Solutions of Homework 9

1 *Let F be a projective transformation of* $\mathbf{RP}$ *such that*

$$[x' : y'] = F([x : y]) = [2x + 3y : 3x + 2y] .$$

Let $P = [6 : 2]$ be a point on $\mathbf{RP}$.

a) Find the affine coordinate u_P of this point, and find the affine coordinate u'_P of the point $F(P)$.

b) Find a point A such that $A = F(\infty)$.

c) Find a point B such that $F(B) = \infty$.

a) We have for a point P, $P = [6 : 2]$, hence $u_P = \frac{x_P}{y_p} = 3$.

The projective transformation F transforms every point $u = x/y$ to the point

$$u' = \frac{x'}{y'} = F(u) = F\left(\frac{x}{y}\right) = \frac{2x + 3y}{3x + 2y} = \frac{2\frac{x}{y} + 3}{3\frac{x}{y} + 2} = \frac{2u + 3}{3u + 2} .$$

$u_P = \frac{x_P}{y_P} = \frac{6}{2} = 3$, hence $u_P \mapsto u'_P = \frac{2u_P+3}{3u_P+2} = \frac{2\cdot 3+3}{3\cdot 3+2} = \frac{9}{11}$.

Another solution: $P = [6 : 2]$, hence

$$P' = F(P) = F([6 : 2]) = F([2x + 3y : 3x + 2y])\big|_{[x:y]=[6:2]} =$$

$$[2 \cdot 6 + 3 \cdot 2 : 3 \cdot 6 + 2 \cdot 2] = [18 : 22] \Rightarrow u_{P'} = \frac{x'}{y'} = \frac{18}{22} = \frac{9}{11} .$$

b) $\infty = [1 : 0]$, $A = F(\infty) = F([2x + 3y : 3x + 2y])\big|_{[x:y]=[1:0]} = [2 : 3]$. The affine coordinate of the point $F(\infty)$ is equal to $u_A = \frac{2}{3}$.

Another solution $F(\infty) = \frac{2u+3}{3u+2}\big|_{u=\infty} = \frac{2}{3}$ (with some abuse of notations).

c) The projective transformation $F\colon F(u) = \frac{2u+3}{3u+2}$. Hence the point B with affine coordinate $u_B = -\frac{2}{3}$:

$$F(u_B) = \frac{2u_B + 3}{3u_B + 2} = \frac{-\frac{4}{3} + 2}{0} = \infty .$$

Another solution $F([x : y]) = [2x + 3y : 3x + 2y]$. We see that for point $B = [2 : -3]$,

$$F[x : y] = [2 \cdot 2 + 3 \cdot (-3) : 3 \cdot 2 + 2 \cdot (-3)] = [-5 : 0] = \infty .$$

Another solution The projective transformation F is generated by the matrix $K = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$. The inverse projective transformation F^{-1} is generated by the inverse matrix

$$K^{-1} = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}^{-1} = -\frac{1}{5}\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix} \Rightarrow B = F^{-1}(\infty) = F_{K^{-1}}([1 : 0]) = [2 : -3] .$$

2 *Consider the projective transformation*

$$F: \quad F([x:y]) = [y:x]\,.$$

Write down this transformation in affine coordinate.

Show by straightforward calculation, that the cross-ratio (A,B,C,D) of four points on the projective line is the invariant of this projective transformation.

