Homework 9—10. Solutions.

1. *Find coordinate basis vectors, first quadratic form, unit normal vector field, shape operator and Gaussian and mean curvatures for*

a) sphere of the radius R: $x^2+y^2+z^2=R^2$,

$$\mathbf{r}(\theta,\varphi)\qquad \begin{cases} x=R\sin\theta\cos\varphi\\ y=R\sin\theta\sin\varphi\\ z=R\cos\theta\end{cases}\qquad (0\leq\varphi<2\pi, 0\leq\theta\leq\pi)\,,$$

b) cylinder $x^2+y^2=R^2$,

$$\mathbf{r}(h,\varphi)\qquad \begin{cases} x=R\cos\varphi\\ y=R\sin\varphi\\ z=h\end{cases}\qquad (0\leq\varphi<2\pi, -\infty<h<\infty)$$

c) cone $x^2+y^2-k^2z^2=0$,

$$\mathbf{r}(h,\varphi)\qquad \begin{cases} x=kh\cos\varphi\\ y=kh\sin\varphi\\ z=h\end{cases}\qquad (0\leq\varphi<2\pi, -\infty<h<\infty)$$

d) graph of the function $z=F(x,y)$

$$\mathbf{r}(u,,v)\qquad \begin{cases} x=u\\ y=v\\ z=uv\end{cases}\qquad (-\infty<u<\infty, -\infty<v<\infty)$$

in the case if $F(u,v)=Au^2+2Buv+Cv^2$

Put down the special case when $F(u,v)=auv$ *(saddle).*

For the case d) you have to calculate First quadratic form, shape operator and curvatures only at origin.

Solution

a) *SPHERE (of radius R)* $x^2+y^2+z^2=R^2$ *:*

$$\mathbf{r}(\theta,\varphi)\qquad \begin{cases} x=R\sin\theta\cos\varphi\\ y=R\sin\theta\sin\varphi\\ z=R\cos\theta\end{cases}$$

$(0\leq\varphi<2\pi,\ 0\leq\theta\leq\pi)$,

$$\mathbf{r}_\theta=\frac{\partial\mathbf{r}(\varphi,\theta)}{\partial\theta}=\begin{pmatrix}R\cos\theta\cos\varphi\\ R\cos\theta\sin\varphi\\ -R\sin\theta\end{pmatrix},\ \mathbf{r}_\varphi=\frac{\partial\mathbf{r}(\varphi,\theta)}{\partial\varphi}=\begin{pmatrix}-R\sin\theta\sin\varphi\\ R\sin\theta\cos\varphi\\ 0\end{pmatrix}$$

$$\mathbf{n}(\theta,\varphi)=\frac{\mathbf{r}(\theta,\varphi)}{R}=\begin{pmatrix}\sin\theta\cos\varphi\\ \sin\theta\sin\varphi\\ \cos\theta\end{pmatrix}\tag{1}$$

(Sometimes we denote $\mathbf{r}_\theta$ by ∂_θ and $\mathbf{r}_\varphi$ by ∂_φ.)

Check that $\mathbf{n}(\theta,\varphi)$ is indeed unit normal vector (in fact this is obvious from geometric considerations):

$$(\mathbf{n},\mathbf{n})=\sin^2\theta(\cos^2\varphi+\sin^2\varphi)+\cos^2\theta=1\,,$$

$$(\mathbf{n},\mathbf{r}_\theta)=R\sin\theta\cos\theta(\cos^2\varphi+\sin^2\varphi)-R\sin\theta\cos\theta=0,\ \ (\mathbf{n},\mathbf{r}_\varphi)=R\sin^2\theta(-\cos\varphi\sin\varphi+\cos\varphi\sin\varphi)=0\,.$$

Unit normal vector is defined up to a sign; $-\mathbf{n}$ is unit normal vector too.

Calculate now first quadratic form. $(\mathbf{r}_\theta, \mathbf{r}_\theta) = R^2\cos^2\theta(\cos^2\varphi + \sin^2\varphi) + R^2\sin^2\theta = R^2$, $(\mathbf{r}_\theta, \mathbf{r}_\varphi) = 0$, $(\mathbf{r}_\varphi, \mathbf{r}_\varphi) = R^2\sin^2\theta(\sin^2\varphi + \cos^2\varphi) = R^2\sin^2\theta$. Thus

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_\theta, \mathbf{r}_\theta) & (\mathbf{r}_\theta, \mathbf{r}_\varphi) \\ (\mathbf{r}_\varphi, \mathbf{r}_\theta) & (\mathbf{r}_\varphi, \mathbf{r}_\varphi) \end{pmatrix} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix}$$

$$dl^2 = G_{11}d\theta^2 + 2G_{12}d\theta d\varphi + G_{22}d\varphi^2 = R^2 d\theta^2 + R^2\sin^2\theta d\varphi^2$$

