**Solutions of problem 1 of Homework 6**

**1**

*Calculate the derivatives of the functions* $f = x^2 + y^2$ , $g = y^2 - x^2$ *and* $h = q \log |r| = q \log \left( \sqrt{x^2 + y^2} \right)$ *(q is a constant) along vector fields* $\mathbf{A} = x\partial_x + y\partial_y$ *and* $\mathbf{B} = x\partial_y - y\partial_x$*,*

*a) calculating directional derivatives* $\partial_{\mathbf{A}} f, \partial_{\mathbf{A}} g, \partial_{\mathbf{A}} h,\ \partial_{\mathbf{B}} f, \partial_{\mathbf{B}} g, \partial_{\mathbf{B}} h$

*b) calculating* $df(\mathbf{A}), dg(\mathbf{A}), dh(\mathbf{A}), df(\mathbf{B}), dg(\mathbf{B}), dh(\mathbf{B})$*.*

We can do this exercise or using the formula for directional derivative or using the 1-form, differential of function: $\partial_{\mathbf{A}} f = df(\mathbf{A})$.

a) First do using directional derivatives:

$\partial_{\mathbf{A}} f = A_x \frac{\partial f}{\partial x} + A_y \frac{\partial f}{\partial y} = x \cdot 2x + y \cdot 2y = 2(x^2 + y^2)$,

$\partial_{\mathbf{A}} g = A_x \frac{\partial g}{\partial x} + A_y \frac{\partial g}{\partial y} = x \cdot (-2x) + y \cdot 2y = 2(y^2 - x^2)$,

$\partial_{\mathbf{A}} h = x \frac{\partial h}{\partial x} + y \frac{\partial h}{\partial y} = \frac{x^2 q}{x^2+y^2} + \frac{y^2 q}{x^2+y^2} = q$

$\partial_{\mathbf{B}} f = B_x \frac{\partial f}{\partial x} + B_y \frac{\partial f}{\partial y} = -y \cdot 2x + x \cdot 2y = 0$,

$\partial_{\mathbf{B}} g = -y \frac{\partial g}{\partial x} + x \frac{\partial g}{\partial y} = -y \cdot (-2x) + x \cdot 2y = 4xy$

$\partial_{\mathbf{B}} h = -y \frac{\partial h}{\partial x} + x \frac{\partial h}{\partial y} = \frac{-xyq}{x^2+y^2} + \frac{xyq}{x^2+y^2} = 0$

b) Now calculate using 1-form using the fact that $\partial_{\mathbf{A}} f = df(\mathbf{A})$:

We have that $df = d(x^2 + y^2) = 2xdx + 2ydy$, $dg = d\left(y^2 - x^2\right) = g_x dx + g_y dy = (2ydy - 2xdx)$, $dh = d\left(q \log \sqrt{x^2 + y^2}\right) = h_x dx + h_y dy = \frac{qxdx + qydy}{x^2+y^2}$.

Hence

$\partial_{\mathbf{A}} f = df(\mathbf{A}) = (2xdx + 2ydy)(x\partial_x + y\partial_y) = 2x^2 dx(\partial_x) + 2y^2 dy(\partial_y) = 2x^2 + 2y^2$,

$\partial_{\mathbf{A}} g = dg(\mathbf{A}) = (2ydy - 2xdx)((x\partial_x + y\partial_y)) = 2ydy(y\partial_y) - 2xdx(x\partial_x) = 2y^2 - 2x^2$.

$\partial_{\mathbf{A}} h = dh(\mathbf{A}) = \frac{qxdx+qydy}{x^2+y^2} (x\partial_x + y\partial_y) = \frac{qxdx(x\partial_x)+qydy(y\partial_y)}{x^2+y^2} = \frac{qx^2+qy^2}{x^2+y^2} = q$

$\partial_{\mathbf{B}} f = df(\mathbf{A}) = (2xdx + 2ydy)(-y\partial_x + x\partial_y) = -2xydx(\partial_x) + 2xydy(\partial_y) = 0$,

$\partial_{\mathbf{B}} g = dg(\mathbf{A}) = (2ydy - 2xdx)((x\partial_y - y\partial_x)) = 2ydy(x\partial_y) - 2xdx(-y\partial_x) = 2xy + 2xy = 4xy$.

