Introduction to Geometry (20222)

2012

COURSEWORK

This assignment counts for 20% of your marks.

Solutions are due at noon on 19 April

Write solutions in the provided spaces.

STUDENTS'S NAME:

Academic Advisor (Tutor):

1

a) Let (x^1, x^2, x^3) be coordinates of the vector $\mathbf{x}$, and (y^1, y^2, y^3) be coordinates of the vector $\mathbf{y}$ in $\mathbf{R}^3$.

Does the formula $(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^2 + x^2y^3 + x^3y^2 + x^3y^3$ define a scalar product on $\mathbf{R}^3$? Justify your answer.

b) Let $\mathbf{x}, \mathbf{y}$ be two vectors in the Euclidean space $\mathbf{E}^2$ such that the length of the vector $\mathbf{x}$ is equal to 1, the length of the vector $\mathbf{y}$ is equal to 29 and scalar product of these vectors is equal to 21.

Find a vector $\mathbf{e}$ in $\mathbf{E}^2$ (express it through the vectors $\mathbf{x}$ and $\mathbf{y}$) such that the following conditions hold

i) an ordered pair $\{\mathbf{e}, \mathbf{x}\}$ is an orthonormal basis in $\mathbf{E}^2$,

ii) the vector $\mathbf{e}$ has an obtuse angle with the vector $\mathbf{y}$.

(c) Consider the matrix $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Calculate the matrix A^2 in the case if $\theta = \frac{\pi}{4}$.

Calculate the matrix A^{18} in the case if $\theta = \frac{\pi}{6}$.

Find all 2×2 orthogonal matrices A such that

$$2A^3 = \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} .$$

2

a) Consider vector $\mathbf{a} = 2\mathbf{e} + 3\mathbf{f} + 6\mathbf{g}$, where $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ is an orthonormal basis in $\mathbf{E}^3$.

Show that the angle θ between vectors $\mathbf{a}$ and $\mathbf{g}$ belongs to the interval $\left(\frac{\pi}{6}, \frac{\pi}{4}\right)$.

Find a unit vector $\mathbf{b}$ such that this vector is orthogonal to the vectors $\mathbf{a}$ and $\mathbf{g}$, and the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{g}\}$ has the same orientation as the basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$.

Calculate the angle between vectors $\mathbf{b}$ and $\mathbf{e}$.

b) In oriented Euclidean space $\mathbf{E}^3$ consider the following function of three vectors:

$$F(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = (\mathbf{X}, \mathbf{Y} \times \mathbf{Z}),$$

where $(\, , \,)$ is the scalar product and $\mathbf{Y} \times \mathbf{Z}$ is the vector product in $\mathbf{E}^3$.

Show that $F(\mathbf{X}, \mathbf{X}, \mathbf{Z}) = 0$ for arbitrary vectors $\mathbf{X}$ and $\mathbf{Z}$.

Deduce that $F(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = -F(\mathbf{Y}, \mathbf{X}, \mathbf{Z})$ for arbitrary vectors $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$.

What is the geometrical meaning of the function F?

c) Give an example of at least one parallelogram in the plane such that all its vertices have integer coordinates, its area is equal to 1 and one of the sides has length greater than 2012.

Describe all parallelograms $ABCD$ in the coordinate plane which obey the following conditions: its area is equal to 1 and its vertices have integer coordinates such that vertex A is at the origin and vertex B has coordinates $(2, 3)$.

3

We consider in this question 3-dimensional oriented Euclidean space. We suppose that $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ is an orthonormal basis in this space which defines the orientation.

a) Let P be a linear orthogonal operator acting in $\mathbf{E}^3$, such that it preserves the orientation of $\mathbf{E}^3$ and the following relations hold:

$$P(\mathbf{e}) = \cos\theta\ \mathbf{e} + \sin\theta\ \mathbf{f}, \quad P(\mathbf{g}) = \varepsilon\mathbf{g}\,,$$

where θ is an arbitrary angle and $\varepsilon = \pm 1$.

Write down the matrix of operator P in the basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$.

(*You have to consider both cases* $\varepsilon = 1$ *and* $\varepsilon = -1$.)

b) We know that due to the Euler Theorem linear operator P considered above is rotation operator.

Find the axis of this rotation.

(You have to consider both cases $\varepsilon = 1$ and $\varepsilon = -1$.)

c) Let P be a linear operator acting in $\mathbf{E}^3$, such that $P(\mathbf{e}) = \mathbf{f}$, $P(\mathbf{f}) = \mathbf{g}$ and $P(\mathbf{g}) = \mathbf{e}$.

Show that P is a rotation operator.

Find the axis and the angle of the rotation.

4

a) Given a vector field $\mathbf{G} = ar\frac{\partial}{\partial r} + b\frac{\partial}{\partial \varphi}$ in polar coordinates express it in Cartesian coordinates ($x = r\cos\varphi$, $y = r\sin\varphi$).

Consider the function $f = r^3 \cos 3\varphi$ and the vector fields $\mathbf{A} = x\partial_x + y\partial_y$, $\mathbf{B} = x\partial_y - y\partial_x$. Calculate $\partial_{\mathbf{A}} f$, $\partial_{\mathbf{B}} f$. Express the answers in polar and Cartesian coordinates.

b) Let C be the upper half of the circle with centre at the point $(R, 0)$ which is tangent to the y-axis. Write down an equation of this circle. Choose any parameterisation of this curve and calculate the integral $\int_C \omega$ if i) $\omega = x^2 dy$ and if ii) $\omega = x^2 dy + 2xydx$.

How does your answer depend on a choice of parameterisation?

Explain why 1-form $\omega = x^2 dy$ is not an exact form.

c) Consider the curve in $\mathbf{E}^2$ defined by the equation $r(2 - \cos\varphi) = 3$ in polar coordinates.

Show that the sum of the distances between the points $F_1 = (0, 0)$ and $F_2 = (2, 0)$, and an arbitrary point of this curve is constant, i.e. the curve is an ellipse and points F_1, F_2 are its foci.

Find the integral of the 1-form $\omega = xdy - ydx$ over this curve. (Here as usual x, y are Cartesian coordinates $x = r\cos\varphi, y = r\sin\varphi$.)