Homework 3. Solutions

1 *a) Show explicitly that matrix* $A_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$ *is an orthogonal matrix.*

b) Show explicitly that under the transformation $\{\mathbf{e}', \mathbf{f}'\} = \{\mathbf{e}_1, \mathbf{f}'\} A_\varphi$ *an orthonormal basis transforms to an orthonormal one.*

c) Show that for orthogonal matrix A_φ *the following relations are satisfied:*

$$A_\varphi^{-1} = A_\varphi^{T} = A_{-\varphi}\,, \qquad A_{\varphi+\theta} = A_\varphi \cdot A_\theta\ .$$

a) Check straightforwardly that $A_\varphi^T \cdot A = I$ (this is definition of orthogonal matrix):

$$A_\varphi^T \cdot A = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} =$$

$$\begin{pmatrix} \cos^2\varphi + \sin^2\varphi & -\cos\varphi\sin\varphi + \sin\varphi\cos\varphi \\ -\sin\varphi\cos\varphi + \cos\varphi\sin\varphi & \sin^2\varphi + \cos^2\varphi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

b) Let $\{\mathbf{e}, \mathbf{f}\}$ be an orthonormal basis, i.e. scalar products $(\mathbf{e}, \mathbf{e}) = 1$ and $(\mathbf{e}, \mathbf{f}) = 0$. Then

$$\{\mathbf{e}', \mathbf{f}'\} = \{\mathbf{e}, \mathbf{f}\} A_\varphi = \{\mathbf{e}, \mathbf{f}\} \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\,, \text{ i.e. } \begin{cases} \mathbf{e}' = \cos\varphi\mathbf{e} + \sin\varphi\mathbf{f} \\ \mathbf{f}' = -\sin\varphi\mathbf{e} + \cos\varphi\mathbf{f} \end{cases}.$$

We have to check that $\{\mathbf{e}', \mathbf{f}'\}$ is also orthonormal basis, i.e. scalar products $(\mathbf{e}', \mathbf{e}) = (\mathbf{f}', \mathbf{f}') = 1$ and $(\mathbf{e}', \mathbf{f}') = 0$. Calculate:

$$(\mathbf{e}', \mathbf{e}') = (\cos\varphi\mathbf{e} + \sin\varphi\mathbf{f}, \cos\varphi\mathbf{e} + \sin\varphi\mathbf{f}) = \cos^2\varphi(\mathbf{e}, \mathbf{e}) + 2\cos\varphi\sin\varphi(\mathbf{e}, \mathbf{f}) + \sin^2\varphi(\mathbf{f}, \mathbf{f}) =$$

$$\cos^2\varphi \cdot 1 + 2\cos\varphi\sin\varphi \cdot 0 + \sin^2\varphi \cdot 1 = 1\,,$$

$$(\mathbf{f}', \mathbf{f}') = (-\sin\varphi\mathbf{e} + \cos\varphi\mathbf{f}_2, -\sin\varphi\mathbf{e} + \cos\varphi\mathbf{f}) = \sin^2\varphi(\mathbf{e}, \mathbf{e}) - 2\cos\varphi\sin\varphi(\mathbf{e}, \mathbf{f}) + \cos^2\varphi(\mathbf{f}, \mathbf{f}) =$$

$$\cos^2\varphi \cdot 1 + 2\cos\varphi\sin\varphi \cdot 0 + \sin^2\varphi \cdot 1 = 1\,,$$

and

$$(\mathbf{e}', \mathbf{f}') = (\cos\varphi\mathbf{e} + \sin\varphi\mathbf{f}, -\sin\varphi\mathbf{e} + \cos\varphi\mathbf{f}) =$$

$$-\cos\varphi\sin\varphi(\mathbf{e}, \mathbf{e}) + (\cos^2\varphi - \sin^2\varphi)(\mathbf{e}, \mathbf{f}) + \sin\varphi\cos\varphi(\mathbf{f}, \mathbf{f}) = -\cos\varphi\sin\varphi + \sin\varphi\cos\varphi = 0\,.$$

c) We have that $A_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$. Then calculate inverse matrix A_φ^{-1}. One can see that $A_\varphi^{T} = A_\varphi^{-1} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$, because $A_\varphi^{T} A_\varphi = I$. On the other hand $\cos\varphi = \cos(-\varphi)$ and $\sin\varphi = -\sin(-\varphi)$. Hence

$$A_\varphi^{T} = A_\varphi^{-1} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} = \begin{pmatrix} \cos(-\varphi) & -\sin(-\varphi) \\ \sin(-\varphi) & \cos(-\varphi) \end{pmatrix} = A_{-\varphi}\,.$$

