Returning to very beatiful story of telephone calls with David in 90..

—-Wolf inteval

We know that fifth $= \frac{3}{2}$ and on the piano it is equal to $\frac{3}{2}$ if we have 12 fifths it will take 7 octaves we come to:

$$\left(\frac{3}{2}\right)^{12} \approx 2^7 \Leftrightarrow \frac{3^{12}}{2^{12+7}} = \frac{3^{12}}{2^{19}} = \frac{531441}{524288} \approx 1\,.$$

About thirty years ago we played with David phoning to people to 531441 and 524288

In fact this is related with continuous fraction for 'quinta': Let α be a number such that

$$\alpha\colon \quad 3 = 2^\alpha$$

Then continuos fraction of α gives a good approximation to α by rational numbers:

$$\alpha = [M_1, M_2, M_3, M_4, \ldots]$$

We have $\alpha = M_1 + \ldots$, i.e.

$$2^\alpha = 2^{M_1+\ldots} = 3\,,$$

i.e.

$$2^{M_1} < 3\,, \quad 2^{M_1+1} > 3 \Rightarrow M_1 = 1\,.$$

Then $\alpha = M_1 + \frac{1}{M_2+\ldots} = 1 + \frac{1}{M_2+\ldots}$, i.e.

$$2^\alpha = 2^{1+\frac{1}{M_2+\ldots}} = 3\,,$$

i.e.

$$2^{\frac{1}{M_2+\ldots}} = \frac{3}{2} \Rightarrow \left(\frac{3}{2}\right)^{M_2+\ldots} = 2\,,$$

i.e.

$$\left(\frac{3}{2}\right)^{M_2} < 2\,, \quad \text{but} \quad \left(\frac{3}{2}\right)^{M_2+1} > 2$$

i.e.

$$3^{M_2} < 2^{M_2+1}\,, \text{ but } 3^{M_2+1} > 2^{M_2+2} \Rightarrow M_2 = 1\,.$$

Then $\alpha = M_1 + \frac{1}{M_2+\frac{1}{M_3+\ldots}} = 1 + \frac{1}{1+\frac{1}{M_3+\ldots}}$, i.e.

$$2^\alpha = 2^{1+\frac{1}{1+\frac{1}{M_3+\ldots}}} = 3\,,$$

i.e.

$$2^{\frac{1}{1+\frac{1}{M_3+\ldots}}} = \frac{3}{2} \Rightarrow \left(\frac{3}{2}\right)^{1+\frac{1}{M_3+\ldots}} = 2\,, \Rightarrow \left(\frac{3}{2}\right)^{\frac{1}{M_3+\ldots}} = \frac{4}{3}\,, \Rightarrow \frac{3}{2} = \left(\frac{4}{3}\right)^{M_3+\ldots}\,,$$

i.e.

$$\left(\frac{4}{3}\right)^{M_3} < \frac{3}{2}\,, \quad \text{but } \left(\frac{4}{3}\right)^{M_3+1} > \frac{3}{2}\,,$$

i.e.

$$2^{2M_3+1} < 3^{M_3+1}\,, \quad \text{but } 2^{2M_3+3} > 3^{M_3+2} \Rightarrow M_3 = 1\,,$$

Then $\alpha = M_1 + \frac{1}{M_2+\frac{1}{M_3+\frac{1}{M_4+\dots}}} = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{M_4+\dots}}}$, i.e.

$$2^\alpha = 2^{1+\frac{1}{1+\frac{1}{1+\frac{1}{M_4+\dots}}}} = 3\,,$$

i.e.

$$2^{\frac{1}{1+\frac{1}{1+\frac{1}{M_4+\dots}}}} = \frac{3}{2} \Rightarrow \left(\frac{3}{2}\right)^{1+\frac{1}{1+\frac{1}{M_4+\dots}}} = 2\,, \Rightarrow \left(\frac{3}{2}\right)^{\frac{1}{1+\frac{1}{M_4+\dots}}} = \frac{4}{3}\,, \Rightarrow \frac{3}{2} = \left(\frac{4}{3}\right)^{1+\frac{1}{M_4+\dots}}\,,$$

i.e.

