Taylor series expansion formula

We calculate here

$$\Phi^*(t_1 G_1 + \ldots + t_n G_n)$$

Suppose that

$$S(x,l) = S(x) + \varphi(x)l + \frac{1}{2}A(x)l^2 + \frac{1}{6}T(x)l^3 + \ldots$$

Recall that for an arbitrary $g(y)$

$$\left[\Phi^*(g)\right](x) = g(y(x)) + S(x, l(x)) - y(x)l(x)\,,$$

where functions $y(x) = y_g(x), l(x) = l_g(x)$ can be defined from equations

$$y = \frac{\partial S(x,l)}{\partial l}\,, \quad l = \frac{\partial g(y)}{\partial y}\,.$$

We have that for $g \equiv 0$, $\Phi^*(g) = S(x)$, and

$$y_0(x) = \frac{\partial S(x,l)}{\partial l}||\varphi_{l=0} = \varphi(x)\,.$$

We have also that

$$\Phi(g + \varepsilon G) = \varepsilon G(y_g(x))\,.$$

1 Let t: $\quad t^2 = 0$. Then

$$\Phi(tG(y)) = tG\left(y_0(x)\right) = tG(\varphi(x))\,. \tag{1}$$

2 Calculate $\Phi^*(t_1 G_1 + t_2 G_2$ assuming that for parameters t_1, t_2

$$t_1^2 = t_2^2 = 0\,.$$

$$\Phi^*(t_1 G_1 + t_2 G_2 = \Phi^*(t_1 G_1) + t_2 G_2\left(y_{t_1 G_1}(x)\right)\,,$$

and

$$y(x) = y_{t_1 G_1}(x) = \left(\varphi(x) + A(x)l + \frac{1}{2}T(x)l^2\right)\big|_{l=t_1 G_1(y)} = \varphi(x) + t_1 A(x) G_1'(y) =$$

$$\varphi(x) + t_1 A(x) G_1'(\varphi(x))\,, \quad \text{since } t_1^2 = 0\,.$$

Hence using (1) we come to

$$\Phi^*(t_1G_1+t_2G_2 = \Phi^*(t_1G_1)+\mathbf{t}_2G_2\left(y_{t_1G_1}(x)\right) = t_1G_1(\varphi(x))+t_2G_2\left(\varphi(x)+t_1A(x)G_1'(\varphi(x))\right) =$$

$$t_1G_1(y)+t_2G_2(y)+t_2t_1G_2'(y)A(x)G_1'(y)\big|_{y=\varphi(x)}\,. \qquad (2)$$

3 Now calculate $\Phi^*(t_1G_1+t_2G_2+t_3G_3)$ assuming that

$$t_1^2 = t_2^2 = t_3^2 = 0\,.(3a)$$

$$\Phi^*(t_1G_1+t_2G_2+t_3G_3) = \Phi^*(t_1G_1+t_2G_2)+t_3G_3\left(y_{t_1G_1+t_2G_2}\right)\,.$$

We have using nilpotency conditions (3a) that

$$y = y_{t_1G_1+t_2G_2} = (\varphi+Al+\frac{1}{2}Tl^2)_{l=t_1G_1'+t_2G_2'} =$$

$$\varphi+A\left(t_1G_1'(y)+t_2G_2'(y)\right)+\frac{1}{2}T\left(t_1G_1'(y)+t_2G_2'(y)\right)^2 =$$

$$\varphi+A\left(t_1G_1'(y=\varphi+\ldots)+t_2G_2'(y=\varphi+\ldots)\right)+\frac{1}{2}T\left(t_1G_1'(\varphi)+t_2G_2'(\varphi)\right)^2$$

$$y+A\left(t_1G_1'(y+At_2G_2'(y))+t_2G_2'(y+At_1G_1'(y))\right)+T(x)t_1t_2G_1'(y)G_2'(y)\big|_{y=\varphi(x)} =$$

$$y+A(x)(t_1G_1'(y)+t_2G_2'(y))+t_1t_2A^2(x)\left(G_1'(y)G_2(y)'\right)'+T(x)t_1t_2G_1'(y)G_2'(y)$$

and using (2) we come to

$$\Phi^*(t_1G_1+t_2G_2+t_3G_3) = \Phi^*(t_1G_1+t_2G_2)+t_3G_3\left(y_{t_1G_1+t_2G_2}\right) =$$

$$t_1G_1(y)+t_2G_2(y)+t_2t_1G_2'(y)A(x)G_1'(y)$$

$$t_3G_3\Big(y+At_1G_1'(y)+t_2G_2'(y))+t_1t_2A^2\left(G_1'(y)G_2(y)'\right)'+Tt_1t_2G_1'(y)G_2'(y)\Big)$$

$$t_1G_1(y)+t_2G_2(y)+t_2t_1G_2'(y)A(x)G_1'(y)\big|_{y=\varphi(x)}+t_3G_3(y)+$$

$$t_3G_3'\Big(At_1G_1'(y)+t_2G_2'(y))+t_1t_2A^2\left(G_1'(y)G_2(y)'\right)'+Tt_1t_2G_1'(y)G_2'(y)\Big)+$$

$$\frac{1}{2}t_3G_3''\left(At_1G_1'(y)+t_2G_2'(y)\right)^2\big|_{y=\varphi(x)}\,.$$

Calculating we come to the answer:

$$\Phi^*(t_1G_1+t_2G_2+t_3G_3) = t_1G_1(y)+t_2G_2(y)+t_3G_3(y)+$$

$$A(x)\left(t_2t_1G_2'(y)G_1'(y)+t_3t_1G_3'(y)G_1'(y)+t_3t_2G_3'(y)G_2'(y)\right)+$$

$$t_3t_2t_1A^2(x)\left(G_1'(y)G_2'(y)G_3'(y)\right)'+T(x)t_3t_2t_1G_1'(y)G_2'(y)G_3(y)'\big|_{y=\varphi(x)}$$

Is it possible to recognize $S(x,l)$ by expansion of $\Phi^*(t_mG_m)$?