Homework 7

1 Let (M, G) be a Riemannian manifold. Let C be a curve on M starting at the point $\mathbf{p}_1$ and ending at the point $\mathbf{p}_2$.

Define an operator $P_C: T_{\mathbf{p}_1}M \to T_{\mathbf{p}_2}M$.

Explain why the parallel transport P_C is a linear orthogonal operator.

Let the points $\mathbf{p}_1$ and $\mathbf{p}_2$ coincide, so that C is a closed curve.

Let $\mathbf{a}$ be a vector attached at the point $\mathbf{p}_1$, and $\mathbf{b} = P_C(\mathbf{a})$.

Consider operator P_C^2. Suppose that $P_C(\mathbf{a}) = \mathbf{b}$ and $P_C^2(\mathbf{a}) = -\mathbf{a}$. Show that vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal to each other. (*Exam question 2016*)

2 Consider plane $\mathbf{R}^2$ equipped with Riemannian metric $G = \sigma(x, y)(dx^2 + dy^2)$.

Consider in this Riemannian manifold upper half-circle equipped with two different parameterisations

$$C_1\colon \begin{cases} x = R\cos t \\ y = R\sin t \end{cases}, \quad 0 \le t \le \pi, \qquad C_2\colon \begin{cases} x = R\cos 2t \\ y = R\sin 2t \end{cases}, \quad 0 \le t \le \frac{\pi}{2}$$

Write down explicitly equations of motion defining parallel transport for the curve C_1.‘

Show explictily that operator of parallel transport is not changed if we change C_1 on C_2.

3 Consider the Lagrangian of a free particle $L = \frac{1}{2} g_{ik} \dot{x}^i \dot{x}^k$ for Riemannian manifold with a metric $G = g_{ik} dx^i dx^k$. Write down the Euler-Lagrange equations of motion for this Lagrangian and compare them with differential equations for geodesics on this Riemannian manifold.

In fact show that

$$\underbrace{\frac{\partial L}{\partial x^i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}}_{\text{Euler-Lagrange equations}} \quad \Leftrightarrow \quad \underbrace{\frac{d^2 x^i}{dt^2} + \Gamma^i_{km} \dot{x}^k \dot{x}^m = 0}_{\text{Equations for geodesics}} , \tag{1}$$

where

$$\Gamma^i_{km} = \frac{1}{2} g^{ij} \left(\frac{\partial g_{jk}}{\partial x^m} + \frac{\partial g_{jm}}{\partial x^k} - \frac{\partial g_{km}}{\partial x^j} \right) . \tag{2}$$

4 Write down the Lagrangian of a free particle $L = \frac{1}{2} g_{ik} \dot{x}^i \dot{x}^k$ and using the Euler-Lagrange equations for this Lagrangian calculate the Christoffel symbols (the Christoffel symbols of the Levi-Civita connection) for

a) for the sphere of radius R

b) for the Lobachevsky plane

Compare with the results that you obtained using straightforwardly formula (2) or using formulae for induced connection.

5 Find geodesics on cylinder

a) using straightforwardly equations for geodesics,

b) using the fact that geodesic is shortest.

6* Find geodesics on sphere and cylinder

a) using straightforwardly equations for geodesics,

b) using the fact that geodesic is shortest.

7 *Great circle is a geodesic.*

Every geodesic is a great circle.

Are these statements correct?

Make on the base of these statements correct statements and justify them.

8 On the unit sphere $x^2 + y^2 + z^2 = 1$ in $\mathbf{E}^3$ consider the curve C defined by the equation $\cos\theta - \sin\theta\sin\varphi = 0$ in spherical coordinates.

Show that in the process of parallel transport along the curve C an arbitrary tangent vector to the curve remains tangent to the curve. (*Exam question 2016*)