<u>**Three hours**</u>

THE UNIVERSITY OF MANCHESTER

RIEMANNIAN GEOMETRY

01 June 2017

14:00 – 17:00

Answer **ALL FIVE** questions in Section A (50 marks in total).
Answer **TWO** of the THREE questions in Section B (30 marks in total).
Answer **ALL TWO** questions in Section C (40 marks in total).
If more than TWO questions in Section B are attempted, the credit will be given for the best TWO answers.

Electronic calculators may <u>not</u> be used.

Throughout the paper, where the index notation is used, the <u>Einstein summation convention</u> over repeated indices is applied if it is not explicitly stated otherwise.

Feedback on the Exam31082, 41082,61082

(10 credit students (31082) have to answer only the sections A and B, and the examination time 14:00–16:00).

SECTION A

Answer **ALL** FIVE questions

A1.

(a) *Explain what is meant by saying that a Riemannian manifold is locally Euclidean.*

(b) *Consider the surface of the cylinder $x^2 + y^2 = R^2$ in $\mathbf{E}^3$*

$$\mathbf{r}(h,\varphi)\colon \quad \begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \\ z = h \end{cases} , \quad -\infty < h < \infty\,, \quad 0 \le \varphi < 2\pi\,.$$

Calculate the Riemannian metric on this surface induced by the Euclidean metric on $\mathbf{E}^3$.

Show that this surface is locally Euclidean.

Is this surface isometric to $\mathbf{E}^2$?

[10 marks]

Riemannian manifold is locally Euclidean if in a vicinity of every point there exists an open neighbourhood such that it is isometric to a domain in Eucldean plane. In other words in a vicinity of every point there exist local coordinates $(u^{,}, \dots, u^n)$ such that in these coordinates Riemannian metric G has an appearance

$$G = (du^1)^2 + \cdots + (du^n)^2\,.$$

No problem with definirion, just some students did not emphasize, that here Euclidean coordinates are local coordinates, they are defined in the vicinity of an **arbitrary point** (marks were not decreased if students did not use the terminology "open neighbourhood").

Almost all students calculated induced Riemannian metric on cylindre $G = R^2 d\varphi^2 + dh^2$, and suggested right locally Euclidean coordinates:

$$u = R\varphi, v = h\,, \qquad du^2 + dv^2 = R^2 d\varphi^2 + dh^2\,. \tag{A1.1}$$

Two (or three) students instead this problem solved, another problem to find local Euclidean coordinates on the cone. However the question was asked about cylindre (and calculations fo cylindre are much easier than for cone.)

Surface of cylindre is not isometric to $\mathbf{E}^2$. (It is is isometric only locally to $\mathbf{E}^2$, coordinates (A1.1) are local coordinates.)

A2.

(a) Explain what is meant by an affine connection on a manifold.

Let ∇ be an affine connection on a 2-dimensional manifold M such that in local coordinates (u, v) we have that $\nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial u} = \frac{\partial}{\partial v}$.

Calculate the Christoffel symbols Γ^u_{uu} and Γ^v_{uu}.

(b) Let ∇ be an affine connection on a 2-dimensional manifold M such that, in local coordinates (x, y), all Christoffel symbols vanish except $\Gamma^x_{xx} = xy$, $\Gamma^y_{xx} = -1$ and $\Gamma^y_{yy} = y$. Show that for the vector field $\mathbf{X} = \partial_x + x\partial_y$,

$$\nabla_{\mathbf{X}}\mathbf{X} = xy\mathbf{X}\,.$$

[10 marks]

Affine connection on M is the *operation* ∇ which assigns to every vector field $\mathbf{X}$ a linear map $\nabla_{\mathbf{X}}$ on the space of vector fields: $\nabla_{\mathbf{X}}(\lambda\mathbf{Y} + \mu\mathbf{Z}) = \lambda\nabla_{\mathbf{X}}\mathbf{Y} + \mu\nabla_{\mathbf{X}}\mathbf{Z}$ $(\lambda, \mu \in \mathbf{R})$, which satisfies the following additional conditions:

1.For arbitrary (smooth) functions f, g on M

$$\nabla_{f\mathbf{X}+\mathbf{gY}}(\mathbf{Z}) = f\nabla_{\mathbf{X}}(\mathbf{Z}) + g\nabla_{\mathbf{Y}}(\mathbf{Z}) \qquad (C(M)\text{-linearity})$$

2. For arbitrary function f

$$\nabla_{\mathbf{X}}(f\mathbf{Y}) = (\nabla_{\mathbf{X}}f)\,\mathbf{Y} + f\nabla_{\mathbf{X}}(\mathbf{Y}) \qquad (\text{Leibnitz rule}) \qquad (A2.1)$$

($\nabla_{\mathbf{X}}f$ is just usual derivative of a function f along vector field: $\nabla_{\mathbf{X}}f = \partial_{\mathbf{X}}f$.)

