On canonical isomorphisms $TT^*M = T^*TM = T^*T^*M$

Let M be manifold. Establish and study canonical isomorphisms $TT^*M = T^*TM = T^*T^*M$

Calculations in coordinates

It may sounds surprising but calculations in coordinates are transparent and illuminating.

First consider local coordinates on TM and T^*M corresponding to local coordinates (x^i) on M.

Local coordinates for TM are (x^i, t^j): every vector $\mathbf{r} \in TM$ is a vector $t^i \frac{\partial}{\partial x^i}$, $t^i(\mathbf{r}) = dx^i(\mathbf{r})$. If $\tilde{x}^\mu = \tilde{x}^\mu(x^i)$ are new local coordinates on M then

$$d\tilde{x}^\mu \left(t^i \frac{\partial}{\partial x^i} \right) = \frac{\partial \tilde{x}^\mu(x^i)}{\partial x^i} dx^i \left(t^i \frac{\partial}{\partial x^i} \right) = \frac{\partial \tilde{x}^\mu(x^i)}{\partial x^i} t^i \,.$$

Hence changing of local coordinates in TM is

$$(x^i, t^j) \mapsto (\tilde{x}^\mu, \tilde{t}^\nu), \quad \tilde{x}^\mu = \tilde{x}^\mu(x^i),\ \tilde{t}^\mu = \begin{pmatrix} \mu \\ i \end{pmatrix} t^i, \tag{1}$$

where we denote $\frac{\partial \tilde{x}^\mu(x^i)}{\partial x^i}$ by $\begin{pmatrix} \mu \\ i \end{pmatrix}$.

Respectively local coordinates for T^*M are (x^i, p_j). For every 1-form $\omega \in T^*M$ $p_i = \omega\left(\frac{\partial}{\partial x^i}\right)$. Under changing of local coordinates on M $\tilde{x}^\mu = \tilde{x}^\mu(x^i)$, coordinates (p_i) change to new coordinates (p_μ):

$$p_\mu = w\left(\frac{\partial}{\partial \tilde{x}^\mu}\right) = w\left(\frac{\partial x^i(\tilde{x}^\mu)}{\partial \tilde{x}^\mu} \frac{\partial}{\partial x^i}\right) = \frac{\partial x^i(\tilde{x}^\mu)}{\partial \tilde{x}^\mu} p_i$$

Hence changing of local coordinates in T^*M is

$$(x^i, p_k) \mapsto (\tilde{x}^\mu, \tilde{p}_\nu), \quad \tilde{x}^\mu = \tilde{x}^\mu(x^i),\ p_\mu = \begin{pmatrix} i \\ \mu \end{pmatrix} p_i, \tag{2}$$

where we denote $\frac{\partial x^i(\tilde{x}^\mu)}{\partial \tilde{x}^\mu}$ by $\begin{pmatrix} i \\ \mu \end{pmatrix}$

Now using (1),(2) we define coordinates on the spaces TT^*M, T^*TM and T^*T^*M.

The space TT^*M is tangent space to the space T^*M. The local coordinates on TT^*M corresponding to local coordinates (x^i, p_j) on T^*M are coordinates $(x^i, p_j; \xi^k, \rho_m)$; $\xi^k = dx^i(\mathbf{r}), \rho_m = dp_m(\mathbf{r})$. Under changing of local coordinates (x^i) to coordinates $\tilde{x}^\mu = \tilde{x}^\mu(x^i)$