Creation and annihilation operators

Let L be Hilbert space, and M a space with meausre such that $L = L^2(M)$. We define generalised operator function

$$a(\xi) =$$

$$\begin{pmatrix} 0 & \delta(y,\xi) & 0 & 0 & \dots & \\ 0 & 0 & \sqrt{2}\delta(x_1,y_1)\delta(y_2,\xi) & 0 & 0 & \dots \\ 0 & 0 & 0 & \sqrt{3}\delta(x_1,y_1)\delta(x_2,y_2)\delta(y_3,\xi) & \dots & \\ \dots & \dots & \dots & \dots & \dots & \end{pmatrix}$$

and its adjoint

$$a^*(\xi) =$$

$$\begin{pmatrix} 0 & 0 & 0 & \dots \\ \delta(x,\xi) & 0 & 0 & \dots \\ 0 & \sqrt{2}\delta(x_1,y_1)\delta(x_2,\xi) & 0 & \dots \\ 0 & 0 & \sqrt{3}\delta(x_1,y_1)\delta(x_2,y_2)\delta(x_3,\xi) & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

If $f = f(\xi)$ is function in L then

$$a_f = \int a(\xi) f(\xi) d\xi =$$

$$\begin{pmatrix} 0 & f(y) & 0 & 0 & \dots & \\ 0 & \sqrt{2}\delta(x_1,y_1)f(y_2) & 0 & 0 & \dots & \\ 0 & 0 & \sqrt{3}\delta(x_1,y_1)\delta(x_2,y_2)f(y_3) & 0 & \dots & \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

and

$$a_f^* = \int a^*(\xi) f(\xi) d\xi =$$

$$\begin{pmatrix} 0 & 0 & 0 & \dots \\ f(x) & 0 & 0 & \dots \\ 0 & \sqrt{2}\delta(x_1,y_1)f(x_2) & 0 & \dots \\ 0 & 0 & \sqrt{3}\delta(x_1,y_1)\delta(x_2,y_2)f(x_3) & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

and

$$a_f\Psi = \begin{pmatrix} \int K_1(y)f(y)dy \\ \sqrt{2}\int K_2(x_1,y)f(y)dy \\ \sqrt{3}\int K_3(x_1,x_2,y)f(y)dy \\ \dots \end{pmatrix}, \quad a_f^*\Psi = \begin{pmatrix} 0 \\ f(x_1)K_0 \\ \sqrt{2}K_1(x_1)f(x_2) \\ \sqrt{3}K_2(x_1,x_2)f(x_3) \\ \dots \end{pmatrix}$$

for the vector Ψ

$$\Psi = \begin{pmatrix} K_0 \\ K_1(x_1) \\ K_2(x_1,x_2) \\ K_3(x_1,x_2,x_3) \\ \dots \end{pmatrix},$$

in $\mathcal{F}$.

Now project this on the subspaces of symmetric and antisymmetric functions.

Let P_B be the projection of $\mathcal{F}$ on subspace $\mathcal{F}_{\mathcal{B}}$ of symmetric states, and P_F be the projection of $\mathcal{F}$ on subspace $\mathcal{F}_{\mathcal{F}}$ of antisymmetric states.

`We define now creation and annihilation operators`

$$a_B(f) = P_B a_f P_B \, , \; a_B^*(f) = P_B a_f^* P_B \, , \; a_F(f) = P_F a_f P_F \, , \; a_F^*(f) = P_F a_f^* P_F \, ,$$

One can see that

$$[a_B(f), a_B(g)] = [a_B^*(f), a_B^*(g)] = 0 \, , \quad \text{and} \;\; [a_B(f), a_B^*(g)] = \int f(x)g(x)dx$$

Compare with operators

$$a|N >= \sqrt{N}|N-1 > a^+|N >= \sqrt{N+1}|N+1 >$$

Calculate commutators for bosons.

We have that

$$a_B(f)a_B^*(g) \begin{pmatrix} K_0 \\ K_1(x_1) \\ K_2(x_1,x_2) \\ K_3(x_1,x_2,x_3) \\ K_4(x_1,x_2,x_3,x_4) \\ \dots \end{pmatrix} = a_B(f)P_B \begin{pmatrix} 0 \\ K_0 g(x_1) \\ \sqrt{2}K_1(x_1)g(x_2) \\ \sqrt{3}K_2(x_1,x_2)g(x_3) \\ 2K_3(x_1,x_2,x_3)g(x_4) \\ \sqrt{5}K_4(x_1,x_2,x_3,x_4)g(x_5) \\ \dots \end{pmatrix} =$$

$$a_B(f) \begin{pmatrix} 0 \\ g(x_1)K_0 \\ \sqrt{2}\left(\frac{K_1(x_1)g(x_2)+K_1(x_2)g(x_1)}{2}\right) \\ \sqrt{3}\left(\frac{K_2(x_1,x_2)g(x_3)+K_2(x_1,x_3)g(x_2)+K_2(x_3,x_2)g(x_1)}{3}\right) \\ \sqrt{4}\left(\frac{K_3(x_1,x_2,x_3)g(x_4)+K_3(x_1,x_2,x_4)g(x_3)K_3(x_1,x_4,x_3)g(x_2)+K_3(x_4,x_2,x_3)g(x_1)+}{4}\right) \\ \dots \end{pmatrix} =$$

