

Homework 5. Solutions

1 Calculate the Christoffel symbols of the canonical flat connection in \mathbf{E}^3 in

a) cylindrical coordinates ($x = r \cos \varphi, y = r \sin \varphi, z = h$),

b) spherical coordinates.

(For the case of sphere try to make calculations at least for components $\Gamma_{rr}^r, \Gamma_{r\theta}^r, \Gamma_{r\varphi}^r, \Gamma_{\theta\theta}^r, \dots, \Gamma_{\varphi\varphi}^r$)

Remark One can calculate Christoffel symbols using Levi-Civita Theorem (Homework 5). There is a third way to calculate Christoffel symbols: It is using approach of Lagrangian. This is may be the easiest and most elegant way. (see the Homework 6)

In cylindrical coordinates (r, φ, h) we have

$$\begin{cases} x = r \cos \varphi \\ y = r \sin \varphi \\ z = h \end{cases} \quad \text{and} \quad \begin{cases} r = \sqrt{x^2 + y^2} \\ \varphi = \arctan \frac{y}{x} \\ h = z \end{cases}$$

We know that in Cartesian coordinates all Christoffel symbols vanish. Hence in cylindrical coordinates (see in detail lecture notes):

$$\begin{aligned} \Gamma_{rr}^r &= \frac{\partial^2 x}{\partial^2 r} \frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial^2 r} \frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial^2 r} \frac{\partial r}{\partial z} = 0, \\ \Gamma_{r\varphi}^r &= \Gamma_{\varphi r}^r = \frac{\partial^2 x}{\partial r \partial \varphi} \frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial r \partial \varphi} \frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial r \partial \varphi} \frac{\partial r}{\partial z} = -\sin \varphi \cos \varphi + \sin \varphi \cos \varphi = 0, \\ \Gamma_{\varphi\varphi}^r &= \frac{\partial^2 x}{\partial^2 \varphi} \frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial^2 \varphi} \frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial^2 \varphi} \frac{\partial r}{\partial z} = -x \frac{x}{r} - y \frac{y}{r} = -r, \\ \Gamma_{rr}^\varphi &= \frac{\partial^2 x}{\partial^2 r} \frac{\partial \varphi}{\partial x} + \frac{\partial^2 y}{\partial^2 r} \frac{\partial \varphi}{\partial y} + \frac{\partial^2 z}{\partial^2 r} \frac{\partial \varphi}{\partial z} = 0, \\ \Gamma_{\varphi r}^\varphi &= \Gamma_{r\varphi}^\varphi = \frac{\partial^2 x}{\partial r \partial \varphi} \frac{\partial \varphi}{\partial x} + \frac{\partial^2 y}{\partial r \partial \varphi} \frac{\partial \varphi}{\partial y} + \frac{\partial^2 z}{\partial r \partial \varphi} \frac{\partial \varphi}{\partial z} = -\sin \varphi \frac{-y}{r^2} + \cos \varphi \frac{x}{r^2} = \frac{1}{r}, \\ \Gamma_{\varphi\varphi}^\varphi &= \frac{\partial^2 x}{\partial^2 \varphi} \frac{\partial \varphi}{\partial x} + \frac{\partial^2 y}{\partial^2 \varphi} \frac{\partial \varphi}{\partial y} + \frac{\partial^2 z}{\partial^2 \varphi} \frac{\partial \varphi}{\partial z} = -x \frac{-x}{r^2} - y \frac{y}{r^2} = 0. \end{aligned}$$

All symbols $\Gamma_{\cdot h}^\cdot, \Gamma_h^\cdot$ vanish

$$\Gamma_{rh}^r = \Gamma_{hr}^r = \Gamma_{hh}^r = \Gamma_{\varphi h}^r = \Gamma_{h\varphi}^r = \Gamma_{hr}^\varphi = \Gamma_{h\varphi}^\varphi = \dots = 0$$

since $\frac{\partial^2 x}{\partial h \partial \dots} = \frac{\partial^2 y}{\partial h \partial \dots} = \frac{\partial^2 z}{\partial h \partial \dots} = 0$

For all symbols $\Gamma_{\cdot\cdot}^h, \Gamma_{\cdot\cdot}^h = \frac{\partial^2 z}{\partial \cdot \partial \cdot}$ since $\frac{\partial h}{\partial x} = \frac{\partial h}{\partial y} = 0$ and $\frac{\partial h}{\partial z} = 1$. On the other hand all $\frac{\partial^2 z}{\partial \cdot \partial \cdot}$ vanish. Hence all symbols $\Gamma_{\cdot\cdot}^h$ vanish. ■

b) spherical coordinates

$$\begin{cases} x = r \sin \theta \cos \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \theta \end{cases} \quad \begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}} \\ \varphi = \arctan \frac{y}{x} \end{cases}$$

We already know the fast way to calculate Christoffel symbol using Lagrangian of free particle and this method work for a flat connection since flat connection is a Levi-Civita connection for Euclidean metric

So perform now brute force calculations only for some components. (Then later (in homework 6) we will calculate using very quickly Lagrangian of free particle.)

$$\Gamma_{rr}^r = 0 \text{ since } \frac{\partial^2 x^i}{\partial^2 r} = 0.$$

$$\Gamma_{r\theta}^r = \Gamma_{\theta r}^r = \frac{\partial^2 x}{\partial r \partial \theta} \frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial r \partial \theta} \frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial r \partial \theta} \frac{\partial r}{\partial z} = \cos \theta \cos \varphi \frac{x}{r} + \cos \theta \sin \varphi \frac{y}{r} - \sin \theta \frac{z}{r} = 0,$$

$$\Gamma_{\theta\theta}^r = \frac{\partial^2 x}{\partial^2 \theta} \frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial^2 \theta} \frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial^2 \theta} \frac{\partial r}{\partial z} = -r \sin \theta \cos \varphi \frac{x}{r} - r \sin \theta \sin \varphi \frac{y}{r} - r \cos \theta \frac{z}{r} = -r$$

$$\Gamma_{r\varphi}^r = \Gamma_{\varphi r}^r = \frac{\partial^2 x}{\partial r \partial \varphi} \frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial r \partial \varphi} \frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial r \partial \varphi} \frac{\partial r}{\partial z} = -\sin \theta \sin \varphi \frac{x}{r} + \sin \theta \cos \varphi \frac{y}{r} = 0$$

and so on....

