We want to model the sound propagation.

Consider N identical strings which may a big circle.

$$L = \sum_k \frac{m\dot{x}_k^2}{2} + \sum \frac{k(x_k - x_{k+1})^2}{2} =$$

$$\frac{m\dot{x}_0^2}{2} + \frac{m\dot{x}_1^2}{2} + \frac{m\dot{x}_2^2}{2} + \dots \frac{m\dot{x}_{N-1}^2}{2} + \frac{m\dot{x}_N^2}{2} +$$

$$\frac{k(x_1 - x_0)^2}{2} + \frac{k(x_2 - x_1)^2}{2} + \dots \frac{k(x_{N-1} - x_N)^2}{2} + \frac{k(x_N - x_0)^2}{2} =$$

where summation gous over the finite ring $Z \backslash NZ$.

One can write it like

$$L = \sum_k \frac{m\dot{x}_k^2}{2} + \frac{k}{2} \sum M_{km} x^k x^m$$

where $N \times N$ matrix M is equal to

$$M = \begin{pmatrix} 2 & -1 & 0 & 0 & 0\dots & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0\dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0\dots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1\dots & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2\dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots\dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & -1\dots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0\dots & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & 0\dots & 0 & -1 & 2 \end{pmatrix},$$

(see the blog on 28 October 2019.) This is the matrix on the cylinder.

Coordinates on configuration space.

Remark The dimension of configuartion space is $N+1$, however it is natural to embedd this space in the space with $N+2$ coordinates $\eta_0, \eta_1, \dots, \eta_N, s$:

$$\eta_i = x_i - x_{i-1}\,, (\eta_0 = x_0 - x_N) \quad , s = \frac{x_0 + x_1 + x_2 + \dots + x_N}{N+1}\,.$$

the constraint is $\eta_0 + \eta_1 + \dots + \eta_N = 0$.

We try to do it in other way: we will choose new coordinates *not destroying the circle symmetry*:

Consider new coordinates $\{\xi^0, \xi^2, \dots \xi^N\}$:

$$\eta_i = x_i - x_{i-1} + s\,, (\eta_0 = x_0 - x_N) \quad ,$$

i.e.

$$\begin{cases} \xi^0 = \eta^0 + s = x^0 - x^N + \frac{x^0+x^1+\dots+x^N}{N+1} \\ \xi^1 = \eta^1 + s = x^1 - x^0 + \frac{x^0+x^1+\dots+x^N}{N+1} \\ \dots \\ \xi^N = \eta^N + s = x^N - x^{N-1} + \frac{x^0+x^1+\dots+x^N}{N+1} \end{cases}$$

Notice that these equation are covariant with respect to action of rotation of circle:

$$\xi^i \mapsto \xi^{i+k} \Leftrightarrow x^i \mapsto x^{i+k}$$

This facilitates the calculations. It suiffice to express x^0 via ξ, anf then apply symmetry. Perform not difficult but enjoyable calculations* We have that

$$x_0 = x_0\,, x_1 = \eta_1 + x_0\,, x_2 = \eta_2 + \eta_1 + x_0\,, \dots\,, x_N = \eta_N + \dots + \eta_1 + x_0$$

Summing these equations we come to

$$x^0 = -\frac{1}{N+1}\left(N\eta_1 + (N-1)\eta_2 + \dots + 2\eta_{N-1} + \eta_N\right) + s =$$

$$x^0 = -\frac{1}{N+1}\left(N\left(\xi^1 - s\right) + (N-1)\left(\xi^2 - s\right) + \dots + 2\left(\xi^{N-1} - s\right) + \left(\xi^N - s\right)\right) + s =$$

$$-\frac{1}{N+1}\left(N\xi^1 + (N-1)\xi^2 + \dots + 2\xi^{N-1} + \xi^N\right) + \frac{N+2}{2}s\,.$$

Thus since

$$s = \frac{x_0 + \dots + x_N}{N+1} = \frac{\xi^0 + \dots + \xi^N}{N+1}$$

we come to

$$x^0 = \frac{N+2}{2(N+1)} \times$$

$$\left(\xi_0 + \left(1 - \frac{2N}{N+2}\right)\xi^1 + \left(1 - \frac{2(N-1)}{N+2}\right)\xi^2 + \left(1 - \frac{2(N-2)}{N+2}\right)\xi^3 + \dots + \left(1 - \frac{2}{N+2}\right)\xi^N\right)$$

To calculate other x^i for $i \neq 0$ we use group symmetries. We have that if

$$x^0 = \alpha_k\xi^k = \alpha_0\xi^0 + \alpha_1\xi^1 + \alpha_2\xi^2 + \dots + \alpha_N\xi^N$$

then

$$x^i = \alpha_{k-i}\colon \quad \begin{cases} x^0 = \alpha_0\xi^0 + \alpha_1\xi^1 + \alpha_2\xi^2 + \dots + \alpha_N\xi^N \\ x^1 = \alpha_N\xi^0 + \alpha_0\xi^1 + \alpha_1\xi^2 + \dots + \alpha_{N-1}\xi^N \\ x^2 = \alpha_{N-1}\xi^0 + \alpha_N\xi^1 + \alpha_0\xi^2 + \dots + \alpha_{N-2}\xi^N \\ \dots \\ x^N = \alpha_1\xi^0 + \alpha_2\xi^1 + \alpha_3\xi^2 + \dots + \alpha_0\xi^N \end{cases}$$

Here we usses symmetries, but calculations are still difficult. We try to go in another way choosing orthogonal transformations of coorrdinates (see the next blog)

* Intermediate expressions may not respect the rotational symmetry, but final answers will respect