Homework 4. Solutions

1 *Calculate the Christoffel symbols of the canonical flat connection in* $\mathbf{E}^3$ *in*

a) cylindrical coordinates $(x = r\cos\varphi, y = r\sin\varphi, z = h)$*,*

b) spherical coordinates.

(For the case of sphere try to make calculations at least for components $\Gamma^r_{rr}, \Gamma^r_{r\theta}, \Gamma^r_{r\varphi}, \Gamma^r_{\theta\theta}, \ldots, \Gamma^r_{\varphi\varphi}$*)*

Remark One can calculate Christoffel symbols using Levi-Civita Theorem (Homework 5). There is a third way to calculate Christoffel symbols: It is using approach of Lagrangian. This is may be the easiest and most elegant way. (see the Homework 6)

In cylindrical coordinates (r, φ, h) we have

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = h \end{cases} \quad \text{and} \begin{cases} r = \sqrt{x^2+y^2} \\ \varphi = arctan\frac{y}{x} \\ h = z \end{cases}$$

We know that in Cartesian coordinates all Christoffel symbols vanish. Hence in cylindrical coordinates (see in detail lecture notes):

$$\Gamma^r_{rr} = \frac{\partial^2 x}{\partial^2 r}\frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial^2 r}\frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial^2 r}\frac{\partial r}{\partial z} = 0\,,$$

$$\Gamma^r_{r\varphi} = \Gamma^r_{\varphi r} = \frac{\partial^2 x}{\partial r\partial\varphi}\frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial r\partial\varphi}\frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial r\partial\varphi}\frac{\partial r}{\partial z} = -\sin\varphi\cos\varphi + \sin\varphi\cos\varphi = 0\,.$$

$$\Gamma^r_{\varphi\varphi} = \frac{\partial^2 x}{\partial^2 \varphi}\frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial^2 \varphi}\frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial^2 \varphi}\frac{\partial r}{\partial z} = -x\frac{x}{r} - y\frac{y}{r} = -r\,.$$

$$\Gamma^\varphi_{rr} = \frac{\partial^2 x}{\partial^2 r}\frac{\partial \varphi}{\partial x} + \frac{\partial^2 y}{\partial^2 r}\frac{\partial \varphi}{\partial y} + \frac{\partial^2 z}{\partial^2 r}\frac{\partial \varphi}{\partial z} = 0\,.$$

$$\Gamma^\varphi_{\varphi r} = \Gamma^\varphi_{r\varphi} = \frac{\partial^2 x}{\partial r\partial\varphi}\frac{\partial \varphi}{\partial x} + \frac{\partial^2 y}{\partial r\partial\varphi}\frac{\partial \varphi}{\partial y} + \frac{\partial^2 z}{\partial r\partial\varphi}\frac{\partial \varphi}{\partial z} = -\sin\varphi\frac{-y}{r^2} + \cos\varphi\frac{x}{r^2} = \frac{1}{r}$$

$$\Gamma^\varphi_{\varphi\varphi} = \frac{\partial^2 x}{\partial^2 \varphi}\frac{\partial \varphi}{\partial x} + \frac{\partial^2 y}{\partial^2 \varphi}\frac{\partial \varphi}{\partial y} + \frac{\partial^2 z}{\partial^2 \varphi}\frac{\partial \varphi}{\partial z} = -x\frac{-x}{r^2} - y\frac{y}{r^2} = 0\,.$$

All symbols $\Gamma^{\cdot}_{\cdot h}, \Gamma^{\cdot}_{h\cdot}$ vanish

$$\Gamma^r_{rh} = \Gamma^r_{hr} = \Gamma^r_{hh} = \Gamma^r_{\varphi h} = \Gamma^r_{h\varphi} = \Gamma^\varphi_{hr} = dots = 00$$

since $\frac{\partial^2 x}{\partial h\partial\ldots} = \frac{\partial^2 y}{\partial h\partial\ldots} = \frac{\partial^2 z}{\partial h\partial\ldots} = 0$

For all symbols $\Gamma^h_{\cdot\cdot}$ $\Gamma^h_{\cdot\cdot} = \frac{\partial^2 z}{\partial\cdot\partial\cdot}$ since $\frac{\partial h}{\partial x} = \frac{\partial h}{\partial y} = 0$ and $\frac{\partial h}{\partial y} = 1$. On the other hand all $\frac{\partial^2 z}{\partial\cdot\partial\cdot}$ vanish. Hence all symbols $\Gamma^h_{\cdot\cdot}$ vanish. ■

b) spherical coordinates

$$\begin{cases} x = r\sin\cos\varphi \\ y = r\sin\sin\varphi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \arccos\frac{z}{\sqrt{x^2+y^2+z^2}} \\ \varphi = \arctan\frac{y}{x} \end{cases}$$

We already know the fast way to calculate Christoffel symbol using Lagrangian of free particle and this method work for a flat connection since flat connection is a Levi-Civita connection for Euclidean metric

So perform now brute force calculations only for some components. (Then later (in homework 6) we will calculate using very quickly Lagrangian of free particle.)

$\Gamma^r_{rr} = 0$ since $\frac{\partial^2 x^i}{\partial^2 r} = 0$.

$$\Gamma^r_{r\theta} = \Gamma^r_{\theta r} = \frac{\partial^2 x}{\partial r \partial \theta}\frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial r \partial \theta}\frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial r \partial \theta}\frac{\partial r}{\partial z} = \cos\theta\cos\varphi\frac{x}{r} + \cos\theta\sin\varphi\frac{y}{r} - \sin\theta\frac{z}{r} = 0\,,$$

$$\Gamma^r_{\theta\theta} = \frac{\partial^2 x}{\partial^2 \theta}\frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial^2 \theta}\frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial^2 \theta}\frac{\partial r}{\partial z} = -r\sin\theta\cos\varphi\frac{x}{r} - r\sin\theta\sin\varphi\frac{y}{r} - r\cos\theta\frac{z}{r} = -r$$

$$\Gamma^r_{r\varphi} = \Gamma^r_{\varphi r} = \frac{\partial^2 x}{\partial r \partial \varphi}\frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial r \partial \varphi}\frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial r \partial \varphi}\frac{\partial r}{\partial z} = -\sin\theta\sin\varphi\frac{x}{r} + \sin\theta\cos\varphi\frac{y}{r} = 0$$

and so on....

