FEEDBACK EXAM[1]

RIEMANNIAN GEOMETRY, (M31082, 41082,61082) SPRING 2014

ANSWER ANY THREE OF QUESTIONS 1—4 AND QUESTION 5

All questions are worth 20 marks

For M31082 fifth question is excluded

Electronic calculators may not be used

1.

(a) *Explain what is meant by saying that G is a Riemannian metric on a manifold M.*

Consider the upper half plane ($y > 0$) in $\mathbf{R}^2$ equipped with the Riemannian metric $G = \sigma(x,y)(dx^2 + dy^2)$.

Explain why $\sigma(x,y) > 0$.

Consider in this Riemannian manifold a curve C such that

$$C\colon \quad \begin{cases} x = 1 \\ y = a + t \end{cases} \quad , \qquad 0 \leq t \leq 1\,,\ (a > 0)\,.$$

Find the length of this curve in the case $\sigma(x,y) = \frac{1}{y^2}$ (the Lobachevsky metric).

[6 marks]

(b) *Consider a surface (the upper sheet of a cone) in $\mathbf{E}^3$*

$$\mathbf{r}(h,\varphi)\colon \quad \begin{cases} x = 2h\cos\varphi \\ y = 2h\sin\varphi \\ z = h \end{cases} \quad , \qquad h > 0, 0 \leq \varphi < 2\pi\,.$$

Calculate the Riemannian metric on this surface induced by the canonical metric on Euclidean space $\mathbf{E}^3$.

Show that this surface is locally Euclidean.

[6 marks]

(c) *Consider a Riemannian manifold M^n with a metric $G = g_{ik}dx^i dx^k$.*
Write down the formula for the volume element on M^n (area element for $n = 2$).

[1] *This text is not the text of solutions of exam papers! Here we will discuss the solutions of the exampapers.*

*Find the volume of a domain $0 < h < H$ of the cone considered in the part (***b***).*

It is well-known that the metric $G = \frac{dx^2+dy^2}{y^2}$ of the Lobachevsky plane is not locally Euclidean. However show that there exist coordinates $u = u(x,y)$, $v = v(x,y)$ such that in these coordinates the area element is equal to $dudv$.

`[8 marks]`

a) No specific problems. Just one note: calculating length of the curve some students confuse when writing the formula for the length of curve.

b) No specific problems.

c) Formula for volume element in M^n is $\sqrt{\det g}dx^1 \dots dx^n$. This is right answer. (One may wright down the answer as $\sqrt{\det g}dx^1 \wedge \dots \wedge dx^n$. This demonstrates that student knows the stuff related with differential forms.)

This subquestion was very easy, but surprisingly many students wrote wrong answer. *In particular some students wrote something like: $\sqrt{\det g}dx^i dx^k$—this is wrong!!!*

Calculating the volume of a domain (area) was not difficult question and students mostly have no problems. Just a remark: even if you have not much time to check is your answer right or wrong, please use common sence that answer has to be reasonable. E.g. answer ....H^4 cannot be true since area of domain is proportional to H^2!!!

The last subquestion was not very easy but it is not very difficult too. It just needs careful understanding of trasnformation rules for metric and volume form under changing of coordinates. E.g. consider new coordinates $\begin{cases} u = x \\ v = \frac{1}{y} \end{cases}$, then $x = u, y = \frac{1}{v}$ and for metric

$$G = \frac{dx^2 + dy^2}{y^2} = \frac{du^2 + \left(-\frac{dv}{v^2}\right)^2}{\frac{1}{v^2}} = v^2 du^2 + \frac{1}{v^2} dv^2 ,$$

We see that in new coordinates metric is such that

$$G = \begin{pmatrix} v^2 & 0 \\ 0 & \frac{1}{v^2} \end{pmatrix} ,$$

and $\det G = 1$, i.e. in these coordinates volume form is $dudv$.

