Homework 6

We consider here the realisation of Lobachevsky plane (hyperbolic plane) as upper half of Euclidean plane $\{(x,y)\colon y>0\}$ *with the metric* $G=\frac{dx^2+dy^2}{y^2}$.

1 Calculate Christoffel symbols of Levi-Civita connection on Lobachevsky plane

2 Show that vertical lines $x=a$ are geodesics (non-parameterised) on Lobachevsky plane.

3 Let $\mathbf{r}=\mathbf{r}(t)$ be an arbitrary geodesic on Lobachevsky plane. Show that magnitudes $I=I=\frac{v_x}{y^2}$ and $E=\frac{v_x^2+v_y^2}{2y^2}$ are preserved along geodesics.

4 Show that the following transformations are isometries of Lobachevsky plane:

a) horizontal translation $\mathbf{r}\to\mathbf{r}+\mathbf{a}$ where $\mathbf{a}=(a,0)$,

b) homothety: $\mathbf{r}\to\lambda\mathbf{r}$ $(\lambda>0)$,

* c) inversion with the centre at the points of the line $y=0$*:

$$\mathbf{r}\to\mathbf{a}+\frac{\mathbf{r}-\mathbf{a}}{|\mathbf{r}-\mathbf{a}|^2} \text{ where } \mathbf{a}=(a,0)\colon \quad \begin{cases} x'=a+\frac{x-a}{(x-a)^2+y^2} \\ y'=\frac{y}{(x-a)^2+y^2} \end{cases}.$$

5* *Show that upper arcs of semicircles* $(x-a)^2+y^2=R^2, y>0$ *are (non-parametersied) geodesics.*

(You can do this exercise solving explicitly differential equations for geodesics, or using integrals of motion obtained in exercise 3, or (and this is most beautiful) use inversion transformation and the results of exercise 2.)

6 Show that parallel transport along the given curve does not depend on parameterisation of the curve.

7 Consider the vector $\mathbf{X}=-\frac{\partial}{\partial x}$ attached at the point (0,a) of the Lobachevsky plane and the arc of circle C: $\begin{cases} x(t)=a\cos t \\ y(t)=a\sin t \end{cases}$, $\frac{\pi}{2}\leq t\leq\frac{3\pi}{4}$ in the Lobachevsky plane. Find parallel transport of the vector $\mathbf{X}$ along the curve C.

(You may use the fact that C is an arc of geodesic (non-parameterised) geodesic.)

8 Find geodesics on the sphere in Euclidean space.

9 Consider a sphere of the radius R in $\mathbf{E}^3$ and an arbitrary vector $\mathbf{X}$ attached at the point (θ_0,φ_0) and tangent to this sphere.

What will be the result of parallel transport of the vector $\mathbf{X}$ along the following closed curves on the sphere

a) $C_1\colon\varphi(t)=\varphi_0,\varphi+\pi$ (two meridians), b) $C_2\colon\theta(t)=\theta_0$ (latitude.)

(θ,φ are spherical coordinates.)

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* This line is called *absolute*.