Riemannian Geometry

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1 Riemannian manifolds

1.1 Manifolds. Tensors. (Recalling)

I recall briefly basics of manifolds and tensor fields on manifolds.

An n-dimensional manifold is a space such that in a vicinity of any point one can consider local coordinates $\{x^1, \ldots, x^n\}$ (charts). One can consider different local coordinates. If both coordinates $\{x^1, \ldots, x^n\}$, $\{y^1, \ldots, y^n\}$ are defined in a vicinity of the given point then they are related by bijective transition functions (functions defined on domains in $\mathbf{R}^n$ and taking values in $\mathbf{R}^n$).

$$
\begin{cases}
x^{1'} = x^{1'}(x^1, \ldots, x^n) \\
x^{2'} = x^{2'}(x^1, \ldots, x^n) \\
\ldots \\
x^{n-1'} = x^{n-1'}(x^1, \ldots, x^n) \\
x^{n'} = x^{n'}(x^1, \ldots, x^n)
\end{cases}
$$

We say that manifold is *differentiable* or *smooth* if transition functions are diffeomorphisms, i.e. they are smooth and rank of Jacobian is equal to k, i.e.

$$
\det \begin{pmatrix}
\frac{\partial x^{1'}}{\partial x^1} & \frac{\partial x^{1'}}{\partial x^2} \cdots & \frac{\partial x^{1'}}{\partial x^n} \\
\frac{\partial x^{2'}}{\partial x^1} & \frac{\partial x^{2'}}{\partial x^2} \cdots & \frac{\partial x^{2'}}{\partial x^n} \\
\ldots \\
\frac{\partial x^{n'}}{\partial x^1} & \frac{\partial x^{n'}}{\partial x^2} \cdots & \frac{\partial x^{n'}}{\partial x^n}
\end{pmatrix} \neq 0 \tag{1.1}
$$

A good example of manifold is an open domain D in n-dimensional vector space $\mathbf{R}^n$. Cartesian coordinates on $\mathbf{R}^n$ define global coordinates on D. On the other hand one can consider an arbitrary local coordinates in different domains in $\mathbf{R}^n$.

E.g. one can consider polar coordinates $\{r, \varphi\}$ in a domain $D = \{x, y \colon y > 0\}$ of $\mathbf{R}^2$ (or in other domain of $\mathbf{R}^2$) defined by standard formulae:

$$
\begin{cases}
x = r \cos \varphi \\
y = r \sin \varphi
\end{cases},
$$

$$
\det \begin{pmatrix}
\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\
\frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi}
\end{pmatrix} = \det \begin{pmatrix}
\cos \varphi & -r \sin \varphi \\
\sin \varphi & r \cos \varphi
\end{pmatrix} = r \tag{1.2}
$$

or one can consider spherical coordinates $\{r, \theta, \varphi\}$ in a domain $D = \{x, y, z \colon x > 0, y > 0, z > 0\}$ of $\mathbf{R}^3$ (or in other domain of $\mathbf{R}^3$) defined by standard formulae

$$\begin{cases} x = r \sin \theta \cos \varphi \\ y = r \sin \theta \sin \varphi \\ z = r \cos \theta \end{cases},$$

,

$$\det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{pmatrix} = \det \begin{pmatrix} \sin \theta \cos \varphi & r \cos \theta \cos \varphi & -r \sin \theta \sin \varphi \\ \sin \theta \sin \varphi & r \cos \theta \sin \varphi & r \sin \theta \cos \varphi \\ \cos \theta & -r \sin \theta & 0 \end{pmatrix} = r^2 \sin \theta$$

(1.3)

Choosing domain where polar (spherical) coordinates are well-defined we have to be award that coordinates have to be well-defined and transition functions (1.1) have to be diffeomorphisms.

Examples of manifolds: $\mathbf{R}^n$, Circle S^1, Sphere S^2, in general sphere S^n, torus $S^1 \times S^1$, cylinder, cone, ...

We also have to recall briefly what are tensors on manifold.

Tensors on Manifold

For every point $\mathbf{p}$ on manifold M one can consider tangent vector space $T_{\mathbf{p}}M$—the space of vectors tangent to the manifold at the point M.

Tangent vector $\mathbf{A} = A^i \frac{\partial}{\partial x^i}$. Under changing of coordinates it transforms as follows:

$$\mathbf{A} = A^i \frac{\partial}{\partial x^i} = A^i \frac{\partial x^{m'}}{\partial x^i} \frac{\partial}{\partial x^{m'}} = A^{m'} \frac{\partial}{\partial x^{m'}}$$

Hence

$$A^{i'} = \frac{\partial x^{i'}}{\partial x^i} A^i \tag{1.4}$$

Consider also cotangent space $T_{\mathbf{p}}^* M$ (for every point $\mathbf{p}$ on manifold M)—space of linear functions on tangent vectors, i.e. space of 1-forms which sometimes are called *covectors*.:

One-form (covector) $\omega = \omega_i dx^i$ transforms as follows

$$\omega = \omega_m dx^m = \omega_m \frac{\partial x^m}{\partial x^{m'}} dx^{m'} = \omega_{m'} dx^{m'}.$$

Hence

$$\omega_{m'} = \frac{\partial x^m}{\partial x^{m'}}\omega_m \,. \tag{1.5}$$

Tensors:

One can consider contravariant tensors of the rank p

$$T = T^{i_1 i_2 \ldots i_p} \frac{\partial}{\partial x^{i_1}} \otimes \frac{\partial}{\partial x^{i_2}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}}$$

with components $\{T^{i_1 i_2 \ldots i_k}\}$.

One can consider covariant tensors of the rank q

$$S = S_{j_1 j_2 \ldots j_q} dx^{j_1} \otimes dx^{j_2} \otimes \ldots dx^{j_q}$$

with components $\{S_{j_1 j_2 \ldots j_q}\}$.

One can also consider mixed tensors:

$$Q = Q^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q} \frac{\partial}{\partial x^{i_1}} \otimes \frac{\partial}{\partial x^{i_2}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes dx^{j_2} \otimes \ldots dx^{j_q}$$

with components $\{Q^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}\}$. We call these tensors *tensors of the type* $\begin{pmatrix} p \\ q \end{pmatrix}$.

Tensors of the type $\begin{pmatrix} p \\ 0 \end{pmatrix}$ are called *contravariant tensors of the rank p*.

Tensors of the type $\begin{pmatrix} 0 \\ q \end{pmatrix}$ are called *covariant tensors of the rank q*.

Having in mind (1.4) and (1.5) we come to the rule of transformation for tensors of the type $\begin{pmatrix} p \\ q \end{pmatrix}$:

$$Q^{i'_1 i'_2 \ldots i'_p}_{j'_1 j'_2 \ldots j'_q} = \frac{\partial x^{i'_1}}{\partial x^{i_1}} \frac{\partial x^{i'_2}}{\partial x^{i_2}} \cdots \frac{\partial x^{i'_p}}{\partial x^{i_p}} \frac{\partial x^{j_1}}{\partial x^{j'_1}} \frac{\partial x^{j_2}}{\partial x^{j'_2}} \cdots \frac{\partial x^{j_q}}{\partial x^{j'_q}} Q^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q} \tag{1.6}$$

E.g. if S_{ik} is a covariant tensor of rank 2 (tensor of the type $\begin{pmatrix} p \\ q \end{pmatrix}$) then

$$S_{i'k'} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^k}{\partial x^{k'}} S_{ik} \,. \tag{1.7}$$

If A^i_k is a tensor of rank $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (linear operator on $T_{\mathbf{p}}M$) then

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$$A^{i'}_{k'} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^k}{\partial x^{k'}} A^i_k$$

Remark Transformations formulae (1.4)—(1.7) define vectors, covectors and in generally any tensor fields in components. E.g. covariant tensor (covariant tensor field) of the rank 2 can be defined as matrix S_{ik} (matrix valued function $S_{ik}(x)$) such that under changing of coordinates $\{x^1, x^2, \ldots, x^n\} \mapsto \{x^{1'}, x^{2'}, \ldots, x^{n'}\}$ S_{ik} change by the rule (1.7).

Remark *Einstein summation rules*

In our lectures we always use so called *Einstein summation convention.* it implies that when an index occurs more than once in the same expression in upper and in.. postitions.., the expression is implicitly summed over all possible values for that index. Sometimes it is called dummy indices summation rule.

1.2 Riemannian manifold—manifold equipped with Riemannian metric

Definition The Riemannian manifold is a manifold equipped with a Riemannian metric.

The Riemannian metric on the manifold M defines the length of the tangent vectors and the length of the curves.

Definition Riemannian metric G on n-dimensional manifold M^n defines for every point $\mathbf{p} \in M$ the scalar product of tangent vectors in the tangent space $T_{\mathbf{p}}M$ smoothly depending on the point $\mathbf{p}$.

It means that in every coordinate system $(x^1, \ldots, x^n)$ a metric $G = g_{ik}dx^i dx^k$ is defined by a matrix valued smooth function $g_{ik}(x)$ $(i = 1, \ldots, n; k = 1, \ldots n)$ such that for any two vectors

$$\mathbf{A} = A^i(x)\frac{\partial}{\partial x^i}, \ \mathbf{B} = B^i(x)\frac{\partial}{\partial x^i},$$

tangent to the manifold M at the point $\mathbf{p}$ with coordinates $x = (x^1, x^2, \ldots, x^n)$ $(\mathbf{A}, \mathbf{B} \in T_{\mathbf{p}}M)$ the scalar product is equal to:

$$\langle \mathbf{A}, \mathbf{B}\rangle_G\big|_{\mathbf{p}} = G(\mathbf{A}, \mathbf{B})\big|_{\mathbf{p}} = A^i(x)g_{ik}(x)B^k(x) =$$

$$\left(A^1 \ldots A^n \right) \begin{pmatrix} g_{11}(x) & \ldots & g_{1n}(x) \\ \ldots & \ldots & \ldots \\ g_{n1}(x) & \ldots & g_{nn}(x) \end{pmatrix} \begin{pmatrix} B^1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ B^n \end{pmatrix} \tag{1.8}$$

where

- $G(\mathbf{A}, \mathbf{B}) = G(\mathbf{B}, \mathbf{A})$, i.e. $g_{ik}(x) = g_{ki}(x)$ (symmetricity condition)

- $G(\mathbf{A}, \mathbf{A}) > 0$ if $\mathbf{A} \neq \mathbf{0}$, i.e.

  $g_{ik}(x)u^i u^k \geq 0$, $g_{ik}(x)u^i u^k = 0$ iff $u^1 = \cdots = u^n = 0$ (positive-definiteness)

- $G(\mathbf{A}, \mathbf{B})\big|_{\mathbf{p}=x}$, i.e. $g_{ik}(x)$ are smooth functions.

One can say that Riemannian metric is defined by symmetric covariant smooth tensor field G of the rank 2 which defines scalar product in the tangent spaces $T_{\mathbf{p}}M$ smoothly depending on the point $\mathbf{p}$. *Components of tensor field G in coordinate system are matrix valued functions* $g_{ik}(x)$:

$$G = g_{ik}(x)dx^i \otimes dx^k . \tag{1.9}$$

The matrix $||g_{ik}||$ of components of the metric G we also sometimes denote by G.

Rule of transformation for entries of matrix $g_{ik}(x)$

$g_{ik}(x)$-entries of the matrix $||g_{ik}||$ are components of tensor field G in a given coordinate system.

How do these components transform under transformation of coordinates $\{x^i\} \mapsto \{x^{i'}\}$?

$$G = g_{ik}dx^i \otimes dx^k = g_{ik}\left(\frac{\partial x^i}{\partial x^{i'}}dx^{i'} \right) \otimes \left(\frac{\partial x^k}{\partial x^{k'}}dx^{k'} \right) =$$

$$\frac{\partial x^i}{\partial x^{i'}}g_{ik}\frac{\partial x^k}{\partial x^{k'}}dx^{i'} \otimes dx^{k'} = g_{i'k'}dx^{i'} \otimes dx^{k'}$$

Hence

$$g_{i'k'} = \frac{\partial x^i}{\partial x^{i'}}g_{ik}\frac{\partial x^k}{\partial x^{k'}} . \tag{1.10}$$

One can derive transformations formulae also using general formulae (1.7) for tensors.

Important remark

$$g_{ik} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k} \right\rangle \qquad (1.11)$$

(See (1.40)).

Later by some abuse of notations we sometimes omit the sign of tensor product and write a metric just as

$$G = g_{ik}(x)dx^i dx^k\,.$$

Examples

- $\mathbf{R}^n$ with canonical coordinates $\{x^i\}$ and with metric

$$G = (dx^1)^2 + (dx^2)^2 + \cdots + (dx^n)^2$$

$G = ||g_{ik}|| = \mathrm{diag}\,[1, 1, \ldots, 1]$

Recall that this is a basis example of n-dimensional Euclidean space, where scalar product is defined by the formula:

$$G(\mathbf{X}, \mathbf{Y}) = \langle \mathbf{X}, \mathbf{Y} \rangle = g_{ik}X^iY^k = X^1Y^1 + X^2Y^2 + \cdots + X^nY^n\,.$$

In the general case if $G = ||g_{ik}||$ is an arbitrary symmetric positive-definite metric then

$$G(\mathbf{X}, \mathbf{Y}) = \langle \mathbf{X}, \mathbf{Y} \rangle = g_{ik}X^iY^k\,.$$

One can show that there exist a new basis $\{\mathbf{e}_i\}$ such that in this basis

$$G(\mathbf{e}_i, \mathbf{e}_k) = \delta_{ik}\,.$$

This basis is called orthonormal basis. (See the Lecture notes in Geometry)

Scalar product in vector space defines the *same* scalar product at all the points. In general case for Riemannian manifold scalar product depends on the point.

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- $\mathbf{R}^2$ with polar coordinates in the domain $y > 0$ ($x = r\cos\varphi, y = r\sin\varphi$):

  $dx = \cos\varphi dr - r\sin\varphi d\varphi, dy = \sin\varphi dr + r\cos\varphi d\varphi$. In new coordinates the Riemannian metric $G = dx^2 + dy^2$ will have the following appearance:

  $$G = (dx)^2 + (dy)^2 = (\cos\varphi dr - r\sin\varphi d\varphi)^2 + (\sin\varphi dr + r\cos\varphi d\varphi)^2 = dr^2 + r^2(d\varphi)^2$$

  We see that for matrix $G = L\|g_{ik}\|$

  $$\underbrace{G = \begin{pmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{in cartesian coordinates}}, \qquad \underbrace{G = \begin{pmatrix} g_{rr} & g_{r\varphi} \\ g_{\varphi r} & g_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}}_{\text{in polar coordinates}}$$

- Circle

  Interval $[0, 2\pi)$ in the line $0 \le x < 2\pi$ with Riemannian metric

  $$G = a^2 dx^2 \tag{1.12}$$

  Renaming $x \mapsto \varphi$ we come to habitual formula for metric for circle of the radius a: $x^2 + y^2 = a^2$ embedded in the Euclidean space $\mathbf{E}^2$:

  $$G = a^2 d\varphi^2 \qquad \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \end{cases}, 0 \le \varphi < 2\pi, \tag{1.13}$$

- Cylinder surface

  Domain in $\mathbf{R}^2$ $D = \{(x, y): , \ 0 \le x < 2\pi$ with Riemannian metric

  $$G = a^2 dx^2 + dy^2 \tag{1.14}$$

  We see that renaming variables $x \mapsto \varphi$, $y \mapsto h$ we come to habitual, familiar formulae for metric in standard polar coordinates for cylinder surface of the radius a embedded in the Euclidean space $\mathbf{E}^3$:

  $$G = a^2 d\varphi^2 + dh^2 \qquad \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases}, 0 \le \varphi < 2\pi, -\infty < h < \infty$$

  $$\tag{1.15}$$

- Sphere

    Domain in $\mathbf{R}^2$, $0 < x < 2\pi$, $0 < y < \pi$ with metric $G = dy^2 + \sin^2 y dx^2$
    We see that renaming variables $x \mapsto \varphi$, $y \mapsto h$ we come to habitual,
    familiar formulae for metric in standard spherical coordinates for sphere
    $x^2 + y^2 + z^2 = a^2$ of the radius a embedded in the Euclidean space $\mathbf{E}^3$:

$$G = a^2 d\theta^2 + a^2 \sin^2 \theta d\varphi^2 \qquad \begin{cases} x = a \sin\theta\cos\varphi \\ y = a \sin\theta\sin\varphi z = a\cos\theta \end{cases} , 0 \le \varphi < 2\pi, -\infty < h < \infty \tag{1.16}$$

1.3 Length of the curve

Let $\gamma\colon\ x^i = x^i(t), (i = 1, \ldots, n))\ (a \le t \le b)$ be a curve on the Riemannian
manifold (M, G).

At the every point of the curve the velocity vector (tangent vector) is
defined:

$$\mathbf{v}(t) = \begin{pmatrix} \dot{x}^1(t) \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}^n(t) \end{pmatrix} \tag{1.17}$$

The length of velocity vector $\mathbf{v} \in T_x M$ (vector $\mathbf{v}$ is tangent to the manifold
M at the point x) equals to

$$|\mathbf{v}|_x = \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle_G}\Big|_x = \sqrt{g_{ik}v^i v^k}\Big|_x = \sqrt{g_{ik}\frac{dx^i(t)}{dt}\frac{dx^k(t)}{dt}}\Big|_x\Big|_x \tag{1.18}$$

The length of the curve is defined by the integral of the length of velocity
vector:

$$L_\gamma = \int_a^b \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle_G\big|_{x(t)}}dt = \int_a^b \sqrt{g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t)}dt \tag{1.19}$$

Bearing in mind that metric (1.9) defines the length we often write metric
in the following form

$$ds^2 = g_{ik}dx^i dx^k \tag{1.20}$$

For example consider 2-dimensional Riemannian manifold with metric

$$||g_{ik}(u, v)|| = \begin{pmatrix} g_{11}(u, v) & g_{12}(u, v) \\ g_{21}(u, v) & g_{22}(u, v) \end{pmatrix}.$$

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Then

$$G = ds^2 = g_{ik} du^i dv^k = g_{11}(u,v) du^2 + 2g_{12}(u,v) du dv + g_{22}(u,v) dv^2$$

The length of the curve $\gamma\colon u = u(t), v = v(t)$, where $t_0 \leq t \leq t_1$ according to (1.19) is equal to

$$L_\gamma = \int_{t_0}^{t_1} \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \int_{t_0}^{t_1} \sqrt{g_{ik}(x)\dot{x}^i \dot{x}^k} = \tag{1.21}$$

$$\int_{t_0}^{t_1} \sqrt{g_{11}\left(u\left(t\right),v\left(t\right)\right)u_t^2 + 2g_{12}\left(u\left(t\right),v\left(t\right)\right)u_t v_t + g_{22}\left(u\left(t\right),v\left(t\right)\right)v_t^2}\,dt \tag{1.22}$$

The length of the curve defined by the formula(1.19) sure obeys the natural conditions

- It coincides with the usual length in the Euclidean space $\mathbf{E}^n$ ($\mathbf{R}^n$ with standard metric $G = (dx^1)^2 + \cdots + (dx^n)^2$ in cartesian coordinates). E.g. for 3-dimensional Euclidean space

$$L_\gamma = \int_a^b \sqrt{g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t)}\,dt = \int_a^b \sqrt{(\dot{x}^1(t))^2 + (\dot{x}^2(t))^2 + (\dot{x}^3(t))^2}\,dt \tag{1.23}$$

- It does not depend on parameterisation of the curve

$$L_\gamma = \int_a^b \sqrt{g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t)}\,dt = \int_{a'}^{b'} \sqrt{g_{ik}(x(\tau))\dot{x}^i(\tau)\dot{x}^k(\tau)}\,d\tau, \tag{1.24}$$

where $x^i(\tau) = x^i(t(\tau))$, $a' \leq \tau \leq b'$ while $a \leq t \leq b$.

- It *does not depend on coordinates on Riemannian manifold M*

$$L_\gamma = \int_a^b \sqrt{g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t)}\,dt = \int_a^b \sqrt{g_{i'k'}(x'(t))\dot{x}^{i'}(t)\dot{x}^{k'}(t)}\,dt \tag{1.25}$$

- It is additive: if a curve $\gamma = \gamma_1 + \gamma$, i.e. $\gamma\colon x^i(t), a \leq t \leq b$, $\gamma_1\colon x^i(t), a \leq t \leq c$ and $\gamma_2\colon x^i(t), c \leq t \leq b$ where a point c belongs to the interval (a,b) then

$$L_\gamma = L_{\gamma^1} + L_{\gamma^2}, \text{ i.e.}$$

$$\int_a^b \sqrt{g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t)}dt =$$

$$\int_a^c \sqrt{g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t)}dt + \int_c^b \sqrt{g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t)}dt \qquad (1.26)$$

Conditions (1.23) and (1.26) evidently are obeyed.

Condition (1.24) follows from the fact that

$$\dot{x}^i(\tau) = \frac{dx(t(\tau))}{d\tau} = \frac{dx(t(\tau))}{dt}\frac{dt}{d\tau} = \dot{x}^i(t)\frac{dt}{d\tau}\,.$$

Bearing in mind the above formula we have

$$\int_{a'}^{b'} \sqrt{g_{ik}(x(\tau))\dot{x}^i(\tau)\dot{x}^k(\tau)}d\tau = \int_{a'}^{b'} \sqrt{g_{ik}(x(t(\tau)))\dot{x}^i(t)\dot{x}^k(t)\left(\frac{dt}{d\tau}^2\right)}d\tau =$$

$$\int_{a'}^{b'} \sqrt{g_{ik}(x(t(\tau)))\dot{x}^i(t)\dot{x}^k(t)}\left|\frac{dt}{d\tau}\right|d\tau = \int_a^b \sqrt{g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t)}dt = L_\gamma$$

Condition (1.25) follows from the condition (1.77):

$$\int_a^b \sqrt{g_{i'k'}(x'(t))\dot{x}^{i'}(t)\dot{x}^{k'}(t)}dt = \int_a^b \sqrt{g_{ik}(x(t))\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^k}{\partial x^{k'}}\dot{x}^{i'}(t)\dot{x}^{k'}(t)}dt =$$

$$\int_a^b \sqrt{g_{ik}(x(t))\left(\frac{\partial x^i}{\partial x^{i'}}\dot{x}^{i'}(t)\right)\left(\frac{\partial x^k}{\partial x^{k'}}\dot{x}^{k'}(t)\right)}dt = \int_a^b \sqrt{g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t)}dt\,.$$

$$(1.27)$$

1.4 Riemannian structure on the surfaces embedded in Euclidean space

Let M be a surface embedded in Euclidean space. Let G be Riemannian structure on the manifold M.

Let $\mathbf{X}, \mathbf{Y}$ be two vectors tangent to the surface M at a point $\mathbf{p} \in M$. An External Observer calculate this scalar product viewing these two vectors as vectors in $\mathbf{E}^3$ attached at the point $\mathbf{p} \in \mathbf{E}^3$ using scalar product in $\mathbf{E}^3$. An Internal Observer will calculate the scalar product viewing these two vectors as vectors tangent to the surface M using the Riemannian metric G (see the formula (1.40)). Respectively

If L is a curve in M then an External Observer consider this curve as a curve in $\mathbf{E}^3$, calculate the modulus of velocity vector (speed) and the length of the curve using Euclidean scalar product of ambient space. An Internal Observer ("an aunt") will define the modulus of the velocity vector and the length of the curve using Riemannian metric.

1.4.1 Induced Riemannian metric

Definition We say that Riemannian metric on the embedded surface is induced by the Euclidean structure of the ambient space if External and Internal Observers come to the same results calculating scalar product of tangent vectors.

In this case modulus of velocity vector (speed) and the length of the curve is the same for External and Internal observer.

Let $\mathbf{r} = \mathbf{r}(u, v)$ be parameterisation of the surface M embedded in the Euclidean space:

$$\mathbf{r}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}$$

Tangent vectors at any point (u, v) are

$$\mathbf{r}_u = \frac{\partial \mathbf{r}(u, v)}{\partial u} = \begin{pmatrix} \frac{\partial x(u,v)}{\partial u} \\ \frac{\partial y(u,v)}{\partial u} \\ \frac{\partial z(u,v)}{\partial u} \end{pmatrix} = \frac{\partial x(u, v)}{\partial u} \frac{\partial}{\partial x} + \frac{\partial y(u, v)}{\partial u} \frac{\partial}{\partial y} + \frac{\partial z(u, v)}{\partial u} \frac{\partial}{\partial z}$$

Here as always x, y, z are Cartesian coordinates in $\mathbf{E}^3$. We denote by $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ basic vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$. Scalar product in $\mathbf{E}^3$ is defined as usual

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}, \qquad \text{(orthonormal basis)},$$

$$\langle \mathbf{X}, \mathbf{Y} \rangle = \langle X^1 \mathbf{e}_x + X^2 \mathbf{e}_y + X^3 \mathbf{e}_z, Y^1 \mathbf{e}_x + Y^2 \mathbf{e}_y + Y^3 \mathbf{e}_z \rangle = X^1 Y^1 + X^2 Y^2 + X^3 Y^3.$$
$$(1.28)$$

Remark It is convenient sometimes to denote parameters (u, v) as (u^1, u^2) or u^α $(\alpha = 1, 2)$ and to write $\mathbf{r} = \mathbf{r}(u^1, u^2)$ or $\mathbf{r} = \mathbf{r}(u^\alpha)$ $(\alpha = 1, 2)$ instead $\mathbf{r} = \mathbf{r}(u, v)$

For example consider surface defined by the equation $z - F(x, y) = 0$. It can be parameterised:

$$\mathbf{r}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix} = \begin{pmatrix} u \\ v \\ F(u, v) \end{pmatrix} \qquad (1.29)$$

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$$\mathbf{r}_u = \begin{pmatrix} 1 \\ 0 \\ F_u(u,v) \end{pmatrix}, \quad \mathbf{r}_v = \begin{pmatrix} 0 \\ 1 \\ F_v(u,v) \end{pmatrix} \tag{1.30}$$

Example Consider sphere of radius a $(a > 0)$:

$$\mathbf{r}(\theta, \varphi) = \begin{pmatrix} x(\theta, \varphi) \\ y(\theta, \varphi) \\ z(\theta, \varphi) \end{pmatrix} = \begin{pmatrix} a \sin\theta \cos\varphi \\ a \sin\theta \sin\varphi \\ a \cos\theta \end{pmatrix} . \tag{1.31}$$

$$\mathbf{r}_\theta = \begin{pmatrix} a\cos\theta\cos\varphi \\ a\cos\theta\sin\varphi \\ -a\sin\theta \end{pmatrix}, \quad \mathbf{r}_\varphi = \begin{pmatrix} -R\sin\theta\sin\varphi \\ R\sin\theta\cos\varphi \\ 0 \end{pmatrix} \tag{1.32}$$

Recall the description of tangent vectors by External and Internal Observers.

