Appendices to Geometry lectures

Here I put appendices to the lecture notes on Geometry
(this file is under construction)
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1 Appendices to Differential forms

1.1 Differential 1-form in arbitrary coordinates

Why differential forms? Why so strange notations for vector fields.

If we use the technique of differential forms we in fact do not care about what coordinates we work in: calculations are the same in arbitrary coordinates.

Consider first some examples

Example (Polar coordinates) Consider polar coordinates in $\mathbf{E}^2$:

$$\begin{cases} x(r,\varphi) = r\cos\varphi \\ y(r,\varphi) = r\sin\varphi \end{cases} \qquad (0 \leq \varphi < 2\pi, 0 \leq r < \infty),$$

We have that for basic 1-forms

$$dr = r_x dx + r_y dy = \frac{x}{\sqrt{x^2+y^2}}dx + \frac{y}{\sqrt{x^2+y^2}}dy = \frac{xdx + ydy}{r} \tag{1.1}$$

Respectively

$$dx = x_r dr + x_\varphi d\varphi = \cos\varphi dr - r\sin\varphi d\varphi$$

and

$$dy = y_r dr + y_\varphi d\varphi = \sin\varphi dr + r\cos\varphi d\varphi \tag{1.2}$$

For basic vector fields

$$\partial_r = \frac{\partial x}{\partial r}\partial_x + \frac{\partial y}{\partial r}\partial_y = \cos\varphi\partial_x + \sin\varphi\partial_y = \frac{x\partial_x + y\partial_y}{r},$$

$$\partial_\varphi = \frac{\partial x}{\partial \varphi}\partial_x + \frac{\partial y}{\partial \varphi}\partial_y = -r\sin\varphi\partial_x + r\cos\varphi\partial_y = x\partial_y - y\partial_x, \tag{1.3}$$

Example Calculate the value of forms $\omega = xdx + ydy$ and $\sigma = xdy - ydx$ on vector fields $\mathbf{A} = x\partial_x + y\partial_y$, $\mathbf{B} = x\partial_y - y\partial_x$. Perform calculations in Cartesian and in polar coordinates.

In Cartesian coordinates:

$$\omega(\mathbf{A}) = xdx(x\partial_x + y\partial_y) + ydy(x\partial_x + y\partial_y) = x^2 + y^2,\ \omega(\mathbf{B}) = xdx(\mathbf{B}) + ydy(\mathbf{B}) = 0,$$

$$\sigma(\mathbf{A}) = xdy(\mathbf{A}) - ydx(\mathbf{A}) = 0,\ \sigma(\mathbf{B}) = xdy(\mathbf{B}) - ydx(\mathbf{B}) = x^2 + y^2\,.$$

Now perform calculations in polar coordinates. According to relation (1.1)

$$\omega = xdx + ydy = rdr,\ \sigma = xdy - ydx = r^2 d\varphi$$

and according to relations (1.3) and (**??**)

$$\mathbf{A} = x\partial_x + y\partial_y = r\partial_r,\ \mathbf{B} = x\partial_y - y\partial_x = \partial_\varphi$$

Hence $\omega(\mathbf{A}) = rdr(\mathbf{A}) = r^2 = x^2 + y^2, \omega(\mathbf{B} = rdr(\partial_\varphi) = 0,$

$$\sigma(\mathbf{A}) = r^2 d\varphi(r\partial_r) = 0, \quad \sigma(\mathbf{B}) = r^2 d\varphi(\partial_\varphi) = r^2 = x^2 + y^2\,.$$

Answers coincide.

1.1.1 Calculations in arbitrary coordinates

Consider an arbitrary (local) coordinates $u^1, \dots, u^n$ on $\mathbf{E}^n$: $u^i = u^i(x^1, \dots, x^n)$, $i = 1, \dots, n$. Show first that

$$du^i = \sum_{k=1}^n \frac{\partial u^i(x^1, \dots, x^n)}{\partial x^k} dx^k\,. \tag{1.4}$$

It is enough to check it on basic fields:

$$du^i\left(\frac{\partial}{\partial x^m}\right) = \partial_{\left(\frac{\partial}{\partial x^m}\right)} u^i = \frac{\partial u^i(x^1, \dots, x^n)}{x^m} = \sum_{k=1}^n \frac{\partial u^i(x^1, \dots, x^n)}{\partial x^k} dx^k \left(\left(\frac{\partial}{\partial x^m}\right)\right)\,.$$

because (see (**??**)):

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}\,. \tag{1.5}$$

(We rewrite the formula (**??**) using new notations ∂_i instead $\mathbf{e}_i$). In the previous formula (**??**) we considered *cartesian* coordinates.

