Homework 1. Solutions

1 *Show that the set of vectors* $\{\mathbf{a}_1, \mathbf{a}_2 \ldots, \mathbf{a}_m\}$ *in vector space* V *is linearly dependent if at least one of these vectors is equal to zero.*

WLOG suppose that $\mathbf{a}_1 = 0$. Then

$$\lambda \mathbf{a}_1 + 0 \cdot \mathbf{a}_2 + \ldots + 0 \cdot \mathbf{a}_n = 0$$

where λ is an arbitrary real number. We see that there exists a linear combinations of vectors $\{\mathbf{a}_1, \mathbf{a}_2 \ldots, \mathbf{a}_m\}$ which is equal to zero and one of the coefficients $\{\lambda, 0, \ldots, 0\}$ could be equal to non-zero. Hence vectors $\{\mathbf{a}_1, \mathbf{a}_2 \ldots, \mathbf{a}_m\}$ are linearly dependent.

2 *Show that any three vectors* $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ *in* $\mathbf{R}^2$ *are linearly dependent.* We will show it straightforwardly here.

Let three vectors

$$\begin{aligned} \mathbf{x}_1 &= (a^1, a^2) \\ \mathbf{x}_2 &= (b^1, b^2) \\ \mathbf{x}_3 &= (c^1, c^2) \end{aligned}$$

be linearly independent. If vector $\mathbf{x}_1 = (a_1, a_2) = 0$ then nothing to prove.(See exercise 1). Let $\mathbf{x}_1 \neq 0$.WLOG suppose $a_1 \neq 0$. Consider

$$\begin{aligned} \mathbf{x}'_2 &= \mathbf{x}_2 - \tfrac{b_1}{a_1}\mathbf{x}_1 = (b^1, b^2) - \tfrac{b_1}{a_1}(a_1, a_2) = (0, b'_2) \\ \mathbf{x}'_3 &= \mathbf{x}_3 - \tfrac{c_1}{a_1}\mathbf{x}_1 = (c^1, c^2) - \tfrac{c_1}{a_1}(a_1, a_2) = (0, c'_2) \end{aligned}$$

We see that vectors $\mathbf{x}'_2, \mathbf{x}'_3$ are proportional—i.e. they are linearly dependent: there exist $\mu_2 \neq 0$ or $\mu_3 \neq 0$ such that $\mu_2 \mathbf{x}'_2 + \mu_3 \mathbf{x}'_3 = 0$ E.g. we can take $\mu_2 = c'_2$, $\mu_3 = -b'_2$ if $c'_2 \neq 0$ or $b'_2 \neq 0$ (if $c'_2 = b'_2 \neq 0$ then we can take coefficients μ_1, μ_2 any real numbers.) We have:

$$0 = \mu_2 \mathbf{x}'_2 + \mu_3 \mathbf{x}'_3 = \mu_2 \left(\mathbf{x}_3 - \frac{c_1}{a_1}\mathbf{x}_1 \right) + \mu_3 \left(\mathbf{x}_3 - \frac{c_1}{a_1}\mathbf{x}_1 \right) = \mu_2 \mathbf{x}_2 + \mu_3 \mathbf{x}_3 - \left(\frac{\mu_2 b_1}{a_1} + \frac{\mu_3 c_1}{a_1} \right) \mathbf{x}_1 = 0,$$

where $\mu_2 \neq 0$ or $\mu_3 \neq 0$. Hence vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linearly dependent. ∎

(Compare with the solution of general statement in the next exercise.)

3 *Let* 3 *vectors* $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ *in vector space* V *can be expressed as a linear combination of* 2 *vectors* $\{\mathbf{a}, \mathbf{b}\}$ *of this vector space, i.e. 3 vectors* $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ *belong to the span of 2 vectors* $\{\mathbf{a}, \mathbf{b}\}$. *Prove that three vectors* $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ *are linearly dependent.*

Let

$$\begin{cases} \mathbf{x}_1 = \lambda_1 \mathbf{a} + \mu_1 \mathbf{b} \\ \mathbf{x}_2 = \lambda_2 \mathbf{a} + \mu_2 \mathbf{b} \\ \mathbf{x}_3 = \lambda_3 \mathbf{a} + \mu_3 \mathbf{b} \end{cases} \tag{1}$$

If one of vectors is equal to zero then nothing to prove (See previous exercise).