Almost all students gave correct definition. Just one comment: Connection is an operation on vector fields, not vector spaces!

Almost all students calculated correctly Γ^u_{uu} and Γ^v_{uv}:

$$\nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial u} = \frac{\partial}{\partial v} = \Gamma^u_{uu}\partial_u + \Gamma^v_{uu}\partial_v \Rightarrow \Gamma^u_{uu} = 0\,, \Gamma^v_{uu} = 1\,.$$

Now about calculation of $\nabla_{\mathbf{X}}\mathbf{X}$:

Using the information about Christoffel symbols and the properties of connection we have:

$$\nabla_{\mathbf{X}}\mathbf{X} = \nabla_{\partial_x + x\partial_y}(\partial_x + x\partial_y) = \nabla_{\partial_x}\partial_x + x\nabla_{\partial_y}\partial_x + \nabla_{\partial_x}(x\partial_y) + x\partial_y(x\partial_y) =$$

$$\nabla_{\partial_x}\partial_x + x\nabla_{\partial_y}\partial_x + \partial_y + x\nabla_{\partial_x}(\partial_y) + x^2\partial_y(\partial_y) =$$

$$\Gamma^x_{xx}\partial_x + \Gamma^y_{xx}\partial_y + x\Gamma^x_{yx}\partial_x + x\Gamma^y_{yx}\partial_y + \partial_y + x\Gamma^x_{xy}\partial_x + x\Gamma^y_{xy}\partial_y + x^2\Gamma^x_{yy}\partial_x + x^2\Gamma^y_{yy}\partial_y =$$

$$xy\partial_x - \partial_y + 0 + 0 + \partial_y + 0 + 0 + 0 + x^2y\partial_y = xy(\partial_x + x\partial_x) = xy\mathbf{X}\,.$$

About half of students did these calculations properly. Students who failed these calculations, failed because they did not use properly the Leibnitz rule (A2.1).

A3.

(a) Formulate the Levi-Civita Theorem. In particular, write down the Christoffel symbols Γ^i_{km} in terms of a Riemannian metric $G = g_{ik}(x)dx^i dx^k$.

(b) Calculate the Christoffel symbol $\Gamma^r_{\varphi\varphi}$ of the metric $G = dr^2 + r^2 d\varphi^2$. (It is the Euclidean metric of $\mathbf{E}^2$ written in polar coordinates.)

[10 marks]

Let M be a Riemannian manifold with metric $G = g_{ik}dx^i dx^k$. Let ∇ be a symmetric connection on M, i.e. its Christophel symbols Γ^m_{ik} satisfies the condition: $\Gamma^m_{ik} = \Gamma^m_{ki}$. We say that symmetric connection ∇ is Levi Civita connection if it preserves scalar product, i.e. if for arbitrary vectors $\mathbf{Y}, \mathbf{Z}$ at an arbitrary point

$$\nabla_{\mathbf{X}} < \mathbf{Y}, \mathbf{Z} > = < \nabla_{\mathbf{X}}(\mathbf{Y}), \mathbf{Z} > + < \mathbf{Y}, \nabla_{\mathbf{X}}(\mathbf{Z}) > \,. \qquad (A3.1)$$

Levi-Civita Theorem claims that on the Riemannian manifold (M, G) there exists uniquely defined Levi Civita connection. In local coordinates Christoffel symbols of this connection have the following appearance:

$$\Gamma^m_{ik}(x) = \frac{1}{2}g^{mn}(x)\left(\frac{\partial g_{in}(x)}{\partial x^k} + \frac{\partial g_{kn}(x)}{\partial x^i} - \frac{\partial g_{ik}(x)}{\partial x^n}\right)\,. \qquad (A3.2)$$

This was little bit surprising, but many students had troubles to fomulate this bookwork result.

In particular many students did not explain that connection has to be symmetric, and what it means and few students just ingnored the condition (A3.1) of compatibility. Few students did not mention the uniqueness of this connection.