$\partial_{\mathbf{A}} h = dh(\mathbf{A}) = \frac{qxdx+qydy}{x^2+y^2} (-y\partial_x + x\partial_y) = \frac{qxdx(-y\partial_x)+qydy(x\partial_y)}{x^2+y^2} = \frac{-qxy+qxy}{x^2+y^2} = 0$.

**2**

*Perform the calculations of the previous exercise using polar coordinates.*

For basic fields $\partial_r, \partial_\varphi$ in polar coordinates $r, \varphi$ ($r = x\cos\varphi, y = r\sin\varphi$) we have that

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\varphi\partial_x + \sin\varphi\partial_y = \frac{x}{r}\partial_x + \frac{y}{r}\partial_y = \frac{x\partial_x + y\partial_y}{r} = \frac{\mathbf{A}}{r} \Rightarrow \mathbf{A} = r\partial_r \tag{1}$$

and

$$\frac{\partial}{\partial \varphi} = \frac{\partial x}{\partial \varphi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \varphi}\frac{\partial}{\partial y} = -r\sin\varphi\partial_x + r\cos\varphi\partial_y = -y\partial_x + x\partial_y \Rightarrow \mathbf{B} = \partial_\varphi \tag{2}$$

We see that fields $\mathbf{A}, \mathbf{B}$ have very simple expression in polar coordinates. Now calculations become almost immediate because in polar coordinates $f = x^2 + y^2 = r^2$, $g = y^2 - x^2 = r^2(\sin^2\varphi - \cos^\varphi) = -r^2\cos 2\varphi$ and $h = q\log r$ and

$$\partial_A f = r\partial_r r^2 = 2r^2 = 2(x^2 + y^2)\,,$$

$$\partial_A g = r\partial_r(-r^2\cos 2\varphi) = -2r^2\cos 2\varphi = 2(y^2 - x^2),\ \ \partial_A h = r\partial_r(q\log r) = q\,.$$

For field $B$ we have that: $\partial_{\mathbf{B}} = \partial_\varphi$, hence

$$\partial_{\mathbf{B}} f = \partial_{\mathbf{B}} g = \partial_{\mathbf{B}} h = 0\,.$$

since the functions $f$ and $h$ do not depend on $\varphi$. For the function $g = y^2 - x^2 = -r^2 \cos 2\varphi$ we have:

$$\partial_{\mathbf{B}} g = \partial_\varphi(-r^2 \cos 2\varphi) = 2r^2 \sin 2\varphi = 4r^2 \sin\varphi \cos\varphi = 4r^2 \left(\frac{y}{r}\right) \cdot \left(\frac{x}{r}\right) = 4xy.$$

**3**

*Consider a function $f = x^4 - y^4$.*

*Calculate the value of 1-form $\omega = df$ on the vector field $\mathbf{B} = x\partial_y - y\partial_x$.*

*Express this 1-form $\omega$ in polar coordinates $r, \varphi$ $(x = r\cos\varphi, y = r\sin\varphi)$.*

$\omega = df = 4x^3dx - 4y^3dy$, $\omega(\mathbf{B}) = 4(x^3dx - y^3dy)(x\partial_y - yp_x) = 4x^3dx(-y\partial_y) - 4y^3dy(x\partial_y) = -4x^3y - 4y^3x = -4xy(x^2 + y^2$ since $dx(\partial_x) = dy(\partial_y) = 1$ and $dx(\partial_y) = dy(\partial_x) = 0$.

One may express differential 1-form $\omega = df = 4x^3dx - 4y^3dy$ straightforwardly in polar coordinates. Instead using "brute force" express function $f$ in polar coordinates then calculate $\omega = df$:

$$f = x^4 - y^4 = (x^2 + y^2)(x^2 - y^2) = r^2(r^2\cos^2\varphi - r^2\sin^2\varphi) = r^4\cos^2\varphi - r^2\sin^2\varphi = r^4\cos 2\varphi\,,$$

hence $\omega = df = d(r^4\cos 2\varphi) = 4r^3\cos 2\varphi dr - 2r^4 \sin 2\varphi d\varphi$.

