Homework 8. Solutions

1 *Show that vertical lines $x=a$ are geodesics (non-parameterised) on Lobachevsky plane.*

We consider here the realisation of Lobachevsky plane (hyperbolic plane) as upper half of Euclidean plane $\{(x,y)\colon y>0\}$ with the metric $G=\frac{dx^2+dy^2}{y^2}$.

Consider second order differential equations defining geodesics with initial conditions such that "horisontal" velocity equals to zero: (we uses the information from Homework 6 or from Lecture notes about Christoffels for Lobachevsky plane: $\Gamma^x_{xx}=0, \Gamma^x_{xy}=\Gamma^x_{yx}=-\frac{1}{y}$, $\Gamma^x_{yy}=0$, $\Gamma^y_{xx}=\frac{1}{y}, \Gamma^y_{xy}=\Gamma^y_{yx}=0, \Gamma^y_{yy}=-\frac{1}{y}$.)

$$\begin{cases} \ddot{x}-\frac{2\dot{x}\dot{y}}{y}=0 \\ \ddot{y}+\frac{\dot{x}^2}{y}-\frac{\dot{y}^2}{y}=0 \\ x(t)\big|_{t=t_0}=x_0, \dot{x}(t)\big|_{t=t_0}=0 \\ y(t)\big|_{t=t_0}=y_0, \dot{y}(t)\big|_{t=t_0}=\dot{y}_0 \end{cases}$$

This equation has a solution and it is unique. One can see that if we put $x(t)\equiv 0$, i.e. curve is vertical then we come to the equation $\ddot{y}-\frac{\dot{y}^2}{y}=0$. Solution of these equation gives curve $x=x_0$, $y=y(t)\colon \ddot{y}-\frac{\dot{y}^2}{y}=0$. The image of this curve clearly is vertical ray $x=x_0, y>0$.

2 *Consider a vertical ray $C\colon x(t)=1, y(t)=1+t$, $0\leq t<\infty$ on the Lobachevsky plane.*

Find the parallel transport $\mathbf{X}(t)$ of the vector $\mathbf{X}_0=\partial_y$ attached at the initial point $(1,1)$ along the ray C at an arbitrary point of the ray.

Find the parallel transport $\mathbf{Y}(t)$ of the vector $\mathbf{Y}_0=\partial_x+\partial_y$ attached at the same initial point $(1,1)$ along the ray C at an arbitrary point of the ray. (Exam question, 2013.)

Since vertical ray is geodesic then during parallel transport vector $\mathbf{X}(t)$ remains proportional to velocity vector. Hence $\mathbf{X}(t)=k(t)\partial_y$. On the other hand during parallel transport its length is not changed, since the connection is Levi-Civita connection. i.e. scalar product

$$\langle\mathbf{X}(t),\mathbf{X}(t)\rangle=\langle k(t)\partial_y,k(t)\partial_y\rangle=\frac{k^2(t)}{(y_0+t)^2}=\frac{k^2(t)}{(1+t)^2}=Constant$$

At the moment $t=0$ it is equal to $\frac{1}{1}=1$. We have $\frac{k^2(t)}{(1+t)^2}=1$, i.e. $k(t)=\pm(1+t)$

Since at $t=0$, $k=1$ we choose sign $+$ and $k(t)=1+t$. We come to $\mathbf{X}(t)=(1+t)\partial_y$

During parallel transport of two vectors along Levi-Civita connection not only their lengths but angles between them is not changed too.

Consider vector $\mathbf{Z}=\partial_x$. it is orthogonal to vector $\mathbf{X}$. Hence during parallel transport it will remain orthogonal. Hence $\mathbf{Z}(t)=k'(t)\partial_x$ since vectors ∂_x,∂_y are orthogonal to each other at any point of the Lobachevsky plane. The length fo the vector $\mathbf{Z}(t)$ is preserved too. Hence it has to be equal always to 1 since at $t=0$ it is equal to 1. We come to $\mathbf{Z}(t)=(1+t)\partial_x$.

Now by linearity of parallel transport $\mathbf{Y}(t)=\mathbf{X}(t)+\mathbf{Z}(t)=(1+t)(\partial_x+\partial_y)$.

