Riemannian Geometry

it is a draft of Lecture Notes of H.M. Khudaverdian.
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1 Riemannian manifolds

1.1 Manifolds. Tensors. (Recollection)

1.1.1 Manifolds

I recall briefly basics of manifolds and tensor fields on manifolds.

An n-dimensional manifold $M = M^n$ is a space[1]

such that in a vicinity of an arbitrary point one can consider local coordinates $\{x^1, \dots, x^n\}$. (We say that in a vicinity of this point a manifold M is covered by local coordinates $\{x^1, \dots, x^n\}$). One can consider different local coordinates. If coordinates $\{x^1, \dots, x^n\}$ and $\{x^{1'}, \dots, x^{n'}\}$ both are defined in a vicinity of the given point then they are related by *bijective transition functions* which are defined on domains in $\mathbf{R}^n$ and taking values also in $\mathbf{R}^n$:

$$\begin{cases} x^{1'} = x^{1'}(x^1, \dots, x^n) \\ x^{2'} = x^{2'}(x^1, \dots, x^n) \\ \dots \\ x^{n-1'} = x^{n-1'}(x^1, \dots, x^n) \\ x^{n'} = x^{n'}(x^1, \dots, x^n) \end{cases} \tag{1.1}$$

We say that n-dimensional manifold is *differentiable* or *smooth* if all transition functions are diffeomorphisms, i.e. they are smooth. Invertability implies that Jacobian matrix is non-degenerate:

$$\det \begin{pmatrix} \frac{\partial x^{1'}}{\partial x^1} & \frac{\partial x^{1'}}{\partial x^2} \dots & \frac{\partial x^{1'}}{\partial x^n} \\ \frac{\partial x^{2'}}{\partial x^1} & \frac{\partial x^{2'}}{\partial x^2} \dots & \frac{\partial x^{2'}}{\partial x^n} \\ \dots & & \\ \frac{\partial x^{n'}}{\partial x^1} & \frac{\partial x^{n'}}{\partial x^2} \dots & \frac{\partial x^{n'}}{\partial x^n} \end{pmatrix} \neq 0\,. \tag{1.2}$$

(If bijective function $x^{i'} = x^{i'}(x^i)$ is smooth function, and its inverse, the transition function $x^i = x^i(x^i)$ is also smooth function, then matrices $||\frac{\partial x^{i'}}{\partial x^i}||$ and $||\frac{\partial x^i}{\partial x^{i'}}||$ are both well defined, hence condition (1.2) is obeyed.

[1] A space M is a topological space, i.e. it is covered by a collection $\mathcal{F}$ of sets, which are called *open* sets. This collection obeys the following axioms

i) the union of an arbitrary set of open sets is an open set

ii) the intersection of finite number of open sets is an open set

iii) the whole space M and the empty set $\emptyset$ are open sets

Example

open domain in $\mathbf{E}^n$

A good example of manifold is an open domain D in n-dimensional vector space $\mathbf{R}^n$. Cartesian coordinates on $\mathbf{R}^n$ define global coordinates on D. On the other hand one can consider an arbitrary local coordinates in different domains in $\mathbf{R}^n$. E.g. one can consider polar coordinates $\{r, \varphi\}$ in a domain $D = \{x, y \colon y > 0\}$ of $\mathbf{R}^2$ defined by standard formulae:

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}, \tag{1.3}$$

$$\det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{pmatrix} = \det\begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix} = r \tag{1.4}$$

or one can consider spherical coordinates $\{r, \theta, \varphi\}$ in a domain $D = \{x, y, z \colon x > 0, y > 0, z > 0\}$ of $\mathbf{R}^3$ (or in other domain of $\mathbf{R}^3$) defined by standard formulae

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases},$$

,

$$\det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{pmatrix} = \det\begin{pmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} = r^2\sin\theta \tag{1.5}$$

Choosing domain where polar (spherical) coordinates are well-defined we have to be aware that coordinates have to be well-defined and transition functions (1.1) have to obey condition (1.2), i.e. they have to be diffeomorphisms. E.g. for domain D in example (1.3) Jacobian (1.4) does not vanish if and only if $r > 0$ in D.

Consider another examples of manifolds, and local coordinates on manifolds.

Example

Circle S^1 in $\mathbf{E}^2$

Consider circle $x^2 + y^2 = R^2$ of radius R in $\mathbf{E}^2$.

One can consider on the circle different local coordinates

i) *polar coordinate* φ:

$$\begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \end{cases} \quad , \qquad 0 < \varphi < 2\pi$$

(this coordinate is defined on all the circle except a point $(R, 0)$),

ii) *another polar coordinate* φ':

$$\begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \end{cases} \quad , \qquad -\pi < \varphi < \pi \,,$$

this coordinate is defined on all the circle except a point $(-R, 0)$,

iii) stereographic coordinate t with respect to north pole of the circle

$$\begin{cases} x = \frac{2R^2t}{t^2+R^2} \\ y = R\frac{t^2-R^2}{t^2+R^2} \end{cases} \quad , \quad t = \frac{Rx}{R-y} \,, \tag{1.6}$$

this coordinate is defined at all the circle except the north pole,

iiii) stereographic coordinate t' with respect to south pole of the circle

$$\begin{cases} x = \frac{2R^2t'}{t'^2+R^2} \\ z = R\frac{R^2-t'^2}{t'^2+R^2} \end{cases} \quad , \quad t' = \frac{Rx}{R+y} \,,$$

this coordinate is defined at all the points except the south pole.

We considered four different local coordinates on the circle S^1. Write down some transition functions (1.1) between these coordinates

- polar coordinate φ coincide with polar coordinate φ' in the domain $x^2 + y^2 > 0$, and in the domain $x^2 + y^2 < 0$ $\varphi' = \varphi - 2\pi$.

- Transition function from polar coordinate φ to stereographic coordinates t is $t = \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)$,

- transition function from stereographic coordinate t to stereographic coordinate t' is

$$t' = \frac{R^2}{t} \,,$$

(see Homework 0.)

Example

Sphere S^2 in $\mathbf{E}^3$

Consider sphere $x^2+y^2+z^2=R^2$ of radius a in $\mathbf{E}^3$.
One can consider on the sphere different local coordinates
i) *spherical coordinates on domain of sphere θ,φ:*

$$\begin{cases} x = a\sin\theta\cos\varphi \\ y = a\sin\theta\sin\varphi \\ z = a\cos\theta \end{cases} , \quad 0<\theta<\pi\,, -\pi<\varphi<\pi$$

ii) stereographic coordinates u,v with respect to north pole of the sphere

$$\begin{cases} x = \frac{2a^2u}{a^2+u^2+v^2} \\ y = \frac{2a^2v}{a^2+u^2+v^2} \\ z = a\frac{u^2+v^2-a^2}{a^2+u^2+v^2} \end{cases} , \quad \frac{x}{u}=\frac{y}{v}=\frac{a-z}{a}, \quad \begin{cases} u = \frac{ax}{a-z} \\ v = \frac{ay}{a-z} \end{cases} .$$

iii) stereographic coordinates u',v' with respect to south pole of the sphere

$$\begin{cases} x = \frac{2a^2u'}{a^2+u'^2+v'^2} \\ y = \frac{2a^2v}{a^2+u'^2+v'^2} \\ z = a\frac{a^2-u'^2-v'^2}{a^2+u^2+v^2} \end{cases} , \quad \frac{x}{u'}=\frac{y}{v'}=\frac{a+z}{a}, \quad \begin{cases} u' = \frac{ax}{a+z} \\ v' = \frac{ay}{a+z} \end{cases} .$$

(see also Homework 0)

Spherical coordinates are defined elsewhere except poles and the meridians $y=0, x\leq 0$.