1.4.2 Internal and external coordinates of tangent vector

Tangent plane

Let $\mathbf{p}$ be an arbitrary point on the surface M. Consider the plane formed by the vectors which are adjusted to the point $\mathbf{p}$ and tangent to the surface M. We call this plane *plane tangent to M at the point $\mathbf{p}$* and denote it by $T_{\mathbf{p}}M$.

Let $\mathbf{r} = \mathbf{r}(u, v)$ be a parameterisation of the surface M. For a point $\mathbf{p} \in M$ one can consider a basis in the tangent plane $T_p M$ adjusted to the parameters u, v. Every vector $\mathbf{X} \in T_p M$ can be expanded over this basis:

$$\mathbf{X} = X_u \mathbf{r}_u + X_v \mathbf{r}_v, \tag{1.33}$$

where X_u, X_v are coefficients, components of the vector $\mathbf{X}$.

Internal Observer views the basis vector $\mathbf{r}_u \in T_p M$, as a velocity vector for the curve $u = u_0 + t, v = v_0$, where (u_0, v_0) are coordinates of the point p. Respectively the basis vector $\mathbf{r}_v \in T_p M$ for an Internal Observer, is velocity vector for the curve $u = u_0, v = v_0 + t$, where (u_0, v_0) are coordinates of the point p.

Let $\mathbf{r} = \mathbf{r}(t)$ be a curve belonging to the surface C, which passes through the point $\mathbf{p}$, $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ and $\mathbf{p} = \mathbf{r}(t_0)$. Then vector

$$\mathbf{r}_t = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}(u(t), v(t))}{dt} \tag{1.34}$$

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belongs to the tangent plane T_pM.

Note that for the vector (1.34) components X_u, X_v are equal to $X_u = u_t, X_v = v_t$ because

$$\mathbf{r}_t = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}(u(t), v(t))}{dt} = u_t\mathbf{r}_u + v_t\mathbf{r}_v \qquad (1.35)$$

An External Observer describes the vector $\mathbf{r}_t$ as a vector in $\mathbf{E}^3$ attached at the point $\mathbf{p}$. The Internal Observer describes this vector as a vector which has components (u_t, v_t) in the basis $\mathbf{r}_u, \mathbf{r}_v$ according to the formula (1.35).

In general consider an arbitrary tangent vector $\mathbf{X} \in T_{\mathbf{p}}M$. Denote $X_u = a$, $X_v = b$

$$\mathbf{X} = X_u\mathbf{r}_u + X_v\mathbf{r}_v = a\mathbf{r}_u + b\mathbf{r}_v = a\begin{pmatrix} x_u(u, v) \\ y_u(u, v) \\ z_u(u, v) \end{pmatrix} + b\begin{pmatrix} x_v(u, v) \\ x_v(u, v) \\ x_v(u, v) \end{pmatrix} = \begin{pmatrix} ax_u(u, v) + bx_v(u, v) \\ ay_u(u, v) + by_v(u, v) \\ az_u(u, v) + bz_v(u, v) \end{pmatrix}$$
$$(1.36)$$

The last column in this formula represents the three components of the vector $\mathbf{X}$ in the ambient space.

The pair (a, b) can be considered as *internal coordinates of the tangent vector* $\mathbf{X}$. An Internal Observer, Ant living on the surface, deals with the vector $\mathbf{X}$ in terms of coordinates (a, b). External observer which contemplates the surface embedded in three-dimensional ambient space deals with vector $\mathbf{X}$ as with vector with three external coordinates (see the last right column in the formula (1.36).)

In condensed notation instead denoting coordinates by (u, v) we often denote them by $u^\alpha = (u^1, u^2)$. Respectively we denote by

$$\mathbf{r}_\alpha = \frac{d\mathbf{r}}{du^\alpha}, \quad \mathbf{r}_u = \mathbf{r}_1, \mathbf{r}_v = \mathbf{r}_2$$

The formula (1.36) for tangent vector field will have the following appearance:

$$\mathbf{X} = X^\alpha\mathbf{r}_\alpha = X^1\mathbf{r}_1 + X^2\mathbf{r}_2, \quad (X^1 = X_u, X^2 = X_v) \qquad (1.37)$$

When using condensed notations we usually omit explicit summation symbols. E.g. we write $u^\alpha\mathbf{r}_\alpha$ instead $\sum_{i=1}^2 u^\alpha\mathbf{r}_\alpha$ or $u^1\mathbf{r}_1 + u^2\mathbf{r}_2$

One can consider also differentials $du^\alpha = (du^1, du^2)$:

$$du^\alpha(\mathbf{r}_\beta) = \delta^\alpha_\beta: \quad du^1(\mathbf{r}_1) = du^2(\mathbf{r}_2) = 1, \, du^1(\mathbf{r}_2) = du^2(\mathbf{r}_1) = 0 \qquad (1.38)$$

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1.4.3 Explicit formulae for induced Riemannian metric (First Quadratic form)

Now we are ready to define the Riemannian metric on the surface induced by metric (scalar product) in ambient Euclidean space.

Let $M: \mathbf{r} = \mathbf{r}(u, v)$ be a surface embedded in $\mathbf{E}^3$. Then since the scalar product of vectors $\mathbf{r}_u, \mathbf{r}_v$ has to be the same in the ambient space and in the Riemannian metric G we have due to (1.11) that

$$\|g\|_M = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} = \begin{pmatrix} \langle \mathbf{r}_u, \mathbf{r}_u \rangle & \langle \mathbf{r}_u, \mathbf{r}_v \rangle \\ l\langle \mathbf{r}_u, \mathbf{r}_v \rangle & \langle \mathbf{r}_v, \mathbf{r}_v \rangle \end{pmatrix}, \quad g_{\alpha\beta} = \langle \mathbf{r}_\alpha, \mathbf{r}_\beta \rangle,$$

$$G_M = g_{\alpha\beta} du^\alpha du^\beta = g_{11} du^2 + 2g_{12} du dv + g_{22} dv^2 \tag{1.39}$$

where $(\,,\,)$ is a scalar product in Euclidean space.

The formula (1.39) is the formula for induced Riemannian metric on the surface—First Quadratic Form.

If $\mathbf{X}, \mathbf{Y}$ are two tangent vectors in the tangent plane T_pC then $G(\mathbf{X}, \mathbf{Y})$ at the point p is equal to scalar product of vectors $\mathbf{X}, \mathbf{Y}$:

$$(\mathbf{X}, \mathbf{Y}) = (X^1\mathbf{r}_1 + X^2\mathbf{r}_2, Y^1\mathbf{r}_1 + Y^2\mathbf{r}_2) = \tag{1.40}$$

$$X^1(\mathbf{r}_1, \mathbf{r}_1)Y^1 + X^1(\mathbf{r}_1, \mathbf{r}_2)Y^2 + X^2(\mathbf{r}_2, \mathbf{r}_1)Y^1 + X^2(\mathbf{r}_2, \mathbf{r}_2)Y^2 =$$

$$X^\alpha(\mathbf{r}_\alpha, \mathbf{r}_\beta)Y^\beta = X^\alpha g_{\alpha\beta} Y^\beta = G(\mathbf{X}, \mathbf{Y})$$

We can come to this formula just transforming differentials. In carteisan coordinates $\langle \mathbf{X}, \mathbf{Y} \rangle = X^1 Y^1 + X^2 Y^2 + X^3 Y^3$, i.e. the Euclidean metric in cartesian coordinates is given by

$$G_{\mathbf{E}^3} = (dx)^2 + (dy)^2 + (dz)^2. \tag{1.41}$$

The condition that Riemannian metric (1.39) is induced by Euclidean scalar product means that

$$G_{\mathbf{E}^3}\big|_{\mathbf{r}=\mathbf{r}(u,v)} = \left((dx)^2 + (dy)^2 + (dz)^2\right)\big|_{\mathbf{r}=\mathbf{r}(u,v)} = G_M = g_{\alpha\beta} du^\alpha du^\beta \tag{1.42}$$

i.e. $\left((dx)^2 + (dy)^2 + (dz)^2\right)\big|_{\mathbf{r}=\mathbf{r}(u,v)} =$

$$\left(\frac{\partial x(u,v)}{\partial u}du + \frac{\partial x(u,v)}{\partial u}du\right)^2 + \left(\frac{\partial x(u,v)}{\partial u}du + \frac{\partial x(u,v)}{\partial u}du\right)^2 + \left(\frac{\partial x(u,v)}{\partial u}du + \frac{\partial x(u,v)}{\partial u}du\right)^2 =$$

$$G_M = g_{\alpha\beta}du^\alpha du^\beta = g_{11}du^2 + 2g_{12}dudv + g_{22}dv^2 = \qquad (1.43)$$

We come to the same formula.

(See the examples of calculations in the next subsection.)

Check explicitly again that length of the tangent vectors and of the curves calculating by External observer (i.e. using Euclidean metric (1.41)) *is the same* as calculating by Internal Observer, aunt (i.e. using the induced Riemannian metric (1.39))

Consider a vector $\mathbf{X} = X^\alpha \mathbf{r}_\alpha = a\mathbf{r}_u + b\mathbf{r}_v$ tangent to the surface M.

Calculate its length by External and Internal Observer.

The square of the length $|\mathbf{X}|$ of this vector calculated by External observer (he calculates using the scalar product in $\mathbf{E}^3$) equals to

$$|\mathbf{X}|^2 = \langle \mathbf{X}, \mathbf{X} \rangle = \langle \mathbf{r}_u + b\mathbf{r}_v, a\mathbf{r}_u + b\mathbf{r}_v \rangle = a^2 \langle \mathbf{r}_u, \mathbf{r}_u \rangle + 2ab \langle \mathbf{r}_u, \mathbf{r}_v \rangle + b^2 \langle \mathbf{r}_v, \mathbf{r}_v \rangle \tag{1.44}$$

where $\langle\,,\,\rangle$ is a scalar product in $\mathbf{E}^3$.

The internal observer will calculate the length using Riemannian metric (1.39):

$$G(\mathbf{X}, \mathbf{X}) = \begin{pmatrix} a, & b \end{pmatrix} \cdot \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & G_{22} \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = g_{11}a^2 + 2g_{12}ab + g_{22}b^2 \qquad (1.45)$$

External observer (person living in ambient space $\mathbf{E}^3$) calculate the length of the tangent vector using formula (1.44). An ant living on the surface calculate length of this vector in internal coordinates using formula (1.45). External observer deals with external coordinates of the vector, ant on the surface with internal coordinates. They come to the same answer.

Let $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ $a \le t \le b$ be a curve on the surface.

Velocity of this curve at the point $\mathbf{r}(u(t), v(t))$ is equal to

$$\mathbf{v} = \mathbf{X} = \xi\mathbf{r}_u + \eta\mathbf{r}_v \text{where } \xi = u_t, \eta = v_t : \quad \mathbf{v} = \tfrac{d\mathbf{r}(t)}{dt} = u_t\mathbf{r}_u + v_t\mathbf{r}_v\,.$$

The length of the curve is equal to

$$L = \int_a^b |\mathbf{v}(t)|dt = \int_a^b \sqrt{\langle \mathbf{v}(t), \mathbf{v}(t) \rangle_{\mathbf{E}^3}}dt = \int_a^b \sqrt{\langle u_t\mathbf{r}_u + v_t\mathbf{r}_v, u_t\mathbf{r}_u + v_t\mathbf{r}_v \rangle_{\mathbf{E}^3}}dt = \tag{1.46}$$

$$\int_a^b \sqrt{\langle \mathbf{r}_u, \mathbf{r}_u \rangle_{\mathbf{E}^3} u_t^2 + 2\langle \mathbf{r}_u, \mathbf{r}_v \rangle_{\mathbf{E}^3} u_t v_t + \langle \mathbf{r}_v, \mathbf{r}_v \rangle_{\mathbf{E}^3} v_t^2}d\tau =$$

$$\int_a^b \sqrt{g_{11}u_t^2 + 2g_{12}u_t v_t + g_{22}v_t^2}\,dt \qquad (1.47)$$

An external observer will calculate the length of the curve using (1.46). An ant living on the surface calculate length of the curve using (1.47) using Riemannian metric on the surface:

$$ds^2 = g_{ik}du^i du^k = g_{11}du^2 + 2g_{12}dudv + g_{22}dv^2 \qquad (1.48)$$

They will come to the same answer.

1.4.4 Induced Riemannian metrics. Examples.

We consider here examples of calculating induced Riemanian metric on some quadratic surfaces in $\mathbf{E}^3$. using calculations for tangent vectors (see (1.39)) or explicitly in terms of differentials (see (1.42) and (1.43)).

First of all consider the general case when a surface M is defined by the equation $z - F(x,y) = 0$. One can consider the following parameterisation of this surface:

$$\mathbf{r}(u,v)\colon \quad \begin{cases} x = u \\ y = v \\ z = F(u,v) \end{cases} \qquad (1.49)$$

Then

$$\mathbf{r}_u = \begin{pmatrix} 1 \\ 0 \\ F_u \end{pmatrix} \quad \mathbf{r}_v = \begin{pmatrix} 0 \\ 1 \\ F_v \end{pmatrix} \qquad (1.50)$$

,

$$(\mathbf{r}_u, \mathbf{r}_u) = 1 + F_u^2, \quad (\mathbf{r}_u, \mathbf{r}_v) = F_u F_v, \quad (\mathbf{r}_v, \mathbf{r}_v) = 1 + F_v^2$$

and induced Riemannina metric (first quadratic form) (1.39) is equal to

$$||g_{\alpha\beta}|| = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_u, \mathbf{r}_u) & (\mathbf{r}_u, \mathbf{r}_v) \\ (\mathbf{r}_u, \mathbf{r}_v) & (\mathbf{r}_v, \mathbf{r}_v) \end{pmatrix} = \begin{pmatrix} 1 + F_u^2 & F_u F_v \\ F_u F_v & 1 + F_v^2 \end{pmatrix} \qquad (1.51)$$

$$G_M = ds^2 = (1 + F_u^2)du^2 + 2F_u F_v dudv + (1 + F_v^2)dv^2 \qquad (1.52)$$

and the length of the curve $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ on C ($a \le t \le b$) can be calculated by the formula:

$$L = \int \int_a^b \sqrt{(1 + F_u^2)u_t^2 + 2F_u F_v u_t v_t + (1 + F_v)^2 v_t^2}\,dt$$

One can calculate (1.53) explicitly using (1.42):

$$G_M = \left(dx^2 + dy^2 + dz^2\right)\big|_{x=u,y=v,z=F(u,v)} = (du)^2 + (dv)^2 + (F_u du + F_v dv)^2 =$$

$$= (1 + F_u^2)du^2 + 2F_u F_v du dv + (1 + F_v^2)dv^2 . \tag{1.53}$$

<center><i>Cylinder</i></center>

Cylinder is given by the equation $x^2 + y^2 = a^2$. One can consider the following parameterisation of this surface:

$$\mathbf{r}(h, \varphi): \quad \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases} \tag{1.54}$$

We have $G_{cylinder} = \left(dx^2 + dy^2 + dz^2\right)\big|_{x=a\cos\varphi, y=a\sin\varphi, z=h} =$

$$= (-a\sin\varphi d\varphi)^2 + (a\cos\varphi d\varphi)^2 + dh^2 = a^2 d\varphi^2 + dh^2 \tag{1.55}$$

The same formula in terms of scalar product of tangent vectors:

$$\mathbf{r}_h = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \mathbf{r}_\varphi = \begin{pmatrix} -a\sin\varphi \\ a\cos\varphi \\ 0 \end{pmatrix} \tag{1.56}$$

,

$$(\mathbf{r}_h, \mathbf{r}_h) = 1, \quad (\mathbf{r}_h, \mathbf{r}_\varphi) = 0, \quad (\mathbf{r}_\varphi, \mathbf{r}_\varphi) = a^2$$

and

$$\|g_{\alpha\beta}\| = \begin{pmatrix} (\mathbf{r}_u, \mathbf{r}_u) & (\mathbf{r}_u, \mathbf{r}_v) \\ (\mathbf{r}_u, \mathbf{r}_v) & (\mathbf{r}_v, \mathbf{r}_v) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a^2 \end{pmatrix} ,$$

$$G = dh^2 + a^2 d\varphi^2 \tag{1.57}$$

and the length of the curve $\mathbf{r}(t) = \mathbf{r}(h(t), \varphi(t))$ on the cylinder ($a \le t \le b$) can be calculated by the formula:

$$L = \int_a^b \sqrt{h_t^2 + a^2 \varphi_t^2}\, dt \tag{1.58}$$

<center><i>Cone</i></center>

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Cone is given by the equation $x^2 + y^2 - k^2 z^2 = 0$. One can consider the following parameterisation of this surface:

$$\mathbf{r}(h, \varphi): \quad \begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases} \tag{1.59}$$

Calculate induced Riemannian metric:
We have

$$G_{conus} = \left(dx^2 + dy^2 + dz^2\right)\big|_{x=kh\cos\varphi, y=kh\sin\varphi, z=h} =$$

$$(k\cos\varphi dh - kh\sin\varphi d\varphi)^2 + (k\sin\varphi dh + kh\cos\varphi d\varphi)^2 + dh^2$$

$$G_{conus} = k^2 h^2 d\varphi^2 + (1+k^2)dh^2, \quad ||g_{\alpha\beta}|| = \begin{pmatrix} 1+k^2 & 0 \\ 0 & k^2 h^2 \end{pmatrix} \tag{1.60}$$

The length of the curve $\mathbf{r}(t) = \mathbf{r}(h(t), \varphi(t))$ on the cone $(a \le t \le b)$ can be calculated by the formula:

$$L = \int_a^b \sqrt{(1+k^2)h_t^2 + k^2 h^2 \varphi_t^2} dt \tag{1.61}$$

Sphere

Sphere is given by the equation $x^2 + y^2 + z^2 = a^2$. Consider the following (standard) parameterisation of this surface:

$$\mathbf{r}(\theta, \varphi): \quad \begin{cases} x = a\sin\theta\cos\varphi \\ y = a\sin\theta\sin\varphi \\ z = a\cos\theta \end{cases} \tag{1.62}$$

Calculate induced Riemannian metric (first quadratic form)

$$G_{S^2} = \left(dx^2 + dy^2 + dz^2\right)\big|_{x=a\sin\theta\cos\varphi, y=a\sin\theta\sin\varphi, z=a\cos\theta} =$$

$$(a\cos\theta\cos\varphi d\theta - a\sin\theta\sin\varphi d\varphi)^2 + (a\cos\theta\sin\varphi d\theta + a\sin\theta\cos\varphi d\varphi)^2 + (-a\sin\theta d\theta)^2 =$$

$$a^2\cos^2\theta d\theta^2 + a^2\sin^2\theta d\varphi^2 + a^2\sin^2\theta d\theta^2 =$$

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$$, \qquad = a^2 d\theta^2 + a^2 \sin^2\theta d\varphi^2, \qquad ||g_{\alpha\beta}|| = \begin{pmatrix} a^2 & 0 \\ 0 & a^2\sin^2\theta \end{pmatrix} \tag{1.63}$$

One comes to the same answer calculating scalar product of tangent vectors:

$$\mathbf{r}_\theta = \begin{pmatrix} a\cos\theta\cos\varphi \\ a\cos\theta\sin\varphi \\ -a\sin\theta \end{pmatrix} \qquad \mathbf{r}_\varphi = \begin{pmatrix} -a\sin\theta\sin\varphi \\ a\sin\theta\cos\varphi \\ 0 \end{pmatrix} \tag{1.64}$$

$$,\qquad (\mathbf{r}_\theta, \mathbf{r}_\theta) = a^2, \quad (\mathbf{r}_h, \mathbf{r}_\varphi) = 0, \quad (\mathbf{r}_\varphi, \mathbf{r}_\varphi) = a^2\sin^2\theta$$

and

$$||g|| = \begin{pmatrix} (\mathbf{r}_u, \mathbf{r}_u) & (\mathbf{r}_u, \mathbf{r}_v) \\ (\mathbf{r}_u, \mathbf{r}_v) & (\mathbf{r}_v, \mathbf{r}_v) \end{pmatrix} =$$

$$\begin{pmatrix} a^2 & 0 \\ 0 & a^2\sin^2\theta \end{pmatrix}, \quad G_{S^2} = ds^2 = a^2 d\theta^2 + a^2\sin^2\theta d\varphi^2$$

The length of the curve $\mathbf{r}(t) = \mathbf{r}(\theta(t), \varphi(t))$ on the sphere of the radius a ($a \le t \le b$) can be calculated by the formula:

$$L = \int_a^b a\sqrt{\theta_t^2 + \sin^2\theta \cdot \varphi_t^2}\, dt \tag{1.65}$$

We considered all quadratic surfaces except paraboloid $z = x^2 - y^2$.
It can be rewritten as $z = xy$ () and is call sometimes "saddle""
$$Saddle\ (paraboloid)$$

Saddle is given by the equation $z - xy = 0$.
Why paraboloid?
Exercise Show that the equation of saddle can be rewritten as $z = x^2 - y^2$

(This surface is a ruled surface containing lines...)
Consider the following (standard) parameterisation of this surface:

$$\mathbf{r}(u, v): \quad \begin{cases} x = u \\ y = v \\ z = uv \end{cases} \tag{1.66}$$

Calculate induced metric:

$$G_{saddle} = \left(dx^2 + dy^2 + dz^2\right)\big|_{x = u\cos\varphi, y = v\sin\varphi, z = uv} =$$

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$$du^2 + dv^2 + (udv + vdu)^2 =$$

$$G_{saddle} = (1 + v^2)du^2 + 2uvdudv + (1 + u^2)dv^2 \,. \tag{1.67}$$

The length of the curve $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ on the sphere of the radius a ($a \le t \le b$) can be calculated by the formula:

$$L = \int_a^b a\sqrt{(1 + v^2)u_t^2 + 2uvu_tv_t + (1 + u^2)v_t^2}\,dt \tag{1.68}$$

One-sheeted and two-sheeted hyperboloids.

Consider surface given by the equation

$$x^2 + y^2 - z^2 = c$$

If $c = 0$ it is a cone. We considered it already above.

If $c > 0$ it is one-sheeted hyperboloid—connected surface in $\mathbf{E}^3$ If $c < 0$ it is two-sheeted hyperboloid— a surface with two sheets $z > 0$ and $z < 0$ L

Consider these cases separately.

1) *One-sheeted hyperboloid:* $x^2 + y^2 - z^2 = a^2$. It is ruled surface. It contains two family of lines....

One-sheeted hyperboloid is given by the equation $x^2 + y^2 - z^2 = a^2$. it is convenient to choose parameterisation:

$$\mathbf{r}(\theta, \varphi)\colon \quad \begin{cases} x = a\cosh\theta\cos\varphi \\ y = a\cosh\theta\sin\varphi \\ z = a\sinh\theta \end{cases} \tag{1.69}$$

$$x^2 + y^2 - z^2 = a^2\cosh^2\theta - a^2\sinh^2\theta = a^2$$

Compare the calculations with calculations for sphere! We changed functions $\cos, \sin$ on $\cosh, \sinh$

Induced Riemannian metric (first quadratic form).

$$G_{HyperbolI} = \left(dx^2 + dy^2 + dz^2\right)\big|_{x=a\cosh\theta\cos\varphi, y=a\cosh\theta\sin\varphi, z=a\sinh\theta} =$$

$$(a\sinh\theta\cos\varphi d\theta - a\cosh\theta\sin\varphi d\varphi)^2 + (a\sinh\theta\sin\varphi d\theta + a\cosh\theta\cos\varphi d\varphi)^2 + (a\cosh\theta d\theta)^2 =$$

$$a^2\sinh^2\theta d\theta^2 + a^2\cosh^2\theta d\varphi^2 + a^2\cosh^2\theta d\theta^2 =$$

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$$, \qquad = a^2(1+2\sinh^2\theta)d\theta^2 + a^2\cosh^2\theta d\varphi^2\,, \qquad ||g_{\alpha\beta}|| = \begin{pmatrix} 1+2\sinh^2\theta & 0 \\ 0 & \cosh^2\theta \end{pmatrix}$$

$$(1.70)$$

For two-sheeted hyperboloid calculatiosn will be very similar.

1) *Two-sheeted hyperboloid*: $z^2 - x^2 - y^2 = a^2$. It is not ruled surface! it is convenient to choose parameterisation:

$$\mathbf{r}(\theta,\varphi)\colon \quad \begin{cases} x = a\sinh\theta\cos\varphi \\ y = a\sinh\theta\sin\varphi \\ z = a\cosh\theta \end{cases} \qquad (1.71)$$

$$z^2 - x^2 - y^2 = a^2\cosh^\theta - a^2\sinh^2\theta = a^2$$

Compare the calculations with calculations for sphere! We changed functions $\cos, \sin$ on $\cosh, \sinh$

Induced Riemannian metric (first quadratic form).

$$G_{HyperbolI} = \left(dx^2 + dy^2 + dz^2\right)\big|_{x=a\sinh\theta\cos\varphi, y=a\sinh\theta\sin\varphi, z=a\cosh\theta} =$$

$$(a\cosh\theta\cos\varphi d\theta - a\sinh\theta\sin\varphi d\varphi)^2 + (a\cosh\theta\sin\varphi d\theta + a\sinh\theta\cos\varphi d\varphi)^2 + (a\sinh\theta d\theta)^2 =$$

$$a^2\cosh^2\theta d\theta^2 + a^2\sinh^2\theta d\varphi^2 + a^2\sinh^2\theta d\theta^2 =$$

$$, \qquad = a^2(1+2\sinh^2\theta)d\theta^2 + a^2\sinh^2\theta d\varphi^2\,, \qquad ||g_{\alpha\beta}|| = \begin{pmatrix} 1+2\sinh^2\theta & 0 \\ 0 & \sinh^2\theta \end{pmatrix}$$

$$(1.72)$$

We calculated examples of induced Rieamnnina structure embedded in Euclidean space almost for all quadratic surfaces.

