Introduction to Geometry (20222)

COURSEWORK 2018

Discussions

`Here we discuss the solutions of the coursework. (The solutions of coursework problems with returned courseworks are in the Reception)`

12 April 2018

1

a) *Let* (x^1, x^2, x^3) *be coordinates of the vector* $\mathbf{x}$*, and* (y^1, y^2, y^3) *be coordinates of the vector* $\mathbf{y}$ *in* $\mathbf{R}^3$.

Does the formula $(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^3 + x^3y^2$ *define a scalar product on* $\mathbf{R}^3$*? Justify your answer.*

b) *Consider the matrix* $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Calculate the matrix A^2 *in the case, if* $\theta = \frac{\pi}{4}$.

Calculate the matrix A^{18} *in the case, if* $\theta = \frac{\pi}{108}$.

Calculate the matrix $A^T \circ A^{2018} \circ A^T$ *in the case, if* $\theta = \frac{\pi}{14}$ *(here* A^T *is a transposed matrix.)*

Find all 2×2 *orthogonal matrices* A *such that*

$$2A^3 = \begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix} .$$

c) *In oriented Euclidean space* $\mathbf{E}^3$ *consider the following linear operator*

$$A(\mathbf{x}) = \mathbf{x} - \mathbf{a} \times (\mathbf{a} \times \mathbf{x}) ,$$

where the vector $\mathbf{a} = \mathbf{e} + 2\mathbf{f} + 2\mathbf{g}$*. Here* $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ *is an orthonormal basis in* $\mathbf{E}^3$ *defining orientation, and* $\times$ *is the vector product.*

Find the eigenvectors of operator A*. (Describe eigenvectors via the basis vectors* $\mathbf{e}, \mathbf{f}, \mathbf{g}$*.)*

Calculate the trace and determinant of the operator A.

a) Students have not problem solving this subquestion. On the other hand answering this question almost all the students first showed that linearity and symmetricity axioms are obeyed, and the last axiom (positive definitness axiom) is failed. Thus they came to the conclusion that this form does not define the scalar product. To come to this answer *one does not need to check all the axioms, it is enough just to show that one of axioms is failed.*

b) Answering this subquestions students have mostly problems with the last part of this subquestion, to find the cube root of the matrix A in the space of orthogonal matrices. Many students noted that the matrix $\sqrt[3]{A}$ is a matrix of rotation on the angle φ such that $3\varphi = 2\pi k + \frac{\pi}{6}$. however only few students obtained the implication of this fact, that *there are just three matrices obeying this condition.* (See the solution file.)

c) This was difficult question. Many students noted that vector $\mathbf{a}$ is eigenvector of this operator with eigenvalue 1, however only few students noted that an arbitrary vector which is orthogonal to vector $\mathbf{a}$ is eigenvector of this operator with eigenvalue 10, and the space of eigenvectors with eigenvalue 10 is 2-dimensional vector space. This immediately leads to the fact that one can choose a basis such that one vector of this basis has eigenvalue 1, and other two vectors of this basis have eigenvalue 10. This implies that $\det A = 1 \times 10 \times 10 = 100$ and $\mathrm{Tr}A = 1 + 10 + 10 = 21$ (see also the solutions file.) Instead many students calculated the matrix of the operator in the basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$: $A = \begin{pmatrix} 9 & -2 & -2 \\ -2 & 6 & 4 \\ -2 & -4 & 6 \end{pmatrix}$ and then they were trying straightforwardly to find eigenvectors using just linear algebra algorithms without geometrical considerations. This solution is also right solution (in spite of the fact that it is much longer) but unfortunately some students who were trying to calculate eigenvectors using the standard linear algebra technique without geometry, did not overcome the problems during these calcualtions. Two words on this solution: characteristic polynomial

$$P(\lambda) = \det \begin{pmatrix} 9-\lambda & -2 & -2 \\ -2 & 6-\lambda & 4 \\ -2 & -4 & 6-\lambda \end{pmatrix} = (1-\lambda)(\lambda-10)^2 = 0\,.$$

Many students who decided to find eigenvectors in this way found correctly eigenvalues $\lambda_1 = 1, \lambda_2 = 10$, but then some students had problems to find eigenvectors: the second root of this equation has the multiplicity 2, and this is why the space of eigenvectors with eigenvalue 10 is 2-dimensional. (To find a basis in this subspace you need to consider 2 linearly independent vectors which are orthogonal to the vector $\mathbf{a}$.)

