Homework 4

1 Calculate the Christoffel symbols of the canonical flat connection in $\mathbf{E}^3$ in

a) cylindrical coordinates ($x = r\cos\varphi, y = r\sin\varphi, z = h$),

b) spherical coordinates.

(For the case b) try to make calculations at least for components $\Gamma^r_{rr}, \Gamma^r_{r\theta}, \Gamma^r_{r\varphi}, \Gamma^r_{\theta\theta}, \dots, \Gamma^r_{\varphi\varphi}$)

2 Let ∇ be an affine connection on a 2-dimensional manifold M such that in local coordinates (u, v) it is given that $\Gamma^u_{uv} = v$, $\Gamma^v_{uv} = 0$.

Calculate the vector field $\nabla_{\frac{\partial}{\partial u}}\left(u\frac{\partial}{\partial v}\right)$.

3 Let ∇ be an affine connection on the 2-dimensional manifold M such that in local coordinates (u, v)

$$\nabla_{\frac{\partial}{\partial u}}\left(u\frac{\partial}{\partial v}\right) = (1+u^2)\frac{\partial}{\partial v} + u\frac{\partial}{\partial u}\,.$$

Calculate the Christoffel symbols Γ^u_{uv} and Γ^v_{uv} of this connection.

4 a) Consider a connection such that its Christoffel symbols are symmetric in a given coordinate system: $\Gamma^i_{km} = \Gamma^i_{mk}$.

Show that they are symmetric in an arbitrary coordinate system.

b*) Show that the Christoffel symbols of connection ∇ are symmetric (in any coordinate system) if and only if

$$\nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X}, \mathbf{Y}] = 0\,,$$

for arbitrary vector fields $\mathbf{X}, \mathbf{Y}$.

c)* Consider for an arbitrary connection the following operation on the vector fields:

$$S(\mathbf{X}, \mathbf{Y}) = \nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X}, \mathbf{Y}]$$

and find its properties.

5 Consider the surface M in the Euclidean space $\mathbf{E}^n$. Calculate the induced connection in the following cases

a) $M = S^1$ in $\mathbf{E}^2$,

b) M— parabola $y = x^2$ in $\mathbf{E}^2$,

c) cylinder in $\mathbf{E}^3$.

d) cone in $\mathbf{E}^3$.

e) sphere in $\mathbf{E}^3$.

f) saddle $z = xy$ in $\mathbf{E}^3$

6 Let ∇_1, ∇_2 be two different connections. Let ${}^{(1)}\Gamma^i_{km}$ and ${}^{(2)}\Gamma^i_{km}$ be the Christoffel symbols of connections ∇_1 and ∇_2 respectively.

a) Find the transformation law for the object : $T^i_{km} = {}^{(1)}\Gamma^i_{km} - {}^{(2)}\Gamma^i_{km}$ under a change of coordinates. Show that it is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ tensor.

b)$^{*?}$ Consider an operation $\nabla_1 - \nabla_2$ on vector fields and find its properties.

7 * a) Consider $t_m = \Gamma^i_{im}$. Show that the transformation law for t_m is

$$t_{m'} = \frac{\partial\, x^m}{\partial x^{m'}} t_m + \frac{\partial^2 x^r}{\partial x^{m'} \partial x^{k'}} \frac{\partial x^{k'}}{\partial x^r}\,.$$

b) † Show that this law can be written as

$$t_{m'} = \frac{\partial\, x^m}{\partial x^{m'}} t_m + \frac{\partial}{\partial x^{m'}} \left(\log\det \left(\frac{\partial x}{\partial x'} \right) \right)\,.$$