Homework 1. Solutions

1 *Show that the condition of non-degeneracy for a symmetric matrix $||g_{ik}||$ follows from the condition that this matrix is positive-definite.*

Solution Suppose $\det g = 0$, i.e. g is degenerate matrix (rows and columns of the matrix are linear dependent). Then there exists non-zero vector $\mathbf{x} = (x^1, x^2)$ such that $g_{ik}x^k = 0$, hence $g_{ik}x^i x^k = 0$ for $\mathbf{x} \neq 0$. Contradiction to the condition of positive-definiteness.

2 *Let (u, v) be local coordinates on 2-dimensional Riemannian manifold M. Let Riemannian metric be given in these local coordinates by the matrix*

$$||g_{ik}|| = \begin{pmatrix} A(u,v) & B(u,v) \\ C(u,v) & D(u,v) \end{pmatrix}, \tag{2}$$

where $A(u,v), B(u,v), C(u,v), D(u,v)$ are smooth functions. Show that the following conditions are fulfilled:

a) $B(u,v) = C(u,v)$,

b) $A(u,v)D(u,v) - B(u,v)C(u,v) \neq 0$,

c) $A(u,v) > 0$,

d) $A(u,v)D(u,v) - B(u,v)C(u,v) > 0$.*

e) Show that conditions a), c) and d) are necessary and sufficient conditions for matrix $||g_{ik}||$ to define locally a Riemannian metric.*

Solution

Consider Riemannian scalar product $G(\mathbf{X}, \mathbf{Y}) = g_{ik}X^iY^k$.

a) The condition that $G(\mathbf{X}, \mathbf{Y}) = G(\mathbf{Y}, \mathbf{X})$ means that $g_{ik} = g_{ki}$, i.e. $B(u, v = C(u, v))$.

b) $\det G = A(u,v)D(u,v) - B(u,v)C(u,v) = AD - B^2 \neq 0$ since it is non-degenerate (see the solution of exercise 1)

c) Consider quadratic form $G(\mathbf{x}, \mathbf{x}) = g_{ik}x^ix^k = Ax^2 + 2Bxy + Dy^2$. (We already know that $B = C$) Positive -definiteness means that $G(\mathbf{x}, \mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$. In particular if we put $\mathbf{x} = (1, 0)$ we come to $G(\mathbf{x}, \mathbf{x}) = A > 0$. Thus $A > 0$.

d) Consider quadratic form $G(\mathbf{x}, \mathbf{x}) = g_{ik}x^ix^k = Ax^2 + 2Bxy + Dy^2$. We have an identity

$$G(\mathbf{x}, \mathbf{x}) = g_{ik}x^ix^k = Ax^2 + 2Bxy + Dy^2 = \frac{(Ax + By)^2 + (AD - B^2)y^2}{A}. \tag{2}$$

*We already know that $A > 0$ (take $\mathbf{x} = (x, 0)$). Now take $\mathbf{x} = (x, y)$: $Ax + By = 0$ (e.g. $\mathbf{x}$=(-B,A)) we come to $G(\mathbf{x}, \mathbf{x}) = \frac{(AD-B^2)y^2}{A} > 0$. Hence $(AD - B^2) = \det G > 0$ *.*

e) it follows from condition a) that matrix (1) is symmetric. It follows from condition (2) and equation (2) that $G(\mathbf{x}, \mathbf{x}) > 0$ for any non-zero vector $\mathbf{x}$.

3 *Consider two-dimensional Riemannian manifold with Euclidean metric $G = dx^2 + dy^2$. How this metric will transform under arbitrary linear transformation $\begin{cases} x = ax' + by' \\ y = cx' + dy' \end{cases}$?*

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* *This special trick works good for dimension is $n = 2$. We could notice that A and $AD - B^2$ are principal main minors of the matrix G. In the general case (if G is $n \times n$ symmetric matrix) using triangular transformations one can show that quadratic form $A(\mathbf{X}, \mathbf{X}) = a_{ik}x^ix^k$ (and respectively) is positive-definite if and only if all the leading principal minors Δ_k are positive (leading Principal minor Δ_k of the matrix A is a determinant of the matrix formed by first k columns and first k rows of the matrix A.). In this case matrix G_{ik} of bilinear form is transformed to unity matrix.*

Solution: Perform straightforward calculations: $dx = adx' + bdy'$ *and* $dy = cdx' + dy'$. *Hence*

$$G = dx^2 + dy^2 = (adx' + bdy')^2 + (cdx' + dy')^2 = (a^2 + c^2)(dx')^2 + 2(ab + cd)dx'dy' + (b^2 + d^2)(dy')^2 .$$

In coordinates (x, y) $||g_{ik}|| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *and in coordinates* (x', y') $||g'_{ik}|| = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix}$.