2 Let ∇ be an affine connection on a 2-dimensional manifold M such that in local coordinates (u, v) it is given that $\Gamma_{uv}^u = v$, $\Gamma_{uv}^v = 0$.

Calculate the vector field $\nabla_{\frac{\partial}{\partial u}} (u \frac{\partial}{\partial v})$.

Using the properties of connection and definition of Christoffel symbols have

$$\nabla_{\frac{\partial}{\partial u}} \left(u \frac{\partial}{\partial v} \right) = \partial_{\frac{\partial}{\partial u}} (u) \frac{\partial}{\partial v} + u \nabla_{\frac{\partial}{\partial u}} \left(\frac{\partial}{\partial v} \right) =$$

$$\frac{\partial}{\partial v} + u \left(\Gamma_{uv}^u \frac{\partial}{\partial u} + \Gamma_{uv}^v \frac{\partial}{\partial v} \right) = \frac{\partial}{\partial v} + u \left(v \frac{\partial}{\partial u} + 0 \right) = \frac{\partial}{\partial v} + uv \frac{\partial}{\partial u}.$$

3 Let ∇ be an affine connection on the 2-dimensional manifold M such that in local coordinates (u, v)

$$\nabla_{\frac{\partial}{\partial u}} \left(u \frac{\partial}{\partial v} \right) = (1 + u^2) \frac{\partial}{\partial v} + u \frac{\partial}{\partial u}.$$

Calculate the Christoffel symbols Γ_{uv}^u and Γ_{uv}^v of this connection.

Using the properties of connection we have $\nabla_{\frac{\partial}{\partial u}} (u \frac{\partial}{\partial v}) = u \nabla_{\frac{\partial}{\partial u}} (\frac{\partial}{\partial v}) +$

$$\partial_{\frac{\partial}{\partial u}} (u) \frac{\partial}{\partial v} = u \left(\Gamma_{uv}^u \frac{\partial}{\partial u} + \Gamma_{uv}^v \frac{\partial}{\partial v} \right) + 1 \cdot \frac{\partial}{\partial v} = (1 + u \Gamma_{uv}^v) \frac{\partial}{\partial v} + u \Gamma_{uv}^u \frac{\partial}{\partial u} = (1 + u^2) \frac{\partial}{\partial v} + u \frac{\partial}{\partial u}.$$

Hence $1 + u^2 = 1 + u \Gamma_{uv}^v$ and $u \Gamma_{uv}^u = u$, i.e. $\Gamma_{uv}^v = u$ and $\Gamma_{uv}^u = 1$. ■

4 a) Consider a connection such that its Christoffel symbols are symmetric in a given coordinate system:
 $\Gamma_{km}^i = \Gamma_{mk}^i$.

Show that they are symmetric in an arbitrary coordinate system.

b*) Show that the Christoffel symbols of connection ∇ are symmetric (in any coordinate system) if and only if

$$\nabla_{\mathbf{X}} \mathbf{Y} - \nabla_{\mathbf{Y}} \mathbf{X} - [\mathbf{X}, \mathbf{Y}] = 0,$$

for arbitrary vector fields \mathbf{X}, \mathbf{Y} .

c)* Consider for an arbitrary connection the following operation on the vector fields:

$$S(\mathbf{X}, \mathbf{Y}) = \nabla_{\mathbf{X}} \mathbf{Y} - \nabla_{\mathbf{Y}} \mathbf{X} - [\mathbf{X}, \mathbf{Y}]$$

and find its properties.

Solution

a) Let $\Gamma_{km}^i = \Gamma_{mk}^i$. We have to prove that $\Gamma_{k'm'}^{i'} = \Gamma_{m'k'}^{i'}$

We have

$$\Gamma_{k'm'}^{i'} = \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^k}{\partial x^{k'}} \frac{\partial x^m}{\partial x^{m'}} \Gamma_{km}^i + \frac{\partial x^r}{\partial x^{k'}} \frac{\partial x^{i'}}{\partial x^{m'}} \frac{\partial x^r}{\partial x^r}. \quad (1)$$

Hence

$$\Gamma_{m'k'}^{i'} = \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^m}{\partial x^{m'}} \frac{\partial x^k}{\partial x^{k'}} \Gamma_{mk}^i + \frac{\partial x^r}{\partial x^{m'}} \frac{\partial x^{i'}}{\partial x^{k'}} \frac{\partial x^r}{\partial x^r}$$

But $\Gamma_{km}^i = \Gamma_{mk}^i$ and $\frac{\partial x^r}{\partial x^{m'}} \frac{\partial x^{i'}}{\partial x^{k'}} = \frac{\partial x^r}{\partial x^{k'}} \frac{\partial x^{i'}}{\partial x^{m'}}$. Hence

$$\Gamma_{m'k'}^{i'} = \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^m}{\partial x^{m'}} \frac{\partial x^k}{\partial x^{k'}} \Gamma_{mk}^i + \frac{\partial x^r}{\partial x^{m'}} \frac{\partial x^{i'}}{\partial x^{k'}} \frac{\partial x^r}{\partial x^r} = \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^m}{\partial x^{m'}} \frac{\partial x^k}{\partial x^{k'}} \Gamma_{km}^i + \frac{\partial x^r}{\partial x^{k'}} \frac{\partial x^{i'}}{\partial x^{m'}} \frac{\partial x^r}{\partial x^r} = \Gamma_{k'm'}^{i'}.$$

b) The relation

$$\nabla_{\mathbf{X}} \mathbf{Y} - \nabla_{\mathbf{Y}} \mathbf{X} - [\mathbf{X}, \mathbf{Y}] = 0$$

holds for all fields if and only if it holds for all basic fields. One can easily check it using axioms of connection (see the next part). Consider $\mathbf{X} = \frac{\partial}{\partial x^i}$, $\mathbf{Y} = \frac{\partial}{\partial x^j}$ then since $[\partial_i, \partial_j] = 0$ we have that

$$\nabla_{\mathbf{X}} \mathbf{Y} - \nabla_{\mathbf{Y}} \mathbf{X} - [\mathbf{X}, \mathbf{Y}] = \nabla_i \partial_j - \nabla_j \partial_i = \Gamma_{ij}^k \partial_k - \Gamma_{ji}^k \partial_k = (\Gamma_{ij}^k - \Gamma_{ji}^k) \partial_k = 0$$

