Homework 4. Solutions

In the all exercises except 5 and 6, vectors belong to 3-dimensional Euclidean space equipped with orientation.

1 *Prove that vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *are linearly independent if and only if* $\mathbf{a} \times \mathbf{b} \neq 0$.

Solution Let vectors $\mathbf{a} \times \mathbf{b} \neq 0$. Prove that this implies that vectors are linearly independent. Suppose that these vectors are linearly dependent and come to contradiction. Indeed let $\mathbf{a}, \mathbf{b}$ are linearly dependent. Then they are collinear (proportional): $\mathbf{a} = \lambda \mathbf{b}$ or if $\mathbf{b} = 0$, then $\mathbf{b} = \lambda \mathbf{a}$. WLOG suppose that $\mathbf{a} = \lambda \mathbf{b}$. We have due to linearity and anticommutativity of vector product (see the axioms defining the vector rpoduct) that

$$\mathbf{a} \times \mathbf{b} = (\lambda \mathbf{b}) \times \mathbf{b} = \lambda(\mathbf{b} \times \mathbf{b}) = -\lambda(\mathbf{b} \times \mathbf{b}) = -(\lambda \mathbf{b}) \times \mathbf{b} = -\mathbf{a} \times \mathbf{b} \Rightarrow \mathbf{a} \times \mathbf{b} = 0\,.\text{Contradiction.}$$

Hence vectors $\mathbf{a}, \mathbf{b}$ are linearly independent.

Now prove that if vectors $\mathbf{a}, \mathbf{b}$ are linearly independent then $\mathbf{a} \times \mathbf{b} \neq 0$. These vectors are both not equal to zero, since they are linearly independent. Expand the vector $\mathbf{b}$ as a sum of two vectors $\mathbf{b} = \mathbf{b}_{||} + \mathbf{b}_{\perp}$, where vector $\mathbf{b}_{||}$ is collinear (proportional) to the vector $\mathbf{a}$ and vector $\mathbf{b}_{\perp}$ is orthogonal to the vector $\mathbf{a}$. To do this consider $\mathbf{b} = \lambda \mathbf{a} + (\mathbf{b} - \lambda \mathbf{a})$ and choose λ such that $\mathbf{b}_{||} = \lambda \mathbf{a}$, $\mathbf{b}_{\perp} = (\mathbf{b} - \lambda \mathbf{a})$ and $\mathbf{a}$ is orthogonal to $\mathbf{b}_{\perp}$ (($\mathbf{a}, \mathbf{b}_{\perp}$)=0). Taking scalar product with $\mathbf{a}$ we come to

$$(\mathbf{a}, \mathbf{b}) = (\mathbf{a}, \lambda \mathbf{a} + (\mathbf{b} - \lambda \mathbf{a})) = (\mathbf{a}, \lambda \mathbf{a}) = \lambda(\mathbf{a}, \mathbf{a}) \Rightarrow \lambda = \frac{(\mathbf{a}, \mathbf{b})}{(\mathbf{a}, \mathbf{a})}.$$

We come to decomposition of vector $\mathbf{b}$ on the vector collinear to the vector $\mathbf{a}$ and the vector orthogonal to the vector $\mathbf{a}$:

$$\mathbf{b} = \underbrace{\lambda \mathbf{a}}_{\mathbf{b}_{||}} + \underbrace{(\mathbf{b} - \lambda \mathbf{a})}_{\mathbf{b}_{\perp}} \quad \text{with } \lambda = \lambda = \frac{(\mathbf{a}, \mathbf{b})}{(\mathbf{a}, \mathbf{a})}$$

Note that $\mathbf{b}_{\perp} \neq 0$, since if $\mathbf{b}_{\perp} \neq 0 = \mathbf{b} - \lambda \mathbf{a} = 0$ then $\mathbf{b} = \lambda \mathbf{a}$, i.e. vectors $\mathbf{a}, \mathbf{b}$ are linearly dependent.

