Statements on Haar measure

Let G be a group and let μ be a left-invariant meausure defined by the volume-form—-left-invariant n-form ω:

$$\forall g \in G\,, \quad L_g \omega = \omega\,,$$

i.e.

$$\forall g \in G\,, \quad \omega\left(\left[g^{-1}x(t)\right]\right) = \omega\left([x(t)]\right)$$

If

$$\int_G f\omega = \int f(x)\mu(dx)$$

then

$$\int f L_g \omega = \int f(x)\mu(dx') = \int f(gx)\mu(dx)\,.$$

§2 `Construction of Haar measure`

Let G be a group. We assume that it is differentiable manifold. Consider the volume form

$$\omega_0 = c dy^1 \dots dy^n$$

at the unity. Now using left-invariance we define the form at the arbitrary element.

Let $x = x_0$ be a point of G. We consider a map

$$L_{x_0^{-1}}\colon \quad y = x_0^{-1} \circ x \tag{2.1}$$

This map sends a vicinity of the point x_0 to a vicinity of unity.

We define the form ω at the point x_0 as the pull-back of the form ω_0:

$$\omega\big|_{x_0} = L^*_{x_0^{-1}} \omega_0\,,$$

For arbitrary n vectors $\mathbf{t}_1, \dots, t_n \in T_{x_0}G$,

$$\omega(x_0, \mathbf{t}_1, \dots, \mathbf{t}_n) = \omega\left(e, \left(L_{x_0^{-1}}\right)_* \mathbf{t}_1, \dots \left(L_{x_0^{-1}}\right)_* \mathbf{t}_n,\right)$$

In particular

$$\omega\left(x_0, \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right) = \omega\left(y(x_0), \frac{\partial y^i}{\partial x^1}\frac{\partial}{\partial y^i}, \dots, \frac{\partial y^i}{\partial x^n}\frac{\partial}{\partial y^i},\right) = \det\left(\frac{\partial y^i}{\partial x^j}\right)\big|_{x_0}\,,$$

where $y = y(x)$ is a map defined by equation (2.1).

`Useful formula` which related apperanace of the form at the point x_0 and e:

$$\omega = \omega(y(x)) \det\left(\frac{\partial y}{\partial x}\right) dx^1 \wedge \dots \wedge dx^n$$

Example Consider groups G_1 of translations of $\mathbf{R}$:

$$G_1 = \{G_1 \ni g = \alpha \colon g \circ x = \alpha \circ x = x + \alpha\}\,,$$

G_2 of affine transformations of $\mathbf{R}$:

$$G_2 = \{G_2 \ni g = (a, \alpha) \colon\,, \quad \forall x \in \mathbf{R}\,, \quad g \circ x = (a, \alpha) \circ x = ax + \alpha\}\,,$$

and G_3 of projective transformation of $\mathbf{R}P$:

$$G_3 = \left\{ G_3 \ni g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \colon \quad \forall [x:y] \in \mathbf{R}P g \circ x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ [ax+by : cx+dy] \right\}\,,$$

Calculate Haar meausre of these groups.

1) For $G_1, \mu = d\alpha$

2) For G_2.

Considee a form

$$\omega_g \colon \quad \omega_g = Ad_g \omega : \quad Ad_g \omega([x(t)]) = \omega(g^{-1}[x(t)]g) \quad f \mapsto \int_G f \omega_g = \int f(gxg^{-1}) \mu(dx)\,.$$

This is also left-invariant volume form:

$$L_h \omega_g = L_h \omega\left([x(t)]\right) = \omega_g\left([h^{-1}x(t)]\right) = \omega\left([g^{-1}h^{-1}x(t)g]\right) = \omega\left(hg^{-1}[x(t)g]\right) =$$

$$\omega\left([x(t)g]\right) = \omega\left([g^{-1}x(t)g^{-1}]\right) = \omega_g\left([x(t)]\right)\,.$$

Hence

$$\omega_g = \Delta(g)\omega\,.$$

We see that $\Delta(g)$ is the determinant of the linear operator Ad_g acting at the unity:

$$\Delta(g) = \det Ad_g\,.$$

Hence Δ is homomorphism of G into subgroup of $\mathbf{R}^*$ ($\mathbf{R}^* = \mathbf{R}\backslash\{0\}$).

Theorem If G is compact connected group then $\Delta \equiv 1$, i.e. it is unimodular.

`Proof` Image of Δ is compact? subgroup of $\mathbf{R}^*$, hence it is $\{1\}$.

How to construct right invariant measure?

Statement A form

$$\sigma(x) = \omega(x)\Delta(x^{-1})\,.$$

is the right invariant form.

`Proof` Show it explicitly using the invariance of the initial volume form with respect to the left action:

$$\omega_R([x(t)g^{-1}]) = \omega([x(t)]g^{-1})\Delta\left(\left(xg^{-1}\right)^{-1}\right) =$$

$$\omega\left(g^{-1}\left([x(t)]\,g^{-1}\right)g\right)\Delta(x^{-1}) =$$

$$\omega\left(g^{-1}\,[x(t)]\right)\Delta(x^{-1}) = \omega\left([x(t)]\right)\Delta(x^{-1}) = \omega_R(x)\,.$$

Remark If we define $\Delta(g) \to \Delta(g^{-1}) = \frac{\omega(gx)}{\omega(x)}$ then

$$\omega_R(x) = \omega(x)\Delta(x)\,.$$

Examples

1. Volume of $GL(n, \mathbf{R})$

$$\omega = (\det A_{ik})^x \prod_{i,k} dA_{ik}$$

Theorem on meause of the product Let $G = P \times Q$, and let $\mu_P(dx)$ be Haar meausure on P, and $\mu_Q(dy)$ be Haar meausure on Q, then the Haar measure on G is defined by

$$\int_G f(g)\mu(dg) = \int_{P\times Q} f(xy)\Delta(y)\mu_P(dx)\mu_Q(dy)\,,$$

where Δ be module of G