Homework 4

As usual all the exercises marked by † are not compulsory.

In the all exercises except 5 and 6, vectors belong to 3-dimensional Euclidean space equipped with orientation.

1 Prove that vectors $\mathbf{a}$ and $\mathbf{b}$ are linear independent if and only if $\mathbf{a}\times\mathbf{b}\neq 0$.

2 Vectors $\mathbf{a}$ and $\mathbf{b}$ are linear independent. Consider the vector $\mathbf{c}=\mathbf{a}\times\mathbf{b}$. Prove that the ordered set $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ is a basis in $\mathbf{E}^3$.

3 Students John and Sarah calculate vector product $\mathbf{a}\times\mathbf{b}$ of two vectors using two different orthonormal bases in the Euclidean space $\mathbf{E}^3$, $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ and $\{\mathbf{e}'_1,\mathbf{e}'_2,\mathbf{e}'_3\}$. John expands the vectors with respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$. Sarah expands the vectors with respect to the basis $\{\mathbf{e}'_1,\mathbf{e}'_2,\mathbf{e}'_3\}$. For two arbitrary vectors $\mathbf{a},\mathbf{b}\in\mathbf{E}^3$

$$\mathbf{a}=a_1\mathbf{e}_1+a_2\mathbf{e}_2+a_3\mathbf{e}_3=a'_1\mathbf{e}'_1+a'_2\mathbf{e}'_2+a'_3\mathbf{e}'_3\,,$$

$$\mathbf{b}=b_1\mathbf{e}_1+b_2\mathbf{e}_2+b_3\mathbf{e}_3=b'_1\mathbf{e}'_1+b'_2\mathbf{e}'_2+b'_3\mathbf{e}'_3\,.$$

John and Sarah both use so called "determinant" formula. Are their answers the same?

$$\mathbf{a}\times\mathbf{b}=\underbrace{\det\begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{pmatrix}}_{\text{John's calculations}}\stackrel{?}{=}\underbrace{\det\begin{pmatrix}\mathbf{e}'_1 & \mathbf{e}'_2 & \mathbf{e}'_3\\ a'_1 & a'_2 & a'_3\\ b'_1 & b'_2 & b'_3\end{pmatrix}}_{\text{Sarah's calculations}}$$

4 Calculate the area of parallelograms formed by the vectors $\mathbf{a},\mathbf{b}$ if

a) $\mathbf{a}=(1,2,3),\mathbf{b}=(1,0,1)$;
b) $\mathbf{a}=(2,2,3),\mathbf{b}=(1,1,1)$;
c) $\mathbf{a}=(5,8,4),\mathbf{b}=(10,16,8)$.
d) $\mathbf{a}=(3,4,0),\mathbf{b}=(5,17,0)$.

5 Let $\mathbf{a},\mathbf{b}$ be two vectors in the 2-dimensional Euclidean space $\mathbf{E}^2$. Calculate the area of the parallelogram formed by these vectors if

a) $\mathbf{a}=(2,3)$, $\mathbf{b}=(5,9)$
b) $\mathbf{a}=(17,12)$, $\mathbf{b}=(7,5)$
c) $\mathbf{a}=(41,29)$, $\mathbf{b}=(99,70)$

6† Do you see any relation between fractions $\frac{3}{2},\frac{17}{12},\frac{41}{29},\frac{99}{70}$ (see the exercises 5b) and 5c) above) and the number... $\sqrt{2}$? Can you continue the sequence of these fractions? (*Hint: Consider the squares of these fractions.*)

7 Show that for any two vectors $\mathbf{a},\mathbf{b}\in\mathbf{E}^3$ the following identity is satisfied

$$(\mathbf{a},\mathbf{a})(\mathbf{b},\mathbf{b})=(\mathbf{a},\mathbf{b})^2+(\mathbf{a}\times\mathbf{b},\mathbf{a}\times\mathbf{b})\,.$$

Write down this identity in components.

† Compare this identity with CBS inequality . (See the problem 7 in the Homework 2).

8 Find a vector $\mathbf{n}$ such that the following conditions hold:
1) It has a unit length
2) It is orthogonal to the vectors $\mathbf{a} = (1, 2, 3)$ and $\mathbf{b} = (1, 3, 2)$.
3) An ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{n}\}$ has an orientation opposite to the orientation of the orthonormal basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ which defines the orientation of the Euclidean space.

9 Volume of parallelepiped $V(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{a}, \mathbf{b} \times \mathbf{c})$, formed by the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ equals to zero if and only if these vectors are linearly dependent. Prove it.

10 Vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal unit vectors. Calculate the length of the vector $\mathbf{c} \times (\mathbf{a} \times \mathbf{b})$, where $\mathbf{c} = \mathbf{a} + \mathbf{b}$.
11 Show that in general $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$. (Associativity law is not obeyed)

12 † a) Show that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a}, \mathbf{c}) - \mathbf{c}(\mathbf{a}, \mathbf{b})$
† b) Show that

$$\mathbf{a} \times \mathbf{b} \times \mathbf{c} + \mathbf{b} \times \mathbf{c} \times \mathbf{a} + \mathbf{c} \times \mathbf{a} \times \mathbf{b} = 0 \quad \text{(Jacobi identity)}\,.$$