We see that commutator for basic fields $\nabla_{\mathbf{X}} \mathbf{Y} - \nabla_{\mathbf{Y}} \mathbf{X} - [\mathbf{X}, \mathbf{Y}] = 0$ if and only if $\Gamma_{ij}^k - \Gamma_{ji}^k = 0$.

c) One can easily check it by straightforward calculations or using axioms for connection that $S(\mathbf{X}, \mathbf{Y})$ is a vector-valued bilinear form on vectors. In particular $S(f\mathbf{X}, \mathbf{Y}) = fS(\mathbf{X}, \mathbf{Y})$ for an arbitrary (smooth) function. Show this just using axioms defining connection:

$$\begin{aligned} S(f\mathbf{X}, \mathbf{Y}) &= \nabla_{f\mathbf{X}} \mathbf{Y} - \nabla_{\mathbf{Y}} (f\mathbf{X}) - [f\mathbf{X}, \mathbf{Y}] = f\nabla_{\mathbf{X}} \mathbf{Y} - f\nabla_{\mathbf{Y}} \mathbf{X} - \partial_{\mathbf{Y}} f\mathbf{X} + [\mathbf{Y}, f\mathbf{X}] = \\ &= f\nabla_{\mathbf{X}} \mathbf{Y} - f\nabla_{\mathbf{Y}} \mathbf{X} - (\partial_{\mathbf{Y}} f)\mathbf{X} + \partial_{\mathbf{Y}} f\mathbf{X} + f[\mathbf{Y}, \mathbf{X}] = f(\nabla_{\mathbf{X}} \mathbf{Y} - \nabla_{\mathbf{Y}} \mathbf{X} - [\mathbf{X}, \mathbf{Y}]) = fS(\mathbf{X}, \mathbf{Y}) \end{aligned}$$

5 Consider the surface M in the Euclidean space \mathbf{E}^n . Calculate the induced connection in the following cases

- a) $M = S^1$ in \mathbf{E}^2 ,
- b) M — parabola $y = x^2$ in \mathbf{E}^2 ,
- c) cylinder in \mathbf{E}^3 .
- d) cone in \mathbf{E}^3 .
- e) sphere in \mathbf{E}^3 .
- f) saddle $z = xy$ in \mathbf{E}^3

Solution.

a) Consider polar coordinate on S^1 , $x = R \cos \varphi$, $y = R \sin \varphi$. We have to define the connection on S^1 induced by the canonical flat connection on \mathbf{E}^2 . It suffices to define $\nabla_{\frac{\partial}{\partial \varphi}} \frac{\partial}{\partial \varphi} = \Gamma_{\varphi\varphi}^{\varphi} \frac{\partial}{\partial \varphi}$.

Recall the general rule. Let $\mathbf{r}(u^\alpha)$: $x^i = x^i(u^\alpha)$ is embedded surface in Euclidean space \mathbf{E}^n . The basic vectors $\frac{\partial}{\partial u^\alpha} = \frac{\partial \mathbf{r}(u)}{\partial u^\alpha}$. To take the induced covariant derivative $\nabla_{\mathbf{X}} \mathbf{Y}$ for two tangent vectors \mathbf{X}, \mathbf{Y} we take a usual derivative of vector \mathbf{Y} along vector \mathbf{X} (the derivative with respect to canonical flat connection: in Cartesian coordinates is just usual derivatives of components) then we take the tangent component of the answer, since in general derivative of vector \mathbf{Y} along vector \mathbf{X} is not tangent to surface:

$$\nabla_{\frac{\partial}{\partial u^\alpha}} \frac{\partial}{\partial u^\beta} = \Gamma_{\alpha\beta}^\gamma \frac{\partial}{\partial u^\gamma} = \left(\nabla_{\frac{\partial}{\partial u^\alpha}}^{\text{(canonical)}} \frac{\partial}{\partial u^\beta} \right)_{\text{tangent}} = \left(\frac{\partial^2 \mathbf{r}(u)}{\partial u^\alpha \partial u^\beta} \right)_{\text{tangent}}$$

($\nabla_{\text{canonical}} \partial_\alpha \frac{\partial}{\partial u^\beta}$) is just usual derivative in Euclidean space since for canonical connection all Christoffel symbols vanish.)

In the case of 1-dimensional manifold, curve it is just tangential acceleration!:

$$\nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial u} = \Gamma_{uu}^u \frac{\partial}{\partial u} = \left(\nabla_{\partial_u}^{(\text{canonical})} \frac{\partial}{\partial u} \right)_{\text{tangent}} = \left(\frac{d^2 \mathbf{r}(u)}{du^2} \right)_{\text{tangent}} = \mathbf{a}_{\text{tangent}}$$

