Two hours

THE UNIVERSITY OF MANCHESTER

RIEMANNIAN GEOMETRY. MOCK EXAMINATION

XX May—XX June 2017

XX:00 – XX:00

Answer **ALL FIVE** questions in Section A (50 marks in total).
Answer **TWO** of the THREE questions in Section B (30 marks in total).
If more than TWO questions in Section B are attempted, the credit will be given for the best TWO answers.

Electronic calculators may not be used.

Throughout the paper, where the index notation is used, the Einstein summation convention over repeated indices is applied if it is not explicitly stated otherwise.

SECTION A

Answer **ALL** FIVE questions

A1.

(a) Explain what is meant by saying that G is a Riemannian metric on a manifold M.

(b) Consider the upper half plane ($y > 0$) in $\mathbf{R}^2$ equipped with the Riemannian metric $G = \sigma(x,y)(dx^2 + dy^2)$.

Explain why $\sigma(x,y) > 0$.

Consider in this Riemannian manifold a curve C such that

$$C\colon \quad \begin{cases} x = 1 \\ y = a + t \end{cases} \quad , \quad 0 \le t \le 1\,, \ (a > 0)\,.$$

Find the length of this curve in the case if $\sigma(x,y) = \frac{1}{y^2}$ (the Lobachevsky metric).

[10 marks]

A2.

(a) Explain what is meant by saying that a Riemannian manifold is locally Euclidean.

(b) Consider a surface (the upper sheet of a cone) in $\mathbf{E}^3$

$$\mathbf{r}(h,\varphi)\colon \quad \begin{cases} x = 2h\cos\varphi \\ y = 2h\sin\varphi \\ z = h \end{cases} \quad , \quad h > 0, 0 \le \varphi < 2\pi\,.$$

Calculate the Riemannian metric on this surface induced by the canonical metric on Euclidean space $\mathbf{E}^3$.

Show that this surface is locally Euclidean.

[10 marks]

A3.

(a) Explain what is meant by an affine connection on a manifold.

Give the definition of the canonical flat connection on the Euclidean space $\mathbf{E}^n$.

(b) Calculate the Christoffel symbols Γ^r_{rr} and $\Gamma^r_{\varphi\varphi}$ of the canonical flat connection in the Euclidean space $\mathbf{E}^2$, where r, φ are polar coordinates ($x = r\cos\varphi, y = r\sin\varphi$).

[10 marks]

A4.

(a) Let M be a surface embedded in the Euclidean space $\mathbf{E}^3$. Let $\mathbf{e}, \mathbf{f}, \mathbf{n}$ be three vector fields defined on the points of this surface such that they form an orthonormal basis at any point, so that the vectors $\mathbf{e}, \mathbf{f}$ are tangent to the surface and the vector $\mathbf{n}$ is orthogonal to the surface. Consider the derivation formula

$$d\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\\ \mathbf{n}\end{pmatrix} = \begin{pmatrix}0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{pmatrix}\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\\ \mathbf{n}\end{pmatrix},$$

where a, b and c are 1-forms on the surface M.
Express the mean curvature and the Gaussian curvature of M in terms of these 1-forms and vector fields.

(b) Show that Gaussian curvature vanishes in the case if 1-forms b and c vanish on vector field $\mathbf{e}$.

[10 marks]

A5.

(a) State the relation between the Riemann curvature tensor of the Levi-Civita connection of a surface in $\mathbf{E}^3$ and its Gaussian curvature K.

(b) Let M be a surface $\mathbf{r} = \mathbf{r}(u, v)$ in $\mathbf{E}^3$, such that at the given point $\mathbf{p}$ Gaussian curvature $K = 1$, and the induced Riemannian metric is equal to $G = du^2 + dv^2$ at this point.

Calculate all components of the Riemannian curvature tensor R_{ikmn} in coordinates u, v at the point $\mathbf{p}$.

Show that induced Riemannian metric cannot be equal identically to $du^2 + dv^2$ in a vicinity of the point $\mathbf{p}$.

[10 marks]

SECTION B

Answer **TWO** of the THREE questions

B6.

(a) Consider the plane $\mathbf{R}^2$ with standard coordinates (x, y) equipped with the Riemannian metric
$$G = \frac{dx^2 + dy^2}{(1 + x^2 + y^2)^2} \ .$$
Calculate the area S_a of the domain $x^2 + y^2 \leq a^2$.

(b) Find the limit S_a when $a \to \infty$.

Show that there is no isometry between the plane with this Riemannian metric and the Euclidean plane $\mathbf{E}^2$.

[15 marks]

B7.

(a) Consider the open disc $u^2 + v^2 < 1$ with the Riemannian metric
$$G = \frac{4(du^2 + dv^2)}{(1 - u^2 - v^2)^2} \, ,$$
(Poincaré disc).
Show that all Christoffel symbols of the Levi-Civita connection of this Riemannian manifold vanish at the point $u = v = 0$.

(b) Let ∇' be a symmetric connection on the Poincaré disc such that all Christoffel symbols of this connection in coordinates (u, v) vanish identically (at all points).
Show that the connection ∇' does not preserve the metric of the Poincaré disc.

[15 marks]

B8.

(a) Consider the Lobachevsky plane as an upper half plane ($y > 0$) in $\mathbf{R}^2$ equipped with the metric $G = \frac{dx^2+dy^2}{y^2}$.

Consider a vertical ray $C\colon x(t) = 1, y(t) = 1 + t$, $0 \leq t < \infty$ on the Lobachevsky plane.

Find the parallel transport $\mathbf{X}(t)$ of the vector $\mathbf{X}_0 = \partial_y$ attached at the initial point $(1, 1)$ along the ray C at an arbitrary point of the ray.

Find the parallel transport $\mathbf{Y}(t)$ of the vector $\mathbf{Y}_0 = \partial_x + \partial_y$ attached at the same initial point $(1, 1)$ along the ray C at an arbitrary point of the ray.
(You may use the fact that the vertical ray C is a geodesic in the Lobachevsky plane.)

[15 marks]

END OF EXAMINATION PAPER