The operation of taking differential can be performed in an arbitrary coordinates in a same way as in Cartesian coordinates.

**4**

*Calculate the value of 1-form $\omega = xdy - ydx$ on the vector fields $\mathbf{A} = r\partial_r$ and $\mathbf{B} = \partial_\varphi$. Perform calculations in Cartesian and polar coordinates).*

We know that $r\partial_r = x\partial_x + y\partial_y$ and $\partial_\varphi = x\partial_y - y\partial_x$ (see formulae (1,2) in the solution of exercise 2). Hence in Cartesian coordinates $\mathbf{A} = x\partial_x + y\partial_y$ and $\mathbf{B} = x\partial_y - y\partial_x$

$$\omega(\mathbf{A}) = (xdy - ydx)(x\partial_x + y\partial_y) = -xydx(\partial_x) + xydy(\partial_y) = -xy + xy = 0\,,$$

$$\omega(\mathbf{B}) = (xdy - ydx)(x\partial_y - y\partial_x) = x^2dx(\partial_x) + y^2dy(\partial_y) = x^2 + y^2\,.$$

Now perform calculations in polar coordinates:

$$\omega = xdy - ydx = r\cos\varphi d(r\sin\varphi) - r\sin\varphi d(r\cos\varphi) = r\cos\varphi(\sin\varphi dr + r\cos\varphi d\varphi) - r\sin\varphi(\cos\varphi dr - r\sin\varphi d\varphi) = r^2(\cos^2\varphi +$$

Hence in polar coordinates

$$\omega(\mathbf{A}) = r^2 d\varphi(\partial_r) = 0, \quad \omega(\mathbf{B}) = r^2 d\varphi(\partial_\varphi) = r^2\,.$$

$(dr(\partial_r) = d\varphi(\partial_\varphi) = 1, dr(\partial_\varphi) = d\varphi(\partial_r) = 0)$.

We see that calculations are much more transparent in polar coordinates.

**5**

*Let $f$ be a function on $\mathbf{E}^2$ given by $f(r, \varphi) = r^2 \sin 2\varphi$, where $r, \varphi$ are polar coordinates in $\mathbf{E}^2$.*

*Calculate the 1-form $\omega = df$.*

*Calculate the value of the 1-form $\omega = df$ on the vector field $\mathbf{X} = r^2\partial_r + r\partial_\varphi$.*

*Express the 1-form $\omega$ in Cartesian coordinates $x, y$.*

$\omega = 2r\sin 2\varphi dr + 2r^2\cos 2\varphi d\varphi$.

The value of the form $\omega = df$ on the vector field $\mathbf{X} = r^2\partial_r + r\partial_\varphi$ is equal to

$$\omega(\mathbf{A}) = \left(2r\sin 2\varphi dr + 2r^2\cos 2\varphi d\varphi\right)\left(r^2\partial_r + r\partial_\varphi\right) = 2r^3\sin 2\varphi dr(\partial_r) + 2r^3\cos 2\varphi d\varphi(\partial_\varphi) = 2r^3(\sin 2\varphi + \cos 2\varphi)\,. \blacksquare$$

because $dr(\partial_r) = 1, dr(\partial_\varphi) = 0$ and $dr(\partial_\varphi) = 0, d\varphi(\partial_\varphi) = 1$.

*Another solution*

$$\omega(\mathbf{X}) = df(\mathbf{X}) = \partial_{\mathbf{X}} f = \left(r^2 \frac{\partial}{\partial r} + r \frac{\partial}{\partial_\varphi}\right)(r^2 \sin 2\varphi) = r^2 \cdot 2r \sin 2\varphi + r \cdot 2r^2 \cos 2\varphi = 2r^3(\sin 2\varphi + \cos 2\varphi)\,.$$

To express the form $\omega$ in Cartesian coordinates it is easier to express $f$ in Cartesian coordinates and then to calculate $\omega = df$:

$$f = r^2 \sin 2\varphi = (x^2 + y^2)(2\cos\varphi \sin\varphi) = (x^2 + y^2) 2 \left(\frac{x}{r}\right) \cdot \left(\frac{y}{r}\right) = 2xy\,.$$

Hence $\omega = d(2xy) = 2ydx + 2xdy$.