Stereographical coordinates (u,v) are defined elsewhere except north pole; stereographic coordinates (u',v') are defined elsewhere except south pole.

One can consider transition function between these different coordinates.E.g. transition functions from spherical coordinates i) to stereographic coordinates (u,v) are

$$\begin{cases} u = \frac{ax}{a-z} = \frac{a\sin\theta\cos\varphi}{1-\cos\theta} = a\mathrm{cotan}\,\frac{\theta}{2}\cos\varphi \\ v = \frac{ay}{a-z} = \frac{a\sin\theta\sin\varphi}{1-\cos\theta} = a\mathrm{cotan}\,\frac{\theta}{2}\sin\varphi \end{cases} ,$$

and transition function from stereographic coordinates u,v to stereographic coordinates (u',v') are

$$\begin{cases} u' = \frac{a^2u}{u^2+v^2} \\ v' = \frac{a^2v}{u^2+v^2} \end{cases} ,$$

(see Homework 0.)

Remark

† One very important property of stereographic projetion which we do not use in this course but it is too beautiful not to mention it: under stereographic projection all points of the circle of radius $R = 1$ with rational coordinates x and y and only these points transform to rational points on line. Thus we come to Pythagorean triples $a^2 + b^2 = c^2$. The same is for unit sphere: the stereographic projection establishes one-one correspondence between points on the unit sphere with rational coordinates and rational points on the plane.

1.1.2 Tensors on Manifold

tangent vector and tangent vector space

Tangent vector at the given point can be considered as a derivation of function at this point.

For an arbitrary (smooth) function f defined in a vicinity of a given point $\mathbf{p}$ a tangent vector $\mathbf{A}(x) = A^i(x)\frac{\partial}{\partial x^i}$ defines the directional derivative of this function

$$\mathbf{A}\colon \quad f \mapsto \partial_{\mathbf{A}} f\big|_{\mathbf{p}} = A^i(x)\frac{\partial f}{\partial x^i}\big|_{\mathbf{p}}\,.$$

Using the chain rule one can see that under changing of coordinates it transforms as follows:

$$\mathbf{A} = A^i(x)\frac{\partial}{\partial x^i} = A^i(x)\frac{\partial x^{i'}(x)}{\partial x^i}\frac{\partial}{\partial x^{i'}} = A^{i'}(x'(x))\frac{\partial}{\partial x^{i'}}\,,$$

i.e.

$$A^{i'}(x') = \frac{\partial x^{i'}}{\partial x^i}A^i(x)\,. \tag{1.7}$$

This leads as to the following equivalent definition of the tangent vector.

Definition Let $M = M^n$ be n-dimensional manifold, and $\mathbf{p}$ the point on it. To define a vector $\mathbf{A}$ tangent to the manifold at the point $\mathbf{p}$ we assign to an arbitrary given local coordinates $\{x^i\}$ the array $\{A^i\}$ $(i = 1, \ldots, n)$ of numbers (components) such that under changing of local coordinates this array transforms according to equation (1.7):

`coordinates`		`components of vector`	
$\{x^i\}$	$\to$	$\{A^i\}$	
			such that $A^{i'} = \frac{\partial x^{i'}(x)}{\partial x^i}\big|_{\mathbf{p}}A^i\,.$ (1.8)
$\{x^{i'}\}$	$\to$	$\{A^{i'}\}$	

Definition Tangent vector space $T_{\mathbf{p}}M$ at the point $\mathbf{p}$ is the space of vectors tangent to the manifold at the point M.

1 *-form (covector) in a given point*

We defined above vectors of tangent space $T_{\mathbf{p}}M$. Now we consider dual obejcts: we consider cotangent space $T^*_{\mathbf{p}}M$ (for every point $\mathbf{p}$ on manifold M)— space of linear functions on tangent vectors, i.e. space of 1-forms which sometimes are called *covectors.*:

Linear function, 1-form $\omega = \omega_i dx^i$ is a function on tangent vectors:

$$T_{\mathbf{p}}M \ni \mathbf{A} = A^i \frac{\partial}{\partial x^i}\,, \omega\left(\mathbf{A}\right) = \omega_m dx^m \left(A^i \frac{\partial}{\partial x^i}\right) = \omega_m A^i \underbrace{dx^i \left(\frac{\partial}{\partial x^m}\right)}_{\delta_i^m} = \omega_m A^m\,.$$

If we consider new coordinates $x^{i'} = x^{i'}(x)$, then

$$\omega = \omega_i dx^i = \omega_i \left(\frac{\partial x^i}{\partial x^{i'}} dx^{i'}\right) = \underbrace{\omega_i \frac{\partial x^i}{\partial x^{i'}}}_{\omega_{i'}} dx^{i'}$$

i.e., nne-form (covector) $\omega = \omega_i(x) dx^i$ transforms as follows

$$\omega_{m'}(x') = \frac{\partial x^m(x')}{\partial x^{m'}} \omega_m(x)\,. \tag{1.9}$$

Differential form sometimes is called *covector*.

In the same way as for vectors we may give definition of covectors in the following way:

Definition Let $M = M^n$ be n-dimensional manifold, and $\mathbf{p}$ the point on it. To define a *covector* $\mathbf{A}$ at the point $\mathbf{p}$, (the linear function on tangent vectors at $\mathbf{p}$) we assign to an arbitrary given local coordinates $\{x^i\}$ the collection $\{\omega_i\}$ $(i = 1, \ldots, n)$ of numbers (components) such that under changing of local coordinates this collection transforms according to equation (1.9):

$$\begin{array}{ccc} \texttt{coordinates} & & \texttt{components of covector} \\ \{x^i\} & \rightarrow & \{\omega_i\} \\ \{x^{i'}\} & \rightarrow & \{\omega_{i'}\} \end{array} \quad \text{such that}\, \omega_{i'} = \frac{\partial x^i(x')}{\partial x^{i'}}\Big|_{\mathbf{p}} \omega_i\,. \tag{1.10}$$

Remark Notice the difference between formulae (1.7) and (1.9). In formulae (1.7), (1.8) transformation is performed by matrix of derivatives

$\partial x^{i'}\partial x^i$ from coordinates x^i to the new coordinates $x^{i'}$, and in formula (1.9) transformation is performed by the *inverse* matrix, matrix of derivatives $\partial x^i \partial x^{i'}$ from new coordinates $x^{i'}$ to the initial coordinates x^i.

Tensors:

Definition Consider geometrical object such that in arbitrary local coordinates (x^i) it is given by components

$$Q = \left\{Q^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q}\right\} \quad, i_1, \dots, i_p; j_1, \dots, j_q = 1, 2, \dots, n\,,$$

and under changing of coordinates this object is transformed in the following way:

$$Q^{i'_1 i'_2 \dots i'_p}_{j'_1 j'_2 \dots j'_q}(x') = \frac{\partial x^{i'_1}}{\partial x^{i_1}} \frac{\partial x^{i'_2}}{\partial x^{i_2}} \dots \frac{\partial x^{i'_p}}{\partial x^{i_p}} \frac{\partial x^{j_1}}{\partial x^{j'_1}} \frac{\partial x^{j_2}}{\partial x^{j'_2}} \dots \frac{\partial x^{j_q}}{\partial x^{j'_q}} Q^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q}(x)\,. \qquad (1.11)$$

We say that this is *p-times contravariant, q-times covariant tensor of valence* $\begin{pmatrix} p \\ q \end{pmatrix}$*, or shorter, tensors of the type* $\begin{pmatrix} p \\ q \end{pmatrix}$.