Quadratic surface is a surface defined by the equation

$$Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz + ex + fy + dz + c = 0$$

One can see that any quadratic surface by affine transformation can be transformed to one of these surfaces

- cylinder (elliptic cylinder) $x^2 + y^2 = 1$

- hyperbolic cylinder: $x^2 - y^2 = 1$)

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- parabolic cylinder $z = x^2$

- paraboloid $x^2 + y^2 = z$

- hyperbolic paraboloid $x^2 - y^2 = z$

- cone $x^2 + y^2 - z^2 = 0$

- sphere $x^2 + y^2 + z^2 = 1$

- one-sheeted hyperboloid

- two-sheeted hyperboloid

(We exclude degenerate cases such as "point" $x^2 + y^2 + z^2 = 0$, planes, e.t.c.)

1.4.5 *Induced metric on two-sheeted hyperboloid embedded in pseudo-Euclidean space.

Consider two-sheeted hyperboloid embedded in pseudo-Euclidean space with pseudo-scalar product defined by bilinear form

$$\langle \mathbf{X}, \mathbf{Y} \rangle_{pseudo} = X^1 Y^1 + X^2 Y^2 - X^3 Y^3 \qquad (1.73)$$

The "pseudoscalar" product is bilinear, symmetric. It is defined by non-degenerate matrix. But it is not positive-definite: The "pseudo-length" of vectors $\mathbf{X} = (a\cos\varphi, a\sin\varphi, \pm a)$ equals to zero:

$$\mathbf{X} = (a\cos\varphi, a\sin\varphi, \pm a) \Rightarrow \langle \mathbf{X}, \mathbf{X} \rangle_{pseudo} = 0, \qquad (1.74)$$

This is not scalar product. The pseudo-Riemannian metric is:

$$G_{pseudo} = dx^2 + dy^2 - dz^2 \qquad (1.75)$$

it turns out that the following remarkable fact occurs:

Proposition *The pseudo-Riemannian metric* (1.75) *in the ambient* 3-*dimensional pseudo-Euclidean space induces Riemannian metric on two-sheeted hyperboloid* $x^2 + y^2 - z^2 = 1$.

Show it. repeat the calculations above for two-sheeted hyperboloid changing in the ambient space Riemannian metric $G = dx^2 + dy^2 + dz^2$ on pseudo-Riemannian $dx^2 + dy^2 - dz^2$:

Using (1.71) and (1.75) we come now to

$$G = \left(dx^2 + dy^2 - dz^2\right)\big|_{x=a\sinh\theta\cos\varphi,\,y=a\sinh\theta\sin\varphi,\,z=a\cosh\theta} =$$

$$(a\cosh\theta\cos\varphi d\theta - a\sinh\theta\sin\varphi d\varphi)^2 + (a\cosh\theta\sin\varphi d\theta + a\sinh\theta\cos\varphi d\varphi)^2 - (a\sinh\theta d\theta)^2 =$$

$$a^2\cosh^2\theta d\theta^2 + a^2\sinh^2\theta d\varphi^2 - a^2\sinh^2\theta d\theta^2$$

$$,\qquad G_L = a^2 d\theta^2 + a^2\sinh^2\theta d\varphi^2\,,\qquad \|g_{\alpha\beta}\| = \begin{pmatrix} 1 & 0 \\ 0 & \sinh^2\theta \end{pmatrix} \qquad (1.76)$$

The two-sheeted hyperboloid equipped with this metric is called hyperbolic or Lobachevsky plane.

Now express Riemannian metric in stereographic coordinates.

Calculations are very similar to the case of stereographic coordinates of 2-sphere $x^2 + y^2 + z^2 = 1$. (See homework 1). Centre of projection $(0, 0, -1)$: For stereographic coordinates u, v we have $\frac{u}{x} = \frac{y}{v} = \frac{1}{1+z}$. We come to

$$\begin{cases} u = \frac{x}{1+z} \\ v = \frac{y}{1+z} \end{cases} \quad,\qquad \begin{cases} x = \frac{2u}{1-u^2-v^2} \\ y = \frac{2v}{1-u^2-v^2} \\ z = \frac{u^2+v^2+1}{1-u^2-v^2} \end{cases} \qquad (4)$$

The image of upper-sheet is an open disc $u^2 + v^2 = 1$ since $u^2 + v^2 = \frac{x^2+y^2}{(1+z)^2} = \frac{z^2-1}{(1+z)^2} = \frac{z-1}{z+1}$. Since for upper sheet $z > 1$ then $0 \le \frac{z-1}{z+1} < 1$.

$$G = (dx^2 + dy^2 - dz^2)\big|_{x=x(u,v),\,y=y(u,v),\,z=z(u,v)} = \left(d\left(\frac{2u}{1-u^2-v^2}\right)\right)^2 +$$

$$\left(d\left(\frac{2v}{1-u^2-v^2}\right)\right)^2 - \left(d\left(\frac{u^2+v^2+1}{1-u^2-v^2}\right)\right)^2 =$$

(Compare with calculations for sphere $x^2 + y^2 + z^2 = 1$). We have $G =$

$$\left(\frac{2du}{1-u^2-v^2} + \frac{2u(2udu+2vdv)}{(1-u^2-v^2)^2}\right)^2 + \left(\frac{2dv}{1-u^2-v^2} + \frac{2v(2udu+2vdv)}{(1-u^2-v^2)^2}\right)^2$$

$$-\left(\frac{2udu+2vdv}{1-u^2-v^2} + \frac{(u^2+v^2+1)(2udu+2vdv)}{(1-u^2-v^2)^2}\right)^2 = \frac{4(du)^2 + 4(dv)^2}{(1+u^2+v^2)^2}$$

To continue calculations it is convenient to denote by $s = 1 - u^2 - v^2$. We come to

$$G = \frac{4}{s^4}\left[\left((1+u^2-v^2)du + 2uvdv\right)^2 + \left((1+v^2-u^2)du + 2uvdu\right)^2 - 4(udu+vdv)^2\right] =$$

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$$\frac{4}{s^4}\left[\left(\left(1+u^2-v^2\right)^2+4u^2v^2-4u^2\right)du^2+(u\leftrightarrow v)+\left(4uv\left(1+u^2-v^2\right)+4uv\left(1+v^2-u^2\right)-8uv\right)du\right.$$

$$\frac{4}{s^4}\left[s^2du^2+s^2dv^2+0\right]=\frac{4du^2+4dv^2}{s^2}=\frac{4du^2+4dv^2}{(1-u^2-v^2)^2}\,.$$

1.5 Isometries of Riemanian manifolds.

Definition Let $(M_1,G_{(1)})$, $(M_2,G_{(2)})$ be two Riemannian manifolds—manifolds equipped with Riemannian metric G_1 and G_2 respectively.

We say that these Riemannian manifolds are *isometric* if there exists a diffeomorphism F (one-one smooth map) such that

$$F^*G_{(2)}=G_{(1)}\,,$$

which means the following:

Let $\mathbf{p}_1$ be an arbitrary point on manifold M_1 and $\mathbf{p}_2\in M_2$ be its image:$F(\mathbf{p}_1)=\mathbf{p}_2$. Let $\{x^i\}$ be coordinates in a vicinity of a point $\mathbf{p}_1\in M_1$ and $\{y^a\}$ be coordinates in a vicinity of a point $\mathbf{p}_2\in M_2$. Let Riemannian metrics G_1 on M_1 has local expression $G_{(1)}=g_{(1)ik}(x)dx^idx^k$ in coordinates $\{x^i\}$ and respectively Riemannian metrics $G_{(2)}$ has local expression $G_2=g_{(2)ab}(y)dy^ady^b$ in coordinates $\{y^i\}$ on M_2. Then

$$g_{(1)ik}(x)dx^idx^k=g_{(2)ab}(y)dy^ady^b=g_{(2)ab}(y(x))\frac{\partial y^a(x)}{\partial x^i}dx^i\frac{\partial y^b(x)}{\partial x^k}dx^k\quad(1.77)$$

i.e.

$$g_{(1)ik}(x)=\frac{\partial y^a(x)}{\partial x^i}g_{(2)ab}(y(x))\frac{\partial y^b(x)}{\partial x^k}\,.\quad(1.78)$$

where $y^i=y^i(x)$ is local expression for diffeomorphism F.

Definition We say that two Riemannian manifolds $(M_1,G_{(1)})$, $(M_2,G_{(2)})$ are *locally isometric* if the following conditions hold:

For arbitrary point $\mathbf{p}_1$ on the manifold M_1 there exists a point $\mathbf{p}_2$ on the manifold M_2 such that there exist coordinates $\{x^i\}$ in a vicinity of a point $\mathbf{p}_1\in M_1$ and coordinates $\{y^a\}$ in a vicinity of a point $\mathbf{p}_2\in M_2$ such that local expression for metric $G_{(1)}$ on M_1 in coordinates $\{x^i\}$ and local expression for metric $G_{(2)}$ on M_2 in coordinates $\{y^a\}$ are related via formulae (1.77), (1.27).

Locally isometric Riemannian manifolds have not to be diffeomorphic. E.g. Euclidean plane is locally isometric to cylinder, but they are not diffeomorphic.

1.5.1 Examples of local isometries

Consider examples.

Example 1 Cylinder-Cone—Plane,

Riemannian metric on cylinder is $G_{cylinder} = a^2 d\varphi^2 + dh^2$ and on the cone $G_{conus} = k^2 h^2 d\varphi^2 + (1 + k^2)dh^2$ (see formulae (1.55) and (1.60)).

To show that cylinder is isometric to Euclidean plan we have to find new local coordinates u, v on the cylinder such that in these coordinates the metric on the cylinder equals to $du^2 + dv^2$. If we put $u = a\varphi$, $v = h$ then

$$du^2 + dv^2 = d(a\varphi)^2 + dh^2 = a^2 d\varphi^2 + dh^2 = G_{cylinder}. \qquad (1.79)$$

Thus we prove the local isometry. Of course the coordinate $u = \varphi$ is not global coordinate on the surface of cylinder. It is evident that cylinder and plane *are not globally isometric* since there are no diffeomorphism of cylinder on the plane. (They are different non-homeomorphic topological spaces.)

Now show that cone is locally isometric to the plane.

This means that we have to find local coordinates u, v on the cone such that in these coordinates induced metric $G|_c$ on cone would have the appearance $G|_c = du^2 + dv^2$.

First of all calculate the metric on cone in natural coordinates h, φ where

$$\mathbf{r}(h, \varphi): \begin{cases} x = kh \cos\varphi \\ y = kh \sin\varphi \\ z = h \end{cases}.$$

$(x^2 + y^2 - k^2 z^2 = k^2 h^2 \cos^2\varphi + k^2 h^2 \sin^2\varphi - k^2 h^2 = k^2 h^2 - k^2 h^2 = 0$.

Calculate metric G_c on the cone in coordinates h, φ induced with the Euclidean metric $G = dx^2 + dy^2 + dz^2$:

$$G_c = \left(dx^2 + dy^2 + dz^2\right)\big|_{x=kh\cos\varphi, y=kh\sin\varphi, z=h} = (k\cos\varphi dh - kh\sin\varphi d\varphi)^2 +$$

$$(k\sin\varphi dh + kh\cos\varphi d\varphi)^2 + dh^2 = (k^2 + 1)dh^2 + k^2 h^2 d\varphi^2.$$

In analogy with polar coordinates try to find new local coordinates u, v such that $\begin{cases} u = \alpha h \cos\beta\varphi \\ v = \alpha h \sin\beta\varphi \end{cases}$, where α, β are parameters. We come to $du^2 + dv^2 =$

$(\alpha\cos\beta\varphi dh - \alpha\beta h\sin\beta\varphi d\varphi)^2 + (\alpha\sin\beta\varphi dh + \alpha\beta h\cos\beta\varphi d\varphi)^2 = \alpha^2 dh^2 + \alpha^2\beta^2 h^2 d\varphi^2$.

Comparing with the metric on the cone $G_c = (1 + k^2)dh^2 + k^2h^2d\varphi^2$ we see that if we put $\alpha = k$ and $\beta = \frac{k}{\sqrt{1+k^2}}$ then $du^2 + dv^2 = \alpha^2dh^2 + \alpha^2\beta^2h^2d\varphi^2 = (1 + k^2)dh^2 + k^2h^2d\varphi^2$.

Thus in new local coordinates

$$\begin{cases} u = \sqrt{k^2 + 1}h\cos\frac{k}{\sqrt{k^2+1}}\varphi \\ v = \sqrt{k^2 + 1}h\sin\frac{k}{\sqrt{k^2+1}}\varphi \end{cases}$$

induced metric on the cone becomes $G|_c = du^2 + dv^2$, i.e. cone locally is isometric to the Euclidean plane ∎

Of course these coordinates are local.— Cone and plane are not homeomorphic, thus they are not globally isometric.

One can show also that

2) Plane with metric $\frac{a(dx^2 + dy^2)}{(1+x^2+y^2)^2}$ is isometric to the sphere with radius $R(a) = ...$

3) Disc with metric $\frac{du^2 + dv^2}{(1-u^2-v^2)^2}$ is isometric to half plane with metric $\frac{dx^2 + dy^2}{4y^2}$.

(see exercises in Homeworks and Coursework.)

1.6 Volume element in Riemannian manifold

The volume element in n-dimensional Riemannian manifold with metric $G = g_{ik}dx^i dx^k$ is defined by the formula

$$\sqrt{\det g_{ik}}\, dx^1 dx^2 \dots dx^n \tag{1.80}$$

If D is a domain in the n-dimensional Riemannian manifold with metric $G = g_{ik}dx^i$ then its volume is equal to to the integral of volume element over this domain.

$$V(D) = \int_D \sqrt{\det g_{ik}}\, dx^1 dx^2 \dots dx^n \tag{1.81}$$

Remark Students who know the concept of exterior forms can read the volume element as

$$\sqrt{\det g_{ik}}\, dx^1 \wedge dx^2 \wedge \dots \wedge dx^n \tag{1.82}$$

Note that in the case of $n = 1$ volume is just the length, in the case if $n = 2$ it is area.

1.6.1 Volume of parallelepiped

Note that the formula (1.80) gives the volume of n-dimensional parallelepiped. Show this. Let $\mathbf{E}^n$ be Euclidean vector space with orthonormal basis $\{\mathbf{e}_i\}$. Let $\mathbf{v}_i$ be an arbitrary basis in this vector space (vectors $\mathbf{v}_i$ in general have not unit length and are not orthogonal to each other). Consider n-parallelepiped spanned by vectors $\{\mathbf{v}_i\}$:

$$\Pi_{\mathbf{v}_i} : \mathbf{r} = t^i \mathbf{v}_i, 0 \le t^i \le 1.$$

The volume of this parallelepiped equals to

$$Vol(\Pi_{\mathbf{v}_i}) = \det \|a_i^m\|, \tag{1.83}$$

where $A = \|a_i^m\|$ is transition matrix, $\mathbf{v}_i = \mathbf{e}_m a_i^m$. On the other hand

$$\mathbf{r} = x^i \mathbf{e}_i = t^m \mathbf{v}_m, \text{ hence } x^i = a_m^i t^m, \text{ where } \mathbf{v}_m = \mathbf{e}_i a_m^i.$$

Let $G = (dx^1)^2 + \cdots + (dx^n)^2 = g_{ik} dt^i dt^k$ be usual Euclidean metric in new coordinates t^i. The

$$G = (dx^1)^2 + \cdots + (dx^n)^2 = dx^i \delta_{ik} dx^k = dt^i \frac{\partial x^{i'}}{\partial t^i} \delta_{i'k'} \frac{\partial x^{k'}}{\partial t^k} dt^k.$$

Since $\frac{\partial x^{i'}}{\partial t^i} = a_i^{i'}$ then

$$g_{ik} = \sum_{i'} a_i^{i'} a_k^{i'} \Rightarrow \det g = (\det A)^2, \det g = \sqrt{\det A}.$$

and according to the formula (1.81)

$$Vol(\Pi_{\mathbf{v}_i}) = \int_{0 \le t^i \le 1} \sqrt{\det g}\, dt^1 dt^2 \ldots dt^n = \det A.$$

We come to (1.83).

Perform these calculations in detail for 3-dimensional case.

E.g. if Euclidean space is 3-dimensional then the parallelepiped spanned by basis vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$

$$\Pi_{\mathbf{a},\mathbf{b},\mathbf{c}} = t^1 \mathbf{a} + t^2 \mathbf{b} + t^3 \mathbf{c}, 0 \le t^1, t^2, t^3 \le 1.$$

Volume of parallelepiped equals to

$$Vol\Pi_{\mathbf{a},\mathbf{b},\mathbf{c}} = \det \begin{pmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{pmatrix}, \tag{1.84}$$

and

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{pmatrix} \begin{pmatrix} t^1 \\ t^2 \\ t^3 \end{pmatrix}.$$

The Riemannian metric in new coordinates (t^1, t^2, t^3) equals to

$$dx^2 + dy^2 + dz^2 = \begin{pmatrix} dt^1 & dt^2 & dt^3 \end{pmatrix} \begin{pmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{pmatrix} \begin{pmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{pmatrix} \begin{pmatrix} dt^1 \\ dt^2 \\ dt^3 \end{pmatrix},$$

i.e. in coordinates t^i Riemannian metric $G = g_{ik} dt^i dt^k$ where

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{pmatrix} \begin{pmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{pmatrix}$$

i.e.

$$\sqrt{\det g_{ik}} = \det \begin{pmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{pmatrix} = Vol\Pi_{\mathbf{a},\mathbf{b},\mathbf{c}}. \tag{1.85}$$

1.6.2 Invariance of volume element under changing of coordinates

Prove that volume element is invariant under coordinate transformations, i.e. if $y^1, \ldots, y^n$ are new coordinates: $x^1 = x^1(y^1, \ldots, y^n), x^2 = x^2(y^1, \ldots, y^n)\ldots,$

$$x^i = x^i(y^p), i = 1, \ldots, n \quad , p = 1, \ldots, n$$

and $\tilde{g}_{pq}(y)$ matrix of the metric in new coordinates:

$$\tilde{g}_{pq}(y) = \frac{\partial x^i}{\partial y^p} g_{ik}(x(y)) \frac{\partial x^k}{\partial y^q}. \tag{1.86}$$

Then

$$\sqrt{\det g_{ik}(x)}\, dx^1 dx^2 \ldots dx^n = \sqrt{\det \tilde{g}_{pq}(y)}\, dy^1 dy^2 \ldots dy^n \tag{1.87}$$

This follows from (1.86). Namely

$$\sqrt{\det g_{ik}(y)}\, dy^1 dy^2 \ldots dy^n = \sqrt{\det \left(\frac{\partial x^i}{\partial y^p} g_{ik}(x(y)) \frac{\partial x^k}{\partial y^q} \right)}\, dy^1 dy^2 \ldots dy^n$$

Using the fact that $\det(ABC) = \det A \cdot \det B \cdot \det C$ and $\det \left(\frac{\partial x^i}{\partial y^p} \right) = \det \left(\frac{\partial x^k}{\partial y^q} \right)$[1] we see that from the formula above follows:

$$\sqrt{\det g_{ik}(y)}\, dy^1 dy^2 \ldots dy^n = \sqrt{\det \left(\frac{\partial x^i}{\partial y^p} g_{ik}(x(y)) \frac{\partial x^k}{\partial y^q} \right)}\, dy^1 dy^2 \ldots dy^n =$$

$$\sqrt{\left(\det \left(\frac{\partial x^i}{\partial y^p} \right) \right)^2}\, \sqrt{\det g_{ik}(x(y))}\, dy^1 dy^2 \ldots dy^n =$$

$$\sqrt{\det g_{ik}(x(y))}\, \det \left(\frac{\partial x^i}{\partial y^p} \right) dy^1 dy^2 \ldots dy^n = \qquad (1.88)$$

Now note that

$$\det \left(\frac{\partial x^i}{\partial y^p} \right) dy^1 dy^2 \ldots dy^n = dx^1 \ldots dx^n$$

according to the formula for changing coordinates in n-dimensional integral [2]. Hence

$$\sqrt{\det g_{ik}(x(y))}\, \det \left(\frac{\partial x^i}{\partial y^p} \right) dy^1 dy^2 \ldots dy^n = \sqrt{\det g_{ik}(x(y))}\, dx^1 dx^2 \ldots dx^n \quad (1.89)$$

Thus we come to (1.87).

1.6.3   Examples of calculating volume element

Consider first very simple example: Volume element of plane in cartesian coordinates, metric $g = dx^2 + dy^2$. Volume element is equal to

$$\sqrt{\det g}\, dxdy = \sqrt{\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}\, dxdy = dxdy$$

---

[1]determinant of matrix does not change if we change the matrix on the adjoint, i.e. change columns on rows.

[2]Determinant of the matrix $\left(\frac{\partial x^i}{\partial y^p} \right)$ of changing of coordinates is called sometimes Jacobian. Here we consider the case if Jacobian is positive. If Jacobian is negative then formulae above remain valid just the symbol of modulus appears.

Volume of the domain D is equal to

$$V(D) = \int_D \sqrt{\det g} \, dx dy = \int_D dx dy$$

If we go to polar coordinates:

$$x = r \cos \varphi, y = r \sin \varphi \tag{1.90}$$

Then we have for metric:
$$G = dr^2 + r^2 d\varphi^2$$

because

$$dx^2 + dy^2 = (dr \cos \varphi - r \sin \varphi d\varphi)^2 + (dr \sin \varphi + r \cos \varphi d\varphi)^2 = dr^2 + r^2 d\varphi^2 \tag{1.91}$$

Volume element in polar coordinates is equal to

$$\sqrt{\det g} \, dr d\varphi = \sqrt{\det \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}} dr d\varphi = dr d\varphi \, .$$

Lobachesvky plane.
In coordinates x, y ($y > 0$) metric $G = \frac{dx^2 + dy^2}{y^2}$, the corresponding matrix $G = \begin{pmatrix} 1/y^2 & 0 \\ 0 & 1/y^2 \end{pmatrix}$. Volume element is equal to $\sqrt{\det g} \, dx dy = \frac{dx dy}{y^2}$.

Sphere in stereographic coordinates Consider the two dimensional plane with Riemannian metrics

$$G = \frac{4(du^2 + dv^2)}{(1 + u^2 + v^2)^2} \tag{1.92}$$

(It is isometric to the sphere without North pole in stereographic coordinates)

Calculate its volume element and volume. It is easy to see that:

$$G = \begin{pmatrix} \frac{1}{(1+u^2+v^2)^2} & 0 \\ 0 & \frac{1}{(1+u^2+v^2)^2} \end{pmatrix} \qquad \det g = \frac{1}{(1 + u^2 + v^2)^4} \tag{1.93}$$

and volume element is equal to $\sqrt{\det g} \, du dv = \frac{du dv}{(1+u^2+v^2)^2}$

The volume (area) of plane will be:

$$\int \frac{dudv}{(1+u^2+v^2)^2} = \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} \frac{du}{(1+u^2+v^2)^2} = \frac{\pi}{2} \int_{-\infty}^{\infty} \frac{du}{(1+u)^{3/2}} = 4\pi$$

We see that in coordinates (u, v) calculation of the integral is not very easy.

One can consider volume form in polar coordinates $u = r\cos\varphi, v = r\sin\varphi$. Then it is easy to see that according to (1.91) we have for the metric $G = \frac{du^2+dv^2}{(1+u^2+v^2)^2} = \frac{dr^2+r^2 d\varphi^2}{(1+r^2)^2}$ and volume form is equal to $\sqrt{\det g}drd\varphi = \frac{rdrd\varphi}{(1+r^2)^2}$
Now calculation of integral becomes easy:

$$V = \int \frac{rdrd\varphi}{(1+r^2)^2} = 2\pi \int_0^{\infty} \frac{rdr}{(1+r^2)^2} = \pi \int_0^{\infty} \frac{du}{(1+u)^2} = \pi \, .$$

Segment of the sphere.
Consider sphere of the radius a in Euclidean space with standard Riemanian metric

$$a^2 d\theta^2 + a^2 \sin^2\theta d\varphi^2$$

This metric is nothing but first quadratic form on the sphere (see (1.4.4)). The volume element is

$$\sqrt{\det g}d\theta d\varphi = \sqrt{\det \begin{pmatrix} a^2 & 0 \\ 0 & a^2\sin\theta \end{pmatrix}}d\theta d\varphi = a^2\sin\theta d\theta d\varphi$$

Now calculate the volume of the segment of the sphere between two parallel planes, i.e. domain restricted by parallels $\theta_1 \leq \theta \leq \theta_0$: Denote by h be the height of this segment. One can see that

$$h = a\cos\theta_0 - a\cos\theta_1 = a(\cos\theta_0 - a\cos\theta_1)$$

There is remarkable formula which express the area of segment via the height h:

$$V = \int_{\theta_1\leq\theta\leq\theta_0} \left(a^2\sin\theta\right)d\theta d\varphi = \int_{\theta^0}^{\theta^1} \left(\int_0^{2\pi} \left(a^2\sin\theta\right)d\varphi\right)d\theta =$$

$$\int_{\theta^1}^{\theta^0} 2\pi a^2\sin\theta d\theta = 2\pi a^2(\cos\theta_0 - \cos\theta_1) = 2\pi a(a\cos\theta_0 - a\cos\theta_1) = 2\pi ah$$

$$(1.94)$$

E.g. for all the sphere $h = 2a$. We come to $S = 4\pi a^2$. It is remarkable formula: area of the segment is a polynomial function of radius of the sphere and height (Compare with formula for length of the arc of the circle)

2 Covariant differentiaion. Connection. Levi Civita Connection on Riemannian manifold

2.1 Differentiation of vector field along the vector field.— Affine connection

How to differentiate vector fields on a (smooth)manifold M?

