}_z \,.$$

Consider ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$. The transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is $T = \begin{pmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{pmatrix}$:

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{e_x}, \mathbf{e_y}, \mathbf{e_z})\mathbf{T} = (\mathbf{e_x}, \mathbf{e_y}, \mathbf{e_z}) \begin{pmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{pmatrix}$$

One can see that the ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ also is a basis, (i.e. these three vectors are linear independent) if and only if transition matrix is not degenerate, i.e.

$$\det T \neq 0 \,. \tag{1.25}$$

We see that the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ has the same orientation as the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ if

$$\det T > 0 \,. \tag{1.26}$$

We see that the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ has the orientation opposite to the orientation of the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ if

$$\det T < 0 \,. \tag{1.27}$$

Exercise Show that bases $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and $\{-\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ have opposite orientation but bases $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and $\{\mathbf{e}_y, \mathbf{e}_x, -\mathbf{e}_z\}$ have the same orientation.

Relations (1.26),(1.27) define equivalence relations in the set of bases. Orientation is equivalence class of bases. There are two orientations, every basis has the same orientation as a given basis or opposite orientation.

If two bases $\{\mathbf{e}_i\}$, $\{\mathbf{e}_{i'}\}$ have the same orientation then they can be transformed to each other by continuous transformation, i.e. there exist one-parametric family of bases $\{\mathbf{e}_i(t)\}$ such that $0 \le t \le 1$ and $\{\mathbf{e}_i(t)\}|_{t=0} = \{\mathbf{e}_i\}$, $\{\mathbf{e}_i(t)\}|_{t=1} = \{\mathbf{e}_{i'}\}$. (All functions $\mathbf{e}_i(t)$ are continuous) In the case of three-dimensional space the following statement is true (Euler Theorem): *Let* $\{\mathbf{e}_i\}, \{\mathbf{e}_{i'}\}$ $(i = 1, 2, 3)$ *be two orthonormal bases in* $\mathbf{E}^3$ *which have the same orientation. Then there exists an axis* $\mathbf{n}$ *such that basis* $\{\mathbf{e}_i\}$ *transforms to the basis* $\{\mathbf{e}_{i'}\}$ *under rotation around the axis.*

We say that Euclidean space is equipped with orientation if we consider in this space only orthonormal bases which have the same orientation.

Remark Note that in the example above we considered in $\mathbf{E}^3$ *arbitrary bases* not necessarily orthonormal bases.

1.4 Linear operator in $\mathbf{E}^3$ preservinig orientation is a rotation

Let P be a linear operator in vector space $\mathbf{R}^n$. Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ be an arbitrary basis in $\mathbf{R}^n$. Considering the action of P on basis vectors we come to vectors $(\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n)$:

$$\mathbf{e}'_1 = P(\mathbf{e}_1), \mathbf{e}'_2 = P(\mathbf{e}_2) \dots, \mathbf{e}'_n = P(\mathbf{e}_n) \tag{1.28}$$

If operator P is non-degenerate ($\det P \neq 0$) then ordered n-tuple $\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n\}$ is a basis too.

Non-degenerate linear operator maps the basis to another basis.

Definition. Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ be an arbitrary basis in $\mathbf{R}^n$. Consider the basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n\}$, where $\mathbf{e}'_i = P(\mathbf{e}_i)$ We say that non-degenerate linear operator P ($\det P \neq 0$) preserves orientation if bases $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ and $\{e'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n\}$, where $\mathbf{e}'_i = P(\mathbf{e}_i)$ have the same orientation. In this case $\det P > 0$.

We say that linear operator P changes orientation if bases $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n\}$ have opposite orientation. In this case $\det P < 0$.

It is easy to see that this definition is correct: The property of operator P to preserve orientation does not depend on choosing a basis. If bases

$\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n\}$, where $\mathbf{e}'_i = P(\mathbf{e}_i)$ have the same (opposite) orientation, then for an another basis $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_n\}$ in $\mathbf{R}^n$, the bases $\{\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_n\}$ and $\{\mathbf{f}'_1, \mathbf{f}'_2, \dots, \mathbf{f}'_n\}$, where $\mathbf{f}'_i = P(\mathbf{f}_i)$ have the same (opposite) orientation also.

In other words we say that non-degenerate linear operator P preserves orientation if it maps vectors of an arbitrary basis to the vectors of another basis which have the same orientation as an initial basis. We say that non-degenerate linear operator P changes orientation if it maps vectors of an arbitrary basis to the vectors of another basis which have the orientation opposite an orientation of initial basis.

Example Let $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ be an orthonormal basis in $\mathbf{E}^3$. Consider linear operator P such that

$$P(\mathbf{e}_x) = \mathbf{e}_y, P(\mathbf{e}_y) = -\mathbf{e}_x, P(\mathbf{e}_z) = \mathbf{e}_z\,.$$

This operator maps orthonormal basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the basis $\{\mathbf{e}_y, -\mathbf{e}_x, \mathbf{e}_z\}$ which is orthonormal too. Both bases have the same orientation. Hence the operator P is linear operator preserving orientation. In this case it is orthogonal operator, because it maps orthonormal basis to the orthonormal one. One can see that P is rotation operator: Under the action of operator P vectors in $\mathbf{E}^3$ rotate on the angle $\frac{\pi}{2}$ about the axis Oz. The vectors $\lambda\mathbf{e}_z$ collinear (proportional) to the vector $\mathbf{e}_z$. are eigenvectors of this operator: $P\mathbf{e}_z = \mathbf{e}_z$. The axis is a line spanned by the vector $\mathbf{e}_z$.

One can show that in Euclidean vector space $\mathbf{E}^3$ every orthogonal operator which preserves orientation is a rotation.

Theorem (Euler Theorem). Let P be an linear orthogonal operator in $\mathbf{E}^3$ preserving orientation. Then it is a rotation operator about some axis passing through the origin.

(The proof of this theorem see in the solutions of Homework 2, Exercise 7)

How to find an axis of rotation? Vectors which belong to axis (starting at origin) are *eigenvectors* of P. They all are proportional each other. Eigenvalue of these vectors is equal to 1— Rotation does not change the vectors which belong to axis. Hence

Claim To find an axis we have to find eigenvector of the operator P with eigenvalue 1.

In the example above vector $\mathbf{e}_z$ was the eigenvector of the operator P. Consider more interesting example:

Example Let $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ be an orthonormal basis in $\mathbf{E}^3$. Consider linear operator P such that

$$P(\mathbf{e}_x) = \mathbf{e}_z, P(\mathbf{e}_y) = -\mathbf{e}_y, P(\mathbf{e}_z) = \mathbf{e}_x\,. \tag{1.29}$$

This operator maps orthonormal basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the basis $\{\mathbf{e}_z, -\mathbf{e}_y, \mathbf{e}_x\}$ which is orthonormal too. Both bases have the same orientation. Hence the operator P is linear orthogonal operator preserving orientation. According to the Euler Theorem it is a rotation operator about an axis. Find this axis. Let vector $\mathbf{N}$ (starting at origin) belongs to the axis. Then

$$P\mathbf{N} = \mathbf{N}\,. \tag{1.30}$$

$\mathbf{N}$ is eigenvector of the operator P. Its eigenvalue is equal to 1. To find an axis of rotation (1.29) we have to find an eigenvector (1.30). It is easy to see that vector $\mathbf{e}_x + \mathbf{e}_z$ obeys the condition (1.30):

$$P(\mathbf{e}_x + \mathbf{e}_z) = \mathbf{e}_x + \mathbf{e}_z$$

We see that eigenvector of P is an arbitrary vector proportional (collinear) to the vector $\mathbf{e}_x + \mathbf{e}_z$. These vectors span the line $\lambda(\mathbf{e}_x + \mathbf{e}_y)$–axis of rotation. We see that the axis of rotation is the line spanned by the eigenvectors which is a bisectrix of the angle between Ox and Oz axis.

1.5 Vector product in $\mathbf{E}^3$

Now we give a definition of vector product of vectors in Euclidean space equipped with orientation.

Definition Vector product of two vectors $\mathbf{a} = a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z$ and $\mathbf{b} = a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z$ is a vector $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ (sometimes it is denoted by $\mathbf{c} = [\mathbf{a}, \mathbf{b}]$)such that

- vector $\mathbf{c}$ is orthogonal to the plane spanned by the vectors $\mathbf{a}$ and $\mathbf{b}$.
- length of the vector $\mathbf{c}$ is equal to the area of parallelogram formed by the vectors $\mathbf{a}$ and $\mathbf{b}$, ($|\mathbf{c}| = |\mathbf{a}||\mathbf{b}||\sin\varphi|$, where φ is an angle between vectors $\mathbf{a}, \mathbf{b}$).

- Direction of the vector $\mathbf{c}$ is defined by the condition that the triple of vectors $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ have the same orientation as a basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$.

  The last condition in particularly means that $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$, because triples $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ and $(\mathbf{b}, \mathbf{a}, \mathbf{c})$ have opposite orientation.

One can see that

$$\begin{array}{ccc} \mathbf{e}_x \times \mathbf{e}_x = 0, & \mathbf{e}_x \times \mathbf{e}_y = \mathbf{e}_z, & \mathbf{e}_x \times \mathbf{e}_z = -\mathbf{e}_y \\ \mathbf{e}_y \times \mathbf{e}_x = -\mathbf{e}_z, & \mathbf{e}_y \times \mathbf{e}_y = 0, & \mathbf{e}_y \times \mathbf{e}_z = \mathbf{e}_x \\ \mathbf{e}_z \times \mathbf{e}_x = \mathbf{e}_y, & \mathbf{e}_z \times \mathbf{e}_y = -\mathbf{e}_x, & \mathbf{e}_z \times \mathbf{e}_z = 0 \end{array} \tag{1.31}$$

Vector product obeys the following conditions:

$$\begin{array}{c} \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \quad \text{(antysymmetricity conditions)} \\ (\lambda\mathbf{a} + \mu\mathbf{b}) \times \mathbf{c}) = \lambda(\mathbf{a} \times \mathbf{c}) + \mu(\mathbf{b} \times \mathbf{c}) \quad \text{(linearity condition)} \\ ((\mathbf{a} \times \mathbf{b}) \times \mathbf{c}) + ((\mathbf{b} \times \mathbf{c}) \times \mathbf{a}) + ((\mathbf{c} \times \mathbf{a}) \times \mathbf{b}) \quad \text{(Jacoby identity)} \end{array} \tag{1.32}$$

It follows from (1.32) and linearity that for vectors $\mathbf{a} = a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z$, $\mathbf{b} = b_x\mathbf{e}_x + b_y\mathbf{e}_y + b_z\mathbf{e}_z$

$$\mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y)\mathbf{e}_x + (a_z b_x - a_x b_z)\mathbf{e}_y + (a_x b_y - a_y b_x)\mathbf{e}_z\,. \tag{1.33}$$

It is convenient to represent this formula in the following way:

$$\mathbf{a} \times \mathbf{b} = \det \begin{pmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{pmatrix} \tag{1.34}$$

Exercise Show that $\mathbf{a} \times \mathbf{b} = 0$ if and only if vectors $\mathbf{a}$ and $\mathbf{b}$ are colinear.

The vector product of two vectors is related with area of parallelogram. What about a volume of parallelepiped formed by three vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$? One can see that it is equal to scalar product of one of the vectors on the vector product of two other vectors:

$$V(\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}) = (\mathbf{a}, [\mathbf{b} \times \mathbf{c}]) = \left(a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z, \det \begin{pmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{pmatrix} \right) \tag{1.35}$$

$$= (a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z, (b_y c_z - b_z c_y)\mathbf{e}_x + (b_z c_x - b_x c_z)\mathbf{e}_y + (b_x c_y - b_y c_x)\mathbf{e}_z) =$$

$$a_x(b_y c_z - b_z c_y) + a_y(b_z c_x - b_x c_z) + a_z(b_x c_y - b_y c_x) = \begin{pmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{pmatrix}$$

We come to beautiful and useful formula:

$$(\mathbf{a}, [\mathbf{b} \times \mathbf{c}]) = \begin{pmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{pmatrix} \tag{1.36}$$

1.6 Planes and lines in $\mathbf{E}^3$

Equation of the plane in $\mathbf{E}^3$

$$ax + by + cz = d \tag{1.37}$$

It intersects with x-axis at the point $(d/a, 0, 0)$, with y-axis at the point $(0, d/b, 0)$ and with z-axis at the point $(0, 0, d/c)$. It is parallel to x-axis if $a = 0$ $(d \neq 0)$, it is parallel y-axis if $b = 0$ $(d \neq 0)$ and it is parallel to z-axis if $c = 0$ $(d \neq 0)$.

Let $\mathbf{r}_0 = (x_0, y_0, z_0)$ be an arbitrary point on the plane α defined by the equation (1.37). Then $ax_0 + by_0 + cz_0 = d$. One can rewrite the equation in the form:

$$ax+by+cz = ax_0+by_0+cz_0 = d \qquad \text{or} \quad a(x-x_0)+b(y-y_0)+c(z-z_0) = 0 \tag{1.38}$$

Geometrical meaning of this equation is following: Vector $\mathbf{N} = (a, b, c)$ is orthogonal to the vector $\mathbf{r} - \mathbf{r}_0$, where $\mathbf{r} = (x, y, z)$, $\mathbf{r}_0 = (x_0, y_0, z_0)$:

$$\mathbf{r} \in \alpha \Leftrightarrow (\mathbf{N}, \mathbf{r} - \mathbf{r}_0) = 0. \tag{1.39}$$

Normal equation of the plane

Coefficients A, B, C in the equation of the plane are defined up to a multiplier: transformation $\mathbf{N} \mapsto \lambda\mathbf{N}$, $d \mapsto \lambda d$ does not change a plane.

Definition We say that equation $Ax+By+Cz = D$ is a normal equation of the plane if $A^2 + B^2 + C^2 = 1$, i.e. if a vector $\mathbf{n} = (A, B, C)$ is unit vector.

(Compare with definition of normal equation of the line)

In the same way as for line one can prove the following Proposition

Proposition. *If $Ax + By + Cz = D$ is a normal equation of the plane α ($A^2 + B^2 + C^2 = 1$) then distance between an arbitrary point $\mathbf{r} = (x, y, z)$ and a plane α up to a sign is equal to $Ax + By + Cz - D$:*

$$d(\mathbf{r}, \alpha) = |Ax + By + Cz - D| \qquad \text{if} \quad \mathbf{r} = (x, y, z) \tag{1.40}$$

Parametric equation of plane in $\mathbf{E}^3$

Recall that line spanned by the direction vector $\mathbf{v}$ attached at the point $\mathbf{r}_0$ is given by parametric equation $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$ or in components $\begin{cases} x = x_0 + tv_x \\ y = y_0 + tv_y \end{cases}$ Analogously the plane spanned by vectors $\mathbf{a}, \mathbf{b}$ attached at the given point $\mathbf{r}_0$ is given by *parametric equation*

$$\mathbf{r}(u, v) = \mathbf{r}_0 + u\mathbf{a} + v\mathbf{b} \text{ or in components } \begin{cases} x = x_0 + ua_x + vb_x \\ y = y_0 + ua_y + vb_y \\ z = z_0 + ua_z + vb_z \end{cases} \tag{1.41}$$

where $u, v \in \mathbf{R}$ are parameters, $-\infty < u < \infty$, $-\infty < v < \infty$ If we exclude two parameters u, v from these three equations we come to one (1.37)-type equation of the plane. The following exercise is very instructive (See also homework 2)

Exercise Write down parametric and standard equation of the plane which passes via three given points.

Recall analogous exercise for line in $\mathbf{E}^2$ (see (1.19)): Line passing via two points $\mathbf{r}_0 = (x_0, y_0)$, $\mathbf{r}_1 = (x_1, y_1)$ has the equation:

$$\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_1},$$

Geometrical meaning is obvious: vector $\mathbf{r} - \mathbf{r}_0$ is colinear to the vector $\mathbf{r}_1 - \mathbf{r}_0$. To write parametric equation of this line note that vector $\mathbf{N} = (x_1 - x_0, y - y_0)$ attached at the point $\mathbf{r}_0 = (x_0, y_0)$ is direction vector. Hence parametric equation is $x(t) = x_0 + t(x_1 - x_0), y(t) = y_0 + t(y_1 - y_0)$.

Now try to write the equation of the plane α such that points $\mathbf{r}_1 = (x_1, y_1, z_1)$, $\mathbf{r}_2 = (x_2, y_2, z_2)$, $\mathbf{r}_3 = (x_3, y_3, z_3)$ belong to the plane.

First write parametric equation. The plane α which we have to define is spanned by the vectors $\mathbf{a} = \mathbf{r}_2 - \mathbf{r}_1$, $\mathbf{b} = \mathbf{r}_3 - \mathbf{r}_1$ attached at the point $\mathbf{r}_1$.

Hence we have parametric equation:

$$\mathbf{r}(u,v)=\mathbf{r}_1+u(\mathbf{r}_1-\mathbf{r}_1)+v(\mathbf{r}_3-\mathbf{r}_1); \qquad \begin{cases} x=x_1+u(x_2-x_1)+v(x_3-x_1) \\ y=y_1+u(y_2-y_1)+v(y_3-y_1) \\ z=z_1+u(z_2-z_1)+v(z_3-z_1) \end{cases} \tag{1.42}$$

Now how to write down (1.37)-type equation of this plane. Using "brute force" we can exclude parameters u, v from these equations. But it is not beautiful. Do it also in another way.

Consider vector $\mathbf{N}=\mathbf{a}\times\mathbf{b}=(\mathbf{r}_2-\mathbf{r}_1)\times(\mathbf{r}_3-\mathbf{r}_1)$. This vector is orthogonal to the plane α. Hence the point $\mathbf{r}$ belongs to the plane if and only if the vector $\mathbf{r}-\mathbf{r}_1$ is orthogonal to the vector $\mathbf{N}$. Translate this geometrical sentence on the language of formulae. We come to the equation:

$$\mathbf{r}\in\alpha \iff 0=(\mathbf{r}-\mathbf{r}_1,\mathbf{N})=(\mathbf{r}-\mathbf{r}_1,(\mathbf{r}_2-\mathbf{r}_1)\times(\mathbf{r}_3-\mathbf{r}_1)]$$

Using the formula (1.36) we see that

$$\mathbf{r}\in\alpha \iff \det\begin{pmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{pmatrix}=0 \tag{1.43}$$

This is an equation of the plane α.

You see, that two equations, parametric equation (1.42) and equation (1.43) describe the same plane [2]

Line in $\mathbf{E}^3$

Line l is $\mathbf{E}^3$ is uniquely defined by two points $\mathbf{r}_0,\mathbf{r}_1$ or by a point $\mathbf{r}_0$ and direction vector $\mathbf{v}=(v_x,v_y,v_z)$. Its parametric equation is

$$\mathbf{r}(t)=\mathbf{r}_0+t\mathbf{N}, \quad \text{in coordinates:} \quad \begin{cases} x(t)=x_0+tv_x \\ y(t)=y_0+tv_y \\ z(t)=x_0+tv_z \end{cases} \tag{1.44}$$

[2]Using little bit more advanced methods we see that in parametric equation (1.42) plane α is the image of some operator A from two-dimensional space of parameters to three dimensional space $\mathbf{E}^3$. On the other hand image of operator A is orthogonal to the kernel of adjoint operator A^+. Kernel of adjoint operator is spanned by the vector $\mathbf{N}=\mathbf{a}\times\mathbf{b}=(\mathbf{r}_2-\mathbf{r}_1)\times(\mathbf{r}_3-\mathbf{r}_1)$. The equation (1.42) defines plane as image of operator A. The equation (1.43) defines plane as subspace orthogonal to the kernel of adjoint operator A^+.

or if we exclude t we come to simultaneous

$$\frac{x-x_0}{N_x} = \frac{y-y_0}{N_y} = \frac{z-z_0}{N_z} \tag{1.45}$$

equations: If point $\mathbf{r}_1 = (x_1, y_1, z_1)$ belongs to the line l , then one can consider $\mathbf{N} = (x_1 - x_0, y_1 - y_0, z_1 - z_0)$ as direction vector of the line.

Sometimes it is convenient to consider line as intersection of two planes, respectively define line via two simultaneous equations defining planes.

2 Differential forms in $\mathbf{E}^2$ and $\mathbf{E}^3$

2.1 Tangent vectors, curves, velocity vectors on the curve

Tangent vector is a vector $\mathbf{v}$ applied at the given point $\mathbf{p} \in \mathbf{E}^3$.

The set of all tangent vectors at the given point $\mathbf{p}$ is a vector space. It is called tangent space of $\mathbf{E}^3$ at the point $\mathbf{p}$ and it is denoted $T_{\mathbf{p}}(\mathbf{E}^3)$.

One can consider *vector field* on $\mathbf{E}^3$, i.e.a function which assigns to every point $\mathbf{p}$ vector $\mathbf{v}(\mathbf{p}) \in T_{\mathbf{p}}(\mathbf{R}^3)$.

It is instructive to study the conception of tangent vectors and vector fields on the curves and surfaces embedded in $\mathbf{E}^3$. We begin with curves.

A curve in $\mathbf{E}^n$ with parameter $t \in (a, b)$ is a continuous map

$$C\colon \qquad (a,b) \to \mathbf{E}^n \qquad \mathbf{r}(t) = (x^1(t), \dots, x^n(t)), \quad a < t < b \tag{2.1}$$

For example consider in $\mathbf{E}^2$ the curve

$$C\colon \qquad (0, 2\pi) \to \mathbf{E}^2 \qquad \mathbf{r}(t) = (R\cos t, R\sin t),\ 0 \le t < 2\pi \tag{2.2}$$

The image of this curve is the circle of the radius R. It can be defined by the equation:

$$x^2 + y^2 = R^2 \tag{2.3}$$

To distinguish between curve and its image we say that curve C in (2.1) is *parameterised* curve or *path*. We will call the image of the curve *unparameterised curve* (see for details the next subsection). It is very useful to think about parameter t as a "time" and consider parameterised curve like *point moving along a curve*. Unparameterised curve is the trajectory of the moving point. The using of word "curve" without adjective "parameterised" or "nonparameterised" sometimes is ambiguous.

Vectors tangent to curve—velocity vector

Let $\mathbf{r}(t) \quad \mathbf{r} = \mathbf{r}(t)$ be a curve in $\mathbf{E}^n$.
Velocity $\mathbf{v}(t)$ it is the vector

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \big(\dot{x}^1(t), \dots, \dots \dot{x}^n(t)\big) = \big(v^1(t), \dots, v^n(t)\big) \tag{2.4}$$

in $\mathbf{E}^n$. Velocity vector is *tangent vector to the curve.*

Let $C\colon \mathbf{r} = \mathbf{r}(t)$ be a curve and $\mathbf{r}_0 = \mathbf{r}(t_0)$ any given point on it. Then the set of all vectors tangent to the curve at the point $\mathbf{r}_0 = \mathbf{r}(t_0)$ is one-dimensional vector space $T_{\mathbf{r}_0}C$. It is linear subspace in vector space $T_{\mathbf{r}_0}C$. The points of the tangent space $T_{\mathbf{r}_0}C$ are the points of tangent line.

In the next section we will return to curves and consider them in more details.

Remark We consider only smooth and regular curves. Curve $\mathbf{r}(t) = (x^1(t), \dots, x^n(t))$ is called smooth if all functions $x^i(t)$, $(i = 1, 2, \dots, n)$ are smooth functions (Function is called smooth if it has derivatives of arbitrary order.) Curve $\mathbf{r}(t)$ is called regular if velocity vector $\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt}$ is not equal to zero at all t. By default we consider simple curves, i.e. curves which have no intersection points.