`Caution: this tensor possess` n^{p+q} `components.`

Sometimes it is useful to view $\begin{pmatrix} p \\ q \end{pmatrix}$-tensor as

$$Q = Q^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q} \frac{\partial}{\partial x^{i_1}} \otimes \frac{\partial}{\partial x^{i_2}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_p}} dx^{j_1} \otimes dx^{j_2} \otimes \dots \otimes dx^{j_q}$$

(Compare with definition of vector: $\mathbf{A} = A^i \frac{\partial}{\partial x^i}$ and covector (1-form) $\omega = \omega_i dx^i$).

Examples

Note that vector field (1.7) is nothing but tensor field of valency $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and 1-form (1.9) is nothing but tensor field of valency $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

One can consider *contravariant* tensors of the rank p

$$T = T^{i_1 i_2 \dots i_p}(x) \frac{\partial}{\partial x^{i_1}} \otimes \frac{\partial}{\partial x^{i_2}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_k}}$$

with components $\{T^{i_1 i_2 \dots i_k}\}(x)$. Under changing of coordinates $(x^1, \dots, x^n) \to (x^{1'}, \dots, x^{n'})$ (see (1.1)) they transform as follows:

$$T^{i'_1 i'_2 \dots i'_p}(x') = \frac{\partial x^{i'_1}}{\partial x^{i_1}} \frac{\partial x^{i'_2}}{\partial x^{i_2}} \dots \frac{\partial x^{i'_p}}{\partial x^{i_p}} T^{i_1 i_2 \dots i_p}(x)\,. \qquad (1.12)$$

One can consider *covariant* tensors of the rank q

$$S = S_{j_1 j_2 \dots j_q} dx^{j_1} \otimes dx^{j_2} \otimes \dots dx^{j_q}$$

with components $\{S_{j_1 j_2 \dots j_q}\}$. Under changing of coordinates $(x^1, \dots, x^n) \to (x^{1'}, \dots, x^{n'})$ they transform as follows:

$$S_{j'_1 j'_2 \dots j'_q}(x') = \frac{\partial x^{i_1}}{\partial x^{i'_1}} \frac{\partial x^{i_2}}{\partial x^{i'_2}} \dots \frac{\partial x^{i_p}}{\partial x^{i'_p}} S_{j_1 j_2 \dots j_q}(x)\,.$$

E.g. if S_{ik} is a covariant tensor of rank 2 then

$$S_{i'k'}(x') = \frac{\partial x^i(x')}{\partial x^{i'}} \frac{\partial x^k(x')}{\partial x^{k'}} S_{ik}(x)\,. \tag{1.13}$$

[**Example**] *linear operator and bilinear form*
Consider in Linear space V the following two geometrical objects

- Bilinear form $B(\mathbf{x}, \mathbf{y})$

$$B(\lambda \mathbf{x} + \mu \mathbf{x}', \mathbf{y}) = \lambda B(\mathbf{x}, \mathbf{y}) + \mu B(\mathbf{x}', \mathbf{y})\,.$$

  In an arbitrary basis $\{\mathbf{e}_i\}$ the bilinear form is presented by the matrrix

$$B_{ik} = B(e_i, \mathbf{e}_k)$$

- Linear operator
  $A \colon V \to V$,

$$A(\lambda \mathbf{x} + \mu \mathbf{x}') = \lambda A(\mathbf{x}) + \mu A(\mathbf{x}')\,.$$

  In an arbitrary basis $\{\mathbf{e}_i\}$ the linear operator A is presented by the matrrix

$$P_i^k P(\mathbf{e}_i) = P_i^k \mathbf{e}_k$$

  These both objexts, bilinear form and linear operator, are both presented in an arbitrary basis by 2×2 marices. However they are `different` objects!

Consider an arbitrary linear changing of coordinates from coordinates $\{x^i\}$ to new linear coordinates $\{x^{i'}\}$:

$$x^{i'} = Q^{i'}_k x^k \,, \text{respectively} x^i = P^i_{i'} x^{i'} \,, \tag{1.14}$$

(matrix $Q^{i'}_i$ is inverse to the matrix $P^i_{i'}$: $Q^{i'}_k P^k_{j'} = \delta^{i'}_{j'}$).

Bilinear form B transforms as

$$B_{i'k'} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^k}{\partial x^{k'}} = P^i_{i'} P^k_{k'} B_{ik}$$

Bilinear form is the covariant tensor of the valency $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$. Under the same linear changing of coordinates (1.14) linear operator A transforms as

$$A^{i'}_{k'} = \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^k}{\partial x^{k'}} A^i_k = Q^{i'}_i P^k_{k'} A^i_k \,.$$

Linear operator is the tensor of the of the valency $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Remark One can calulate determinant of the linear operator, however the determinant of bilinear form is not well-defined. In its turn one can consider symmetric bilinear form, which is represented by symmetrical matrix, however one cannot consider symmetric linear operator in linear space: it is ill-defined notion

Remark *Einstein summation rules*

In our lectures we always use so called *Einstein summation convention.* it implies that when an index occurs twice in the same expression in upper and in lower postitions, then the expression is implicitly summed over all possible values for that index. Sometimes it is called dummy indices summation rule.

Using Einstein summation rules we avoid to write bulky expressions. Later we will see that these notations are really very effective. E.g. equation (1.7) in 'standard' notations will appear as

$$\text{for every } i' = 1, \dots, n \;\; A^{i'}(x') = \sum_{i=1}^{n} \frac{\partial x^{i'}}{\partial x^i} A^i(x) \,.$$

1.2 Riemannian manifold

1.2.1 Riemannian manifold— manifold equipped with Riemannian metric

Definition The Riemannian manifold (M, G) is a manifold equipped with a Riemannian metric.

The Riemannian metric G on the manifold M defines the length of the tangent vectors and the length of the curves.

Definition Riemannian metric G on n-dimensional manifold M^n defines for every point $\mathbf{p} \in M$ the scalar product of tangent vectors in the tangent space $T_{\mathbf{p}}M$ smoothly depending on the point $\mathbf{p}$.

It means that in every coordinate system $(x^1, \dots, x^n)$ a metric $G = g_{ik}dx^i dx^k$ is defined by a matrix valued smooth function $g_{ik}(x)$ ($i = 1, \dots, n; k = 1, \dots n$) such that for any two vectors

$$\mathbf{A} = A^i(x)\frac{\partial}{\partial x^i}, \ \mathbf{B} = B^i(x)\frac{\partial}{\partial x^i},$$

tangent to the manifold M at the point $\mathbf{p}$ with coordinates $x = (x^1, x^2, \dots, x^n)$ ($\mathbf{A}, \mathbf{B} \in T_{\mathbf{p}}M$) the scalar product is equal to:

$$\langle \mathbf{A}, \mathbf{B} \rangle_G \big|_{\mathbf{p}} = G(\mathbf{A}, \mathbf{B})\big|_{\mathbf{p}} = A^i(x) g_{ik}(x) B^k(x) =$$

$$\begin{pmatrix} A^1 \dots A^n \end{pmatrix} \begin{pmatrix} g_{11}(x) & \dots & g_{1n}(x) \\ \dots & \dots & \dots \\ g_{n1}(x) & \dots & g_{nn}(x) \end{pmatrix} \begin{pmatrix} B^1 \\ \cdot \\ \cdot \\ \cdot \\ B^n \end{pmatrix} \tag{1.15}$$

where

- $G(\mathbf{A}, \mathbf{B}) = G(\mathbf{B}, \mathbf{A})$, i.e. $g_{ik}(x) = g_{ki}(x)$ (symmetricity condition)
- $G(\mathbf{A}, \mathbf{A}) > 0$ if $\mathbf{A} \neq \mathbf{0}$, i.e.

