**Feedback**

RIEMANNIAN GEOMETRY(31082, 41082,61082. Spring 2016)

ANSWER ANY THREE OF QUESTIONS 1—4 AND QUESTION 5
All questions are worth 20 marks

For 31082 fifth question is excluded
Electronic calculators may not be used

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**1.**

**(a)** Explain what is meant by saying that $G$ is a Riemannian metric on a manifold $M$.

Consider the plane $\mathbf{R}^2$ with standard coordinates $(x, y)$ equipped with Riemannian metric $G = \sigma(x, y)(dx^2 + dy^2)$.
Explain why $\sigma(x, y) > 0$.

Let $\mathbf{A}$ and $\mathbf{B}$ be two arbitrary vectors attached at some point of this plane. Explain why the cosine of the angle between these vectors does not depend on a choice of a function $\sigma(x, y)$.

In the plane with the metric $G$ as above, consider the circle $C$ defined by the equation $x^2 + y^2 = R^2$ and calculate its length in the case $\sigma(x, y) = e^{-x^2-y^2}$.

[8 marks]

**(b)** Consider the plane $\mathbf{R}^2$ with standard coordinates $(x, y)$ equipped with Riemannian metric $G = e^{-x^2-y^2}(dx^2 + dy^2)$.

Write down the formula for the area element in this metric in coordinates $(x, y)$ and in polar coordinates $(r, \varphi)$ $(x = r\cos\varphi, y = r\sin\varphi)$.
Calculate the area of the disc, $x^2 + y^2 \leq R^2$ (in the given metric).

In the case when $R = 1$ give an example of another metric $G' = \sigma(x, y)(dx^2 + dy^2)$ such that the area of the disc $x^2 + y^2 \leq 1$ will be the same as for the metric $G$.

[8 marks]

**(c)** Explain what is meant by saying that a Riemannian manifold is locally Euclidean.

Consider in Euclidean space $\mathbf{E}^3$ the surface

$$\begin{cases} x = 3h\cos\varphi \\ y = 3h\sin\varphi \\ z = h \end{cases} , \qquad h > 0\,, \quad 0 \leq \varphi < 2\pi\,,$$

(the upper sheet of the cone).
It is known that the induced Riemannian metric on this surface is given by the formula $G = 10dh^2 + 9h^2 d\varphi^2$.
Show that this surface is locally Euclidean.

[4 marks]

**Discussion of the first question**

**a)** Students have no special problems answering bookwork part of this question.

For the length of the circle: the caclulation are:

$$L = \int_{t_0}^{t_1} \sqrt{g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t)dt} = \int_0^{2\pi} \sqrt{\sigma(x(t), y(t))(x_t^2 + y_t^2)} =$$

$$= \int_0^{2\pi} \sqrt{\sigma(x(t), y(t))(R^2 \sin^2 t + R^2 \cos^2 t)} = 2\pi \cdot \sqrt{e^{-R^2} R^2} = 2\pi R e^{-\frac{R^2}{2}} .$$

(We choose parameterisation $x(t) = R\cos t, y(t) = R\sin t$, $0 \leq t < 2\pi$.)

Some students confused the caclulations of lenght with integral appearing in calculations for volume (area) (see the part **(b)** of this question) [1].

**b)** Calculation of area: We come in polar coordinates to the answer

$$S = \int_0^R \int_0^{2\pi} (\sqrt{\det G} dr d\varphi = \int_0^R r dr \int_0^{2\pi} d\varphi e^{-r^2} d\varphi =$$

$$2\pi \int_0^R e^{-r^2} r dr = \pi \int_0^{R^2} e^{-u} du = \pi(1 - e^{-R^2}) .$$

Students who performed these calculations did these calculations good [2].

The last subquestion was not easy. Only few students answered it right. They choose the metric $G_C = C(dx^2 + dy^2)$, tunning constant $C$ in a such way that area of circle remains the same:

$$\pi(1 - e^{-1}) = C\pi(\text{area of the disc with respect to the metric}) \Rightarrow C = \frac{1}{1 - e^{-1} = \frac{e}{e-1}} .$$

[1] To avoid mistakes in calculation it is worth to note that in the case if $R$ is very small the length is equal 'approximmately' to $2\pi R$(the length of the circle in standard Euclidean metric ) up to terms proportional $R^3$; this is in agreement with the fact that metric $G$ coincides with stadnard Euclidean metric $dx^2 + dy^2$ up to terms proportional to $R^2$.

