Solutions of Homework 6

1 *A point moves in* $\mathbf{E}^2$ *along an ellipse with the law of motion* $x = a\cos t$, $y = b\sin t$, $0 \le t < 2\pi$, $(0 < b < a)$. *Find the velocity and acceleration vectors. Find the points of the ellipse where the angle between velocity and acceleration vectors is acute. Find the points where speed attains its maximum value.*

Calculate velocity and acceleration vectors

$$\mathbf{v} = \mathbf{r}_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} -a\sin t \\ b\cos t \end{pmatrix}, \ \mathbf{a} = \mathbf{r}_{tt} = \begin{pmatrix} x_{tt} \\ y_{tt} \end{pmatrix} = \begin{pmatrix} -a\cos t \\ -b\sin t \end{pmatrix}.$$

We see that acceleration is collinear to $\mathbf{r}$: $\mathbf{a} = -\mathbf{r}$.

The scalar product of these vectors is equal to $(\mathbf{v}, \mathbf{a}) = |\mathbf{v}||\mathbf{a}|\cos\alpha = v_x a_x + v_y a_y = (a^2 - b^2)\sin t\cos t$, where α is angle between velocity and acceleration vectors.

Speed is increasing $\Leftrightarrow$ angle α is acute $\Leftrightarrow$ $(\mathbf{v}, \mathbf{a}) > 0 \Leftrightarrow \sin t\cos t > 0 \Leftrightarrow 0 \le t \le \pi/2$ or $\pi < t < 3\pi/2$.

Speed is decreasing $\Leftrightarrow$ angle α is obtuse $\Leftrightarrow$ $(\mathbf{v}, \mathbf{a}) < 0 \Leftrightarrow \sin t\cos t < 0 \Leftrightarrow \pi/2 \le t \le \pi$ or $3\pi/2 < t < 2\pi$.

Speed is attains its maximum when $t = \frac{\pi}{2}, \frac{3\pi}{2}$ and speed attains its minimum when $t = 0, \pi$.

(At these points acceleration is orthogonal to velocity vector and scalar product is equal to zero).

2 *Find a natural parameter for the line* $y = kx + b$

Consider an arbitrary parameterisation of the line, e.g. $\begin{cases} x = t \\ y = kt + b \end{cases}, \ -\infty < t < \infty$

We know that natural parameter $s(t)$ measures the length of the arc of the curve between a point $\mathbf{r}(t)$ and initial point. Take a point $(0, k)$ as initial point.

$$s(t) = \text{length of the interval of the line between point } (0, k) \text{ and point } (t, kt + b)$$

If α is angle between the line and x-axis then $s(t) = t/\cos\alpha$. $\cos\alpha = \frac{1}{\sqrt{1+k^2}}$. Hence $s(t) = t\sqrt{1 + k^2}$. One comes to the same answer making straigthforward integration:

$$s(t) = \int_0^t \sqrt{x_\tau^2 + y_\tau^2}\, d\tau = \int_0^t \sqrt{1 + k^2}\, d\tau = t\sqrt{1 + k^2}\,.$$

If we take another point as a initial point then natural parameter will change on a constant: E.g. if we take an initial point $(1, k + b)$ $(t = 1)$ then a new natural parameter:

$$s'(t) = \int_1^t \sqrt{x_\tau^2 + y_\tau^2}\, d\tau = \int_1^t \sqrt{1 + k^2}\, d\tau = \sqrt{1 + k^2}(t - 1) = s(t) - \sqrt{1 + k^2}\,.$$

Usually if a curve $\mathbf{r}(t)$ is given for parameters $t \in [t_1, t_2]$ one takes as initial a point $\mathbf{r}(t_1)$ and

$$s(t) = \int_0^t \sqrt{x_\tau^2 + y_\tau^2}\, d\tau\,.$$

3 *Consider the following curve (a helix):*

$$\mathbf{r}(t)\colon \quad \begin{cases} x(t) = R\cos t \\ y(t) = R\sin t \\ z(t) = ct \end{cases}.$$

Show that the tangential acceleration is equal to zero.