  $g_{ik}(x)u^i u^k \geq 0$, $g_{ik}(x)u^i u^k = 0$ iff $u^1 = \dots = u^n = 0$ (positive-definiteness)
- $G(\mathbf{A}, \mathbf{B})\big|_{\mathbf{p}=x}$, i.e. $g_{ik}(x)$ are smooth functions.

The matrix $||g_{ik}||$ of components of the metric G we also sometimes denote by G.

Now we establish rule of transformation for entries of matrix $g_{ik}(x)$, of metric G.

Notice that an arbitrary matrix entry g_{ik} is nothing but scalar product of vectors ∂_i, ∂_k at the given point:

$$g_{ik}(x) = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k} \right\rangle, \quad \text{in coordinates } (x^1, \dots, x^n) \tag{1.16}$$

Use this formula for establishing rule of transformations of $g_{ik}(x)$. In the new coordinates $x^{i'} = (x^{1'}, \dots, x^{n'})$ according this formula we have that

$$g_{i'k'}(x') = \left\langle \frac{\partial}{\partial x^{i'}}, \frac{\partial}{\partial x^{k'}} \right\rangle, \quad \text{in coordinates } (x^1, \dots, x^n)\,.$$

Now using chain rule, linearity of scalar product and formula (1.16) we see that

$$g_{i'k'}(x') = \left\langle \frac{\partial}{\partial x^{i'}}, \frac{\partial}{\partial x^{k'}} \right\rangle = \left\langle \frac{\partial x^i}{\partial x^{i'}} \frac{\partial}{\partial x^i}, \frac{\partial x^k}{\partial x^{k'}} \frac{\partial}{\partial x^k} \right\rangle$$
$$= \frac{\partial x^i}{\partial x^{i'}} \underbrace{\left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k} \right\rangle}_{g_{ik}(x)} \frac{\partial x^k}{\partial x^{k'}} = \frac{\partial x^i}{\partial x^{i'}} g_{ik}(x) \frac{\partial x^k}{\partial x^{k'}} \tag{1.17}$$

This transformation law implies that g_{ik} entries of matrix $||g_{ik}||$ are components of *covariant tensor field* $G = g_{ik} dx^i dx^k$ *of rank 2*(see equation (1.13)).

One can say that Riemannian metric is defined by symmetric covariant smooth tensor field G *of the rank 2 which defines scalar product in the tangent spaces* $T_{\mathbf{p}}M$ *smoothly depending on the point* $\mathbf{p}$. *Components of tensor field* G *in coordinate system are functions* $g_{ik}(x)$:

$$G = g_{ik}(x) dx^i \otimes dx^k\,,$$
$$\langle \mathbf{A}, \mathbf{B} \rangle = G(\mathbf{A}, \mathbf{B}) = g_{ik}(x) dx^i \otimes dx^k\, (\mathbf{A}, \mathbf{B})\,. \tag{1.18}$$

In practice it is more convenient to perform transformation of metric G under changing of coordinates in the following way:

$$G = g_{ik} dx^i \otimes dx^k = g_{ik} \left(\frac{\partial x^i}{\partial x^{i'}} dx^{i'} \right) \otimes \left(\frac{\partial x^k}{\partial x^{k'}} dx^{k'} \right) =$$

$$\frac{\partial x^i}{\partial x^{i'}} g_{ik} \frac{\partial x^k}{\partial x^{k'}} dx^{i'} \otimes dx^{k'} = g_{i'k'} dx^{i'} \otimes dx^{k'} \text{ , hence } g_{i'k'} = \frac{\partial x^i}{\partial x^{i'}} g_{ik} \frac{\partial x^k}{\partial x^{k'}} \,. \quad (1.19)$$

We come to transfomation rule (1.17).

Later by some abuse of notations we sometimes omit the sign of tensor product and write a metric just as

$$G = g_{ik}(x) dx^i dx^k \,.$$

1.2.2 Examples

- $\mathbf{R}^n$ with canonical coordinates $\{x^i\}$ and with metric

  $$G = (dx^1)^2 + (dx^2)^2 + \cdots + (dx^n)^2$$

  $G = ||g_{ik}|| = \text{diag } [1, 1, \ldots, 1]$

  Recall that this is a basis example of n-dimensional Euclidean space $\mathbf{E}^n$, where scalar product is defined by the formula:

  $$G(\mathbf{X}, \mathbf{Y}) = \langle \mathbf{X}, \mathbf{Y} \rangle = g_{ik} X^i Y^k = X^1 Y^1 + X^2 Y^2 + \cdots + X^n Y^n \,.$$

  In the general case if $G = ||g_{ik}||$ is an arbitrary symmetric positive-definite metric then $G(\mathbf{X}, \mathbf{Y}) = \langle \mathbf{X}, \mathbf{Y} \rangle = g_{ik} X^i Y^k$. One can show that there exists a new basis $\{\mathbf{e}_i\}$ such that in this basis $G(\mathbf{e}_i, \mathbf{e}_k) = \delta_{ik}$. This basis is called orthonormal basis. (See the Lecture notes in Geometry)

  Scalar product in vector space defines the *same* scalar product at all the points. In general case for Riemannian manifold scalar product depends on a point. In Riemannian manifold we consider arbitrary transformations from local coordinates to new local coordinates.

- Euclidean space $\mathbf{E}^2$ with polar coordinates in the domain $y > 0$ ($x = r\cos\varphi, y = r\sin\varphi$):

  $dx = \cos\varphi dr - r\sin\varphi d\varphi, dy = \sin\varphi dr + r\cos\varphi d\varphi$. In new coordinates the Riemannian metric $G = dx^2 + dy^2$ will have the following appearance:

  $$G = (dx)^2 + (dy)^2 = (\cos\varphi dr - r\sin\varphi d\varphi)^2 + (\sin\varphi dr + r\cos\varphi d\varphi)^2 = dr^2 + r^2 (d\varphi)^2$$

  We see that for matrix $G = ||g_{ik}||$

  $$\underbrace{G = \begin{pmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{in Cartesian coordinates}}, \qquad \underbrace{G = \begin{pmatrix} g_{rr} & g_{r\varphi} \\ g_{\varphi r} & g_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}}_{\text{in polar coordinates}}$$

- `Circle (without a point)`

  Interval $[0,2\pi)$ in the line $0 \leq x < 2\pi$ with Riemannian metric

  $$G = a^2 dx^2 \tag{1.20}$$

  Renaming $x \mapsto \varphi$ we come to habitual formula for metric for circle without a point of the radius a: $x^2+y^2 = a^2$ embedded in the Euclidean space $\mathbf{E}^2$:

  $$G = a^2 d\varphi^2 \qquad \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \end{cases}, 0 < \varphi < 2\pi, \quad \text{or} \quad -\pi < \varphi < \pi\,. \tag{1.21}$$

  Rewrite this metric in stereographic coordinate t:

  $$G = a^2 d\varphi^2 = 4a^4 dt^2 (a^2+t^2)^2\,, \quad \text{where } t = \frac{ax}{a-y} = \frac{a^2\cos\varphi}{a - a\sin\varphi} = \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)\,. \tag{1.22}$$

  (See (1.6) and Homeworks 0 and 2.)

