For real case I know it from childhood, but for the complex case it is little bit fun: Today I did the proof which works for both cases.

CBH claims that

$$\langle f,f\rangle\langle g,g\rangle = ||f||^2||g||^2 \geq |\langle f,g\rangle|^2$$

Consider the following polynomial on $z = x + iy$:

$$P(z) = ||f||^2||zf+g||^2 = \langle f,f\rangle\langle zf+g, zf+g\rangle$$

It is not negative hence we have

$$0 \leq P(z) = ||f||^2||fz+g||^2 = ||f||^2\,\langle fz+g\,,fz+g\rangle =$$

$$= ||f||^2\,\langle\langle f,f\rangle\,z\bar{z} + z\langle f,g\rangle + \langle g,f\rangle\bar{z} + \langle g,g\rangle\rangle = ||f||^2\left(||f||^2|z|^2 + z\langle f,g\rangle + \langle g,f\rangle\bar{z} + |< g,f >|^2\right) =$$

$$\Big\langle ||f||^2z + \langle g,f\rangle\ ,||f||^2z + \langle g,f\rangle\Big\rangle$$