Homework 0

In this Homework we will just recall some necessary background material from linear algebra

1 Show that the vectors $\{\mathbf{a}_1, \mathbf{a}_2 \dots, \mathbf{a}_m\}$ in vector space V are linearly dependent if at least one of these vectors is equal to zero.

2 Show that the ordered set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of vectors

$$\begin{array}{ll} \mathbf{e}_1 & = (1, \quad 2, \quad 3) \\ \mathbf{e}_2 & = (0, \quad 1, \quad 2) \\ \mathbf{e}_3 & = (0, \quad 0, \quad 1) \end{array}$$

is a basis in $\mathbf{R}^3$.

3 Show that any three vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ in $\mathbf{R}^2$ are linearly dependent.

4 Let 3 vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ in vector space V belong to the span of 2 vectors $\{\mathbf{a}, \mathbf{b}\}$ of this vector space, i.e. vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are expressible as linear combinations of vectors $\mathbf{a}$ and $\mathbf{b}$. Prove that vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ are linearly dependent.

5 Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a basis in 3-dimensional vector space V.
Show that all vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are not equal to zero.
Show that an arbitrary vector $\mathbf{x}$ can be expressed via this basis.

6 Let $\{\mathbf{a}, \mathbf{b}\}$ be two vectors in the vector space V such that
i) these vectors are linearly independent
ii) for an arbitrary vector $\mathbf{x} \in V$ vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{x}\}$ are linearly dependent.
What is a dimension of the vector space V?
Is an ordered set $\{\mathbf{a}, \mathbf{b}\}$ a basis in the vector space V?

7 Consider the vector space V of all polynomials of order ≤ 2:

$$V = \{ax^2 + bx + c, a, b, c, \in \mathbf{R}\}\,.$$

a) Show that the polynomials $\{1, x, x^2\}$ are linearly independent.
b) Show that for arbitrary p, q, the polynomials $\{1, x, x^2 + px + q\}$ are linearly independent.
c)† Show that arbitrary four polynomials in this space are linearly dependent.
What is a dimension of this vector space?

8 Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a basis of 3-dimensional vector space V. Is a set of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ a basis of V in the case if
a) $\mathbf{e}'_1 = \mathbf{e}_2$, $\mathbf{e}'_2 = \mathbf{e}_1$, $\mathbf{e}'_3 = \mathbf{e}_3$;
b) $\mathbf{e}'_1 = \mathbf{e}_1$, $\mathbf{e}'_2 = \mathbf{e}_1 + 3\mathbf{e}_3$, $\mathbf{e}'_3 = \mathbf{e}_3$;
c) $\mathbf{e}'_1 = \mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{e}'_2 = 3\mathbf{e}_1 - 3\mathbf{e}_2$, $\mathbf{e}'_3 = \mathbf{e}_3$;
d) $\mathbf{e}'_1 = \mathbf{e}_2$, $\mathbf{e}'_2 = \mathbf{e}_1$, $\mathbf{e}'_3 = \mathbf{e}_1 + \mathbf{e}_2 + \lambda\mathbf{e}_3$ (where λ is an arbitrary coefficient)?