Homework 2. Solutions

1

a) Show that $(\mathbf{x},\mathbf{y})=x^1y^1+x^2y^2+x^3y^3$ *defines a scalar product in* $\mathbf{R}^3$.

b) Show that $\langle\mathbf{x},\mathbf{y}\rangle=x^1y^1+x^2y^2$ *does not define a scalar product in* $\mathbf{R}^3$.

c) Show that $(\mathbf{x},\mathbf{y})=x^1y^1+x^2y^2-x^3y^3$ *does not define a scalar product in* $\mathbf{R}^3$.

d) Show that $(\mathbf{x},\mathbf{y})=x^1y^1+3x^2y^2+5x^3y^3$ *defines a scalar product in* $\mathbf{R}^3$.

e) Show that $(\mathbf{x},\mathbf{y})=x^1y^2+x^2y^1+x^3y^3$ *does not define a scalar product in* $\mathbf{R}^3$.

f †*) Find necessary and sufficient conditions for entries* a,b,c *of symmetrical matrix* $\begin{pmatrix} a & b\\ b & c\end{pmatrix}$ *such that the formula*

$$(\mathbf{x},\mathbf{y})=(\,x^1,x^2\,)\begin{pmatrix} a & b\\ b & c\end{pmatrix}\begin{pmatrix} y^1\\ y^2\end{pmatrix}=ax^1y^1+b(x^1y^2+x^2y^1)+cx^2y^2$$

defines scalar product in $\mathbf{R}^2$.

Recall that scalar product on a vector space V is a function $B(\mathbf{x},\mathbf{y})=(\mathbf{x},\mathbf{y})$ on a pair of vectors which takes real values and satisfies the the following conditions:

1) $B(\mathbf{x},\mathbf{y})=B(\mathbf{y},\mathbf{x})$ (symmetricity condition)

2) $B(\lambda\mathbf{x}+\mu\mathbf{y},\mathbf{z})=\lambda B(\mathbf{x},\mathbf{z})+\mu B(\mathbf{y},\mathbf{z})$ (linearity condition (with respect to the first argument))

3) $B(\mathbf{x},\mathbf{x})\geq 0$, $B(\mathbf{x},\mathbf{x})=0\Leftrightarrow\mathbf{x}=0$ (positive-definiteness condition)

(The linearity condition with respect to the second argument follows from the conditions 2) and 1))

a) Check all these conditions for $(\mathbf{x},\mathbf{y})=x^1y^1+x^2y^2+x^3y^3$:

1) $(\mathbf{y},\mathbf{x})=y^1x^1+y^2x^2+y^3x^3=x^1y^1+x^2y^2+x^3y^3=(\mathbf{x},\mathbf{y})$. Hence it is symmetrical.

2) $(\lambda\mathbf{x}+\mu\mathbf{y},\mathbf{z})=(\lambda x^1+\mu y^1)z^1+(\lambda x^2+\mu y^2)z^2+(\lambda x^3+\mu y^3)z^3=$
$=\lambda(x^1z^1+x^2z^2+x^3z^3)+\mu(y^1z^1+y^2z^2+y^3z^3)=\lambda(\mathbf{x},\mathbf{y})+\mu(\mathbf{y},\mathbf{z})$. Hence it is linear.

3) $(\mathbf{x},\mathbf{x})=(x^1)^2+(x^2)^2+(x^3)^2\geq 0$. It is non-negative. If $\mathbf{x}=0$ then $(\mathbf{x},\mathbf{x})=0$. If $(\mathbf{x},\mathbf{x})=(x^1)^2+(x^2)^2+(x^3)^2=0$, then $x^1=x^2=x^3=0$, i.e. $\mathbf{x}=0$. This we proved positive-definiteness.

