Solutions of Homework 7

(Only first four pages contain solutions of compulsory exercises.)

1

Calculate the integral of the form $\omega = e^{-y}dx + \sin x dy$ over the segment of straight line which connects the points $A = (1,1)$, $B = (2,3)$. At what extent an answer depends on a chosen parameterisation?

Choose any parameterisation of this segment, e.g. $x = 1+t, y = 1+2t$, $0 \leq t \leq 1$. Then $\mathbf{v} = (v_x, v_y) = (1,2)$ $(x_t = 1, y_t = 2)$ and

$$\int_C e^{-y}dx + \sin x dy = \int_0^1 \left(e^{-(1+2t)}x_t + \sin(1+t)y_t\right)dt = \int_0^1 \left(e^{-(1+2t)} + 2\sin(1+t)\right)dt\,.$$

What happens if we choose another parameterisation, i.e. consider reparameterisation $t = t(\tau)$. Answer remains the same if the reparameterisation does not change orientation i.e. $t'(\tau) > 0$. This means that starting and ending points of the curve remain the same. Answer is multiplied on -1 in other case: if the reparameterisation changes orientation i.e. $t'(\tau) < 0$.

2

Calculate the integral of the form $\omega = xdy$ over the following curves
a) closed curve $x^2 + y^2 = 12y$.
b) arc of the ellipse $x^2 + y^2/9 = 1$ defined by the condition $y \geq 0$.

a) Consider closed curve $x^2 + y^2 = 12y$. We have

$$0 = x^2 + y^2 - 12y = x^2 + (y-6)^2 - 36\,.$$

That is this curve is a circle of the radius 6 with a centre at the point $(0,6)$. The parametric equation of this circle is

$$\begin{cases} x = 6\cos t \\ y = 6 + 6\sin t \end{cases}, \quad 0 \leq t \leq 2\pi\,.$$

We have that

$$\mathbf{v} = \begin{pmatrix} -6\sin t \\ 6\cos t \end{pmatrix} \quad \text{and} \quad \omega(\mathbf{v}) = xdy(v_x\partial_x + v_y\partial_y) = xv_y = 6x(t)\cdot 6\cos t = 36\cos^2 t\,,$$

$$\int_C \omega = \int_0^{2\pi} \omega(\mathbf{v}(t))dt = \int_0^{2\pi} 36\cos^2 t\, dt = 36 \cdot \frac{2\pi}{2} = 36\pi\,.$$

So for an arbitrary parameterisation answer will be $\pm 36\pi$. (36π if orientation is the same and -36π if opposite) E.g. if we change parameterisation above on the parameterisation $\tau = -t$ then integral will change a sign, since this reparameterisation changes the orientation of the circle.

b) For the the arc of the ellipse $x^2 + y^2/9 = 1, y \geq 0$ choose a parameterisation: $\begin{cases} x = \cos t \\ y = 3\sin t \end{cases}, 0 \leq t \leq \pi$. Then $\mathbf{v} = (-\sin t, 3\cos t)$ and

$$\int_C \omega = \int_0^{\pi} \omega(\mathbf{v})dt = \int_0^{\pi} x(t)y_t dt = \int_0^{\pi 2} 3\cos t\cos t dt = \int_0^{\pi} 3\cos^2 t dt = 3\pi/2$$

So for an arbitrary parameterisation answer will be $\pm 3\pi/2$, sign is depending on orientation of parameterisation. E.g. if we change parameterisation above on the parameterisation $\tau = -t$ then integral will change a sign, since this reparameterisation changes the orientation of the ellipse.

3

Calculate the integral of the form $\omega = 5xdy + 4ydx$ over the upper arc of the unit circle which passes through the point $A = (4,0)$ and the point $B = (2,0)$.

a) The equation of the arc is $(x-3)^2 + y^2 = 1, y \geq 0$. We know that answer up to a sign does not depend on parameterisation. Choose an arbitrary parameterisation of this curve, e.g.

$$\begin{cases} x = 3 + \cos t \\ y = \sin t \end{cases}, \quad 0 \leq t \leq \pi\,.$$