Remark May be you have temptation to find coordinates $\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$ such that $G = \frac{dx^2+dy^2}{y^2} = du^2 + dv^2$? Yes, if such coordinates exist then everything is alright: If $G = du^2 + dv^2$ then sure $\det G = 1$. But in this case *one cannot find local coordinates*

$\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$ *such that in these coordinates* $G = \frac{dx^2+dy^2}{y^2} = du^2 + dv^2$. (Why[2]?)

Looking for coordinates u, v in which metric becomes locally Euclidean is a wrong way, we just try to obey to much less restricted condition: $\det G = 1$ in new cooridinates u, v. and area form is equal to $\sqrt{\det G} du dv = du dv$.

2.

(a) *Explain what is meant by an affine connection on a manifold.*

Let ∇ *be an affine connection on a 2-dimensional manifold* M *such that in local coordinates* (u, v) *all Christoffel symbols vanish except* $\Gamma^u_{vv} = u$ *and* $\Gamma^v_{uu} = v$. *Calculate the vector field* $\nabla_{\mathbf{X}}\mathbf{X}$, *where* $\mathbf{X} = \frac{\partial}{\partial u} + u\frac{\partial}{\partial v}$.

[5 marks]

(b) *Explain what is meant by the induced connection on a surface in Euclidean space.*

Calculate the induced connection on the cylindrical surface in $\mathbf{E}^3$

$$\mathbf{r}(h, \varphi): \quad \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases} .$$

[6 marks]

(c) *Give a detailed formulation of the Levi-Civita Theorem. In particular write down the expression for the Christoffel symbols* Γ^i_{km} *of the Levi-Civita connection in terms of the Riemannian metric* $G = g_{ik}(x)dx^i dx^k$.

Prove that the induced connection on a surface $\mathbf{r} = \mathbf{r}(u, v)$ *in* $\mathbf{E}^3$ *is equal to the Levi-Civita connection of the Riemannnian metric induced by the canonical metric on Euclidean space* $\mathbf{E}^3$.

A Riemannian metric in local coordinates u, v *is equal to* $G = e^{-u^2-v^2}(du^2 + dv^2)$. *Calculate the Christoffel symbols of the Levi-Civita connection at the point* $u = v = 0$.

[9 marks]

a) When listing the axioms defining connection, in the last axiom:

$$\nabla_{\mathbf{X}}(f\mathbf{Y}) = f\nabla_{\mathbf{X}}\mathbf{Y} + (\nabla_{\mathbf{X}} f)\mathbf{Y} \quad \text{(Leibnitz rule)}$$

[2] if these coordinates exist then we come to locally Euclidean space with locally Euclidean geometry. (Another way to see it is just to note that curvature is not zero.)

it is important to note that the operation $\nabla_{\mathbf{X}} f$ is just derivation of function f along vector field $\mathbf{X}$: $\nabla_{\mathbf{X}} f = \partial_{\mathbf{X}} f$ (or you can just write $\partial_{\mathbf{X}} f$ instead $\nabla_{\mathbf{X}} f$ in the previous formula.)

b) In the definition of induced connection ∇

$$\nabla_M : \quad \nabla^M_{\mathbf{X}} \mathbf{Y} = \left(\nabla^{\text{can.flat}}_{\mathbf{X}} \mathbf{Y}\right)_{\text{tangent}} ,$$

it is very important to note that canonical flat connection is defined by condition that Christoffel symbols of canonical flat connection vanish in Cartesian coordinates. Some students wrote something like

'canonical flat connection is defined by condition that Christoffel symbols of canonical flat connection vanish (in arbitrary?) '

This is wrong (or unclear) statement! Chrsitoffel symbols Γ^i_{km} are not coefficients of a tensor. If $\Gamma^i_{km} \equiv 0$ in some coordinates this does not mean that it has to vanish in arbitrary coordinates. On the other hand if Christoffel symbols vanish in some Cartesian coordinates then they vanish in an arbitrary Cartesian coordinates too (see for detail the lecture notes.)

c)

When formulating Levi-Civita Theorem it is very important to note that Levi-Civita connection is *symmetric connection* and it is uniquely defined by Riemannian metric in the class of symmetric connections which are compatible with metric, i.e. symmetric connections which preserve scalar product which is induced by metric:

$$\partial_{\mathbf{X}} < \mathbf{Y}, \mathbf{Z} > = < \nabla_{\mathbf{X}}(\mathbf{Y}), \mathbf{Z} > + < \mathbf{Y}, \nabla_{\mathbf{X}}(\mathbf{Z}) > .$$

(see lecture notes for details).