All conditions are checked. Hence $B(\mathbf{x},\mathbf{y})=x^1y^1+x^2y^2+x^3y^3$ is indeed a scalar product in $\mathbf{R}^3$

Remark Note that x^1,x^2,x^3—are components of the vector, do not be confused with exponents!

b) Show that $B(\mathbf{x},\mathbf{y})=x^1y^1+x^2y^2$ *does not define scalar product* in $\mathbf{R}^3$.

To see that the formula $(\mathbf{x},\mathbf{y})=x^1y^1+x^2y^2$ *does not* define scalar product check the condition 3) of positive-definiteness: $(\mathbf{x},\mathbf{x})=(x^1)^2+(x^2)^2$ may take zero values for $\mathbf{x}\neq 0$. E.g. if $\mathbf{x}=(0,0,-1)$ $(\mathbf{x},\mathbf{x})=0$, in spite of the fact that $\mathbf{x}\neq 0$. The condition 3) of positive-definiteness is not satisfied. Hence it is not scalar product.

c) Show that $B(\mathbf{x},\mathbf{y})=x^1y^1+x^2y^2-x^3y^3$ *does not define scalar product* in $\mathbf{R}^3$.

To see that the formula $(\mathbf{x},\mathbf{y})=x^1y^1+x^2y^2-x^3y^3$ *does not* define scalar product check the condition 3): $(\mathbf{x},\mathbf{x})=(x^1)^2+(x^2)^2-(x^3)^2$ may take negative values. E.g. if $\mathbf{x}=(0,0,-1)$ $(\mathbf{x},\mathbf{x})=-1<0$. The condition 3) of positive-definiteness is not satisfied. Hence it is not scalar product.

d) Now show that $(\mathbf{x},\mathbf{y})=x^1y^1+3x^2y^2+5x^3y^3$ is a scalar product in $\mathbf{R}^3$.

We need to check all the conditions above for scalar product for $(\mathbf{x},\mathbf{y})=x^1y^1+3x^2y^2+5x^3y^3$:

1) $(\mathbf{y},\mathbf{x})=y^1x^1+3y^2x^2+5y^3x^3=x^1y^1+3x^2y^2+5x^3y^3=(\mathbf{x},\mathbf{y})$. Hence it is symmetrical.

2) $(\lambda\mathbf{x}+\mu\mathbf{y},\mathbf{z})=(\lambda x^1+\mu y^1)z^1+3(\lambda x^2+\mu y^2)z^2+5(\lambda x^3+\mu y^3)z^3=$
$=\lambda(x^1z^1+3x^2z^2+5x^3z^3)+\mu(y^1z^1+3y^2z^2+5y^3z^3)=\lambda(\mathbf{x},\mathbf{y})+\mu(\mathbf{y},\mathbf{z})$. Hence it is linear.

3) $(\mathbf{x},\mathbf{x})=(x^1)^2+3(x^2)^2+5(x^3)^2\geq 0$. It is non-negative. If $\mathbf{x}=0$ then obviously $(\mathbf{x},\mathbf{x})=0$. If $(\mathbf{x},\mathbf{x})=(x^1)^2+3(x^2)^2+5(x^3)^2=0$, then $x^1=x^2=x^3=0$. Hence it is positive-definite.

All conditions are checked. Hence $(\mathbf{x},\mathbf{y}) = x^1y^1 + 3x^2y^2 + 5x^3y^3$ is indeed a scalar product in $\mathbf{R}^3$

e) Show that $B(\mathbf{x},\mathbf{y}) = x^1y^2 + x^2y^1 + x^3y^3$ *does not define scalar product* in $\mathbf{R}^3$.

To see that the formula $(\mathbf{x},\mathbf{y}) = x^1y^2 + x^2y^1 + x^3y^3$ *does not* define scalar product check the condition 3): $(\mathbf{x},\mathbf{x}) = 2x^1x^2 + (x^3)^2$ may take negative values. E.g. if $\mathbf{x} = (1,-1,0)$ $(\mathbf{x},\mathbf{x}) = -2 < 0$. The condition 3) of positive-definiteness is not satisfied. Hence it is not scalar product.

f) †)

The condition of linearity and symmetricity for the bilinear form

$$B(\mathbf{x},\mathbf{y}) = (\, x^1, x^2\,) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = ax^1y^1 + b(x^1y^2 + x^2y^1) + cx^2y^2$$

are evidently obeyed.