Recall the differentiation of functions on a (smooth)manifold M.

Let $\mathbf{X} = \mathbf{X^i}(\mathbf{x})\mathbf{e_i}(\mathbf{x}) = \frac{\partial}{\partial \mathbf{x^i}}$ be a vector field on M. Recall that vector field [3] $\mathbf{X} = \mathbf{X^i}\mathbf{e_i}$ defines at the every point x_0 an infinitesimal curve: $x^i(t) = x_0^i + tX^i$ (More exactly the equivalence class $[\gamma(t)]_{\mathbf{X}}$ of curves $x^i(t) = x_0^i + tX^i + \dots$).

Let f be an arbitrary (smooth) function on M and $\mathbf{X} = X^i\frac{\partial}{\partial x^i}$. Then derivative of function f along vector field $\mathbf{X} = X^i\frac{\partial}{\partial x^i}$ is equal to

$$\partial_{\mathbf{X}}f = \nabla_{\mathbf{X}}f = X^i\frac{\partial f}{\partial x^i}$$

The geometrical meaning of this definition is following: If $\mathbf{X}$ is a velocity vector of the curve $x^i(t)$ at the point $x_0^i = x^i(t)$ at the "time" $t = 0$ then the value of the derivative $\nabla_{\mathbf{X}}f$ at the point $x_0^i = x^i(0)$ is equal just to the derivative by t of the function $f(x^i(t))$ at the "time" $t = 0$:

$$\text{if} \quad X^i(x)\big|_{x_0=x(0)} = \frac{dx^i(t)}{dt}\big|_{t=0}, \quad \text{then} \quad \nabla_{\mathbf{X}}f\big|_{x^i=x^i(0)} = \frac{d}{dt}f\left(x^i(t)\right)\big|_{t=0}$$

(2.1)

Remark In the course of Geometry and Differentiable Manifolds the operator of taking derivation of function along the vector field was denoted by "$\partial_{\mathbf{X}}f$". In this course we prefer to denote it by "$\nabla_{\mathbf{X}}f$" to have the uniform notation for both operators of taking derivation of functions and vector fields along the vector field.

One can see that the operation ∇_X on the space $C^\infty(M)$ (space of smooth functions on the manifold) satisfies the following conditions:

- $\nabla_{\mathbf{X}}(\lambda f + \mu g) = a\nabla_{\mathbf{X}}f + b\nabla_{\mathbf{X}}g$ where $\lambda, \mu \in \mathbf{R}$ (linearity over numbers)

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[3] here like always we suppose by default the summation over repeated indices. E.g. $\mathbf{X} = X^i\mathbf{e}_i$ is nothing but $\mathbf{X} = \sum_{i=1}^n X^i\mathbf{e}_i$

- $\nabla_{h\mathbf{X}+g\mathbf{Y}}(f) = h\nabla_{\mathbf{X}}(f) + g\nabla_{\mathbf{Y}}(f)$ (linearity over the space of functions)

- $\nabla_{\mathbf{X}}(\lambda f g) = f\nabla_{\mathbf{X}}(\lambda g) + g\nabla_{\mathbf{X}}(\lambda f)$ (Leibnitz rule)

$$(2.2)$$

Remark One can prove that these properties characterize vector fields:operator on smooth functions obeying the conditions above is a vector field. (May be you remember a statement similar to this from the course of Differentiable Manifolds.)

How to define differentiation of vector fields along vector fields.

The formula (2.1) cannot be generalised straightforwardly because vectors at the point x_0 and $x_0 + tX$ are vectors from different vector spaces. (We cannot substract the vector from one vector space from the vector from the another vector space, because *apriori* we cannot compare vectors from different vector space. One have to define an operation of transport of vectors from the space $T_{x_0}M$ to the point $T_{x_0+tX}M$ defining the transport from the point $T_{x_0}M$ to the point $T_{x_0+tX}M$).

Try to define the operation ∇ on vector fields such that conditions (2.2) above be satisfied.

2.1.1 Definition of connection. Christoffel symbols of connection

Definition Affine connection on M is the *operation* ∇ which assigns to every vector field $\mathbf{X}$ a linear map, (but not necessarily $C(M)$-linear map!) (i.e. a map which is linear over numbers not necessarily over functions) $\nabla_{\mathbf{X}}$ on the space $\mathcal{O}(M)$ of vector fields:

$$\nabla_{\mathbf{X}}(\lambda\mathbf{Y} + \mu\mathbf{Z}) = \lambda\nabla_{\mathbf{X}}\mathbf{Y} + \mu\nabla_{\mathbf{X}}\mathbf{Z}, \qquad \text{for every } \lambda, \mu \in \mathbf{R} \qquad (2.3)$$

(Compare the first condition in (2.2)).
which satisfies the following conditions:

- for arbitrary (smooth) functions f, g on M

$$\nabla_{f\mathbf{X}+\mathbf{g}\mathbf{Y}}(\mathbf{Z}) = f\nabla_{\mathbf{X}}(\mathbf{Z}) + g\nabla_{\mathbf{Y}}(\mathbf{Z}) \qquad (C(M)\text{-linearity}) \qquad (2.4)$$

(compare with second condition in (2.2))

- for arbitrary function f

$$\nabla_{\mathbf{X}}\left(f\mathbf{Y}\right) = \left(\nabla_{\mathbf{X}}f\right)\mathbf{Y} + f\nabla_{\mathbf{X}}\left(\mathbf{Y}\right) \qquad \text{(Leibnitz rule)} \qquad (2.5)$$

Recall that $\nabla_{\mathbf{X}}f$ is just usual derivative of a function f along vector field: $\nabla_{\mathbf{X}}f = \partial_{\mathbf{X}}f$.

(Compare with Leibnitz rule in (2.2)).

The operation $\nabla_{\mathbf{X}}\mathbf{Y}$ is called covariant derivative of vector field $\mathbf{Y}$ along the vector field $\mathbf{X}$.

Write down explicit formulae in a given local coordinates $\{x^i\}$ ($i = 1, 2, \ldots, n$) on manifold M.

Let

$$\mathbf{X} = X^i\mathbf{e}_i = X^i\frac{\partial}{\partial x^i} \quad \mathbf{Y} = Y^i\mathbf{e}_i = Y^i\frac{\partial}{\partial x^i}$$

The basis vector fields $\frac{\partial}{x^i}$ we denote sometimes by ∂_i sometimes by $\mathbf{e}_i$

Using properties above one can see that

$$\nabla_{\mathbf{X}}\mathbf{Y} = \nabla_{X^i\partial_i}Y^k\partial_k = X^i\left(\nabla_i\left(Y^k\partial_k\right)\right), \qquad \text{where } \nabla_i = \nabla_{\partial_i} \qquad (2.6)$$

Then according to (2.4)

$$\nabla_i\left(Y^k\partial_k\right) = \nabla_i\left(Y^k\right)\partial_k + Y^k\nabla_i\partial_k$$

Decompose the vector field $\nabla_i\partial_k$ over the basis ∂_i:

$$\nabla_i\partial_k = \Gamma_{ik}^m\partial_m \qquad (2.7)$$

and

$$\nabla_i\left(Y^k\partial_k\right) = \frac{\partial Y^k(x)}{\partial x^i}\partial_k + Y^k\Gamma_{ik}^m\partial_m, \qquad (2.8)$$

$$\nabla_{\mathbf{X}}\mathbf{Y} = X^i\frac{\partial Y^m(x)}{\partial x^i}\partial_m + X^iY^k\Gamma_{ik}^m\partial_m, \qquad (2.9)$$

In components

$$\left(\nabla_{\mathbf{X}}\mathbf{Y}\right)^m = X^i\left(\frac{\partial Y^m(x)}{\partial x^i} + Y^k\Gamma_{ik}^m\right) \qquad (2.10)$$

Coefficients $\{\Gamma_{ik}^m\}$ are called *Christoffel symbols* in coordinates $\{x^i\}$. These coefficients define covariant derivative—**connection**.

If operation of taking covariant derivative is given we say that the connection is given on the manifold. Later it will be explained why we us the word "connection"

We see from the formula above that to define covariant derivative of vector fields, connection, we have to define Christoffel symbols in local coordinates.

2.1.2 Canonical flat affine connection

It follows from the properties of connection that it is suffice to define connection at vector fields which form basis at the every point using (2.7), i.e. to define Christoffel symbols of this connection.

Example Consider n-dimensional Euclidean space $\mathbf{E}^n$ with cartesian coordinates $\{x^1, \ldots, x^n\}$.

Define connection such that all Christoffel symbols are equal to zero in these cartesian coordinates $\{x^i\}$.

$$\nabla_{\mathbf{e}_i} \mathbf{e}_k = \Gamma_{ik}^m \mathbf{e}_m = 0, \quad \Gamma_{ik}^m = 0 \tag{2.11}$$

Does this mean that Christoffel symbols are equal to zero in an arbitrary cartesian coordinates if they equal to zero in given cartesian coordinates?

Does this mean that Christoffel symbols of this connection equal to zero in arbitrary coordinates system?

To answer these questions note that the relations (2.11) mean that

$$\nabla_{\mathbf{X}} \mathbf{Y} = X^m \frac{\partial Y^i}{\partial x^m} \frac{\partial}{\partial x^i} \tag{2.12}$$

in coordinates $\{x^i\}$

Consider an arbitrary new coordinates $x^{i'} = x^{i'}(x^1, \ldots, x^n)$. Recall the transformation rule for an arbitrary vector field (see subsection 1.1)

$$\mathbf{R} = R^m \frac{\partial}{\partial x^m} = R^m \frac{\partial x^{m'}}{\partial x^m} \frac{\partial}{\partial x^{m'}}, \quad \text{i.e.} R^{m'} = \frac{\partial x^{m'}}{\partial x^m} R^m, \text{and}, R^m = \frac{\partial x^m}{\partial x^{m'}} R^{m'}.$$

Hence we have from (2.12) that

$$\nabla_{\mathbf{X}} \mathbf{Y} = X^m \frac{\partial Y^i}{\partial x^m} \frac{\partial}{\partial x^i} = X^m \frac{\partial}{\partial x^m} \left(Y^i\right) \frac{\partial}{\partial x^i} = X^m \frac{\partial x^{m'}}{\partial x^m} \frac{\partial}{\partial x^{m'}} \left(\frac{\partial x^i}{\partial x^{i'}} Y^{i'}\right) \frac{\partial}{\partial x^i} =$$

$$X^{m'} \frac{\partial}{\partial x^{m'}} \left(\frac{\partial x^i}{\partial x^{i'}} Y^{i'}\right) \frac{\partial}{\partial x^i} = X^{m'} \frac{\partial}{\partial x^{m'}} \left(Y^{i'}\right) \frac{\partial x^i}{\partial x^{i'}} \frac{\partial}{\partial x^i} + X^{m'} \frac{\partial^2 x^i}{\partial x^{m'} \partial x^{i'}} \left(Y^{i'}\right) \frac{\partial}{\partial x^i} =$$

$$X^{m'} \frac{\partial Y^{i'}}{\partial x^{m'}} \frac{\partial}{\partial x^{i'}} + \underbrace{X^{m'} \frac{\partial^2 x^i}{\partial x^{m'} \partial x^{i'}} Y^{i'} \frac{\partial}{\partial x^i}}_{\text{an additional term}} = \qquad (2.13)$$

We see that an additional term equals to zero for arbitrary vector fields $\mathbf{X}, \mathbf{Y}$ if and only if the relations between new and old coordinates are linear:

$$\frac{\partial^2 x^i}{\partial x^{m'} \partial x^{i'}} = 0, \text{ i.e. } x^i = b^i + a^i_k x^k \qquad (2.14)$$

Comparing formulae (2.14) and (2.12) we come to simple but very important

Proposition *Let all Christoffel symbols of a given connection be equal to zero in a given coordinate system $\{x^i\}$. Then all Christoffel symbols of this connection are equal to zero in an arbitrary coordinate system $\{x^{i'}\}$ such that the relations between new and old coordinates are linear:*

$$x^{i'} = b^i + a^i_k x^k \qquad (2.15)$$

If transformation to new coordinate system is not linear, i.e. $\frac{\partial^2 x^i}{\partial x^{m'} \partial x^{i'}} \neq 0$ then Christoffel symbols of this connection in general are not equal to zero in new coordinate system $\{x^{i'}\}$.

Definition We call connection ∇ flat if there exists coordinate system such that all Christoffel symbols of this connection are equal to zero in a given coordinate system.

In particular connection (2.11) has zero Christoffel symbols in arbitrary cartesian coordinates.

Corollary Connection has zero Christoffel symbols in arbitrary cartesian coordinates if it has zero Christoffel symbols in a given cartesian coordinates.

Hence the following definition is correct:

Definition A connection on $\mathbf{E}^n$ which Christoffel symbols vanish in cartesian coordinates is called *canonical flat connection.*

Remark Canonical flat connection in Euclidean space is uniquely defined, sincce cartesian coordinates are defined globally. On the other hand on arbitrary manifold one can define flat connection locally just choosing any arbitrary *local* coordinates and define *locally flat connection* by condition that Christoffel symbols vanish in these local coordinates. We will study this question after learning transformation law for Christoffel symbols.

2.1.3 Transformation of Christoffel symbols for an arbitrary connection

Let ∇ be a connection on manifold M. Let $\{\Gamma^i_{km}\}$ be Christoffel symbols of this connection in given local coordinates $\{x^i\}$. Then according (2.7) and (2.8) we have

$$\nabla_{\mathbf{X}}\mathbf{Y} = X^m \frac{\partial Y^i}{\partial x^m}\frac{\partial}{\partial x^i} + X^m \Gamma^i_{mk} Y^k \frac{\partial}{\partial x^i},$$

and in particularly

$$\Gamma^i_{mk}\partial_i = \nabla_{\partial_m}\partial_k$$

Use this relation to calculate Christoffel symbols in new coordinates $x^{i'}$

$$\Gamma^{i'}_{m'k'}\partial_{i'} = \nabla_{\partial'_m}\partial_{k'}$$

We have that $\partial'_m = \frac{\partial}{\partial x^{m'}} = \frac{\partial x^m}{\partial x^{m'}}\frac{\partial}{\partial x^m} = \frac{\partial x^m}{\partial x^{m'}}\partial_m$. Hence due to properties (2.4), (2.5) we have

$$\Gamma^{i'}_{m'k'}\partial_{i'} = \nabla_{\partial'_m}\partial_{k'} = \nabla_{\partial'_m}\left(\frac{\partial x^k}{\partial x^{k'}}\partial_k\right) = \left(\frac{\partial x^k}{\partial x^{k'}}\right)\nabla_{\partial'_m}\partial_k + \frac{\partial}{\partial x^{m'}}\left(\frac{\partial x^k}{\partial x^{k'}}\right)\partial_k =$$

$$\left(\frac{\partial x^k}{\partial x^{k'}}\right)\nabla_{\frac{\partial x^m}{\partial x^{m'}}\partial_m}\partial_k + \frac{\partial^2 x^k}{\partial x^{m'}\partial x^{k'}}\partial_k = \frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\nabla_{\partial_m}\partial_k + \frac{\partial^2 x^k}{\partial x^{m'}\partial x^{k'}}\partial_k$$

$$\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\Gamma^i_{mk}\partial_i + \frac{\partial^2 x^k}{\partial x^{m'}\partial x^{k'}}\partial_k = \frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\Gamma^i_{mk}\frac{\partial x^{i'}}{\partial x^i}\partial_{i'} + \frac{\partial^2 x^k}{\partial x^{m'}\partial x^{k'}}\frac{\partial x^{i'}}{\partial x^k}\partial_{i'}$$

Comparing the first and the last term in this formula we come to the transformation law:

If $\{\Gamma^i_{km}\}$ are Christoffel symbols of the connection ∇ in local coordinates $\{x^i\}$ and $\{\Gamma^{i'}_{k'm'}\}$ are Christoffel symbols of this connection in new local coordinates $\{x^{i'}\}$ then

$$\Gamma^{i'}_{k'm'} = \frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^i}\Gamma^i_{mk} + \frac{\partial^2 x^k}{\partial x^{m'}\partial x^{k'}}\frac{\partial x^{i'}}{\partial x^k} \tag{2.16}$$

Remark Christoffel symbols do not transform as tensor. If the second term is equal to zero, i.e. transformation of coordinates are linear (see the Proposition on flat connections) then the transformation rule above is the the same as a transformation rule for tensors of the type $\begin{pmatrix}1\\2\end{pmatrix}$ (see the formula

(1.6)). In general case this is not true. Christoffel symbols does not transform as tensor under arbitrary non-linear coordinate transformation: see the second term in the formula above.

Example Consider a connection (2.11) in $\mathbf{E}^2$. It is a flat connection. Calculate Christoffel symbols of this connection in polar coordinates

$$\begin{cases} x = r\cos\varphi \\ y = y\sin\varphi \end{cases} \qquad \begin{cases} r = \sqrt{x^2 + y^2} \\ \varphi = \arctan\frac{y}{x} \end{cases} \tag{2.17}$$

Write down Jacobians of transformations—matrices of partial derivatives:

$$\begin{pmatrix} x_r & y_r \\ x_\varphi & y_\varphi \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -r\sin\varphi & r\cos\varphi \end{pmatrix}, \qquad \begin{pmatrix} r_x & \varphi_x \\ r_y & \varphi_y \end{pmatrix} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & -\frac{y}{x^2+y^2} \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{x}{x^2+y^2} \end{pmatrix} \tag{2.18}$$

According (2.16) and since Chrsitoffel symbols are equal to zero in cartesian coordinates (x, y) we have

$$\Gamma^{i'}_{k'm'} = \frac{\partial^2 x^k}{\partial x^{m'} \partial x^{k'}} \frac{\partial x^{i'}}{\partial x^k}, \tag{2.19}$$

where $(x^1, x^2) = (x, y)$ and $(x^{1'}, x^{2'}) = (r, \varphi)$. Now using (2.18) we have

$$\Gamma^r_{rr} = \frac{\partial^2 x}{\partial r \partial r} \frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial r \partial r} \frac{\partial r}{\partial y} = 0$$

$$\Gamma^r_{r\varphi} = \Gamma^r_{\varphi r} = \frac{\partial^2 x}{\partial r \partial \varphi} \frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial r \partial \varphi} \frac{\partial r}{\partial y} = -\sin\varphi\cos\varphi + \sin\varphi\cos\varphi = 0\,.$$

$$\Gamma^r_{\varphi\varphi} = \frac{\partial^2 x}{\partial \varphi \partial \varphi} \frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial r \partial \varphi} \frac{\partial r}{\partial y} = -x\frac{x}{r} - y\frac{y}{r} = -r\,.$$

$$\Gamma^\varphi_{rr} = \frac{\partial^2 x}{\partial r \partial r} \frac{\partial \varphi}{\partial x} + \frac{\partial^2 y}{\partial r \partial r} \frac{\partial \varphi}{\partial y} = 0\,.$$

$$\Gamma^\varphi_{\varphi r} = \Gamma^\varphi_{r\varphi} = \frac{\partial^2 x}{\partial r \partial \varphi} \frac{\partial \varphi}{\partial x} + \frac{\partial^2 y}{\partial r \partial \varphi} \frac{\partial \varphi}{\partial y} = -\sin\varphi\frac{-y}{r^2} + \cos\varphi\frac{x}{r^2} = \frac{1}{r}$$

$$\Gamma^\varphi_{\varphi\varphi} = \frac{\partial^2 x}{\partial \varphi \partial \varphi} \frac{\partial \varphi}{\partial x} + \frac{\partial^2 y}{\partial \varphi \partial \varphi} \frac{\partial \varphi}{\partial y} = -x\frac{-x}{r^2} - y\frac{y}{r^2} = 0\,. \tag{2.20}$$

Hence we have that the covariant derivative (2.12) in polar coordinates has the following appearance

$$\nabla_r \partial_r = \Gamma^r_{rr}\partial_r + \Gamma^\varphi_{rr}\partial_\varphi = 0\,,, \qquad \nabla_r \partial_\varphi = \Gamma^r_{r\varphi}\partial_r + \Gamma^\varphi_{r\varphi}\partial_\varphi = \frac{\partial_\varphi}{r}$$

$$\nabla_\varphi \partial_r = \Gamma^r_{\varphi r}\partial_r + \Gamma^\varphi_{\varphi r}\partial_\varphi = \frac{\partial_\varphi}{r}, \qquad \nabla_\varphi \partial_\varphi = \Gamma^r_{\varphi\varphi}\partial_r + \Gamma^\varphi_{\varphi\varphi}\partial_\varphi = -r\partial_r \quad (2.21)$$

Remark Later when we study geodesics we will learn a very quick method to calculate Christoffel symbols.

2.1.4 * **Global aspects of existence of connection**

We defined connection as an operation on vector fields obeying the special axioms (see the subsubsection 2.1.1). Then we showed that in a given coordinates connection is defined by Christoffel symbols. On the other hand we know that in general coordinates on manifold are not defined globally. (We had not this trouble in Euclidean space where there are globally defined cartesian coordinates.)

- How to define connection globally using local coordinates?

- Does there exist at least one globally defined connection?

- Does there exist globally defined flat connection?

These questions are not naive questions. Answer on first and second questions is "Yes". It sounds bizzare but answer on the first question is not "Yes" [4]

Global definition of connection

The formula (2.16) defines the transformation for Christoffer symbols if we go from one coordinates to another.

Let $\{(x^i_\alpha), U_\alpha\}$ be an atlas of charts on the manifold M.

If connection ∇ is defined on the manifold M then it defines in any chart (local coordinates) (x^i_α) Christoffer symbols which we denote by $_{(\alpha)}\Gamma^i_{km}$. If (x^i_α), $(x^{i'}_{(\beta)})$ are different local coordinates in a vicinity of a given point then according to (2.16)

$$_{(\beta)}\Gamma^{i'}_{k'm'} = \frac{\partial x^k_{(\alpha)}}{\partial x^{k'}_{(\beta)}}\frac{\partial x^m_{(\alpha)}}{\partial x^{m'}_{(\beta)}}\frac{\partial x^{i'}_{(\beta)}}{\partial x^i_{(\alpha)}}\,_{(\beta)}\Gamma^{(\alpha)i}_{mk} + \frac{\partial^2 x^k_{(\alpha)}}{\partial x^{m'}_{(\beta)}\partial x^{k'}_{(\beta)}}\frac{\partial x^{i'}_{(\beta)}}{\partial x^k_{(\alpha)}} \qquad (2.22)$$

Definition Let $\{(x^i_\alpha), U_\alpha\}$ be an atlas of charts on the manifold M

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[4]Topology of the manifold can be an obstruction to existence of global flat connection. E.g. it does not exist on sphere S^n if $n > 1$.

We say that the collection of Christoffel symbols $\{\Gamma^{(\alpha)i}_{km}\}$ defines globally a connection on the manifold M in this atlas if for every two local coordinates $(x^i_{(\alpha)})$, $(x^i_{(\beta)})$ from this atlas the transformation rules (2.22) are obeyed.

Using partition of unity one can prove the existence of global connection constructing it in explicit way. Let $\{(x^i_\alpha), U_\alpha\}$ $(\alpha = 1, 2, \ldots, N)$ be a finite atlas on the manifold M and let $\{\rho_\alpha\}$ be a partition of unity adjusted to this atlas. Denote by $^{(\alpha)}\Gamma^i_{km}$ local connection defined in domain U_α such that its components in these coordinates are equal to zero. Denote by $^{(\alpha)}_{(\beta)}\Gamma^i_{km}$ Christoffel symbols of this local connection in coordinates $(x^i_{(\beta)})$ $(^{(\alpha)}_{(\beta)}\Gamma^i_{km} = 0)$. Now one can define globally the connection by the formula:

$$_{(\beta)}\Gamma^i_{km}(\mathbf{x}) = \sum_\alpha \rho_\alpha(\mathbf{x}) \, ^{(\alpha)}_{(\beta)}\Gamma^i_{km}(\mathbf{x}) = \sum_\alpha \rho_\alpha(\mathbf{x}) \frac{\partial x^i_{(\beta)}}{\partial x^{i'}_{(\alpha)}} \frac{\partial^2 x^{i'}_{(\alpha)}(\mathbf{x})}{\partial x^k_{(\beta)} \partial x^m_{(\beta)}} \,. \qquad (2.23)$$

This connection in general is not flat connection[5]

2.2 Connection induced on the surfaces

Let M be a manifold (surface) embedded in Euclidean space[6]. Canonical flat connection on $\mathbf{E}^N$ induces the connection on surface in the following way.

Let $\mathbf{X}, \mathbf{Y}$ be tangent vector fields to the surface M and $\nabla^{\mathrm{can.flat}}$ a canonical flat connection in $\mathbf{E}^N$. In general

$$\mathbf{Z} = \nabla^{\mathrm{can.flat}}_{\mathbf{X}}\mathbf{Y} \quad \text{is not tangent to manifold } M \qquad (2.24)$$

Consider its decomposition on two vector fields:

$$\mathbf{Z} = \mathbf{Z}_{tangent} + \mathbf{Z}_\perp, \nabla^{\mathrm{can.flat}}_{\mathbf{X}}, \mathbf{Y} = \left(\nabla^{\mathrm{can.flat}}_{\mathbf{X}}\mathbf{Y}\right)_{tangent} + \left(\nabla^{\mathrm{can.flat}}_{\mathbf{X}}\mathbf{Y}\right)_\perp \,, \quad (2.25)$$

where $\mathbf{Z}_\perp$ is a component of vector which is orthogonal to the surface M and $\mathbf{Z}_{||}$ is a component which is tangent to the surface. Define an induced connection ∇^M on the surface M by the following formula

$$\nabla^M: \qquad \nabla^M_{\mathbf{X}}\mathbf{Y} := \left(\nabla^{\mathrm{can.flat}}_{\mathbf{X}}\mathbf{Y}\right)_{tangent} \qquad (2.26)$$

Remark One can imply this construction for an arbitrary connection in $\mathbf{E}^N$.

