Homework 1

1 Show that $(\mathbf{x},\mathbf{y}) = x^1y^1 + x^2y^2 + x^3y^3$ is a scalar product in $\mathbf{R}^3$.
Show that $(\mathbf{x},\mathbf{y}) = x^1y^1 + x^2y^2 - x^3y^3$ *does not define scalar product* in $\mathbf{R}^3$.

2† Prove the Cauchy–Bunyakovsky–Schwarz inequality

$$(\mathbf{x},\mathbf{y})^2 \leq (\mathbf{x},\mathbf{x})(\mathbf{y},\mathbf{y}),$$

where $\mathbf{x},\mathbf{y}$ are arbitrary two vectors and $(\ ,\ )$ is a scalar product in Euclidean space.

Hint: For any two given vectors $\mathbf{x},\mathbf{y}$ *consider the quadratic polynomial* $At^2+2Bt+C$ *where* $A = (\mathbf{x},\mathbf{x})$, $B = (\mathbf{x},\mathbf{y})$, $C = (\mathbf{y},\mathbf{y})^2$. *Show that this polynomial has at most one real root and consider its discriminant.*

3 Consider a matrix $T_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$.
Show that $T_\varphi^{-1} = T_\varphi^{+} = T_{-\varphi}$.
Show that $T_{\varphi+\theta} = T_\varphi \cdot T_\theta$.

4 Show that under the transformation $(\mathbf{e}_1', \mathbf{e}_2') = (\mathbf{e}_1, \mathbf{e}_2)\, T_\varphi$ an orthonormal basis transforms to an orthonormal one.

How coordinates of vectors change if we rotate the orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2)$ on the angle $\varphi = \frac{\pi}{3}$ anticlockwise?

5 Find standard (normal) and parametric equations of the line passing through the point $(2,3)$ and making an angle $\varphi = 30^\circ$ with x-axis.

6 Find an equation of the line passing through the point $(0,1)$ and which is orthogonal to the line $y - 2x = 0$.

7 Calculate the distance between the point (x_0, y_0) and the line $y - kx = b$ using
a) geometrical methods
b) "brute force": just the minimum of the distance between the point (x_0, y_0) and an arbitrary point on the line, i.e. minimum of the function: $\sqrt{(x-x_0)^2 + (y-y_0)^2}$ with $y = kx + b$.

8 Calculate the distance between the point $A = (1,1)$ and the line $x + 2y = 1$

9 Write down an equation of the line passing via point $A = (x_0, y_0)$ which is tangent to the circle $(x-a)^2 + (y-b)^2 = R^2$. How many solutions does this problem have?
(You could consider for simplicity only the case $a = b = 0$).

10 Find the locus formed by centres of segments of the length 1, such that their endpoints lie on the axes OX, OY.