The length of the curve $\mathbf{r}(t) = \mathbf{r}(\theta(t), \varphi(t))$ with $\theta = \theta(t), \varphi = \varphi(t)$, $t_1 \le t \le t_2$ is given by the integral:

$$\int_{t_1}^{t_2} |\mathbf{v}(t)|dt = \int_{t_1}^{t_2} \sqrt{G_{11}\dot\theta^2 + 2G_{12}\dot\theta\dot\varphi + G_{22}\dot\varphi^2}dt = \int_{t_1}^{t_2} \sqrt{R^2\dot\theta^2 + R^2\sin^2\theta\dot\varphi^2}dt \tag{1b}$$

Now calculate shape operator and Gaussian and mean curvatures for sphere:

By the definition (see lecture notes) the action of shape operator on any tangent vector $\mathbf{v}$ is given by the formula $S\mathbf{v} = -\partial_{\mathbf{v}}\mathbf{n}$. We know that for sphere $\mathbf{n} = \frac{\mathbf{r}}{R}$ (see the equations (1) above). Hence for basis vectors $\mathbf{r}_\theta = \partial_\theta, \mathbf{r}_\varphi = \partial_\varphi$ we have

$$S\mathbf{r}_\theta = -\partial_\theta\mathbf{n}(\theta,\varphi) = -\partial_\theta\left(\frac{\mathbf{r}(\theta,\varphi)}{R}\right) = -\left(\frac{\partial_\theta\mathbf{r}(\theta,\varphi)}{R}\right) = -\frac{\mathbf{r}_\theta}{R}$$

and

$$S\mathbf{r}_\varphi = -\partial_\varphi\mathbf{n}(\theta,\varphi) = -\partial_\varphi\left(\frac{\mathbf{r}(\theta,\varphi)}{R}\right) = -\left(\frac{\partial_\varphi\mathbf{r}(\theta,\varphi)}{R}\right) = -\frac{\mathbf{r}_\varphi}{R}$$

We see that shape operator is equal to $S = -\frac{I}{R}$, where I is an identity operator. Its matrix in the basis $\partial_\theta, \partial_\varphi$ is equal to

$$-\begin{pmatrix} \frac{1}{R} & 0 \\ 0 & \frac{1}{R} \end{pmatrix}.$$

In the case if we choose the opposite direction for unit normal vector then we will come to the same answer just with changing the signs: if $\mathbf{n} \to -\mathbf{n}$, $S \to -S$.

We see that principal curvatures, i.e. eigenvalues of shape operator are the same:

$$\lambda_1 = \lambda_2 = -\frac{1}{R}, \text{i.e. } \kappa_1 = \kappa_2 = -\frac{1}{R}$$

(if we choose the opposite sign for $\mathbf{n}$ then $\kappa_1 = \kappa_2 = \frac{1}{R}$). Thus we can calculate Gaussian and mean curvature: Gaussian curvature

$$K = \kappa_1 \cdot \kappa_2 = \det S = \frac{1}{R^2}.$$

Mean curvature

$$H = \kappa_1 + \kappa_2 = \operatorname{Tr} S = -\frac{2}{R}.$$

If we choose the opposite sign for $\mathbf{n}$ then $S \to -S$, principal curvatures change the sign, Gaussian curvature $K = \kappa_1 \cdot \kappa_2$ does not change but mean curvature $H = \kappa_1 + \kappa_2$ will change the sign: if $\mathbf{n} \to -\mathbf{n}$ then $H = \frac{2}{R}$.

b) CYLINDER $x^2 + y^2 = R^2$

$$\mathbf{r}(h,\varphi) \qquad \begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \\ z = h \end{cases} \qquad (0 \le \varphi < 2\pi, -\infty < h < \infty)$$

$$\mathbf{r}_\varphi\big|_{\varphi,h} = \frac{\partial\mathbf{r}(\varphi,h)}{\partial\varphi} = \begin{pmatrix} -R\sin\varphi \\ R\cos\varphi \\ 0 \end{pmatrix}, \ \mathbf{r}_h\big|_{\varphi,h} = \frac{\partial\mathbf{r}(\varphi,h)}{\partial h} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ \mathbf{n}(\varphi,h) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} \tag{2}$$

Sometimes we denote $\mathbf{r}_\varphi$ by ∂_φ and $\mathbf{r}_h$ by ∂_h.

Check that $\mathbf{n}(\varphi, h)$ is indeed unit normal vector:

$$(\mathbf{n}, \mathbf{n}) = \cos^2\varphi + \sin^2\varphi = 1, \ \ (\mathbf{n}, \mathbf{r}_\varphi) = R\cos\varphi\sin\varphi(-1+1) = 0, \ \ (\mathbf{n}, \mathbf{r}_h) = 0$$

Unit normal vector is defined up to a sign; $-\mathbf{n}$ is unit normal vector too.

