Geometry of Differential operators

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This is very preliminary version of lectures which I plan to have on the School to Workshop. Lectures contain textbook staff+ something that I did with Ted Voronov (mostly the first part) and something that I did with Adam Biggs (end of the second part)).

Contents

1 Differential operators and algebra of densities

1.1 Differential operators on functions

M is a manifold.

We say that Δ is an operator on functions on M of the order $\leq n$ if for an arbitrary function g $\Delta_g = g \circ \Delta - \Delta \circ g$ is an operator of the order $\leq n-1$. (non-zero operator L has order 0 if it is linear with respect to algebra of functions:$L(fg) = fL(g)$)

$\mathbf{X}(fg) = f\mathbf{X}(g) + g\mathbf{X}(f)$. Every linear operator that obeys this identity is a vector field. One can se that L is first order operator if $L = \mathbf{X} + R$ where $R = L(1)$ is a function (scalar). What about higher order operators?

If Δ is n-th order operator on functions then

$$\Delta = S^{i_1 i_2 \dots i_n} \partial_{i_1} \dots \partial_{i_n} + T^{i_1 i_2 \dots i_{n-1}} \partial_{i_1} \dots \partial_{i_{n-1}} + \dots$$

The first term $S^{i_1 i_2 \dots i_n} \partial_{i_1} \dots \partial_{i_n}$ transforms in the following way under changing of coordinates:

$$S^{i_1 i_2 \dots i_n} = \frac{\partial x^{i'_1}}{\partial x^{i_1}} \frac{\partial x^{i'_2}}{\partial x^{i_2}} \dots \frac{\partial x^{i'_n}}{\partial x^{i_n}} S^{i_1 i_2 \dots i_n}$$

This is symmetric n-th order contravariant tensor*principal symbol* of operator Δ.

What about the next terms?

Consider this question in more detail for second order operators

1.1.1 Second order operators on functions and connections

Let Δ be second order operator. In local coordinates $\Delta = S^{ij} \partial_i \partial_j + T^i \partial_i$.

We already know that first order operator is vector field+scalar. Use it. Consider a scalar product $\langle \quad , \quad \rangle_{\boldsymbol{\rho}} = \int_M fg\boldsymbol{\rho}$, where $\boldsymbol{\rho} = \rho(x)|Dx|$ is an arbitrary volume form. Then with respect to this scalar product consider conjugate operator:

$$\Delta^+ = \frac{1}{\rho} \partial_i \partial_j S^{ij} - \frac{1}{\rho} \partial_i T^i + R\,.$$

We see that

$$\Delta^+ - \Delta = 2\partial_r S^{ri} + 2S^{ik} \partial_k \log \rho \partial_i - 2T^i \partial_i + \dots$$

is first order operator. Hence

$$K^i = 2\partial_r S^{ri} + 2S^{ik}\partial_k \log\rho\partial_i - 2T^i\partial_i$$

is the vector field.

$\gamma_k = \partial_k \log\rho$ defines connection...

What is it a connection? It defines derivative of an arbitrary volume form:

$$\nabla_{\mathbf{X}}(f\boldsymbol{\rho}') = \partial_{\mathbf{X}} f\boldsymbol{\rho}' + f\nabla_{\mathbf{X}}\boldsymbol{\rho}'$$

Denote by γ_k: $\quad \gamma_k = \nabla_k|Dx|$ in given coordinates. Then

$$\nabla_{\mathbf{X}}(f\boldsymbol{\rho}') = \nabla_{\mathbf{X}}(f\rho'(x)|Dx|) = X^i\left(\partial_i(f\rho') + \gamma_i f\rho'(x)\right)|Dx|\,.$$

An arbitrary volume $\boldsymbol{\rho}$ form defines flat connection ∇:

$$\nabla_{\mathbf{X}}\colon\ \nabla_{\boldsymbol{\rho}}(\boldsymbol{\rho}') = \partial_{\mathbf{X}}\left(\frac{\boldsymbol{\rho}'}{\boldsymbol{\rho}}\right)$$

with $\Gamma_k = -\partial_k \log\boldsymbol{\rho}$.

