Homework 2. Solutions

Let $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ be an orthonormal basis in $\mathbf{E}^3$. Consider the following ordered triples:
a) $\{\mathbf{e}_x, \mathbf{e}_x + 2\mathbf{e}_y, 5\mathbf{e}_z\}$,
b) $\{\mathbf{e}_y, \mathbf{e}_x, 5\mathbf{e}_z\}$,
c) $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$,
d) $\{\mathbf{e}_y, \mathbf{e}_x, -5\mathbf{e}_z\}$,
e) $\{\frac{\sqrt{3}}{2}\mathbf{e}_x + \frac{1}{2}\mathbf{e}_y, -\frac{1}{2}\mathbf{e}_x + \frac{\sqrt{3}}{2}\mathbf{e}_y, \mathbf{e}_z\}$,
f) $\{\mathbf{e}_y, \mathbf{e}_x, -\mathbf{e}_z\}$.

1 *Show that all triples a),b),c),d),e),f) are bases.*

Solution:
One can easy see that transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the basis $\{\mathbf{e}_x, \mathbf{e}_x + 2\mathbf{e}_y, 5\mathbf{e}_z\}$ is

$$T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix} \tag{1a)}$$

Indeed $(\mathbf{e}_x, \mathbf{e}_x + 2\mathbf{e}_y, 5\mathbf{e}_z) = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)T$: $(\mathbf{e}_x, \mathbf{e}_x + 2\mathbf{e}_y, 5\mathbf{e}_z) = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.

Analogously for the case b) transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the basis $\{\mathbf{e}_y, \mathbf{e}_x, 5\mathbf{e}_z\}$ is

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \text{because } (\mathbf{e}_y, \mathbf{e}_x, 5\mathbf{e}_z) = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \tag{1b)}$$

for the case c) transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$ is

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{because } (\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z) = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1c)}$$

for the case d) transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the basis $\{\mathbf{e}_y, \mathbf{e}_x, -5\mathbf{e}_z\}$ is

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -5 \end{pmatrix}, \text{because } (\mathbf{e}_y, \mathbf{e}_x, -5\mathbf{e}_z) = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -5 \end{pmatrix}, \tag{1d)}$$

for the case e) transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the basis $\{\frac{\sqrt{3}}{2}\mathbf{e}_x + \frac{1}{2}\mathbf{e}_y, -\frac{1}{2}\mathbf{e}_x + \frac{\sqrt{3}}{2}\mathbf{e}_y, \mathbf{e}_z\}$ is

$$T = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{because } \left(\frac{\sqrt{3}}{2}\mathbf{e}_x + \frac{1}{2}\mathbf{e}_y, -\frac{1}{2}\mathbf{e}_x + \frac{\sqrt{3}}{2}\mathbf{e}_y, \mathbf{e}_z\right) = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1e)}$$

and for the case f) transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the basis $\{\mathbf{e}_y, \mathbf{e}_x, -\mathbf{e}_z\}$ is

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \text{because } (\mathbf{e}_y, \mathbf{e}_x, -\mathbf{e}_z) = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{1f)}$$

We see that all these matrices are non-degenerated, because they have non-zero determinant:
$\det T = 10$ in (1a)),
$\det T = -5$ in (1b)),

$\det T = -1$ in (1c)),
$\det T = 5$ in (1d),
$\det T = 1$ in (1e)),
$\det T = 1$ in (1f)).
Hence ordered triples a),b),c),d),e),f) are bases.

2 *Show that the bases a), d), e) and f) have the same orientation as the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and the bases b) and c) have the orientation opposite to the orientation of the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$.*

Solution: In the exercise 1) we calculated already the determinants of transition matrices for all the bases a),b),c),d),e),f). In the cases a),d), e), f) determinants of transition matrices were positive numbers (see the end of solution of the problem 1). Hence bases a),d), e), f) have the same orientation as the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$.