Then in the same way as in the previous exercise $\mathbf{v} = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$ and

$$\omega(\mathbf{v}(t)) = (5xdy+4ydx)(x_t\partial_x+y_t\partial_y)) = 5xy_t+4yx_t = 5(3+\cos t)\cos t+4\sin t(-\sin t) = 15\cos t+5\cos^2 t-4\sin^2 t\,,$$

$$\int_C \omega = \int_0^\pi \omega(\mathbf{v})dt = \int_0^\pi (15\cos t + 5\cos^2 t - 4\sin^2 t)dt = 15\cdot 0 + 5\cdot\frac{1}{2}\pi - 4\cdot\frac{1}{2}\pi = \frac{\pi}{2}\,.$$

So for an arbitrary parameterisation answer will be $\pm\frac{\pi}{2}$. ($\frac{\pi}{2}$ if orientation is the same and $-\frac{\pi}{2}$ if opposite)

Exact forms

4

Calculate the integral $\int_C \omega$ where $\omega = xdx + ydy$ and C is

a) the straight line segment $x = t, y = 1-t$, $0 \leq t \leq 1$

b) the segment of parabola $x = t$, $y = 1 - t^n$, $0 \leq t \leq 1$, $n = 2, 3, 4, \dots$

c) **an arbitrary** *curve starting at the point $(0,1)$ and ending at the point $((1,0)$.*

For any of these curves we can perform calculations naively just using definition of integral E.g. for the curve a)

$$\int_C \omega = \int_0^1 (x(t)x_t + y(t)y_t)dt = \int_0^1 (t + (1-t)(-1))dt = \int_0^1 (2t-1)dt = 0\,,$$

for the curve b) if $n = 2$

$$\int_C \omega = \int_0^1 (x(t)x_t + y(t)y_t)dt = \int_0^1 (x(t)x_t + y(t)y_t)dt = \int_0^1 (t + (1-t^2)(-2t))dt = \int_0^1 (2t^3 - 3t^2)dt = 0\,,$$

for the curve b) in general case:

$$\int_C \omega = \int_0^1 (x(t)x_t + y(t)y_t)dt = \int_0^1 (x(t)x_t + y(t)y_t)dt =$$

$$\int_0^1 (t + (1-t^n)(-nt^{n-1}))dt = \int_0^1 (t - nt^{n-1} + nt^{2n-1})dt = 0\,.$$

But there is another nice way to calculate these integrals. We immediately come to these results in a clear and elegant way if we use the fact that $\omega = xdx + ydy$ is an **exact form**, i.e. $\omega = df$ where $f = \frac{x^2+y^2}{2}$. Indeed using Theorem we see that for an arbitrary curve starting at the point $A = (0,1)$ and ending at the point $B = (1,0)$

$$\int_C \omega = \int_C df = f(x,y)|_A^B = f(1,0) - f(0,1) = 0\,.$$

5

Show that the form 1*-form* $\omega = 3x^2ydx + x^3dy$ *is an exact* 1*-form. Calculate integral of this form over the curves considered in exercises 2) and 3)*

One can see that $\omega = 3xydx + x^3dy = d(x^3y)$ $(d(x^3y) = \frac{\partial(x^3y)}{\partial x}dx + \frac{\partial(x^3y)}{\partial y}dy = 3x^2ydx + x^3dy.)$
It is an exact form.

Integral of this exact form over the circle $x^2 + y^2 = 12y$ (exercise 2a) equals to zero, since it is closed curve: starting and ending points coincide.

Integral of this exact form over the arc of the ellipse $x^2 + y/9 = 1(exercise 2b), y \geq 0$ and the integral over arc of the unit circle $x^2 + y^2 = 1, y > 0$ both are equal zero in spite of the fact that these curves are not closed. The reason is that the function $f = x^3y$ $(df = \omega)$ vanishes at starting and ending points of these curves.

The integral of this form over arc of the unit circle starting at the point $A = (4, 0)$ and ending at the point $(2, 0)$ (see the exercise 3) is equal to $\int_C \omega = f|_B^A = f(1, 0) = f(0, 1) = 0$ because $f = x^2y$ and $f(1, 0) = f(0, 1) = 0$. Answer is equal to zero. Hence it does not depend on orientation of the curve.