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[5] See for detail the text: "Global affine connection on manifold" " in my homepage: "www.maths.mancheser.ac.uk/khudian" in subdirectory Etudes/Geometry

[6] We know that every n-dimensional manifodl can be embedded in $2n + 1$-dimensional Euclidean space

2.2.1 Calculation of induced connection on surfaces in $\mathbf{E}^3$.

Let $\mathbf{r} = \mathbf{r}(u, v)$ be a surface in $\mathbf{E}^3$. Let $\nabla^{\text{can.flat}}$ be a flat connection in $\mathbf{E}^3$. Then

$$\nabla^M: \qquad \nabla^M_{\mathbf{X}} \mathbf{Y} := \left(\nabla^{\text{can.flat}}_{\mathbf{X}} \mathbf{Y}\right)_{||} = \nabla^{\text{can.flat}}_{\mathbf{X}} \mathbf{Y} - \mathbf{n}(\nabla^{\text{can.flat}}_{\mathbf{X}} \mathbf{Y}, \mathbf{n}), \quad (2.27)$$

where $\mathbf{n}$ is normal unit vector field to M. Consider a special example

Example (Induced connection on sphere) Consider a sphere of the radius R in $\mathbf{E}^3$:

$$\mathbf{r}(\theta, \varphi): \quad \begin{cases} x = R \sin\theta \cos\varphi \\ y = R \sin\theta \sin\varphi \\ z = R \cos\theta \end{cases}$$

then

$$\mathbf{r}_\theta = \begin{pmatrix} R\cos\theta\cos\varphi \\ R\cos\theta\sin\varphi \\ -R\sin\theta \end{pmatrix}, \mathbf{r}_\varphi = \begin{pmatrix} -R\sin\theta\sin\varphi \\ R\sin\theta\cos\varphi \\ 0 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} sin\theta\cos\varphi \\ sin\theta\sin\varphi \\ \cos\theta \end{pmatrix},$$

where $\mathbf{r}_\theta = \frac{\partial \mathbf{r}(\theta,\varphi)}{\partial\theta}$, $\mathbf{r}_\varphi = \frac{\partial \mathbf{r}(\theta,\varphi)}{\partial\varphi}$ are basic tangent vectors and $\mathbf{n}$ is normal unit vector.

Calculate an induced connection ∇ on the sphere.

First calculate $\nabla_{\partial_\theta} \partial_\theta$.

$$\nabla_{\partial_\theta} \partial_\theta = \left(\frac{\partial \mathbf{r}_\theta}{\partial\theta}\right)_{tangent} = (\mathbf{r}_{\theta\theta})_{tangent}.$$

On the other hand one can see that $\mathbf{r}_{\theta\theta} = \begin{pmatrix} -Rsin\theta\cos\varphi \\ -Rsin\theta\sin\varphi \\ -R\cos\theta \end{pmatrix} = -R\mathbf{n}$ is proportional to normal vector, i.e. $(\mathbf{r}_{\theta\theta})_{tangent} = 0$. We come to

$$\nabla_{\partial_\theta} \partial_\theta = (\mathbf{r}_{\theta\theta})_{tangent} = 0 \Rightarrow \Gamma^\theta_{\theta\theta} = \Gamma^\varphi_{\theta\theta} = 0. \qquad (2.28)$$

Now calculate $\nabla_{\partial_\theta} \partial_\varphi$ and $\nabla_{\partial_\varphi} \partial_\theta$.

$$\nabla_{\partial_\theta} \partial_\varphi = \left(\frac{\partial \mathbf{r}_\varphi}{\partial\theta}\right)_{tangent} = (\mathbf{r}_{\theta\varphi})_{tangent}, \nabla_{\partial_\varphi} \partial_\theta = \left(\frac{\partial \mathbf{r}_\theta}{\partial\varphi}\right)_{tangent} = (\mathbf{r}_{\varphi\theta})_{tangent}$$

We have

$$\nabla_{\partial_\theta}\partial_\varphi = \nabla_{\partial_\varphi}\partial_\theta = (\mathbf{r}_{\varphi\theta})_{tangent} = \begin{pmatrix} -R\cos\theta\sin\varphi \\ R\cos\theta\cos\varphi \\ 0 \end{pmatrix}_{tangent}.$$

We see that the vector $\mathbf{r}_{\varphi\theta}$ is orthogonal to $\mathbf{n}$:

$$\langle \mathbf{r}_{\varphi\theta}, \mathbf{n}\rangle = -R\cos\theta\sin\varphi\sin\theta\cos\varphi + R\cos\theta\cos\varphi\sin\theta\sin\varphi = 0.$$

Hence

$$\nabla_{\partial_\theta}\partial_\varphi = \nabla_{\partial_\varphi}\partial_\theta = (\mathbf{r}_{\varphi\theta})_{tangent} = \mathbf{r}_{\varphi\theta} = \begin{pmatrix} -R\cos\theta\sin\varphi \\ R\cos\theta\cos\varphi \\ 0 \end{pmatrix} = \cotan\theta\,\mathbf{r}_\varphi\,.$$

We come to

$$\nabla_{\partial_\theta}\partial_\varphi = \nabla_{\partial_\varphi}\partial_\theta = \cotan\theta\,\partial_\varphi \Rightarrow \Gamma^\theta_{\theta\varphi} = \Gamma^\theta_{\varphi\theta} = 0,\ \Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = \cotan\theta \quad (2.29)$$

Finally calculate $\nabla_{\partial_\varphi}\partial_\varphi$

$$\nabla_{\partial_\varphi}\partial_\varphi = (\mathbf{r}_{\varphi\varphi})_{tangent} = \left(\begin{pmatrix} -R\sin\theta\cos\varphi \\ -R\sin\theta\sin\varphi \\ 0 \end{pmatrix}\right)_{tangent}$$

Projecting on the tangent vectors to the sphere (see (2.27)) we have

$$\nabla_{\partial_\varphi}\partial_\varphi = (\mathbf{r}_{\varphi\varphi})_{tangent} = \mathbf{r}_{\varphi\varphi} - \mathbf{n}\langle\mathbf{n}, \mathbf{r}_\varphi\rangle =$$

$$\begin{pmatrix} -R\sin\theta\cos\varphi \\ -R\sin\theta\sin\varphi \\ 0 \end{pmatrix} - \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}(-R\sin\theta\cos\varphi\sin\theta\cos\varphi - R\sin\theta\sin\varphi\sin\theta\sin\varphi) =$$

$$- \sin\theta\cos\theta\begin{pmatrix} R\cos\theta\cos\varphi \\ R\cos\theta\sin\varphi \\ -R\sin\theta \end{pmatrix} = -\sin\theta\cos\theta\,\mathbf{r}_\theta,$$

i.e.

$$\nabla_{\partial_\varphi}\partial_\varphi = -\sin\theta\cos\theta\,\mathbf{r}_\theta \Rightarrow \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta,\ \Gamma^\varphi_{\varphi\varphi} = \Gamma^\varphi_{\varphi\varphi} = 0\,. \qquad (2.30)$$

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2.3 Levi-Civita connection

2.3.1 Symmetric connection

definition. We say that connection is symmetric if its Christoffel symbols Γ^i_{km} are symmetric with respect to lower indices

$$\Gamma^i_{km} = \Gamma^i_{mk} \tag{2.31}$$

The canonical flat connection and induced connections considered above are symmetric connections.

Invariant definition of symmetric connection
A connection ∇ is symmetric if for an arbitrary vector fields $\mathbf{X}, \mathbf{Y}$

$$\nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X}, \mathbf{Y}] = 0 \tag{2.32}$$

If we apply this definition to basic fields ∂_k, ∂_m which commute: $[\partial_k, \partial_m] = 0$ we come to the condition

$$\nabla_{\partial_k}\partial_m - \nabla_{\partial_m}\partial_k = \Gamma^i_{mk}\partial_i - \Gamma^i_{km}\partial_i = 0$$

and this is the condition (2.31).

2.3.2 Levi-Civita connection. Theorem and Explicit formulae

Let (M, G) be a Riemannian manifold.
   Definition. Theorem
   A symmetric connection ∇ is called Levi-Civita connection if it is compatible with metric, i.e. if it preserves scalar product:

$$\partial_{\mathbf{X}}\langle \mathbf{Y}, \mathbf{Z}\rangle = \langle \nabla_{\mathbf{X}}\mathbf{Y}, \mathbf{Z}\rangle + \langle \mathbf{Y}, \nabla_{\mathbf{X}}\mathbf{Z}\rangle \tag{2.33}$$

for arbitrary vector fields $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$.
   There exists unique levi-Civita connection on the Riemannian manifold.
   In local coordinates Christoffel symbols of Levi-Civita connection are given by the following formulae:

$$\Gamma^i_{mk} = \frac{1}{2}g^{ij}\left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^m} - \frac{\partial g_{mk}}{\partial x^j}\right). \tag{2.34}$$

where $G = g_{ik}dx^i dx^k$ is Riemannian metric in local coordinates and $||g^{ik}||$ is the matrix inverse to the matrix $||g_{ik}||$.

Proof

Suppose that this connection exists and Γ^i_{mk} are its Christoffel symbols. Consider vector fields $\mathbf{X} = \partial_m, \mathbf{Y} = \partial_i$ and $\mathbf{Z} = \partial_k$ in (2.33). We have that

$$\partial_m g_{ik} = \langle \Gamma^r_{mi}\partial_r, \partial_k \rangle + \langle \partial_i, \Gamma^r_{mk}\partial_r \rangle = \Gamma^r_{mi}g_{rk} + g_{ir}\Gamma^r_{mk}. \qquad (2.35)$$

for arbitrary indices m, i, k.

Denote by $\Gamma_{mik} = \Gamma^r_{mi}g_{rk}$ we come to

$$\partial_m g_{ik} = \Gamma_{mik} + \Gamma_{mki}, \text{ i.e.}$$

Now using the symmetricity $\Gamma_{mik} = \Gamma_{imk}$ since $\Gamma^k_{mi} = \Gamma^k_{im}$ we have

$$\Gamma_{mik} = \partial_m g_{ik} - \Gamma_{mki} = \partial_m g_{ik} - \Gamma_{kmi} = \partial_m g_{ik} - (\partial_k g_{mi} - \Gamma_{kim}) =$$

$$\partial_m g_{ik} - \partial_k g_{mi} + \Gamma_{kim} = \partial_m g_{ik} - \partial_k g_{mi} + \Gamma_{ikm} = \partial_m g_{ik} - \partial_k g_{mi} + (\partial_i g_{km} - \Gamma_{imk}) =$$

$$\partial_m g_{ik} - \partial_k g_{mi} + \partial_i g_{km} - \Gamma_{mik}.$$

Hence

$$\Gamma_{mik} = \frac{1}{2}(\partial_m g_{ik} + \partial_i g_{mk} - \partial_k g_{mi}) \Rightarrow \Gamma^k_{im} = \frac{1}{2}g^{kr}(\partial_m g_{ir} + \partial_i g_{mr} - \partial_r g_{mi}) \quad (2.36)$$

We see that if this connection exists then it is given by the formula(2.34).

On the other hand one can see that (2.34) obeys the condition (2.35). We prove the uniqueness and existence.

since $\nabla_{\partial_i}\partial_k = \Gamma^m_{ik}\partial_m$.

Consider examples.

2.3.3 Levi-Civita connection on 2-dimensional Riemannian manifold with metric $G = a du^2 + b dv^2$.

Example Consider 2-dimensional manifold with Riemannian metrics

$$G = a(u,v)du^2 + b(u,v)dv^2, \qquad G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} a(u,v) & 0 \\ 0 & b(u,v) \end{pmatrix}$$

Calculate Christoffel symbols of Levi Civita connection.

Using (2.36) we see that:

$$\Gamma_{111} = \tfrac{1}{2}\left(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}\right) = \tfrac{1}{2}\partial_1 g_{11} = \tfrac{1}{2}a_u$$

$$\Gamma_{211} = \Gamma_{121} = \tfrac{1}{2}\left(\partial_1 g_{12} + \partial_2 g_{11} - \partial_1 g_{12}\right) = \tfrac{1}{2}\partial_2 g_{11} = \tfrac{1}{2}a_v$$

$$\Gamma_{221} = \tfrac{1}{2}\left(\partial_2 g_{12} + \partial_2 g_{12} - \partial_1 g_{22}\right) = -\tfrac{1}{2}\partial_1 g_{22} = -\tfrac{1}{2}b_u$$

$$\Gamma_{112} = \tfrac{1}{2}\left(\partial_1 g_{12} + \partial_1 g_{12} - \partial_2 g_{11}\right) = -\tfrac{1}{2}\partial_2 g_{11} = -\tfrac{1}{2}a_v \tag{2.37}$$

$$\Gamma_{122} = \Gamma_{212} = \tfrac{1}{2}\left(\partial_2 g_{21} + \partial_1 g_{22} - \partial_2 g_{21}\right) = \tfrac{1}{2}\partial_1 g_{22} = \tfrac{1}{2}b_u$$

$$\Gamma_{222} = \tfrac{1}{2}\left(\partial_2 g_{22} + \partial_2 g_{22} - \partial_2 g_{22}\right) = \tfrac{1}{2}\partial_2 g_{22} = \tfrac{1}{2}b_v$$

To calculate $\Gamma^i_{km} = g^{ir}\Gamma_{kmr}$ note that for the metric $a(u,v)du^2 + b(u,v)dv^2$

$$G^{-1} = \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{a(u,v)} & 0 \\ 0 & \frac{1}{b(u,v)} \end{pmatrix}$$

Hence

$$\Gamma^1_{11} = g^{11}\Gamma_{111} = \frac{a_u}{2a}, \qquad \Gamma^1_{21} = \Gamma^1_{12} = g^{11}\Gamma_{121} = \frac{a_v}{2a}, \qquad \Gamma^1_{22} = g^{11}\Gamma_{221} = \frac{-b_u}{2a}$$

$$\Gamma^2_{11} = g^{22}\Gamma_{112} = \frac{-a_v}{2b}, \qquad \Gamma^2_{21} = \Gamma^2_{12} = g^{22}\Gamma_{122} = \frac{b_u}{2b}, \qquad \Gamma^2_{22} = g^{22}\Gamma_{222} = \frac{b_v}{2b} \tag{2.38}$$

2.3.4 Example of the sphere again

Calculate Levi-Civita connection on the sphere.

On the sphere first quadratic form (Riemannian metric) $G = R^2 d\theta^2 + R^2 \sin^2\theta d\varphi^2$. Hence we use calculations from the previous example with $a(\theta,\varphi) = R^2, b(\theta,\varphi) = R^2\sin^2\theta$ $(u = \theta, v = \varphi)$. Note that $a_\theta = a_\varphi = b_\varphi = 0$. Hence only non-trivial components of Γ will be:

$$\Gamma^\theta_{\varphi\varphi} = \frac{-b_\theta}{2a} = \frac{-\sin 2\theta}{2}, \qquad \left(\Gamma_{\varphi\varphi\theta} = \frac{-R^2\sin 2\theta}{2}\right), \tag{2.39}$$

$$\Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = \frac{b_\theta}{2b} = \frac{\cos\theta}{\sin\theta} \qquad \left(\Gamma_{\theta\varphi\varphi} = \frac{R^2\sin 2\theta}{2}\right) \tag{2.40}$$

All other components are equal to zero:

$$\Gamma^{\theta}_{\theta\theta} = \Gamma^{\theta}_{\theta\varphi} = \Gamma^{\theta}_{\varphi\theta} = \Gamma^{\varphi}_{\theta\theta} = \Gamma^{\varphi}_{\varphi\varphi} = 0$$

Remark Note that Christoffel symbols of Levi-Civita connection on the sphere coincide with Christoffel symbols of induced connection calculated in the subsection "Connection induced on surfaces". later we will understand the geometrical meaning of this fact.

3 Parallel transport and geodesics

3.1 Parallel transport

3.1.1 Definition

Let M be a manifold equipped with affine connection ∇.

Definition Let C: $\mathbf{x}(t)$ with coordinates $x^i = x^i(t)$, $t_0 \leq t \leq t_1$ be a curve on the manifold M. Let $\mathbf{X} = \mathbf{X}(t_0)$ be a vector attached at the initial point $\mathbf{x}_0$ with coordinates $x^i(t_0)$ of the curve C, i.e. $\mathbf{X}(t_0) \in T_{\mathbf{x}_0}M$ is a vector tangent to the manifold M at the point $\mathbf{x}_0$ with coordinates $x^i(t_0)$.

We say that $\mathbf{X}(t)$, $t_0 \leq t \leq t_1$ is a parallel transport of the vector $\mathbf{X}(t_0) \in T_{\mathbf{x}_0}M$ along the curve C: $x^i = x^i(t), t_0 \leq t \leq t_1$ if

- For an arbitrary t, $t_0 \leq t \leq t$, vector $\mathbf{X} = \mathbf{X}(t)$, $(\mathbf{X}(t)|_{t=t_0} = \mathbf{X}(t_0))$ is a vector attached at the point $\mathbf{x}(t)$ of the curve C, i.e. $\mathbf{X}(t)$ is a vector tangent to the manifold M at the point $\mathbf{x}(t)$ of the curve C.

- The covariant derivative of $\mathbf{X}(t)$ along the curve C equals to zero:

$$\frac{\nabla \mathbf{X}}{dt} = \nabla_{\mathbf{v}} \mathbf{X} = 0 \,. \tag{3.1}$$

In components: if $X^m(t)$ are components of the vector field $\mathbf{X}(t)$ and $v^m(t)$ are components of the velocity vector $\mathbf{v}$ of the curve C ,

$$\mathbf{X}(t) = X^m(t) \frac{\partial}{\partial x^m}|_{\mathbf{x}(t)}, \quad \mathbf{v} = \frac{d\mathbf{x}(t)}{dt} = \frac{dx^i}{dt} \frac{\partial}{\partial x^m}|_{\mathbf{x}(t)}$$

then the condition (3.1) can be rewritten as

$$\frac{dX^i(t)}{dt} + v^k(t)\Gamma^i_{km}(x^i(t))X^m(t) \equiv 0 \,. \tag{3.2}$$

Remark We say sometimes that $\mathbf{X}(t)$ is *covariantly constant along the curve C* If $\mathbf{X}(t)$ is parallel transport of the vector $\mathbf{X}$ along the curve C. If we consider Euclidean space with canonical flat connection then in Cartesian coordinates Christoffel symbols vanish and parallel transport is nothing but $\frac{d\mathbf{X}}{dt} = \nabla_{\mathbf{v}}\mathbf{X} = 0$, $\mathbf{X}(t)$ is constant vector.

Remark Compare this definition of parallel transport with the definition which we consider in the course of "Introduction to Geometry" where we consider parallel transport of the vector along the curve on the surface embedded in $\mathbf{E}^3$ and define parallel transport by the condition, that only orthogonal component of the vector changes during parallel transport, i.e. $\frac{d\mathbf{X}(t)}{dt}$ is a vector orthogonal to the surface.

3.1.2　*Parallel transport is a linear map

Consider two different points $\mathbf{x}_0, \mathbf{x}_1$ on the manifold M with connection ∇. Let C be a curve $\mathbf{x}(t)$ joining these points. The parallel transport (3.1) $\mathbf{X}(t)$ defines the map between tangent vectors at the point $\mathbf{x}_0$ and tangent vectors at the point $\mathbf{x}_1$. This map depends on the curve C. Parallel transport along different curves joining the same points is in general different (if we are not in Euclidean space).

On the other hand parallel transport is a linear map of tangent spaces which *does not depend on the parameterisation of the curve* joining these points.

Proposition *Let $\mathbf{X}(t), t_0 \leq t \leq t_1$ be a parallel transport of the vector $\mathbf{X}(t_0) \in T_{\mathbf{x}_0}M$ along the curve C: $\mathbf{x} = \mathbf{x}(t), t_0 \leq t \leq t_1$, joining the points $\mathbf{x}_0 = \mathbf{x}(t_0)$ and $\mathbf{x}_1 = \mathbf{x}(t_1)$. Then the map*

$$\tau_C: \quad T_{\mathbf{x}_0}M \ni \mathbf{X}(t_0) \longrightarrow \mathbf{X}(t_1) \in T_{\mathbf{x}_1}M \tag{3.3}$$

is a linear map from the vector space $T_{\mathbf{x}_0}M$ to the vector space $T_{\mathbf{x}_1}M$ which does not depend on the parametersiation of the curve C.

The fact that it is a linear map follows immediately from the construction.

The fact that the map (3.3) does not depend on the parameterisation follows from the differential equation (3.2) also.

3.2　Geodesics

3.2.1　Definition. Geodesic on Riemannian manifold.

Let M be manifold equipped with connection ∇.

Definition A parameterised curve C: $\quad x^i = x^i(t)$ is called geodesic if velocity vector $\mathbf{v}(t)$: $v^i(t) = \frac{dx^i(t)}{dt}$ is covariantly constant along this curve, i.e. it remains parallel along the curve:

$$\nabla_{\mathbf{v}}\mathbf{v} = \frac{\nabla \mathbf{v}}{dt} = \frac{dv^i(t)}{dt} + v^k(t)\Gamma^i_{km}(x(t))v^m(t) = 0, \;\; i.e. \qquad (3.4)$$

$$\frac{d^2 x^i(t)}{dt^2} + \frac{dx^k(t)}{dt}\Gamma^i_{km}(x(t))\frac{dx^m(t)}{dt} = 0\,.$$

These are linear second order differential equations. One can prove that this equations have solution and it is unique[7]for an arbitrary initial data $(x^i(t_0) = x^i_0, \dot{x}^i(t_0) = \dot{x}^i_0.$)

Geodesics defined with Levi-Civita connection on the Riemannian manifold is called geodesic on Riemannian manifold. We mostly consider geodesics on Riemannian manifolds.

Proposition *If C: $\mathbf{x}(t)$ is a geodesics on Riemannian manifold then the length of velocity vector is preserved along the geodesic.*

Proof Since the connection is Levi-Civita connection then it preserves scalar product of tangent vectors, (see (2.33)) in particularly the length of the velocity vector $\mathbf{v}$:

$$\mathbf{v} = \frac{d\mathbf{x}(t)}{dt} \Rightarrow \frac{d}{dt}\langle \mathbf{v}, \mathbf{v} \rangle = \left\langle \frac{\nabla \mathbf{v}}{dt}, \mathbf{v} \right\rangle + \left\langle \mathbf{v}, \frac{\nabla \mathbf{v}}{dt} \right\rangle = 0 \Rightarrow |\mathbf{v}| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = 0\,.$$
$$(3.5)$$

In fact one can see that if $\mathbf{X}(t)$ is a parallel transport of an arbitrary vector (not necessary tangent to the curve) along an arbitrary curve C on the Riemannian manifold then the length of the vector $\mathbf{X}(t)$ does not change, since the connection is Levi-Civita connection preserves scalar product:

$$\frac{d}{dt}\langle \mathbf{X}(t), \mathbf{X}(t) \rangle = 2\left\langle \frac{\nabla \mathbf{X}(t)}{dt}, \mathbf{X}(t) \right\rangle = 0 \Rightarrow |\mathbf{X}(t)| = \sqrt{\langle \mathbf{X}(t), \mathbf{X}(t) \rangle} = const.$$
$$(3.6)$$

3.2.2   Un-parameterised geodesic

We defined a geodesic as a parameterised curve such that the velocity vector is covariantly constant along the curve.

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[7]this is true under additional technical conditions which we do not discuss here

What happens if we change the parameterisation of the curve?

Another question: Suppose a tangent vector to the curve remains tangent during parallel transport. Is it true that this curve (in a suitable parameterisation) becomes geodesic?

Definition We call un-parameterised curve geodesic if under suitable parameterisation it obeys the equation (3.4) for geodesics.

Let C–be un-parameterised geodesic. Then the following statement is valid.

Proposition *A curve C (un-parameterised) is geodesic if an only if a tangent non-zero vector remains tangent during parallel transport.*

Proof. Let C be un-parameterised geodesic. Let $x^i = x^i(t)$ be an arbitrary parameterisation of this curve (not necessary the parametrisation such that equations (3.4) obey!). Since C is geodesic there exists a reparameterisation $t = t(\tau)$ such that $x_0^i(\tau) = x^i(t(\tau))$ obeys the equations (3.4) for geodesics.

Consider velocity vector $v^i(t) = \frac{dx^i(t)}{dt}$. In general the equations (3.4) are not obeyed for the velocity vector. (They are obeyed for the velocity vector in the right parameteristaion $x_0^i(\tau) = x^i(t(\tau))$.)

Prove that nevertheless the velocity vector $v^i(t) = \frac{dx^i(t)}{dt}$ remains tangent during parallel transport. Consider differential equation (3.4) of parameterised geodesic $x_0^i(\tau) = x^i(t(\tau))$:

$$\frac{d}{d\tau}\left(v_0^i(\tau)\right) + v_0^k(\tau)\Gamma_{km}^i v_0^m(\tau) = 0 \,. \tag{3.7}$$

Express in this diff. equation the velocity $v_0^i(\tau)$ via the velocity $v^i(t)$ in the initial parameterisation:

$$v_0^i(\tau) = \frac{dx_0^i(\tau)}{d\tau} = \frac{dt(\tau)}{d\tau}\frac{dx(t)}{dt} = t_\tau v^i(t) \,.$$

We have

$$\frac{d}{d\tau}\left(v_0^i(\tau)\right) + v_0^k(\tau)\Gamma_{km}^i v_0^m(\tau) = t_\tau\frac{d}{dt}\left(t_\tau v^i(t)\right) + t_\tau v^k(t)\Gamma_{km}^i t_\tau v^m(t) = 0 \,.$$

Dividing by t_τ and denoting by $a(t)$ the function $t_\tau = \frac{dt}{d\tau}$: $a(t) = \frac{dt}{d\tau}$ we come to

$$\frac{d}{dt}\left(a(t)v^i(t)\right) + v^k(t)\Gamma_{km}^i\left(a(t)v^m(\mathbf{t})\right) = \frac{\nabla\left((a(t)\mathbf{v}(t))\right)}{dt} = 0 \,. \tag{3.8}$$

We see that the vector field $a(t)\mathbf{v}(t)$ is covariantly constant on the geodesic, i.e it is parallel transport of the initial vector $a(t_0)\mathbf{v}(t_0)$, where t_0 is an initial point of the curve. Hence the vector field $\frac{a(t)\mathbf{v}(t)}{a(t_0)}$ is a a parallel transport of the vector $\mathbf{v}(t_0)$.

