Solutions of Homework 3

In all exercises we assume by default that Riemannian metric on embedded surfaces is induced by the Euclidean metric.

1 *Consider plane* $\mathbf{R}^2$ *with Riemannian metric given in Cartesian coordinates* (x, y) *by the formula*

$$G = \frac{a\left((dx)^2 + (dy)^2\right)}{(1 + x^2 + y^2)^2}\ , \tag{1}$$

and a sphere S_r $x^2 + y^2 + z^2 = r^2$ *(of the radius* r*) in the Euclidean space* $\mathbf{E}^3$.

Consider the following map F *from the plane* $\mathbf{R}^2$ *to the sphere*

$$F(x, y)\colon \begin{cases} u = rx \\ v = ry \end{cases},$$

where (u, v) *are stereographic coordinates of the sphere* $(u = \frac{rx}{r-z},\ v = \frac{ry}{r-z})$.

The map F *is a diffeomorphism of* $\mathbf{R}^2$ *on the sphere without North pole (the point* N *with coordinates* $x = 0, y = 0, z = r$*).*

a) Write down the Riemannian metric on the sphere in stereographic coordinates.

b) Find a value of parameter a *such that* F_p *is isometry of the plane* $\mathbf{R}^2$ *equipped with Riemannian metric (1) and* $S_r \backslash N$.

a) The Riemannian metric on the sphere in stereographic coordinates is $G_s = \frac{4r^4(du^2+dv^2}{(r^2+u^2+v^2)^2}$ (see lecture notes or Homework 2).

b) Calculate F^*G_s under the diffeomorphism F_p:

$$F^*\left(\frac{4r^4(du^2 + dv^2}{(r^2 + u^2 + v^2)^2}\right) = \left(\frac{4r^4(du^2 + dv^2}{(r^2 + u^2 + v^2)^2}\right)\big|_{u=rx,v=ry} = \left(\frac{4r^4(r^2dx^2 + r^2dy^2}{(r^2 + r^2x^2 + r^2y^2)^2}\right) = \frac{4r^2(dx^2 + dy^2}{(1 + x^2 + y^2)^2}$$

Hence F is isometry if $a = 4r^2$.

2 a) *Show that surface of the cone* $\begin{cases} x^2 + y^2 - k^2z^2 = 0 \\ z > 0 \end{cases}$ *in* $\mathbf{E}^3$ *is locally Euclidean Riemannian surface (locally isometric to Euclidean plane.*

Solution.

This means that we have to find local coordinates u, v on the cone such that in these coordinates induced metric $G|_c$ on cone would have the appearance $G|_c = du^2 + dv^2$.

First of all calculate the metric on cone in coordinates h, φ where

$$\mathbf{r}(h, \varphi)\colon \begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases}.$$

$(x^2 + y^2 - k^2z^2 = k^2h^2\cos^2\varphi + k^2h^2\sin^2\varphi - k^2h^2 = k^2h^2 - k^2h^2 = 0.$

Calculate metric G_c on the cone in coordinates h, φ induced with the Euclidean metric $G = dx^2 + dy^2 + dz^2$:

$$G_c = \left(dx^2 + dy^2 + dz^2\right)\big|_{x=kh\cos\varphi,y=kh\sin\varphi,z=h} = (k\cos\varphi dh - kh\sin\varphi d\varphi)^2 +$$
$$(k\sin\varphi dh + kh\cos\varphi d\varphi)^2 + dh^2 = (k^2 + 1)dh^2 + k^2h^2d\varphi^2\,.$$

In analogy with polar coordinates try to find new local coordinates u, v such that $\begin{cases} u = \alpha h\cos\beta\varphi \\ v = \alpha h\sin\beta\varphi \end{cases}$, where α, β are parameters. We come to

$$du^2 + dv^2 = (\alpha\cos\beta\varphi dh - \alpha\beta h\sin\beta\varphi d\varphi)^2 + (\alpha\sin\beta\varphi dh + \alpha\beta h\cos\beta\varphi d\varphi)^2 = \alpha^2dh^2 + \alpha^2\beta^2h^2d\varphi^2.$$

Comparing with the metric on the cone $G_c = (1+k^2)dh^2 + k^2h^2d\varphi^2$ we see that if we put $\alpha = k$ and $\beta = \frac{k}{\sqrt{1+k^2}}$ then $du^2 + dv^2 = \alpha^2 dh^2 + \alpha^2\beta^2 h^2 d\varphi^2 = (1+k^2)dh^2 + k^2h^2d\varphi^2$.