Now calculate vector product $\mathbf{a} \times \mathbf{b}$ using linearity axiom:

$$\mathbf{a} \times \mathbf{b} = \mathbf{a} \times (\mathbf{b}_{||} + \mathbf{b}_{\perp}) = \mathbf{a} \times \mathbf{b}_{||} + \mathbf{a} \times \mathbf{b}_{\perp} = 0 + \mathbf{a} \times \mathbf{b}_{\perp}\,.$$

Since vectors $\mathbf{a}, \mathbf{b}_{\perp}$ are orthogonal then using axiom about vector product of orthogonal vectors we see that $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a} \times \mathbf{b}_{\perp}| = |\mathbf{a}||\mathbf{b}_{\perp}| \neq 0$, since $\mathbf{a}\,\mathbf{b}_{\perp} \neq 0$. Hence $\mathbf{a} \times \mathbf{b} \neq 0$. ■

2 Vectors $\mathbf{a}$ and $\mathbf{b}$ are linear independent. Consider the vector $\mathbf{c} = \mathbf{a} \times \mathbf{b}$. Prove that the ordered set $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is a basis in $\mathbf{E}^3$.

Solution In the previous exercise we already proved that vector $\mathbf{c} = \mathbf{a} \times \mathbf{b} \neq 0$ if vectors $\mathbf{a}, \mathbf{b}$ are linear independent. To prove that the ordered set $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is a basis we need to prove that these three vectors are linearly independent. Suppose $\lambda \mathbf{a} + \mu \mathbf{b} + \nu \mathbf{c} = 0$. We need to prove that all coefficients equal to zero, thus we will prove linear independence. Take the scalar product of the relation $\lambda \mathbf{a} + \mu \mathbf{b} + \nu \mathbf{c} = 0$ on $\mathbf{c}$. Since according to the first axiom vector $\mathbf{c}$ is orthogonal to the vectors $\mathbf{a}, \mathbf{b}$ we come to

$$0 = (\mathbf{c}, 0) = (\mathbf{c}, \lambda \mathbf{a} + \mu \mathbf{b} + \nu \mathbf{c}) = \lambda(\mathbf{c}, \mathbf{a}) + \mu(\mathbf{c}, \mathbf{b}) + \nu(\mathbf{c}, \mathbf{c}) = \nu |\mathbf{c}|^2$$

Hence $\nu = 0$ since $\mathbf{c} \neq 0$. Hence $\lambda = \mu = 0$ also since $\mathbf{a}, \mathbf{b}$ are linear independent vectors. ■

Remark Note that the condition $\mathbf{c} = \mathbf{a} \times \mathbf{b} \neq 0$ immediately implies that vectors $\{\mathbf{a}, \mathbf{b}\,\mathbf{c}\}$ are linearly independent.

3 Students John and Sarah calculate vector product $\mathbf{a} \times \mathbf{b}$ of two vectors using two different orthonormal bases in the Euclidean space $\mathbf{E}^3$, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$. John expands the vectors with respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Sarah expands the vectors with respect to the basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$. For two arbitrary vectors $\mathbf{a}, \mathbf{b} \in \mathbf{E}^3$

$$\mathbf{a} = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3 = a'_1 \mathbf{e}'_1 + a'_2 \mathbf{e}'_2 + a'_3 \mathbf{e}'_3\,,$$

$$\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 = b'_1\mathbf{e}'_1 + b'_2\mathbf{e}'_2 + b'_3\mathbf{e}'_3 \,.$$

John and Sarah both use so called "determinant" formula. Are their answers the same?

$$\mathbf{a}\times\mathbf{b} = \underbrace{\det\begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}}_{\text{John's calculations}} \stackrel{?}{=} \underbrace{\det\begin{pmatrix} \mathbf{e}'_1 & \mathbf{e}'_2 & \mathbf{e}'_3 \\ a'_1 & a'_2 & a'_3 \\ b'_1 & b'_2 & b'_3 \end{pmatrix}}_{\text{Sarah's calculations}}$$

Solution: In the case if bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ have the same orientation, then answer will be the same. If bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, e'_3\}$ have opposite orientation then the answer of John will differ from the answer of Sarah by sign. Explain why.

