

Riemannian Geometry

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1 Riemannian manifolds

1.1 Manifolds. Tensors. (Recollection)

1.1.1 Manifolds

I recall briefly basics of manifolds and tensor fields on manifolds.

An n -dimensional manifold is a space such that in a vicinity of an arbitrary point one can consider local coordinates $\{x^1, \dots, x^n\}$. One can consider different local coordinates. If coordinates $\{x^1, \dots, x^n\}$ and $\{x^{1'}, \dots, x^{n'}\}$ both are defined in a vicinity of the given point then they are related by *bijective transition functions* which are defined on domains in \mathbf{R}^n and taking values also in \mathbf{R}^n :

$$\begin{cases} x^{1'} = x^{1'}(x^1, \dots, x^n) \\ x^{2'} = x^{2'}(x^1, \dots, x^n) \\ \dots \\ x^{n-1'} = x^{n-1'}(x^1, \dots, x^n) \\ x^{n'} = x^{n'}(x^1, \dots, x^n) \end{cases} \quad (1.1)$$

We say that n -dimensional manifold is *differentiable* or *smooth* if all transition functions are diffeomorphisms, i.e. they are smooth. Invertability implies that Jacobian matrix is non-degenerate:

$$\det \begin{pmatrix} \frac{\partial x^{1'}}{\partial x^1} & \frac{\partial x^{1'}}{\partial x^2} & \dots & \frac{\partial x^{1'}}{\partial x^n} \\ \frac{\partial x^{2'}}{\partial x^1} & \frac{\partial x^{2'}}{\partial x^2} & \dots & \frac{\partial x^{2'}}{\partial x^n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial x^{n'}}{\partial x^1} & \frac{\partial x^{n'}}{\partial x^2} & \dots & \frac{\partial x^{n'}}{\partial x^n} \end{pmatrix} \neq 0. \quad (1.2)$$

(If bijective function $x^{i'} = x^{i'}(x^i)$ is smooth function, and its inverse, the transition function $x^i = x^i(x^{i'})$ is also smooth function, then matrices $\|\frac{\partial x^{i'}}{\partial x^i}\|$ and $\|\frac{\partial x^i}{\partial x^{i'}}\|$ are both well defined, hence condition (1.2) is obeyed.

Example

open domain in \mathbf{E}^n

A good example of manifold is an open domain D in n -dimensional vector space \mathbf{R}^n . Cartesian coordinates on \mathbf{R}^n define global coordinates on D . On the other hand one can consider an arbitrary local coordinates in different domains in \mathbf{R}^n . E.g. one can consider polar coordinates $\{r, \varphi\}$ in a domain

$D = \{x, y : y > 0\}$ of \mathbf{R}^2 defined by standard formulae:

$$\begin{cases} x = r \cos \varphi \\ y = r \sin \varphi \end{cases}, \quad (1.3)$$

$$\det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{pmatrix} = \det \begin{pmatrix} \cos \varphi & -r \sin \varphi \\ \sin \varphi & r \cos \varphi \end{pmatrix} = r \quad (1.4)$$

or one can consider spherical coordinates $\{r, \theta, \varphi\}$ in a domain $D = \{x, y, z : x > 0, y > 0, z > 0\}$ of \mathbf{R}^3 (or in other domain of \mathbf{R}^3) defined by standard formulae

$$\begin{cases} x = r \sin \theta \cos \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \theta \end{cases},$$

$$\det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{pmatrix} = \det \begin{pmatrix} \sin \theta \cos \varphi & r \cos \theta \cos \varphi & -r \sin \theta \sin \varphi \\ \sin \theta \sin \varphi & r \cos \theta \sin \varphi & r \sin \theta \cos \varphi \\ \cos \theta & -r \sin \theta & 0 \end{pmatrix} = r^2 \sin \theta \quad (1.5)$$

Choosing domain where polar (spherical) coordinates are well-defined we have to be aware that coordinates have to be well-defined and transition functions (1.1) have to obey condition (1.2), i.e. they have to be diffeomorphisms. E.g. for domain D in example (1.3) Jacobian (1.4) does not vanish if and only if $r > 0$ in D .

Consider another examples of manifolds, and local coordinates on manifolds.

Example

Circle S^1 in \mathbf{E}^2

Consider circle $x^2 + y^2 = R^2$ of radius R in \mathbf{E}^2 .

One can consider on the circle different local coordinates

i) *polar coordinate φ :*

$$\begin{cases} x = R \cos \varphi \\ y = R \sin \varphi \end{cases}, \quad 0 < \varphi < 2\pi$$

(this coordinate is defined on all the circle except a point $(R, 0)$),

ii) *another polar coordinate φ' :*

$$\begin{cases} x = R \cos \varphi \\ y = R \sin \varphi \end{cases}, \quad -\pi < \varphi < \pi,$$

this coordinate is defined on all the circle except a point $(-R, 0)$,

iii) stereographic coordinate t with respect to north pole of the circle

$$\begin{cases} x = \frac{2R^2 t}{t^2 + R^2} \\ y = R \frac{t^2 - R^2}{t^2 + R^2} \end{cases}, \quad t = \frac{Rx}{R - y}, \quad (1.6)$$

this coordinate is defined at all the circle except the north pole,

iiii) stereographic coordinate t' with respect to south pole of the circle

$$\begin{cases} x = \frac{2R^2 t'}{t'^2 + R^2} \\ z = R \frac{R^2 - t'^2}{t'^2 + R^2} \end{cases}, \quad t' = \frac{Rx}{R + z},$$

this coordinate is defined at all the points except the south pole.

One can consider transition function between these different coordinates.

E.g. polar coordinate φ coincide with polar coordinate φ' in the domain $x^2 + y^2 > 0$, and in the domain $x^2 + y^2 < 0$ $\varphi' = \varphi - 2\pi$.

Transition function from polar coordinate i) to stereographic coordinates t is $t = \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)$, and transition function from stereographic coordinate t to stereographic coordinate t' is

$$t' = \frac{R^2}{t},$$

(see Homework 0.)

Example

Sphere S^2 in \mathbf{E}^3

Consider sphere $x^2 + y^2 + z^2 = R^2$ of radius a in \mathbf{E}^3 .

One can consider on the sphere different local coordinates

i) *spherical coordinates on domain of sphere θ, φ :*

$$\begin{cases} x = a \sin \theta \cos \varphi \\ y = a \sin \theta \sin \varphi \\ z = a \cos \theta \end{cases}, \quad 0 < \theta < \pi, -\pi < \varphi < \pi$$

ii) stereographic coordinates u, v with respect to north pole of the sphere

$$\begin{cases} x = \frac{2a^2u}{a^2+u^2+v^2} \\ y = \frac{2a^2v}{a^2+u^2+v^2} \\ z = a \frac{u^2+v^2-a^2}{a^2+u^2+v^2} \end{cases}, \quad \frac{x}{u} = \frac{y}{v} = \frac{a-z}{a}, \quad \begin{cases} u = \frac{ax}{a-z} \\ v = \frac{ay}{a-z} \end{cases}.$$

iii) stereographic coordinates u', v' with respect to south pole of the sphere

$$\begin{cases} x = \frac{2a^2u'}{a^2+u'^2+v'^2} \\ y = \frac{2a^2v'}{a^2+u'^2+v'^2} \\ z = a \frac{a^2-u'^2-v'^2}{a^2+u'^2+v'^2} \end{cases}, \quad \frac{x}{u'} = \frac{y}{v'} = \frac{a+z}{a}, \quad \begin{cases} u' = \frac{ax}{a+z} \\ v' = \frac{ay}{a+z} \end{cases}.$$

(see also Homework 0)

Spherical coordinates are defined elsewhere except poles and the meridians $y = 0, x \leq 0$. Stereographical coordinates (u, v) are defined elsewhere except north pole; stereographic coordinates (u', v') are defined elsewhere except south pole. One can consider transition function between these different coordinates. E.g. transition functions from spherical coordinates i) to stereographic coordinates (u, v) are

$$\begin{cases} u = \frac{ax}{a-z} = \frac{a \sin \theta \cos \varphi}{1 - \cos \theta} = a \cotan \frac{\theta}{2} \cos \varphi \\ v = \frac{ay}{a-z} = \frac{a \sin \theta \sin \varphi}{1 - \cos \theta} = a \cotan \frac{\theta}{2} \sin \varphi \end{cases},$$

and transition function from stereographic coordinates u, v to stereographic coordinates (u', v') are

$$\begin{cases} u' = \frac{a^2u}{u^2+v^2} \\ v' = \frac{a^2v}{u^2+v^2} \end{cases},$$

(see Homework 0.)

[†] One very important property of stereographic projection which we do not use in this course but it is too beautiful not to mention it: under stereographic projection all points on the circle $x^2 + y^2 = 1$ with rational coordinates x and y and only these points transform to rational points on line. Thus we come to Pythagorean triples $a^2 + b^2 = c^2$.

1.1.2 Tensors on Manifold

Recall briefly what are tensors on manifold. For every point \mathbf{p} on manifold M one can consider tangent vector space $T_{\mathbf{p}}M$ — the space of vectors tangent to the manifold at the point M .

Tangent vector $\mathbf{A}(x) = A^i(x) \frac{\partial}{\partial x^i}$. Under changing of coordinates it transforms as follows:

$$\mathbf{A} = A^i(x) \frac{\partial}{\partial x^i} = A^i(x) \frac{\partial x^{m'}(x)}{\partial x^i} \frac{\partial}{\partial x^{m'}} = A^{m'}(x'(x)) \frac{\partial}{\partial x^{m'}}.$$

Hence

$$A^{i'}(x') = \frac{\partial x^{i'}(x)}{\partial x^i} A^i(x). \quad (1.7)$$

Consider also cotangent space $T_{\mathbf{p}}^*M$ (for every point \mathbf{p} on manifold M)—space of linear functions on tangent vectors, i.e. space of 1-forms which sometimes are called *covectors*:

One-form (covector) $\omega = \omega_i(x) dx^i$ transforms as follows

$$\omega = \omega_m(x) dx^m = \omega_m \frac{\partial x^m(x')}{\partial x^{m'}} dx^{m'} = \omega_{m'}(x') dx^{m'}.$$

Hence

$$\omega_{m'}(x') = \frac{\partial x^m(x')}{\partial x^{m'}} \omega_m(x). \quad (1.8)$$

Differential form sometimes is called *covector*.

Tensors:

Definition Consider geometrical object such that in arbitrary local coordinates (x^i) it is given by components

$$Q = \left\{ Q_{j_1 j_2 \dots j_q}^{i_1 i_2 \dots i_p} \frac{\partial}{\partial x^{i_1}}(x) \right\}, i_1, \dots, i_p; j_1, \dots, j_q = 1, 2, \dots, n,$$

and under changing of coordinates this object is transformed in the following way:

$$Q_{j'_1 j'_2 \dots j'_q}^{i'_1 i'_2 \dots i'_p}(x') = \frac{\partial x^{i'_1}}{\partial x^{i_1}} \frac{\partial x^{i'_2}}{\partial x^{i_2}} \dots \frac{\partial x^{i'_p}}{\partial x^{i_p}} \frac{\partial x^{j_1}}{\partial x^{j'_1}} \frac{\partial x^{j_2}}{\partial x^{j'_2}} \dots \frac{\partial x^{j_q}}{\partial x^{j'_q}} Q_{j_1 j_2 \dots j_q}^{i_1 i_2 \dots i_p}(x). \quad (1.9)$$

We say that this is *p-times contravariant, q-times covariant tensor of valence $\begin{pmatrix} p \\ q \end{pmatrix}$* , or shorter, *tensors of the type $\begin{pmatrix} p \\ q \end{pmatrix}$* .

