## Homework 5

**1** Calculate the Christoffel symbols of the canonical flat connection in $\mathbf{E}^3$ in

a) cylindrical coordinates ($x = r\cos\varphi, y = r\sin\varphi, z = h$),

b) spherical coordinates.

(For the case b) try to make calculations at least for components $\Gamma^r_{rr}, \Gamma^r_{r\theta}, \Gamma^r_{r\varphi}, \Gamma^r_{\theta\theta}, \dots, \Gamma^r_{\varphi\varphi}$)

**2** Let $\nabla$ be an affine connection on a 2-dimensional manifold $M$ such that in local coordinates $(u, v)$ it is given that $\Gamma^u_{uv} = v$, $\Gamma^v_{uv} = 0$.

Calculate the vector field $\nabla_{\frac{\partial}{\partial u}}\left(u\frac{\partial}{\partial v}\right)$.

**3** Let $\nabla$ be an affine connection on the 2-dimensional manifold $M$ such that in local coordinates $(u, v)$

$$\nabla_{\frac{\partial}{\partial u}}\left(u\frac{\partial}{\partial v}\right) = (1+u^2)\frac{\partial}{\partial v} + u\frac{\partial}{\partial u}\,.$$

Calculate the Christoffel symbols $\Gamma^u_{uv}$ and $\Gamma^v_{uv}$ of this connection.

**4** a) Consider a connection such that its Christoffel symbols are symmetric in a given coordinate system: $\Gamma^i_{km} = \Gamma^i_{mk}$.

Show that they are symmetric in an arbitrary coordinate system.

b*) Show that the Christoffel symbols of connection $\nabla$ are symmetric (in any coordinate system) if and only if

$$\nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X}, \mathbf{Y}] = 0\,,$$

for arbitrary vector fields $\mathbf{X}, \mathbf{Y}$.

c)* Consider for an arbitrary connection the following operation on the vector fields:

$$S(\mathbf{X}, \mathbf{Y}) = \nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X}, \mathbf{Y}]$$

and find its properties.

**5** Consider the surface $M$ in the Euclidean space $\mathbf{E}^n$. Calculate the induced connection in the following cases

a) $M = S^1$ in $\mathbf{E}^2$,

b) $M$— parabola $y = x^2$ in $\mathbf{E}^2$,

c) cylinder in $\mathbf{E}^3$.

d) cone in $\mathbf{E}^3$.

e) sphere in $\mathbf{E}^3$.

f) saddle $z = xy$ in $\mathbf{E}^3$

**6** Let $\nabla_1, \nabla_2$ be two different connections. Let ${}^{(1)}\Gamma^i_{km}$ and ${}^{(2)}\Gamma^i_{km}$ be the Christoffel symbols of connections $\nabla_1$ and $\nabla_2$ respectively.

a) Find the transformation law for the object : $T^i_{km} = {}^{(1)}\Gamma^i_{km} - {}^{(2)}\Gamma^i_{km}$ under a change of coordinates. Show that it is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ tensor.

b)$^{*?}$ Consider an operation $\nabla_1 - \nabla_2$ on vector fields and find its properties.

**7** $^*$ a) Consider $t_m = \Gamma^i_{im}$. Show that the transformation law for $t_m$ is

$$t_{m'} = \frac{\partial\, x^m}{\partial x^{m'}} t_m + \frac{\partial^2 x^r}{\partial x^{m'} \partial x^{k'}} \frac{\partial x^{k'}}{\partial x^r} \,.$$

b) $^\dagger$ Show that this law can be written as

$$t_{m'} = \frac{\partial\, x^m}{\partial x^{m'}} t_m + \frac{\partial}{\partial x^{m'}} \left( \log \det \left( \frac{\partial x}{\partial x'} \right) \right) \,.$$