Homework 9. Solutions.

1 *On the sphere $x^2+y^2+z^2=R^2$ in $\mathbf{E}^3$ consider a circle C which is the intersection of the sphere with the plane $z=R-h$, $0<h<R$*

Let $\mathbf{X}$ be an arbitrary vector tangent to the sphere at a point of C.

Find the angle between $\mathbf{X}$ and the result of parallel transport of $\mathbf{X}$ along C.

The circle C is a boundary of the sphere segment of the height H. The area of this domain is equal to $2\pi Rh$. The Gaussian curvature of sphere iis equal to $\frac{K=1}{R^2}$. Hence due to Theorem we see that vector $\mathbf{X}$ through parallel transport rotates on the angle $KS=\frac{2\pi R}{h}$.

2 *Write down components of curvature tensor in terms of Christoffel symbols.*

$$R^i_{kmn}\partial_i=\mathcal{R}(\partial_m,\partial_n)\partial_k=\nabla_m\left(\nabla_n\partial_k\right)-(m\leftrightarrow n)=\nabla_m(\Gamma^r_{nk}\partial_r)-(m\leftrightarrow n)=\left(\partial_m\Gamma^i_{nk}+\Gamma^i_{mr}\Gamma^r_{nk}\right)\partial_i-(m\leftrightarrow n)$$

i.e.

$$R^i_{kmn}=\partial_m\Gamma^i_{nk}+\Gamma^i_{mr}\Gamma^r_{nk}-\partial_n\Gamma^i_{mk}-\Gamma^i_{nr}\Gamma^r_{mk}\,.$$

(See also lecture notes).

3 *Let ∇ be a connection on n-dimensional manifold M and $\{R^i{}_{rmn}\}$ be the components of the curvature tensor of a connection ∇ in local coordinates $(x^1,x^2,\dots,x^n)$.*

a) For arbitrary vector fields $\mathbf{A},\mathbf{B}$ and $\mathbf{D}$ calculate the vector field

$$\left(\nabla_{\mathbf{A}}\nabla_{\mathbf{B}}-\nabla_{\mathbf{B}}\nabla_{\mathbf{A}}\right)\mathbf{D}-\nabla_{\mathbf{C}}\mathbf{D}\,,$$

where the vector field $\mathbf{C}$ is a commutator of vector fields $\mathbf{A}$ and $\mathbf{B}$:

$$\mathbf{C}=C^i\frac{\partial}{\partial x^i}=[\mathbf{A},\mathbf{B}]=\left(A^m\frac{\partial B^i(x)}{\partial x^m}-B^m\frac{\partial A^i(x)}{\partial x^m}\right)\frac{\partial}{\partial x^i}\,. \tag{1.0}$$

b) Calculate the vector field

$$\left(\nabla_{\mathbf{A}}\nabla_{\mathbf{B}}-\nabla_{\mathbf{B}}\nabla_{\mathbf{A}}\right)\mathbf{D}$$

in the case if for vector fields $\mathbf{A}$ and $\mathbf{B}$ components A^i and B^m are constants (in the local coordinates $(x^1,\dots,x^n)$)

(You have to express the answers in terms of components of the vector fields and components of the curvature tensor $R^i{}_{rmn}$.)

a) According to the definition of the curvature tensor for every vector fields $\mathbf{X}=X^m\partial_m,\mathbf{Y}=Y^m\partial_m$ and $\mathbf{Z}=Z^m\partial_m$ we have that $\mathcal{R}(\mathbf{X},\mathbf{Y})\mathbf{Z}=$

$$\mathcal{R}(X^m\partial_m,Y^n\partial_n)(Z^r\partial_r)=\left(\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}-\nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}-\nabla_{[\mathbf{X},\mathbf{Y}]}\right)\mathbf{Z}=Z^rR^i{}_{rmn}X^mY^n\partial_i\,.$$