In the cases b) and c) determinants of transition matrices were negative numbers (see the end of solution of the problem 1). Hence bases b) and c) have the orientation opposite to the orientation of the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$.

3 *Let $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ be an arbitrary basis in $\mathbf{E}^3$. Show that the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ either has the same orientation as the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$, or the same orientation as the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$.*

Solution:. Bases $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$, $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$ have opposite orientation: $(\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z) = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)T$, where $T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\det T = -1 < 0$ (see 1c).

Let T_1 be transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ and T_2 be transition matrix from the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$ to the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$:

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)T_1, \quad (\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z)T_2\,.$$

We see that $T_1 = T \cdot T_2$:

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z)T_2 = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)TT_2 = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)T_1$$

If $\det T_2 > 0$ then the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ have the same orientation as the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$. If $\det T_2 < 0$ then $\det T_1 = \det T \cdot \det T_2 > 0$ because $\det T = -1 < 0$. Hence the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ have the same orientation as the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$

Bases $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$ have opposite orientations. Hence they belong to different classes of bases (with respect to orientation). There are two classes. Hence the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ belongs to the same equivalence class of the bases to which belongs the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ or the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ belongs to the same equivalence class of the bases to which belongs the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$.

Arbitrary basis has the same orientation as the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and the orientation opposite to the orientation of the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$ or vice versa: it has the same orientation as the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$ and the orientation opposite to the orientation of the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$

4 *Show that bases c), e) and f) are orthonormal bases and bases a), b) and d) are not orthonormal bases.*

Solution To show that basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal basis we have to show that all basis vectors have the unit length and they are orthogonal each other, i.e. we have to check that following relations are satisfied:

$$(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \tag{4.1}$$

(See the subsection 1.1 in lecture notes)

It is given that the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ is an orthonormal basis, i.e. for this basis conditions (4.1) are satisfied. This immediately implies that the bases a), b) and d) are not orthonormal bases, because for all these bases the length of the last vector is not equal to 1. It is equal to 5.

Consider the basis c): $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$. It is reordering of vectors of initial basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. It is trivial to see that conditions (4.1) are satisfied. (we know already that the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ is an orthonormal basis). Hence it is orthonormal basis.

It is also very easy to see that the basis f) $\{\mathbf{e}_y, \mathbf{e}_x, -\mathbf{e}_z\}$ is an orthonormal one.

To see that basis e): $\{\frac{\sqrt{3}}{2}\mathbf{e}_x + \frac{1}{2}\mathbf{e}_y, -\frac{1}{2}\mathbf{e}_x + \frac{\sqrt{3}}{2}\mathbf{e}_y, \mathbf{e}_z\}$ is an orthonormal basis we need to do some calculations:

$\left(\frac{\sqrt{3}}{2}\mathbf{e}_x + \frac{1}{2}\mathbf{e}_y, \frac{\sqrt{3}}{2}\mathbf{e}_x + \frac{1}{2}\mathbf{e}_y\right) = \frac{3}{4}(\mathbf{e}_x, \mathbf{e}_x) + \frac{\sqrt{3}}{4}(\mathbf{e}_y, \mathbf{e}_x) + \frac{\sqrt{3}}{4}(\mathbf{e}_x, \mathbf{e}_y) + \frac{1}{4}(\mathbf{e}_y, \mathbf{e}_y) = \frac{3}{4} + 0 + 0 + \frac{1}{4} = 1$

$\left(\frac{\sqrt{3}}{2}\mathbf{e}_x + \frac{1}{2}\mathbf{e}_y, \frac{-1}{2}\mathbf{e}_x + \frac{\sqrt{3}}{2}\mathbf{e}_y\right) = \frac{-\sqrt{3}}{4}(\mathbf{e}_x, \mathbf{e}_x) + \frac{3}{4}(\mathbf{e}_y, \mathbf{e}_x) - \frac{1}{4}(\mathbf{e}_x, \mathbf{e}_y) + \frac{\sqrt{3}}{4}(\mathbf{e}_y, \mathbf{e}_y) = -\frac{3}{4} + 0 - 0 + \frac{3}{4} = 0,$