Show that the formula above is valid in an *arbitrary coordinates*.

One can see using chain rule that

$$\frac{\partial}{\partial u^i} = \frac{\partial x^1}{\partial u^i}\frac{\partial}{\partial x^1} + \frac{\partial x^2}{\partial u^i}\frac{\partial}{\partial x^2} + \dots + \frac{\partial x^n}{\partial u^i}\frac{\partial}{\partial x^n} = \sum_{k=1}^n \frac{\partial x^k}{\partial u^i}\frac{\partial}{\partial x^k} \tag{1.6}$$

Calculate the value of differential form du^i on vector field $\frac{\partial}{\partial u^j}$ using (1.4) and (1.6):

$$du^i\left(\frac{\partial}{\partial u^j}\right) = \sum_{k=1}^{n} \frac{\partial u^i(x^1,\dots,x^n)}{\partial x^k} dx^k \left(\sum_{r=1}^{n} \frac{\partial x^r}{\partial u^j}\frac{\partial}{\partial x^r}\right) = \tag{1.7}$$

$$\sum_{k,r=1}^{n} \frac{\partial u^i(x^1,\dots,x^n)}{\partial x^k}\frac{\partial x^r(u^1,\dots,u^n)}{\partial u^j} dx^k\left(\frac{\partial}{\partial x^r}\right) =$$

$$\sum_{k,r=1}^{n} \frac{\partial u^i(x^1,\dots,x^n)}{\partial x^k}\frac{\partial x^r(u^1,\dots,u^n)}{\partial u^j}\delta^k_r = \sum_{k=1}^{n} \frac{\partial x^k}{\partial u^j}\frac{\partial u^i}{\partial x^k} = \delta^j_i$$

We come to

$$du^i\left(\frac{\partial}{\partial u^j}\right) = \delta^i_j = \begin{cases} 1 \text{ if } i=j \\ 0 \text{ if } i\neq j \end{cases} . \tag{1.8}$$

We see that formula (1.5) has the same appearance in arbitrary coordinates. In other words it is invariant with respect to an arbitrary transformation of coordinates.

Exercise Check straightforwardly the invariance of the definition (**??**). In coordinates $(u^1,\dots,u^n)$

Solution We have to show that the formula (**??**) does not changed under changing of coordinates $u^i = u^i(x^1,\dots,x^n)$.

$$df = \sum_{i=1}^{n} \frac{\partial f(x^1,\dots,x^n)}{\partial x^i} dx^i = \sum_{i=1,k}^{n} \frac{\partial f(x^1,\dots,x^n)}{\partial x^i}\frac{\partial x^i}{\partial u^k} du^k == \sum_{i=1}^{n} \frac{\partial f}{\partial u^k} du^k ,$$

because $\sum_{i=1}^{n} \frac{\partial f(x^1,\dots,x^n)}{\partial x^i}\frac{\partial x^i}{\partial u^k} = \frac{\partial f}{\partial u^k}$

Example

Consider more in detail $\mathbf{E}^2$. (For $\mathbf{E}^3$ considerations are the same, just calculations little bit more complicated) Let u, v be an arbitrary coordinates in $\mathbf{E}^2$, $u = u(x,y), v = v(x,y)$.

$$du = \frac{\partial u(x,y)}{\partial x}dx + \frac{\partial u(x,y)}{\partial y}dy, \qquad dv = \frac{\partial v(x,y)}{\partial x}dx + \frac{\partial v(x,y)}{\partial y}dy \tag{1.9}$$

and

$$\partial_u = \frac{\partial x(u,v)}{\partial u}\partial_x + \frac{\partial y(u,v)}{\partial u}\partial_y, \;\; \partial_v = \frac{\partial x(u,v)}{\partial v}\partial_x + \frac{\partial y(u,v)}{\partial v}\partial_y \tag{1.10}$$

(As always sometimes we use notation ∂_u instead $\frac{\partial}{\partial u}$, ∂_x instead $\frac{\partial}{\partial x}$ e.t.c.)
Then

$$
\begin{gathered}
du(\partial_u) = 1, du(\partial_v) = 0 \\
dv(\partial_w) = 0, dv(\partial_v) = 1
\end{gathered}
\tag{1.11}
$$