Now prove that $A_{\varphi+\theta} = A_\varphi \cdot A_\theta$:

$$A_\varphi \cdot A_\theta = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} (\cos\varphi\cos\theta - \sin\varphi\sin\theta) & -(\cos\varphi\sin\theta + \sin\varphi\cos \\ (\cos\varphi\sin\theta + \sin\varphi\cos\theta) & (\cos\varphi\cos\theta - \sin\varphi\sin \end{pmatrix}$$

$$\begin{pmatrix} \cos(\varphi+\theta) & -\sin(\varphi+\theta) \\ \sin(\varphi+\theta) & \cos(\varphi+\theta) \end{pmatrix} = A_{\varphi+\theta}$$

Remark Geometrical meaning of this relation is that composition of "rotations" on angle φ and θ is "rotation" on angle $\varphi + \theta$.

2 *Let* $\mathbf{e}, \mathbf{f}$ *be orthonormal basis in Euclidean space* $\mathbf{E}^2$*. Consider a vector*

$$\mathbf{n}_\varphi = \mathbf{e}\cos\varphi + \mathbf{f}\sin\varphi\,.$$

Let A *be a linear orthogonal operator acting on the space* $\mathbf{E}^2$ *such that* $A(\mathbf{e}) = \mathbf{n}_\varphi$*.*

We know that $\det A = \pm 1$ *since* A *is orthogonal operator.*

In the case if $\det A = 1$*, find the image* $A(\mathbf{f})$ *of vector* $\mathbf{f}$ *and an image* $A(\mathbf{x})$ *of an arbitrary vector* $\mathbf{x} = a\mathbf{e} + b\mathbf{f}$*, write down the matrix of operator* A *in the basis* $\mathbf{e}, \mathbf{f}$ *and explain geometrical meaning of the operator* A*.*

† *How the answer will change if* $\det A = -1$*?*

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be transition matrix of operator A in the orthonormal basis $\{\mathbf{e}, \mathbf{f}\}$:

$$\{\mathbf{e}', \mathbf{f}'\} = \{\mathbf{e}, \mathbf{f}\}A = \{\mathbf{e}, \mathbf{f}\} \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \begin{cases} \mathbf{e}' = a\mathbf{e} + c\mathbf{f} \\ \mathbf{f}' = b\mathbf{e} + c\mathbf{f} \end{cases}.$$

New basis is also orthonormal. We have that $\mathbf{e}' = \mathbf{n}_\varphi = \mathbf{e}\cos\varphi + \mathbf{f}\sin\varphi$, hence matrix of the orthogonal operator A in orthonormal basis $\{\mathbf{e}, \mathbf{f}\}$ is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\varphi & b \\ \sin\varphi & d \end{pmatrix}$$

Matrix of orthogonal operator in orthonormal basis is an orthogonal matrix. Hence $\begin{pmatrix} \cos\varphi & b \\ \sin\varphi & d \end{pmatrix}$ is orthogonal matrix, i.e.

$$\begin{cases} b\cos\varphi + d\sin\varphi = 0 \\ b^2 + d^2 = 1 \end{cases}.$$

Put $b = \sin\psi, d = \cos\psi$, then bearing in mind the condition that $\det A = d\cos\varphi - b\sin\varphi = 1$, we come to equations

$$\begin{cases} b\cos\varphi + d\sin\varphi = \sin\psi\cos\varphi + \cos\psi\sin\varphi = \sin(\varphi+\psi) = 0 \\ d\cos\varphi - b\sin\varphi = \cos\psi\cos\varphi - \sin\psi\sin\varphi = \cos(\varphi+\psi) = 1 \end{cases},$$