2 *Let ∇ be an affine connection on a 2-dimensional manifold M such that in local coordinates (u, v) it is given that $\Gamma^u_{uv} = v$, $\Gamma^v_{uv} = 0$.*

Calculate the vector field $\nabla_{\frac{\partial}{\partial u}}\left(u\frac{\partial}{\partial v}\right)$.

Using the properties of connection and definition of Christoffel symbols have

$$\nabla_{\frac{\partial}{\partial u}}\left(u\frac{\partial}{\partial v}\right) = \partial_{\frac{\partial}{\partial u}}(u)\frac{\partial}{\partial v} + u\nabla_{\frac{\partial}{\partial u}}\left(\frac{\partial}{\partial v}\right) =$$

$$\frac{\partial}{\partial v} + u\left(\Gamma^u_{uv}\frac{\partial}{\partial u} + \Gamma^v_{uv}\frac{\partial}{\partial v}\right) = \frac{\partial}{\partial v} + u\left(v\frac{\partial}{\partial u} + 0\right) = \frac{\partial}{\partial v} + uv\frac{\partial}{\partial u}\,.$$

3 *Let ∇ be an affine connection on the 2-dimensional manifold M such that in local coordinates (u, v)*

$$\nabla_{\frac{\partial}{\partial u}}\left(u\frac{\partial}{\partial v}\right) = (1+u^2)\frac{\partial}{\partial v} + u\frac{\partial}{\partial u}\,.$$

Calculate the Christoffel symbols Γ^u_{uv} and Γ^v_{uv} of this connection.

Using the properties of connection we have $\nabla_{\frac{\partial}{\partial u}}\left(u\frac{\partial}{\partial v}\right) = u\nabla_{\frac{\partial}{\partial u}}\left(\frac{\partial}{\partial v}\right) +$

$$\partial_{\frac{\partial}{\partial u}}(u)\frac{\partial}{\partial v} = u\left(\Gamma^u_{uv}\frac{\partial}{\partial u} + \Gamma^v_{uv}\frac{\partial}{\partial v}\right) + 1\cdot\frac{\partial}{\partial v} = (1 + u\Gamma^v_{uv})\frac{\partial}{\partial v} + u\Gamma^u_{uv}\frac{\partial}{\partial u} = \left(1+u^2\right)\frac{\partial}{\partial v} + u\frac{\partial}{\partial u}.$$

Hence $1 + u^2 = 1 + u\Gamma^v_{uv}$ and $u\Gamma^u_{uv} = u$, i.e. $\Gamma^v_{uv} = u$ and $\Gamma^u_{uv} = 1$. ■

4 a) *Consider a connection such that its Christoffel symbols are symmetric in a given coordinate system: $\Gamma^i_{km} = \Gamma^i_{mk}$.*

Show that they are symmetric in an arbitrary coordinate system.

b*) Show that the Christoffel symbols of connection ∇ are symmetric (in any coordinate system) if and only if

$$\nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X}, \mathbf{Y}] = 0\,,$$

for arbitrary vector fields $\mathbf{X}, \mathbf{Y}$.

c)* Consider for an arbitrary connection the following operation on the vector fields:

$$S(\mathbf{X}, \mathbf{Y}) = \nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X}, \mathbf{Y}]$$

and find its properties.

Solution

a) Let $\Gamma^i_{km} = \Gamma^i_{mk}$. We have to prove that $\Gamma^{i'}_{k'm'} = \Gamma^{i'}_{m'k'}$

We have

$$\Gamma^{i'}_{k'm'} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\Gamma^i_{km} + \frac{\partial x^r}{\partial x^{k'}\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^r}\,. \tag{1}$$

Hence

$$\Gamma^{i'}_{m'k'} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^m}{\partial x^{m'}}\frac{\partial x^k}{\partial x^{k'}}\Gamma^i_{mk} + \frac{\partial x^r}{\partial x^{m'}\partial x^{k'}}\frac{\partial x^{i'}}{\partial x^r}$$

But $\Gamma^i_{km} = \Gamma^i_{mk}$ and $\frac{\partial x^r}{\partial x^{m'}\partial x^{k'}} = \frac{\partial x^r}{\partial x^{k'}\partial x^{m'}}$. Hence

$$\Gamma^{i'}_{m'k'} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^m}{\partial x^{m'}}\frac{\partial x^k}{\partial x^{k'}}\Gamma^i_{mk} + \frac{\partial x^r}{\partial x^{m'}\partial x^{k'}}\frac{\partial x^{i'}}{\partial x^r} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^m}{\partial x^{m'}}\frac{\partial x^k}{\partial x^{k'}}\Gamma^i_{km} + \frac{\partial x^r}{\partial x^{k'}\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^r} = \Gamma^{i'}_{k'm'}\,.$$

b) The relation

$$\nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X},\mathbf{Y}] = 0$$

holds for all fields if and only if it holds for all basic fields. One can easy check it using axioms of connection (see the next part). Consider $\mathbf{X} = \frac{\partial}{\partial x^i}, \mathbf{Y} = \frac{\partial}{\partial x^j}$ then since $[\partial_i, \partial_j] = 0$ we have that

$$\nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X},\mathbf{Y}] = \nabla_i\partial_j - \nabla_j\partial_i = \Gamma^k_{ij}\partial_k - \Gamma^k_{ji}\partial_k = (\Gamma^k_{ij} - \Gamma^k_{ji})\partial_k = 0$$

We see that commutator for basic fields $\nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X},\mathbf{Y}] = 0$ if and only if $\Gamma^k_{ij} - \Gamma^k_{ji} = 0$.

