Homework 2. Solutions

1 *Let (M, G) be 2-dimensional Riemannian manifold with Riemannian metric G such that in local coordinates (u, v) it has apperance*

$$G = A(u,v)du^2 + 2B(u,v)dudv + C(u,v)dv^2\,, ||g_{ik}|| = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

Consider vector fields $\mathbf{A} = t\frac{\partial}{\partial u}+r\frac{\partial}{\partial v}$ and $\mathbf{B} = r\frac{\partial}{\partial u}-t\frac{\partial}{\partial v}$ where t, r are arbitrary coefficients.

a) Calculate the scalar product $\langle \mathbf{A}, \mathbf{B}\rangle_G$ in the case if u, v are conformal coordinates.

b) Show that condition

$$\langle \mathbf{A}, \mathbf{B}\rangle_G = 0\,, \quad \textit{for arbitrary } t, r \in \mathbf{R}$$

implies that u, v are conformal coordinates.

a) If coordinates u, v are conformal, then by definition

$$G = \sigma(u,v)\left(du^2 + dv^2\right)\,, \quad ||g_{ik}|| = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

and

$$\langle \mathbf{A}, \mathbf{B}\rangle_G = \left\langle t\frac{\partial}{\partial u} + r\frac{\partial}{\partial v}, r\frac{\partial}{\partial u} - t\frac{\partial}{\partial v}\right\rangle_G = \begin{pmatrix} t & r \end{pmatrix} \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} \begin{pmatrix} r \\ -t \end{pmatrix} = 0\,.$$

Now suppose $\langle \mathbf{A}, \mathbf{B}\rangle_G = 0$. Thus

$$\langle \mathbf{A}, \mathbf{B}\rangle_G = \left\langle t\frac{\partial}{\partial u} + r\frac{\partial}{\partial v}, r\frac{\partial}{\partial u} - t\frac{\partial}{\partial v}\right\rangle_G = \begin{pmatrix} t & r \end{pmatrix} \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} r \\ -t \end{pmatrix} = (A-C)tr + B(r^2-t^2) = 0\,.$$

Now condition $t = 0$ implies that $B = 0$, and condition implies $t = r$ that $A = C$, thus $G = A(du^2 + dv^2)$, i.e. u, v are conformal coordinates.

2 *Write down the standard Euclidean metric on $\mathbf{E}^2$ in polar coordinates*

$$dx^2+dy^2 = d(r\cos\varphi)^2+d(r\sin\varphi)^2 = (-r\sin\varphi d\varphi+\cos\varphi dr)^2+(r\cos\varphi d\varphi+\sin\varphi\varphi dr)^2 = dr^2+r^2d\varphi^2\,.$$

(See also lecture notes.)

3 *Consider the Riemannian metric on the circle of the radius R induced by the Euclidean metric on the ambient plane.*

a) Express it using polar angle as a coordinate on the circle.

b) Express the same metric using stereographic coordinate t obtained by stereographic projection of the circle on the line, passing through its centre.

a) using the angle: In this case parametric equation of circle is $\begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \end{cases}$. Then

$$G = (dx^2 + dy^2)\big|_{x=R\cos\varphi, y=R\sin\varphi} = (d\cos\varphi)^2 + (d\sin\varphi)^2 = R^2 d\varphi^2\,.$$

b) Consider stereographic coordinate with repect to North pole. One can do it straightforwardly using results of Homework 0 (or lecture notes):

$$\begin{cases} x = \frac{2tR^2}{R^2+t^2} \\ y = R\frac{t^2-R^2}{t^2+R^2} = R\left(1 - \frac{2R^2}{t^2+R^2}\right) \end{cases}.$$

Hence

$$G = (dx^2 + dy^2)\big|_{x=x(t), y=y(t)} = \left(d\left(\frac{2tR^2}{R^2+t^2}\right)\right)^2 + \left(d\left(\frac{t^2-R^2}{R^2+t^2}R\right)\right)^2 =$$

$$\left(\frac{2R^2dt}{R^2+t^2} - \frac{4t^2R^2dt}{(R^2+t^2)^2}\right)^2 + \left(-\frac{4R^2tdt}{(t^2+R^2)^2}\right)^2 == \frac{4R^4dt^2}{(R^2+t^2)^2} \blacksquare$$