This follows from the general formula but it is good exercise to repeat the previous calculations for this case:

$$
du(\partial_u) = \left(\frac{\partial u(x,y)}{\partial x}dx + \frac{\partial u(x,y)}{\partial y}dy\right)\left(\frac{\partial x(u,v)}{\partial u}\partial_x + \frac{\partial y(u,v)}{\partial u}\partial_y\right) =
$$

$$
\frac{\partial u(x,y)}{\partial x}\frac{\partial x(u,v)}{\partial u} + \frac{\partial u(x,y)}{\partial y}\frac{\partial y(u,v)}{\partial u} = \frac{\partial x(u,v)}{\partial u}\frac{\partial u(x,y)}{\partial x} + \frac{\partial y(u,v)}{\partial u}\frac{\partial u(x,y)}{\partial y} = 1
$$

We just apply chain rule to the function $u = u(x,y) = u(x(u,v), y(u,v))$:

Analogously

$$
du(\partial_v) = \left(\frac{\partial u(x,y)}{\partial x}dx + \frac{\partial u(x,y)}{\partial y}dy\right)\left(\frac{\partial x(u,v)}{\partial v}\partial_x + \frac{\partial y(u,v)}{\partial v}\partial_y\right)
$$

$$
\frac{\partial u(x,y)}{\partial x}\frac{\partial x(u,v)}{\partial v} + \frac{\partial u(x,y)}{\partial y}\frac{\partial y(u,v)}{\partial v} = \frac{\partial x(u,v)}{\partial v}\frac{\partial u(x,y)}{\partial x} + \frac{\partial y(u,v)}{\partial v}\frac{\partial u(x,y)}{\partial y} = 0
$$

The same calculations for dv.

1.1.2 Calculations in polar coordinates

Example. Let $f = x^4 - y^4$ and vector field $\mathbf{A} = r\partial_r$. Calculate 1-form $\omega = df$ and $\omega(\mathbf{A})$.
We have $\omega = df = 4x^3dx - 4y^3dy$. One has transforms form from Cartesian coordinates to polar or vector field from polar coordinates to Cartesian.
In Cartesian coordinates: $\mathbf{A} = r\frac{\partial}{\partial r} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$. Hence $\omega(\mathbf{A}) = df(\mathbf{A}) =$

$$
(4x^3dx - 4y^3dy)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) = 4x^3dx\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) - 4y^3dy\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) = 4x^4 - 4y^4\,.
$$

Or using (**??**), $\omega(A) = df(\mathbf{A}) = \partial_{\mathbf{A}} f = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)(x^4 - y^4) = 4x^4 - 4y^4$

In polar coordinates $f = x^4 - y^4 = (x^2 - y^2)(x^2 + y^2) = r^2(r^2\cos^{\varphi} - r^2\sin^2\varphi) = r^4\cos 2\varphi$, $\omega = df = 4r^3\cos 2\varphi dr - 2r^4\sin 2\varphi d\varphi$, and $\omega(\mathbf{A}) = \omega(r\partial_r) = 4r^4\cos 2\varphi$ since $dr(\partial_r) = 1, d\varphi(\partial_r) = 0$. Or using (**??**)

$$\omega(\mathbf{A}) = df(\mathbf{A}) = \partial_A f = r\frac{\partial}{\partial r}\left(r^4\cos\varphi\right) = 4r^4\cos 2\varphi\,.$$

Example Calculate the value of form $\omega = \frac{xdy - ydx}{x^2+y^2}$ on the vector field $\mathbf{A} = \partial_\varphi$. $\partial_{\mathbf{A}}F = r\frac{\partial}{\partial r}(r^4\cos 2\varphi) = 4r^4\cos 2\varphi = 4(x^4 - y^4)$. Or using 1-forms: We have to transform form from Cartesian coordinates to polar or vector field from polar to Cartesian.

$$\frac{xdy - ydx}{x^2 + y^2} = d\varphi, \quad \omega(\mathbf{A}) = d\varphi(\partial_\varphi) = 1$$

or

$$\partial_\varphi = x\partial_y - y\partial_x,\ \omega(\mathbf{A}) = \frac{xdy(x\partial_y - y\partial_x) - ydx(x\partial_y - y\partial_x)}{x^2 + y^2} = 1\,.$$