Now calculate $\Gamma^r_{\varphi\varphi}$. We have that

$$G = \begin{pmatrix} g_{rr} & g_{r\varphi} \\ g_{\varphi r} & g_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \quad G^{-1} = \begin{pmatrix} g^{rr} & g^{r\varphi} \\ g^{\varphi r} & g^{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}, \tag{A3.3}$$

and using (A3.2) we come to

$$\Gamma^r_{\varphi\varphi} = \frac{1}{2} g^{rr}(x) \left(\frac{\partial g_{\varphi r}(x)}{\partial \varphi} + \frac{\partial g_{r\varphi}(x)}{\partial \varphi} - \frac{\partial g_{\varphi\varphi}(x)}{\partial r} \right) = \frac{1}{2} \cdot 1 \cdot \left(- \frac{\partial g_{\varphi\varphi}(x)}{\partial r} \right) = \frac{1}{2} \left(- \frac{\partial}{\partial r} \left(r^2 \right) \right) = -r\,.$$

Another solution One can calculate $\Gamma^r_{\varphi\varphi}$ using the fact that it is canonical flat connection of $\mathbf{E}^2$ in polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$. Using the fact that Chrsitoffel symbpols of canonical flat connection vanish in Cartesian coordinates, we come to

$$\Gamma^r_{\varphi\varphi} = \frac{\partial^2 x}{\partial\varphi\partial\varphi} \frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial\varphi\partial\varphi} \frac{\partial r}{\partial y} = -r\cos\varphi \cdot \frac{x}{r} - r\sin\varphi \cdot \frac{y}{r} = -r\cos^2\varphi - r\sin^2\varphi = -r\,. \tag{A3.4}$$

Many students did this exercise using Levi-Civita formula. (A3.2). Finding metric few students calculated wrong G^{-1} (see (A3.3)).

Some students calculated $\Gamma^r_{\varphi\varphi}$ in a second way, using formula for flat connection. Sure this solution was accepted, but students recieve full mark if they justified the use of formula (A3.4) (this formula is true since the connection is canonical flat connection, whith vanishing Christoffel symbols in Cartesian coordinates.)

A4.

(a) *Consider the sphere of radius R in Euclidean space $\mathbf{E}^3$*

$$\mathbf{r}(\theta, \varphi) \colon \quad \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases}.$$

Let $\mathbf{e}, \mathbf{f}$ be unit vectors in the directions of the vectors $\mathbf{r}_\theta = \frac{\partial \mathbf{r}}{\partial \theta}$ and $\mathbf{r}_\varphi = \frac{\partial \mathbf{r}}{\partial \varphi}$, and let $\mathbf{n}$ be a unit normal vector to the sphere.
Express these vectors explicitly.

(b) *For the obtained orthonormal basis $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ calculate the 1-forms a, b and c in the derivation formula*

$$d \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix}.$$

[10 marks]

This is standard bookwork question from lecture notes.

Students did it alright.

Some students did not justify why $\mathbf{e}, \mathbf{f}, \mathbf{n}$ is an orthonormal basis

A5.

(a) *State the relation between the Riemannian curvature tensor of the Levi-Civita connection of a surface in $\mathbf{E}^3$ and its Gaussian curvature.*

(b) *For the sphere of radius ρ in $\mathbf{E}^3$ calculate all components of the Riemannian curvature tensor R_{ikmn} in spherical coordinates θ, φ at the points of the equator ($\theta = \frac{\pi}{2}$).*

Explain why the sphere is not locally Euclidean.

[10 marks]

Let M be a surface in Euclidean space $\mathbf{E}^3$. Let K be Gaussian curvature of this surface, and let R_{ikmn} be Riemann curvature tensor of Levi-Civita connection. Then

$$\frac{R}{2} = \frac{R_{1212}}{\det g} = K\,, \tag{A5.1}$$

where g_{ik} is tensor of induced Riemann metric on the surface, R—scalar curvature ($R = R^i_{rin} g^{rn}$).

Almost all students wrote this bookwork statement, but many students did not do it in a complete way. For example many students just wrote the formula (A5.1) without explaining the geometrical meaning of terms of this formula.

The solution of next exercise is based on this formula: using symmetry properties $R_{ikmn} = -R_{kimn} = -R_{iknm}$ we have that

$$R_{1111} = R_{1112} = R_{1121} = R_{1211} = R_{2111} = R_{2222} = R_{2212} = R_{2221} = R_{1222} = R_{2122} = R_{1122} = R_{2211} = 0\,.$$

It remains to calculate $R_{1212}, R_{1221}, R_{2112}, R_{2121}$ We know that Gaussian curvature of the sphere is equal to

$$K = \frac{1}{\rho^2}\,. \tag{A5.2}$$

Since $G = R^2(d\theta^2 + \sin^2\theta d\varphi^2)$, $\det g = \rho^4 \sin^2\theta$, we have that

$$R_{1212} = -R_{1221} = -R_{2112} = R_{2121} = K \det g = \rho^2 \sin^2\theta = \rho^2\,.$$

at the equator.

It was astonishing that few students did not know (or could not use) formula (A5.2) for Gaussian curvature of the sphere.

