Solutions
1

(**1a**) (**3+1+2+2**).

Riemannian metric G on n-dimensional manifold M^n defines for every point $\mathbf{p} \in M$ the scalar product of tangent vectors in the tangent space $T_{\mathbf{p}}M$ smoothly depending on the point $\mathbf{p}$. It means that in every coordinate system $(x^1, \dots, x^n)$ a metric $G = g_{ik}dx^i dx^k$ is defined by a matrix valued function $g_{ik}(x)$ $(i = 1, \dots, n; k = 1, \dots n)$ such that for any two vectors $\mathbf{A} = A^i(x)\frac{\partial}{\partial x^i}$, $\mathbf{B} = B^i(x)\frac{\partial}{\partial x^i}$, tangent to the manifold M at the point $\mathbf{p}$ with coordinates $x = (x^1, x^2, \dots, x^n)$ $(\mathbf{A}, \mathbf{B} \in T_{\mathbf{p}}M)$ the scalar product is equal to:

$$\langle \mathbf{A}, \mathbf{B} \rangle_G \big|_{\mathbf{p}} = G(\mathbf{A}, \mathbf{B})\big|_{\mathbf{p}} = A^i(x) g_{ik}(x) B^k(x)\,,$$

where

1. $G(\mathbf{A}, \mathbf{B}) = G(\mathbf{B}, \mathbf{A})$, i.e. $g_{ik}(x) = g_{ki}(x)$ (symmetricity condition)
2. $G(\mathbf{A}, \mathbf{A}) > 0$ if $\mathbf{A} \neq \mathbf{0}$, i.e.
   $g_{ik}(x)u^i u^k \geq 0$, $g_{ik}(x)u^i u^k = 0$ iff $u^1 = \ldots = u^n = 0$ (positive-definiteness)
3. $G(\mathbf{A}, \mathbf{B})\big|_{\mathbf{p}=x}$,i.e. $g_{ik}(x)$ are smooth functions. $\blacksquare$

This follows from positive definitness of metric. E.g. for the vector $A = \partial_x$ at the point (x, y), $\langle \mathbf{A}, \mathbf{A} \rangle = \sigma(x, y) > 0$.

Calculate cosine of angle. Calculating we see that

$$\cos \angle(\mathbf{A}, \mathbf{B}) = \frac{\langle \mathbf{A}, \mathbf{B} \rangle}{\sqrt{\langle \mathbf{A}, \mathbf{A} \rangle}\sqrt{\langle \mathbf{B}, \mathbf{B} \rangle}} = \frac{\sigma(x,y)(A_x B_x + A_y B_y)}{\sqrt{\sigma(x,y)(A_x A_x + A_y A_y)}\sqrt{\sigma(x,y)(B_x B_x + B_y B_y)}}$$

$$= \frac{\sigma(x,y)(A_x B_x + A_y B_y)}{\sigma(x,y)\sqrt{(A_x A_x + A_y A_y)}\sqrt{(B_x B_x + B_y B_y)}} = \frac{(A_x B_x + A_y B_y)}{\sqrt{(A_x A_x + A_y A_y)}\sqrt{(B_x B_x + B_y B_y)}}\,,$$

i.e. answer does not depend on σ,

Consider an arbitrary parametrisation of the circle, e.g. $x(t) = R\cos t, y(t) = R\sin t$, $0 \leq t < 2\pi$. We have

$$L = \int_{t_0}^{t_1} \sqrt{g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t)dt} = \int_0^{2\pi} \sqrt{\sigma(x(t), y(t))(x_t^2 + y_t^2)} = \int_0^{2\pi} \sqrt{\sigma(x(t), y(t))(R^2 \sin^2 t + R^2$$

$2\pi \cdot \sqrt{e^{-R^2}R^2} = 2\pi R e^{-\frac{R^2}{2}}$.

(**1b**) (**3+2+3**) Volume element in cordinates (x, y) is equal to

$$dv = \sqrt{\det G} dr d\varphi = \sqrt{\det \begin{pmatrix} e^{-x^2-y^2} & 0 \\ 0 & e^{-x^2-y^2} \end{pmatrix}} dxdy = \sqrt{e^{-2x^2-2y^2}}dxdy = e^{-x^2-y^2}dxdy$$

Riemannian metric $G = e^{-x^2-y^2}(dx^2 + dy^2)$ in polar coordinates becomes

$$G = e^{-r^2\cos^2\varphi - r^2\sin^2\varphi}\left((dr\cos\varphi - r\sin\varphi d\varphi)^2 + (dr\sin\varphi + r\cos\varphi d\varphi)^2\right) = e^{-r^2}(dr^2 + r^2 d\varphi^2), \blacksquare$$

Volume element in polar coordinates is equal to

$$dv = \sqrt{\det G} dr d\varphi = \sqrt{\det \begin{pmatrix} e^{-r^2} & 0 \\ 0 & r^2 e^{-r^2} \end{pmatrix}} dr d\varphi = \sqrt{e^{-2r^2} r^2} dr d\varphi = e^{-r^2} r dr d\varphi\,.$$

To calculate the area of the disc, interior of the circle $x^2 + y^2 = R^2$ it is convenient to use polar coordinates.

$$S = \int_0^R \int_0^{2\pi} (\sqrt{\det G} dr d\varphi = \int_0^R r dr \int_0^{2\pi} d\varphi e^{-r^2} d\varphi =$$

$$2\pi \int_0^R e^{-r^2} r dr = \pi \int_0^{R^2} e^{-u} du = \pi(1 - e^{-R^2})\,.$$

One can consider just metric proportional to standard Euclidean metric, $G = C(dx^2 + dy^2)$. In this metric the area of the disc $x^2 + y^2 \leq 1$ is equal to $C\pi$. Hence if we choose coefficient $C = 1 - \frac{1}{e}$, then area will be the same.