The general answer on this question is: symmetric matrix is positive-definite if and only if all principal minors are positive. For matrix under consideration it means that conditions $a > 0$ and $ac - b^2 > 0$ are necessary and sufficient conditions.

Give a proof for this special case.

Check the positive-definiteness condition.

For $\mathbf{x} = (1,0)$ $B(\mathbf{x},\mathbf{x}) = a$. Hence $a > 0$ is necessary condition. Now consider

$$B(\mathbf{x},\mathbf{x}) = a(x^1)^2 + 2bx^1x^2 + c(x^2)^2 = \frac{(ax^1 + bx^2)^2 + (ac - b^2)(x^2)^2}{a} \geq 0 \Leftrightarrow ac - b^2 \geq 0$$

We see that $B(\mathbf{x},\mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$ iff $a > 0$ and $(ac - b^2) > 0$.

2 *a) Let* $\mathbf{e},\mathbf{f}$ *and* $\mathbf{g}$ *be three vectors in* 3*-dimensional Euclidean space* $\mathbf{E}^3$ *such that all these vectors have unit length and they are pairwise orthogonal. Show explicitly that the ordered set of these vectors* $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$ *is a basis.*

b) Let $\mathbf{a},\mathbf{b}$ *and* $\mathbf{c}$ *be three vectors in* 3*-dimensional Euclidean space* $\mathbf{E}^3$ *such that vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *have unit length, and are orthogonal to each other and vector* $\mathbf{c}$ *has length* $\sqrt{3}$ *and it forms an angle* $\varphi = \arccos\frac{1}{\sqrt{3}}$ *with vectors* $\mathbf{a}$ *and* $\mathbf{b}$.

Show that an ordered set $\{\mathbf{a},\mathbf{b},\mathbf{c}-\mathbf{a}-\mathbf{b}\}$ *of vectors is an orthonormal basis in* $\mathbf{E}^3$.

a) The space is 3-dimensional. Hence to show that $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$ is a basis it suffices to show that vectors $(\mathbf{e},\mathbf{f},\mathbf{g})$ are linearly independent. Suppose $c_1\mathbf{e} + c_2\mathbf{f} + c_3\mathbf{g} = 0$. Take scalar product of this equation on the vector $\mathbf{e}$. Since vectors $\mathbf{e},\mathbf{f}$ and $\mathbf{g}$ have unit length and they are pairwise orthogonal then

$$(c_1\mathbf{e} + c_2\mathbf{f} + c_3\mathbf{g}, \mathbf{e}) = c_1(\mathbf{e},\mathbf{e}) + c_2(\mathbf{f},\mathbf{e}) + c_3(\mathbf{g},\mathbf{e}) = c_1\cdot 1 + c_2\cdot 0 + c_3\cdot 0 = c_1 = 0\,.$$

In the same way we prove that $c_2 = c_3 = 0$. Hence vectors $(\mathbf{e},\mathbf{f},\mathbf{g})$ are linearly independent.

b) Since vectors $\mathbf{a}$ and b have unit length and they are orthogonal to each other then $(\mathbf{a},\mathbf{a}) = (\mathbf{b},\mathbf{b}) = 1$ and $(\mathbf{a},\mathbf{b}) = 0$. Since angle φ between vectors $\mathbf{a}$ and $\mathbf{c}$ equals to $\arccos\frac{1}{\sqrt{3}}$ and length of vector $\mathbf{c}$ equals to $\sqrt{3}$ then

$$(\mathbf{a},\mathbf{c}) = |\mathbf{a}||\mathbf{c}|\cos\varphi = 1\cdot\cdot\sqrt{3}\cdot\frac{1}{\sqrt{3}} = 1.$$