Let third student, say David enters, the "game". David knows that formulae of John and Sarah both obey to axioms defining vector product (see the lecture notes). Without paying attention on formulae of John and Sarah he just uses the axioms defining vector product: He will consider the direction orthogonal to the plane spanned by vectors $\mathbf{a}, \mathbf{b}$ and take the vector which length equal to the area of parallelogram. One thing that David also have to do it is to choose the direction of this vector. It is here where the question of orientation of bases becomes crucial.

Suppose David uses an orthonoromal basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ defining the orientation, which have the same orientation as the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ which John uses.

According of the fifth axiom he chooses the direction of the vector $\mathbf{c}$ in a such way that bases $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ and $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ have the same orientation.

Now the answer is clear: if bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ (of John) and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ (of Sarah) have the same orientation then all three bases of David, John and Sarah will have the same orientation, hence all three answers will coincide: all bases $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ (calculation of vector product), $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ (David's basis) $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ (John's basis) and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ (Sarah's basis) have the same orinetation.

If bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ (of John) and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ (of Sarah) have opposite orientation then answer of David will coincide with answer of John and it will have the opposite sign with answer of Sarah:

Indeed in this case the bases $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ (David's basis) $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ (John's basis) will have the same orientation, hence the bases $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ (of Sarah) will have opposite orientation. Hence calculations of vector product in the basis which Sarah is using lead to the answer $-\mathbf{c}$: in this case the bases $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ (of Sarah) will have the same orientation.

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4 *Calculate the area of parallelograms formed by the vectors* $\mathbf{a}, \mathbf{b}$ *if*

a) $\mathbf{a} = (1, 2, 3), \mathbf{b} = (1, 0, 1)$*;*

b) $\mathbf{a} = (2, 2, 3), \mathbf{b} = (1, 1, 1)$*;*

c) $\mathbf{a} = (5, 8, 4), \mathbf{b} = (10, 16, 8)$.

d) $\mathbf{a} = (3, 4, 0), \mathbf{b} = (5, 17, 0)$.

Solution

Area of parallelogram formed by the vectors $\mathbf{a}, \mathbf{b}$ is equal to the length of the vector $\mathbf{c} = \mathbf{a}\times\mathbf{b}$.

$$\mathbf{c} = \mathbf{a}\times\mathbf{b} = (a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z)\times(b_x\mathbf{e}_x + b_y\mathbf{e}_y + b_z\mathbf{e}_z) = a_xb_x\mathbf{e}_x\times\mathbf{e}_x + a_xb_y\mathbf{e}_x\times\mathbf{e}_y + a_xb_z\mathbf{e}_x\times\mathbf{e}_z +$$

$$a_yb_x\mathbf{e}_y\times\mathbf{e}_x + a_yb_y\mathbf{e}_y\times\mathbf{e}_y + a_yb_z\mathbf{e}_y\times\mathbf{e}_z + a_zb_x\mathbf{e}_z\times\mathbf{e}_x + a_zb_y\mathbf{e}_z\times\mathbf{e}_y + a_zb_z\mathbf{e}_z\times\mathbf{e}_z =$$

$$(a_xb_y - a_yb_x)\mathbf{e}_z + (a_yb_z - a_zb_y)\mathbf{e}_x + (a_zb_x - a_xb_z)\mathbf{e}_y$$

$$|\mathbf{c}| = \sqrt{(a_yb_z - a_zb_y)^2 + (a_zb_x - a_xb_z)^2 + (a_xb_y - a_yb_x)^2}$$

a) $S = |\mathbf{a}\times\mathbf{b}| = |-2\mathbf{e}_z + 2\mathbf{e}_x + 2\mathbf{e}_y|$, $S = \sqrt{4+4+4} = 2\sqrt{3}$.

