## Homework 2

**1**

a) Show that $(\mathbf{x},\mathbf{y}) = x^1y^1 + x^2y^2 + x^3y^3$ defines a scalar product in $\mathbf{R}^3$.
b) Show that $\langle\mathbf{x},\mathbf{y}\rangle = x^1y^1 + x^2y^2$ *does not define a scalar product* in $\mathbf{R}^3$.
c) Show that $(\mathbf{x},\mathbf{y}) = x^1y^1 + x^2y^2 - x^3y^3$ *does not define a scalar product* in $\mathbf{R}^3$.
d) Show that $(\mathbf{x},\mathbf{y}) = x^1y^1 + 3x^2y^2 + 5x^3y^3$ defines a scalar product in $\mathbf{R}^3$.
e) Show that $(\mathbf{x},\mathbf{y}) = x^1y^2 + x^2y^1 + x^3y^3$ *does not define a scalar product* in $\mathbf{R}^3$.

f$^\dagger$) Find necessary and sufficient conditions for entries $a, b, c$ of symmetrical matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ such that the formula

$$(\mathbf{x},\mathbf{y}) = (\, x^1, x^2\,) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = ax^1y^1 + b(x^1y^2 + x^2y^1) + cx^2y^2$$

defines scalar product in $\mathbf{R}^2$.

**2** The matrix $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ obeys the conditions $A^T A = I$ . Show that
a) $\det A = \pm 1$
b) if $\det A = 1$ then there exists an angle $\varphi : 0 \leq \varphi < 2\pi$ such that $A = A_\varphi$ where

$$A_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \text{ (rotation matrix)}$$

c) if $\det A = -1$ then then there exists an angle $\varphi : 0 \leq \varphi < 2\pi$ such that $A = A_\varphi R$, where $R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (a reflection matrix).

**3** Show that for matrix $A_\varphi$ defined in the previous exercise the following relations are satisfied:

$$A_\varphi^{-1} = A_\varphi^T = A_{-\varphi}\,, \qquad A_{\varphi+\theta} = A_\varphi \cdot A_\theta\ .$$

**4** Show explicitly that under the transformation $(\mathbf{e}'_1, \mathbf{e}'_2) = (\mathbf{e}_1, \mathbf{e}_2)\, A_\varphi$ an orthonormal basis transforms to an orthonormal one.

How coordinates of vectors change if we rotate the orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2)$ on the angle $\varphi = \frac{\pi}{3}$ anticlockwise?

**5** Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal basis of Euclidean space $\mathbf{E}^3$. Consider the ordered set of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ which is expressed via basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as in the exercise 7 of the Homework 1.

Write down explicitly transition matrix from the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to the ordered set of the vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$. What is the rank of this matrix? Is this matrix orthogonal?
Find out is the ordered set of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ a basis in $\mathbf{E}^3$. Is this basis an orthonormal basis of $\mathbf{E}^3$?
(you have to consider all cases a),b) c) and d)).

**6**. Show that an arbitrary orthogonal transformation of two-dimensional Euclidean space can be considered as a composition of reflections.

**7**$^\dagger$ Prove the Cauchy–Bunyakovsky–Schwarz inequality

$$(\mathbf{x},\mathbf{y})^2 \leq (\mathbf{x},\mathbf{x})(\mathbf{y},\mathbf{y})\,,$$

where $\mathbf{x},\mathbf{y}$ are arbitrary two vectors and ( , ) is a scalar product in Euclidean space.

*Hint: For any two given vectors $\mathbf{x},\mathbf{y}$ consider the quadratic polynomial $At^2 + 2Bt + C$ where $A = (\mathbf{x},\mathbf{x})$, $B = (\mathbf{x},\mathbf{y})$, $C = (\mathbf{y},\mathbf{y})$. Show that this polynomial has at most one real root and consider its discriminant.*