c) One can easy check it by straightforward calculations or using axioms for connection that $S(\mathbf{X},\mathbf{Y})$ is a vector-valued bilinear form on vectors. In particularly $S(f\mathbf{X}, Y) = fS(\mathbf{X},\mathbf{Y})$ for an arbitrary (smooth) function. Show this just using axioms defining connection:

$$S(f\mathbf{X}, Y) = \nabla_{f\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}(f\mathbf{X}) - [f\mathbf{X},\mathbf{Y}] = f\nabla_X\mathbf{Y} - f\nabla_{\mathbf{Y}}\mathbf{X} - \partial_{\mathbf{Y}}f\mathbf{X} + [\mathbf{Y}, f\mathbf{X}] =$$

$$f\nabla_X\mathbf{Y} - f\nabla_{\mathbf{Y}}\mathbf{X} - (\partial_{\mathbf{Y}}f)\mathbf{X} + \partial_{\mathbf{Y}}f\mathbf{X} + f\,[\mathbf{Y},\mathbf{X}] = f(\nabla_X\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X},\mathbf{Y}]) = fS(\mathbf{X},\mathbf{Y})$$

5 *Consider the surface M in the Euclidean space $\mathbf{E}^n$. Calculate the induced connection in the following cases*

a) $M = S^1$ in $\mathbf{E}^2$,
b) M— parabola $y = x^2$ in $\mathbf{E}^2$,
c) cylinder in $\mathbf{E}^3$.
d) cone in $\mathbf{E}^3$.
e) sphere in $\mathbf{E}^3$.
f) saddle $z = xy$ in $\mathbf{E}^3$

Solution.

a) Consider polar coordinate on S^1, $x = R\cos\varphi, y = R\sin\varphi$. We have to define the connection on S^1 induced by the canonical flat connection on $\mathbf{E}^2$. It suffices to define $\nabla_{\frac{\partial}{\partial\varphi}}\frac{\partial}{\partial\varphi} = \Gamma^\varphi_{\varphi\varphi}\partial_\varphi$.

Recall the general rule. Let $\mathbf{r}(u^\alpha)$: $x^i = x^i(u^\alpha)$ is embedded surface in Euclidean space $\mathbf{E}^n$. The basic vectors $\frac{\partial}{\partial u^\alpha} = \frac{\partial\mathbf{r}(u)}{\partial u^\alpha}$. To take the induced covariant derivative $\nabla_{\mathbf{X}}\mathbf{Y}$ for two tangent vectors $\mathbf{X},\mathbf{Y}$ we take a usual derivative of vector $\mathbf{Y}$ along vector $\mathbf{X}$ (the derivative with respect to canonical flat connection: in Cartesian coordiantes is just usual derivatives of components) then we take the tangent component of the answer, since in general derivative of vector $\mathbf{Y}$ along vector $\mathbf{X}$ is not tangent to surface:

$$\nabla_{\frac{\partial}{\partial u^\alpha}}\frac{\partial}{\partial u^\beta} = \Gamma^\gamma_{\alpha\beta}\frac{\partial}{\partial u^\gamma} = \left(\nabla^{(\text{canonical})}_{\partial_\alpha}\frac{\partial}{\partial u^\beta}\right)_{\text{tangent}} = \left(\frac{\partial^2\mathbf{r}(u)}{\partial u^\alpha\partial u^\beta}\right)_{\text{tangent}}$$

$(\nabla_{\text{canonical } \partial_\alpha} \frac{\partial}{\partial u^\beta})$ is just usual derivative in Euclidean space since for canonical connection all Christoffel symbols vanish.)

In the case of 1-dimensional manifold, curve it is just tangential acceleration!:

$$\nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial u} = \Gamma^u_{uu} \frac{\partial}{\partial u} = \left(\nabla^{(\text{canonical})}_{\partial_u} \frac{\partial}{\partial u}\right)_{\text{tangent}} = \left(\frac{d^2\mathbf{r}(u)}{du^2}\right)_{\text{tangent}} = \mathbf{a}_{\text{tangent}}$$

For the circle S^1, $(x = R\cos\varphi, y = R\sin\varphi)$, in $\mathbf{E}^2$. We have

$$\mathbf{r}_\varphi = \frac{\partial}{\partial \varphi} = \frac{\partial x}{\partial \varphi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \varphi}\frac{\partial}{\partial y} = -R\sin\varphi\frac{\partial}{\partial x} + R\cos\varphi\frac{\partial}{\partial y}\,,$$

$$\nabla_{\frac{\partial}{\partial \varphi}} \frac{\partial}{\partial \varphi} = \Gamma^\varphi_{\varphi\varphi}\partial_\varphi = \left(\nabla^{(\text{canonic.})}_{\partial_\varphi} \partial_\varphi\right)_{\text{tangent}} = \left(\frac{\partial}{\partial \varphi}\mathbf{r}_\varphi\right)_{\text{tangent}} =$$

$$\left(\frac{\partial}{\partial \varphi}\left(-R\sin\varphi\right)\frac{\partial}{\partial x} + \frac{\partial}{\partial \varphi}\left(R\cos\varphi\right)\frac{\partial}{\partial y}\right)_{\text{tangent}} = \left(-R\cos\varphi\frac{\partial}{\partial x} - R\sin\varphi\frac{\partial}{\partial y}\right)_{\text{tangent}} = 0,$$

since the vector $-R\cos\varphi\frac{\partial}{\partial x} - R\sin\varphi\frac{\partial}{\partial y}$ is orthogonal to the tangent vector $\mathbf{r}_\varphi$. In other words it means that acceleration is centripetal: tangential acceleration equals to zero.

