Solutions of Homework 3

In all exercises we assume by default that Riemannian metric on embedded surfaces is induced by the Euclidean metric.

1 a) *Show that surface of the cone* $\begin{cases} x^2+y^2-k^2z^2=0 \\ z>0 \end{cases}$ *in* $\mathbf{E}^3$ *is locally Euclidean Riemannian surface (locally isometric to Euclidean plane.*

Solution.

This means that we have to find local coordinates u, v on the cone such that in these coordinates induced metric $G|_c$ on cone would have the appearance $G|_c = du^2+dv^2$.

First of all calculate the metric on cone in coordinates h, φ where

$$\mathbf{r}(h,\varphi)\colon \begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases}.$$

$(x^2+y^2-k^2z^2 = k^2h^2\cos^2\varphi + k^2h^2\sin^2\varphi - k^2h^2 = k^2h^2-k^2h^2 = 0.$

Calculate metric G_c on the cone in coordinates h, φ induced with the Euclidean metric $G = dx^2+dy^2+dz^2$:

$$G_c = \left(dx^2+dy^2+dz^2\right)\big|_{x=kh\cos\varphi, y=kh\sin\varphi, z=h} = (k\cos\varphi dh - kh\sin\varphi d\varphi)^2+$$

$$(k\sin\varphi dh + kh\cos\varphi d\varphi)^2 + dh^2 = (k^2+1)dh^2 + k^2h^2d\varphi^2\,.$$

In analogy with polar coordinates try to find new local coordinates u, v such that $\begin{cases} u = \alpha h\cos\beta\varphi \\ v = \alpha h\sin\beta\varphi \end{cases}$, where α, β are parameters. We come to

$$du^2+dv^2 = (\alpha\cos\beta\varphi dh - \alpha\beta h\sin\beta\varphi d\varphi)^2 + (\alpha\sin\beta\varphi dh + \alpha\beta h\cos\beta\varphi d\varphi)^2 = \alpha^2dh^2+\alpha^2\beta^2h^2d\varphi^2.$$

Comparing with the metric on the cone $G_c = (1+k^2)dh^2+k^2h^2d\varphi^2$ we see that if we put $\alpha = k$ and $\beta = \frac{k}{\sqrt{1+k^2}}$ then $du^2+dv^2 = \alpha^2dh^2+\alpha^2\beta^2h^2d\varphi^2 = (1+k^2)dh^2+k^2h^2d\varphi^2$.

Thus in new local coordinates

$$\begin{cases} u = \sqrt{k^2+1}h\cos\frac{k}{\sqrt{k^2+1}}\varphi \\ v = \sqrt{k^2+1}h\sin\frac{k}{\sqrt{k^2+1}}\varphi \end{cases}$$

induced metric on the cone becomes $G|_c = du^2+dv^2$, i.e. surface of the cone is locally isometric to the Euclidean plane (is locally Euclidean Riemannian surface). ■

2 a) *a) Consider the conic surface C defined by the equation $x^2+y^2-z^2=0$ in $\mathbf{E}^3$. Consider a part of this conic surface between planes $z=0$ and $z=H>0$ and remove the line $z=-x, y=0$ from this part of conic surface C. We come to the surface D defined by the conditions*

$$\begin{cases} x^2+y^2-z^2=0 \\ 0<z<H \\ y\neq 0 \quad \text{if } x<0 \end{cases}.$$

Find a domain D' in Euclidean plane such that it is isometric to the surface D.

b) Find a shortest distance between points $A=(1,0,1)$ and $B=(-1,0,1)$, between points A and $D=(0,1,1)$ for an ant living on the conic surface C.

Solution.