Let u_A, u_B, u_C, u_D be affine coordiantes of points A,B,C,D and let $u_{A'}, u_{B'}, u_{C'}, u_{D'}$ be affine coordinates of points A', B', C', D', where $A' = F(A)$, $B' = F(B)$ and $D' = F(D)$, $A' = F(A)$. We have $F(u) = F\left(\frac{x}{y}\right) = \frac{y}{x} = \frac{1}{\frac{x}{y}} = \frac{1}{u}$, hence $u_{A'} = \frac{1}{u_A}$, $u_{B'} = \frac{1}{u_B}$, $u_{C'} = \frac{1}{u_C}$, $u_{D'} = \frac{1}{u_D}$. We have $(A', B', C', D') = \frac{(u'_A - u'_C)(u'_B - u'_D)}{(u'_A - u'_D)(u'_B - u'_C)} =$

$$\frac{\left(\frac{1}{u_A} - \frac{1}{u_C}\right)\left(\frac{1}{u_B} - \frac{1}{u_D}\right)}{\left(\frac{1}{u_A} - \frac{1}{u_D}\right)\left(\frac{1}{u_B} - \frac{1}{u_C}\right)} = \frac{\frac{(u_C - u_A)(u_D - u_B)}{u_A u_C u_B u_D}}{\frac{(u_D - u_A)(u_C - u_B)}{u_A u_D u_B u_C}} = \frac{(u_A - u_C)(u_B - u_D)}{(u_A - u_D)(u_B - u_C)} = (A,B,C,D)\,.$$

3 *Four points $A,B,C,D \in \mathbf{RP}^2$ are given in homogeneous coordinates by*

$$A = [2:-1:1]\,, \quad B = [15:-10:5]\,, \quad C = \left[1:-\frac{4}{5}:\frac{1}{5}\right]\,, \quad D = [2:0:2]\,.$$

a) Show that these points are collinear.

b) Calculate their cross-ratio.

Remark Cross-ratio of four points on the plane has a sence only if these four points are collinear, since in this case it is the invariant of projective transformations. It is why we first have to check are the points collinear or no, and only then we have to calculate their cross-ratio.

a) Consider the affine coordinates $u = \frac{x}{z}, y = \frac{y}{z}$ of these points:

$$\begin{pmatrix} u_A \\ v_A \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} u_B \\ v_B \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} u_C \\ v_C \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \end{pmatrix}, \quad \begin{pmatrix} u_D \\ v_D \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (3.1)$$

We see that affine coordinates u, v of these four points obey equation $u + v = 1$. Thus these four points belong to the affine line $u + v = 1$ on the projective plane.

Another way to see that points are collinear

Recall that for three points A,B,C on projective plane we may consider the matrix T_{ABC}, such that columns of this matrix are homogeneous coordinates of points A, B and C. Matrix T_{ABC} is degenerate if and only if the points A,B,C are collinear. (Degenerate $\Leftrightarrow$ $\det T_{ABC} = 0 \Leftrightarrow$ matrix is not invertible $\Leftrightarrow$ columns of this matrix are linearly dependent

$\Leftrightarrow$ rows of this matrix are linearly dependent) Every column of matrix T_{ABC} represents the corresponding point. Multiplying the columns on non-zero numbers does not change the condition of degeneracy of matrices, and this column operation does not change the homogeneous coordinates of points.

We have that $A = [2 : -1 : 1]$,

$$B = [15 : -10 : 5] = [3 : -2 : 1]\,, C = \left[1 : -\frac{4}{5} : \frac{1}{5}\right] = [5 : -4 : 1]\,, D = [2 : 0 : 2] = [1 : 0 : 1]\,.$$

Check degeneracy of matrices T_{ABC} and T_{ABD}. We see that $T_{ABC} = \begin{pmatrix} 2 & 3 & 5 \\ -1 & -2 & -4 \\ 1 & 1 & 1 \end{pmatrix}$, and $T_{ABD} = \begin{pmatrix} 2 & 3 & 1 \\ -1 & -2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$. Both these matrices are degenerate. One can check it calulating their determinants, and checking that they vanish: $\det T_{ABC} = 2{\cdot}2 - 3{\cdot}3 + 5{\cdot}1 = 0$ and $\det T_{ABD} = 2 \cdot (-2) - 3 \cdot (-1) + 1 \cdot 1 = 0$; or you can see it in another way: for both matrices the third row is the sum of the first and the second rows; or you can see that studying columns of these matrices.