Find a natural parameter of this curve.

Calculate velocity and acceleration vectors:

$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \begin{pmatrix} -R\sin t \\ R\cos t \\ c \end{pmatrix}, |\mathbf{v}| = \sqrt{R^2 + c^2}, \mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \frac{d^2\mathbf{r}(t)}{dt^2} = \begin{pmatrix} -R\cos t \\ R\sin t \\ 0 \end{pmatrix}, |\mathbf{a}| = R.$$

The scalar (inner) product of velocity and acceleration vectors is equal to zero: $(\mathbf{v}(t), \mathbf{a}(t)) = 0$, i.e. these vectors are orthogonal. Hence the projection of acceleration vector on velocity vector (tangential vector to the curve) is equal to zero. Thus tangential velocity is equal to zero. (Note that speed $|\mathbf{v}|$ is constant. This also implies that tangential acceleration is equal to zero.)

Remark One can see that helix belongs to the surface of cylinder $x^2 + y^2 = R^2$ and acceleration is orthogonal to surface of the cylinder. This remark is essential to understand the geometry of cylinder (see the next homework).

Now calculate a normal parameter $s(t) =$ length of the arc of the helix from the point $\mathbf{r}(t_1)$ till point $\mathbf{r}(t)$. Take $t_1 = 0$ One can calculate the length taking integral

$$s(t) = \int_0^t |\mathbf{v}(\tau)| d\tau = \int_0^t \sqrt{x_t^2 + y_t^2 + z_t^2} dt.$$

On the other hand we note that speed is constant $|\mathbf{v}|^2 = (\mathbf{v}, \mathbf{v}) = R^2 + c^2$, i.e. $|\mathbf{v}| = \sqrt{R^2 + c^2}$. Thus we do not need to calculate the integral, natural parameter $s(t) = |\mathbf{v}| t = \sqrt{R^2 + c^2} t$.

4 *Find a natural parameter for the parabola $x = t, y = t^2$.*

$$s(t) = \{\text{length of the arc of the curve for parameter less or equal to } t\} =$$

$$\int_0^t \sqrt{\left(\frac{dx(\tau)}{d\tau}\right)^2 + \left(\frac{dy(\tau)}{d\tau}\right)^2} d\tau = \int_0^t \sqrt{1 + 4\tau^2} d\tau = \frac{t\sqrt{1 + 4t^2}}{2} + \frac{1}{4} \log\left(2t + \sqrt{1 + 4t^2}\right)$$

We calculated this integral using the following substitution:

Denote by $I = \int_0^t \sqrt{1 + 4\tau^2} d\tau$. Then integrating by parts we come to:

$$I = t\sqrt{1 + 4t^2} - \int \frac{4\tau^2}{\sqrt{1 + 4\tau^2}} d\tau = t\sqrt{1 + 4t^2} - I + \int \frac{1}{\sqrt{1 + 4\tau^2}} d\tau.$$

Hence $I = \frac{t\sqrt{1+4t^2}}{2} + \frac{1}{2} \int \frac{1}{\sqrt{1+4\tau^2}} d\tau$. and we come to the answer.

Remark We see that in general case natural parameter is not so easy to calculate. But its notion is very important for studying properties of curves.

5 *Calculate the curvature of the parabola $x = t, y = at^2$ $(a > 0)$ at an arbitrary point. Sure we cannot use the results of previous exercise.*

Find an equation of the circle which has a second order touching with this parabola at its vertex (the point $(0, 0)$).

**(not compulsory problem) Let s be a natural parameter on this parabola. Show that the integral $\int_{-\infty}^{\infty} k(s) ds$ of the curvature $k(s)$ over the parabola is equal to π.*

Sure it is not practical to use the results of previous exercise for calculating the curvature.