Much more efficient to use explciitly polar coordinates. Cosnidering the triangle NOP where $N = (0, R)$ is North pole, $P = (t, 0)$ (see Homework 0) we come to

$$t = \tan\left(\frac{\varphi}{2} + \frac{\pi}{4}\right) \Rightarrow \varphi = 2\mathrm{arctan}\left(\frac{t}{R}\right) - \frac{\pi}{2}\,,$$

where φ is angular coordinate of the point on the circle. Hence

$$G = R^2 d\varphi^2 = R^2\left[d\left(2\mathrm{arctan}\left(\frac{t}{R}\right) - \frac{\pi}{2}\right)\right]^2 = 4R^2\frac{\left(\frac{dt}{R}\right)^2}{\left(1+\frac{t}{R}\right)^2} = \frac{4R^2dt^2}{(R^2+t^2)^2}\,.$$

Another solution We can perform these calculations Using the fact that stereographic projection is restriction of inversion with the radius $R\sqrt{2}$

5 *Consider the Riemannian metric on the sphere of the radius R induced by the Euclidean metric on the ambient 3-dimensional space.*

a) Express it using spherical coordinates on the sphere.

b) Express the same metric using stereographic coordinates u, v obtained by stereographic projection of the sphere on the plane, passing through its centre.

Solution

Riemannian metric of Euclidean space is $G = dx^2 + dy^2 + dz^2$.

a) using the spherical coordinates: In this case parametric equation of sphere is $\begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases}$. Then

$$G = (dx^2 + dy^2 + dz^2)\big|_{x=R\sin\theta\cos\varphi, y=R\sin\theta\sin\varphi, z=R\cos\theta} =$$

$$R^2\left((d\sin\theta\cos\varphi)\right)^2 + R^2\left((d\sin\theta\sin\varphi)\right)^2 + R^2\left((d\cos\theta)\right)^2 =$$

$$R^2\left(\cos\theta\cos\varphi d\theta - \sin\theta\sin\varphi d\varphi\right)^2 + R^2\left(\cos\theta\sin\varphi d\theta + \sin\theta\cos\varphi d\varphi\right)^2 + R^2\left(-\sin\theta d\theta\right)^2 =$$

$$R^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) . \tag{1}$$

b) in stereographic coordinates using stereographic coordinates u, v with respect to the North pole (see Homework 0) we have after explicit (but may be long) cacluclations: $G = (dx^2 + dy^2 + dz^2)\big|_{x=x(u,v), y=y(u,v), z=z(u,v)} =$

$$\left(d\left(\frac{2uR^2}{R^2+u^2+v^2}\right)\right)^2 + \left(d\left(\frac{2vR^2}{R^2+u^2+v^2}\right)\right)^2 + \left(d\left(1 - \frac{2R^2}{R^2+u^2+v^2}\right)R\right)^2 =$$

$$R^4\left(\frac{2du}{R^2+u^2+v^2} - \frac{2u(2udu+2vdv)}{(R^2+u^2+v^2)^2}\right)^2 + R^4\left(\frac{2dv}{R^2+u^2+v^2} - \frac{2v(2udu+2vdv)}{(R^2+u^2+v^2)^2}\right)^2 + \frac{16R^6(udu+v}{(R^2+u^2+v}$$

$$\frac{4R^4}{(R^2+u^2+v^2)^2}\left[\left(du - \frac{2u(udu+vdv)}{R^2+u^2+v^2}\right)^2 + \left(dv - \frac{2v(udu+vdv)}{R^2+u^2+v^2}\right)^2 + \frac{4R^2(udu+vdv)^2}{(R^2+u^2+v^2)^2}\right] =$$

$$\frac{4R^4(du^2+dv^2)}{(R^2+u^2+v^2)^2} \tag{2}$$

It is more efficient to use expression for metric in spherical coordinates (see above). Indeed if θ, φ spherical coordinates, and u, v stereographic coordnates then one can see that

$$\begin{cases} u = \frac{Rx}{R-z} = \frac{R\sin\theta\cos\varphi}{1-\cos\theta} = R\cos\varphi\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = R\mathrm{cotan}\,\frac{\theta}{2}\cos\varphi \\ v = \frac{Ry}{R-z} = \frac{R\sin\theta\sin\varphi}{1-\cos\theta} = R\sin\varphi\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = R\mathrm{cotan}\,\frac{\theta}{2}\sin\varphi \end{cases}$$

i.e.

$$\begin{cases} \mathrm{cotan}\,\frac{\theta}{2} = \frac{\sqrt{u^2+v^2}}{R} \\ \tan\varphi = \frac{v}{u} \end{cases}$$