Returning to our second order operator we see that

$$T^i = \partial_r S^{ri} + S^{ik}\partial_k \log\rho - \frac{1}{2}K^i = \partial_r S^{ri} + \Gamma^i - \frac{1}{2}K^i = \partial_r S^{ri} + \gamma^i$$

Fact $T^i - \partial_r S^{ri}$ is *upper connection* or in the other words: An arbitrary second order operator Δ has an appearance:

$$\Delta f = \partial_i\left(S^{ik}\partial_k f\right) - \gamma^i\partial_i + R$$

where γ^i upper connection, R scalar. We see that connection appears....

Example In Riemannian manifold one can consider connection $\gamma_i = -\Gamma^k_{ik}$. This connection with Riemannian metrics defines the well-known Beltrami-Laplace operator:

$$\Delta_{B.L} f = \partial_i\left(g^{ik\partial_k f}\right) - \gamma^i\partial_i\,. \tag{1.1}$$

1.2 Algebra of densities.

We will go to densities. (This is very useful.) Density $\boldsymbol{s} = s(x)|Dx|^\lambda$ of the weight λ. Under changing of coordinates it is multiplied on λ-th power of Jacobian. (M is orientable manifold with chosen class of orientation.)

Examples of densities
Functions –densities of weight $\lambda = 0$.
Volume forms–densities of the weight $\lambda = 1$
Wave function–densities of the weight $\lambda = \frac{1}{2}$
Schwarzian of diffeomorphism F–densities of the weight $\lambda = 2$...

1.2.1 Lie derivative of densities

In local coordinates

$$\mathcal{L}_{\mathbf{X}}(\boldsymbol{s}) = \left(X^i \partial_i s(x) + \lambda \partial_i X^i s(x)\right) |Dx|$$

One can easy check its invariance. If $\lambda = 1$ we come to divergence

$$\operatorname{div}_{\boldsymbol{\rho}} \mathbf{X} = \frac{1}{\boldsymbol{\rho}} \mathcal{L}_{\mathbf{X}}(\boldsymbol{\rho}) = \left(X^i \partial_i \rho(x) + \lambda \partial_i X^i \rho(x)\right) |Dx| = \frac{1}{\rho(x)} \partial_1 \left(\rho(x) X^i\right)$$

Exercise In Riemannian case where $\rho = \sqrt{\det g}$. Compare with covariant divergence.

Exercise For Beltrami Laplace operator

$$\Delta_{B.L} f = \operatorname{div}_{\boldsymbol{\rho}} \mathrm{gradf} = \frac{1}{\rho(x)} \partial_1 \left(\rho(x) g^{ik} \partial_k f\right)$$

1.2.2 Algebra of densities. Scalar product. Extended manifold $\widehat{M}$.

Density on M is a polynomial function on $\widehat{M}$.

Local coordinates on $\widehat{M}$- (x^i, t). Globally defined operator

$$\widehat{\lambda} = t \frac{\partial}{\partial t}$$

Fact *This is first order operator, vector field on $\widehat{M}$.*

1.2.3 Conjugate operators. Vector fields on $\widehat{M}$

Vector fields of weight δ:

$$\widehat{\mathbf{X}} = t^{\delta} \left(X^i \partial_i + X^0 \widehat{\lambda}\right)$$

definition

$$\mathrm{div}\,\mathbf{X} = -\left(\widehat{\mathbf{X}}^{+} + \widehat{\mathbf{X}}\right) = t^{\delta}\left(\partial_i X^i \partial_i + (\delta - 1))X^0\right)$$

Fact: Lie derivative—-divergence less vector field:

$$\widehat{\mathbf{X}} = X^i \partial_i - \widehat{\lambda}\partial_i \mathbf{X}^i$$

