Homework 8

1. Find coordinate basis vectors, first quadratic form and unit normal vector field for the following surfaces:

a) sphere of the radius R:

$$\mathbf{r}(\varphi,\theta) \qquad \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases} \qquad (0 \le \varphi < 2\pi, 0 \le \theta \le \pi)\,, \tag{1}$$

b) cylinder

$$\mathbf{r}(\varphi,h) \qquad \begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \\ z = h \end{cases} \qquad (0 \le \varphi < 2\pi, -\infty < h < \infty) \tag{2}$$

c) graph of the function $z = F(x,y)$

$$\mathbf{r}(u,,v) \qquad \begin{cases} x = u \\ y = v \\ z = F(u,v) \end{cases} \qquad (-\infty < u < \infty, -\infty < v < \infty) \tag{3}$$

in the case if $F(u,v) = F = Au^2 + 2Buv + Cv^2$.

Put down the special case of saddle when $F = uv$.

2. Consider helix $\mathbf{r}(t)$: $\begin{cases} x(t) = R\cos t \\ y(t) = R\sin t \\ z(t) = ct \end{cases}$.

Show that this helix belongs to cylinder surface $x^2 + y^2 = R^2$.

Using first quadratic form calculate length of this curve ($0 \le t \le t_0$). (Compare with problem 4 from Homework 7.)

3. a) Consider on the sphere (1) the following curves:

C_1: $x = R\cos t,\ y = R\sin t,\ z = 0,\ 0 \le t < 2\pi$ (Equator),

C_2: $x = R\cos t,\ y = 0,\ z = R\sin t\ 0 \le t < \pi$ ("Greenwich" Meridian),

C_3: $x = R\sin\theta_0\cos t,\ y = R\sin\theta_0\sin t,\ z = R\cos\theta_0\ \ 0 \le t < 2\pi$ (Circle of constant latitude)

Sketch these curves. Calculate length of these curves considering them in the ambient Euclidean space. Calculate length of these curves using first quadratic form.

4. Calculate the shape operator for an arbitrary point of the sphere (1).

Recall the notion of *normal curvature* of a curve on a surface.

Let C be an arbitrary curve on the sphere of radius R. Show that the normal curvature of the curve C at an arbitrary point is equal to $1/R$ (up to a sign).

5. a) Calculate the shape operator for an arbitrary point of the cylinder (2).

b) Consider on the cylinder (2) the following curves:

C_1: $x = R\cos t,\ y = R\sin t,\ z = h_0$ (circle),

C_2: $x = R\cos t,\ y = R\sin t,\ z = vt$ (helix),

C_3: $x = R\cos\varphi_0,\ y = R\sin\varphi_0,\ z = t$ (straight line).

Calculate the normal curvatures of these curves.

c) What values the normal curvature of an arbitrary curve on the cylinder of radius R can take?

6. Calculate the shape operator for the surface (3) at the point $u = v = 0$.

Put down the shape operator for this surface at the point $u = v = 0$ in the special case $F = uv$ (a "saddle")

7 Calculate principal curvatures, Gaussian and mean curvature

a) at the points of the sphere of radius R

b) at the points of cylinder surface of radius R

c) at the point $u = v = 0$ of the surface (3).

d) at the point $u = v = 0$ of the saddle.

8 Assume that the action of the shape operator at the tangent coordinate vectors ∂_u, ∂_v at the given point $\mathbf{p}$ of the surface $\mathbf{r} = \mathbf{r}(u, v)$ is defined by the relations: $S(\partial_u) = 2\partial_u + 2\partial_v$ and $S(\partial_v) = -\partial_u + 5\partial_v$. Calculate principal curvatures, Gaussian and mean curvatures of the surface at this point.

9 Consider on the sphere (1) the points $A = (1, 0, 0)$, $B = (0, 1, 0)$ and $C = (0, 0, 1)$ and arcs of great circles AB, BC and CA.

Find the image $\mathbf{A}_2$ of the vector $\mathbf{A}_1$ under parallel transport along the closed curve ABC.

†**10** Show that there are two straight lines which pass through the point $(3, 4, 12)$ on the saddle $z = xy$ and lie on this saddle.

Show that this is true for an arbitrary point of the saddle.