It was sad to realise that two or three students were trying to explain the wrong statement, why Riemann curvature tensor vanishes at equator—- this is wrong, and this is in contradiction with formulae (A5.1) and (A5.2).

Now the last question, about why sphere is not locally Eucldean. Suppose it is locally Eucldean. Then $G = du^2 + dv^2$ in suitable local coordinates, i.e. due to Levi-Civita formula, Christoffel symbols vanish identically in these coordinates. This implies that curvature tensor vanishes also, and formula (A5.1) implies that Gaussian curvature vanishes too. Contradiction.

Less than half of students answered this subquestion.

SECTION B

Answer **TWO** of the THREE questions

B6.

(a) Let $G = dx^2 + 2pdxdy + dy^2$ be a Riemannian metric on $\mathbf{R}^2$, where x, y are standard Cartesian coordinates, and p is a real constant.

Show that $|p| < 1$.

(b) Calculate the area of the disc $x^2 + y^2 \leq R^2$ with respect to this metric.

[15 marks]

The condition that $p^2 < 1$ can be deduced in many different ways.

May be the quickest one is to consider the length of vector fields $\partial_x \pm \partial_y$:

$|\partial_x \pm \partial_y|^2 = \langle \partial_x \pm \partial_y \, , \, \partial_x \pm \partial_y \rangle = \langle \partial_x, \partial_x \rangle \pm 2 \langle \partial_x, \partial_y \rangle + \langle \partial_y, \partial_y \rangle = 1 \pm 2p + 1 = 2(1 \pm p) > 0 \Rightarrow |p| < 1\,.$

Some students just considered just vector field $\partial_x + \partial_y$. Few students answered completely considering both vector fields, the vector field $\partial_x + \partial_y$ and the vector field $\partial_x - \partial_y$

On the other hand many students have deduce condition $|p| < 1$ using the fact that $\det G = (1 - p^2)$. One can see that if condition $\det G > 0$ does not hold then volume becomes negative, and this is the proof that this condition is necessary condition. Students received partial marks for this considerations.

Calculating area of the disc is much simpler than many students did:

$$\text{Area} = \int_{x^2+y^2 \leq R^2} \sqrt{\det G} dxdy = \int_{x^2+y^2 \leq R^2} \sqrt{1-p^2} dxdy = \sqrt{1-p^2} \int_{x^2+y^2 \leq R^2} dxdy =$$
$$\sqrt{1-p^2} \cdot \text{ Area of circle of radius } R \text{ in Eulcidean plane} = \pi R^2 \sqrt{1-p^2}\,.$$

This is all. One does not need to calculate integral, just you have to use the standard formula $S = \pi R^2$.

Many students did confusing calculations trying to calculate the integral. Few students were trying to calculate this in polar coordinates. Good exercise, but it needs a time. One student did all these caclulations properly.

B7.

(a) *Consider the upper half-plane $y > 0$ with the Riemannian metric*

$$G = \frac{dx^2 + dy^2}{y^2}$$

(the Lobachevsky plane).

Consider in the Lobachevsky plane the domain D defined by

$$D = \{x, y: \quad x^2 + y^2 \geq 1\,,\ 0 \leq x \leq a\}\,,$$

where a is a parameter ($0 < a < 1$).

It is known that the area of the domain D is equal to $\frac{\pi}{6}$.

Find the value of the parameter a.

(b) *Consider the points $A_t = (0, t)$ and $B_t = (a, t)$ on the vertical rays delimiting the domain D. Show that the distance between these points tends to 0 if $t \to \infty$.*

[15 marks]

Calculate the area (another way to calculate this area see in the Remark 2 above):

$$\text{Area} = \int_{x^2+y^2 \geq 11, 0 \leq x \leq a} \sqrt{\det G} dxdy$$

We have $G = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}$, i.e. $\det G = \frac{1}{y^4}$ and

$$\text{Area} = \int_{x^2+y^2>1, 0\leq x\leq a} \sqrt{\det G} dxdy = \int_{x^2+y^2\geq 1, 0\leq x\leq a} \frac{1}{y^2} dxdy = \int_0^a \left(\int_{\sqrt{1-x^2}}^{\infty} \frac{dy}{y^2} \right) dx =$$

$$\int_0^a \left(\left(-\frac{1}{y} \right) |_{\sqrt{1-x^2}}^{\infty} \right) dx = \int_0^a \frac{dx}{\sqrt{1-x^2}}\,.$$