Exercise A connection on M defines lifting

$$\mathbf{X} \mapsto \widehat{\mathbf{X}}_\gamma = X^i \partial_i + \gamma_i X^i \widehat{\lambda}$$

Thus we come to

$$\mathrm{div}_{\,\gamma} = \widehat{\mathbf{X}}_\gamma$$

remark The divergence possesses curvature:

$$\mathrm{div}_{\,\gamma}\,[\mathbf{X}, \mathbf{Y}] = \partial_{\mathbf{X}} \mathrm{div}_{\,\gamma} \mathbf{Y} - \partial_{\mathbf{Y}} \mathrm{div}_{\,\gamma} \mathbf{X} + \mathcal{F}(\mathbf{X}, \mathbf{Y})$$

where $\mathcal{F}_{ik} = \partial_i \gamma_k - \partial_k \gamma_i$ is a curvature of connection. (Recall that Ricci tensor for general affine connection is not symmetric.)

Exercise: An arbitrary vector field $\widehat{\mathbf{X}}$ is a sum of Lie derivative and vertical vector fields.

One can consider another lifting $\mathbf{X} \mapsto \mathcal{L}_{\mathbf{X}}$...... Difference of these two liftings is $\widehat{\lambda}\mathrm{div}_{\,\gamma}\mathbf{X}$

1.3 Second order operators on $\widehat{M}$

Operator pencil $\{\Delta_\lambda\}$—operator on $\widehat{M}$ of the order...??? (it is polynomial on λ).

Consider useful example. Let $\boldsymbol{\rho}$ be an arbitrary volume form and S^{ik} second order principal symbol. One can define the pencil of operators:

$$\Delta_\lambda \colon \Delta_\lambda \sigma = \boldsymbol{\rho}^{\lambda - 1} \partial_i \left[\boldsymbol{\rho} S^{ik} \partial_k \left(\frac{\boldsymbol{s}}{\boldsymbol{\rho}^\lambda}\right)\right] =$$

$$S^{ik}\partial_i\partial_k + \left(\partial_r S^{ri} + (2\lambda - 1)\Gamma^i\right)\partial_i + \left(\lambda \partial_i \Gamma^i + \lambda(\lambda - 1)\Gamma^i \Gamma_i\right) \tag{1.2}$$

where $\Gamma_i = -\partial_i \log \rho(x)$ is flat connection defined by the volume form.

One can see that this is self-adjoint operator: $\Delta^+\lambda = \Delta_{1-\lambda}$ or in terms of operator on $\widehat{M}$:

$$\widehat{\Delta} = S^{ik}\partial_i\partial_k + \left(\partial_r S^{ri} + (2\widehat{\lambda}-1)\Gamma^i\right)\partial_i + \left(\widehat{\lambda}\partial_i\Gamma^i + \widehat{\lambda}(\widehat{\lambda}-1)\Gamma^i\Gamma_i\right)$$

$\widehat{\Delta}^+ = \widehat{\Delta}$ Note that the operator $(2\widehat{\lambda}-1)\mathcal{L}_{\mathbf{X}}$ is second order self-adjoint operator on M too.

One can show that we come to the operator: $\widehat{\Delta} + (2\widehat{\lambda}-1)\mathcal{L}_{\mathbf{X}} =$

$$S^{ik}\partial_i\partial_k + \left(\partial_r S^{ri} + (2\lambda-1)\Gamma^i\right)\partial_i + \left(\lambda\partial_i\Gamma^i + \lambda(\lambda-1)\Gamma^i\Gamma_i\right) + (2\widehat{\lambda}-1)X^i\partial_i + \widehat{\lambda}(2\widehat{\lambda}-1)\partial_i X^i =$$

$$S^{ik}\partial_i\partial_k + \left(\partial_r S^{ri} + (2\lambda-1)\gamma^i\right)\partial_i + \left(\lambda\partial_i\gamma^i + \lambda(\lambda-1)\theta\right)$$

where

$$\gamma^i = \Gamma^i + X^i, \qquad \theta = (\Gamma_i\Gamma^i + 2\Gamma_i X^i) + 2\mathrm{div}_{\,\Gamma}\mathbf{X}$$

In the case if symbol is invertible

$$\theta = (\Gamma_i\Gamma^i + 2\Gamma_i X^i) + 2\mathrm{div}_{\,\Gamma}\mathbf{X} = \gamma_i\gamma^i + 2\mathrm{div}_{\,\Gamma}\mathbf{X} - \mathbf{X}^2\,.$$

This is basic example....

