# Homework 1

**1** Show that the vectors $\{\mathbf{a}_1, \mathbf{a}_2 \dots, \mathbf{a}_m\}$ in vector space $V$ are linearly dependent if at least one of these vectors is equal to zero.

**2** Show that any three vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ in $\mathbf{R}^2$ are linearly dependent.

**3** Let 3 vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ in vector space $V$ belong to the span of 2 vectors $\{\mathbf{a}, \mathbf{b}\}$ of this vector space, i.e. vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are expressible as linear combinations of vectors $\mathbf{a}$ and $\mathbf{b}$. Prove that vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ are linearly dependent.

**4** Let $\{\mathbf{a}, \mathbf{b}\}$ be two vectors in the vector space $V$ such that

i) these vectors are linearly independent

ii) for an arbitrary vector $\mathbf{x} \in V$ vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{x}\}$ are linearly dependent.

What is a dimension of the vector space $V$?

Is an ordered set $\{\mathbf{a}, \mathbf{b}\}$ a basis in the vector space $V$?

**5** Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a basis in 3-dimensional vector space $V$. Show that

a) all vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are not equal to zero.

b) an arbitrary vector $\mathbf{a} \in V$ can be expressed as a linearly combination of the basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in a unique way, i.e.$\mathbf{a} = a^1\mathbf{e}_1 + a^2\mathbf{e}_2 + a^3\mathbf{e}_3$ and if

$$\mathbf{a} = a^1\mathbf{e}_1 + a^2\mathbf{e}_2 + a^3\mathbf{e}_3 = a^{1'}\mathbf{e}_1 + a^{2'}\mathbf{e}_2 + a^{3'}\mathbf{e}_3 \text{ then } a^1 = a^{1'}, a^2 = a^{2'}, a^3 = a^{3'} .$$

**Remark** The following statement is very useful: *Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ be an ordered set of vectors in the vector space $V$ such that an arbitrary vector $\mathbf{x} \in V$ can be expressed as a linear combination of the vectors $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ in a unique way. Then one can show that $V$ is an $n$-dimensional vector space and an ordered set $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is a basis in $V$.*

**6**$^\dagger$ Show that the ordered set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \dots, \mathbf{e}_n\}$ of vectors is a basis in $\mathbf{R}^n$ in the case if

$$\begin{array}{ll}
\mathbf{e}_1 & = (1, \quad 2, \quad 3, \quad 4, \dots, \quad n) \\
\mathbf{e}_2 & = (0, \quad 1, \quad 2, \quad 3, \dots, \quad n-1) \\
\mathbf{e}_3 & = (0, \quad 0, \quad 1, \quad 2, \dots, \quad n-2) \\
\dots & \\
\mathbf{e}_n & = (0, \quad 0, \quad 0, \quad 0, \dots, \quad 1)
\end{array}$$

**7** Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a basis of 3-dimensional vector space $V$. Is a set of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ a basis of $V$ in the case if

a) $\mathbf{e}'_1 = \mathbf{e}_2$, $\mathbf{e}'_2 = \mathbf{e}_1$, $\mathbf{e}'_3 = \mathbf{e}_3$;

b) $\mathbf{e}'_1 = \mathbf{e}_1$, $\mathbf{e}'_2 = \mathbf{e}_1 + 3\mathbf{e}_3$, $\mathbf{e}'_3 = \mathbf{e}_3$;

c) $\mathbf{e}'_1 = \mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{e}'_2 = 3\mathbf{e}_1 - 3\mathbf{e}_2$, $\mathbf{e}'_3 = \mathbf{e}_3$;

d) $\mathbf{e}'_1 = \mathbf{e}_2$, $\mathbf{e}'_2 = \mathbf{e}_1$, $\mathbf{e}'_3 = \mathbf{e}_1 + \mathbf{e}_2 + \lambda\mathbf{e}_3$ (where $\lambda$ is an arbitrary coefficient)?