$\left(\frac{\sqrt{3}}{2}\mathbf{e}_x + \frac{1}{2}\mathbf{e}_y, \mathbf{e}_z\right) = \frac{\sqrt{3}}{4}(\mathbf{e}_x, \mathbf{e}_z) + \frac{1}{2}(\mathbf{e}_y, \mathbf{e}_z) = 0 + 0 = 0,$

$\left(-\frac{1}{2}\mathbf{e}_x + \frac{\sqrt{3}}{2}\mathbf{e}_y, -\frac{1}{2}\mathbf{e}_x + \frac{\sqrt{3}}{2}\mathbf{e}_y\right) = \frac{1}{4} + \frac{3}{4} = 1,$

$\left(-\frac{1}{2}\mathbf{e}_x + \frac{\sqrt{3}}{2}\mathbf{e}_y, \mathbf{e}_z\right) = 0 + 0 = 0,$

$\{\mathbf{e}_z, \mathbf{e}_z\} = 1.$

We see that relations (4.1) hold. It is orthonormal basis.

(it is easy to see that in this case transition matrix rotates basic vectors $\{\mathbf{e}_x, \mathbf{e}_y\}$ on the angle $\frac{\pi}{6}$.

Another solution: one have to check that transition matrix satisfies the condition $TT^t = I$ (I is identity matrix).

5 *Consider the linear operator P defined by the conditions*

$$P(\mathbf{e}_x) = \mathbf{e}_y, P(\mathbf{e}_y) = -\mathbf{e}_x, P(\mathbf{e}_z) = \mathbf{e}_z\,. \tag{5.1}$$

Show that this operator is a rotation (find the axis and the angle of rotation)

Solution

It is easy to see that operator P preserves orientation and the vector $\mathbf{e}_z$ is an eigenvector: $P\mathbf{e}_z = \mathbf{e}_z$. We see that operator P rotates the vectors around axis z.

Vector $\mathbf{e}_x$ rotated on the angle φ (anticlockwise) transforms to the vector $\mathbf{e}_x \cos\varphi + \mathbf{e}_y \sin\varphi$.

Vector $\mathbf{e}_y$ rotated on the angle φ (anticlockwise) transforms to the vector $-\mathbf{e}_x \sin\varphi + \mathbf{e}_y \cos\varphi$.

Comparing with (5.1) we see that operator P rotates vectors on the angle $\frac{\pi}{2}$.

on the angle $\frac{\pi}{2}$.

6 *Solve the previous problem if $P(\mathbf{e}_x) = \mathbf{e}_y, P(\mathbf{e}_y) = \mathbf{e}_x, P(\mathbf{e}_z) = -\mathbf{e}_z$.*

Solution

In this case too, orientation is preserved, because bases $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and $\{\mathbf{e}_y, \mathbf{e}_x, -\mathbf{e}_z\}$ have the same orientation (see the equation (1f))). Hence P is rotation. To find axis of rotation, we have to find an eigenvector with eigenvalue 1. (Vectors directed along axes does not change under this linear transformation.) It is easy to see that the vector

$$\mathbf{N} = \mathbf{e}_x + \mathbf{e}_y$$

is an eigenvector as well as all vectors $\lambda\mathbf{N}$: $P(\mathbf{N}) = P(\mathbf{e}_x + \mathbf{e}_y) = \mathbf{e}_y + \mathbf{e}_x = \mathbf{N}$.