Theorem (Vor. Kh.2003) Let $\Delta \in \mathcal{D}^{(2)}_{\lambda_0}(M)$ be second order operator defined on densities of weight λ_0. In the case if $\lambda_0 \neq 0, 1, \frac{1}{2}$ there exists unique operator pencil Δ_λ of the order 2, i.e. the second order operator $\widehat{\Delta}$ on $\widehat{M}$ such that

- $\widehat{\Delta}\big|_{\lambda=\lambda_0} = \Delta_{\lambda_0}$.
- $\widehat{\Delta} = \widehat{\Delta}^+$, i.e. $\Delta^+_\lambda = \Delta_{1-\lambda}$
- $\widehat{\Delta}1 = 0$.

The Lie derivative of self-adjoint operator is self-adjoint hence:

$$\mathrm{ad}_{\mathbf{K}} \circ T_{\lambda,\mu} = T_{\lambda,\mu}\mathrm{ad}_{\mathbf{K}}$$

This implies **Corollary**. There exists equivariant map

$$\Delta_\lambda \xrightarrow{T_{\lambda\mu}} \Delta_\lambda \tag{1.3}$$

for $\lambda, \mu \neq 0, 1/2, 1$. The "bare hand" proof is difficult....

In general case this map has very ugly appearance

If an operator $\Delta_\lambda \in \mathcal{D}_\lambda(M)$ is given in local coordinates by the expression $\Delta_\lambda = A^{ij}(x)\partial_i\partial_j + A^i(x)\partial_i + A(x)$ then its image $T_{\lambda,\mu}(\Delta_\lambda) = \Delta_\mu \in \mathcal{D}_\mu(M)$ is given in the same local coordinates by the expression $\Delta_\mu = B^{ij}(x)\partial_i\partial_j + B^i(x)\partial_i + B(x)$ where

$$\begin{cases} B^{ij} & = A^{ij}\,, \\ B^i & = \frac{2\mu-1}{2\lambda-1}A^i + \frac{2(\lambda-\mu)}{2\lambda-1}\partial_j A^{ji}\,, \\ B & = \frac{\mu(\mu-1)}{\lambda(\lambda-1)}A + \frac{\mu(\lambda-\mu)}{(2\lambda-1)(\lambda-1)}\left(\partial_j A^j - \partial_i\partial_j A^{ij}\right)\,. \end{cases} \tag{1.4}$$

At the exceptional cases $\lambda, \mu = 0, \frac{1}{2}, 1$, non-isomorphic modules occur.

Very beautiful example (Matthonet, Lecompte:)

$$T_{\lambda,\mu}\left(\mathcal{L}^{(\lambda)}_{\mathbf{X}} \circ \mathcal{L}^{(\lambda)}_{\mathbf{Y}}\right) = \mathcal{L}^{(\mu)}_{\mathbf{X}} \circ \mathcal{L}^{(\mu)}_{\mathbf{Y}} + \frac{\mu-\lambda}{2\lambda-1}\mathcal{L}_{[\mathbf{X},\mathbf{Y}]}$$

