

Homework 5

1 Calculate the Christoffel symbols of the canonical flat connection in \mathbf{E}^3 in

a) cylindrical coordinates $(x = r \cos \varphi, y = r \sin \varphi, z = h)$,

b) spherical coordinates.

(For the case b) try to make calculations at least for components $\Gamma_{rr}^r, \Gamma_{r\theta}^r, \Gamma_{r\varphi}^r, \Gamma_{\theta\theta}^r, \dots, \Gamma_{\varphi\varphi}^r$)

2 Let ∇ be an affine connection on a 2-dimensional manifold M such that in local coordinates (u, v) it is given that $\Gamma_{uv}^u = v, \Gamma_{uv}^v = 0$.

Calculate the vector field $\nabla_{\frac{\partial}{\partial u}} \left(u \frac{\partial}{\partial v} \right)$.

3 Let ∇ be an affine connection on the 2-dimensional manifold M such that in local coordinates (u, v)

$$\nabla_{\frac{\partial}{\partial u}} \left(u \frac{\partial}{\partial v} \right) = (1 + u^2) \frac{\partial}{\partial v} + u \frac{\partial}{\partial u}.$$

Calculate the Christoffel symbols Γ_{uv}^u and Γ_{uv}^v of this connection.

4 Let ∇ be an affine connection on a 2-dimensional manifold M such that, in local coordinates (x, y) , all Christoffel symbols vanish except $\Gamma_{xx}^x = xy, \Gamma_{xx}^y = -1$ and $\Gamma_{yy}^y = y$. Show that for the vector field $\mathbf{X} = \partial_x + x\partial_y$,

$$\nabla_{\mathbf{X}} \mathbf{X} = xy \mathbf{X}.$$

5 a) Consider a connection such that its Christoffel symbols are symmetric in a given coordinate system: $\Gamma_{km}^i = \Gamma_{mk}^i$.

Show that they are symmetric in an arbitrary coordinate system.

b*) Show that the Christoffel symbols of connection ∇ are symmetric (in any coordinate system) if and only if

$$\nabla_{\mathbf{X}} \mathbf{Y} - \nabla_{\mathbf{Y}} \mathbf{X} - [\mathbf{X}, \mathbf{Y}] = 0,$$

for arbitrary vector fields \mathbf{X}, \mathbf{Y} .

c)† Consider for an arbitrary connection the following operation on the vector fields:

$$S(\mathbf{X}, \mathbf{Y}) = \nabla_{\mathbf{X}} \mathbf{Y} - \nabla_{\mathbf{Y}} \mathbf{X} - [\mathbf{X}, \mathbf{Y}]$$

and find its properties⁽¹⁾.

⁽¹⁾ This equation defines $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ -tensor, *torsion*.

6 Consider the surface M in the Euclidean space \mathbf{E}^n . Calculate the induced connection in the following cases

- a) $M = S^1$ in \mathbf{E}^2 ,
- b) M — parabola $y = x^2$ in \mathbf{E}^2 ,
- c) cylinder in \mathbf{E}^3 .
- d) cone in \mathbf{E}^3 .
- e) sphere in \mathbf{E}^3 .
- f) saddle $z = xy$ in \mathbf{E}^3

7 Let ∇_1, ∇_2 be two different connections. Let ${}^{(1)}\Gamma_{km}^i$ and ${}^{(2)}\Gamma_{km}^i$ be the Christoffel symbols of connections ∇_1 and ∇_2 respectively.

a) Find the transformation law for the object : $T_{km}^i = {}^{(1)}\Gamma_{km}^i - {}^{(2)}\Gamma_{km}^i$ under a change of coordinates. Show that it is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ tensor.

8[†] Let \mathbf{K}, \mathbf{X} be vector fields on manifold M , and ∇ connection. Consider the operation

$$\mathbf{K}, \mathbf{X} \mapsto A_{\mathbf{K}}(\mathbf{X}) = \nabla_{\mathbf{K}}\mathbf{X} - \mathcal{L}_{\mathbf{K}}\mathbf{X}, (\mathcal{L} \text{ is a Lie derivative, } \mathcal{L}_{\mathbf{X}}\mathbf{Y} = [\mathbf{X}, \mathbf{Y}]) \quad (1)$$

a) Show that for an arbitrary function f , $A_{\mathbf{K}}(f\mathbf{X}) = fA_{\mathbf{K}}(\mathbf{X})$. (2)

This condition implies that equation (1) defines *linear operation on tangent vectors*, i.e. it is well defined on tangent vectors (not vector fields) and it is linear²⁾.

b) Show that linear operator $A_{\mathbf{K}}$ is equal to

$$A_{\mathbf{K}}(\mathbf{X}) = \nabla_{\mathbf{X}}\mathbf{K} + S(\mathbf{K}, \mathbf{X}),$$

where S the torsion of connection (see exercise 5c) above).

²⁾ In other words for a given vector \mathbf{X}_0 tangent to manifold M at the given point \mathbf{p} , $\mathbf{X}_0 \in T_{\mathbf{p}}M$, consider *an arbitrary vector field* \mathbf{X} passing via this vector, i.e, such that value of vector field at the given point \mathbf{p} coincides with the vector $\mathbf{X}_0, \mathbf{X}|_{\mathbf{p}} = \mathbf{X}_0$. Condition (2) tells that the answer at the point \mathbf{p} does not depend on a choice of vector field passing through vector \mathbf{X}_0 . It depends only on the value of this vector field at the point \mathbf{p}_0 . Indeed let two vector fields $\mathbf{X}, \tilde{\mathbf{X}}$ coincide at the point \mathbf{p} , i.e. the vector field $\tilde{\mathbf{X}} - \mathbf{X}$ vanishes at the point \mathbf{p} . Moreover Hadamard lemma (it states: if smooth function g vanishes at the origin, then $g = \sum_i x^i h_i(x)$, where $h_i(x)$ are also smooth.) tells that in this case vector field $\tilde{\mathbf{X}} - \mathbf{X}$ is linear combination of vector fields with coefficients vanishing at the point \mathbf{p} : $\tilde{\mathbf{X}} - \mathbf{X} = \sum_a h_a(x)\mathbf{T}_a$, where all $h_a(x)$ vanish at the point \mathbf{p} . Hence due to (2)

$$A_{\mathbf{K}}(\tilde{\mathbf{X}} - \mathbf{X})|_{\mathbf{p}} = A_{\mathbf{K}}\left(\sum_a h_a(x)\mathbf{T}_a\right)|_{\mathbf{p}} = \sum_a h_a(x)|_{\mathbf{p}} A_{\mathbf{K}}(\mathbf{T}_a) = \mathbf{0}.$$