Calculate now first quadratic form. $(\mathbf{r}_\varphi, \mathbf{r}_\varphi) = R^2(\sin^2\varphi + \cos^2\varphi) = R^2$, $(\mathbf{r}_\varphi, \mathbf{r}_h) = 0$, $(\mathbf{r}_h, \mathbf{r}_h) = 1$. Thus

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} G_{hh} & G_{h\varphi} \\ G_{\varphi h} & G_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_h, \mathbf{r}_h) & (\mathbf{r}_h, \mathbf{r}_\varphi) \\ (\mathbf{r}_\varphi, \mathbf{r}_\varphi) & (\mathbf{r}_\varphi, \mathbf{r}_\varphi) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & R^2 \end{pmatrix}$$

$$dl^2 = G_{11}dh^2 + 2G_{12}dhd\varphi + G_{22}d\varphi^2 = dh^2 + R^2 d\varphi^2 \,.$$

The length of the curve $\mathbf{r}(t) = \mathbf{r}(\varphi(t), h(t))$ with $\varphi = \varphi(t), h = h(t)$, $t_1 \le t \le t_2$ is given by the integral:

$$\int_{t_1}^{t_2} \sqrt{G_{11}\dot{h}^2 + 2G_{12}\dot{\varphi}\dot{h} + G_{22}\dot{\varphi}^2} dt = \int_{t_1}^{t_2} \sqrt{\dot{h}^2 + R^2\dot{\varphi}^2} dt, \tag{2b}$$

Now calculate shape operator Gaussian and mean curvatures for cylinder.

To calculate the shape operator for the cylinder we use results of calculations above of vectors $\mathbf{r}_h, \mathbf{r}_\varphi$ and of unit normal vector $\mathbf{n}(\varphi, h)$ (see the equations (2) above). By the definition the action of shape operator on any tangent vector $\mathbf{v}$ is given by the formula $S\mathbf{v} = -\partial_\mathbf{v}\mathbf{n}$. Hence for basis vectors $\mathbf{r}_\varphi = \partial_\varphi, \mathbf{r}_h = \partial_h$ we have

$$S\mathbf{r}_h = -\partial_h \mathbf{n}(\varphi, h) = -\partial_h \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} = 0$$

and

$$S\mathbf{r}_\varphi = -\partial_\varphi \mathbf{n}(\varphi, h) = -\partial_\varphi \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} \sin\varphi \\ -\cos\varphi \\ 0 \end{pmatrix} = -\frac{\mathbf{r}_\varphi}{R}$$

(Recall that $\mathbf{n}(h, \varphi) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$ and $\mathbf{r}_\varphi = \begin{pmatrix} -R\sin\varphi \\ R\cos\varphi \\ 0 \end{pmatrix}$ (See the equations (2) above.)

For an arbitrary tangent vector $\mathbf{X} = a\mathbf{r}_h + b\mathbf{r}_\varphi$, $S\mathbf{X} = -\frac{b\mathbf{r}_\varphi}{R}$. Shape operator transforms tangent vectors to tangent vectors. Its matrix in the basis $\mathbf{r}_h, \mathbf{r}_\varphi$ equals to

$$-\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{R} \end{pmatrix}$$

In the case if we choose the opposite direction for unit normal vector then we will come to the same answer just with changing the signs: if $\mathbf{n} \to -\mathbf{n}$, $S \to -S$.

We see that principal curvatures, i.e. eigenvalues of shape operator are:

$$\kappa_1 = 0, \kappa_2 = -\frac{1}{R}$$

(if we choose the opposite sign for $\mathbf{n}$ then $\kappa_1 = \kappa_2 = \frac{1}{R}$). Thus we can calculate Gaussian and mean curvature: Gaussian curvature

$$K = \kappa_1 \cdot \kappa_2 = \det S = 0 \,.$$

Mean curvature

$$H = \kappa_1 + \kappa_2 = \mathrm{Tr}\, S = -\frac{1}{R} \,.$$

If we choose the opposite sign for $\mathbf{n}$ then $S \to -S$, principal curvatures change the sign, Gaussian curvature $K = \kappa_1 \cdot \kappa_2$ remains the same but mean curvature $H = \kappa_1 + \kappa_2$ will change the sign: if $\mathbf{n} \to -\mathbf{n}$ then $H = \frac{1}{R}$.

b) CONE $x^2+y^2-k^2z^2=0$

$$\mathbf{r}(h,\varphi) \qquad \begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases} \qquad (0\le\varphi<2\pi, -\infty<h<\infty) \tag{3}$$

$$\mathbf{r}_h\big|_{\varphi,h} = \frac{\partial\mathbf{r}(\varphi,h)}{\partial h} = \begin{pmatrix} k\cos\varphi \\ k\sin\varphi \\ 1 \end{pmatrix},\ \mathbf{r}_\varphi\big|_{\varphi,h} = \frac{\partial\mathbf{r}(\varphi,h)}{\partial\varphi} = \begin{pmatrix} -kh\sin\varphi \\ kh\cos\varphi \\ 0 \end{pmatrix}.$$

Sometimes we denote $\mathbf{r}_\varphi$ by ∂_φ and $\mathbf{r}_h$ by ∂_h.