1.2 †Differential 2-forms (in $\mathbf{E}^2$)

1.2.1 2-form–area of parallelogram

We give first general ideas about what is it differential k-form ($k = 2, 3$)

1-form is a linear function on vectors:

$$\omega(\mathbf{A}):\ \ \omega(\lambda\mathbf{A} + \mu\mathbf{B}) = \lambda\omega(\mathbf{A}) + \mu\omega(\mathbf{B})\,,$$

2-form is a bilinear function on two vectors:

$$\omega(\mathbf{A},\mathbf{K}):\ \ \omega(\lambda\mathbf{A}+\mu\mathbf{B},\mathbf{K}) = \lambda\omega(\mathbf{A},K)+\mu\omega(\mathbf{B},K)\,, \omega(\mathbf{K},\lambda\mathbf{A}+\mu\mathbf{B}) = \lambda\omega(\mathbf{K},\mathbf{A})+\mu\omega(\mathbf{K},\mathbf{B})$$

which obey to the following condition

$$\omega(\mathbf{A},\mathbf{B}) = -\omega(\mathbf{B}.\mathbf{A}) \tag{1.12}$$

This condition implies that the value of of 2-form on vectors $\mathbf{A}, \mathbf{B}$ is proportional to the area of parallelogram $\Pi_{\mathbf{A},\mathbf{B}}$ formed vy these vectors. Explain it on a simple example.

Consider differential 2-form $dx \wedge dy$ in $\mathbf{E}^2$:

$$dx \wedge dy\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = 1$$

(In the same way as 1-forms dx, dy are basic forms for 1-form.)

Linearity conditions and condition (1.12) imply that for an arbitrary 2-form ω in $\mathbf{E}^2$ $\omega = a(x,y)dx \wedge dy$.

Take two vector fields $\mathbf{A}, \mathbf{B}$, $\mathbf{A} = A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y}$, Then due to conditions (1.12) above we have

$$\omega(\mathbf{A}, \mathbf{B}) = \omega\left(A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y}, B_x \frac{\partial}{\partial x} + B_y \frac{\partial}{\partial y}\right) =$$

$$A_x B_x \omega(\partial_x, \partial_x) A_x B_y \omega(\partial_x, \partial_y) A_y B_x \omega(\partial_y, \partial_x) A_y B_y \omega(\partial_y, \partial_y) =$$

$$a\left(A_x B_x \underbrace{dx \wedge dy(\partial_x, \partial_x)}_{=0} A_x B_y \underbrace{dx \wedge dy(\partial_x, \partial_y)}_{=1} A_y B_x \underbrace{dx \wedge dy(\partial_y, \partial_x)}_{=-1} A_y B_y \underbrace{dx \wedge dy(\partial_y, \partial_y)}_{=0}\right) =$$

$$a(A_x B_y - A_y B_x) = a \cdot \text{area of parallelogram } \Pi_{\mathbf{A},\mathbf{B}} = a \det \begin{pmatrix} A_x & A_y \\ B_x & B_y \end{pmatrix}$$

In a analogous way 3-forms are related with volume of parallelipiped, ... k-form with volume of k-parallelipiped...

1.2.2 Wedge product

We considered detailed definition of 1-forms. Now we give some formal approach to describe 2-forms. Differential forms on $\mathbf{E}^2$ is an expression obtained by adding and multiplying functions and differentials dx, dy. These operations obey usual associativity and distributivity laws but multiplications is not moreover of one-forms on each other is *anticommutative*:

$$\omega \wedge \omega' = -\omega' \wedge \omega \quad \text{if } \omega, \omega' \text{ are 1-forms} \tag{1.13}$$

In particular

$$dx \wedge dy = -dy \wedge dx, dx \wedge dx = 0, dy \wedge dy = 0 \tag{1.14}$$

Example If $\omega = xdy + zdx$ and $\rho = dz + ydx$ then

$$\omega \wedge \rho = (xdy + zdx) \wedge (dz + ydx) = xdy \wedge dz + zdx \wedge dz + xydy \wedge dx$$

and

$$\rho \wedge \omega = (dz + ydx) \wedge (xdy + zdx) = xdz \wedge dy + zdz \wedge dx + xydx \wedge dy = -\omega \wedge \rho$$

