Solutions 8

1. *Find unit normal vector field, the Weingarten (shape) operator, principal curvatures and the Gaussian curvature for the cylinder $x^2+y^2=a^2$.*

(See also solution in lecture notes.)

We have for cylinder

$$\mathbf{r}(h,\varphi)\qquad \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases}\qquad (0\leq\varphi<2\pi, -\infty<h<\infty)$$

$$\mathbf{r}_\varphi\big|_{\varphi,h} = \frac{\partial\mathbf{r}(\varphi,h)}{\partial\varphi} = \begin{pmatrix} -a\sin\varphi \\ a\cos\varphi \\ 0 \end{pmatrix},\ \mathbf{r}_h\big|_{\varphi,h} = \frac{\partial\mathbf{r}(\varphi,h)}{\partial h} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ \mathbf{n}(\varphi,h) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} \tag{2}$$

Sometimes we denote $\mathbf{r}_\varphi$ by ∂_φ and $\mathbf{r}_h$ by ∂_h.

Check that $\mathbf{n}(\varphi,h)$ is indeed unit normal vector:

$$(\mathbf{n},\mathbf{n}) = \cos^2\varphi+\sin^2\varphi = 1,\ \ (\mathbf{n},\mathbf{r}_\varphi) = a\cos\varphi\sin\varphi(-1+1) = 0,\ \ (\mathbf{n},\mathbf{r}_h) = 0$$

Unit normal vector is defined up to a sign; $-\mathbf{n}$ is unit normal vector too.

Now calculate shape operator Gaussian and mean curvatures for cylinder.

To calculate the shape operator for the cylinder we use results of calculations above of vectors $\mathbf{r}_h, \mathbf{r}_\varphi$ and of unit normal vector $\mathbf{n}(\varphi,h)$ (see the equations (2) above). By the definition the action of shape operator on any tangent vector $\mathbf{v}$ is given by the formula $S\mathbf{v} = -\partial_{\mathbf{v}}\mathbf{n}$. Hence for basis vectors $\mathbf{r}_\varphi = \partial_\varphi, \mathbf{r}_h = \partial_h$ we have

$$S\mathbf{r}_h = -\partial_h\mathbf{n}(\varphi,h) = -\partial_h \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} = 0$$

and

$$S\mathbf{r}_\varphi = -\partial_\varphi\mathbf{n}(\varphi,h) = -\partial_\varphi \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} \sin\varphi \\ -\cos\varphi \\ 0 \end{pmatrix} = -\frac{\mathbf{r}_\varphi}{a}$$

(Recall that $\mathbf{n}(h,\varphi) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$ and $\mathbf{r}_\varphi = \begin{pmatrix} -a\sin\varphi \\ a\cos\varphi \\ 0 \end{pmatrix}$ (See the equations (2) above.)

For an arbitrary tangent vector $\mathbf{X} = a\mathbf{r}_h + b\mathbf{r}_\varphi$, $S\mathbf{X} = -\frac{b\mathbf{r}_\varphi}{a}$. Shape operator transforms tangent vectors to tangent vectors. Its matrix in the basis $\mathbf{r}_h, \mathbf{r}_\varphi$ equals to

$$-\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{a} \end{pmatrix}$$

In the case if we choose the opposite direction for unit normal vector then we will come to the same answer just with changing the signs: if $\mathbf{n} \to -\mathbf{n}$, $S \to -S$.

We see that principal curvatures, i.e. eigenvalues of shape operator are:

$$\kappa_1 = 0, \kappa_2 = -\frac{1}{a}$$

(if we choose the opposite sign for $\mathbf{n}$ then $\kappa_1 = \kappa_2 = \frac{1}{a}$). Thus we can calculate Gaussian and mean curvature: Gaussian curvature

$$K = \kappa_1 \cdot \kappa_2 = \det S = 0\,.$$

Remark The vanishing of Gaussian curvature follows from the fact that $S(\mathbf{r}_h) = 0$. This relation means that operator is degenerate (its matrix has zero column). Hence $K = \det S = 0$.