[2] To avoid mistakes it is useful to note that in the right answer:

$$S = \pi(1 - e^{-R^2}) = \pi \left(1 - \left(1 - R^2 + o(R^2)\right)\right) = \pi R^2(1 + 0(R^2)) .$$

We see coicidence with the area in standard Euclidean metric up to higher orders. (Compare with comment to calculation for length in part **a**).

There is also another "natural" solution: one can deform initial metric such that integral does not change, e.g. consider new metric which is equal to $G = e^{-r^2}(1 + k\sin\varphi)rdrd\varphi$ for $|k| < 1$. One can see that in this metric area of an arbitrary disc $x^2 + y^2 \le R^2$ (not only the disc $x^2 + y^2 \le 1$ )with centre at origin does not depend on $k$.

**c)** Almost no problem with this question. In the definition of locally Euclidean Riemannian manifold it is important to emphasize that coordinates in which metric has appearance $(du^1)^2 + \cdots + (du^n)^2$ are *local coordinates.*

## 2.

**(a)** Explain what is meant by an affine connection on a manifold.

Let $\nabla$ be an affine connection on a 2-dimensional manifold $M$ in local coordinates $(u, v)$. It is known that $\nabla_{\frac{\partial}{\partial u}} \left(u\frac{\partial}{\partial u}\right) = \frac{\partial}{\partial u} + u\frac{\partial}{\partial v}$.
Calculate the Christoffel symbols $\Gamma^u_{uu}$ and $\Gamma^v_{uu}$.

`[5 marks]`

**(b)** Explain what is meant by the induced connection on a surface in Euclidean space.

Consider a sphere in $\mathbf{E}^3$:

$$\mathbf{r}(\theta,\varphi):\qquad \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases} .$$

Let $\nabla$ be the induced connection on the sphere.

Calculate the Christoffel symbols $\Gamma^\theta_{\theta\varphi}$, $\Gamma^\varphi_{\theta\varphi}$, $\Gamma^\theta_{\varphi\theta}$ and $\Gamma^\varphi_{\varphi\theta}$.

`[6 marks]`

**(c)** Give a detailed formulation of the Levi-Civita Theorem. In particular write down the expression for the Christoffel symbols $\Gamma^i_{km}$ in terms of a Riemannian metric $G = g_{ik}(x)dx^i dx^k$.

Consider the open disc $u^2 + v^2 < 1$ with the Riemannian metric

$$G = \frac{4(du^2 + dv^2)}{(1 - u^2 - v^2)^2},$$

(Poincaré disc).
Show that all Christoffel symbols of the Levi-Civita connection of this Riemannian manifold vanish at the point $u = v = 0$.

Let $\nabla'$ be a symmetric connection on the Poincaré disc such that all Christoffel symbols of this connection in coordinates $(u, v)$ vanish identically (at all points).

Show that the connection $\nabla'$ does not preserve the metric of the Poincaré disc.
(You may wish to consider the vector field $\mathbf{A} = \frac{\partial}{\partial u}$ .)

[9 marks]

## Discussion of the second question

**a)** In the list of axioms it is very important the axiom

$$\nabla_{\mathbf{X}}(f\mathbf{Y}) = \partial_{\mathbf{X}} f\mathbf{Y} + f\nabla_{\mathbf{X}}\mathbf{Y}$$

(Leibnitz rule). It is important that first term $\partial_{\mathbf{X}}$ is just directional derivative of function.

Calculations of Christoffel symbols: in generall it was alright, just some students did mistakes during these calculations.

**b)** In the definition of induced connection it is important to note that *it is induced from canonical flat connection in the ambient Euclidean space* $\mathbf{E}^3$.

**c)** Stating Levi–Civita Theorem it is very important to note that the connection $\nabla$ is symmetric, and that there exists *unique symmetric connection which preserves scalar product.*

Almost all students who did these calculations, explained why derivatives of metric vanish at origin, and they argued correctly that Christoffel symbols vanish at origin in coordinates $u, v$ since derivatives of metric at origin vanish due to Levi-Civita formula.