$$\begin{pmatrix} K_0 \int f(y)g(y)dy \\ \sqrt{2}\sqrt{2}\int\left(\frac{K_1(x_1)g(y)+K_1(y)g(x_1)}{2}\right) f(y)dy \\ \sqrt{3}\sqrt{3}\int\left(\frac{K_2(x_1,x_2)g(y)+K_2(x_1,y)g(x_2)+K_2(y,x_2)g(x_1)}{3}\right) f(y)dy \\ \sqrt{4}\sqrt{4}\int\left(\frac{K_3(x_1,x_2,x_3)g(y)+K_3(x_1,x_2,y)g(x_3)K_3(x_1,y,x_3)g(x_2)+K_3(y,x_2,x_3)g(x_1)+}{4}\right) f(y)dy \\ \dots \end{pmatrix} =$$

$$
\left(\begin{matrix}
K_0 \int f(y)g(y)dy \\
K_1(x_1) \int g(y)f(y)dy + \\
K_2(x_1,x_2) \int g(y)f(y)dy + \\
K_3(x_1,x_2,x_3) \int g(y)f(y)dy + \\
\cdots
\end{matrix}\right. ,
$$

$$
\left.\begin{matrix}
g(x_1) \int K_1(y)f(y)dy \\
g(x_2) \int K_2(x_1,y)f(y)dy + g(x_1) \int K_2(x_2,y)f(y)dy \\
g(x_3) \int K_3(x_1,x_2,y)f(y)dy + g(x_1) \int K_3(x_3,x_2,y)f(y)dy + g(x_2) \int K_3(x_1,x_3,y)f(y)dy + \\
\cdots
\end{matrix}\right)
$$

and

$$
a_B^*(g)a_B(f)\begin{pmatrix} K_0 \\ K_1(x_1) \\ K_2(x_1,x_2) \\ K_3(x_1,x_2,x_3) \\ K_4(x_1,x_2,x_3,x_4) \\ \cdots \end{pmatrix} = a_B^*(g)\begin{pmatrix} \int K_1(y)f(y)dy \\ \sqrt{2}\int K_2(x_1,y)f(y) \\ \sqrt{3}\int K_3(x_1,x_2,y)f(y)dy \\ 2\int K_4(x_1,x_2,x_3,y)f(y)dy \\ \cdots \end{pmatrix} =
$$

$$
P_B\begin{pmatrix} 0 \\ g(x_1)\int K_1(y)f(y)dy \\ \sqrt{2}\sqrt{2}\int K_2(x_1,y)f(y)g(x_2) \\ \sqrt{3}\sqrt{3}\int K_3(x_1,x_2,y)f(y)dyg(x_3) \\ 2\cdot 2\int K_4(x_1,x_2,x_3,y)f(y)dyg(x_4) \\ \cdots \end{pmatrix} =
$$

$$
\left(\begin{matrix}
0 \\
g(x_1)\int K_1(y)f(y)dy \\
\int K_2(x_1,y)f(y)g(x_2) + \int K_2(x_2,y)f(y)g(x_1) \\
\int K_3(x_1,x_2,y)f(y)dyg(x_3) + \int K_3(x_3,x_2,y)f(y)dyg(x_1) + \\
\int K_4(x_1,x_2,x_3,y)f(y)dyg(x_4) + \int K_4(x_4,x_2,x_3,y)f(y)dyg(x_1) + \\
\cdots
\end{matrix}\right.
$$

$$
\left.\begin{matrix}
0 \\
0 \\
0 \\
\int K_3(x_1,x_3,y)f(y)dyg(x_2) \\
\int K_4(x_1,x_4,x_3,y)f(y)dyg(x_2) + \int K_4(x_1,x_2,x_4,y)f(y)dyg(x_3) \\
\cdots
\end{matrix}\right).
$$

Thus we see that

$$
a_B(f)a_B^*(g) - a_B^*(g)a_B(f) = \int g(y)f(y)dy\,.
$$

Useful formulae

Let Φ be so called vacuum vector, i.e.

$$\Phi = \begin{pmatrix} 1 \\ 0 \\ \\ \dots \end{pmatrix}$$

Then for every vector

$$\Psi = \begin{pmatrix} K_0 \\ K_1(x_1) \\ K_2(x_1, x_2) \\ K_3(x_1, x_2, x_3) \\ K_4(x_1, x_2, x_3, x_4) \\ \dots \end{pmatrix} =$$

$$\left(\sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \int K_n(x_1, x_2, \dots, x_n) a_B^*(x_1) a_B^*(x_2) \dots a_B^*(x_n) dx^1 dx^2 \dots dx^n \right) \Phi_0 \, .$$

The same formula holds for fermionic case.