We can calculate the 1-form $\omega = 3x^2ydx + x^3dy$ just using formulae for changing of cooridnates: $x = r\cos\varphi, y = r\sin\varphi$ and the formulae for dx, dy $(dx = \cos\varphi dr - r\sin\varphi d\varphi,\ dy = \sin\varphi dr + r\sin\varphi d\varphi)$. This is not the nicest way. Much better to use the fact that $\omega = df$ is an exact form, calculate a function f in polar coordinates (which is much easier) and then calculate $\omega = df$ in polar coordinates:

We have $\omega = 3x^2ydx + x^3dy = d(x^3y)$, $f = x^3y = (r\cos\varphi)^3 r\sin\varphi = r^4\cos^3\varphi\sin\varphi = 2r^2\cos^2\varphi\sin 2\varphi$ $r^4(2\cos^2\varphi - 1)\sin 2\varphi + r^4\sin 2\varphi = r^4\left(\frac{1}{2}\sin 4\varphi + \sin 2\varphi\right)$. Hence

$$\omega = df = d\left(r^4\left(\frac{1}{2}\sin 4\varphi + \sin 2\varphi\right)\right) = 4r^3\left(\frac{1}{2}\sin 4\varphi + \sin 2\varphi\right)dr + 2r^4(\cos 4\varphi + \cos 2\varphi)\,d\varphi\,.$$

6.

Consider the following differential 1*-forms:*
a) xdx *, b)* xdy *c)* $xdx + ydy$ *, d)*$xdy + ydx$*, e)* $xdy - ydx$
f) $x^4dy + 4x^3ydx$ *, g)* $xdy + ydx + dz$ *, h)* $xdy - ydx + dz$ *.*
a) Show that 1*-forms a), c), d), f) and g) are exact forms*
b)Why 1*-forms b), e) and h) are not exact?*

a) It is an exact form since $xdx = df$ where $f = \frac{x^2}{2} + c$, where c is a constant.

b) Suppose $\omega = xdy$ is an exact form: $\omega = df = f_xdx + f_ydy$. Hence $f_x = 0, f_y = x$. We see that $f_{xy} = \frac{\partial}{\partial x}f_y = 1$. On the other hand $f_{yx} = \frac{\partial}{\partial y}f_x = f_{xy} = 0$. Contradiction.

Another solution; There is another way to show why $\omega = xdy$ is not an exact form. We already calculated that the integral of the form $\omega = xdy$ over the closed circle $x^2 + y^2 = 12y$ is equal to $36\pi \neq 0$. (see the exercise 2a) and its solution above)Hence ω is not exact, since the integral of an exact form over an arbitrary closed curve is equal to zero.

c) It is an exact form since $xdx + ydy = d\left(\frac{x^2+y^2}{2} + c\right)$, ($c$ is a constant).

d)It is an exact form since $xdy + ydx = d(xy + c)$,where c is a constant.

e)Suppose $\omega = xdy - ydx$ is an exact form: $\omega = df = f_xdx + f_ydy$. Hence $f_x = -y, f_y = x$. We see that $f_{xy} = 1$. On the other hand $f_{yx} = f_{xy} = -1$. Contradiction.

f) It is an exact form since $x^4dy + 4x^3ydx = d(x^4y + c)$, where c is a constant.

g) It is an exact form since $xdy + ydx + dz = d(xy + z + c)$, where c is a constant.

h) Suppose $\omega = xdy - ydx + dz$ is an exact form: $\omega = df = f_xdx + f_ydy + f_zdz$. Hence $f_x = -y, f_y = x, f_z = 1$. We see that $f_{xy} = 1$. On the other hand $f_{yx} = f_{xy} = -1$. Contradiction.

7

Consider 1-form $\omega = xdy + aydx$ where a is a constant.

a) Find an integral of this form over a closed curve defined by equation $x^2 + y^2 - 4x - 4y + 7 = 0$.
b) Explain why the form ω is exact if $a = 1$
b) Explain why the form ω is not exact if $a \neq 1$

a) We see that the closed curve is a unit circle with centre at the point $(2, 2)$:

$$0 = x^2 + y^2 - 4x - 4y + 7 = (x-2)^2 + (y-2)^2 - 1\,.$$

Choose a parameterisation $\begin{cases} x = 2 + \cos t \\ y = 2 + \sin t \end{cases}$. We see that velocity vector $\mathbf{v} = x_t\partial_x + y_t\partial_y == -\sin t\partial_x + \cos t\partial_y$ and

$$\omega(\mathbf{v}(t)) = (xdy + aydx)(-\sin t\partial_x + \cos t\partial_y) = x(t)\cos t - ay(t)\sin t = (2+\cos t)\cos t - a(2+\sin t)\sin t\,.$$

