Homework 6

1 Calculate the derivatives of the functions $f = x^2+y^2$, $g = y^2-x^2$ and $h = q\log|r| = q\log\left(\sqrt{x^2+y^2}\right)$ (q is a constant) along vector fields $\mathbf{A} = x\partial_x + y\partial_y$ and $\mathbf{B} = x\partial_y - y\partial_x$

a) calculating directional derivatives $\partial_{\mathbf{A}} f, \partial_{\mathbf{A}} g, \partial_{\mathbf{A}} h$, $\partial_{\mathbf{B}} f, \partial_{\mathbf{B}} g, \partial_{\mathbf{B}} h$

b) calculating $df(\mathbf{A}), dg(\mathbf{A}), dh(\mathbf{A}), df(\mathbf{B}), dg(\mathbf{B}), dh(\mathbf{B})$.

2 Perform the calculations of the previous exercise in polar coordinates.

3 Calculate the integrals of the form $\omega = \sin y\, dx$ over the following three curves. Compare answers.

C_1: $\mathbf{r}(t)$ $\begin{cases} x = 2t^2 - 1 \\ y = t \end{cases}$, $0 < t < 1$, $\qquad$ C_2: $\mathbf{r}(t)$ $\begin{cases} x = 8t^2 - 1 \\ y = 2t \end{cases}$, $0 < t < 1/2$,

C_3: $\mathbf{r}(t)$ $\begin{cases} x = \cos 2t \\ y = \cos t \end{cases}$, $0 < t < \frac{\pi}{2}$

4 Calculate the integral of the form $\omega = e^{-y}dx + \sin x dy$ over the segment of straight line which connects the points $A = (1,1)$, $B = (2,3)$. At what extent an answer depends on a chosen parameterisation?

5 Calculate the integral of the form $\omega = xdy$ over the following curves

a) upper arc of the unit circle which passes through the point $A = (1,0)$ and the point $B = (0,1)$.

b) closed curve $x^2 + y^2 = 2x$

c) arc of the ellipse $x^2 + y^2/9 = 1$ defined by the condition $y \geq 0$.

At what extent the answer depends on the choice of parameterisation?

Exact forms

6 Calculate the integral $\int_C \omega$ where $\omega = xdx + ydy$ and C is

a) the straight line segment $x = t, y = 1 - t$, $0 \leq t \leq 1$

b) the segment of parabola $x = t$, $y = 1 - t^n$, $0 \leq t \leq 1$, $n = 2, 3, 4, \ldots$

c) for **an arbitrary** curve starting at the point $(0,1)$ and ending at the point $((1,0)$.

7 Show that the form 1-form $\omega = 3x^2ydx + x^3dy$ is an exact 1-form.

a) Calculate integral of this form over the curves considered in exercise 5).

b) Write down the 1-form ω in polar coordinates.

8. Consider 1-forms

a) xdx , b) xdy c) $xdx + ydy$, d)$xdy + ydx$, e) $xdy - ydx$

f) $x^4dy + 4x^3ydx$, g) $xdy + ydx + dz$, h) $xdy - ydx + dz$.

a) Show that 1-forms a), c), d), f) and g) are exact forms

b) Why 1-forms b), e) and and h) are not exact?

All the exercises below are not compulsory

9† Consider one-form

$$\omega = \frac{xdy - ydx}{x^2 + y^2} \tag{1}$$

This form is defined in $\mathbf{E}^2 \setminus 0$.

Calculate differential of this form.

Write down this form in polar coordinates

Find a function f such that $\omega = df$.

Is this function defined in the same domain as ω?

10† Calculate the integral of the form $\omega = \frac{xdy-ydx}{x^2+y^2}$ over the curves

a) circle $x^2 + y^2 = 1$

b) circle $(x-3)^2 + y^2 = 1$

c) ellipse $\frac{x^2}{9} + \frac{x^2}{16} = 1$

11† What values can take the integral $\int_C \omega$ if C is an arbitrary curve starting at the point $(0,1)$ and ending at the point $((1,0)$ and $\omega = \frac{xdy-ydx}{x^2+y^2}$.

12† Let $\omega = a(x,y)dx + b(x,y)dy$ be a closed form in $\mathbf{E}^2$, $d\omega = 0$.

Consider the function

$$f(x,y) = x\int_0^1 a(tx,ty)dt + y\int_0^1 b(tx,ty)dt \tag{2}$$

Show that

$$\omega = df\,.$$

This proves that an arbitrary closed form in $\mathbf{E}^2$ is an exact form.

Why we cannot apply the formula (2) to the form ω defined by the expression (1)?