Another solution One can deform initial metric such that integral does not change, e.g. consider new metric which is equal to $G = e^{-r^2}(1 + k \sin\varphi) r dr d\varphi$ for $|k| < 1$. One can see that in this metric area of an arbitrary circle with centre at origin does not depend on k.

(**1c**) (**2+2**).

n-dimensional Riemannian manifold (M, G) is locally Euclidean Riemannian manifold, if for every point $\mathbf{p} \in M$ there exists an open neighboorhood D (domain) containing this point, $\mathbf{p} \in D$ such that D is isometric to a domain in Euclidean plane. In other words in a vicinity of every point $\mathbf{p}$ there exist local coordinates $u^1, \ldots, u^n$ such that Riemannian metric G in these coordinates has an appearance

$$G = du^i \delta_{ik} du^k = (du^1)^2 + \ldots + (du^n)^2\,.$$

For upper sheet of conic surface consider new local parameteriation. In analogy with polar coordinates try to find new local coordinates u, v such that $\begin{cases} u = \alpha h \cos\beta\varphi \\ v = \alpha h \sin\beta\varphi \end{cases}$, where α, β are parameters. We come to

$$du^2 + dv^2 = (\alpha \cos\beta\varphi dh - \alpha\beta h \sin\beta\varphi d\varphi)^2 + (\alpha \sin\beta\varphi dh + \alpha\beta h \cos\beta\varphi d\varphi)^2 = \alpha^2 dh^2 + \alpha^2\beta^2 h^2 d\varphi^2.$$

Comparing this metric with the metric $G_c = 10dh^2 + 9h^2 d\varphi^2$ on the cone we see that if we put $\alpha = \sqrt{10}$ and $\beta = \frac{3}{\sqrt{10}}$ then $du^2 + dv^2 = \alpha^2 dh^2 + \alpha^2\beta^2 h^2 d\varphi^2 = 10dh^2 + 9h^2 d\varphi^2$. Thus in new local coordinates $u = \sqrt{10} h \cos\frac{3\sqrt{10}}{10}\varphi$, $v = \sqrt{10} h \sin\frac{3\sqrt{10}}{10}\varphi$ metric on the upper sheet of this cone becomes $du^2 + dv^2$, i.e. it is locally isometric to the Euclidean plane $\blacksquare$

2

(**2a**) (**3+2**).

Affine connection on M is the *operation* ∇ which assigns to every vector field $\mathbf{X}$ a linear map $\nabla_{\mathbf{X}}$ on the space of vector fields: $\nabla_{\mathbf{X}}(\lambda\mathbf{Y}+\mu\mathbf{Z}) = \lambda\nabla_{\mathbf{X}}\mathbf{Y}+\mu\nabla_{\mathbf{X}}\mathbf{Z}$ $(\lambda,\mu\in\mathbf{R})$, which satisfies the following additional conditions:

1. For arbitrary (smooth) functions f,g on M

$$\nabla_{f\mathbf{X}+\mathbf{g}\mathbf{Y}}(\mathbf{Z}) = f\nabla_{\mathbf{X}}(\mathbf{Z}) + g\nabla_{\mathbf{Y}}(\mathbf{Z}) \qquad (C(M)\text{-linearity})$$

2 For arbitrary function f

$$\nabla_{\mathbf{X}}(f\mathbf{Y}) = (\nabla_{\mathbf{X}}f)\,\mathbf{Y} + f\nabla_{\mathbf{X}}(\mathbf{Y}) \qquad (\text{Leibnitz rule})$$

($\nabla_{\mathbf{X}}f$ is just usual derivative of a function f along vector field: $\nabla_{\mathbf{X}}f = \partial_{\mathbf{X}}f$.) $\blacksquare$

using properties of ∇ and definition of Christoffel symbol we have that $\nabla_{\partial_u}(u\partial_u) = \partial_u + u\nabla_{\partial_u}\partial_u = \partial_u + u\partial_v$, and $\nabla_{\partial_u}(\partial_u) = (\Gamma^u_{uu}\partial_u + \Gamma^v_{uu}\partial_v)$. We see that $\Gamma^u_{uu} = 0$ and $\Gamma^v_{uu} = 1$.

(**2b**) (**3+3**). Let M be surface embedded in $\mathbf{E}^3$. Let $\nabla^{\text{can.flat}}$ be canonical flat connection in $\mathbf{E}^3$ (It is defined by the condition that its Christoffel symbols vanish in Cartesian coordinates on $\mathbf{E}^3$: $\nabla^{\text{can.flat}}_{\mathbf{X}}Y = X^i\frac{\partial Y^m}{\partial x^i}\frac{\partial}{\partial x^m}$.) The induced connection $\nabla^{(M)}$ is defined in the following way: for arbitrary vector fields $\mathbf{X},\mathbf{Y}$ tangent to the surface M, $\nabla^M_{\mathbf{X}}\mathbf{Y}$ equals to the projection on the tangent space of the vector field $\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}$:

$$\nabla^M_{\mathbf{X}}\mathbf{Y} = \left(\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}\right)_{\text{tangent}},$$

where $\mathbf{A}_{\text{tangent}}$ is a projection of the vector A attached at the point of the surface on the tangent space: $\mathbf{A}_{\perp} = \mathbf{A} - \mathbf{n}(\mathbf{A},\mathbf{n})$, ($\mathbf{n}$ is normal unit vector field to the surface.)

