Homework 2

Let $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ be an orthonormal basis in $\mathbf{E}^3$. Consider the following ordered triples:
a) $\{\mathbf{e}_x, \mathbf{e}_x + 2\mathbf{e}_y, 5\mathbf{e}_z\}$,
b) $\{\mathbf{e}_y, \mathbf{e}_x, 5\mathbf{e}_z\}$,
c) $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$,
d) $\{\mathbf{e}_y, \mathbf{e}_x, -5\mathbf{e}_z\}$,
e) $\{\frac{\sqrt{3}}{2}\mathbf{e}_x + \frac{1}{2}\mathbf{e}_y, -\frac{1}{2}\mathbf{e}_x + \frac{\sqrt{3}}{2}\mathbf{e}_y, \mathbf{e}_z\}$,
f) $\{\mathbf{e}_y, \mathbf{e}_x, -\mathbf{e}_z\}$.

1 Show that all triples a),b),c),d),e),f) are bases.

2 Show that the bases a), d), e) and f) have the same orientation as the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and the bases b) and c) have the orientation opposite to the orientation of the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$.

3 Let $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ be an arbitrary basis in $\mathbf{E}^3$. Show that the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ either has the same orientation as the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$, or the same orientation as the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$.

4 Show that bases c), e) and f) are orthonormal bases and bases a), b) and d) are not orthonormal bases.

5 Consider the linear operator P defined by the conditions

$$P(\mathbf{e}_x) = \mathbf{e}_y, P(\mathbf{e}_y) = -\mathbf{e}_x, P(\mathbf{e}_z) = \mathbf{e}_z\,.$$

Show that this operator is a rotation (find the axis and the angle of rotation)

6 Solve the previous problem if $P(\mathbf{e}_x) = \mathbf{e}_y, P(\mathbf{e}_y) = \mathbf{e}_x, P(\mathbf{e}_z) = -\mathbf{e}_z$.

7† Show that an arbitrary orthogonal transformation that preserves an orientation of $\mathbf{E}^3$ is a rotation. (Euler Theorem)

8 Calculate the area of parallelograms formed by the vectors $\mathbf{a}, \mathbf{b}$ if
a) $\mathbf{a} = (1, 2, 3), \mathbf{b} = (1, 0, 1)$;
b) $\mathbf{a} = (2, 2, 3), \mathbf{b} = (1, 1, 1)$;
c) $\mathbf{a} = (5, 8, 4), \mathbf{b} = (10, 16, 8)$.

9 Prove the inequality $(ad - bc)^2 \leq (a^2 + b^2)(c^2 + d^2)$
a) by a direct calculation
b) considering vector product of vectors $\mathbf{x} = a\mathbf{e}_x + b\mathbf{e}_y$ and vectors $\mathbf{y} = c\mathbf{e}_x + d\mathbf{e}_y$

10 Show that for any two vectors $\mathbf{a}, \mathbf{b} \in \mathbf{E}^3$ the following identity is satisfied

$$(\mathbf{a}, \mathbf{a})(\mathbf{b}, \mathbf{b}) = (\mathbf{a}, \mathbf{b})^2 + (\mathbf{a} \times \mathbf{b}, \mathbf{a} \times \mathbf{b})\,.$$

Write down this identity in components.
Compare this identity with CBS inequality from the previous homework.