Homework 3. Solutions

1 *Let $\{\mathbf{e}_x, \mathbf{e}_y\}$ be an orthonormal basis in $\mathbf{E}^2$. Consider the following ordered pairs:*

a) $\{\mathbf{e}_y, \mathbf{e}_x\}$

b) $\{\mathbf{e}_y, -\mathbf{e}_x\}$

c) $\{\frac{\sqrt{2}}{2}\mathbf{e}_x + \frac{\sqrt{2}}{2}\mathbf{e}_y, -\frac{\sqrt{2}}{2}\mathbf{e}_x + \frac{\sqrt{2}}{2}\mathbf{e}_y\}$

d) $\{\frac{\sqrt{3}}{2}\mathbf{e}_x + \frac{1}{2}\mathbf{e}_y, \frac{1}{2}\mathbf{e}_x - \frac{\sqrt{3}}{2}\mathbf{e}_y\}$

Show that all these ordered pairs are orthonormal bases in $\mathbf{E}^2$.

Find amongst them the bases which have the same orientation as the orientation of the basis $\{\mathbf{e}_x, \mathbf{e}_y\}$.

Find amongst them the bases which have the orientation opposite to the orientation of the basis $\{\mathbf{e}_x, \mathbf{e}_y\}$.

Solution:

First check that all the bases are orthonormal. For the bases a) and b) this is obvious: both vectors have unit length and they are orthogonal to each other.

Check orthogonality condition for the basis d). (For the basis a) all calculations are analogous). We have to check that vectors $\mathbf{a} = \frac{\sqrt{3}}{2}\mathbf{e}_x + \frac{1}{2}\mathbf{e}_y$ and $\mathbf{b} = \frac{1}{2}\mathbf{e}_x - \frac{\sqrt{3}}{2}\mathbf{e}_y$ have both unit length and are orthogonal to each other. For calculations we use the fact that initial basis is orthonormal too, i.e. vectors $\mathbf{e}_x, \mathbf{e}_y$ have unit length: scalar products $(\mathbf{e}_x, \mathbf{e}_x)$, $(\mathbf{e}_y, \mathbf{e}_y)$ both are equal to 1 and these vectors are orthogonal: scalar product $(\mathbf{e}_x, \mathbf{e}_y)$ is equal to zero. Calculate scalar products:

$$(\mathbf{a}, \mathbf{a}) = \left(\frac{\sqrt{3}}{2}\mathbf{e}_x + \frac{1}{2}\mathbf{e}_y, \frac{\sqrt{3}}{2}\mathbf{e}_x + \frac{1}{2}\mathbf{e}_y\right) = \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}(\mathbf{e}_x, \mathbf{e}_x) + \frac{1}{2}\cdot\frac{\sqrt{3}}{2}(\mathbf{e}_y, \mathbf{e}_x) + \frac{\sqrt{3}}{2}\cdot\frac{1}{2}(\mathbf{e}_x, \mathbf{e}_y) + \frac{1}{2}\cdot\frac{1}{2}(\mathbf{e}_y, \mathbf{e}_y) =$$

$$\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\cdot 1 + \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\cdot 0 + \frac{\sqrt{3}}{2}\cdot\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot\frac{1}{2}\cdot 1 = \frac{3}{4} + \frac{1}{4} = 1. \tag{1}$$

We see that scalar product $(\mathbf{a}, \mathbf{a})$ is equal to 1. This means that $|\mathbf{a}| = 1$.

Analogously we show that the length of the vector $\mathbf{b}$ is equal to 1:

$$(\mathbf{b}, \mathbf{b}) = \left(\frac{1}{2}\mathbf{e}_x - \frac{\sqrt{3}}{2}\mathbf{e}_y, \frac{1}{2}\mathbf{e}_x - \frac{\sqrt{3}}{2}\mathbf{e}_y\right) = \frac{1}{2}\cdot\frac{1}{2}\cdot 1 - \frac{\sqrt{3}}{2}\cdot\frac{1}{2}\cdot 0 - \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\cdot 0 + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\cdot 1 = 1\,. \tag{2}$$

