Homework 8. Solutions

Almost all exercises of this homework are considered in detail in the subsection 4.4 of the lecture notes (see the subsubsections: "Derivation formulae", Gauss condition (structure equations), "Geometrical meaning and Weingarten operator in terms of derivation formulae", "Gaussian and mean curvature in terms of derivation formulae."

1) *Let M be a surface embedded in Euclidean space $\mathbf{E}^3$. We say that the triple of vector fields $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ is adjusted to the surface M if $\mathbf{e}, \mathbf{f}, \mathbf{n}$ be three vector fields defined on the points of this surface such that they form an orthonormal basis at any point, so that the vectors $\mathbf{e}, \mathbf{f}$ are tangent to the surface and the vector $\mathbf{n}$ is orthogonal to the surface.*

Consider the derivation formulae for adjusted vector fields $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$:

$$d \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix}, \tag{1}$$

where a, b, c are 1-forms on the surface M.

Write down the explicit expression for connection, Weingarten operator, the mean curvature and the Gaussian curvature of M in terms of 1-forms a, b, c and vector fields $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$.

Solution (see also lecture notes):

Induced connection

Let ∇ be the connection induced by the canonical flat connection on the surface M. Then according to derivation formulae for every tangent vector $\mathbf{X}$

$$\nabla_{\mathbf{X}}\mathbf{e} = (\partial_{\mathbf{X}}\mathbf{e})_{\text{tangent}} = (d\mathbf{e}(\mathbf{X}))_{\text{tangent}} = (a(\mathbf{X})\mathbf{f} + b(\mathbf{X})\mathbf{n})_{\text{tangent}} = a(\mathbf{X})\mathbf{f}\,.$$

and

$$\nabla_{\mathbf{X}}\mathbf{f} = (\partial_{\mathbf{X}}\mathbf{f})_{\text{tangent}} = (d\mathbf{f}(\mathbf{X}))_{\text{tangent}} = (-a(\mathbf{X})\mathbf{e} + c(\mathbf{X})\mathbf{n})_{\text{tangent}} = -a(\mathbf{X})\mathbf{e}\,.$$

In particular

$$\begin{array}{ll} \nabla_{\mathbf{e}}\mathbf{e} = a(\mathbf{e})\mathbf{f} & \nabla_{\mathbf{f}}\mathbf{e} = a(\mathbf{f})\mathbf{f} \\ \nabla_{\mathbf{e}}\mathbf{f} = -a(\mathbf{e})\mathbf{e} & \nabla_{\mathbf{f}}\mathbf{f} = -a(\mathbf{f})\mathbf{e} \end{array}$$

Weingarten operator

Let S be Weingarten operator: $S\mathbf{X} = -\partial_{\mathbf{X}}\mathbf{n}$. Then

$$S\mathbf{X} = -\partial_{\mathbf{X}}\mathbf{n} = -d\mathbf{n}(\mathbf{X}) = -\left(-b(X)\mathbf{e} - c(\mathbf{X})\mathbf{f}\right) = b(\mathbf{X})\mathbf{e} + c(\mathbf{X})f$$

since $d\mathbf{n} = -b\mathbf{e} - c\mathbf{e}$ due to derivation formulae. In particular

$$S(\mathbf{e}) = b(\mathbf{e})\mathbf{e} + c(\mathbf{e})\mathbf{f}\,, S(\mathbf{f}) = b(\mathbf{f})\mathbf{e} + c(\mathbf{f})\mathbf{f}$$

and the matrix of the Weingarten operator in the basis $\{\mathbf{e}, \mathbf{f}\}$ is

$$S = \begin{pmatrix} b(\mathbf{e}) & c(\mathbf{e}) \\ b(\mathbf{f}) & c(\mathbf{f}) \end{pmatrix}$$

Curvatures We have that Gaussian curvature

$$K = \det S = \det \begin{pmatrix} b(\mathbf{e}) & c(\mathbf{e}) \\ b(\mathbf{f}) & c(\mathbf{f}) \end{pmatrix} = b(\mathbf{e})c(\mathbf{f}) - b(\mathbf{f})c(\mathbf{e}),$$

and Mean curvature

$$H = \operatorname{Tr} S = \operatorname{Tr} \begin{pmatrix} b(\mathbf{e}) & c(\mathbf{e}) \\ b(\mathbf{f}) & c(\mathbf{f}) \end{pmatrix} = b(\mathbf{e}) + c(\mathbf{f}),$$

Remark* Note that Gaussian curvature $K = b(\mathbf{e})c(\mathbf{f}) - b(\mathbf{f})c(\mathbf{e}) = b \wedge c(\mathbf{e}, \mathbf{f}) = da(\mathbf{e}, \mathbf{f})$ due to Gauss condition. This is very important to deduce the formula of rotation of the vector during parallel transport along the closed curve.