Since velocity vector remains tangent to the curve during parallel transport, hence an arbitrary tangent vector, $\mathbf{X} = c\mathbf{v}(t)$ remains tangent during parallel transport.

One can easy prove the converse implication too: if velocity vector of the curve $x^i(t)$ remains tangent vector during parallel transport then there exists a parameterisation such that equations (3.4) are obeyed. Indeed suppose that the velocity vector $v^i(t)$ remains tangent vector during parallel transport, i.e. there exists a function $a(t)$ such that the vector field $\mathbf{X}(t) = a(t)\mathbf{v}(t)$ obeys the equations (3.8). Then choosing a parameter τ such that $\frac{dt(\tau)}{d\tau} = a(t)$ we come to the parameterisation such that the equations (3.7) are obeyed. ∎

Remark One can consider more compact proof using the fact that the map (3.3) of parallel transport does not depend on parameterisation.

Remark One can see that if $x^i = x^i(t)$ is geodesic in an arbitrary parameterisation and $s = s(t)$ is a natural parameter (which defines the length of the curve) then $x^i(t(s))$ is parameterised geodesic.

3.3 Geodesics and Lagrangians of "free" particle on Riemannian manifold.

3.3.1 Lagrangian and Euler-Lagrange equations

A function $L = L(x, \dot{x})$ on points and velocity vectors on manifold M is a *Lagrangian* on manifold M.

We assign to Lagrangian $L = L(x, \dot{x})$ the following second order differential equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{\partial L}{\partial x^i} \tag{3.9}$$

In detail

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{\partial^2 L}{\partial x^m \partial \dot{x}^i}\dot{x}^m + \frac{\partial^2 L}{\partial \dot{x}^m \partial \dot{x}^i}\ddot{x}^m = \frac{\partial L}{\partial x^i}. \tag{3.10}$$

These equations are called *Euler-Lagrange equations* of the Lagrangian L. We will explain later the variational origin of these equations [8].

3.3.2 Lagrangian of "free" particle

Let (M, G), $G = g_{ik}dx^i dx^k$ be a Riemannian manifold.

Definition We say that *Lagrangian* $L = L(x, \dot{x})$ is the Lagrangian of a "free" particle on the Riemannian manifold M if

$$L = \frac{g_{ik}\dot{x}^i\dot{x}^k}{2} \qquad (3.11)$$

Example "Free" particle in Euclidean space. Consider $\mathbf{E}^3$ with standard metric $G = dx^2 + dy^2 + dz^2$

$$L = \frac{g_{ik}\dot{x}^i\dot{x}^k}{2} = \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{2} \qquad (3.12)$$

Note that this is the Lagrangian that describes the dynamics od free particle.

Example "Free" particle on sphere.

The metric on the sphere of radius R is $G = R^2 d\theta^2 + R^2 \sin^2\theta d\varphi^2$. Respectively for the Lagrnagian of "free" particle we have

$$L = \frac{g_{ik}\dot{x}^i\dot{x}^k}{2} = \frac{R^2\dot{\theta}^2 + R^2\sin^2\theta\dot{\varphi}^2}{2} \qquad (3.13)$$

3.3.3 Equations of geodesics and Euler-Lagrange equations

Theorem. *Euler-Lagrange equations of Lagrangian of free particle are equivalent to the second order differential equations for geodesics.*

This Theorem makes very easy calculations for Christoffel indices.

This Theorem can be proved by direct calculations.

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[8]To every mechanical system one can put in correspondence a Lagrangian on configuration space. The dynamics of the system is described by Euler-Lagrange equations. The advantage of Lagrangian approach is that it works in an arbitrary coordinate system: Euler-Lagrange equations are invariant with respect to changing of coordinates since they arise from variational principe.

Calculate Euler-Lagrange equations (3.9) for the Lagrangian (3.11):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{d}{dt}\left(\frac{\partial\left(\frac{g_{mk}\dot{x}^m\dot{x}^k}{2}\right)}{\partial \dot{x}^i}\right) = \frac{d}{dt}\left(g_{ik}\dot{x}^k\right) = g_{ik}\ddot{x}^{\ k} + \frac{\partial g_{ik}}{\partial x^m}\dot{x}^m\dot{x}^k$$

and

$$\frac{\partial L}{\partial x^i} = \frac{\partial\left(\frac{g_{mk}\dot{x}^m\dot{x}^k}{2}\right)}{\partial x^i} = \frac{1}{2}\frac{\partial g_{mk}}{\partial x^i}\dot{x}^m\dot{x}^k\,.$$

Hence we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = g_{ik}\ddot{x}^{\ k} + \frac{\partial g_{ik}}{\partial x^m}\dot{x}^m\dot{x}^k = \frac{\partial L}{\partial x^i} = \frac{1}{2}\frac{\partial g_{mk}}{\partial x^i}\dot{x}^m\dot{x}^k\,,$$

i.e.

$$g_{ik}\ddot{x}^{\ k} + \partial_m g_{ik}\dot{x}^m\dot{x}^k = \frac{1}{2}\partial_i g_{mk}\dot{x}^m\dot{x}^k\,.$$

Note that $\partial_m g_{ik}\dot{x}^m\dot{x}^k = \frac{1}{2}\left(\partial_m g_{ik}\dot{x}^m\dot{x}^k + \partial_k g_{im}\dot{x}^m\dot{x}^k\right)$. Hence we come to equation:

$$g_{ik}\frac{d^2 x^k}{dt^2} + \frac{1}{2}\left(\partial_m g_{ik} + \partial_k g_{im} - \partial_i g_{mk}\right)\dot{x}^m\dot{x}^k$$

Multiplying on the inverse matrix g^{ik} we come

$$\frac{d^2 x^i}{dt^2} + \frac{1}{2}g^{ij}\left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^m} - \frac{\partial g_{mk}}{\partial x^j}\right)\frac{dx^m}{dt}\frac{dx^k}{dt} = 0\,. \qquad (3.14)$$

We recognize here Christoffel symbols of Levi-Civita connection (see (2.34)) and we rewrite this equation as

$$\frac{d^2 x^i}{dt^2} + \frac{dx^m}{dt}\Gamma^i_{mk}\frac{dx^k}{dt} = 0\,. \qquad (3.15)$$

This is nothing but the equation (3.4).

Applications of this Theorem: calculation of Christoffel symbols of Levi-Civita connection.

3.3.4 Examples of calculations of Christoffel symbols and geodesics using Lagrangians.

Consider two examples: We calculate Levi-Civita connection on sphere in $\mathbf{E}^3$ and on Lobachevsky plane using Lagrangians and find geodesics.

1) *Sphere of the radius R in $\mathbf{E}^3$*:

Lagrangian of "free" particle on the sphere is given by (3.13):

$$L = \frac{R^2\dot\theta^2 + R^2\sin^2\theta\,\dot\varphi^2}{2}$$

Euler-Lagrange equations defining geodesics are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) - \frac{\partial L}{\partial\theta} = \frac{d}{dt}\left(R^2\dot\theta\right) - R^2\sin\theta\cos\theta\,\dot\varphi^2 \Rightarrow \ddot\theta - \sin\theta\cos\theta\,\dot\varphi^2 = 0\,,$$
$$(3.16)$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\varphi}\right) - \frac{\partial L}{\partial\varphi} = \frac{d}{dt}\left(R^2\sin^2\theta\,\dot\varphi\right) = 0 \Rightarrow \ddot\varphi + \cotan\theta\,\dot\theta\,\dot\varphi = 0\,.$$

Comparing Euler-Lagrange equations with equations for geodesic in terms of Christoffel symbols:

$$\ddot\theta + \Gamma^\theta_{\theta\theta}\dot\theta^2 + 2\Gamma^\theta_{\theta\varphi}\dot\theta\dot\varphi + \Gamma^\theta_{\varphi\varphi}\dot\varphi^2 = 0,$$

$$\ddot\varphi + \Gamma^\varphi_{\theta\theta}\dot\theta^2 + 2\Gamma^\varphi_{\theta\varphi}\dot\theta\dot\varphi + \Gamma^\varphi_{\varphi\varphi}\dot\varphi^2 = 0$$

we come to

$$\Gamma^\theta_{\theta\theta} = \Gamma^\theta_{\theta\varphi} = \Gamma^\theta_{\varphi\theta} = 0\,, \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta\,, \qquad (3.17)$$

$$\Gamma^\varphi_{\theta\theta} = \Gamma^\varphi_{\varphi\varphi} = 0,\ \Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = \cotan\theta\,. \qquad (3.18)$$

(Compare with previous calculations for connection in subsections 2.2.1 and 2.3.4)

To find geodesics one have to solve second order differential equations (3.16):

One can see that the great circles: $\varphi = \varphi_0$, $\theta = \theta_0 + t$ are solutions of second order differential equations (3.16) with initial conditions

$$\theta(t)\big|_{t=0} = \theta_0, \dot\theta(t)\big|_{t=0} = 1, \quad \varphi(t)\big|_{t=0} = \varphi_0, \dot\theta(t)\big|_{t=0} = 0\,. \qquad (3.19)$$

The rotation of the sphere is isometry, which does not change Levi-Civta connection. Hence an arbitrary great circle is geodesic.

Prove that an arbitrary geodesic is an arc of great circle. Let the curve $\theta = \theta(t), \varphi = \varphi(t)$, $0 \leq t \leq t_1$ be geodesic. Rotating the sphere we can come to the curve $\theta = \theta'(t), \varphi = \varphi'(t)$, $0 \leq t \leq t_1$ such that velocity vector at the initial time is direccted along meridian, i.e. initial conditions are

$$\theta'(t)\big|_{t=0} = \theta_0, \dot{\theta}'(t)\big|_{t=0} = a, \quad \varphi'(t)\big|_{t=0} = \varphi_0, \dot{\varphi}'(t)\big|_{t=0} = 0. \qquad (3.20)$$

(Compare with initial conditions (3.19)) Second order differential equations with boundary conditions for coordinates and velocities at $t = 0$ have unique solution. The solutions of second order differential equations (3.16) with initial conditions (3.20) is a curve $\theta'(t) = \theta_0 + at$, $\varphi'(t) = \varphi_0$. It is great circle. Hence initial curve the geodesic $\theta = \theta(t), \varphi = \varphi(t)$, $0 \leq t \leq t_1$ is an arc of great circle too.

We proved that all geodesics are great circles.

2) *Lobachevsky plane.*

Lagrangian of "free" particle on the Lobachevsky plane with metric $G = \frac{dx^2 + dy^2}{y^2}$ is

$$L = \frac{1}{2}\frac{\dot{x}^2 + \dot{y}^2}{y^2}.$$

Euler-Lagrange equations are

$$\frac{\partial L}{\partial x} = 0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{d}{dt}\left(\frac{\dot{x}}{y^2}\right) = \frac{\ddot{x}}{y^2} - \frac{2\dot{x}\dot{y}}{y^3}, \text{i.e.} \quad \ddot{x} - \frac{2\dot{x}\dot{y}}{y} = 0\,,$$

$$\frac{\partial L}{\partial y} = -\frac{\dot{x}^2 + \dot{y}^2}{y^3} = \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = \frac{d}{dt}\left(\frac{\dot{y}}{y^2}\right) = \frac{\ddot{y}}{y^2} - \frac{2\dot{y}^2}{y^3}, \text{i.e.} \quad \ddot{y} + \frac{\dot{x}^2}{y} - \frac{\dot{y}^2}{y} = 0\,.$$

Comparing these equations with equations for geodesics: $\ddot{x}^i - \dot{x}^k\Gamma^i_{km}\dot{x}^m = 0$ $(i = 1, 2,\ x = x^1, y = x^2)$ we come to

$$\Gamma^x_{xx} = 0, \Gamma^x_{xy} = \Gamma^x_{yx} = -\frac{1}{y}, \Gamma^x_{yy} = 0, \Gamma^y_{xx} = \frac{1}{y}, \Gamma^y_{xy} = \Gamma^y_{yx} = 0, \Gamma^y_{yy} = -\frac{1}{y}. \blacksquare$$

In a similar way as for a sphere one can find geodesics on Lobachevsky plane. First we note that vertical rays are geodesics. Then using the inversions with centre on the absolute one can see that arcs of the circles with centre at the absolute ($y = 0$) are geodesics too. (See homework 6).

3.3.5 Variational principe and Euler-Lagrange equations

Here very briefly we will explain how Euler-Lagrange equations follow from variational principe.

Let M be a manifold (not necessarily Riemannian) and $L = L(x^i, \dot{x}^i)$ be a Lagrangian on it.

Denote my $\mathcal{M}_{\mathbf{x}_1,t_1}^{\mathbf{x}_2,t_2}$ the space of curves (paths) such that they start at the point $\mathbf{x}_1$ at the "time" $t = t_1$ and end at the point $\mathbf{x}_2$ at the "time" $t = t_2$:

$$\mathcal{M}_{\mathbf{x}_1,t_1}^{\mathbf{x}_2,t_2} = \{C: \ \mathbf{x}(t), t_1 \leq t \leq t_2, \ \mathbf{x}(t_1) = \mathbf{x}_1, \mathbf{x}(t_2) = \mathbf{x}_2\}. \tag{3.21}$$

Consider the following functional S on the space $\mathcal{M}_{\mathbf{x}_1,t_1}^{\mathbf{x}_2,t_2}$:

$$S\left[\mathbf{x}(t)\right] = \int_{t_1}^{t_2} L\left(x^i(t), \dot{x}^i(t)\right) dt. \tag{3.22}$$

for every curve $\mathbf{x}(t) \in \mathcal{M}_{\mathbf{x}_1,t_1}^{\mathbf{x}_2,t_2}$.

This functional is called *action* functional.

Theorem *Let functional S attaints the minimal value on the path $\mathbf{x}_0(t) \in \mathcal{M}_{\mathbf{x}_1,t_1}^{\mathbf{x}_2,t_2}$, i.e.*

$$\forall \mathbf{x}(t) \in \mathcal{M}_{\mathbf{x}_1,t_1}^{\mathbf{x}_2,t_2} \quad S[\mathbf{x}_0(t)] \leq S[\mathbf{x}(t)]. \tag{3.23}$$

Then the path $\mathbf{x}_0(t)$ is a solution of Euler-Lagrange equations of the Lagrangian L:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{\partial L}{\partial x^i} \text{ if } \mathbf{x}(t) = \mathbf{x}_0(t). \tag{3.24}$$

Remark The path $\mathbf{x}(t)$ sometimes is called *extremal* of the action functional (3.22).

We will use this Theorem to show that the geodesics are in some sense shortest curves [9].

---

[9] The statement of this Theorem is enough for our purposes. In fact in classical mechanics another more useful statement is used: the path $\mathbf{x}_0(t)$ is a solution of Euler-Lagrange equations of the Lagrangian L if and only if it is the stationary "point" of the action functional (3.22), i.e.

$$S[\mathbf{x}_0(t) + \delta\mathbf{x}(t)] - S[\mathbf{x}_0(t) + \delta\mathbf{x}(t)] = 0(\delta\mathbf{x}(t)) \tag{3.25}$$

for an arbitrary infinitesimal variation of the path $\mathbf{x}_0(t)$: $\delta\mathbf{x}_{(}t_1) = \delta\mathbf{x}_{(}t_2) = 0$.

3.4 Geodesics and shortest distance.

Many of you know that geodesics are in some sense shortest curves. We will give an exact meaning to this statement and prove it using variational principe:

Let M be a Riemannian manifold.

Theorem *Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two points on M. The shortest curve which joins these points is an arc of geodesic.*

Let C be a geodesic on M and $\mathbf{x}_1 \in C$. Then for an arbitrary point $\mathbf{x}_2 \in C$ which is close to the point $\mathbf{x}_1$ the arc of geodesic joining the points $\mathbf{x}_1, \mathbf{x}_2$ is a shortest curve between these points[10] .

This Theorem makes a bridge between two different approach to geodesic: the shortest disntance and parallel transport of velocity vector.

Sketch a proof:

Consider the following two Lagrangians: Lagrangian of a "free " particle $L_{\text{free}} = \frac{g_{ik}(x)\dot{x}^i\dot{x}^k}{2}$ and the length Lagrangian

$$L_{\text{length}}(x, \dot{x}) = \sqrt{g_{ik}(x)\dot{x}^i\dot{\mathbf{x}}^k} = \sqrt{2L_{\text{free}}}.$$

If C: $x^i(t), t_1 \leq t \leq t_2$ is a curve on M then

$$\text{Length of the curve } C =$$

$$\int_{t_1}^{t_2} L_{\text{length}}(x^i(t), \dot{x}^i(t))dt = \int_{t_1}^{t_2} \sqrt{g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t)}dt. \qquad (3.26)$$

The proof of the Theorem follows from the following observation:

Observation Euler-Lagrange equations for the length functional (3.26) are equivalent to the Euler-Lagrange equations for action functional (3.22). This means that extremals of the length functional and action functionals coincide.

Indeed it follows from this observation and the variational principe that the shortest curves obey the Euler-Lagrange equations for the action functional. We showed before that Euler-Lagrange equations for action functional (3.22) define geodesics. Hence the shortest curves are geodesics.

---

[10]More precisely: for every point $\mathbf{x}_1 \in C$ there exists a ball $B_\delta(\mathbf{x}_1)$ such that for an arbitrary point $\mathbf{x}_2 \in C \cap B_\delta(\mathbf{x}_1)$ the arc of geodesic joining the points $\mathbf{x}_1, \mathbf{x}_2$ is a shortest curve between these points.

One can check the observation by direct calculation: Calculate Euler-Lagrange equations for the Lagrangian $L_{\text{length}} = \sqrt{g_{ik}(x)\dot{x}^i\dot{x}^k} = \sqrt{2L_{\text{free}}}$:

$$\frac{d}{dt}\left(\frac{\partial L_{\text{length}}}{\partial \dot{x}^i}\right) - \frac{\partial L_{\text{length}}}{\partial x^i} = \frac{d}{dt}\left(\frac{1}{\sqrt{g_{ik}\dot{x}^i\dot{x}^k}}g_{ik}\dot{x}^k\right) - \frac{1}{2\sqrt{g_{ik}\dot{x}^i\dot{x}^k}}\frac{\partial g_{km}\dot{x}^k\dot{x}^m}{\partial x^i}$$

$$= \frac{d}{dt}\left(\frac{1}{L_{\text{length}}}\frac{\partial L_{\text{free}}}{\partial \dot{x}^i}\right) - \frac{1}{L_{\text{length}}}\frac{\partial L_{\text{free}}}{\partial x^i} = 0\,. \qquad (3.27)$$

To facilitate calculations note that the length functional (3.26) is reparameterisation invariant. Choose the natural parameter $s(t)$ or a parameter proportional to the natural parameter on the curve $x^i(t)$. We come to $L_{\text{length}} = const$ and it follows from (3.27) that

$$\frac{d}{dt}\left(\frac{\partial L_{\text{length}}}{\partial \dot{x}^i}\right) - \frac{\partial L_{\text{length}}}{\partial x^i} = \frac{1}{L_{\text{length}}}\left(\frac{d}{dt}\left(\frac{\partial L_{\text{free}}}{\partial \dot{x}^i}\right) - \frac{\partial L_{\text{free}}}{\partial x^i}\right) = 0\,.$$

We prove that Euler-Lagrange equations for length and action Lagrangians coincide. ∎

In the Euclidean space straight lines are the shortest distances between two points. On the other hand their velocity vectors are constant. We realise now that in general Riemannian manifold the role of geodesic is twofold also: they are locally shortest and have covariantly constant velocity vectors.

3.4.1 Again geodesics for sphere and Lobachevsky plane

The fact that geodesics are shortest gives us another tool to calculate geodesics.

Consider again examples of sphere and Lobachevsky plane and find geodesics using the fact that they are shortest. The fact that geodesics are locally the shortest curves

Consider again sphere in $\mathbf{E}^3$ with the radius R: Coordinates θ, φ, induced Riemannian metrics (first quadratic form):

$$G = R^2(d\theta^2 + \sin^2\theta d\varphi^2)\,. \qquad (3.28)$$

Consider two arbitrary points A and B on the sphere. Let (θ_0, φ_0) be coordinates of the point A and (θ_1, φ_1) be coordinates of the point B

Let C_{AB} be a curve which connects these points: $C_{AB}: \theta(t), \varphi(t)$ such that $\theta(t_0) = \theta_0, \theta(t_1) = \theta_1$, $\varphi(t_0) = \theta_0, \theta(t_1) = \theta_1$ then:

$$L_{C_{AB}} = \int R\sqrt{\theta_t^2 + \sin^2 \theta(t)\varphi_t^2} dt \tag{3.29}$$

Suppose that points A and B have the same latitude, i.e. if (θ_0, φ_0) are coordinates of the point A and (θ_1, φ_1) are coordinates of the point B then $\varphi_0 = \varphi_1$ (if it is not the fact then we can come to this condition rotating the sphere)

Now it is easy to see that an arc of meridian, the curve $\varphi = \varphi_0$ is geodesics: Indeed consider an arbitrary curve $\theta(t), \varphi(t)$ which connects the points A, B: $\theta(t_0) = \theta(t_1) = \theta_0$, $\varphi(t_0) = \varphi(t_1) = \varphi_0$. Compare its length with the length of the meridian which connects the points A, B:

$$\int_{t_0}^{t_1} R\sqrt{\theta_t^2 + \sin^2 \theta \varphi_t^2} dt \geq R \int_{t_0}^{t_1} \sqrt{\theta_t^2} dt = R \int_{t_0}^{t_1} \theta_t dt = R(\theta_1 - \theta_0) \tag{3.30}$$

Thus we see that the great circle joining points A, B is the shortest. *The great circles on sphere are geodesics.* It corresponds to geometrical intuition: The geodesics on the sphere are the circles of intersection of the sphere with the plane which crosses the centre.

Geodesics on Lobachevsky plane
Riemannian metric on Lobachevsky plane:

$$G = \frac{dx^2 + dy^2}{y^2} \tag{3.31}$$

The length of the curve $\gamma: x = x(t, y = y(t))$ is equal to

$$L = \int \sqrt{\frac{x_t^2 + y_t^2}{y^2(t)}} dt$$

In particularly the length of the vertical interval $[1, \varepsilon]$ tends to infinity if $\varepsilon \to 0$:

$$L = \int \sqrt{\frac{x_t^2 + y_t^2}{y^2(t)}} dt = \int_\varepsilon^1 \sqrt{\frac{1}{t^2}} dt = \log \frac{1}{\varepsilon}$$

One can see that the distance from every point to the line $y = 0$ is equal to infinity. This motivates the fact that the line $y = 0$ is called *absolute.*

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Consider two points $A = (x_0, y_0)$, $B = (x_1, y_1)$ on Lobachevsky plane.

It is easy to see that vertical lines are geodesics of Lobachevsky plane.

Namely let points A, B are on the ray $x = x_0$. Let C_{AB} be an arc of the ray $x = x_0$ which joins these points: C_{AB}: $x = x_0, y = y_0 + t$ Then it is easy to see that the length of the curve C_{AB} is less or equal than the length of the arbitrary curve $x = x(t), y = y(t)$ which joins these points: $x(t)\big|_{t=0} = x_0, y(t)\big|_{t=0} = y_0, x(t)\big|_{t=t_1} = x_0, y(t)\big|_{t=t_1} = y_1$:

$$\int_0^t \sqrt{\frac{x_t^2 + y_t^2}{y^2(t)}} dt \geq \int_0^t \sqrt{\frac{y_t^2}{y^2(t)}} dt = \int_{y_0}^{y_1} \frac{dt}{t} dt = \log \frac{y_1}{y_0} = \text{length of } C_{AB}$$

Hence C_{AB} is shortest. We prove that vertical rays are geodesics.

Consider now the case if $x_0 \neq x_1$. Find geodesics which connects two points A, B which are not on the same vertical ray. Consider semicircle which passes these two points such that its centre is on the absolute. We prove that it is a geodesic.

Proof Let coordinates of the centre of the circle are $(a, 0)$. Then consider polar coordinates (r, φ):

$$x = a + r \cos \varphi, y = r \sin \varphi \tag{3.32}$$

In these polar coordinates r-coordinate of the semicircle is constant.

Find Lobachevsky metric in these coordinates: $dx = -r \sin \varphi d\varphi + \cos \varphi dr$, $dy = r \cos \varphi d\varphi + \sin \varphi dr$, $dx^2 + dy^2 = dr^2 + r^2 d\varphi^2$. Hence:

$$G = \frac{dx^2 + dy^2}{y^2} = \frac{dr^2 + r^2 d\varphi^2}{r^2 \sin^2 \varphi} == \frac{d\varphi^2}{\sin^2 \varphi} + \frac{dr^2}{r^2 \sin^2 \varphi} \tag{3.33}$$

We see that the length of the arbitrary curve which connects points A, B is greater or equal to the length of the arc of the circle:

$$L_{AB} = \int_{t_0}^{t_1} \sqrt{\frac{\varphi_t^2}{\sin^2 \varphi} + \frac{r_t^2}{r^2 \sin^2 \varphi}} dt \geq \int_{t_0}^{t_1} \sqrt{\frac{\varphi_t^2}{\sin^2 \varphi}} dt = \tag{3.34}$$

$$\int_{t_0}^{t_1} \frac{\varphi_t}{\sin \varphi} dt = \int_{\varphi_0}^{\varphi_1} \frac{d\varphi}{\sin \varphi} = \log \frac{\tan \varphi_1}{\tan \varphi_1}$$

The proof is finished.

3.5   *Integrals of motions and geodesics.

We see how useful in Riemannian geometry to use the Lagrangian approach.

To solve and study solutions of Lagrangian equations (in particular geodesics which are solutions of Euler-Lagrange equations for Lagrangian of free particle) it is very useful to use integrals of motion

3.5.1 *Integral of motion for arbitrary Lagrangian $L(x, \dot{x})$

Let $L = L(x, \dot{x})$ be a Lagrangian, the function of point and velocity vectors on manifold M (the function on tangent bundle TM).