b) $S = |\mathbf{a}\times\mathbf{b}|$. $\mathbf{a}\times\mathbf{b} = -\mathbf{e}_x + \mathbf{e}_y$, $S = \sqrt{1+1} = \sqrt{2}$

* In the case if one of vectors equal to zero and vectors do not span plane then on can see that all three students John, Sarah and David will come to the answer: zero.

c) Vectors $\mathbf{a} = (5, 8, 4), \mathbf{b} = (10, 16, 8)$ are collinear, hence $\mathbf{a} \times \mathbf{b} = 0$, $S = 0$.
d) $S = |\mathbf{a} \times \mathbf{b}|$. $\mathbf{a} \times \mathbf{b} = 31\mathbf{e}_z$, $S = 31$

5 *Let $\mathbf{a}, \mathbf{b}$ be two vectors in the 2-dimensional Euclidean space $\mathbf{E}^2$. Calculate the area of the parallelogram formed by these vectors if*
a) $\mathbf{a} = (2, 3)$, $\mathbf{b} = (5, 9)$
b) $\mathbf{a} = (17, 12)$, $\mathbf{b} = (7, 5)$
c) $\mathbf{a} = (41, 29)$, $\mathbf{b} = (99, 70)$

Solution a) $S(\mathbf{a}, \mathbf{b}) = \left|\det \begin{pmatrix} 2 & 3 \\ 5 & 9 \end{pmatrix}\right| = |18 - 15| = 3.$
b) $S(\mathbf{a}, \mathbf{b}) = \left|\det \begin{pmatrix} 17 & 12 \\ 7 & 5 \end{pmatrix}\right| = |85 - 84| = 1.$
c) $S(\mathbf{a}, \mathbf{b}) = \left|\det \begin{pmatrix} 41 & 29 \\ 99 & 70 \end{pmatrix}\right| = |2870 - 2871| = 1.$

6[†] *Do you see any relation between fractions $\frac{3}{2}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}$ (see the exercises 5b) and 5c) above) and the number... $\sqrt{2}$? Can you continue the sequence of these fractions? (Hint: Consider the squares of these fractions.)*

One can consider continuous fraction $\sqrt{2} = 1 + \frac{1}{2+\frac{1}{2+\frac{1}{2+\dots}}}$, Consider approximations: $a_0 = 1$, $a_1 = 1 + \frac{1}{2} = \frac{3}{2}$, $a_2 = 1 + \frac{1}{2+\frac{1}{2}} = \frac{7}{5}$, and so on we come to the sequence of fractions:

$$a_k = \frac{p_k}{q_k} \text{ where } p_0 = q_0 = 1, q_{k+1} = p_k + q_k, p_k = 2q_k + p_k\,.$$

One can see that $\left|\frac{p_k}{q_k} - \frac{p_{k+1}}{q_{k+1}}\right| = \frac{1}{q_k q_{k+1}}$ which is just another manifestation of the fact that the area of the parallelogram formed by the vectors $\mathbf{a} = (p_k, q_k), \mathbf{b} = p_{k+1}, q_{k+1}$equals to 1. Vectors $\mathbf{a} = (p_k, q_k), \mathbf{b} = p_{k+1}, q_{k+1}$ form the parallelograms which become longer and longer but all have the same area.
7 *Show that for any two vectors $\mathbf{a}, \mathbf{b} \in \mathbf{E}^3$ the following identity is satisfied*

$$(\mathbf{a}, \mathbf{a})(\mathbf{b}, \mathbf{b}) = (\mathbf{a}, \mathbf{b})^2 + (\mathbf{a} \times \mathbf{b}, \mathbf{a} \times \mathbf{b})\,.$$

Write down this identity in components.
Compare this identity with CBS inequality . See the problem 7 in the Homework 2.