Hence for vector fields $\mathbf{A},\mathbf{B}$ and $\mathbf{C}=[\mathbf{A},\mathbf{B}]$ we have that

$$\left(\nabla_{\mathbf{A}}\nabla_{\mathbf{B}}-\nabla_{\mathbf{B}}\nabla_{\mathbf{A}}-\nabla_{[\mathbf{A},\mathbf{B}]}\right)\mathbf{D}=\left(\nabla_{\mathbf{A}}\nabla_{\mathbf{B}}-\nabla_{\mathbf{B}}\nabla_{\mathbf{A}}\right)\mathbf{D}-\nabla_{\mathbf{C}}\mathbf{D}=\mathcal{R}(\mathbf{A},\mathbf{B})\mathbf{D}=D^rR^i{}_{rmn}A^mB^n\partial_i\,. \tag{1.1}$$

b) in the case if in the local coordinates $(x^1,\dots,x^n)$ for vector fields $\mathbf{A}$ and $\mathbf{B}$ components A^i and B^m are constants then the commutator of these vector fields $\mathbf{C}=[\mathbf{A},\mathbf{B}]$ vanishes: $\mathbf{C}=0$ (see the formula (1.0)). Hence according to the formula (1.1) above we have that

$$\left(\nabla_{\mathbf{A}}\nabla_{\mathbf{B}}-\nabla_{\mathbf{B}}\nabla_{\mathbf{A}}\right)\mathbf{D}-\nabla_{\mathbf{C}}\mathbf{D}=\left(\nabla_{\mathbf{A}}\nabla_{\mathbf{B}}-\nabla_{\mathbf{B}}\nabla_{\mathbf{A}}\right)\mathbf{D}=\mathcal{R}(\mathbf{A},\mathbf{B})\mathbf{D}=D^rR^i{}_{rmn}A^mB^n\partial_i\,.$$

4 *For every of the statements below prove it or show that it is wrong considering counterexample.*

a) If there exist coordinates u,v such that Riemannian metric G at the given point $\mathbf{p}$ is equal to $G=du^2+dv^2$ in these coordinates, then curvature of Levi-Civita connection at the point $\mathbf{p}$ vanishes.

b) If all derivatives of components of Riemannian metric in coordinates u, v vanish at the given point with coordinates (u_0, v_0):

$$\frac{\partial g_{ik}(u,v)}{\partial u}\Big|_{u=u_0,v=v_0} = \frac{\partial g_{ik}(u,v)}{\partial v}\Big|_{u=u_0,v=v_0} = 0\,,$$

then curvature of Levi-Civita connection at this point vanishes

c) If all first and second derivatives of components of Riemannian metric

$$\frac{\partial g_{ik}(u,v)}{\partial u}, \frac{\partial g_{ik}(u,v)}{\partial v}, \frac{\partial^2 g_{ik}(u,v)}{\partial u^2}, \frac{\partial^2 g_{ik}(u,v)}{\partial u \partial v}, \frac{\partial^2 g_{ik}(u,v)}{\partial v^2},$$

vanish at the given point then curvature of Levi-Civita connection vanishes at this point.

First and second statements are wrong. The thrid statement is true.

Counterexample to the first statement: Consider on the unit sphere metric $G = d\theta^2 + \sin^2\theta d\varphi^2$.

At the points of equator (but not in their neighborhood!!!!) this metric is Euclidean and first derivatives of components vanish, but curvature is not vanished.

Counterexample to the second statement: Consider surface $\mathbf{r} = \mathbf{r}(u,v)\colon x = u, y = v, z = F(x,y)$. Induced Riemannian metric $G = g_{\alpha\beta}du^\alpha du^\beta = g_{uu}du^2 + 2g_{uv}dudv + g_{vv}dv^2$

$$G = \begin{pmatrix} 1+F_u^2 & F_uF_v \\ F_uF_v & 1+F_v^2 \end{pmatrix}$$

One can see that at the points of extremum of function F first derivatives of Riemannian metric vanish, this Christoffel symbols vanish at the extrema in coordinates u, v, but the curvature in general does not vanish (It is proportional to $F_{uu}F_{vv} - F_{uv}^2$. It is convenient to consider the case of sphere: $F = \sqrt{R - u^2 - v^2}$.