2.2 Reparameterisation

One can move along trajectory with different velocities, i.e. one can consider different parameterisation. E.g. consider

$$C_1\colon \qquad \begin{cases} x(t) = t \\ y(t) = t^2 \end{cases} \quad 0 < t < 1 \qquad C_2\colon \qquad \begin{cases} x(t) = \sin t \\ y(t) = \sin^2 t \end{cases} \quad 0 < t < \frac{\pi}{2} \tag{2.5}$$

Images of these two parameterised curves are the same. In both cases point moves along a piece of the same parabola but with different velocities.

Definition

Two smooth curves $C_1\colon \quad \mathbf{r}_1(t)\colon (a_1, b_1) \to \mathbf{E}^n$ and $C_2\colon \quad \mathbf{r}_2(\tau)\colon (a_2, b_2) \to \mathbf{E}^n$ are called equivalent if there exists reparameterisation map:

$$t(\tau)\colon (a_2, b_2) \to (a_1, b_1),$$

such that

$$r_2(\tau) = r_1(t(\tau)) \tag{2.6}$$

Reparameterisation $t(\tau)$ is diffeomorphism, i.e. function $t(\tau)$ has derivatives of all orders and first derivative $t'(\tau)$ is not equal to zero.

E.g. curves in (2.5) are equivalent because a map $\varphi(t) = \sin t$ transforms first curve to the second.

Equivalence class of equivalent parameterised curves is called non-parameterised curve.

It is useful sometimes to distinguish curves in the same equivalence class which differ by orientation.

Definition Let curves C_1, C_2 be two equivalent curves. We say that they have same orientation (parameterisations $\mathbf{r}_1(t$ and $\mathbf{r}_(\tau)$ have the same orientation) if reparameterisation $t = t(\tau)$ has positive derivative, $t'(\tau) > 0$. We say that they have opposite orientation (parameterisations $\mathbf{r}_1(t$ and $\mathbf{r}_(\tau)$ have the opposite orientation) if reparameterisation $t = t(\tau)$ has negative derivative, $t'(\tau) < 0$.

Changing orientation means changing the direction of "walking" around the curve.

Equivalence class of equivalent curves splits on two subclasses with respect to orientation.

Non-formally: Two curves are equivalent curves (belong to the same equivalence class) if these parameterised curves (paths) have the same images. Two equivalent curves have the same image. They define the same set of points in $\mathbf{E}^n$. Different parameters correspond to moving along curve with different velocity. Two equivalent curves have different orientation If two parameterisations correspond to moving along the curve in different directions then these parameterisations define opposite orientation.

What happens with velocity vector if we change parameterisation? It changes its value, but it can change its direction only on opposite (If these parameterisations have opposite orientation of the curve):

$$\mathbf{v}(\tau) = \frac{d\mathbf{r}_2(\tau)}{d\tau} = \frac{d\mathbf{r}(t(\tau))}{d\tau} = \frac{dt(\tau)}{d\tau} \cdot \frac{d\mathbf{r}(t)}{dt}\Big|_{t=t(\tau)} \tag{2.7}$$

Or shortly: $\mathbf{v}(\tau)\big|_\tau = t_\tau(\tau)\mathbf{v}(t)\big|_{t=t(\tau)}$

We see that velocity vector is multiplied on the coefficient (depending on the point of the curve), i.e. velocity vector changes on the collinear vector.

Example Consider following curves in $\mathbf{E}^2$:

$$C_1\colon \quad \begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases}, 0 < \theta < \pi, \qquad C_2\colon \quad \begin{cases} x = u \\ y = \sqrt{1-u^2} \end{cases}, -1 < u < 1, \tag{2.8}$$

$$\begin{cases} x = \tan t \\ y = \frac{\sqrt{\cos 2t}}{\cos t} \end{cases}, -\frac{\pi}{4} < t < \frac{\pi}{4} \tag{2.9}$$

These three parameterised curves,(paths) define the same non-parameterised curve: the upper piece of the circle: $x^2 + y^2 = 1, y > 0$. The reparameterisation $u(\theta) = \cos\theta$ transforms the second curve to the first one.

The reparameterisation $u(\theta) = \cos\theta$ transforms the second curve to the first one.

The reparameterisation $u(\theta) = \tan t$ transforms the second curve to the third one one.

Curves C_1, C_2 have opposite orientation because $u'(\theta) < 0$. Curves C_2, C_3 have the same orientation, because $u'(t) > 0$. Curves C_1 and C_2 have opposite orientations too (Why?).

In the first case point moves with constant pace $|\mathbf{v}(\theta)| = 1$ anti clock-wise "from right to left" from the point $A = (1, 0)$ to the point $B = (-1, 0)$. In the second case pace is not constant, but $v_x = 1$ is constant. Point moves clock-wise "from left to right", from the point $B = (-1, 0)$ to the point $A = (1, 0)$. In the third case point also moves clock-wise "from the left to right".

There are other examples in the Homeworks.

2.3 0-forms and 1-forms

Most of considerations of this and next subsections can be considered only for $\mathbf{E}^2$ or $\mathbf{E}^3$. Compulsory material for differential forms is only for $\mathbf{E}^2$, $\mathbf{E}^3$.

0-form on $\mathbf{E}^n$ it is just function on $\mathbf{E}^n$ (all functions under consideration are differentiable)

Now we define 1-forms.

Definition Differential 1-form ω on $\mathbf{E}^n$ is a function on tangent vectors of $\mathbf{E}^n$, such that it is linear at each point:

$$\omega(\mathbf{r}, \lambda\mathbf{v}_1 + \mu\mathbf{v}_2) = \lambda\omega(\mathbf{r}, \mathbf{v}_1) + \mu\omega(\mathbf{r}, \mathbf{v}_2)\,. \tag{2.10}$$

Here $\mathbf{v}_1, \mathbf{v}_2$ are vectors tangent to $\mathbf{E}^n$ at the point $\mathbf{r}$, ($\mathbf{v}_1, \mathbf{v}_2 \in T_x\mathbf{E}^n$) (We recall that vector tangent at the point $\mathbf{r}$ means vector attached at the point $\mathbf{r}$). We suppose that ω is smooth function on points $\mathbf{r}$.

If $\mathbf{X}(\mathbf{r})$ is vector field and ω-1-form then evaluating ω on $\mathbf{X}(\mathbf{r})$ we come to the function $w(\mathbf{r}, \mathbf{X}(\mathbf{r}))$ on $\mathbf{E}^3$.

Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be a basis in $\mathbf{E}^n$ and $(x^1, \dots, x^n)$ corresponding coordinates: an arbitrary point with coordinates $(x^1, \dots, x^n)$ is assigned to the vector $\mathbf{r} = x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + \dots x^n\mathbf{e}_n$ starting at the origin.

Translating basis vectors $\mathbf{e}_i$ $(i = 1, \dots, n)$ from the origin to other points of $\mathbf{E}^n$ we come to vector field which we also denote $\mathbf{e}_i$ $(i = 1, \dots, n)$.

Let ω be an 1-form on $\mathbf{E}^n$. Consider an arbitrary vector field $\mathbf{A}(\mathbf{r}) = \mathbf{A}(x^1, \dots, x^n)$:

$$\mathbf{A}(\mathbf{r}) = A^1(\mathbf{r})\mathbf{e}_1 + \dots + A^n(\mathbf{r})\mathbf{e}_n = \sum_{i=1}^n A^i(\mathbf{r})\mathbf{e}_i \tag{2.11}$$

Then by linearity

$$\omega(\mathbf{r}, \mathbf{A}(\mathbf{r})) = \omega(\mathbf{r}, A^1(\mathbf{r})\mathbf{e}_1 + \dots + A^n(\mathbf{r})\mathbf{e}_n) = A^1\omega(\mathbf{r}, \mathbf{e}_1) + \dots + A^n\omega(\mathbf{r}, \mathbf{e}_n)$$

Consider *basic* differential forms $dx^1, dx^2, \dots, dx^n$ such that

$$dx^i(\mathbf{e}_j) = \delta^i_j = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}\,. \tag{2.12}$$

Then it is easy to see that

$$dx^1(\mathbf{A}) = \mathbf{A}^1, dx^2(\mathbf{A}) = \mathbf{A}^2, ..., \text{i.e.} dx^i(\mathbf{A}) = A^i$$

Hence

$$\omega(\mathbf{r}, \mathbf{A}(\mathbf{r})) = \left(\omega_1(\mathbf{r})dx^1 + \omega_2(\mathbf{r})dx^2 + \dots + \omega_n(\mathbf{r})dx^n\right)(\mathbf{A}(\mathbf{r}))$$

where components $\omega_i(\mathbf{r}) = \omega(\mathbf{r}, \mathbf{e}_i)$.

In the same way as an arbitrary vector field on $\mathbf{E}^n$ can be expanded over the basis $\{\mathbf{e}_i\}$ (see (2.11)), an arbitrary differential 1-form ω can be expanded over the basis forms(2.12)

$$\omega = \omega_1(x^1, \dots, x^n)dx^1 + \omega_2(x^1, \dots, x^n)x^2 + \dots + \omega_n(x^1, \dots, x^n)dx^n \,.$$

Example Consider in $\mathbf{E}^3$ a basis $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ and corresponding coordinates (x, y, z). Then

$$\begin{array}{l} dx(\mathbf{e}_x) = 1, dx(\mathbf{e}_y) = 0, dx(\mathbf{e}_z) = 0 \\ dy(\mathbf{e}_x) = 0, dy(\mathbf{e}_y) = 1, dy(\mathbf{e}_z) = 0 \\ dz(\mathbf{e}_x) = 0, dz(\mathbf{e}_y) = 0, dz(\mathbf{e}_z) = 1 \end{array} \tag{2.13}$$

The value of a differential 1-form $\omega = a(x, y, z)dx + b(x, y, z)dy + c(x, y, z)dz$ on vector field $\mathbf{X} = A(x, y, z)\mathbf{e}_x + B(x, y, z)\mathbf{e}_y + C(x, y, z)\mathbf{e}_z$ is equal to

$$\omega(\mathbf{r}, \mathbf{X}) = a(x, y, z)dx(\mathbf{X}) + b(x, y, z)dx(\mathbf{X}) + c(x, y, z)dx(\mathbf{X}) =$$

$$a(x, y, z)A(x, y, z) + b(x, y, z)B(x, y, z) + c(x, y, z)C(x, y, z)$$

It is very useful to introduce new notation for vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$. To introduce this new notation we need a short recalling of derivational derivative of functions.

Vectors—directional derivatives of functions

Let $\mathbf{R}$ be a vector in $\mathbf{E}^n$ tangent to the point $\mathbf{r} = \mathbf{r}_0$ (attached at a point $\mathbf{r} = \mathbf{r}_0$). Define the operation of derivative of an arbitrary (differentiable) function at the point $\mathbf{r}_0$ along the vector $\mathbf{R}$— directional derivative of function f along the vector $\mathbf{R}$

Definition

Let $\mathbf{r}(t)$ be a curve such that

- $\mathbf{r}(t)\big|_{t=0} = \mathbf{r}_0$
- Velocity vector of the curve at the point $\mathbf{r}_0$ is equal to $\mathbf{R}$: $\frac{d\mathbf{r}(t)}{dt}\big|_{t=0} = \mathbf{R}$

Then directional derivative of function f with respect to the vector $\mathbf{R}$ at the point $\mathbf{r}_0$ $\partial_{\mathbf{R}} f\big|_{\mathbf{r}_0}$ is defined by the relation

$$\partial_{\mathbf{R}} f\big|_{\mathbf{r}_0} = \frac{d}{dt}\left(f\left(\mathbf{r}(t)\right)\right)\big|_{t=0} \,. \tag{2.14}$$

Using chain rule one come from this definition to the following important formula for the directional derivative:

$$\text{If } \mathbf{R} = \sum_{i=1}^{n} R^i \mathbf{e}_i \text{ then } \partial_{\mathbf{R}} f\big|_{\mathbf{r}_0} = \sum_{i=1}^{n} R^i \frac{\partial}{\partial x^i} f(x^1, \dots, x^n)\big|_{\mathbf{r}=\mathbf{r}_0} \tag{2.15}$$

It follows form this formula that

One can assign to every vector $\mathbf{R} = \sum_{i=1}^{n} R^i \mathbf{e}_i$ *the operation* $\partial_{\mathbf{R}} = R^1 \frac{\partial}{\partial x^1} + R^2 \frac{\partial}{\partial x^2} + \dots + R^n \frac{\partial}{\partial x^n}$ *of taking directional derivative:*

$$\mathbf{R} = \sum_{i=1}^{n} R^i \mathbf{e}_i \mapsto \partial_{\mathbf{R}} = \sum_{i=1}^{n} R^i \frac{\partial}{\partial x^i} \tag{2.16}$$

Vector $\mathbf{e}_x$ we will denote sometimes by ∂_x, vectors $\mathbf{e}_y, \mathbf{e}_z$ by ∂_y, ∂_z respectively. The symbols $\partial_x, \partial_y, \partial_z$ correspond to partial derivative with respect to coordinate x or y or z . Later we see that these new notations are very illuminating when we deal with arbitrary coordinates, such as polar coordinates or spherical coordinates, The conception of orthonormal basis is ill-defined in arbitrary coordinates, but one can still consider the corresponding partial derivatives. Vector fields $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ (or in new notation $\partial_x, \partial_y, \partial_z$) can be considered as a basis[3] in the space of all vector fields on $\mathbf{E}^3$.

An arbitrary vector field (2.11) can be rewritten in the following way:

$$\mathbf{A}(\mathbf{r}) = A^1(\mathbf{r})\mathbf{e}_1 + \dots + A^n(\mathbf{r})\mathbf{e}_n = A^1(\mathbf{r})\frac{\partial}{\partial x^1} + A^2(\mathbf{r})\frac{\partial}{\partial x^2} + \dots + A^n(\mathbf{r})\frac{\partial}{\partial x^n} \tag{2.17}$$

Differential on 0*-forms*

Now we introduce very important operation: Differential d which acts on 0-forms and transforms them to 1 forms.

$$\boxed{\begin{array}{l}\text{Differential}\\ \text{0-forms}\end{array}} \xrightarrow{d} \boxed{\begin{array}{l}\text{Differential}\\ \text{1-forms}\end{array}}$$

Later we will learn how differential acts on 1-forms transforming them to 2-forms.

[3]Coefficients of expansion are functions, elements of algebra of functions, not numbers ,elements of field. To be more careful, these vector fields are basis of the *module* of vector fields on $\mathbf{E}^3$

Definition Let $f = f(x)$-be 0-form, i.e. function on $\mathbf{E}^n$. Then

$$df = \sum_{i=1}^{n} \frac{\partial f(x^1, \dots, x^n)}{\partial x^i} dx^i \,. \tag{2.18}$$

The value of 1-form df on an arbitrary vector field (2.17) is equal to

$$df(\mathbf{A}) = \sum_{i=1}^{n} \frac{\partial f(x^1, \dots, x^n)}{x^i} dx^i(\mathbf{A}) = \sum_{i=1}^{n} \frac{\partial f(x^1, \dots, x^n)}{x^i} A^i = \partial_{\mathbf{A}} f \tag{2.19}$$

We see that *value of differential of* 0*-form* f *on an arbitrary vector field* $\mathbf{A}$ *is equal to directional derivative of function* f *with respect to the vector* $\mathbf{A}$.

The formula (2.19) defines df in invariant way without using coordinate expansions. Later we check straightforwardly the coordinate-invariance of the definition (2.18).

Exercise Check that

$$dx^i(\mathbf{A}) = \partial_{\mathbf{A}} x^i \tag{2.20}$$

Example If $f = f(x, y)$ is a function $(0 - form)$ on $\mathbf{E}^2$ then

$$df = \frac{\partial f(x,y)}{\partial x} dx + \frac{\partial f(x,y)}{\partial y} dy$$

and for an arbitrary vector field $\mathbf{A} = \mathbf{A} = A_x \mathbf{e}_x + \mathbf{A}_y \mathbf{e}_y = A_x(x, y)\partial_x + A_y(x, y)\partial_y$

$$df(\mathbf{A}) = \frac{\partial f(x,y)}{\partial x} dx(\mathbf{A}) + A_y(x,y) \frac{\partial f(x,y)}{\partial y} dy(\mathbf{A}) =$$

$$\mathbf{A}_x(x,y) \frac{\partial f(x,y)}{\partial x} + A_y(x,y) \frac{\partial f(x,y)}{\partial y} = \partial_{\mathbf{A}} f \,.$$

Example Find the value of 1-form $\omega = df$ on the vector field $\mathbf{A} = x\partial_x + y\partial_y$ if $f = \sin(x^2 + y^2)$.

$\omega(\mathbf{A}) = df(\mathbf{A})$. One can calculate it using formula (2.18) or using formula (2.19).

Solution (using (2.18)*)*:

$$\omega = df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 2x \cos(x^2 + y^2) dx + 2y \cos(x^2 + y^2) dy \,.$$

$$\omega(\mathbf{A}) = 2x\cos(x^2+y^2)dx(\mathbf{A}) + 2y\cos(x^2+y^2)dy(\mathbf{A}) =$$
$$2x\cos(x^2+y^2)A_x + 2y\cos(x^2+y^2)dA_y = 2(x^2+y^2)\cos(x^2+y^2)\,,$$

Another solution (using (2.19)*)*

$$df(\mathbf{A}) = \partial_{\mathbf{A}} f = A_x \frac{\partial f}{\partial x} + A_x \frac{\partial f}{\partial y} = 2(x^2+y^2)\cos(x^2+y^2)\,.$$

See other examples in Homeworks.

2.4 Differential 1-form in arbitrary coordinates

Why differential forms? Why so strange notations for vector fields.

Now we see that working with differential forms we in fact do not care about what coordinates we work in. And notations ∂_i intorduced for vector fields becomes adequate.

One of advantages of the technique of differential form is that calculations are the same in arbitrary coordinates.

Consider an arbitrary (local) coordinates $u^1, \dots, u^n$ on $\mathbf{E}^n$: $u^i = u^i(x^1, \dots, x^n)$, $i = 1, \dots, n$. Show first that

$$du^i = \sum_{k=1}^{n} \frac{\partial u^i(x^1, \dots, x^n)}{\partial x^k} dx^k\,. \tag{2.21}$$

It is enough to check it on basic fields:

$$du^i\left(\frac{\partial}{\partial x^m}\right) = \partial_{\left(\frac{\partial}{\partial x^m}\right)} u^i = \frac{\partial u^i(x^1, \dots, x^n)}{x^m} = \sum_{k=1}^{n} \frac{\partial u^i(x^1, \dots, x^n)}{\partial x^k} dx^k\left(\left(\frac{\partial}{\partial x^m}\right)\right) = \left(\frac{\partial}{\partial x^m}\right)\,.$$

because (see (2.12)):

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}\,. \tag{2.22}$$

(We rewrite the formula (2.12) using new notations ∂_i instead $\mathbf{e}_i$). In the previous formula (2.12) we considered *cartesian* coordinates.

Show that the formula above is valid in an *arbitrary coordinates.*

One can see using chain rule that

$$\frac{\partial}{\partial u^i} = \frac{\partial x^1}{\partial u^i}\frac{\partial}{\partial x^1} + \frac{\partial x^2}{\partial u^i}\frac{\partial}{\partial x^2} + \cdots + \frac{\partial x^n}{\partial u^i}\frac{\partial}{\partial x^n} = \sum_{k=1}^{n} \frac{\partial x^k}{\partial u^i}\frac{\partial}{\partial x^k} \tag{2.23}$$

Calculate the value of differential form du^i on vector field $\frac{\partial}{\partial u^j}$ using (2.21) and (2.23):

$$du^i\left(\frac{\partial}{\partial u^j}\right) = \sum_{k=1}^{n} \frac{\partial u^i(x^1,\dots,x^n)}{\partial x^k} dx^k \left(\sum_{r=1}^{n} \frac{\partial x^r}{\partial u^j}\frac{\partial}{\partial x^r}\right) = \tag{2.24}$$

$$\sum_{k,r=1}^{n} \frac{\partial u^i(x^1,\dots,x^n)}{\partial x^k}\frac{\partial x^r(u^1,\dots,u^n)}{\partial u^j} dx^k\left(\frac{\partial}{\partial x^r}\right) =$$

$$\sum_{k,r=1}^{n} \frac{\partial u^i(x^1,\dots,x^n)}{\partial x^k}\frac{\partial x^r(u^1,\dots,u^n)}{\partial u^j}\delta_r^k = \sum_{k=1}^{n}\frac{\partial x^k}{\partial u^j}\frac{\partial u^i}{\partial x^k} = \delta_i^j$$

We come to

$$du^i\left(\frac{\partial}{\partial u^j}\right) = \delta_j^i = \begin{cases} 1 \text{ if } i=j \\ 0 \text{ if } i\neq j \end{cases} . \tag{2.25}$$

We see that formula (2.22) has the same appearance in arbitrary coordinates. In other words it is invariant with respect to an arbitrary transformation of coordinates.

Exercise Check straightforwardly the invariance of the definition (2.18). In coordinates $(u^1,\dots,u^n)$

Solution We have to show that the formula (2.18) does not changed under changing of coordinates $u^i = u^i(x^1,\dots,x^n)$.

$$df = \sum_{i=1}^{n} \frac{\partial f(x^1,\dots,x^n)}{\partial x^i} dx^i = \sum_{i=1,k}^{n} \frac{\partial f(x^1,\dots,x^n)}{\partial x^i}\frac{\partial x^i}{\partial u^k} du^k == \sum_{i=1}^{n}\frac{\partial f}{\partial u^k} du^k ,$$

because $\sum_{i=1}^n \frac{\partial f(x^1,\dots,x^n)}{\partial x^i}\frac{\partial x^i}{\partial u^k} = \frac{\partial f}{\partial u^k}$

Example

Consider more in detail $\mathbf{E}^2$. (For $\mathbf{E}^3$ considerations are the same, just calculations little bit more complicated) Let u, v be an arbitrary coordinates in $\mathbf{E}^2$, $u = u(x,y), v = v(x,y)$.

$$du = \frac{\partial u(x,y)}{\partial x}dx + \frac{\partial u(x,y)}{\partial y}dy, \qquad dv = \frac{\partial v(x,y)}{\partial x}dx + \frac{\partial v(x,y)}{\partial y}dy \tag{2.26}$$

and

$$\partial_u = \frac{\partial x(u,v)}{\partial u}\partial_x + \frac{\partial y(u,v)}{\partial u}\partial_y, \;\; \partial_v = \frac{\partial x(u,v)}{\partial v}\partial_x + \frac{\partial y(u,v)}{\partial v}\partial_y \tag{2.27}$$

(As always sometimes we use notation ∂_u instead $\frac{\partial}{\partial u}$, ∂_x instead $\frac{\partial}{\partial x}$ e.t.c.) Then

$$du(\partial_u) = 1, du(\partial_v) = 0$$
$$dv(\partial_w) = 0, dv(\partial_v) = 1 \tag{2.28}$$

This follows from the general formula but it is good exercise to repeat the previous calculations for this case:

$$du(\partial_u) = \left(\frac{\partial u(x,y)}{\partial x}dx + \frac{\partial u(x,y)}{\partial y}dy\right)\left(\frac{\partial x(u,v)}{\partial u}\partial_x + \frac{\partial y(u,v)}{\partial u}\partial_y\right) =$$

$$\frac{\partial u(x,y)}{\partial x}\frac{\partial x(u,v)}{\partial u} + \frac{\partial u(x,y)}{\partial y}\frac{\partial y(u,v)}{\partial u} = \frac{\partial x(u,v)}{\partial u}\frac{\partial u(x,y)}{\partial x} + \frac{\partial y(u,v)}{\partial u}\frac{\partial u(x,y)}{\partial y} = 1$$

We just apply chain rule to the function $u = u(x,y) = u(x(u,v), y(u,v))$:

Analogously

$$du(\partial_v) = \left(\frac{\partial u(x,y)}{\partial x}dx + \frac{\partial u(x,y)}{\partial y}dy\right)\left(\frac{\partial x(u,v)}{\partial v}\partial_x + \frac{\partial y(u,v)}{\partial v}\partial_y\right)$$

$$\frac{\partial u(x,y)}{\partial x}\frac{\partial x(u,v)}{\partial v} + \frac{\partial u(x,y)}{\partial y}\frac{\partial y(u,v)}{\partial v} = \frac{\partial x(u,v)}{\partial v}\frac{\partial u(x,y)}{\partial x} + \frac{\partial y(u,v)}{\partial v}\frac{\partial u(x,y)}{\partial y} = 0$$

The same calculations for dv.