Doing any of these considerations we come to the conclusion that both matrices T_{ABC} and T_{ABD} are degenerate matrices, hence points A, B, C as well points A, B, D are collinear. Hence all these four points are collinear.

Remark When checking degeneracy of matrices we also can to consider instead matrix, its transpose.

b) Now when we checked that these four points are collinear, we can caluclate their cross-ratio.

Take an arbitrary affine coordinate, for example the coordinate u of points A, B, C, D (see equation (3.1))

$$(A, B, C, D) = \frac{(u_A - u_C)(u_B - u_D)}{(u_A - u_D)(u_B - u_C)} = \frac{(2-5)(3-1)}{(2-1)(3-5)} = \frac{-6}{-2} = 3\,. \tag{3.2}$$

Remark Cross-ratio of four arbitrary collinear points does not depend on a choice of affine coordinate, since cross-ratio is an invariant of projective transformations. E.g. if we choose a coordinate v instead coordinate u in equation (3.1) we come to the same answer as in (3.2):

$$(A, B, C, D) = \frac{(v_A - v_C)(v_B - v_D)}{(v_A - v_D)(v_B - v_C)} = \frac{(-1-(-4))(-2-0)}{(-1-0)(-2-(-4))} = \frac{-6}{-2} = 3\,. \tag{3.2}$$

Remark Just one *precaution*. Choosing an arbitrary affine coordinate be aware that this coordinate takes different values at points. For example if for four distinct collinear points you will consider a coordinate such that it takes the same value at all the points, the numerator and denumerator of fraction defining cross-ratio will be equal to zero, and you will not calculate the cross-ratio using this coordinate.

4 *Three points $A, B, C \in \mathbf{RP}^2$ are given in homogeneous coordinates by*

$$A = [6:2:2]\,, B = [15:5:1]\,, C = [18:6:3]\,.$$

Show that these points are collinear. Find a point D on projective plane $\mathbf{RP^2}$ such that the cross-ratio $(A, B, C, D) = -1$.

Consider the affine coordinates $u = \frac{x}{z}, v = \frac{y}{z}$ of these points:

$$\begin{pmatrix} u_A \\ v_A \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} u_B \\ v_B \end{pmatrix} = \begin{pmatrix} 15 \\ 5 \end{pmatrix}, \quad \begin{pmatrix} u_C \\ v_C \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}. \tag{4.1}$$

(Compare with equation (3.1))

We see that affine coordinates u, v of these three points obey equation $u = 3v$. Thus these three points belong to the affine line $u = 3v$ on the projective plane.

Another way to see that points are collinear `Of course the above checkinig was easy and short, but consider also another way to check collinearity of these three points`

In the same way as in the previous exercise consider for points A, B and C the matrix T_{ABC}. We have

$$A = [6:2:2] = [3:1:1]\,, B = [15:5:1]\,, C = [18:6:3] = [6:2:1]\,.$$

We see that the matrix $T_{ABC} = \begin{pmatrix} 3 & 15 & 6 \\ 1 & 5 & 2 \\ 1 & 1 & 1 \end{pmatrix}$ is degenerate since the first and second rows are proportional. One can see it also in other way calculating the determinant: $\det T_{ABC} = 3 \cdot 3 - 15 \cdot (-1) + 6 \cdot (-4) = 0$. Hence points A, B, C are collinear.

Now find a point D on the projective plane such that the cross-ratio $(A, B, C, D) = -1$.