Examples

Note that vector field (1.7) is nothing but tensor field of valency $\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and 1-form (1.8) is nothing but tensor field of valency $\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

One can consider *contravariant* tensors of the rank p

$$T = T^{i_1 i_2 \dots i_p}(x) \frac{\partial}{\partial x^{i_1}} \otimes \frac{\partial}{\partial x^{i_2}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_p}}$$

with components $\{T^{i_1 i_2 \dots i_p}\}(x)$. Under changing of coordinates $(x^1, \dots, x^n) \rightarrow (x'^1, \dots, x'^n)$ (see (1.1)) they transform as follows:

$$T^{i'_1 i'_2 \dots i'_p}(x') = \frac{\partial x^{i'_1}}{\partial x^{i_1}} \frac{\partial x^{i'_2}}{\partial x^{i_2}} \dots \frac{\partial x^{i'_p}}{\partial x^{i_p}} T^{i_1 i_2 \dots i_p}(x). \quad (1.10)$$

One can consider *covariant* tensors of the rank q

$$S = S_{j_1 j_2 \dots j_q} dx^{j_1} \otimes dx^{j_2} \otimes \dots \otimes dx^{j_q}$$

with components $\{S_{j_1 j_2 \dots j_q}\}$. Under changing of coordinates $(x^1, \dots, x^n) \rightarrow (x'^1, \dots, x'^n)$ they transform as follows:

$$S_{j'_1 j'_2 \dots j'_q}(x') = \frac{\partial x^{j_1}}{\partial x^{j'_1}} \frac{\partial x^{j_2}}{\partial x^{j'_2}} \dots \frac{\partial x^{j_q}}{\partial x^{j'_q}} S_{j_1 j_2 \dots j_q}(x).$$

E.g. if S_{ik} is a covariant tensor of rank 2 then

$$S_{i'k'}(x') = \frac{\partial x^i(x')}{\partial x^{i'}} \frac{\partial x^k(x')}{\partial x^{k'}} S_{ik}(x). \quad (1.11)$$

If A_k^i is a tensor of rank $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (linear operator on $T_{\mathbf{p}}M$) then

$$A_{k'}^{i'}(x') = \frac{\partial x^{i'}(x')}{\partial x^i} \frac{\partial x^k(x')}{\partial x^{k'}} A_k^i(x).$$

If S_{ik}^m is a tensor of the type $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ then

$$S_{i'k'}^{m'} = \frac{\partial x^{m'}}{\partial x^m} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^k}{\partial x^{k'}} S_{ik}^m(x). \quad (1.12)$$

Transformations formulae (1.7)—(1.12) define vectors, covectors and in generally any tensor fields in components.

Remark *Einstein summation rules*

In our lectures we always use so called *Einstein summation convention*. it implies that when an index occurs twice in the same expression in upper and in lower positions, then the expression is implicitly summed over all possible values for that index. Sometimes it is called dummy indices summation rule.

Looking on righthand sides of equations (1.7)—(1.12) we see that Einstein rules help essentially to write down formulae correctly.

1.2 Riemannian manifold

1.2.1 Riemannian manifold— manifold equipped with Riemannian metric

Definition The Riemannian manifold (M, G) is a manifold equipped with a Riemannian metric.

The Riemannian metric G on the manifold M defines the length of the tangent vectors and the length of the curves.

Definition Riemannian metric G on n -dimensional manifold M^n defines for every point $\mathbf{p} \in M$ the scalar product of tangent vectors in the tangent space $T_{\mathbf{p}}M$ smoothly depending on the point \mathbf{p} .

It means that in every coordinate system (x^1, \dots, x^n) a metric $G = g_{ik} dx^i dx^k$ is defined by a matrix valued smooth function $g_{ik}(x)$ ($i = 1, \dots, n; k = 1, \dots, n$) such that for any two vectors

$$\mathbf{A} = A^i(x) \frac{\partial}{\partial x^i}, \quad \mathbf{B} = B^i(x) \frac{\partial}{\partial x^i},$$

tangent to the manifold M at the point \mathbf{p} with coordinates $x = (x^1, x^2, \dots, x^n)$ ($\mathbf{A}, \mathbf{B} \in T_{\mathbf{p}}M$) the scalar product is equal to:

$$\langle \mathbf{A}, \mathbf{B} \rangle_G \big|_{\mathbf{p}} = G(\mathbf{A}, \mathbf{B}) \big|_{\mathbf{p}} = A^i(x) g_{ik}(x) B^k(x) =$$

$$(A^1 \dots A^n) \begin{pmatrix} g_{11}(x) & \dots & g_{1n}(x) \\ \dots & \dots & \dots \\ g_{n1}(x) & \dots & g_{nn}(x) \end{pmatrix} \begin{pmatrix} B^1 \\ \cdot \\ \cdot \\ \cdot \\ B^n \end{pmatrix} \quad (1.13)$$

where

- $G(\mathbf{A}, \mathbf{B}) = G(\mathbf{B}, \mathbf{A})$, i.e. $g_{ik}(x) = g_{ki}(x)$ (symmetricity condition)
- $G(\mathbf{A}, \mathbf{A}) > 0$ if $\mathbf{A} \neq \mathbf{0}$, i.e.
 $g_{ik}(x)u^i u^k \geq 0$, $g_{ik}(x)u^i u^k = 0$ iff $u^1 = \dots = u^n = 0$ (positive-definiteness)
- $G(\mathbf{A}, \mathbf{B})|_{\mathbf{p}=x}$, i.e. $g_{ik}(x)$ are smooth functions.

The matrix $\|g_{ik}\|$ of components of the metric G we also sometimes denote by G .

Now we establish rule of transformation for entries of matrix $g_{ik}(x)$, of metric G .

Notice that an arbitrary matrix entry g_{ik} is nothing but scalar product of vectors ∂_i, ∂_k at the given point:

$$g_{ik}(x) = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k} \right\rangle, \quad \text{in coordinates } (x^1, \dots, x^n) \quad (1.14)$$

Use this formula for establishing rule of transformations of $g_{ik}(x)$. In the new coordinates $x^{i'} = (x^1, \dots, x^n)$ according this formula we have that

$$g_{i'k'}(x') = \left\langle \frac{\partial}{\partial x^{i'}}, \frac{\partial}{\partial x^{k'}} \right\rangle, \quad \text{in coordinates } (x^1, \dots, x^n).$$

Now using chain rule, linearity of scalar product and formula (1.14) we see that

$$\begin{aligned} g_{i'k'}(x') &= \left\langle \frac{\partial}{\partial x^{i'}}, \frac{\partial}{\partial x^{k'}} \right\rangle = \left\langle \frac{\partial x^i}{\partial x^{i'}} \frac{\partial}{\partial x^i}, \frac{\partial x^k}{\partial x^{k'}} \frac{\partial}{\partial x^k} \right\rangle \\ &= \frac{\partial x^i}{\partial x^{i'}} \underbrace{\left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k} \right\rangle}_{g_{ik}(x)} \frac{\partial x^k}{\partial x^{k'}} = \frac{\partial x^i}{\partial x^{i'}} g_{ik}(x) \frac{\partial x^k}{\partial x^{k'}} \end{aligned} \quad (1.15)$$

This transformation law implies that g_{ik} entries of matrix $\|g_{ik}\|$ are components of *covariant tensor field* $G = g_{ik}dx^i dx^k$ of rank 2 (see equation (1.11)).

One can say that *Riemannian metric* is defined by symmetric covariant smooth tensor field G of the rank 2 which defines scalar product in the tangent spaces $T_{\mathbf{p}}M$ smoothly depending on the point \mathbf{p} . Components of tensor field G in coordinate system are functions $g_{ik}(x)$:

$$G = g_{ik}(x)dx^i \otimes dx^k,$$

$$\langle \mathbf{A}, \mathbf{B} \rangle = G(\mathbf{A}, \mathbf{B}) = g_{ik}(x) dx^i \otimes dx^k (\mathbf{A}, \mathbf{B}) . \quad (1.16)$$

In practice it is more convenient to perform transformation of metric G under changing of coordinates in the following way:

$$G = g_{ik} dx^i \otimes dx^k = g_{ik} \left(\frac{\partial x^i}{\partial x^{i'}} dx^{i'} \right) \otimes \left(\frac{\partial x^k}{\partial x^{k'}} dx^{k'} \right) =$$

$$\frac{\partial x^i}{\partial x^{i'}} g_{ik} \frac{\partial x^k}{\partial x^{k'}} dx^{i'} \otimes dx^{k'} = g_{i'k'} dx^{i'} \otimes dx^{k'} , \text{ hence } g_{i'k'} = \frac{\partial x^i}{\partial x^{i'}} g_{ik} \frac{\partial x^k}{\partial x^{k'}} . \quad (1.17)$$

We come to transformation rule (1.15).

Later by some abuse of notations we sometimes omit the sign of tensor product and write a metric just as

$$G = g_{ik}(x) dx^i dx^k .$$

1.2.2 Examples

- \mathbf{R}^n with canonical coordinates $\{x^i\}$ and with metric

$$G = (dx^1)^2 + (dx^2)^2 + \dots + (dx^n)^2$$

$$G = ||g_{ik}|| = \text{diag} [1, 1, \dots, 1]$$

Recall that this is a basis example of n -dimensional Euclidean space \mathbf{E}^n , where scalar product is defined by the formula:

$$G(\mathbf{X}, \mathbf{Y}) = \langle \mathbf{X}, \mathbf{Y} \rangle = g_{ik} X^i Y^k = X^1 Y^1 + X^2 Y^2 + \dots + X^n Y^n .$$

In the general case if $G = ||g_{ik}||$ is an arbitrary symmetric positive-definite metric then $G(\mathbf{X}, \mathbf{Y}) = \langle \mathbf{X}, \mathbf{Y} \rangle = g_{ik} X^i Y^k$. One can show that there exists a new basis $\{\mathbf{e}_i\}$ such that in this basis $G(\mathbf{e}_i, \mathbf{e}_k) = \delta_{ik}$. This basis is called orthonormal basis. (See the Lecture notes in Geometry)

Scalar product in vector space defines the *same* scalar product at all the points. In general case for Riemannian manifold scalar product depends on a point. In Riemannian manifold we consider arbitrary transformations from local coordinates to new local coordinates.

- Euclidean space \mathbf{E}^2 with polar coordinates in the domain $y > 0$ ($x = r \cos \varphi, y = r \sin \varphi$):

$dx = \cos \varphi dr - r \sin \varphi d\varphi$, $dy = \sin \varphi dr + r \cos \varphi d\varphi$. In new coordinates the Riemannian metric $G = dx^2 + dy^2$ will have the following appearance:

$$G = (dx)^2 + (dy)^2 = (\cos \varphi dr - r \sin \varphi d\varphi)^2 + (\sin \varphi dr + r \cos \varphi d\varphi)^2 = dr^2 + r^2 (d\varphi)^2$$

We see that for matrix $G = ||g_{ik}||$

$$\underbrace{G = \begin{pmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{in Cartesian coordinates}}, \quad \underbrace{G = \begin{pmatrix} g_{rr} & g_{r\varphi} \\ g_{\varphi r} & g_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}}_{\text{in polar coordinates}}$$

- Circle

Interval $[0, 2\pi)$ in the line $0 \leq x < 2\pi$ with Riemannian metric

$$G = a^2 dx^2 \tag{1.18}$$

Renaming $x \mapsto \varphi$ we come to habitual formula for metric for circle of the radius a : $x^2 + y^2 = a^2$ embedded in the Euclidean space \mathbf{E}^2 :

$$G = a^2 d\varphi^2 \quad \begin{cases} x = a \cos \varphi \\ y = a \sin \varphi \end{cases}, 0 < \varphi < 2\pi, \quad \text{or} \quad -\pi < \varphi < \pi. \tag{1.19}$$

Rewrite this metric in stereographic coordinate t :

$$G = a^2 d\varphi^2 = 4a^4 dt^2 (a^2 + t^2)^{-2}, \quad \text{where } t = \frac{ax}{a - y} = \frac{a^2 \cos \varphi}{a - a \sin \varphi} = \tan \left(\frac{\pi}{4} + \frac{\varphi}{2} \right). \tag{1.20}$$

(See (1.6) and Homeworks 0 and 2.)

- Cylinder surface

Consider domain in \mathbf{R}^2 , $D = \{(x, y) : 0 \leq x < 2\pi\}$ with Riemannian metric

$$G = a^2 dx^2 + dy^2 \tag{1.21}$$

We see that renaming variables $x \mapsto \varphi$, $y \mapsto h$ we come to habitual, familiar formulae for metric in standard polar coordinates for cylinder

surface of the radius a embedded in the Euclidean space \mathbf{E}^3 :

$$G = a^2 d\varphi^2 + dh^2 \quad \begin{cases} x = a \cos \varphi \\ y = a \sin \varphi \\ z = h \end{cases}, 0 < \varphi < 2\pi, -\infty < h < \infty \quad (1.22)$$

(Coordinate φ is well defined for $-\pi < \varphi < \pi$ also.)

- Sphere

Consider domain in \mathbf{R}^2 , $0 < x < 2\pi$, $0 < y < \pi$ with metric $G = dy^2 + \sin^2 y dx^2$. We see that renaming variables $x \mapsto \varphi$, $y \mapsto h$ we come to habitual, familiar formulae for metric in standard spherical coordinates for sphere $x^2 + y^2 + z^2 = a^2$ of the radius a embedded in the Euclidean space \mathbf{E}^3 :

$$G = a^2 d\theta^2 + a^2 \sin^2 \theta d\varphi^2 \quad \begin{cases} x = a \sin \theta \cos \varphi \\ y = a \sin \theta \sin \varphi \\ z = a \cos \theta \end{cases}, \quad 0 < \theta < \pi, 0 < \varphi < 2\pi. \quad (1.23)$$

(See examples also in the Homeworks.)