For the circle S^1 , ($x = R \cos \varphi$, $y = R \sin \varphi$), in \mathbf{E}^2 . We have

$$\begin{aligned} \mathbf{r}_\varphi &= \frac{\partial}{\partial \varphi} = \frac{\partial x}{\partial \varphi} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \varphi} \frac{\partial}{\partial y} = -R \sin \varphi \frac{\partial}{\partial x} + R \cos \varphi \frac{\partial}{\partial y}, \\ \nabla_{\frac{\partial}{\partial \varphi}} \frac{\partial}{\partial \varphi} &= \Gamma_{\varphi\varphi}^\varphi \partial_\varphi = \left(\nabla_{\partial_\varphi}^{(\text{canonic.})} \partial_\varphi \right)_{\text{tangent}} = \left(\frac{\partial}{\partial \varphi} \mathbf{r}_\varphi \right)_{\text{tangent}} = \\ &= \left(\frac{\partial}{\partial \varphi} (-R \sin \varphi) \frac{\partial}{\partial x} + \frac{\partial}{\partial \varphi} (R \cos \varphi) \frac{\partial}{\partial y} \right)_{\text{tangent}} = \left(-R \cos \varphi \frac{\partial}{\partial x} - R \sin \varphi \frac{\partial}{\partial y} \right)_{\text{tangent}} = 0, \end{aligned}$$

since the vector $-R \cos \varphi \frac{\partial}{\partial x} - R \sin \varphi \frac{\partial}{\partial y}$ is orthogonal to the tangent vector \mathbf{r}_φ . In other words it means that acceleration is centripetal: tangential acceleration equals to zero.

We see that in coordinate φ , $\Gamma_{\varphi\varphi}^\varphi = 0$. ■

Additional work: Perform calculation of Christoffel symbol in stereographic coordinate t :

$$x = \frac{2tR^2}{R^2 + t^2}, y = \frac{R(t^2 - R^2)}{t^2 + R^2}.$$

In this case

$$\begin{aligned} \mathbf{r}_t &= \frac{\partial}{\partial t} = \frac{\partial x}{\partial t} \frac{\partial}{\partial x} + \frac{\partial y}{\partial t} \frac{\partial}{\partial y} = \frac{2R^2}{(R^2 + t^2)^2} \left((R^2 - t^2) \frac{\partial}{\partial x} + 2tR \frac{\partial}{\partial y} \right), \\ \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} &= \Gamma_{tt}^t \partial_t = \left(\nabla_{\partial_t}^{(\text{canonic.})} \partial_t \right)_{\text{tangent}} = \left(\frac{\partial}{\partial t} \mathbf{r}_t \right)_{\text{tangent}} = (\mathbf{r}_{tt})_{\text{tangent}} = \\ &= \left(-\frac{4t}{t^2 + R^2} \mathbf{r}_t + \frac{2R^2}{(R^2 + t^2)^2} \left(-2t \frac{\partial}{\partial x} + 2R \frac{\partial}{\partial y} \right) \right)_{\text{tangent}} \end{aligned}$$

In this case \mathbf{r}_{tt} is not orthogonal to velocity: to calculate $(\mathbf{r}_{tt})_{\text{tangent}}$ we need to extract its orthogonal component:

$$(\mathbf{r}_{tt})_{\text{tangent}} = \mathbf{r}_{tt} - \langle \mathbf{r}_{tt}, \mathbf{n}_t \rangle \mathbf{n}$$

We have

$$\mathbf{n}_t = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{1}{R^2 + t^2} (2tR \partial_x + (t^2 - R^2) \partial_y),$$

where $\langle \mathbf{r}_t, \mathbf{n} \rangle = 0$. Hence $\langle \mathbf{r}_{tt}, \mathbf{n}_t \rangle = \frac{-4R^3}{(t^2 + R^2)^2}$ and

$$\begin{aligned} (\mathbf{r}_{tt})_{\text{tangent}} &= \mathbf{r}_{tt} - \langle \mathbf{r}_{tt}, \mathbf{n}_t \rangle \mathbf{n} = \\ &= \left(-\frac{4t}{t^2 + R^2} \mathbf{r}_t + \frac{2R^2}{(R^2 + t^2)^2} \left(-2t \frac{\partial}{\partial x} + 2R \frac{\partial}{\partial y} \right) \right) + \frac{4R^3}{(t^2 + R^2)^2} \cdot \frac{1}{R^2 + t^2} (2tR \partial_x + (t^2 - R^2) \partial_y) = \frac{-2t}{t^2 + R^2} \mathbf{r}_t \end{aligned}$$

We come to the answer:

$$\nabla_{\partial_t} \partial_t = \frac{-2t}{t^2 + R^2} \partial_t, \quad \text{i.e. } \Gamma_{tt}^t = \frac{-2t}{t^2 + R^2}$$

Of course we could calculate the Christoffel symbol in stereographic coordinates just using the fact that we already know the Christoffel symbol in polar coordinates: $\Gamma_{\varphi\varphi}^\varphi = 0$, hence

$$\Gamma_{tt}^t = \frac{dt}{d\varphi} \frac{d\varphi}{dx} \frac{d\varphi}{dx} \Gamma_{\varphi\varphi}^\varphi + \frac{d^2\varphi}{dt^2} \frac{dt}{d\varphi} = \frac{d^2\varphi}{dt^2} \frac{dt}{d\varphi}$$

It is easy to see that $t = R \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)$, i.e. $\varphi = 2 \arctan \frac{t}{R} - \frac{\pi}{2}$ and

$$\Gamma_{tt}^t = \frac{d^2\varphi}{dt^2} \frac{dt}{d\varphi} = \frac{\frac{d^2\varphi}{dt^2}}{\frac{d\varphi}{dt}} = -\frac{2t}{t^2 + R^2}.$$

b) For parabola $x = t, y = t^2$

$$\mathbf{r}_t = \frac{\partial}{\partial t} = \frac{\partial x}{\partial t} \frac{\partial}{\partial x} + \frac{\partial y}{\partial t} \frac{\partial}{\partial y} = \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial y},$$

$$\nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} = \Gamma_{tt}^t \partial_t = \left(\nabla_{\partial_t}^{(\text{canonic.})} \partial_t \right)_{\text{tangent}} = \left(\frac{\partial}{\partial t} \mathbf{r}_t \right)_{\text{tangent}} = (\mathbf{r}_{tt})_{\text{tangent}} = \left(2 \frac{\partial}{\partial y} \right)_{\text{tangent}}$$