We see that in coordinate φ, $\Gamma^\varphi_{\varphi\varphi} = 0$. $\blacksquare$

Additional work: Perform calculation of Christoffel symbol in stereographic coordinate t:

$$x = \frac{2tR^2}{R^2 + t^2}, y = \frac{R(t^2 - R^2)}{t^2 + R^2}\,.$$

In this case

$$\mathbf{r}_t = \frac{\partial}{\partial t} = \frac{\partial x}{\partial t}\frac{\partial}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial}{\partial y} = \frac{2R^2}{(R^2 + t^2)^2}\left((R^2 - t^2)\frac{\partial}{\partial x} + 2tR\frac{\partial}{\partial x}\right)\,,$$

$$\nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} = \Gamma^t_{tt}\partial_t = \left(\nabla^{(\text{canonic.})}_{\partial_t} \partial_t\right)_{\text{tangent}} = \left(\frac{\partial}{\partial t}\mathbf{r}_t\right)_{\text{tangent}} = (\mathbf{r}_{tt})_{\text{tangent}} =$$

$$\left(-\frac{4t}{t^2 + R^2}\mathbf{r}_t + \frac{2R^2}{(R^2 + t^2)^2}\left(-2t\frac{\partial}{\partial x} + 2R\frac{\partial}{\partial y}\right)\right)_{\text{tangent}}$$

In this case $\mathbf{r}_{tt}$ is not orthogonal to velocity: to calculate $(\mathbf{r}_{tt})_{\text{tangent}}$ we need to extract its orthogonal component:

$$(\mathbf{r}_{tt})_{\text{tangent}} = \mathbf{r}_{tt} - \langle \mathbf{r}_{tt}, \mathbf{n}_t \rangle \mathbf{n}$$

We have

$$\mathbf{n}_t = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{1}{R^2 + t^2}\left(2tR\partial_x + (t^2 - R^2)\partial_y\right)\,,$$

where $\langle \mathbf{r}_t, \mathbf{n} \rangle = 0$. Hence $\langle \mathbf{r}_{tt}, \mathbf{n}_t \rangle = \frac{-4R^3}{(t^2+R^2)^2}$ and

$$(\mathbf{r}_{tt})_{\text{tangent}} = \mathbf{r}_{tt} - \langle \mathbf{r}_{tt}, \mathbf{n}_t \rangle \mathbf{n} =$$

$$\left(-\frac{4t}{t^2 + R^2}\mathbf{r}_t + \frac{2R^2}{(R^2 + t^2)^2}\left(-2t\frac{\partial}{\partial x} + 2R\frac{\partial}{\partial y}\right)\right) + \frac{4R^3}{(t^2 + R^2)^2} \cdot \frac{1}{R^2 + t^2}\left(2tR\partial_x + (t^2 - R^2)\partial_y\right) = \frac{-2t}{t^2 + R^2}\mathbf{r}_t$$

We come to the answer:

$$\nabla_{\partial_t}\partial_t = \frac{-2t}{t^2 + R^2}\partial_t, \quad \text{i.e.}\Gamma^{\mathrm{t}}_{\mathrm{tt}} = \frac{-2\mathrm{t}}{\mathrm{t}^2 + \mathrm{R}^2}$$

Of course we could calculate the Christoffel symbol in stereographic coordinates just using the fact that we already know the Christoffel symbol in polar coordinates: $\Gamma^{\varphi}_{\varphi\varphi} = 0$, hence

$$\Gamma^t_{tt} = \frac{dt}{d\varphi}\frac{d\varphi}{dx}\frac{d\varphi}{dx}\Gamma^{\varphi}_{\varphi\varphi} + \frac{d^2\varphi}{dt^2}\frac{dt}{d\varphi} = \frac{d^2\varphi}{dt^2}\frac{dt}{d\varphi}$$

It is easy to see that $t = R\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)$, i.e. $\varphi = 2\arctan\frac{t}{R} - \frac{\pi}{2}$ and

$$\Gamma^t_{tt} = \frac{d^2\varphi}{dt^2}\frac{dt}{d\varphi} = \frac{\frac{d^2\varphi}{dt^2}}{\frac{d\varphi}{dt}} = -\frac{2t}{t^2+R^2}\,.$$

b) For parabola $x = t, y = t^2$

$$\mathbf{r}_t = \frac{\partial}{\partial t} = \frac{\partial x}{\partial t}\frac{\partial}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial}{\partial y} = \frac{\partial}{\partial x} + 2t\frac{\partial}{\partial y}\,,$$

$$\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t} = \Gamma^t_{tt}\partial_t = \left(\nabla^{(\text{canonic.})}_{\partial_t}\partial_t\right)_{\text{tangent}} = \left(\frac{\partial}{\partial t}\mathbf{r}_t\right)_{\text{tangent}} = (\mathbf{r}_{tt})_{\text{tangent}} = \left(2\frac{\partial}{\partial y}\right)_{\text{tangent}}$$

To calculate $(\mathbf{r}_{tt})_{\text{tangent}}$ we need to extract its orthogonal component: $(\mathbf{r}_{tt})_{\text{tangent}} = \mathbf{r}_{tt} - \langle \mathbf{r}_{tt}, \mathbf{n}_t\rangle\mathbf{n}$, where $\mathbf{n}$ is an orthogonal unit vector: $\langle \mathbf{n}, \mathbf{r}_t\rangle = 0, \langle \mathbf{n}, \mathbf{n}\rangle = 1$:

$$\mathbf{n}_t = \frac{1}{\sqrt{1+4t^2}}\left(-2t\partial_x + \partial_y\right)\,.$$

We have

$$(\mathbf{r}_{tt})_{\text{tangent}} = \mathbf{r}_{tt} - \langle \mathbf{r}_{tt}, \mathbf{n}_t\rangle\mathbf{n} = 2\partial_y - \left\langle 2\partial_y, \frac{1}{\sqrt{1+4t^2}}\left(-2t\partial_x + \partial_y\right)\right\rangle \frac{1}{\sqrt{1+4t^2}}\left(-2t\partial_x + \partial_y\right) =$$

$$\frac{4t}{1+4t^2}\partial_x + \frac{8t^2}{1+4t^2}\partial_y = \frac{4t}{1+4t^2}\left(\partial_x + 2t\partial_y\right) = \frac{4t}{1+4t^2}\partial_t$$

We come to the answer:

$$\nabla_{\partial_t}\partial_t = \frac{4t}{1+4t^2}\partial_t, \quad \text{i.e.}\Gamma^{\text{t}}_{\text{tt}} = \frac{4\text{t}}{1+4t^2}$$

Remark Do not be surprised by resemblance of the answer to the answer for circle in stereographic coordinates.

c) *Cylinder*

$\mathbf{r}(h,\varphi)$: $\begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases}$.