To calulate this integral it is useful to use the substitution $x = \sin\theta$,. We come to

$$\text{Area} = \int_0^a \frac{dx}{\sqrt{1-x^2}} = \int_0^{\theta=\theta(a)} \frac{d\sin\theta}{\sqrt{1-\sin^2\theta}} = \int_0^{\theta=\theta(a)} \frac{\cos\theta d\theta}{\cos\theta} = \int_0^{\theta=\theta(a)} d\theta = \theta(a)\,.$$

Thus we see that

$$\text{Area} = \frac{\pi}{6} = \theta \Rightarrow a = \sin\theta = \sin\frac{\pi}{6} = \frac{1}{2}\,. \qquad (B7.1)$$

The distance between points $A_t = (0, t)$

and $B_t = (a.t)$ is less or equal to the length of the horizontal segment

$$A_tB_t\colon \begin{cases} x = \tau \\ y = t \end{cases} \qquad 0 \le \tau \le \tau\,.$$

One can easy calculate the length of this segment:

$$|A_tB_t| = \int_0^a \sqrt{\frac{x_\tau^2 + y_\tau^2}{y^2}}d\tau = \int_0^a \sqrt{\frac{1+0}{t^2}}d\tau = \frac{a}{t}\,.$$

We see that if $t \to \infty$ then $|A_tB_t| \to 0$. Hence distance between points A_t, B_t tends to zero if $t \to 0$ since

$$\text{distance between points } A_t,\, B_t \le \text{the length of the segment } A_tB_t = \frac{a}{t} \to 0 \text{ if } t \to \infty\,. \tag{B7.2}$$

This question students answered good. Some students have problems calculating the integral

$$\int_0^a \frac{dx}{\sqrt{1-x^2}}dx\,.$$

On the final step, two (or three) students came to the wrong answer since they did not know that $\sin \pi/6 = \frac{1}{2}$. (If you forget it just draw equilaterla triangle, and draw its height, and you immediately will see that $\sin \pi = \frac{1}{2}$.)

Remark 1 The distance between these points is in fact the length of the geodesic, passing via points A_t, B_t. Students did not need to calculate the distance, however they needed to show that it tends to zero, and this follows from (B7.2).

Remark 2 The fact that distance between points A_t and B_t tends to zero opens way to the following very beautiful calculation for area (B7.1). Consider points $A = (0, 1)$, $B = (a, \sqrt{1-a^2})$. One can see that the domain considered above is the interior of the isocseless triangle ABC, here the point C is at infinity. (Sides of this triangle are geodesics.) One can see that $\angle A = \frac{\pi}{2}$, $\angle B = \arccos a$ and $\angle C = 0$. Since the curvature of Lobachevsky plane is equal to -1 hence we see that

$$Area = \pi - (\angle A + \angle B + \angle C) = \pi - \left(\frac{\pi}{2} + \text{arccosa} + 0\right) = \frac{\pi}{2} - \arccos\ a = \frac{\pi}{6}\,.$$

Thus $\arccos a = \frac{\pi}{3}$ and $a = \cos\frac{\pi}{3} = \frac{1}{2}$. We come to (B7.1). Nice is not it?

Sure this solution uses some facts which was not supposed that students have to know. I am very happy that one student solved the problem in this way.

B8.

(a) Let M be a surface in $\mathbf{E}^3$ with the Riemannian metric induced by the Euclidean metric on $\mathbf{E}^3$.

Prove that the induced connection on the surface M is equal to the Levi-Civita connection of the induced Riemannian metric.

(b) Explain why the acceleration vector of an arbitrary parameterised geodesic on a surface M is orthogonal to the surface.

Deduce that all geodesics of the sphere $S^2\colon\ x^2+y^2+z^2=1$ are great circles.

Let C be a geodesic on the sphere S^2, which passes through the North pole $N=(0,0,1)$. Show that the image of the geodesic C under the stereographic projection on the plane $z=0$ with respect to N is a straight line.

[15 marks]

The induced connection on the surfaces is defined by equation

$$\nabla^{\text{induced}}\colon \nabla^{\text{induced}}_{\mathbf{X}}\mathbf{Y} = \left(\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}\right)_{\text{tangent}} ,$$

where $\nabla^{\text{can.flat}}$ is canonical flat connection in $\mathbf{E}^3$, which in Cartesian coordiantes look very simple: $\nabla^{\mathbf{E}^3}_{\mathbf{X}}\mathbf{Y} = \partial_{\mathbf{X}}\mathbf{Y}$, and $\mathbf{Z}_{\text{tangent}}$ means the projection of the vector $\mathbf{Z}$ on the surface: $\mathbf{Z} = \mathbf{Z}_{\text{tangent}} + \mathbf{Z}_{\text{normal}}$.