4 *Consider two-dimensional Riemannian manifold with Riemannian metric* $G = du^2 + 2bdudv + dv^2$, *where* b *is a constant.*

a) Show that $b^2 < 1$

b) Find new coordinates x, y *such that under a "triangular" linear transformation* $\begin{cases} x = u + \beta v \\ y = \delta v \end{cases}$ *metric* G *transforms to the Euclidean metric* $dx^2 + dy^2$. *(Find parameters* β, δ *of this linear transformation)*

c) Write down the metric $G = du^2 + 2bdudv + dv^2$ *in new coordinates* r, θ *where* $\begin{cases} u = r\cos\theta \\ y = r\sin\theta \end{cases}$

a) Solution: Matrix of the metric G $\begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix}$ *is positive definite, hence* $\det g = 1 - b^2 > 0$, *i.e.* $b^2 < 1$.

Another solution: for any non-zero vector $\mathbf{x}$, $G(\mathbf{x}, \mathbf{x}) > 0$. *Consider* $\mathbf{x} = (t, 1)$. *Then for an arbitrary* t $(t, 1) \neq 0$ *and* $G(\mathbf{x}, \mathbf{x}) = t^2 + 2bt + 1 > 0$. *Hence polynomial* $t^2 + 2bt + 1$ *has no real roots, i.e.* $b^2 < 1$.

One can see that the condition $b^2 < 1$ *is not only necessary but it is sufficient condition for* G *to be a metric.*

b) Solution:

Consider triangular transformation $\begin{cases} x = u + \beta v \\ y = \delta v \end{cases}$. *Then*

$$G = dx^2 + dy^2 = (du + \beta dv)^2 + \delta^2 dv^2 = du^2 + 2\beta dv + (\beta^2 + \delta^2)dv^2 = du^2 + 2bdudv + dv^2$$

if we put $\beta = b$ *and* $\delta = \sqrt{1 - b^2}$. *(*$b^2 < 1$ *according 4a). We see that with linear triangular transformation metric* $u^2 + 2bdudv + dv^2$ *can be transformed to Pythagorean one.*

c) Solution: If $\begin{cases} u = r\cos\theta \\ y = r\sin\theta \end{cases}$ *then*

$$G = du^2 + 2bdudv + dv^2 = (\cos\theta dr - r\sin\theta d\theta)^2 + 2b(\cos\theta dr - r\sin\theta d\theta)(\sin\theta dr + r\cos\theta d\theta) + (\sin\theta dr + r\cos\theta d\theta)^2 =$$

$$(\cos^2\theta + 2b\sin\theta\cos\theta + \sin^2\theta)dr^2 + (-2r\cos\theta\sin\theta + 2br\cos^2\theta - 2br\sin^2\theta + 2r\cos\theta\sin\theta)drd\theta +$$

$$(r^2\sin^2\theta - 2br^2\sin\theta\cos\theta + r^2\cos^2\theta)d\theta^2 =$$

$$(1 + b\sin 2\theta)dr^2 + 2br\cos 2\theta drd\theta + r^2(1 - b\sin 2\theta)d\theta^2 .$$

In the case $b = 0$ *we come to standard Pythagorean metric in polar coordinates.*

5 *Let* γ *be a curve in Riemannian manifold given in local coordinates by parametric equation* $x^i = x^i(t)$, $t_1 \leq t \leq t_2$. *Show that the length of this curve*

$$L = \int_{t_1}^{t_2} \sqrt{g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t)}dt$$

does not change under arbitrary reparameterisation $t = t(\tau)$.

Solution: See the end of Subsection 1.3 of Lecture notes.

6 * *Show that* $G = dx^2 + dy^2 + cdz^2$ *in* $\mathbf{R}^3$ *defines Riemannian metric iff* $c > 0$.

* *Find null-vectors of pesudo-Riemannian metric* G *if* $c < 0$.

It is symmetric matrix which is positive definite iff $c > 0$. *If* $c < 0$ *condition of positive-definiteness is failed, but matrix* G *is still non-degenerate. Null-vector* $\mathbf{X} = (x, y, z)$: $G_{ik}X^iX^k = x^2 + y^2 + cz^2 = 0$, *i.e. vector* $\mathbf{X}$ *belongs to the cone* $x^2 + y^2 - cz^2 = 0$.