Our beautiful proof:

$$\mathcal{L}^{(\lambda)}_{\mathbf{X}} \circ \mathcal{L}^{(\lambda)}_{\mathbf{Y}} = \frac{1}{2}\left[\mathcal{L}^{(\lambda)}_{\mathbf{X}} \circ \mathcal{L}^{(\lambda)}_{\mathbf{Y}} + \mathcal{L}^{(\lambda)}_{\mathbf{X}} \circ \mathcal{L}^{(\lambda)}_{\mathbf{Y}}\right]\frac{1}{2}\left[\mathcal{L}^{(\lambda)}_{\mathbf{X}} \circ \mathcal{L}^{(\lambda)}_{\mathbf{Y}} - \mathcal{L}^{(\lambda)}_{\mathbf{X}} \circ \mathcal{L}^{(\lambda)}_{\mathbf{Y}}\right] +$$

The first operator is self-conjugate, the second antiself conjugate hence we draw the following slf-conjugate pencil through this operator

$$\widehat{\Delta} = \frac{1}{2}\left[\mathcal{L}_{\mathbf{X}} \circ \mathcal{L}_{\mathbf{Y}} + \mathcal{L}_{\mathbf{X}} \circ \mathcal{L}_{\mathbf{Y}}\right] + \frac{1}{2}\frac{2\widehat{\lambda}-1}{2\lambda-1}\left[\mathcal{L}_{\mathbf{X}} \circ \mathcal{L}_{\mathbf{Y}} - \mathcal{L}_{\mathbf{X}} \circ \mathcal{L}_{\mathbf{Y}}\right] =$$

$$\frac{1}{2}\left[\mathcal{L}_{\mathbf{X}} \circ \mathcal{L}_{\mathbf{Y}} + \mathcal{L}_{\mathbf{X}} \circ \mathcal{L}_{\mathbf{Y}}\right] + \frac{2\widehat{\lambda}-1}{4\lambda-2}\mathcal{L}_{[\mathbf{X},\mathbf{Y}]}$$

We see that $\widehat{\Delta}\big|_{\widehat{\lambda}=\lambda} = \mathcal{L}^{(\lambda)}_{\mathbf{X}} \circ \mathcal{L}^{(\lambda)}_{\mathbf{Y}}$ and

$$\widehat{\Delta}\big|_{\widehat{\lambda}=\mu} = \frac{1}{2}\left[\mathcal{L}^{(\mu)}_{\mathbf{X}} \circ \mathcal{L}^{(\mu)}_{\mathbf{Y}} + \mathcal{L}^{(\mu)}_{\mathbf{X}} \circ \mathcal{L}^{(\mu)}_{\mathbf{Y}}\right] + \frac{1}{2}\left[\mathcal{L}^{(\mu)}_{\mathbf{X}} \circ \mathcal{L}^{(\mu)}_{\mathbf{Y}} - \mathcal{L}^{(\mu)}_{\mathbf{X}} \circ \mathcal{L}^{(\mu)}_{\mathbf{Y}}\right] - \frac{1}{2}\mathcal{L}^{(\mu)}_{[\mathbf{X},\mathbf{Y}]} + \frac{2\mu-1}{4\lambda-2}\mathcal{L}^{(\mu)}_{[\mathbf{X},\mathbf{Y}]}$$

$$= \mathcal{L}^{(\mu)}_{\mathbf{X}} \circ \mathcal{L}^{(\mu)}_{\mathbf{Y}} + \frac{\mu-\lambda}{2\lambda-1}\mathcal{L}^{(\mu)}_{[\mathbf{X},\mathbf{Y}]} \blacksquare$$

Prove od the theorem. First we construct Beltrami-Laplas pencil (1.2) such that second order terms coicide, then add the operator $(2\widehat{\lambda}-1)\mathcal{L}_{\mathbf{X}}$ choosing $\mathbf{X}$ such that first terms coicide, then add scalar. Thus we come to self-adjoint pencil $\widehat{\Delta}$ passing through the operator.

Most important to prove the uniqueness (Without uniqueness we will not have the equivariant map (1.3)).

