For real case I know it from childhood, but for the complex case it is little bit fun: Today I did the proof which works for both cases.

CBH claims that

$$\langle f, f\rangle\langle g, g\rangle = ||f||^2||g||^2 \geq |\langle f, g\rangle|^2$$

Consider the following polynomial on $z = x + iy$:

$$P(z) = ||f||^2||zf + g||^2 = \langle f, f\rangle\langle zf + g, zf + g\rangle$$

It is not negative hence we have

$$0 \leq P(z) = ||f||^2||fz + g||^2 = ||f||^2 \langle fz + g\, , fz + g\rangle =$$

$$= ||f||^2 \langle\langle f, f\rangle\, z\bar{z} + z\langle f, g\rangle + \langle g, f\rangle\bar{z} + \langle g, g\rangle\rangle = ||f||^2 \left(||f||^2|z|^2 + z\langle f, g\rangle + \langle g, f\rangle\bar{z} + ||g||^2\right) =$$

$$\left\langle ||f||^2 z + \langle g, f\rangle\ , ||f||^2 z + \langle g, f\rangle\right\rangle + ||f||^2||g||^2 - |\langle f, g\rangle|^2\ . \qquad (*)$$

This implies that

$$||f||^2||g||^2 \geq |\langle f, g\rangle|^2\ , \qquad (**)$$

and

$$||f||^2||g||^2 = |\langle f, g\rangle|^2 \Leftrightarrow f||g\,.$$

Indeed if $f = 0$, then this is obvious. Suppose that $f \neq 0$. and choose $z = -\frac{\langle f,g\rangle}{||f||^2}$. Then the inequality (*) implies that

$$P(z)\big|_{z=-\frac{\langle f,g\rangle}{||f||^2}} = \left\langle ||f||^2 z + \langle g, f\rangle\ , ||f||^2 z + \langle g, f\rangle\right\rangle\big|_{z=-\frac{\langle f,g\rangle}{||f||^2}} + ||f||^2||g||^2 - |\langle f, g\rangle|^2 = ||f||^2||g||^2 - |\langle f, g$$