i.e. we come to $\psi = -\varphi + 2\pi k$. Matrix of operator A in the basis $\{\mathbf{e}, \mathbf{f}\}$ is equal to $\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$. A is operator of rotation on the angle φ (see the section 1.7.1. in lecture notes). $A(\mathbf{f}) = b\mathbf{e} + d\mathbf{f} = -\sin\varphi\mathbf{e} + \cos\varphi\mathbf{f}$. For arbitrary vector $\mathbf{x}$ we have that

$$A(\mathbf{x}) = A(x^1\mathbf{e} + x^2\mathbf{f}) = x^1A(\mathbf{e}) + x^2A(\mathbf{f}) = x^1(\mathbf{e}\cos\varphi + \mathbf{f}\sin\varphi) + x^2(-\mathbf{e}\sin\varphi + \mathbf{f}\cos\varphi) =$$

$$(x^1\cos\varphi - x^2\sin\varphi)\mathbf{e} + (x^1\sin\varphi + x^2\cos\varphi)\mathbf{f}\,,$$

or in the other way: $A(\mathbf{x}) = A(x^1\mathbf{e} + x^2\mathbf{f}) =$

$$= A\left(\{\mathbf{e}, \mathbf{f}\}\begin{pmatrix} x^1 \\ x^2 \end{pmatrix}\right) = \{\mathbf{e}, \mathbf{f}\}\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \{\mathbf{e}, \mathbf{f}\}\begin{pmatrix} x^1\cos\varphi - x^2\sin\varphi \\ x^1\sin\varphi + x^2\cos\varphi \end{pmatrix}.$$

† One can see that in the case if $\det A = -1$, then A is the operator of reflection with respect to the line directed along the vector $\mathbf{n}_{\frac{\varphi}{2}} = \cos\frac{\varphi}{2}\mathbf{e} + \sin\frac{\varphi}{2}\mathbf{f}$.

Remark Note that condition that one can find an angle φ such that $A(\mathbf{e}) = \cos\varphi\mathbf{e} + \sin\varphi\mathbf{f}$ is automaticall fullfilled for orthogonal operator. In fact solving this problem we repeated the calculation of matrix of orthogonal operator in $\mathbf{E}^2$ (see lecture notes, subsection 1.7.1)

3 *Let* $\mathbf{e}, \mathbf{f}$ *be an orthonormal basis in Euclidean space* $\mathbf{E}^2$.

Consider a vector $\mathbf{N} = \mathbf{e} + \mathbf{f}$ *in* $\mathbf{E}^2$.

Let A be an orthogonal operator acting on the space $\mathbf{E}^2$ *such that* $A\mathbf{N} = \mathbf{N}$. *($\mathbf{N}$ is eigenvector of A with eigenvalue 1.) Suppose that A is not identity operator.*

a) Find an action of operator A on the vector $\mathbf{R} = \mathbf{e} - \mathbf{f}$ *in* $\mathbf{E}^2$.

b) Explain geometrical meaning of the operator A.

c) Write down the matrix of operator A in the basis $\mathbf{e}, \mathbf{f}$.

a) Let $A(\mathbf{R}) = a\mathbf{e} + b\mathbf{f}$. Vectors $\mathbf{N}$ and $\mathbf{R}$ are orthogonal to each other:

$$(\mathbf{N}, \mathbf{R}) = (\mathbf{e} + \mathbf{f}, \mathbf{e} - \mathbf{f}) = (\mathbf{e}, \mathbf{e}) - \mathbf{f}, \mathbf{f} = 1 - 1 = 0\,,$$

Hence the vectors $A(\mathbf{N})$ and $A(\mathbf{R})$ have to be orthogonal to each other also, since orthogonal operator does not change the scalar product.

Hence vector $A(\mathbf{R})$ has to be proportional to the vector $\mathbf{R}$ also, i.e. $A(\mathbf{R}) = a\mathbf{R}$. The length of the vector is not changed under othogonal transformation, hence $a = \pm 1$. If $a = 1$, i.e. $A(\mathbf{R}) = \mathbf{R}$ we see that operator A is identical on two linear independent vectors: $A(\mathbf{R}) = \mathbf{R}$, $A(\mathbf{N}) = \mathbf{N}$ hence it is identical on their span, i.e. $A = \mathbf{id}$. On the other hand we know that A is not identity operator. Hence $a = -1$. We come to the conclusion that $A(\mathbf{R}) = -\mathbf{R}$.

