Composition of morphisms

For Lagrangian $L = L(x, \dot{x})$ we know that

$$\frac{\partial \mathcal{S}(x,y,t)}{\partial y} = q$$

and this implies that

$$\partial \mathcal{S}(x,y,t) \partial x = -p$$

One can see it reversing the time. More carefully this means the following:

$$\mathcal{S}(t,x,y) = \int_0^t d\tau L(x(\tau), \dot{x}(\tau)) d\tau \,,$$

where $x(\tau)$ solution of equations of motion, $x(\tau = 0) = x$, $x(\tau = t) = y$. (We suppose that L does not depend on time) We have that

$$\mathcal{S}(t+\delta t, x+\delta x, y+\delta y) = \mathcal{S}(t,x,y) = \int_0^{t+\delta t} d\tau L\left(x\left(\tau\right) + \varepsilon h\left(\tau\right), \dot{x}\left(\tau\right)\right) d\tau \,,$$

where

$$h(0) = \delta x \,,$$

and

$$x(t+\delta t) + \varepsilon h(t+\delta t) = y(t) + \delta y \,, \text{i.e. } \dot{\text{x}}(\text{t}) + \varepsilon \text{h}(\text{t}) = \delta \text{y}$$

$$\frac{\partial S(q,t)}{\partial t} = H\left(\frac{\partial S}{\partial q}, q\right) = H\left(\frac{x, \partial S}{\partial x}\right) \,.$$