The domain D on the cone can be parameterised as (see the previous exeercise for $k = 1$)

$$\mathbf{r}(h,\varphi)\colon \begin{cases} x = h\cos\varphi \\ y = h\sin\varphi \\ z = h \end{cases} \quad 0 < h < H, -\pi < \varphi < \pi$$

Notice that we consider the range of φ $(-\pi,\pi)$ not $(0,2\pi)$ as usual, since the removed line $z = -x, y = 0$ corresponds to the points with $\varphi = \pm\pi$. If we would remove the line $z = x, y = 0$, then the parameterisation $0 < \varphi < 2\pi$ would be valid.

Using the results of previous exercise for $k = 1$ consider new local coordinates

$$u, v\colon \quad \begin{cases} u = \sqrt{2}h\cos\frac{\varphi}{\sqrt{2}} \\ v = \sqrt{2}h\sin\frac{\varphi}{\sqrt{2}} \end{cases} \quad 0 < h < H, -\pi < \varphi < \pi\,.$$

In these coordinates metric $G = du^2 + dv^2$. Consider Euclidean plane with Cartesian coordinates u, v and new polar coordinates R, θ:

$$(R,\theta)\colon \quad \begin{cases} R = \sqrt{u^2+v^2} = \sqrt{2}h \\ \theta = \frac{\varphi}{\sqrt{2}} \end{cases} \quad 0 < z < h, -\pi < \varphi < \pi\,.$$

We see that for the angle $\varphi \mapsto \theta = \varphi\sqrt{2}$, for u, v, $0 < R = \sqrt{u^2+v^2} < h\sqrt{2}$. We come to the domain, sector D' in $\mathbf{E}^2$ such that

$$0 < R < \sqrt{2}H,\ , 0 < \theta < \frac{2\pi}{\sqrt{2}}\,,$$

where R, θ are polar coordinates on the plane. Cartesian coordinates on this plane are $u = R\cos\theta, v = R\sin\theta$. It is what happens with the cone when we use scissors!. We come to the sector of the circle of the radius $R = H\sqrt{2}$ with the arc $L = 2\pi H$.

To find the shortest distance between points A, B on the cone we find this distance on the domain D' defined above, since D' has metric of Euclidean plane. The points A, B and the origin O (strictly speaking their images on D') make the isosceles triangle $\triangle OAB$ with $OA = OB = \sqrt{2}$, and the angle $\angle A0B = \frac{\pi}{\sqrt{2}}$ The distance $|AB| = 2|0A|\sin\frac{\angle AOB}{2} = 2\sqrt{2}\sin\frac{\pi}{2\sqrt{2}}$.

The "naive" distance (trip around the circle) equals to $\pi > 2\sqrt{2}\sin\frac{\pi}{2\sqrt{2}}$.

Remark Of course the point B on the cone is already removed, since it was belonged to the cutted line $z = -x$. But we can take instead the point B' which belongs to domain D and is infinitesimally close to the point B. (B belongs to the closure of the open domain D'.)

Analogous calculations for the points A and C. The points A, C and the origin O on the domain D' make the isosceles triangle $\triangle OAC$ with $OA = OB = \sqrt{2}$, and the angle $\angle A0B = \frac{\pi}{2\sqrt{2}}$. The distance $|AC| = 2|0A|\sin\frac{\angle AOC}{2} = 2\sqrt{2}\sin\frac{\pi}{4\sqrt{2}}$. The "naive" distance (trip around the circle) equals to $\pi/2$.

3 *Consider plane with Riemannian metric given in Cartesian coordinates (x, y) by the formula*

$$G = \frac{a\left((dx)^2 + (dy)^2\right)}{(1+x^2+y^2)^2}\,,$$

and a sphere of the radius r in the Euclidean space $\mathbf{E}^3$. Find r such that this palne is isometric to the sphere without north pole. (You may use the formula for Riemannian metric on the sphere in stereographic coordinates.)

Recall that for sphere of radius R, $x^2 + y^2 + z^2 = R^2$ in stereographic coordinates u, v:

$$\begin{cases} u = \frac{Rx}{R-z} \\ v = \frac{Ry}{R-z} \end{cases}, x^2 + y^2 + z^2 = R^2$$

and induced Riemannian metric is $G = 4R^4 \frac{du^2+dv^2}{(R^2+u^2+v^2)^2}$.