Levi-Civita Theorem claims that on the Riemannian manifold (M, G) there exists uniquely defined Levi Civita connection. In local coordinates Christoffel symbols of this connection have the following appearance:

$$\Gamma^m_{ik}(x) = \frac{1}{2} g^{mn}(x) \left(\frac{\partial g_{in}(x)}{\partial x^k} + \frac{\partial g_{kn}(x)}{\partial x^i} - \frac{\partial g_{ik}(x)}{\partial x^n} \right) . \blacksquare \qquad (*)$$

Metric $G = \begin{pmatrix} e^{-u^2-v^2} & 0 \\ 0 & e^{-u^2-v^2} \end{pmatrix}$. One can see that first derivatives of metric G vanish at the point $u = v = 0$. E.g.

$$\frac{\partial g_{uu}}{\partial u}\Big|_{u=v=0} = \frac{\partial e^{-u^2-v^2}}{\partial u}\Big|_{u=v=0} = -2ue^{-u^2-v^2}\Big|_{u=v=0} = 0 .$$

Hence it follows form formula (*) which expresses Cristoffel symbols of Levi-Civita connection via coefficients of metric that Christoffel symbols vanish at origin.

Some students "deduced" this noticing that at the point $u = v = 0$

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (**)$$

and claiming that this fact implies vanishing of Cristoffel symbols at origin. Yes formula (**) is true, but this does not imply that Christoffel symbols vanish at the point $u = v = 0$! E.g. if in a vicinity of origin $G = \begin{pmatrix} 1+u & 0 \\ 0 & 1 \end{pmatrix}$ then the condition (**) holds but $\frac{\partial g_{uu}}{\partial u}\Big|_{u=v=0} = 1 \neq 0$! and

$$\Gamma^u_{uu}\Big|_{u=v=0} = \frac{1}{2} g^{uu}\left(-\frac{\partial g_{uu}}{\partial u}\right)\Big|_{u=v=0} = -\frac{1}{2} \neq 0\,.$$

(see also the discussion of the question 4**c**.)

3.

(a) *Define a geodesic on a Riemannian manifold as a parameterised curve.*

Write down the differential equations for geodesics in terms of the Christoffel symbols.

Explain why the great circles are the geodesics on the sphere.

[7 marks]

(b) *Explain what is meant by the Lagrangian of a "free" particle on a Riemannian manifold.*

Explain what is the relation between the Lagrangian of a free particle and the differential equations for geodesics.

Calculate the Christoffel symbols on the Lobachevsky plane.
(You may use the Lagrangian of a "free" particle on this plane $L = \frac{1}{2}\frac{\dot{x}^2+\dot{y}^2}{y^2}$.)

Consider an arbitrary geodesic $\mathbf{r} = (x(t), y(t))$ on the Lobachevsky plane.
Show that the magnitude $I(t) = \frac{\dot{x}(t)}{y^2(t)}$ is preserved along the geodesic.

[8 marks]

(c) *Consider a plane $\mathbf{R}^2$ equipped with the Riemannian metric $G = \frac{4R^4(du^2+dv^2)}{(R^2+u^2+v^2)^2}$.*
We know that it is isometric to the sphere of radius R in $\mathbf{E}^3$ (without the North pole) in stereographic coordinates $u = \frac{Rx}{R-z}$, $v = \frac{Ry}{R-z}$, $(x^2 + y^2 + z^2 = R^2)$.

Consider the parallel transport of the vector $\mathbf{A} = \partial_u$ attached at the point $u = R, v = 0$ along the circle $u^2 + v^2 = R^2$ with respect to this Riemannian metric.
Show that during the parallel transport along this circle it will always be orthogonal to this circle.

[5 marks]

a) Students who answered these questions had problems to explain why great circles are geodesics on the sphere. They had two choices

1-st choice: to analyse differential equations defining geodesics and to study their solutions. (See lecture notes.)