If we omit the condition of positive-definiteness for Riemannian metric we come to so called *Pseudoriemannian metric*. Manifold equipped with pseudoriemannian metric is called pseudoriemannian manifold. Pseudoriemannian manifolds appear in applications in the special and general relativity theory.

In pseudoriemannian space scalar product (\mathbf{X}, \mathbf{X}) may take an arbitrary real values: it can be positive, negative, it can be equal to zero. Vectors \mathbf{X} such that $(\mathbf{X}, \mathbf{X}) = 0$ are called null-vectors.

For example consider 4-dimensional linear space \mathbf{R}^4 with pseudometric

$$G = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2.$$

For an arbitrary vector $\mathbf{X} = (a^0, a^1, a^2, a^3)$ scalar product (\mathbf{X}, \mathbf{X}) is positive if $(a^0)^2 > (a^1)^2 + (a^2)^2 + (a^3)^2$, and it is negative if $(a^0)^2 < (a^1)^2 + (a^2)^2 + (a^3)^2$, and \mathbf{X} is null-vector if $(a^0)^2 = (a^1)^2 + (a^2)^2 + (a^3)^2$. It is so called Minkovski space. The coordinate x^0 plays a role of the time: $x^0 = ct$, where c is the value of the speed of the light. Vectors \mathbf{X} such that $(\mathbf{X}, \mathbf{X}) > 0$ are called time-like vectors and they called space-like vectors if $(\mathbf{X}, \mathbf{X}) < 0$.

1.2.3 Scalar product \rightarrow Length of tangent vectors and angle between them

The Riemannian metric defines scalar product of tangent vectors attached at the given point. Hence it defines the length of tangent vectors and angle between them. If $\mathbf{X} = X^m \frac{\partial}{\partial x^m}$, $\mathbf{Y} = Y^m \frac{\partial}{\partial x^m}$ are two tangent vectors at the given point \mathbf{p} of Riemannian manifold with coordinates x^1, \dots, x^n , then we have that lengths of these vectors equal to

$$|\mathbf{X}| = \sqrt{\langle \mathbf{X}, \mathbf{X} \rangle} = \sqrt{g_{ik}(x) X^i X^k}, \quad |\mathbf{Y}| = \sqrt{\langle \mathbf{Y}, \mathbf{Y} \rangle} = \sqrt{g_{ik}(x) Y^i Y^k}, \quad (1.24)$$

and an ‘angle’ θ between these vectors is defined by the relation

$$\cos \theta = \frac{\langle \mathbf{X}, \mathbf{Y} \rangle}{|\mathbf{X}| \cdot |\mathbf{Y}|} = \frac{g_{ik} X^i Y^k}{\sqrt{g_{ik}(x) X^i X^k} \sqrt{g_{ik}(x) Y^i Y^k}} \quad (1.25)$$

Remark We say ‘angle’ but we calculate just cosinus of angle.

Example Let M be 3-dimensional Riemannian manifold, and $\mathbf{p} \in M$ a point in it. Suppose that the manifold M is equipped with local coordinates x, y, z in a vicinity of this point, and the expression of Riemannian metric in these local coordinates is

$$G = \frac{dx^2 + dy^2 + dz^2}{(1 + x^2 + y^2 + z^2)^2}. \quad (1.26)$$

Consider the vectors $\mathbf{X} = a\partial_x + b\partial_y + c\partial_z$ and $\mathbf{Y} = p\partial_x + q\partial_y + r\partial_z$, attached at the point \mathbf{p} , with coordinates $x = 2, y = 2, z = 1$. Find the lengths of vectors \mathbf{X} and \mathbf{Y} and find cosinus of the angle between these vectors.

We see that matrix of Riemannian metric is

$$||g_{ik}(x)|| = \begin{pmatrix} \frac{1}{(1+x^2+y^2+z^2)^2} & 0 & 0 \\ 0 & \frac{1}{(1+x^2+y^2+z^2)^2} & 0 \\ 0 & 0 & \frac{1}{(1+x^2+y^2+z^2)^2} \end{pmatrix} \text{ i. e. } g_{ik}(x, y, z) = \frac{\delta_{ik}}{(1 + x^2 + y^2 + z^2)^2},$$

where $g_{ik}(x)$ are entries of matrix: $G = g_{ik}(x) dx^i dx^k$, (δ_{ik} is Kronecker symbol: $\delta_{ik} = 1$ if $i = k$ and it vanishes otherwise).

According to formulae above

$$|\mathbf{X}| = \sqrt{\langle \mathbf{X}, \mathbf{X} \rangle} = \sqrt{g_{ik}(x, y, z) X^i X^k} \Big|_{\mathbf{p}} = \sqrt{\frac{\delta_{ik} X^i X^k}{(1 + x^2 + y^2 + z^2)^2}} \Big|_{x=2, y=2, z=1} =$$

$$\sqrt{\frac{a^2 + b^2 + c^2}{(1 + 2^2 + 2^2 + 1^2)^2}} = \frac{\sqrt{a^2 + b^2 + c^2}}{10},$$

$$|\mathbf{Y}| = \sqrt{\langle \mathbf{Y}, \mathbf{Y} \rangle} = \sqrt{g_{ik}(x, y, z) Y^i Y^k} \Big|_{\mathbf{p}} = \sqrt{\frac{\delta_{ik} Y^i Y^k}{(1 + x^2 + y^2 + z^2)^2}} \Big|_{x=2, y=2, z=1} =$$

$$\sqrt{\frac{p^2 + q^2 + r^2}{(1 + 2^2 + 2^2 + 1^2)^2}} = \frac{\sqrt{p^2 + q^2 + r^2}}{10},$$

and

$$\cos \theta = \frac{\langle \mathbf{X}, \mathbf{Y} \rangle}{|\mathbf{X}| |\mathbf{Y}|} = \frac{g_{ik}(x, y, z) X^i Y^k \Big|_{\mathbf{p}}}{\sqrt{g_{pq}(x, y, z) X^p X^q} \sqrt{g_{rs}(x, y, z) Y^r Y^s}} = \frac{\frac{\delta_{ik} X^i Y^k}{(1 + x^2 + y^2 + z^2)^2}}{|\mathbf{X}| |\mathbf{Y}|}$$

$$= \frac{\frac{ap + bq + cr}{(1 + 2^2 + 2^2 + 1^2)^2}}{\frac{\sqrt{a^2 + b^2 + c^2}}{10} \frac{\sqrt{p^2 + q^2 + r^2}}{10}} = \frac{ap + bq + cr}{\sqrt{a^2 + b^2 + c^2} \sqrt{p^2 + q^2 + r^2}}.$$

This example is related with the notion of so called *conformally euclidean metric* (see paragraph 1.2.5).

1.2.4 Length of curves

Let $\gamma: x^i = x^i(t), (i = 1, \dots, n)$ ($a \leq t \leq b$) be a curve on the Riemannian manifold (M, G) .

At the every point of the curve the velocity vector (tangent vector) is defined:

$$\mathbf{v}(t) = \begin{pmatrix} \dot{x}^1(t) \\ \vdots \\ \dot{x}^n(t) \end{pmatrix}$$

The length of velocity vector $\mathbf{v} \in T_x M$ (vector \mathbf{v} is tangent to the manifold M at the point x) equals to

$$|\mathbf{v}|_x = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle_G} \Big|_x = \sqrt{g_{ik} v^i v^k} \Big|_x = \sqrt{g_{ik} \frac{dx^i(t)}{dt} \frac{dx^k(t)}{dt}} \Big|_x.$$

For an arbitrary curve its length is equal to the integral of the length of velocity vector:

$$L_\gamma = \int_a^b \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle_G|_{x(t)}} dt = \int_a^b \sqrt{g_{ik}(x(t)) \dot{x}^i(t) \dot{x}^k(t)} dt. \quad (1.27)$$

Bearing in mind that metric (1.16) defines the length we often write metric in the following form

$$G = ds^2 = g_{ik} dx^i dx^k$$

For example consider 2-dimensional Riemannian manifold with metric

$$||g_{ik}(u, v)|| = \begin{pmatrix} g_{11}(u, v) & g_{12}(u, v) \\ g_{21}(u, v) & g_{22}(u, v) \end{pmatrix}.$$

Then

$$G = ds^2 = g_{ik} du^i dv^k = g_{11}(u, v) du^2 + 2g_{12}(u, v) du dv + g_{22}(u, v) dv^2.$$

The length of the curve $\gamma: u = u(t), v = v(t)$, where $t_0 \leq t \leq t_1$ according to (1.27) is equal to $L_\gamma = \int_{t_0}^{t_1} \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \int_{t_0}^{t_1} \sqrt{g_{ik}(x) \dot{x}^i \dot{x}^k} =$

$$\int_{t_0}^{t_1} \sqrt{g_{11}(u(t), v(t)) u_t^2 + 2g_{12}(u(t), v(t)) u_t v_t + g_{22}(u(t), v(t)) v_t^2} dt. \quad (1.28)$$

The length of curves defined by the formula(1.27) obeys the following natural conditions

- It coincides with the usual length in the Euclidean space \mathbf{E}^n (\mathbf{R}^n with standard metric $G = (dx^1)^2 + \dots + (dx^n)^2$ in Cartesian coordinates). E.g. for 3-dimensional Euclidean space

$$L_\gamma = \int_a^b \sqrt{g_{ik}(x(t)) \dot{x}^i(t) \dot{x}^k(t)} dt = \int_a^b \sqrt{(\dot{x}^1(t))^2 + (\dot{x}^2(t))^2 + (\dot{x}^3(t))^2} dt$$

- It does not depend on parameterisation of the curve

$$L_\gamma = \int_a^b \sqrt{g_{ik}(x(t)) \dot{x}^i(t) \dot{x}^k(t)} dt = \int_{a'}^{b'} \sqrt{g_{ik}(x(\tau)) \dot{x}^i(\tau) \dot{x}^k(\tau)} d\tau,$$

($x^i(\tau) = x^i(t(\tau))$, $a' \leq \tau \leq b'$ while $a \leq t \leq b$) since under changing of parameterisation

$$\dot{x}^i(\tau) = \frac{dx^i(t(\tau))}{d\tau} = \frac{dx^i(t(\tau))}{dt} \frac{dt}{d\tau} = \dot{x}^i(t) \frac{dt}{d\tau}.$$

- It does not depend on coordinates on Riemannian manifold M

$$L_\gamma = \int_a^b \sqrt{g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t)}dt = \int_a^b \sqrt{g_{i'k'}(x'(t))\dot{x}^{i'}(t)\dot{x}^{k'}(t)}dt.$$

This immediately follows from transformation rule (1.69) for Riemannian metric:

$$g_{i'k'}\dot{x}^{i'}(t)\dot{x}^{k'}(t) = g_{ik}\left(\frac{\partial x^i}{\partial x^{i'}}\dot{x}^{i'}(t)\right)\left(\frac{\partial x^k}{\partial x^{k'}}\dot{x}^{k'}(t)\right)g_{ik}\dot{x}^i(t)\dot{x}^k(t).$$

- It is additive: length of the sum of two curves is equal to the sum of their lengths. If a curve $\gamma = \gamma_1 + \gamma_2$, i.e. $\gamma: x^i(t), a \leq t \leq b$, $\gamma_1: x^i(t), a \leq t \leq c$ and $\gamma_2: x^i(t), c \leq t \leq b$ where a point c belongs to the interval (a, b) then $L_\gamma = L_{\gamma_1} + L_{\gamma_2}$.

One can show that formula (1.27) for length is defined uniquely by these conditions. More precisely one can show under some technical conditions one may show that any local additive functional on curves which does not depend on coordinates and parameterisation, and depends on derivatives of curves of order ≤ 1 is equal to (1.27) up to a constant multiplier. To feel the taste of this statement you may do the following exercise:

Exercise Let $A = A\left(x(t), y(t), \frac{dx(t)}{dt}, \frac{dy(t)}{dt}\right)$ be a function such that an integral $L = \int A\left(x(t), y(t), \frac{dx(t)}{dt}, \frac{dy(t)}{dt}\right) dt$ over an arbitrary curve γ in \mathbf{E}^2 does not change under reparameterisation of this curve and under an arbitrary isometry, i.e. translation and rotation of the curve. Then one can easily show (show it!) that

$$A\left(x(t), y(t), \frac{dx(t)}{dt}, \frac{dy(t)}{dt}\right) = c\sqrt{\left(\frac{dx(t)}{dt}\right)^2 + \left(\frac{dy(t)}{dt}\right)^2},$$

where c is a constant, i.e. it is a usual length up to a multiplier

1.2.5 Conformally Euclidean metric

Let (M, G) be a Riemannian manifold.

