Homework 3

In all exercises we assume by default that Riemannian metric on embedded surfaces is induced by the Euclidean metric.

1 Show that surface of the cone $\begin{cases} x^2+y^2-k^2z^2=0 \\ z>0 \end{cases}$ in $\mathbf{E}^3$ is locally isometric to Euclidean plane.

2 a) Consider the conic surface C defined by the equation $x^2+y^2-z^2=0$ in $\mathbf{E}^3$. Consider a part of this conic surface between planes $z=0$ and $z=H>0$, and remove the line $z=y, x=0$ from this part of conic surface C. We come to the surface D defined by the conditions

$$\begin{cases} x^2+y^2-z^2=0 \\ 0<z<H \\ y\neq 0 \quad \text{if } x>0 \end{cases} .$$

Find a domain D' in Euclidean plane such that it is isometric to the surface D.

b) Find a shortest distance between points $A=(1,0,1)$ and $B=(-1,0,1)$ for an ant living on the conic surface C.

3 Consider plane with Riemannian metric given in cartesian coordinates (x,y) by the formula

$$G=\frac{a\left((dx)^2+(dy)^2\right)}{(1+x^2+y^2)^2} ,$$

and a sphere of S_r^2 of the radius r in the Euclidean space $\mathbf{E}^3$. Find a value of the parameter r such that this plane is locally isometric to the sphere S_r^2. Justify you answer. (You may use the formula for Riemannian metric on the sphere in stereographic coordinates.)

4 Consider catenoid: $x^2+y^2=\cosh^2 z$ and helicoid: $y-x\tan z=0$.

Find induced Riemannian metrics on these surfaces.

Show that these surfaces are locally isometric.

5 a) Consider the domain D on the cone $x^2+y^2-k^2z^2$ defined by the condition $0<z<H$. Find an area of this domain using induced Riemannian metric. Compare with the answer when using standard formulae.

6 Find an area of 2-dimensional sphere of radius R using explicit formulae for induced Riemannian metric in stereographic coordinates.

7 Show that two spheres of different radii in Euclidean space are not isometric to each other.

8 Find new local coordinates $u=u(x,y), v=v(x,y)$ in Euclidean space $\mathbf{E}^2$ such that $du^2+dv^2=dx^2+dy^2$. (You may assume that functions $u(x,y,v(x,y)$ are linear: $u=a+bx+cy$, $v=e+dx+fy$, where a,b,c,d are constants.)

Show that the transformation is a composition of translation, rotation and reflection.

† Will answer change if we allow arbitrary (not only linear functions $u(x,y), v(x,y)$)?

9 Let D be a domain in Lobachevsky plane which is lying between lines $x = a, x = -a$ and outside of the disc $x^2 + y^2 = 1$, $(0 < a < 1)$: $D = \{(x,y)\colon\ |x| < a, x^2 + y^2 > 1\}$,

a) Find the area of this domain.

b) Find the angles between lines and arc of the circle.

Lobachevsky plane, i.e. hyperbolic plane is the upper half plane with Riemannian metric $\frac{dx^2+dy^2}{y^2}$ *in cartesian coordinates* x, y *(*$y > 0$*).*

10† Find a volume of n-dimensional sphere of radius a. (You may use Riemannian metric in stereographic coordinates, or you may do it in other way... You just have to calculate the answer.)