- `Cylinder surface (without a line)`

  Consider domain in $\mathbf{R}^2$, $D = \{(x,y)\colon,\ 0 \leq x < 2\pi$ with Riemannian metric

  $$G = a^2 dx^2 + dy^2 \tag{1.23}$$

  We see that renaming variables $x \mapsto \varphi$, $y \mapsto h$ we come to habitual, familiar formulae for metric in standard polar coordinates for cylinder surface of the radius a without a line embedded in the Euclidean space $\mathbf{E}^3$:

  $$G = a^2 d\varphi^2 + dh^2 \qquad \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases}, 0 < \varphi < 2\pi, -\infty < h < \infty \tag{1.24}$$

  (Coordinate φ is well defined for $-\pi < \varphi < \pi$ also.)

- `Sphere without...`

  Consider domain in $\mathbf{R}^2$, $0 < x < 2\pi$, $0 < y < \pi$ with metric $G = dy^2 + \sin^2 y dx^2$ We see that renaming variables $x \mapsto \varphi$, $y \mapsto h$ we come to habitual, familiar formulae for metric in standard spherical coordinates for sphere $x^2+y^2+z^2=a^2$ of the radius a embedded in the Euclidean space $\mathbf{E}^3$:

$$G = a^2 d\theta^2 + a^2 \sin^2\theta d\varphi^2 \qquad \begin{cases} x = a\sin\theta\cos\varphi \\ y = a\sin\theta\sin\varphi \\ z = a\cos\theta \end{cases} \quad , \quad 0<\theta<\pi\,,\, 0<\varphi<2\pi\,. \tag{1.25}$$

(See examples also in the Homeworks.)

If we omit the condition of positive-definiteness for Riemannian metric we come to so called *Pseudoriemannian metric*. Manifold equipped with pseudoriemannian metric is called pseudoriemannian manifold. Pseudoriemannian manifolds appear in applications in the special and general relativity theory.

In pseudoriemanninan space scalar product $(\mathbf{X},\mathbf{X})$ may take an arbitrary real values: it can be positive, negative, it can be equal to zero. Vectors $\mathbf{X}$ such that $(\mathbf{X},\mathbf{X})=0$ are called null-vectors.

For example consider 4-dimensional linear space $\mathbf{R}^4$ with pseudometric

$$G = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2\,.$$

For an arbitrary vector $\mathbf{X}=(a^0,a^1,a^2,a^3)$ scalar product $(\mathbf{X},\mathbf{X})$ is positive if $(a^0)^2 > (a_1)^2+(a_2)^2+(a_3)^2$, and it is negative if $(a^0)^2 < (a_1)^2+(a_2)^2+(a_3)^2$, and $\mathbf{X}$ is null-vector if $(a^0)^2 = (a_1)^2+(a_2)^2+(a_3)^2$. It is so called Minkovski space. The coordinate x^0 plays a role of the time: $x^0=ct$, where c is the value of the speed of the light. Vectors $\mathbf{X}$ such that $(\mathbf{X},\mathbf{X})>0$ are called time-like vectors and they called space-like vectors if $(\mathbf{X},\mathbf{X})<0$.

1.2.3 Scalar product $\rightarrow$ Length of tangent vectors and angle between them

The Riemannian metric defines scalar product of tangent vectors attached at the given point. Hence it defines the length of tangent vectors and angle between them. If $\mathbf{X}=X^m\frac{\partial}{\partial x^m}$, $\mathbf{Y}=Y^m\frac{\partial}{\partial x^m}$ are two tangent vectors at the given point $\mathbf{p}$ of Riemannian manifold with coordinates $x^1,\dots,x^n$, then we have that lengths of these vectors equal to

$$|\mathbf{X}| = \sqrt{\langle \mathbf{X},\mathbf{X}\rangle} = \sqrt{g_{ik}(x)X^iX^k}, \quad |\mathbf{Y}| = \sqrt{\langle \mathbf{Y},\mathbf{Y}\rangle} = \sqrt{g_{ik}(x)Y^iY^k}, \tag{1.26}$$

and an 'angle' θ between these vectors is defined by the relation

$$\cos\theta = \frac{\langle \mathbf{X},\mathbf{Y}\rangle}{|\mathbf{X}|\cdot|\mathbf{Y}|} = \frac{g_{ik}X^iY^k}{\sqrt{g_{ik}(x)X^iX^k}\sqrt{g_{ik}(x)Y^iY^k}} \tag{1.27}$$

Remark We say 'angle' but we calculate just cosinus of angle.

Example Let M be 3-dimensional Riemannian manifold, and $\mathbf{p} \in M$ a point in it. Suppose that the manifold M is equipped with local coordinates x, y, z in a vicinity of this point, and the expression of Riemannian metric in these local coordinates is

$$G = \frac{dx^2 + dy^2 + dz^2}{(1 + x^2 + y^2 + z^2)^2} \,. \tag{1.28}$$

Consider the vectors $\mathbf{X} = a\partial_x + b\partial_y + c\partial_z$ and $\mathbf{Y} = p\partial_x + q\partial_y + r\partial_z$, attached at the point $\mathbf{p}$, with coordinates $x = 2, y = 2, z = 1$. Find the lengths of vectors $\mathbf{X}$ and $\mathbf{Y}$ and find cosinus of the angle between these vectors.

We see that matrix of Riemannian metric is

$$||g_{ik(x)}|| = \begin{pmatrix} \frac{1}{(1+x^2+y^2+z^2)^2} & 0 & 0 \\ 0 & \frac{1}{(1+x^2+y^2+z^2)^2} & 0 \\ 0 & 0 & \frac{1}{(1+x^2+y^2+z^2)^2} \end{pmatrix} \text{ i. e. } g_{ik}(x, y, z) = \frac{\delta_{ik}}{(1 + x^2 + y^2 + z^2)^2} \,,$$

where $g_{ik}(x)$ are entries of matrix: $G = g_{ik}(x)dx^i dx^k$, (δ_{ik} is Kronecker symbol: $\delta_{ik} = 1$ if $i = k$ and it vanishes otherwise).