Let ∇^{induced} be induced connection on a surface M in $\mathbf{E}^3$ given by equations $\mathbf{r} = \mathbf{r}(u,v)$. Considering this connection on the basic vectors $\mathbf{r}_h, \mathbf{r}_v$ we see that it is symmetric connection. Indeed $\mathbf{r}_{uv} = \mathbf{r}_{vu}$ hence

$$\nabla^{\text{induced}}_{\partial_u}\partial_v = (\mathbf{r}_{uv})_{\text{tangent}} = (\mathbf{r}_{vu})_{\text{tangent}} = \nabla^{\text{induced}}_{\partial_v}\partial_u \,. \Rightarrow \Gamma^{\cdots}_{uv} = \Gamma^{\cdots}_{vu} \,.$$

Prove that this connection preserves scalar product on M. For arbitrary tangent vector fields $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ we have

$$\partial_{\mathbf{X}}\langle \mathbf{Y}, \mathbf{Z}\rangle_{\mathbf{E}^3} = \langle \nabla^{\text{can. flat}}_{\mathbf{X}}\mathbf{Y}, \mathbf{Z}\rangle_{\mathbf{E}^3} + \langle \mathbf{Y}, \nabla^{\text{can. flat}}_{\mathbf{X}}\mathbf{Z}\rangle_{\mathbf{E}^3} \,.$$

since canonical flat connection in $\mathbf{E}^3$ preserves Euclidean metric in $\mathbf{E}^3$ (it is evident in Cartesian coordinates). Now project this equation on the surface M. If $\mathbf{A}$ is an arbitrary vector attached to the surface and $\mathbf{A}_{\text{tangent}}$ is its projection on the tangent space to the surface, then for every tangent vector $\mathbf{B}$ scalar product $\langle \mathbf{A}, \mathbf{B}\rangle_{\mathbf{E}^3}$ equals to the scalar product $\langle \mathbf{A}_{\text{tangent}}, \mathbf{B}\rangle_{\mathbf{E}^3} = \langle \mathbf{A}_{\text{tangent}}, \mathbf{B}\rangle_M$ since vector $\mathbf{A} - \mathbf{A}_{\text{tangent}}$ is orthogonal to the surface. Hence we deduce from that

$$\partial_{\mathbf{X}}\langle \mathbf{Y}, \mathbf{Z}\rangle_M = \langle \left(\nabla^{\text{can. flat}}_{\mathbf{X}}\mathbf{Y}\right)_{\text{tangent}}, \mathbf{Z}\rangle_{\mathbf{E}^3} + \langle \mathbf{Y}, \left(\nabla^{\text{can. flat}}_{\mathbf{X}}\mathbf{Z}\right)_{\text{tangent}}\rangle_{\mathbf{E}^3} =$$

$$\langle \nabla^{\text{induced}}_{\mathbf{X}} \mathbf{Y}, \mathbf{Z} \rangle_M + \langle \mathbf{Y}, \nabla^{\text{induced}}_{\mathbf{X}} \mathbf{Z} \rangle_M \,.$$

We see that induced connection is symmetric connection which preserves the induced metric. Hence due to Levi-Civita Theorem it is unique and is expressed as in the formula (A3.2).

This is standard bookwork question, and students have no special problems with this question.

Unfortunately many students struggled with the next question (explain why the acceleration vector of an arbitrary parameterised geodesic on a surface M is orthogonal to the surface) in spite of the fact that its answer is just a corollary of the Theorem above. The differential equation for geodesic $\nabla_{\mathbf{v}} \mathbf{v} = 0$, where ∇ is the Levi-Civita connection induced on the manifold M and $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ velocity vector. Levi-Civita connection = the induced connection induced on the manifold M (we just proved it): $\nabla_{\mathbf{X}} \mathbf{Y} = \left(\nabla^{\text{can.flat}}_{\mathbf{X}} \mathbf{Y}\right)_{\text{tangent}}$. In particular Cartesian coordiantes $\nabla^{\text{can.flat}}_{\mathbf{X}} \mathbf{Y}$ is just usual derivative $\partial_{\mathbf{X}} \mathbf{Y}$. We have:

$$\nabla_{\mathbf{v}} \mathbf{v} = 0 = (\partial_{\mathbf{v}} \mathbf{v})_{\text{tangent}} = \left(\frac{d^2\mathbf{r}}{dt^2}\right)_{\text{tangent}} = 0.$$

This means the acceleration vector $\mathbf{a}(t) = \frac{d^2\mathbf{r}(t)}{dt^2}$ is orthogonal to the surface M.

On the base of this answer students had to show that all geodesics of the sphere are great circles. Why? We just showed that acceleration vector $\mathbf{a}(t) = \frac{d^2\mathbf{r}(t)}{dt^2}$ is orthogonal to the surface, which is now the sphere $x^2 + y^2 + z^2 = R^2$.