For sphere $\partial_\varphi = \mathbf{r}_\varphi = \frac{\partial\mathbf{r}}{\partial\varphi} = \begin{pmatrix} -R\sin\theta\sin\varphi \\ R\sin\theta\cos\varphi \\ 0 \end{pmatrix}$ and $\nabla^{\text{can.flat}}_{\partial_\theta}\partial_\theta = \frac{\partial^2\mathbf{r}}{\partial\theta\partial\varphi} = \begin{pmatrix} -R\cos\theta\sin\varphi \\ R\cos\theta\cos\varphi \\ 0 \end{pmatrix}$.
This vector is tangent to the surface —it is proportional to the vector $\mathbf{r}_\varphi$: $\mathbf{r}_{\theta\varphi} = \text{cotan}\,\theta\mathbf{r}_\varphi$. Hence $\nabla_{\partial_\theta}\partial_\varphi = \text{cotan}\,\theta\mathbf{r}_\varphi = \nabla_{\partial_\varphi}\partial_\theta$ since $\mathbf{r}_{\theta\varphi} = \mathbf{r}_{\varphi\theta}$. We come to

$$\Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = \text{cotan}\,\theta\,, \quad \Gamma^\theta_{\theta\varphi} = \Gamma^\theta_{\varphi\theta} = 0\,.$$

(**2c**) (**3+3+3**). Let M be a Riemannian manifold with metric $G = g_{ik}dx^idx^k$. Let ∇ be a symmetric connection on M, i.e. its Christophel symbols Γ^m_{ik} satisfies the condition: $\Gamma^m_{ik} = \Gamma^m_{ki}$. We say that symmetric connection ∇ is Levi Civita connection if it preserves scalar product, i.e. if for arbitrary vectors $\mathbf{Y},\mathbf{Z}$ at an arbitrary point

$$\nabla_{\mathbf{X}} < \mathbf{Y},\mathbf{Z} > = < \nabla_{\mathbf{X}}(\mathbf{Y}),\mathbf{Z} > + < \mathbf{Y},\nabla_{\mathbf{X}}(\mathbf{Z}) > .$$

Levi-Civita Theorem claims that on the Riemannian manifold (M,G) there exists uniquely defined Levi Civita connection. In local coordinates Christoffel symbols of this connection have the following appearance:

$$\Gamma^m_{ik}(x) = \frac{1}{2}g^{mn}(x)\left(\frac{\partial g_{in}(x)}{\partial x^k} + \frac{\partial g_{kn}(x)}{\partial x^i} - \frac{\partial g_{ik}(x)}{\partial x^n}\right). \blacksquare$$

Consider the expansion of metric coefficients in series in a vicinity of point $u=v=0$. Mamtrix entries of metric are $\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \sigma(u,v) & 0 \\ 0 & \sigma(u,v) \end{pmatrix}$, where

$$\sigma(u,v) = \frac{4}{(1-u^2-v^2)^2)} = (4\left(1+u^2+v^2+O(u^2,v^2)\right)^2 = 4+4u^2+4v^2+O(u^2,v^2)\,.$$

Hence in particular differential of the function σ vanishes at the point $u=v=0$, i.e. all partial derivatives $\frac{\partial g_{11}}{\partial u}, \frac{\partial g_{11}}{\partial v} \dots, \frac{\partial g_{11}}{\partial u}, \frac{\partial g_{11}}{\partial v}$ vaniush at $u=v=0$. Hence it follows from Levi-Civita formula (1) that all Christoffel symbols vanish in coordinates (u,v) at the point $u=v=0$.

To show that ∇' does not preserve scalar product, consider vector field $\mathbf{A} = \partial_u$. The condition that the scalar product $\langle \mathbf{A}, \mathbf{A} \rangle$ is preserved with respect to the connection ∇' means that for an arbitrary vector field $\mathbf{B}$, $\partial_{\mathbf{B}} \langle \mathbf{A}, \mathbf{A} \rangle = \langle \nabla'_B \mathbf{A}, \mathbf{A} \rangle + \langle \mathbf{A}, \nabla'_{\mathbf{B}} \mathbf{A} \rangle = 2 \langle \nabla'_{\mathbf{B}} \mathbf{A}, \mathbf{A} \rangle$. Choose $\mathbf{B} = \mathbf{A} = \partial_u$ also. Then $\langle \mathbf{A}, \mathbf{A} \rangle = \langle \partial_u, \partial_u \rangle = \frac{4}{(1-u^2-v^2)^2}$, and $\nabla'_{\mathbf{B}} \mathbf{A} = \nabla'_{\partial_u} \partial_u = 0$. We have that $2\langle \nabla'_{\mathbf{B}} \mathbf{A}, \mathbf{A} \rangle = 0$, but $\partial_{\mathbf{B}} \langle \mathbf{A}, \mathbf{A} \rangle = \partial_u \langle \partial_u, \partial_u \rangle = \frac{\partial}{\partial u} \left(\frac{4}{(1-u^2-v^2)^2} \right)$ obviously does not vanish for all u, v. Contradiction.

3

(**3a**) (**3+2+3**).