Definition We say that metric G is locally conformally Euclidean in a vicinity of the point \mathbf{p} if in a vicinity of this point there exist local coordinates $\{x^i\}$ such that in these coordinates metric has an appearance

$$G = \sigma(x)\delta_{ik}dx^i dx^k = \sigma(x)\left((dx^1)^2 + \dots + (dx^n)^2\right), \quad (1.29)$$

i.e. it is proportional to ‘Euclidean metric’. We call coordinates $\{x^i\}$ *conformal* coordinates or *isothermic* coordinates if condition (1.29) holds.

We say that metric is conformally Euclidean if it is locally conformally Euclidean in the vicinity of every point. We say that Riemannian manifold

(M, G) is conformally Euclidean if the metric G on it is conformally Euclidean¹.

It is evident that coefficient $\sigma(x)$ in (1.29) has to be positive. It is convenient sometimes to denote it as $\sigma(x) = e^{\lambda(x)}$,

$$G = \sigma(x) \delta_{ik} dx^i dx^k = e^{\lambda(x)} \delta_{ik} dx^i dx^k, \quad (1.30)$$

One can see that if metric is conformally Euclidean in a vicinity of some point \mathbf{p} , then the angle between vectors, more precisely the cosinus of the angle between vectors attached at this point (see equation (1.25)) is the same as for Euclidean metric. Indeed, let G be conformally Euclidean metric and let x^i be local coordinates such that the metric has an appearance (1.29) in these coordinates. Let \mathbf{X}, \mathbf{Y} be two non-vanishing vectors $\mathbf{X} = X^m(x) \frac{\partial}{\partial x^m}$, $\mathbf{Y} = Y^m(x) \frac{\partial}{\partial x^m}$ ($|\mathbf{X}| \neq 0, |\mathbf{Y}| \neq 0$) attached at a same point. Then

$$\begin{aligned} \cos \theta &= \frac{\langle \mathbf{X}, \mathbf{Y} \rangle}{|\mathbf{X}| \cdot |\mathbf{Y}|} = \frac{g_{ik} X^i Y^k}{\sqrt{g_{ik}(x) X^i X^k} \sqrt{g_{ik}(x) Y^i Y^k}} = \\ &= \frac{\lambda(x) \delta_{ik} X^i Y^k}{\sqrt{\lambda(x) \delta_{ik}(x) X^i X^k} \sqrt{\lambda(x) \delta_{ik}(x) Y^i Y^k}} = \frac{\sum_k X^k Y^k}{\sqrt{\sum_k (x) X^k X^k} \sqrt{\sum_k Y^k Y^k}}. \end{aligned} \quad (1.31)$$

It is instructive to recall the example considered in subsection 1.2.3), where Riemannian metric in a vicinity of a point had an appearance (1.26). This is example of Riemannian manifold which is locally conformally Euclidean in a vicinity of a point \mathbf{p} .

1.3 Riemannian structure on the surfaces embedded in Euclidean space

Let M be a surface embedded in Euclidean space. Let G be Riemannian structure on the manifold M .

Let \mathbf{X}, \mathbf{Y} be two vectors tangent to the surface M at a point $\mathbf{p} \in M$. An External Observer calculate this scalar product viewing these two vectors as vectors in \mathbf{E}^3 attached at the point $\mathbf{p} \in \mathbf{E}^3$ using scalar product in \mathbf{E}^3 . An Internal Observer will calculate the scalar product viewing these two vectors

¹One can show that arbitrary 2-dimensional Riemannian manifold is conformally Euclidean, i.e. in a vicinity of arbitrary point there exist conformal coordinates.

as vectors tangent to the surface M using the Riemannian metric G (see the formula (1.36)). Respectively

If L is a curve in M then an External Observer consider this curve as a curve in \mathbf{E}^3 , calculate the modulus of velocity vector (speed) and the length of the curve using Euclidean scalar product of ambient space. An Internal Observer ("an ant") will define the modulus of the velocity vector and the length of the curve using Riemannian metric.

Definition Let M be a surface embedded in the Euclidean space. We say that metric G_M on the surface is induced by the Euclidean metric if the scalar product of arbitrary two vectors $\mathbf{A}, \mathbf{B} \in T_{\mathbf{p}}M$ calculated in terms of the metric G equals to Euclidean scalar product of these two vectors:

$$\langle \mathbf{A}, \mathbf{B} \rangle_{G_M} = \langle \mathbf{A}, \mathbf{B} \rangle_{G_{\text{Euclidean}}} \quad (1.32)$$

In other words we say that Riemannian metric on the embedded surface is induced by the Euclidean structure of the ambient space if External and Internal Observers come to the same results calculating scalar product of vectors tangent to the surface.

In this case modulus of velocity vector (speed) and the length of the curve is the same for External and Internal Observer.

1.3.1 Internal and external observers

Tangent vectors, coordinate tangent vectors

Here we recall basic notions from the course of Geometry which we will need here.

Let $\mathbf{r} = \mathbf{r}(u, v)$ be parameterisation of the surface M embedded in the Euclidean space:

$$\mathbf{r}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}$$

Here as always x, y, z are Cartesian coordinates in \mathbf{E}^3 .

Let \mathbf{p} be an arbitrary point on the surface M . Consider the plane formed by the vectors which are adjusted to the point \mathbf{p} and tangent to the surface M . We call this plane *plane tangent to M at the point \mathbf{p}* and denote it by $T_{\mathbf{p}}M$.

For a point $\mathbf{p} \in M$ one can consider a basis in the tangent plane $T_{\mathbf{p}}M$ adjusted to the parameters u, v .

Tangent basis vectors at any point (u, v) are

$$\mathbf{r}_u = \frac{\partial \mathbf{r}(u, v)}{\partial u} = \begin{pmatrix} \frac{\partial x(u, v)}{\partial u} \\ \frac{\partial y(u, v)}{\partial u} \\ \frac{\partial z(u, v)}{\partial u} \end{pmatrix} = \frac{\partial x(u, v)}{\partial u} \frac{\partial}{\partial x} + \frac{\partial y(u, v)}{\partial u} \frac{\partial}{\partial y} + \frac{\partial z(u, v)}{\partial u} \frac{\partial}{\partial z}$$

Definition We call basis vectors $\mathbf{r}_u, \mathbf{r}_v$ adjusted to parameters (coordinates) u, v *coordinate basis vectors*

Every vector $\mathbf{X} \in T_p M$ can be expanded over the basis of coordinate basis vectors:

$$\mathbf{X} = X_u \mathbf{r}_u + X_v \mathbf{r}_v,$$

where X_u, X_v are coefficients, components of the vector \mathbf{X} .

Internal Observer views the basis vector $\mathbf{r}_u \in T_p M$, as a velocity vector for the curve $u = u_0 + t, v = v_0$, where (u_0, v_0) are coordinates of the point \mathbf{p} , and Internal Observer denotes this vector ∂_u . Respectively the basis vector $\mathbf{r}_v \in T_p M$ for an Internal Observer, is velocity vector for the curve $u = u_0, v = v_0 + t$, where (u_0, v_0) are coordinates of the point \mathbf{p} , and Internal Observer denotes this vector ∂_v .

Example Consider sphere of radius R in \mathbf{E}^3 , $x^2 + y^2 + z^2 = R^2$. In spherical coordinates

$$\mathbf{E}^3 \ni \mathbf{r} = \mathbf{r}(\theta, \varphi) \begin{cases} x = R \sin \theta \cos \varphi \\ y = R \sin \theta \sin \varphi \\ z = R \cos \theta \end{cases},$$

these coordinates are well-defined for $0 < \theta < \frac{\pi}{2}$ and $0 < \varphi < 2\pi$. For coordinate basis vectors \mathbf{r}_θ and \mathbf{r}_φ we have:

$$\mathbf{r}_\theta = \frac{\partial \mathbf{r}(\theta, \varphi)}{\partial \theta} = \frac{\partial}{\partial \theta} \begin{pmatrix} x(\theta, \varphi) \\ y(\theta, \varphi) \\ z(\theta, \varphi) \end{pmatrix} = \frac{\partial}{\partial \theta} \begin{pmatrix} R \sin \theta \cos \varphi \\ R \sin \theta \sin \varphi \\ R \cos \theta \end{pmatrix} =$$

$$\begin{pmatrix} R \cos \theta \cos \varphi \\ R \cos \theta \sin \varphi \\ -R \sin \theta \end{pmatrix} = R \cos \theta \cos \varphi \frac{\partial}{\partial x} + R \cos \theta \sin \varphi \frac{\partial}{\partial y} - R \sin \theta \frac{\partial}{\partial z},$$

and respectively

$$\mathbf{r}_\varphi = \frac{\partial \mathbf{r}(\theta, \varphi)}{\partial \varphi} = \frac{\partial}{\partial \varphi} \begin{pmatrix} x(\theta, \varphi) \\ y(\theta, \varphi) \\ z(\theta, \varphi) \end{pmatrix} = \frac{\partial}{\partial \varphi} \begin{pmatrix} R \sin \theta \cos \varphi \\ R \sin \theta \sin \varphi \\ R \cos \theta \end{pmatrix} =$$

$$\begin{pmatrix} -R \sin \theta \sin \varphi \\ R \sin \theta \cos \varphi \\ 0 \end{pmatrix} = -R \sin \theta \sin \varphi \frac{\partial}{\partial x} + R \sin \theta \cos \varphi \frac{\partial}{\partial y}. \quad (1.33)$$

Here is a table how observers look at the objects on sphere:

	INTERNAL OBSERVER	EXTERNAL OBSERVER
point on S^2	2 coordinates θ, φ	3 coordinates $\mathbf{r} = \mathbf{r}(\theta, \varphi)$
curve on S^2	$\theta(t), \varphi(t)$	$\mathbf{r}(t) = \mathbf{r}(\theta(t), \varphi(t))$
coordinate tangent vectors to S^2	$\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \varphi}$	$\mathbf{r}_\theta, \mathbf{r}_\varphi$
tangent vector to S^2	$a \frac{\partial}{\partial \theta} + b \frac{\partial}{\partial \varphi}$	$A \frac{\partial}{\partial x} + B \frac{\partial}{\partial y} + C \frac{\partial}{\partial z} = a \mathbf{r}_\theta + b \mathbf{r}_\varphi$

Explicit formulae for induced Riemannian metric (First Quadratic form)

Now we are ready to write down the explicit formulae for the Riemannian metric on the surface induced by metric (scalar product) in ambient Euclidean space (see the Definition (1.32)). We will return to induced metric again in next paragraph 1.3.2.

Let $M: \mathbf{r} = \mathbf{r}(u, v)$ be a surface embedded in \mathbf{E}^3 .

The formula (1.32) means that scalar products of basic vectors $\mathbf{r}_u = \partial_u, \mathbf{r}_v = \partial_v$ has to be the same calculated on the surface or in the ambient space, i.e. calculated by Internal observer, or by External observer. For example scalar product $\langle \partial_u, \partial_v \rangle_M = g_{uv}$ calculated by the Internal Observer is the same as a scalar product $\langle \mathbf{r}_u, \mathbf{r}_v \rangle_{\mathbf{E}^3}$ calculated by the External Observer, scalar product $\langle \partial_v, \partial_v \rangle_M = g_{vv}$ calculated by the Internal Observer is the same as a scalar product $\langle \mathbf{r}_v, \mathbf{r}_v \rangle_{\mathbf{E}^3}$ calculated by the External Observer and so on:

$$G = \begin{pmatrix} g_{uu} & g_{uv} \\ g_{vu} & g_{vv} \end{pmatrix} = \begin{pmatrix} \langle \partial_u, \partial_u \rangle & \langle \partial_u, \partial_v \rangle \\ \langle \partial_v, \partial_u \rangle & \langle \partial_v, \partial_v \rangle \end{pmatrix} = \begin{pmatrix} \langle \mathbf{r}_u, \mathbf{r}_u \rangle_{\mathbf{E}^3} & \langle \mathbf{r}_u, \mathbf{r}_v \rangle_{\mathbf{E}^3} \\ \langle \mathbf{r}_v, \mathbf{r}_u \rangle_{\mathbf{E}^3} & \langle \mathbf{r}_v, \mathbf{r}_v \rangle_{\mathbf{E}^3} \end{pmatrix} \quad (1.34)$$

where as usual we denote by $\langle \cdot, \cdot \rangle_{\mathbf{E}^3}$ the scalar product in the ambient Euclidean space.