It is much more practical to use the formula for curvature in arbitrary parameterisation:

$$k(t) = \frac{|x_t y_{tt} - y_t x_{tt}|}{(x_t^2 + y_t^2)^{3/2}} = \frac{|1 \cdot 2a - 2at \cdot 0|}{(1^2 + (2at)^2)^{3/2}} = \frac{2a}{(1 + a^2t^2)^{3/2}}, \ (a > 0).$$

We see that the curvature at the point (t, at^2) is equal to $k(t) = \frac{2a}{(1+a^2t^2)^{3/2}}$ $(a > 0)$.

(Curvature is positive by definition. If $a < 0$, then $k(t) = \frac{-2a}{(1+a^2t^2)^{3/2}}$).

Now we find an equation of the circle which has a second order touching with this parabola at its vertex (the point $(0, 0)$). Note that curvature at the vertex is equal to $k(t)|_{t=0} = 2a$. Hence the radius of the circle which ahs second order touching is equal to

$$R = \frac{1}{2a}.$$

To find equation of this circle note that the circle which has second order touching to parabola at the vertex passes trough the vertex (point $(0, 0)$) and is tangent to x-axis. The radius of this circle is equal to $R = \frac{1}{2a}$. Hence equation of the circle is

$$(x - R^2) + y^2 = 0, \text{ where } R = \frac{1}{2a}$$

(See in more detail the example in the subsection of lecture notes "Second order touching")

To calculate $\int k(s)ds$, where s is natural parameter, better to return to an arbitrary parameterisation:

$$\int k(s)ds = \int k(s(t))\frac{ds(t)}{dt}dt = \int k(t)|\mathbf{v}(t)|dt$$

One can see that

$$k(t)|\mathbf{v}(t)| = \frac{d}{dt}\varphi(t),$$

where $\varphi(t)$ is the angle between the velocity vector and a given direction (see in detail lecture notes subsection 3.8). One can see this also by straightforward calculation:

$$\pm k(t)|\mathbf{v}(t)| = \frac{x_t y_{tt} - x_{tt} y_t}{x_t^2 + y_t^2} = \frac{d}{dt} \arctan \frac{y_t}{x_t}$$

Hence $\int k(s)ds = \varphi|_{-\infty}^{+\infty} = \pi$.

5a *For any function $f = f(x)$ one can consider its graph as not-parameterised curve C_f. Calculate curvature of the curve C_f at any point $(x, f(x))$.*

Find a radius of circle which has second order touching with the curve C_f at the point $(x, f(x))$.

One can choose parameteristaion: $\mathbf{r}(t)$: $\begin{cases} x = t \\ y = f(t) \end{cases}$.

Then $\mathbf{v}(t) = \begin{pmatrix} 1 \\ f'(t) \end{pmatrix}$, $\mathbf{a}(t) = \begin{pmatrix} 0 \\ f''(t) \end{pmatrix}$ and we have for the curvature that

$$k(t) = \frac{|x_t y_{tt} - y_t x_{tt}|}{(x_t^2 + y_t^2)^{\frac{3}{2}}} = \frac{|f''(t)|}{(1 + f'^2(t))^{\frac{3}{2}}}$$

Radius of the circle which has second order touching with the curve C_f at the point $(x, f(x))$ is equal to

$$R(x) = \frac{1}{k(x)} = \frac{(1 + f'^2(x))^{\frac{3}{2}}}{|f''(x)|}$$

6 Consider the parabola

$$\mathbf{r}(t): \begin{cases} x = v_x t \\ y = v_y t - \frac{gt^2}{2} \end{cases}.$$

(It is path of the point moving under the gravity force with initial velocity $\mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$.) Calculate the curvature at the vertex of this parabola.

To calculate the curvature one has to perform the same calculations as in the exercise 5:

$$k(t) = \frac{|x_t y_{tt} - y_t x_{tt}|}{(x_t^2 + y_t^2)^{3/2}} = \frac{|v_x \cdot (-g)|}{((v_x^2 + (v_y - gt)^2))^{3/2}}$$

In the vertex of this parabola vertical component of velocity is equal to zero. Hence curvature at the vertex is equal to

$$k = \frac{|v_x \cdot (-g)|}{((v_x^2 + (v_y - gt)^2))^{3/2}} |v_y = gt = g\mathbf{v}_x^2$$

The answer in fact immediately follows from considerations of classical mechanics: If curvature in the vertex is equal to k then radius of the circle which has second order touching is equal to $R = \frac{1}{k}$ and centripetal acceleration is equal to $a = \frac{v_x^2}{R}$. On the other hand $a = g$. Hence $R = \frac{v_x^2}{g}$ and $k = \frac{g}{v_x^2}$.