To calculate the normal unit vector field $\mathbf{n}(h,\varphi)$ note that the vector $\mathbf{N}(h,\varphi) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix}$ is orthogonal to the surface of the cone: $(\mathbf{N}\, r_h) = (\mathbf{N},\mathbf{r}_\varphi) = 0$ and its length equals to $|\mathbf{N}| = \sqrt{k^2+1}$. Hence normal unit vector field equals to

$$\mathbf{n}(h,\varphi) = \frac{\mathbf{N}(h,\varphi)}{\sqrt{k^2+1}} = \frac{1}{\sqrt{k^2+1}}\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix}$$

It is indeed normal unit vector field: $(\mathbf{n},\mathbf{n}) = \frac{\cos^2\varphi}{k^2+1} + \frac{\sin^2\varphi}{k^2+1} + \frac{k^2}{k^2+1} = 1$, $(\mathbf{n},\mathbf{r}_\varphi) = \frac{1}{\sqrt{k^2+1}}(\cos\varphi\cdot(-kh\sin\varphi) + \sin\varphi\cdot(+kh\cos\varphi)) = 0$, and $(\mathbf{n},\mathbf{r}_h) = \frac{1}{\sqrt{k^2+1}}(\cos\varphi\cdot(kh\cos\varphi) + \sin\varphi\cdot k\sin\varphi - k) = 0$. Unit normal vector is defined up to a sign; $-\mathbf{n}$ is unit normal vector too.

Calculate now first quadratic form. $(\mathbf{r}_h,\mathbf{r}_h) = k^2\cos^2\varphi + k^2\sin^2\varphi + 1 = k^2+1$, $(\mathbf{r}_h,\mathbf{r}_\varphi) = (\mathbf{r}_\varphi,\mathbf{r}_h) = k^2h\cos\varphi(-\sin\varphi) + k^2h\sin\varphi\cos\varphi = 0$, $(\mathbf{r}_\varphi,\mathbf{r}_\varphi) = k^2h^2\sin^2\varphi + k^2h^2\cos^2\varphi = k^2h^2$, Thus

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} G_{hh} & G_{h\varphi} \\ G_{\varphi h} & G_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_h,\mathbf{r}_h) & (\mathbf{r}_h,\mathbf{r}_\varphi) \\ (\mathbf{r}_\varphi,\mathbf{r}_h) & (\mathbf{r}_\varphi,\mathbf{r}_\varphi) \end{pmatrix} = \begin{pmatrix} k^2+1 & 0 \\ 0 & k^2h^2 \end{pmatrix}$$

$$dl^2 = G_{hh}dh^2 + 2G_{h\varphi}dhd\varphi + G_{\varphi\varphi}d\varphi^2 = (k^2+1)dh^2 + k^2h^2d\varphi^2R^2\,,$$

The length of the curve $\mathbf{r}(t) = \mathbf{r}(h(t),\varphi(t))$ with $\varphi=\varphi(t), h=h(t)$, $t_1\le t\le t_2$ is given by the integral:

$$\int_{t_1}^{t_2}\sqrt{G_{11}\dot h^2 + 2G_{12}\dot h\dot\varphi + G_{22}\dot\varphi^2}dt = \int_{t_1}^{t_2}\sqrt{(k^2+1)\dot h^2 + k^2h(t)^2\dot\varphi^2}dt\,. \tag{3b}$$

To calculate the shape operator for the cone we use the results of calculations of vectors $\mathbf{r}_h,\mathbf{r}_\varphi$ and of unit normal vector $\mathbf{n}(\varphi,h)$ (see the equations (3) above.) By the definition the action of shape operator on any tangent vector $\mathbf{v}$ is given by the formula $S\mathbf{v} = -\partial_\mathbf{v}S$. Hence for basis vectors $\mathbf{r}_h=\partial_h, \mathbf{r}_\varphi=\partial_\varphi$

$$S\mathbf{r}_h = -\partial_h\mathbf{n}(\varphi,h) = -\frac{1}{\sqrt{k^2+1}}\partial_h\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix} = 0$$

and

$$S\mathbf{r}_\varphi = -\partial_\varphi\mathbf{n}(\varphi,h) = -\frac{1}{\sqrt{k^2+1}}\partial_\varphi\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix} = \frac{1}{\sqrt{k^2+1}}\begin{pmatrix} \sin\varphi \\ -\cos\varphi \\ 0 \end{pmatrix} =$$

$$-\frac{1}{k\sqrt{k^2+1}}\frac{\mathbf{r}_\varphi}{h} \text{ since } \mathbf{r}_\varphi = \begin{pmatrix} -kh\sin\varphi \\ kh\cos\varphi \\ 0 \end{pmatrix}.$$