These coordinates establish diffeomorphism between sphere without north pole. and plane

If we consider new coordinates $x = Ru, y = Rv$ then

$$G = 4R^4 \frac{du^2 + 4dv^2}{(R^2+u^2+v^2)^2} = 4R^4 \frac{R^2dx^2 + R^2dy^2}{(R^2+R^2x^2+R^2y^2)^2} = \frac{4R^2dx^2 + 4R^2dy^2}{(1+x^2+y^2)^2} .$$

So the answer is the following: The plane with metric $G = \frac{a\left((dx)^2+(dy)^2\right)}{(1+x^2+y^2)^2}$ is isometric to the sphere of the radius $R = \frac{\sqrt{a}}{2}$ without north pole.

4 *Consider catenoid: $x^2 + y^2 = \cosh^2 z$ and helicoid: $y - x\tan z = 0$.*

Find induced Riemannian metrics on these surfaces.

Show that these surfaces are locally isometric.

Write down following parametric equations for catenoid and helicoid.

Catenoid is the surface of resolution:

$$\mathbf{r}(t,\varphi)\colon \quad \begin{cases} x = f(t)\cos\varphi \\ y = f(t)\sin\varphi \\ z = t \end{cases}$$

for $f(t) = cosht$, i.e.

$$\mathbf{r}(t,\varphi)\colon \quad \begin{cases} x = \cosh t\cos\varphi \\ y = \cosh t\sin\varphi \\ z = t \end{cases} \qquad \text{(catenoid)}$$

$(x^2 + y^2 - \cosh^2 z = 0)$.

We come to helicoid If we rotate the horisontal line and move it in vertical direction with constant speeds *:

$$\mathbf{r}(u,\theta)\colon \quad \begin{cases} x = t\cos\varphi \\ y = t\sin\varphi \\ z = \varphi \end{cases} \qquad \text{(helicoid)}$$

Calculate induced Riemannian structures:

$$G_{cat} = (dx^2 + dy^2 + dz^2)\big|_{x=\cosh t\cos\varphi, y=\cosh t\sin\varphi, z=t} =$$

$$(\sinh t\cos\varphi dt - \cosh t\sin\varphi d\varphi)^2 + (\sinh t\sin\varphi dt + \cosh t\cos\varphi d\varphi)^2 + dt^2 =$$

$$(1+\sinh^2 t)dt^2 + \cosh^2 t d\varphi^2 = \cosh^2 t(dt^2 + d\varphi^2) . \tag{2}$$

$$G_{hel} = (dx^2 + dy^2 + dz^2)\big|_{x=t\cos\varphi, y=t\sin\varphi, z=\varphi} =$$

$$(dt\cos\varphi - t\sin\varphi d\varphi)^2 + (dt\sin\varphi + t\cos\varphi d\varphi)^2 + d\varphi^2 =$$

$$dt^2 + t^2d\varphi^2 + d\varphi^2 = dt^2 + (1+t^2)d\varphi^2 . \tag{3}$$

Compare Riemannian metrics (2) and (3). We see that if we consider in (3) $t \mapsto \sinh t$ we come to (2):

$$G_{helicoid} = \left(dt^2 + (1+t^2)d\varphi^2\right)_{t\mapsto\sinh t} = (d\sinh t)^2 + (1+\sinh^2 t)d\varphi^2 = \cosh^2 t(dt^2 + d\varphi^2) = G_{hel} .$$

Résumé: We see that transformation $t \mapsto \sinh t$ and identification of second local coordinate φ for catenoind and helicoid implies the identification of metrics. We nay consider domain $0 < \varphi < 2\pi$ of catenoid and domain $0 < \varphi < 2\pi$ of helicoid and these domains are isometric:

$$\text{Domain of catenoid} \begin{cases} x^2 + y^2 = \cosh^2 z \\ y \neq 0 \text{ if } x > 0 \end{cases} \quad \text{is isometric to the domain of helicoid} \quad \begin{cases} y = x\tan z \\ 0 < z < 2\pi \end{cases}$$

* look in Wikipedia for images

You could find very beautiful picture how helicoid isometrically can be transformd to catenoid (see Wikipedia).