Example In this example we apply formulae (2.26), (2.27) for polar coordinates on $\mathbf{E}^2$. Recall that for polar coordinates (r, φ)

$$\begin{cases} x(r,\varphi) = r\cos\varphi \\ y(r,\varphi) = r\sin\varphi \end{cases} \qquad (0 \le \varphi < 2\pi, 0 \le r < \infty),$$

Respectively

$$\begin{cases} r(x,y) = \sqrt{x^2+y^2} \\ \varphi = \arctan\frac{y}{x} \end{cases} . \tag{2.29}$$

We have that for basic 1-forms

$$dr = r_x dx + r_y dy = \frac{x}{\sqrt{x^2+y^2}}dx + \frac{y}{\sqrt{x^2+y^2}}dy = \frac{xdx + ydy}{r}$$

and

$$d\varphi = \varphi_x dx + \varphi_y dy = \frac{-ydx}{x^2+y^2} + \frac{xdy}{x^2+y^2}dx = \frac{xdy - ydx}{r^2}$$

Respectively

$$dx = x_r dr + x_\varphi d\varphi = \cos\varphi dr - r\sin\varphi d\varphi$$

and

$$dy = y_r dr + y_\varphi d\varphi = \sin\varphi dr + r\cos\varphi d\varphi \tag{2.30}$$

For basic vector field

$$\partial_r = \frac{\partial x}{\partial r}\partial_x + \frac{\partial y}{\partial r}\partial_y = \cos\varphi\partial_x + \sin\varphi\partial_y = \frac{x\partial_x + y\partial_y}{r},$$

$$\partial_\varphi = \frac{\partial x}{\partial\varphi}\partial_x + \frac{\partial y}{\partial\varphi}\partial_y = -r\sin\varphi\partial_x + r\cos\varphi\partial_y = x\partial_y - y\partial_x,$$

respectively

$$\partial_x = \frac{\partial r}{\partial x}\partial_r + \frac{\partial\varphi}{\partial x}\partial_\varphi = \frac{x}{r}\partial_r - \frac{y}{r^2}\partial_\varphi$$

and

$$\partial_y = \frac{\partial r}{\partial y}\partial_r + \frac{\partial\varphi}{\partial y}\partial_\varphi = \frac{y}{r}\partial_r + \frac{x}{r^2}\partial_\varphi \tag{2.31}$$

Example Calculate the value of forms $\omega_1 = xdx + ydy$, $\omega_2 = xdy - ydx$ on vector fields $\mathbf{A} = x\partial_x + y\partial_y$, $\mathbf{B} = x\partial_y - y\partial_x$. Perform calculations in cartesian and in polar coordinates.

In cartesian coordinates:

$$\omega_1(\mathbf{A}) = xdx(x\partial_x + y\partial_y) + ydy(x\partial_x + y\partial_y) = x^2 + y^2,\ \omega_1(\mathbf{B}) = xdx(\mathbf{B}) + ydy(\mathbf{B}) = 0,$$

$$\omega_2(\mathbf{A}) = xdy(\mathbf{A}) - ydx(\mathbf{A}) = 0,\ \omega_2(\mathbf{B}) = xdy(\mathbf{B}) - ydx(\mathbf{B}) = x^2 + y^2\,.$$

Now perform calculations in polar coordinates. According to calculations in previous example we have that

$$\omega_1 = xdx + ydy = rdr,\ \omega_2 = xdy - ydx = r^2 d\varphi$$

and

$$\mathbf{A} = x\partial_x + y\partial_y = r\partial_r,\ \mathbf{B} = x\partial_y - y\partial_x = \partial_\varphi$$

Hence $\omega_1(\mathbf{A}) = rdr(\mathbf{A}) = r^2, \omega_1(\mathbf{B} = rdr(\partial_\varphi) = 0$,

$$\omega_2(\mathbf{A}) = r^2 d\varphi(r\partial_r) = 0, \quad \omega_2(\mathbf{B}) = r^2 d\varphi(\partial_\varphi) = r^2$$

Example Calculate the value of form $\omega = \frac{xdy - ydx}{x^2+y^2}$ on the vector field $\mathbf{A} = \partial_\varphi$. We have to transform form from cartesian coordinates to polar or vector field from polar to cartesian.

$$\frac{xdy - ydx}{x^2 + y^2} = d\varphi, \quad \omega(\mathbf{A}) = d\varphi(\partial_\varphi) = 1$$

or

$$\partial_\varphi = x\partial_y - y\partial_x,\ \omega(\mathbf{A}) = \frac{xdy(x\partial_y - y\partial_x) - ydx(x\partial_y - y\partial_x)}{x^2 + y^2} = 1\,.$$

Cylindrical and spherical coordinates

- Cylindrical coordinates in $\mathbf{E}^3$

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = h \end{cases} \qquad (0 \le \varphi < 2\pi, 0 \le r < \infty) \tag{2.32}$$

- Spherical coordinates in $\mathbf{E}^3$

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\cos\varphi \\ z = r\cos\theta \end{cases} \quad (0 \le \varphi < 2\pi, 0 \le r < \infty) - - - \text{cylindrical coordiantes in } \mathbf{E}^3 \tag{2.33}$$

Example (Basis vectors and forms for cylindrical coordinates)

Consider cylindrical coordinates in $\mathbf{E}^3$: $u = r, v = \varphi, w = h$. Then calculating partial derivatives we come to

$$\begin{cases} \partial_r = \frac{\partial x}{\partial r}\partial_x + \frac{\partial y}{\partial r}\partial_y + \frac{\partial z}{\partial r}\partial_z = \cos\varphi\partial_x + \sin\varphi\partial_y \\ \\ \partial_\varphi = \frac{\partial x}{\partial \varphi}\partial_x + \frac{\partial y}{\partial \varphi}\partial_y + \frac{\partial z}{\partial \varphi}\partial_z = -r\sin\varphi\partial_x + r\cos\varphi\partial_y \\ \\ \partial_h = \frac{\partial x}{\partial h}\partial_x + \frac{\partial y}{\partial h}\partial_y + \frac{\partial z}{\partial h}\partial_z = \partial_z \end{cases} \tag{2.34}$$

Basic forms are $dr, d\varphi, dh$ and

$$\begin{gathered} dr(\partial_r) = 1, dr(\partial_\varphi) = 0, dr(\partial_h) = 0 \\ d\varphi(\partial_r) = 0, d\varphi(\partial_\varphi) = 1, d\varphi(\partial_h) = 0 \\ dh(\partial_r) = 0, dh(\partial_\varphi) = 0, dh(\partial_h) = 1 \end{gathered} \tag{2.35}$$

Example (Basis vectors for spheric coordinates)

Consider spheric coordinates in $\mathbf{E}^3$: $u = r, v = \theta, w = \varphi$. Then calculating partial derivatives we come to

$$\begin{cases} \partial_r = \frac{\partial x}{\partial r}\partial_x + \frac{\partial y}{\partial r}\partial_y + \frac{\partial z}{\partial r}\partial_z = \sin\theta\cos\varphi\partial_x + \sin\theta\sin\varphi\partial_y + \cos\theta\partial_z \\ \\ \partial_\theta = \frac{\partial x}{\partial \theta}\partial_x + \frac{\partial y}{\partial \theta}\partial_y + \frac{\partial z}{\partial \theta}\partial_z = r\cos\theta\cos\varphi\partial_x + r\cos\theta\sin\varphi\partial_y - r\sin\theta\partial_z \\ \\ \partial_\varphi = \frac{\partial x}{\partial \varphi}\partial_x + \frac{\partial y}{\partial \varphi}\partial_y + \frac{\partial z}{\partial \varphi}\partial_z = -r\cos\theta\sin\varphi\partial_x + r\sin\theta\cos\varphi\partial_y \end{cases} \tag{2.36}$$

Basic forms are $dr, d\theta, d\varphi$ and

$$\begin{gathered} dr(\partial_r) = 1, dr(\partial_\theta) = 0, dr(\partial_\varphi) = 0 \\ d\theta(\partial_r) = 0, d\theta(\partial_\theta) = 1, d\theta(\partial_\varphi) = 0 \\ d\varphi(\partial_r) = 0, d\varphi(\partial_\theta) = 0, d\varphi(\partial_\varphi) = 1 \end{gathered} \tag{2.37}$$

2.5 Integration of differential 1-forms over curves

Let $\omega = \omega_1(x^1, \dots, x^n)dx^1 + \dots + \omega_1(x^1, \dots, x^n)dx^n = \sum_{i=1}^n \omega_i dx^i$ be an arbitrary 1-form in $\mathbf{E}^n$

and C: $\mathbf{r} = \mathbf{r}(t), t_1 \le t \le t_2$ be an arbitrary smooth curve in $\mathbf{E}^n$.

One can consider the value of one form ω on the velocity vector field $\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt}$ of the curve:

$$\omega(\mathbf{v}(t)) = \sum_{i=1}^n \omega_i\left(x^1(t), \dots, x^n(t))dx^i(\mathbf{v}(t)\right) = \sum_{i=1}^n \omega_i\left(x^1(t), \dots, x^n(t)\right)\frac{dx^i(t)}{dt}$$

We define now integral of 1-form ω over the curve C.

Definition The integral of the form $\omega = \omega_1(x^1, \dots, x^n)dx^1 + \dots + \omega_1(x^1, \dots, x^n)dx^n$ over the curve C: $\mathbf{r} = \mathbf{r}(t)$ $t_1 \le t \le t_2$ is equal to the integral of the function $\omega(\mathbf{v}(t))$ over the interval $t_1 \le t \le t_2$:

$$\int_C \omega = \int_{t_1}^{t_2} \omega(\mathbf{v}(t))dt = \int_{t_1}^{t_2} \left(\sum_{i=1}^n \omega_i\left(x^1(t), \dots, x^n(t)\right)\frac{dx^i(t)}{dt}\right)dt\,. \tag{2.38}$$

The remarkable property of this construction is that $\int_C \omega$ does not depend on coordinates, and up to a sign on parameterisation. Before formulating and proving this statement consider examples.

Example
Let

$$\omega = \omega_1(x,y)dx + \omega_2(x,y)dy$$

be 1-form in $\mathbf{E}^2$ (x, y–are usual cartesian coordinates). Let C: $\mathbf{r} = \mathbf{r}(t)$ $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$, $t_1 \leq t \leq t_2$ be a curve in $\mathbf{E}^2$.

Consider velocity vector field of this curve

$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} = \begin{pmatrix} x_t(t) \\ y_t(t) \end{pmatrix} = x_t\partial_x + y_t\partial_y \tag{2.39}$$

($x_t = \frac{dx(t)}{dt}$, $y_t = \frac{dy(t)}{dt}$).

One can consider the value of one form ω on the velocity vector field $\mathbf{v}(t)$ of the curve: $\omega(\mathbf{v}) = \omega_1 dx(\mathbf{v}) + \omega_2 dy(\mathbf{v}) =$

$$\omega_1(x(t), y(t))x_t(t) + \omega_2(x(t), y(t))y_t(t)\,.$$

The integral of the form $\omega = \omega_1(x,y)dx + \omega_2(x,y)dy$ over the curve C: $\mathbf{r} = \mathbf{r}(t)$ $t_1 \leq t \leq t_2$ is equal to the integral of the function $\omega(\mathbf{v}(t))$ over the interval $t_1 \leq t \leq t_2$:

$$\int_C \omega = \int_{t_1}^{t_2} \omega(\mathbf{v}(t))dt = \int_{t_1}^{t_2} \left(\omega_1(x(t), y(t))\frac{dx(t)}{dt} + \omega_2(x(t), y(t))\frac{dy(t)}{dt}\right) dt\,. \tag{2.40}$$

Example Consider an integral of the form $\omega = 3dy + 3y^2dx$ over the curve C: $\mathbf{r}(t)$ $\begin{cases} x = \cos t \\ y = \sin t \end{cases}$, $0 \leq t \leq \pi/2$. (C is the arc of the circle $x^2 + y^2 = 1$ defined by conditions $x, y \geq 0$).

Velocity vector $\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} = \begin{pmatrix} x_t(t) \\ y_t(t) \end{pmatrix} = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$. The value of the form on velocity vector is equal to

$$\omega(\mathbf{v}(t)) = 3y^2(t)v_x(t) + 3v_y(t) = 3\sin^2 t(-\sin t) + 3\cos t = 3\cos t - 3\sin^3 t$$

and

$$\int_C \omega = \int_0^{\frac{\pi}{2}} w(\mathbf{v}(t))dt = \int_0^{\frac{\pi}{2}} (3\cos t - 3\sin^3 t)dt = 3\left(\sin t + \cos t - \frac{\cos^3 t}{3}\right)\Big|_0^{\frac{\pi}{2}}$$

Example

Consider the following curve:

$$C\colon \begin{cases} x = R\cos\Omega t \\ y = R\sin\Omega t \qquad 0 \le t \le \frac{2\pi}{\Omega},\ c = L\Omega,\ (\Omega > 0) \\ z = ct \end{cases}$$

One can see that image of this curve is the helix. If $0 \le t \le \frac{2\pi}{\Omega}$ and $c = L\Omega$ then it has the hight $h = 2\pi L$ and it makes one circle around z-axis.

Calculate the integral of 1-form $\omega = xdy + dz$ over this helix.

Velocity vector $\mathbf{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = \begin{pmatrix} -R\Omega\sin\Omega t \\ R\Omega\cos\Omega t \\ c \end{pmatrix}$

The value of differential form ω on velocity vector $\mathbf{v}$ is equal to

$$\omega(\mathbf{v}) = \omega_x dx(\mathbf{v}) + \omega_y dy(\mathbf{v}) + \omega_z dz(\mathbf{v}) = xdy(\mathbf{v}) + dz(\mathbf{v}) = x(t)v_y(t) + v_z(t) =$$

$$R\cos\Omega t \cdot R\Omega\cos\Omega t + c = R^2\Omega^2\cos^2 tdt + \frac{L}{\Omega} \quad \left(c = \frac{L}{\Omega}\right).$$

Then

$$\int_C \omega = \int_{t_1}^{t_2} \omega(\mathbf{v}(t))dt = \int_0^{\frac{2\pi}{\Omega}} \left(R^2\Omega\cos^2 tdt + \frac{L}{\Omega}\right) dt = \pi R^2 + 2\pi L\,.$$

Remark: Focus your attention, please, on the fact that changing angular velocity Ω, i.e. changing the reparameterisation of the curve C does not change the image of the curve and it does not change the value of the integral.

For another examples see Homeworks and Coursework.

The Integral of form over curve does not depend on coordinates on $\mathbf{E}^2$. At what extent the integral of form over curve depends on parameterisation of the curve?

Proposition The integral $\int_C \omega$ does not depend on the coordinates on $\mathbf{E}^2$. It does not depend (up to a sign) on parameterisation of the curve: if $C\colon \quad \mathbf{r} = \mathbf{r}(t) \quad t_1 \le t \le t_2$ is a curve and $t = t(\tau)$ is reparameterisation, i.e. new curve $C'\colon \quad \mathbf{r}'(\tau) = \mathbf{r}(t(\tau)) \quad \tau_1 \le \tau \le \tau_2$, then $\int_C \omega = \pm \int_C' \omega$:

$$\int_C \omega = \int_C' \omega, \quad \text{if orientaion is not changed, i.e. if } t'(\tau) > 0 \qquad (2.41)$$

and

$$\int_C \omega = -\int_C' \omega, \quad \text{if orientaion is changed, i.e. if } t'(\tau) < 0 \tag{2.42}$$

Proof of the Proposition Show that integral does not depend (up to a sign) on the parameterisation of the curve. Let $t(\tau)$ ($\tau_1 \leq t \leq \tau_2$) be reparameterisation. We come to the new curve C': $\mathbf{r}'(\tau) = \mathbf{r}(t(\tau))$. Note that the new velocity vector $\mathbf{v}'(\tau) = \frac{dr(t(\tau))}{d\tau} = t'(\tau)\mathbf{v}(t(\tau))$. Hence $\omega(\mathbf{v}'(\tau)) = w(\mathbf{v}(t(\tau)))t'(\tau)$. For the new curve C'

$$\int_{C'} \omega = \int_{\tau_1}^{\tau_2} \omega(\mathbf{v}'(\tau))d\tau = \int_{\tau_1}^{\tau_2} \omega(\mathbf{v}(t(\tau)))\frac{dt(\tau)}{d\tau}d\tau = \int_{t(\tau_1)}^{t(\tau_2)} \omega(\mathbf{v}(t))dt$$

$t(\tau_1) = t_1$, $t(\tau_2) = t_2$ if reparameterisation does not change orientation and $t(\tau_1) = t_2$, $t(\tau_2) = t_1$ if reparameterisation changes orientation.

Hence $\int_{C'} w = \int_{t_1}^{t_2)} \omega(\mathbf{v}(t))dt = \int_C \omega$ if orientation is not changed and $\int_{C'} w = \int_{t_2}^{t_1)} \omega(\mathbf{v}(t))dt = -\int_{t_1}^{t_2)} \omega(\mathbf{v}(t))dt = -\int_C w$ is orientation is changed.

Example Consider 1-form $\omega = xdy - ydx$ and curve C—- upper half of the circle $x^2 + y^2 = R^2$, $(y \geq 0)$.

We define the image of the curve not the parameterised curve. Consider different parameterisations of this curve:

$$\mathbf{r}_1(t)\colon \begin{cases} x = R\cos t \\ y = R\sin t \end{cases}, 0 \leq t \leq \pi\,,\ \mathbf{r}_2(t)\colon \begin{cases} x = R\cos\Omega t \\ y = R\sin\Omega t \end{cases}, 0 \leq t \leq \frac{\pi}{\Omega}\,, (\Omega > 0)$$

and

$$\mathbf{r}_3(t)\colon \begin{cases} x = t \\ y = \sqrt{R^2 - t^2} \end{cases}, 0 \leq t \leq R,, \tag{2.43}$$

All these curves are the same image. If $\Omega = 1$ the second curve coincides with the first one. First and second curve have the same orientation (reparameterisation $t \mapsto \Omega t$) The third curve have orientation opposite to first and second (reparameterisation $t \mapsto \cos t$).

Calculate integrals $\int_{C_1} \omega$, $\int_{C_2} \omega$, $\int_{C_3} \omega$

$$\int_{C_1} \omega = \int_0^\pi (xy_t - yx_t)dt = \int_0^\pi (R^2\cos^2 t + R^2\sin^2 t)dt = \pi R^2$$

$$\int_{C_2} \omega = \int_0^{\frac{\pi}{\Omega}} (xy_t - yx_t)dt = \int_0^{\pi} (R^2\Omega\cos^2\Omega t + R^2\Omega\sin^2\Omega t)dt = \pi R^2 \,.$$

These answers coincide: both parameterisation have the same orientation. Note that these integrals is much nicer to calculate in polar coordinates: Recall that in polar coordinates

$$\omega = xdy - ydx = r\cos\varphi d(r\cos\varphi) - r\sin\varphi d(r\cos\varphi) = r^2 d\varphi$$

Hence

$$\int_{C_1} \omega = \int_0^{\pi} (r^2\varphi_t)dt = \pi R^2, \ \int_{C_2} \omega = \int_0^{\pi/\Omega} (r^2\varphi_t)dt = \pi R^2 \,.$$

For the third parameterisation:

$$\int_{C_3} \omega = \int_0^{R} (xy_t - yx_t)dt = \int_0^1 \left(t\left(\frac{-t}{\sqrt{R^2-t^2}}\right) - \sqrt{R^2-t^2}\right) dt =$$

$$-R^2 \int_0^R \frac{dt}{\sqrt{R^2-t^2}} = -R^2 \int_0^1 \frac{du}{\sqrt{1-u^2}} = -\pi R^2$$

We see that the sign is changed.

For other examples see Homeworks.

2.6 Integral over curve of exact form

1-form ω is called exact if there exists a function f such that $\omega = df$.

Theorem
Let ω be an exact 1-form in $\mathbf{E}^n$, $\omega = df$.
Then the integral of this form over an arbitrary curve C: $\quad \mathbf{r} = \mathbf{r}(t) \quad t_1 \leq t \leq t_2$ is defined by the ending and starting point of the curve:

$$\int_C \omega = f\big|_{\partial C} = f(\mathbf{r}_2) - f(\mathbf{r}_1) \,, \quad \mathbf{r}_1 = \mathbf{r}(t_1), \mathbf{r}_2 = \mathbf{r}(t_2) \,. \tag{2.44}$$

Proof: $\int_C df = \int_{t_1}^{t_2} df(\mathbf{v}(t)) = \int_{t_1}^{t_2} \frac{d}{dt} f(\mathbf{r}(t))dt = f(\mathbf{r}(t))|_{t_1}^{t_2}$.

Example Calculate an integral of the form $\omega = 3x^2(1+y)dx + x^3dy$ over the arc of the semicircle $x^2 + y^2 = 1, y \geq 0$.

One can calculate the integral using just the formula (2.40): Choose a parameterisation of C,e.g., $x = \cos t, y = \sin t$, then $\mathbf{v}(t) = -\sin t\partial_x + \cos t\partial_x$ and $\omega(\mathbf{v}(t)) = (3x^2(1+y)dx + x^3 dy)(-\sin t\partial_x + \cos t\partial_y) = -3\cos^2 t(1+\sin t)\sin t + \cos^3 t \cdot \cos t$ and

$$\int_C w = \int_0^\pi (-3\cos^2 t \sin t - 3\cos^2 t \sin^2 t + \cos^4 t)dt = ...$$

Calculations are little bit long.

But for the form $\omega = 3x^2(1+y)dx + x^3 dy$ one can calculate the integral in much more efficient way noting that it is an exact form:

$$\omega = 3x^2(1+y)dx + x^3 dy = d\left(x^3(1+y)\right) \tag{2.45}$$

Hence it follows from the Theorem that

$$\int_C w = f(\mathbf{r}(\pi)) - f(\mathbf{r}(0)) = x^3(1+y)\Big|_{x=1,y=0}^{x=-1,y=0} = -2 \tag{2.46}$$

Corollary The integral of an exact form over an arbitrary closed curve is equal to zero.

Proof. According to the Theorem $\int_C \omega = \int_C df = f\big|_{\partial C} = 0$, because the starting and ending points of closed curve coincide.

Example. Show that the integral of 1-form $\omega = x^5 dy + 5x^4 y dx$ over the ellipse $x^2 + \frac{y^2}{9} = 1$.

The form $\omega = x^5 dy + 5x^4 y dx$ is exact form because $\omega = x^5 dy + 5x^4 y dx = d(x^5 y)$. Hence the integral over ellipse is equal to zero, because it is a closed curve.

2.7 Differential 2-forms in $\mathbf{E}^2$

We considered detailed definition of 1-forms. Now we give some formal approach to describe 2-forms.