2. *Find unit normal vector field, the Weingarten (shape) operator, principal curvatures and the Gaussian curvature for the sphere of the radius R: $x^2 + y^2 + z^2 = R^2$.*

`For sphere there is an elegant and short solution which works in arbitrary parameterisation (see subsection 4.1.4 of lecture notes) Here we give straight-forward solution, not the shortest one, however...`

(See also lecture notes)

SPHERE of radius R:

$$\mathbf{r}(\theta, \varphi) \qquad \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases}$$

$$(0 \le \varphi < 2\pi,\ 0 \le \theta \le \pi),$$

$$\mathbf{r}_\theta = \frac{\partial \mathbf{r}(\varphi, \theta)}{\partial \theta} = \begin{pmatrix} R\cos\theta\cos\varphi \\ R\cos\theta\sin\varphi \\ -R\sin\theta \end{pmatrix},\ \mathbf{r}_\varphi = \frac{\partial \mathbf{r}(\varphi, \theta)}{\partial \varphi} = \begin{pmatrix} -R\sin\theta\sin\varphi \\ R\sin\theta\cos\varphi \\ 0 \end{pmatrix}$$

$$\mathbf{n}(\theta, \varphi) = \frac{\mathbf{r}(\theta, \varphi)}{R} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} \tag{1}$$

(Sometimes we denote $\mathbf{r}_\theta$ by ∂_θ and $\mathbf{r}_\varphi$ by ∂_φ.)

Check that $\mathbf{n}(\theta, \varphi)$ is indeed unit normal vector (in fact this is obvious from geometric considerations):

$$(\mathbf{n}, \mathbf{n}) = \sin^2\theta(\cos^2\varphi + \sin^2\varphi) + \cos^2\theta = 1\,,$$

$(\mathbf{n}, \mathbf{r}_\theta) = R\sin\theta\cos\theta(\cos^2\varphi + \sin^2\varphi) - R\sin\theta\cos\theta = 0, \ (\mathbf{n}, \mathbf{r}_\varphi) = R\sin^2\theta(-\cos\varphi\sin\varphi + \cos\varphi\sin\varphi) =$

One can check it in a more elegant way: The equation of sphere $(\mathbf{r}, \mathbf{r}) = 1$. Differentiating this relation by θ and by φ we will come to condition that vectors $\mathbf{r}_\theta$ and $\mathbf{r}_\varphi$ are orthogonal to radius-vector $\mathbf{r}$:

$$\frac{\partial}{\partial\theta}(\mathbf{r}, \mathbf{r}) = 0 = 2(\mathbf{r}_\theta, \mathbf{r}) \quad \frac{\partial}{\partial\varphi}(\mathbf{r}, \mathbf{r}) = 0 = 2(\mathbf{r}_\varphi, \mathbf{r})\,.$$

Vector $\mathbf{r}$ has a length R. Hence $\mathbf{n} = \pm\frac{\mathbf{r}}{R}$. (See also these calculations in lecture notes). Unit normal vector is defined up to a sign; $-\mathbf{n}$ is unit normal vector too.

Now calculate shape operator and Gaussian and mean curvatures for sphere:

By the definition (see lecture notes) the action of shape operator on any tangent vector $\mathbf{v}$ is given by the formula $S\mathbf{v} = -\partial_{\mathbf{v}}\mathbf{n}$. We know that for sphere $\mathbf{n} = \frac{\mathbf{r}}{R}$ (see the equations (1) above). Hence for basis vectors $\mathbf{r}_\theta = \partial_\theta, \mathbf{r}_\varphi = \partial_\varphi$ we have

$$S\mathbf{r}_\theta = -\partial_\theta\mathbf{n}(\theta,\varphi) = -\partial_\theta\left(\frac{\mathbf{r}(\theta,\varphi)}{R}\right) = -\left(\frac{\partial_\theta\mathbf{r}(\theta,\varphi)}{R}\right) = -\frac{\mathbf{r}_\theta}{R}$$

and

$$S\mathbf{r}_\varphi = -\partial_\varphi\mathbf{n}(\theta,\varphi) = -\partial_\varphi\left(\frac{\mathbf{r}(\theta,\varphi)}{R}\right) = -\left(\frac{\partial_\varphi\mathbf{r}(\theta,\varphi)}{R}\right) = -\frac{\mathbf{r}_\varphi}{R}$$

We see that shape operator is equal to $S = -\frac{I}{R}$, where I is an identity operator. Its matrix in the basis $\partial_\theta, \partial_\varphi$ is equal to

$$-\begin{pmatrix} \frac{1}{R} & 0 \\ 0 & \frac{1}{R} \end{pmatrix}.$$

In the case if we choose the opposite direction for unit normal vector then we will come to the same answer just with changing the signs: if $\mathbf{n} \to -\mathbf{n}$, $S \to -S$.

We see that principal curvatures, i.e. eigenvalues of shape operator are the same:

$$\lambda_1 = \lambda_2 = -\frac{1}{R}, \text{i.e. } \kappa_1 = \kappa_2 = -\frac{1}{R}$$

(if we choose the opposite sign for $\mathbf{n}$ then $\kappa_1 = \kappa_2 = \frac{1}{R}$). Thus we can calculate Gaussian and mean curvature: Gaussian curvature

$$K = \kappa_1 \cdot \kappa_2 = \det S = \frac{1}{R^2}\,.$$

3 *Find the Weingarten (shape) operator and the Gaussian curvature for the saddle* $z = xy$ *at the point* $x = y = 0$.