5 a) *Consider the domain D on the cone $x^2+y^2-k^2z^2$ defined by the condition $0<z<H$. Find an area of this domain using induced Riemannian metric. Compare with the answer when using standard formulae.*

We have cone with height H with radius $R=kH$ ($k>0$).

First of all standard answer: The area of cone (of surface of cone) is area of the sector with the radius $\sqrt{H^2+R^2}$ and length of the arc $2\pi R$:

$$S=\frac{1}{2}\cdot\sqrt{R^2+H^2}\cdot 2\pi R=\pi R\sqrt{H^2+R^2}=\pi k\sqrt{1+k^2}H^2\,.$$

Now calculate this are using Riemannian geometry. It follows from the result of the exercise (2) that volume form on the cone equals

$$d\sigma=\sqrt{\det G}dh\wedge d\varphi=k\sqrt{1+k^2}dh\wedge d\varphi$$

since $G=\begin{pmatrix}k^2+1 & 0\\ 0 & k^2h^2\end{pmatrix}$ Hence

$$S=\int_{0<h<H}\sqrt{\det G}dh\wedge d\varphi=\int_{0<h<H}k\sqrt{1+k^2}dh\wedge d\varphi=2\pi k\sqrt{1+k^2}\int_0^H hdh=\pi k\sqrt{1+k^2}H^2\,.$$

(Compare with standard calculations).

6 *Find an area of 2-dimensional sphere of radius a using explicit formulae for induced Riemannian metric in stereographic coordinates.*

Riemannian metric for sphere (without point) in stereographic coordinates is $G=\frac{4R^4du^2+4R^4dv^2}{(R^2+u^2+v^2)^2}$. We already know that doing transformation $u\mapsto ru, v\mapsto Rv$ we come to the expression

$$G=\frac{4R^2du^2+4R^2dv^2}{(1+u^2+v^2)^2}$$

(see the exercise 3.)

$$G=\begin{pmatrix}\frac{4R^2}{(1+u^2+v^2)^2} & 0\\ 0 & \frac{4R^2}{(1+u^2+v^2)^2}\end{pmatrix},\ \det G=\frac{16R^4}{(1+u^2+v^2)^4}$$

Hence the volume (area)of the sphere equals to

$$S=\int_{\mathbf{R}^2}\sqrt{\det G}dudv=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{4R^2}{(1+u^2+v^2)^2}\right)dudv$$

Choosing polar coordinates $u=r\cos\varphi, v=r\sin\varphi$ we come to

$$S=\int_0^{\infty}\int_0^{2\pi}\left(\frac{4R^2}{(1+r^2)^2}\right)d\varphi rdr=8\pi R^2\int_0^{\infty}\frac{rdr}{1+r^2}=4\pi R^2\,.$$

7 *Show that two spheres of different radii in Euclidean space are not isometric to each other.*

Suppose that these two spheres of different radii are isometric (globally). This means that their volume is the same. Contradiction. (In fact two spheres of different radii are not isometric even locally, since they have different curvatures.)