Differential forms on $\mathbf{E}^2$ is an expression obtained by adding and multiplying functions and differentials dx, dy. These operations obey usual associativity and distributivity laws but multiplications is not moreover of one-forms on each other is *anticommutative*:

$$\omega \wedge \omega' = -\omega' \wedge \omega \quad \text{if } \omega, \omega' \text{ are 1-forms} \tag{2.47}$$

In particular

$$dx \wedge dy = -dy \wedge dx, dx \wedge dx = 0, dy \wedge dy = 0 \tag{2.48}$$

Example If $\omega = xdy + zdx$ and $\rho = dz + ydx$ then

$$\omega \wedge \rho = (xdy + zdx) \wedge (dz + ydx) = xdy \wedge dz + zdx \wedge dz + xydy \wedge dx$$

and

$$\rho \wedge \omega = (dz + ydx) \wedge (xdy + zdx) = xdz \wedge dy + zdz \wedge dx + xydx \wedge dy = -\omega \wedge \rho$$

Changing of coordinates. If $\omega = a(x,y)dx \wedge dy$ be two form and $x = x(u,v), y = y(u,v)$ new coordinates then $dx = x_u du + x_v dv$, $dy = y_u du + y_v dv$ ($x_u = \frac{\partial x(u,v)}{\partial u}$, $x_v = \frac{\partial x(u,v)}{\partial v}$, $y_u = \frac{\partial y(u,v)}{\partial u}$, $y_v = \frac{\partial y(u,v)}{\partial v}$). and

$$a(x,y)dx \wedge dy = a\left(x(u,v), y(u,v)\right)(x_u du + x_v dv) \wedge (y_u du + y_v dv) = \tag{2.49}$$

$$a\left(x(u,v), y(u,v)\right)(x_u du + x_v dv)\left(x_u y_v du \wedge dv + x_v y_u dv \wedge du\right) =$$
$$a\left(x(u,v), y(u,v)\right)(x_u y_v - x_v y_u) du \wedge dv$$

Example Let $w = dx \wedge dy$ then in polar coordinates $x = r\cos\varphi, y = r\sin\varphi$

$$dx \wedge dy = (\cos\varphi dr - r\sin\varphi d\varphi) \wedge (\sin\varphi dr + r\cos\varphi d\varphi) = rdr \wedge d\varphi \tag{2.50}$$

2.8 0-forms (functions) $\xrightarrow{d}$ 1-forms $\xrightarrow{d}$ 2-forms

We introduced differential d of functions (0-forms) which transform them to 1-form. It obeys the following condition:

- d: is linear operator: $d(\lambda f + \mu g) = \lambda df + \mu dg$
- $d(fg) = df \cdot g + f \cdot dg$

Now we introduce differential on 1-forms such that

- d: is linear operator on 1-forms also
- $d(fw) = df \wedge w + fdw$
- $ddf = 0$

Remark Sometimes differential d is called *exterior differential.*

Perform calculations using this definition and (2.47):

$$d\omega = d(\omega_1 dx + \omega_2 dy) = dw_1 \wedge dx + dw_2 \wedge dy = \left(\frac{\partial \omega_1(x,y)}{\partial x} dx + \frac{\partial \omega_1(x,y)}{\partial y} dy\right) \wedge dx +$$

$$\left(\frac{\partial \omega_2(x,y)}{\partial x} dx + \frac{\partial \omega_2(x,y)}{\partial y} dy\right) \wedge dy = \left(\frac{\partial \omega_2(x,y)}{\partial x} - \frac{\partial \omega_1(x,y)}{\partial y}\right) dx \wedge dy$$

Example Consider 1-form $\omega = xdy$. Then $d\omega = d(xdy) = dx \wedge dy$. It is instructive to perform calculations for $d\omega$ in polar coordinates. Note that

$$\omega = xdy = \frac{1}{2}(xdy - ydx) + \frac{1}{2}(xdy + ydx) = \frac{1}{2} r^2 d\varphi + d\left(\frac{xy}{4}\right) = \frac{1}{2} r^2 d\varphi + d\left(\frac{r^2 \sin 2\varphi}{4}\right)$$

Hence in polar coordinates

$$d\omega = d\left(\frac{1}{2} r^2 d\varphi + d\left(\frac{r^2 \sin 2\varphi}{4}\right)\right) = r dr \wedge d\varphi \tag{2.51}$$

because $d^2 = 0$. Answers coincide, $dx \wedge dy = r dr \wedge d\varphi$.

2.9 Exact and closed forms

We know that it is very easy to integrate exact 1-forms over curves (se the subsection "Integral over curve of exact form")

How to know is the 1-form exact or no?

Definition We say that one form ω is *closed* if two form $d\omega$ is equal to zero.

Example One-form $xdy + ydx$ is closed because $d(xdy + ydx) = 0$.

(See other examples in the Homeworks.)

It is evident that exact 1-form is closed:

$$\omega = d\rho \Rightarrow d\omega = d(d\rho) = d \circ d\rho = 0 \tag{2.52}$$

We see that the condition that form is closed is necessary condition that form is exact.

So if $d\omega \neq 0$, i.e. the form is not closed, then it is not exact.

Is this condition sufficient? Is it true that a closed form is exact?

In general the answer is: *No.*

E.g. we considered differential 2-form

$$\omega = \frac{xdy - ydx}{x^2 + y^2} \tag{2.53}$$

defined in $\mathbf{E}^2\backslash 0$. It is closed, but it is not exact (See solution of Homework 5).

How to recognize for 1-form w is it exact or no?

Inverse statement (Poincaré lemma) is true if 1-form is well-defined in $\mathbf{E}^2$:

A closed 1-form ω in $\mathbf{E}^n$ is exact if it is well-defined at all points of $\mathbf{E}^n$, i.e. if it is differentiable function at all points of $\mathbf{E}^n$.

Sketch a proof for 1-form in $\mathbf{E}^2$: if ω is defined in whole $\mathbf{E}^2$ then consider the function

$$F(\mathbf{r}) = \int_{C_\mathbf{r}} \omega \tag{2.54}$$

where we denote by $C_\mathbf{r}$ an arbitrary curve which starts at origin and ends at the point $\mathbf{r}$. It is easy to see that the integral is well-defined and one can prove that $\omega = df$.

The explicit formula for the function (2.54) is the following: If $\omega = a(x,y)dx + b(x,y)dy$ then

$$F(x,y) = \int_0^1 (a(tx,ty)x + b(tx,ty)y)\, dt \tag{2.55}$$

Exercise Check by straightforward calculation that $\omega = dF$.

2.10 Integration of two-forms. Area of the domain

We know that 1-form is a linear function on tangent vectors. If $\mathbf{A}, \mathbf{B}$ are two vectors attached at the point $\mathbf{r}_0$, i.e. tangent to this point and ω, ρ are two 1-forms then one defines the value of $\omega \wedge \rho$ on $\mathbf{A}, B$ by the formula

$$\omega \wedge \rho(\mathbf{A}, \mathbf{B}) = \omega(\mathbf{A})\rho(B) - \omega(B)\rho(A) \tag{2.56}$$

We come to bilinear anisymmetric function on tangent vectors. If $\sigma = a(x,y)dx \wedge dy$ is an arbitrary two form then this form defines bilinear form on pair of tangent vectors: $\sigma(\mathbf{A}, \mathbf{B}) =$

$$a(x,y)dx \wedge dy(A,B) = a(x,y)\left(dx(\mathbf{A})dy(\mathbf{B}) - dx(\mathbf{B})dy(\mathbf{A})\right) = a(x,y)(A_xB_y - A_yB_y) \tag{2.57}$$

One can see that in the case if $a = 1$ then right hand side of this formula is nothing but the area of parallelogram spanned by the vectors $\mathbf{A}, \mathbf{B}$.

This leads to the conception of integral of form over domain.

Let $\omega = a(x)dx \wedge dy$ be a two form and D be a domain in $\mathbf{E}^2$. Then by definition

$$\int_D w = \int_D a(x,y)dxdy \tag{2.58}$$

If $\omega = dx \wedge dy$ then

$$\int_D w = \int_D (x,y)dxdy = \text{Area of the domain } D \tag{2.59}$$

The advantage of these formulae is that we do not care about coordinates[4]

Example Let D be a domain defined by the conditions

$$\begin{cases} x^2 + y^2 \leq 1 \\ y \geq 0 \end{cases} \tag{2.61}$$

Calculate $\int_D dx \wedge dy$.

$\int_D dx \wedge dy = \int_D dxdy =$ area of the $D = \frac{\pi}{2}$.

If we consider polar coordinates then according (2.50)

$$dx \wedge dy = rdr \wedge d\varphi$$

Hence $\int_D dx \wedge dy = \int_D rdr \wedge d\varphi = \int_D rdrd\varphi = \int_0^1 \left(\int_0^\pi d\varphi\right) rdr = \pi \int_0^1 rdr = \pi/2$.

Another example

Example Let D be a domain in $\mathbf{E}^2$ defined by the conditions

$$\begin{cases} \frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} \leq 1 \\ y \geq 0 \end{cases} \tag{2.62}$$

D is domain restricted by upper half of the ellipse and x-axis. Ellipse has the centre at the point $(c, 0)$. Its area is equal to $S = \int_D dx \wedge dy$. Consider new variables x', y': $x = c + ax', y = by'$. In new variables domain D becomes the domain from the previous example:

$$\frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} = {x'}^2 + {y'}^2$$

and $dx \wedge dy = abdx' \wedge dy'$. Hence

$$S = \int_{\frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} \leq 1, y \geq 0} dx \wedge dy = ab \int_{x'^2 + y'^2 \leq 1, y' \geq 0} dx' \wedge dy' = \frac{\pi ab}{2} \tag{2.63}$$

---

[4] If we consider changing of coordinates then jacobian appears: If u, v are new coordinates, $x = x(u,v)$, $y = y(u,v)$ are new coordinates then

$$\int a(x,y)dxdy = \int a(x(u,v), y(u,v)) \det \begin{pmatrix} x_u & x_v \\ x_u & x_v \end{pmatrix} dudv \tag{2.60}$$

In formula(2.58) it appears under as a part of coefficient of differential form.

Theorem 2 (Green formula) Let ω be 2-form such that $\omega = d\omega'$ and D be a domain–interior of the closed curve C. Then

$$\int_D w = \int_C w' \tag{2.64}$$

3 Curves in $\mathbf{E}^3$. Curvature

3.1 Curves. Velocity and acceleration vectors

We already study velocity vector of curves. Consider now acceleration vector $\mathbf{a} = \frac{d^2\mathbf{r}(t)}{dt^2}$. For curve in $\mathbf{r} = \mathbf{r}(t)$ $\mathbf{E}^n$ we have

$$\mathbf{v} = \frac{d\mathbf{r}(t)}{dt},\ v^i = \frac{dx^i(t)}{dt},\ (i = 1, 2, \dots, n)\ ,$$

and

$$\mathbf{a} = \frac{d\mathbf{v}(t)}{dt} = \frac{d^2\mathbf{r}(t)}{dt^2}, a^i = \frac{d^2x^i(t)}{dt^2}, (i = 1, 2, \dots, n)\ . \tag{3.1}$$

Velocity vector $\mathbf{v}(t)$ is tangent to the curve. In general acceleration vector is not tangent to the curve. One can consider decomposition of acceleration vector $\mathbf{a}$ on tangential and normal component:

$$\mathbf{a} = \mathbf{a}_{tangent} + \mathbf{a}_\perp, \tag{3.2}$$

where $\mathbf{a}_{tangent}$ is the vector tangent to the curve (collinear to velocity vector) and $\mathbf{a}_\perp$ is orthogonal to the tangent vector (orthogonal to the velocity vector). The vector $\mathbf{a}_\perp$ is called normal acceleration vector of the curve [5].

Example Consider a curve

$$C\colon \quad \begin{cases} x = R\cos\Omega t \\ y = R\sin\Omega t \end{cases} , \tag{3.3}$$

If we consider parameter t as a time then we have the point which moves over circle of the radius R with angular velocity Ω. We see that

$$\mathbf{v} = \begin{pmatrix} -R\Omega\sin\Omega t \\ R\Omega\cos\Omega t \end{pmatrix}, \mathbf{a} = -\begin{pmatrix} R\Omega^2\cos\Omega t \\ R\Omega^2\sin\Omega t \end{pmatrix} = -\Omega^2\mathbf{r}(t)$$

[5] Component of acceleration orthogonal to the velocity vector sometimes is called also *centripetal acceleration*

Speed is constant: $|\mathbf{v}| = R\Omega$. Acceleration is perpendicular to the velocity. (It is just *centripetal acceleration.*)

What happens if speed is increasing, or decreasing, i.e. if angular velocity is not constant? One can see that in this case tangential acceleration is not equal to zero, i.e. the velocity and acceleration are not orthogonal to each other.

Analyze the meaning of an angle between velocity and acceleration vectors for an arbitrary parameterised curve $\mathbf{r} = \mathbf{r}(t)$. For this purpose consider the equation for speed: $|\mathbf{v}|^2 = (\mathbf{v}, \mathbf{v})$ and differentiate it:

$$\frac{d|\mathbf{v}|^2}{dt} = \frac{d}{dt}(\mathbf{v}(t), \mathbf{v}(t)) = 2(\mathbf{v}(t), \mathbf{a}(t)) = |\mathbf{v}(t)||\mathbf{a}(t)| \cos\theta(t) = (\mathbf{v}(t), \mathbf{a}_{tangent}(t)) \tag{3.4}$$

where θ is an angle between velocity vector and acceleration vector.

Suppose that parameter t is just time. We see from this formula that if point moves along the curve $\mathbf{r}(t)$ then

- speed is increasing in time if and only if the angle between velocity and acceleration vector is acute, i.e. tangential acceleration has the same direction as a velocity vector:

$$\frac{d|\mathbf{v}|^2}{dt} > 0 \Leftrightarrow (\mathbf{v}, \mathbf{a}) > 0 \Leftrightarrow \cos\theta > 0 \Leftrightarrow \mathbf{a}_{tang} = \lambda\mathbf{v} \text{ with } \lambda > 0\,. \tag{3.5}$$

- speed is decreasing in time if and only if the angle between velocity and acceleration vector is obtuse, i.e. tangential acceleration has the direction opposite to the direction of a velocity vector.

$$\frac{d|\mathbf{v}|^2}{dt} < 0 \Leftrightarrow (\mathbf{v}, \mathbf{a}) < 0 \Leftrightarrow \cos\theta < 0 \Leftrightarrow \mathbf{a}_{tang} = \lambda\mathbf{v} \text{ with } \lambda < 0\,. \tag{3.6}$$

- speed is constant in time if and only if the velocity and acceleration vectors are orthogonal to each other, i..e. tangential acceleration is equal to zero.

$$\frac{d|\mathbf{v}|^2}{dt} = 0 \Leftrightarrow (\mathbf{v}, \mathbf{a}) = 0 \Leftrightarrow \cos\theta = 0 \Leftrightarrow \mathbf{a}_{tang} = 0\,. \tag{3.7}$$

Example Consider the curve $\mathbf{r}(t)$: $\begin{cases} x(t) = v_x t \\ y(t) = v_y t - \frac{gt^2}{2} \end{cases}$ It is path of the point moving under the gravity force with initial velocity $\mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$. One can see that the curve is parabola: $y = \left(\frac{v_y}{v_x}\right) x - \left(\frac{gv_y^2}{v_x^2}\right) x^2$. We have that $\mathbf{v}(t) = \begin{pmatrix} v_x \\ v_y - gt \end{pmatrix}$ and acceleration vector $\mathbf{a} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$. Suppose that $v_y > 0$. $(\mathbf{v}, \mathbf{a}) = -g(v_y - gt)$. Then at the highest point (vertex of the parabola) $(t = v_y/g)$ acceleration is orthogonal to the velocity. For $t < v_y/g$ angle between acceleration and velocity vectors is obtuse. Speed is decreasing. For $t > v_y/g$ angle between acceleration and velocity vectors is acute. Speed is increasing.

3.2 Behaviour of acceleration vector under reparameterisation

How acceleration vector changes under changing of parameterisation of the curve?

Let C: $\mathbf{r} = \mathbf{r}(t), t_1 \leq t \leq t_2$ be a curve and $t = t(\tau)$ reparametrisation of this curve. We know that for new parameterised curve C': $\mathbf{r}'(\tau) = \mathbf{r}(t(\tau)), \tau_1 \leq \tau \leq \tau_2$ velocity vector $\mathbf{v}'(\tau)$ is collinear to the velocity vector $\mathbf{v}(t)$ (see (2.7)):

$$\mathbf{v}'(\tau) = \frac{d\mathbf{r}'(\tau)}{d\tau} = \frac{d\mathbf{r}(t(\tau))}{d\tau} = \frac{dt(\tau)}{d\tau} \frac{d\mathbf{r}(t(\tau))}{dt} = t_\tau \mathbf{v}(t(\tau))$$

Taking second derivative we see that for acceleration vector:

$$\mathbf{a}'(\tau) = \frac{d^2\mathbf{r}'(\tau)}{d\tau^2} = \frac{d\mathbf{v}'(\tau)}{d\tau} = \frac{d}{d\tau}\left(t_\tau \mathbf{v}(t(\tau))\right) = t_{\tau\tau}\mathbf{v}(t(\tau)) + t_\tau^2 \mathbf{a}(t(\tau)) \quad (3.8)$$

Under reparameterisation acceleration vector in general changes its direction: new acceleration vector becomes linear combination of old velocity and acceleration vectors: direction of acceleration vector does not remain unchanged [6].

[6] The plane spanned by velocity and acceleration vectors remains unchanged.(This plane is called osculating plane.)

We observed this phenomenon already when we considered the moving along the curve with different velocities (see (3.5), (3.6) and (3.7)).

We know that acceleration vector can be decomposed on tangential and normal components (see (3.2)). Study how tangential and normal components change under reparameterisation.

Decompose left and right hand sides of the equation (3.8) on tangential and orthogonal components:

$$\mathbf{a}'(\tau)_{tangent} + \mathbf{a}'(\tau)_{\perp} = t_{\tau\tau}\mathbf{v}(t) + t_{\tau}^2 \left(\mathbf{a}(t)_{tangent} + \mathbf{a}(t)_{\perp}\right)$$

Then comparing tangential and orthogonal components we see that new tangential acceleration is equal to

$$\mathbf{a}'(\tau)_{tangent} = t_{\tau\tau}\mathbf{v}(t) + t_{\tau}^2\mathbf{a}(t)_{tangent} \tag{3.9}$$

and normal acceleration is equal to

$$\mathbf{a}'(\tau)_{\perp} = t_{\tau}^2\mathbf{a}(t)_{\perp} \tag{3.10}$$

The magnitude of normal (centripetal) acceleration under changing of parameterisation is multiplied on the t_{τ}^2. Now recall that magnitude of velocity vector under reparameterisation is multiplied on t_{τ}. We come to very interesting and important observation:

Observation

$$\text{The magnitude } \frac{|\mathbf{a}_{\perp}|}{|\mathbf{v}^2|} \text{ remains unchanged under reparameterisation.} \tag{3.11}$$

We come to the expression which is independent of parameterisation: it must have deep mechanical and geometrical meaning. We see later that it is nothing but curvature.

3.3 Length of the curve

If $\mathbf{r}(t), a \leq t \leq b$ is a parameterisation of the curve L and $\mathbf{v}(t)$ velocity vector then length of the curve is equal to the integral of of $|\mathbf{v}(t)|$ over curve:

$$\text{Length of the curve } L = \int_a^b |\mathbf{v}(t)|dt = \tag{3.12}$$

$$\int_a^b \sqrt{\left(\frac{dx^1(t)}{dt}\right)^2 + \left(\frac{dx^2(t)}{dt}\right)^2 + \cdots + \left(\frac{dx^n(t)}{dt}\right)^2} dt\,.$$

Note that formula above is *reparameterisation* invariant. The length of the image of the curve does not depend on parameterisation. This corresponds to our intuition.

Proof Consider curve $\mathbf{r}_1 = \mathbf{r}_1(t)$, $a_1 \leq t \leq b_1$. Let $t = t(\tau)$, $a_2 < \tau < b_2$ be another parameterisation of the curve $\mathbf{r} = \mathbf{r}(t)$, In other words we have two different parameterised curves $\mathbf{r}_1 = \mathbf{r}_1(t), a_1 \leq t \leq b_1$ and $\mathbf{r}_2 = \mathbf{r}_1(t(\tau))$, $a_2 \leq \tau \leq \mathbf{b}_2$ such that their images coincide (See (2.6)). Then under reparameterisation velocity vector is multiplied on t_τ

$$\mathbf{v}_2(\tau) = \frac{d\mathbf{r}_2}{d\tau} = \frac{dt}{d\tau}\frac{d\mathbf{r}_1}{dt} = t_\tau(\tau)\mathbf{v}_1(t(\tau))$$

Hence

$$L_1 = \int_{a_1}^{b_1} |\mathbf{v}_1(t)|dt = \int_{a_2}^{b_2} |\mathbf{v}_1(t)|\frac{dt(\tau)}{d\tau}d\tau = \int_{a_2}^{b_2} |t_\tau \mathbf{v}_1(t)|d\tau = \int_{a_2}^{b_2} |\mathbf{v}_2(\tau)|d\tau = L_2\,, \tag{3.13}$$

i.e. length of the curve does not change under reparameterisation.

If C: $\mathbf{r} = \mathbf{r}(t)$ $t_1 \leq t \leq t_2$ is a curve in $\mathbf{E}^2$ then its length is equal to

$$L_C = \int_{t_1}^{t_2} |\mathbf{v}(t)|dt = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx(t)}{dt}\right)^2 + \left(\frac{dy(t)}{dt}\right)^2} dt \tag{3.14}$$

3.4 Natural parameterisation of the curves

Non-parameterised curve can be parameterised in many different ways.

Is there any distinguished parameterisation? Yes, it is.

Definition A natural parameter $s = s(t)$ on the curve $\mathbf{r} = r(t)$ is a parameter which defines the length of the arc of the curve between initial point $\mathbf{r}(t_1)$ and the point $\mathbf{r}(t)$.

If a natural parameter s is chosen we say that a curve $\mathbf{r} = \mathbf{r}(s)$ is given in natural parameterisation.

Write down explicit formulae for natural parameter.

Let $C : \mathbf{r}(t), a < t < b$ be a curve in $\mathbf{E}^n$. As always we suppose that it is smooth and regular curve: (i.e. $\mathbf{r}(t)$ has derivatives of arbitrary order, and velocity vector $\mathbf{v} \neq 0$.

Then it follows from (3.12) that

$$s(t) = \{\text{length of the arc of the curve between points } \mathbf{r}(a) \text{ and } \mathbf{r}(t)\} \tag{3.15}$$

$$= \int_a^t |\mathbf{v}(\tau)| d\tau =$$

$$= \int_a^t \sqrt{\left(\frac{dx^1(\tau)}{d\tau}\right)^2 + \left(\frac{dx^2(\tau)}{d\tau}\right)^2 + \cdots + \left(\frac{dx^n(\tau)}{d\tau}\right)^2} d\tau \,. \qquad (3.16)$$

(As always we suppose that it is smooth and regular curve: (i.e. $\mathbf{r}(t)$ has derivatives of arbitrary order, and velocity vector $\mathbf{v} \neq 0$.)

Example Consider circle: $x = R\cos t, y = R\sin t$ in $\mathbf{E}^2$. Then we come to the obvious answer

$$s(t) = \{\text{length of the arc of the circle between points } \mathbf{r}(0) \text{ and } \mathbf{r}(t)\} = Rt =$$

$$\int_0^t \sqrt{\left(\frac{dx(\tau)}{d\tau}\right)^2 + \left(\frac{dy(\tau)}{d\tau}\right)^2} d\tau = \int_0^t \sqrt{R^2 \sin^2 \tau + R^2 \cos^2 \tau} d\tau = \int_a^t R d\tau = Rt$$

$s = Rt$. Hence in natural parameterisation $x = R\cos\frac{s}{R}$, $y = R\sin\frac{s}{R}$.