`Another remark`: few students supposed that if $\mathbf{x}$ is an eigenvector of the operator A then $A(\mathbf{x}) = \mathbf{x}$. Yes if $A(\mathbf{x}) = \mathbf{x}$ then $\mathbf{x}$ is an eigenvector, with eigenvalue 1 ($\mathbf{x} \neq 0$), but the condition that eigenvalue is equal to 1 is not necessary condition. The confusion is may be coming from the fact that the Euler theorem tells that an orthogonal operators preserving orientation have eigenvector with eigenvalue $\lambda = 1$ (it is the eigenvector which defines the axis of rotation.) Euler Theorem has nothing to do with this example: this linear operator is not an orthogonal operator. In general eigenvector may have arbitrary eigenvalueis, e.g. in the example above we have eigenvectors with eigenvalues 1 and 10.

2

a) *Consider a vector* $\mathbf{a} = 2\mathbf{e} + 3\mathbf{f} + 6\mathbf{g}$*, where* $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ *is an orthonormal basis in* $\mathbf{E}^3$*. Show that the angle* θ *between vectors* $\mathbf{a}$ *and* $\mathbf{g}$ *belongs to the interval* $\left(\frac{\pi}{6}, \frac{\pi}{4}\right)$*.*

Find a unit vector $\mathbf{b}$ *such that this vector is orthogonal to vectors* $\mathbf{a}$ *and* $\mathbf{g}$*, and the basis* $\{\mathbf{a}, \mathbf{b}, \mathbf{g}\}$ *has the same orientation as the basis* $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$*.*

Calculate the angle between vectors $\mathbf{b}$ *and* $\mathbf{e}$*.*

b) *In oriented Euclidean space* $\mathbf{E}^3$ *consider the following function of three vectors:*

$$F(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = (\mathbf{X}, \mathbf{Y} \times \mathbf{Z}) \,,$$

where $(\ ,\ )$ *is the scalar product and* $\mathbf{Y} \times \mathbf{Z}$ *is the vector product in* $\mathbf{E}^3$*.*

Show that $F(\mathbf{X}, \mathbf{X}, \mathbf{Z}) = 0$ *for arbitrary vectors* $\mathbf{X}$ *and* $\mathbf{Z}$*.*

Deduce, that $F(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = -F(\mathbf{Y}, \mathbf{X}, \mathbf{Z})$ *for arbitrary vectors* $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$*.*

What is the geometrical meaning of the function F*?*

c) *Let* $ABCD$ *be a rhombus (parallelogram with equal sides) such that*

i) vertex A *is at the origin*

ii) the diagonal AC *belongs to the line* $y = x$*.*

iii) vertex B *has integer coordinates.*

Find the area of this rhombus, if the vertex B *has coordinates* $(20, 21)$*. Justify your answer.*

Find all the rhombi, which obey the conditions i), ii) and iii) above, and which have area $S = 25$.

a) Many students first calculated the unit vector $\mathbf{b}$ up to a sign using the fact that it is orthogonal to vectors $\mathbf{a}$ and $\mathbf{g}$; some of these students have stopped calculations here, some of students have chosen the right sign using the orientation arguments. Sure students who were taking into considerations orientation arguments came to the right solution. However there is another more elegant way: instead doing this two steps procedure it is enough to consider just the vector $\mathbf{x} = \frac{\mathbf{g}\times\mathbf{a}}{|\mathbf{g}\times\mathbf{g}|}$. This vector obeys all the conditions including orientation conditions, due to the properties of vector product (see the solutions file.).