According to formulae above

$$|\mathbf{X}| = \sqrt{\langle \mathbf{X}, \mathbf{X} \rangle} = \sqrt{g_{ik}(x, y, z)X^i X^k}\Big|_{\mathbf{p}} = \sqrt{\frac{\delta_{ik} X^i X^k}{(1 + x^2 + y^2 + z^2)^2}}\Big|_{x=2,y=2,z=1} =$$

$$\sqrt{\frac{a^2 + b^2 + c^2}{(1 + 2^2 + 2^2 + 1^2)^2}} = \frac{\sqrt{a^2 + b^2 + c^2}}{10} \,,$$

$$|\mathbf{Y}| = \sqrt{\langle \mathbf{Y}, \mathbf{Y} \rangle} = \sqrt{g_{ik}(x, y, z)Y^i Y^k}\Big|_{\mathbf{p}} = \sqrt{\frac{\delta_{ik} Y^i Y^k}{(1 + x^2 + y^2 + z^2)^2}}\Big|_{x=2,y=2,z=1} =$$

$$\sqrt{\frac{p^2 + q^2 + r^2}{(1 + 2^2 + 2^2 + 1^2)^2}} = \frac{\sqrt{p^2 + q^2 + r^2}}{10} \,,$$

and

$$\cos\theta = \frac{\langle \mathbf{X}, \mathbf{Y} \rangle}{|\mathbf{X}||\mathbf{Y}|} = \frac{g_{ik}(x, y, z)X^i Y^k\big|_{\mathbf{p}}}{\sqrt{g_{pq}(x, y, z)X^p X^q}\sqrt{g_{rs}(x, y, z)Y^r Y^s}} = \frac{\frac{\delta_{ik} X^i Y^k}{(1+x^2+y^2+z^2)^2}}{|\mathbf{X}||\mathbf{Y}|}$$

$$= \frac{\frac{ap+bq+cr}{(1+2^2+2^2+1)^2}}{\frac{\sqrt{a^2+b^2+c^2}}{10}\frac{\sqrt{p^2+q^2+r^2}}{10}} == \frac{ap+bq+cr}{\sqrt{a^2+b^2+c^2}\sqrt{p^2+q^2+r^2}} .$$

This example is related with the notion of so called *conformally euclidean metric* (see the next paragraph, 1.2.4).

1.2.4 Conformally Euclidean metric

Let (M, G) be a Riemannian manifold.

Definition We say that metric G is locally conformally Euclidean in a vicinity of the point $\mathbf{p}$ if in a vicinity of this point there exist local coordinates $\{x^i\}$ such that in these coordinates metric has an appearance

$$G = \sigma(x)\delta_{ik}dx^i dx^k = \sigma(x)\left((dx^1)^2 + \cdots + (dx^n)^2\right) , \tag{1.29}$$

i.e. it is proportional to 'Euclidean metric'. We call coordinates $\{x^i\}$ *conformall* coordinates or *isothermic* coordinates if condition (1.29) holds.

We say that metric is conformally Euclidean if it is locally conformally Euclidean in the vicinity of every point. We say that Riemannian manifold $(M.G)$ is conformally Euclidean if the metric G on it is conformally Euclidean

One can see that if metric is conformally Euclidean in a vicinity of some point $\mathbf{p}$, then the angle between vectors, more precisely the cosinus of the angle between vectors attached at this point (see equation (1.27)) is the same as for Eucldean metric. Indeed, let G be conformally Euclidean metric and let x^i be local coordinates such that the metric has an appearance (1.29) in these coordinates. Let $\mathbf{X}, \mathbf{Y}$ be two non-vanishing vectors $\mathbf{X} = X^m(x)\frac{\partial}{\partial x^m}$, $\mathbf{Y} = Y^m(x)\frac{\partial}{\partial x^m}$ $(|\mathbf{X}| \neq 0, |\mathbf{Y}| \neq 0)$ attached at a same point. Then

$$\cos\theta = \frac{\langle \mathbf{X}, \mathbf{Y}\rangle}{|\mathbf{X}| \cdot |\mathbf{Y}|} = \frac{g_{ik}X^iY^k}{\sqrt{g_{ik}(x)X^iX^k}\sqrt{g_{ik}(x)Y^iY^k}} =$$

$$\frac{\sigma(x)\delta_{ik}X^iY^k}{\sqrt{\sigma(x)\delta_{ik}(x)X^iX^k}\sqrt{\sigma(x)\delta_{ik}(x)Y^iY^k}} = \frac{\sum_k X^kY^k}{\sqrt{\sum_k(x)X^kX^k}\sqrt{\sum_k Y^kY^k}} . \tag{1.30}$$

(Note that coefficient σ in equation (1.29) has to be positive.)

Remark One can show that the condition of 'preserving the angles' is not only necessary condition but it is also sufficient condition for metric to be conformally Euclidean (see the problem 1 in Homework 2).

Now consider examples.

First It is instructive to recall the example considered in previous subsection 1.2.3), where Riemannian metric in a vicinity of a point had an appearance (1.28) This is example of Riemannian manifold which is locally conformally Euclidean in a vicinity of a point $\mathbf{p}$.

Another

Example Consider the surface of cylinder with the metric

$$G = a^2 d\varphi^2 + dh^2 \tag{1.31}$$

(see equation (1.25)). In a vicinity of every point one can conisder coordinates $\begin{cases} u = a\varphi \\ v = h \end{cases}$. It is evident that in these coordinates $G = du^2 + dv^2$, i.e. this Riemannian manifold is conformally Euclidean.

Remark. In fact we proved more: for metric of cylinder in coordinates u, v, the coefficient $\sigma(x) \equiv 1$, i,e. in these coordinates metric is not only *locally conformally Euclidean*, but also it is *locally Euclidean*. We will study this question later in details. (see paragraph "Locally Euclidean Riemannian manifold" later).

Later we consider also another important examples.

It is important, that the following Theorem takes a place:

Theorem (Gauss) *Every 2-dimensional Riemannian manifold is locally conformally Euclidean, i.e. for arbitrary 2-dimensional Riemannian manifold, in a vicinity of arbitrary point, there exist coordinates u', v' such that in these coordinates Riemannian metric*

$$G = A(u,v)du^2 + 2B(u,v)dudv + C(u,v)dv^2 = \sigma(u',v')\left(du'^2\right) \tag{1.32}$$

We will not prove this theorem footnotethe proof is easy and almost evident for anaylitical manifolds, and it is hard for smooth manifolds, but consider many examples of 2-dimensional Riemannian manifolds, with suitable conformall coordinates.

1.2.5 Length of curves

Let γ: $x^i = x^i(t), (i = 1, \ldots, n))$ $(a \leq t \leq b)$ be a curve on the Riemannian manifold (M, G).

At the every point of the curve the velocity vector (tangent vector) is defined:

$$\mathbf{v}(t) = \begin{pmatrix} \dot{x}^1(t) \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}^n(t) \end{pmatrix} = \frac{dx^i(t)}{dt}\frac{\partial}{\partial x^i}$$

Remark Note that $\mathbf{v}(t)$ is a vector; check transformation rules:

$$\frac{dx^i(t)}{dt}\frac{\partial}{\partial x^i} = \frac{dx^i(t)}{dt}\frac{\partial x^{i'}}{\partial x^i}\frac{\partial}{\partial x^{i'}} = \frac{\partial x^{i'}}{\partial x^i}\frac{dx^i(t)}{dt}\frac{\partial}{\partial x^{i'}} = \frac{dx^{i'}(t)}{dt}\frac{\partial}{\partial x^{i'}} .$$

The length of velocity vector $\mathbf{v} \in T_xM$ (vector $\mathbf{v}$ is tangent to the manifold M at the point x) equals to

$$|\mathbf{v}|_x = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle_G \big|_x} = \sqrt{g_{ik} v^i v^k}\big|_x = \sqrt{g_{ik}\frac{dx^i(t)}{dt}\frac{dx^k(t)}{dt}}\big|_x .$$

For an arbitrary curve its length is equal to the integral of the length of velocity vector:

$$L_\gamma = \int_a^b \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle_G \big|_{x(t)}}dt = \int_a^b \sqrt{g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t)}dt . \tag{1.33}$$

Bearing in mind that metric (1.18) defines the length we often write metric in the following form

$$G = ds^2 = g_{ik}dx^i dx^k$$

Example 1 Consider 2-dimensional Riemannian manifold with metric

$$||g_{ik}(u, v)|| = \begin{pmatrix} g_{11}(u, v) & g_{12}(u, v) \\ g_{21}(u, v) & g_{22}(u, v) \end{pmatrix} .$$

