Homework 7

1 Show that great circles are geodesics on sphere. Do it

a) using the fact that for geodesic, acceleration is orthogonal to the surface.

b*) using straightforwardly equations for geodesics

c) using the fact that geodesic is shortest.

2 Consider in $\mathbf{E}^3$ a vector $\mathbf{X} = \frac{\partial}{\partial y}$ attached at the point $\mathbf{p}$: $\left(x = \frac{R}{2}, y = 0, z = \frac{\sqrt{3}R}{2}\right)$.

Consider also a sphere $x^2 + y^2 + z^2 = R^2$ in $\mathbf{E}^3$ and the following two curves: a curve C_1 which is the intersections of this sphere with plane $y = 0$ and a curve C_1 which is the intersections of this sphere with the plane $z = \frac{\sqrt{3}R}{2}$. Both curves C_1, C_2 pass through the point $\mathbf{p}$.

Show that the vector $\mathbf{X}$ is tangent to the sphere and express this vector in spherical coordinates.

* Describe the parallel transport of the vector $\mathbf{X}$ along these closed curves.

What will be the result of parallel transport of the vector $\mathbf{X}$ along these closed curves?

3 Show that vertical lines $x = a$ are geodesics (non-parameterised) on Lobachevsky plane [1)].

4 Show that the following transformations are isometries of Lobachevsky plane:

a) horizontal translation $\mathbf{r} \to \mathbf{r} + \mathbf{a}$ where $\mathbf{a} = (a, 0)$,

b) homothety: $\mathbf{r} \to \lambda\mathbf{r}$ $(\lambda > 0)$,

* c) inversion with the centre at the points of the absolute (the line $x = 0$):

$$\mathbf{r} \to \mathbf{a} + \frac{\mathbf{r} - \mathbf{a}}{|\mathbf{r} - \mathbf{a}|^2} \text{ where } \mathbf{a} = (a, 0): \quad \begin{cases} x' = a + \frac{x-a}{(x-a)^2+y^2} \\ y' = \frac{y}{(x-a)^2+y^2} \end{cases}.$$

5* Show that upper arcs of semicircles $(x - a)^2 + y^2 = R^2, y > 0$ are (non-parametersied) geodesics.

6 Let ABC be triangle formed by geodesics on the sphere of the radius R. Express the area of this triangle via its angles.

Do the previous exercise for the triangle on the Lobachevsky plane.

7 Let $\mathbf{X}(t)$ be parallel transport of the vector $\mathbf{X}$ along the curve on the surface M embedded in $\mathbf{E}^3$, i.e. $\nabla_{\mathbf{v}}\mathbf{X} = 0$, where $\mathbf{v}$ is a velocity vector of the curve C and ∇ Levi-Civita connection (induced connection) on the surface. Compare this condition $\nabla_{\mathbf{v}}\mathbf{X} = 0$ (for internal observer) with the condition for external observe that for the vector $\mathbf{X}(t)$ $\frac{d\mathbf{X}(t)}{dt}$ is orthogonal to the surface[2)].

[1)] We consider here the realisation of Lobachevsky plane (hyperbolic plane) as upper half of Euclidean plane $\{(x, y): y > 0\}$ with the metric $G = \frac{dx^2+dy^2}{y^2}$. The line $x = 0$ is called *absolute*.

[2)] We defined parallel transport in Geometry course using the second condition

8* Let $\mathbf{r} = \mathbf{r}(t)$ be an arbitrary geodesic on the Riemannian manifold. Show that magnitudes $I = g_{ik}\dot{x}^i\dot{x}^k$ is preserved along geodesic.

Let $\mathbf{r} = \mathbf{r}(t)$ be an arbitrary geodesic on the sphere. Show that magnitudes $I = \sin^2\theta\dot{\varphi}$ and $E = \frac{\sin^2\theta\dot{\varphi}^2+\dot{\theta}^2}{2}$ are preserved along geodesics.

Let $\mathbf{r} = \mathbf{r}(t)$ be an arbitrary geodesic on Lobachevsky plane. Show that magnitudes $I = \frac{v_x^2}{y^2}$ and $E = \frac{v_x^2+v_y^2}{2y^2}$ are preserved along geodesics.