Remark It is convenient sometimes to denote parameters (u, v) as (u^1, u^2) or u^α ($\alpha = 1, 2$) and to write $\mathbf{r} = \mathbf{r}(u^1, u^2)$ or $\mathbf{r} = \mathbf{r}(u^\alpha)$ ($\alpha = 1, 2$) instead $\mathbf{r} = \mathbf{r}(u, v)$

In these notations:

$$G_M = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} = \begin{pmatrix} \langle \mathbf{r}_u, \mathbf{r}_u \rangle_{\mathbf{E}^3} & \langle \mathbf{r}_u, \mathbf{r}_v \rangle_{\mathbf{E}^3} \\ \langle \mathbf{r}_u, \mathbf{r}_v \rangle_{\mathbf{E}^3} & \langle \mathbf{r}_v, \mathbf{r}_v \rangle_{\mathbf{E}^3} \end{pmatrix}, \quad g_{\alpha\beta} = \langle \mathbf{r}_\alpha, \mathbf{r}_\beta \rangle,$$

$$G_M = g_{\alpha\beta} du^\alpha du^\beta = g_{11} du^2 + 2g_{12} dudv + g_{22} dv^2 \quad (1.35)$$

where (\cdot, \cdot) is a scalar product in Euclidean space.

The formula (1.35) is the formula for induced Riemannian metric on the surface $\mathbf{r} = \mathbf{r}(u, v)$ ².

If \mathbf{X}, \mathbf{Y} are two tangent vectors in the tangent plane $T_p C$ then $G(\mathbf{X}, \mathbf{Y})$ at the point p is equal to scalar product of vectors \mathbf{X}, \mathbf{Y} :

$$(\mathbf{X}, \mathbf{Y}) = (X^1 \mathbf{r}_1 + X^2 \mathbf{r}_2, Y^1 \mathbf{r}_1 + Y^2 \mathbf{r}_2) = \quad (1.36)$$

$$\begin{aligned} & X^1(\mathbf{r}_1, \mathbf{r}_1)Y^1 + X^1(\mathbf{r}_1, \mathbf{r}_2)Y^2 + X^2(\mathbf{r}_2, \mathbf{r}_1)Y^1 + X^2(\mathbf{r}_2, \mathbf{r}_2)Y^2 = \\ & X^\alpha(\mathbf{r}_\alpha, \mathbf{r}_\beta)Y^\beta = X^\alpha g_{\alpha\beta} Y^\beta = G(\mathbf{X}, \mathbf{Y}) \end{aligned}$$

1.3.2 Formulae for induced metric

We obtained (1.35) from equation (1.32).

We can do these calculations in a little bit other way.

The Riemannian structure of Euclidean space— standard Euclidean metric in Euclidean coordinates is given by

$$G_{\mathbf{E}^3} = (dx)^2 + (dy)^2 + (dz)^2. \quad (1.37)$$

Then the induced metric (1.32) on the surface M defined by equation $\mathbf{r} = \mathbf{r}(u, v)$ is equal to

$$G_M = G_{\mathbf{E}^3} \big|_{\mathbf{r}=\mathbf{r}(u,v)} = ((dx)^2 + (dy)^2 + (dz)^2) \big|_{\mathbf{r}=\mathbf{r}(u,v)} = G_M = g_{\alpha\beta} du^\alpha du^\beta \quad (1.38)$$

i.e. $((dx)^2 + (dy)^2 + (dz)^2) \big|_{\mathbf{r}=\mathbf{r}(u,v)} =$

$$\begin{aligned} & \left(\frac{\partial x(u,v)}{\partial u} du + \frac{\partial x(u,v)}{\partial v} dv \right)^2 + \left(\frac{\partial y(u,v)}{\partial u} du + \frac{\partial y(u,v)}{\partial v} dv \right)^2 + \left(\frac{\partial z(u,v)}{\partial u} du + \frac{\partial z(u,v)}{\partial v} dv \right)^2 = \\ & (x_u^2 + y_u^2 + z_u^2) du^2 + 2(x_u x_v + y_u y_v + z_u z_v) dudv + (x_v^2 + y_v^2 + z_v^2) dv^2 \end{aligned}$$

We see that

$$G_M = g_{\alpha\beta} du^\alpha du^\beta = g_{11} du^2 + 2g_{12} dudv + g_{22} dv^2, \quad (1.39)$$

²it is called sometimes First Quadratic Form of this surface.

where for matrix $||g_{\alpha\beta}||$, $(\alpha, \beta = 1, 2)$,

$$||g_{\alpha\beta}|| = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} g_{uu} & g_{uv} \\ g_{vu} & g_{vv} \end{pmatrix} = \begin{pmatrix} (x_u^2 + y_u^2 + z_u^2) & (x_u x_v + y_u y_v + z_u z_v) \\ (x_u x_v + y_u y_v + z_u z_v) & (x_v^2 + y_v^2 + z_v^2) \end{pmatrix} = \begin{pmatrix} \langle \mathbf{r}_u, \mathbf{r}_u \rangle_{\mathbf{E}^3} & \langle \mathbf{r}_u, \mathbf{r}_v \rangle_{\mathbf{E}^3} \\ \langle \mathbf{r}_v, \mathbf{r}_u \rangle_{\mathbf{E}^3} & \langle \mathbf{r}_v, \mathbf{r}_v \rangle_{\mathbf{E}^3} \end{pmatrix}. \quad (1.40)$$

We come to same formula (1.35).

Example Consider again sphere of radius R in \mathbf{E}^3 , $x^2 + y^2 + z^2 = R^2$ in stereographic coordinates. We calculated coordinate tangent vectors to this sphere in (1.33). Now calculate induced Riemannian metric:

$$\begin{aligned} G_{S^2} &= (dx^2 + dy^2 + dz^2) \big|_{x=R \sin \theta \cos \varphi, y=R \sin \theta \sin \varphi, z=R \cos \theta} = \\ &= [d(R \sin \theta \cos \varphi)]^2 + [d(R \sin \theta \sin \varphi)]^2 + [d(R \cos \theta)]^2 = \\ &= [R \cos \theta \cos \varphi d\theta - R \sin \theta \sin \varphi d\varphi]^2 + [R \cos \theta \sin \varphi d\theta + R \sin \theta \cos \varphi d\varphi]^2 + [-R \sin \theta d\theta]^2 = \\ &= (R^2 \sin^2 \theta \sin^2 \varphi + R^2 \sin^2 \theta \cos^2 \varphi) d\varphi^2 + (R^2 \cos^2 \theta \cos^2 \varphi + R^2 \sin^2 \theta) d\theta^2 = R^2 d\theta^2 + R^2 \sin^2 \theta d\varphi^2. \end{aligned}$$

We see that

$$G_{S^2} = R^2 d\theta^2 + R^2 \sin^2 \theta d\varphi^2, ||g_{\alpha\beta}|| = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} g_{\theta\theta} & g_{\theta\varphi} \\ g_{\varphi\theta} & g_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2 \theta \end{pmatrix}. \quad (1.41)$$

Remark Sometimes it is useful to use the following “condensed” notations. We denote Cartesian coordinates (x, y, z) of Euclidean space by x^i , $(i = 1, 2, 3)$. Let surface M be given in local parameterisation $x^i = x^i(u^\alpha)$. Riemannian metric of Euclidean space (1.37) has appearance

$$G_{\mathbf{E}} = dx^i \delta_{ik} dx^k. \quad (1.42)$$

and calculations (1.38) — (1.40) for induced metric (1.38) has appearance

$$G_M = dx^i \delta_{ik} dx^k \big|_{x^i = x^i(u^\alpha)} = \frac{\partial x^i(u)}{\partial u^\alpha} \delta_{ik} \frac{\partial x^k(u)}{\partial u^\beta} du^\alpha du^\beta = g_{\alpha\beta}(u) du^\alpha du^\beta \quad (1.43)$$

(See also remark above before equation (1.35)). One can rewrite (1.43) in the following way:

$$g_{\alpha\beta} = \frac{\partial x^i(u)}{\partial u^\alpha} \delta_{ij} \frac{\partial x^j(u)}{\partial u^\beta}. \quad (\alpha, \beta = 1, 2,).$$

It is instructive to come to this equation straightforwardly from equation (1.32) and definition (1.16). We have that due to (1.32)

$$g_{\alpha\beta} = g_{\pi\rho} dx^\pi dx^\rho \left(\frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial u^\beta} \right) = G_M \left(\frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial u^\beta} \right) = G_{\mathbf{E}^3}(\mathbf{r}_\alpha, \mathbf{r}_\beta) =$$

$$\delta_{pq} dx^p dx^q \left(\frac{\partial x^i}{\partial u^\alpha} \frac{\partial}{\partial x^i}, \frac{\partial x^j}{\partial u^\beta} \frac{\partial}{\partial x^j} \right) = \frac{\partial x^i}{\partial u^\alpha} \left[\delta_{pq} dx^p dx^q \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \right] \frac{\partial x^j}{\partial u^\beta} = \frac{\partial x^i}{\partial u^\alpha} \delta_{ij} \frac{\partial x^j}{\partial u^\beta}.$$

Representation (1.43) in condensed notations is very useful. It is easy to see that this formula works for arbitrary dimensions, i.e. if we have m -dimensional manifold embedded in n -dimensional Euclidean space. We just have to suppose that in this case $i = 1, \dots, n$ and $\alpha = 1, \dots, m$; manifold is given by parameterisation $x^i = x^i(u^\alpha)$ ($\alpha = 1, \dots, m$). Moreover in the case if manifold is embedded not in Euclidean space but in an arbitrary Riemannian space then one can see that we come to the induced metric

$$G_M = dx^i g_{ik}((x(u))) dx^k \Big|_{x^i=x^i(u^\alpha)} = \frac{\partial x^i(u)}{\partial u^\alpha} g_{ik}((x(u))) \frac{\partial x^k(u)}{\partial u^\beta} du^\alpha du^\beta = g_{\alpha\beta}(x(u)) du^\alpha du^\beta$$

Check explicitly again that length of the tangent vectors and curves on the surface calculating by External observer (i.e. using Euclidean metric (1.37)) *is the same* as calculating by Internal Observer, ant (i.e. using the induced Riemannian metric (1.35), (1.39)). Let $\mathbf{X} = X^\alpha \mathbf{r}_\alpha = a \mathbf{r}_u + b \mathbf{r}_v$ be a vector tangent to the surface M . The square of the length $|\mathbf{X}|$ of this vector calculated by External observer (he calculates using the scalar product in \mathbf{E}^3) equals to

$$|\mathbf{X}|^2 = \langle \mathbf{X}, \mathbf{X} \rangle = \langle \mathbf{r}_u + b \mathbf{r}_v, a \mathbf{r}_u + b \mathbf{r}_v \rangle = a^2 \langle \mathbf{r}_u, \mathbf{r}_u \rangle + 2ab \langle \mathbf{r}_u, \mathbf{r}_v \rangle + b^2 \langle \mathbf{r}_v, \mathbf{r}_v \rangle \quad (1.44)$$

where \langle, \rangle is a scalar product in \mathbf{E}^3 . The internal observer will calculate the length using Riemannian metric (1.35) (1.39):

$$G(\mathbf{X}, \mathbf{X}) = \begin{pmatrix} a & b \end{pmatrix} \cdot \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = g_{11}a^2 + 2g_{12}ab + g_{22}b^2 \quad (1.45)$$

External observer (person living in ambient space \mathbf{E}^3) calculate the length of the tangent vector using formula (1.44). An ant living on the surface calculate length of this vector in internal coordinates using formula (1.45). External observer deals with external coordinates of the vector, ant on the surface with internal coordinates. They come to the same answer.

Let $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ $a \leq t \leq b$ be a curve on the surface.

Velocity of this curve at the point $\mathbf{r}(u(t), v(t))$ is equal to

$$\mathbf{v} = \mathbf{X} = \xi \mathbf{r}_u + \eta \mathbf{r}_v \text{ where } \xi = u_t, \eta = v_t: \quad \mathbf{v} = \frac{d\mathbf{r}(t)}{dt} = u_t \mathbf{r}_u + v_t \mathbf{r}_v.$$

The length of the curve is equal to

$$L = \int_a^b |\mathbf{v}(t)| dt = \int_a^b \sqrt{\langle \mathbf{v}(t), \mathbf{v}(t) \rangle_{\mathbf{E}^3}} dt = \int_a^b \sqrt{\langle u_t \mathbf{r}_u + v_t \mathbf{r}_v, u_t \mathbf{r}_u + v_t \mathbf{r}_v \rangle_{\mathbf{E}^3}} dt = \quad (1.46)$$

$$\int_a^b \sqrt{\langle \mathbf{r}_u, \mathbf{r}_u \rangle_{\mathbf{E}^3} u_t^2 + 2 \langle \mathbf{r}_u, \mathbf{r}_v \rangle_{\mathbf{E}^3} u_t v_t + \langle \mathbf{r}_v, \mathbf{r}_v \rangle_{\mathbf{E}^3} v_t^2} d\tau = \quad (1.47)$$

$$\int_a^b \sqrt{g_{11} u_t^2 + 2g_{12} u_t v_t + g_{22} v_t^2} dt$$

An external observer will calculate the length of the curve using (1.46). An ant living on the surface calculate length of the curve using (1.47) using Riemannian metric on the surface. They will come to the same answer.