We see that for an arbitrary tangent vector $\mathbf{X} = a\mathbf{r}_h + b\mathbf{r}_\varphi$ $S\mathbf{X} = S(a\mathbf{r}_h + b\mathbf{r}_\varphi) = -\frac{b}{kh\sqrt{k^2+1}}\mathbf{r}_\varphi$. Shape operator transforms tangent vectors to tangent vectors. Its matrix in the basis $\mathbf{r}_h,\mathbf{r}_\varphi$ equals to

$$-\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{hk\sqrt{1+k^2}} \end{pmatrix}$$

In the case if we choose the opposite direction for unit normal vector then we will come to the same answer just with changing the signs: if $\mathbf{n} \to -\mathbf{n}$, $S \to -S$. We see that principal curvatures, i.e. eigenvalues of shape operator are:

$$\kappa_1 = 0, \kappa_2 = -\frac{1}{hk\sqrt{1+k^2}}$$

(if we choose the opposite sign for $\mathbf{n}$ then $\kappa_1 = \kappa_2 = \frac{1}{hk\sqrt{1+k^2}}$). Thus we can calculate Gaussian and mean curvature: Gaussian curvature

$$K = \kappa_1 \cdot \kappa_2 = \det S = 0\,.$$

Mean curvature

$$H = \kappa_1 + \kappa_2 = \operatorname{Tr} S = -\frac{1}{hk\sqrt{1+k^2}}\,.$$

If we choose the opposite sign for $\mathbf{n}$ then $S \to -S$, principal curvatures change the sign, Gaussian curvature $K = \kappa_+ \cdot \kappa_-$ remains the same but mean curvature $H = \kappa_+ + \kappa_-$ will change the sign: if $\mathbf{n} \to -\mathbf{n}$ then $H = \frac{1}{hk\sqrt{1+k^2}}$.

d) GRAPH OF THE FUNCTION $z = F(x, y)$

$$\mathbf{r}(u,v) \qquad \begin{cases} x = u \\ y = v \\ z = F(u,v) \end{cases} \qquad (-\infty < u < \infty, -\infty < v < \infty) \tag{4}$$

in the case if $F(u,v) = Au^2 + 2Buv + Cv^2$

$$\mathbf{r}_u\big|_{u,v} = \frac{\partial \mathbf{r}(u,v)}{\partial u} = \begin{pmatrix} 1 \\ 0 \\ F_u \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2Au + 2Bv \end{pmatrix}, \ \mathbf{r}_u\big|_{u=v=0} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbf{r}_v\big|_{u,v} = \frac{\partial \mathbf{r}(u,v)}{\partial v} = \begin{pmatrix} 0 \\ 1 \\ F_v \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2Bu + 2Cv \end{pmatrix}, \ \mathbf{r}_v\big|_{u=v=0} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

$$\mathbf{n}(u,v) = \frac{1}{\sqrt{1+F_u^2+F_v^2}} \begin{pmatrix} -F_u \\ -F_v \\ 1 \end{pmatrix}, \ \mathbf{n}(u,v)\big|_{u=v=0} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Sometimes we denote $\mathbf{r}_u$ by ∂_u and $\mathbf{r}_v$ by ∂_v.

Check that $\mathbf{n}(u,v)$ is indeed unit normal vector: $(\mathbf{n},\mathbf{n}) = \frac{1}{1+F_u^2+F_v^2}(F_u^2 + F_v^2 + 1) = 1$, $(\mathbf{n}, \mathbf{r}_u) = \frac{1}{\sqrt{1+F_u^2+F_v^2}}(F_u - F_u) = 0$, $(\mathbf{n}, \mathbf{r}_v) = \frac{1}{\sqrt{1+F_u^2+F_v^2}}(F_v - F_v) = 0$. Calculate now first quadratic form. $(\mathbf{r}_u, \mathbf{r}_u) = 1 + F_u^2$, $(\mathbf{r}_u, \mathbf{r}_v) = F_u F_v$, $(\mathbf{r}_v, \mathbf{r}_v) = 1 + F_v^2$. Thus

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_u, \mathbf{r}_u) & (\mathbf{r}_u, \mathbf{r}_v) \\ (\mathbf{r}_v, \mathbf{r}_u) & (\mathbf{r}_v, \mathbf{r}_v) \end{pmatrix} = \begin{pmatrix} 1 + F_u^2 & F_u F_v \\ F_u F_v & 1 + F_v^2 \end{pmatrix}$$

$$dl^2 = G_{11} d\varphi^2 + 2G_{12} dd\varphi dh + G_{22} dh^2 = (1 + F_u^2)du^2 + 2F_u F_v du dv + (1 + F_v^2)dv^2$$

At the point $u = v = 0$, $F_u = F_v = 0$ and

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_u, \mathbf{r}_u) & (\mathbf{r}_u, \mathbf{r}_v) \\ (\mathbf{r}_v, \mathbf{r}_u) & (\mathbf{r}_v, \mathbf{r}_v) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ dl^2 = du^2 + dv^2$$

