Homework 10

1 On the sphere $x^2+y^2+z^2=R^2$ in $\mathbf{E}^3$ consider a circle C which is the intersection of the sphere with the plane $z=R-h$, $0<h<R$

Let $\mathbf{X}$ be an arbitrary vector tangent to the sphere at a point of C.

Find the angle between $\mathbf{X}$ and the result of parallel transport of $\mathbf{X}$ along C.

2 On two-dimensional Riemannian manifold with coordinates x^1, x^2 consider the vector fields $\mathbf{A}=\frac{\partial}{\partial x^1}$, $\mathbf{B}=\frac{\partial}{\partial x^2}$, $\mathbf{X}=(1+x^1x^2)\frac{\partial}{\partial x^2}$, and the vector field $\mathbf{Y}=(\nabla_{\mathbf{A}}\nabla_{\mathbf{B}}-\nabla_{\mathbf{B}}\nabla_{\mathbf{A}})\,\mathbf{X}$, where ∇ is a connection. Calculate the value of the field $\mathbf{Y}$ at the point $x^1=x^2=0$ if the curvature tensor of the connection ∇ is such that $R^1{}_{212}=1$ and $R^2{}_{212}=0$ at this point.

3 Write down components of curvature tensor in terms of Christoffel symbols.

4 *For every of the statements below prove it or show that it is wrong considering counterexample.*

a) If there exist coordinates u, v such that Riemannian metric G at the given point $\mathbf{p}$ is equal to $G=du^2+dv^2$ in these coordinates, then curvature of Levi-Civita connection at the point $\mathbf{p}$ vanishes.

b*) If all first derivatives of components of Riemannian metric in coordinates u, v vanish at the given point with coordinates (u_0, v_0):

$$\frac{\partial g_{ik}(u,v)}{\partial u}\Big|_{u=u_0,v=v_0}=\frac{\partial g_{ik}(u,v)}{\partial v}\Big|_{u=u_0,v=v_0}=0\,,$$

then curvature of Levi-Civita connection also vanishes at this point.

c) If all first and second derivatives of components of Riemannian metric

$$\frac{\partial g_{ik}(u,v)}{\partial u}\,,\ \frac{\partial g_{ik}(u,v)}{\partial v}\,,\ \frac{\partial^2 g_{ik}(u,v)}{\partial u^2}\,,\ \frac{\partial^2 g_{ik}(u,v)}{\partial u\partial v}\,,\ \frac{\partial^2 g_{ik}(u,v)}{\partial v^2}\,,$$

vanish at the given point then curvature of Levi-Civita connection also vanishes at this point.

5 State the relation between the Riemann curvature tensor of the Levi-Civita connection of a surface in $\mathbf{E}^3$ and its Gaussian curvature K. Deduce *the Theorema Egregium* from this statement.

Let M be a surface $\mathbf{r}=\mathbf{r}(u,v)$ in $\mathbf{E}^3$, such that at the given point $\mathbf{p}$ Gaussian curvature $K=1$, and the induced Riemannian metric is equal to $G=du^2+dv^2$ at this point.

Calculate all components of the Riemannian curvature tensor R_{ikmn} in coordinates u, v at the point $\mathbf{p}$.

Show that induced Riemannian metric cannot be equal identically to du^2+dv^2 in a vicinity of the point $\mathbf{p}$.

6 Using relation between Gaussian curvature and Riemann curvature tensor for Levi-Civita connection, write down all components $\{R_{ikmn}\}$ of Riemann curvature tensor for sphere of radius ρ in spherical coordinates.