Remark Note that $v_x = \sqrt{\frac{g}{k}} = \sqrt{Rg}$. if we take $R \approx 6400km$ (radius of the Earth) then $v_x \approx 8km\,\mathrm{sec}$— if a point has this velocity then it will become satellite of the Earth (we ignore resistance of atmosphere).

7 Find a curvature at an arbitrary point of the helix considered in Exercise 3.

In this exercise we have to calculate curvature of the curve in three-dimensional Euclidean space. So we need to use the formula

$$k(t) = \frac{\text{Area of parallelogram formed by vectors } \mathbf{v}, \mathbf{a}}{|\mathbf{v}|^3} = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$$

We already calculated velocity and acceleration vectors for helix (see exercise 3)

Acceleration is orthogonal to velocity vector. Hence

$$|\mathbf{v} \times \mathbf{a}| = |\mathbf{v}| \cdot |\mathbf{a}| = R\sqrt{R^2 + c^2}\,.$$

and curvature is equal to

$$k = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{|v||\mathbf{a}|}{|\mathbf{v}|^3} = \frac{|\mathbf{a}|}{|\mathbf{v}|^2} = \frac{R}{R^2 + c^2} \qquad (*).$$

Remark Note that $k \to \frac{1}{R}$ if $c \to 0$ and $k \to 0$ if $c \to \infty$.

Remark Note that we could calculate curvature using the formula $k = \frac{|\mathbf{a}_{norm}|}{|\mathbf{v}|^2}$. We already know that tangential acceleration is equal to zero, hence $\mathbf{a} = \mathbf{a}_{norm}$ and

$$k = \frac{|\mathbf{a}_{norm}|}{|\mathbf{v}|^2} = k = \frac{|\mathbf{a}|}{|\mathbf{v}|^2}$$

We come to the formula (*).

8 *The curve C in $\mathbf{E}^3$ is given by the parameterisation $x = t,\ y = t^2,\ z = t^3,\ 0 \le t \le 2$. Find the velocity and acceleration vectors for this curve.*

Consider the plane α given by the equation $3x - 3y + z = 1$.

Prove that α is the plane spanned by the velocity and acceleration vectors at the point $(1, 1, 1)$ of the curve C.

Velocity vector for this curve is equal to $\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix}$ and acceleration vector $\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \begin{pmatrix} 0 \\ 2 \\ 6t \end{pmatrix}$.

Consider point $\mathbf{r} = (1,1,1)$. It belongs to the curve C: $\mathbf{r}(t)|_{t=1} = (1,1,1)$. Consider velocity and acceleration vectors attached at this point: $\mathbf{v}(t)|_{t=1} = (1,2,3)$ and $\mathbf{a}(t)|_{t=1} = (0,2,6)$. Vector $\mathbf{v}(t)|_{t=1} = (1,2,3)$ starts at the point $(1,1,1)$ belonging to the plane α: $3-3+1 = 1$ and ending at the point $(1+1, 1+2, 1+3)$. This point also belongs to the plane: $3\cdot 2 - 3\cdot 3 + 4 = 1$. Hence the vector $\mathbf{v}(t)|_{t=1}$ belongs to the plane α.

Vector $\mathbf{a}(t)|_{t=1} = (0,2,6)$ starts at the point $(1,1,1)$ belonging to the plane α and ending at the point $(1+0, 1+2, 1+6)$. This point also belongs to the plane: $3\cdot 1 - 3\cdot 3 + 7 = 1$. Hence the vector $\mathbf{a}(t)|_{t=1}$ belongs to the plane α also. These two vectors are linear independent because $a_x = 0$. Hence they span the plane α.