The length of the curve $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ with $u = u(t), v = v(t)$ can be calculated by the integral:

$$\int_{t_1}^{t_2} \sqrt{G_{11}\dot{u}^2 + 2G_{12}\dot{u}\dot{v} + G_{22}\dot{v}^2}dt = \int_{t_1}^{t_2} \sqrt{(1 + F_u^2)\dot{u}^2 + 2F_u F_v \dot{u}\dot{v} + (1 + F_v^2)\dot{v}^2}dt \tag{4a}$$

Special case of saddle: In the special case of saddle we just take $F = auv$ in previous formulae. In particular normal for normal unit vector we have

$$\mathbf{n}(u,v) = \frac{1}{\sqrt{1+a^2v^2+a^2u^2}}\begin{pmatrix} -av \\ -au \\ 1 \end{pmatrix}, \ \mathbf{n}(u,v)\big|_{u=v=0} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and first quadratic form is equal to

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_u, \mathbf{r}_u) & (\mathbf{r}_u, \mathbf{r}_v) \\ (\mathbf{r}_v, \mathbf{r}_u) & (\mathbf{r}_v, \mathbf{r}_v) \end{pmatrix} = \begin{pmatrix} 1+F_u^2 & F_uF_v \\ F_uF_v & 1+F_v^2 \end{pmatrix} = \begin{pmatrix} 1+v^2 & vu \\ vu & 1+u^2 \end{pmatrix},$$

$$dl^2 = G_{11}d\varphi^2 + 2G_{12}dd\varphi dh + G_{22}dh^2 = (1+a^2v^2)du^2 + 2a^2uvdudv + (1+a^2u^2)dv^2 \,.$$

To calculate shape operator we use results of calculations vectors $\mathbf{r}_u, \mathbf{r}_v$ and for unit normal vector $\mathbf{n}(u,v)$. We do calculations only at origin. For basic vectors $\mathbf{r}_u = \partial_u, \mathbf{r}_v = \partial_v$ we have

$$S\mathbf{r}_u = -\frac{\partial \mathbf{n}(u,v)}{\partial u}_{|u=v=0} = -\partial_u \left(\frac{1}{\sqrt{1+F_u^2+F_v^2}} \begin{pmatrix} -F_u \\ -F_v \\ 1 \end{pmatrix} \right)_{|u=v=0} =$$

$$\left(\frac{1}{\sqrt{1+F_u^2+F_v^2}} \right)_{|u=v=0} \begin{pmatrix} F_{uu} \\ F_{uv} \\ 1 \end{pmatrix}_{|u=v=0} = \begin{pmatrix} 2A \\ 2B \\ 0 \end{pmatrix} = 2A\mathbf{r}_u + 2B\mathbf{r}_v$$

and $S\mathbf{r}_v = -\partial_v\left(\mathbf{n}(u,v)\right)_{|u=v=0} =$

$$-\partial_v \left(\frac{1}{\sqrt{1+F_u^2+F_v^2}} \begin{pmatrix} -F_u \\ -F_v \\ 1 \end{pmatrix} \right)_{|u=v=0} = \left(\frac{1}{\sqrt{1+F_u^2+F_v^2}} \right)_{|u=v=0} \begin{pmatrix} F_{vu} \\ F_{vv} \\ 1 \end{pmatrix}_{|u=v=0} = \begin{pmatrix} 2B \\ 2C \\ 0 \end{pmatrix} = 2B\mathbf{r}_u + 2C\mathbf{r}_v$$

The matrix of the shape operator in the basis $\mathbf{r}_u, \mathbf{r}_v$ is $\begin{pmatrix} 2A & 2B \\ 2B & 2C \end{pmatrix}$. Gaussian curvature at origin is equal to $\det S = 4AC - 4B^2$ and mean curvature is equal to $H = \mathrm{tr} S = 2A + 2C$. (Mean curvature as always is defined up to a sign.)

In the case of saddle $F = auv$, i.e. $A = C = 0, B = \frac{a}{2}$. The shape operator at the point $u = v = 0$ equals to

$$\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}$$

Gaussian curvature $K = k_- \cdot k_+ = \det S = -a^2$ and mean curvature $H = (k_- + k_+) = \mathrm{Tr}\, S = 0$.

2 *Consider surface defined by equation $z - Ax^2 - Ay^2 = 0$. (See the exercise 1d) above.) Show that this is a saddle: you have to show that under the rotation on the angle $\varphi = \frac{\pi}{4}$ with respect to z-axis it becomes a surface $z - axy = 0$. Find relation between parameters A and a.*

Matrix of rotation on the angle $\frac{\pi}{4}$ (in the basis of vectors $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$) directed along x, y and z axis is $P = \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} & 0 \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Under the rotation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to P\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{x-y}{\sqrt{2}} \\ \frac{x+y}{\sqrt{2}} \\ z \end{pmatrix}$$

Hence $Ax^2 - Ay^2 \to A\left(\frac{x-y}{\sqrt{2}}\right)^2 - A\left(\frac{x+y}{\sqrt{2}}\right)^2 = -2Axy$, $a = -2A$.