Let $\mathbf{x} = \mathbf{x}(t)$, $t_1 \le t \le t_2$ be a parameterisation of curve C beginning at the point ∂_1 and ending at the point ∂_2. such that $\mathbf{x}(t_1) = \mathbf{p}_1$ and $\mathbf{x}(t_2) = \mathbf{p}_2$. Let $\mathbf{X} = \mathbf{X}_1 \in T_{\mathbf{p}_1} M$ be an arbitrary tangent vector at the initial point $\mathbf{p}_1$ of the curve C. (The vector $\mathbf{X}$ is not necessarily tangent to the curve C.) We say that $\mathbf{X}(t)$, $t_1 \le t \le t_2$ is a parallel transport of the vector $\mathbf{X}_1 \in T_{\partial_1} M$ along the curve C: $\mathbf{x} = \mathbf{x}(t), t_1 \le t \le t_2$, and the vector field $\mathbf{X}_2 = \mathbf{X}(t_2)$ attached at the point $\mathbf{p}_2$ is the result of parallel transport of the vector $\mathbf{X}_1$ from the point $\mathbf{p}_1$ to the point $\mathbf{p}_2$ along the curve C if the following conditions obey

i) For an arbitrary t, $t_1 \le t \le t_2$, vector $\mathbf{X}(t)$, ($\mathbf{X}_1 = \mathbf{X}(t_1)$) is a tangent vector attached at the point $\mathbf{x}(t)$, i.e. $\mathbf{X}(t)$ is a vector tangent to the manifold M at the point $\mathbf{x}(t)$ of the curve C, $\mathbf{X}(t) \in T_{\mathbf{x}(t)} M$, and in particularly the vector $\mathbf{X}_2 = \mathbf{X}(t_2) \in T_{\mathbf{p}_2} M$

ii) The covariant derivative of $\mathbf{X}(t)$ along the curve C equals to zero: $\frac{\nabla \mathbf{X}(t)}{dt} = \nabla_{\mathbf{v}} \mathbf{X} = 0$. If $X^m(t)$ are components of the vector field $\mathbf{X}(t)$ and $v^m(t)$ are components of the velocity vector $\mathbf{v}$ of the curve C, then the equation $\frac{\nabla \mathbf{X}}{dt} = \nabla_{\mathbf{v}} \mathbf{X} = 0$ has in the components the following appearance:

$$\frac{dX^i(t)}{dt} + v^k(t) \Gamma^i_{km}(x^i(t)) X^m(t) \equiv 0\,, \qquad t_1 \le t \le t_2 \tag{1}$$

Here ∇ is Levi-Civita connection.

One can say that the parallel transport of the vector $\mathbf{X}_1 \in T_{\mathbf{p}_1} M$ along a curve C is a vector $\mathbf{X}_2 \in T_{\mathbf{p}_2} M$ which is obtained from the solution of differentiao equation (1) with intial condition $\mathbf{X}(t)\big|_{t=t_1}$: $\mathbf{X}_2 = \mathbf{X}(t)\big|_{t=t_2}$.

We define parallel transport using a parameterisation of the curve C, One can show that the parallel transport $\mathbf{X} \to \mathbf{X}'$ remains the same if we choose another parameterisation of the curve C starting at the point $\mathbf{p}_1$, i.e. if new parameterisation has the same orientation.

The differential equation (1) defining parallel transport is a linear equation, and the operator $P_C : T_{p_1}M \ni \mathbf{X}_1 \to \mathbf{X}_2 \in T_{p_2}M$ is linear operator. The connection ∇ defining parallel transport is Levi-Civita connection, it preserves scalar product:

$$\frac{d}{dt}\langle \mathbf{X}(t), \mathbf{X}(t)\rangle_{\mathbf{x}(t)} = \partial_{\mathbf{v}}\langle \mathbf{X}(t), \mathbf{X}(t)\rangle_{\mathbf{x}(t)} = 2\langle \nabla_{\mathbf{v}} X(t), \mathbf{X}(t)\rangle_{\mathbf{x}(t)} = 0\,,$$

due to the equation $\nabla_{\mathbf{v}} dX = 0$. We see that linear operator P_C of parallel transport, parallel transport preserves scalar product, i.e. it is an orthogonal operator.

Since parallel transport preserves scalar product, hence $\langle \mathbf{a}, \mathbf{b}\rangle_{\mathbf{p}_1} = \langle P_C(\mathbf{a}), P_C(\mathbf{b})\rangle_{\mathbf{p}_1} = \langle \mathbf{b}, -\mathbf{a}\rangle_{\mathbf{p}_1} = -\langle \mathbf{b}, \mathbf{a}\rangle_{\mathbf{p}_1} = -\langle \mathbf{a}, \mathbf{b}\rangle_{\mathbf{p}_1}$. Hence $\langle \mathbf{a}, \mathbf{b}\rangle_{\mathbf{p}_1} = 0$ ∎.

(3b) (2+1+3)) Differentail equation for geodesics $\mathbf{x} = \mathbf{x}(t)$ on Riemannian manifold are the following secod order differential equations

$$\nabla_{\mathbf{v}}\mathbf{v} = \frac{\nabla \mathbf{v}}{dt} = \frac{dv^i(t)}{dt} + v^k(t)\Gamma^i_{km}(x^i(t))v^m(t) = 0\,, \text{where } v^i(t) = \tfrac{dx^i(t)}{dt}, \tag{2}$$

where Γ^i_{km} are Christoffel symbols of Levi-Civita connection.