Remark If we change an initial point then a natural parameter changes on a constant.

For example if we choose as a initial point for the circle above a point $\mathbf{r}(t_1)$ for $t_1 = -\frac{\pi}{2}$, then the length of the arc between points $\mathbf{r}(-\frac{\pi}{2})$ and $\mathbf{r}(0)$ is equal to $R\frac{\pi}{2}$ and

$$s'(t) = s(t) + R\frac{\pi}{2} \,.$$

Another

Example Consider arc of the parabola $x = t, y = t^2, 0 < t < 1$:

$$s(t) = \{\text{length of the arc of the curve for parameter less or equal to } t\} = \qquad (3.17)$$

$$\int_0^t \sqrt{\left(\frac{dx(\tau)}{d\tau}\right)^2 + \left(\frac{dy(\tau)}{d\tau}\right)^2} d\tau =$$

$$\int_0^t \sqrt{1 + 4\tau^2} d\tau = \frac{t\sqrt{1+4t^2}}{2} + \frac{1}{4} \log\left(2t + \sqrt{1+4t^2}\right)$$

The first example was very simple. The second is harder to calculate [7]. In general case natural parameter is not so easy to calculate. But its notion is very important for studying properties of curves.

Natural parameterisation is distinguished. Later we will often use the following very important property of natural parameterisation:

Proposition *If a curve is given in natural parameterisation then*

- *the speed is equal to* 1

$$(\mathbf{v}(s), \mathbf{v}(s)) \equiv 1, \quad \text{i.e. } |\mathbf{v}(s)| \equiv 1\,, \tag{3.18}$$

- *acceleration is orthogonal to velocity, i.e. tangential acceleration is equal to zero:*

$$(\mathbf{v}(s), \mathbf{a}(s)) = 0\,, \quad \text{i.e. } \mathbf{a}_{tangent} = 0\,. \tag{3.19}$$

Proof: For an arbitrary parameterisation $|\mathbf{v}(t)| = \frac{dL(t)}{dt}$, where $L(t)$ is a length of the curve. In the case of natural parameter $L(s) = s$, i.e. $|\mathbf{v}(t)| = \frac{dL(t)}{dt} = 1$. We come to the first relation.

The second relation means that value of the speed does not change (see (3.4) and (3.7)).

3.5 Curvature. Curvature of curves in $\mathbf{E}^2$

How to find invariants of non-parameterised curve, i.e. magnitudes which depend on the points of non-parameterised curve but which do not depend on parameterisation?

Answer at the first sight looks very simple: Consider the distinguished natural parameterisation $\mathbf{r} = \mathbf{r}(s)$ of the curve. Then arbitrary functions on $x^i(s)$ and its derivatives do not depend on parameterisation. But the problem is that it is not easy to calculate

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[7] Denote by $I = \int_0^t \sqrt{1+4\tau^2}d\tau$. Then integrating by parts we come to:

$$I = t\sqrt{1+4t^2} - \int \frac{4\tau^2}{\sqrt{1+4\tau^2}}d\tau = t\sqrt{1+4t^2} - I + \int \frac{1}{\sqrt{1+4\tau^2}}d\tau\,.$$

Hence

$$I = \frac{t\sqrt{1+4t^2}}{2} + \frac{1}{2}\int \frac{1}{\sqrt{1+4\tau^2}}d\tau\,.$$

and we come to the answer.

natural parameter explicitly (See e.g. calculations of natural parameter for parabola in the previous subsection). So it is preferable to know how to construct these magnitudes in arbitrary parameterisation, i.e. construct functions $f(\frac{dx^i}{dt}, \frac{d^2x^i}{dt^2}, \dots)$ such that they *do not depend on parameterisation.*

We define now curvature. First formulate reasonable conditions on curvature:

- it has to be a function of the points of the curve
- it does not depend on parameterisation
- curvature of the line must be equal to zero
- curvature of the circle with radius R must be equal to $1/R$

We first give definition of curvature in natural parameterisation. Then study how to calculate it for a curve in an arbitrary parameterisation.

For a given non-parameterised curve consider natural parameterisation $\mathbf{r} = \mathbf{r}(s)$. We know already that velocity vector has length 1 and acceleration vector is orthogonal to curve in natural parameterisation (see (3.18) and (3.19)). It is just normal (centripetal) acceleration.

Definition. The curvature of the curve in a given point is equal to the modulus (length) of acceleration vector (normal acceleration) in natural parameterisation. Namely, let $\mathbf{r}(s)$ be natural parameterisation of this curve. Then curvature at every point $\mathbf{r}(s)$ of the curve is equal to the length of acceleration vector:

$$k = |\mathbf{a}(s)|, \qquad \mathbf{a}(s) = \frac{d^2\mathbf{r}(s)}{ds^2} \tag{3.20}$$

First check that it corresponds to our intuition (see reasonable conditions above)

It does not depend on parameterisation by definition.

It is evident that for the line in normal parameterisation $x^i(s) = x_0^i + b^i s$ ($\sum b^i b^i = 1$) the acceleration is equal to zero.

Now check that the formula (3.20) gives a natural answer for circle. For circle of radius R in natural parameterisation

$$\mathbf{r} = \mathbf{r}(s) = (x(s), y(s)), \quad \text{where} \quad x(s) = R\cos\frac{s}{R}, \quad y(s) = R\sin\frac{s}{R}$$

(length of the arc of the angle θ of the circle is equal to $s = R\theta$.) Then

$$\mathbf{a}(s) = \frac{d\mathbf{r}^2(s)}{ds^2} = \left(-\frac{1}{R}\cos\frac{s}{R}, -\frac{1}{R}\sin\frac{s}{R}\right)$$

and for curvature

$$k = |\mathbf{a}(s)| = \frac{1}{R} \tag{3.21}$$

we come to the answer which agrees with our intuition.

Geometrical meaning of curvature: We will consider this question later in more detail. But even now it is easy to see from this example that $\frac{1}{k}$ is *just a radius of the circle which has second order touching to curve.*(See subsection "Second order contact")

3.6 Curvature of curve in an arbitrary parameterisation.

Let curve be given in an arbitrary parameterisation. How to calculate curvature. One way is to go to natural parameterisation. But in general it is very difficult (see the example of parabola in the subsection "Natural parameterisation").

We do it in another more elegant way.

Return to our observation (3.11) in the end of the subsection "Velocity and acceleration vectors". Consider for any curve $\mathbf{r}(t)$ ratio of the modulus of normal acceleration and square of velocity vectors. We denote this ratio temporarily by the letter k'

$$k' = \frac{|\mathbf{a}_\perp|}{|\mathbf{v}|^2} = \frac{|\mathbf{a}_\perp|}{(\mathbf{v}, \mathbf{v})}$$

Our aim is to prove that k' is just the curvature k: $k' = k$. Do it.

Note that in natural parameterisation speed is equal to 1 and acceleration is orthogonal to curve: $\mathbf{a} = \mathbf{a}_\perp$, $|\mathbf{v}| = 1$. Hence in natural parameterisation k' is equal just to modulus of acceleration vector, i.e. to the curvature. But on the other hand we proved already that k' is reparameterisation invariant. Hence $k' = k$. We come to the following result:

Proposition Curvature of the curve in terms of an arbitrary parameterisation $\mathbf{r} = \mathbf{r}(t)$ is given by the formula:

$$k = \frac{|\mathbf{a}_\perp(t)|}{|\mathbf{v}(t)|^2}, \tag{3.22}$$

where $\mathbf{v}(t) = d\mathbf{r}(t)/dt$ is velocity vector and $\mathbf{a}_\perp(t)$ is normal acceleration.

Advantage of this formula is that it is given in an arbitrary parameterisation. Disadvantage of this formula is that we still do not know how to calculate $\mathbf{a}_\perp(t)$.

Do now the next step: Note that

$$\frac{|\mathbf{a}_{norm}(t)|}{|\mathbf{v}|^2} = \frac{|\mathbf{a}_\perp(t)|\cdot|\mathbf{v}|}{|\mathbf{v}|^3} = \frac{\text{Area of parallelogram spanned by the vectors } \mathbf{a},\mathbf{v}}{|\mathbf{v}|^3} \tag{3.23}$$

The last formula is practical. We already know how to calculate area of parallelogram spanned by the vectors $\mathbf{a},\mathbf{v}$ in $\mathbf{E}^3$ and in $\mathbf{E}^2$: In $\mathbf{E}^3$ it is just given by vector product: $S = |\mathbf{v}\times\mathbf{a}|$. In $\mathbf{E}^2$, $S = |v_x a_y - v_y a_x|$ because $\mathbf{v}\times\mathbf{a} = (v_x\mathbf{e}_x + v_y\mathbf{e}_y)\times(a_x\mathbf{e}_x + a_y\mathbf{e}_y) = (v_x a_y - v_y a_x)\mathbf{e}_z$ if curve is in $\mathbf{E}^2$.

In general case if curve is in $\mathbf{E}^n$ then to calculate the area S of parallelogram note that $S = |\mathbf{v}||\mathbf{a}||\sin\theta|$ where $|\mathbf{v}||\mathbf{a}|\cos\theta = (\mathbf{v},\mathbf{a})$. Hence $S = |\mathbf{v}||\mathbf{a}|\sqrt{1-\cos^2\theta} = \sqrt{\mathbf{v}^2\mathbf{a}^2 - (\mathbf{v}\cdot\mathbf{a})^2}$.

Using (3.23) we rewrite the formula (3.22) and come to

Proposition *Let C be non-parameterised curve (equivalence class of parameterised curves). Let $\mathbf{r}(t)$ be any parameterisation of this curve. Then curvature at the point $\mathbf{r}(t_0)$ of the curve is equal to the area of parallelogram formed by the vectors $\mathbf{a},\mathbf{v}$ at this point divided by the cube of the speed $\mathbf{v}$.*

$$k\big|_{\mathbf{r}=\mathbf{r}(t_0)} = \frac{|\mathbf{a}_\perp(t)||\mathbf{v}|}{|\mathbf{v}(t)|^3} =$$

$$\frac{\text{Area of parallelogram formed by the vectors } \mathbf{v} \text{ and } \mathbf{a}}{\text{Cube of the speed}} = \frac{\sqrt{\mathbf{v}^2\mathbf{a}^2 - (\mathbf{v}\cdot\mathbf{a})^2}}{|\mathbf{v}|^3} \tag{3.24}$$

In the special cases of $\mathbf{E}^3$ and $\mathbf{E}^2$ we have

$$\text{In } \mathbf{E}^3 \qquad k = \frac{|\mathbf{v}(t)\times\mathbf{a}(t)|}{|\mathbf{v}(t)|^3}, \tag{3.25}$$

$$\text{and in } \mathbf{E}^2 \text{ we come to} \qquad k = \frac{|\mathbf{v}(t)\times\mathbf{a}(t)|}{|\mathbf{v}(t)|^3} = \frac{|v_x a_y - v_y a_x|}{(v_x^2+v_y^2)^{\frac{3}{2}}} \tag{3.26}$$

or more explicit formula:

$$k = \frac{|x_t y_{tt} - y_t x_{tt}|}{(x_t^2 + y_t^2)^{\frac{3}{2}}} \tag{3.27}$$

This is workable formula.

Remark 1. Of course one can come to formulae (3.24), (3.25) and (3.26) by "brute force" making straightforward attack. Instead considering explicitly natural parameterisation of the curve we just try to rewrite the formula in definition (3.20) in arbitrary parameterisation using chain rule. The calculations are not transparent. Try to do it.

Consider examples of calculating curvature for curves in $\mathbf{E}^2$.

Example. Consider circle of the radius R, $x^2 + y^2 = R^2$. Take any parameterisation, e.g. $x = R\cos t, y = R\sin t$. Then $\mathbf{v} = (-R\cos t, R\sin t)$, $\mathbf{a} = (-R\sin t, -R\cos t)$. Applying the formula (3.27) we come to

$$k = \frac{|x_t y_{tt} - y_t x_{tt}|}{(x_t^2 + y_t^2)^{\frac{3}{2}}} = \frac{|R^2\cos^2 t + R^2\sin^2 t|}{(R^2\cos^2 t + R^2\sin^2 t)^{\frac{3}{2}}} = \frac{R^2}{R^3} = \frac{1}{R}$$

Example Consider ellipse

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2}$$

. Choose a parameterisation, e.g. $\mathbf{r}(t)\colon \begin{cases} x = x_0 + a\cos t \\ y = y_0 + b\sin t \end{cases}$, $0 \leq t < 2\pi$. Then $\mathbf{v}(t) = \begin{pmatrix} -a\sin t \\ b\cos t \end{pmatrix}$, $\mathbf{a}(t) = \begin{pmatrix} -a\cos t \\ -b\sin t \end{pmatrix}$ and for curvature we have

$$k = \frac{|x_t y_{tt} - y_t x_{tt}|}{(x_t^2 + y_t^2)^{\frac{3}{2}}} = k = \frac{|ab\sin^2 t + ab\cos^2 t|}{(a^2\sin^2 t + b^2\cos^2 t)^{\frac{3}{2}}} = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{\frac{3}{2}}}\,. \tag{3.28}$$

Exercise Calculate curvatures at the points $(x_0 \pm a, y_0)$ and $(x_0, y_0 \pm b)$.

Example For any function $f = f(x)$ one can consider its graph as not-parameterised curve C_f. Calculate curvature of the curve C_f at any point $(x, f(x))$.

One can choose parameteristaion: $\mathbf{r}(t)\colon \quad \begin{cases} x = t \\ y = f(t) \end{cases}$.

Then $\mathbf{v}(t) = \begin{pmatrix} 1 \\ f'(t) \end{pmatrix}$, $\mathbf{a}(t) = \begin{pmatrix} 0 \\ f''(t) \end{pmatrix}$ and we have for the curvature that

$$k = \frac{|x_t y_{tt} - y_t x_{tt}|}{(x_t^2 + y_t^2)^{\frac{3}{2}}} = k = \frac{|f''(t)|}{(1 + f'(t)^2)^{\frac{3}{2}}} \tag{3.29}$$

3.7 Second order contact (touching) of curves

Let C_1, C_2 be two curves in $\mathbf{E}^2$. For simplicity we here consider only curves in $\mathbf{E}^2$.

Definition Two non-parameterised curves C_1, C_2 have second order contact (touching) at the point $\mathbf{r}_0$ if

- They coincide at the point $\mathbf{r}_0$
- they have the same tangent line at this point
- they have the same curvature at the point $\mathbf{r}_0$

If $\mathbf{r}_1(t), \mathbf{r}_2(t)$ are an arbitrary parameterisations of these curves such that $\mathbf{r}_1(t_0) = \mathbf{r}_2(t_0) = \mathbf{r}_0$ then the condition that they have the same tangent line means that velocity vectors $\mathbf{v}_1(t), \mathbf{v}_2(t)$ are collinear at the point t_0.

(As always we assume that curves under considerations are smooth and regular, i.e. $x(t), y(t)$ are smooth functions and velocity vector $\mathbf{v}(t) \neq 0$.)

Example Consider two curves C_f, C_g—graphs of the functions f_1, f_2. Recall that curvature of the graph of the function f at the point $(x, y = f(x))$ is equal to (see (3.29))

$$k(x) = \frac{f''(x)}{(1 + f'(x))^{\frac{3}{2}}} \tag{3.30}$$

Then condition of the second order touching at the point $\mathbf{r}_0 = (x_0, y_0)$ means that

$$\begin{cases} \text{They coincide at the point } \mathbf{r}_0\text{: } f(x_0) = g(x_0) \\ \text{They have the same tangent line at this point: } f'(x_0) = g'(x_0) \\ \text{They have the same curvature at the point } \mathbf{r}_0\text{:} \frac{f''(x_0)}{(1+f'(x_0))^{\frac{3}{2}}} = \frac{g''(x_0)}{(1+g'(x_0))^{\frac{3}{2}}}, \text{ i.e. } f''(x_0) = g''(x_0) \end{cases}$$

We see that second order touching means that difference of the functions in vicinity of the point x_0 is of order $o((x-x_0)^2)$. Indeed due to Taylor formula

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2 + \dots \\ g(x) &= g(x_0) + g'(x_0)(x-x_0) + \frac{1}{2}g''(x_0)(x-x_0)^2 + \dots \end{aligned} \tag{3.31}$$

where we denote by dots terms which are $o(x-x_0)^2$. (They say that $f(x) = o(x-x_0)^n$ if $\lim_{x\to x_0} \frac{f(x)}{(x-x_0)^n} = 0$).

Hence

$$f(x) - g(x) = o(x-x_0)^2 \tag{3.32}$$

because $f(x_0) = g(x_0), f(x_0) = g(x_0), f'(x_0) = g'(x_0)$ and $f''(x_0) = g''(x_0)$

In general case if two curves have second order contact then in the vicinity of the contact point one can consider these curves as a graphs of the functions $y = f(x)$ (or $x = f(y)$).

To clarify geometrical meaning of second order touching consider the case where one of the curves is a circle. Then second order touching means that curvature of one of these curves is equal to $1/R$, where R is a radius of the circle.

We see that to calculate the radius of the circle which has the second order touching with the given curve at the given point we have to calculate the curvature of this curve at this given point.

Example. Let C_1 be parabola $y = ax^2$ and C_2 be a circle. Suppose these curves have second order contact at the vertex of the parabola: point $0, 0$.

Calculate the curvature of the parabola at the vertex. Curvature at the vertex is equal to $k(t)|_{t=0} = 2a$ (see Homework). Hence the radius of the circle which has second order touching is equal to

$$R = \frac{1}{2a}.$$

To find equation of this circle note that the circle which has second order touching to parabola at the vertex passes trough the vertex (point $(0, 0)$) and is tangent to x-axis. The radius of this circle is equal to $R = \frac{1}{2a}$. Hence equation of the circle is

$$(x - R^2) + y^2 = 0, \text{ where } R = \frac{1}{2a}$$

One comes to the same answer by the following detailed analysis:

Consider equation of a circle: $(x-x_0)^2 + (y-y_0)^2 = R^2$. The condition that curves coincide at the point $(0, 0)$ means that $x_0^2 + y_0^2 = R^2$. x-axis is tangent to parabola at the vertex. Hence it is tangent to the circle too. Hence $y_0^2 = R^2$ and $x_0 = 0$. We see that an equation of the circle is $x^2 + (y - R)^2 = R^2$. The circle $x^2 + (y - R)^2 = x^2 + y^2 - 2yR = 0$ in the vicinity of the point $(0, 0)$ can be considered as a graph of the function $y = R - \sqrt{R^2 - x^2}$.. The condition that functions $y = ax^2$ and $y = R - \sqrt{R^2 - x^2}$ have second order contact means that

$$R - \sqrt{R^2 - x^2} = ax^2 + \text{terms of the order less that } x^2 \,.$$

But

$$R - \sqrt{R^2 - x^2} = R - R\sqrt{1 - \frac{x^2}{2R^2}} = R - R\left(1 - \frac{x^2}{2R^2} + o(x^2)\right) = \frac{x^2}{2R} + o(x^2) \,.$$

Comparing we see that $a = \frac{1}{2R}$ and $\frac{1}{R} = 2a$. But curvature of the parabola at the vertex is equal to $k = 2a$ (if $a > 0$). We see that $k = \frac{1}{R}$.

3.8 Integral of curvature over planar curve.

We consider in this paragraph the following problem: Let $C = \mathbf{r}(t)$ be a planar curve, i.e. a curve in $\mathbf{E}^2$.

Let $\mathbf{n}(\mathbf{r}(t)$ be a unit normal vector field to the curve, i.e. $\mathbf{n}$ is orthogonal to the curve (velocity vector) and it has unit length.

E.g. if $\mathbf{r}(t) : x(t) = R\cos t, y(t) = R\sin t$, then $\mathbf{n}(\mathbf{r}(t)) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

If point moves along the curve $\mathbf{r}(t), t_1 \leq t \leq t_2$ then velocity vector and vector field $\mathbf{n}(t)$ rotate on the same angle. It turns out that this angle is expressed via integral of curvature over the curve...

Try to analyze the situation:

Proposition Let C: $\mathbf{r}(t)$ be a curve in $\mathbf{E}^2$, $\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt}$, velocity vector, $k(\mathbf{r}(t))$— curvature and $\mathbf{n}(t)$ unit normal vector field. Denote by $\varphi(t)$ the angle between normal vector $\mathbf{n}(t)$ and x-axis.

Then

$$\frac{d\mathbf{n}(t)}{dt} = \pm k(\mathbf{r}(t))\mathbf{v}(t) \tag{3.33}$$

$$\frac{d\varphi(t)}{dt} = \pm k(\mathbf{r}(t))|\mathbf{v}(t)| \tag{3.34}$$

(Sign depends on the orientation of the pair of vectors $(\mathbf{v}, \mathbf{n})$)

Note that the second statement of the Proposition has a clear geometrical meaning: If C is a circle of the radius R then RHS of (3.34) is equal to $\frac{v}{R}$. It is just angular velocity $d\varphi/dt$.

To prove this Proposition note that $(\mathbf{n}, \mathbf{n}) = 1$. Hence

$$0 = \frac{d}{dt}(\mathbf{n}(t), \mathbf{n}(t)) = 2\left(\frac{d\mathbf{n}(t)}{dt}, \mathbf{n}(t)\right) \,,$$

i.e. vector $\frac{d\mathbf{n}(t)}{dt}$ is orthogonal to the vector $\mathbf{n}$. This means that $\frac{d\mathbf{n}(t)}{dt}$ is collinear to $\mathbf{v}(t)$, because curve is planar. We have $\frac{d\mathbf{n}(t)}{dt} = \kappa(\mathbf{r}(t))\mathbf{v}(t)$ where κ is a coefficient. Show that the coefficient κ is just equal to curvature k (up to a sign). Clearly $(\mathbf{n}, \mathbf{v}) = 0$ because these vectors are orthogonal. Hence

$$0 = \frac{d}{dt}(\mathbf{n}(t), \mathbf{v}(t)) = \left(\frac{d\mathbf{n}(t)}{dt}, \mathbf{v}(t)\right) + \left(\mathbf{n}(t), \frac{d\mathbf{v}(t)}{dt}\right) =$$

$$(\kappa(\mathbf{r}(t))\mathbf{v}(t), \mathbf{v}(t)) + (\mathbf{n}(t), \mathbf{a}(t)) = \kappa(\mathbf{r}(t))|\mathbf{v}(t)|^2 + (\mathbf{n}, \mathbf{a}_\perp),$$

because $(\mathbf{n}(t), \mathbf{a}(t)) = (\mathbf{n}, \mathbf{a}_\perp)$. But $(\mathbf{n}, \mathbf{a}_\perp)$ is just centripetal acceleration: $(\mathbf{n}, \mathbf{a}_\perp) = \pm|\mathbf{a}_\perp|$ and curvature is equal to $|\mathbf{a}_\perp|/|\mathbf{v}|^2$. Hence we come to $\kappa(\mathbf{r}(t)) = \pm\frac{|\mathbf{a}_\perp|}{|\mathbf{v}|^2} = \pm k$. Thus we prove (3.33).

To prove (3.34) consider expansion of vectors $\mathbf{n}(t), \mathbf{v}(t)$ over basis vectors ∂_x, ∂_y. We see that

$$\mathbf{n}(t) = \cos\varphi(t)\partial_x + \sin\varphi(t)\partial_y \text{ and } \mathbf{v}(t) = |\mathbf{v}(t)|\left(-\sin\varphi(t)\partial_x + \cos\varphi(t)\partial_y\right) \tag{3.35}$$

Differentiating $\mathbf{n}(t)$ by t we come to $\frac{d\mathbf{n}(t)}{dt} = \frac{d\varphi(t)}{dt}\left(-\sin\varphi(t)\partial_x + \cos\varphi(t)\partial_y\right) = \frac{d\varphi(t)}{dt}\frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$. Comparing this equation with equation (3.33) we come to (3.34).