Analogously $(\mathbf{b},\mathbf{c}) = 1$ too. Hence scalar product of vector $\mathbf{c}-\mathbf{a}-\mathbf{b}$ with vector $\mathbf{a}$ equals to $(\mathbf{c}-\mathbf{a}-\mathbf{b},\mathbf{a}) = 1 - 1 - 0 = 0$, i.e. vector $\mathbf{c}-\mathbf{a}-\mathbf{b}$ is orthogonal to the vector $\mathbf{a}$. In the same way we prove that vector $\mathbf{c}-\mathbf{a}-\mathbf{b}$ is orthogonal to the vector $\mathbf{b}$. Hence we proved that all vectors $\mathbf{a},\mathbf{b}$ and $\mathbf{c}-\mathbf{a}-\mathbf{b}$ are pairwise orthogonal to each other. To see that $\{\mathbf{a},\mathbf{b},\mathbf{c}-\mathbf{a}-\mathbf{b}\}$ is orthonormal basis it remains to prove that vector $\mathbf{c}-\mathbf{a}-\mathbf{b}$ is unit vector. This is the fact since

$$(\mathbf{c}-\mathbf{a}-\mathbf{b},\mathbf{c}-\mathbf{a}-\mathbf{b}) = (\mathbf{c},\mathbf{c}) + (\mathbf{a},\mathbf{a}) + (\mathbf{b},\mathbf{b}) - 2(\mathbf{c},\mathbf{a}) - 2(\mathbf{c},b) + 2(\mathbf{a},\mathbf{b}) = \sqrt{3}\cdot\sqrt{3} + 1 + 1 - 2\cdot 1 - 2\cdot 1 = 0. \blacksquare$$

3 *a) Show explicitly that matrix* $A_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$ *is an orthogonal matrix.*

b) Show explicitly that under the transformation $(\mathbf{e}'_1, \mathbf{e}'_2) = (\mathbf{e}_1, \mathbf{e}_2)\, A_\varphi$ *an orthonormal basis transforms to an orthonormal one.*

c) Show that for orthogonal matrix A_φ *the following relations are satisfied:*

$$A_\varphi^{-1} = A_\varphi^{T} = A_{-\varphi}\,, \qquad A_{\varphi+\theta} = A_\varphi \cdot A_\theta\ .$$

a) Check straightforwardly that $A_\varphi^T \cdot A = I$ (this is definition of orthogonal matrix):

$$A_\varphi^T \cdot A = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = \begin{pmatrix} \cos^2\varphi + \sin^2\varphi & -\cos\varphi\sin\varphi + \sin\varphi\cos\varphi \\ -\sin\varphi\cos\varphi + \cos\varphi\sin\varphi & \sin^2\varphi + \cos^2\varphi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \blacksquare$$

b) We have to check that scalar products $(\mathbf{e}'_1, \mathbf{e}'_1) = (\mathbf{e}'_2, \mathbf{e}'_2) = 1$ and $(\mathbf{e}'_1, \mathbf{e}'_2) = 0$. Calculate.

$$(\mathbf{e}'_1, \mathbf{e}'_1) = (\cos\varphi\mathbf{e}_1 + \sin\varphi\mathbf{e}_2, \cos\varphi\mathbf{e}_1 + \sin\varphi\mathbf{e}_2) = \cos^2\varphi(\mathbf{e}_1, \mathbf{e}_1) + 2\cos\varphi\sin\varphi(\mathbf{e}_1, \mathbf{e}_2) + \sin^2\varphi(\mathbf{e}_2, \mathbf{e}_2) =$$

$$\cos^2\varphi \cdot 1 + 2\cos\varphi\sin\varphi \cdot 0 + \sin^2\varphi \cdot 1 = 1\,.$$

