Chasles' Theorem

Sure it has hudnred idfferent proofs. 17 years ago I did something around it (see screw.tex in Etudes.) Today I realised that one can shorter the orthogonal operators in Geometry, and by the way to prove the Schall Theorem.

Let A be an orthogonal operator in $\mathbf{E}^2$

Let $\mathbf{e}, \mathbf{f}$ be an rbiotrary orthonormal basis. Then due to orthogonality $\mathbf{e}' = A(\mathbf{e}) = \cos\varphi\mathbf{e} + \sin\varphi\mathbf{f}$ (presering of the length)

Again due to orthogonality $\mathbf{f}' = A(\mathbf{f})$ is orthogonal to $\mathbf{e}'$.

Hence

$$\mathbf{f}' = A(\mathbf{f}) = -\sin\varphi\mathbf{e} + \cos\varphi\mathbf{f}\,, \quad A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

or

$$\mathbf{f}' = \tilde{A}(\mathbf{f}) = \sin\varphi\mathbf{e} - \cos\varphi\mathbf{f}\,, \quad \tilde{A} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}$$

In the first case we have *rotation.* In the second case A changes orientation ($\det A = -1$). In this case the vector $\mathbf{e}$ rotates on the angle φ and the vector $\mathbf{f}$ rotates on the angle φ and then is multiplied on (-1).

Thus one can see that the operator $\tilde{A}$ has two eigenvectors:

$$\begin{cases} \tilde{\mathbf{e}} = \cos\frac{\varphi}{2}\mathbf{e} + \sin\frac{\varphi}{2}\mathbf{f}\,, & \tilde{A}(\tilde{\mathbf{e}}) = \mathbf{e} \\ \tilde{\mathbf{f}} = -\sin\frac{\varphi}{2}\mathbf{e} + \cos\frac{\varphi}{2}\mathbf{f}\,, & \tilde{A}(\tilde{\mathbf{f}}) = -\mathbf{f} \end{cases}$$

For arbitrary vector $\mathbf{a}$,

$$\mathbf{a} = \mathbf{a}_{||} + \mathbf{a}_{\perp}\,, \quad \tilde{A}(\mathbf{a}_{||}) = \mathbf{a}_{||} \;\; \tilde{A}(\mathbf{a}_{\perp}) = -\mathbf{a}_{\perp}$$

Now we are ready to formulate the Theorem:

Theorem Let F be an isometry of $\mathbf{E}^2$.

then it is translation or rotation or Chasles.

`Proof` Choose origin. One can prove (not evident) that

$$F(\mathbf{x}) = A(\mathbf{x}) + \mathbf{b}\,.$$

If $A = 1$, then F is translation.

Let $A \neq 1$, but $\det A = 1$, then one can choose $\mathbf{r}$ such that

$$A(\mathbf{r}) + \mathbf{b} - \mathbf{r} = 0, i.e. \mathbf{y}' = A(\mathbf{x}')$$

for $\mathbf{x} \to \mathbf{x} + \mathbf{r}$.

Now consider the last case if $\det A = -1$.

Then for the vector

$$A\left(\frac{\mathbf{b}_\perp}{2} + \mathbf{x}_{||}\right) + \mathbf{b} = \frac{\mathbf{b}_\perp}{2} + \mathbf{x}_{||} + \mathbf{c}_{||},$$

i.e. F is Shall.

This can be doe much shorter. Let A be orthogonal operator with $\det A = -1$. Let

$$\mathbf{y} = A(\mathbf{x}) + \mathbf{b}$$

Then

$$\mathbf{y} = A\left(\mathbf{x} - \frac{\mathbf{b}_\perp}{2} + \frac{\mathbf{b}_\perp}{2}\right) + \mathbf{b}_{||} + \mathbf{b}_\perp = A\left(\mathbf{x} - \frac{\mathbf{b}_\perp}{2}\right) + \mathbf{b}_{||} + \frac{\mathbf{b}_\perp}{2},$$

i.e.

$$\mathbf{y}' = \mathbf{y} - \frac{\mathbf{b}_\perp}{2} = A\left(\mathbf{x} - \frac{\mathbf{b}_\perp}{2}\right) + \mathbf{b}_{||} = A(\mathbf{x}') + \mathbf{b}_{||} = -\mathbf{x}'_\perp + \mathbf{x}'_{||} + \mathbf{b}_{||}.$$

This is Chales.