1.3.3 Induced Riemannian metrics. Examples.

We consider already an example of induced Riemannian metric on sphere in spherical coordinates. Now we consider here other examples of induced Riemannian metric on some surfaces in \mathbf{E}^3 . using calculations for tangent vectors (see (1.35)) or explicitly in terms of differentials (see (1.38) and (1.39)).

First of all consider the general case when a surface M is defined by the equation $z - F(x, y) = 0$. One can consider the following parameterisation of this surface:

$$\mathbf{r}(u, v): \quad \begin{cases} x = u \\ y = v \\ z = F(u, v) \end{cases} \quad (1.48)$$

Then coordinate tangent vectors $\mathbf{r}_u, \mathbf{r}_v$ are

$$\mathbf{r}_u = \begin{pmatrix} 1 \\ 0 \\ F_u \end{pmatrix} \quad \mathbf{r}_v = \begin{pmatrix} 0 \\ 1 \\ F_v \end{pmatrix} \quad (1.49)$$

,

$$(\mathbf{r}_u, \mathbf{r}_u) = 1 + F_u^2, \quad (\mathbf{r}_u, \mathbf{r}_v) = F_u F_v, \quad (\mathbf{r}_v, \mathbf{r}_v) = 1 + F_v^2$$

and induced Riemannian metric (first quadratic form) (1.35) is equal to

$$\|g_{\alpha\beta}\| = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_u, \mathbf{r}_u) & (\mathbf{r}_u, \mathbf{r}_v) \\ (\mathbf{r}_u, \mathbf{r}_v) & (\mathbf{r}_v, \mathbf{r}_v) \end{pmatrix} = \begin{pmatrix} 1 + F_u^2 & F_u F_v \\ F_u F_v & 1 + F_v^2 \end{pmatrix} \quad (1.50)$$

$$G_M = ds^2 = (1 + F_u^2)du^2 + 2F_u F_v dudv + (1 + F_v^2)dv^2 \quad (1.51)$$

and the length of the curve $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ on C ($a \leq t \leq b$) can be calculated by the formula:

$$L = \int_a^b \int_a^b \sqrt{(1 + F_u^2)u_t^2 + 2F_u F_v u_t v_t + (1 + F_v^2)v_t^2} dt$$

One can calculate (1.51) explicitly using (1.38):

$$\begin{aligned} G_M &= (dx^2 + dy^2 + dz^2) \big|_{x=u, y=v, z=F(u,v)} = (du)^2 + (dv)^2 + (F_u du + F_v dv)^2 = \\ &= (1 + F_u^2)du^2 + 2F_u F_v dudv + (1 + F_v^2)dv^2. \end{aligned} \quad (1.52)$$

Cylinder

Cylinder is given by the equation $x^2 + y^2 = a^2$. One can consider the following parameterisation of this surface:

$$\mathbf{r}(h, \varphi) : \begin{cases} x = a \cos \varphi \\ y = a \sin \varphi \\ z = h \end{cases} \quad (1.53)$$

$$\begin{aligned} \text{We have } G_{cylinder} &= \iota^* G_{\mathbf{E}^3} = (dx^2 + dy^2 + dz^2) \big|_{x=a \cos \varphi, y=a \sin \varphi, z=h} = \\ &= (-a \sin \varphi d\varphi)^2 + (a \cos \varphi d\varphi)^2 + dh^2 = a^2 d\varphi^2 + dh^2 \end{aligned} \quad (1.54)$$

The same formula in terms of scalar product of tangent vectors:

$$\text{coordinate basis vectors } \mathbf{r}_h = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{r}_\varphi = \begin{pmatrix} -a \sin \varphi \\ a \cos \varphi \\ 0 \end{pmatrix} \quad (1.55)$$

,

$$(\mathbf{r}_h, \mathbf{r}_h) = 1, \quad (\mathbf{r}_h, \mathbf{r}_\varphi) = 0, \quad (\mathbf{r}_\varphi, \mathbf{r}_\varphi) = a^2$$

and

$$\|g_{\alpha\beta}\| = \begin{pmatrix} (\mathbf{r}_u, \mathbf{r}_u) & (\mathbf{r}_u, \mathbf{r}_v) \\ (\mathbf{r}_u, \mathbf{r}_v) & (\mathbf{r}_v, \mathbf{r}_v) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a^2 \end{pmatrix},$$

$$G = dh^2 + a^2 d\varphi^2 \quad (1.56)$$

and the length of the curve $\mathbf{r}(t) = \mathbf{r}(h(t), \varphi(t))$ on the cylinder ($a \leq t \leq b$) can be calculated by the formula:

$$L = \int_a^b \sqrt{h_t^2 + a^2 \varphi_t^2} dt \quad (1.57)$$

Cone

Cone is given by the equation $x^2 + y^2 - k^2 z^2 = 0$. One can consider the following parameterisation of this surface:

$$\mathbf{r}(h, \varphi): \quad \begin{cases} x = kh \cos \varphi \\ y = kh \sin \varphi \\ z = h \end{cases} \quad (1.58)$$

Calculate induced Riemannian metric:

We have

$$\begin{aligned} G_{conus} &= \iota^* G_{\mathbf{E}^3} = (dx^2 + dy^2 + dz^2) \big|_{x=kh \cos \varphi, y=kh \sin \varphi, z=h} = \\ &= (k \cos \varphi dh - kh \sin \varphi d\varphi)^2 + (k \sin \varphi dh + kh \cos \varphi d\varphi)^2 + dh^2 \\ G_{conus} &= k^2 h^2 d\varphi^2 + (1 + k^2) dh^2, \quad \|g_{\alpha\beta}\| = \begin{pmatrix} 1 + k^2 & 0 \\ 0 & k^2 h^2 \end{pmatrix} \end{aligned} \quad (1.59)$$

The length of the curve $\mathbf{r}(t) = \mathbf{r}(h(t), \varphi(t))$ on the cone ($a \leq t \leq b$) can be calculated by the formula:

$$L = \int_a^b \sqrt{(1 + k^2) h_t^2 + k^2 h^2 \varphi_t^2} dt \quad (1.60)$$

Circle (again)

Circle of radius R is given by the equation $x^2 + y^2 = R^2$. Consider standard parameterisation φ of this surface:

$$\mathbf{r}(\varphi): \quad \begin{cases} x = R \cos \varphi \\ y = R \sin \varphi \end{cases}$$

Calculate induced Riemannian metric (first quadratic form)

$$G_{S^1} = \iota^* G_{\mathbf{E}^3} = (dx^2 + dy^2) \Big|_{x=R \cos \varphi, y=R \sin \varphi} = \\ (-R \sin \varphi d\varphi)^2 + (R \cos \varphi d\varphi)^2 = (R^2 \cos^2 \varphi + R^2 \sin^2 \varphi) d\varphi^2 = R^2 d\varphi^2.$$

One can consider stereographic coordinates on the circle (see Example in the subsection 1.1) A point x, y : $x^2 + y^2 = R^2$ has stereographic coordinate t if points $(0, 1)$ (north pole), the point (x, y) and the point $(t, 0)$ belong to the same line, i.e. $\frac{x}{t} = \frac{R-y}{R}$, i.e.

$$t = \frac{Rx}{R-y}, \quad \begin{cases} x = \frac{2tR^2}{R^2+t^2} \\ y = \frac{t^2-R^2}{t^2+R^2} R \end{cases} \quad \text{since } x^2 + y^2 = R^2.$$

Induced metric in coordinate t is

$$G = (dx^2 + dy^2) \Big|_{x=x(t), y=y(t)} = \left(d \left(\frac{2tR^2}{R^2+t^2} \right) \right)^2 + \left(d \left(\frac{t^2-R^2}{R^2+t^2} R \right) \right)^2 = \\ \left(\frac{2R^2 dt}{R^2+t^2} - \frac{4t^2 R^2 dt}{(R^2+t^2)^2} \right)^2 + \left(-\frac{4R^2 t dt}{(t^2+R^2)^2} \right)^2 = \frac{4R^4 dt^2}{(R^2+t^2)^2}. \quad (1.61)$$

(See for detail Homework 2³.)

Remark Stereographic coordinates very often are preferable since they define birational equivalence between circle and line.

Sphere (again...)

Sphere of radius R is given by the equation $x^2 + y^2 + z^2 = R^2$. Consider first stereographic coordinates

$$\mathbf{r}(\theta, \varphi): \quad \begin{cases} x = R \sin \theta \cos \varphi \\ y = R \sin \theta \sin \varphi \\ z = R \cos \theta \end{cases} \quad (1.62)$$

We already calculated the coordinate basis in (1.33) and we calculated induced Riemannian metric in (1.41):

$$, \quad G_{S^2} = R^2 d\theta^2 + R^2 \sin^2 \theta d\varphi^2, \quad \|g_{\alpha\beta}\| = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2 \theta \end{pmatrix} \quad (1.63)$$

³One can also obtain this formula in a very beautiful way using inversion (see Appendices)

One comes to the same answer calculating scalar product of coordinate tangent vectors:

$$\text{coordinate tangent vectors are } \mathbf{r}_\theta = \begin{pmatrix} R \cos \theta \cos \varphi \\ R \cos \theta \sin \varphi \\ -R \sin \theta \end{pmatrix}, \quad \mathbf{r}_\varphi = \begin{pmatrix} -R \sin \theta \sin \varphi \\ R \sin \theta \cos \varphi \\ 0 \end{pmatrix}$$

,

$$(\mathbf{r}_\theta, \mathbf{r}_\theta) = R^2, \quad (\mathbf{r}_\theta, \mathbf{r}_\varphi) = 0, \quad (\mathbf{r}_\varphi, \mathbf{r}_\varphi) = R^2 \sin^2 \theta$$

and

$$\|g\| = \begin{pmatrix} (\mathbf{r}_u, \mathbf{r}_u) & (\mathbf{r}_u, \mathbf{r}_v) \\ (\mathbf{r}_u, \mathbf{r}_v) & (\mathbf{r}_v, \mathbf{r}_v) \end{pmatrix} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2 \theta \end{pmatrix}, \quad G_{S^2} = ds^2 = R^2 d\theta^2 + R^2 \sin^2 \theta d\varphi^2$$

The length of the curve $\mathbf{r}(t) = \mathbf{r}(\theta(t), \varphi(t))$ on the sphere of the radius a ($a \leq t \leq b$) can be calculated by the formula:

$$L = \int_a^b R \sqrt{\theta_t^2 + \sin^2 \theta \cdot \varphi_t^2} dt \quad (1.64)$$

One can consider on sphere as well as on a circle stereographic coordinates:

$$\begin{cases} u = \frac{Rx}{R-z} \\ v = \frac{Ry}{R-z} \end{cases}, \quad \begin{cases} x = \frac{2uR^2}{R^2+u^2+v^2} \\ y = \frac{2vR^2}{R^2+u^2+v^2} \\ z = \frac{u^2+v^2-R^2}{u^2+v^2+R^2} R \end{cases} \quad (1.65)$$

In these coordinates Riemannian metric is

$$\begin{aligned} G &= (dx^2 + dy^2 + dz^2)|_{x=x(u,v), y=y(u,v), z=z(u,v)} = \\ &= \left(d \left(\frac{2uR^2}{R^2 + u^2 + v^2} \right) \right)^2 + \left(d \left(\frac{2vR^2}{R^2 + u^2 + v^2} \right) \right)^2 + \left(d \left(1 - \frac{2R^2}{R^2 + u^2 + v^2} \right) R \right)^2 = \\ &= \frac{4R^4(du^2 + dv^2)}{(R^2 + u^2 + v^2)^2}. \end{aligned} \quad (1.66)$$

(See for detail Homework 2⁴).

Notice that we showed that metric on sphere is conformally Euclidean.

⁴Another beautiful deduction of this formula see in Appendices (Inversion)

Saddle (paraboloid)

Consider paraboloid $z = x^2 - y^2$. It can be rewritten as $z = axy$ and it is called sometimes “saddle” (rotation on the angle $\varphi = \pi/4$ transforms $z = x^2 - y^2$ onto $z = 2xy$.) We considered this example in homework 3. Paraboloid and saddle they are ruled surfaces which are formed by lines.

Examples of other quadratic surfaces see in Appendix.

1.4 Isometries of Riemannian manifolds.

1.4.1 Riemannian metric induced by map

Let M be a manifold, and let $(N, G(N))$ be a Riemannian manifold. Let F be a map from M to N ,

$$F: M \longrightarrow N$$

We do not suppose that F is diffeomorphism, we even do not suppose that manifolds M and N have the same dimension⁵.