Solution
Let θ be an angle between vectors $\mathbf{a}, \mathbf{b}$. Then

$$(\mathbf{a}, \mathbf{a})(\mathbf{b}, \mathbf{b}) = |\mathbf{a}|^2|\mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2(\cos^2\theta + \sin^2\theta) = \underbrace{|\mathbf{a}|^2|\mathbf{b}|^2\cos^2\theta}_{\text{scalar product}} + \underbrace{|\mathbf{a}|^2|\mathbf{b}|^2\sin^2\theta}_{\text{Area of parallelogram = vector product}}$$

$$= (\mathbf{a}, \mathbf{b})^2 + |\mathbf{a} \times \mathbf{b}|^2 \qquad (10.1)$$

In components:

$$(a_x^2 + a_y^2 + a_z^2)(b_x^2 + b_y^2 + b_z^2) = (a_x b_x + a_y b_y + a_z b_z)^2 + (a_x b_y - a_y b_x)^2 + (a_y b_z - a_z b_y)^2 + (a_z b_x - a_x b_z)^2 \quad (10.2)$$

Notice that for $n = 2, 3$ this identity is more strong statement than CBS inequality: $(a_x^2 + a_y^2 + a_z^2)(b_x^2 + b_y^2 + b_z^2) \geq (a_x b_x + a_y b_y + a_z b_z)^2$. CBS inequality $(\mathbf{a}, \mathbf{a})(\mathbf{b}, \mathbf{b}) \geq |\mathbf{a}|^2|\mathbf{b}|^2$ follows from the identity (1.10).
The proof of the identity (10.2) becomes more complicated if we use only algebraical methods.

8 *Find a vector* $\mathbf{n}$ *such that the following conditions hold:*
1) It has a unit length
2) It is orthogonal to the vectors $\mathbf{a} = (1, 2, 3)$ *and* $\mathbf{b} = (1, 3, 2)$.

3) An ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{n}\}$ has an orientation opposite to the orientation of the orthonormal basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ which defines the orientation of the Euclidean space.

Solution: Consider a vector $\mathbf{N} = \mathbf{a} \times \mathbf{b}$ and a vector $\frac{\mathbf{N}}{|\mathbf{N}|}$. The vector $\mathbf{N}$ is orthogonal to vectors $\mathbf{a}, \mathbf{b}$ (vector product) and a vector $\frac{\mathbf{N}}{|\mathbf{N}|}$ is a unit vector. It remains to solve the problem of orientation. Both vectors $\pm\frac{\mathbf{N}}{|\mathbf{N}|}$ are unit vectors which are orthogonal to vectors $\mathbf{a}, \mathbf{b}$. On the other hand the ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{N}\}$ is a basis and this basis has the same orientation as a basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. This follows from the axioms defining the vector product and the fact that vectors $\mathbf{N} \neq 0$, i.e. the ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{N}\}$ is a basis. Hence the ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{n}\}$ where $\mathbf{n} = -\frac{\mathbf{N}}{|\mathbf{N}|}$ has an orientation opposite to the orientation of the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$.

The vector

$$\mathbf{n} = -\frac{\mathbf{N}}{|\mathbf{N}|} = -\frac{(\mathbf{e}_x + 2\mathbf{e}_y + 3\mathbf{e}_z) \times (\mathbf{e}_x + 3\mathbf{e}_y + 2\mathbf{e}_z)}{|\mathbf{N}|} = \frac{5\mathbf{e}_x - \mathbf{e}_y - \mathbf{e}_z}{3\sqrt{3}}\,.$$

9 *Volume of parallelepiped $V(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{a}, \mathbf{b} \times \mathbf{c})$, formed by the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ equals to zero if and only if these vectors are linearly dependent. Prove it.*

Solution: Let vectors $\{\mathbf{a}, \mathbf{b}\, c\}$ are linearly dependent, i.e. $\lambda\mathbf{a} + \mu\mathbf{b} + \tau\mathbf{c} = 0$. If $\lambda \neq 0$, then $\mathbf{a}$ belongs to the span of vectors $\mathbf{c}, \mathbf{b}$: $\mathbf{a} = \lambda'\mathbf{b} + \mu'\mathbf{c}$ and

$$V(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{a}, \mathbf{b} \times \mathbf{c}) = (\lambda'\mathbf{b} + \mu'\mathbf{c}, \mathbf{b} \times \mathbf{c}) = \lambda'(\mathbf{b}, \mathbf{b} \times \mathbf{c}) + \mu'(\mathbf{c}, \mathbf{b} \times \mathbf{c}) = 0\,.$$

