Homework 5

1 Calculate the Christoffel symbols of the canonical flat connection in $\mathbf{E}^3$ in

a) cylindrical coordinates ($x = r\cos\varphi, y = r\sin\varphi, z = h$),

b) spherical coordinates.

(For the case b) try to make calculations at least for components $\Gamma^r_{rr}, \Gamma^r_{r\theta}, \Gamma^r_{r\varphi}, \Gamma^r_{\theta\theta}, \dots, \Gamma^r_{\varphi\varphi}$)

2 Let ∇ be an affine connection on a 2-dimensional manifold M such that in local coordinates (u, v) it is given that $\Gamma^u_{uv} = v$, $\Gamma^v_{uv} = 0$.

Calculate the vector field $\nabla_{\frac{\partial}{\partial u}} \left(u\frac{\partial}{\partial v}\right)$.

3 Let ∇ be an affine connection on the 2-dimensional manifold M such that in local coordinates (u, v)

$$\nabla_{\frac{\partial}{\partial u}} \left(u\frac{\partial}{\partial v}\right) = (1+u^2)\frac{\partial}{\partial v} + u\frac{\partial}{\partial u}\,.$$

Calculate the Christoffel symbols Γ^u_{uv} and Γ^v_{uv} of this connection.

4 Let ∇ be an affine connection on a 2-dimensional manifold M such that, in local coordinates (x, y), all Christoffel symbols vanish except $\Gamma^x_{xx} = xy$, $\Gamma^y_{xx} = -1$ and $\Gamma^y_{yy} = y$. Show that for the vector field $\mathbf{X} = \partial_x + x\partial_y$,

$$\nabla_{\mathbf{X}}\mathbf{X} = xy\mathbf{X}\,.$$

5 a) Consider a connection such that its Christoffel symbols are symmetric in a given coordinate system: $\Gamma^i_{km} = \Gamma^i_{mk}$.

Show that they are symmetric in an arbitrary coordinate system.

b*) Show that the Christoffel symbols of connection ∇ are symmetric (in any coordinate system) if and only if

$$\nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X}, \mathbf{Y}] = 0\,,$$

for arbitrary vector fields $\mathbf{X}, \mathbf{Y}$.

c)† Consider for an arbitrary connection the following operation on the vector fields:

$$S(\mathbf{X}, \mathbf{Y}) = \nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X}, \mathbf{Y}]$$

and find its properties[(1)].

---

(1) This equation defines $\begin{pmatrix}1\\2\end{pmatrix}$- tensor, *torsion*.

6 Consider the surface M in the Euclidean space $\mathbf{E}^n$. Calculate the induced connection in the following cases

a) $M = S^1$ in $\mathbf{E}^2$,

b) M— parabola $y = x^2$ in $\mathbf{E}^2$,

c) cylinder in $\mathbf{E}^3$.

d) cone in $\mathbf{E}^3$.

e) sphere in $\mathbf{E}^3$.

f) saddle $z = xy$ in $\mathbf{E}^3$

7 Let ∇_1, ∇_2 be two different connections. Let ${}^{(1)}\Gamma^i_{km}$ and ${}^{(2)}\Gamma^i_{km}$ be the Christoffel symbols of connections ∇_1 and ∇_2 respectively.

a) Find the transformation law for the object : $T^i_{km} = {}^{(1)}\Gamma^i_{km} - {}^{(2)}\Gamma^i_{km}$ under a change of coordinates. Show that it is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ tensor.

8† Let $\mathbf{K}$, $\mathbf{X}$ be vector fields on manifold M, and ∇ connection. Consider the operation

$$\mathbf{K}\,,\mathbf{X} \mapsto A_{\mathbf{K}}(\mathbf{X}) = \nabla_{\mathbf{K}}\mathbf{X} - \mathcal{L}_{\mathbf{K}}\mathbf{X}\,, (\mathcal{L} \text{ is a Lie derivative,} \quad \mathcal{L}_{\mathbf{X}}\mathbf{Y} = [\mathbf{X}, Y]) \tag{1}$$

$$\text{a) Show that for an arbitrary function } f\,, A_{\mathbf{K}}(f\mathbf{X}) = f\mathbf{A}_{\mathbf{K}}(\mathbf{X})\,. \tag{2}$$

This condition implies that equation (1) defines *linear operation on tangent vectors*, i.e. it is well defined on tangent vectors (not vector fields) and it is linear[2)].

b) Show that linear operator $A_{\mathbf{K}}$ is equal to

$$A_{\mathbf{K}}(\mathbf{X}) = \nabla_{\mathbf{X}}\mathbf{K} + S(\mathbf{K}, \mathbf{X})\,,$$

where S the torsion of connection (see exercise 5c) above).

[2)] In other words for a given vector $\mathbf{X}_0$ tangent to manifold M at the given point $\mathbf{p}$, $\mathbf{X}_0 \in T_{\mathbf{p}}M$, consider *an arbitrary vector field* $\mathbf{X}$ passing via this vector, i,e, such that value of vector field at the given point $\mathbf{p}$ coincides with the vector $\mathbf{X}_0$,$\mathbf{X}\big|_{\mathbf{p}} = \mathbf{X}_0$. Condition (2) tells that the answer at the point $\mathbf{p}$ does not depend on a choice of vector field passing through vector $\mathbf{X}_0$. It depends only on the value of this vector field at the point $\mathbf{p}_0$. Indeed let two vector fields $\mathbf{X}, \tilde{\mathbf{X}}$ coincide at the point $\mathbf{p}$, i.e. the vector field $\tilde{\mathbf{X}} - \mathbf{X}$ vanishes at the point $\mathbf{p}$. Moreover Hadamard lemma (it states: if smooth function g vanishes at the origin, then $g = \sum_i x^i h_i(x)$, where $h_i(x)$ are also smooth.). tells that in this case vector field $\tilde{\mathbf{X}} - \mathbf{X}$ is linear combination of vector fields with coefficients vanishing at the point $\mathbf{p}$: $\tilde{\mathbf{X}} - \mathbf{X} = \sum_a h_a(x)\mathbf{T_a}$, where all $h_a(x)$ vanish at the point $\mathbf{p}$. Hence due to (2)

$$A_{\mathbf{K}}(\tilde{X} - \mathbf{X})|_{\mathbf{p}} = A_{\mathbf{K}}\left(\sum_a h_a(x)\mathbf{T_a}\right)|_{\mathbf{p}} = \sum_a h_a(x)|_{\mathbf{p}} A_{\mathbf{K}}(\mathbf{T_a}) = \mathbf{0}\,.$$