Do calculations for the GRAPH OF THE FUNCTION $z = Ax^2 + 2Bxy + Cy^2$:

$$\mathbf{r}(u,v) \qquad \begin{cases} x = u \\ y = v \\ z = F(u,v) \end{cases} \qquad (-\infty < u < \infty, -\infty < v < \infty) \tag{4}$$

in the case if $F(u,v) = Au^2 + 2Buv + Cv^2$

$$\mathbf{r}_u\big|_{u,v} = \frac{\partial \mathbf{r}(u,v)}{\partial u} = \begin{pmatrix} 1 \\ 0 \\ F_u \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2Au + 2Bv \end{pmatrix}, \ \mathbf{r}_u\big|_{u=v=0} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbf{r}_v\big|_{u,v} = \frac{\partial \mathbf{r}(u,v)}{\partial v} = \begin{pmatrix} 0 \\ 1 \\ F_v \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2Bu + 2Cv \end{pmatrix}, \ \mathbf{r}_v\big|_{u=v=0} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

$$\mathbf{n}(u,v) = \frac{1}{\sqrt{1 + F_u^2 + F_v^2}} \begin{pmatrix} -F_u \\ -F_v \\ 1 \end{pmatrix}, \ \mathbf{n}(u,v)\big|_{u=v=0} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Sometimes we denote $\mathbf{r}_u$ by ∂_u and $\mathbf{r}_v$ by ∂_v.

Check that $\mathbf{n}(u,v)$ is indeed unit normal vector: $(\mathbf{n},\mathbf{n}) = \frac{1}{1+F_u^2+F_v^2}(F_u^2 + F_v^2 + 1) = 1$, $(\mathbf{n},\mathbf{r}_u) = \frac{1}{\sqrt{1+F_u^2+F_v^2}}(F_u - F_u) = 0$, $(\mathbf{n},\mathbf{r}_v) = \frac{1}{\sqrt{1+F_u^2+F_v^2}}(F_v - F_v) = 0$. To calculate shape operator we use results of calculations vectors $\mathbf{r}_u, \mathbf{r}_v$ and for unit normal vector $\mathbf{n}(u,v)$. We do calculations only at origin. For basic vectors $\mathbf{r}_u = \partial_u, \mathbf{r}_v = \partial_v$ we have

$$S\mathbf{r}_u = -\frac{\partial \mathbf{n}(u,v)}{\partial u}_{|u=v=0} = -\partial_u \left(\frac{1}{\sqrt{1 + F_u^2 + F_v^2}} \begin{pmatrix} -F_u \\ -F_v \\ 1 \end{pmatrix} \right)_{|u=v=0} =$$

$$\left(\frac{1}{\sqrt{1 + F_u^2 + F_v^2}} \right)_{|u=v=0} \begin{pmatrix} F_{uu} \\ F_{uv} \\ 1 \end{pmatrix}_{|u=v=0} = \begin{pmatrix} 2A \\ 2B \\ 0 \end{pmatrix} = 2A\mathbf{r}_u + 2B\mathbf{r}_v$$

and $S\mathbf{r}_v = -\partial_v \left(\mathbf{n}(u,v)\right)_{|u=v=0} =$

$$-\partial_v \left(\frac{1}{\sqrt{1 + F_u^2 + F_v^2}} \begin{pmatrix} -F_u \\ -F_v \\ 1 \end{pmatrix} \right)_{|u=v=0} = \left(\frac{1}{\sqrt{1 + F_u^2 + F_v^2}} \right)_{|u=v=0} \begin{pmatrix} F_{vu} \\ F_{vv} \\ 1 \end{pmatrix}_{|u=v=0} = \begin{pmatrix} 2B \\ 2C \\ 0 \end{pmatrix} = 2B\mathbf{r}_u$$

The matrix of the shape operator in the basis $\mathbf{r}_u, \mathbf{r}_v$ is $\begin{pmatrix} 2A & 2B \\ 2B & 2C \end{pmatrix}$. Gaussian curvature at origin is equal to $\det S = 4AC - 4B^2$

For the saddle we have to put $A = C = 0$ and $B = \frac{1}{2}$. In particular for saddle $K = -1$.

4 *Let D be a domain on a sphere of radius R such that its area is equal to one eighth of the area of the sphere. Calculate the integral of Gaussian curvature of the sphere over the domain D.*

What is a result of parallel transport of the vector, tangent to the sphere, along the curve $C = \partial D$?