Changing of coordinates. If $\omega = a(x,y)dx \wedge dy$ be two form and $x = x(u,v), y = y(u,v)$ new coordinates then $dx = x_u du + x_v dv$, $dy = y_u du + y_v dv$ ($x_u = \frac{\partial x(u,v)}{\partial u}$, $x_v = \frac{\partial x(u,v)}{\partial v}$, $y_u = \frac{\partial y(u,v)}{\partial u}$, $y_v = \frac{\partial y(u,v)}{\partial v}$). and

$$a(x,y)dx \wedge dy = a\left(x(u,v), y(u,v)\right)(x_u du + x_v dv) \wedge (y_u du + y_v dv) = \quad (1.15)$$

$$a\left(x(u,v), y(u,v)\right)(x_u du + x_v dv)\left(x_u y_v du \wedge dv + x_v y_u dv \wedge du\right) =$$

$$a\left(x(u,v), y(u,v)\right)(x_u y_v - x_v y_u) du \wedge dv$$

Example Let $\omega = dx \wedge dy$ then in polar coordinates $x = r\cos\varphi, y = r\sin\varphi$

$$dx \wedge dy = (\cos\varphi dr - r\sin\varphi d\varphi) \wedge (\sin\varphi dr + r\cos\varphi d\varphi) = r dr \wedge d\varphi \quad (1.16)$$

1.2.3 0-forms (functions) $\xrightarrow{d}$ 1-forms $\xrightarrow{d}$ 2-forms

We introduced differential d of functions (0-forms) which transform them to 1-form. It obeys the following condition:

- d: is linear operator: $d(\lambda f + \mu g) = \lambda df + \mu dg$
- $d(fg) = df \cdot g + f \cdot dg$

Now we introduce differential on 1-forms such that

- d: is linear operator on 1-forms also
- $d(f\omega) = df \wedge \omega + f d\omega$
- $ddf = 0$

Remark Sometimes differential d is called *exterior differential.*

Perform calculations using this definition and (1.13):

$$d\omega = d(\omega_1 dx + \omega_2 dy) = dw_1 \wedge dx + dw_2 \wedge dy = \left(\frac{\partial \omega_1(x,y)}{\partial x} dx + \frac{\partial \omega_1(x,y)}{\partial y} dy\right) \wedge dx +$$

$$\left(\frac{\partial \omega_2(x,y)}{\partial x} dx + \frac{\partial \omega_2(x,y)}{\partial y} dy\right) \wedge dy = \left(\frac{\partial \omega_2(x,y)}{\partial x} - \frac{\partial \omega_1(x,y)}{\partial y}\right) dx \wedge dy$$

Example Consider 1-form $\omega = x dy$. Then $d\omega = d(x dy) = dx \wedge dy$.

1.2.4 Exact and closed forms

We know that it is very easy to integrate exact 1-forms over curves (see the subsection "Integral over curve of exact form")

How to know is the 1-form exact or no?

Definition We say that one form ω is *closed* if two form $d\omega$ is equal to zero.

Example 1-form $xdy + ydx$ is closed because $d(xdy + ydx) = 0$.

It is evident that exact 1-form is closed:

$$\omega = d\rho \Rightarrow d\omega = d(d\rho) = d \circ d\rho = 0 \tag{1.17}$$

We see that the condition that form is closed is necessary condition that form is exact.

So if $d\omega \neq 0$, i.e. the form is not closed, then it is not exact.

Is this condition sufficient? Is it true that a closed form is exact?

In general the answer is: *No*.

E.g. we considered differential 2-form

$$\omega = \frac{xdy - ydx}{x^2 + y^2} \tag{1.18}$$

defined in $\mathbf{E}^2 \backslash 0$. It is closed, but it is not exact (See non-compulsory exercises 11,12,13 in the Homework 6).

How to recognize for 1-form ω is it exact or no?

Inverse statement (Poincaré lemma) is true if 1-form is well-defined in $\mathbf{E}^2$:

A closed 1-form ω in $\mathbf{E}^n$ is exact if it is well-defined at all points of $\mathbf{E}^n$, i.e. if it is differentiable function at all points of $\mathbf{E}^n$.

Sketch a proof for 1-form in $\mathbf{E}^2$: if ω is defined in whole $\mathbf{E}^2$ then consider the function

$$F(\mathbf{r}) = \int_{C_\mathbf{r}} \omega \tag{1.19}$$

where we denote by $C_\mathbf{r}$ an arbitrary curve which starts at origin and ends at the point $\mathbf{r}$. It is easy to see that the integral is well-defined and one can prove that $\omega = df$.