The appearance of sign factor in previous formulae related with the fact that normal vector field is defined up to a sign factor $\mathbf{n} \to -\mathbf{n}$.

It is useful to write formulae (3.33), (3.34) in explicit way. Let $\mathbf{r}(t)\colon x(t), y(t)$ be a parameterisation of the curve. Then $\mathbf{v}(t) = \begin{pmatrix} x_t \\ y_t \end{pmatrix}$ velocity vector. One can define normal vector field as

$$\mathbf{n}(t) = \frac{1}{\sqrt{x_t^2 + y_t^2}}\begin{pmatrix} -y_t \\ x_t \end{pmatrix} \tag{3.36}$$

or changing the sign as

$$\mathbf{n}(t) = \frac{1}{\sqrt{x_t^2 + y_t^2}}\begin{pmatrix} y_t \\ -x_t \end{pmatrix} \tag{3.37}$$

If we consider (3.36) for normal vector field then

$$\frac{d\mathbf{n}(t)}{dt} = \frac{x_{tt}y_t - y_{tt}x_t}{(x_t^2 + y_t^2)^{\frac{3}{2}}}\begin{pmatrix} x_t \\ y_t \end{pmatrix} \tag{3.38}$$

Recalling that $k = \frac{|x_{tt}y_t - y_{tt}|x_t}{(x_t^2+y_t^2)^{\frac{3}{2}}}$ we come to (3.33). For the angle we have

$$\frac{d\varphi}{dt} = \frac{x_t y_{tt} - y_t x_{tt}}{(x_t^2 + y_t^2)^{\frac{3}{2}}}\sqrt{x_t^2 + y_t^2} = \frac{x_t y_{tt} - y_t x_{tt}}{(x_t^2 + y_t^2)} \tag{3.39}$$

This follows from the considerations above but it can be also calculated straightforwardly.

Remark Note that last two formulae do not possess indefenity in sign.

This Proposition has very important application. Consider just two examples:

Consider upper half part of the ellipse $x^2/a^2 + y^2/b^2 = 1, y \geq 0$. We already know that curvature at the point $x = a\cos t, y = b\cos t$ of the ellipse is equal to

$$k = \frac{ab}{(a^2 \sin t^2 t + b^2 \cos^2 t)^{3/2}}$$

and speed is equal to $\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}$ Apply formula (3.34) of Proposition. The curvature is not equal to zero at all the point. Hence the sign in the (3.34) is the same for all the points, i.e.

$$\pi = \int_0^\pi d\varphi(t)dt = \pm \int_0^\pi k(\mathbf{r}(t))|\mathbf{v}(t)| = \tag{3.40}$$

$$\int_0^\pi \frac{ab}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\, dt = \int_0^\pi \frac{abdt}{a^2 \sin^2 t + b^2 \cos^2 t}\,.$$

We calculated this integral using geometrical considerations: left hand side represents the angle of rotation of normal unit vector and this angle is equal to π. Try to calculate the last integral straightforwardly: it is not easy exercise in calculus.

Another example: Let $\mathbf{r} = \mathbf{r}(t), x = x(t), y = y(t), t_1 \leq t \leq t_2$ be a closed curve in $\mathbf{E}^2$ ($\mathbf{r}(t_1) = \mathbf{r}(t_2)$.) We suppose that it possesses self-intersections points. We cannot us a formula (3.34) for integration because in general curvature may vanish at some points, but we still can use the formula (3.39). The rotation of the angle φ is equal to $2\pi n$, (n-is called winding number of the curve). Hence according to (3.39) see that

$$\int_{t_1}^{t_2} \frac{x_t y_{tt} - x_t y_{tt}}{x_t^2 + y_t^2} = 2\pi n$$

or

$$\frac{1}{2\pi} \int_{t_1}^{t_2} \frac{x_t y_{tt} - x_t y_{tt}}{x_t^2 + y_t^2} = n \tag{3.41}$$

The integrand us equal to the curvature multiplied by the speed (up to a sign). Left hand side is integral of continuous function divided by transcendent number π. The geometry tells us that the answer must to be equal to integer number.

4 Surfaces in $\mathbf{E}^3$. Curvatures and Shape operator.

In this section we study surfaces in $\mathbf{E}^3$. One can define surfaces by equation $F(x, y, z)$ or by parametric equation

$$\mathbf{r}(u, v)\colon \begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases}, \tag{4.1}$$

Example the equation $x^2+y^2 = R^2$ defines cylinder (cylindrical surface). z-axis is the axis of this cylinder, R is radius of this cylinder. One can define this cylinder by the parametric equation

$$\mathbf{r}(\varphi, h)\colon \begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \\ z = h \end{cases}, \tag{4.2}$$

where φ is the angle $0 \leq \varphi < 2\pi$ and $-\infty < h < \infty$ takes arbitrary real values.

Example sphere $x^2 + y^2 + z^2 = R^2$:

$$\mathbf{r}(\theta, \varphi)\colon \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases}, 0 \leq \theta \leq \pi,\ 0 \leq \varphi \leq 2\pi \tag{4.3}$$

Example graph of the surface $z = F(x, y)$:

$$\mathbf{r}(u, v)\colon \begin{cases} x = u \\ y = v \\ z = F(u, v) \end{cases}, -\infty < u < \infty,\ -\infty < v < \infty \tag{4.4}$$

It is interesting to consider this example when $F = uv$ we come to the surface saddle:

$$\mathbf{r}(u, v)\colon \begin{cases} x = u \\ y = v \\ z = uv \end{cases}, -\infty < u < \infty,\ -\infty < v < \infty \tag{4.5}$$

4.1 Coordinate basis, tangent plane to the surface.

Coordinate basis vectors are $\mathbf{r}_u = \partial_u, \mathbf{r}_v = \partial_v$. At the any point $\mathbf{p}$, $\mathbf{p} = \mathbf{r}(u, v)$ these vectors span the plane, (two-dimensional linear space) $T_{\mathbf{p}}M$ in three dimensional vector space $T_{\mathbf{p}}E^3$.

$$T_{\mathbf{p}}M = \{\lambda\mathbf{r}_u + \mu\mathbf{r}_v, \lambda, \mu \in \mathbf{R}\}, \qquad T_{\mathbf{p}}\,\text{subspace in}\,T_{\mathbf{p}}E^3 \tag{4.6}$$

E.g. consider the point $\mathbf{p} = (R, 0, 0)$ on the cylinder (4.2). Then $\mathbf{p} = \mathbf{r}(\varphi, h)$ for $\varphi = 0, h = 0$. Coordinate basis vectors are

$$\mathbf{r}_\varphi = \begin{pmatrix} -R\sin\varphi \\ R\cos\varphi \\ 0 \end{pmatrix}, \qquad \mathbf{r}_h = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{4.7}$$

or in other notations

$$\mathbf{r}_\varphi = -R\sin\varphi\partial_x + R\cos\varphi\partial_y, \qquad \mathbf{r}_h = \partial_z \tag{4.8}$$

At the point $\mathbf{p} = (R, 0, 0)$ they are are equal to the vectors ∂_y and ∂_z respectively attached at this point. Tangent plane at the point $\mathbf{p}$ is the plane passing through the point $\mathbf{p}$ spanned by the vectors ∂_y and ∂_z.

4.2 Curves on surfaces. Length of the curve. Internal and external point of the view. First Quadratic Form

Let $M\colon \mathbf{r} = \mathbf{r}(u, v)$ be a surface and C curve on this surface, i.e. $C\colon \mathbf{r}(t) = \mathbf{r}(u(t, v(t)))$.

Consider an arbitrary point $\mathbf{p} = \mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ at this curve.

- $T_{\mathbf{p}}E^3$—three-dimensional tangent space to the point $\mathbf{p}$,
- $T_{\mathbf{p}}M$—two dimensional linear space tangent to the surface at the point $\mathbf{p}$, spanned by the tangent vectors ∂_u, ∂_v
- $T_{\mathbf{p}}M$—one dimensional linear space tangent to the curve at the point $\mathbf{p}$ spanned by the velocity vector $\mathbf{v}(t)$.

$$\mathbf{v}(t) = \frac{d\mathbf{r}(u(t), v(t))}{dt} = u_t\frac{\partial \mathbf{r}}{\partial u} + v_t\frac{\partial \mathbf{r}}{\partial v} = u_t\mathbf{r}_u + v_t\mathbf{r}_v \tag{4.9}$$

These tangent spaces form flag of subspaces $T_{\mathbf{p}}C < T_{\mathbf{p}}M < T_{\mathbf{p}}E^3$.

How to calculate the length of the arc of the curve: $C\colon \mathbf{r}(t) = \mathbf{r}(u(t, v(t)))$, $t_1 \leq t_2$. External and internal observer do it in different ways. External observer just looks at the curve as the curve in ambient space. He uses the formula (3.12):

$$L = \text{Length of the curve } L = \int_a^b |\mathbf{v}(t)|dt = \int_a^b \sqrt{\left(\frac{dx^1(t)}{dt}\right)^2 + \left(\frac{dx^2(t)}{dt}\right)^2 + \left(\frac{dx^3(t)}{dt}\right)^2}\, dt\,. \tag{4.10}$$

What about internal observer?

To understand how he (internal observer) can calculate the lenght of the curve we have to introduce *first quadratic form.*

Definition First quadratic form defines length of the tangent vector to the surface in internal coordinates and length of the curves on the surface.

The first quadratic form at the point $\mathbf{r} = \mathbf{r}(u, v)$ is defined by symmetric matrix:

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_u, \mathbf{r}_u) & (\mathbf{r}_u, \mathbf{r}_v) \\ (\mathbf{r}_u, \mathbf{r}_v) & (\mathbf{r}_v, \mathbf{r}_v) \end{pmatrix}, \tag{4.11}$$

where (,) is a scalar product.

E.g. calculate the first quadratic form for the cylinder (4.2). Using (4.7), (4.8) we come to

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_\varphi, \mathbf{r}_\varphi) & (\mathbf{r}_\varphi, \mathbf{r}_h) \\ (\mathbf{r}_h, \mathbf{r}_\varphi) & (\mathbf{r}_h, \mathbf{r}_h) \end{pmatrix} = \begin{pmatrix} R^2 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.12}$$

(See this example and other examples in Homework 5)

Let $\mathbf{X} = a\mathbf{r}_u + b\mathbf{r}_v$ be a vector tangent to the surface M at the point $\mathbf{r}(u, v)$. Then the length of this vector is defined by the scalar product $(\mathbf{X}, \mathbf{X})$:

$$|\mathbf{X}|^2 = (\mathbf{X}, \mathbf{X}) = (a\mathbf{r}_u + b\mathbf{r}_v, a\mathbf{r}_u + b\mathbf{r}_v) = a^2(\mathbf{r}_u, \mathbf{r}_u) + 2ab(\mathbf{r}_u, \mathbf{r}_v) + b^2(\mathbf{r}_v, \mathbf{r}_v) \tag{4.13}$$

It is just equal to the value of the first quadratic form on this tangent vector:

$$(\mathbf{X}, \mathbf{X}) = G(\mathbf{X}, \mathbf{X}) = \begin{pmatrix} a, & b \end{pmatrix} \cdot \begin{pmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = G_{11}a^2 + 2G_{12}ab + G_{22}b^2 \tag{4.14}$$

External observer (person living in ambient space $\mathbf{E}^3$) calculate the length of the tangent vector using formula (4.13). An ant living on the surface (internal observer) calculate length of this vector in internal coordinates using formula (4.14). External observer deals with external coordinates of the vector, ant on the surface with internal coordinates.

If $\mathbf{X}, \mathbf{Y}$ are two tangent vectors in the tangent plane T_pC then $G(\mathbf{X}, \mathbf{Y})$ at the point p is equal to scalar product of vectors $\mathbf{X}, \mathbf{Y}$: $(\mathbf{X}, \mathbf{Y}) = (X^1\mathbf{r}_1 + X^2\mathbf{r}_2, Y^1\mathbf{r}_1 + Y^2\mathbf{r}_2) = X^1(\mathbf{r}_1, \mathbf{r}_1)Y^1 + X^1(\mathbf{r}_1, \mathbf{r}_2)Y^2 + X^2(\mathbf{r}_2, \mathbf{r}_1)Y^1 + X^2(\mathbf{r}_2, \mathbf{r}_2)Y^2 = X^\alpha(\mathbf{r}_\alpha, \mathbf{r}_\beta)Y^\beta = X^\alpha G_{\alpha\beta} Y^\beta = G(\mathbf{X}, \mathbf{Y})$. We identify quadratic forms and corresponding symmetric bilinear forms. Bilinear symmetric form $B(\mathbf{X}, \mathbf{Y}) = B(\mathbf{Y}, \mathbf{X})$ defines quadratic form $Q(\mathbf{X}) = B(\mathbf{X}, \mathbf{X})$. Quadratic form satisfies the condition $Q(\lambda\mathbf{X}) = \lambda^2 Q(\mathbf{X})$ and so called parallelogram condition

$$Q(\mathbf{X} + \mathbf{Y}) + Q(\mathbf{X} - \mathbf{Y}) = 2Q(\mathbf{X}) + 2Q(\mathbf{Y}) \tag{4.15}$$

First quadratic form and length of the curve

Let $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ $a \le t \le b$ be a curve on the surface.

The first quadratic form measures the length of velocity vector at every point of this curve. Thus we come to the formula for length of the curve.

Velocity of this curve at the point $\mathbf{r}(u(t), v(t))$ is equal to $\mathbf{v} = \frac{d\mathbf{r}(t)}{dt} = u_t\mathbf{r}_u + v_t\mathbf{r}_v$. The length of the curve is equal to

$$L = \int_a^b |\mathbf{v}(t)| dt = \int_a^b \sqrt{(\mathbf{v}(t), \mathbf{v}(t))} dt = \int_a^b \sqrt{(u_t\mathbf{r}_u + v_t\mathbf{r}_v, u_t\mathbf{r}_u + v_t\mathbf{r}_v)} dt = \tag{4.16}$$

$$\int_a^b \sqrt{(\mathbf{r}_u, \mathbf{r}_u)u_t^2 + 2(\mathbf{r}_u, \mathbf{r}_v)u_t v_t + (\mathbf{r}_v, \mathbf{r}_v)v_t^2} d\tau =$$

$$\int_a^b \sqrt{G_{11}u_t^2 + 2G_{12}u_t v_t + G_{22}v_t^2} dt\,. \tag{4.17}$$

An external observer will calculate the length of the curve using (4.13). An ant living on the surface calculate length of the curve via first quadratic form using (4.17): first quadratic form defines Riemannian metric on the surface:

$$ds^2 = G_{11}du^2 + 2G_{12}dudv + G_{22}dv^2 \tag{4.18}$$

Example Consider the curve

$$\mathbf{r}(t) \quad \begin{cases} x = R\cos t \\ y = R\sin t \\ z = vt \end{cases}, \quad 0 \le t \le 1$$

on the cylinder (4.2) (helix). The coordinates of this curve on the cylinder (internal coordinates) are

$$\begin{cases} \varphi(t) = t \\ h(t) = vt \end{cases}.$$

To calculate the length of this curve the external observer will perform the calculations

$$L = \int_0^1 \sqrt{x_t^2 + y_t^2 + z_t^2} dt \int_0^1 \sqrt{R^2 \sin^2 t + R^2 \cos^2 t + v^2} dt = \int_0^1 \sqrt{R^2 + v^2} dt = \sqrt{R^2 + v^2}.$$

An internal observer ("ant") uses quadratic form (4.12) and perform the following calculations:

$$L = \int_0^1 \sqrt{G_{11}\varphi_t^2 + 2G_{12}\varphi_t h_t + G_{22} h_t^2} dt = \int_0^1 \sqrt{R^2\varphi_t^2 + h_t^2} dt = \int_0^1 \sqrt{R^2 + v^2} dt = \sqrt{R^2 + v^2}\,.$$

The answer will be the same. (See this and other examples in Homework 7).

4.3 Unit normal vector to surface

We define unit normal vector field for surfaces in $\mathbf{E}^3$.

Consider vector field defined on the points of surface.

Definition Let $M\colon\ \mathbf{r} = \mathbf{r}(u, v)$ be a surface in $\mathbf{E}^3$. We say that vector $\mathbf{n}(u, v)$ is *normal unit vector* at the point $\mathbf{p} = \mathbf{r}(u, v)$ of the surface M if it has unit length $|\mathbf{n}| = 1$, and it is orthogonal to the surface, i.e. it is orthogonal to the tangent plane $T_{\mathbf{p}}M$. This means that it is orthogonal to any tangent vector $\xi \in T_pM$, i.e. it is orthogonal to the coordinate vectors $\mathbf{r}_u = \partial_u$, $\mathbf{r}_v = \partial_v$ at the point $\mathbf{p}$.

$$\mathbf{n}\colon (\mathbf{n}, \mathbf{r}_u) = (\mathbf{n}, \mathbf{r}_v) = 0, (\mathbf{n}, \mathbf{n}) = 1\,. \tag{4.19}$$

Write down this equation in components:

If surface is given by equation $\mathbf{r}(u, v)\colon \begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases}$ then

$$\mathbf{r}_u = \begin{pmatrix} x_u \\ y_u \\ z_u \end{pmatrix}, \ \mathbf{r}_v = \begin{pmatrix} x_v \\ y_v \\ z_v \end{pmatrix},$$

and $\mathbf{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$ is unit normal vector. Then writing the previous conditions in components we come to

$$(\mathbf{n}, \mathbf{r}_u) = n_x x_u + n_y y_u + n_z z_v = 0,\ (\mathbf{n}, \mathbf{r}_v) = n_x x_v + n_y y_v + n_z z_v = 0,\ (\mathbf{n}, \mathbf{n}) = n_x^2 + n_y^2 + n_z^2 = 1$$

Normal unit vector is defined up to a sign. At any point there are two normal unit vetors: the transformation $\mathbf{n} \to -\mathbf{n}$ transforms normal unit vector to normal unit vector.

Vector field defined at the points of the surface is called normal unit vector field if any vector is normal unit vector.

In simple cases one can guess how to find unit normal vector field using geometrical intuition and just check that conditions above are satisfied. E.g. for sphere (4.3) $\mathbf{r}$ is orthogonal to the surface, hence

$$\mathbf{n}(\theta, \varphi) = \frac{\mathbf{r}(\theta, \varphi)}{R} = \pm \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

For cylinder (4.2) it is easy to see that at any point (φ, h) (4.2), $\mathbf{r}\colon x = R\cos\varphi, y = R\sin\varphi, z = h$, a normal unit vector is equal to

$$\mathbf{n}(\varphi, h) = \pm \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} \tag{4.20}$$

Indeed it is easy to see that the conditions (4.19) are satisfied.

In general case one can define $\mathbf{n}(u, v)$ in two steps using vector product formula:

$$\mathbf{n}(u, v) = \frac{\mathbf{N}(u, v)}{|\mathbf{N}(u, v)|} \quad \text{where} \quad \mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v \tag{4.21}$$

Indeed by definition of vector product vector field $\mathbf{N}(u, v)$ is orthogonal to $\mathbf{r}_u$ and $\mathbf{r}_v$, i.e. it is orthogonal to the surface. Dividing $\mathbf{N}$ on the length we come to unit normal vector field $\mathbf{n}(u, v)$ at the point $\mathbf{r}(u, v)$.

4.4 Curves on surfaces—normal acceleration and normal curvature

We know already how to measure the length of the curve belonging to the given surface. What about curvature? Answering this question we will be able to study curvature of the surface.

We know that curvature of curve is nothing but the magnitude of a normal acceleration of particle which moves along the curve with unit speed: $k = \frac{|\mathbf{a}_\perp|}{|\mathbf{v}|}$. This helps us to understand the meaning of surfaces curvature.

Let M: $\mathbf{r} = \mathbf{r}(u, v)$ be a surface and C: $u = u(t), v = v(t)$, i.e. $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$, be curve on the surface M. Consider an arbitrary point $\mathbf{p} = \mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ on this curve and velocity and acceleration vectors $\mathbf{v} = \frac{d\mathbf{r}(t)}{dt}$, $\mathbf{a} = \frac{d^2\mathbf{r}(t)}{dt^2}$ at this point.

Consider at any point $\mathbf{p}$ of the curve the following basis $\{\mathbf{v}, \mathbf{f}, \ \mathbf{n}\}$, where

- velocity vector $\mathbf{v}$ is tangent to the curve
- the vector $\mathbf{f}$ is the vector tangent to the surface but orthogonal to the vector $\mathbf{v}$.
- $\mathbf{n}$ is the unit normal vector to the surface, i.e.it is orthogonal to vectors $\mathbf{v}$ and $\mathbf{f}$.

Decompose acceleration vector over three directions, i.e. over three one-dimensional spaces spanned by vectors $\mathbf{v}, \mathbf{f}$ and $\mathbf{n}$:

$$\mathbf{a} = \mathbf{a}_{\text{orthogonal to surface}} + \mathbf{a}_{\text{tang.to surf. and orthog. to curve}} + \mathbf{a}_{\text{tangent to curve}} \tag{4.22}$$

The vector $\mathbf{a}_{\text{orthogonal to surface}}$ which is collinear to normal unit vector $\mathbf{n}$, will be called *vector of normal acceleration of the curve on the surface.* We denote it by $\mathbf{a}_n$.

The vector $\mathbf{a}_{\text{tang.to surf. and orthog. to curve}}$, collinear to unit vector $\mathbf{f}_C$ will be called *vector of geodesic acceleration* . We denote it by $\mathbf{a}_{geod}$.

The vector $\mathbf{a}_{\text{tangent to curve}}$, collinear to velocity vector $\mathbf{v}$, is just *vector of tangential acceleration* . We denote it $\mathbf{a}_{tang}$. We can rewrite (4.22) as

$$\mathbf{a} = \mathbf{a}_n + \mathbf{a}_{geod} + \mathbf{a}_{tang} \tag{4.23}$$

Study the expansion (4.23). Both vectors $\mathbf{a}_n$ and $\mathbf{a}_{geod}$ are orthogonal to the curve. The vector $\mathbf{a}_{geod}$ is orthogonal to the curve but it is tangent to the surface. The vector $\mathbf{a}_n$ is orthogonal not only to the curve. It is orthogonal to the surface.

The vector $\mathbf{a}_{geod} + \mathbf{a}_n = \mathbf{a}_\perp$ is orthogonal to the curve. It is the vector of normal acceleration of the curve.

Remark Please note that when we consider the curves on the surface it could arise the confusion between the vector $\mathbf{a}_n$—normal acceleration of the curve on the surface and the vector $\mathbf{a}_\perp$ of normal acceleration of the curve (see (3.2)).

When we decompose in (4.23) the acceleration vector $\mathbf{a}$ in the sum of three vectors $\mathbf{a}_n, \mathbf{a}_{geod}$ and $\mathbf{a}_{tang}$ then the vector $\mathbf{a}_n$, *the normal acceleration of the curve on the surface* is orthogonal to the surface not only to the curve. The vector

$$\mathbf{a}_\perp = \mathbf{a}_n + \mathbf{a}_{geod},$$

is orthogonal only to the curve and in general it is not orthogonal to the surface (if $\mathbf{a}_{geod} \neq 0$). It is the normal acceleration of the curve. It depends only on the curve. The normal acceleration $\mathbf{a}_n$ of the curve on the surface which is orthogonal to the surface depends on the surface where the curve lies.