Hence we have that

$$\int_C \omega = \int_0^{2\pi} \omega(\mathbf{v}(t))dt = \int_0^{2\pi} \left(2\cos t + \cos^2 t - 2a\sin t - a\sin^2 t\right) dt = \frac{2\pi(1-a)}{2} = (1-a)\pi\,,$$

since $\int_0^{2\pi}\cos t dt = \int_0^{2\pi}\sin t dt = 0$ and $\int_0^{2\pi}\cos^2 t dt = \int_0^{2\pi}\sin^2 t dt = \pi$.

b) If $a = 1$ then the form $\omega = xdy + ydx = d(xy)$ is an exact 1-form. In this case the integral over an arbitrary closed curve vanishes. Sure it vanishes over the circle above too.

c) If $a \neq 1$ then the form $\omega = xdy + aydx$ is not exact. Indeed we have shown above that the integral of ω over the circle $x^2 + y^2 + 4x - 4y + 7$ is equal to $\pi(1-a)$. On the other hand $\pi(1-a) \neq 0$ if $a \neq 1$. Hence $\omega = xdy + aydx$ is not exact for $a \neq 1$ since for an exact form the integral over an arbitrary closed curve is equal to zero.

Remark Please note that if the integral of the 1-form ω over the given closed curve C vanishes then this does not mean that the form ω is necessarily an exact form. (The inverse is true!)

8 *

Calculate the integral of the form $\sigma = \frac{xdy-ydx}{x^2}$ over the curve $x^2 + y^2 = 4x + 4y + 7$ consdered in the previous exercise.

The curve $x^2 + y^2 - 4x - 4y + 7$ is a unit circle with centre at the point $(2, 2)$. This is in the quadrant $x > 0, y > 0$. The form $\sigma = \frac{xdy-ydx}{x^2} = d\left(-\frac{y}{x}\right)$ is well-defined exact form on this circle. Hence the integral of this form over the circle is equal to zero.

All the exercises below are not compulsory

9†

Consider one-form

$$\omega = \frac{xdy - ydx}{x^2 + y^2} \tag{1}$$

This form is defined in $\mathbf{E}^2 \setminus 0$.

Calculate differential of this form.
Write down this form in polar coordinates
Find a function f such that $\omega = df$.
Is this function defined in the same domain as ω?

First calculate differential in cartesian coordinates with "brute force"

$$d\omega = d\left(\frac{xdy - ydx}{x^2+y^2}\right) = \frac{d(xdy - ydx)}{x^2+y^2} - (xdy - ydx)\wedge d\left(\frac{1}{x^2+y^2}\right) = \frac{2dx\wedge dy}{x^2+y^2} +$$

$$\frac{(xdy-ydx)\wedge d(x^2+y^2)}{(x^2+y^2)^2}=\frac{2dx\wedge dy}{x^2+y^2}+\frac{(xdy-ydx)\wedge(2xdx+2ydy)}{(x^2+y^2)^2}=$$

$$\frac{2dx\wedge dy}{x^2+y^2}+\frac{2x^2dy\wedge dx+2y^2dy\wedge dx}{(x^2+y^2)^2}=0\,.$$

Much more illuminating to write down this form in polar coordinates then calculate its differential. We know already that $xdy-ydx=r^2d\varphi$. Indeed

$dx=d(r\cos\varphi)=\cos\varphi dr-r\sin\varphi d\varphi=\frac{x}{r}dr-yd\varphi$ and $dy=d(r\sin\varphi)=\sin\varphi dr+r\cos\varphi d\varphi=\frac{y}{r}dr+xd\varphi$.

Hence

$$xdy-ydx=x\left(\frac{y}{r}dr+xd\varphi\right)-y\left(\frac{x}{r}dr-yd\varphi\right)=(x^2+y^2)d\varphi\ \text{ and }\ \frac{xdy-ydx}{x^2+y^2}=d\varphi$$

Hence the form is closed.