Thus using expression (1) for metric in spherical coordinates we come to the same answer (2):

$$G = R^2(d\theta^2 + \sin^2\theta d\varphi^2) = R^2\left[\left(2d\left(\mathrm{arccotan}\,\frac{\sqrt{u^2+v^2}}{R}\right)\right)^2 + \sin^2\theta\left(d\left(\arctan\frac{v}{u}\right)\right)^2\right] =$$

$$R^2\left[\left[2\frac{d\left(\frac{\sqrt{u^2+v^2}}{R}\right)}{1+\frac{u^2+v^2}{R^2}}\right]^2+4\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}\left[\frac{udv-vdu}{u^2+v^2}\right]^2\right]=$$

$$R^2\left[\frac{4R^2(udu+vdv)^2}{(u^2+v^2)(R^2+u^2+v^2)}+4\frac{1}{1+\frac{u^2+v^2}{R^2}}\left[1-\frac{1}{1+\frac{u^2+v^2}{R^2}}\right]\left[\frac{udv-vdu}{u^2+v^2}\right]^2\right]=$$

$$\frac{4R^4(udu+vdv)^2}{(u^2+v^2)(R^2+u^2+v^2)^2}+\frac{4R^4}{(R^2+u^2+v^2)}\frac{(udv-vdu)^2}{(u^2+v^2)^2}{}^2=\frac{4R^4(du^2+dv^2)}{(R^2+u^2+v^2)^2}\ \blacksquare$$

Another solution One can avoid this straightforward long caluclations, just noting that stereographic projection is the restriction of inversion, of radius $\sqrt{2}R$. This immediately implies the answer.

4 a) *Let (u,v) be local coordinates on 2-dimensional Riemannian manifold (M,G) such that Riemannian metric has an appearance $G=du^2+u^2dv^2$ in these coordinates. Show that there exist local coordinates x,y such that such that $G=dx^2+dy^2$.*

b) Let (u,v) be local coordinates on 2-dimensional Riemannian manifold (M,G) such that Riemannian metric has an appearance $G=du^2+\sin^2 u dv^2$ in these coordinates.

Do there exist coordinates x,y such that $G=dx^2+dy^2$?

a) Consider new coordinates x,y such that $\begin{cases} x=u\cos v\\ y=u\sin v\end{cases}$. We see (comparing with polar coordinates) that

$$dx^2+dy^2=[d(u\cos v)]^2+[d(u\sin v)]^2=du^2+u^2dv^2\,.$$

b) Answer: 'No'.

Suppose tht there exist coordinates $\begin{cases} x=f(u,v)\\ y=g(u,v\end{cases}$ such that $dx^2+dy^2=du^2+\sin^2 udv^2$. This implies that on the sphere of radius $R=1$ there exist coordinates $\begin{cases} x=f(\theta,\varphi)\\ y=g(\theta,\varphi\end{cases}$

$$dx^2+dy^2=d\theta^2+\sin^2\theta d\varphi^2\,.$$

This contradicst to the fact that sphere has curvature.

5 *Consider an upper half-plain ($y>0$) in $\mathbf{R}^2$ equipped with Riemannian metric*

$$G=\sigma(x,y)(dx^2+dy^2)\,,\qquad (1)$$

a) Show that $\sigma>0$,

Consider two vectors $\mathbf{A}=2\partial_x$ and $\mathbf{B}=12\partial_x+5\partial_y$ attached at the point $(x,y)=(1,2)$,

b) calculate the cosine of the angle between these vectors, and show that the answer does not depend on the choice of the function $\sigma(x,y)$.

c) Calculate the lengths of these vectors in the case if

$$\sigma = \frac{1}{y^2}\,, \qquad \textit{(hyperbolic (Lobachevsky) metric)} \qquad (2)\,,$$

d) Calculate the length of the segments $x = a+t, y = b$, and $x = a, y = b+t$, $0 \le t \le 1$ if condition (2) is obeyed.

e) (exam question) Consider two curves L_1 and L_2 in upper half-plane (1) such that

$$L_1 = \begin{cases} x = f(t) \\ y = g(t) \end{cases}, \quad \text{and } L_2 \begin{cases} x = g(t) \\ y = f(t) \end{cases}, \qquad 0 \le t \le 1\,,$$

where $f(t), g(t)$ are arbitrary functions ($f(t) > 0, g(t) > 0$).

