18.01.2012

One calculation of one determinant.

Let $\|a_{ik}\|$ be $n \times n$ matrix.
Calculate determinant of $(n+1) \times (n+1)$ matrix

$$\left(\begin{array}{c|c} a_{ik} & u_i \\ \hline V_k & 0 \end{array}\right)$$

Calculation:

$$\det \begin{pmatrix} a_{ik} & u_i \\ V_k & 0 \end{pmatrix} = \lim_{\varepsilon \to 0} \begin{pmatrix} a_{ik} & u_i \\ V_k & \varepsilon \end{pmatrix} =$$

$$= \lim_{\varepsilon \to 0} \det \left(a_{ik} - \frac{u_i V_k}{\varepsilon} \right) \varepsilon = \quad \left(\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A - BD^{-1}C) \det D \right)$$

$$= \lim_{\varepsilon \to 0} \left[\det(a_{ik}) \det \left(1 - \frac{a^{im} u_m V_k}{\varepsilon} \right) \varepsilon \right] =$$

$$= \lim_{\varepsilon \to 0} \left[\det a \left(1 + \operatorname{Tr} \left(\frac{a^{im} V_m u_k}{\varepsilon} \right) + \sum_{k > 1} \frac{L_k}{\varepsilon^k} \right) \varepsilon \right] =$$

$$= \det a \; a^{im} V_m u_i \quad (a_{ir} a^{rm} = \delta_i^m)$$

(We do not need to calculate higher ~~tr~~ traces L_k)

Geom. meaning: If $a_{ik} = a_{ki}$, $a_{ik} x^i x^k = 0$ quadric in $\mathbb{P}^{n+1}$,
then $\det \begin{pmatrix} a_{ik} & u_k \\ u_i & 0 \end{pmatrix} = 0$ defines the pencil

of lines $u_i x^i = 0$

which are tangent to the quadric

$$a_{ik} x^i x^k = 0.$$