Euler-Lagrange second order differential equations ($\frac{\partial L}{\partial x^i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right)$) for the Lagrangian L of the "free" particle on the Riemannian manifold $L = L(x, \dot{x}) = \frac{1}{2}g_{ik}(x)\dot{x}^i\dot{x}^k$ are equivalent to the second order differential equations (2) for parameterised geodesics for this Riemannian manifold. ∎

Euler-Lagrange equations for Lagrangian $L = \frac{\dot{x}^2+\dot{y}^2}{2y^2}$ are:

$$\frac{\partial L}{\partial x} = 0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{d}{dt}\left(\frac{\dot{x}}{y^2}\right) = \frac{\ddot{x}}{y^2} - \frac{2\dot{x}\dot{y}}{y^3}, \text{i.e.}\quad \ddot{x} - \frac{2\dot{x}\dot{y}}{y} = 0\,,$$

$$\frac{\partial L}{\partial y} = -\frac{\dot{x}^2+\dot{y}^2}{y^3} = \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = \frac{d}{dt}\left(\frac{\dot{y}}{y^2}\right) = \frac{\ddot{y}}{y^2} - \frac{2\dot{y}^2}{y^3}, \text{i.e.}\quad \ddot{y} + \frac{\dot{x}^2}{y} - \frac{\dot{y}^2}{y} = 0\,.$$

Comparing these equations with equations for geodesics: $\ddot{x}^i + \dot{x}^k\Gamma^i_{km}\dot{x}^m = 0$ ($i = 1, 2$, $x = x^1, y = x^2$) we come to

$$\Gamma^x_{xx} = 0, \Gamma^x_{xy} = \Gamma^x_{yx} = -\frac{1}{y}, \Gamma^x_{yy} = 0, \Gamma^y_{xx} = \frac{1}{y}, \Gamma^y_{xy} = \Gamma^y_{yx} = 0, \Gamma^y_{yy} = -\frac{1}{y}\,. \blacksquare$$

(3c) (3+3).

Let C be a great circle on the sphere. Consider particle moving along the great circle with a constant speed, i.e. $\mathbf{r} = \mathbf{r}(t)$ is a parameterisation of great circle C such that $|\mathbf{v}(t)| = consta$. Show that $\mathbf{r}(t)$ is a geodesic, i.e. $\nabla_{\mathbf{v}}\mathbf{v} = 0$. The acceleration vector $\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \partial_{\mathbf{v}}\mathbf{v}$ is orthogonal to surface. Hence for the connection ∇^{S^2} induced on the surface $\nabla^{S^2}_{\mathbf{v}}\mathbf{v} = 0$. Since Levi-Civita connection coincides with induced connection on the surface we have that $\nabla_{\mathbf{v}}\mathbf{v} = 0$ for Levi-Civita conenction also, i.e. $\mathbf{r}(t)$ is a geodesic.

For unit sphere in spherical coordinates $x = \sin\theta\cos\varphi, y = \sin\theta\sin\varphi, z = \cos\theta$. Hence the curve C belongs to the intersection of the plane $y - x = 0$ with the sphere $x^2 + y^2 + z^2 = 1$. This is the great circle. They are non-parameterised geodesics, hemce tangent vector to curve C remains tanget during parallel transport along the curve.

4

4a (3+3).

Let S be the shape (Weingarten) operator: $S\mathbf{X} = -\partial_{\mathbf{X}}\mathbf{n}$ for an arbitrary tangent vector $\mathbf{X}$. From the derivation equation $d\mathbf{n} = -b\mathbf{e} - c\mathbf{f}$ it follows that $S\mathbf{X} = -\partial_{\mathbf{X}}\mathbf{n} = b(\mathbf{X})\mathbf{e}+c(\mathbf{X})\mathbf{f}$. In particularly it means that for basic vectors $\mathbf{e},\mathbf{f}$ we have $S\mathbf{e} = b(\mathbf{e})\mathbf{e}+c(\mathbf{e})\mathbf{f}$ and $S\mathbf{f} = b(\mathbf{f})\mathbf{e}+c(\mathbf{f})\mathbf{f}$. A matrix of the shape operator in the basis $\{\mathbf{e},\mathbf{f}\}$ is $\begin{pmatrix} b(\mathbf{e}) & b(\mathbf{f}) \\ c(\mathbf{e}) & c(\mathbf{f}) \end{pmatrix}$. Hence mean curvature $H = \mathrm{Tr}S = b(\mathbf{e}) + c(\mathbf{f})$ and

$$\text{Gaussian curvature}\, K = \det S = b(\mathbf{e})c(\mathbf{f}) - b(\mathbf{f})c(\mathbf{e}) = b \wedge c(\mathbf{e},\mathbf{f})\,. \blacksquare$$

Consider derivation equations $d\mathbf{e} = a\mathbf{f} + b\mathbf{n}$. Hence $d^2\mathbf{e} = 0 = d(a\mathbf{f} + b\mathbf{n}) = da\mathbf{f} - a \wedge d\mathbf{f} + db\mathbf{n} - b\wedge d\mathbf{n} = da\mathbf{f} - a\wedge(-a\mathbf{e}+c\mathbf{n}) + db\mathbf{n} - b\wedge(-b\mathbf{e}-c\mathbf{f}) = (da+b\wedge c)\mathbf{f} + (db-a\wedge c)\mathbf{n} = 0$. Hence components are equal to zero. In particular $da + b \wedge c = 0$. $\blacksquare$