Vectors $\mathbf{e}_z$ and $\mathbf{e}_x - \mathbf{e}_y$ are orthogonal to the axis. They span linear subspace (plane) orthogonal to the axis of rotation. These vectors are eigenvectors of the operator P with eigenvalue -1: $P(\mathbf{e}_z) = -\mathbf{e}_z$,

$P(\mathbf{e}_x - \mathbf{e}_y) = -(\mathbf{e}_x - \mathbf{e}_y)$. Hence all vectors of this plane are eigenvectors with eigenvalue -1, i.e. operator P is rotation around axis $\mathbf{e}_x + \_y$ on the angle $\frac{\pi}{2}$ *

7† *Show that an arbitrary orthogonal transformation that preserves an orientation of* $\mathbf{E}^3$ *is a rotation. (Euler Theorem)*

Let P be linear orthogonal transformation: i.e. for an arbitrary vector $\mathbf{a} = a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z$

$$P(\mathbf{a}) = P(a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z) = \mathbf{a} = a_x\mathbf{e}'_x + a_y\mathbf{e}'_y + a_z\mathbf{e}'_z$$

where the new basis $\{\mathbf{e}'_x, \mathbf{e}'_y, \mathbf{e}'_z\}$ is the orthonormal basis as well as the initial basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$.

First solution: Note that for any two vectors of the same length one can consider linear transformation—reflection which transforms the first vector to the second one.

Consider any reflection which transforms basis vector $\mathbf{e}_x$ to the basis vector $\mathbf{e}_{x'}$, then reflection which does not move vector $\mathbf{e}_{x'}$ and transforms basis vector $\mathbf{e}_y$ to the basis vector $\mathbf{e}_{y'}$. If it is not enough we can in principal consider the third reflection. We see that every linear transformation P can be considered as a reflection or a composition of two or three reflections. But determinant of every reflection is equal to -1. On the other hand these bases have the same orientation. Hence we see that composition O_1O_2 of two reflections transforms basis $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ to basis $\mathbf{e}_{x'}, \mathbf{e}_{y'}, \mathbf{e}_{z'}$. Let α_1 be invariant plane of reflection O_1 and α_2 be invariant plane of reflection O_2. Then a line which is intersection of these planes is an axis. ∎

8 *Calculate the area of parallelograms formed by the vectors* $\mathbf{a}, \mathbf{b}$ *if*

a) $\mathbf{a} = (1, 2, 3), \mathbf{b} = (1, 0, 1)$;
b) $\mathbf{a} = (2, 2, 3), \mathbf{b} = (1, 1, 1)$;
c) $\mathbf{a} = (5, 8, 4), \mathbf{b} = (10, 16, 8)$.

Area of parallelogram formed by the vectors $\mathbf{a}, \mathbf{b}$ is equal to the length of the vector $\mathbf{c} = \mathbf{a} \times \mathbf{b}$.

$$\mathbf{c} = \alpha \times \mathbf{b} = (a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z) \times (b_x\mathbf{e}_x + b_y\mathbf{e}_y + b_z\mathbf{e}_z) = a_xb_x\mathbf{e}_x \times \mathbf{e}_x + a_xb_y\mathbf{e}_x \times \mathbf{e}_y + a_xb_z\mathbf{e}_x \times \mathbf{e}_z +$$

$$a_yb_x\mathbf{e}_y \times \mathbf{e}_x + a_yb_y\mathbf{e}_y \times \mathbf{e}_y + a_yb_z\mathbf{e}_y \times \mathbf{e}_z + a_zb_x\mathbf{e}_z \times \mathbf{e}_x + a_zb_y\mathbf{e}_z \times \mathbf{e}_y + a_zb_z\mathbf{e}_z \times \mathbf{e}_z =$$

$$(a_xb_y - a_yb_x)\mathbf{e}_z + (a_yb_z - a_zb_y)\mathbf{e}_x + (a_zb_x - a_xb_z)\mathbf{e}_y$$

a) $S = |\mathbf{a} \times \mathbf{b}| = |-2\mathbf{e}_z + 2\mathbf{e}_x + 2\mathbf{e}_y|$, $S = \sqrt{4+4+4} = 2\sqrt{3}$.
b) $S = |\mathbf{a} \times \mathbf{b}|$. $\mathbf{a} \times \mathbf{b} = -\mathbf{e}_x + \mathbf{e}_y$, $S = \sqrt{1+1} = \sqrt{2}$
c) Vectors $\mathbf{a} = (5, 8, 4), \mathbf{b} = (10, 16, 8)$ are colinear, hence $\mathbf{a} \times \mathbf{b} = 0$, $S = 0$.