$\mathbf{x}_1 \neq 0$. WLOG suppose that $\lambda_1 \neq 0$. Thus vector $\mathbf{a}$ can be expressed as a linear combination of vectors $\mathbf{x}_1$ and $\mathbf{b}$:

$$\mathbf{a} = \frac{1}{\lambda_1}\mathbf{x}_1 - \frac{\mu_1}{\lambda_1}\mathbf{b} \tag{2}$$

. (If $\lambda_1 = 0$ then $\mu \neq 0$ and we express the vector b as a linearly combination of vectors $\mathbf{x}_1$ and $\mathbf{a}$). Then using the relation (2) we express vector $\mathbf{x}_2$ as linear combinations of vectors $\mathbf{a}$ and $\mathbf{x}_1$:

$$\mathbf{x}_2 = \lambda_2 \left(\frac{1}{\lambda_1}\mathbf{x}_1 - \frac{\mu_1}{\lambda_1}\mathbf{b} \right) + \mu_2 \mathbf{b} = \lambda'_2 \mathbf{x}_1 + \mu'_2 \mathbf{b} \tag{3}$$

If $\mu'_2 = 0$ then everything is proved: vectors $\mathbf{x}_1, \mathbf{x}_2$ are linearly dependent, hence vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linearly dependent too. If $\mu'_2 \neq 0$ we express vector $\mathbf{b}$ via vectors $\mathbf{x}_1$ and $\mathbf{x}_2$:

$$\mathbf{b} = -\frac{1}{\mu_1}\mathbf{x}_2 - \frac{\lambda'}{\mu'}\mathbf{x}_1 \tag{4}$$

and using relations (4) and (2) we express vector $\mathbf{x}_3$ in (1) as a linear combinations of vectors $\mathbf{x}_1$ and $\mathbf{x}_2$, thus proving that vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ are linearly dependent.

$$\mathbf{x}_3 = \lambda_3 \mathbf{a} + \mu_3 \mathbf{b} = \lambda_3 \left(-\frac{1}{\lambda_1}\mathbf{x}_1 - \frac{\mu_1}{\lambda_1}\mathbf{b} \right) + \mu_3 \left(\frac{1}{\mu_1}\mathbf{x}_2 - \frac{\lambda'}{\mu'}\mathbf{x}_1 \right) =$$

$$\lambda_3 \left(-\frac{1}{\lambda_1}\mathbf{x}_1 - \frac{\mu_1}{\lambda_1} \left(\frac{1}{\mu_1}\mathbf{x}_2 - \frac{\lambda'}{\mu'}\mathbf{x}_1 \right) \right) + \mu_3 \left(\frac{1}{\mu_1}\mathbf{x}_2 - \frac{\lambda'}{\mu'}\mathbf{x}_1 \right) = \lambda_3''\mathbf{x}_1 + \mu_3''\mathbf{x}_2$$

Vector $\mathbf{x}_3$ is a linear combination of vectors $\mathbf{x}_2, \mathbf{x}_3$. Hence vectors $\mathbf{x}_1 y, \mathbf{x}_2, \mathbf{x}_3$ are linearly dependent.

In a similar way one can prove that any $m+1$ vectors are linearly dependent if they belong to the span of m vectors

4 *Let* $\{\mathbf{a}, \mathbf{b}\}$ *be two vectors in the linear space* V *such that*
i) these vectors are linearly independent
ii) for an arbitrary vector $\mathbf{x} \in V$ *vectors* $\{\mathbf{a}, \mathbf{b}, \mathbf{x}\}$ *are linearly dependent.*
What is a dimension of the vector space V*?*
Is an ordered set $\{\mathbf{a}, \mathbf{b}\}$ *a basis in the vector space* V*?*

Recall that the dimension of vector space V is equal to n if there exist n linearly independent vectors and any $n+1$ vectors are linearly dependent. Show that the dimension of the vector space under consideration is equal to 2.

On one hand there exist two linearly dependent vectors $\mathbf{a}$ and $\mathbf{b}$. This means that dimension of V is greater or equal than 2: $\dim V \geq 2$.

To prove that $\dim V = 2$ it remains to prove that any three vectors are linearly dependent.

