Solutions of Homework 7

1 *A point moves in $\mathbf{E}^2$ along a parabola with the law of motion $x = t$, $y = t - t^2$, $-\infty < t < \infty$. Find the velocity and acceleration vectors. Find the points of the parabola where the angle between velocity and acceleration vectors is acute. Find the points where speed attains its minimum value.*

Calculate velocity and acceleration vectors

$$\mathbf{v} = \mathbf{r}_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} 1 \\ 1-2t \end{pmatrix}, \ \mathbf{a} = \mathbf{r}_{tt} = \begin{pmatrix} x_{tt} \\ y_{tt} \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

The scalar product of these vectors is equal to $(\mathbf{v}, \mathbf{a}) = |\mathbf{v}||\mathbf{a}| \cos\alpha = v_x a_x + v_y a_y = -2(1-2t) = 4t - 2$, where α is angle between velocity and acceleration vectors.

Speed is increasing $\Leftrightarrow$ angle α is acute $\Leftrightarrow$ $(\mathbf{v}, \mathbf{a}) > 0 \Leftrightarrow 4t - 2 > 0 \Leftrightarrow t > \frac{1}{2}$.

Speed is decreasing $\Leftrightarrow$ angle α is obtuse $\Leftrightarrow$ $(\mathbf{v}, \mathbf{a}) < 0 \Leftrightarrow 4t - 2 < 0 \Leftrightarrow t < \frac{1}{2}$.

Speed is attains its minimum when $t = \frac{1}{2}$ (at vertex of parabola) At these points acceleration is orthogonal to velocity vector and scalar product is equal to zero.

2 *A point moves in $\mathbf{E}^2$ along an ellipse with the law of motion $x = a\cos t$, $y = b\sin t$, $0 \leq t < 2\pi$, $(0 < b < a)$. Find the velocity and acceleration vectors. Find the points of the ellipse where the angle between velocity and acceleration vectors is acute. Find the points where speed attains its maximum value.*

Calculate velocity and acceleration vectors

$$\mathbf{v} = \mathbf{r}_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} -a\sin t \\ b\cos t \end{pmatrix}, \ \mathbf{a} = \mathbf{r}_{tt} = \begin{pmatrix} x_{tt} \\ y_{tt} \end{pmatrix} = \begin{pmatrix} -a\cos t \\ -b\sin t \end{pmatrix}.$$

We see that acceleration is collinear to $\mathbf{r}$: $\mathbf{a} = -\mathbf{r}$.

The scalar product of these vectors is equal to $(\mathbf{v}, \mathbf{a}) = |\mathbf{v}||\mathbf{a}| \cos\alpha = v_x a_x + v_y a_y = (a^2 - b^2)\sin t \cos t$, where α is angle between velocity and acceleration vectors.

Speed is increasing $\Leftrightarrow$ angle α is acute $\Leftrightarrow$ $(\mathbf{v}, \mathbf{a}) > 0 \Leftrightarrow \sin t\cos t > 0 \Leftrightarrow 0 \leq t \leq \pi/2$ or $\pi < t < 3\pi/2$.

Speed is decreasing $\Leftrightarrow$ angle α is obtuse $\Leftrightarrow$ $(\mathbf{v}, \mathbf{a}) < 0 \Leftrightarrow \sin t\cos t < 0 \Leftrightarrow \pi/2 \leq t \leq \pi$ or $3\pi/2 < t < 2\pi$.

Speed is attains its maximum when $t = \frac{\pi}{2}, \frac{3\pi}{2}$ and speed attains its minimum when $t = 0, \pi$.

(At these points acceleration is orthogonal to velocity vector and scalar product is equal to zero).

3 *Find a natural parameter for the following interval of the straight line*

$$C\colon \begin{cases} x = t \\ y = 2t + 1 \end{cases}, \ 0 < t < \infty$$

We know that natural parameter $s(t)$ measures the length of the arc of the curve between a point $\mathbf{r}(t)$ and initial point. Take a point $t = 0$: $x = 0, y = 1$ as initial point.

$$s(t) = \text{length of the interval of the line between point } (0,1) \text{ and point } (t, 2t+1)$$

If α is angle between the line and x-axis then $s(t) = t/\cos\alpha$. $\cos\alpha = \frac{1}{\sqrt{1+2^2}} = \frac{1}{\sqrt{5}}$. Hence $s(t) = t\sqrt{5}$. One comes to the same answer making straightforward integration:

$$s(t) = \int_0^t \sqrt{x_\tau^2 + y_\tau^2}\, d\tau = \int_0^t \sqrt{1 + 2^2}\, d\tau = t\sqrt{5}\,.$$