Thus in new local coordinates

$$\begin{cases} u = \sqrt{k^2+1}h\cos\frac{k}{\sqrt{k^2+1}}\varphi \\ v = \sqrt{k^2+1}h\sin\frac{k}{\sqrt{k^2+1}}\varphi \end{cases}$$

induced metric on the cone becomes $G|_c = du^2 + dv^2$, i.e. surface of the cone is locally isometric to the Euclidean plane (is locally Euclidean Riemannian surface). ■

3 a) *a) Consider the conic surface C defined by the equation $x^2+y^2-z^2 = 0$ in $\mathbf{E}^3$. Consider a part of this conic surface between planes $z = 0$ and $z = H > 0$ and remove the line $z = -x, y = 0$ from this part of conic surface C. We come to the surface D defined by the conditions*

$$\begin{cases} x^2+y^2-z^2 = 0 \\ 0 < z < H \\ y \neq 0 \quad \text{if } x < 0 \end{cases}.$$

Find a domain D' in Euclidean plane such that it is isometric to the surface D, that is there exists isometry $F\colon D \to D'$.

b) Find a shortest distance between points $A = (1,0,1)$ and $B = (-1,0,1)$, between points A and $E = (0,1,1)$ for an ant living on the conic surface C.

Solution.

The domain D on the cone can be parameterised as (see the previous exeercise for $k = 1$)

$$\mathbf{r}(h,\varphi)\colon \begin{cases} x = h\cos\varphi \\ y = h\sin\varphi \\ z = h \end{cases} \quad 0 < h < H, -\pi < \varphi < \pi$$

Notice that we cosnider the range of φ $(-\pi,\pi)$ not $(0,2\pi)$ as usual, since the removed line $z = -x, y = 0$ corresponds to the points with $\varphi = \pm\pi$. If we would remove the line $z = x, y = 0$, then the parameterisation $0 < \varphi < 2\pi$ would be valid.

Using the results of previous exercise for $k = 1$ consider new local coordinates

$$u, v\colon \quad \begin{cases} u = \sqrt{2}h\cos\frac{\varphi}{\sqrt{2}} \\ v = \sqrt{2}h\sin\frac{\varphi}{\sqrt{2}} \end{cases} \quad 0 < h < H, -\pi < \varphi < \pi\,.$$

In these coordinates metric $G = du^2 + dv^2$ and these coordinates are global coordinates in the surface $D =$ which is the piece of the conic surface C. It is easy to see from this equation that the domain D is isometric to the angular sector in $\mathbf{R}^2$ with with radius $R = \sqrt{2}h$ and the angle $\left(-\frac{\pi}{\sqrt{2}}, \frac{\pi}{\sqrt{2}}\right)$:

$$D'\colon (x,y)\text{such that in polar coordinates} \begin{cases} 0 < R < h\sqrt{2} \\ -\frac{\pi}{\sqrt{2}} < \theta < \frac{\pi}{\sqrt{2}} \end{cases}$$

The isometry $u = x, v = y$ estabilshes the diffeomorphism between D' and D since the metric in both cases is Euclidean.

To find the shortest distance between points A, B and points A, D on the cone we find the distance between images of these points on the domain D' defined above, since D' has metric of Euclidean plane. We identify points A, B, D on the cone with ther images on the domain D'. The points A, B and the origin O (strictly speaking their images on D') make the isosceles triangle $\triangle OAB$ with $OA = OB = \sqrt{2}$, and the angle $\angle A0B = \frac{\pi}{\sqrt{2}}$ The distance $|AB| = 2|0A|\sin\frac{\angle AOB}{2} = 2\sqrt{2}\sin\frac{\pi}{2\sqrt{2}}$.

The "naive" distance (trip around the circle) equals to $\pi > 2\sqrt{2}\sin\frac{\pi}{2\sqrt{2}}$.