Definition We say that the function $F = F(q, \dot{q})$ on TM is *integral of motion* for Lagrangian $L = L(x, \dot{x})$ if for any curve $q = q(t)$ which is the solution of Euler-Lagrange equations of motions the magnitude $I(t) = F(x(t), \dot{x}(t))$ is preserved along this curve:

$$F(x(t), \dot{x}(t)) = const \text{ if } x(t) \text{ is a solution of Euler-Lagrange equations(3.9).}$$
$$(3.35)$$

In other words

$$\frac{d}{dt}\left(F(x(t), \dot{x}(t))\right) = 0 \text{ if } x^i(t)\colon \quad \frac{d}{dt}\left(\frac{\partial L}{\dot{\partial} x^i}\right) - \frac{\partial L}{\partial x^i} = 0 \,. \qquad (3.36)$$

3.5.2 *Basic examples of Integrals of motion: Generalised momentum and Energy

Let $L(x^i, \dot{x}^i)$ does not depend on the coordinate x^1. $L = L(x^2, \ldots, x^n, \dot{x}^1, \dot{x}^2, \ldots, \dot{x}^n)$. Then the function

$$F_1(x, \dot{x}) = \frac{\partial L}{\partial \dot{x}^1}$$

is integral of motion. (In the case if $L(x^i, \dot{x}^i)$ does not depend on the coordinate x^i. the function $F_i(x, \dot{x}) = \frac{\partial L}{\partial x^i}$ will be integral of motion.)

Proof is simple. Check the condition (3.36): Euler-Lagrange equations of motion are:

$$\frac{d}{dt}\left(\frac{\partial L}{\dot{\partial} x^i}\right) - \frac{\partial L}{\partial x^i} = 0 \quad (i = 1, 2, \ldots, n)$$

We see that exactly first equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\dot{\partial} x^1}\right) = \frac{d}{dt}F_1(q, \dot{q}) = 0 \quad \text{since } \tfrac{\partial L}{\partial x^1} = 0, .$$

(if $L(x^i, \dot{x}^i)$ does not depend on the coordinate x^i then the function $F_i(x, \dot{x}) = \frac{\partial L}{\dot{x}^1}$ is integral of motion since $i - th$ equation is exactly the condition $\dot{F}_i = 0$.)

The integral of motion $F_i = \frac{\partial L}{\partial x^i}$ is called sometimes *generalised momentum*.

Another very important example of integral of motion is: energy.

$$E(x, \dot{x}) = \dot{x}^i \frac{\partial L}{\partial \dot{x}^i} - L. \qquad (3.37)$$

One can check by direct calculation that it is indeed integral of motion. Using Euler Lagrange equations $\frac{d}{dt}\left(\frac{\partial L}{\partial x^i}\right) - \frac{\partial L}{\partial x^i}$ we have:

$$\frac{d}{dt} E(x(t), \dot{x}(t)) = \frac{d}{dt}\left(\dot{x}^i \frac{\partial L}{\partial \dot{x}^i} - L\right) = \frac{\partial L}{\partial \dot{x}^i}\frac{d\dot{x}^i}{dt} + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right)\dot{x}^i - \frac{dL}{dt} =$$

$$\frac{\partial L}{\partial \dot{x}^i}\frac{d\dot{x}^i}{dt} + \frac{\partial L}{\partial x^i}\frac{dx^i}{dt} - \frac{dL(x, \dot{x})}{dt} = \frac{dL(x, \dot{x})}{dt} - \frac{dL(x, \dot{x})}{dt} = 0.$$

3.5.3   *Integrals of motion for geodesics

Apply the integral of motions for studying geodesics.

The Lagrangian of "free" particle $L_{\text{free}} = \frac{g_{ik}(x)\dot{x}^i\dot{x}^k}{2}$. For Lagrangian of free particle solution of Euler-Lagrange equations of motions are geodesics.

If $F = F(x, \dot{x})$ is the integral of motion of the free Lagrangian $L_{\text{free}} = \frac{g_{ik}(x)\dot{x}^i\dot{x}^k}{2}$ then the condition (3.35) means that the magnitude $I(t) = F(x^i(t), \dot{x}^i(t))$ is preserved along the geodesics:

$$I(t) = F(x^i(t), \dot{x}^i(t)) = const, \text{i.e.} \frac{d}{dt}I(t) = 0 \text{ if } x^i(t) \text{ is geodesic.} \qquad (3.38)$$

Consider examples of integrals of motion for free Lagrangian, i.e. magnitudes which preserve along the geodesics:

Example 1 Note that for an arbitrary "free" Lagrangian Energy integral (3.37) is an integral of motion:

$$E = \dot{x}^i \frac{\partial L}{\partial \dot{x}^i} - L = \dot{x}^i \frac{\partial\left(\frac{g_{pq}(x)\dot{x}^p\dot{x}^q}{2}\right)}{\partial \dot{x}^i} - \frac{g_{ik}(x)\dot{x}^i\dot{x}^k}{2} =$$

$$\dot{x}^i g_{iq}(x)\dot{x}^q - \frac{g_{ik}(x)\dot{x}^i\dot{x}^k}{2} = \frac{g_{ik}(x)\dot{x}^i\dot{x}^k}{2}. \qquad (3.39)$$

This is an integral of motion for an arbitrary Riemannian metric. It is preserved on an arbitrary geodesic

$$\frac{dE(t)}{dt} = \frac{d}{dt}\left(\frac{1}{2}g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t)\right) = 0.$$

In fact we already know this integral of motion: Energy (3.39) is proportional to the square of the length of velocity vector:

$$|\mathbf{v}| = \sqrt{g_{ik}(x)\dot{x}^i\dot{x}^k} = \sqrt{2E}\,. \qquad (3.40)$$

We already proved that velocity vector is preserved along the geodesic (see the Proposition in the subsection 3.2.1 and its proof (3.5).)

Example 2 Consider Riemannian metric $G = adu^2 + bdv^2$ (see also calculations in subsection 2.3.3) in the case if $a = a(u)$, $b = b(u)$, i.e. coefficients do not depend on the second coordinate v:

$$G = a(u)du^2 + b(u)dv^2,\ \ L_{\text{free}} = \frac{1}{2}\left(a(u)\dot{u}^2 + b(u)\dot{v}^2\right) \qquad (3.41)$$

We see that Lagrangian does not depend on the second coordinate v hence the magnitude

$$F = \frac{\partial L_{\text{free}}}{\partial \dot{v}} = b(u)\dot{v} \qquad (3.42)$$

is preserved along geodesic. It is integral of motion because Euler-Lagrange equation for coordinate v is

$$\frac{d}{dt}\frac{\partial L_{\text{free}}}{\partial \dot{v}} - \frac{\partial L_{\text{free}}}{\partial v} = \frac{d}{dt}\frac{\partial L_{\text{free}}}{\partial \dot{v}} = \frac{d}{dt}F = 0 \ \ \text{since} \ \frac{\partial L_{\text{free}}}{\partial v} = 0..$$

In fact all revolution surfaces which we consider here (cylinder, cone, sphere,...) have Riemannian metric of this type. Indeed consider further examples.

Example (sphere)

Sphere of the radius R in $\mathbf{E}^3$. Riemannian metric: $G = Rd\theta^2 + R^2\sin^2\theta d\varphi^2$ and $L_{\text{free}} = \frac{1}{2}\left(R^2\dot{\theta}^2 + R^2\sin^2\theta\dot{\varphi}^2\right)$ It is the case (3.41) for $u = \theta, v = \varphi$, $b(u) = R^2\sin^2\theta$ The integral of motion is

$$F = \frac{\partial L_{\text{free}}}{\partial \dot{\varphi}} = R^2\sin^2\theta\dot{\varphi}$$

Example (cone)

Consider cone $\begin{cases} x = ah\cos\varphi \\ y = ah\sin\varphi \\ z = bh \end{cases}$. Riemannian metric:

$$G = d(ah\cos\varphi)^2 + d(ah\sin\varphi)^2 + (dbh)^2 = (a^2 + b^2)dh^2 + a^2h^2d\varphi^2\,.$$

and free Lagrangian

$$L_{\text{free}} = \frac{(a^2 + b^2)\dot{h}^2 + a^2 h^2 \dot{\varphi}^2}{2} \, .$$

The integral of motion is

$$F = \frac{\partial L_{\text{free}}}{\partial \dot{\varphi}} = a^2 h^2 \dot{\varphi}.$$

Example (general surface of revolution)
Consider a surface of revolution in $\mathbf{E}^3$:

$$\mathbf{r}(h, \varphi): \quad \begin{cases} x = f(h) \cos \varphi \\ y = f(h) \sin \varphi \\ z = h \end{cases} \quad (f(h) > 0) \qquad (3.43)$$

(In the case $f(h) = R$ it is cylinder, in the case $f(h) = kh$ it is a cone, in the case $f(h) = \sqrt{R^2 - h^2}$ it is a sphere, in the case $f(h) = \sqrt{R^2 + h^2}$ it is one-sheeted hyperboloid, in the case $z = \cos h$ it is catenoid,...)

For the surface of revolution (3.43)

$$G = d(f(h) \cos \varphi)^2 + d(f(h) \sin \varphi)^2 + (dh)^2 = (f'(h) \cos \varphi dh - f(h) \sin \varphi d\varphi)^2 +$$

$$(f'(h) \sin \varphi dh + f(h) \cos \varphi d\varphi)^2 + dh^2 = (1 + f'^2(h)) dh^2 + f^2(h) d\varphi^2 \, .$$

The "free" Lagrangian of the surface of revolution is

$$L_{\text{free}} = \frac{\left(1 + f'^2(h)\right) \dot{h}^2 + f^2(h) \dot{\varphi}^2}{2} \, .$$

and the integral of motion is

$$F = \frac{\partial L_{\text{free}}}{\partial \dot{\varphi}} = f^2(h) \dot{\varphi}.$$

3.5.4 *Using integral of motions to calculate geodesics

Integrals of motions may be very useful to calculate geodesics. The equations for geodesics are second order differential equations. If we know integrals of motions they help us to solve these equations. Consider just an example.

For Lobachevsky plane the free Lagrangian $L = \frac{\dot{x}^2 + \dot{y}^2}{2y^2}$. We already calculated geodesics in the subsection 3.3.4. Geodesics are solutions of second order Euler-Lagrange equations for the Lagrangian $L = \frac{\dot{x}^2 + \dot{y}^2}{2y^2}$ (see the subsection 3.3.4)

$$\begin{cases} \ddot{x} - \frac{2\dot{x}\dot{y}}{y} = 0 \\ \ddot{y} + \frac{\dot{x}^2}{y} - \frac{\dot{y}^2}{y} = 0 \end{cases}$$

It is not so easy to solve these differential equations.

For Lobachevsky plane we know two integrals of motions:

$$E = L = \frac{\dot{x}^2 + \dot{y}^2}{2y^2}, \quad \text{and } F = \frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{y^2}. \tag{3.44}$$

These both integrals preserve in time: if $x(t), y(t)$ is geodesics then

$$\begin{cases} F = \frac{\dot{x}(t)}{y(t)^2} \\ E = \frac{\dot{x}(t)^2 + \dot{y}(t)^2}{2y(t)^2} = C_2 \end{cases} \Rightarrow \begin{cases} \dot{x} = C_1 y^2 \\ \dot{y} = \pm\sqrt{2C_2 y^2 - C_1^2 y^4} \end{cases}$$

These are first order differential equations. It is much easier to solve these equations in general case than initial second order differential equations.

4   Surfaces in $\mathbf{E}^3$

Now equipped by the knowledge of Riemannian geometry we consider surfaces in $\mathbf{E}^3$. We reconsider again conceptions of Shape (Weingarten) operator, Gaussian and mean curvatures, focusing attention on the the fact what properties are internal and what properties are external.
  .

4.1   Induced metric on surfaces.

Recall here again induced metric (see for detail subsection 1.4)

If surface $M\colon \mathbf{r} = \mathbf{r}(u,v)$ is embedded in $\mathbf{E}^3$ then induced Riemannian metric G_M is defined by the formulae

$$\langle \mathbf{X}, \mathbf{Y} \rangle = G_M(\mathbf{X}, \mathbf{Y}) = G(\mathbf{X}, \mathbf{Y}), \tag{4.1}$$

where G is Euclidean metric in $\mathbf{E}^3$:

$$G_M = dx^2 + dy^2 + dz^2 \big|_{\mathbf{r}=\mathbf{r}(u,v)} = \sum_{i=1}^{3} (dx^i)^2 \big|_{\mathbf{r}=\mathbf{r}(u,v)} = \sum_{i=1}^{3} \left(\frac{\partial x^i}{\partial u^\alpha} du^\alpha \right)^2$$

$$= \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^i}{\partial u^\beta} du^\alpha du^\beta \,,$$

i.e.

$$G_M = g_{\alpha\beta} du^\alpha, \ \text{where} \ g_{\alpha\beta} = \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^i}{\partial u^\beta} du^\alpha du^\beta \,.$$

We use notations x, y, z or x^i ($i = 1, 2, 3$) for Cartesian coordinates in $\mathbf{E}^3$, u, v or u^α ($\alpha = 1, 2$) for coordinates on the surface. We usually omit summation symbol over dummy indices. For coordinate tangent vectors

$$\underbrace{\frac{\partial}{\partial u_\alpha}}_{\text{Internal observer}} = \underbrace{\mathbf{r}_\alpha = \frac{\partial x^i}{\partial u^\alpha} \frac{\partial}{\partial x^i}}_{\text{External observer}}$$

We have already plenty examples in the subsection 1.4. In particular for scalar product

$$g_{\alpha\beta} = \left\langle \frac{\partial}{u_\alpha}, \frac{\partial}{u_\beta} \right\rangle = x^i \alpha x^i \beta \,. \langle \mathbf{r}_\alpha, \mathbf{r}_\beta \rangle \,. \tag{4.2}$$

4.2 Induced connection

We already studied before induced connection on manifolds embedded in $\mathbf{E}^n$ (see the subsection 2.2).

We do it again in a more detail for surfaces in $\mathbf{E}^3$.

Let $\nabla^{\text{can.flat}}$ be a canonical flat connection in Euclidean space $\mathbf{E}^3$, i.e. a connection such that its Christoffel symbols vanish in cartesian coordinates, i.e.

$$\nabla_{\mathbf{X}}^{\text{can.flat}} \mathbf{Y} = \partial_{\mathbf{X}} \mathbf{Y} = \mathbf{X}^m(x) \frac{\partial}{x^m} Y^i(x) \frac{\partial}{\partial x^i} \tag{4.3}$$

in Cartesian coordinates. (Of course in arbitrary curvilinear coordinates Christoffel symbols of canonical flat connection may have non-zero components, but in Cartesian coordinates Christoffel symbols vanish. It is why we sometimes use notation $\partial_{\mathbf{X}}$ instead $\nabla_{\mathbf{X}}^{\text{can.flat}} \mathbf{Y}$.)

Induced connection ∇ on the surface can be defined by the relation

$$\nabla_{\mathbf{X}} \mathbf{Y} = \left(\nabla_{\mathbf{X}}^{\text{can.flat}} \mathbf{Y} \right)_{\text{tangent}} \tag{4.4}$$

(See also the subsection (2.2)) One can show that this formula indeed defines the connection on M.

Calculate Christoffel symbols of this connection using the formulae above and the expressions for basic vectors $\partial_\alpha = \frac{\partial}{\partial u^\alpha}$, $\partial_\beta = \frac{\partial}{\partial u^\beta}$ for external observer.

We have

$$\nabla_\alpha \partial_\beta = \Gamma^\gamma_{\alpha\beta} \partial_\gamma = \left(\nabla^{\text{can.flat}}_{\partial_\alpha} \partial_\beta \right)_{\text{tangent}} = \left(\nabla^{\text{can.flat}}_{\partial_\alpha} \mathbf{r}_\beta \right)_{\text{tangent}} \qquad (4.5)$$

i.e.

$$\Gamma^\gamma_{\alpha\beta} \mathbf{r}_\gamma = \left(\frac{\partial^2 \mathbf{r}}{\partial u^\alpha \partial u^\beta} \right)_{\text{tangent}}, \qquad (4.6)$$

Before continuing calculations note that we see already that the induced connection is *symmetric* one.

Now continue to calculate. Any vector attached at the surface can be decomposed as a sum of tangent vector and the vector orthogonal to the surface:

$$\frac{\partial^2 \mathbf{r}}{\partial u^\alpha \partial u^\beta} = \left(\frac{\partial^2 \mathbf{r}}{\partial u^\alpha \partial u^\beta} \right)_{\text{tangent}} + \mathbf{N}$$

Hence

$$\Gamma^\gamma_{\alpha\beta} \mathbf{r}_\gamma = \left(\frac{\partial^2 \mathbf{r}}{\partial u^\alpha \partial u^\beta} \right)_{\text{tangent}} = \frac{\partial^2 \mathbf{r}}{\partial u^\alpha \partial u^\beta} - \mathbf{N} = \mathbf{r}_{\alpha\beta} - \mathbf{N} \qquad (4.7)$$

Take scalar product of both parts of this equation on the tangent vector vector $\mathbf{r}_\pi$, we come to:

$$\Gamma^\gamma_{\alpha\beta} \langle \mathbf{r}_\gamma, \mathbf{r}_\pi \rangle = \langle \mathbf{r}_{\alpha\beta}, \mathbf{r}_\pi \rangle + \langle \mathbf{N}, \mathbf{r}_\pi \rangle.$$

We have that $\mathbf{N} \perp \mathbf{r}_\pi$, hence $\langle \mathbf{N}, \mathbf{r}_\pi \rangle = 0$. Using the fact that $\langle \mathbf{r}_\gamma, \mathbf{r}_\pi \rangle = g_{\gamma\pi}$ are entries of the matrix $||G_M||$ of the induced Riemannian metric (see the equation (4.2)) we come to

$$\Gamma^\gamma_{\alpha\beta} g_{\gamma\pi} = \langle \mathbf{r}_{\alpha\beta}, \mathbf{r}_\pi \rangle = x^i_{\alpha\beta} x^i_\pi.$$

Multiplying both sides of this equation on inverse matrix we come to :

$$\Gamma^\gamma_{\alpha\beta} g_{\gamma\pi} g^{\pi\rho} = \Gamma^\gamma_{\alpha\beta} \delta^\rho_\gamma = \Gamma^\rho_{\alpha\beta} = x^i_{\alpha\beta} x^i_\pi g^{\pi\rho}.$$

We come to the explicit formula for the Christoffel symbols of the induced connection in terms of parametric equations $x^i(u^\alpha)$ and induced metric $g_{\alpha\beta}$:

$$\Gamma^\gamma_{\alpha\beta} = x^i_{\alpha\beta} x^i_\pi g^{\pi\gamma}. \qquad (4.8)$$

4.2.1 Induced connection—the Levi Civita connection of induced metric.

Now you may ask a question: is there another way to obtain induced connection (may be different one) on the manifold $x^i = x^i(u^\alpha)$. Recall that one can consider Levi-Civita connection of Riemannian metric $G_M = g_{\alpha\beta}du^\alpha du^\beta$ with Christoffel symbols:

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2}g^{\gamma\pi}\left(\frac{\partial g_{\pi\alpha}}{\partial u^\beta} + \frac{\partial g_{\pi\beta}}{\partial u^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial u^\pi}\right) \qquad (4.9)$$

(see the subsection about Levi-Civita connection)

Natural question arises: may be these connections are the same?—-Yes, it is

Proposition *The connection ∇ induced through the flat canonical connection on the surface M coincides with Levi-Civita connection. In other words Christoffel symbols* (4.2) *coincide with Christoffel symbols* (4.9).. The proof of this Proposition follows from the following claim

Claim The induced connection ∇ is symmetric connection which is compatible with metric.

Indeed we proved that Levi-Civita connection (4.9) is a unique symmetric connection which is compatible with metric. Hence every connection which is compatible wit metric equals to Levi-Civita connection (4.9).

It remains to prove the Claim. When calculating Christoffel symbols of induced connection we already noted that its Christoffel symbols are symmetric. It remains to prove that induced connection is compatible with induced metric, i.e. covariant derivative preserves scalar product of tangent vectors. To prove this note that covariant derivative with respect to flat canonical metric preserves metric in Euclidean space:

$$\partial_{\mathbf{X}}\langle\mathbf{Y},\mathbf{Z}\rangle = \langle\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y},\mathbf{Z}\rangle + \langle\mathbf{Y},\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Z}\rangle$$

On the other hand for an arbitrary tangent vector $\mathbf{R}$

$$\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{R} = \left(\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{R}\right)_{\text{tangent}} + \text{vector orthogonal to the surface}$$

and $\left(\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{R}\right)_{\text{tangent}} = \nabla_{\mathbf{X}}\mathbf{R}$, where ∇ is the induced connection. Hence

$$\partial_{\mathbf{X}}\langle\mathbf{Y},\mathbf{Z}\rangle = \langle\nabla_{\mathbf{X}}\mathbf{Y} + \dots,\mathbf{Z}\rangle + \langle\mathbf{Y},\nabla_{\mathbf{X}}\mathbf{Z} + \dots\rangle = \langle\nabla_{\mathbf{X}}\mathbf{Y},\mathbf{Z}\rangle + \langle\mathbf{Y},\nabla_{\mathbf{X}}\mathbf{Z}\rangle,$$

where we denote by ... vectors orthogonal to the surface. Thus ∇ preserves induced metric. The claim is proved.

Note that we come to two different definitions for induced connections and prove that they same. The second connection is totally described in terms of metric on the surface, i.e. in terms of *Internal Observer*. The first connection is explicitly depended on the embedding functions. It is described in terms of *External Observer*.

We continue to consider different geometrical structures form the point of view of External and Internal Observers.

We will try to prove the remarkable theorem Gauß's *Theorema Egregium* that the Gaussian curvature of the surface (which you already learned in the course of geometry) in fact depends only on induced metric...

4.3 Weingarten (shape) operator and second quadratic form

4.3.1 Recalling Weingarten operator

Continue to play with formulae [11].

Recall the Weingarten (shape) operator which acts on tangent vectors:

$$S\mathbf{X} = -\partial_{\mathbf{X}}\mathbf{n}\,, \tag{4.10}$$

where we denote by $\mathbf{n}$-unit normal vector field at the points of the surface M: $\langle \mathbf{n}, \mathbf{r}_\alpha \rangle = 0$, $\langle \mathbf{n}, \mathbf{r}_\alpha \rangle = 1$.

Now we realise that the derivative $\partial_{\mathbf{X}}\mathbf{R}$ of vector field with respect to another vector field is not a well-defined object: we need a connection. The formula $\partial_{\mathbf{X}}\mathbf{R}$ in Cartesian coordinates, is nothing but the derivative with respect to flat canonical connection: If we work only in Cartesian coordinates we do not need to distinguish between $\partial_{\mathbf{X}}\mathbf{R}$ and $\nabla_{\mathbf{X}}^{\mathrm{can.flat}}\mathbf{R}$. Sometimes with some abuse of notations we will use $\partial_{\mathbf{X}}\mathbf{R}$ instead $\nabla_{\mathbf{X}}^{\mathrm{can.flat}}\mathbf{R}$, but never forget: this can be done only in Cartesian coordinates where Christoffel symbols of flat canonical connection vanish:

$$\partial_{\mathbf{X}}\mathbf{R} = \nabla_{\mathbf{X}}^{\mathrm{can.flat}}\mathbf{R} \quad \text{in Cartesian coordinates}\,.$$

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[11]In some sense differential geometry it is when we write down the formulae expressing the geometrical facts, differentiate these formulae then reveal the geometrical meaning of the new obtained formulae e.t.c.

So the rigorous definition of Weingarten operator is

$$S\mathbf{X} = -\nabla_{\mathbf{X}}^{\text{can.flat}}\mathbf{n}\,, \tag{4.11}$$

but we often use the former one (equation (4.10)) just remembering that this can be done only in Cartesian coordinates.

Recall that the fact that Weingarten operator S maps tangent vectors to tangent vectors follows from the property: $\langle \mathbf{n}, \mathbf{X} \rangle = 0 \Rightarrow \mathbf{X}$ is tangent to the surface.

Indeed:

$$0 = \partial_{\mathbf{X}}\langle \mathbf{n}, \mathbf{n} \rangle = 2\langle \partial_{\mathbf{X}}\mathbf{n}, \mathbf{n} \rangle = -2\langle S\mathbf{X}, \mathbf{n} \rangle = 0 \Rightarrow S\mathbf{X} \text{ is tangent to the surface}$$

Recall also that normal unit vector is defined up to a sign, $\mathbf{n} \to -\mathbf{n}$. On the other hand if $\mathbf{n}$ is chosen then $\mathbf{S}$ is defined uniquely.

4.3.2 Second quadratic form

We define now the new object: *second quadratic form*

Definition. For two tangent vectors $\mathbf{X}, \mathbf{Y}$ $A(\mathbf{X}, \mathbf{Y})$ is defined such that

$$\left(\nabla_{\mathbf{X}}^{\text{can.flat}}\mathbf{Y}\right)_{\perp} = A(\mathbf{X}, \mathbf{Y})\mathbf{n} \tag{4.12}$$

i.e. we take orthogonal component of the derivative of $\mathbf{Y}$ with respect to $\mathbf{X}$.

This definition seems to be very vague: to evaluate covariant derivative we have to consider not a vector $\mathbf{Y}$ at a given point but the vector field. In fact one can see that $A(\mathbf{X}, \mathbf{Y})$ does depend only on the value of $\mathbf{Y}$ at the given point.

Indeed it follows from the definition of second quadratic form and from the properties of Weingarten operator that

$$A(\mathbf{X}, \mathbf{Y}) = \left\langle \left(\nabla_{\mathbf{X}}^{\text{can.flat}}\mathbf{Y}\right)_{\perp}, \mathbf{n} \right\rangle = \left\langle \nabla_{\mathbf{X}}^{\text{can.flat}}\mathbf{Y}, \mathbf{n} \right\rangle =$$

$$\partial_{\mathbf{X}}\langle \mathbf{Y}, \mathbf{n} \rangle - \left\langle \mathbf{Y}, \nabla_{\mathbf{X}}^{\text{can.flat}}\mathbf{n} \right\rangle = \langle S(\mathbf{X}), \mathbf{Y} \rangle \tag{4.13}$$

We proved that second quadratic form depends only on value of vector field $\mathbf{Y}$ at the given poit and we established the relation between second quadratic form and Weingarten operator.