If we take another point as a initial point then natural parameter will change on a constant: E.g. if we take an initial point $(1,3)$ $(t = 1)$ then a new natural parameter:

$$s'(t) = \int_1^t \sqrt{x_\tau^2 + y_\tau^2}\, d\tau = \int_1^t \sqrt{5}\, d\tau = \sqrt{5}(t-1) = s(t) - \sqrt{5}\,.$$

Usually if a curve $\mathbf{r}(t)$ is given for parameters $t \in [t_1, t_2]$ one takes as initial a point $\mathbf{r}(t_1)$ and

$$s(t) = \int_0^t \sqrt{x_\tau^2 + y_\tau^2} d\tau \,.$$

4 *Consider the following curve (a helix):*

$$\mathbf{r}(t)\colon \quad \begin{cases} x(t) = R\cos t \\ y(t) = R\sin t \\ z(t) = ct \end{cases} .$$

Find a natural parameter of this curve. What can you say about the acceleration of this curve? (In particular show that the tangential acceleration is equal to zero.)

Calculate a normal parameter $s(t)$ = length of the arc of the helix from the point $\mathbf{r}(t_1)$ till point $\mathbf{r}(t)$. Take $t_1 = 0$ One can calculate the length taking integral

$$s(t) = \int_0^t |\mathbf{v}(\tau)| d\tau = \int_0^t \sqrt{x_t^2 + y_t^2 + z_t^2} dt.$$

On the other hand we note that speed is constant $|\mathbf{v}|^2 = (\mathbf{v}, \mathbf{v}) = R^2 + c^2$, i.e. $|\mathbf{v}| = \sqrt{x_t^2 + y_t^2 + z_t^2} = \sqrt{R^2 + c^2}$. Thus we do not need to calculate the integral: natural parameter $s(t) = |\mathbf{v}|t = \sqrt{R^2 + c^2}t$.

Calculate velocity and acceleration vectors:

$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \begin{pmatrix} -R\sin t \\ R\cos t \\ c \end{pmatrix} , |\mathbf{v}| = \sqrt{R^2 + c^2} \,, \mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \frac{d^2\mathbf{r}(t)}{dt^2} = \begin{pmatrix} -R\cos t \\ R\sin t \\ 0 \end{pmatrix} , |\mathbf{a}| = R \,.$$

The scalar (inner) product of velocity and acceleration vectors is equal to zero: $(\mathbf{v}(t), \mathbf{a}(t)) = 0$, i.e. these vectors are orthogonal. Hence the projection of acceleration vector on velocity vector (tangential vector to the curve) is equal to zero. Thus tangential acceleration is equal to zero. (Note that speed $|\mathbf{v}|$ is constant. This also implies that tangential acceleration is equal to zero.)

Remark One can see that helix belongs to the surface of cylinder $x^2 + y^2 = R^2$ and acceleration is orthogonal to surface of the cylinder. This remark is essential to understand the geometry of cylinder (see the next homework).

5 *Calculate the curvature of the parabola $x = t, y = at^2$ $(a > 0)$ at an arbitrary point.*

†*(not compulsory problem) Let s be a natural parameter on this parabola. Show that the integral $\int_{-\infty}^{\infty} k(s) ds$ of the curvature $k(s)$ over the parabola is equal to π.*

Sure it is not practical to use the results of previous exercise for calculating the curvature.

It is much more practical to use the formula for curvature in arbitrary parameterisation:

$$k(t) = \frac{|x_t y_{tt} - y_t x_{tt}|}{(x_t^2 + y_t^2)^{3/2}} = \frac{|1 \cdot 2a - 2at \cdot 0|}{(1^2 + (2at)^2)^{3/2}} = \frac{2a}{(1 + 4a^2t^2)^{3/2}}, \quad (a > 0) \,.$$

We see that the curvature at the point (t, at^2) is equal to $k(t) = \frac{2a}{(1+a^2t^2)^{3/2}}$ $(a > 0)$.

(Curvature is positive by definition. If $a < 0$, then $k(t) = \frac{-2a}{(1+4a^2t^2)^{3/2}}$).

† To calculate $\int k(s) ds$, where s is natural parameter, better to return to an arbitrary parameterisation:

$$\int k(s) ds = \int k(s(t)) \frac{ds(t)}{dt} dt = \int k(t) |\mathbf{v}(t)| dt$$

One can see that

$$k(t)|\mathbf{v}(t)| = \frac{d}{dt}\varphi(t),$$

where $\varphi(t)$ is the angle between the velocity vector and a given direction (see in detail appendix to lecture notes). One can see this also by straightforward calculation:

$$\pm k(t)|\mathbf{v}(t)| = \frac{x_t y_{tt} - x_{tt} y_t}{x_t^2 + y_t^2} = \frac{d}{dt}\arctan\frac{y_t}{x_t}$$

Hence $\int k(s)ds = \varphi|_{-\infty}^{+\infty} = \pi$.