To calculate $(\mathbf{r}_{tt})_{\text{tangent}}$ we need to extract its orthogonal component: $(\mathbf{r}_{tt})_{\text{tangent}} = \mathbf{r}_{tt} - \langle \mathbf{r}_{tt}, \mathbf{n}_t \rangle \mathbf{n}$, where \mathbf{n} is an orthogonal unit vector: $\langle \mathbf{n}, \mathbf{r}_t \rangle = 0, \langle \mathbf{n}, \mathbf{n} \rangle = 1$:

$$\mathbf{n}_t = \frac{1}{\sqrt{1+4t^2}} (-2t\partial_x + \partial_y).$$

We have

$$\begin{aligned} (\mathbf{r}_{tt})_{\text{tangent}} &= \mathbf{r}_{tt} - \langle \mathbf{r}_{tt}, \mathbf{n}_t \rangle \mathbf{n} = 2\partial_y - \left\langle 2\partial_y, \frac{1}{\sqrt{1+4t^2}} (-2t\partial_x + \partial_y) \right\rangle \frac{1}{\sqrt{1+4t^2}} (-2t\partial_x + \partial_y) = \\ &= \frac{4t}{1+4t^2} \partial_x + \frac{8t^2}{1+4t^2} \partial_y = \frac{4t}{1+4t^2} (\partial_x + 2t\partial_y) = \frac{4t}{1+4t^2} \partial_t \end{aligned}$$

We come to the answer:

$$\nabla_{\partial_t} \partial_t = \frac{4t}{1+4t^2} \partial_t, \quad \text{i.e. } \Gamma_{tt}^t = \frac{4t}{1+4t^2}$$

Remark Do not be surprised by resemblance of the answer to the answer for circle in stereographic coordinates.

c) *Cylinder*

$$\mathbf{r}(h, \varphi): \begin{cases} x = a \cos \varphi \\ y = a \sin \varphi \\ z = h \end{cases}$$

$$\partial_h = \mathbf{r}_h = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \partial_\varphi = \mathbf{r}_\varphi = \begin{pmatrix} -a \sin \varphi \\ a \cos \varphi \\ 0 \end{pmatrix}$$

Calculate

$$\nabla_{\partial_h} \partial_h = \Gamma_{hh}^h \partial_h + \Gamma_{hh}^\varphi \partial_\varphi = \left(\frac{\partial^2 \mathbf{r}}{\partial h^2} \right)_{\text{tangent}} = 0 \text{ since } \mathbf{r}_{hh} = 0.$$

Hence $\Gamma_{hh}^h = \Gamma_{hh}^\varphi = 0$

$$\nabla_{\partial_h} \partial_\varphi = \nabla_{\partial_\varphi} \partial_h = \Gamma_{h\varphi}^h \partial_h + \Gamma_{h\varphi}^\varphi \partial_\varphi = \left(\frac{\partial^2 \mathbf{r}}{\partial h \partial \varphi} \right)_{\text{tangent}} = 0 \text{ since } \mathbf{r}_{h\varphi} = 0$$

Hence $\Gamma_{h\varphi}^h = \Gamma_{\varphi h}^h = \Gamma_{h\varphi}^\varphi = \Gamma_{\varphi h}^\varphi = 0$.

$$\nabla_{\partial_\varphi} \partial_\varphi = \Gamma_{\varphi\varphi}^h \partial_h + \Gamma_{\varphi\varphi}^\varphi \partial_\varphi = \left(\frac{\partial^2 \mathbf{r}}{\partial \varphi \partial \varphi} \right)_{\text{tangent}} = \left(\begin{pmatrix} -a \cos \varphi \\ -a \sin \varphi \\ 0 \end{pmatrix} \right)_{\text{tangent}} = 0$$

since the vector $\mathbf{r}_{\varphi\varphi} = \begin{pmatrix} -a \cos \varphi \\ -a \sin \varphi \\ 0 \end{pmatrix}$ is orthogonal to the surface of cylinder. Hence $\Gamma_{h\varphi}^h = \Gamma_{\varphi h}^h = \Gamma_{h\varphi}^\varphi = \Gamma_{\varphi h}^\varphi = 0$

We see that for cylinder all Christoffel symbols in cylindrical coordinates vanish. This is not big surprise: in cylindrical coordinates metric equals $dh^2 = a^2 d\varphi^2$. This due to Levi-Civita theorem one can see that Levi-Civita connection which is equal to induced connection vanishes since all coefficients are constants.

d) *Cone*

For cone: $x^2 + y^2 = k^2 z^2$ we have $\mathbf{r}(h, \varphi) = \begin{cases} x = kh \cos \varphi \\ y = kh \sin \varphi \\ z = h \end{cases}$

$$\frac{\partial}{\partial h} = \mathbf{r}_h = \begin{pmatrix} k \cos \varphi \\ k \sin \varphi \\ 1 \end{pmatrix}, \quad \frac{\partial}{\partial \varphi} = \mathbf{r}_\varphi = \begin{pmatrix} -kh \sin \varphi \\ kh \cos \varphi \\ 0 \end{pmatrix}, \quad \mathbf{n} = \frac{1}{\sqrt{1+k^2}} \begin{pmatrix} \cos \varphi \\ \sin \varphi \\ -k \end{pmatrix}$$

We have $\mathbf{r}_{hh} = 0$, hence $\nabla_{\partial_h} \partial_h = 0$. i.e. $\Gamma_{hh}^h = \Gamma_{hh}^\varphi = 0$.