If at the given point first and second derivatves of metric vanish then due to Levi-Civita formula Christoffely symbols and their first derivatives vanish. This imply that cuirvature vanish too.

5 *Using relation between Gaussian curvature and Riemann curvature tensor for Levi-Civita connection, write down all components $\{R_{ikmn}\}$ of Riemann curvature tensor for sphere of radius R in spehrical coordinates.*

Using symmetry properties $R_{ikmn} = -R_{kimn} = -R_{iknm}$ we have that

$$R_{1111} = R_{1112} = R_{1121} = R_{1211} = R_{2111} = R_{2222} = R_{2212} = R_{2221} = R_{1222} = R_{2122} = R_{1122} = R_{2211} = 0\,.$$

It remains to calculate $R_{1212}, R_{1221}, R_{2112}, R_{2121}$ We have that

$$\frac{R_{1212}}{\det g} = K = \frac{1}{R^2},.$$

Since $G = R^2(d\theta^2 + \sin^2\theta d\varphi^2)$, $\det g = R^4\sin^2\theta$, we have that

$$R_{1212} = -R_{1221} = -R_{2112} = R_{2121} = K\det g = R^2\sin^2\theta\,.$$

6* *Consider a surface M in $\mathbf{E}^3$ defined by the equation*

$$\begin{cases} x = u \\ y = v \\ z = F(u,v) \end{cases} \qquad (2.0)\,.$$

Calculate explicitly the component R_{1212} of the Riemannian curvature tensor at the point $\mathbf{p}$ with coordinates $u = v = 0$ in the case if $F(u,v) = \frac{1}{2}(au^2 + 2buv + bv^2)$, where a, b, c are parameters.

Consider a point $\mathbf{p}$ *on the surface* M *with coordinates* $u = x_0, v = y_0$ *such that* (x_0, y_0) *is a point of local extremum for the function* F.

Using the results of previous exercise calculate the component R_{1212} *of the Riemannian curvature tensor at the point* $\mathbf{p}$.

*Solution**

The function $F(u,v) = F(u,v) = \frac{1}{2}(au^2 + 2buv + bv^2)$ *obeys the conditions that* $F_u = F_V = 0$ *at the origin, the point* $u = v = 0$. *We perform the calculation for an arbitrary function* $F(u,v)$ *which obeys the conditions* $F_u = F_V = 0$ *at the origin, the point* $u = v = 0$, *i.e. the point* $u = v = 0$ *is stationary point for the function* F. *Geometrically this means that we consider the surface such that the plane* OXY *is tangent to the plane at the origin and the axis OZ is orthogonal to the surface at this point.*

First of all recall the expression for Riemannian metric for the surface (2.0):

$$G = \begin{pmatrix} 1 + F_u^2 & F_u F_v \\ F_v F_u & 1 + F_v^2 \end{pmatrix} \tag{2.1}$$

Note that Riemannian metric g_{ik} *in (2.1) is defined by unity matrix* $g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $g_{uu} = g_{vv} = 1$, $g_{uv} = g_{vu} = 0$ *at the point* $u = v = 0$ *since* $\mathbf{p}$ *is stationary point* ($F_u = F_v = 0$ *at the point* $\mathbf{p}$): $G = \begin{pmatrix} 1 + F_u^2 & F_u F_v \\ F_v F_u & 1 + F_v^2 \end{pmatrix} \big|_{\mathbf{p}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. *Hence the components of the tensor* $R^i{}_{kmn}$ and $R_{ikmn} = g_{ij} R^j{}_{kmn}$ *at the point* $\mathbf{p}$ *are the same.*

The components of $R^i{}_{kmn}$ are defined by the formula

$$R^i{}_{kmn} = \partial_m \Gamma^i_{nk} + \Gamma^i_{mp} \Gamma^p_{nk} - \partial_n \Gamma^i_{mk} - \Gamma^i_{np} \Gamma^p_{mk} \tag{2.2}$$