$\partial_h = \mathbf{r}_h = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\partial_\varphi = \mathbf{r}_\varphi = \begin{pmatrix} -a\sin\varphi \\ a\cos\varphi \\ 0 \end{pmatrix}$

Calculate

$$\nabla_{\partial_h}\partial_h = \Gamma^h_{hh}\partial_h + \Gamma^{\varphi}_{hh}\partial_\varphi = \left(\frac{\partial^2\mathbf{r}}{\partial h^2}\right)_{\text{tangent}} = 0 \,\text{since}\; \mathbf{r}_{hh} = 0.$$

Hence $\Gamma^h_{hh} = \Gamma^{\varphi}_{hh} = 0$

$$\nabla_{\partial_h}\partial_\varphi = \nabla_{\partial_\varphi}\partial_h = \Gamma^h_{h\varphi}\partial_h + \Gamma^{\varphi}_{h\varphi}\partial_\varphi = \left(\frac{\partial^2\mathbf{r}}{\partial h\partial\varphi}\right)_{\text{tangent}} = 0 \,\text{since}\; \mathbf{r}_{h\varphi} = 0$$

Hence $\Gamma^h_{h\varphi} = \Gamma^h_{\varphi h} = \Gamma^\varphi_{h\varphi} = \Gamma^\varphi_{\varphi h} = 0$.

$$\nabla_{\partial_\varphi}\partial_\varphi = \Gamma^h_{\varphi\varphi}\partial_h + \Gamma^\varphi_{\varphi\varphi}\partial_\varphi = \left(\frac{\partial^2\mathbf{r}}{\partial\varphi\partial\varphi}\right)_{\text{tangent}} = \left(\begin{pmatrix} -a\cos\varphi \\ -a\sin\varphi \\ 0 \end{pmatrix}\right)_{\text{tangent}} = 0$$

since the vector $\mathbf{r}_{\varphi\varphi} = \begin{pmatrix} -a\cos\varphi \\ -a\sin\varphi \\ 0 \end{pmatrix}$ is orthogonal to the surface of cylinder. Hence $\Gamma^h_{h\varphi} = \Gamma^h_{\varphi h} = \Gamma^\varphi_{h\varphi} = \Gamma^\varphi_{\varphi h} = 0$

We see that for cylinder all Christoffel symbols in cylindrical coordinates vanish. This is not big surprise: in cylindrical coordinates metric equals $dh^2 = a^2 d\varphi^2$. This due to Levi-Civita theorem one can see that Levi-Civita connection which is equal to induced connection vanishes since all coefficients are constants.

d) *Cone*

For cone: $x^2 + y^2 = k^2 z^2$ we have

$$\mathbf{r}(h,\varphi) = \begin{cases} x = kh\cos\ \varphi \\ y = kh\sin\varphi \\ z = h \end{cases}$$

$$\frac{\partial}{\partial h} = \mathbf{r}_h = \begin{pmatrix} k\cos\varphi \\ k\sin\varphi \\ 1 \end{pmatrix}, \ \frac{\partial}{\partial\varphi} = \mathbf{r}_\varphi = \begin{pmatrix} -kh\sin\varphi \\ kh\cos\varphi \\ 0 \end{pmatrix}, \ \mathbf{n} = \frac{1}{\sqrt{1+k^2}}\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix}$$

We have $\mathbf{r}_{hh} = 0$, hence $\nabla_{\partial_h}\partial_h = 0$. i.e. $\Gamma^h_{hh} = \Gamma^\varphi_{hh} = 0$.

We have that $\mathbf{r}_{h\varphi} = \mathbf{r}_{\varphi h} = \begin{pmatrix} -k\sin\varphi \\ k\cos\varphi \\ 0 \end{pmatrix} = \frac{\mathbf{r}_\varphi}{h}$, i.e. $\nabla_{\partial_h}\partial_\varphi = \nabla_{\partial_\varphi}\partial_h = \frac{\mathbf{r}_\varphi}{h}$:

$$\Gamma^\varphi_{h\varphi} = \Gamma^\varphi_{\varphi,h} = \frac{1}{h}, \quad \Gamma^h_{h\varphi} = \Gamma^h_{\varphi h}.$$

Now calculate $\mathbf{r}_{\varphi\varphi}$: $\mathbf{r}_{\varphi\varphi} = \begin{pmatrix} -kh\cos\varphi \\ -kh\sin\varphi \\ 0 \end{pmatrix}$. This vector is neither tangent to the cone nor orthogonal to the cone:..... *see the continuation after 16 April: this is coursework assignment.*

e) *Sphere*

For the sphere $\mathbf{r}(\theta,\varphi)$: $\begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases}$, we have

$$\frac{\partial}{\partial\theta} = \mathbf{r}_\theta = \begin{pmatrix} R\cos\theta\cos\varphi \\ R\cos\theta\sin\varphi \\ -R\sin\theta \end{pmatrix}, \ \frac{\partial}{\partial\varphi} = \mathbf{r}_\varphi = \begin{pmatrix} -R\sin\theta\sin\varphi \\ R\sin\theta\cos\varphi \\ 0 \end{pmatrix}, \ \mathbf{n} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

Calculate

$$\nabla_{\partial_\theta}\partial_\theta = \Gamma^\theta_{\theta\theta}\partial_\theta + \Gamma^\varphi_{\theta\theta}\partial_\varphi = \left(\frac{\partial^2\mathbf{r}}{\partial\theta^2}\right)_{\text{tangent}} = 0$$

since $\frac{\partial^2\mathbf{r}}{\partial\theta^2} = -R\mathbf{n}$ is orthogonal to the sphere. Hence $\Gamma^\theta_{\theta\theta} = \Gamma^\varphi_{\theta\theta} = 0$.