3 *Find a parameterisation of vertical lines in the Lobachevsky plane such that they become parameterised geodesics.*

We know also that vertical line is geodesic. Let $x=x_0, y=f(t)$ be right parameterisation, i.e. parameterisation such that veclotiy vector remains velocity vector during parallel transport. Velocity vector $\mathbf{v}(t)=\begin{pmatrix}0\\f_t\end{pmatrix}$. Its length is equal to $\sqrt{\frac{x_t^2+y_t^2}{y^2}}=\sqrt{\frac{0+f_t^2}{f^2}}=\frac{f_t}{f}$ and it has remain the same. Hence $\frac{f_t}{f}=c$, i.e. $f(t)=Ae^{ct}$. We see that $x=x_0, y=ae^{ct}$ is parameterised geodesic. (One can see that differential equation of geodesics are obeyed (see the exercise 2)).

4 *Consider the plane* $\mathbf{R}^2$ *with Cartesian coordinates and with Riemannian metric*

$$G = \frac{4R^2(du^2+dv^2)}{(R^2+u^2+v^2)^2)}.$$

Show that all lines passing through the origin ($u = v = 0$) and only these lines are geodesics of the Levi-Civita connection of this metric.

Give examples of other geodesics.

† *Find all geodesics of this metric.*

(You may use the fact that this Rimeannian manifold is isometric to the sphere without North pole.)

$\mathbf{R}^2$ with this Riemannian metric is isometric to the sphere of raidus R without North Pole. One can say in other way: u, v are stereographic coordiantes on the sphere and the metric G is just the metric of the sphere.

The straight lines passing through the origin is $u = kv$ or $v = ku$, are the images of great cirles which pass through North Pole: it is evident form the definition, but one can see it also from the formulae for stereographic coordiantes:

$$u = \frac{Rx}{R-z}, \quad v = \frac{Rx}{R-z}$$

We have $\frac{u}{v} = k$, hence $\frac{x}{y=k}$. Hence the line $u = kv$ is the image of the curve $x = ky$ on the sphere. This curve is intersection of the plane $x = ky$ with sphere, it is a great circle.

Now we know that all geodesics are images of great circles under stereographic projection. E.g. if we take an equatior on the sphere: $x^2+y^2 = R^2$, $z = 0$, then we come to $u = x, v = y$, i.e. $u^2+v^2 = R^2$ is aq geodesic.

Remark Other circles with centre at origin will not be geodesics, since they are image of the circles which are intersection of the plane $z = a \neq 0$ with a sphere, and this is not a great circle.

† Find all geodesics. (For simplicity we consider only the case $R = 1$.) Any great circle on sphere is the intersection of the plane passing thorugh origin and the sphere. Hence it is given ny equations $\begin{cases} Ax+By+Cz=0 \\ x^2+y^2+z^2=1 \end{cases}$ Using formulae for stereographic projections we see that in coordiantes u, v these equations will be rewritten in the following way:

$$A\frac{2u}{1+u^2+v^2} + B\frac{2v}{1+u^2+v^2} + C\frac{u^2+v^2-1}{1+u^2+v^2} = 0\,,$$

i.e. they are lines passing through origin if $C = 0$ and in the case if $C \neq 0$ we come to circles such that

$$(u+a)^2 + (v+b)^2 = 1 + a^2 + b^2\,.$$

, we see that geodesics are lines passing through origin and circles such that $R = \sqrt{1+d^2}$, where d is a distance between origin and the centre of the circle and R is radiis of the circle.

5* *Let* $\mathbf{X}(t)$ *be parallel transport of the vector* $\mathbf{X}$ *along the curve on the surface* M *embedded in* $\mathbf{E}^3$*, i.e.* $\nabla_{\mathbf{v}}\mathbf{X} = 0$*, where* $\mathbf{v}$ *is a velocity vector of the curve* C *and* ∇ *Levi-Civita connection of the metric induced on the surface. Compare the condition* $\nabla_{\mathbf{v}}\mathbf{X} = 0$ *(this is condition of parallel transport for internal observer) with the condition that for the vector* $\mathbf{X}(t)$*, the derivative* $\frac{d\mathbf{X}(t)}{dt}$ *is orthogonal to the surface (this is condition of parallel transport for external observer)*[2]*.*

Do these two conditions coincide, i.e. do they imply the same parallel transport?