b) Operator A is reflection operator with respect to the line directed along the vector $\mathbf{N}$.

c) We have that $\mathbf{e} = \frac{\mathbf{N}+\mathbf{R}}{2}$ and $\mathbf{f} = \frac{\mathbf{N}-\mathbf{R}}{2}$. Hence

$$A(\mathbf{e}) = A\left(\frac{\mathbf{N}+\mathbf{R}}{2}\right) = \frac{\mathbf{N}-\mathbf{R}}{2} = \mathbf{f}\,, A(\mathbf{f}) = A\left(\frac{\mathbf{N}-\mathbf{R}}{2}\right) = \frac{\mathbf{N}+\mathbf{R}}{2} = \mathbf{e}\,,$$

i.e. the matrix of operator A in the bases $\{\mathbf{e},\mathbf{f}\}$ is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

4 *Let* $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$ *be an orthonormal basis in Euclidean space* $\mathbf{E}^3$*. Consider a linear operator* P *in* $\mathbf{E}^3$ *such that*

$$\mathbf{e}' = P(\mathbf{e}) = \mathbf{e}, \quad \mathbf{f}' = P(\mathbf{f}) = \frac{\sqrt{2}}{2}\mathbf{f} + \frac{\sqrt{2}}{2}\mathbf{g}, \quad \mathbf{g}' = P(\mathbf{g}) = -\frac{\sqrt{2}}{2}\mathbf{f} + \frac{\sqrt{2}}{2}\mathbf{g}\,.$$

Write down the matrix of operator P *in the basis* $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$ *to the order*

Show that P *is an orthogonal operator.*

Show that orthogonal operator P *preserves the orientation of* $\mathbf{E}^3$*.*

Find an axis of the rotation and the angle of the rotation.

The matrix of operator P in the basis $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$ is the trnasition matrix from basis $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$ to the basis $\{\mathbf{e}',\mathbf{f}',\mathbf{g}'\} = \{P(\mathbf{e}),P(\mathbf{f}),P(\mathbf{g})\}$. We have

$$\{\mathbf{e}',\mathbf{f}',\mathbf{g}'\} = \{P(\mathbf{e}),P(\mathbf{f}),P(\mathbf{g})\} = \left\{\mathbf{e}, \frac{\sqrt{2}}{2}\mathbf{f} + \frac{\sqrt{2}}{2}\mathbf{g}, -\frac{\sqrt{2}}{2}\mathbf{f} + \frac{\sqrt{2}}{2}\mathbf{g}\right\} =$$

$$\{\mathbf{e},\mathbf{f},\mathbf{g}\}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \tag{1}$$

One can see that the matrix in (1) is invertible. The triple $\{\mathbf{e}',\mathbf{f}',\mathbf{g}'\}$ is a basis. It is easy to see that the new basis $\{\mathbf{e}',\mathbf{f}',\mathbf{g}'\}$ is orthonormal basis since the former basis $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$ is orthonormal one: $(\mathbf{e}',\mathbf{e}') = (\mathbf{f}',\mathbf{f}') = (\mathbf{g}',\mathbf{g}') = 1$ and $(\mathbf{e}',\mathbf{f}') = (\mathbf{e}',\mathbf{g}') = (\mathbf{f}',\mathbf{g}') = 0$. Linear operator P is orthogonal operator and its matrix in orthonormal basis is orthogonal matrix operator.

One can check the condition of orthogonality of matrix in equation (1) straightforwardly:

$$P^T \cdot P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We see that $\det P = 1$, hence he linear operator P does not change orientation.

One can see from expression (1) that operator P rotates space $\mathbf{E}^3$ with rspect to the axis directed along the vector $\mathbf{e}$ on the angle $\frac{\pi}{4}$:

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ 0 & \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix}$$

5 *Consider a linear operator P_1 in $\mathbf{E}^3$ such that it transforms the orthonormal basis $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$ into the orthonormal basis $\{\mathbf{f},\mathbf{e},\mathbf{g}\}$:*

$$P_1(\mathbf{e}) = \mathbf{f}\,,\quad P_1(\mathbf{f}) = \mathbf{e}\,,\quad P_1(\mathbf{g}) = \mathbf{g}\,.$$

Consider also a linear orthogonal operator P_2 such that it is the reflection operator with respect to the plane spanned by vectors $\mathbf{e}$ and $\mathbf{f}$.