Notice that at the point $\mathbf{p}$ *not only the matrix of the metric* g_{ik} *equals to unity matrix* $g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, *but more: Christoffel symbols vanish at this point in coordinates* u, v *since the derivatives of metric at this point vanish. (Why they vanish: see in detail the file "The solution of the problem 5 in the coursework, revisited.")*

Hence to calculate $R^i{}_{kmn}$ *at the point* $\mathbf{p}$ *one can consider more simple formula:*

$$R^i{}_{kmn}|_{\mathbf{p}} = \partial_m \Gamma^i_{nk}|_{\mathbf{p}} - \partial_n \Gamma^i_{mk}|_{\mathbf{p}} \tag{2.3}$$

Try to calculate in a more "economical" way. Due to Levi-Civita formula

$$\Gamma^i_{mk} = \frac{1}{2} g^{ij} \left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^m} - \frac{\partial g_{mk}}{\partial x^j} \right) \tag{2.4}$$

Since metric g_{ik} *equals to unity matrix* $g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *at the point* $\mathbf{p}$ *hence* g^{ij} *is unity matrix also:*

$$g^{ik}|_{\mathbf{p}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \delta^{ik} \tag{2.5}$$

(We denote δ^{ik} *the unity matrix: all diagonal components equal to 1, all other components equal to zero. (It is so called Kronecker symbols)) Moreover we know also that all the first derivatives of the metric vanish at the point* $\mathbf{p}$:

$$\frac{\partial g_{ik}}{\partial x^m}|_{\mathbf{p}} = 0\,. \tag{2.6}$$

* *The solution below is almost the same as the proof of the Proposition about the relation between Gaussian and scalar curvature for surfaces in* $\mathbf{E}^3$ *(see the subsection 5.2.3 of lecture notes)*

Hence it follows from the formulae (2.5) and (2.6) that for an arbitrary indices i, j, k, m, n

$$\frac{\partial}{\partial x^i}\left(g^{km}\frac{\partial g_{pr}}{\partial x^j}\right)\Big|_{\mathbf{p}} = \frac{\partial g^{km}}{\partial x^i}\Big|_{\mathbf{p}}\frac{\partial g_{pr}}{\partial x^j}\Big|_{\mathbf{p}} + g^{km}\big|_{\mathbf{p}}\frac{\partial^2 g_{pr}}{\partial x^i \partial x^j}\Big|_{\mathbf{p}} = \delta^{km}\frac{\partial^2 g_{pr}}{\partial x^i \partial x^j}\Big|_{\mathbf{p}}$$

In particular it follows from this formula that

$$\partial_n \Gamma^i_{mk}|_{\mathbf{p}} = \frac{\partial}{\partial x^n}\left(\frac{1}{2}g^{ij}\left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^m} - \frac{\partial g_{mk}}{\partial x^j}\right)\right)\Big|_{\mathbf{p}} = \frac{1}{2}\delta^{ij}\left(\frac{\partial^2 g_{jm}}{\partial x^n \partial x^k} + \frac{\partial^2 g_{jk}}{\partial x^n \partial x^m} - \frac{\partial^2 g_{mk}}{\partial x^n \partial x^j}\right)\Big|_{\mathbf{p}} \qquad (2.7)$$

Now we are ready to calculate $\partial_n \Gamma^i_{mk}|_{\mathbf{p}}$ using the last formula (2.7) and the formula (2.1) for the metric. Remember that we want to calculate $R^1{}_{212}$ which is equal at the point $\mathbf{p}$ according to (2.3) to

$$R^1{}_{212}|_{\mathbf{p}} = \partial_1 \Gamma^1_{22}|_{\mathbf{p}} - \partial_2 \Gamma^1_{12}|_{\mathbf{p}} \qquad (2.8)$$

and according to (2.7)