9 *Prove the inequality* $(ad - bc)^2 \leq (a^2 + b^2)(c^2 + d^2)$

a) by a direct calculation
b) considering vector product of vectors $\mathbf{x} = a\mathbf{e}_x + b\mathbf{e}_y$ *and vectors* $\mathbf{y} = c\mathbf{e}_x + d\mathbf{e}_y$

Solution

a) $-2adbc \leq a^2c^2 + b^2d^2$ because $(ac + bd)^2 \geq 0$. Hence $(ad - bc)^2 = a^2d^2 + b^2c^2 - 2adbc \leq a^2d^2 + b^2c^2 + a^2c^2 + b^2d^2 = (a^2 + b^2)(c^2 + d^2)$.

b) $\mathbf{x} \times \mathbf{y} = (ad - bc)\mathbf{e}_z$. Length of this vector is the area of the parallelogram spanned by the vectors $\mathbf{x}, \mathbf{y}$. On the other hand area of parallelogram is less or equal than area of the rectangle with the same sides: $S = |ad - bc| = |\mathbf{x}||\mathbf{y}| \sin\varphi \leq |\mathbf{x}||\mathbf{y}| = \sqrt{(a^2 + b^2)(c^2 + d^2)}$

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* If linear operator P rotates vectors on an arbitrary angle $\varphi \neq 0$, $\varphi \neq \pi$, then there is only one eigenvector (up to multiplier) of the rotation–it is axis of rotation. Eigenvalue of this vector is equal to 1. If P rotates on the angle $\varphi = 0$ then all vectors are eigenvectors with eigenvalue 1–P is identity operator.

10 *Show that for any two vectors* $\mathbf{a}, \mathbf{b} \in \mathbf{E}^3$ *the following identity is satisfied*

$$(\mathbf{a}, \mathbf{a})(\mathbf{b}, \mathbf{b}) = (\mathbf{a}, \mathbf{b})^2 + (\mathbf{a} \times \mathbf{b}, \mathbf{a} \times \mathbf{b})\,.$$

Write down this identity in components.
Compare this identity with CBS inequality from the previous homework.

Solution
Let φ be an angle between vectors $\mathbf{a}, \mathbf{b}$. Then

$$(\mathbf{a}, \mathbf{a})(\mathbf{b}, \mathbf{b}) = |\mathbf{a}|^2|\mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2(\cos^2\varphi + \sin^2\varphi) = (\mathbf{a}, \mathbf{b})^2 + |\mathbf{a} \times \mathbf{b}|^2 \tag{10.1}$$

In components:

$$(a_x^2 + a_y^2 + a_z^2)(b_x^2 + b_y^2 + b_z^2) = (a_x b_x + a_y b_y + a_z b_z)^2 + (a_x b_y - a_y b_x)^2 + (a_y b_z - a_z b_y)^2 + (a_z b_x - a_x b_z)^2 \tag{10.2}$$

Notice that for $n = 2, 3$ this identity is more strong statement than CBS inequality: $(a_x^2 + a_y^2 + a_z^2)(b_x^2 + b_y^2 + b_z^2) \geq (a_x b_x + a_y b_y + a_z b_z)^2$. CBS inequality $(\mathbf{a}, \mathbf{a})(\mathbf{b}, \mathbf{b}) \geq |\mathbf{a}|^2|\mathbf{b}|^2$ follows from the identity (1.10).

The proof of the identity (10.2) becomes more complicated if we use only algebraical methods.