Then

$$G = ds^2 = g_{ik}du^i dv^k = g_{11}(u, v)du^2 + 2g_{12}(u, v)dudv + g_{22}(u, v)dv^2 .$$

The length of the curve γ: $u = u(t), v = v(t)$, where $t_0 \le t \le t_1$ according to (1.33) is equal to $L_\gamma = \int_{t_0}^{t_1} \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \int_{t_0}^{t_1} \sqrt{g_{ik}(x)\dot{x}^i\dot{x}^k} =$

$$\int_{t_0}^{t_1} \sqrt{g_{11}\left(u\left(t\right), v\left(t\right)\right) u_t^2 + 2g_{12}\left(u\left(t\right), v\left(t\right)\right) u_t v_t + g_{22}\left(u\left(t\right), v\left(t\right)\right) v_t^2 dt} . \tag{1.34}$$

Example Consider Lobachevsky (hyperbolic) plane. We consider upper-half model of Lobachevsky (hyperbolic) plane:

$$G = \frac{dx^2 + dy^2}{y^2}\,, \quad (y > 0)$$

Consider in Lobachevsky plane the curve C: $\begin{cases} x = x_0 \\ y = t \end{cases}$, $a < t < b$ and claculate its length:

$$L_C = \int_a^b \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \int_a^b \sqrt{g_{ik}(x)\dot{x}^i\dot{x}^k} =$$

$$\int_a^b \sqrt{g_{11}\left(x\left(t\right), y\left(t\right)\right) x_t^2 + 2g_{12}\left(x\left(t\right), y\left(t\right)\right) x_t y_t + g_{22}\left(x\left(t\right), y\left(t\right)\right) y_t^2}dt =$$

$$\int_a^b \sqrt{\frac{1}{y^2}\left(x_t^2 + y_t^2\right)}dt = \int_a^b \sqrt{\frac{1}{t^2}\left(0+1\right)}dt = \int_a^b \frac{dt}{t} = \left|\log \frac{a}{b}\right|\,.$$

The length of curves defined by the formula(1.33) obeys the following natural conditions

- It coincides with the usual length in the Euclidean space $\mathbf{E}^n$ ($\mathbf{R}^n$ with standard metric $G = (dx^1)^2 + \cdots + (dx^n)^2$ in Cartesian coordinates). E.g. for 3-dimensional Euclidean space

$$L_\gamma = \int_a^b \sqrt{g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t)}dt = \int_a^b \sqrt{(\dot{x}^1(t))^2 + (\dot{x}^2(t))^2 + (\dot{x}^3(t))^2}dt$$

- It does not depend on parameterisation of the curve

$$L_\gamma = \int_a^b \sqrt{g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t)}dt = \int_{a'}^{b'} \sqrt{g_{ik}(x(\tau))\dot{x}^i(\tau)\dot{x}^k(\tau)}d\tau\,,$$

  ($x^i(\tau) = x^i(t(\tau))$, $a' \leq \tau \leq b'$ while $a \leq t \leq b$) since under changing of parameterisation

$$\dot{x}^i(\tau) = \frac{dx(t(\tau))}{d\tau} = \frac{dx(t(\tau))}{dt}\frac{dt}{d\tau} = \dot{x}^i(t)\frac{dt}{d\tau}\,.$$

- It *does not depend on coordinates on Riemannian manifold M*

$$L_\gamma = \int_a^b \sqrt{g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t)}dt = \int_a^b \sqrt{g_{i'k'}(x'(t))\dot{x}^{i'}(t)\dot{x}^{k'}(t)}dt\,.$$

  This immediately follows from transformation rule (1.19) for Riemannian metric:

$$g_{i'k'}\dot{x}^{i'}(t)\dot{x}^{k'}(t) = g_{ik}\left(\frac{\partial x^i}{\partial x^{i'}(t)}\dot{x}^{i'}(t)\right)\left(\frac{\partial x^k}{\partial x^{k'}}\dot{x}^{k'}(t)\right) g_{ik}\,\dot{x}^i(t)\dot{x}^k(t)\,.$$

- It is additive: length of the sum of two curves is equal to the sum of their lengths. If a curve $\gamma = \gamma_1 + \gamma$, i.e. $\gamma\colon x^i(t), a \le t \le b$, $\gamma_1\colon x^i(t), a \le t \le c$ and $\gamma_2\colon x^i(t), c \le t \le b$ where a point c belongs to the interval (a,b) then $L_\gamma = L_{\gamma^1} + L_{\gamma^2}$.

One can show that formula (1.33) for length is defined uniquely (up to a constant multiplier) by these conditions. More precisely one can show under some technical conditions one may show that any local additive functional on curves which does not depend on coordinates and parameterisation, and depends on derivatives of curves of order ≤ 1 is equal to (1.33) up to a constant multiplier. To feel the taste of this statement you may do the following exercise:

Exercise Let $A = A\left(x(t), y(t), \frac{dx(t)}{dt}, \frac{dy(t)}{dt}\right)$ be a function such that an integral $L = \int A\left(x(t), y(t), \frac{dx(t)}{dt}, \frac{dy(t)}{dt}\right) dt$ over an arbitrary curve γ in $\mathbf{E}^2$ does not change under reparameterisation of this curve and under an arbitrary isometry, i.e. translation and rotation of the curve. Then one can easy show (show it!) that

$$A\left(x(t), y(t), \frac{dx(t)}{dt}, \frac{dy(t)}{dt}\right) = c\sqrt{\left(\frac{dx(t)}{dt}\right)^2 + \left(\frac{dy(t)}{dt}\right)^2},$$

where c is a constant, i.e. it is a usual length up to a multiplier

1.3 Riemannian structure on the surfaces embedded in Euclidean space

Let M be a surface embedded in Euclidean space. Let G be Riemannian structure on the manifold M.

Let $\mathbf{X}, \mathbf{Y}$ be two vectors tangent to the surface M at a point $\mathbf{p} \in M$. An External Observer calculate this scalar product viewing these two vectors as vectors in $\mathbf{E}^3$ attached at the point $\mathbf{p} \in \mathbf{E}^3$ using scalar product in $\mathbf{E}^3$. An Internal Observer will calculate the scalar product viewing these two vectors as vectors tangent to the surface M using the Riemannian metric G (see the formula (1.15)). Respectively

If L is a curve in M then an External Observer consider this curve as a curve in $\mathbf{E}^3$, calculate the modulus of velocity vector (speed) and the length of the curve using Euclidean scalar product of ambient space. An Internal Observer ("an ant") will define the modulus of the velocity vector and the length of the curve using Riemannian metric.

Definition Let M be a surface embedded in the Euclidean space. We say that metric G_M on the surface is induced by the Euclidean metric if the scalar product of arbitrary two vectors $\mathbf{A}, \mathbf{B} \in T_\mathbf{p}M$ calculated in terms of the metric G equals to Euclidean scalar product of these two vectors:

$$\langle \mathbf{A}, \mathbf{B} \rangle_{G_M} = \langle \mathbf{A}, \mathbf{B} \rangle_{G_{\text{Euclidean}}} \tag{1.35}$$

In other words we say that Riemannian metric on the embedded surface is induced by the Euclidean structure of the ambient space if External and Internal Observers come to the same results calculating scalar product of vectors tangent to the surface.

In this case modulus of velocity vector (speed) and the length of the curve is the same for External and Internal Observer.