2) *Show that in derivation formulae $da + b \wedge c$=0.*

Solution

Recall that a, b, c are 1-forms, $\mathbf{e}, f, \mathbf{n}$ are vector valued functions (0-forms) and $d\mathbf{e}, d\mathbf{f}, d\mathbf{n}$ are vector valued 1-forms. (We use the simple identity that $ddf = 0$ and the fact that for 1-form $\omega \wedge \omega = 0$.) We have from derivation formulae that

$$d^2\mathbf{e} = 0 = d(a\mathbf{f} + b\mathbf{n}) = da\mathbf{f} - a \wedge d\mathbf{f} + db\mathbf{n} - b \wedge d\mathbf{n} =$$

$$da\,\mathbf{f} - a \wedge (-a\mathbf{e} + c\mathbf{n}) + db\,\mathbf{n} - b \wedge (-b\mathbf{e} - c\mathbf{f}) =$$

$$(da + b \wedge c)\mathbf{f} + (a \wedge a + b \wedge b)\mathbf{e} + (db - a \wedge c)\mathbf{n} = (da + b \wedge c)\mathbf{f} + (db + c \wedge a)\mathbf{n} = 0\,.$$

We see that $(da + b \wedge c)\mathbf{f} + (db + c \wedge a)\mathbf{n} = 0$. Hence components of the left hand side equal to zero: $(da + b \wedge c) = 0\,,\ (db + c \wedge a) = 0$. In particular $da + b \wedge c = 0$ ■.

3) *Find explicitly a triple of vector fields $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ adjusted to the surface M if M is*
a) cylinder
b) cone
c) sphere
Solution.
See the detailed solution of this and of the next exercise in the Lecture Notes (section 4.4).

One can consider different adjusted triples. In this solution we just consider an example of the adjusted triple

4) *Using results of the previous exercise find explicit expression for derivation formulae (1) in the case if the surface M is a) cylinder, b) cone, c) sphere*

Deduce from these results the formulae for Gaussian and mean curvature for cylinder, cone and sphere
See the detailed solution of this and of the previous exercise in the lecture notes (section 4.4).

5) *a) Find explicitly a triple of vector fields $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ adjusted to the surface M if a Riemannian metric on a surface M is given by formula $G = a(u, v)du^2 + b(u, v)dv^2$.*
b*) Calculate 1-form a in derivation formulae in the special case if $a = b = \sigma(u, v)$ (conformal metric). Calculate Gaussian curvature. (It is convenient to use notation $\sigma = e^\Phi$).
Solution.
a) E.g. one may take

$$\mathbf{e} = \frac{1}{\sqrt{a}}\frac{\partial}{\partial u}\,, \mathbf{e} = \frac{1}{\sqrt{b}}\frac{\partial}{\partial u}$$

and $\mathbf{n}$ its vector product. It is evident that these vectors form orthonormal basis. Of course one may consider another examples (see for detail exercise 6.)

b)* In the case $a = b = \sigma$ we come to isothermal coordinates. See detailed calculations in Lecture Notes (section 4).

6)[*] *Let $\{\mathbf{e},\mathbf{f},\mathbf{n}\}$ and $\{\tilde{\mathbf{e}},\tilde{\mathbf{f}},\tilde{\mathbf{n}}\}$ be two triples of vector fields adjusted to the surface M.*

What is the relation between these triples?

How 1-forms a,b,c in derivation formulae will change if we will change $\{\mathbf{e},\mathbf{f},\mathbf{n}\}$ to $\{\tilde{\mathbf{e}},\tilde{\mathbf{f}},\tilde{\mathbf{n}}\}$?

Show that 2-form da and 2-form $b\wedge c$ are independent on the choice of the adjusted triple.

Solution The normal unit vector $\mathbf{n}$ is defined up to a sign. Hence $\tilde{\mathbf{n}}=\pm\mathbf{n}$. Choose $\tilde{\mathbf{n}}=\mathbf{n}$ (In the case $\mathbf{n}=-\mathbf{n}$ solution is rather the same.) It is easy to see that transition matrix from the basis to $\{\mathbf{e},\mathbf{f}\}$ $\{\tilde{\mathbf{e}},\tilde{\mathbf{f}}\}$ is orthogonal matrix. Hence transition matrix is matrix of rotation or rotation+ reflection. We consider the case when it is rotation (if we add reflection nothing essential will change in the solution). We see that at an arbitrary point $\mathbf{r}=\mathbf{r}(u,v)\in M$

$$\{\tilde{\mathbf{e}}(\mathbf{r}),\tilde{\mathbf{f}}(\mathbf{r})\}=\{\mathbf{e}(\mathbf{r}),\mathbf{f}(\mathbf{r})\}\begin{pmatrix}\cos\Phi(\mathbf{r}) & \sin\Phi(\mathbf{r})\\ -\sin\Phi(\mathbf{r}) & \cos\Phi(\mathbf{r})\end{pmatrix},\ \ \mathbf{n}=\mathbf{n}\,. \tag{$*$}$$