**6**

*Consider* 1*-forms* $\omega = df$ *and* $\sigma = dg$ *such that*

$$f(x, y) + ig(x, y) = (x + iy)^3\,.$$

*Find the values of these* 1*-forms on vector field* $\mathbf{Y} = r\partial_r + \partial_\varphi$.

One may try to use the fact that

$$f + ig = (x + iy)^3 = x^3 - 3xy^2 + i(3xy^2 - y^3) \Rightarrow f = x^3 - 3xy^2, g = 3xy^2 - y^3$$

and perform calculations. We come to hard calculations.

There is more nice solution using complex variables. If $z = x + iy = re^{i\varphi}$, then

$$f + ig = z^3 = r^3 e^{3i\varphi} = r^3 \cos 3\varphi + ir^3 \sin 3\varphi \Rightarrow f = r^3 \cos 3\varphi, g = r^3 \sin 3\varphi\,.$$

Hence

$$\omega(\mathbf{Y}) = df(\mathbf{Y}) = \partial_{\mathbf{Y}}(f) = \left(r\frac{\partial}{\partial r} + \frac{\partial}{\partial \varphi}\right) r^3 \cos 3\varphi = 3r^3 \cos 3\varphi - 3r^3 \sin 3\varphi\,,$$

and

$$\sigma(\mathbf{Y}) = dg(\mathbf{Y}) = \partial_{\mathbf{Y}}(f) = \left(r\frac{\partial}{\partial r} + \frac{\partial}{\partial \varphi}\right) r^3 \sin 3\varphi = 3r^3 \sin 3\varphi + 3r^3 \cos 3\varphi\,.$$

**7**

*Calculate the integrals of the form* $\omega = \sin y\, dx$ *over the following three curves. Compare answers.*

$C_1$: $\mathbf{r}(t)$ $\begin{cases} x = 2t^2 - 1 \\ y = t \end{cases}$, $0 < t < 1$, $C_2$: $\mathbf{r}(t)$ $\begin{cases} x = 8t^2 - 1 \\ y = 2t \end{cases}$, $0 < t < 1/2$,

$C_3$: $\mathbf{r}(t)$ $\begin{cases} x = \cos 2t \\ y = \cos t \end{cases}$, $0 < t < \frac{\pi}{2}$

For any curve $\mathbf{r}(t), t_1 < t < t_2$

$$\int_C \omega = \int_C \sin y dx = \int_C \sin y dx(\mathbf{v}) = \int_{t_1}^{t_2} \sin y(t) \frac{dx(t)}{dt} dt$$

where $\mathbf{v} = (x_t, y_t)$.

For the first curve $x_t = 4t$ and

$$\int_{C_1} \omega = \int_0^1 4t \sin t dt = 4(-t\cos t + \sin t)\Big|_0^1 = -4\cos 1 + 4\sin 1$$

For the second curve $x_t = 16t$ and

$$\int_{C_2} \omega = \int_0^{1/2} 16t \sin 2t dt = 4(-2t\cos 2t + \sin 2t)\big|_0^{1/2} = -4\cos 1 + 4\sin 1$$

Answer is the same. Non-surprising. The second curve is reparameterised first curve ($t \mapsto 2t$) and reparameterisation preserves the orientation.

For the third curve $x_t = -2\sin 2t dt$ and

$$\int_{C_3} w = \int_0^{\frac{\pi}{2}} (-2\sin 2t)\sin(\cos t)dt = -4\left(\cos t\cos(\cos t) - \sin(\cos t)\right)\big|_0^{\pi/2} = 4\cos 1 - 4\sin 1$$

Answer is the same up to a sign: This curve is reparameterised first curve ($t \mapsto \cos t$) and reparameterisation changes the orientation, because $(\cos t)' = -\sin t < 0$ on the interval $(0, \pi/2)$.

**Resumé**: In these three examples an integral over the same (non-parameteresed) curve was considered. All the answers are the same up to a sign. Sign changes if reparameterisation changes an orientation.