$$(\mathbf{e}'_2, \mathbf{e}'_2) = (-\sin\varphi\mathbf{e}_1 + \cos\varphi\mathbf{e}_2, -\sin\varphi\mathbf{e}_1 + \cos\varphi\mathbf{e}_2) = \sin^2\varphi(\mathbf{e}_1, \mathbf{e}_1) - 2\cos\varphi\sin\varphi(\mathbf{e}_1, \mathbf{e}_2) + \cos^2\varphi(\mathbf{e}_2, \mathbf{e}_2) = 1\,,$$

and

$$(\mathbf{e}'_1, \mathbf{e}'_1) = (\cos\varphi\mathbf{e}_1 + \sin\varphi\mathbf{e}_2, -\sin\varphi\mathbf{e}_1 + \cos\varphi\mathbf{e}_2) = -\cos\varphi\sin\varphi(\mathbf{e}_1, \mathbf{e}_1) + (\cos^2\varphi - \sin^2\varphi)(\mathbf{e}_1, \mathbf{e}_2) + \sin\varphi\cos\varphi(\mathbf{e}_2, \mathbf{e}_2) = 0\,. \blacksquare$$

c) We have that $A_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$. Then calculate inverse matrix A_φ^{-1}. One can see that $A_\varphi^{T} = A_\varphi^{-1} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$, because $A_\varphi^{T} A_\varphi = I$ (see equation (1) above). On the other hand $\cos\varphi = \cos(-\varphi)$ and $\sin\varphi = -\sin(-\varphi)$. Hence

$$A_\varphi^{T} = A_\varphi^{-1} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} = \begin{pmatrix} \cos(-\varphi) & -\sin(-\varphi) \\ \sin(-\varphi) & \cos(-\varphi) \end{pmatrix} = A_{-\varphi}\,.$$

Now prove that $A_{\varphi+\theta} = A_\varphi \cdot A_\theta$:

$$A_\varphi \cdot A_\theta = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} (\cos\varphi\cos\theta - \sin\varphi\sin\theta) & -(\cos\varphi\sin\theta + \sin\varphi\cos\theta) \\ (\cos\varphi\sin\theta + \sin\varphi\cos\theta) & (\cos\varphi\cos\theta - \sin\varphi\sin\theta) \end{pmatrix} =$$

$$\begin{pmatrix} \cos(\varphi+\theta) & -\sin(\varphi+\theta) \\ \sin(\varphi+\theta) & \cos(\varphi+\theta) \end{pmatrix} = A_{\varphi+\theta}$$

Remark Geometrical meaning of this relation is that composition of “rotations” on angle φ and θ is “rotation” on angle $\varphi + \theta$.

4 *Let* $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ *be an orthonormal basis of Euclidean space* $\mathbf{E}^3$*. Consider the ordered set of vectors* $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ *which is expressed via basis* $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ *as in the exercise 7 of the Homework 1.*

Write down explicitly transition matrix from the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ *to the ordered set of the vectors* $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$*. What is the rank of this matrix? Is this matrix orthogonal?*

Find out is the ordered set of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ a basis in $\mathbf{E}^3$. Is this basis an orthonormal basis of $\mathbf{E}^3$?

(you have to consider all cases a),b) c) and d)).

Case a) The ordered set $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\} = \{\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_3\}$ is evidently orthonormal basis. Transition matrix $T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $(\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3) = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)T$. This matrix is non-degenerate, its rank is equal to 3 ($\det T = 1 \neq 0$). It is orthogonal because both bases are orthonormal.

Case b) The ordered set $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\} = \{\mathbf{e}_1, \mathbf{e}_1 + 3\mathbf{e}_3, \mathbf{e}_3\}$ is not a basis because vectors are linear dependent: $\mathbf{e}'_1 - \mathbf{e}'_2 + 3\mathbf{e}'_3 = 0$. Transition matrix $T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 1 \end{pmatrix}$, $(\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3) = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)T$. This matrix is degenerate, its rank ≤ 2. One can see it noting that rows are linear dependent or noting that $\det T = 0$. Vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ are linear dependent. On the other hand vectors $\{\mathbf{e}'_1, \mathbf{e}'_2\}$ are linear independent. Hence rank of the matrix T is equal to 2. This matrix is not orthogonal.