$$\left(\frac{4}{3}\right)^{\frac{1}{M_4+\dots}} = \frac{3}{2}\cdot\frac{3}{4} \Rightarrow \left(\frac{9}{8}\right)^{M_4+\dots} = \frac{4}{3}\,,$$

i.e.

$$\left(\frac{9}{8}\right)^{M_4} < \frac{4}{3}\,, \quad \text{but } \left(\frac{9}{8}\right)^{M_4+1} > \frac{4}{3}\,,$$

i.e.

$$3^{2M_4+1} < 2^{3M_4+2}\,, \quad \text{but } 3^{2M_4+3} > 2^{3M_4+5} \Rightarrow M_4 = 2\,,$$

Indeed if $M_4 = 2$ then $3^5 = 243 < 2^8 = 256$, but if $M_4 = 3$ then $3^7 = 729 \times 3 = 2187 > 2^11 = 1024 \times 2 = 2048$.

Continue: $\alpha = M_1 + \frac{1}{M_2+\frac{1}{M_3+\frac{1}{M_4+\frac{1}{M_5+\dots}}}} = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{2+\frac{1}{M_5+\dots}}}}$, i.e.

$$2^\alpha = 2^{1+\frac{1}{1+\frac{1}{1+\frac{1}{2+\frac{1}{M_5+\dots}}}}} = 3\,,$$

i.e.

$$2^{\frac{1}{1+\frac{1}{1+\frac{1}{2+\frac{1}{M_5+\dots}}}}} = \frac{3}{2} \Rightarrow \left(\frac{3}{2}\right)^{1+\frac{1}{1+\frac{1}{2+\frac{1}{M_5+\dots}}}} = 2\,, \Rightarrow \left(\frac{3}{2}\right)^{\frac{1}{1+\frac{1}{2+\frac{1}{M_5+\dots}}}} = \frac{4}{3}\,, \Rightarrow \frac{3}{2} = \left(\frac{4}{3}\right)^{1+\frac{1}{2+\frac{1}{M_5+\dots}}}\,,$$

i.e.

$$\left(\frac{4}{3}\right)^{\frac{1}{2+\frac{1}{M_5+\dots}}} = \frac{3}{2}\cdot\frac{3}{4} \Rightarrow \left(\frac{9}{8}\right)^{2+\frac{1}{M_5+\dots}} = \frac{4}{3}\,, \Rightarrow \left(\frac{9}{8}\right)^{\frac{1}{M_5+\dots}} = \frac{4}{3}\cdot\frac{64}{81} \Rightarrow \left(\frac{256}{243}\right)^{M_5+\dots} = \frac{9}{8}$$

i.e.
$$\left(\frac{256}{243}\right)^{M_5}<\frac{9}{8}\,,\quad \text{but}\ \left(\frac{256}{243}\right)^{M_5+1}>\frac{9}{8}\,,$$
i.e.
$$2^{8M_5+3}<3^{5M_5+2}\,,\quad \text{but}\ 2^{8M_5+11}>3^{5M_5+7}\Rightarrow M_5=2\,,$$
Indeed if $M_5=2$ then $2^{19}=524288<3^{12}=531441$ (these are famous phone numbers!!!!) , but if $M_5=3$ then
$$2^{27}=134217728>3^{17}=129140163$$

Calculate next? Let us try:
Repeat recurrently:
We already have:
$\alpha=M_1+\frac{1}{M_2+\frac{1}{M_3+\frac{1}{M_4+\frac{1}{M_5+\frac{1}{M_6+\dots}}}}}=1+\frac{1}{1+\frac{1}{1+\frac{1}{2+\frac{1}{2+\frac{1}{M_6+\dots}}}}}$, ‘ i.e.