**8**

*Calculate the integrals of the form $\omega = xdy - ydx$ over the following three curves. Compare answers.*

$$C_1\colon \mathbf{r}(t)\ \begin{cases} x = R\cos t \\ y = R\sin t \end{cases},\ 0 < t < \pi, \quad C_2\colon \mathbf{r}(t)\ \begin{cases} x = R\cos 4t \\ y = R\sin 4t \end{cases},\ 0 < t < \frac{\pi}{4}$$

$$\text{and}\quad C_3\colon \mathbf{r}(t)\ \begin{cases} x = Rt \\ y = R\sqrt{1-t^2} \end{cases},\ -1 \le t \le 1.$$

In the same way like for the previous integral

$$\int_C \omega = \int_{t_1}^{t_2} \omega(\mathbf{v}(t))dt = \int_{t_1}^{t_2} (xdy - ydx)(x_t\partial_x + y_t\partial_y)dt = \int_{t_1}^{t_2} (-y(t)x_t(t) + x(t)y_t(t))dt\,,$$

where $\mathbf{v} = (x_t, y_t)$ is velocity vector: $dx(\partial_x) = dy(\partial_y) = 1$, $dx(\partial_y) = dy(\partial_x) = 0$. )

For the first curve $C_1$ we have $\mathbf{v}(t) = (-R\sin t, R\cos t)$ and $\int_{C_1}\omega = \int_0^\pi (xdy - ydx)(-R\sin t\partial_x + R\cos t\partial_y) =$

$$\int_0^\pi (R\cos t dy - R\sin t dx)(-R\sin t\partial_x + R\cos t\partial_y) = \int_0^\pi (R^2\cos^2 t + R^2\sin^2 t)dt = \int_0^\pi R^2 \cdot dt = \pi R^2\,.$$

For the second curve $C_2$ we have $\mathbf{v}(t) = (-4R\sin 4t, 4R\cos 4t)$ and $\int_{C_2}\omega = \int_0^{\frac{\pi}{4}} (xdy - ydx)(-4R\sin 4t\partial_x + 4R\cos 4t\partial_y) =$

$$\int_0^{\frac{\pi}{4}} (R\cos 4t dy - R\sin 4t dx)(-4R\sin 4t\partial_x + 4R\cos 4t\partial_y) = \int_0^{\frac{\pi}{4}} (4R^2\cos^2 4t + 4R^2\sin^2 4t)dt = \int_0^{\frac{\pi}{4}} 4R^2 \cdot dt = \pi R^2\,.$$

Answer is the same. The second curve is reparameterised first curve ($t \mapsto 4t$) and reparameterisation preserves the orientation: $(4t)' = 4 > 0$.

For the third curve $C_3$ we have $\mathbf{v}(t) = \left(-R, -\frac{Rt}{\sqrt{1-t^2}}\right)$ and $\omega(\mathbf{v}(t)) = (xdy - ydx)(v_x\partial_x + v + y\partial_y) =$

$$= \left(Rtdy - R\sqrt{1-t^2}dx\right)\left(R\partial_x - \frac{Rt}{\sqrt{1-t^2}}\partial_y\right) = -R^2\sqrt{1-t^2} - \frac{R^2t^2}{\sqrt{1-t^2}} = -\frac{R^2}{1-t^2}\,.$$

Hence

$$\int_{C_3} \omega = \int_0^1 \omega(\mathbf{v}(t)) = \int_0^1 \left(-\frac{R^2}{\sqrt{1-t^2}}\right)dt = -R^2\int_0^1 \frac{dt}{\sqrt{1-t^2}} = -\pi R^2\,.$$

Answer is the same up to a sign: This curve is reparameterised first curve. If we put $t = \cos\tau$ then third curve $C_3$ will transform to the first curve $C_1$. This reparameterisation changes the orientation, because $(\cos t)' = -\sin t < 0$ on the interval $(0, \pi/2)$.

**Resumé**: In these three examples was considered an integral over the same (non-parameteresed) half-circle. All the answers are the same up to a sign. Sign changes if reparameterisation changes an orientation.