$$\partial_1 \Gamma^1_{22} = \frac{1}{2}\delta^{1j}\left(\frac{\partial^2 g_{j2}}{\partial x^1 \partial x^2} + \frac{\partial^2 g_{j2}}{\partial x^1 \partial x^2} - \frac{\partial^2 g_{22}}{\partial x^1 \partial x^j}\right)\Big|_{\mathbf{p}} = \frac{1}{2}\left(\frac{\partial^2 g_{12}}{\partial x^1 \partial x^2} + \frac{\partial^2 g_{12}}{\partial x^1 \partial x^2} - \frac{\partial^2 g_{22}}{\partial x^1 \partial x^1}\right)\Big|_{\mathbf{p}} = \frac{\partial^2 g_{12}}{\partial x^1 \partial x^2} - \frac{1}{2}\frac{\partial^2 g_{22}}{\partial x^1 \partial x^1}\Big|_{\mathbf{p}},$$

$$\partial_2 \Gamma^1_{12} = \frac{1}{2}\delta^{1j}\left(\frac{\partial^2 g_{j1}}{\partial x^2 \partial x^2} + \frac{\partial^2 g_{j2}}{\partial x^2 \partial x^1} - \frac{\partial^2 g_{12}}{\partial x^2 \partial x^j}\right)\Big|_{\mathbf{p}} = \frac{1}{2}\left(\frac{\partial^2 g_{11}}{\partial x^2 \partial x^2} + \frac{\partial^2 g_{12}}{\partial x^2 \partial x^1} - \frac{\partial^2 g_{12}}{\partial x^2 \partial x^1}\right)\Big|_{\mathbf{p}} = \frac{1}{2}\frac{\partial^2 g_{11}}{\partial x^2 \partial x^2}\Big|_{\mathbf{p}}, \qquad (2.9)$$

Hence

$$R_{1212}|_{\mathbf{p}} = R^1{}_{212}|_{\mathbf{p}} = \partial_1 \Gamma^1_{22}|_{\mathbf{p}} - \partial_2 \Gamma^1_{12}|_{\mathbf{p}} = \left(\frac{\partial^2 g_{12}}{\partial x^1 \partial x^2} - \frac{1}{2}\frac{\partial^2 g_{22}}{\partial x^1 \partial x^1} - \frac{1}{2}\frac{\partial^2 g_{11}}{\partial x^2 \partial x^2}\right)\Big|_{\mathbf{p}} \qquad (2.10)$$

Now using (2.1) calculate second derivatives $\frac{\partial^2 g_{12}}{\partial x^1 \partial x^2}$, $\frac{\partial^2 g_{11}}{\partial x^2 \partial x^2}$ and $\frac{\partial^2 g_{22}}{\partial x^1 \partial x^1}$:

$$\frac{\partial^2 g_{12}}{\partial x^1 \partial x^2}\Big|_{\mathbf{p}} = \frac{\partial^2 (F_u F_v)}{\partial u \partial v}\Big|_{\mathbf{p}} = \frac{\partial}{\partial u}\left(F_u F_{vv} + F_{uv} F_v\right)|_{\mathbf{p}} = \left(F_{uu} F_{vv} + F^2_{uv}\right)|_{\mathbf{p}}.$$

since $F_u = F_v = 0$ at the point $\mathbf{p}$ $(x^1 = u, x^2 = v)$. Analogoulsy

$$\frac{\partial^2 g_{11}}{\partial x^2 \partial x^2}\Big|_{\mathbf{p}} = \frac{\partial^2 (1 + F^2_u)}{\partial v \partial v}\Big|_{\mathbf{p}} = \frac{\partial}{\partial v}\left(2F_u F_{uv}\right)|_{\mathbf{p}} = 2F^2_{uv}|_{\mathbf{p}}$$

and

$$\frac{\partial^2 g_{22}}{\partial x^1 \partial x^1}\Big|_{\mathbf{p}} = \frac{\partial^2 (1 + F^2_v)}{\partial u \partial u}\Big|_{\mathbf{p}} = \frac{\partial}{\partial u}\left(2F_v F_{uv}\right)|_{\mathbf{p}} = 2F^2_{uv}|_{\mathbf{p}},$$