We can define F^*G , a *Riemannian metric on manifold M induced by the map $F: M \rightarrow N$* .

Describe the metric F^*G on M in local coordinates.

Let x^a , ($a = 1, \dots, m$) be local coordinates on m -dimensional manifold M in a vicinity of some point \mathbf{p}_M on M . Consider a point $\mathbf{p}_N = F(\mathbf{p}_M)$ on manifold N and let y^i , ($i = 1, \dots, n$) be local coordinates on n -dimensional manifold N in a vicinity of point \mathbf{p}_N . If in local coordinates y^i , Riemannian metric $G^{(N)}$ on N has appearance

$$G^{(N)} = g_{ij}^{(N)}(y) dy^i dy^j,$$

then in local coordinates x^a , Riemannian metric $F^*(G^{(N)})$ on M has appearance

$$F^*(G^{(N)}) = g_{ab}^{(M)}(x) dx^a dx^b = dx^a \frac{\partial y^j(x)}{\partial x^a} g_{ab}^{(N)}(y(x)) \frac{\partial y^j(x)}{\partial x^b} dx^b, \quad ,$$

i.e.

$$g_{ab}^{(M)}(x) = \frac{\partial y^j(x)}{\partial x^a} g_{ab}^{(N)}(y(x)) \frac{\partial y^j(x)}{\partial x^b}, \quad (1.67)$$

where $y^i = y^i(x^a)$ is expression of map F in local coordinates x^a and y^i ,

⁵We just suppose that F is differentiable map, i.e. local expressions for F are smooth functions

Remark The metric $F^*(G^{(N)})$ on M is called *pull-back* of metric $G^{(N)}$ under the map $F: M \rightarrow N$.

Example The induced metric on surfaces in \mathbf{E}^3 is a special example of this general construction. Indeed embedding $\iota: M \rightarrow \mathbf{E}^3$ is a map from points of 2-dimensional manifold M to points of 3-dimensional Euclidean space \mathbf{E}^3 . Applying (1.67) to this map we come to formulae (1.39) for induced metric:

$$G_M = \iota^* G_{\mathbf{E}^3} = \iota^*(dx^2 + dy^2 + dz^2) =$$

$$G_{\mathbf{E}^3}|_{\mathbf{r}=\mathbf{r}(u,v)} = ((dx)^2 + (dy)^2 + (dz)^2)|_{\mathbf{r}=\mathbf{r}(u,v)} = G_M = g_{\alpha\beta} du^\alpha du^\beta$$

(or another manifestation of this formula, equation (1.43)).

1.4.2 Diffeomorphism, which is an isometry

Let $(M_1, G_{(1)})$, $(M_2, G_{(2)})$ be two Riemannian manifolds— manifolds equipped with Riemannian metric $G_{(1)}$ and $G_{(2)}$ respectively.

Loosely speaking isometry is the diffeomorphism of Riemannian manifolds which preserves the distance.

Definition Let F be a diffeomorphisms (one-one smooth map with smooth inverse) of manifold M_1 on manifold M_2 .

We say that diffeomorphism F is an isometry of Riemannian manifolds $(M_1, G_{(1)})$ and $(M_2, G_{(2)})$ if it preserves the metrics, i.e. $G_{(1)}$ is pull-back of $G_{(2)}$:

$$F^* G_{(2)} = G_{(1)}. \quad (1.68)$$

According (1.67) this means that

$$F^* \left(g_{(2)ab}(y) dy^a dy^b \right) = g_{(2)ab}(y) dy^a dy^b|_{y=y(x)} =$$

$$g_{(2)ab}(y(x)) \frac{\partial y^a(x)}{\partial x^i} dx^i \frac{\partial y^b(x)}{\partial x^k} dx^k = g_{(1)ik}(x) dx^i dx^k,$$

i.e.

$$g_{(1)ik}(x) = \frac{\partial y^a(x)}{\partial x^i} g_{(2)ab}(y(x)) \frac{\partial y^b(x)}{\partial x^k}, \quad (1.69)$$

where $y^a = y^a(x)$ is local expression for diffeomorphism F . We say that diffeomorphism F is *isometry* of Riemannian manifolds $(M_1, G_{(1)})$ and $M_2, G_{(2)}$. The difference of this equation with equation (1.67) is that F in (1.67) was

just a differentiable map, which is not a diffeomorphism. In (1.69) diffeomorphism F establishes one-one correspondence between local coordinates on manifolds M_1 and M_2 . The left hand side of equation (1.69) can be considered as a local expression of metric $G_{(2)}$ in coordinates x^i on M_2 and the right hand side of this equation is local expression of metric $G_{(1)}$ in coordinates x^i on M_1 . Diffeomorphism F identifies manifolds M_1 and M_2 and it can be considered as changing of coordinates.

Example Consider surface of cylinder C , $x^2 + y^2 = a^2$ in \mathbf{E}^3 with induced Riemannian metric $G_C = a^2 d\varphi^2 + dh^2$ (see equations (1.53) and (1.54)). If we remove the line l : $x = a, y = 0$ from the cylinder surface C we come to surface $C' = C \setminus l$. Consider a map F of this surface in Euclidean space E^2 with Cartesian coordinates u, v (with standard Euclidean metric $G_{Eucl} = du^2 + dv^2$):

$$F: \begin{cases} u = a\varphi \\ v = h \end{cases} \quad 0 < \varphi < 2\pi. \quad (1.70)$$

One can see that F is the diffeomorphism of C' on the domain $0 < u < 2\pi a$ in \mathbf{E}^2 and this diffeomorphism is an isometry: it transforms the metric G_{Eucl} on Euclidean space in metric G_C on cylinder, i.e. pull-back condition (1.68) is obeyed:

$$F^* G_{Eucl} = F^* (du^2 + dv^2) = (du^2 + dv^2) \big|_{u=a\varphi, v=h} = a^2 d\varphi^2 + dh^2 = G_1.$$

We see that cylinder surface with removed line is isometric to domain in \mathbf{E}^2 and the map F establishes this isometry.

Remark Notice that if F is diffeomorphism of manifold M_1 on a Riemannian manifold $(M_2, G_{(2)})$, then it defines Riemannian structure, the pull-back $G_1 = F^*(G_{(2)})$ on M_1 , and F is isometry of Riemannian manifold $(M_1, G_{(1)})$ on Riemannian manifold $(M_2, G_{(2)})$.

1.4.3 Isometries of Riemannian manifold on itself

Definition Let (M, G) be a Riemannian manifold. We say that a diffeomorphism F is an isometry of Riemannian manifold on itself if it preserves the metric, i.e. $F^*G = G$. In local coordinates this means that

$$g_{ik}(x) = g_{pq}(x'(x)) \frac{\partial x^p(x')}{\partial x^i} \frac{\partial x^q(x')}{\partial x^k}, \quad (1.71)$$

where $x' = x'(x)$ is a local expression for diffeomorphism F . **Example** Let \mathbf{E}^2 be Euclidean plane with metric $dx^2 + dy^2$ in Cartesian coordinates x, y .

Consider the transformation

$$\begin{cases} x' = p + ax + by \\ y' = q + cx + dy \end{cases}$$

is isometry if and only if the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an orthogonal matrix, i.e. if the transformation above is combination of translation, rotation and reflection.

Another example **Example** Consider Lobachevsky (hyperbolic) plane: an upper half-plane ($y > 0$) in \mathbf{R}^2 equipped with Riemannian metric

$$G = \frac{dx^2 + dy^2}{y^2},$$

One can see that the map

$$\begin{cases} x = \lambda x' \\ y = \lambda y' \end{cases}, (\lambda > 0)$$

($\lambda > 0$) is an isometry of the Lobachevsky plane on itself. Are there other isometries? Yes there are (See the discussion of these questions in Homeworks.)

1.4.4 Locally Euclidean Riemannian manifolds

It is useful to formulate the local isometry condition between Riemannian manifold and Euclidean space. A neighbourhood of every point of n -dimensional manifold is diffeomorphic to \mathbf{R}^n . Let as usual \mathbf{E}^n be n -dimensional Euclidean space, i.e. \mathbf{R}^n with standard Riemannian metric $G = dx^i \delta_{ik} dx^k = (dx^1)^2 + \dots + (dx^n)^2$ in Cartesian coordinates (x^1, \dots, x^n) .

Definition We say that n -dimensional Riemannian manifold (M, G) is locally isometric to Euclidean space \mathbf{E}^n , i.e. it is locally Euclidean Riemannian manifold, if for every point $\mathbf{p} \in M$ there exists an open neighborhood D (domain) containing this point, $\mathbf{p} \in D$ such that D is isometric to a domain in Euclidean plane. In other words in a vicinity of every point \mathbf{p} there exist local coordinates u^1, \dots, u^n such that Riemannian metric G in these coordinates has an appearance

$$G = du^i \delta_{ik} du^k = (du^1)^2 + \dots + (du^n)^2. \quad (1.72)$$

Consider examples.

Example Consider again cylinder surface..

We know that cylinder is not diffeomorphic to plane (cylinder surface is $S^1 \times \mathbf{R}$, $\mathbf{E}^2 = \mathbf{R} \times \mathbf{R}$, and circle is not diffeomorphic to line). In the previous subsection we cutted the line from cylindre. Thus we came to surface diffeomorphic to plane. We established that this surface is isometric to Euclidean plane. (See equation (1.70) and considerations above.) Local isometry of cylinder to the Euclidean plane, i.e. the fact that it is locally Euclidean Riemannian surface immediately follows from the fact that under changing of local coordinates $u = a\varphi, v = h$ in equation (1.70), the standard Euclidean metric $du^2 + dv^2$ transforms to the metric $G_{cylinder} = a^2 d\varphi^2 + dh^2$ on cylinder.

Remark Strictly speaking we consider all the points except the points on the cutted line (with coordinate $\varphi = 0$). On the other hand for the points on cuttg line we can consider instead coordinate φ another coordinate $\varphi' = \varphi - \pi$, $-\pi < \varphi' < \pi$, and we will come to the same answer. In this case the cutted line will be the line $\varphi' = \pi$.

Example Now show that cone is locally Euclidean Riemannian surface, i.e, it is locally isometric to the Euclidean plane. This means that we have to find local coordinates u, v on the cone such that in these coordinates induced metric $G|_c$ on cone would have the appearance $G|_c = du^2 + dv^2$. Recall calculations of the metric on cone in coordinates h, φ where

$$\mathbf{r}(h, \varphi): \begin{cases} x = kh \cos \varphi \\ y = kh \sin \varphi \\ z = h \end{cases},$$

$x^2 + y^2 - k^2 z^2 = k^2 h^2 \cos^2 \varphi + k^2 h^2 \sin^2 \varphi - k^2 h^2 = k^2 h^2 - k^2 h^2 = 0$. We have that metric G_c on the cone in coordinates h, φ induced with the Euclidean metric $G = dx^2 + dy^2 + dz^2$ is equal to

$$G_c = (dx^2 + dy^2 + dz^2) \big|_{x=kh \cos \varphi, y=kh \sin \varphi, z=h} = (k \cos \varphi dh - kh \sin \varphi d\varphi)^2 + (k \sin \varphi dh + kh \cos \varphi d\varphi)^2 + dh^2 = (k^2 + 1)dh^2 + k^2 h^2 d\varphi^2.$$

In analogy with polar coordinates try to find new local coordinates u, v such that $\begin{cases} u = \alpha h \cos \beta \varphi \\ v = \alpha h \sin \beta \varphi \end{cases}$, where α, β are parameters. We come to $du^2 + dv^2 =$

$$(\alpha \cos \beta \varphi dh - \alpha \beta h \sin \beta \varphi d\varphi)^2 + (\alpha \sin \beta \varphi dh + \alpha \beta h \cos \beta \varphi d\varphi)^2 = \alpha^2 dh^2 + \alpha^2 \beta^2 h^2 d\varphi^2.$$

Comparing with the metric on the cone $G_c = (1+k^2)dh^2 + k^2h^2d\varphi^2$ we see that if we put $\alpha = \sqrt{k^2 + 1}$ and $\beta = \frac{k}{\sqrt{1+k^2}}$ then $du^2 + dv^2 = \alpha^2 dh^2 + \alpha^2 \beta^2 h^2 d\varphi^2 = (1+k^2)dh^2 + k^2h^2d\varphi^2$.

Thus in new local coordinates

$$\begin{cases} u = \sqrt{k^2 + 1}h \cos \frac{k}{\sqrt{k^2+1}}\varphi \\ v = \sqrt{k^2 + 1}h \sin \frac{k}{\sqrt{k^2+1}}\varphi \end{cases}$$

induced metric on the cone becomes $G|_c = du^2 + dv^2$, i.e. cone locally is isometric to the Euclidean plane ■

Of course these coordinates are local.— Cone and plane are not homeomorphic, thus they are not globally isometric.