8 *Find new local coordinates $u = u(x,y), v = v(x,y)$ in Euclidean space $\mathbf{E}^2$ such that $du^2 + dv^2 = dx^2 + dy^2$ (transformation is linear: $u = a + bx + cy$, $v = e + dx + fy$)*

* *Will answer change if we allow arbitrary (not only linear transformations?)*

If $u = a + bx + cy$, $v = c + dx + fy$ then

$$du^2 + dv^2 = (bdx + cdy)^2 + (ddx + fdy)^2 = (b^2 + d^2)dx^2 + 2(bc + df)dxdy + (c^2 + f^2)dy^2 = dx^2 + dy^2$$

This means that $b^2 + d^2 = c^2 + f^2 = 1$ and $bc + df = 0$, i.e. for matrix $A = \begin{pmatrix} b & d \\ c & f \end{pmatrix}$ rows have length 1 and they are orthogonal, i.e. the matrix A is orthogonal: $AA^T = I$. We come to the answer: In the class of linear transformations the transformation that preserves the Euclidean metric is a translation and orthogonal transformation, i.e. translations, rotations and reflections.

* It is very interesting to answer the question: how look general transformations which preserve the metric? Answer: any transformation preserving Euclidean metric is linear transformation, i.e. there is no rotation "depending on point." There are many different and beautiful and illuminating proofs of this fact which is true for any dimensions. We consider here not the best one:

Let $u = u(x,y), v = v(x,y)$ be transformation such that $du^2 + dv^2 = dx^2 + dy^2$, i.e. according to calculations above

$$(u_x^2 + v_x^2 \quad u_x u_y + v_x v_y u_x u_y + v_x v_y \quad v_x^2 + v_y^2) = (1 \quad 00 \quad 1)$$

Then orthogonality condition for matrix will imply that condition that

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} \cos\Psi(x,y) & -\sin\Psi(x,y) \\ \sin\Psi(x,y) & \cos\Psi(x,y) \end{pmatrix}$$

(if determinant is negative we compensate by transformation $u \mapsto -u$.)

Thus the function $F(x,y) = u(x,y) + iv(x,y)$ is holomorphic function (Cauchy-Riemann conditions $u_x = v_y$, $u_y + v_x = 0$). The condition $u_x^2 + v_x^2 = 1$ means that the modulus of the analytical function $F' = \frac{\partial F}{\partial z}$ equals to 1. Thus $F' = const$ and $F = e^{i\varphi}z + c$, i.e. we have rotation on angle φ and translation. (There is no differential rotation!)

(In the case if determinant is negative then we come to antiholomorphic transformation: $F = a\bar{z} + c$, i.e. reflection, rotation and translation.)

9 *Let D be a domain in Lobachevsky plane which is lying between lines $x = a, x = -a$ and outside of the disc $x^2 + y^2 = 1$, $(0 < a < 1)$: $D = \{(x,y)\colon\ |x| < a, x^2 + y^2 > 1\}$,*

a) Find the area of this domain.

b) Find the angles between lines and arc of the circle.*

Lobachevsky plane, i.e. hyperbolic plane is the upper half plane with Riemannian metric $\frac{dx^2+dy^2}{y^2}$ in cartesian coordinates x,y $(y > 0)$.

a) We have $G = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}$. We see that $\sqrt{\det G} = \frac{1}{y^2}$. Hence

$$S = \int_{x^2+y^2\geq 1, -a\leq x\leq a} \sqrt{\det G}dxdy = \int_{x^2+y^2\geq 1, -a\leq x\leq a} \frac{1}{y^2}dxdy = \int_{-a}^{a} \left(\int_{\sqrt{1-x^2}}^{\infty} \frac{dy}{y^2} \right) dx =$$

$$\int_{-a}^{a} \frac{dx}{\sqrt{1-x^2}} = 2\arcsin a\,.$$

This has a deep geometrical meaning[*]!