It remains to show that these vectors are orthogonal , i.e. their scala product is equal to zero:

$$(\mathbf{a}, \mathbf{b}) = \left(\frac{\sqrt{3}}{2}\mathbf{e}_x + \frac{1}{2}\mathbf{e}_y, \frac{1}{2}\mathbf{e}_x - \frac{\sqrt{3}}{2}\mathbf{e}_y\right) = \frac{\sqrt{3}}{2}\cdot\frac{1}{2}(\mathbf{e}_x, \mathbf{e}_x) + \frac{1}{2}\cdot\frac{1}{2}(\mathbf{e}_y, \mathbf{e}_x) + -\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}(\mathbf{e}_x, \mathbf{e}_y) - \frac{1}{2}\cdot\frac{\sqrt{3}}{2}(\mathbf{e}_y, \mathbf{e}_y) =$$

$$\frac{\sqrt{3}}{2}\cdot\frac{1}{2}\cdot 1 + \frac{1}{2}\cdot\frac{1}{2}\cdot 0 + -\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\cdot 0 - \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\cdot 1 = 0, \tag{3}$$

i.e. these vectors are orthogonal

Orthonormality conditions for the basis a) could be checked in the same way.

Remark You could ask a question: how comes we call these pairs bases without checking the condition that they are the bases. The point is that if two vectors are not equal to zero and are orthogonal each other (and this was checked) this implies that they are not linearly dependent. (Why?: see the footnote to the lecture notes at the subsection 1.6) Hence the ordered pair of these two vectors form a basis.

Now find orientation of these bases with respect to the basis $\{\mathbf{e}_x, \mathbf{e}_y\}$. (We already show that all ordered pairs are bases.)

Case a) One can easy see that transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y\}$ to the basis $\{\mathbf{e}_y, \mathbf{e}_x\}$ is

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1a)}$$

Indeed $(\mathbf{e}_y, \mathbf{e}_x) = (\mathbf{e}_x, \mathbf{e}_y)T$: $(\mathbf{e}_y, \mathbf{e}_x) = (\mathbf{e}_x, \mathbf{e}_y)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Calculate determinant of transition matrix $\det T = -1 < 0$. Hence this basis has an orientation opposite to the orientation of the basis $\{\mathbf{e}_x, \mathbf{e}_y\}$ (The fact that $\det T \neq 0$ makes us to double check that this ordered pair is a basis. Of course in this case it is obvious.).

Case b) Analogously for the case b) one can easy see that transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y\}$ to the basis $\{\mathbf{e}_y, -\mathbf{e}_x\}$ is

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Indeed $(\mathbf{e}_y, \mathbf{e}_x) = (\mathbf{e}_x, \mathbf{e}_y)T$: $(\mathbf{e}_y, \mathbf{e}_x) = (\mathbf{e}_x, \mathbf{e}_y) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Calculate determinant of transition matrix $\det T = 1 > 0$. Hence this basis has the same orientation as the basis $\{\mathbf{e}_x, \mathbf{e}_y\}$. (The fact that $\det T \neq 0$ makes us to double check that this ordered pair is a basis. Of course in this case it is obvious.)

Case c) Analogously for the case c) one can easy see that transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y\}$ to the basis $\{\frac{\sqrt{2}}{2}\mathbf{e}_x + \frac{\sqrt{2}}{2}\mathbf{e}_y, -\frac{\sqrt{2}}{2}\mathbf{e}_x + \frac{\sqrt{2}}{2}\mathbf{e}_y\}$ is

$$T = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

Calculate determinant of transition matrix $\det T = 1 > 0$. Hence this basis has the same orientation as the basis $\{\mathbf{e}_x, \mathbf{e}_y\}$ (The fact that $\det T \neq 0$ makes us to double check that this ordered pair is a basis.)

Case d) and finally for the case d) one can easy see that transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y\}$ to the basis $\{\frac{\sqrt{3}}{2}\mathbf{e}_x + \frac{1}{2}\mathbf{e}_y, \frac{1}{2}\mathbf{e}_x - \frac{\sqrt{3}}{2}\mathbf{e}_y\}$ is

$$T = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$$

Calculate determinant of transition matrix $\det T = -1 < 0$. Hence this basis has an orientation opposite to the orientation of the basis $\{\mathbf{e}_x, \mathbf{e}_y\}$ (The fact that $\det T \neq 0$ makes us to double check that this ordered pair is a basis.)