The Gaussian curvature $K = \frac{1}{R^2}$, it is constant. Hence

$$\int_D K d\sigma = K \cdot \int_D d\sigma = K \cdot \text{area of the domain } D = \frac{1}{8} K \cdot \text{area of the domain sphere} = \frac{1}{8} \cdot \frac{1}{R^2} \cdot 4\pi R^2 = \frac{\pi}{2}$$

Thus the result of parallel transport if the vector along the curve is the rotation on the angle

$$\angle \Phi = \int_D K d\sigma = \frac{\pi}{2},$$

i.e. vector will rotate to the orthogonal vector.

5 *Assume that the action of the shape operator at the tangent coordinate vectors $\mathbf{r}_u = \partial_u$, $\mathbf{r}_v = \partial_v$ at the given point $\mathbf{p}$ of the surface $\mathbf{r} = \mathbf{r}(u, v)$ is defined by the relations: $S(\partial_u) = 2\partial_u + 2\partial_v$ and $S(\partial_v) = -\partial_u + 5\partial_v$. Calculate principal curvatures and Gaussian of the surface at this point.*

We see that the matrix of the shape operator in the basis ∂_u, ∂_v is equal to

$$S = \begin{pmatrix} 2 & -1 \\ 2 & 5 \end{pmatrix}$$

Hence Gaussian curvature $K = \det S = 12$ and mean curvature $H = \operatorname{Tr} S = 7$. To calculate principal curvatures k_1, k_2 note that

$$\begin{cases} k_1 + k_2 = H = 7 \\ k_1 \cdot k_2 = K = 12 \end{cases}$$

Hence $k_1 = 3, k_2 = 4$; k_1, k_2 are eigenvalues of the shape operator.

`Question` Shape operator has to be the symmetrical operator. Does this condition obey?

6 *Consider a surface M, the upper sheet of the cone in $\mathbf{E}^3$*

$$\mathbf{r}(h, \varphi)\colon \quad \begin{cases} x = 3h\cos\varphi \\ y = 3h\sin\varphi \\ z = 4h \end{cases}, \quad h > 0\,,\ 0 \le \varphi < 2\pi\,.$$

Calculate Weingarten operator at points of this cone and show that Gaussian curvature vanishes at all the points of this surface.

Let C_1 be a closed curve on this surface which is the boundary of a compact oriented domain $D \subset M$.

Let C_2 be a circle which is the intersection of the plane $z = h_0$ ($h_0 > 0$) with the surface M.

Show that the parallel transport along the closed curve C_1 is the identical transformation.

Show that the parallel transport along the closed curve C_2 is the rotation through a non-zero angle.

Calculate this angle and explain why the fact that the angle is not equal to zero does not contadict to the Theorem on parallel transport.

First calculate explicitly Weingarten operator and show that in all the points of the surface M, the Gaussian curvature vanishes. We have for coordinate tangent vectors:

$$\mathbf{r}_h = \frac{\partial \mathbf{r}}{\partial h} = \begin{pmatrix} 3\cos\varphi \\ 3\sin\varphi \\ 4 \end{pmatrix}, \quad \mathbf{r}_\varphi = \frac{\partial \mathbf{r}}{\partial \varphi} = \begin{pmatrix} -3h\sin\varphi \\ 3h\cos\varphi \\ 0 \end{pmatrix},$$

One can see that vector $\mathbf{N} = \begin{pmatrix} 4\cos\varphi \\ 4\sin\varphi \\ -3 \end{pmatrix}$ is orthogonal two vectors $\mathbf{r}_h$ and $\mathbf{r}_\varphi$, and its length is equal to $\sqrt{4^2+3^2} = 5$. Hence the vector field

$$\mathbf{n}(h,\varphi) = \frac{1}{5}\begin{pmatrix} 4\cos\varphi \\ 4\sin\varphi \\ -3 \end{pmatrix}$$

is unit normal vector field for the conical surface M.

Now consider action of shape operator on tangent vectors. We see that

$$S(\mathbf{r}_h) = -\partial_{\mathbf{r}_h}\mathbf{n}(h,\varphi) = -\frac{\partial \mathbf{n}(h,\varphi}{\partial h} = 0\,,$$

It is enough: we do not need to calculate the action of shape oeprator on the second basic vector. It follows from this equation that shape operator is degenerate, and $\det S = 0$. Hence the Gaussian curvature of the conical surface M vanishes.

The fact that the Gaussian curvature of the conical surface vanishes follows also from the Theorema Egregium

The point that answer differens fro the curves C_1 and C_2 is that the vertex of conical surface (which does not belong to upper-sheet) is a singular point. The curve C_2 is nto the boundary of the compact domain.

Up-rolling the conical surface we see that the angle of rotation is equal to

$$\Theta = \frac{2\pi\cdot 3h}{\sqrt{16h^2+9h^2}} = \frac{6\pi h}{5h} = \frac{6}{5}\pi\,.$$