This is my may be last blog before beginning the Spring semester. Here I use one construction of orthogonal matrices, to diaognalise $N+1=2^k$ interacting oscyllators. The idea is taken from Z_2 addition which I "rediscovered" about 30 years ago.

Here I will continue to try to find good coordinates for $N+1$ strings which are joined to the ring, but now I will try to do it in more clever way.

Consider again the Lagrangian of $N+1$ particles which are joind in the ring:

$$L=\sum_k \frac{m\dot{x}_k^2}{2}+\sum \frac{k(x_k-x_{k+1})^2}{2}=$$

$$\frac{m\dot{x}_0^2}{2}+\frac{m\dot{x}_1^2}{2}+\frac{m\dot{x}_2^2}{2}+\dots\frac{m\dot{x}_{N-1}^2}{2}+\frac{m\dot{x}_N^2}{2}+$$

$$\frac{k(x_1-x_0)^2}{2}+\frac{k(x_2-x_1)^2}{2}+\dots\frac{k(x_{N-1}-x_N)^2}{2}r+=\frac{k(x_N-x_0)^2}{2}\,.$$

Potential energy is quadratic form of the corank 1. Indeed for vector the potential energy vanishes for vectors such that all components are equal:

$$\mathbf{x}=\sum_{i=0}^N x^i\mathbf{e}_i=a\sum_{i=0}^N \mathbf{e}_i\,.$$

Hence one can choose the new orthonormal basis $\{\mathbf{f_i}\}$ such that the dfirst vector of this basis is proportional to the vector $\sum_i \mathbf{e}_i$, i.e. there exists an orthonormal matrix P such that its first row has the same componens. This matrix transforms coordinates to new coordinates $u^0,u^1,u^2,\dots,u^N$ such that

$$\begin{pmatrix} x^0\\ x^1\\ \dots\\ x^N\end{pmatrix}=\underbrace{\frac{1}{\sqrt{N+1}}\begin{pmatrix} 1 & p_{01}\dots & p_{0N}\\ 1 & p_{11}\dots & p_{1N}\\ \dots & \dots\dots & \dots\\ 1 & p_{N1}\dots & p_{NN}\end{pmatrix}}_{\text{orthonormal matrix}}\begin{pmatrix} u^0\\ u^1\\ \dots\\ u^N\end{pmatrix}=$$

such that in these coordinates Lagrangian splits on two Lagrangians: Lagrangian of free particle with coordinate u_0 and the Lagrangian of N non-interacting oscillators

$$\underbrace{\frac{m\dot{u}_0^2}{2}+}_{\text{free particle}}\quad+\sum_{i=1}^N\left(\frac{m\dot{u}_i^2}{2}+\frac{k_i^2u_i^2}{2}+\right)\,,$$

where $k_i = k\lambda_i$ and λ_i are non-zero eigenvectors of the matrix of potential enery:

$$U = M_{ik}x^i x^k\,, M = \begin{pmatrix} 2 & -1 & 0 & 0 & 0\dots & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0\dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0\dots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1\dots & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2\dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots\dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & -1\dots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0\dots & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & 0\dots & 0 & -1 & 2 \end{pmatrix},$$

We have that

$$P^* M P =$$

$$\begin{pmatrix} 1 & 1\dots & 1 \\ p_{01} & p_{11}\dots & p_{N1} \\ \dots & \dots\dots & \dots \\ p_{0N} & p_{1N}\dots & p_{NN} \end{pmatrix} \begin{pmatrix} 2 & -1 & 0\dots & 0 & -1 \\ -1 & 2 & -1\dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -1 & 0 & 0\dots & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & p_{01}\dots & p_{0N} \\ 1 & p_{11}\dots & p_{1N} \\ \dots & \dots\dots & \dots \\ 1 & p_{N1}\dots & p_{NN} \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & 0\dots & 0 & 0 \\ 0 & \lambda_1 & 0\dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0\dots & 0 & \lambda_N \end{pmatrix}$$

Orthogonal matrices

The problem is to find the orthonormal basis, such that one of the vectors go along the vector $\sim_i \mathbf{e}_i$ and this basis "respects symmetry"

It turns out that the question becomes easy for $N+1 = 2^k$. In this case one can construct orthogonal matrix which contains only $\pm$, and the idea of this comes from the quesstion that I was solving in 1990*

One can easy to find orthogonal matirices which diagonalise the matrix of potential energy in the case Now calculate eigenvalues. I managed to do it very elegantly in the case $N+1 = 2^k$. In this case we can inductively define orthonormal matrix P which contains only ± 1: If we have $2^k \times 2^k$ orthonormal matrix P_k then we define $2^{k+1} \times 2^{k+1}$ orthonormal matrix P_{k+1} as

$$P_{k+1} = \frac{1}{\sqrt{2}} \begin{pmatrix} P_k & P_k \\ P_k & -P_k \end{pmatrix}$$

Thus we have :

$$P_0 = +\,, \quad P_1 = \begin{pmatrix} P_0 & P_0 \\ P_0 & -P_0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} + & + \\ + & - \end{pmatrix},$$

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* trying to find solution of problem: there are n positive integeres, $a_1, a_2, \dots, a_n$. Every player can take any number a_i and transform it to the humber that is less $a_i \mapsto a_i - k$. The winner is who do the last step.

$$P_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} P_1 & P_1 \\ P_1 & -P_1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \begin{pmatrix} + & + \\ + & - \end{pmatrix} & \begin{pmatrix} + & + \\ + & - \end{pmatrix} \\ \begin{pmatrix} + & + \\ + & - \end{pmatrix} & -\begin{pmatrix} + & + \\ + & - \end{pmatrix} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{pmatrix}$$

and so on....

$$P_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} P_2 & P_2 \\ P_2 & -P_2 \end{pmatrix} =$$

$$\frac{1}{2\sqrt{2}} \begin{pmatrix} \begin{pmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{pmatrix} & \begin{pmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{pmatrix} \\ \begin{pmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{pmatrix} & -\begin{pmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{pmatrix} \end{pmatrix} =$$

$$\frac{1}{2\sqrt{2}} \begin{pmatrix} + & + & + & + & + & + & + & + \\ + & - & + & - & + & - & + & - \\ + & + & - & - & + & + & - & - \\ + & - & - & + & + & - & - & + \\ + & + & + & + & - & - & - & - \\ + & - & + & - & - & + & - & + \\ + & + & - & - & - & - & + & + \\ + & - & - & + & - & + & + & - \end{pmatrix}.$$

One can calulcate eigenvalues using rows of the matirx P.