Introduction to Geometry (20222)

2013

COURSEWORK

This assignment counts for 20% of your marks.

Solutions are due by 19-th April

Write solutions in the provided spaces.

STUDENTS'S NAME:

Academic Advisor (Tutor):

1

a) Let (x^1, x^2, x^3) be coordinates of the vector $\mathbf{x}$, and (y^1, y^2, y^3) be coordinates of the vector $\mathbf{y}$ in $\mathbf{R}^3$.

Does the formula $(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^3 + x^3y^2$ define a scalar product on $\mathbf{R}^3$?

Justify your answer.

b) Let $\mathbf{x}, \mathbf{y}$ be two vectors in the Euclidean space $\mathbf{E}^2$ such that the length of the vector $\mathbf{x}$ is equal to 1, the length of the vector $\mathbf{y}$ is equal to 25 and scalar product of these vectors is equal to 7.

Find a vector $\mathbf{e}$ in $\mathbf{E}^2$ (express it through the vectors $\mathbf{x}$ and $\mathbf{y}$) such that the following conditions hold:

i) an ordered pair $\{\mathbf{e}, \mathbf{x}\}$ is an orthonormal basis in $\mathbf{E}^2$,

ii) the vector $\mathbf{e}$ has an acute angle with the vector $\mathbf{y}$.

(c) Consider the matrix $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Calculate the matrix A^2 in the case if $\theta = \frac{\pi}{4}$.

Calculate the matrix A^{12} in the case if $\theta = \frac{\pi}{6}$.

Calculate the matrix A^{2013} in the case if $\theta = \frac{\pi}{11}$.

Find all 2×2 orthogonal matrices A such that

$$2A^3 = \begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}.$$

2

a) Consider vector $\mathbf{a} = 2\mathbf{e} + 3\mathbf{f} + 6\mathbf{g}$, where $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ is an orthonormal basis in $\mathbf{E}^3$.

Show that the angle θ between vectors $\mathbf{a}$ and $\mathbf{g}$ belongs to the interval $\left(\frac{\pi}{6}, \frac{\pi}{4}\right)$.

Find a unit vector $\mathbf{b}$ such that this vector is orthogonal to vectors $\mathbf{a}$ and $\mathbf{g}$, and the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{g}\}$ has the same orientation as the basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$.

Calculate the angle between vectors $\mathbf{b}$ and $\mathbf{e}$.

b) In oriented Euclidean space $\mathbf{E}^3$ consider the following function of three vectors:

$$F(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = (\mathbf{X}, \mathbf{Y} \times \mathbf{Z}) \,,$$

where $(\ ,\ )$ is the scalar product and $\mathbf{Y} \times \mathbf{Z}$ is the vector product in $\mathbf{E}^3$.

Show that $F(\mathbf{X}, \mathbf{X}, \mathbf{Z}) = 0$ for arbitrary vectors $\mathbf{X}$ and $\mathbf{Z}$.

Deduce that $F(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = -F(\mathbf{Y}, \mathbf{X}, \mathbf{Z})$ for arbitrary vectors $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$.

What is the geometrical meaning of the function F?

c) Let $ABCD$ be a rhombus (parallelogram with equal sides) such that

i) vertex A is at the origin

ii) the diagonal AC belongs to the line $y = x$.

iii) vertex B has integer coordinates.

Find the area of this rhombus if the vertex B has coordinates $(21, 20)$. Justify your answer.

Find all the rhombi which obey the conditions above and which have area $S = 25$.

3

We consider in this question 3-dimensional Euclidean space $\mathbf{E}^3$. We suppose that $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ is an orthonormal basis in this space.

a) Let P be a linear orthogonal operator acting in $\mathbf{E}^3$, such that it preserves the orientation of $\mathbf{E}^3$ and the following relations hold:

$$P(\mathbf{e}) = \cos\theta \ \mathbf{e} + \sin\theta \ \mathbf{f}, \quad P(\mathbf{g}) = \varepsilon \mathbf{g}\,,$$

where θ is an arbitrary angle and $\varepsilon = \pm 1$.

Write down the matrix of operator P in the basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$.

(*You have to consider separately both cases* $\varepsilon = 1$ *and* $\varepsilon = -1$.)

b) We know that due to the Euler Theorem linear operator P considered above is rotation operator.

Find the axis and an angle of this rotation.

(You have to consider separately both cases $\varepsilon = 1$ and $\varepsilon = -1$.)

c) Let P be a linear operator acting in $\mathbf{E}^3$, such that $P(\mathbf{e}) = \mathbf{f}$, $P(\mathbf{f}) = \mathbf{g}$ and $P(\mathbf{g}) = \mathbf{e}$.

Show that P is a rotation operator.

Find the axis and an angle of the rotation.

4

a) Given a vector field $\mathbf{G} = ar\frac{\partial}{\partial r} + b\frac{\partial}{\partial \varphi}$ in polar coordinates express it in Cartesian coordinates ($x = r\cos\varphi$, $y = r\sin\varphi$).

Consider the function $f = r^2\cos 2\varphi$ and the vector fields $\mathbf{A} = x\partial_x + y\partial_y$, $\mathbf{B} = x\partial_y - y\partial_x$. Calculate $\partial_{\mathbf{A}} f$, $\partial_{\mathbf{B}} f$. Express the answers in polar and in Cartesian coordinates.

Let $F = F(x, y), G = G(x, y)$ be two functions on $\mathbf{E}^2$ such that $F + iG = (x + iy)^3$.

Calculate the values of 1-forms $\omega = dF$ and $\sigma = dG$ on the vector field $\mathbf{A} = r\partial_r + \partial_\varphi$.

b) Consider the circle $\mathbf{r}(t)$: $\begin{cases} x = R\cos t \\ y = R\sin t \end{cases}$, $\quad 0 \le t < 2\pi$.

Calculate the integral of 1-form $\omega = x^2 dy$ over this circle.

Give an example of another parametersiation of this circle such that the integral changes the sign.

c) Consider the curve in $\mathbf{E}^2$ defined by the equation $r(2 - \cos\varphi) = 3$ in polar coordinates.

Show that the sum of the distances between the points $F_1 = (0, 0)$ and $F_2 = (2, 0)$, and an arbitrary point of this curve is constant, i.e. the curve is an ellipse and points F_1, F_2 are its foci.