If $\widehat{\Delta}'$ is an arbitrary operator which obeys the condition of theorem then

$$\widehat{\Delta}' = \widehat{\Delta} + (\widehat{\lambda} - \lambda_0)(L_1 + \widehat{\lambda} F)$$

Step by step we prove that it vanishes.

1.4 Canonical pencil in general case

:

$$\Delta = t^\delta \left[S^{ik}\partial_i\partial_k + \left(\partial_r S^{ri} + (2\lambda + \delta - 1)\gamma^i\right)\partial_i + \left(\lambda\partial_i\gamma^i + \lambda(\lambda = \delta - 1)\theta\right)\right]$$

- $\mathbf{S} = t^\delta S^{ab}(x) = S^{ab}(x)\mathcal{D}x^\delta$ is symmetric contravariant tensor field-density of the weight δ. Under changing of local coordinates $x^{a'} = x^{a'}(x^a)$ it transforms in the following way:

$$S^{a'b'} = J^{-\delta} x_a^{a'} x_b^{b'} S^{ab} \,,$$

- γ^a is a symbol of upper connection-density of weight δ. Under changing of local coordinates $x^{a'} = x^{a'}(x^a)$ it transforms in the following way:

$$\gamma^{a'} = J^{-\delta} x_a^{a'} \left(\gamma^a + S^{ab}\partial_b \log J\right) \,,$$

- and θ transforms in the following way:

$$\theta' = J^{-\delta}\left(\theta + 2\gamma^a\partial_a \log J + \partial_a \log J \, S^{ab}\partial_b \log J\right)$$

  ($\theta = \gamma^a\gamma_a +$ scalar in the case if symbol is invertible)

1.5 Special cases.

Consider $\lambda = \frac{1-\delta}{2}$. Then

$$\Delta = t^\delta \left[S^{ik}\partial_i\partial_k + \partial_r S^{ri}\partial_i + \frac{1-\delta}{2}\left(\partial_i\gamma^i + \frac{\delta - 1}{2}\theta\right)\right]$$

Example. $\delta = 0$. S^{ik} defines Poisson structure. —Batalin-Vilkovisky formalism.

Next example $\delta = 2$, $n = 1$. $S = 1$ (invariant) We come to

$$\Delta = t^2 \left[\partial_x^2 - \frac{1}{2}U(x)\right] = |Dx|^2 \left[\partial_x^2 - \frac{1}{2}U(x)\right] ,$$

$$\Delta \left(\Psi(x)|Dx|^{-\frac{1}{2}}\right) = \left(\Psi_{xx}(x) - \frac{1}{2}U(x)\Psi(x)\right)|Dx|^{\frac{3}{2}} . \tag{1.5}$$

This operator leads us to .... Schwarzian.

2 Schwarzian , Projective geometry

See how operator (1.5)transforms the operator above under diffeomorphisms.

If $f = y(x)$ is diffeomorphism then

$$\Delta^f \left(\Psi(x)|Dx|^2\right) = \left\{|Dy|^2 \left[\partial_y^2 - \frac{1}{2}U(y)\right]\left[\Psi\left(x(y)\right)|Dx|^{-\frac{1}{2}}\right]\right\}\big|_{y=y(x)} =$$

$$\left\{|Dy|^2 \left[\partial_y^2 - \frac{1}{2}U(y)\right]\left[\Psi\left(x(y)\right)x_y^{-\frac{1}{2}}|Dy|^{-\frac{1}{2}}\right]\right\}\big|_{y=y(x)} =$$

$$\left[\Psi_{xx}(x)x_y^{\frac{3}{2}} + \frac{3}{4}\Psi(x)x_y^{-\frac{5}{2}}x_{yy}^2 - \frac{1}{2}\Psi(x)x_y^{-\frac{3}{2}}x_{yyy} - \frac{1}{2}\left[U(y(x)) +\right]\Psi(x)\right] y_x^{\frac{3}{2}}|Dx|^{\frac{3}{2}} =$$