2*Let $\{\mathbf{e}, \mathbf{f}\}$ be a basis in two-dimensional linear space V. Consider an ordered pair $\{\mathbf{a}, \mathbf{b}\}$ such that*

$$\mathbf{a} = \mathbf{f}, \ \mathbf{b} = \gamma\mathbf{e} + \mu\mathbf{f}\,,$$

where γ, μ are arbitrary real numbers.

Find values γ, μ such that an ordered pair $\{\mathbf{a}, \mathbf{b}\}$ is a basis and this basis has the same orientation as the basis $\{\mathbf{e}, \mathbf{f}\}$.

Solution: Transition matrix T from the basis $\{\mathbf{e}, \mathbf{f}\}$ to the ordered pair $\{\mathbf{a}, \mathbf{b}\}$ is matrix $T = \begin{pmatrix} 0 & \gamma \\ 1 & \mu \end{pmatrix}$: $\{\mathbf{a}, \mathbf{b}\} = \{\mathbf{e}, \mathbf{f}\} \begin{pmatrix} 0 & \gamma \\ 1 & \mu \end{pmatrix}$. Its determinant is equal to $-\gamma$. Hence the ordered pair $\{\mathbf{a}, \mathbf{b}\}$ is a basis if and only if $\gamma \neq 0$. (This can be done in other way: one can see that if $\gamma = 0$ then vectors $\mathbf{a} = \mathbf{f}$ and $\mathbf{b} = \mu\mathbf{f}$ are linear dependent, and If $\gamma \neq 0$ then vectors $\mathbf{a} = \mathbf{f}$ and $\mathbf{b} = \mu\mathbf{f}$ are linear independent.)

If $\det T = -\gamma > 0$, i.e. $\gamma < 0$ then the bases $\{\mathbf{a}, \mathbf{b}\}$ and $\{\mathbf{e}_x, \mathbf{e}_y\}$ have the same orientation.

If $\det T = -\gamma < 0$, i.e. $\gamma > 0$ then the bases $\{\mathbf{a}, \mathbf{b}\}$ and $\{\mathbf{e}_x, \mathbf{e}_y\}$ have opposite orientation.

3 *Let $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ be an arbitrary basis in $\mathbf{E}^3$. Show that the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ either has the same orientation as the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$, or the same orientation as the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$.*

Solution:. Bases $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$, $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$ have opposite orientation: $(\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z) = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)T$, where $T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\det T = -1 < 0$.

Let T_1 be transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ and T_2 be transition matrix from the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$ to the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$:

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)T_1, \quad (\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z)T_2\,.$$

We see that $T_1 = T \cdot T_2$:

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z)T_2 = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)TT_2 = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)T_1 \text{ } i.e. \text{ } TT_2 = T_1\,.$$

If $\det T_2 > 0$ then the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ have the same orientation as the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$. If $\det T_2 < 0$ then $\det T_1 = \det(TT_2) = \det T \cdot \det T_2 > 0$ because $\det T = -1 < 0$. Hence the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ have the same orientation as the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$

In other words bases $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$ have opposite orientations. Hence they belong to different classes of bases (with respect to orientation). There are two classes. Hence the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ belongs to the same equivalence class of the bases to which belongs the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ or the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ belongs to the same equivalence class of the bases to which belongs the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$.

Arbitrary basis has the same orientation as the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and the orientation opposite to the orientation of the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$ or vice versa: it has the same orientation as the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$ and the orientation opposite to the orientation of the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$

4 *Let $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ be an orthonormal basis in $\mathbf{E}^3$. Consider the following ordered triples:*
a) $\{\mathbf{e}_x, \mathbf{e}_x + 2\mathbf{e}_y, 5\mathbf{e}_z\}$,
b) $\{\mathbf{e}_y, \mathbf{e}_x, 5\mathbf{e}_z\}$,
c) $\{\mathbf{e}_y, \mathbf{e}_x, -5\mathbf{e}_z\}$,
d) $\{\frac{\sqrt{3}}{2}\mathbf{e}_x + \frac{1}{2}\mathbf{e}_y, -\frac{1}{2}\mathbf{e}_x + \frac{\sqrt{3}}{2}\mathbf{e}_y, \mathbf{e}_z\}$,
e) $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$,
f) $\{\mathbf{e}_y, \mathbf{e}_x, -\mathbf{e}_z\}$.
Show that all these ordered triples a),b),c),d),e),f) are bases.
Show that the bases a), c), d) and f) have the same orientation as the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$, and the bases b) and e) have the orientation opposite to the orientation of the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$.