[2] *We defined parallel transport in Geometry course using this condition*

We know that Levi-Civita connection on surfaces coicides with induced conenction. We have by definition of induced connection that

$$\nabla_{\mathbf{v}}\mathbf{X} = \left(\nabla^{\text{can. flat}}_{\mathbf{v}}\mathbf{X}\right) = (\partial_{\mathbf{v}}\mathbf{X})_{\text{tangent}} = \left(\frac{d\mathbf{X}(t)}{dt}\right)_{\text{tangent}}$$

Hence $\nabla_{\mathbf{v}}\mathbf{X} = 0$ if and only if the derivative $\frac{d\mathbf{X}(t)}{dt}$ is orthogonal to the surface.

6 *On the sphere $x^2 + y^2 + z^2 = R^2$ of radius R in $\mathbf{E}^3$ consider the following three closed curves.*

a) the triangle $\triangle ABC$ with vertices at the points $A = (0,0,1)$, $B = (0,1,0)$ and $C = (1,0,0)$. The edges of triangle are geodesics.

b) the triangle $\triangle ABC$ with vertices at the points $A = (0,0,1)$, $B = (0,\cos\varphi,\sin\varphi)$ and $C = (1,0,0)$, $0 < \varphi < \frac{\pi}{2}$ The edges of triangle are geodesics.

c) the curve $\theta = \theta_0$ (line of constant latitude).

Consider the result of parallel transport of the vectors tangent to sphere over these closed curves.

In the case a) angle of rotation will be $\frac{\pi}{2}$: e,g, take the vector ∂_y at the point A and see how it transforms during parallel transport. This vector is tangent to the sphere. (The angle of rotation is the same for all the tangent vectors.)

The vector ∂_y during parallel transport along the arc AB will remain tangent to this arc, since it is an arc of geodesic, and it will preserve its length. Hence at the point B it will become the vector ∂_z.

The vector ∂_z during parallel transport along the arc of geodesic BC will remain always orthogonal to this arc, since at the initial point it was orthogonal to the arc, i.e. it will remian the same vector (in the ambient space $\mathbf{E}^3$), and at the point C it will remain ∂_z,

Then the same reasoning for the curve CA: during parallel transport along arc of geodesic CA it will remain always tangent to the curve, and finally it will be the vector ∂_x. We see that the vector ∂_y becomes the vector ∂_x after parallel transport.

Notice that angle of rotation

$$\frac{\pi}{2} = \frac{\text{area of the triangle } ABC}{R^2} = K \cdot \text{area of the triangle } ABC\,.$$

For the case b) doing the same considerations we come to the fact that vector rotates on the angle φ.

One can take initial vector $\mathbf{A} = \cos\varphi\partial_x + \sin\varphi\partial_y$. During parapllel transport this vector will remain tangent to the arc AB since it is geodesic, and its length will not change. The result of parallel trasport at the point B will be the vector ∂_z. The vector ∂_z during parallel transport along the arc of geodesic BC will remain always orthogonal to this arc, and in the same way as in the case a) we will come to the conclusion that at the point C it will remain ∂_z. Then applying the same reasoning as in the case a) we will come to the conclusion that finally it will be the vector ∂_x. We see that the vector ∂_y becomes the vector ∂_x after parallel transport.

Notice that angle of rotation

$$\frac{\varphi}{2} = \frac{\text{area of the triangle } ABC}{R^2} = K \cdot \text{area of the triangle } ABC\,.$$

In the case c) the closed curve C is not geodesic, and we have to apply the Theorem

The curve C is the boundary of the segment of the sphere D:

$$D = \{x, y, z \colon R\cos\theta_0 \le z \le R\}$$

The height of this domain is equal to $h = R(1-\cos\theta_0)$, and The area of this domain is equal to $S = 2\pi Rh = 2\pi R^2(1-\cos\theta_0)$. Due to the Theorem of parallel transport we have for angle of rotation $\Delta\Phi$

$$\Delta\Phi = \int_{R\cos\theta_0 \le z \le R} K d\sigma = K \cdot \text{Area of the segment} = 2\pi R^2(1-\cos\theta_0)\mathbf{R}^2 = 2\pi(1-\cos\theta_0)\,.$$