Consider the vector $\mathbf{M}(t) = \mathbf{v}(t) \times \mathbf{r}(t)$. We see that $\frac{d}{dt}\mathbf{M}(\mathbf{t}) = \frac{d}{dt}(\mathbf{v} \times \mathbf{r}) = (\mathbf{a} \times \mathbf{r}) + (\mathbf{v} \times \mathbf{v}) = 0$, since acceleration vector $\mathbf{a}(t) = \frac{d^2\mathbf{r}(t)}{dt^2}$ which is orthogonal to the sphere is proportional to the $\mathbf{r}$. Hence vector $\mathbf{M}(t)$ is preserved: $\mathbf{M}(t) \equiv \mathbf{M}_0$. Hence $\mathbf{r}(t)$ is orthogonal to the vector $\mathbf{M}_0$ for any t. Hence $\mathbf{r}(t)$ belongs to the plane orthogonal to the $\mathbf{M}_0$ which is passes through the centre of the sphere. Thus $\mathbf{r}(t)$ is the great circle. On the other hand if point moves with constant velocity over the great circle then its acceleration is orthogonal to the surface, hence $\nabla_{\mathbf{v}} \mathbf{v} = 0$, i.e. the equation of geodesic is obeyed.

This is also bookwork question but many students failed to answer this question.

And finally the last subquestion about geodesics on the sphere passing through North pole and its image under stereographic projection.

The geodesic C belongs to the plane π passing trough the North Pole, and the centre of the sphere. CXonsider the line l intersection of the plane π with the equatorial plane $z = 0$. By the definiiton of the stereographic projection, the geodesic C transforms to this line. This question was unexpectably hard for many students. Many students instead this simple geometrical consideration were trying to do the confusing calculations with formulae for stereographic projections.

SECTION C

Answer **ALL** TWO questions

C9.

(a) Let $\mathbf{K}$ *be a Killing vector field on a Riemannian manifold* M.

Consider the following bilinear form $S_{\mathbf{K}}$ *on vector fields:*

$$S_{\mathbf{K}}(\mathbf{X},\mathbf{Y}) = G\left(\nabla_{\mathbf{X}}\mathbf{K}\,,\mathbf{Y}\right)\,.$$

Here $\mathbf{K}$ *is a Killing vector field and* $\mathbf{X},\mathbf{Y}$ *are arbitrary vector fields,* G *is the Riemannian metric, and* ∇ *is the Levi-Civita connection of this metric.*

Show that this relation defines an antisymmetric tensor field, i.e.,

$$S_{\mathbf{K}}(\mathbf{X}\,,\mathbf{Y}) = -S_{\mathbf{K}}(\mathbf{Y}\,,\mathbf{X})\,. \qquad (C9.1)$$

(b) Show that $\mathbf{K} = x\partial_x + y\partial_y$ *is a Killing vector field for the Lobachevsky plane (realised as the upper half-plane).*

Calculate all components of the tensor field $S_{\mathbf{K}}$.

(You may use without proof the Christoffel symbols $\Gamma^y_{xx} = \frac{1}{y}$ *and* $\Gamma^y_{xy} = 0$ *for the Lobachevsky plane.)*

[15 marks]

First part of this question was bookwork question, students prepared it well. On the other hand calculation of components S_{xx}, S_{xy}, S_{yx} and S_{yy} of tensor fields was problem for many students. From the result (C9.1) which was proved by almost all stdudents it immediately follows that

$$S_{xx} = S(\partial_x,\partial_x) = 0\,,\quad S_{yy} = S(\partial_y,\partial_y)0\,,\quad \text{and} S_{xy} = -S_{yx}\,. \qquad (C9.2)$$

This means that we have to calculate only one component S_{xy}:

$$S_{xy} = S(\partial_x,\partial_y) = \langle\nabla_{\partial_x}\mathbf{K},\partial_y\rangle = \langle\nabla_{\partial_x}(x\partial_x + y\partial_y),\partial_y\rangle$$

Now calculating this component note that $\langle\partial_x,\partial_y\rangle = 0$ and $\langle\partial_y,\partial_y\rangle = \frac{1}{y^2}$, and $\nabla_{\partial_x}(x\partial_x + y\partial_y) = \partial_x + x\nabla_{\partial_x}\partial_x + y\nabla_{\partial_x}\partial_y$ Hence we have

$$S_{xy} = S(\partial_x,\partial_y) = \langle\nabla_{\partial_x}\mathbf{K},\partial_y\rangle = \langle\nabla_{\partial_x}(x\partial_x + y\partial_y),\partial_y\rangle = \langle x\Gamma^y_{xx}\partial_y + y\Gamma^y_{xy}\partial_y,\partial_y\rangle$$