Note that if two metrics G, G' are proportional, $G' = \sigma(\mathbf{x})G$, i.e. $g'_{ik} = \sigma(x)g_{ik}$ then the angles calculated with respect to these metrics are the same:

$$\cos \mathbf{t}\angle(\mathbf{X},\mathbf{Y}) = \frac{G'(\mathbf{X},\mathbf{Y})}{\sqrt{\mathbf{t}G(\mathbf{X},\mathbf{X})}\sqrt{\mathbf{t}G(\mathbf{Y},\mathbf{Y})}} = \frac{\sigma G(\mathbf{X},\mathbf{Y})}{\sqrt{\sigma G(\mathbf{X},\mathbf{X})}\sqrt{\sigma G(\mathbf{Y},\mathbf{Y})}} = \frac{\sigma}{\sigma}\frac{G(\mathbf{X},\mathbf{Y})}{\sqrt{G(\mathbf{X},\mathbf{X})}\sqrt{G(\mathbf{Y},\mathbf{Y})}} = \cos\angle(\mathbf{X},\mathbf{Y})$$

(Two proportional metrics are called conformally equivalent).

Notice that Lobachevsky metric $G = \frac{dx^2+dy^2}{y^2} = \frac{1}{y^2}(dx^2+dy^2)$ is proportional to the Euclidean metric dx^2+dy^2. Hence the angles will be the same as in the Euclidean metric. (The difference is that straigh lines in Euclidean metric are not “straigh lines” in Lobachevsky plane)

10[†] *Find a volume of n-dimensional sphere of radius a. (You may use Riemannian metric in stereographic coordinates, or you may do it in other way... You just have to calculate the answer.)*

Denote by σ_n the volume of n-dimensional unit sphere embedded in Euclidean space $\mathbf{E}^{n+1}$. Then the volume of n-dimensional sphere of the radius R is equal to $\sigma_n R^n$. Now consider the integral

$$I = \int_{-\infty}^{\infty} e^{-t^2}dt = \sqrt{\pi}.$$

For any integer k consider

$$I^k = \pi^{\frac{k}{2}} = \left(\int_{-\infty}^{\infty} e^{-t^2}dt\right)^k = \int_{\mathbf{E}^k} e^{-x_1^2-x_2^2-\dots-x_k^2}dx_1dx_2\dots dx_k\,.$$

Make changing of variables in the volume form $dx_1 \wedge dx_2 \wedge \dots \wedge dx_k$. Since integrand depend only on the radius we can rewrite the integral above as

$$\int_{\mathbf{E}^k} e^{-x_1^2-x_2^2-\dots-x_k^2}dx_1dx_2\dots dx_k = \int_{\mathbf{E}^k} e^{-r^2}r^{k-1}\sigma_{k-1}dr\,,$$

where σ_{k-1} is a volume of the unit sphere in dimension $k-1$. (Here is the trick!) We have the identity:

$$\pi^{\frac{k}{2}} = \sigma_{k-1}\int_0^{\infty} e^{-r^2}r^{k-1}dr$$

To calculate this integral consider $r^2 = t$ we come to

$$\int_0^{\infty} e^{-r^2}r^{k-1}dr = \frac{1}{2}\int_0^{\infty} e^{-t}t^{\frac{k}{2}-1}dt = \frac{1}{2}\Gamma\left(\frac{k}{2}\right)\,.$$

We come to

$$\pi^{\frac{k}{2}} = \sigma_{k-1}\int_0^{\infty} e^{-r^2}r^{k-1}dr = \frac{\sigma_{k-1}}{2}\Gamma\left(\frac{k}{2}\right)\,.$$

Thus

$$\sigma_{k-1} = \frac{2\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)}.$$

Recall that $\Gamma(x)$ can be calculated for all $\frac{k}{2}$ using the following recurrent formulae:

1. $\Gamma(n+1) = n!$
2. $\Gamma(x+1) = x\Gamma(x)$
3. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, $(\Gamma(x)\Gamma(1-x) = \pi\sin\pi x)$.

E.g. the volume of the 15-dimensional unit sphere in $\mathbf{E}^{16}$ equals to $\sigma_{15} = \frac{2\pi^8}{\Gamma(8)} = \frac{2\pi^6}{7!}$

[*] there is a remarkable formula that for an arbitrary triangle sum of its angles minus π is equal to integral of curvature over area of triangle: in the case if curvature if constant (this is the case for sphere and hyperbolic plane) it is just proportional to area of triangle.)