The point D has to be collinear to the points A, B, C. It is only in this case that the cross-ratio of four points is well defined (see the remark just before solution of the previous exercise.) Since the point D belongs to the same line as the points A, B, C hence according equation (4.1) the affine coordinates of the point D has to obey relation $u_D = 3v_D$. We see that $\begin{pmatrix} u_D \\ v_D \end{pmatrix} = \begin{pmatrix} 3x \\ x \end{pmatrix}$, where x is some number. Now use the fact that the cross-ratio (A, B, C, D) is equal to -1. Choose an arbitrary affine coordinate, e.g. the coordinate v. We have

$$(A, B, C, D) = \frac{(v_A - v_C)(v_B - v_D)}{(v_A - v_D)(v_B - v_C)} = \frac{(1-2)(5-x)(-2-0)}{(1-x)(5-2)} = \frac{x-5}{3-3x} = -1 \Rightarrow x = -1\,.$$

Hence the coordinate $v_D = -1$, hence the coordinate $u_D = -3$. The point D has affine coordinates $u = 3, v = 1$.

5 *Let A, B, C, D be four collinear points on projective plane $\mathbf{RP}^2$.*
Let $(A, B, C, D) = \lambda$. Calculate (B, A, C, D), (A, B, D, C) and (B, A, D, C).

We have that

$$(A, B, C, D) = \lambda = \frac{(u_A - u_C)(u_B - u_D)}{(u_A - u_D)(u_B - u_C)} \Rightarrow (B, A, C, D) = \frac{(u_B - u_C)(u_A - u_D)}{(u_B - u_D)(u_A - u_C)} =$$

$$\frac{1}{\frac{(u_A-u_C)(u_B-u_D)}{(u_A-u_D)(u_B-u_C)}} = \frac{1}{(A, B, C, D)} = \frac{1}{\lambda}.$$

In the same way:

$$(A, B, D, C) = \frac{(u_A - u_D)(u_B - u_C)}{(u_A - u_C)(u_B - u_D)} = \frac{1}{(A, B, C, D)} = \frac{1}{\lambda}$$

and

$$(B, A, D, C) = \frac{(u_B - u_D)(u_A - u_C)}{(u_B - u_C)(u_A - u_D)} = (A, B, C, D) = \lambda.$$

6 *On the projective line are given two points $A = [3 : 3]$ and $B = [7 : 1]$.*
Find a point P on the projective line such that the ratio

$$(A, B, P) = \frac{u_A - u_P}{u_B - u_P} = -2.$$

Let F be a projective transformation such that $F([x : y]) = [x + y : x]$.

Consider also a point $Q = \infty$ and find images A', B', P', Q' of the points A, B, P, Q under the projective transformation F:

$$A' = F(A), B' = F(B), P' = F(P), Q' = F(Q).$$

Calculate the cross ratio (A', B', P', Q'),

Explain why the ratio (A, B, P) is equal to the cross-ratio (A', B', P', Q').

We have that the affine coordinate $u_A = 1$ and $u_B = 7$. Then

$$(A, B, P) = \frac{u_A - u_P}{u_B - u_P} = \frac{1 - u_P}{7 - u_P} = -2 \Rightarrow u_P = 5.$$

We have that $F(u) = F\left(\frac{x}{y}\right) = \frac{x+y}{x} = 1 + \frac{y}{x} = 1 + \frac{1}{u}$. Hence $u_{A'} = F(u_A) = 1 + \frac{1}{1} = 2$, $u_{B'} = F(u_B) = 1 + \frac{1}{7} = \frac{8}{7}$, $u_{P'} = F(u_P) = \frac{5+1}{5} = \frac{6}{5}$, $u_{Q'} = F(u_Q) = \frac{\infty+1}{\infty} = 1$.

Calculate

$$(A', B', P', Q') = \frac{(u_{A'} - u_{P'})(u_{B'} - u_{Q'})}{(u_{A'} - u_{Q'})(u_{B'} - u_{P'})} = \frac{\left(2 - \frac{6}{5}\right)\left(\frac{8}{7} - 1\right)}{(2 - 1)\left(\frac{8}{7} - \frac{6}{5}\right)} = -2.$$

The answers coincide since the poitn Q was at the infinity:

$$(A, B, P) = (A, B, P, \infty) = (A', B', P', Q').$$