We know that the curvature of the curve is equal to the magnitude of normal acceleration of the curve divided on the square of the speed (see (3.22)). We have:

$$\text{curvature of the curve } k = \frac{|\mathbf{a}_\perp|}{|\mathbf{v}|^2} = \frac{|\mathbf{a}_n + \mathbf{a}_{geod}|}{|\mathbf{v}|^2}.$$

The vectors $\mathbf{a}_n$ and $\mathbf{a}_{geod}$ transform under reparameterisation in the same way as a vector $\mathbf{a}_\perp$ (see (3.11)). If $t \to t(\tau)$ then

$$\mathbf{a}'_\perp(\tau) = t_\tau^2 \mathbf{a}_\perp \text{ and } \mathbf{a}'_n(\tau) = t_\tau^2 \mathbf{a}_n(t), \quad \mathbf{a}'_{geod}(\tau) = t_\tau^2 \mathbf{a}_{geod}(t) \tag{4.24}$$

where $\mathbf{a}'(\tau) = \frac{d^2}{d\tau^2}\mathbf{r}(t(\tau)) = t_\tau^2 \mathbf{a} + t_{\tau\tau}\mathbf{v}$ (see (3.9), (3.10), (3.8)). Hence the magnitudes

$$\frac{|\mathbf{a}_{geod}|}{|\mathbf{v}|^2} \text{ and } \qquad \frac{|\mathbf{a}_n|}{|\mathbf{v}|^2} \tag{4.25}$$

are reparameterisation invariant as well as magnitude $k = \frac{|\mathbf{a}_\perp|}{|\mathbf{v}|^2} = \frac{|\mathbf{a}_n + \mathbf{a}_{geod}|}{|\mathbf{v}|^2}$.

Multiply left and right hand sides of the equation (4.23) on unit normal vector $\mathbf{n}$. Then $(\mathbf{a}_{tang}, \mathbf{n}) = (\mathbf{a}_{geod}, \mathbf{n}) = 0$ because vectors $\mathbf{a}_{geod}$ and $\mathbf{a}_{tang}$ are orthogonal to the vector $\mathbf{n}$. We come to the relation

$$\mathbf{a}_n = (\mathbf{n}, \mathbf{a})\mathbf{n} \text{ and}\,, |\mathbf{a}_n| = |(\mathbf{a}, \mathbf{n})|\,. \tag{4.26}$$

Or in other words scalar product $(\mathbf{n}, \mathbf{a})$ is equal to $|\mathbf{a}_n|$ (up to a sign).

Definition Let C be a curve on the surface M. Let $\mathbf{v}$, $\mathbf{a}$ be velocity and acceleration vectors at the given point of this curve and $\mathbf{n}$ be normal unit vector at this point. Then

$$\kappa_n = \frac{(\mathbf{n}, \mathbf{a})}{(\mathbf{v}, \mathbf{v})} \tag{4.27}$$

is called *normal curvature of the curve* C on the surface M at the point $\mathbf{p}$.

Remark One can see from (4.26) and definition (4.27) that

$$|\kappa_n| = \frac{|\mathbf{a}_n|}{(\mathbf{v}, \mathbf{v})}, \tag{4.28}$$

i.e. up to a sign normal curvature is equal to modulus of normal acceleration divided on the square of speed (Compare with formula (3.23) for usual curvature.)

It follows from (4.24), (4.25) and (4.27) (or (4.28)) that for any curve on the surface the modulus of the normal curvature is less or equal than usual curvature.

$$|\kappa_n| \leq k \tag{4.29}$$

Indeed we have for usual curvature

$$k = \frac{\mathbf{a}_\perp}{|\mathbf{v}|^2} = \frac{|\mathbf{a}_{geod} + \mathbf{a}_{normal}|}{|\mathbf{v}|^2} = \sqrt{\frac{\mathbf{a}_{geod}^2 + \mathbf{a}_{norm}^2}{|\mathbf{v}|^2}} \geq \frac{|\mathbf{a}_{normal}|}{|\mathbf{v}|^2} = |\kappa_n| \tag{4.30}$$

Normal curvature is a positive or negative real number. (Usual curvature is non-negative real number). Normal curvature changes a sign if $\mathbf{n} \to -\mathbf{n}$.

Remark We obtained in (4.25) that the magnitude $\frac{|\mathbf{a}_{geod}|}{|\mathbf{v}|^2}$ is reparameterisation invariant. It defines so called *geodesic curvature* $\kappa_{geod} = \frac{|\mathbf{a}_{geod}|}{|\mathbf{v}|^2}$. We see that usual curvature k, normal curvature κ and geodesic curvature κ_{geod} are related by the formula

$$k^2 = \kappa_{geod}^2 + \kappa_{normal}^2 \tag{4.31}$$

Example Consider an arbitrary curve $C\colon h = h(t), \varphi = \varphi(t)$ on the cylinder

$$\mathbf{r}(\varphi, h)\colon \begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \\ z = h \end{cases}$$

Pick any point $\mathbf{p}$ on this curve and find normal acceleration vector at this point of this curve.

Without loss of generality suppose that point $\mathbf{p}$ is just a point $(R, 0, 0)$. Note that vector $\mathbf{e}_x$ attached at the point $(R, 0, 0)$ is unit vector orthogonal to the surface of cylinder, i.e. $\mathbf{e}_x = -\mathbf{n}$ at the point $\mathbf{p} = (R, 0, 0)$.

RemarkUnit vector, as well as normal curvature is defined up to a sign. It is convenient for us to choose $\mathbf{n} = -\mathbf{e}_x$, not $\mathbf{n} = \mathbf{e}_x$.

Vectors $\mathbf{e}_y, \mathbf{e}_z$ are tangent to the surface of cylinder. At the point $\mathbf{p} = (R, 0, 0)$ $\varphi = 0, h = 0$.

We have

$$\mathbf{v} = \frac{d\mathbf{r}(t)}{dt} = \frac{dx(t)}{dt}\mathbf{e}_x + \frac{dy(t)}{dt}\mathbf{e}_y + \frac{dz(t)}{dt}\mathbf{e}_z =$$

$$R\frac{d\cos\varphi(t)}{dt}\mathbf{e}_x + R\frac{d\sin\varphi(t)}{dt}\mathbf{e}_y + \frac{dh(t)}{dt}\mathbf{e}_z = -R\sin\varphi\dot{\varphi}\mathbf{e}_x + R\cos\varphi\dot{\varphi}\mathbf{e}_y + \dot{h}\mathbf{e}_z$$

$$\text{Thus } \mathbf{v} = R\dot{\varphi}\mathbf{e}_y + \dot{h}\mathbf{e}_z \text{ at the point } \mathbf{p} = (R, 0, 0)\,. \tag{4.32}$$

For acceleration vector

$$\mathbf{a} = \frac{d^2\mathbf{r}(t)}{dt^2} = \frac{d^2x(t)}{dt^2}\mathbf{e}_x + \frac{d^2y(t)}{dt^2}\mathbf{e}_y + \frac{d^2z(t)}{dt^2}\mathbf{e}_z = R\frac{d^2\cos\varphi(t)}{dt^2}\mathbf{e}_x + R\frac{d^2\sin\varphi(t)}{dt^2}\mathbf{e}_y + \frac{d^2h(t)}{dt^2}\mathbf{e}_z =$$

$$R\left(-(\dot{\varphi})^2\cos\varphi - \ddot{\varphi}\sin\varphi\right)\mathbf{e}_x + R\left(-(\dot{\varphi})^2\sin\varphi + \ddot{\varphi}\cos\varphi\right)\mathbf{e}_y + \ddot{h}\mathbf{e}_z = \ddot{\varphi}R\mathbf{e}_y + \ddot{h}\mathbf{e}_z - (\dot{\varphi})^2R\mathbf{e}_x$$

at the point $\mathbf{p} = (R, 0, 0)$ where $\cos\varphi = 0, \sin\varphi = 1$. We see that

$$\mathbf{a} = \underbrace{\ddot{\varphi}R\mathbf{e}_y + \ddot{h}\mathbf{e}_z}_{\text{tangent to the surface}} - \underbrace{(\dot{\varphi})^2R\mathbf{e}_x}_{\text{normal to the surface}} \tag{4.33}$$

We see that $\mathbf{a}_n = (\dot{\varphi})^2R\mathbf{e}_x$. Comparing with velocity vector (4.32) we see that

$$\mathbf{a}_n = \frac{\mathbf{v}^2_{horizontal}}{R}\mathbf{n} \tag{4.34}$$

We see that for any curve on the cylinder $x^2 + y^2 = R^2$ the normal curvature $\frac{(\mathbf{a}_n, \mathbf{n})}{|\mathbf{v}|^2}$ (see (4.27)) is equal to

$$\frac{(\mathbf{a}_n, \mathbf{n})}{|\mathbf{v}|^2} = \frac{R\dot{\varphi}^2}{R^2\dot{\varphi}^2 + \dot{h}^2} \tag{4.35}$$

and it obeys relations

$$0 \leq \kappa_{normal} \leq \frac{1}{R}$$

depending of the curve. E.g. if the curve on the cylinder is a straight line $x = x_0, y = y_0, z = t$ then $\mathbf{a} = 0$ and normal curvature of this curve is equal to the naught as well as usual curvature.

If the curve is circle $x = R\cos t, y = R\sin t, z = z_0$ then normal curvature of this curve as well as usual curvature is equal to $\frac{1}{R}$.

Remark Very important conclusion from this example is
normal curvature of the cylinder of the radius R takes values in the interval $\left(0, \frac{1}{R}\right)$. It cannot be greater than $\frac{1}{R}$
Note that we can consider on cylinder very curly curve of very big curvature. The normal cuirvature at the points of this curve will be still less than $\frac{1}{R}$.

At any point of the surface normal curvature in general depends on the curve but it takes values in the restricted interval.

E.g. for the sphere of radius R one can see that normal curvature at any point is equal to $\frac{1}{R}$ independent of curve. In spite of this fact the usual curvature of curve can be very big [8]. If we consider the circle of very small radius r on the sphere then its usual curvature is equal to $k = \frac{1}{r}$ and $k \to \infty$ if $r \to 0$ So we see that one can define curvature of surface in terms of normal curvature.

Definition-Proposition

One can consider different curves passing through an arbitrary point $\mathbf{p}$ on the surface M. The normal curvature of these curves at the point $\mathbf{p}$ takes values in the interval (a, b)

$$a \leq k_n \leq b \tag{4.36}$$

The numbers $\kappa_- = a$, $\kappa_+ = b$ are called principal curvatures of the surface M at the point $\mathbf{p}$.

- *Gaussian curvature* K of the surface M at a point $\mathbf{p}$ is equal to the product of principal curvatures.

$$K = \kappa_- \kappa_+ \tag{4.37}$$

[8] It is the geodesic curvature of the curves which characterises its curvature with respect to the curve. The relation between usual geodesic and normal curvature is given by the formula (4.31).

- *Mean curvature* K of the surface M at a point S is equal to the sum of the principal curvatures, i.e. trace of shape operator S:

$$H = \kappa_- + \kappa_+ \tag{4.38}$$

Example. We see that for a cylinder of the radius R normal curvature takes values in the interval $\left(0, \frac{1}{R}\right)$: $0 \leq \kappa_n \leq \frac{1}{R}$.

Hence $\kappa_- = 0$ and $\kappa_+ = \frac{1}{R}$. Gaussian curvature is equal to $K = \kappa_- \cdot \kappa_+ = 0$ and mean curvature is equal to $H = \kappa_- + \kappa_+ = \frac{1}{R}$.

4.5 Shape operator on the surface

Let $M\colon \mathbf{r} = \mathbf{r}(u,v)$ be a surface and $\mathbf{L}(u,v)$ be an arbitrary (not necessarily unit normal) vector field at the points of the surface M. We define at every point $\mathbf{p} = \mathbf{r}(u,v)$ a linear operator K_L acting on the vectors tangent to the surface M such that its value is equal to the derivative of vector field $\mathbf{L}(u,v)$ along vector $\boldsymbol{\xi}$

$$K_{\mathbf{L}}\colon\ \boldsymbol{\xi} \in T_{\mathbf{p}}M \mapsto K_{\mathbf{L}}(\boldsymbol{\xi}) = \partial_{\boldsymbol{\xi}} L = \xi_u \frac{\partial \mathbf{L}(u,v)}{\partial u} + \xi_v \frac{\partial \mathbf{L}(u,v)}{\partial v}\,, \tag{4.39}$$

ξ_u, ξ_v are components of vector $\boldsymbol{\xi}$

$$\boldsymbol{\xi} = \xi_u \mathbf{r}_u + \xi_v \mathbf{r}_v \tag{4.40}$$

The vector $K_{\mathbf{L}}\boldsymbol{\xi} \in T_{\mathbf{p}}\mathbf{E}^3$ in general is not a vector tangent to the surface C and $K_{\mathbf{L}}$ is linear operator from the space $T_{\mathbf{p}}M$ in the space $T_{\mathbf{p}}\mathbf{E}^3$ of all vectors in $\mathbf{E}^3$ attached at the point $\mathbf{p}$

It turns out that in the case if vector field $\mathbf{L}(u,v)$ is unit normal vector field then operator $K_{\mathbf{L}}$ takes values in vectors tangent to M and it is very important geometric properties.

Definition-Proposition Let $\mathbf{n}(u,v)$ be a unit normal vector field to the surface M. Then operator

$$S\colon = L_{-\mathbf{n}}\colon\ S(\boldsymbol{\xi}) = \partial_{\boldsymbol{\xi}}(-\mathbf{n}) = -\xi_u \frac{\partial \mathbf{n}(u,v)}{\partial u} - \xi_v \frac{\partial \mathbf{n}(u,v)}{\partial v} \tag{4.41}$$

maps tangent vectors to the tangent vectors:

$$S\colon T_{\mathbf{p}}M \to T_{\mathbf{p}}M \text{ for every } \boldsymbol{\xi} \in T_{\mathbf{p}}M, \qquad S(\boldsymbol{\xi}) \in T_{\mathbf{p}}M \tag{4.42}$$

This operator is called *shape operator*.

Remark The sign " $-$ " seems to be senseless: if $\mathbf{n}$ is unit normal vector field then $-\mathbf{n}$ is normal vector field too. Later we will see why it is convenient (see the proof of the Proposition below).

Show that the property (4.42) is indeed obeyed, i.e. vector $\boldsymbol{\xi}' = S(\boldsymbol{\xi})$ is tangent to surface. Consider derivative of scalar product $(\mathbf{n}, \mathbf{n})$ with respect to the vector field $\boldsymbol{\xi}$. We have that $(\mathbf{n}, \mathbf{n}) = 1$. Hence

$$\partial_{\boldsymbol{\xi}}(\mathbf{n}, \mathbf{n}) = 0 = \partial_{\boldsymbol{\xi}}(\mathbf{n}, \mathbf{n}) = (\partial_{\boldsymbol{\xi}}\mathbf{n}, \mathbf{n}) + (\mathbf{n}, \partial_{\boldsymbol{\xi}}\mathbf{n}) = 2(\partial_{\boldsymbol{\xi}}\mathbf{n}, \mathbf{n})\,.$$

Hence $(\partial_{\boldsymbol{\xi}}\mathbf{n}, \mathbf{n}) = -(S(\boldsymbol{\xi}), \mathbf{n}) = -(\boldsymbol{\xi}', \mathbf{n}) = 0$, i.e. vector $\partial_{\boldsymbol{\xi}}\mathbf{n} = -\boldsymbol{\xi}'$ is orthogonal to the vector $\mathbf{n}$. This means that vector $\boldsymbol{\xi}'$ is tangent to the surface.

We show now that normal acceleration of a curve on the surface and normal curvature are expressed in terms of shape operator.

Let $C\colon \mathbf{r}(t)$ be a curve on the surface M, $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$. Let $\mathbf{v} = \mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt}$, $\mathbf{a} = \mathbf{a}(t) = \frac{d^2r(t)}{dt^2}$ be velocity and acceleration vectors respectively. Recall that

$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \dot{x}\mathbf{e}_x + \dot{y}\mathbf{e}_y + \dot{z}\mathbf{e}_z = \frac{d\mathbf{r}(u(t), v(t))}{dt} = \dot{u}\mathbf{r}_u + \dot{v}\mathbf{r}_v \tag{4.43}$$

be velocity vector; $\dot{u}, \dot{v}$ are internal components of the velocity vector with respect to the basis $\{\mathbf{r}_u = \partial_u, \mathbf{r}_v = \partial_v\}$ and $\dot{x}, \dot{y}, \dot{z}$, are external components velocity vectors with respect to the basis $\{\mathbf{e}_x = \partial_x, \mathbf{e}_y = \partial_y, \mathbf{e}_z = \partial_z\}$. As always we denote by $\mathbf{n}$ normal unit vector.

Proposition *The normal acceleration* (4.22) *at an arbitrary point* $\mathbf{p} = \mathbf{r}(u(t_0), v(t_0))$ *of the curve* C *on the surface* M *is defined by the scalar product of the velocity vector* $\mathbf{v}$ *of the curve at the point* $\mathbf{p}$ *on the value of the shape operator on the velocity vector:*

$$\mathbf{a}_n = a_n\mathbf{n} = (\mathbf{v}, S\mathbf{v})\,\mathbf{n} \tag{4.44}$$

and normal curvature (4.27) *is equal to*

$$\kappa_n = \frac{(\mathbf{n}, \mathbf{a})}{(\mathbf{v}, \mathbf{v})} = \frac{(\mathbf{v}, S\mathbf{v})}{(\mathbf{v}, \mathbf{v})} \tag{4.45}$$

Proof of the Proposition. According to (4.26) we have

$$\mathbf{a}_n = (\mathbf{n}, \mathbf{a})\mathbf{n} = \mathbf{n}\left(\mathbf{n}, \frac{d}{dt}\mathbf{v}(t)\right)\mathbf{n} = \mathbf{n}\frac{d}{dt}\left(\mathbf{n}, \mathbf{v}(t)\right) - \mathbf{n}\left(\frac{d}{dt}\mathbf{n}(u(t), v(t)), \mathbf{v}(t)\right)$$

$$= 0 + \left(-\partial_{\mathbf{v}}\mathbf{n}, \mathbf{v}\right)\mathbf{n} = (S\mathbf{v}, \mathbf{v})\mathbf{n}$$

This proves Proposition.

Examples of shape operator see in the Homework 7.

4.6 Principal curvatures, Gaussian and mean curvatures and shape operator

Now we express principal curvatures, Gaussian and mean curvature intorduced in terms of normal curvature (see the subsection 4.4) in terms of shape operator.

Let $\mathbf{p}$ be an arbitrary point of the surface M and S be shape operator at this point. S is symmetric operator: $(S\mathbf{a}, \mathbf{b}) = (\mathbf{b}, S\mathbf{a})$. Hence it has eigenvctors. Consider eigenvalues $\lambda_\pm$ and eigenvectors $\mathbf{l}_\pm$ of the shape operator S

$$\mathbf{l}_+, \mathbf{l}_- \in T_{\mathbf{p}}M, \qquad S\,\mathbf{l}_+ = \kappa_+\mathbf{l}_+, \qquad S\,\mathbf{l}_- = \kappa_-\mathbf{l}_-, \tag{4.46}$$

One can see that $\lambda_-(\mathbf{l}_-, \mathbf{l}_+) = (S\mathbf{l}_-, \mathbf{l}_+) = (\mathbf{l}_-, S\mathbf{l}_+) = \lambda_+(\mathbf{l}_-, \mathbf{l}_+)$. It follows from this relation that eigenvectors are orthogonal ($(\mathbf{l}_-, \mathbf{l}_+) = 0$) if $\lambda_- \neq \lambda_+$ If $\lambda_- = \lambda_+$ then all vectors are eigenvectors. One can choose in this case $\mathbf{l}_-$, $\mathbf{l}_+$ to be orthogonal.

In fact eigenvalues of shape operator are just principal curvatures of surface!

Let $\mathbf{p}$ be an arbitrary point on the surface M and C be an arbitrary curve passing through the point $\mathbf{p}$. If velocity vector $\mathbf{v}$ is collinear to the eigenvector $\mathbf{l}_+$ then normal curvature of the curve C at the point $\mathbf{p}$ according to Proposition is equal to

$$\kappa_{normal} = \frac{(\mathbf{v}, S\mathbf{v})}{(\mathbf{v}, \mathbf{v})} = \frac{(\mathbf{l}_+, S\mathbf{l}_+)}{(\mathbf{l}_+, \mathbf{l}_+)} = \frac{(\mathbf{l}_+, \kappa_+\mathbf{l}_+)}{(\mathbf{l}_+, \mathbf{l}_+)} = \lambda_+ \,.$$

Analogously if velocity vector $\mathbf{v}$ is colinear to the eigenvector $\mathbf{l}_-$ then normal curvature of the curve C at the point $\mathbf{p}$ is equal to

$$\kappa_{normal} = \frac{(\mathbf{v}, S\mathbf{v})}{(\mathbf{v}, \mathbf{v})} = \frac{(\mathbf{l}_-, S\mathbf{l}_-)}{(\mathbf{l}_-, \mathbf{l}_-)} = \frac{(\mathbf{l}_-, \kappa_-\mathbf{l}_-)}{(\mathbf{l}_-, \mathbf{l}_-)} = \lambda_- \,.$$

In the general case if $\mathbf{v} = v_+\mathbf{l}_+ + v_-\mathbf{l}_-$ is expansion of velocity vector with respect to the basis of eigenvectors then we have for normal curvature

$$k_{normal} = \frac{(\mathbf{v}, S\mathbf{v})}{(\mathbf{v}, \mathbf{v})} = \frac{(v_+\mathbf{l}_+ + v_-\mathbf{l}_-, \lambda_+ v_+\mathbf{l}_+ + \lambda_- v_-\mathbf{l}_-)}{(v_+\mathbf{l}_+ + v_-\mathbf{l}_-, v_+\mathbf{l}_+ + v_-\mathbf{l}_-)} = \tag{4.47}$$

$$\frac{\lambda_+ v_+^2 + \lambda_- v_-^2}{v_+^2 + v_-^2}$$

Hence we come to the conclusion that

$$\lambda_- \leq \kappa_{normal} = \frac{\kappa_+ v_+^2 + \kappa_- v_-^2}{v +^2 + v_-^2} \leq \lambda_+ \tag{4.48}$$

We see that normal curvature varies in the interval (λ_-, λ_+).

Now remember the definition of principal curvatures from the subsection 4.4: we see that λ_-, λ_+ are just principal curvatures.

Proposition Eigenvalues of shape operator are principal curvatures:

$$\lambda_+ = \kappa_+, \quad \lambda_- = \kappa_-$$

Eigenvectors $\mathbf{l}_1, \mathbf{l}_2$ define the two orthogonal directions such that curves directed along these vectors have normal curvature equal to the principal curvatures κ_+, κ_-.

These directions are called principal directions.

In the basis $\mathbf{l}_-, \mathbf{l}_+$ the matrix of shape operator is diagonal:

$$S = \begin{pmatrix} \kappa_+ & 0 \\ 0 & \kappa_- \end{pmatrix} \tag{4.49}$$

In the previous subsection we introduced Gaussian curvature $K = \kappa_- \kappa_+$ and mean curvature $H = \kappa_- + \kappa_+$. We see that we are able to express Gaussian curvature and mean curvatures

It follows form (4.49) that Gaussian curvature equals to determinant of shape operator. and mean curvature equals to Trace of this operator.

Summarize all the relations between normal curvature, shape operator and Gaussian and mean curvature.

