Homework 2

1 Consider an upper half-plain ($y > 0$) in $\mathbf{R}^2$ equipped with Riemannian metric

$$G = \sigma(x, y)(dx^2 + dy^2)\,,$$

a) Show that $\sigma > 0$,

b) In the case if $\sigma = \frac{1}{y^2}$ (the Lobachevsky metric) calculate the lengths of vectors $\mathbf{A} = 2\partial_x$ and $\mathbf{B} = 12\partial_x + 5\partial_y$ attached at the point $(x, y) = (1, 2)$,

c) calculate the cosine of the angle between the vectors $\mathbf{A}$ and $\mathbf{B}$ and show that the answer does not depend on the choice of the function $\sigma(x, y)$,

d) Calculate the length of the segments $x = a+t, y = b$, and $x = a, y = b+t$, $0 \leq t \leq 1$ in the case if $\sigma = \frac{1}{y^2}$ (Lobachevsky plane)

e) Suppose $\sigma(x, y) = \frac{1}{(1+x^2+y^2)^2}$. Consider two curves L_1 and L_2 in upper half-plane such that

$$L_1 = \begin{cases} x = f(t) \\ y = g(t) \end{cases}, \quad \text{and } L_2 \begin{cases} x = g(t) \\ y = f(t) \end{cases}, \qquad 0 \leq t \leq 1\,,$$

where $f(t), g(t)$ are arbitrary functions ($f(t) > 0, g(t) > 0$).

Show that these curves have the same length in the case if $\sigma(x, y) = \frac{1}{(1+x^2+y^2)^2}$.

2 Consider the Riemannian metric on the circle of the radius R induced by the Euclidean metric on the ambient plane.

a) Express it using polar angle as a coordinate on the circle.

b) Express the same metric using stereographic coordinate t obtained by stereographic projection of the circle on the line, passing through its centre.

3 Consider the Riemannian metric on the sphere of the radius R induced by the Euclidean metric on the ambient 3-dimensional space.

a) Express it using spherical coordinates on the sphere.

b) Express the same metric using stereographic coordinates u, v obtained by stereographic projection of the sphere on the plane, passing through its centre.

In Exercises 2,3 we can use results of Homework 1— where stereographic coordinates were calculated. Calculating metric on the sphere in stereographic coordinates one can escape boring caclulations using inversion.

4 Consider the surface L which is the upper sheet of two-sheeted hyperboloid in $\mathbf{R}^3$:

$$L\colon \qquad z^2 - x^2 - y^2 = 1, \quad z > 0\,,$$

a) Find parametric equation of the surface L using hyperbolic functions $\cosh, \sinh$ following an analogy with spherical coordinates on the sphere.

b) Consider the stereographic projection of the surface L on the plane OXY, i.e. the central projection on the plane $z=0$ with the centre at the point $(0,0,-1)$.

Show that the image of projection of the surface L is the open disc $x^2+y^2<1$ in the plane OXY.

5* Consider the pseudo-Euclidean metric on $\mathbf{R}^3$ given by the formula

$$ds^2 = dx^2 + dy^2 - dz^2 \,. \tag{1}$$

Calculate the induced metric on the surface L considered in the Exercise 5, and show that it is a Riemannian metric (it is positive-definite).

Perform calculations in spherical-like coordinates (see Exercise 5a) above), and in stereographic coordinates (see exercise 5b) above).

Remark *The surface L sometimes is called pseudo-sphere. The Riemannian metric on this surface sometimes is called Lobachevsky (hyperbolic) metric. The surface L with this metric realises Lobachevsky (hyperbolic) geometry, where Euclid's 5-th Axiom fails. This Riemannian manifold (manifold+Riemannian metric) is called Lobachevsky (hyperbolic) plane. In stereographic coordinates Lobachevsky plane is realised as an open disc* $u^2+v^2<1$ *in* $\mathbf{E}^2$. *It is so called Poincare model of Lobachevsky geometry. On the other hand Lobachevsky (hyperbolic plane) can be realised as a upper half-plane with metric*

$$G = \frac{dx^2+dy^2}{y^2}\,, \tag{2}$$

(see the exercise 1 above). In the next exercise we will see how to compare these two realisations. of Lobachevsky plane.

6 * In the exercises 4 and 5 it was shown that pseudo-Euclidean metric (1) in $\mathbf{R}^3$ induces Riemanian metric on two-sheeted hyperboloid $z^2-x^2-y^2=1$. Show that it is not true for one-sheeted hyperboloid: metric on one-sheeted hyperboloid $x^2+y^2-z^2=1$ in $\mathbf{R}^3$ is not Riemannian if it is induced with the pseudo-Euclidean metric (1).

7* In exercise 5 we realised Lobachevsky plane as a disc $u^2+v^2<1$. Find new coordinates x,y such that in these coordinates Lobachevsky plane (hyperbolic plane) can be considered as an upper half plane $\{(x,y)\colon\ y>0\}$ in $\mathbf{E}^2$ and write down explicitly Riemannian metric in these coordinates.

Hint: *You may use complex coordinates:* $z=x+iy, \bar z = x-iy, \omega = u+iv, \bar w = u-iv$, *and consider a holomorphic transformation:* $\omega = \frac{1+iz}{1-iz} \Leftrightarrow z = i\frac{1-\omega}{1+\omega}$, *which transforms the open disc* $w\bar w<1$ *onto the upper plane* $\mathbf{Im} z>0$.

Remark Later by default we will call "Lobachevsky (hyperbolic) plane" the realisation of Lobachevsky plane as an half-upper plane in $\mathbf{E}^2$ with coordinates x,y $(y>0)$. (Riemannian metric is given by equation (2)).