4b (2+3+3). For the saddle $\mathbf{r}(u,v)$: consider coordinate vectors $\mathbf{r}_u = \begin{pmatrix} 1 \\ 0 \\ kv \end{pmatrix}$, $\mathbf{r}_v = \begin{pmatrix} 0 \\ 1 \\ ku \end{pmatrix}$. We may choose unit vector field $\mathbf{e} = \frac{\mathbf{r}_u}{|\mathbf{r}_u|} = \frac{1}{\sqrt{1+k^2v^2}} \begin{pmatrix} 1 \\ 0 \\ kv \end{pmatrix}$. The vector fields $\mathbf{r}_u$ and $\mathbf{r}_v$ are not orthogonal. To find second unit field $\mathbf{f}$ tangent to the saddle, we first consider normal unit vector field, a field $\mathbf{n} = \frac{1}{\sqrt{1+k^2(u^2+v^2)}} \begin{pmatrix} -kv \\ -ku \\ 1 \end{pmatrix}$. Indeed this field is obviously unit field which is orthogonal to coordinate vector fields $\mathbf{r}_u$ and $\mathbf{r}_v$, hence this is orthogonal to surface. Now we define vector field $\mathbf{f} = \mathbf{n} \times \mathbf{e} =$:

$$\frac{1}{\sqrt{1+k^2(u^2+v^2)}} \begin{pmatrix} -kv \\ -ku \\ 1 \end{pmatrix} \times \frac{1}{\sqrt{1+k^2v^2}} \begin{pmatrix} 1 \\ 0 \\ kv \end{pmatrix} = \frac{1}{\sqrt{(1+k^2(u^2+v^2))(1+k^2v^2)}} \begin{pmatrix} -k^2uv \\ 1+k^2v^2 \\ ku \end{pmatrix}$$

(Vector field $\mathbf{f}$ is linear combination of coordinate vector fields $\mathbf{r}_u$ and $\mathbf{r}_v$.)

Notice that at the origin $(u = v = 0)$ we have

$$\mathbf{e} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{f} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Now it is easy to calculate $d\mathbf{e}, d\mathbf{f}$ and $d\mathbf{n}$ at the origin. We use the fact that differentials of the functions $\frac{1}{\sqrt{1+k^2v^2}}$,, $\frac{1}{\sqrt{1+k^2u^2+k^2v^2}}$, v^2, uv vanish at the point $u = v = 0$, i.e. all partial derivatives of these functions vanish at the origin. This simplifies all the calculations:

$$d\mathbf{e}\big|_{u=v=0} = d\left(\frac{1}{\sqrt{1+k^2v^2}}\begin{pmatrix}1\\0\\kv\end{pmatrix}\right)\big|_{u=v=0} = \begin{pmatrix}0\\0\\1\end{pmatrix} kdv\,,$$

and

$$d\mathbf{n}\big|_{u=v=0} = d\left(\frac{1}{\sqrt{1+k^2u^2+k^2v^2}}\begin{pmatrix}-kv\\-ku\\1\end{pmatrix}\right)\big|_{u=v=0} = -\begin{pmatrix}1\\0\\0\end{pmatrix} kdv - \begin{pmatrix}0\\1\\0\end{pmatrix} kdu\,,$$

We see that at the point $u = v = 0$, $d\mathbf{e} = \mathbf{n}kdv$ and $d\mathbf{n} = -\mathbf{e}kdv - \mathbf{f}kdu$. Hence in derivation formulae at the origin differential forms $a = 0$, $b = kdv$ and $c = kdu$ (we calculate them only at the origin!) Hence according to derivation formulae at origin $d\mathbf{f} = -a\mathbf{e} + c\mathbf{n} = \mathbf{n}kdu$ (*Sure we may easy to calculate* $d\mathbf{f}$ *at origin straightforwardly*). Now calculate Gaussian curvature:

$$K = b(\mathbf{e})c(\mathbf{f}) - b(\mathbf{f})c(\mathbf{e}) = 0 \cdot 0 - k \cdot k = -k^2$$

since at the origin $\mathbf{e} = \partial_u, \mathbf{f} = \partial_v$ and $b(\mathbf{e}) = c(\mathbf{f}) = 0$, $b(\mathbf{f}) = c(\mathbf{e}) = kdu(\partial_u) = kdv(\partial_v) = k$.

4c (2+2+2) The formula for Gaussian curvature in conformal coordinates tells that

$$K = -\frac{1}{2\sigma}\Delta \log \sigma = -\frac{1}{2\sigma}\left(\frac{\partial^2\sigma(u,v)}{\partial u^2} + \frac{\partial^2\sigma(u,v)}{\partial v^2}\right).$$

Calculate Gaussian curvature if metric $G = \frac{du^2+dv^2}{v^2}$ i.e. $\sigma(u,v) = \frac{1}{v^2}$.

$$K = -\frac{1}{2\sigma}(\log\sigma)_{vv} = -\frac{v^2}{2}\log\left(\frac{1}{v^2}\right)_{vv} = \frac{v^2}{2}2(\log v)_{vv} = v^2\left(\frac{1}{v}\right)_v = -1\,.$$

We see that the curvature is equal to -1.

If surface is locally Euclidean, i.e. one can introduce local coordinates x, y such that $G = dx^2 + dy^2$ then obviously Gaussian curvature $K = 0 \neq -1$. Contradiction.

Remark It may happen that some students will guess (without calculation of curvature) that this surface is Lobachevsky plane, hence it has negative curvature. They will have credits.

Solution of the additional question

5 (for students who earn 15 credits)

(a 2+1+7)

Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ be an arbitrary vector fields on the manifold equipped with affine connection ∇. Consider the operation which assigns to the vector fields $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$ the new vector field: $\mathcal{R}(\mathbf{X},\mathbf{Y})\mathbf{Z} = \left(\nabla_\mathbf{X}\nabla_\mathbf{Y} - \nabla_\mathbf{Y}\nabla_\mathbf{X} - \nabla_{[\mathbf{X},\mathbf{Y}]}\right)\mathbf{Z}$. One can show that it is $C^\infty(M)$-linear operation with respect to vector fields $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$, i.e. for an arbitrary functions f, g, h, $\mathcal{R}(f\mathbf{X}, g\mathbf{Y})(h\mathbf{Z}) = fgh\mathcal{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z}$. Thus it it defines the tensor field of the type $\begin{pmatrix}1\\3\end{pmatrix}$: If $\mathbf{X} = X^i\partial_i, \mathbf{X} = X^i\partial_i, \mathbf{X} = X^i\partial_i$ then

$$\mathcal{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z} = \mathcal{R}(X^m\partial_m, Y^n\partial_n)(Z^r\partial_r) = Z^r R^i_{rmn} X^m Y^n$$

where we denote by R^i_{rmn} the components of the tensor $\mathcal{R}$ in the coordinate basis ∂_i $R^i_{rmn}\partial_i = \mathcal{R}(\partial_m, \partial_n)\partial_r$. This $\begin{pmatrix}1\\3\end{pmatrix}$ tensor field is called curvature tensor of the connection ∇.