6 Consider the parabola

$$\mathbf{r}(t)\colon \begin{cases} x = v_x t \\ y = v_y t - \frac{gt^2}{2} \end{cases}.$$

(It is path of the point moving under the gravity force with initial velocity $\mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$.) Calculate the curvature at the vertex of this parabola.

To calculate the curvature one has to perform the same calculations as in the exercise 5:

$$k(t) = \frac{|x_t y_{tt} - y_t x_{tt}|}{(x_t^2 + y_t^2)^{3/2}} = \frac{|v_x \cdot (-g)|}{((v_x^2 + (v_y - gt)^2))^{3/2}}$$

In the vertex of this parabola vertical component of velocity is equal to zero. Hence curvature at the vertex is equal to

$$k = \frac{|v_x \cdot (-g)|}{((v_x^2 + (v_y - gt)^2))^{3/2}}|_{v_y = gt} = \frac{g}{v_x^2}$$

The answer in fact immediately follows from considerations of classical mechanics: If curvature in the vertex is equal to k then radius of the circle which has second order touching is equal to $R = \frac{1}{k}$ and centripetal acceleration is equal to $a = \frac{v_x^2}{R}$. On the other hand $a = g$. Hence $R = \frac{v_x^2}{g}$ and $k = \frac{g}{v_x^2}$.

Remark Note that $v_x = \sqrt{\frac{g}{k}} = \sqrt{Rg}$. if we take $R \approx 6400km$ (radius of the Earth) then $v_x \approx 8km\,\mathrm{sec}$— if a point has this velocity then it will become satellite of the Earth (we ignore resistance of atmosphere).

7 *Consider the ellipse $x = a\cos t, y = b\sin t$ $(a, b > 0,\ 0 \le t < 2\pi)$ in $\mathbf{E}^2$. Calculate the curvature $k(t)$ at an arbitrary point of this ellipse.*

† *Find the radius of a circle which has second order touching with the ellipse at the point $(0, b)$.*

For the ellipse $\mathbf{r}(t)\colon x = a\cos t, y = b\sin t$ velocity vector $\mathbf{v}(t) = (-a\sin t, b\cos t)$, acceleration vector $\mathbf{a}(t) = (-a\cos t, -b\sin t)$ and for curvature

$$k(t) = \frac{|v_x a_y - v_y a_x|}{(v_x^2 + v_y^2)^{3/2}} = \frac{ab\sin^2 t + ab\cos^2 t}{(a^2\cos^2 t + b^2\sin^2 t)^{\frac{3}{2}}} = \frac{ab}{(a^2\cos^2 t + b^2\sin^2 t)^{\frac{3}{2}}}.$$

† The value of parameter t at the point $(0, b)$ is $t = \frac{\pi}{2}$. The curvature of the ellipse at the point $(0, b)$ is equal to $k(t)|t = \frac{\pi}{2} = \frac{ab}{(b^2)^{\frac{3}{2}}} = \frac{a}{b^2}$. The circle has the same curvature $k = \frac{1}{R}$. Hence its radius is equal to $\frac{b^2}{a}$.

8 Find a curvature at an arbitrary point of the helix considered in Exercise 3.

In this exercise we have to calculate curvature of the curve in three-dimensional Euclidean space. So we need to use the formula

$$k(t) = \frac{\text{Area of parallelogram formed by vectors } \mathbf{v}, \mathbf{a}}{|\mathbf{v}|^3} = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$$

We already calculated velocity and acceleration vectors for helix (see exercise 3)

Acceleration is orthogonal to velocity vector. Hence

$$|\mathbf{v} \times \mathbf{a}| = |\mathbf{v}| \cdot |\mathbf{a}| = R\sqrt{R^2 + c^2}\,.$$

and curvature is equal to

$$k = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{|v||\mathbf{a}|}{|\mathbf{v}|^3} = \frac{|\mathbf{a}|}{|\mathbf{v}|^2} = \frac{R}{R^2 + c^2} \qquad (*).$$

Remark Note that $k \to \frac{1}{R}$ if $c \to 0$ and $k \to 0$ if $c \to \infty$.

Remark Note that we could calculate curvature using the formula $k = \frac{|\mathbf{a}_\perp|}{|\mathbf{v}|^2}$. We already know that tangential acceleration is equal to zero, hence $\mathbf{a} = \mathbf{a}_{norm}$ and

$$k = \frac{|\mathbf{a}_\perp|}{|\mathbf{v}|^2} = k = \frac{|\mathbf{a}|}{|\mathbf{v}|^2}$$

We come to the formula (*).