We have that $\mathbf{r}_{h\varphi} = \mathbf{r}_{\varphi h} = \begin{pmatrix} -k \sin \varphi \\ k \cos \varphi \\ 0 \end{pmatrix} = \frac{\mathbf{r}_\varphi}{h}$, i.e. $\nabla_{\partial_h} \partial_\varphi = \nabla_{\partial_\varphi} \partial_h = \frac{\mathbf{r}_\varphi}{h}$:

$$\Gamma_{h\varphi}^\varphi = \Gamma_{\varphi h}^\varphi = \frac{1}{h}, \quad \Gamma_{h\varphi}^h = \Gamma_{\varphi h}^h.$$

Now calculate $\mathbf{r}_{\varphi\varphi}$: $\mathbf{r}_{\varphi\varphi} = \begin{pmatrix} -kh \cos \varphi \\ -kh \sin \varphi \\ 0 \end{pmatrix}$. This vector is neither tangent to the cone nor orthogonal to the cone: $0 \neq \langle \mathbf{r}_{\varphi\varphi}, \mathbf{n} \rangle = -\frac{kh}{\sqrt{1+k^2}}$. Hence we have consider its decomposition:

$$\mathbf{r}_{\varphi\varphi} = \underbrace{\mathbf{r}_{\varphi\varphi} - \langle \mathbf{r}_{\varphi\varphi}, \mathbf{n} \rangle \mathbf{n}}_{\text{tangent component}} + \underbrace{\langle \mathbf{r}_{\varphi\varphi}, \mathbf{n} \rangle \mathbf{n}}_{\text{orthogonal component}}$$

Hence we have

$$\begin{aligned} \nabla_{\partial_\varphi} \partial_\varphi &= (\mathbf{r}_{\varphi\varphi})_{\text{tangent}} = \mathbf{r}_{\varphi\varphi} - \langle \mathbf{r}_{\varphi\varphi}, \mathbf{n} \rangle \mathbf{n} = \mathbf{r}_{\varphi\varphi} + \frac{kh}{\sqrt{1+k^2}} \mathbf{n} = \\ &= \begin{pmatrix} -kh \cos \varphi \\ -kh \sin \varphi \\ 0 \end{pmatrix} + \frac{kh}{1+k^2} \begin{pmatrix} \cos \varphi \\ \sin \varphi \\ -k \end{pmatrix} = -\frac{hk^2}{1+k^2} \begin{pmatrix} k \cos \varphi \\ k \sin \varphi \\ 1 \end{pmatrix} = -\frac{hk^2}{1+k^2} \mathbf{r}_h, \end{aligned}$$

i.e.

$$\Gamma_{\varphi\varphi}^h = -\frac{hk^2}{1+k^2}, \quad \Gamma_{\varphi\varphi}^\varphi = 0.$$

e) *Sphere*

For the sphere $\mathbf{r}(\theta, \varphi)$: $\begin{cases} x = R \sin \theta \cos \varphi \\ y = R \sin \theta \sin \varphi \\ z = R \cos \theta \end{cases}$, we have

$$\frac{\partial}{\partial \theta} = \mathbf{r}_\theta = \begin{pmatrix} R \cos \theta \cos \varphi \\ R \cos \theta \sin \varphi \\ -R \sin \theta \end{pmatrix}, \quad \frac{\partial}{\partial \varphi} = \mathbf{r}_\varphi = \begin{pmatrix} -R \sin \theta \sin \varphi \\ R \sin \theta \cos \varphi \\ 0 \end{pmatrix}, \quad \mathbf{n} = \begin{pmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{pmatrix}$$

Calculate

$$\nabla_{\partial_\theta} \partial_\theta = \Gamma_{\theta\theta}^\theta \partial_\theta + \Gamma_{\theta\theta}^\varphi \partial_\varphi = \left(\frac{\partial^2 \mathbf{r}}{\partial \theta^2} \right)_{\text{tangent}} = 0$$

since $\frac{\partial^2 \mathbf{r}}{\partial \theta^2} = -R\mathbf{n}$ is orthogonal to the sphere. Hence $\Gamma_{\theta\theta}^\theta = \Gamma_{\theta\theta}^\varphi = 0$.

Now calculate

$$\nabla_{\partial_\theta} \partial_\varphi = \Gamma_{\theta\varphi}^\theta \partial_\theta + \Gamma_{\theta\varphi}^\varphi \partial_\varphi = \left(\frac{\partial^2 \mathbf{r}}{\partial \theta \partial \varphi} \right)_{\text{tangent}}.$$

We have

$$\frac{\partial^2 \mathbf{r}}{\partial \theta \partial \varphi} = \cotan \theta \mathbf{r}_\varphi,$$

hence

$$\nabla_{\partial_\theta} \partial_\varphi = \Gamma_{\theta\varphi}^\theta \partial_\theta + \Gamma_{\theta\varphi}^\varphi \partial_\varphi = \left(\frac{\partial^2 \mathbf{r}}{\partial \theta \partial \varphi} \right)_{\text{tangent}} = \cotan \theta \mathbf{r}_\varphi, \text{ i.e.}$$

$$\Gamma_{\theta\varphi}^\theta = 0, \Gamma_{\theta\varphi}^\varphi = \cotan \theta$$

Now calculate

$$\nabla_{\partial_\varphi} \partial_\theta = \Gamma_{\varphi\theta}^\theta \partial_\theta + \Gamma_{\varphi\theta}^\varphi \partial_\varphi = \left(\frac{\partial^2 \mathbf{r}}{\partial \varphi \partial \theta} \right)_{\text{tangent}}.$$

We have

$$\frac{\partial^2 \mathbf{r}}{\partial \varphi \partial \theta} = \cotan \theta \mathbf{r}_\varphi,$$

hence

$$\nabla_{\partial_\varphi} \partial_\theta = \Gamma_{\varphi\theta}^\theta \partial_\theta + \Gamma_{\varphi\theta}^\varphi \partial_\varphi = \left(\frac{\partial^2 \mathbf{r}}{\partial \varphi \partial \theta} \right)_{\text{tangent}} = \cotan \theta \mathbf{r}_\varphi, \text{ i.e.}$$

$\Gamma_{\varphi\theta}^\theta = 0, \Gamma_{\varphi\theta}^\varphi = \cotan \theta$. Of course we did not need to perform these calculations: since ∇ is symmetric connection and $\nabla_{\partial_\varphi} \partial_\theta = \nabla_{\partial_\theta} \partial_\varphi$, i.e.