1.3.1 Internal and external observers

Tangent vectors, coordinate tangent vectors

Here we recall basic notions from the course of Geometry which we will need here.

Let $\mathbf{r} = \mathbf{r}(u, v)$ be parameterisation of the surface M embedded in the Euclidean space:

$$\mathbf{r}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}$$

Here as always x, y, z are Cartesian coordinates in $\mathbf{E}^3$.

Let $\mathbf{p}$ be an arbitrary point on the surface M. Consider the plane formed by the vectors which are adjusted to the point $\mathbf{p}$ and tangent to the surface M. We call this plane *plane tangent to M at the point* $\mathbf{p}$ and denote it by $T_{\mathbf{p}}M$.

For a point $\mathbf{p} \in M$ one can consider a basis in the tangent plane T_pM adjusted to the parameters u, v.

Tangent basis vectors at any point (u, v) are

$$\mathbf{r}_u = \frac{\partial \mathbf{r}(u, v)}{\partial u} = \begin{pmatrix} \frac{\partial x(u,v)}{\partial u} \\ \frac{\partial y(u,v)}{\partial u} \\ \frac{\partial z(u,v)}{\partial u} \end{pmatrix} = \frac{\partial x(u, v)}{\partial u} \frac{\partial}{\partial x} + \frac{\partial y(u, v)}{\partial u} \frac{\partial}{\partial y} + \frac{\partial z(u, v)}{\partial u} \frac{\partial}{\partial z}$$

and

$$\mathbf{r}_v = \frac{\partial \mathbf{r}(u, v)}{\partial v} = \begin{pmatrix} \frac{\partial x(u,v)}{\partial v} \\ \frac{\partial y(u,v)}{\partial v} \\ \frac{\partial z(u,v)}{\partial v} \end{pmatrix} = \frac{\partial x(u, v)}{\partial v} \frac{\partial}{\partial x} + \frac{\partial y(u, v)}{\partial v} \frac{\partial}{\partial y} + \frac{\partial z(u, v)}{\partial v} \frac{\partial}{\partial z}$$

Definition We call basis vectors $\mathbf{r}_u, \mathbf{r}_v$ adjusted to parameters (coordinates) u, v *coordinate basis vectors*

Every vector $\mathbf{X} \in T_pM$ can be expanded over the basis of coordinate basis vectors:

$$\mathbf{X} = X_u \mathbf{r}_u + X_v \mathbf{r}_v,$$

where X_u, X_v are coefficients, components of the vector $\mathbf{X}$.

Internal Observer views the basis vector $\mathbf{r}_u \in T_pM$ as the vector ∂_u. Why? The vector $\mathbf{r}_u$ attached at the point $\mathbf{p}$ is a velocity vector for the curve $\gamma_{\mathbf{r}_u}(t)\colon \begin{cases} u = u_0 + t \\ v = v_0 \end{cases}$ starting at the point $\mathbf{p}$ ((u_0, v_0) are coordinates of the point $\mathbf{p}$). If $f = f(u, v)$ is a function on the surface M, then one can see that directional derivative of this function along a vector $\mathbf{r}_u$ is defined by $\frac{\partial}{\partial u}$:

$$\partial_u f(u,v)\big|_{\mathbf{p}} = \frac{d}{dt} f\left(\gamma_{\mathbf{r}_u}(t)\right) = \frac{d}{dt} f\left(u_0 + t, v_0\right) .$$

Respectively the basis vector $\mathbf{r}_v \in T_pM$ for an Internal Observer, is velocity vector for the curve $\gamma_{\mathbf{r}_v}(t)\colon \begin{cases} u = u_0 \\ v = v_0 + t \end{cases}$ starting at the point $\mathbf{p}$ and Internal Observer denotes this vector ∂_v:

$$\partial_v f(u,v)\big|_{\mathbf{p}} = \frac{d}{dt} f\left(\gamma_{\mathbf{r}_v}(t)\right) = \frac{d}{dt} f\left(u_0, v_0 + t\right) .$$

For an arbitrary vector $\mathbf{X}$ which is tangent to surface M at the point $\mathbf{p}$, ($\mathbf{X} \in T_pM$)

`External observer`	`Internal observer`
$\mathbf{X} = a\mathbf{r}_u + b\mathbf{r}_v$	$\mathbf{X} = a\partial_u + b\partial_v$

Example Consider sphere of radius R in $\mathbf{E}^3$, $x^2 + y^2 + z^2 = R^2$. In spherical coordinates

$$\mathbf{E}^3 \ni \mathbf{r} = \mathbf{r}(\theta, \varphi) \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ x = R\cos\theta \end{cases} ,$$

these coordinates are well-defined for $0 < \theta < \frac{\pi}{2}$ and $0 < \varphi < 2\pi$. For coordinate basis vectors $\mathbf{r}_\theta$ and $\mathbf{r}_\varphi$ we have:

$$\mathbf{r}_\theta = \frac{\partial \mathbf{r}(\theta,\varphi)}{\partial \theta} = \frac{\partial}{\partial\theta}\begin{pmatrix} x(\theta,\varphi) \\ y(\theta,\varphi) \\ z(\theta,\varphi) \end{pmatrix} = \frac{\partial}{\partial\theta}\begin{pmatrix} R\sin\theta\cos\varphi \\ R\sin\theta\sin\varphi \\ R\cos\theta \end{pmatrix} =$$

$$
\begin{pmatrix} R\cos\theta\cos\varphi \\ R\cos\theta\sin\varphi \\ -R\sin\theta \end{pmatrix} = R\cos\theta\cos\varphi\frac{\partial}{\partial x} + R\cos\theta\sin\varphi\frac{\partial}{\partial y} - R\sin\theta\frac{\partial}{\partial z}\,,
$$

and respectively

$$
\mathbf{r}_\varphi = \frac{\partial \mathbf{r}(\theta,\varphi)}{\partial\theta} = \frac{\partial}{\partial\varphi}\begin{pmatrix} x(\theta,\varphi) \\ y(\theta,\varphi) \\ z(\theta,\varphi) \end{pmatrix} = \frac{\partial}{\partial\varphi}\begin{pmatrix} R\sin\theta\cos\varphi \\ R\sin\theta\sin\varphi \\ R\cos\theta \end{pmatrix} =
$$

$$
\begin{pmatrix} -R\sin\theta\sin\varphi \\ R\sin\theta\cos\varphi \\ 0 \end{pmatrix} = -R\sin\theta\sin\varphi\frac{\partial}{\partial x} + R\sin\theta\cos\varphi\frac{\partial}{\partial y}\,. \tag{1.36}
$$

Here is a table how observers look at the objects on sphere:

	`INTERNAL OBSERVER`	`EXTERNAL OBSERVER`
point on S^2	2 coordinates θ,φ	3 coordinates $\mathbf{r}=\mathbf{r}(\theta,\varphi)$
curve on S^2	$\theta(t),\varphi(t)$	$\mathbf{r}(t)=\mathbf{r}(\theta(t,\varphi(t)))$
coordinate tangent vectors to S^2	$\frac{\partial}{\partial\theta}\,,\frac{\partial}{\partial\varphi}$	$\mathbf{r}_\theta\,,\quad \mathbf{r}_\varphi$
tangent vector to S^2	$a\frac{\partial}{\partial\theta}+b\frac{\partial}{\partial\varphi}$	$A\frac{\partial}{\partial x}+B\frac{\partial}{\partial y}+C\frac{\partial}{\partial z}=a\mathbf{r}_\theta+b\mathbf{r}_\varphi$