2-nd choice: to to note that for the magnitude $\mathbf{r} \times \mathbf{v}$ is preserved on geodesic and to imply from this fact that great circles are geodesic. (See lecture notes.)

The second way is nicer and much more shorter. By some reasons all students (who tried to do this question) have chosen the first way and only few of them did it succesfully till end.

b) No special problems

c) This was not easy question. To solve the problem you have first to note that the circle $u^2 + v^2 = R^2$ is an image of great circle $\begin{cases} x^2 + y^2 = R^2 \\ z = 0 \end{cases}$ on the sphere. Hence it is geodesic with respect to the metric

$$G = \frac{4R^4(du^2 + dv^2)}{(R^2 + u^2 + v^2)^2} \qquad (***)$$

Since $u^2 + v^2 = R^2$ is geodesic hence any tangent vector remains tangent during parallel transport. In particular the vector ∂_v (not the vector ∂_u) which is tangent at the point $u = R, v = 0$ during parallel transport will remain tangent.

The vector ∂_u is orthogonal to the vector ∂_v at the initial point. Hence it will always remain orthogonal during parallel transport,since parpallel transport along Levi-Civita connection does not change scalar product.

Remark You have clearly realise that for metric (***) two vectors are orthogonal in Euclidean metric if and only if they are orthogonal in the metric (***). (Why?)

4.

(a) *Consider the sphere of radius R in Euclidean space $\mathbf{E}^3$*

$$\mathbf{r}(\theta,\varphi): \quad \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases} .$$

Let $\mathbf{e}, \mathbf{f}$ be unit vectors in the directions of the vectors $\mathbf{r}_\theta = \frac{\partial \mathbf{r}}{\partial\theta}$ and $\mathbf{r}_\varphi = \frac{\partial \mathbf{r}}{\partial\varphi}$ and $\mathbf{n}$ be a unit normal vector to the sphere.
Express these vectors explicitly.

For the obtained orthonormal basis $\{\mathbf{e},\mathbf{f},\mathbf{n}\}$ calculate the 1-forms a, b and c in the derivation formula

$$d\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\\ \mathbf{n}\end{pmatrix} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0\end{pmatrix}\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\\ \mathbf{n}\end{pmatrix} .$$

Deduce from these calculations the mean curvature and the Gaussian curvature of the sphere.

[7 marks]

(b) *Give a definition of the curvature tensor for a manifold equipped with a connection.*

On two-dimensional Riemannian manifold with coordinates x^1, x^2 consider the vector fields $\mathbf{A} = \frac{\partial}{\partial x^1}$, $\mathbf{B} = \frac{\partial}{\partial x^2}$, $\mathbf{X} = (1+x^1x^2)\frac{\partial}{\partial x^2}$, and the vector field $\mathbf{Y} = (\nabla_\mathbf{A}\nabla_\mathbf{B} - \nabla_\mathbf{B}\nabla_\mathbf{A})\,\mathbf{X}$, where ∇ is a connection.
Calculate the value of the field $\mathbf{Y}$ at the point $x^1 = x^2 = 0$ if the curvature tensor of the connection ∇ is such that $R^1{}_{212} = 1$ and $R^2{}_{212} = 0$ at this point.

[5 marks]

(c) *State the relation between the Riemann curvature tensor of the Levi-Civita connection of a surface in $\mathbf{E}^3$ and its Gaussian curvature K.*

Explain why the sphere is not a locally Euclidean Riemannian manifold.

On the sphere of radius R give an example of local coordinates in the vicinity of an arbitrary point $\mathbf{p}$ such that in these coordinates standard Riemannian metric of the sphere is equal to $du^2 + dv^2$ at this point $\mathbf{p}$.