Show first that arbitrary vector $\mathbf{x} \in V$ can be expressed via vectors $\mathbf{a}, \mathbf{b}$. Indeed vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{x}\}$ are linearly dependent, hence

$$\mu_1 \mathbf{a} + \mu_2 \mathbf{b} + \mu_3 \mathbf{x} = 0, \quad \text{where } \mu_1 \neq 0, \text{ or } \mu_2 \neq 0 \text{ or } \mu_3 \neq 0$$

If $\mu_3 = 0$ then $\mu_1 \neq 0$, or $\mu_2 \neq 0$ and $\mu_1 \mathbf{a} + \mu_2 \mathbf{b} = 0$, i.e. vectors $\mathbf{a}, \mathbf{b}$ are linearly dependent. Contradiction. Hence $\mu_3 \neq 0$, that is a vector $\mathbf{x}$ can be expressed as a linear combination of vectors $\mathbf{a}, \mathbf{b}$, i.e. it belongs to the span of the vectors $(\mathbf{a}, \mathbf{b})$.

Let $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ be a set of arbitrary 3 vectors. We just proved that any of these vectors belong to the span of the vectors $\{\mathbf{a}, \mathbf{b}\}$. Hence according to previous exercise these three vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ are linearly dependent. Thus we proved that any three vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ are linearly dependent.

Hence the dimension of the space V is equal to 2.

The vectors $\{\mathbf{a}, \mathbf{b}\}$ are two linearly independent vectors in 2-dimensional vector space V. Hence it is a basis.

5 *Let* $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ *be a basis in* 3*-dimensional vector space* V*. Show that*
a) all vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ *are not equal to zero.*
b) an arbitrary vector $\mathbf{a} \in V$ *can be expressed as a linear combination of the basis vectors* $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ *in a unique way, i.e. if*

$$\mathbf{a} = a^1\mathbf{e}_1 + a^2\mathbf{e}_2 + a^3\mathbf{e}_3 = a^{1'}\mathbf{e}_1 + a^{2'}\mathbf{e}_2 + a^{3'}\mathbf{e}_3 \text{ then } a^1 = a^{1'},\, a^2 = a^{2'},\, a^3 = a^{3'} . \tag{5}$$

a) Suppose one of these vectors is equal to zero: $\mathbf{e}_1 = 0$. Then the vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are linearly dependent. (See the exercise 1).

b) First prove the uniqueness of expansion (5) then the existence. Let $\mathbf{a}$ be an arbitrary vector in V. Suppose

$$\mathbf{a} = a^1\mathbf{e}_1 + a^2\mathbf{e}_2 + a^3\mathbf{e}_3 = a^{1'}\mathbf{e}_1 + a^{2'}\mathbf{e}_2 + a^{3'}\mathbf{e}_3 .$$

Then

$$0 = \mathbf{a} - \mathbf{a} = (a^1 - a1')\mathbf{e}_1 + (a^2 - a^{2'})\mathbf{e}_2 + (a^3 - a^{3'})\mathbf{e}_3 .$$

On the other hand vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are linearly independent. Hence all coefficients $(a^1 - a^{1'}), (a^2 - a^{2'}), (a^3 - a^{3'})$ are equal to zero:

$$a^1 - a^{1'} = a^2 - a^{2'} = a^3 - a^{3'} = 0, \text{ i.e. } a^1 = a^{1'}, a^2 = a^{2'}, a^3 = a^{3'}.$$

According to definition of basis 4 vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{a}\}$ are linearly dependent. Hence vector $\mathbf{a}$ can be expressed via the vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Indeed there exist coefficients $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that

$$\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 + \lambda_3 \mathbf{e}_3 + \lambda_4 \mathbf{a} = 0 \tag{6}$$

and at least one of these coefficients is not equal to zero.

Prove that $\lambda_4 \neq 0$. Suppose $\lambda_4 = 0$. Then it follows from (6) that vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are linearly dependent. Contradiction. Hence $\lambda_4 \neq 0$ and $\mathbf{a}$ can be expressed via $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$:

$$\mathbf{a} = -\frac{\lambda_1}{\lambda_4}\mathbf{e}_1 - \frac{\lambda_2}{\lambda_4}\mathbf{e}_2 - \frac{\lambda_3}{\lambda_4}\mathbf{e}_3$$

