**one fact in symplectic linear algebra**`Guill.Sternberg`

Let $V$ be finite dimensional symplectifc vector space.

Let $A$ be linear transformation of symplectic space $V$ such that

$$\langle Au, v\rangle + \langle u, Av\rangle = 0\,, \tag{1}$$

i.e. $A \in sp(V)$. Here $\langle\_,\_\rangle$ is antisymmetric non-degenerate bilinear form, the symplectic scalar product.

One can consider bilinear form:

$$H(u,v) = P(u,v)$$

We see that it is symmetric. (This form plays the role of Hamiltonian.)

Conisder generalised eigen spaces of $A$:

$$V_\lambda = \{\mathbf{x} \in V\colon \quad (A-\lambda)^\infty x = 0\}$$

("Infinity" means the enough big integer. More carefully one has to write instead this formula the following

$$V_\lambda = \{\mathbf{x} \in V\colon \quad \text{there exists natural } K \text{ such that } (A-\lambda)^k x = 0\}$$

$k$ in fact depends on $\mathbf{x}$, however it is less than dimension of $V$, hence one can write:

$$V_\lambda = \{\mathbf{x} \in V\colon \quad (A-\lambda)^N x = 0\}\,, \quad \text{where } N > \dim V$$

Proposition

a)

$$V_\lambda = \left((A+\lambda)^N\, V\right)^\perp$$

b)

$$\dim V_\lambda = \dim V_{-\lambda}\,.$$

Recall that for an arbitrary operator $P$ in finite dimensional space $V$

$$\dim\,(\operatorname{Im} P)^\perp = \dim\,(\ker P)$$

and

$$V_\lambda = \ker\,(A-\lambda)^N$$

hence b) follows from a) since

$$\dim V_{-\lambda} = \dim\ker\,(A+\lambda)^N = \dim\,\left(\operatorname{Im}\,(A+\lambda)^N\right)^\perp \dim V_\lambda\,.$$

Now prove a)

Note that condition (1) implies that

$$\langle (A-\lambda)u, v\rangle = -\langle u, (A+\lambda)v\rangle \tag{2}$$

Take an arbitrary $x \in V$. If $x \in V_\lambda$ then for arbitrary $\mathbf{y} \in \operatorname{Im}(A+\lambda)$ we have

$$\langle x, y\rangle = \langle x, (A+\lambda)^N v\rangle = (-1)^N \langle (A-\lambda)^N x, y\rangle = 0 \Rightarrow V_\lambda \subseteq \left((A+\lambda)^N V\right)^\perp$$

In its turn suppose that for an arbitrary $x \in V$, $x \in \left((A+\lambda)^N V\right)^\perp$, i.e. for an arbitrary $v \in V$

$$\langle x, (A+\lambda)^N v\rangle = 0$$

Now using (2) we have that

$$\langle x, (A+\lambda)^N v\rangle = 0 = (-1)^N \langle (A-\lambda)^N x, v\rangle = 0\,.$$

Since this equation holds for an arbitrary $v \in V$, then non-degeneracy of scalar product on $V$ implies that $(A-\lambda)^N x = 0$, i.e. $x \in V_\lambda$. Hence we have proved that $\left((A+\lambda)^N V\right)^\perp \subseteq V_\lambda$ also. Hence we proved proposition.

**Theorem**. Let $X$ be an arbitrary Lagrangian plane in $V$.

*a) The set of Lagrangian planes which are transversal to $X$ is in one-one correspondence with set of , linear operators $P$, such that*

$$(Pu, v) + (u, PV) = (u, v)\,, \tag{3a}$$

*and*

$$\ker P = X\,. \tag{3b}$$

*b) The set of $\mathcal{L}_X$ of Lagrangian planes which is transversal to the plane given Lagrangian plane $X$ is an affine space. The vector space associated with the affine space $\mathcal{L}_X$ is the vector space of symmetric bilinear forms on the factor space $V \backslash X$. In particular*

$$\dim \mathcal{L}_X = \frac{nn+1}{2}\,, \quad (\dim V = 2n)\,.$$

**Remark** Condition (3a) means that operator $P = \frac{1}{2} + A$, where $A$ belongs to $sp(V)$.

`Proof`

Indeed suppose that $Y$ is Lagrangian surface which is transversal to $X$. Define

$$P\colon\ P(u) = P(\mathbf{x} + \mathbf{y}) = \mathbf{y}\,,$$

where $u = \mathbf{x} + \mathbf{y}$ is the expansion of vector over Lagrangian surfaces $X$ and $Y$:

$$V = X \oplus Y\,, V \ni u = \mathbf{x} + \mathbf{y}\,,\ \mathbf{x} \in X\,, \mathbf{y} \in Y\,.$$

One can see that condition (3b) evidently holds. Check condition (3a). For vectors $u = \mathbf{x} + \mathbf{y}$ and $\mathbf{v} = \mathbf{x}' + \mathbf{y}'$

$$\langle Pu, v\rangle + \langle Pu, v\rangle = \langle \mathbf{y}, \mathbf{x}' + \mathbf{y}'\rangle + \langle \mathbf{x} + \mathbf{y}, \mathbf{y}'\rangle = \langle \mathbf{y}, \mathbf{x}'\rangle + \langle \mathbf{x}, \mathbf{y}'\rangle = \langle \mathbf{x} + \mathbf{y}, \mathbf{x}' + \mathbf{y}'\rangle = \langle u, v\rangle\,,$$

i.e. condition (3a) holds also.

Now suppose that $P$ is linear operator which obeys conditions (3a) and (3b).

Consider subspace $Y = \mathrm{Im}P$. Vectors of $Y = \mathrm{Im}P$ are eigenvectors with eigenvalue 1:

$X = V_0$ and $Y = V_1$. It is easy to see that equation (3a) implies that $Y$ is Lagrnagian. Indeed $\mathrm{dim}Y = n$ ($\mathrm{dim}V = N = 2n$) and [1]:

$$\langle \mathbf{y}, \mathbf{y}'\rangle = \langle P(\mathbf{y}), \mathbf{y}'\rangle + \langle \mathbf{y}, P(\mathbf{y}')\rangle = 2\langle \mathbf{y}, \mathbf{y}'\rangle \Rightarrow \langle \mathbf{y}, \mathbf{y}'\rangle = 0\,.$$

Now study the set $\mathcal{L}_X$. It is set of projectors $P$ which obey to conditions (3a) and (3b). Condition 3a) means that $P - \frac{1}{2}$ belongs to Lie algebra of the group of linear symplectic transfromations. and condition (3b) means that the corresponding 'Hamiltonian' is vanished on $X$. Hence i

[1] in fact proof follows from Proposition applied to the operator $A = P - \frac{1}{2}$