3 *Show that there are two straight lines which pass through the point* $(3, 4, 12)$ *on the saddle* $z = xy$ *and lie on this saddle.*

† *Show that this is true for an arbitrary point of the saddle.*

Let $\mathbf{r}_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ be an arbitrary point on the saddle: $z_0 = x_0 y_0$.

Consider the following two lines: the line

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{a}, \text{ where } \mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ y_0 \end{pmatrix}, \text{ i.e. } \begin{cases} x = x_0 + t \\ y = y_0 \\ z = z_0 + y_0 t \end{cases}$$

and the line

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{b}, \text{ where } \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ x_0 \end{pmatrix}, \text{ i.e. } \begin{cases} x = x_0 \\ y = y_0 + t \\ z = z_0 + x_0 t \end{cases}$$

It is easy to check that these both lines belong to the saddle: $xy = (x_0 + t)y_0 = z_0 + ty_0 = z$ and $xy = x_0(y_0 + t) = z_0 + x_0 t$.

One the other hand it is easy to see that it is all: If $\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$ is an arbitrary straight line on the saddle passing through the point (x_0, y_0, z_0), $x_0 y_0 = z_0$ then

$$xy = (x_0 + at)(y_0 + bt) = z = z_0 + ct \text{ for all } t$$

Hence $ab = 0$. Thus $a = 0$ or $b = 0$. We see that through the an arbitrary point on the saddle pass exactly two straight lines.

4 *Consider helix* $\mathbf{r}(t)$: $\begin{cases} x(t) = a\cos t \\ y(t) = a\sin t \\ z(t) = ct \end{cases}$. *Show that this helix belongs to cylinder surface* $x^2 + y^2 = a^2$. *Using first quadratic form on the surface of cylindre or in a different way a) calculate length of the helix* $(0 \le t \le t_0)$.

b) what are relations between principal curvatures of cylinder and curvature of helix?

Solution This helix belongs to cylinder surface $x^2 + y^2 = a^2$ because $x^2 + y^2 = a^2$ on the points of the helix.

For the helix internal coordinates are $\varphi = \varphi(t) = t$ and $h = h(t) = ct$ $(x = R\cos\varphi, y = R\sin\varphi, z = h)$. Use First Quadratic form which we obtained in the previous exercise (see equation (2b). We come to

$$L = \int_0^{t_0} \sqrt{G_{11}\dot{h}^2 + 2G_{12}\dot{\varphi}\dot{h} + G_{22}\dot{\varphi}^2}dt = \int_0^{t_0} \sqrt{a^2\dot{\varphi}^2 + \dot{h}^2}dt = \int_0^{t_0} \sqrt{a^2 + c^2}dt = t_0\sqrt{a^2 + c^2}$$

Of course the answer can be obtained without integration: speed is constant, hence $L = |\mathbf{v}|t = t\sqrt{a^2 + c^2}$. This is the calculations of the Internal observer. The external observer will calculate using the coordinates x, y, z: $|\mathbf{v}| = \sqrt{x_t^2 + y_t^2 + z_t^2} = (a^2\cos^2 + a^2\sin^2 t + c^2) = \sqrt{a^2 + c^2}$ and will come to the same answer.

b) Note that curvature of the helix equals to $k = \frac{a}{a^2+c^2}$. Curvatue of helix varies between zero and $\frac{1}{a}$, i.e. principal curvatures of cylinder surface:

$$0 \le \frac{a}{a^2 + c^2} \le \frac{1}{a}.$$

5 *Assume that the action of the shape operator at the tangent coordinate vectors* $\mathbf{r}_u = \partial_u$*,* $\mathbf{r}_v = \partial_v$ *at the given point* $\mathbf{p}$ *of the surface* $\mathbf{r} = \mathbf{r}(u, v)$ *is defined by the relations:* $S(\partial_u) = 2\partial_u + 2\partial_v$ *and* $S(\partial_v) = -\partial_u + 5\partial_v$*. Calculate principal curvatures, Gaussian and mean curvatures of the surface at this point.*

Solution We see that the matrix of the shape operator in the basis ∂_u, ∂_v is equal to

$$S = \begin{pmatrix} 2 & -1 \\ 2 & 5 \end{pmatrix}$$

Hence Gaussian curvature $K = \det S = 12$ and mean curvature $H = \operatorname{Tr} S = 7$. To calculate principal curvatures k_1, k_2 note that

$$\begin{cases} k_1 + k_2 = H = 7 \\ k_1 \cdot k_2 = K = 12 \end{cases}$$

Hence $k_1 = 3, k_2 = 4$; k_1, k_2 are eigenvalues of the shape operator.

(We assume that $a \geq 0$).