$$2^\alpha=2^{1+\frac{1}{1+\frac{1}{1+\frac{1}{2+\frac{1}{2+\frac{1}{M_6+\dots}}}}}}=3\,,$$
i.e.
$$2^{\frac{1}{1+\frac{1}{1+\frac{1}{2+\frac{1}{2+\frac{1}{M_6+\dots}}}}}}=\frac{3}{2}\Rightarrow\left(\frac{3}{2}\right)^{1+\frac{1}{1+\frac{1}{2+\frac{1}{2+\frac{1}{M_6+\dots}}}}}=2\,,\Rightarrow\left(\frac{3}{2}\right)^{\frac{1}{1+\frac{1}{2+\frac{1}{2+\frac{1}{M_6+\dots}}}}}=\frac{4}{3}\,,\Rightarrow$$
$$\frac{3}{2}=\left(\frac{4}{3}\right)^{1+\frac{1}{2+\frac{1}{2+\frac{1}{M_6+\dots}}}}\,,$$
i.e.
$$\left(\frac{4}{3}\right)^{\frac{1}{2+\frac{1}{2+\frac{1}{M_6+\dots}}}}=\frac{3}{2}\cdot\frac{3}{4}\Rightarrow\left(\frac{9}{8}\right)^{2+\frac{1}{2+\frac{1}{M_6+\dots}}}=\frac{4}{3}\,,\Rightarrow$$
$$\left(\frac{9}{8}\right)^{\frac{1}{2+\frac{1}{M_6+\dots}}}=\frac{4}{3}\cdot\frac{64}{81}\Rightarrow\left(\frac{256}{243}\right)^{2+\frac{1}{M_6+\dots}}=\frac{9}{8}\,,\Rightarrow\left(\frac{256}{243}\right)^{\frac{1}{M_6+\dots}}=\frac{9}{8}\cdot\left(\frac{243}{256}\right)^2\Rightarrow$$
$$\frac{256}{243}=\left(\frac{9\cdot 243^2}{8\cdot 256^2}\right)^{M_6+\dots}=\left(\frac{3^{12}}{2^{19}}\right)^{M_6+\dots}$$
i.e.
$$\left(\frac{3^{12}}{2^{19}}\right)^{M_6}<\frac{256}{243}\quad \text{but}\ \left(\frac{3^{12}}{2^{19}}\right)^{M_6+1}>\frac{256}{243}$$
i.e.
$$3^{12M_6+5}<2^{19M_6+8}\quad \text{but}\ 3^{12M_6+17}>2^{19M_6+27}\Rightarrow M_5=2\,,$$

Indeed if $M_6 = 3$ then

$$3^{17} = 129140163 < 2^{27} = 134217728\,,$$

Repeat recurrently:
The rules for the fraction:

$$\begin{gathered} 3^{M_2} < 2^{M_2} \\ 2^{2M_3+1} < 3M_3 + 1 \\ 3^{2M_4+1} < 2^{3M_4+2} \\ 2^{8M_5+3} < 35M_5 + 2 \\ 3^{12M_6+5} < 2^{19M_6+8} \\ \dots \end{gathered}$$

Aproximations:

$$\log_2 3 = [1, 1, 1, 2, 2, \dots] = 1 + \cfrac{1}{1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{2 + \dots}}}}$$

1) $\log_2 3 \approx 1$
There are two note in octave.
2) $\log_2 3 \approx 1 + \frac{1}{1} = 2$
3) $\log_2 3 \approx 1 + \frac{1}{1 + \frac{1}{1}} = \frac{3}{2}$
There are two notes: C and G.
4) $\log_2 3 \approx 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2}}} = \frac{8}{5}$
There are 5 notes, the third is quinta
5) $\log_2 3 \approx 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{2}}}} = \frac{19}{12}$