On the other hand still few students were trying to explain the vanishing of Christoffel symbols at origin using "statement" that metric at the origin is constant. This is empty statement like statement 'function is constant at the given point' (Compare with statement: 'function is constant in a vicinity of given point up to terms of the order....')

The lasst part of this question was difficult. Only few students managed to do it right.

Solution: To show that $\nabla'$ does not preserve scalar product, consider vector field $\mathbf{A} = \partial_u$. The condition that the scalar product $\langle \mathbf{A}, \mathbf{A} \rangle$ is preserved with respect to the connection $\nabla'$ means that for an arbitrary vector field $\mathbf{B}$, $\partial_{\mathbf{B}}\langle \mathbf{A}, \mathbf{A} \rangle = \langle \nabla'_B \mathbf{A}, \mathbf{A} \rangle + \langle \mathbf{A}, \nabla'_{\mathbf{B}} \mathbf{A} \rangle = 2\langle \nabla'_{\mathbf{B}} \mathbf{A}, \mathbf{A} \rangle$. Choose $\mathbf{B} = \mathbf{A} = \partial_u$ also. Then $\langle \mathbf{A}, \mathbf{A} \rangle = \langle \partial_u, \partial_u \rangle = \frac{4}{(1-u^2-v^2)^2}$, and $\nabla'_{\mathbf{B}} \mathbf{A} = \nabla'_{\partial_u} \partial_u = 0$. We have that $2\langle \nabla'_{\mathbf{B}} \mathbf{A}, \mathbf{A} \rangle = 0$, but $\partial_{\mathbf{B}}\langle \mathbf{A}, \mathbf{A} \rangle = \partial_u \langle \partial_u, \partial_u \rangle = \frac{\partial}{\partial u}\left(\frac{4}{(1-u^2-v^2)^2}\right)$ obviously does not vanish for all $u, v$. Contradiction.

Some students trying to attack this problem in the following right direction: they were trying to show that existence of connection $\nabla'$ is in contradiction with Levi-Civita Theorem.

**3.**

**(a)** Let $(M, G)$ be a Riemannian manifold.
Let $C$ be a curve on $M$ starting at the point $\mathbf{p}_1$ and ending at the point $\mathbf{p}_2$.

Explain what is meant by the parallel transport $P_C$ along the curve $C$.

Explain why the parallel transport $P_C$ is a linear orthogonal operator.

Let the points $\mathbf{p}_1$ and $\mathbf{p}_2$ coincide, so that $C$ is a closed curve.

Let $\mathbf{a}$ be a vector tangent to the curve at the point $\mathbf{p}_1$, and $\mathbf{b} = P_C(\mathbf{a})$.
Suppose that $P_C(\mathbf{b}) = -\mathbf{a}$.
Show that vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal to each other.

[8 marks]

**(b)** Write down the differential equation for geodesics of a Riemannian manifold in terms of Christoffel symbols.

Explain the relation between the Lagrangian of a free particle on a Riemannian manifold and the differential equations for geodesics.

Calculate the Christoffel symbols on the Lobachevsky plane.

(You may use the Lagrangian of a free particle on the Lobachevsky plane $L = \frac{1}{2}\frac{\dot{x}^2+\dot{y}^2}{y^2}$.)

[6 marks]

**(c)** Explain why great circles are geodesics on a sphere in $\mathbf{E}^3$.

On the unit sphere $x^2 + y^2 + z^2 = 1$ in $\mathbf{E}^3$ consider the curve $C$ defined by the equation $\cos\theta - \sin\theta \sin\varphi = 0$ in spherical coordinates.

Show that in the process of parallel transport along the curve $C$ an arbitrary tangent vector to the curve remains tangent to the curve.

[6 marks]

## Discussion of the third question

**a)** The answer on bookwork part of this question was almost alright. Students have problem explaining that orthogonality of operator follows from the Levi-Civita Theorem, (Levi-Civita connection preserves scalar product).