Remark Of course the point B on the cone is already removed, since it was belonged to the cutted line $z=-x$. But we can take instead the point B' which belongs to domain D and is infinitesimally close to the point B. (B belongs to the closure of the open domain D'.)

Analogous calculations for the points A and E. The points A, E and the origin O on the domain D' make the isosceles triangle $\triangle OAE$ with $OA = OB = \sqrt{2}$, and the angle $\angle A0B = \frac{\pi}{2\sqrt{2}}$. The distance $|AE| = 2|0A|\sin\frac{\angle AOC}{2} = 2\sqrt{2}\sin\frac{\pi}{4\sqrt{2}}$. The "naive" distance (trip around the circle) equals to $\pi/2$.

4 *Find diffeomorphism F: of Euclidean space such that F: $\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$ is isometry, in other words $du^2 + dv^2 = dx^2 + dy^2$. (You may assume that functions $u(x,y), v(x,y)$ are linear: $u = a + bx + cy$, $v = c + dx + fy$, where a, b, c, d are constants.)*

Show that the transformation is a composition of translation, rotation and reflection.

† *Will the answer change if we allow arbitrary (not only linear functions $u(x,y), v(x,y)$)?*

If $u = a + bx + cy$, $v = c + dx + fy$ then

$$du^2 + dv^2 = (bdx + cdy)^2 + (ddx + fdy)^2 = (b^2 + d^2)dx^2 + 2(bc + df)dxdy + (c^2 + f^2)dy^2 = dx^2 + dy^2 \,.$$

This means that $b^2 + d^2 = c^2 + f^2 = 1$ and $bc + df = 0$, i.e. for matrix $A = \begin{pmatrix} b & d \\ c & f \end{pmatrix}$ rows have length 1 and they are orthogonal, i.e. the matrix A is orthogonal: $AA^T = I$. We come to the answer: In the class of linear transformations the transformation that preserves the Euclidean metric is a translation and orthogonal transformation, i.e. translations, rotations and reflections.

† It is very interesting to answer the question: how look general transformations which preserve the metric? Answer: any transformation preserving Euclidean metric is linear transformation, i.e. there is no rotation "depending on point." There are many different and beautiful and illuminating proofs of this fact which is true for any dimensions. We consider here not the best one:

Let $u = u(x,y), v = v(x,y)$ be transformation such that $du^2 + dv^2 = dx^2 + dy^2$, i.e. according to calculations above

$$\left(u_x^2 + v_x^2 \quad u_x u_y + v_x v_y u_x u_y + v_x v_y \quad v_x^2 + v_y^2 \right) = \left(1 \quad 00 \quad 1 \right)$$

Then orthogonality condition for matrix will imply that condition that

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} \cos\Psi(x,y) & -\sin\Psi(x,y) \\ \sin\Psi(x,y) & \cos\Psi(x,y) \end{pmatrix}$$

(if determinant is negative we compensate by transformation $u \mapsto -u$.)

Thus the function $F(x,y) = u(x,y) + iv(x,y)$ is holomorphic function (Cauchy-Riemann conditions $u_x = v_y$, $u_y + v_x = 0$). The condition $u_x^2 + v_x^2 = 1$ means that the modulus of the analytical function $F' = \frac{\partial F}{\partial z}$ equals to 1. Thus $F' = const$ and $F = e^{i\varphi}z + c$, i.e. we have rotation on angle φ and translation. (There is no differential rotation!)

(In the case if determinant is negative then we come to antiholomorphic transformation: $F = a\bar{z} + c$, i.e. reflection, rotation and translation.)

5 *Let $\mathbf{K} = K^i(x)\frac{\partial}{\partial x^i}$ be a Killing vector field on Euclidean plane, i.e. a vector field such that it induces infinitesimal isometry of Euclidean space.*

a) Show that

$$\frac{\partial K^i(x)}{\partial x^j} + \frac{\partial K^j(x)}{\partial x^i} = 0 \,,$$

b) Find all Killing vector fields of Euclidean plane $\mathbf{E}^2$

c) Describe all Killing vector fields on Euclidean plane.*

See for solutions lecture notes: