Homework 2

1 Write down the standard Euclidean metric on $\mathbf{E}^2$ in polar coordinates

2 Consider the Riemannian metric on the circle of the radius R induced by the Euclidean metric on the ambient plane.

a) Express it using polar angle as a coordinate on the circle.

b) Express the same metric using stereographic coordinate obtained by stereographic projection of the circle on the line, passing through its centre.

3 Consider the Riemannian metric on the sphere of the radius R induced by the Euclidean metric on the ambient 3-dimensional space.

a) Express it using spherical coordinates on the sphere.

b) Express the same metric using stereographic coordinates u, v obtained by stereographic projection of the sphere on the plane, passing through its centre.

4 a) Let (u, v) be local coordinates on 2-dimensional Riemannian manifold (M, G) such that Riemannian metric has an appearance $G = du^2 + u^2 dv^2$ in these coordinates. Show that there exist local coordinates x, y such that such that $G = dx^2 + dy^2$.

b) Let (u, v) be local coordinates on 2-dimensional Riemannian manifold (M, G) such that Riemannian metric has an appearance $G = du^2 + \sin^2 u dv^2$ in these coordinates.

Do there exist coordinates x, y such that $G = dx^2 + dy^2$?

5 Consider an upper half-plain $(y > 0)$ in $\mathbf{R}^2$ equipped with Riemannian metric

$$G = \sigma(x, y)(dx^2 + dy^2)\,, \tag{1}$$

a) Show that $\sigma > 0$,

Consider two vectors $\mathbf{A} = 2\partial_x$ and $\mathbf{B} = 12\partial_x + 5\partial_y$ attached at the point $(x, y) = (1, 2)$,

b) calculate the cosine of the angle between these vectors, and show that the answer does not depend on the choice of the function $\sigma(x, y)$.

c) Calculate the lengths of these vectors in the case if

$$\sigma = \frac{1}{y^2}\,, \qquad \text{(hyperbolic (Lobachevsky) metric)} \qquad (2)\,,$$

d) Calculate the length of the segments $x = a + t, y = b$, and $x = a, y = b + t$, $0 \leq t \leq 1$ if condition (2) is obeyed.

e) Consider two curves L_1 and L_2 in upper half-plane (1) such that

$$L_1 = \begin{cases} x = f(t) \\ y = g(t) \end{cases}, \quad \text{and } L_2 \begin{cases} x = g(t) \\ y = f(t) \end{cases}, \qquad 0 \leq t \leq 1\,,$$

where $f(t), g(t)$ are arbitrary functions $(f(t) > 0, g(t) > 0)$.

Show that these curves have the same length in the case if $\sigma(x,y) = \frac{1}{(1+x^2+y^2)^2}$.
(*Exam quesstion*)

6 Consider half-plane model of 2-dimensional hyperbolic (Lobachevsky plane): metric

$$G = \frac{dx^2 + dy^2}{y^2}\,.$$

(see also questions 5c) and 5d) above).

a) Show that coordinates u, v such that

$$\begin{cases} x = u^2 - v^2 \\ y = 2uv \end{cases},$$

are conformal coordinates[1].

b) Are polar coordinates r, φ, $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$ conformal coordinates?

[1] coordinates u, v are conformall (isothemric) if Riemannin metric has appearance $\sigma(u,v)(du^2+dv^2)$ in these coordinates. E.g. coordinates in (1) are conformall coordinates.