The answer on the last question: parallel transport preserves scalar product, hence $\langle \mathbf{a}, \mathbf{b}\rangle_{\mathbf{p}_1} = \langle P_C(\mathbf{a}), P_C(\mathbf{b})\rangle_{\mathbf{p}_1} = \langle \mathbf{b}, -\mathbf{a}\rangle_{\mathbf{p}_1} = -\langle \mathbf{b}, \mathbf{a}\rangle_{\mathbf{p}_1} = -\langle \mathbf{a}, \mathbf{b}\rangle_{\mathbf{p}_1}$. Hence $\langle \mathbf{a}, \mathbf{b}\rangle_{\mathbf{p}_1} = 0$ $\blacksquare$.

Beatiul, short but not too easy.

**b)** Answering this question students have no special problems.

**c)** Why geodesics are solutions. There are many ways to show it. One of the ways si the following: Let $C$ be a great circle on the sphere. Consider particle moving along the great circle with a constant speed, i.e. $\mathbf{r} = \mathbf{r}(t)$ is a parameterisation of great circle $C$ such that $|\mathbf{v}(t)| = consta$. Show that $\mathbf{r}(t)$ is a geodesic, i.e. $\nabla_{\mathbf{v}}\mathbf{v} = 0$. The acceleration vector $\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \partial_{\mathbf{v}}\mathbf{v}$ is orthogonal to surface. Hence for the connection $\nabla^{S^2}$ induced on the surface $\nabla^{S^2}_{\mathbf{v}}\mathbf{v} = 0$. Since Levi-Civita connection coincides with induced connection on the surface we have that $\nabla_{\mathbf{v}}\mathbf{v} = 0$ for Levi-Civita conenction also, i.e. $\mathbf{r}(t)$ is a geodesic.

Many students who tried to answer this question (and some of them gave full answer on this question ) decided to deduce it from second order equations of geodesics. Sure in general this is difficult task, but using the symmetry one can consider second order differential equations for geodesics with specially chosen initial conditions such that geodesics is nothing that arc of equator (in spherical coordinates). This is not the easy way to answer this question.

Second part of the question was not easy. Few students notice that the curve $C$, is great circle since it is intersection of sphere with the plane $z-y=0$. The rest is almost obvious.

## 4.

**(a)** Let $M$ be a surface in the Euclidean space $\mathbf{E}^3$. Let $\mathbf{e}, \mathbf{f}, \mathbf{n}$ be three vector fields defined on the points of this surface such that they form an orthonormal basis at any point, so that the vectors $\mathbf{e}, \mathbf{f}$ are tangent to the surface and the vector $\mathbf{n}$ is orthogonal to the surface. Consider the derivation formula

$$d\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\\ \mathbf{n}\end{pmatrix} = \begin{pmatrix}0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{pmatrix}\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\\ \mathbf{n}\end{pmatrix}, \qquad (1)$$

where $a, b$ and $c$ are 1-forms on the surface $M$.
Express the mean curvature and the Gaussian curvature of $M$ in terms of these 1-forms and vector fields.
Show that $da + b \wedge c = 0$.

[6 marks]

**(b)** Consider the surface of a saddle in Euclidean space $\mathbf{E}^3$,

$$\mathbf{r}(u,v)\colon \quad \begin{cases} x = u \\ y = v \\ z = kuv \end{cases} . \quad k \text{ is a parameter, } k \neq 0.$$

Find vector fields $\mathbf{e}, \mathbf{f}, \mathbf{n}$ defined at the points of the saddle such that they form an orthonormal basis at any point, the vectors $\mathbf{e}, \mathbf{f}$ are tangent to the surface and the vector $\mathbf{n}$

is orthogonal to the surface.
For the obtained orthonormal basis $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ calculate the vector 1-forms $d\mathbf{e}, d\mathbf{f}, d\mathbf{n}$ and the 1-forms $a$, $b$ and $c$ at the point $\mathbf{p}$ with coordinates $u = v = 0$.
Deduce from these calculations the Gaussian curvature of the saddle at the point $\mathbf{p}$.

`[8 marks]`

**(c)** Consider a surface $M$ in $\mathbf{E}^3$ with local coordinates $(u, v)$ such that the induced metric of this surface is equal to $G = \sigma(u, v)(du^2 + dv^2)$.
Write down the formula expressing Gaussian curvature of this surface in terms of the function $\sigma(u, v)$. (You do not need to prove this formula.)

Let $\sigma(u, v) = \frac{1}{v^2}$. Calculate the Gaussian curvature of this surface and explain why this surface is not locally Euclidean.

`[6 marks]`

## Discussion of the fourth question

**a)** Students who answered this question have no essential problems.

**b)** This was difficult question, in spite of the fact that in lecture notes it was considered in details a question of calculating derivation formulae and curvature for surface $z = F(x, y)$ at the point of extremum of the function $F$. Only few students overhelmed the problems of calculations. Some students did calculations considering basis of unit vectors which are parallel to vectors $\mathbf{r}_u, \mathbf{r}_v$. These vectors are orthognal only at the origin [3]. Students who performed these 'calculations' received partial credits.

**c)** Using the formula

$$K = -\frac{1}{2\sigma}\Delta \log \sigma = -\frac{1}{2\sigma}\left(\frac{\partial^2\sigma(u,v)}{\partial u^2} + \frac{\partial^2\sigma(u,v)}{\partial v^2}\right).$$

for Gaussian curvature $K$ in isothermal coordiantes we come to

$$K = -\frac{1}{2\sigma}(\log \sigma)_{vv} = -\frac{v^2}{2}\log\left(\frac{1}{v^2}\right)_{vv} = \frac{v^2}{2}2(\log v)_{vv} = v^2\left(\frac{1}{v}\right)_v = -1\,.$$

one come to the answer $K = -1$. Students who performed these caclulations succesfully have no essential problems to explain why this surface is not locally Euclidean: if surface is locally Euclidean, i.e. one can introduce local coordinates $x, y$ such that $G = dx^2 + dy^2$ then obviously Gaussian curvature $K = 0 \neq -1$. Contradiction.

On the other hand some students did mistakes calculating curvature $K$, they came to wrong answer for Gaussian curvature and finally they made confusions in their explanations, why the surface is not Euclidean.

[3] To find orthogonal basis one can take first unit vectore $\mathbf{e}$ which is parallel to $\mathbf{r}_u$, then take unit normal vector $\mathbf{n}$, and then take vector unit vector $\mathbf{f}$ which is parallel to the vector $\mathbf{e} \times \mathbf{n}$.

**The following question is compulsory.**

**5.**

**(a)** Give a definition of curvature tensor of an affine connection.

Deduce the expresssion for the components of the curvature tensor in terms of the Christoffel symbols.

Consider 2-dimensional Riemannian manifold $(M, G)$ with Riemannian metric $G = e^{-ax^2-by^2}(dx^2 + dy^2)$, where $a, b$ are parameters.

Calculate the component $R_{1212}$ of the Riemann curvature tensor at the point $x = y = 0$ of this manifold.

[10 marks]

**(b)** Let $M\colon \mathbf{r} = \mathbf{r}(u, v)$ be a surface in $\mathbf{E}^3$ with induced Riemannian metric $G = \sigma(u, v)(du^2 + dv^2)$.
Using derivation formulae deduce the expression for the Gaussian curvature of the surface $M$ via the function $\sigma(x, y)$.

Give an example of a surface $M'$: $\mathbf{r}' = \mathbf{r}'(u, v)$ such that the induced Riemannian metric on this surface is $G' = \lambda\sigma(u, v)(du^2 + dv^2)$, where $\lambda$ is a constant ($\lambda > 0$).

What is the relation between Gaussian curvature of surfaces $M'$ and $M$?

[10 marks]

## Discussion of the fifth question

This question was difficult. Students who were well prepared answered good on the question **a)** and bookwork part of the question **b)**. Unfortunately students have problem to answer the last part of the question **b**:

The answer is: If $\mathbf{r}' = \sqrt{\lambda}\mathbf{r}(u, v)$ then for every point $(u, v)$ metric $G" = \lambda G$. E.g. $G'_{uu} = (\mathbf{r}'_u, \mathbf{r}'_u) = \lambda(\mathbf{r}_u, \mathbf{r}_u) = \lambda G_{uu}$. Changing $G \to G'$, $\sigma \to \sigma' = \lambda\sigma$ and

$$K' = -\frac{1}{2\sigma'}\Delta \log(\sigma') = \frac{K}{\lambda}\,.$$