Do operators P_1, P_2 preserve orientation?

Does operator $P = P_2 \circ P_1$ preserve orientation?

Find eignevectors of operator P.

Show that P is rotation operator.

Operator P_1 is orthogonal operator since it transforms orthonormal basis to orthonormal one. We have that

$$P_1(\mathbf{e}) = \mathbf{f},\ P_1(\mathbf{f}) = \mathbf{e},\ P_1(\mathbf{g}) = \mathbf{g}\,. \tag{7.1}$$

The transition matrix of basis $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$ to basis $\{\mathbf{f},\mathbf{e},\mathbf{g}\}$ is a matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ since } \{\mathbf{f},\mathbf{e},\mathbf{g}\} = \{\mathbf{e},\mathbf{f},\mathbf{g}\} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Its determinant equals $-1 < 0$. Hence linear operator P_1 changes orientation. It is reflection operator (with respect to the plan spanned by vectors $\mathbf{e}+\mathbf{f}$ and $\mathbf{g}$,

These vectors and their arbitrary linear combinations are eigenvalues of this operator:

$$P(\mathbf{e}+\mathbf{f}) = \mathbf{f}+\mathbf{e} = \mathbf{e}+\mathbf{f}, \qquad P(\mathbf{g}) = \mathbf{g}\,, P(\lambda(\mathbf{e}+\mathbf{f}) + \mu\mathbf{g}) = \lambda(\mathbf{e}+\mathbf{f}) + \mu\mathbf{g}\,.$$

Now consider orthogonal operator P_2. The plane spanned by vectors $\mathbf{e},\mathbf{f}$ remains intact, hence $P_2(\mathbf{e}) = \mathbf{e}$ and $P_2(\mathbf{f}) = \mathbf{f}$. Vector $\mathbf{g}$ transforms to vector $-\mathbf{g}$ since it is orthogonal to vectors $\mathbf{e}$ and $\mathbf{f}$. We have

$$P_2(\mathbf{e}) = \mathbf{e},\ P_2(\mathbf{f}) = \mathbf{f},\ P_2(\mathbf{g}) = -\mathbf{g}\,. \tag{7.2}$$

Vectors $\mathbf{e},\mathbf{f}$ and $\mathbf{g}$ are eigenvectors with eigenvalues $1, 1, -1$ respectively. Matrix of operator P_2 in the basis $\{\mathbf{e},\mathbf{f},g\}$ is equal to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ Determinant of operator P_2 is equal to product of eigenvalues: $\det P = 1 \cdot 1 \cdot (-1) = 1$. (Or you can calculate it using matrix of operator P.)

This orthogonal operator as well as orthogonal operator P_1 does not preserve orientation. Using equations (7.1) and (7.2) we have that for operator $P = P_2 \circ P_1$

$$P(\mathbf{e}) = P_2 \circ P_1(\mathbf{e}) = P_2(\mathbf{f}) = \mathbf{f},\ P(\mathbf{f}) = P_2 \circ P_1(\mathbf{f}) = P_2(\mathbf{e}) = \mathbf{e},\ P(\mathbf{g}) = P_2 \circ P_1(\mathbf{g}) = P_2(\mathbf{g}) = -\mathbf{g}, \ \blacksquare$$

$\det P = \det(P_2 \circ P_1) = \det P_2 \cdot \det P_1 = (-1)(-1) = 1$. P is orthogonal matrix which preserves orientation.

Consider the vector $\mathbf{N} = \mathbf{e} + \mathbf{f}$. This is eigenvector of operator P:

$$P(\mathbf{N}) = P(\mathbf{e} + \mathbf{f}) = \mathbf{f} + \mathbf{e} = \mathbf{N}$$

We see that $\mathbf{N}$ is an eigenvector of non-identical orthogonal operator preserving orientation. Thus axis of rotation is along the vector $\mathbf{N}$. To calculate the angle of rotation notice that vector $\mathbf{g}$ transforms to vector $-g$. Hence the rotation is on the angle π [(1)].

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(1) One can see that an arbitrary vector $\mathbf{a}$ orthogonal to vector $\mathbf{N}$ ("axis") changes to vector $-\mathbf{a}$.