

Riemannian Geometry

2018

COURSEWORK

This assignment counts for 20% of your marks.

Solutions are due by 18 April 3pm

Write solutions in the provided spaces.

STUDENT'S NAME:

1

Consider a surface M , the upper sheet of the cone

$$\mathbf{r}(h, \varphi): \begin{cases} x = h \cos \varphi \\ y = h \sin \varphi \\ z = kh \end{cases}, \quad 0 \leq \varphi < 2\pi, h > 0.$$

Find the length of the shortest curve C which belongs to the surface M , starts at the point $(h, 0, kh)$ and ends at the point $(-h, 0, kh)$.

2

You know that the Riemannian metric on the sphere of radius R in the stereographic coordinates is expressed by the formula

$$G_{\text{stereogr.}} = \frac{4R^4(du^2 + dv^2)}{(R^2 + u^2 + v^2)^2}.$$

a) Give an example of a non-identity transformation of coordinates u, v such that it preserves this metric.

b) Give an example of a non-linear transformation of coordinates u, v such that it preserves this metric.

(*Hint: You may find this transformation considering transformations of the sphere.*)

c) Find the length of the line $v = au$ in \mathbf{R}^2 with respect to this metric.

Why the length of this curve does not depend on the parameter a ?

3.

a) Evaluate the area of the part of the sphere of radius $R = 1$ between the planes given by equations $2x + 2y + z = 1$ and $2x + 2y + z = 2$.

b) Consider the plane \mathbf{R}^2 with standard coordinates (x, y) equipped with Riemannian metric

$$G = (1 + x^2 + y^2)e^{-a^2x^2 - a^2y^2} (dx^2 + dy^2) .$$

Calculate the total area of this plane.

4

a) Consider the points $A = (0, 0, R)$, $B = (R, 0, 0)$ and $C = (R \cos \varphi, R \sin \varphi, 0)$ on the sphere $x^2 + y^2 + z^2 = R^2$ in \mathbf{E}^3 ($0 < \varphi < \pi$). Consider the isosceles triangle ABC on this sphere. (Sides of this triangle are the arcs of great circles joining these points.) Show that:

$$\frac{\text{Area}(\triangle ABC)}{R^2} = \alpha + \beta + \gamma - \pi ,$$

where α, β, γ are angles of this triangle.

b) Consider the upper half-plane $y > 0$ with the Riemannian metric

$$G = \frac{dx^2 + dy^2}{y^2}$$

(the Lobachevsky plane).

In the Lobachevsky plane consider the domain D defined by

$$D = \{x, y: \quad x^2 + y^2 \geq 1, \quad -a \leq x \leq a\} ,$$

where a is a parameter such that $0 < a < 1$.

Find the area of the domain D (with respect to the metric G).

Show that

$$\text{Area of the domain } D = \pi - \beta - \gamma ,$$

where β, γ are angles between the vertical lines $x = \pm a$ and arc of the circle delimiting the domain D .

Consider the points $A_t = (-a, t)$ and $B_t = (a, t)$ on the vertical rays $x = \pm a$ delimiting the domain D . Show that the distance between these points tends to 0 if $t \rightarrow \infty$.

Explain why the domain D can be considered as an isosceles triangle.

Why it can be said that the third angle of this triangle vanishes.

5

a) Let ∇ be an affine connection on the 2-dimensional manifold M such that in local coordinates (u, v) , $\nabla_{\frac{\partial}{\partial u}} \left(u^2 \frac{\partial}{\partial v} \right) = 3u \frac{\partial}{\partial v} + u \frac{\partial}{\partial u}$.

Calculate the Christoffel symbols Γ_{uv}^u and Γ_{uv}^v of this connection.

b) Let ∇ be an arbitrary connection on a manifold M . Show that

$$\cos F \nabla_{\mathbf{A}} (\sin F \mathbf{B}) - \sin F \nabla_{\mathbf{A}} (\cos F \mathbf{B}) = (\partial_{\mathbf{A}} F) \mathbf{B},$$

where \mathbf{A}, \mathbf{B} are arbitrary vector fields and F is an arbitrary function.

c) Let $\Gamma_{km}^{i(1)}$ be the Christoffel symbols of a connection $\nabla^{(1)}$ and $\Gamma_{km}^{i(2)}$ be the Christoffel symbols of a connection $\nabla^{(2)}$. Show, that the linear combinations $\frac{2}{3}\Gamma_{km}^{i(1)} + \frac{1}{3}\Gamma_{km}^{i(2)}$, are Christoffel symbols for some connection.

Explain, why $\frac{1}{2}\Gamma_{km}^{i(1)} + \frac{1}{3}\Gamma_{km}^{i(2)}$ are not Christoffel symbols for any connection.

6

Let M be a surface considered in question 1 (the upper sheet of a cone),

a) Calculate the induced connection on this surface (the connection induced by the canonical flat connection in the ambient Euclidean space: $\nabla_{\mathbf{X}} \mathbf{Y} = (\nabla_{\mathbf{X}}^{\text{can.flat}} \mathbf{Y})_{\text{tangent}}$).

b) Calculate the Riemannian metric on the cone induced by the canonical metric in ambient Euclidean space \mathbf{E}^3 and calculate explicitly the Levi-Civita connection of this metric using the Levi-Civita Theorem.

c) Calculate the Christoffel symbols of Levi-Civita connection using Lagrangian of free particle.

