**Example of two spinors–rotation**

*orthogonal transformation corresponding to vectors are rotations on angle $\pi$. Transformation corresponding to the product of vectors in Clifford algebra looks in $\mathbf{E}^3$ as a rotation on the angle $2\varphi$ around the axis which is orthogonal to these vectors. In the case if dimension is higher than 3, still this has a meaning.*

In the previous blog we considered the group *pin* generated by product of vectors of $V$.

For every vector unit $\mathbf{v} \in V$

$$\mathbf{v}\mathbf{x}\mathbf{v}^{-1} = L_{\mathbf{v}}(\mathbf{x}) = 2(\mathbf{v},\mathbf{x})\mathbf{v} - \mathbf{x}\,,$$

where $Q(\mathbf{x}) = (\mathbf{x},\mathbf{x})$ is quadratic form, which defines Clifford algebra.

The transformation $L_{\mathbf{v}}$ has axis directed along vector $\mathbf{v}$ and all the vectors orthogonal to $\mathbf{v}$ it rotates on the angle $\pi$

Consider the action of two rotations of this type: let $\mathbf{v}_1, \mathbf{v}_2$ be two unit vectors, we calculate

$$\mathbf{v}_1\mathbf{v}_2 x \mathbf{v}_2^{-1}\mathbf{v}_1^{-1} = P(\mathbf{x})$$

Consider how looks operator $P$ first in $\mathbf{E}^3$

**Theorem** The rotation $P_{\mathbf{v}_1\mathbf{v}_2} = L_{\mathbf{v}_1}L_{\mathbf{v}_2}$ in $\mathbf{E}^3$ this is the element of the group *Spin* (*Spin* contains even elements of the group *pin*)

$P_{\mathbf{v}_1\mathbf{v}_2}$ rotates the space around axis which is orthogonal to the plane $\mathbf{v}_1, \mathbf{v}_2$ on the angle $2\varphi$, where $\varphi$ is the angle between vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. which is orthogonal to the plane

Choose orthonormal basis $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$ such that $\mathbf{e}$ is directed along $\mathbf{v}_1$, and vector $\mathbf{g}$ is orthogonal to plane formed by vectors $\mathbf{v}_1, \mathbf{v}_2$. Then

$$L_{\mathbf{v}_1}\colon \quad \begin{cases} L_{\mathbf{v}_1}(\mathbf{e}) = \mathbf{e} \\ L_{\mathbf{v}_1}(\mathbf{f}) = -\mathbf{f} \\ L_{\mathbf{v}_1}(\mathbf{g}) = -\mathbf{g} \end{cases}$$

Now consider the second rotation. It is around the vector $\mathbf{e}' = \mathbf{v}_2$, The corresponding basis $\{\mathbf{e}',\mathbf{f}',\mathbf{g}'\}$, where

$$\begin{cases} \mathbf{e}' = \cos\varphi\mathbf{e} + \sin\varphi\mathbf{f}\,, \\ \mathbf{f}' = -\sin\varphi\mathbf{e} + \cos\varphi\mathbf{f}\,, \\ \mathbf{g}' = \mathbf{g}\,. \end{cases} \Leftrightarrow \begin{cases} \mathbf{e} = \cos\varphi\mathbf{e}' - \sin\varphi\mathbf{f}'\,, \\ \mathbf{f} = \sin\varphi\mathbf{e}' + \cos\varphi\mathbf{f}'\,, \\ \mathbf{g} = \mathbf{g}'\,. \end{cases}$$

For the second rotation we have

$$L_{\mathbf{v}_2}\colon \quad \begin{cases} L_{\mathbf{v}_2}(\mathbf{e}') = \mathbf{e}' \\ L_{\mathbf{v}_1}(\mathbf{f}') = -\mathbf{f}' \\ L_{\mathbf{v}_1}(\mathbf{g}) = -\mathbf{g} \end{cases}$$

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These elements form subspace in the orthogonal group (see Etude about fantom peines....)

