Homework 3.

1 Let $\{\mathbf{e}_x, \mathbf{e}_y\}$ be an orthonormal basis in $\mathbf{E}^2$. Consider the following ordered pairs:

a) $\{\mathbf{e}_y, \mathbf{e}_x\}$
b) $\{\mathbf{e}_y, -\mathbf{e}_x\}$
c) $\{\frac{\sqrt{2}}{2}\mathbf{e}_x + \frac{\sqrt{2}}{2}\mathbf{e}_y, -\frac{\sqrt{2}}{2}\mathbf{e}_x + \frac{\sqrt{2}}{2}\mathbf{e}_y\}$
d) $\{\frac{\sqrt{3}}{2}\mathbf{e}_x + \frac{1}{2}\mathbf{e}_y, \frac{1}{2}\mathbf{e}_x - \frac{\sqrt{3}}{2}\mathbf{e}_y\}$

Show that all these ordered pairs are orthonormal bases in $\mathbf{E}^2$.

Find amongst them the bases which have the same orientation as the orientation of the basis $\{\mathbf{e}_x, \mathbf{e}_y\}$.

Find amongst them the bases which have the orientation opposite to the orientation of the basis $\{\mathbf{e}_x, \mathbf{e}_y\}$.

2 Let $\{\mathbf{e}, \mathbf{f}\}$ be a basis in two-dimensional linear space V. Consider an ordered pair $\{\mathbf{a}, \mathbf{b}\}$ such that

$$\mathbf{a} = \mathbf{f},\ \mathbf{b} = \gamma\mathbf{e} + \mu\mathbf{f}\,,$$

where γ, μ are arbitrary real numbers.

Find values γ, μ such that an ordered pair $\{\mathbf{a}, \mathbf{b}\}$ is a basis and this basis has the same orientation as the basis $\{\mathbf{e}, \mathbf{f}\}$.

3 Let $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ be an arbitrary basis in $\mathbf{E}^3$. Show that the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ either has the same orientation as the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$, or the same orientation as the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$.

4 Let $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ be an orthonormal basis in $\mathbf{E}^3$. Consider the following ordered triples:

a) $\{\mathbf{e}_x, \mathbf{e}_x + 2\mathbf{e}_y, 5\mathbf{e}_z\}$,
b) $\{\mathbf{e}_y, \mathbf{e}_x, 5\mathbf{e}_z\}$,
c) $\{\mathbf{e}_y, \mathbf{e}_x, -5\mathbf{e}_z\}$,
d) $\{\frac{\sqrt{3}}{2}\mathbf{e}_x + \frac{1}{2}\mathbf{e}_y, -\frac{1}{2}\mathbf{e}_x + \frac{\sqrt{3}}{2}\mathbf{e}_y, \mathbf{e}_z\}$,
e) $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$,
f) $\{\mathbf{e}_y, \mathbf{e}_x, -\mathbf{e}_z\}$.

Show that all ordered triples a),b),c),d),e),f) are bases.
Show that the bases a), c), d) and f) have the same orientation as the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$, and the bases b) and e) have the orientation opposite to the orientation of the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$.
Show that bases d), e) and f) are orthonormal bases and bases a), b) and c) are not orthonormal bases.

5 Let $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ be a basis in linear three-dimensional space V.

Consider the following ordered triples: $\{\mathbf{f}, \mathbf{e} + 2\mathbf{f}, 3\mathbf{g}\}$, $\quad\{\mathbf{e}, \mathbf{f}, 2\mathbf{f} + 3\mathbf{g}\}$.
Show that these ordered triples are bases and these bases have opposite orientations.

6 Show that a linear operator P which transforms the orthonormal basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the basis $\{\mathbf{e}_x, \mathbf{e}_z, -\mathbf{e}_y\}$ is a rotation. Find an axis and an angle of this rotation.

What about a linear operator P which transforms the orthonormal basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the basis $\{\mathbf{e}_y, \mathbf{e}_x, -\mathbf{e}_z\}$. Is it a rotation?

7 † (*Euler Theorem*). A linear operator P in $\mathbf{E}^3$ transforms an orthonormal basis to the orthonormal basis with the same orientation. Prove that it is a rotation.