[8 marks]

a) No specific problems.

b) No specific problems.

c) Using the relation between Gaussian and Riemann curvature students have showed that sphere is not locally Euclidean, i.e. there are no local coordinates $\begin{cases} u = u(\theta, \varphi) \\ v = v(\theta, varphi) \end{cases}$ such that in these coordinates

$$G = R^2(d\theta^2 + \sin^2\theta d\varphi^2) = du^2 + dv^2$$

Indeed if $G = du^2 + dv^2$ in a vicinity of a given point then Christoffel symbols vanish in coordinates u, v. This implies that curvature vanishes. Contradiction. Many students (almost everybody who tried this question) answered this part of question satisfactory. This is very good! Psychologically it is not easy after this right answer to show that on the other hand for every point $\mathbf{p}$ on the sphere there exist coordinates such that $G = du^2 + dv^2$... just at this point. But this is true. Indeed e.g.consider spherical coordinates θ, φ such that at the point $\mathbf{p}$, $\theta = \frac{\pi}{2}$, $\varphi = \varphi_9$, i.e. point $\mathbf{p}$ is on the Equator, and consider new local coordinates $u = R\theta$, $v = R\varphi$. Then one can see that at the point $\mathbf{p}$, $G = du^2 + dv^2$. This does not contradict with the condition that curvature is not vanished. (see also the discussion of last subquestion of the question 2.) In fact one can see that this is true for arbitrary Riemannian metric: For every point there are local coordinates such that at this point the matrix of metric is just unity matrix.

Is it difficult question or no. ON one hand not very difficult, but as a matter of fact If I remember right only one student answered this subquestion.

The following question is compulsory.(for 15 credit students)

5.

(a) *Explain what is meant by saying that F is an isometry between two Riemannian manifolds.*

Consider the plane $\mathbf{R}^2$ with coordinates (x, y) and with the Riemannian metric

$$G_{(1)} = \frac{a(dx^2 + dy^2)}{(1 + x^2 + y^2)^2}, \quad (a > 0),$$

and the sphere of radius R (without the North pole) with standard metric G which in stereographic coordinates (u, v) has the appearance

$$G_{(2)} = \frac{4R^4(du^2 + dv^2)}{(R^2 + u^2 + v^2)^2}.$$

Find an isometry F: $\begin{cases} u = u(x, y) \\ v = v(x, y) \end{cases}$ between these two Riemannian manifolds in the case of $a = 4R^2$.

Explain why, in the case where $a \neq 4R^2$, there is no isometry between these Riemannian manifolds.

[10 marks]

(b) *Describe all infinitesimal isometries (Killing vector fields) of the Lobachevsky plane (the upper half plane ($y > 0$) with the metric $G = \frac{dx^2+dy^2}{y^2}$), and deduce equations for the geodesics.*

You may use the fact that translations $\begin{cases} x' = x + a \\ y' = y \end{cases}$, *homotheties* $\begin{cases} x' = \lambda x \\ y' = \lambda y \end{cases}$, $(\lambda > 0)$ *and inversion* $\begin{cases} x' = \frac{x}{x^2+y^2} \\ y' = \frac{y}{x^2+y^2} \end{cases}$ *are isometries of the Lobachevsky plane.*

[10 marks]

a) No specific problems. Students just need the clear understanding of detailed definition of isometry (see lecture notes)

b) Just one comment: when finding geodesics students come to system of simultaneous equations

Geodesics is trajectory of free particle. It is well-defined by three conditions $I_1 = C_1, I_2 = C_2, I_3 = C_3$. We have linear equations on $\dot{x}, \dot{y}$

$$\begin{cases} \dot{x} = y^2 C_1 \\ x\dot{x} + y\dot{y} = y^2 C_2 \\ (y^2 - x^2)\dot{x} - 2xy\dot{y} = y^2 C_3 \end{cases}$$

Compatibility of three linear equations on two variables $\dot{x}, \dot{y}$ implies that

$$\det \begin{pmatrix} 1 & 0 & y^2 C_1 \\ x & y & y^2 C_2 \\ (y^2 - x^2) & -2xy & y^2 C_3 \end{pmatrix} = 0$$

Calculating the determinant we come to

$$y^3 (C_3 - 2xC_2 + C_1(x^2 + y^2)) = 0\,.$$

Nobody explained why the vanishing of determinant implies the equation of geodesics.