6 *On the sphere of the radius* $x^2 + y^2 + z^2 = R^2$ *in* E^3 *consider the triangle* ABC *with vertices at the North Pole and at Equator:* $A = (0, 0, R)$*,* $B = (R, 0, 0)$ *and* $C = (R\cos\varphi, R\sin\varphi, 0)$*. The edges of this triangle are arcs of the meridians and the arc of the Equator.*

Find the result of the parallel transport of vector $\mathbf{X} = \mathbf{e}_x$ *attached at the North pole along the edges of the triangle* ABC*.*

Let $\Delta\phi$ *be angle of rotation of vector* $\mathbf{x}$ *under parallel transport along the triangle* ABC*. calculate the ratio*

$$\frac{\delta\phi}{kS_{\triangle ABC}}$$

where k *is gaussian curvature of the sphere and* $S_{\triangle ABC}$ *is the area of the spherical triangle* ABC*.*

Do it in three steps.

First perform parallel transport of the vector $\mathbf{e}_x$ along the arc AB of the great circle

Consider the vector field $\mathbf{X}(t) = \begin{pmatrix} \cos t \\ 0 \\ -\sin t \end{pmatrix}$ attached at the points of the curve AB: $\mathbf{r}(t) = \begin{pmatrix} R\sin t \\ 0 \\ R\cos t \end{pmatrix}$, $0 \leq t \leq \frac{\pi}{2}$. It is tangent to the sphere. One can see that $\frac{d\mathbf{X}}{dt} = \begin{pmatrix} -\sin t \\ 0 \\ -\cos t \end{pmatrix} = -\frac{\mathbf{r}(t)}{R}$ is colinear to the normal unit vector hence this is parallel transport. At the initial point A, $(t = 0)$ this is the initial vector $\mathbf{X}(0) = \mathbf{e}_x$,at the final point B, $t = \frac{\pi}{2}$ this is the vector $\mathbf{X}\left(\frac{\pi}{2}\right) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = -\mathbf{e}_z$. We see that under parallel transport along the arc AB the vector $\mathbf{e}_x$ tangent to the sphere at the North pole transforms to the vector $\mathbf{e}_z$ tangent to the sphere at the point B.

Second step: parallel transport of the vector $\mathbf{e}_z$ along the arc BC.

The vector $\mathbf{e}_z$ attached at the arbitrary point of the equator is tangent to the sphere at all the . We see that parallel transport of the vector $\mathbf{e}_z$ tangent to the sphere at the point B along the arc of the equator does not change this vector.

Third step: parallel transport of the vector $\mathbf{e}_z$ along the arc CA.

Consider the vector field $\mathbf{X}(t) = \begin{pmatrix} \sin t\cos\varphi \\ \sin t\sin\varphi \\ -\cos t \end{pmatrix}$ attached at the points of the curve CA: $\mathbf{r}(t) = \begin{pmatrix} R\cos t\cos\varphi \\ R\cos t\sin\varphi \\ R\sin t \end{pmatrix}$, $0 \leq t \leq \frac{\pi}{2}$. It is tangent to the sphere: the scalar product $(\mathbf{X}(t), \mathbf{r}(t)) = 0$. The derivative $\frac{d\mathbf{X}}{dt} = \begin{pmatrix} \cos t\cos\varphi \\ \cos t\sin\varphi \\ \sin t \end{pmatrix} = \frac{\mathbf{r}(t)}{R}$ is colinear to the normal unit vector hence this is parallel transport. At the initial point B, ($t = 0$) this is the vector $\mathbf{e}_z$. At the final point A, $t = \frac{\pi}{2}$ this is the vector $\mathbf{X}\left(\frac{\pi}{2}\right) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} = \cos\varphi\mathbf{e}_x + \sin\varphi\mathbf{e}_y$.

So we see that under parallel transport along the spherical triangle ABC the vector $\mathbf{e}_x$ transforms to the vector $\cos\varphi\mathbf{e}_x + \sin\varphi\mathbf{e}_y$. It rotates on the angle $\Delta\Phi = \varphi$.

On the other hand the area of the triangle ABC equals to $\frac{4\pi R^2}{2} \cdot \frac{\varphi}{2\pi} = R^2\varphi$ and Gaussian curvature $K = \frac{1}{R^2}$. Hence $KS = \varphi$. We come to the fact that $\frac{\Delta\Phi}{KS} = \frac{\varphi}{\varphi} = 1$*

7† *On the sphere $x^2 + y^2 + z^2 = R^2$ in $\mathbf{E}^3$ consider the closed curve $\theta = \theta_0, \varphi = t$, $0 \le t < 2\pi$ (latitude) Find the result of parallel transport of the vector tangent to the sphere along this curve.*

See the solution in Appendices to the Lecture notes.

* This is fundamental fact: the angle of rotation of vector due to parallel transport along the boundary of domain D equals to the integral of Gaussian curvature over the domain D.