For the form $\omega=\frac{xdy-ydx}{x^2+y^2}$ one can consider the function $f=\varphi=\arctan\frac{y}{x}$, such that $\omega=df$, but the function f is not well-defined on whole $\mathbf{E}^2$. It is well-defined e.g. we remove the ray $(-\infty,0]$.

Note that ω is defined in $\mathbf{E}^2\setminus 0$, but f is defined on $\mathbf{E}^2\setminus(-\infty,0]$.

On the other hand it is well defined in any domain where we can define one-valued continuous function $f=\varphi$, i.e. the domain does not contain a loop which rotates around origin. (The function $f=\varphi$ is multi-valued function in the domain $\mathbf{R}^2\setminus 0$ which contains loops rotating around origin). E.g. one can see that for an arbitrary convex domain which does not contain the origin, or for an arbitrary domain which does not contain a ray $[-\infty,0]$ a function $f=\varphi$ is well defined one-valued function.

10†

Calculate the integral of the form $\omega=\frac{xdy-ydx}{x^2+y^2}$ *over the curves*

a) circle $x^2+y^2=1$

b) circle $(x-3)^2+y^2=1$

c) ellipse $\frac{x^2}{9}+\frac{x^2}{16}=1$

As it follows from the previous exercise answer equals to $\pm 2\pi$ for the first curve and third curves and it is equal to zero for the second curve.

11†

What values can take the integral $\int_C\omega$ *if* C *is an arbitrary curve starting at the point* $(0,1)$ *and ending at the point* $((1,0)$ *and* $\omega=\frac{xdy-ydx}{x^2+y^2}$.

Answer is the same as in previous exercise: if the curve **does not pass the origin** then the integral is well-defined, It is equal $\frac{\pi}{2}+2\pi n$ if starting point of the curve is $(1,0)$ and ending point is

The integer n depends on the curve.

Remark Please, note that the form $\omega=\frac{xdy-ydx}{x^2+y^2}$ strictly speaking is not exact, because it is not defined for all points (it is not defined at origin) and moreover its "antiderivative" $f=\varphi$ $(\omega=df)$ is not well-defined function.

In the next exercise we show that for 1-forms which are defined in the whole $\mathbf{E}^2$ the exactness coincide with closeness.

12†

Let $\omega=a(x,y)dx+b(x,y)dy$ *be a closed form in* $\mathbf{E}^2$, $d\omega=0$.

Consider the function

$$f(x,y)=x\int_0^1 a(tx,ty)dt+y\int_0^1 b(tx,ty)dt \tag{2}$$

Show that

$$\omega=df\,.$$

This proves that an arbitrary closed form in $\mathbf{E}^2$ is an exact form. (Converse implication is always true.) Why we cannot apply the formula (2) to the form ω defined by the expression (1)?

Perform the calculations: $df = f_x d + f_y dy$.

$$f_x = \int_0^1 a(tx,ty)dt + x\int_0^1 a_x(tx,ty)tdt + y\int_0^1 b_x(tx,ty)tdt\,.$$

and

$$f_y = \int_0^1 b(tx,ty)dt + x\int_0^1 a_y(tx,ty)tdt + y\int_0^1 b_y(tx,ty)tdt\,.$$

On the other hand $d\omega = d(adx + bdy) = (b_x - a_y)dx \wedge dy = 0$. Hence $b_x = a_y$ and

$$f_x = \int_0^1 a(tx,ty)dt + x\int_0^1 a_x(tx,ty)tdt + y\int_0^1 a_y(tx,ty)tdt = \int_0^1 \left(\frac{d}{dt}\left(ta(tx,ty)\right)\right) = ta(tx,ty)\big|_0^1 = a(x,y)\,,$$

because

$$\frac{d}{dt}\left(ta(tx,ty)\right) = a(tx,ty) + xta_x(tx,ty) + yta_y(tx,ty).$$

Analogously

$$f_y = \int_0^1 b(tx,ty)dt + x\int_0^1 b_x(tx,ty)tdt + y\int_0^1 b_y(tx,ty)tdt = \int_0^1 \left(\frac{d}{dt}\left(tb(tx,ty)\right)\right) = tb(tx,ty)\big|_0^1 = b(x,y)\,,$$

We see that $f_x = a(x,y)$ and $f_y = b(x,y)$, i.e. $df = adx + bdy$ $\blacksquare$