Thus we have

$$P = P_{\mathbf{v}_1\mathbf{v}_2} = L_{\mathbf{v}_1} L_{\mathbf{v}_2}:$$

$$\begin{cases} P(\mathbf{e}) = L_{\mathbf{v}_1} L_{\mathbf{v}_2}(\mathbf{e}) = L_{\mathbf{v}_1} L_{\mathbf{v}_2}(\cos\varphi\mathbf{e}' - \sin\varphi\mathbf{f}') = L_{\mathbf{v}_1}(\cos\varphi\mathbf{e}' + \sin\varphi\mathbf{f}') = \\ L_{\mathbf{v}_1}(\cos 2\varphi\mathbf{e} + \sin 2\varphi\mathbf{f}') = \cos 2\varphi\mathbf{e} - \sin 2\varphi\mathbf{f} \\ P(\mathbf{f}) = L_{\mathbf{v}_1} L_{\mathbf{v}_2}(\mathbf{f}) = L_{\mathbf{v}_1} L_{\mathbf{v}_2}(\sin\varphi\mathbf{e}' + \cos\varphi\mathbf{f}') = L_{\mathbf{v}_1}(\sin\varphi\mathbf{e}' - \sin\varphi\mathbf{f}') = \\ L_{\mathbf{v}_1}(\sin 2\varphi\mathbf{e} - \cos 2\varphi\mathbf{f}) = \sin 2\varphi\mathbf{e} + \cos 2\varphi\mathbf{f} \\ P(\mathbf{g}) = L_{\mathbf{v}_1} L_{\mathbf{v}_2}(\mathbf{g}) = L_{\mathbf{v}_1}(-\mathbf{g}) = \mathbf{g} \end{cases}$$

Thus matrix of the operator $P$ in the basis $\{\mathbf{e}, \mathbf{f}.\mathbf{g}\}$ is

$$\begin{pmatrix} \cos 2\varphi & -\sin 2\varphi & 0 \\ \sin 2\varphi & \cos 2\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

sio f is finished.

In the case of arbitrary $\mathbf{E}^n$ the transformation look :

$$\begin{cases} P(\mathbf{e}) = L_{\mathbf{v}_1} L_{\mathbf{v}_2}(\mathbf{e}) = L_{\mathbf{v}_1} L_{\mathbf{v}_2}(\cos\varphi\mathbf{e}' - \sin\varphi\mathbf{f}') = L_{\mathbf{v}_1}(\cos\varphi\mathbf{e}' + \sin\varphi\mathbf{f}') = \\ L_{\mathbf{v}_1}(\cos 2\varphi\mathbf{e} + \sin 2\varphi\mathbf{f}') = \cos 2\varphi\mathbf{e} - \sin 2\varphi\mathbf{f} \\ P(\mathbf{f}) = L_{\mathbf{v}_1} L_{\mathbf{v}_2}(\mathbf{f}) = L_{\mathbf{v}_1} L_{\mathbf{v}_2}(\sin\varphi\mathbf{e}' + \cos\varphi\mathbf{f}') = L_{\mathbf{v}_1}(\sin\varphi\mathbf{e}' - \sin\varphi\mathbf{f}') = \\ L_{\mathbf{v}_1}(\sin 2\varphi\mathbf{e} - \cos 2\varphi\mathbf{f}) = \sin 2\varphi\mathbf{e} + \cos 2\varphi\mathbf{f} \\ P(\mathbf{g}_i) = L_{\mathbf{v}_1} L_{\mathbf{v}_2}(\mathbf{g}) = L_{\mathbf{v}_1}(-\mathbf{g}) = \mathbf{g}_i \end{cases}$$

Here $\{\mathbf{e}, \mathbf{f}, \mathbf{g}_i\}$ orthonormal basis. Matrix of the operator $P$ in the basis $\{\mathbf{e}, \mathbf{f}.\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_{n-2}\}$ $\blacksquare$ is

$$\begin{pmatrix} \cos 2\varphi & -\sin 2\varphi & 0 & 0 & \ldots & 0 \\ \sin 2\varphi & \cos 2\varphi & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & & \ldots & 0 \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$$