Show that these curves have the same length in the case if $\sigma(x,y) = \frac{1}{(1+x^2+y^2)^2}$.

a)$\sigma > 0$ since positive definiteness: e.g. $G(\mathbf{X},\mathbf{X}) = \sigma(x,y) > 0$ if $\mathbf{X} = \partial_x$.

b)

$$|\mathbf{A}| = \sqrt{G(\mathbf{A},\mathbf{A})} = \sqrt{\frac{A_x^2 + A_y^2}{y^2}} = \sqrt{\frac{2^2+0^2}{2^2}} = 1,\ |\mathbf{B}| = \sqrt{G(\mathbf{B},\mathbf{B})} = \sqrt{\frac{B_x^2+B_y^2}{y^2}} = \sqrt{\frac{12^2+5^2}{2^2}} =$$

c)Calculate the cosine for an arbitrary σ: $\cos(\angle(\mathbf{A},\mathbf{B})) = \frac{G(\mathbf{A},\mathbf{B})}{\sqrt{G(\mathbf{A},\mathbf{A})}\sqrt{G(\mathbf{B},\mathbf{B})}} = \frac{\langle \mathbf{A},\mathbf{B}\rangle_G}{|\mathbf{A}||\mathbf{B}|} =$

$$\frac{\sigma(x,y)\left(A_xB_x + A_yB_y\right)}{\sqrt{\sigma(x,y)\left(A_x^2+A_y^2\right)}\sqrt{\sigma(x,y)\left(B_x^2+B_y^2\right)}} = \frac{\left(A_xB_x + A_yB_y\right)}{\sqrt{\left(A_x^2+A_y^2\right)}\sqrt{\left(B_x^2+B_y^2\right)}} = \frac{2\cdot 12 + 0\cdot 5}{1\cdot 2\cdot 13} = \frac{12}{13}\,.$$

d) Length of the first curve is equal to

$$\int_0^1 \sqrt{\frac{x_t^2 + y_t^2}{y^2(t)}}\,dt = \int_0^1 \sqrt{\frac{1+0}{b^2}}\,dt = \frac{1}{b}\,,$$

length of the second curve is equal to

$$\int_0^1 \sqrt{\frac{x_t^2+y_t^2}{y^2(t)}}\,dt = \int_0^1 \sqrt{\frac{0+1}{(b+t)^2}}\,dt = \int_0^1 \frac{1}{b+t}\,dt = \log\left(1+\frac{1}{b}\right)\,.$$

e) If $x \leftrightarrow y$ then metric does not change since $\sigma(x,y) = \sigma(y,x)$: $\sigma(x,y)(dx^2+dy^2) = \sigma(y,x)(dx^2+dy^2)$, and $L_1 \leftrightarrow L_2$. Hence lengths of these curves coincide.

6 *Consider half-plane model of 2-dimensional hyperbolic (Lobachevsky plane): metric*

$$G = \frac{dx^2+dy^2}{y^2}\,.$$

Coordinates x, y are conformal coordinates[1]*. (see also questions 5c) and 5d) above).*

a) Show that coordinates u, v such that

$$\begin{cases} x = u^2 - v^2 \\ y = 2uv \end{cases},$$

are conformal coordinates[1]*.*

b) Are polar coordinates r, φ, $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$ *conformal coordinates?*

We have

$$\frac{dx^2 + dy^2}{y^2} = \frac{\left[d(u^2 - v^2)\right]^2 + \left[d(2uv)\right]^2}{4u^2v^2} = \frac{(2udu - 2vdv)^2 + (2udv + 2vdu)^2}{4u^2v^2} =$$

$$\left(\frac{1}{u^2} + \frac{1}{v^2}\right)(du^2 + dv^2),$$

i.e. these coordinates are conformal. *Another solution*

$$x + iy = z = (u^2 - v^2) + 2iuv = (u + iv)^2 = w^2$$

This is a holomorphic function, hence new coordinates are conformal also:

$$G = \frac{dx^2 + dy^2}{y^2} = \frac{4dzd\bar{z}}{(z - \bar{z})} =$$

hence for arbitrary holomorphic function $z = f(w)$

$$G = \frac{dx^2 + dy^2}{y^2} = \frac{4dzd\bar{z}}{(z - \bar{z})} = \frac{4f_w\bar{f}_{\bar{w}}dwd\bar{w}}{(f(w) - \bar{f}(\bar{w}))}.$$

Now check straightforwardly that polar coordiantes are not conformal:

$$\frac{dx^2 + dy^2}{y^2} = \frac{\left(d\left(r\cos\varphi\right)\right)^2 + \left(d\left(r\sin\varphi\right)\right)^2}{r^2\sin^2\varphi} = \frac{dr^2 + r^2d\varphi^2}{r^2\sin^2\varphi} = \frac{1}{r^2\sin^2\varphi}dr^2 + \frac{1}{\sin^2\varphi}d\varphi^2.$$

[1] *coordinates u, v are conformall (isothemric) if Riemannin metric has appearance $\sigma(u, v)(du^2 + dv^2)$ in these coordinates. E.g. coordinates in (1) are conformall coordinates.*

[1] *coordinates u, v are conformall (isothemric) if Riemannin metric has appearance $\sigma(u, v)(du^2 + dv^2)$ in these coordinates. E.g. coordinates in (1) are conformall coordinates.*