- *Principal curvatures* κ_-, κ_+ of the surface M at the given point $\mathbf{p}$ are eigenvalues of shape operator S acting at the tangent space $T_{\mathbf{p}}M$ (κ_-, κ_+). Corresponding eigenvectors $\mathbf{l}_+, \mathbf{l}_-$ define directions which are called *principal directions*. Principal directions are orthogonal or can be chosen to be orthogonal if $\kappa_- = \kappa_+$. The normal curvature κ_n for an arbitrary curve on the surface M at the point $\mathbf{p}$ varies in the interval (κ_-, κ_+):
$$\kappa_- \leq \kappa_n \leq \kappa_+ \tag{4.50}$$

- *Gaussian curvature* K of the surface M at a point S is equal to the product of principal curvatures, i.e. determinant of shape operator S:
$$K = \kappa_+ \cdot \kappa_- = \det S \tag{4.51}$$

- *Mean curvature* H of the surface M at a point S is equal to the sum of the principal curvatures, i.e. trace of shape operator S:
$$H = \kappa_+ + \kappa_- = \mathrm{Tr}\ S \tag{4.52}$$

One can calculate principal curvatures $\kappa_\pm$ via formula (4.50) and Gaussian and then mean curvatures via formulae (4.51), (4.52)

If shape operator is given in an arbitrary basis then one can easy to calculate Gaussian and mean curvatures and via them to find principal curvatures. (See examples in the Homework 7)

5 Parallel transport; Gauss–Bonnet Theorem

5.1 Concept of parallel transport

Parallel transport of the vectors is one of the fundamental concept of differential geometry. Here we just give some preliminary ideas and formulate the concept of parallel transport for surfaces embedded in Euclidean space.

Let M be a surface $\mathbf{r} = \mathbf{r}(u, v)$ in $\mathbf{E}^3$ and $C\colon \mathbf{r}(t) = \mathbf{r}(u(t), v(t))$, $t_1 \leq t \leq t_2$ be a curve on this surface.

Let $\mathbf{X}_1$ be a vector tangent to the surface at the initial point $p = \mathbf{r}(t_1)$ of the curve $\mathbf{r}(t)$ on the surface: $\mathbf{X}_1 \in T_pM$. Note that $\mathbf{X}_1$ is a vector tangent to the surface, not necessarily to the curve. We define now parallel transport of the vector along the curve C:

Definition Let $\mathbf{X}(t)$ be a family of vectors depending on the parameter t $(t_1 \leq t \leq t_2)$ such that following conditions hold

- For every $t \in [t_1, t_2]$ vector $\mathbf{X}(t)$ is a vector tangent to the surface M at the point $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ of the curve C.
- $\mathbf{X}(t) = \mathbf{X}_1$ for $t = t_1$
- $\frac{d\mathbf{X}(t)}{dt}$ is orthogonal to the surface, i.e.

$$\frac{d\mathbf{X}(t)}{dt} \text{ is collinear to the normal vector } \mathbf{n}(t), \quad \frac{d\mathbf{X}(t)}{dt} = \lambda(t)\mathbf{n}(t) \tag{5.1}$$

Recall that normal vector $\mathbf{n}(t)$ is a vector attached to the point $\mathbf{r}(t)$ of the curve $C\colon \mathbf{r}(t)$. This vector is orthogonal to the surface M.

The condition (5.1) means that only orthogonal component of vector field $\mathbf{X}(t)$ can be changed.

We say that a family $\mathbf{X}(t)$ is a parallel transport of the vector $\mathbf{X}_1$ along a curve $C\colon \mathbf{r}(t)$ on the surface M. The final vector $\mathbf{X}_2 = \mathbf{X}(t_2)$ is the image of the vector $\mathbf{X}_1$ under the parallel transport along the curve C.

Using the relation (5.1) it is easy to see that the scalar product of two vectors remains invariant under parallel transport. In particularly it means that length of the vector does not change. If $\mathbf{X}(t)$, $\mathbf{Y}(t)$ are parallel transports of vectors $\mathbf{X}_1, \mathbf{Y}_1$ then

$$\frac{d}{dt}\left(\mathbf{X}(t), \mathbf{Y}(t)\right) = \left(\frac{d\mathbf{X}(t)}{dt}, \mathbf{Y}(t)\right) + \left(\mathbf{X}(t), \frac{d\mathbf{Y}(t)}{dt}\right) = 0$$

because vector $\frac{d\mathbf{X}(t)}{dt}$ is orthogonal to the vector $\mathbf{Y}(t)$ and vector $\frac{d\mathbf{Y}(t)}{dt}$ is orthogonal to the vector $\mathbf{X}(t)$. In particularly length does not change:

$$\frac{d}{dt}|\mathbf{X}(t)|^2 = \frac{d}{dt}(\mathbf{X}(t), \mathbf{X}(t)) = 2\left(\frac{d\mathbf{X}(t)}{dt}, \mathbf{X}(t)\right) = 2(\lambda(t)\mathbf{n}(t), \mathbf{X}(t)) = 0 \tag{5.2}$$

Remark The relation (5.1) shows how the surface is engaged in the parallel transport. Note that it is non-sense to put the right hand side of the equation (5.1) equal to zero: In general a tangent vector ceased to be tangent to the surface if it is not changed! (E.g. consider the vector which transports along the great circle on the sphere)

In the next paragraph we consider an example of parallel transport of vectors along meridians in the sphere and equator and more interesting examples.

5.2 Parallel transport of vectors tangent to the sphere.

1. In the case if surface is a plane then everything is easy. If vector $\mathbf{X}_1$ is tangent to the plane at the given point, it is tangent at all the points. Vector does not change under parallel transport $\mathbf{X}(t) \equiv \mathbf{X}$.

Consider a case of parallel transport along curves on the sphere.

Consider on the sphere $x^2 + y^2 + z^2 = a^2$ (a is a radius) tangent vectors:

$$\mathbf{r}_\theta = \begin{pmatrix} a\cos\theta\cos\varphi \\ a\cos\theta\sin\varphi \\ -a\sin\theta \end{pmatrix} \qquad \mathbf{r}_\varphi = \begin{pmatrix} -a\sin\theta\sin\varphi \\ a\sin\theta\cos\varphi \\ 0 \end{pmatrix} \tag{5.3}$$

attached at the point $\mathbf{r}(\theta,\varphi)=\begin{pmatrix} a\sin\theta\cos\varphi\\ a\sin\theta\sin\varphi\\ a\cos\theta\end{pmatrix}$. One can see that

$$(\mathbf{r}_\theta,\mathbf{r}_\theta)=a,\quad (\mathbf{r}_h,\mathbf{r}_\varphi)=0,\quad (\mathbf{r}_\varphi,\mathbf{r}_\varphi)=a^2\sin^2\theta$$

It is convenient to introduce vectors which are parallel to these vectors but have unit length:

$$\mathbf{e}_\theta=\frac{\mathbf{r}_\theta}{a},\quad \mathbf{e}_\varphi=\frac{\mathbf{r}_\varphi}{a\sin\theta}\qquad (\mathbf{e}_\theta,\mathbf{e}_\theta)=1,(\mathbf{e}_\theta,\mathbf{e}_\varphi)=0,(\mathbf{e}_\varphi,\mathbf{e}_\varphi)=1\,. \tag{5.4}$$

How these vectors change if we move along parallel (i.e. what is the value of $\frac{\partial\mathbf{e}_\theta}{\partial\varphi}$, $\frac{\partial\mathbf{e}_\varphi}{\partial\varphi}$); how these vectors change if we move along meridians (i.e. what is the value of $\frac{\partial\mathbf{e}_\theta}{\partial\theta}$, $\frac{\partial\mathbf{e}_\varphi}{\partial\theta}$). First of all recall that unit normal vector to the sphere at the point θ,φ is equal to $\frac{\mathbf{r}(\theta,\varphi)}{a}$:

$$\mathbf{n}(\theta,\varphi)=\begin{pmatrix}\sin\theta\cos\varphi\\ \sin\theta\sin\varphi\\ \cos\theta\end{pmatrix}$$

Now calculate:

$$\frac{\partial\mathbf{e}_\theta}{\partial\theta}=\frac{\partial}{\partial\theta}\begin{pmatrix}\cos\theta\cos\varphi\\ \cos\theta\sin\varphi\\ -\sin\theta\end{pmatrix}=\begin{pmatrix}-\sin\theta\cos\varphi\\ -\sin\theta\sin\varphi\\ -\cos\theta\end{pmatrix}=-\mathbf{n} \tag{5.5}$$

,

$$\frac{\partial\mathbf{e}_\theta}{\partial\varphi}=\frac{\partial}{\partial\varphi}\begin{pmatrix}\cos\theta\cos\varphi\\ \cos\theta\sin\varphi\\ -\sin\theta\end{pmatrix}=\begin{pmatrix}-\cos\theta\sin\varphi\\ \cos\theta\cos\varphi\\ 0\end{pmatrix}=\cos\theta\mathbf{e}_\varphi, \tag{5.6}$$

,

$$\frac{\partial\mathbf{e}_\varphi}{\partial\theta}=\frac{\partial}{\partial\theta}\begin{pmatrix}-\sin\varphi\\ \cos\varphi\\ 0\end{pmatrix}=\begin{pmatrix}-\cos\theta\sin\varphi\\ \cos\theta\cos\varphi\\ 0\end{pmatrix}=0, \tag{5.7}$$

$$\frac{\partial\mathbf{e}_\varphi}{\partial\varphi}=\frac{\partial}{\partial\varphi}\begin{pmatrix}-\sin\varphi\\ \cos\varphi\\ 0\end{pmatrix}=\begin{pmatrix}-\cos\varphi\\ -\sin\varphi\\ 0\end{pmatrix}=-\sin\theta\mathbf{n}-\cos\theta\mathbf{e}_\theta, \tag{5.8}$$

Some of these formulaes are intuitively evident: For example formula (5.5) which means that family of the vectors $\mathbf{e}_\theta(\theta)$ is just parallel transport along meridian, because its derivation is equal to $-\mathbf{n}$.

Another intuitively evident example: consider the meridian $\theta(t)=t$, $\varphi(t)=\varphi_0$, $0\leq t\leq\pi$. It is easy to see that the vector field

$$\mathbf{X}(t)=\mathbf{e}_\theta(\theta(t),\varphi_0)=\begin{pmatrix}\cos\theta(t)\cos\varphi_0\\ \cos\theta(t)\sin\varphi_0\\ -\sin\theta(t)\end{pmatrix}$$

attached at the point $(\theta(t),\varphi_0)$ is a parallel transport because for family of vectors $\mathbf{X}(t)$ all the conditions of parallel transport are satisfied. In particular according to (5.5)

$$\frac{d\mathbf{X}(t)}{dt}=\frac{d\theta(t)}{dt}\frac{\partial}{\partial\theta}\begin{pmatrix}\cos\theta\cos\varphi\\ \cos\theta\sin\varphi\\ -\sin\theta\end{pmatrix}=-\mathbf{n}(\theta(t),\varphi_0)$$

Now consider an example which is intuitively not-evident.

Example. Calculate parallel transport of the vector $\mathbf{e}_\varphi$ along the parallel. On the sphere of the radius a consider the parallel

$$\theta(t) = \theta_0, \varphi(t) = t, \quad 0 \le t \le 2\pi \tag{5.9}$$

In cartesian coordinates equation of parallel will be:

$$\mathbf{r}(t) = \begin{pmatrix} a\sin\theta(t)\cos\varphi(t) \\ a\sin\theta(t)\sin\varphi(t) \\ -a\cos\theta(t) \end{pmatrix} = \begin{pmatrix} a\sin\theta_0\cos t \\ a\sin\theta(t)\sin t \\ -a\cos\theta_0 \end{pmatrix}, \quad 0 \le t \le 2\pi \tag{5.10}$$

It is easy to see that the family of the vectors $\mathbf{e}_\varphi(\theta_0, \varphi(t))$ on parallel, is not parallel transport! because $\frac{d\mathbf{e}_\varphi(\theta_0,\varphi(t))}{dt} = \frac{d\mathbf{e}_\varphi(\theta_0,\varphi)}{d\varphi}$ is not equal to zero (see (5.8) above). Let a family of vectors $\mathbf{X}(t)$ be a parallel transport of the vector $\mathbf{e}_\varphi$ along the parallel (5.9): $\mathbf{X}(t) = a(t)\mathbf{e}_\theta(t) + b(t)\mathbf{e}_\varphi(t)$ where $a(t), b(t)$ are components of the tangent vector $\mathbf{X}(t)$ with respect to the basis $\mathbf{e}_\theta, \mathbf{e}_\varphi$ at the point $\theta = \theta_0, \varphi = t$ on the sphere. Initial conditions for coefficients are $a(t)|_{t=0} = 0, b(t)|_{t=0} = 1$ According to the definition of parallel transport and formulae (5.5)—(5.8) we have:

$$\frac{d\mathbf{X}(t)}{dt} = \frac{d\left(a(t)\mathbf{e}_\theta(t) + b(t)\mathbf{e}_\varphi(t)\right)}{dt} = \left(\frac{da(t)}{dt}\right)\mathbf{e}_\theta + a(t)\cos\theta_0\mathbf{e}_\varphi + \frac{db(t)}{dt}\mathbf{e}_\varphi +$$

$$b(t)\left(-\sin\theta_0\mathbf{n} - \cos\theta\mathbf{e}_\theta\right) =$$

$$= \left(\frac{da(t)}{dt} - b(t)\cos\theta_0\right)\mathbf{e}_\theta + \left(\frac{db(t)}{dt} + a(t)\cos\theta_0\right)\mathbf{e}_\varphi - b(t)\sin\theta_0\mathbf{n} \tag{5.11}$$

Under parallel transport only orthogonal component of the vector changes. Hence we come to differential equations

$$\begin{cases} \frac{da(t)}{dt} - wb(t) = 0 \\ \frac{db(t)}{dt} + wa(t) \end{cases} \qquad a(0) = 0, b(0) = 0, w = \cos\theta_0 \tag{5.12}$$

The solution of these equations is $a(t) = \sin wt, b(t) = \cos wt$. We come to the following answer: parallel transport along parallel $\theta = \theta_0$ of the initial vector $\mathbf{e}_\varphi$ is the family

$$\mathbf{X}(t) = \sin wt\,\mathbf{e}_\theta + \cos wt\,\mathbf{e}_\varphi, w = \cos\theta_0 \tag{5.13}$$

During traveling along the parallel $\theta = \theta_0$ the $\mathbf{e}_\theta$ component becomes non-zero At the end of the traveling the initial vector $\mathbf{X}(t)|_{t=0} = \mathbf{e}_\varphi$ becomes $\mathbf{X}(t)|_{t=2\pi} = \sin 2\pi w\mathbf{e}_\theta + \cos 2\pi w\mathbf{e}_\varphi$: **the vector $\mathbf{e}_\varphi$ after woldtrip traveling along the parallel $\theta = \theta_0$ transforms to the vector $\sin(2\pi\cos\theta_0)\mathbf{e}_\theta + \cos(2\pi\cos\theta_0)\mathbf{e}_\varphi$. In particularly this means that the vector $\mathbf{e}_\varphi$ after parallel transport will rotate on the angle**

$$\text{angle of rotation} = 2\pi\cos\theta_0$$

Compare the angle of rotation with the area of the segment of the sphere above the parallel $\theta = \theta_0$. According to the formula (**??**) area of this segment is equal to $S =$

$2\pi ah = 2\pi a^2(1-\cos\theta_0)$. On the other hand Gaussian curvature of the sphere is equal to $\frac{1}{a^2}$. Hence we see that up to the sign angle of rotation is equal to area of the seqment divided on the Gaussian curvature:

$$\Delta\varphi = \pm\frac{S}{K} = \pm 2\pi\cos\theta_0 \tag{5.14}$$

5.3 Parallel transport along a closed curve on arbitrary surface.

The formula above for the parallel transport along parallel on the sphere keeps in the general case.

Theorem Let M be a surface in $\mathbf{E}^3$. Let $\mathbf{r}(t)\colon \mathbf{r}(t), t_1 \leq t \leq t_2, \mathbf{r}(t_1) = \mathbf{r}(t_2)$ be a closed curve on the surface M such that it is a boundary of domain D of the surface M. (We suppose that the domain D is bounded an orientate.) Let $\mathbf{X}(t)$ be a parallel transport of the arbitrary tangent vector along this closed curve. Consider initial and final vectors $\mathbf{X}(t_1), \mathbf{X}(t_2)$. They have the same length according to (5.2).

Theorem *The angle $\Delta\varphi$ between these vectors is equal to the integral of Gaussian curvature over the domain D:*

$$\Delta\varphi = \pm\int_D K d\sigma \tag{5.15}$$

where we denote by $d\sigma$ the element of the area of surface of M.

The calculations above for traveling along the parallel are just example of this Theorem. The integral of Gaussian curvature over the domain above parallel $\theta = \theta_0$ is equal to $K \cdot 2\pi a(1-\cos\theta_0) = \frac{1}{a^2} \cdot 2\pi a^2(1-\cos\theta_0) = 2\pi(1-\cos\theta_0)$. This is equal to the angle of rotation $2\pi\cos\theta_0$ (up to a sign and modulo 2π). Another simple

Example. Consider on the sphere $x^2+y^2+z^2 = a^2$ points $A = (0,0,1)$, $B = (1,0,0)$ and $C = (0,1,0)$. Consider arcs of great circles which connect these points. Consider the vector $\mathbf{e}_x$ attached at the point A. This vector is tangent to the sphere. It is easy to see that under parallel transport along the arc AB it will transform at the point B to the vector $-\mathbf{e}_z$. The vector $-\mathbf{e}_z$ under parallel transport along the arc BC will remain the same vector $-\mathbf{e}_z$. And finally under parallel transport along the arc CA the vector $-\mathbf{e}_z$ will transform at the point A to the vector $-\mathbf{e}_y$. We see that under traveling along the curvilinear triangle ABC vector $\mathbf{e}_x$ becomes the vector $-\mathbf{e}_y$, i.e. it rotates on the angle $\frac{\pi}{2}$. It is just the integral of the curvature $\frac{1}{a^2}$ over the triangle ABC: $K \cdot S = \frac{1}{a^2} \cdot \frac{4\pi a^2}{8} = \frac{\pi}{2}$.

We know that for plnar triangles sum of the angles is equal to π. It turns out that

Corollary Let ABC be a triangle on the surface formed by geodesics. Then

$$\angle A + \angle B + \angle C = \pi + \int_{\triangle ABC} K ds \tag{5.16}$$

The Gaussian curvature measures the difference of π and sum of angles.

The corollary evidently follows form the Theorem. It is of great importance: It gives us tool to measure curvature. (See the tale about ant.)

5.4 Gauss Bonnet Theorem

Consider the integral of curvature over whole closed surface M. According to the Theorem above the answer has to be equal to 0 (modulo 2π), i.e. $2\pi N$ where N is an integer, because this integral is a limit when we consider very small curve. We come to the formula:

$$\int_D K d\sigma = 2\pi N$$

(Compare this formula with formula (3.41)).

What is the value of integer N?

We present now one remarkable Theorem which answers this question and prove this Theorem using the formula (5.16).

Let M be a closed orientable surface.[9] All these surfaces can be classified up to a diffeomorphism. Namely arbitrary closed oriented surface M is diffeomorphic either to sphere (zero holes), or torus (one hole), or pretzel (two holes),... "Number k" of holes is intuitively evident characteristic of the surface. It is related with very important characteristic—Euler characteristic $\chi(M)$ by the following formula:

$$\chi(M) = 2(1 - g(M)), \quad \text{where } g \text{ is number of holes} \tag{5.17}$$

Remark What we have called here "holes" in a surface is often referred to as "handles" attached o the sphere, so that the sphere itself does not have any handles, the torus has one handle, the pretzel has two handles and so on. The number of handles is also called genus.

Euler characteristic appears in many different way. The simplest appearance is the following:

Consider on the surface M an arbitrary set of points (vertices) connected with edges (graph on the surface) such that surface is divided on polygons with (curvilinear sides)—plaquets. ("Map of world")

Denote by P number of plaquets (countries of the map)

Denote by E number of edges (boundaries between countries)

Denote by V number of vertices.

Then it turns out that

$$P - E + V = \chi(M) \tag{5.18}$$

It does not depend on the graph, it depends only on how much holes has surface.

E.g. for every graph on M, $P - E + V = 2$ if M is diffeomorphic to sphere. For every graph on M $P - E + V = 0$ if M is diffeomorphic to torus.

Now we formulate Gauß-Bonnet Theorem.

[9] Closed means compact surface without boundaries. Intuitively orientability means that one can define out and inner side of the surface. In terms of normal vectors orientability means that one can define the continuous field of normal vectors at all the points of M. The direction of normal vectors at any point defines outward direction. Orientable surface is called oriented if the direction of normal vector is chosen.

Let M be closed oriented surface in $\mathbf{E}^3$.

Let $K(p)$ be Gaussian curvature at any point p of this surface.

Recall that sign of Gaussian curvature does not depend on the orientation. If we change direction of normal vector $\mathbf{n} \to -\mathbf{n}$ then both principal curvatures change the sign and Gaussian curvature $K = \det A / \det G$ does not change the sign [10].

Theorem (Gauß -Bonnet) The integral of Gaussian curvature over the closed compact oriented surface M is equal to 2π multiplied by the Euler characteristic of the surface M

$$\frac{1}{2\pi}\int_M K d\sigma = \chi(M) = 2(1 - \text{number of holes}) \tag{5.19}$$

In particular for the surface M diffeomorphic to the sphere $\kappa(M) = 2$, for the surface diffeomorphic to the torus it is equal to 0.

The value of the integral does not change under continuous deformations of surface! It is integer number (up to the factor π) which characterises topology of the surface.

E.g. consider surface M which is diffeomorphic to the sphere. If it is sphere of the radius R then curvature is equal to $\frac{1}{R^2}$, area of the sphere is equal to $4\pi R^2$ and left hand side is equal to $\frac{4\pi}{2\pi} = 2$.

If surface M is an arbitrary surface diffeomorphic to M then metrics and curvature depend from point to the point, Gauß-Bonnet states that integral nevertheless remains unchanged.

Very simple but impressive corollary:

Let M be surface diffeomorphic to sphere in $\mathbf{E}^3$. Then there exists at least one point where Gaussian curvature is positive.

Proof: Suppose it is not right. Then $\int_M K\sqrt{\det g} dudv \leq 0$. On the other hand according to the Theorem it is equal to 4π. Contradiction.

In the first section in the subsection "Integrals of curvature along the plane curve" we proved that the integral of curvature over closed convex curve is equal to 2π. This Theorem seems to be "ancestor" of Gauß-Bonnet Theorem[11].

Proof of Gauß-Bonet Theorem

Consider triangulation of the surface M. Suppose M is covered by N triangles. Then number of edges will be $3N/over2$. If V number of vertices then according to Euler Theorem

$$N - \frac{3N}{2} + V = V - \frac{N}{2} = \chi(M).$$

[10] For an arbitrary point p of the surface M one can always choose cartesian coordinates (x, y, z) such that surface in a vicinity of this spoint is defined by the equation $z = ax^2 + bx^2 + \dots$, where dots means terms of the order higher than 2. Then Gaussian curvature at this point will be equal to ab. If a, b have the same sign then a surfaces looks as paraboloid in the vicinity of the point p. If If a, b have different signs then a surfaces looks as saddle in the vicinity of the point p. Gaussian curvature is positive if $ab > 0$ (case of paraboloid) and negative if $ab < 0$ saddle

[11] Note that there is a following deep difference: Gaussian curvature is internal property of the surface: it does not depend on isometries of surface. Curvature of curve depends on the position of the curve in ambient space.

Calculate the sum of the angles of all triangles. On the one hand it is equal to $2\pi V$. On the other hand according the formula (5.16) it is equal to

$$\sum_{i=1}^{N}\left(\pi+\int_{\triangle_i} K d\sigma\right) = \pi N + \sum_{i=1}^{N}\left(\int_{\triangle_i} K d\sigma\right) = N\pi + \int_M K d\sigma$$

We see that $2\pi V = N\pi + \int_M K d\sigma$, i.e.

$$\int_M K d\sigma = \pi\left(2V - \frac{N}{2}\right) = 2\pi\chi(M) \blacksquare$$