$$\left[\Psi_{xx}(x) + \frac{3}{4}\Psi(x)x_y^{-4}x_{yy}^2 - \frac{1}{2}\Psi(x)x_y^{-3}x_{yyy} - \frac{1}{2}U(y(x))\Psi(x)\right]|Dx|^{\frac{3}{2}} =$$

$$\left[\Psi_{xx}(x) + \frac{3}{4}\Psi(x)x_y^{-4}x_{yy}^2 - \frac{1}{2}\Psi(x)x_y^{-3}x_{yyy} - \frac{1}{2}U(y(x))\Psi(x)\right]|Dx|^{\frac{3}{2}} =$$

$$\left[\Psi_{xx}(x) + \frac{3}{4}\Psi(x)x_y^{-4}x_{yy}^2 - \frac{1}{2}\Psi(x)x_y^{-3}x_{yyy} - \frac{1}{2}\left[U(y(x)) + \left(\frac{x_{yyy}}{x_y} - \frac{3}{2}\frac{x_{yy}^2}{x_y^2}\right)y_x^2\right]\Psi(x)\right]|Dx|^{\frac{3}{2}} =$$

Sorry for these calculations but *Paris vaut bien une messe*:

$$\Delta^f = \Delta - \frac{1}{2}\left(\frac{x_{yyy}}{x_y} - \frac{3}{2}\frac{x_{yy}^2}{x_y^2}\right)|Dy|^2 \tag{2.1}$$

The cocycle

$$\mathcal{S}(f^{-1}) = \left(\frac{x_{yyy}}{x_y} - \frac{3}{2}\frac{x_{yy}^2}{x_y^2}\right)|Dy|^2 \tag{2.2}$$

is Schwarzian.

2.1 Meaning of Schwartzian

2.2 Projective structures on curves in $\mathrm{R}P^n$ and Schwarzian

2.3 Curves in $\mathbf{R}P^n$ and $n+1$-th order operators.

We consider the following construction.

let C: $u^i(x)$ be an arbitrary curve in $\mathbf{R}^{n+1}$. Let $\boldsymbol{\rho}$ be an arbitrary density and ∇ an arbitrary connection.

Let $\mathcal{D} = |Dx|(\partial_x + \gamma)$

We consider the following pencil of operators For an arbitrary density $\boldsymbol{s} = s(x)|Dx|^\lambda$ of weight λ we consider operator

$$\Delta_\lambda(\boldsymbol{s}) = \frac{\det \begin{pmatrix} u^1\boldsymbol{\rho}^\lambda & u^2\boldsymbol{\rho}^\lambda & \dots u^{n+1}\boldsymbol{\rho}^\lambda & \boldsymbol{s}_\lambda \\ \mathcal{D}u^1\boldsymbol{\rho}^\lambda & \mathcal{D}u^2\boldsymbol{\rho}^\lambda & \dots \mathcal{D}u^{n+1}\boldsymbol{\rho}^\lambda & \mathcal{D}\boldsymbol{s}_\lambda \\ \mathcal{D}^2u^1\boldsymbol{\rho}^\lambda & \mathcal{D}^2u^2\boldsymbol{\rho}^\lambda & \dots \mathcal{D}^2u^{n+1}\boldsymbol{\rho}^\lambda & \mathcal{D}^2\boldsymbol{s}_\lambda \\ \dots & \dots & \dots\dots & \dots \\ \mathcal{D}^{n+1}u^1\boldsymbol{\rho}^\lambda & \mathcal{D}^{n+1}u^2\boldsymbol{\rho}^\lambda & \dots \mathcal{D}^2u^{n+1}\boldsymbol{\rho}^\lambda & \mathcal{D}^{n+1}\boldsymbol{s}_\lambda \end{pmatrix}}{\det \begin{pmatrix} u^1\boldsymbol{\rho}^\lambda & u^2\boldsymbol{\rho}^\lambda & \dots u^{n+1}\boldsymbol{\rho}^\lambda \\ \mathcal{D}u^1\boldsymbol{\rho}^\lambda & \mathcal{D}u^2\boldsymbol{\rho}^\lambda & \dots \mathcal{D}u^{n+1}\boldsymbol{\rho}^\lambda \\ \mathcal{D}^2u^1\boldsymbol{\rho}^\lambda & \mathcal{D}^2u^2\boldsymbol{\rho}^\lambda & \dots \mathcal{D}^2u^{n+1}\boldsymbol{\rho}^\lambda \\ \dots & \dots & \dots\dots \\ \mathcal{D}^nu^1\boldsymbol{\rho}^\lambda & \mathcal{D}^nu^2\boldsymbol{\rho}^\lambda & \dots \mathcal{D}^2u^n\boldsymbol{\rho}^\lambda q \end{pmatrix}} \tag{2.3}$$