Show that bases d), e) and f) are orthonormal bases and bases a), b) and c) are not orthonormal bases.

Solution:

Recall that to check that is an ordered triple a basis or no, and to find an orientation of this basis we have to find transition matrix T. If this matrix is non-degenerate, i.e. $\det T \neq 0$ then it transforms basis to a basis. If determinant of transition matrix is positive, then these two bases have the same orientation. If determinant of transition matrix is negative , then these two bases have opposite orientation.

To show that basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal basis we have to show that all basis vectors have the unit length and they are orthogonal each other, i.e. we have to check that following relations are satisfied:

$$(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \tag{4.1}$$

(Another way to check orthonormality of new basis: one have to check that transition matrix is orthogonal, i.e. it satisfies the condition $TT^t = I$ (I is identity matrix), i.e. it is a orthogonal matrix.)

Case a). One can easy see that transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the ordered triples $\{\mathbf{e}_x, \mathbf{e}_x + 2\mathbf{e}_y, 5\mathbf{e}_z\}$ is

$$T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix} \tag{1a))}$$

Indeed $\{\mathbf{e}_x, \mathbf{e}_x+2\mathbf{e}_y, 5\mathbf{e}_z\} = \{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}T$: $\{\mathbf{e}_x, \mathbf{e}_x+2\mathbf{e}_y, 5\mathbf{e}_z\} = \{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.
Calculate determinant of transition matrix: $\det T = 10 \neq 0$. Transition matrix is non-degenerate. Hence the ordered triple $\{\mathbf{e}_x, \mathbf{e}_x+2\mathbf{e}_y, 5\mathbf{e}_z\}$ is a basis. $\det T = 10 > 0$. Hence this new basis has the same orientation as the initial basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. This new basis is not orthogonal. One can see it e.g. checking that the length of the third vector is equal to $5 \neq 1$.

Case b). In this case analogously: transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the ordered triple $\{\mathbf{e}_y, \mathbf{e}_x, 5\mathbf{e}_z\}$ is

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \text{because } \{\mathbf{e}_y, \mathbf{e}_x, 5\mathbf{e}_z\} = \{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix},$$

Calculate determinant of transition matrix: $\det T = -5 \neq 0$. Transition matrix is non-degenerate. Hence the ordered triple $\{\mathbf{e}_y, \mathbf{e}_x, 5\mathbf{e}_z\}$ is a basis. $\det T = -5 < 0$. Hence this new basis has the orientation opposite to the orientation of the initial basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. This new basis is not orthogonal. One can see it e.g. checking that the length of the third vector is equal to $5 \neq 1$.

Case c) Analogously: transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the basis $\{\mathbf{e}_y, \mathbf{e}_x, -5\mathbf{e}_z\}$ is

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -5 \end{pmatrix}, \text{because } \{\mathbf{e}_y, \mathbf{e}_x, -5\mathbf{e}_z\} = \{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -5 \end{pmatrix},$$

Calculate determinant of transition matrix: $\det T = 5 \neq 0$. Transition matrix is non-degenerate. Hence the ordered triple $\{\mathbf{e}_y, \mathbf{e}_x, -5\mathbf{e}_z\}$ is a basis. $\det T = 5 > 0$. Hence this new basis has the same orientation as the initial basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. This new basis is not orthogonal. One can see it e.g. checking that the length of the third vector is equal to $5 \neq 1$.