Calculate R^i_{kmn} in given local coordinates. We have from definition that

$$R^i_{kmn}\partial_i = \nabla_{\partial_m}\nabla_{\partial_n}\partial_k - (m \leftrightarrow n) = \nabla_{\partial_m}\left(\Gamma^r_{nk}\partial_r\right) - (m \leftrightarrow n) =$$

$$\left(\partial_m\Gamma^i_{nk} + \Gamma^i_{mr}\Gamma^r_{nk} - (m \leftrightarrow n)\right)\partial_i\,, \qquad \text{i.e. } R^i_{kmn} = \partial_m\Gamma^i_{nk} + \Gamma^i_{mr}\Gamma^r_{nk} - (m \leftrightarrow n)\,.$$

We have that $G = e^{-au^2-bv^2}(du^2 + dv^2)$. Denote by $F(u, v) = e^{-au^2-bv^2}$. We see that $g_{12} = g_{21} = 0$ and at the point $\mathbf{p}$ with the coordinates $u = v = 0$ $F(u, v) = 1$. Moreover the derivatives $\frac{\partial F(u,v)}{\partial u}$ and $\frac{\partial F(u,v)}{\partial v}$ also vanish at this point: $F_u = -2auF$, $F_v = -2bvF$. Hence calculating connection with Levi-Civita formula:

$$\Gamma^m_{ik}(x) = \frac{1}{2}g^{mn}(x)\left(\frac{\partial g_{in}(x)}{\partial x^k} + \frac{\partial g_{kn}(x)}{\partial x^i} - \frac{\partial g_{ik}(x)}{\partial x^n}\right)\,. \tag{3}$$

we see that Christoffel symbols vanish at the point $\mathbf{p}$ and $g_{ik} = \delta_{ik}$ at the point $\mathbf{p}$. Hence for the formula of curvature tensor we have:

$$R_{1212}|_{\mathbf{p}} = R^1{}_{212} = \left(\partial_1\Gamma^1_{22} - \partial_2\Gamma^1_{12}\right)|_{\mathbf{p}}$$

We need to calculate $\Gamma^1_{22}, \Gamma^1_{12}$ up to the first order in u, v

Since first derivatives F_u, F_v vanish at the point $\mathbf{p}$ and $g_{ik} = \delta_{ik}$ at this point then $\Gamma^i_{km}|_{\mathbf{p}} = \Gamma_{ikm}|_{\mathbf{p}}$. We see that

$$\Gamma^1_{22}|_{\mathbf{p}} = \frac{1}{2}\left(\frac{-\partial g_{22}}{\partial x^1}\right)|_{\mathbf{p}} = -\frac{1}{2}F_u|_{\mathbf{p}} + o^1(u, v),\ \Gamma^1_{12}|_{\mathbf{p}} = \frac{1}{2}\left(\frac{\partial g_{11}}{\partial x^2}\right)|_{\mathbf{p}} = \frac{1}{2}F_v|_{\mathbf{p}} + o^1(u, v)$$

(we denote $x^1 = u, x^2 = v$)

We have finally using formula (3) that

$$R_{1212}|_{\mathbf{p}} = R^1{}_{212} = \partial_1\Gamma^1_{22} - \partial_2\Gamma^1_{12} = \frac{1}{2}\left(-F_{uu} - F_{vv}\right)|_{\mathbf{p}}$$

$F_{uu}|_{\mathbf{p}} = \frac{\partial^2}{\partial u^2}\left(e^{-au^2-bv^2}|_{\mathbf{p}}\right) = \frac{\partial}{\partial u}\left(-2aue^{-au^2-bv^2}|_{\mathbf{p}}\right) = -2a$ and $F_{vv}|_{\mathbf{p}} = \frac{\partial^2}{\partial v^2}\left(e^{-au^2-bv^2}|_{\mathbf{p}}\right) = -2b$

$$R_{1212}|_{\mathbf{p}} = \frac{1}{2}\left(-F_{uu} - F_{vv}\right)|_{\mathbf{p}} = a + b\,.$$

(b 6+2+2)