This operator sends density of weight λ to the density of the weight $\lambda+n+1$

E.g. for curve C: $u(x), v(x)$ in $\mathbf{R}^2$

$$\Delta_\lambda(\boldsymbol{s}) = \frac{\det \begin{pmatrix} u\boldsymbol{\rho}^\lambda & v\boldsymbol{\rho}^\lambda & \boldsymbol{s}_\lambda \\ \mathcal{D}u\boldsymbol{\rho}^\lambda & \mathcal{D}v\boldsymbol{\rho}^\lambda & \mathcal{D}\boldsymbol{s}_\lambda \\ \mathcal{D}^2u\boldsymbol{\rho}^\lambda & \mathcal{D}^2v\boldsymbol{\rho}^\lambda & \mathcal{D}^2\boldsymbol{s}_\lambda \end{pmatrix}}{\det \begin{pmatrix} u\boldsymbol{\rho}^\lambda & v\boldsymbol{\rho}^\lambda \\ \mathcal{D}u\boldsymbol{\rho}^\lambda & \mathcal{D}v\boldsymbol{\rho}^\lambda \end{pmatrix}} \tag{2.4}$$

This operator sends density of weight λ to the density of the weight $\lambda+2$

Densities prportional to linear combination of functions $u^i(t)$ eblong to kernel.

Remark Note that in components all terms proportioanl to γ and $\boldsymbol{\rho}$ dissapear....

Consider the projection of curve C in $\mathbf{R}P^n$.

Denominator is the density of the weight $n\lambda+1+\cdots+n=(n+1)\lambda+\frac{n(n+1}{2)}$. We may choose multiplier such that for $\lambda=-\frac{1}{n+1}$ denominator equals to 1.

We come to Schwarzian in third terms for operator.....

2.4 (Anti)-self conjugate operator of order n on R

.

Example

$$\Delta=t^n\mathcal{D}^n, \qquad \text{where } \mathcal{D}=t^n(\partial_x+\widehat{\lambda}\gamma_{\text{-}}) \tag{2.5}$$

Calculations show that...

Example

$$\Delta=t^{n+\delta'}\left[s(x)\partial_x^n+\frac{n}{2}(s_x+2s\gamma_{\text{-}}\widehat{\lambda}_{n+\delta'})\partial^{n-1}+B_n\partial_x^{n-1}+\dots\right]$$

where

$$B_n=\frac{n(n-1)}{2}\left[\gamma_x^{\text{-}}+s(\gamma_{\text{-}}^2+\tau)\right]+\frac{n(n-1)}{2}\left[\frac{n-2}{6}s_x\gamma_{\text{-}}-\frac{n+1+3\delta'}{6}sU_\gamma-\frac{\delta'^2}{4}s\gamma^2+s\rho\right] \tag{2.6}$$

Here ρ,τ are densities of weight 2, $U_\gamma=\gamma_x-\frac{1}{2}\gamma^2$ is related with Schwarzian "antiderivative" $\widehat{\lambda}_k=\widehat{\lambda}+\frac{k-1}{2}$

We see here how Schwarzian appears in critical dimensions....

2.5 Projective symbol of operators