$$\Gamma_{\varphi\theta}^\theta = \Gamma_{\theta\varphi}^\theta = 0, \Gamma_{\varphi\theta}^\varphi = \Gamma_{\theta\varphi}^\varphi = \cotan \theta.$$

and finally

$$\nabla_{\partial_\varphi} \partial_\varphi = \Gamma_{\varphi\varphi}^\theta \partial_\theta + \Gamma_{\varphi\varphi}^\varphi \partial_\varphi = \left(\frac{\partial^2 \mathbf{r}}{\partial \varphi^2} \right)_{\text{tangent}}.$$

We have

$$\frac{\partial^2 \mathbf{r}}{\partial \varphi^2} = \mathbf{r}_{\varphi\varphi} = \begin{pmatrix} -R \sin \theta \cos \varphi \\ -R \sin \theta \sin \varphi \\ 0 \end{pmatrix}.$$

The vector $\mathbf{r}_{\varphi\varphi}$ is not proportional to normal vector \mathbf{n} , i.e. it is not orthogonal to the sphere; the vector $\mathbf{r}_{\varphi\varphi}$ is not tangent to sphere, i.e. it is not orthogonal to vector \mathbf{n} : $0 \neq \langle \mathbf{r}_{\varphi\varphi}, \mathbf{n} \rangle = -R \sin^2 \theta$. We decompose the vector $\mathbf{r}_{\varphi\varphi}$ on the sum of tangent vector and orthogonal vector:

$$\mathbf{r}_{\varphi\varphi} = \underbrace{\mathbf{r}_{\varphi\varphi} - \mathbf{n} \langle \mathbf{r}_{\varphi\varphi}, \mathbf{n} \rangle}_{\text{tangent vector}} + \mathbf{n} \langle \mathbf{r}_{\varphi\varphi}, \mathbf{n} \rangle,$$

We see that

$$\begin{aligned} \left(\frac{\partial^2 \mathbf{r}}{\partial \varphi^2} \right)_{\text{tangent}} &= \mathbf{r}_{\varphi\varphi} - \mathbf{n} \langle \mathbf{r}_{\varphi\varphi}, \mathbf{n} \rangle = \mathbf{r}_{\varphi\varphi} + R \sin^2 \theta \mathbf{n} = \begin{pmatrix} -R \sin \theta \cos \varphi \\ -R \sin \theta \sin \varphi \\ 0 \end{pmatrix} + R \sin^2 \theta \begin{pmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{pmatrix} = \\ &= \begin{pmatrix} -R \cos^2 \theta \sin \theta \cos \varphi \\ -R \cos^2 \theta \sin \theta \sin \varphi \\ R \sin^2 \theta \cos \theta \end{pmatrix} = -\sin \theta \cos \theta \begin{pmatrix} \cos \theta \cos \varphi \\ \cos \theta \sin \varphi \\ -\sin \theta \end{pmatrix} = -\sin \theta \cos \theta \mathbf{r}_\theta. \end{aligned}$$

We have

$$\nabla_{\partial_\varphi} \partial_\varphi = \Gamma_{\varphi\varphi}^\theta \partial_\theta + \Gamma_{\varphi\varphi}^\varphi \partial_\varphi = \left(\frac{\partial^2 \mathbf{r}}{\partial \varphi \partial \varphi} \right)_{\text{tangent}} = -\sin \theta \cos \theta \mathbf{r}_\theta, \text{ i.e.}$$

$$\Gamma_{\varphi\varphi}^\theta = -\sin \theta \cos \theta, \Gamma_{\varphi\varphi}^\varphi = 0.$$

f) *Saddle*

For saddle $z = xy$: We have $\mathbf{r}(u, v)$: $\begin{cases} x = u \\ y = v \\ z = uv \end{cases}$, $\partial_u = \mathbf{r}_u = \begin{pmatrix} 1 \\ 0 \\ v \end{pmatrix}$, $\partial_v = \mathbf{r}_v = \begin{pmatrix} 0 \\ 1 \\ u \end{pmatrix}$ It will be useful also

to use the normal unit vector $\mathbf{n} = \frac{1}{\sqrt{1+u^2+v^2}} \begin{pmatrix} -v \\ -u \\ 1 \end{pmatrix}$.

Calculate:

$$\nabla_{\partial_u} \partial_u = \Gamma_{uu}^u \partial_u + \Gamma_{uu}^v \partial_v = \left(\frac{\partial^2 \mathbf{r}}{\partial u^2} \right)_{\text{tangent}} = (\mathbf{r}_{uu})_{\text{tangent}} = 0 \text{ since } \mathbf{r}_{uu} = 0.$$

Hence $\Gamma_{uu}^u = \Gamma_{uu}^v = 0$.

Analogously $\Gamma_{vv}^u = \Gamma_{vv}^v = 0$ since $\mathbf{r}_{vv} = 0$.

Now calculate $\Gamma_{uv}^u, \Gamma_{uv}^v, \Gamma_{vu}^u, \Gamma_{vu}^v$:

$$\nabla_{\partial_u} \partial_v = \nabla_{\partial_v} \partial_u = \Gamma_{uv}^u \partial_u + \Gamma_{uv}^v \partial_v = (\mathbf{r}_{uv})_{\text{tangent}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{\text{tangent}}$$

Using normal unit vector \mathbf{n} we have: $(\mathbf{r}_{uv})_{\text{tangent}} = \mathbf{r}_{uv} - \langle \mathbf{r}_{uv}, \mathbf{n} \rangle \mathbf{n} = \Gamma_{uv}^u \partial_u + \Gamma_{uv}^v \partial_v =$

$$\begin{aligned} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{\text{tangent}} &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{1+u^2+v^2}} \begin{pmatrix} -v \\ -u \\ 1 \end{pmatrix} \right\rangle \frac{1}{\sqrt{1+u^2+v^2}} \begin{pmatrix} -v \\ -u \\ 1 \end{pmatrix} = \\ &= \frac{1}{1+u^2+v^2} \begin{pmatrix} v \\ u \\ u^2+v^2 \end{pmatrix} = \frac{v}{1+u^2+v^2} \begin{pmatrix} 1 \\ 0 \\ v \end{pmatrix} + \frac{u}{1+u^2+v^2} \begin{pmatrix} 0 \\ 1 \\ u \end{pmatrix} = \frac{v\mathbf{r}_u + u\mathbf{r}_v}{1+u^2+v^2}. \end{aligned}$$

Hence $\Gamma_{uv}^u = \Gamma_{vu}^u = \frac{v}{1+u^2+v^2}$ and $\Gamma_{uv}^v = \Gamma_{vu}^v = \frac{u}{1+u^2+v^2}$. ■

Sure one may calculate this connection as Levi-Civita connection of the induced Riemannian metric using explicit Levi-Civita formula or using method of Lagrangian of free particle.