Case c) The ordered set $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\} = \{\mathbf{e}_1 - \mathbf{e}_2, 3\mathbf{e}_1 - 3\mathbf{e}_2, \mathbf{e}_3\}$ is not a basis because vectors are linear dependent: $3\mathbf{e}'_1 - \mathbf{e}'_2 = 0$.

One can see it also studying the transition matrix. Transition matrix $T = \begin{pmatrix} 1 & 3 & 0 \\ -1 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $(\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3) = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)T$. This matrix is degenerate, $\det T = 0$. (Its rank ≤ 2. On the other hand second and third row of this matrix are linear independent. Hence rank of the matrix T is equal to 2). This matrix is not orthogonal.

Case d)

The transition matrix from the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to the ordered triple $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\} = \{\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2 + \lambda\mathbf{e}_3\}$ is $T = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & \lambda \end{pmatrix}$, $(\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3) = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)T$

I-st case. $\lambda \neq 0$. The ordered set $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ is a basis because vectors are linear independent (see the exercise 3), This basis is not orthogonal, because the length of vector $\mathbf{e}'_3$ is not equal to 1 ($(\mathbf{e}'_3, \mathbf{e}'_3) = |\mathbf{e}'_3|^2 = 2 + \lambda^2$). This matrix is not orthogonal, because the new basis is not orthonormal.

II-nd case $\lambda = 0$. The ordered set $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ is not a basis because vectors are linear independent: $\mathbf{e}'_1 + \mathbf{e}'_2 - \mathbf{e}'_3 = 0$. The transition matrix T has rank less or equal to 2, because vectors are linear dependent. On the other hand vectors $\mathbf{e}'_1, \mathbf{e}'_2$ are linear independent. Hence the rank of the matrix is equal to 2.

7† Prove the Cauchy–Bunyakovsky–Schwarz inequality

$$(\mathbf{x}, \mathbf{y})^2 \leq (\mathbf{x}, \mathbf{x})(\mathbf{y}, \mathbf{y}),$$

where $\mathbf{x}, \mathbf{y}$ are arbitrary two vectors and $(\ ,\ )$ is a scalar product in Euclidean space.

Hint: For any two given vectors $\mathbf{x}, \mathbf{y}$ *consider the quadratic polynomial* $At^2 + 2Bt + C$ *where* $A = (\mathbf{x}, \mathbf{x})$, $B = (\mathbf{x}, \mathbf{y})$, $C = (\mathbf{y}, \mathbf{y})$. *Show that this polynomial has at most one real root and consider its discriminant.*

Consider quadratic polynomial $P(t) = \sum_{i=1}^n (tx^i + y^i)^2 = At^2 + 2Bt + C$, where $A = \sum_{i=1}^n (x^i)^2 = (\mathbf{x}, \mathbf{x})$, $B = \sum_{i=1}^n (x^i y^i) = (\mathbf{x}, \mathbf{y})$, $C = \sum_{i=1}^n (y^i)^2 = (\mathbf{y}, \mathbf{y})$. We see that equation $P(t) = 0$ has at most one root (and this is the case if only vector $\mathbf{x}$ is collinear to the vector $\mathbf{y}$). This means that discriminant of this equation is less or equal to zero. But discriminant of this equation is equal to $4B^2 - 4AC$. Hence $B^2 \leq AC$. It is just CBS inequality. ($(\mathbf{x}, \mathbf{y})^2 = (\mathbf{x}, \mathbf{x})(\mathbf{y}, \mathbf{y})$, i.e. discriminant is equal to zero $\Leftrightarrow$ vectors $\mathbf{x}$, $\mathbf{y}$ are colinear.