Geometry of diff.equations

§1 Necessary mathematics from Arnold's book

`I began to understand the pages in Arnold on diff. equations.....`

Here it is:

Definition Let ω be 1-form on M which does not vanish. . We say that it is *contact* form if

2-form $d\omega$ is non-degenerate on the plane $\omega = 0$ in TM

Since $d\omega$ is not degenerate on $\omega = 0$ and $\omega \not\equiv 0$ them $\dim M = 2k + 1$.

Theorem Contact form is defined up to a function (Valya had a talk on it!) If ω is contact form and $f \neq 0$ then $f\alpha$ is contact also.

Let $\mathcal{K}$ be a distributions of $2n$-dimensional planes in TM such that ω vanishes on these planes.

We say that this distribution is a contct structure *

Theorem Let N be a submanifold of M which is an integral submanifold (not necessarily maximal) of contact distribution $\mathcal{K}$, i.e. for every point on N the tangent vectors belong to this distribution. Then $\dim N \leq n$, where $\dim M \leq 2n + 1$

`Proof` Let ω be an arbitrary non-zero form which vanishes at $\mathcal{K}$. Since a form ω vanishes on vectors tangent to the manifold N, the form $d\omega$ vanishes on these vectors also:

$$\iota\colon N \subset M, \qquad d\left(\iota^*\omega\big|_N\right) = \iota^* dw\big|_N = 0\,.$$

Hence two arbitrary vectors are orthogonal to each other with respect to this form. If dimension of tangent plane is bigger than n then there exist at least two vectors which are not orthogonal, since $d\omega$ is not degenerate. Now we apply this mathematics to the differential equations.

§2 Geometry of first order equation

Let J^1M be a space of first jets of functions on manifold M. Cooridnates on J^1M are (p_i, q^j, u), where q^j are coordinates on M. Jet of every function $u = u(x)$ has coordinates $\left(p_i = \frac{\partial u}{\partial x q^i}, q^i, u\right)$.

Consider $\mathcal{C}$, the Cartan distribution of $2n$-dimensional planes in J^1M defined by the form $\omega = p_i dq^i - du$

$$\mathcal{C}_{\mathbf{p}} \subset T_{\mathbf{p}} J^1M = \{T_{\mathbf{p}}(J^1M) \ni \mathbf{X}\colon\ \omega(\mathbf{X}) = 0\}\,,$$

Vector field

$$M^i \frac{\partial}{\partial q^i} + N_i \frac{\partial}{\partial p_i} + A\frac{\partial}{\partial u} \text{ belongs to Cartan distribution } \mathcal{C} \text{ if } A = p_i M^i\,.$$

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* One can say that distribution of hyperplanes defines constant structure if an 1-form which vanishes this dstribution is non-degenerated on it

$\mathcal{C}$ is non-integrable distribution. It is a *contact structure* and the form $\omega_C = p_i dq^i = du$ is a contact form since

$$d\omega\big|_{\omega=0} = dp_i \wedge dq^i$$

is non-degenerte form. Consider differential equation,

$$\mathcal{E}\colon F(p,q,u) = 0\,.$$

Differential equation is sumbmanifold of codimension 1 in the space $J^1(M)$.

The Cartan distribution $\mathcal{C}$ of hyperplanes on J^1M defines distribution $\mathcal{C}(\mathcal{E})$ in $T\mathcal{E}$:

$$\mathcal{C}(E) = \mathcal{C} \cap T\mathcal{E}\,.$$

$$\mathbf{X} = M^i\frac{\partial}{\partial q^i} + N_i\frac{\partial}{\partial p_i} + A\frac{\partial}{\partial u} \in \mathcal{C}(\mathcal{E}) \text{ if } A = p_i M^i \& \left(M^i\frac{\partial}{\partial q^i} + N_i\frac{\partial}{\partial p_i} + A\frac{\partial}{\partial u}\right) F(p,q,u)\big|_{F=0} = 0\,.$$

This distribution is not integrable.

The solution of differential equation (1) is the maximal integral of the distribution.

What is the dimension of N?

The considerations of the first paragraph say that

$$\dim N \leq n$$

If N is the n-dimensional submanifold in M. (We come in this case to the n-parametric family of solutions?)

Let N be an arbitrary solution, N is the surface of dimension at least n. Any tangent plane to N belongs to Cartan distribution and is tangent to $\mathcal{E}$. Consider an arbitrary point $\mathbf{p} \in N$. Let $\alpha = \alpha_{\mathbf{p}}$ be the tangent plane. The vectors in tangent plane belong to distribution $\mathcal{C}_{\mathcal{E}} = \mathcal{C} \cap \mathcal{E}$, i.e. they are orthogonal to the covector (1-form) in $2n+1$-dimensional space

$$\omega_C = (1, 0, \dots, 0, -p_1, \dots, -p_n)\,, \qquad \text{(the vector belongs to Cartan distribution } \mathcal{C}\text{)}$$

and to the covector

$$dF = \left(F_u, \frac{\partial F}{\partial q^1}, \dots, \frac{\partial F}{\partial q^n}; \frac{\partial F}{\partial p_1}, \dots \frac{\partial F}{\partial p_n}\right)\,, \qquad \text{(the vector is tangent to the differential equation } \mathcal{E} =$$

$$\alpha_{\mathbf{p}} \ni \mathbf{X} \Leftrightarrow \omega_C(\mathbf{X}) = dF(\mathbf{X}) = 0\,.$$

Definition The point $\mathbf{p}$ is not singular point if tnhe rank of the matrix $\begin{pmatrix}\omega_C\\ dF\end{pmatrix}$ is equal to 2. IN this case the dimension of