Let $G = \sigma(u,v)(du^2 + dv^2)$ be conformal metric. Let $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ be an orthonormal basis adjusted to the surface, then due to derivation formulae and Gauss formula

$$K = b \wedge c(\mathbf{e}, \mathbf{f}) = -da(\mathbf{e}, \mathbf{f})\,, \qquad (*)$$

where a, b, c are 1-forms on the surface such that $d\mathbf{e} = a\mathbf{f} + b\mathbf{n}$, $d\mathbf{f} = -a\mathbf{e} + c\mathbf{n}$, $d\mathbf{n} = -b\mathbf{e} - c\mathbf{f}$, Calculate 1-form a. Choose appropritate adjusted basis. Basic tangent vectors $\mathbf{r}_u, \mathbf{r}_v$ attached at the point $\mathbf{r}(u,v)$ have length $\sqrt{\sigma}$, and they are orthogonal to each other, since $G = \sigma(du^2 + dv^2)$. Choose $\mathbf{e} = \mathbf{r}_u\sqrt{\sigma}$ and $\mathbf{f} = \mathbf{r}_v\sqrt{\sigma}$. Consider vector-valued 1-form $d\mathbf{e}$, $d\mathbf{e} = a\mathbf{f} + b\mathbf{n}$. Taking scalar product with $\mathbf{f}$ we come to the equation $\langle d\mathbf{e}, \mathbf{f}\rangle$. Now calculating in detail we come to

$$a = \langle d\mathbf{e}, \mathbf{f}\rangle = \langle d\left(\frac{\mathbf{r}_u}{\sqrt{\sigma}}\right), \frac{\mathbf{r}_v}{\sqrt{\sigma}}\rangle = \langle \frac{d\mathbf{r}_u}{\sqrt{\sigma}} - d\left(\frac{1}{\sqrt{\sigma}}\right)\mathbf{r}_u, \frac{\mathbf{r}_v}{\sqrt{\sigma}}\rangle = \frac{1}{\sigma}\langle d\mathbf{r}_u, \mathbf{r}_v\rangle\,,$$

since $\langle \mathbf{r}_u, \mathbf{r}_v\rangle = 0$. So it remains to calculate $\langle d\mathbf{r}_u, \mathbf{r}_v\rangle$: $\langle d\mathbf{r}_u, \mathbf{r}_v\rangle = \langle \mathbf{r}_{uu}, \mathbf{r}_v\rangle du + \langle \mathbf{r}_{uv}, \mathbf{r}_v\rangle dv$. We see that

$$\langle \mathbf{r}_{uv}, \mathbf{r}_v\rangle = \frac{1}{2}\frac{\partial}{\partial u}\langle \mathbf{r}_v, \mathbf{r}_v\rangle = \frac{\sigma_u}{2}\,, \text{ and } \langle \mathbf{r}_{uu}, \mathbf{r}_v\rangle = -\langle \mathbf{r}_u, \mathbf{r}_{vu}\rangle = -\frac{1}{2}\frac{\partial}{\partial v}\langle \mathbf{r}_u, \mathbf{r}_u\rangle = -\frac{\sigma_v}{2}$$

Hence

$$a = \frac{1}{\sigma}\langle d\mathbf{r}_u, \mathbf{r}_v\rangle = \frac{1}{2\sigma}(\sigma_u dv - \sigma_v du) = \frac{1}{2}\left(\frac{\partial \log\sigma}{\partial u}dv - \frac{\partial \log\sigma}{\partial v}du\right),\ da = \frac{1}{2\sigma}\Delta\log\sigma$$

Now due to equation (*) Gaussian curvature

$$K = b \wedge c(\mathbf{e}, \mathbf{f}) = -da(\mathbf{e}, \mathbf{f}) = -\frac{1}{2}\Delta\log\sigma du \wedge dv\left(\frac{\mathbf{r}_u}{\sqrt{\sigma}}, \frac{\mathbf{r}_u}{\sqrt{\sigma,}}\right) = -\frac{1}{2\sigma}\Delta\log\sigma \quad \blacksquare$$

If $\mathbf{r}' = \sqrt{\lambda}\mathbf{r}(u,v)$ then for every point (u,v) metric $G" = \lambda G$. E.g. $G'_{uu} = (\mathbf{r}'_u, \mathbf{r}'_u) = \lambda(\mathbf{r}_u, \mathbf{r}_u) = \lambda G_{uu}$. Changing $G \to G'$, $\sigma \to \sigma' = \lambda\sigma$ and

$$K' = -\frac{1}{2\sigma'}\Delta\log(\sigma') = \frac{K}{\lambda}$$

Every question is worth 20 marks

The marks for every subquestions are indicated above in the text of solutions.

Bookwork

$First\ question:$	$(a1, a2, a3) - 3 + 1 + 2$	$(b1) - 2$	$(c1) - 2$	$6 + 2 + 2 = 10$
$Second\ question:$	$(a1) - 3$	$(b1) - 3$	$(c1) - 3$	$3 + 3 + 3 = 9$
$Third\ question:$	$(a1, 2)3 + 2$	$(b1, 2)2 + 1$	$c(1) - 3$	$3 + 2 + 2 + 1 + 3 = 11$
$Fourth\ question:$	$(a1, 2) - 2 + 3$	$(b1)3$	$c(1) - 3$	$2 + 3 + 3 + 3 = 11$
$Fifth\ question:$	$(a1, 2)2 + 1$		$b(1) - 6$	$3 + 6 = 9$

Easy questions

$First\ question:$	$(a2, a3) - 1 + 2$	$b(1) - 2$	$c(1) - 2$	$3 + 2 + 2 = 7$
$Second\ question$	$(a2) - 2$		$c(1) - 3$	$2 + 3 = 5$
$Third\ question$	$a(2)2$	$b(1, 2)2 + 1$		$2 + 2 + 1 = 5$
$Fourth\ question$	$a(1) - 3$	$b(1) - 2$		$2 + 3 = 5$

Difficult questions

$First\ question$	$b(2) - 3$(unseen, unexpected, but not very difficult)
$Second\ question$	$c(3) - 3$little bit hard
$Third\ question$	$a(3) - 3, c(2) - 3$(both unseen not difficult, but beautiful)
$Fourth\ question$	$b2, b3 - 3 + 3$(not difficult but uncustomary)
$Fifth\ question$	$b(2) - 3$(not very difficult but unseen)