Using the fact that $\Gamma^y_{xy} = 0$ we come to

$$S_{xy} = \langle x\Gamma^y_{xx}\partial_y + y\Gamma^y_{xy}\partial_y, \partial_y\rangle = x\Gamma^y_{xx}\langle\partial_y, \partial_y\rangle = \frac{x}{y^3}\,.$$

Now it follows from the antisymmetricity that

$$S = \begin{pmatrix} S_{xx} & S_{xy} \\ S_{yx} & S_{yy} \end{pmatrix} = \begin{pmatrix} 0 & \frac{x}{y^3} \\ -\frac{x}{y^3} & 0 \end{pmatrix}\,.$$

Some students calculating components of this tensor wrote that $S_{xx} = ...$ (something not-zero) in spite of the fact that they proved before that S is antisymmetric tensor, and this means that $S_{xx} = 0$:

$$S_{xx} = S(\partial_x, \partial_x) = -S(\partial_x, \partial_x) = -S_{xx} \Rightarrow S_{xx} = 0\,.$$

Surprisingly many students have problems to explain why $x\partial_x + y\partial_y$ is Killing in spite of the fact that question was discussed in Coursework: vector field $\mathbf{K}$ corresponds to infinitesimal homothety (dilation): $\begin{cases} x \to \lambda x \\ y \to \lambda y \end{cases}$ and homothety obviously does not change the metric $\frac{dx^2+dy^2}{y^2}$.

Some students were trying to check straightforwadly condition $\mathcal{L}_{\mathbf{K}}G = 0$.

C10.

(a) *Let M be a surface in the Euclidean space $\mathbf{E}^3$. Consider a closed curve C on M which is the boundary of a compact oriented domain $D \subset M$. Consider the parallel transport of an arbitrary tangent vector $\mathbf{A}$ along the closed curve C. Show that the result of the parallel transport is the vector obtained from $\mathbf{A}$ by rotation through the angle*

$$\Delta\Phi = \int_D K d\sigma\,, \qquad (C10)$$

where K is the Gaussian curvature and $d\sigma$ is the area element induced by the Riemannian metric on the surface M.

(b) *Consider a surface M, the upper sheet of the cone in $\mathbf{E}^3$*

$$\mathbf{r}(h,\varphi)\colon \quad \begin{cases} x = 3h\cos\varphi \\ y = 3h\sin\varphi \\ z = 4h \end{cases}, \quad h > 0\,,\ 0 \le \varphi < 2\pi\,. \qquad (C10.2)$$

Let C_1 be a closed curve on this surface which is the boundary of a compact oriented domain $D \subset M$.

Let C_2 be a circle which is the intersection of the plane $z = h_0$ ($h_0 > 0$) with the surface M.

Show that the parallel transport along the closed curve C_1 is the identical transformation.

Show that the parallel transport along the closed curve C_2 is the rotation through a non-zero angle.

Calculate this angle.

[25 marks]

The first, bookwork part of this question students did well, but their results to solve the porblems with curves C_1 and C_2 was much worse.

For the curve C_1 answer almost literally follows from the satement of the Theorem. You just have to use the fact that surface of the cone for $h > 0$ is locally Eucldean (I am sure that all students knew it very well). This imples that $K = 0$ and due to (C10.1) we come to the statement that the parallel transport along the closed curve C_1 is the identical transformation.

Question about curve C_2 was really difficult question. It was not expected that many students will answer this question completely, but the result was that only one student gave the complete answer and one or two students gave some ideas which may lead to the answer.

When we say that surface of cone is locally Euclidean we exclude the vertex. The curve C_2 is not a boundary of the compact domain. Consider the surface of the cone under the curve C_2. If we develop the surface of the cone (consider its scan) we will come to the interior of the sector of the circle with radius $R = \sqrt{(4h_0)^2 + (3h_0)^2} = 5h_0$. The length of the curve C_2 is equal to $L = 2\pi \cdot 3h_0 = 6\pi h_0$. Hence we will come to the sector of the circle with the radius $R = 5h_0$ and with the angle Θ such that

$$R\Theta = 5h_0\Theta = \text{length of } C_2 = 6\pi h_0 \,. \Rightarrow \Theta = \frac{6\pi}{5}$$

parallel transport along C_2 is now the parallel transport along the arc of this circle. We see that vector rotates on the angle Θ.

Remark One can say that curvature of surfface of cone is concetrated in teh vertex. It can be described using generalised functions.

END OF EXAMINATION PAPER