The explicit formula for the function (1.19) is the following: If $\omega = a(x,y)dx + b(x,y)dy$ then $F(x,y) = \int_0^1 (a(tx,ty)x + b(tx,ty)y)\,dt$.

Exercise Check by straightforward calculation that $\omega = dF$ (See exercise 14 in Homework 6).

1.2.5 Integration of two-forms. Area of the domain

We know that 1-form is a linear function on tangent vectors. If $\mathbf{A}, \mathbf{B}$ are two vectors attached at the point $\mathbf{r}_0$, i.e. tangent to this point and ω, ρ are two 1-forms then one defines the value of $\omega \wedge \rho$ on $\mathbf{A}, B$ by the formula

$$\omega \wedge \rho(\mathbf{A}, \mathbf{B}) = \omega(\mathbf{A})\rho(B) - \omega(B)\rho(A) \tag{1.20}$$

We come to bilinear anisymmetric function on tangent vectors. If $\sigma = a(x, y)dx \wedge dy$ is an arbitrary two form then this form defines bilinear form on pair of tangent vectors: $\sigma(\mathbf{A}, \mathbf{B}) =$

$$a(x, y)dx \wedge dy(A, B) = a(x, y)\left(dx(\mathbf{A})dy(\mathbf{B}) - dx(\mathbf{B})dy(\mathbf{A})\right) = a(x, y)(A_x B_y - A_y B_y) \tag{1.21}$$

One can see that in the case if $a = 1$ then right hand side of this formula is nothing but the area of parallelogram spanned by the vectors $\mathbf{A}, \mathbf{B}$.

This leads to the conception of integral of form over domain.

Let $\omega = a(x)dx \wedge dy$ be a two form and D be a domain in $\mathbf{E}^2$. Then by definition

$$\int_D \omega = \int_D a(x, y)dxdy \tag{1.22}$$

If $\omega = dx \wedge dy$ then

$$\int_D w = \int_D (x, y)dxdy = \text{Area of the domain } D \tag{1.23}$$

The advantage of these formulae is that we do not care about coordinates[1]

Example Let D be a domain defined by the conditions

$$\begin{cases} x^2 + y^2 \leq 1 \\ y \geq 0 \end{cases} \tag{1.25}$$

Calculate $\int_D dx \wedge dy$.

[1] If we consider changing of coordinates then jacobian appears: If u, v are new coordinates, $x = x(u, v)$, $y = y(u, v)$ are new coordinates then

$$\int a(x, y)dxdy = \int a(x(u, v), y(u, v)) \det \begin{pmatrix} x_u & x_v \\ x_u & x_v \end{pmatrix} dudv \tag{1.24}$$

In formula(1.22) it appears under as a part of coefficient of differential form.

$\int_D dx \wedge dy = \int_D dxdy =$ area of the $D = \frac{\pi}{2}$.

If we consider polar coordinates then according (1.16)

$$dx \wedge dy = rdr \wedge d\varphi$$

Hence $\int_D dx \wedge dy = \int_D rdr \wedge d\varphi = \int_D rdrd\varphi = \int_0^1 \left(\int_0^\pi d\varphi\right) rdr = \pi \int_0^1 rdr = \pi/2$.

Another example

Example Let D be a domain in $\mathbf{E}^2$ defined by the conditions

$$\begin{cases} \frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} \leq 1 \\ y \geq 0 \end{cases} \tag{1.26}$$

D is domain restricted by upper half of the ellipse and x-axis. Ellipse has the centre at the point $(c, 0)$. Its area is equal to $S = \int_D dx \wedge dy$. Consider new variables x', y': $x = c + ax', y = by'$. In new variables domain D becomes the domain from the previous example:

$$\frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} = {x'}^2 + {y'}^2$$

and $dx \wedge dy = abdx' \wedge dy'$. Hence

$$S = \int_{\frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} \leq 1, y \geq 0} dx \wedge dy = ab \int_{x'^2 + y'^2 \leq 1, y' \geq 0} dx' \wedge dy' = \frac{\pi ab}{2} \tag{1.27}$$