Scalar product $(\mathbf{b}, \mathbf{b} \times \mathbf{c})$ equals to zero since $\mathbf{b} \perp \mathbf{b} \times \mathbf{c}$. Analogously scalar product $(\mathbf{c}, \mathbf{b} \times \mathbf{c})$ equals to zero.

If $\lambda = 0$ in the relation $\lambda\mathbf{a} + \mu\mathbf{b} + \tau\mathbf{c} = 0$, then linear dependence of vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ means that vectors $\mathbf{c}$ and $\mathbf{b}$ are proportional to each other. Hence $\mathbf{b} \times \mathbf{c} = 0$ and $V(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{a}, \mathbf{b} \times \mathbf{c}) = (\mathbf{a}, 0) = 0$.

We proved that linear dependence of vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ implies that $V(\mathbf{a}, \mathbf{b}, \mathbf{c}) = 0$.

Now prove the converse implication.

Let $V(\mathbf{a}, \mathbf{b}, \mathbf{c}) = 0$. Assume that vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are linear independent and come to contradiction. In particularly this means that vectors $\mathbf{b}$ and $\mathbf{c}$ are linear independent. They span the plane. The vector $\mathbf{a}$ belongs to this plane since it is orthogonal to vectors $\mathbf{a}, \mathbf{b}$. Hence vector $\mathbf{a}$ belongs to the span of the vectors $\mathbf{b}$ and $\mathbf{c}$. Contradiction.

10 *Vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal unit vectors. Calculate the length of the vector $\mathbf{c} \times (\mathbf{a} \times \mathbf{b})$, where $\mathbf{c} = \mathbf{a} + \mathbf{b}$.*

Vector $\mathbf{a} \times \mathbf{b}$ is unit vector. Vector $\mathbf{c} = \mathbf{a} = \mathbf{b}$ has the length $\sqrt{2}$ and is orthogonal to the vector $\mathbf{a} \times \mathbf{b}$.Hence $|\mathbf{c} \times (\mathbf{a} \times \mathbf{b})| = \sqrt{2} \cdot 1 = \sqrt{2}$.

11 *Show that in general $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$. (Associativity law is not obeyed)*

Consider for example unit vectors $\mathbf{a}, \mathbf{b}$ which are orthogonal to each other and $\mathbf{c} = \mathbf{b}$. Then $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \times (\mathbf{b} \times \mathbf{b}) = 0$ and $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{b}) \times \mathbf{b} \neq 0$ since its length equals to 1.

12† *Show that $\mathbf{a} \times \mathbf{b} \times \mathbf{c} = \mathbf{b}(\mathbf{a}, \mathbf{c}) - \mathbf{c}(\mathbf{a}, \mathbf{b})$ and*

$$\mathbf{a} \times \mathbf{b} \times \mathbf{c} + \mathbf{b} \times \mathbf{c} \times \mathbf{a} + \mathbf{c} \times \mathbf{a} \times \mathbf{b} = 0 \quad \textit{(Jacobi identity)}\,.$$

Solution: e.g. straightforward calculations. They could be little bit simplified if you choose an adjusted basis such that $\mathbf{e}_x$ is proportional to one of the vectors.

The oriented vector space $\mathbf{E}^3$ equipped with vector product becomes an algebra where multiplication $\times$ is anticommutative and not-associative but obeying the Jacobi identity. These algebras are called Lie algebras. The Lie algebra of vectors with cross product is Lie algebra of the Lie group $SO(3)$. It is may be the simplest example of non-trivial Lie algebra[**].

[**] See also in my home-page an etude "Jacobi Identity and intersection of heights of triangle" in subdirectory Etudes/Teaching/Geometry .