3 November 2018

It is long time that I know about Fock remark to Dicrak book. I make different attempt to understand the Fock quasiclassical soltion :

$$\Psi(x,t) = \sqrt{\det\left(\frac{\partial^2 F}{\partial x^i \partial y^j}\right)} e^{\frac{i}{\hbar} S(x,t;x_0,t_0)} \tag{0.1}$$

more precisely the amplitude of this solution, and cannot. Here I try another attempt.

We have

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = \hat{H}\Psi\,, \qquad E\Phi(x,E) = \hat{H}\Phi(x,E)\,, \tag{1}$$

where

$$H = \frac{p^2}{2m} + U(x)\,,$$

$$\Psi(x,t) = \int \Phi(x,E) e^{-\frac{iEt}{\hbar}}\,.$$

We find first quasiclassical solution in Energy coordinates, then do their Legendre transformation to time picture.

We consider

$$\Psi(x_0,t_0;x,t) = \exp\left[\frac{i}{\hbar} S_\hbar(x_0,t_0;x,t)\right]\,, \qquad \Phi(x_0,x,E) = \exp\left[\frac{i}{\hbar}\mathcal{S}_\hbar(x_0 x,E)\right]\,,$$

where $S_\hbar, \mathcal{S}_\hbar$ are formal polynomials on $\hbar$ of inifnite degrees:

$$S_\hbar(x_0,t_0;x,t) = S(x_0,t_0;x,t) + \frac{\hbar}{i}\sigma(x_0,t_0,x,t) + \dots \qquad \mathcal{S}_\hbar(x,E) = \mathcal{S}(x,E) + \frac{\hbar}{i} s(x,E) + \dots$$

Here S ($\mathcal{S}$) is the classical action in time representation (E)-representation (S and $\mathcal{S}$ are reciprocal Legendre transforms):

$$S(x,t) = \mathcal{S}(x,E) - Et\,, \quad \text{with } E\colon t = \frac{\partial \mathcal{S}}{\partial E}\,.$$

The term σ (s) is responcible for the amplitude. The dependence on x_0 is due to initial conditions; H does not depend on time, it is why $\mathcal{S}$ deos not depend on E_0.

The Shrodinger equation becomes

$$i\hbar\frac{\partial \Psi}{\partial t} = i\hbar \frac{\partial}{\partial t} e^{\frac{i}{\hbar}\mathcal{S}_\hbar(x,t)} = \hat{H} e^{\frac{i}{\hbar}\mathcal{S}_\hbar(x,t)} = , \qquad E\Phi(x,E) = E e^{\frac{i}{\hbar}\mathcal{S}_\hbar(x,E)} = \hat{H}\Psi(x,E) =$$

$$\left[\frac{\hbar}{i}\frac{\partial}{\partial x^i}\left(\frac{\hbar}{i}\frac{\partial}{\partial x_i}\right) + U(x)\right] e^{\frac{i}{\hbar}\mathcal{S}_\hbar(x,t)} = \qquad \left[\frac{\hbar}{i}\frac{\partial}{\partial x^i}\left(\frac{\hbar}{i}\frac{\partial}{\partial x_i}\right) + U(x)\right] e^{\frac{i}{\hbar}\mathcal{S}_\hbar(x,E)} =$$

$$\left[\frac{\hbar}{i}\frac{1}{2m}\Delta S_\hbar+\frac{1}{2m}\left(\mathrm{grad}S_\hbar\right)^2+U(x)\right]e^{\frac{i}{\hbar}S_\hbar(x,t)}\,,\qquad \left[\frac{\hbar}{i}\frac{1}{2m}\Delta \mathcal{S}_\hbar+\frac{1}{2m}\left(\mathrm{grad}\mathcal{S}_\hbar\right)^2+U(x)\right]e^{\frac{i}{\hbar}\mathcal{S}_\hbar(x,E)}\,,$$

i.e.

$$\begin{array}{cc} \frac{\hbar}{i}\frac{1}{2m}\Delta S_\hbar(x,t) \quad +\frac{1}{2m}\left(\mathrm{grad}\mathcal{S}_\hbar\left(x,t\right)\right)^2\,, & \frac{\hbar}{i}\frac{1}{2m}\Delta \mathcal{S}_\hbar(x,E) \quad +\frac{1}{2m}\left(\mathrm{grad}\mathcal{S}_\hbar\left(x,E\right)\right)^2 \\ +U=-\frac{\partial S}{\partial t} & +U=E\mathcal{S} \end{array}$$

We have

$$\Delta\mathcal{S}_\hbar=\Delta S(x,E)+\frac{\hbar}{i}s(x,E)+\dots\,,$$

$$\left(\mathrm{grad}\mathcal{S}_\hbar x,E\right)^2=\left(\mathrm{grad}\mathcal{S}\right)^2+2\left(\frac{\hbar}{i}\right)\mathrm{grad}\mathcal{S}\cdot\mathrm{grad}s+\dots$$

The zeroth approximmation is

$$\frac{1}{2m}\left(\mathrm{grad}\mathcal{S}\left(x,t\right)\right)^2+U(x)+S_t=0\,,\quad \frac{1}{2m}\left(\mathrm{grad}\mathcal{S}\left(x,E\right)\right)^2+U(x)=E\,,$$

this is nothing but Hamilton-Jacobi equation

$$H\left(x,\frac{\partial S}{\partial x}\right)+\frac{\partial S}{pt}=0\,,\qquad H\left(x,\frac{\partial S}{\partial x}\right)=E$$

in time (E)-representation. They are related with Legendre transform.

In the first approximation E-representation is simpler than t-representation.

In the first approximation in E we have

$$\frac{1}{2}\Delta\mathcal{S}(x,E)+\mathrm{grad}\mathcal{S}\left(x,E\right)\mathrm{grad}s\left(x,E\right)=0\,,$$

this is the equation on the semidensity, it can be rewritten

$$\mathcal{L}_{\mathbf{V}}\left(s\sqrt{Dx}\right)=V^i\frac{\partial s}{px^i}+\frac{1}{2}\frac{\partial V^i}{\partial x^i}\sqrt{Dx}\qquad \text{for}\quad \mathbf{V}=\mathrm{grad}\mathcal{S}\,.$$

In time representation it will be

The equation for semidensity in E representation is simpler than in time representation.

I can easy to solve this equation at least in one-dimensiona case: Then

$$\frac{1}{2m}\mathcal{S}_x^2+U\mathcal{S}=ES\Rightarrow\mathcal{S}(x,E)=\int_{x_0}^x p(y)dy\,,\quad p(y)=\sqrt{2mE-U(x)}\,,$$

and

$$\frac{1}{2}\mathcal{S}_{xx}+s_x^2=0, i.e. s(x)\sim 1\sqrt{p}(x)$$

In general case it is usefule to express done the solution of the transport euquation, the amplitude $\sigma(x,t)$ in terms of the amplitude $s(x,E)$. Instea doing Legendre transform it is much more illiminating to do the quasiclassi stationary method:

W

We have in the first approximation

$$e^{\frac{i}{\hbar}S_\hbar(x,t)} = \int \left(e^{\frac{i}{\hbar}(S_\hbar(x,E)-Et)}\right) dE = \int A_E(x,E)\left(e^{\frac{i}{\hbar}(S(x,E)-Et)}\right) dE = A_E(x,E(t))\frac{1}{\sqrt{\det\frac{\partial E}{\partial t}}}$$