Theorem 2 (Green formula) Let ω be 2-form such that $\omega = d\omega'$ and D be a domain–interior of the closed curve C. Then

$$\int_D \omega = \int_C \omega' \tag{1.28}$$

2 Appendices to linear algebra

2.1 Gramm matrix, Gramm determinant

Let $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ be m vectors in Euclidean vector space $\mathbf{E}^n$ (where m, n in general are two different positive integers. Consider so called Gramm matrix (Grammian) of these vectors

$$||G_{ik}||: \quad G_{ik} = (\mathbf{a}_i, \mathbf{a}_k)$$

Let a_i^k is a matrix of components of vectors $\{\mathbf{a}_i\}$ in an orthonormal basis $\{\mathbf{e}_i\}$:

$$\mathbf{a}_i = \sum_{k=1}^{n} a_i^k \mathbf{e}_k$$

Then the following very important identity takes place

$$\det G = (\det A)^2 \,. \tag{2.1}$$

Proof is easy. We have

$$(\mathbf{a}_i, \mathbf{a}_j) = \left(\sum_{k=1}^{n} a_i^k \mathbf{e}_k, \sum_{k'=1}^{n} a_j^{k'} \mathbf{e}_{k'}\right) = \left(AA^T\right)_{ij} \Rightarrow \det G = \det\left(AA^T\right) = (\det A)^2$$

Corollary 1. Vectors $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ are linear independent if and only if $\det G > 0$..

Corollary 2. Take $m = 2$. We come to CBS inequality:

$$\det G = (\mathbf{a}, \mathbf{a})(\mathbf{b}, \mathbf{b}) - (\mathbf{a}, \mathbf{b})^2 \geq 0 \,.$$

3 A Tale on Differential Geometry

Once upon a time there was an ant living on a sphere of radius R. One day he asked himself some questions: What is the structure of the Universe (surface) where he lives? Is it a sphere? Is it a torus? Or may be something more sophisticated, e.g. pretzel (a surface with two holes)

Three-dimensional human beings do not need to be mathematicians to distinguish between a sphere torus or pretzel. They just have to look on the surface. But the ant living on two-dimensional surface cannot fly. He cannot look on the surface from outside. How can he judge about what surface he lives on [2]This is not very far from reality: For us human beings it is impossible to have a global look on three-dimensional manifold. We need to develop local methods to understand global properties of our Universe. *Differential Geometry* allows to study global properties of manifold with local tools.?

Our ant loved mathematics and in particular *Differential Geometry*. He liked to draw various triangles, calculate their angles α, β, γ, area $S(\Delta)$. He

[2]*

knew from geometry books that the sum of the angles of a triangle equals π, but for triangles which he drew it was not right!!!!

Finally he understood that the following formula is true: For every triangle

$$\frac{(\alpha+\beta+\gamma-\pi)}{S(\Delta)} = c \tag{1}$$

A constant in the right hand side depended neither on size of triangle nor the triangles location. After hard research he came to conclusion that its Universe can be considered as a sphere embedded in three-dimensional Euclidean space and a constant c is related with radius of this sphere by the relation

$$c = \frac{1}{R^2} \tag{2}$$

...Centuries passed. Men have deformed the sphere of our old ant. They smashed it. It seized to be round, but the ant civilisation survived. Moreover old books survived. New ant mathematicians try to understand the structure of their Universe. They see that formula (1) of the Ancient Ant mathematician is not true. For triangles at different places the right hand side of the formula above is different. Why? If ants could fly and look on the surface from the cosmos they could see how much the sphere has been damaged by humans beings, how much it has been deformed, But the ants cannot fly. On the other hand they adore mathematics and in particular *Differential Geometry.* One day considering for every point very small triangles they introduce so called curvature for every point P as a limit of right hand side of the formula (1) for small triangles:

$$K(P) = \lim_{S(\Delta)\to 0} \frac{(\alpha+\beta+\gamma-\pi)}{S(\Delta)}$$

Ants realise that curvature which can be calculated in every point gives a way to decide where they live on sphere, torus, pretzel... They come to following formula [3] : integral of curvature over the whole Universe (the sphere) has to equal 4π, for torus it must equal 0, for pretzel it equalts -4π...

$$\frac{1}{2\pi}\int K(P)dP = 2\,(1 - \text{number of holes})$$

[3]In human civilisation this formula is called Gauß-Bonet formula. The right hand side of this formula is called Euler characteristics of the surface.