Homework 1. Solutions

1 *Let $G = ||g_{ik}(x)||$ be Riemannian metric on n-dimensional Riemannian manifold M in local coordinates (x^i) $(i = 1, 2, \dots, n)$.*

a) Show that

$$g_{11}(x) > 0\,, g_{22}(x) > 0\,, \dots\,, g_{nn}(x) > 0\,.$$

b) show that condition of non-degeneracy for a symmetric matrix $G = ||g_{ik}||$ ($\det g_{ik} \neq 0$) follows from the condition that this matrix is positive-definite.

Solution

a) Consider vector fields

$$\mathbf{r}_1 = \frac{\partial}{\partial x^1}\,, \mathbf{r}_2 = \frac{\partial}{\partial x^2}\,, \dots\,, \mathbf{r}_n = \frac{\partial}{\partial x^n}\,,$$

Positive definiteness condition tells that

$$G(\mathbf{r}_1, \mathbf{r}_1) = g_{11} > 0\,, G(\mathbf{r}_2, \mathbf{r}_2) = g_{22} > 0\,, \dots\,, G(\mathbf{r}_n, \mathbf{r}_n) = g_{nn} > 0\,,$$

b) Suppose $\det g = 0$, i.e. g is degenerate matrix (rows and columns of the matrix are linear dependent). Then there exists non-zero vector $\mathbf{x} = (x^1, \dots, x^k)$ such that $g_{ik}x^k = 0$, hence $g_{ik}x^i x^k = 0$ for $\mathbf{x} \neq 0$. Contradiction to the condition of positive-definiteness.

2 *Let (u, v) be local coordinates on 2-dimensional Riemannian manifold M. Let Riemannian metric be given in these local coordinates by the matrix*

$$G = ||g_{ik}|| = \begin{pmatrix} A(u,v) & B(u,v) \\ C(u,v) & D(u,v) \end{pmatrix}\,,$$

where $A(u, v), B(u, v), C(u, v), D(u, v)$ are smooth functions. Show that the following conditions are fulfilled:

a) $B(u, v) = C(u, v)$,

b) $A(u, v)D(u, v) - B(u, v)C(u, v) = A(u, v)D(u, v) - B^2(u, v) \neq 0$,

c) $A(u, v) > 0$,

d)† $A(u, v)D(u, v) - B(u, v)C(u, v) = A(u, v)D(u, v) - B^2(u, v) > 0$.

e)† Show that conditions a), c) and d) are necessary and sufficient conditions for matrix $||g_{ik}||$ to define locally a Riemannian metric.

Solution Consider Riemannian scalar product $G(\mathbf{X}, \mathbf{Y}) = g_{ik}X^iY^k$.

a) The condition that $G(\mathbf{X}, \mathbf{Y}) = G(\mathbf{Y}, \mathbf{X})$ means that $g_{ik} = g_{ki}$, i.e. $B(u, v = C(u, v))$.

b) $\det G = A(u, v)D(u, v) - B(u, v)C(u, v) = AD - B^2 \neq 0$ since it is non-degenerate (see the solution of exercise 1)

c) Consider quadratic form $G(\mathbf{x},\mathbf{x}) = g_{ik}x^i x^k = Ax^2+2Bxy+Dy^2$. (We already know that $B = C$) Positive -definiteness means that $G(\mathbf{x},\mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$. In particular if we put $\mathbf{x} = (1,0)$ we come to $G(\mathbf{x},\mathbf{x}) = A > 0$. Thus $A > 0$.

d) Consider quadratic form $G(\mathbf{x},\mathbf{x}) = g_{ik}x^i x^k = Ax^2 + 2Bxy + Dy^2$. We have an identity

$$G(\mathbf{x},\mathbf{x}) = g_{ik}x^i x^k = Ax^2 + 2Bxy + Dy^2 = \frac{(Ax+By)^2 + (AD-B^2)y^2}{A}\,. \tag{1}$$

We already know that $A > 0$ (take $\mathbf{x} = (x,0)$). Now take $\mathbf{x} = (x,y)$: $Ax + By = 0$ (e.g. $\mathbf{x}$=(-B,A)) we come to $G(\mathbf{x},\mathbf{x}) = \frac{(AD-B^2)y^2}{A} > 0$. Hence $(AD - B^2) = \det G > 0$ *.

e) it follows from condition a) that matrix (1) is symmetric. It follows from conditionis (c) and (d) and equation (2) that $G(\mathbf{x},\mathbf{x}) > 0$ for any non-zero vector $\mathbf{x}$.

3 *Consider 2-dimensional Euclidean plane with standard Euclidean metric*

$$G = dx^2 + dy^2\,.$$

a) How this metric will transform under arbitrary affine coordinates transformation

$$\begin{cases} x = ax' + by' + e \\ y = cx' + dy' + f \end{cases}, \quad (a,b,c,d,e,f \in \mathbf{R})\,. \tag{1}$$

b) Find an affine transformation such that metric has the same appearance in new and old coordinates: $G = dx^2 + dy^2 = (dx')^2 + (dy')^2$.

c) How this metric will transform under coordinates transformation

$$x = \frac{u}{u^2+v^2}\,, \quad y = \frac{v}{u^2+v^2}\,, \quad (u,v \neq 0)\,.$$

d)† *Let $x = x(u,v)$, and $y = y(u,v)$ be an arbitrary coordinate transformation such that the metric has the same appearance in new and old coordinates:*

$$G = dx^2 + dy^2 = du^2 + dv^2\,.$$

How does this coordinate transformation look?