b) Almost all students explained correctly why $F(\mathbf{X},\mathbf{X},\mathbf{Z}) = 0$, and then based on the properties of vector product and determinant, they showed that $F(\mathbf{X},\mathbf{Y},\mathbf{Z}) = -F(\mathbf{Y},\mathbf{X},\mathbf{Z})$. On the other hand the second equation has to be deduced from the first one (see question in the coursework and the solutions). Only few students deduced the second equation explicitly from the first equation: The fact that $F(\mathbf{X},X,\mathbf{Z}) = 0$ implies immedaitely the antisymmetricity condition $F(\mathbf{X},\mathbf{Y},\mathbf{Z}) = -F(\mathbf{Y},\mathbf{X},\mathbf{Z})$:

$$F(\mathbf{X},\mathbf{X},\mathbf{Z}) = 0 \Rightarrow 0 = F(\mathbf{X}+\mathbf{Y},\mathbf{X}+\mathbf{Y},\mathbf{Z}) = F(\mathbf{X},\mathbf{X},\mathbf{Z})+F(\mathbf{X},\mathbf{Y},\mathbf{Z})+F(\mathbf{Y},\mathbf{X},\mathbf{Z})+F(\mathbf{Y},\mathbf{Y},\mathbf{Z})$$

$$= F(\mathbf{X},\mathbf{Y},\mathbf{Z}) + F(\mathbf{Y},\mathbf{Y},\mathbf{Z}) = 0\,.$$

It is not very deep trick, but it is elegant, is not it? Almost all students received the full mark for this question, in spite of the fact that only few students presented this short solution.

c) This question was not easy question, and I was happy that students answered it well.

A few students calculated the area of the rhombus using the fact that it is equal to the half of the product of the lengths of the diagonals. It is more convenient to use the determinant formula. (see the file of solutions.)

3

We consider in this question a 3-dimensional Euclidean space. We suppose that $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$ is an orthonormal basis in this space.

a) *Consider an operator P such that*

$$P(\mathbf{e}) = \frac{2}{3}\mathbf{e} + \frac{2}{3}\mathbf{f} + \frac{1}{3}\mathbf{g}\,, P(\mathbf{f}) = -\frac{1}{3}\mathbf{e} + \frac{2}{3}\mathbf{f} - \frac{2}{3}\mathbf{g}\,, P(\mathbf{g}) = -\frac{2}{3}\mathbf{e} + \frac{1}{3}\mathbf{f} + \frac{2}{3}\mathbf{g}\,.$$

Show, that it is an orthogonal operator preserving orientation.

Show, that this operator defines rotation, and find the axis and the angle of this rotation.

b) *Let P be a linear orthogonal operator acting in $\mathbf{E}^3$, such that it preserves the orientation of $\mathbf{E}^3$ and the following relations hold:*

$$P(\mathbf{e}) = \cos\frac{\pi}{5}\ \mathbf{e} + \sin\frac{\pi}{5}\ \mathbf{f}, \quad P(\mathbf{g}) = -\mathbf{g}\,.$$

Write down the matrix of operator P in the basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$.

Show that this operator defines rotation, and find the axis and the angle of this rotation.

c) *Orthogonal operator P obeys the condition*

$$P \neq I\,, \quad \text{and} \quad P^3 = I\,.$$

Show that P is a rotation operator, and calculate the angle of rotation.

a) Almost all students answered correctly this subquestion. Few students had problems to calculate the eigenvector which defines axis.

It is important to notice that to check that operator is rotation operator in $\mathbf{E}^3$ it is not enough just to check that it preserves orientation. You have to show that the operator is orthogonal (e.g. showing that its matrix in orthonormal basis is orthogonal matrix) and it also preserves orientation. Few students have forgotten to check the condition that the matrix of operator is orthogonal, and they claimed that operator is orthogonal based only on the fact that its determinant is equal to 1.

On the other hand few students did the following: they proved correctly that the operator operator is orthogonal and it preserves orientation, were trying straightwforwardly to prove that this

b) This subquestion was done by majority of students. Few students had problems to calculate axis of rotations. It is interesting to note that axis are directed along the bisectrix of the angle $\frac{\pi}{5}$ (see the solutions). Only few students noted it.

Another remark about subquestions a) and b): in both these subquestions, we consider orthogonal oeprator which does not change orientation, (i.e. in terms of matrices, orthogonal matrices with determinant $= 1$). Few students answering these two subquestions did the following: they proved correctly that the operator is orthogonal then they showed by straightforward calculations that at least one eigenvalue is equal to 1. Yes, this is right, but it is the statement of Euler Theorem. Notice also that the fact that the orthogonal operator in suquestion *b*) has two eigenvalues ($\lambda_1 = 1$ and $\lambda_2 = -1$), is related with the fact that the angle of rotation is equal to π: an arbitrary vector which is orthogonal to the axis is an eigenvector with eigenvalue -1.

c) Surprisingly almost all students did this exercise correctly.