Homework 8

1 A point moves in $\mathbf{E}^2$ along an ellipse with the law of motion $x = a\cos t$, $y = b\sin t$, $0 \le t < 2\pi$, $(0 < b < a)$. Find the velocity and acceleration vectors. Find the points of the ellipse where the angle between velocity and acceleration vectors is acute.

Find the points where the speed attains its maximum value.

What is the direction of the acceleration vector at these points?

2 Find a natural parameter for the following interval of the straight line:

$$C\colon \begin{cases} x = t \\ y = 2t+1 \end{cases}, \quad 0 < t < \infty\,.$$

Calculate the curvature of the straight line C.

3 Find a natural parameter for the curve $x^2 + y^2 = 6x + 8y$ in $\mathbf{E}^2$.

Write down the equation of this curve in a natural parameterisation.

Calculate the curvature of this curve.

4 Calculate the curvature of the parabola $x = t, y = mt^2$ $(m > 0)$.

† Let s be a natural parameter on this parabola. Show that $\int_0^\infty k(s)ds = \int_0^\infty k(t)|\mathbf{v}(t)|dt$ and calculate this integral.

5 Consider the parabola

$$\mathbf{r}(t)\colon \begin{cases} x = v_x t \\ y = v_y t - \frac{gt^2}{2} \end{cases}.$$

(It is the path of a point moving under gravity with initial velocity $\mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$.) Calculate the curvature at the vertex of this parabola.

6 Consider the ellipse $x = a\cos t, y = b\sin t$ $(a, b > 0,\ 0 \le t < 2\pi)$ in $\mathbf{E}^2$. Calculate the curvature $k(t)$ of this ellipse.

Find the curvatures and lengths of acceleration vectors at vertices of ellipse (the points $(a, 0), (-a, 0), (0, b), (0, -b)$) of the ellipse).

† Calculate $\int k(s)ds$ over the ellipse, where s is a natural parameter.

7 Consider the following curve (a helix): $\mathbf{r}(t)\colon \begin{cases} x(t) = R\cos\Omega t \\ y(t) = R\sin\Omega t \\ z(t) = ct \end{cases}$.

Find the velocity and acceleration vectors and curvature of this curve.

8 Let C be a curve in $\mathbf{E}^3$ given in parameterisation $\mathbf{r} = \mathbf{r}(t)$. Let $\mathbf{v}(t)$ be velocity vector and $\mathbf{a}(t)$ acceleration vector in this parameterisation. Consider the parallelogram $\Pi_{\mathbf{v},\mathbf{a}}$ formed by vectors $\mathbf{v}(t)$ and $\mathbf{a}(t)$.

Explain how the area of this parallelogram depends on a choice of parameterisation of the curve.

Calculate the curvature at the given point $\mathbf{r} = \mathbf{r}(t_0)$ of the curve C if velocity and acceleration vectors at this point both have length 2 and the angle between them is equal to $\frac{\pi}{6}$.

9 Calculate the curvature of the following curve (latitude on the sphere)

$$\begin{cases} x = R\sin\theta_0\cos\varphi(t) \\ y = R\sin\theta_0\sin\varphi(t) \\ z = R\cos\theta_0 \end{cases}, \text{ where } \varphi(t) = t, 0 \le t < 2\pi\,.$$

10 Show that the curvature of an arbitrary curve on the sphere of the radius R is greater than or equal to $\frac{1}{R}$.