Example and counterexample

Consider domain D in Euclidean plane with two metrics:

$$G_{(1)} = du^2 + \sin^2 v dv^2, \quad \text{and} \quad G_{(2)} = du^2 + \sin^2 u dv^2 \quad (1.73)$$

Thus we have two different Riemannian manifolds $(D, G_{(1)})$ and $(D, G_{(2)})$. Metrics in (1.73) look similar. But... It is easy to see that the first one is locally isometric to Euclidean plane, i.e. it is locally Euclidean Riemannian manifold since $\sin^2 v dv^2 = d(-\cos v)^2$: in new coordinates $u' = u, v' = \cos v$ Riemannian metric $G_{(1)}$ has appearance of standard Euclidean metric:

$$(du')^2 + (dv')^2 = (du)^2 + (d(\cos v))^2 = du^2 + \sin^2 v dv^2 = G_{(1)}.$$

This is not the case for second metric $G_{(2)}$. If we change notations $u \mapsto \theta, v \mapsto \varphi$ then $G_{(2)} = d\theta^2 + \sin^2 \theta d\varphi^2$. This is local expression for Riemannian metric induced on the sphere of radius $R = 1$. Suppose that there exist coordinates $u' = u'(\theta, \varphi), v' = v'(\theta, \varphi)$ such that in these coordinates metric has Euclidean appearance. This means that locally geometry of sphere is as a geometry of Euclidean plane. On the other hand we know from the course of Geometry that this is not the case: sum of angles of triangles on the sphere is not equal to π , sphere cannot be bended without shrinking. Later in this course we will return to this question...

There are plenty other examples:

- 2) Plane with metric $\frac{4R^4(dx^2+dy^2)}{(R^2+x^2+y^2)^2}$ is isometric to the sphere with radius R .
- 3) Disc with metric $\frac{du^2+dv^2}{(1-u^2-v^2)^2}$ is isometric to half plane with metric $\frac{dx^2+dy^2}{4y^2}$.

(see also exercises in Homeworks and Coursework.)

1.5 Volume element in Riemannian manifold

The volume element in n -dimensional Riemannian manifold with metric $G = g_{ik}dx^i dx^k$ is defined by the formula

$$\sqrt{\det g} dx^1 dx^2 \dots dx^n. \quad (1.74)$$

If D is a domain in the n -dimensional Riemannian manifold with metric $G = g_{ik}dx^i dx^k$ then its volume is equal to the integral of volume element over this domain.

$$V(D) = \int_D \sqrt{\det g} dx^1 dx^2 \dots dx^n. \quad (1.75)$$

Note that in the case of $n = 1$ volume is just the length, in the case if $n = 2$ it is area.

Volume form (1.74) is invariant with respect to changing of coordinates⁶. In the next paragraph we will give motivation of this formula.

Remark Students who know the concept of exterior forms can read the volume element as n -form $\sqrt{\det g} dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$

1.5.1 Motivation: Gram formula for volume of parallelepiped

In this short paragraph we consider formulae for volume of n -dimensional parallelepiped, and we explain how formulae (1.74), (1.75) are related with basic formulae in geometry. For simplicity one can consider just the case if $n = 2, 3$.

Let \mathbf{E}^n be Euclidean vector space equipped with orthonormal basis $\{\mathbf{e}_i\}$.

Let $\{\mathbf{a}_i\} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ be an arbitrary row of n -vectors in \mathbf{E}^n . Consider n -dimensional parallelepiped $\Pi_{\{\mathbf{a}_i\}}$ formed by these vectors: $\Pi_{\{\mathbf{a}_i\}}: \mathbf{r} = t^i \mathbf{v}_i, 0 \leq t^i \leq 1$. The volume of this parallelepiped is equal to

$$Vol(\Pi_{\{\mathbf{a}_i\}}) = \det A, \quad (1.76)$$

where the matrix $A = \|\mathbf{a}_i^m\|$ is defined by expansion of vectors $\{\mathbf{a}_i\}$ over orthonormal basis $\{\mathbf{e}_i\}$: $\mathbf{a}_i = \mathbf{e}_m a_m^i$ (Volume vanishes ($Vol(\Pi_{\mathbf{a}_i}) = \det A = 0$) \Leftrightarrow if $\{\mathbf{a}_i\}$ is not a basis.)

Now consider the scalar product (Riemannian metric) in \mathbf{E}^n in the basis $\mathbf{a}_1, \dots, \mathbf{a}_n$:

$$g_{ik} = \langle \mathbf{a}_i, \mathbf{a}_k \rangle, \quad (1.77)$$

⁶see paragraph ?? in Appendices

where $\langle \cdot, \cdot \rangle$ is scalar product in \mathbf{E}^n : $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}$. We see that in (1.77)

$$g_{ij} = \langle \mathbf{a}_i, \mathbf{a}_j \rangle = \left\langle \sum_m a_i^m \mathbf{e}_m, \sum_n a_j^n \mathbf{e}_n \right\rangle = a_i^m \delta_{mn} a_j^n = (A^T \cdot A)_{ij} \Rightarrow \det G = (\det A)^2,$$

where $G = ||g_{ij}||$. Comparing with formula (1.76) we come to formula:

$$Vol(\Pi_{\mathbf{a}_i}) = \sqrt{\det g_{ik}} \quad (1.78)$$

This formula is called Gram formula, and the matrix $G = ||g_{ik}||$ is called Gram matrix for the vectors $\{\mathbf{a}_i\}$. Gram formula justifies equations (1.74) and (1.75) ⁷.

Remark One can easily see that formula (1.78) works for arbitrary n -dimensional parallelogram in m -dimensional space. Indeed if $\alpha_1, \dots, \alpha_n$ are just arbitrary n vectors in m -dimensional Euclidean space then if $n < m$, the formula (1.76) is failed (matrix A is $m \times n$ matrix), but formula (1.78) works. For example the area of parallelogram formed by arbitrary vectors $\mathbf{a}_1, \mathbf{a}_2$ in \mathbf{E}^n is equal to

$$\sqrt{\det \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}} = \sqrt{\det \begin{pmatrix} \langle \mathbf{a}_1, \mathbf{a}_1 \rangle & \langle \mathbf{a}_1, \mathbf{a}_2 \rangle \\ \langle \mathbf{a}_2, \mathbf{a}_1 \rangle & \langle \mathbf{a}_2, \mathbf{a}_2 \rangle \end{pmatrix}}.$$

1.5.2 Examples of calculating volume element

Consider first very simple example: Volume element of plane in Cartesian coordinates, metric $g = dx^2 + dy^2$. Volume element is equal to

$$\sqrt{\det g} dx dy = \sqrt{\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} dx dy = dx dy$$

Volume of the domain D is equal to

$$V(D) = \int_D \sqrt{\det g} dx dy = \int_D dx dy$$

⁷We see that n -dimensional parallelepiped $\Pi_{\{\mathbf{a}_i\}}$ in new coordinates t^i corresponding to the basis $\{\mathbf{a}_i\}$ becomes n -dimensional cube, Standard Euclidean metric $G = dx^i \delta_{ik} dx^k$ (in orthonormal basis $\{e_i\}$) transforms to $G = dx^i \delta_{ik} dx^k = (a_m^i dt^m) \delta_{ik} (a_n^k dt^n) = (A^T A)_{mn} dt^m dt^n$ and

$$\text{Volume} \Pi_{\{\mathbf{a}_i\}} = \int_{\mathbf{x} \in \Pi} dx^1 \dots dx^n = \int_{0 \leq t_i \leq 1} \sqrt{G} dt^1 \dots dt^n = \sqrt{\det G} = \sqrt{\det A^T A}.$$

If we go to polar coordinates:

$$x = r \cos \varphi, y = r \sin \varphi \quad (1.79)$$

Then we have for metric:

$$G = dr^2 + r^2 d\varphi^2$$

because

$$dx^2 + dy^2 = (dr \cos \varphi - r \sin \varphi d\varphi)^2 + (dr \sin \varphi + r \cos \varphi d\varphi)^2 = dr^2 + r^2 d\varphi^2 \quad (1.80)$$

Volume element in polar coordinates is equal to

$$\sqrt{\det g} dr d\varphi = \sqrt{\det \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}} dr d\varphi = r dr d\varphi.$$

Lobachesvsky plane.

In coordinates x, y ($y > 0$) metric $G = \frac{dx^2 + dy^2}{y^2}$, the corresponding matrix $G = \begin{pmatrix} 1/y^2 & 0 \\ 0 & 1/y^2 \end{pmatrix}$. Volume element is equal to $\sqrt{\det g} dx dy = \frac{dx dy}{y^2}$.

Sphere in stereographic coordinates In stereographic coordinates

$$G = \frac{4R^4(du^2 + dv^2)}{(R + u^2 + v^2)^2} \quad (1.81)$$

(It is isometric to the sphere of the radius R without North pole in stereographic coordinates (see the Homeworks.))

Calculate its volume element and volume. It is easy to see that:

$$G = \begin{pmatrix} \frac{4R^4}{(R^2 + u^2 + v^2)^2} & 0 \\ 0 & \frac{4R^4}{(R^2 + u^2 + v^2)^2} \end{pmatrix} \quad \det g = \frac{16R^8}{(R^2 + u^2 + v^2)^4} \quad (1.82)$$

and volume element is equal to $\sqrt{\det g} du dv = \frac{4R^4 du dv}{(R^2 + u^2 + v^2)^2}$

One can calculate volume in coordinates u, v but it is better to consider homothety $u \rightarrow Ru, v \rightarrow Rv$ and polar coordinates: $u = Rr \cos \varphi, v = Rr \sin \varphi$. Then volume form is equal to $\sqrt{\det g} du dv = \frac{4R^4 du dv}{(R^2 + u^2 + v^2)^2} = \frac{4R^2 r dr d\varphi}{(1 + r^2)^2}$.

Now calculation of integral becomes easy:

$$V = \int \frac{4R^2 r dr d\varphi}{(1 + r^2)^2} = 8\pi R^2 \int_0^\infty \frac{r dr}{(1 + r^2)^2} = 4\pi R^2 \int_0^\infty \frac{du}{(1 + u)^2} = 4\pi R^2.$$

Domain in Lobachevsky plane.

Consider in Lobachevsky plane the domain D_a such that

$$D_a = \{x, y: \quad x^2 + y^2 \geq 1, |y| \leq a\}, (|a| \leq 1). \quad (1.83)$$

Remark Note that vertical lines and half-circle are geodesics. One can see that the distance between these lines tends to zero. (We will study it later). If we denote by A a point $(a, \sqrt{1-a^2})$ and by B the point $(-a, \sqrt{1-a^2})$, then the domain D_a can be considered as a ‘triangle’ with vertices at the point A, B, C where C is a point at infinity. The meaning of this remark we will study later.

One can calculate the area of this domain, using area form on Lobachevsky plane

$$V(D_a) = \int_{-a \leq y \leq a, x^2 + y^2 \geq 1} \frac{dx dy}{y^2} = 2 \arcsin a \quad (1.84)$$

(See in detail Homework) We will discuss later the geometrical meaning of this formula.

Segment of the sphere.

Consider sphere of the radius a in Euclidean space with standard Riemannian metric

$$a^2 d\theta^2 + a^2 \sin^2 \theta d\varphi^2$$

This metric is nothing but first quadratic form on the sphere (see (1.3.3)). The volume element is

$$\sqrt{\det g} d\theta d\varphi = \sqrt{\det \begin{pmatrix} a^2 & 0 \\ 0 & a^2 \sin^2 \theta \end{pmatrix}} d\theta d\varphi = a^2 \sin \theta d\theta d\varphi$$

Now calculate the volume of the segment of the sphere between two parallel planes, i.e. domain restricted by parallels $\theta_1 \leq \theta \leq \theta_0$: Denote by h be the height of this segment. One can see that

$$h = a \cos \theta_0 - a \cos \theta_1 = a(\cos \theta_0 - \cos \theta_1)$$

There is remarkable formula which express the area of segment via the height h :

$$V = \int_{\theta_1 \leq \theta \leq \theta_0} (a^2 \sin \theta) d\theta d\varphi = \int_{\theta_0}^{\theta_1} \left(\int_0^{2\pi} (a^2 \sin \theta) d\varphi \right) d\theta =$$

$$\int_{\theta^1}^{\theta^0} 2\pi a^2 \sin \theta d\theta = 2\pi a^2 (\cos \theta_0 - \cos \theta_1) = 2\pi a (a \cos \theta_0 - a \cos \theta_1) = 2\pi a h \quad (1.85)$$

E.g. for all the sphere $h = 2a$. We come to $S = 4\pi a^2$. It is remarkable formula: area of the segment is a polynomial function of radius of the sphere and height (Compare with formula for length of the arc of the circle)