6 Let ∇_1, ∇_2 be two different connections. Let ${}^{(1)}\Gamma_{km}^i$ and ${}^{(2)}\Gamma_{km}^i$ be the Christoffel symbols of connections ∇_1 and ∇_2 respectively.

a) Find the transformation law for the object: $T_{km}^i = {}^{(1)}\Gamma_{km}^i - {}^{(2)}\Gamma_{km}^i$ under a change of coordinates.

Show that it is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ tensor.

b)*? Consider an operation $\nabla_1 - \nabla_2$ on vector fields and find its properties.

Christoffel symbols of both connections transform according the law (1). The second term is the same. Hence it vanishes for their difference:

$$T_{k'm'}^{i'} = {}^{(1)}\Gamma_{k'm'}^{i'} - {}^{(2)}\Gamma_{k'm'}^{i'} = \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^k}{\partial x^{k'}} \frac{\partial x^m}{\partial x^{m'}} \left({}^{(1)}\Gamma_{km}^i - {}^{(2)}\Gamma_{km}^i \right) = \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^k}{\partial x^{k'}} \frac{\partial x^m}{\partial x^{m'}} T_{km}^i$$

We see that $T_{k'm'}^{i'}$ transforms as a tensor of the type $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

b) One can do it in invariant way. Using axioms of connection study $T = \nabla_1 - \nabla_2$ is a vector field. Consider

$$T(\mathbf{X}, \mathbf{Y}) = \nabla_{1\mathbf{X}} \mathbf{Y} - \nabla_{2\mathbf{X}} \mathbf{Y}$$

Show that $T(f\mathbf{X}, \mathbf{Y}) = fT(\mathbf{X}, \mathbf{Y})$ for an arbitrary (smooth) function, i.e. it does not possesses derivatives:

$$T(f\mathbf{X}, \mathbf{Y}) = \nabla_{1f\mathbf{X}} \mathbf{Y} - \nabla_{2f\mathbf{X}} \mathbf{Y} = (\partial_{\mathbf{X}} f) \mathbf{Y} + f \nabla_{1\mathbf{X}} \mathbf{Y} - (\partial_{\mathbf{X}} f) \mathbf{Y} - f \nabla_{2\mathbf{X}} \mathbf{Y} = fT(\mathbf{X}, \mathbf{Y}).$$

7 * a) Consider $t_m = \Gamma_{im}^i$. Show that the transformation law for t_m is

$$t_{m'} = \frac{\partial x^m}{\partial x^{m'}} t_m + \frac{\partial^2 x^r}{\partial x^{m'} \partial x^{k'}} \frac{\partial x^{k'}}{\partial x^r}.$$

b) † Show that this law can be written as

$$t_{m'} = \frac{\partial x^m}{\partial x^{m'}} t_m + \frac{\partial}{\partial x^{m'}} \left(\log \det \left(\frac{\partial x}{\partial x'} \right) \right).$$

Solution. Using transformation law (1) we have

$$t_{m'} = \Gamma_{i'm'}^{i'} = \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^k}{\partial x^{i'}} \frac{\partial x^m}{\partial x^{m'}} \Gamma_{km}^i + \frac{\partial x^r}{\partial x^{i'}} \frac{\partial x^{i'}}{\partial x^{m'}} \frac{\partial x^r}{\partial x^r}$$

We have that $\frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^k}{\partial x^{i'}} = \delta_i^k$. Hence

$$t_{m'} = \Gamma_{i'm'}^{i'} = \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^k}{\partial x^{i'}} \frac{\partial x^m}{\partial x^{m'}} \Gamma_{km}^i + \frac{\partial x^r}{\partial x^{i'}} \frac{\partial x^{i'}}{\partial x^{m'}} \frac{\partial x^r}{\partial x^r} = \delta_i^k \frac{\partial x^m}{\partial x^{m'}} \Gamma_{km}^i + \frac{\partial x^r}{\partial x^{i'}} \frac{\partial x^{i'}}{\partial x^{m'}} \frac{\partial x^r}{\partial x^r} = \frac{\partial x^m}{\partial x^{m'}} t_m + \frac{\partial x^r}{\partial x^{i'}} \frac{\partial x^{i'}}{\partial x^{m'}} \frac{\partial x^r}{\partial x^r}.$$

b) † When calculating $\frac{\partial}{\partial x^{m'}} (\log \det (\frac{\partial x}{\partial x'}))$ use very important formula:

$$\delta \det A = \det A \operatorname{Tr} (A^{-1} \delta A) \rightarrow \delta \log \det A = \operatorname{Tr} (A^{-1} \delta A).$$

Hence

$$\frac{\partial}{\partial x^{m'}} \left(\log \det \left(\frac{\partial x}{\partial x'} \right) \right) = \frac{\partial x^{i'}}{\partial x^r} \frac{\partial^2 x^r}{\partial x^{i'} \partial x^{m'}}$$

and we come to transformation law for (1).

To deduce the formula for $\delta \det A$ notice that

$$\det(A + \delta A) = \det A \det(1 + A^{-1} \delta A)$$

and use the relation: $\det(1 + \delta A) = 1 + \operatorname{Tr} \delta A + O(\delta^2 A)$