We have finally that

$$R_{1212}|_{\mathbf{p}} = R^1{}_{212}|_{\mathbf{p}} = \partial_1 \Gamma^1_{22}|_{\mathbf{p}} - \partial_2 \Gamma^1_{12}|_{\mathbf{p}} = \left(\frac{\partial^2 g_{12}}{\partial x^1 \partial x^2} - \frac{1}{2}\frac{\partial^2 g_{22}}{\partial x^1 \partial x^1} - \frac{1}{2}\frac{\partial^2 g_{11}}{\partial x^2 \partial x^2}\right)\Big|_{\mathbf{p}} = F_{uu} F_{vv} - F^2_{uv}.$$

These calculations and these statements are so good that it is worth to write it as an statements.

The calculations (2.2)—(2.10) lead us to the following statement:

Statement 1 *Let $G = g_{ik} dx^i dx^k$ $(i, k = 1, 2)$ be 2-dimensional Riemannian manifold, and in local coordinates x^i at the give point $\mathbf{p}$ Riemannian metric is obeyed the following conditions:*

a) at the given point $\mathbf{p}$, $G = dx^2 + dy^2$,

b) First derivatives of metric vanish at this point.

Then according to the formulae above for curvaature tensor R^i_{kmn} R^1_{212} is expressed via equation (2.9).

Statement 2 *Consider a surface M in $\mathbf{E}^3$ defined by the equation*

$$\begin{cases} x = u \\ y = v \\ z = F(u, v) \end{cases}.$$

Suppose at the origin, the point $\mathbf{p}$ *with coordinates* $u = v = 0$ $F_u = F_v = 0.$*, i.e. normal unit vector is colinear to* Oz *axis. Then at the point* $\mathbf{p}$

$$R^1{}_{212}|_{\mathbf{p}} = R_{1212}|_{\mathbf{p}} = F_{uu}F_{vv} - F_{uv}^2|_{\mathbf{p}} = \det\begin{pmatrix} F_{uu} & F_{uv} \\ F_{uv} & F_{vv} \end{pmatrix}|_{\mathbf{p}}$$

One can see that the matrix above is nothing but shape operator, and this proves the relation between Gaussian and scalar curvature (see the lecture notes.)

Now very last and simplest step: in the case if $F(u,v) = \frac{1}{2}(au^2 + 2buv + cv^2)$ *we see that*

$$R^1{}_{212}|_{\mathbf{p}} = R_{1212}|_{\mathbf{p}} = F_{uu}F_{vv} - F_{uv}^2|_{\mathbf{p}} = (ac - b^2)\,.$$

Now consider the answer on the last question: by translation it can be trivially reduced to the case considered anove: the equation of the surface can be written as

$$\begin{cases} x = x_0 + u \\ y = y_0 + v \\ z = F(x_0 + u, y_0 + v) \end{cases}.$$

Consider $z = z_0 = F(x_0, y_0)$ *and the function* $\tilde{F}(u,v) = F(x_0 + u, y_0 + v) - z_0$ *we come to the equation*

$$\begin{cases} x = x_0 + u \\ y = y_0 + v \\ z = z_0 + \tilde{F}(u,v) \end{cases}$$

Considering new Cartesian coordinates $x' = x - x_0, y' = y - y_0, z' = z - z_0$ *we come to the equation*

$$\begin{cases} x' = u \\ y' = v \\ z' = \tilde{F}(u,v) \end{cases}$$

in the vicinity of the extremum point with coordinates $u = v = 0$*. This is just the case of Porposition that was studied above (see (2.5) Hence in the extremum point*

$$R^1{}_{212}|_{\mathbf{p}} = \tilde{F}_{uu}\tilde{F}_{vv} - \tilde{F}_{uv}^2|_{u=v=0} = F_{uu}F_{vv} - F_{uv}^2|_{u=x_0, v=y_0}\,.$$