Now calculate

$$\nabla_{\partial_\theta}\partial_\varphi = \Gamma^\theta_{\theta\varphi}\partial_\theta + \Gamma^\varphi_{\theta\varphi}\partial_\varphi = \left(\frac{\partial^2\mathbf{r}}{\partial\theta\partial\varphi}\right)_{\text{tangent}}.$$

We have

$$\frac{\partial^2 \mathbf{r}}{\partial\theta\partial\varphi} = \operatorname{cotan}\theta \mathbf{r}_\varphi,$$

hence

$$\nabla_{\partial_\theta}\partial_\varphi = \Gamma^\theta_{\theta\varphi}\partial_\theta + \Gamma^\varphi_{\theta\varphi}\partial_\varphi = \left(\frac{\partial^2 \mathbf{r}}{\partial\theta\partial\varphi}\right)_{\text{tangent}} = \operatorname{cotan}\theta \mathbf{r}_\varphi, i.e.$$

$\Gamma^\theta_{\theta\varphi} = 0, \Gamma^\varphi_{\theta\varphi} = \operatorname{cotan}\theta$

Now calculate

$$\nabla_{\partial_\varphi}\partial_\theta = \Gamma^\theta_{\varphi\theta}\partial_\theta + \Gamma^\varphi_{\varphi\theta}\partial_\varphi = \left(\frac{\partial^2 \mathbf{r}}{\partial\varphi\partial\theta}\right)_{\text{tangent}}.$$

We have

$$\frac{\partial^2 \mathbf{r}}{\partial\varphi\partial\theta} = \operatorname{cotan}\theta \mathbf{r}_\varphi,$$

hence

$$\nabla_{\partial_\theta}\partial_\varphi = \Gamma^\theta_{\theta\varphi}\partial_\theta + \Gamma^\varphi_{\theta\varphi}\partial_\varphi = \left(\frac{\partial^2 \mathbf{r}}{\partial\theta\partial\varphi}\right)_{\text{tangent}} = \operatorname{cotan}\theta \mathbf{r}_\varphi, i.e.$$

$\Gamma^\theta_{\varphi\theta} = 0, \Gamma^\varphi_{\varphi\theta} = \operatorname{cotan}\theta$. Of course we did not need to perform these calculations: since ∇ is symmetric connection and $\nabla_{\partial_\varphi}\partial_\theta = \nabla_{\partial_\theta}\partial_\varphi$, i.e.

$$\Gamma^\theta_{\varphi\theta} = \Gamma^\theta_{\theta\varphi} = 0 \ \Gamma^\varphi_{\varphi\theta} = \Gamma^\varphi_{\theta\varphi} = \operatorname{cotan}\theta\,.$$

and finally

$$\nabla_{\partial_\varphi}\partial_\varphi = \Gamma^\theta_{\varphi\varphi}\partial_\theta + \Gamma^\varphi_{\varphi\varphi}\partial_\varphi = \left(\frac{\partial^2 \mathbf{r}}{\partial\varphi^2}\right)_{\text{tangent}}.$$

We have

$$\frac{\partial^2 \mathbf{r}}{\partial\varphi^2} = \mathbf{r}_{\varphi\varphi} = \begin{pmatrix} -R\sin\theta\cos\varphi \\ -R\sin\theta\sin\varphi \\ 0 \end{pmatrix}.$$

The vector $\mathbf{r}_{\varphi\varphi}$ is not proportional to normal vector $\mathbf{n}$, i.e. it is not orthogonal to the sphere; the vector $\mathbf{r}_{\varphi\varphi}$ is not tangent to sphere, i.e. it is not orthogonal to vector $\mathbf{n}$: $0 \neq \langle \mathbf{r}_{\varphi\varphi}, \mathbf{n}\rangle = -R\sin^2\theta$. We decompose the vector $\mathbf{r}_{\varphi\varphi}$ on the sum of tangent vector and orthogonal vector:

$$\mathbf{r}_{\varphi\varphi} = \underbrace{\mathbf{r}_{\varphi\varphi} - \mathbf{n}\langle \mathbf{r}_{\varphi\varphi}, \mathbf{n}\rangle}_{\text{tangent vector}} + \mathbf{n}\langle \mathbf{r}_{\varphi\varphi}, \mathbf{n}\rangle,$$

We see that

$$\left(\frac{\partial^2 \mathbf{r}}{\partial\varphi^2}\right)_{\text{tangent}} = \mathbf{r}_{\varphi\varphi} - \mathbf{n}\langle \mathbf{r}_{\varphi\varphi}, \mathbf{n}\rangle = \mathbf{r}_{\varphi\varphi} + R\sin^2\theta\mathbf{n} = \begin{pmatrix} -R\sin\theta\cos\varphi \\ -R\sin\theta\sin\varphi \\ 0 \end{pmatrix} + R\sin^2\theta \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} =$$

$$\begin{pmatrix} -R\cos^2\theta\sin\theta\cos\varphi \\ -R\cos^2\theta\sin\theta\sin\varphi \\ R\sin^2\theta\cos\theta \end{pmatrix} = -\sin\theta\cos\theta \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} = -\sin\theta\cos\theta\mathbf{r}_\theta\,.$$

We have

$$\nabla_{\partial_\varphi}\partial_\varphi = \Gamma^\theta_{\varphi\varphi}\partial_\theta + \Gamma^\varphi_{\varphi\varphi}\partial_\varphi = \left(\frac{\partial^2 \mathbf{r}}{\partial\varphi\partial\varphi}\right)_{\text{tangent}} = -\sin\theta\cos\theta\mathbf{r}_\theta,\ i.e.$$

$\Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta, \Gamma^\varphi_{\varphi\varphi} = 0.$

f) *Saddle*

For saddle $z = xy$: We have $\mathbf{r}(u,v)$: $\begin{cases} x = u \\ y = v \\ z = uv \end{cases}$, $\partial_u = \mathbf{r}_u = \begin{pmatrix} 1 \\ 0 \\ v \end{pmatrix}$, $\partial_v = \mathbf{r}_v = \begin{pmatrix} 0 \\ 1 \\ u \end{pmatrix}$ It will be useful also to use the normal unit vector $\mathbf{n} = \frac{1}{\sqrt{1+u^2+v^2}} \begin{pmatrix} -v \\ -u \\ 1 \end{pmatrix}$.