Case d) Transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the basis $\{\frac{\sqrt{3}}{2}\mathbf{e}_x + \frac{1}{2}\mathbf{e}_y, -\frac{1}{2}\mathbf{e}_x + \frac{\sqrt{3}}{2}\mathbf{e}_y, \mathbf{e}_z\}$ is

$$T = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{because } \left\{\frac{\sqrt{3}}{2}\mathbf{e}_x + \frac{1}{2}\mathbf{e}_y, -\frac{1}{2}\mathbf{e}_x + \frac{\sqrt{3}}{2}\mathbf{e}_y, \mathbf{e}_z\right\} = \{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\} \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

Calculate determinant of transition matrix: $\det T = 1 \neq 0$. Transition matrix is non-degenerate. Hence the ordered triple $\{\frac{\sqrt{3}}{2}\mathbf{e}_x + \frac{1}{2}\mathbf{e}_y, -\frac{1}{2}\mathbf{e}_x + \frac{\sqrt{3}}{2}\mathbf{e}_y, \mathbf{e}_z\}$ is a basis. $\det T = 1 > 0$. Hence this new basis has the same orientation as the initial basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. This new basis orthonormal basis. Indeed the first two vectors have the length 1 and they are orthogonal to each other (See equations (1,2,3) in the previous exercise). The third vector $\mathbf{e}_z$ has length one and it is obviously is orthogonal to first two vectors.

Case e) for the case e) transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the basis $\{\mathbf{e}_y, \mathbf{e}_x, -\mathbf{e}_z\}$ is

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \text{because } \{\mathbf{e}_y, \mathbf{e}_x, -\mathbf{e}_z\} = \{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

Calculate determinant of transition matrix: $\det T = 1 \neq 0$. Transition matrix is non-degenerate. Hence the ordered triple $\{\mathbf{e}_y, \mathbf{e}_x, -\mathbf{e}_z\}$ is a basis. $\det T = 1 > 0$. Hence this new basis has the same orientation as the initial basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. This new basis is obviously orthonormal basis because all the vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ have unit length and they are orthogonal to each other.

Case f) for the case f) transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$ is

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{because } (\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z) = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

Calculate determinant of transition matrix: $\det T = 1 \neq 0$. Transition matrix is non-degenerate. Hence the ordered triple $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$ is a basis. $\det T = -1 > 0$. Hence this new basis has the orientation opposite to the orientation of the initial basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. (Another way to see it: we come to basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$ from the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$ just by swapping the vectors $\mathbf{e}_x$ and $\mathbf{e}_y$.) This new basis is obviously orthonormal basis because all the vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ have unit length and they are orthogonal to each other.

5 *Let $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ be a basis in linear three-dimensional space V.*
Consider the following ordered triples:

$$\{\mathbf{f}, \mathbf{e} + 2\mathbf{f}, 3\mathbf{g}\}, \quad \{\mathbf{e}, \mathbf{f}, 2\mathbf{f} + 3\mathbf{g}\}$$

Show that these ordered triples are bases and these bases have opposite orientations.

To write transition matrix from the basis $\{\mathbf{f}, \mathbf{e} + 2\mathbf{f}, 3\mathbf{g}\}$ to the basis $\{\mathbf{e}, \mathbf{f}, 2\mathbf{f} + 3\mathbf{g}\}$? This is little bit long exercise. We do it in another way:

Consider transition matrices;

T_1—transition matrix from the initial basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ to the basis $\{\mathbf{f}, \mathbf{e} + 2\mathbf{f}, 3\mathbf{g}\}$.

T_2—transition matrix from the initial basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ to the basis $\{\mathbf{e}, \mathbf{f}, 2\mathbf{f} + 3\mathbf{g}\}$ It is easy to see that

$$T_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \qquad \text{and} \qquad T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

We see that determinant of the transition matrix T_1 is negative and the determinant of the transition matrix T_2 is positive. This means that the both ordered triples $\{\mathbf{f}, \mathbf{e} + 2\mathbf{f}, 3\mathbf{g}\}$ and $\{\mathbf{e}, \mathbf{f}, 2\mathbf{f} + 3\mathbf{g}\}$ are bases. The first basis has an orientation opposite to the orientation of the initial basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ and the second basis has the same orientation as the initial basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$. Hence these both bases have opposite orientation $\blacksquare$.

