Introduction to Geometry (20222)

2017

COURSEWORK

This assignment counts for 20% of your marks.

Solutions are due by 30 March, 3pm

Write solutions in the provided spaces.

STUDENT'S NAME:

Academic Advisor (Tutor):

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a) Let (x^1, x^2, x^3) be coordinates of the vector $\mathbf{x}$, and (y^1, y^2, y^3) be coordinates of the vector $\mathbf{y}$ in $\mathbf{R}^3$.

Does the formula $(\mathbf{x}, \mathbf{y}) = x^1y^3 + x^2y^2 + x^3y^1$ define a scalar product on $\mathbf{R}^3$?

Justify your answer.

b) Consider the matrix $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Calculate the matrix A^9 in the case if $\theta = \frac{\pi}{27}$.

Calculate the matrix A^{2017} in the case if $\theta = \frac{\pi}{63}$.

c) Find all 2×2 orthogonal matrices A such that $2A^3 = \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$.

d) In oriented Euclidean space $\mathbf{E}^3$ consider the following linear operator

$$A(\mathbf{x}) = \mathbf{x} - \mathbf{a} \times (\mathbf{a} \times \mathbf{x})\,,$$

where the vector $\mathbf{a} = \frac{3}{13}\mathbf{e} + \frac{4}{13}\mathbf{f} + \frac{12}{13}\mathbf{g}$. Here $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ is an orthonormal basis in $\mathbf{E}^3$ defining orientation, and $\times$ is the vector product.

Find the eigenvectors of operator A. (Describe eigenvectors through basis vectors $\mathbf{e}, \mathbf{f}, \mathbf{g}$.)

Calculate the trace and determinant of the operator A.

e) Let $\{\mathbf{e}, \mathbf{f}\}$ be an orthonormal basis of Euclidean space $\mathbf{E}^2$. Consider a linear operator P such that $\mathbf{a} = P(\mathbf{e}) = 61\mathbf{e} + 12\mathbf{f}$, $\mathbf{b} = P(\mathbf{f}) = 5\mathbf{e} + \mathbf{f}$.

Calculate determinant of the operator P.

Show that P is not an orthogonal operator.

Consider the parallelogram $\Pi_{\mathbf{a},\mathbf{b}}$ spanned by the vectors $\mathbf{a}$ and $\mathbf{b}$ attached at the origin. Find the area of this parallelogram.

Show that the vertices of the parallelogram $\Pi_{\mathbf{a},\mathbf{b}}$ are the only points of $\Pi_{\mathbf{a},\mathbf{b}}$, whose coordinates are both integers.

2

We consider in this question 3-dimensional Euclidean space $\mathbf{E}^3$*. We suppose that* $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ *is an orthonormal basis in this space.*

a) Let P be a linear orthogonal operator acting in $\mathbf{E}^3$ such that its matrix in the basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ has the following appearance

$$P = \frac{1}{7}\begin{pmatrix} 3 & * & 6 \\ -6 & -3 & 2 \\ 2 & -6 & * \end{pmatrix} .$$

Find the entries of the matrix denoted by *.

Show that the operator P preserves orientation.

We know that due to the Euler Theorem the linear operator P considered above is a rotation operator. Find the axis and the angle of this rotation.

b) Let P_1 be a rotation operator on the angle θ around the axis directed along the vector $\mathbf{g}$, and P_2 be a rotation operator on the same angle θ around the axis directed along the vector $\mathbf{e}$:

$$\{\mathbf{e},\, \mathbf{f},\, \mathbf{g}\} \xrightarrow{P_1} \{\cos\theta\mathbf{e} + \sin\theta\mathbf{f},\ -\sin\theta\mathbf{e} + \cos\theta\mathbf{f},\ \mathbf{g}\}\,,$$

$$\{\mathbf{e},\, \mathbf{f},\, \mathbf{g}\} \xrightarrow{P_2} \{\mathbf{e},\ \cos\theta\mathbf{f} + \sin\theta\mathbf{g},\ -\sin\theta\mathbf{f} + \cos\theta\mathbf{g}\}\,.$$

Show that the operator $P = P_1 \circ P_2$ is also a rotation operator. Find the axis of rotation and the angle $\Phi = \Phi(\theta)$ of rotation for the operator P.

Calculate the angle Φ in the case $\theta = \frac{\pi}{2}$.

For $\theta << 1$, $\Phi \approx \sqrt{2}\theta$ [1]. Give an argument, justifying this formula.

[1] i.e. $\Phi(\theta) = \sqrt{2}\theta + O(\theta^2)$. In particular this means that $\lim_{\theta\to 0} \frac{\Phi(\theta)}{\theta} = \sqrt{2}$.

3

a) Consider the curve $\mathbf{r}(t)$: $\begin{cases} x = Rt \\ y = R\sqrt{1-t^2} \end{cases}, \quad 0 \leq t \leq 1.$

Draw the image of this curve.

Give an example of a parameterisation of this curve with opposite orientation.

b) Let f be a function in $\mathbf{E}^2$ given by $f = r^2 \cos 2\varphi$, where r, φ are polar coordinates in $\mathbf{E}^2$ $(x = r\cos\varphi, y = r\sin\varphi)$. Consider vector fields which are given in Cartesian coordinates by $\mathbf{A} = x\partial_x + y\partial_y$, $\mathbf{B} = x\partial_y - y\partial_x$.

Calculate $\partial_{\mathbf{A}} f$, $\partial_{\mathbf{B}} f$.

c) Consider the differential 1-form $\omega = ydx + dy$.

Give an example of a function $G(x, y)$ such that $G \neq 0$ and the differential 1-form $G\omega = G(x, y)(ydx + dy)$ is an exact form. (A 1-form ω is called exact if there exists a function $F(x, y)$ such that $\omega = dF$.)

Justify the answer.