Geometry of diff.equations and Monge cone

Consider $2n+1$-dimension space $J^1(M)$ of first jets of n-dimensional manifold M. Let

$$\mathcal{E}\colon \quad F(u, p_i, q^j) = 0$$

be hypersurface in $J^1(M)$, the differential equation.

Suppose that points of $\mathcal{E}$ are regular.

DEFINITION A point $\mathbf{p} \in \mathcal{E}$ is called *non- characteristic* if tangent hyperplane $T_{\mathbf{p}}M$ and the *contact hyperplane* $\mathcal{C}_{\mathbf{p}}$ (contact hyperplane consists of vectors which are vanished by the contact 1-form $\omega = p_i dq^i - du$) are transversal to each other, i.e. tangent vectors to $\mathcal{E}$ at the point $\mathbf{p}$ and vectors in contact plane span all the tangent space:

$$T_{\mathbf{p}}\mathcal{E} \oplus \mathcal{C}_{\mathbf{p}} = T_{\mathbf{p}}(J^1(M)) \Leftrightarrow \dim \mathcal{C}_{\mathbf{p}} \cap T_{\mathbf{p}}\mathcal{E} = 2n - 1\,.$$

and this is is equivalent to the condition that contact form ω and the form dF are not collinear at the point $\mathbf{p}$ *.

DEFINITION The intersection of contact and tangent planes is called *characteristic plane.*

We denote characteristic plane by $\mathcal{C}(\mathcal{E})_{\mathbf{p}}$

The form $dp_1 \wedge dq^1 + \ldots + dp_n \wedge dq^n$ is non-degenerate on contact hyperplane $\mathcal{C}_{\mathbf{p}}$, and on $2n-1$-dimensional plane $\Pi_{\mathbf{p}} \subset \mathcal{C}_{\mathbf{p}}$. It defines the characteristic direction, the direction that is symplectoorthogonal to all the vectors of the plane $\Pi_{\mathbf{p}}$.

One can see that this direction can be defined by the infinitesimal contact symmetry:

$$\mathbf{X}_F = \frac{\partial F}{\partial p^i}\frac{\partial}{\partial q^i} - \frac{\partial F}{\partial q_i}\frac{\partial}{\partial p_i} - p_i\frac{\partial F}{\partial u}\frac{\partial}{\partial p_i} + p_i\frac{\partial F}{\partial p_i}\frac{\partial}{\partial u} - F\frac{\partial}{\partial u}$$

The contact distribution $\mathcal{C}$, (the distribution of contact planes) defines

Now return to differential equation defined by non-charateristic hypersurface $\mathcal{E}$.

Solution of this equation is maximal integral manifold of distribution $\mathcal{C}(\mathcal{E})$

Let N be n-dimensional manifold s graph of function $u = \varphi(x)$ such that it is the solution of differential equation defined by the surface $\mathcal{E}$, i.e. $N \subset \mathcal{E}$,and for every point $\mathbf{p} \in N$, tangent space $T_{\mathbf{p}}N$ belongs to the characteristic plane $\mathcal{C}(\mathcal{E})_{\mathbf{p}}$: all pints of N belongs to $\mathcal{E}$ and all tangent vectors of N belong to the $T\mathcal{E}$ and to contact plane**. Of course one may consider also generalised solutions......

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* One can easy to check this choosing covectors ω and dF as elemnets of basis.

** Belonging to contact distribution is auto matical since every vector tangent to `manifold` belongs to contact plane

`Boundary condition`: Take the $n-1$-dimensional surface,and Γ on $\mathcal{E}$. Emit from the points of this surface the $n-1$-parametric family of curves which go in characteristic direction. We come locally in the vicinity of the surface Γ to n-dimensional surface, it will be solution.

Example Consider dfferential equation

$$u_x^2 + u_y^2 = 1\,, \quad F(u,p,q,x,y) = p^2 + q^2 - 1\,.$$

in the space (u,p,q,x,y) $(p = u_x, q = u_y)$ consider surface

$$\Gamma\colon \quad \begin{cases} u(\xi) = \varphi(\xi) \\ p(\xi) = \sqrt{1 - \varphi'^2(\xi)} \\ q(\xi) = \varphi'(\xi) \\ x(\xi) = 0 \\ y(\xi) = \xi \end{cases}$$

This surface is generated by the function $\varphi(y)$. It belongs to the differential equation $p^2 + q^2 = 1$. Consider characteristic vector field

$$\frac{1}{2} X_F = p\partial_p + q\partial_q + \partial_u$$

This vector field emit the 1-parametric family of the curves which are solutions of characteristic equations

$$\begin{cases} \dot u = 1 \\ \dot p = 0 \\ \dot q = 0 \\ \dot x = p \\ \dot y = q \end{cases} \text{with boundary conditions} \begin{cases} u(\xi,\tau)\big|_{\tau=0} = \varphi(\xi) \\ p(\xi,\tau)\big|_{\tau=0} = \sqrt{1-\varphi'^2(\xi)} \\ q(\xi,\tau)\big|_{\tau=0} = \varphi'(\xi) \\ x(\xi,\tau)\big|_{\tau=0} = 0 \\ y(\xi,\tau)\big|_{\tau=0} = \xi \end{cases}$$

$$\begin{cases} u(\xi,\tau) = \varphi(\xi) + \tau \\ p(\xi,\tau) = \sqrt{1-\varphi'^2(\xi)} \\ q(\xi,\tau) = \varphi'(\xi) \\ x(\xi,\tau) = \tau\sqrt{1-\varphi'^2(\xi)} \\ y(\xi,\tau) = \xi + \tau\phi'(\xi) \end{cases}$$

The solution is