6† *Show that the ordered set* $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \dots, \mathbf{e}_n\}$ *of vectors is a basis in* $\mathbf{R}^n$ *in the case if*

$$\begin{array}{lllllll}
\mathbf{e}_1 & = (1, & 2, & 3, & 4, \dots, & n) \\
\mathbf{e}_2 & = (0, & 1, & 2, & 3, \dots, & n-1) \\
\mathbf{e}_3 & = (0, & 0, & 1, & 2, \dots, & n-2) \\
\dots & & & & & \\
\mathbf{e}_n & = (0, & 0, & 0, & 0, \dots, & 1)
\end{array}$$

If $\sum \lambda_i \mathbf{e}_i = 0$ then one can see that $\lambda_1 = 0$. This implies that $\lambda_2 = 0$ and so on all coefficeints λ_i vanish. We proved that these n vectors in n-dimensional space $\mathbf{R}^n$ are linear independent. Hence $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \dots, \mathbf{e}_n\}$ is a basis.

7 Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a basis of 3-dimensional vector space V.
Is a set of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ a basis of V in the case if
a) $\mathbf{e}'_1 = \mathbf{e}_2$, $\mathbf{e}'_2 = \mathbf{e}_1$, $\mathbf{e}'_3 = \mathbf{e}_3$;
b) $\mathbf{e}'_1 = \mathbf{e}_1$, $\mathbf{e}'_2 = \mathbf{e}_1 + 3\mathbf{e}_3$, $\mathbf{e}'_3 = \mathbf{e}_3$;
c) $\mathbf{e}'_1 = \mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{e}'_2 = 3\mathbf{e}_1 - 3\mathbf{e}_2$, $\mathbf{e}'_3 = \mathbf{e}_3$;
d) $\mathbf{e}'_1 = \mathbf{e}_2$, $\mathbf{e}'_2 = \mathbf{e}_1$, $\mathbf{e}'_3 = \mathbf{e}_1 + \mathbf{e}_2 + \lambda\mathbf{e}_3$ (where λ is an arbitrary coefficient)?

To analyse the cases we use the definition of basis: 3 vectors in 3-dimensional space form a basis if and only if these vectors are linearly independent.

Case a) Vectors $\mathbf{e}'_1 = \mathbf{e}_2, \mathbf{e}'_2 = \mathbf{e}_1, \mathbf{e}'_3 = \mathbf{e}_3$ are linearly independent, since $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis. Hence $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ is a basis too.

Case b) Vectors $\mathbf{e}'_1 = \mathbf{e}_1, \mathbf{e}'_2 = \mathbf{e}_1 + 3\mathbf{e}_3, \mathbf{e}'_3 = \mathbf{e}_3$ are linearly dependent. Indeed

$$\mathbf{e}'_1 - \mathbf{e}'_2 + 3\mathbf{e}'_3 = \mathbf{e}_1 - (\mathbf{e}_1 + 3\mathbf{e}_3) + 3\mathbf{e}_3 = 0\,.$$

Hence it is not a basis.

Case c) First two vectors $\mathbf{e}'_1 = \mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}'_2 = 3\mathbf{e}_1 - 3\mathbf{e}_2$ are already linearly dependent: $\mathbf{e}'_1 = 3\mathbf{e}'_2$. Hence these three vectors do not form a basis.

Case d) Check are vectors linearly independent or not. Let $c_1\mathbf{e}'_1 + c_2\mathbf{e}'_2 + c_3\mathbf{e}'_3 = 0$, i.e.

$$c_1\mathbf{e}'_1 + c_2\mathbf{e}'_2 + c_3\mathbf{e}'_3 = c_1\mathbf{e}_2 + c_2\mathbf{e}_1 + c_3(\mathbf{e}_1 + \mathbf{e}_2 + \lambda\mathbf{e}_3) = (c_2 + c_3)\mathbf{e}_1 + (c_1 + c_3)\mathbf{e}_2 + c_3\lambda\mathbf{e}_3 = 0\,.$$

I-st case $\lambda \neq 0$. We have $c_2 + c_3 = c_1 + c_3 = \lambda c_3 = 0$. Hence $c_3 = 0, c_1 = 0, c_2 = 0$. These three vectors are linearly independent. This means that ordered triple $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ is a basis.

II-nd case $\lambda = 0$. We have $c_2 + c_3 = c_1 + c_3 = 0c_3 = 0$. Hence c_3 can be an arbitrary number and $c_1 = -c_3, c_2 = -c_3$. c_3 These three vectors are linearly dependent. This means that ordered triple $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ is not a basis.