Riemannian Geometry

2019

COURSEWORK

This assignment counts for 20% of your marks.

Solutions are due by 28 March 3pm

Write solutions in the provided spaces.

STUDENT'S NAME:

1

Consider a surface M, the upper sheet of the cone

$$\mathbf{r}(h,\varphi)\colon \begin{cases} x = h\cos\varphi \\ y = h\sin\varphi \\ z = 2h \end{cases}, \qquad 0 \le \varphi < 2\pi\,,\; h > 0\,. \tag{1}$$

Calculate the Riemannian metric G on this surface induced by the Euclidean metric in $\mathbf{E}^3$ in coordinates (h,φ).

Show that this surface is locally Euclidean by giving an example of local coordinates (u,v), which are Euclidean coordinates.

Find the length of the shortest curve which belongs to the surface M, starts at the point $(h_0, 0, 2h_0)$ and ends at the point $(-h_0, 0, 2h_0)$.

[3 marks]

2

Recall that the Riemannian metric on the sphere of radius R in the stereographic coordinates is expressed by the formula

$$G_{\text{stereogr.}} = \frac{4R^4(du^2+dv^2)}{(R^2+u^2+v^2)^2}\,.$$

(a) Give an example of a non-identity transformation of coordinates u,v which preserves this metric .

(b) Give an example of a non-linear transformation of coordinates u,v which preserves this metric.

(*Hint: You may find this transformation considering transformations of the sphere.*)

(c) Find the length of the line $v = au$ in $\mathbf{R}^2$ with respect to this metric.

Explain why the length of this curve does not depend on a.

[3 marks]

3

Evaluate the area of the part of the sphere of radius $R = 1$ between the planes given by equations $2x+2y+z=1$ and $2x+2y+z=2$.

[1 marks]

4

Consider the plane $\mathbf{R}^2$ with standard coordinates (x, y) equipped with Riemannian metric

$$G = (1 + x^2 + y^2)e^{-a^2x^2-a^2y^2} \left(dx^2 + dy^2\right) .$$

Calculate the total area of this plane.

[1 marks]

5

Consider the upper half-plane $y > 0$ with the Riemannian metric

$$G = \frac{dx^2 + dy^2}{y^2}$$

(the Lobachevsky plane).

Consider in the Lobachevsky plane the domain D defined by

$$D = \{x, y\colon \quad x^2 + y^2 \geq 1\,,\ -a \leq x \leq a\}\,,$$

where a is a parameter $(0 < a < 1)$.

Find the area of the domain D (with respect to the metric G).

[3 marks]

6

(a) Let ∇ be an affine connection on the 2-dimensional manifold M such that in local coordinates (u, v), $\nabla_{\frac{\partial}{\partial u}} \left(u^2 \frac{\partial}{\partial v}\right) = 3u\frac{\partial}{\partial v} + u\frac{\partial}{\partial u}$.

Calculate the Christoffel symbols Γ^u_{uv} and Γ^v_{uv} of this connection.

(b) Let ∇ be an arbitrary connection on a manifold M. Show that

$$\cos F \nabla_{\mathbf{A}}(\sin F\mathbf{B}) - \sin F \nabla_{\mathbf{A}}(\cos F\mathbf{B}) = (\partial_{\mathbf{A}}F)\mathbf{B}\,,$$

where F is an arbitrary function.

(c) Let $\Gamma^{i(1)}_{km}$ be the Christoffel symbols of a connection $\nabla^{(1)}$ and $\Gamma^{i(2)}_{km}$ be the Christoffel symbols of a connection $\nabla^{(2)}$.

Show that the linear combinations $\frac{2}{3}\Gamma^{i(1)}_{km} + \frac{1}{3}\Gamma^{i(2)}_{km}$, are Christoffel symbols for some connection.

Explain why $\frac{1}{2}\Gamma^{i(1)}_{km} + \frac{1}{3}\Gamma^{i(2)}_{km}$ are not Christoffel symbols for some connection.

[3 marks]

7

Let ∇ be a connection in $\mathbf{E}^3$ such that Christoffel symbols of this connection in Cartesian coordinates are the following:

$$\Gamma^i_{km} = \begin{cases} 0 & \text{if at least two of indices coincide: } i=k \text{ or } k=m \text{ or } i=m \\ +1 & \text{if } \{ikm\} \text{ is an even permutation of indices } \{123\} \\ -1 & \text{if } ikm \text{ is an odd permutation of indices } \{123\} \end{cases},$$

e.g. $\Gamma^1_{13} = \Gamma^3_{22} = \Gamma^2_{12} = 0$, $\Gamma^1_{23} = \Gamma^3_{12} = 1$, and $\Gamma^1_{32} = \Gamma^2_{13} = -1$.

Show that this connection preserves the Euclidean scalar product.

[3 marks]

8

Let M be the surface considered in the question **1** (upper sheet of cone),

a) Calculate the induced connection on this surface (the connection induced by canonical flat connection in the ambient Euclidean space: $\nabla_{\mathbf{X}}\mathbf{Y} = \left(\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}\right)_{\text{tangent}}$.)

b) Calculate the Riemannian metric on the cone induced by the canonical metric in ambient Euclidean space $\mathbf{E}^3$ and calculate explicitly the Levi-Civita connection of this metric using the Levi-Civita Theorem.

Compare the answers

[3 marks]