One example of integral

We consider

$$\int_{-\infty}^{\infty} \exp\left[-\frac{x^4}{4}+\frac{x}{\nu}\right] dx\,.$$

(this integral is from witten paper....)

We have for the exponent:

$$F(x)=-\frac{x^4}{4}+\frac{x}{\nu}\,,\ F'(x)=-x^3+\frac{1}{\nu}\,,\ F^{''}(x)=-3x^2\,,\,,F^{'''}(x)=-6x\,,\,,F^{''''}(x)=-6\,,\,,$$

the stationary point $x_0=\nu^{-1/3}$ and

$$F(x)=\frac{3}{4}\nu^{-4/3}-\frac{3}{2}\nu^{-2/3}\left(x-\nu^{-1/3}\right)^2-\nu^{-1/3}\left(x-\nu^{-1/3}\right)^3-\frac{1}{4}\left(x-\nu^{-1/3}\right)^4\,.$$

Thus we see that

$$\int_{-\infty}^{\infty} \exp\left[-\frac{x^4}{4}+\frac{x}{\nu}\right] dx=\int_{-\infty}^{\infty} \exp\left[F(x)\right] dx=$$

$$\exp\left[\frac{3}{4}\nu^{-4/3}\right]\int_{-\infty}^{\infty} \exp\left[-\frac{3}{2}\nu^{-2/3}\left(x-\nu^{-1/3}\right)^2\right] f_\nu(x)dx\,,$$

where

$$f_\nu(x)=\exp\left[-\nu^{-1/3}\left(x-\nu^{-1/3}\right)^3-\frac{1}{4}\left(x-\nu^{-1/3}\right)^4\right]\,.$$

Doing translation we will come to

$$\int_{-\infty}^{\infty} \exp\left[-\frac{x^4}{4}+\frac{x}{\nu}\right] dx=\exp\left[\frac{3}{4}\nu^{-4/3}\right]\int_{-\infty}^{\infty} \exp\left[-\frac{3}{2}\nu^{-2/3}x^2\right]\exp\left[-\nu^{-1/3}x^3-\frac{1}{4}x^4\right] dx$$

On the other hand we know that

$$\int \exp\left[-\frac{x^2}{h}\right] f(x)dx=C_h\exp\left[h\frac{d^2}{dx^2}\right] f(x)\big|_{x=0}$$

prove this formula and calculate C_h. Consider

$$I_n(h)=\int_{-\infty}^{\infty} \exp\left[-\frac{x^2}{h}\right] x^{2n}dx\,.$$

(This integral vanishes if $2n\to 2n+1$.) We have

$$I_n(h)=\int_{-\infty}^{\infty} \exp\left[-\frac{x^2}{h}\right] x^{2n}dx=(x^2=hz)=2\int_{0}^{\infty} \exp\left[-z\right] h^n z^n d\left(\sqrt{h}d\sqrt{z}\right)=$$

$$h^{n+\frac{1}{2}}\int_{0}^{\infty} \exp\left[-z\right] z^{n-\frac{1}{2}}dz=h^{n+\frac{1}{2}}\Gamma\left(n+\frac{1}{2}\right)=\sqrt{h}\frac{\left(\frac{h}{4}\frac{d^2}{dx^2}\right)^n}{n!}x^{2n}=\sqrt{h}\exp\left[\frac{h}{4}\frac{d^2}{dx^2}\right]x^{2n}\big|_{x=0}$$