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We calculate here the wave function of free particle using continual integral and the classial action

Let particle starts at the point x_0, t_0 and ends at the point x_1, t_1 Divide $[t_0, t_1]$ on

$$[t_0, t_1, t_2, \ldots, t_{N-1}, t_N]\,, \qquad (t_N \mapsto t_1)$$

and consider the integral

$$\int\int\int\ldots\int e^{\frac{i}{\hbar}S_{\text{free}}(x_0,t_0;x_1,t_1)}e^{\frac{i}{\hbar}S_{\text{free}}(x_1,t_1;x_2,t_2)}e^{\frac{i}{\hbar}S_{\text{free}}(x_2,t_2;x_3,t_3)}\ldots$$

$$e^{\frac{i}{\hbar}S_{\text{free}}(x_{N-2},t_{N-2};x_{N-1},t_{N-1})}e^{\frac{i}{\hbar}S_{\text{free}}(x_{N-1},t_{N-1};x_N,t_N)}dx^1dx^2dx^3\ldots dx^{N-2}dx^{N-1} =$$

$$\int\prod_{i=1}^{N}\exp\left[\frac{i}{\hbar}S_{\text{free}}(x_{i-1},t_{i-1};x_i,t_i)\right]\prod_{j=1}^{N-1}dx^j\,,$$

where

$$S_{\text{free}}(x_0,t_0;x_1,t_1) = \frac{m(x_1-x_0)^2}{2t}$$

is the classical action.

This we have for the integral

$$I = \int\prod_{i=1}^{N}\exp\left[\frac{i}{\hbar}S_{\text{free}}(x_{i-1},t_{i-1};x_i,t_i)\right]\prod_{j=1}^{N-1}dx^j = \int\prod_{i=1}^{N}\exp\left[\frac{im}{2\varepsilon\hbar}(x_i-x_{i-1})^2\right]\prod_{j=1}^{N-1}dx^j\,,$$

where

$$\varepsilon = \frac{t_N - t_0}{N+1}\,.$$

Proposition Consider the function

$$F = (x_0-x_1)^2 + (x_1-x_2)^2 + \ldots + (x_{N-1}-x_N)^2\,,$$

and transform it tto the expression which is convenient for integration. It is useful to denote $x_i = u_i$.

$$F = (x_0-x_1)^2 + (x_1-x_2)^2 + \ldots = 2(u_1^2 - u_1(x_0+x_2)) + x_0^2 + x_1^2 + \ldots =$$

$$2\left(u_1 - \frac{x_0+x_2}{2}\right)^2 + x_0^2 + x_1^2 - \frac{(x_0+x_1)^2}{2} + \ldots =$$

$$2\left(x_1 - \frac{x_0+x_2}{2}\right)^2 + \frac{(x_0-x_2)^2}{2} + (x_2-x_3)^2 + \ldots =$$

$$2\left(x_1 - \frac{x_0+x_2}{2}\right)^2 + \frac{x_0^2}{2} + \frac{3}{2}u_2^2 + x_3^2 - u_2(x_0+2x_3) + \ldots =$$

$$2\left(x_1 - \frac{x_0+x_2}{2}\right)^2 + \frac{3}{2}\left(x_2 - \frac{x_0+2x_3}{3}\right)^2$$

Now consider any integral. We see that for two consecutive exponents,

$$\int dx_i \exp\left[\frac{im}{2\varepsilon\hbar}(x_i - x_{i-1})^2\right] \exp\left[\frac{im}{2\varepsilon\hbar}(x_{i+1} - x_i)^2\right] =$$

$$\int \exp\left[\frac{im}{2\varepsilon\hbar}\left((u - x_{i-1})^2 + (x_{i+1} - u)^2\right)\right] du =$$

$$\int \exp\left[\frac{im}{\varepsilon\hbar}\left(u - \frac{x_{i-1}+x_{i+1}}{2}\right)^2 + \frac{(x_{i+1}-x_{i-1})^2}{4}\right] du =$$

$$\left(\frac{\pi\varepsilon\hbar im}{}\right) \exp\left[\frac{im}{2\hbar\cdot 2\varepsilon}\left(x_{i+1} - x_{i-1}\right)^2\right]$$

Remark This is "up" to $e^{\frac{i\pi}{4}}$:

$$\int e^{iau^2} du = \sqrt{\frac{\pi}{a}} e^{\frac{i\pi}{4}} .$$

Using this fact continue the calculations:

$$I =$$