6 *Show that a linear operator P which transforms the orthonormal basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the basis $\{\mathbf{e}_x, \mathbf{e}_z, -\mathbf{e}_y\}$ is a rotation. Find an axis and an angle of this rotation.*

What about a linear operator P which transforms the orthonormal basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the basis $\{\mathbf{e}_y, \mathbf{e}_x, -\mathbf{e}_z\}$. Is it a rotation?

Solution

Linear operator P acts on the space $\mathbf{E}^3$ such that

$$P\mathbf{e}_x = \mathbf{e}_x,\ P\mathbf{e}_y = \mathbf{e}_z\ P\mathbf{e}_z = -\mathbf{e}_y, \qquad P(x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z) = x\mathbf{e}_x - z\mathbf{e}_y + y\mathbf{e}_z$$

It rotates the vectors with respect to the axis OX on the angle $\frac{\pi}{2}$.

A linear operator P which transforms the orthonormal basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the basis $\{\mathbf{e}_y, \mathbf{e}_x, -\mathbf{e}_z\}$ acts on the space $\mathbf{E}^3$ such that

$$P\mathbf{e}_x = \mathbf{e}_y,\ P\mathbf{e}_y = \mathbf{e}_x\ P\mathbf{e}_z = -\mathbf{e}_y, \qquad P(x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z) = y\mathbf{e}_x + x\mathbf{e}_y - z\mathbf{e}_z$$

Orientations of the orthonormal bases $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and $\{-\mathbf{e}_x, \mathbf{e}_y, -\mathbf{e}_z\}$ are the same. Due to Euler Theorem tt is a rotation around an axis. An axis—the eigenvector. One can see that it is $\mathbf{N} = \mathbf{e}_x + \mathbf{e}_y$. Linear operator P rotates vectors with respect to the axis $\mathbf{e}_x + \mathbf{e}_y$ on the angle π.

7 † (*Euler Theorem*). *A linear operator P in $\mathbf{E}^3$ transforms an orthonormal basis to the orthonormal basis with the same orientation. Prove that it is a rotation.*

Theorem *An arbitrary orthogonal transformation that preserves an orientation of $\mathbf{E}^3$ is a rotation. (Euler Theorem)*

Proof: Let P be linear orthogonal transformation: i.e. for an arbitrary vector $\mathbf{a} = a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z$

$$P(\mathbf{a}) = P(a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z) = \mathbf{a} = a_x\mathbf{e}'_x + a_y\mathbf{e}'_y + a_z\mathbf{e}'_z$$

where the new basis $\{\mathbf{e}'_x, \mathbf{e}'_y, \mathbf{e}'_z\}$ is the orthonormal basis as well as the initial basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$.

Note that for any two vectors of the same length one can consider linear transformation—reflection which transforms the first vector to the second one. The reflection is obviously orthogonal transformation, since all the lengths are preserved, but it is an orthogonal transformation which changes the orientation. The determinant of the reflection equals to -1.

Consider any reflection which transforms basis vector $\mathbf{e}_x$ to the basis vector $\mathbf{e}_{x'}$, then reflection which does not move vector $\mathbf{e}_{x'}$ and transforms basis vector $\mathbf{e}_y$ to the basis vector $\mathbf{e}_{y'}$. If it is not enough we can in principal consider the third reflection. We see that every orthogonal transformation P can be considered as a reflection or a composition of two or three reflections. But determinant of every reflection is equal to -1. On the other hand these bases have the same orientation. Hence we see that composition O_1O_2 of two reflections transforms basis $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ to basis $\mathbf{e}_{x'}, \mathbf{e}_{y'}, \mathbf{e}_{z'}$. Let α_1 be invariant plane of reflection O_1 and α_2 be invariant plane of reflection O_2. Then a line which is intersection of these planes is an axis. ∎

Another solution: Any non-degenerate matrix has at least one eigenvector. It is easy to prove. Then try to prove that orthogonal operator $P \neq I$ which preserves orientation has exactly one eigenvector (up to multiplication). This is an axis of rotation.

The proof above was in fact the explicit construction of this eigenvector.