* This special trick works good for dimension is $n = 2$. We could notice that A and $AD-B^2$ are principal main minors of the matrix G. In the general case (if G is $n \times n$ symmetric matrix) using triangular transformations one can show that quadratic form $A(\mathbf{X},\mathbf{X}) = a_{ik}x^i x^k$ (and respectively) is positive-definite if and only if all the leading principal minors Δ_k are positive (leading Principal minor Δ_k of the matrix A is a determinant of the matrix formed by first k columns and first k rows of the matrix A.). In this case matrix G_{ik} of bilinear form can be transformed to unity matrix.

Solution

a) Perform straightforward calculations: $dx = adx' + bdy'$ and $dy = cdx' + dy'$. Hence

$$G = dx^2 + dy^2 = (adx' + bdy')^2 + (cdx' + dy')^2 = (a^2 + c^2)(dx')^2 + 2(ab + cd)dx'dy' + (b^2 + d^2)(dy')^2 .$$

In coordinates (x, y) $G = ||g_{ik}|| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and in coordinates (x', y') $||g'_{ik}|| = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix}$.

It is useful to denote the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then we see that

$$G' = A^+ \circ G \circ A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 d \end{pmatrix} . \tag{2}$$

b) It follows from equation (2) that Riemannian metric has the same appearance in new and old coordinates if and only if $A^+ A = 1$, i.e. if and only if A is orthogonal transformation.

Affine transformations which preserve metric are translations and orthogonal transformations. Recalling explicit formulae for orthogonaldd transformation we see that affine transformations preserving metric are

1)

$$\begin{cases} x = x' \cos\varphi + y' \sin\varphi + e \\ y = -x' \sin\varphi + y' \cos\varphi + f \end{cases}, \qquad \text{preserving orientation}$$

and

2)

$$\begin{cases} x = x' \cos\varphi + y' \sin\varphi + e \\ y = x' \sin\varphi - y' \cos\varphi + f \end{cases}, \qquad \text{changing orientation}$$

c) Using the exercise 3 from the Homework 0 we come to

$$G_{ik}(x) dx^i dy^k = dx^2 + dy^2 = \left(\frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv\right)^2 + \left(\frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv\right)^2 =$$

$$\left(\frac{v^2 - u^2}{\rho^4} du - \frac{2uv}{\rho^4} dv\right)^2 + \left(-\frac{2uv}{\rho^4} du + \frac{u^2 - v^2}{\rho^4} dv\right)^2 = \frac{du^2 + dv^2}{\rho^4}, \qquad (\rho^2 = u^2 + v^2) .$$

d† It is very interesting to answer the question: how look general transformations which preserve the metric? Answer: any transformation preserving Euclidean metric is linear transformation, i.e. there is no rotation "depending on point." There are many different and beautiful and illuminating proofs of this fact which is true for any dimensions. We consider here not the best one:

Let $u = u(x,y), v = v(x,y)$ be transformation such that $du^2 + dv^2 = dx^2 + dy^2$, i.e. according to calculations above

$$\begin{pmatrix} u_x^2 + v_x^2 & u_x u_y + v_x v_y \\ u_x u_y + v_x v_y & v_x^2 + v_y^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then orthogonality condition for matrix will imply that condition that

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} \cos\Psi(x,y) & -\sin\Psi(x,y) \\ \sin\Psi(x,y) & \cos\Psi(x,y) \end{pmatrix}$$

(if determinant is negative we compensate by transformation $u \mapsto -u$.)

Thus the function $F(x,y) = u(x,y) + iv(x,y)$ is holomorphic function (Cauchy-Riemann conditions $u_x = v_y$, $u_y + v_x = 0$). The condition $u_x^2 + v_x^2 = 1$ means that the modulus of the analytical function $F' = \frac{\partial F}{\partial z}$ equals to 1. Thus $F' = const$ and $F = e^{i\varphi}z + c$, i.e. we have rotation on angle φ and translation. (There is no differential rotation!)

(In the case if determinant is negative then we come to antiholomorphic transformation: $F = a\bar{z} + c$, i.e. reflection, rotation and translation.)

4 *Consider domain in two-dimensional Riemannian manifold with Riemannian metric $G = du^2 + 2bdudv + dv^2$ in local coordinates u, v, where b is a constant.*

Show that $b^2 < 1$

Solution

Matrix of the metric G $\begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix}$ is positive definite, hence $det g = 1 - b^2 > 0$, i.e. $b^2 < 1$.

Another solution: for any non-zero vector $\mathbf{x}$, $G(\mathbf{x},\mathbf{x}) > 0$. Consider $\mathbf{x} = (t, 1)$. Then for an arbitrary t, $(t,1) \neq 0$ and $G(\mathbf{x},\mathbf{x}) = t^2 + 2bt + 1 > 0$. Hence polynomial $t^2 + 2bt + 1$ has no real roots, i.e. $b^2 < 1$.

One can see that the condition $b^2 < 1$ is not only necessary but it is sufficient condition for G to be a metric.

5 *Let $G = cdu^2 + dudv + dv^2$ be Riemannian metric on 2-dimensional manifold M, where c is a real constant. Show that $c > \frac{1}{4}$.*

(Hint: You may consider the length of a vector $\mathbf{X} = \frac{\partial}{\partial u} + t\frac{\partial}{\partial v}$ where t is an arbitrary real number.)

Solution

The length of the vector $\mathbf{X} = \partial_u + t\partial_v \neq 0$ is equal to $G(\mathbf{X},\mathbf{X}) = c + t + t^2 > 0$ due to positive-definitness. We see that $t^2 + t + c = (t + \frac{1}{2})^2 + \left(c - \frac{1}{4}\right) > 0$ for all t. Hence $c > \frac{1}{4}$.