Calculate:

$$\nabla_{\partial_u}\partial_u = \Gamma^u_{uu}\partial_u + \Gamma^v_{uu}\partial_v = \left(\frac{\partial^2 \mathbf{r}}{\partial u^2}\right)_{\text{tangent}} = (\mathbf{r}_{uu})_{\text{tangent}} = 0 \,\text{since}\; \mathbf{r}_{uu} = 0.$$

Hence $\Gamma^u_{uu} = \Gamma^v_{uu} = 0$.

Analogously $\Gamma^u_{vv} = \Gamma^v_{vv} = 0$ since $\mathbf{r}_{vv} = 0$.

Now calculate $\Gamma^u_{uv}, \Gamma^v_{uv}, \Gamma^u_{vu}, \Gamma^v_{vu}$:

$$\nabla_{\partial_u}\partial_v = \nabla_{\partial_v}\partial_u = \Gamma^u_{uv}\partial_u + \Gamma^v_{uv}\partial_v = (\mathbf{r}_{uv})_{\text{tangent}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{\text{tangent}}$$

Using normal unit vector $\mathbf{n}$ we have: $(\mathbf{r}_{uv})_{\text{tangent}} = \mathbf{r}_{uv} - \langle \mathbf{r}_{uv}, \mathbf{n}\rangle \mathbf{n} = \Gamma^u_{uv}\partial_u + \Gamma^v_{uv}\partial_v =$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{\text{tangent}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{1+u^2+v^2}} \begin{pmatrix} -v \\ -u \\ 1 \end{pmatrix} \right\rangle \frac{1}{\sqrt{1+u^2+v^2}} \begin{pmatrix} -v \\ -u \\ 1 \end{pmatrix} =$$

$$\frac{1}{1+u^2+v^2} \begin{pmatrix} v \\ u \\ u^2+v^2 \end{pmatrix} = \frac{v}{1+u^2+v^2} \begin{pmatrix} 1 \\ 0 \\ v \end{pmatrix} + \frac{u}{1+u^2+v^2} \begin{pmatrix} 0 \\ u \\ u \end{pmatrix} = \frac{v\mathbf{r}_u + u\mathbf{r}_v}{1+u^2+v^2}.$$

Hence $\Gamma^u_{uv} = \Gamma^u_{vu} = \frac{v}{1+u^2+v^2}$ and $\Gamma^v_{uv} = \Gamma^v_{vu} = \frac{u}{1+u^2+v^2}$. ■

Sure one may calculate this connection as Levi-Civita connction of the induced Riemannian metric using explicit Levi-Civita formula or using method of Lagrangian of free particle.

6 *Let ∇_1, ∇_2 be two different connections. Let ${}^{(1)}\Gamma^i_{km}$ and ${}^{(2)}\Gamma^i_{km}$ be the Christoffel symbols of connections ∇_1 and ∇_2 respectively.*

a) Find the transformation law for the object : $T^i_{km} = {}^{(1)}\Gamma^i_{km} - {}^{(2)}\Gamma^i_{km}$ under a change of coordinates. Show that it is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ tensor.

b)? Consider an operation $\nabla_1 - \nabla_2$ on vector fields and find its properties.*

Christoffel symbols of both connections transform according the law (1). The second term is the same. Hence it vanishes for their difference:

$$T^{i'}_{k'm'} = {}^{(1)}\Gamma^{i'}_{k'm'} - {}^{(2)}\Gamma^{i'}_{k'm'} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}} \left({}^{(1)}\Gamma^i_{km} - {}^{(2)}\Gamma^i_{km}\right) = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}} T^i_{km}$$

We see that $T^{i'}_{'km'}$ transforms as a tensor of the type $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

b) One can do it in invariant way. Using axioms of connection study $T = \nabla_1 - \nabla_2$ is a vector field. Consider

$$T(\mathbf{X}, \mathbf{Y}) = \nabla_{1\mathbf{X}}\mathbf{Y} - \nabla_{2\mathbf{X}}\mathbf{Y}$$

Show that $T(f\mathbf{X}, \mathbf{Y}) = fT(\mathbf{X}, \mathbf{Y})$ for an arbitrary (smooth) function, i.e. it does not possesses derivatives:

$$T(f\mathbf{X}, \mathbf{Y}) = \nabla_{1fX}\mathbf{Y} - \nabla_{2fX}\mathbf{Y} = (\partial_{\mathbf{X}} f)\mathbf{Y} + f\nabla_{1\mathbf{X}}\mathbf{Y} - (\partial_{\mathbf{X}} f)\mathbf{Y} - f\nabla_{2\mathbf{X}}\mathbf{Y} = fT(\mathbf{X}, \mathbf{Y}).$$

7 * a) *Consider $t_m = \Gamma^i_{im}$. Show that the transformation law for t_m is*

$$t_{m'} = \frac{\partial\, x^m}{\partial x^{m'}} t_m + \frac{\partial^2 x^r}{\partial x^{m'} \partial x^{k'}} \frac{\partial x^{k'}}{\partial x^r}\,.$$

b) † *Show that this law can be written as*

$$t_{m'} = \frac{\partial\, x^m}{\partial x^{m'}} t_m + \frac{\partial}{\partial x^{m'}} \left(\log\det \left(\frac{\partial x}{\partial x'} \right) \right)\,.$$

Solution. Using transformation law (1) we have

$$t_{m'} = \Gamma^{i'}_{i'm''} = \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^k}{\partial x^{i'}} \frac{\partial x^m}{\partial x^{m'}} \Gamma^i_{km} + \frac{\partial x^r}{\partial x^{i'} \partial x^{m'}} \frac{\partial x^{i'}}{\partial x^r}$$