It is much more convenient to introduce formally $\mathbf{z}(\mathbf{r})=\mathbf{e}(\mathbf{r})+i\mathbf{f}(\mathbf{r})$ and respectively $\tilde{\mathbf{z}}(\mathbf{r})=\tilde{\mathbf{e}}(\mathbf{r})+i\tilde{\mathbf{f}}(\mathbf{r})$. We have that under rotation

$$\tilde{\mathbf{z}}=\tilde{\mathbf{e}}(\mathbf{r})+i\tilde{\mathbf{f}}(\mathbf{r})=(\mathbf{e}\cos\Phi-\mathbf{f}\sin\Phi)+i(\mathbf{e}\sin\Phi+\mathbf{f}\cos\Phi)=$$

$$\mathbf{e}(\cos\Phi+i\sin\Phi)+i\mathbf{f}(\cos\Phi+i\sin\Phi)=(\mathbf{e}+i\mathbf{f})(\cos\Phi+i\sin\Phi)=e^{i\Phi}\mathbf{z}$$

We see that the formula (*) can be rewritten in much more compact way:

$$\tilde{\mathbf{z}}=e^{i\Phi}\mathbf{z}\,. \tag{$**$}$$

It follows from derivation formulae that

$$d\mathbf{z}=d(\mathbf{e}+i\mathbf{f})=a\mathbf{f}+b\mathbf{n}+i(-a\mathbf{e}+c\mathbf{n})=-ia(\mathbf{e}+i\mathbf{f})+(b+ic)\mathbf{n}=-ia\mathbf{z}+(b+ic)\mathbf{n} \tag{$***$}$$

Hence

$$d\tilde{\mathbf{z}}=d(e^{i\Phi}\mathbf{z})=id\Phi e^{i\Phi}\mathbf{z}+e^{i\Phi}d\mathbf{z}=id\Phi\tilde{\mathbf{z}}+e^{i\Phi}(-ia\mathbf{z}+(b+ic)\mathbf{n}=id\Phi\tilde{\mathbf{z}}-ia\tilde{\mathbf{z}}+\mathbf{e}^{i\Phi}(b+ic)\mathbf{n}\,,$$

Thus

$$d\tilde{\mathbf{z}}=-i(a-d\Phi)\mathbf{z}+\mathbf{e}^{i\Phi}(b+ic)\mathbf{n}$$

Comparing with (***) we see that if $\{\mathbf{e},\mathbf{f},\mathbf{n}\}\to\{\tilde{\mathbf{e}},\tilde{\mathbf{f}},\tilde{\mathbf{n}}\}$ then in derivation formulae

$$a\to a-d\Phi,(b+ic)\to\mathbf{e}^{i\Phi}(b+ic),\ \text{i.e.}\ b\to bcos\Phi-c\sin\Phi\,,c\to b\sin\Phi+c\cos\Phi$$

We see that

$$da\to d(a-d\Phi)=da-d^2\Phi=da\ \ \text{does not change}$$

$da+b\wedge c=0$ (Gauss condition (see the exercise 2)). Hence $b\wedge c=-da$ does not change also.

7 *Consider in $\mathbf{E}^3$ a vector $\mathbf{X}=\frac{\partial}{\partial y}$ attached at the point $\mathbf{p}\colon(x=R\cos\theta_0,y=0,z=R\sin\theta_0)$ of the sphere $x^2+y^2+z^2=R^2$ in $\mathbf{E}^3$. Consider on the sphere the following two curves passing via the point $\mathbf{p}$:*

a curve C_1 which is the intersections of this sphere with plane $y=0$ and a curve C_2 which is the intersections of this sphere with the plane $z=R\sin\theta_0$.

Find the result of parallel transport of the vector $\mathbf{X}$ along these closed curves.

Accordind the Theorem of parallel transport along the closed curve the vector $\mathbf{X}$ will rotate on the angle ϕ which is defined by the formula $\phi = \int_D K d\sigma$, where C is the boundary of the domain D. On the other hand for the sphere Gaussian curvature is equal to $1/R^2$. Hence

$$\phi = \frac{\text{Area of } D}{R^2} \, .$$

The curve C_1 is the great circle. It is the boundary of semisphere:

$$\phi_1 = \frac{\text{Area of } D_1}{R^2} = \frac{2\pi R^2}{R^2} = 2\pi \, ,$$

i.e. the angle of rotation for the vector is equal to zero.

The curve C_2 is the boundary of the segment of the height $H = R(1 - \cos\theta_0)$. Hence

$$\phi_2 = \frac{\text{Area of } D_2}{R^2} = \frac{2\pi R H}{R^2} = 2\pi(1 - \cos\theta_0) \, .$$

8) *What will be the result of the parallel transport of an arbitrary tangent vector along the closed curve C on the cone?*

How this result correlates with the fact that the Gaussian connection of the cone equals to zero.

Considering the domain on the plane which is isometric to the surface of the cone one can see that during parallel transport the vector will rotates on the angle $\Delta\Phi \neq 0$. This seems to be in contradiction with the fact that Gaussian curvature of the cone equals to zero since

$$\Delta\Phi = \angle\left(\mathbf{X}, \mathbf{R}_C \mathbf{X}\right) = \int_D K d\sigma \, ,$$

where K is the Gaussian curvature and $d\sigma = \sqrt{\det g} dudv$ is the area element induced by the Riemannian metric on the surface M $(d\sigma = \sqrt{\det g} dudv)$.

In fact in the case of cone there is a singular point–the apex of the cone. In this point curvature is not well-defined.