We have that $\frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^k}{\partial x^{i'}} = \delta^k_i$. Hence

$$t_{m'} = \Gamma^{i'}_{i'm''} = \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^k}{\partial x^{i'}} \frac{\partial x^m}{\partial x^{m'}} \Gamma^i_{km} + \frac{\partial x^r}{\partial x^{i'} \partial x^{m'}} \frac{\partial x^{i'}}{\partial x^r} = \delta^k_i \frac{\partial x^m}{\partial x^{m'}} \Gamma^i_{km} + \frac{\partial x^r}{\partial x^{i'} \partial x^{m'}} \frac{\partial x^{i'}}{\partial x^r} = \frac{\partial x^m}{\partial x^{m'}} t_m + \frac{\partial x^r}{\partial x^{i'} \partial x^{m'}} \frac{\partial x^{i'}}{\partial x^r}\,. \blacksquare$$

b) † When calculating $\frac{\partial}{\partial x^{m'}} \left(\log\det \left(\frac{\partial x}{\partial x'} \right) \right)$ use very important formula:

$$\delta \det A = \det A \,\mathrm{Tr}\, (A^{-1} \delta A) \to \delta \log\det A = \mathrm{Tr}\, (A^{-1} \delta A)\,.$$

Hence

$$\frac{\partial}{\partial x^{m'}} \left(\log\det \left(\frac{\partial x}{\partial x'} \right) \right) = \frac{\partial x^{i'}}{\partial x^r} \frac{\partial^2 x^r}{\partial x^{i'} \partial x^{m'}}$$

and we come to transformation law for (1).

To deduce the formula for $\delta \det A$ notice that

$$\det(A + \delta A) = \det A \det(1 + A^{-1} \delta A)$$

and use the relation: $\det(1 + \delta A) = 1 + \mathrm{Tr}\, \delta A + O(\delta^2 A)$

7*

Let $\mathbf{K}$, $\mathbf{X}$ be vector fields on manifold M, and ∇ connection. Consider the operation

$$\mathbf{K}\,, \mathbf{X} \mapsto A_{\mathbf{K}}(\mathbf{X}) = \nabla_{\mathbf{K}} \mathbf{X} - \mathcal{L}_{\mathbf{K}} \mathbf{X}\,, (\mathcal{L} \textit{ is a Lie derivative}, \quad \mathcal{L}_{\mathbf{X}} \mathbf{Y} = [\mathbf{X}, Y]) \tag{1}$$

$$\textit{a) Show that for an arbitrary function } f\,, A_{\mathbf{K}}(f\mathbf{X}) = f \mathbf{A}_{\mathbf{K}}(\mathbf{X})\,. \tag{2}$$

This condition implies that equation (1) defines linear operaration on tangent vectors, i.e. it is well defined on tangent vectors (not vector fields) and it is linear. In other words for a given vector $\mathbf{X}_0$ tangent to manifold M at the given point $\mathbf{p}$, $\mathbf{X}_0 \in T_{\mathbf{p}} M$, consider an arbitrary vector field $\mathbf{X}$ passing via this vector, i,e, such that value of vector field at the given point $\mathbf{p}$ coincides with the vector $\mathbf{X}_0$,$\mathbf{X}\big|_{\mathbf{p}} = \mathbf{X}_0$. Condition (**) tells that the answer at the point $\mathbf{p}$ does not depend on a choice of vector field passing through vector $\mathbf{X}_0$. It depends only on the value of this vector field at the point $\mathbf{p}_0$. Indeed let two vector fields $\mathbf{X}, \tilde{\mathbf{X}}$ coincide at the point $\mathbf{p}$, i.e. the vector field $\tilde{\mathbf{X}} - \mathbf{X}$ vanishes at the point $\mathbf{p}$. Moreover Hadamard lemma [1]. tells that in

[1] *if smooth function g vanishes at the origin, then $g = \sum_i x^i h_i(x)$, where $h_i(x)$ are also smooth.*

this case vector field $\tilde{\mathbf{X}} - \mathbf{X}$ *is linear combination of vector fields with coefficients vanishing at the point* $\mathbf{p}$*:* $\tilde{\mathbf{X}} - \mathbf{X} = \sum_a h_a(x)\mathbf{T_a}$*, where all* $h_a(x)$ *vanish at the point* $\mathbf{p}$*. Hence due to (2)*

$$A_{\mathbf{K}}(\tilde{X} - \mathbf{X})|_{\mathbf{p}} = A_{\mathbf{K}}\left(\sum_a h_a(x)\mathbf{T_a}\right)|_{\mathbf{p}} = \sum_a h_a(x)|_{\mathbf{p}} A_{\mathbf{K}}(\mathbf{T_a}) = \mathbf{0}\,.$$

b) Show that linear operator $A_{\mathbf{K}}$ *in equation (*) is equal to*

$$A_{\mathbf{K}}(\mathbf{X}) = \nabla_{\mathbf{X}}\mathbf{K} + S(\mathbf{K}, \mathbf{X})\,.$$

c) How does it look this operator for symmetric connection?

Solution:

a) Using definition of connection and properties of commutator we see that

$$A_{\mathbf{K}}(f\mathbf{X}) = \nabla_{\mathbf{K}}(f\mathbf{X}) - \mathcal{L}_{\mathbf{K}}(f\mathbf{X}) = (\partial_{\mathbf{K}} f\mathbf{X} + f\nabla_{\mathbf{K}}\mathbf{X}) - [\mathbf{K}, f\mathbf{X}] =$$

$$(\partial_{\mathbf{K}} f\mathbf{X} + f\nabla_{\mathbf{K}}\mathbf{X}) - f[\mathbf{K}, \mathbf{X}] - (\partial_{\mathbf{K}} f)\mathbf{X} = f(\nabla_{\mathbf{K}}\mathbf{X} - [\mathbf{K}, \mathbf{X}]) = fA_{\mathbf{K}}(\mathbf{X})\,.$$

b) Now prove the relation (2):

$$A_{\mathbf{K}}(f\mathbf{X}) = \nabla_{\mathbf{K}}(f\mathbf{X}) - \mathcal{L}_{\mathbf{K}}(f\mathbf{X}) = (\nabla_{\mathbf{K}}\mathbf{X} - \nabla_{\mathbf{X}}\mathbf{K} + [\mathbf{K}, \mathbf{X}]) + (\nabla_{\mathbf{X}}\mathbf{K} = \nabla_{\mathbf{X}}\mathbf{K} + S(\mathbf{K}, \mathbf{X})$$

c) In the case of symmetric connection $S \equiv 0$, hence

$$A_{\mathbf{K}}(\mathbf{X}) = \nabla_{\mathbf{X}}\mathbf{K}\,.$$