Proposition *The second quadratic form $A(\mathbf{X}, \mathbf{Y})$ is symmetric bilinear form on tangent vectors $\mathbf{X}, \mathbf{Y}$ in a given point.*

$$A \colon T_{\mathbf{p}}M \otimes T_{\mathbf{p}}M \to \mathbf{R}, \quad A(\mathbf{X}, \mathbf{Y}) = A(\mathbf{Y}, \mathbf{X}) = \langle S\mathbf{X}, \mathbf{Y} \rangle. \qquad (4.14)$$

In components

$$A = A_{\alpha\beta} du^{\alpha} du^{\beta} = \langle \mathbf{r}_{\alpha\beta}, \mathbf{n} \rangle = \frac{\partial^2 x^i}{\partial u^{\alpha} \partial u^{\beta}} n^i. \qquad (4.15)$$

and

$$S^{\alpha}_{\beta} = g^{\alpha\pi} A_{\pi\beta} = g^{\alpha\pi} x^i_{\pi\beta} n^i, \qquad (4.16)$$

i.e.

$$A = GS, S = G^{-1}A.$$

Remark The normal unit vector field is defined up to a sign.

4.3.3 Gaussian and mean curvatures

Recall that Gaussian curvature

$$K = \det S$$

and mean curvature

$$H = \operatorname{Tr} S$$

It is easy to see that for Gaussian curvature

$$K = \det S = \det(G^{-1}A) = \frac{\det A}{\det G}.$$

We know already the geometrical meaning of Gaussian and mean curvatures from the point of view of the External Observer:

Gaussian curvature K equals to the product of principal curvatures, and mean curvartures equals to the sum of principal curvatures.

Now we ask a principal question: what bout internal observer, "aunt" living on the surface?

We will show that Gaussian curvature can be expressed in terms of induced Riemannian metric, i.e. it is an internal characteristic of the surface, invariant of isometries.

It is not the case with mean curvature: cylinder is isometric to the plane but it have non-zero mean curvature.

4.3.4 Examples of calculation of Weingarten operator, Second quadratic forms, curvatures for cylinder, cone and sphere.

Cylinder

We already calculated induced Riemannian metric on the cylinder (see (1.57)).

Cylinder is given by the equation $x^2 + y^2 = R^2$. One can consider the following parameterisation of this surface:

$$\mathbf{r}(h, \varphi)\colon \quad \begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \\ z = h \end{cases}, \quad \mathbf{r}_h = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{r}_\varphi = \begin{pmatrix} -R\sin\varphi \\ R\cos\varphi \\ 0 \end{pmatrix}, \quad (4.17)$$

$$G_{cylinder} = (dx^2 + dy^2 + dz^2)\big|_{x = a\cos\varphi,\, y = a\sin\varphi,\, z = h} =$$

$$= (-a\sin\varphi\, d\varphi)^2 + (a\cos\varphi\, d\varphi)^2 + dh^2 = a^2 d\varphi^2 + dh^2\,, \quad ||g_{\alpha\beta}|| = \begin{pmatrix} 1 & 0 \\ 0 & R^2 \end{pmatrix}.$$

Normal unit vector $\mathbf{n} = \pm\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$. Choose $\mathbf{n} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$. Weingarten operator

$$S\partial_h = -\nabla_{\mathbf{r}_h}^{\text{can.flat}}\mathbf{n} = -\partial_{\mathbf{r}_h}\mathbf{n} = -\partial_h\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} = 0\,,$$

$$S\partial_\varphi = -\nabla_{\mathbf{r}_\varphi}^{\text{can.flat}}\mathbf{n} = -\partial_{\mathbf{r}_\varphi}\mathbf{n} = -\partial_\varphi\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} \sin\varphi \\ -\cos\varphi \\ 0 \end{pmatrix} = -\frac{\partial_\varphi}{R}.$$

$$S\begin{pmatrix} \partial_h \\ \partial_\varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{-\partial_\varphi}{R} \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 \\ 0 & \frac{-1}{R} \end{pmatrix}. \quad (4.18)$$

Calculate second quadratic form: $\mathbf{r}_{hh} = \partial_h\mathbf{r}_h = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{r}_{h\varphi} = \mathbf{r}_{\varphi h} =$

$$\partial_h\begin{pmatrix} -R\sin\varphi \\ R\cos\varphi \\ 0 \end{pmatrix} = 0, \quad \mathbf{r}_{\varphi\varphi} = \partial_\varphi\begin{pmatrix} -R\sin\varphi \\ R\cos\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} -R\cos\varphi \\ -R\sin\varphi \\ 0 \end{pmatrix} = -R\mathbf{n}\,.$$

We have

$$A_{\alpha\beta} = \langle \mathbf{r}_{\alpha\beta}, \mathbf{n} \rangle, \quad A = \begin{pmatrix} \langle \mathbf{r}_{hh}, \mathbf{n} \rangle & \langle \mathbf{r}_{h\varphi}, \mathbf{n} \rangle \\ \langle \mathbf{r}_{\varphi h}, \mathbf{n} \rangle & \langle \mathbf{r}_{\varphi\varphi}, \mathbf{n} \rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -R \end{pmatrix}, \qquad (4.19)$$

$$A = GS = \begin{pmatrix} 1 & 0 \\ 0 & R^2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{R} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -R \end{pmatrix},$$

For Gaussian and mean curvatures we have

$$K = \det S = \frac{\det A}{\det G} = \det \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{R} \end{pmatrix} = 0, \qquad (4.20)$$

and mean curvature

$$H = \operatorname{Tr} S = \operatorname{Tr} \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{R} \end{pmatrix} = -\frac{1}{R}. \qquad (4.21)$$

Mean curvature is define up to a sign. If we change $\mathbf{n} \to -\mathbf{n}$ mean curvature $H \to \frac{1}{R}$ and Gaussian curvature will not change.

   Cone
   We already calculated induced Riemannian metric on the cone (see (1.60).
   Cone is given by the equation $x^2 + y^2 - k^2 z^2 = 0$. One can consider the following parameterisation of this surface:

$$\mathbf{r}(h,\varphi): \quad \begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases}, \quad \mathbf{r}_h = \begin{pmatrix} k\cos\varphi \\ k\sin\varphi \\ 1 \end{pmatrix}, \quad \mathbf{r}_\varphi = \begin{pmatrix} -kh\sin\varphi \\ kh\cos\varphi \\ 0 \end{pmatrix},$$
$$(4.22)$$

$$G_{cone} = (dx^2 + dy^2 + dz^2)\big|_{x=kh\cos\varphi, y=kh\sin\varphi, z=h} =$$

$$= (-kh\sin\varphi d\varphi + k\cos\varphi dh)^2 + (kh\cos\varphi d\varphi + k\sin\varphi dh)^2 + dh^2 =$$

$$k^2 h^2 d\varphi^2 + (k^2+1)dh^2, \quad \|g_{\alpha\beta}\| = \begin{pmatrix} k^2+1 & 0 \\ 0 & k^2 h^2 \end{pmatrix}.$$

One can see that $\mathbf{N} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix}$ is orthogonal to the surface: $\mathbf{N} \perp \mathbf{r}_h, \mathbf{r}_\varphi$.

Hence normal unit vector $\mathbf{n} = \pm\frac{1}{\sqrt{1+k^2}} \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix}$. Choose $\mathbf{n} = \frac{1}{\sqrt{1+k^2}} \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix}$.

Weingarten operator

$$S\partial_h = -\nabla^{\text{can.flat}}_{\mathbf{r}_h}\mathbf{n} = -\partial_{\mathbf{r}_h}\mathbf{n} = -\partial_h\left(\frac{1}{\sqrt{1+k^2}}\begin{pmatrix}\cos\varphi\\\sin\varphi\\-k\end{pmatrix}\right) = 0\,,$$

$$S\partial_\varphi = -\nabla^{\text{can.flat}}_{\mathbf{r}_\varphi}\mathbf{n} = -\partial_{\mathbf{r}_\varphi}\mathbf{n} = -\partial_\varphi\left(\frac{1}{\sqrt{1+k^2}}\begin{pmatrix}\cos\varphi\\\sin\varphi\\-k\end{pmatrix}\right) =$$

$$\frac{1}{\sqrt{1+k^2}}\begin{pmatrix}\sin\varphi\\-\cos\varphi\\0\end{pmatrix} = -\frac{\partial_\varphi}{kh\sqrt{k^2+1}}.$$

$$S\begin{pmatrix}\partial_h\\\partial_\varphi\end{pmatrix} = \begin{pmatrix}0\\-\frac{\partial_\varphi}{kh\sqrt{k^2+1}}\end{pmatrix},\quad S = \begin{pmatrix}0 & 0\\0 & \frac{-1}{kh\sqrt{k^2+1}}\end{pmatrix}. \tag{4.23}$$

Calculate second quadratic form: $\mathbf{r}_{hh} = \partial_h\mathbf{r}_h = \begin{pmatrix}0\\0\\0\end{pmatrix}$, $\mathbf{r}_{h\varphi} = \mathbf{r}_{\varphi h} =$

$$\partial_h\begin{pmatrix}-kh\sin\varphi\\kh\cos\varphi\\0\end{pmatrix} = \begin{pmatrix}-k\sin\varphi\\k\cos\varphi\\0\end{pmatrix},\ \mathbf{r}_{\varphi\varphi} = \partial_\varphi\begin{pmatrix}-kh\sin\varphi\\kh\cos\varphi\\0\end{pmatrix} = \begin{pmatrix}-kh\cos\varphi\\-kh\sin\varphi\\0\end{pmatrix}.$$

We have

$$A_{\alpha\beta} = \langle\mathbf{r}_{\alpha\beta},\mathbf{n}\rangle,\ A = \begin{pmatrix}\langle\mathbf{r}_{hh},\mathbf{n}\rangle & \langle\mathbf{r}_{h\varphi},\mathbf{n}\rangle\\\langle\mathbf{r}_{\varphi h},\mathbf{n}\rangle & \langle\mathbf{r}_{\varphi\varphi},\mathbf{n}\rangle\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & -\frac{kh}{\sqrt{1+k^2}}\end{pmatrix}, \tag{4.24}$$

$$A = GS = \begin{pmatrix}k^2+1 & 0\\0 & k^2h^2\end{pmatrix}\begin{pmatrix}0 & 0\\0 & \frac{-1}{kh\sqrt{k^2+1}}\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & \frac{-kh}{\sqrt{k^2+1}}\end{pmatrix},$$

For Gaussian and mean curvatures we have

$$K = \det S = \frac{\det A}{\det G} = \det\begin{pmatrix}0 & 0\\0 & \frac{-1}{kh\sqrt{k^2+1}}\end{pmatrix} = 0\,, \tag{4.25}$$

and mean curvature

$$H = \operatorname{Tr} S = \operatorname{Tr}\begin{pmatrix}0 & 0\\0 & \frac{-1}{kh\sqrt{k^2+1}}\end{pmatrix} = \frac{-1}{kh\sqrt{k^2+1}}\,. \tag{4.26}$$

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Mean curvature is define up to a sign. If we change $\mathbf{n} \to -\mathbf{n}$ mean curvature $H \to \frac{1}{R}$ and Gaussian curvature will not change.

Sphere

Sphere is given by the equation $x^2 + y^2 + z^2 = a^2$. Consider the parameterisation of sphere in spherical coordinates

$$\mathbf{r}(\theta, \varphi): \quad \begin{cases} x = R \sin \theta \cos \varphi \\ y = R \sin \theta \sin \varphi \\ z = R \cos \theta \end{cases} \tag{4.27}$$

We already calculated induced Riemannian metric on the sphere (see (4.3.4)). Recall that

$$\mathbf{r}_\theta = \begin{pmatrix} R \cos \theta \cos \varphi \\ R \cos \theta \sin \varphi \\ -R \sin \theta \end{pmatrix}, \quad \mathbf{r}_\varphi = \begin{pmatrix} -R \sin \theta \sin \varphi \\ R \sin \theta \cos \varphi \\ 0 \end{pmatrix}$$

and

$$G_{S^2} = \left(dx^2 + dy^2 + dz^2 \right) \big|_{x=R \sin \theta \cos \varphi, y=R \sin \theta \sin \varphi, z=R \cos \theta} =$$

$$(R \cos \theta \cos \varphi d\theta - R \sin \theta \sin \varphi d\varphi)^2 + (R \cos \theta \sin \varphi d\theta + R \sin \theta \cos \varphi d\varphi)^2 +$$

$$(-R \sin \theta d\theta)^2 = R^2 \cos^2 \theta d\theta^2 + R^2 \sin^2 \theta d\varphi^2 + R^2 \sin^2 \theta d\theta^2 =$$

$$= R^2 d\theta^2 + R^2 \sin^2 \theta d\varphi^2, \qquad ||g_{\alpha\beta}|| = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2 \theta \end{pmatrix}.$$

For the sphere $\mathbf{r}(\theta, \varphi)$ is orthogonal to the surface. Hence normal unit vector $\mathbf{n}(\theta, \varphi) = \pm \frac{\mathbf{r}(\theta, \varphi)}{R} = \pm \begin{pmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{pmatrix}$. Choose $\mathbf{n} = \frac{\mathbf{r}}{R} = \begin{pmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{pmatrix}$.

Weingarten operator

$$S\partial_\theta = -\nabla_{\mathbf{r}_\theta}^{\text{can.flat}} \mathbf{n} = -\partial_\theta \mathbf{n} = -\partial_\theta \left(\frac{\mathbf{r}}{R} \right) = -\frac{\mathbf{r}_\theta}{R},$$

$$S\partial_\varphi = -\nabla_{\mathbf{r}_\varphi}^{\text{can.flat}} \mathbf{n} = -\partial_\varphi \mathbf{n} = -\partial_\varphi \left(\frac{\mathbf{r}}{R} \right) = -\frac{\mathbf{r}_\varphi}{R}.$$

$$S \begin{pmatrix} \partial_\theta \\ \partial_\varphi \end{pmatrix} = \begin{pmatrix} -\frac{\partial_\theta}{R} \\ -\frac{\partial_\varphi}{R} \end{pmatrix}, \quad S = -\begin{pmatrix} \frac{1}{R} & 0 \\ 0 & \frac{1}{R} \end{pmatrix}. \tag{4.28}$$

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For second quadratic form: $\mathbf{r}_{\theta\theta} = \partial_\theta \mathbf{r}_\theta = \begin{pmatrix} -R\sin\theta\cos\varphi \\ -R\sin\theta\sin\varphi \\ -R\cos\theta \end{pmatrix}$, $\mathbf{r}_{\theta\varphi} = \mathbf{r}_{\varphi\theta} =$

$$\partial_\theta \begin{pmatrix} -R\sin\theta\sin\varphi \\ R\sin\theta\cos\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} -R\cos\theta\sin\varphi \\ R\cos\theta\cos\varphi \\ 0 \end{pmatrix}, \quad \mathbf{r}_{\varphi\varphi} = \partial_\varphi \begin{pmatrix} -R\sin\theta\sin\varphi \\ R\sin\theta\cos\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} -R\sin\theta\cos\varphi \\ -R\sin\theta\sin\varphi \\ 0 \end{pmatrix}.$$

We have

$$A_{\alpha\beta} = \langle \mathbf{r}_{\alpha\beta}, \mathbf{n} \rangle, \ A = \begin{pmatrix} \langle \mathbf{r}_{\theta\theta}, \mathbf{n} \rangle & \langle \mathbf{r}_{\theta\varphi}, \mathbf{n} \rangle \\ \langle \mathbf{r}_{\varphi\theta}, \mathbf{n} \rangle & \langle \mathbf{r}_{\varphi\varphi}, \mathbf{n} \rangle \end{pmatrix} = \begin{pmatrix} -R & 0 \\ 0 & -R\sin^2\theta \end{pmatrix}, \quad (4.29)$$

$$A = GS = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix} \begin{pmatrix} \frac{-1}{R} & 0 \\ 0 & \frac{-1}{R} \end{pmatrix} = -R \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix},$$

For Gaussian and mean curvatures we have

$$K = \det S = \frac{\det A}{\det G} = \det \begin{pmatrix} -\frac{1}{R} & 0 \\ 0 & -\frac{1}{R} \end{pmatrix} = \frac{1}{R^2}, \quad (4.30)$$

and mean curvature

$$H = \operatorname{Tr} S = \operatorname{Tr} \begin{pmatrix} -\frac{1}{R} & 0 \\ 0 & -\frac{1}{R} \end{pmatrix} = -\frac{2}{R}, \quad (4.31)$$

Mean curvature is define up to a sign. If we change $\mathbf{n} \to -\mathbf{n}$ mean curvature $H \to \frac{1}{R}$ and Gaussian curvature will not change.

We see that for the sphere Gaussian curvature is not equal to zero, whilst for cylinder and cone Gaussian curvature equals to zero.

4.4 Parallel transport along closed curves on surfaces. Derivation formulae and *Theorema Egregium*

4.4.1 Formulation of the main result

Let M be a surface in Euclidean space $\mathbf{E}^3$. Consider a closed curve C on M, $M: \mathbf{r} = \mathbf{r}(u, v)$, $C: \mathbf{r} = \mathbf{r}(u(t), v(t))$, $0 \leq t \leq t_1$, $\mathbf{x}(0) = \mathbf{x}(t_1)$. ($u(t), v(t)$ are internal coordinates of the curve C.)

Consider the parallel transport of an arbitrary tangent $\mathbf{X}$ vector along the closed curve C:

$$\mathbf{X}(t) = \underbrace{X^\alpha(t)\frac{\partial}{\partial u_\alpha}\Big|_{u^\alpha(t)}}_{\text{Internal observer}} = \underbrace{X^\alpha(t)\mathbf{r}_\alpha\Big|_{\mathbf{r}(u(t),v(t))}}_{\text{External observer}} , \left(\mathbf{r}_\alpha = \frac{\partial x^i}{\partial u^\alpha}\frac{\partial}{\partial x^i}\right) .$$

$$\mathbf{X}(t): \ \frac{\nabla\mathbf{X}(t)}{dt} = 0, \ 0 \le t \le t_1 ,$$

i.e.

$$\frac{dX^\alpha(t)}{dt} + X^\beta(t)\Gamma^\alpha_{\beta\gamma}(u(t))\frac{du^\gamma(t)}{dt} = 0, \ 0 \le t \le t_1 , \tag{4.32}$$

where ∇ is the connection induced on the surface M by canonical flat connection (see (4.5)), i.e. the Levi-Civita connection of the induced Riemannian metric on the surface M and $\Gamma^\alpha_{\beta\gamma}$ its Christoffel symbols:

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2}g^{\gamma\pi}\left(\frac{\partial g_{\pi\alpha}}{\partial u^\beta} + \frac{\partial g_{\pi\beta}}{\partial u^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial u^\pi}\right) , \text{where } g_{\alpha\beta} = \langle\mathbf{r}_\alpha, \mathbf{r}_\beta\rangle = \frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^i}{\partial u^\beta} \tag{4.33}$$

are components of induced Riemannian metric $G_M = g_{\alpha\beta}du^\alpha$ (see the subsection 4.2.1)

Let $\mathbf{r}(0) = \mathbf{p}$ be a starting (and ending) point of the curve C: $\mathbf{r}(0) = \mathbf{r}(t_1) = \mathbf{p}$. The differential equation (4.32) defines the linear operator

$$R_C: T_\mathbf{p}M \longrightarrow T_\mathbf{p}M \tag{4.34}$$

For any vector $\mathbf{X} \in T_\mathbf{p}M$, its image the vector $R_C\mathbf{X}$ as the solution of the differential equation (4.32) with initial condition $\mathbf{X}(t)\big|_{t=0} = \mathbf{X}$.

On the other hand we know that parallel transport of the vector does not change its length (see (3.6) in the subsection 3.2.1):

$$\langle\mathbf{X}, \mathbf{X}\rangle = \langle R_C\mathbf{X}, R_C\mathbf{X}\rangle \tag{4.35}$$

We see that $\mathbf{R}_C$ is an orthogonal operator in the 2-dimensional vector space $T_\mathbf{p}M$. We know that orthogonal operator preserving orientation is the operator of rotation on some angle Φ.

We see that if $\mathbf{R}_C$ preserves orientation[12] then the action of operator $\mathbf{R}_C$ on vectors is rotation on the angle, i.e. the result of parallel transport along

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[12]We consider the case if the operator R_C preserves orientation. In our considerations we consider only the case if the closed curve C is a boundary of a compact oriented domain $D \subset M$. In this case one can see that operator R_C preserves an orientation.

closed curve is rotation on the angle

$$\Delta\Phi = \Delta\Phi(C) \tag{4.36}$$

which depends on the curve.

The very beautiful question arises: How to calculate this angle $\Delta\Phi(C)$

Theorem *Let M be a surface in Euclidean space $\mathbf{E}^3$. Let C be a closed curve C on M such that C is a boundary of a compact oriented domain $D \subset M$. Consider the parallel transport of an arbitrary tangent vector along the closed curve C. As a result of parallel transport along this closed curve any tangent vector rotates through the angle*

$$\Delta\Phi = \angle\,(\mathbf{X}, \mathbf{R}_C\mathbf{X}) = \int_D K d\sigma\,, \tag{4.37}$$

where K is the Gaussian curvature and $d\sigma = \sqrt{\det g}\,dudv$ is the area element induced by the Riemannian metric on the surface M, i.e. $d\sigma = \sqrt{\det g}\,dudv$.

Remark One can show that the angle of rotation does not depend on initial point of the curve.

Example Consider the closed curve, "latitude" C_{θ_0}: $\theta = \theta_0$ on the sphere of the radius R. Calculations show that

$$\Delta\Phi(C_{\theta_0}) = 2\pi(1 - \cos\theta_0) \tag{4.38}$$

(see the Homework 6). On the other hand the latitude C_{θ_0} is the boundary of the segment D with area $2\pi RH$ where $H = R(1 - \cos\theta_0)$. Hence

$$\angle\,(\mathbf{X}, \mathbf{R}_C\mathbf{X}) = \frac{2\pi RH}{R^2} = \frac{1}{R^2} \cdot \text{area of the segment} = \int_D K d\sigma$$

since Gaussian curvature is equal to $\frac{1}{R^2}$

The proof of this Theorem is the content of the subsections above.

First of all show that this Theorem implies the remarkable corollary.

Let D be a small domain around a given point $\mathbf{p}$, let C its boundary and $\Delta\Phi(D)$ the angle of rotation. Denote by $S(D)$ an area of this domain. Applying the Theorem for the case when area of the domain D tends to zero we we come to the statement that

$$\text{if } S(D) \to 0 \text{ then } \Delta\Phi(D) = \int_D K d\sigma \to K(\mathbf{p})S(D), \text{ i.e.}$$

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$$K(\mathbf{p}) = \lim_{S(D) \to 0} \frac{\Delta \Phi(D)}{S(D)}, \text{ i.e.} \tag{4.39}$$

Now notice that left hand side od this equation defining Gaussian curvature $K(\mathbf{p})$ depends only on Riemannian metric on the surface C. Indeed numerator of LHS is defined by the solution of differential equation (4.32) which depends on Levi-Civita connection depending on the induced Riemannian metric, and denominator is an area depending on Riemannian metric too.

Thus we come to very important

Corollary *Gauß Egregium Theorema*

Gaussian curvature of the surface is invariant of isometries.

In next subsections we develop the technique which itself is very interesting. One of the applications of this technique is the proof of the Theorem (4.37). Thus we will prove Theorema Egregium too.

Later in the fifth section we will give another proof of the Theorema Egregium.

4.4.2 Derivation formulae

Let M be a surface embedded in Euclidean space $\mathbf{E}^3$, $M: \mathbf{r} = \mathbf{r}(u, v)$.

Let $\mathbf{e}, \mathbf{f}, \mathbf{n}$ be three vector fields defined on the points of this surface such that they form an orthonormal basis at any point, so that the vectors $\mathbf{e}, \mathbf{f}$ are tangent to the surface and the vector $\mathbf{n}$ is orthogonal to the surface[13]. Vector fields $\mathbf{e}, \mathbf{f}, \mathbf{n}$ are functions on the surface M:

$$\mathbf{e} = \mathbf{e}(u, v), \ \mathbf{f} = \mathbf{f}(u, v), \mathbf{n} = \mathbf{n}(u, v).$$

Consider 1-forms $d\mathbf{e}, d\mathbf{f}, d\mathbf{n}$:

$$d\mathbf{e} = \frac{\partial \mathbf{e}}{\partial u} du + \frac{\partial \mathbf{e}}{\partial v} dv, \ d\mathbf{f} = \frac{\partial \mathbf{f}}{\partial u} du + \frac{\partial \mathbf{f}}{\partial v} dv, \ d\mathbf{n} = \frac{\partial \mathbf{n}}{\partial u} du + \frac{\partial \mathbf{n}}{\partial v} dv$$

These 1-forms take values in the vectors in $\mathbf{E}^3$, i.e. they are *vector valued* 1-forms. Any vector in $\mathbf{E}^3$ attached at an arbitrary point of the surface can be expanded over the basis $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$. Thus vector valued 1-forms $d\mathbf{e}, d\mathbf{f}, d\mathbf{n}$

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[13]One can say that $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ is an orthonormal basis in $T_{\mathbf{p}}\mathbf{E}^3$ at every point of surface $\mathbf{p} \in M$ such that $\{\mathbf{e}, \mathbf{f}\}$ is an orthonormal basis in $T_{\mathbf{p}}\mathbf{E}^3$ at every point of surface $\mathbf{p} \in M$.

can be expanded in a sum of 1-forms with values in basic vectors $\mathbf{e}, \mathbf{f}, \mathbf{n}$. E.g. for $d\mathbf{e} = \frac{\partial \mathbf{e}}{\partial u} du + \frac{\partial \mathbf{e}}{\partial v} dv$ expanding vectors $\frac{\partial \mathbf{e}}{\partial u}$ and $\frac{\partial \mathbf{e}}{\partial v}$ over basis vectors we come to

$$\frac{\partial \mathbf{e}}{\partial u} = A_1(u,v)\mathbf{e} + B_1(u,v)\mathbf{f} + C_1(u,v)\mathbf{n}, \quad \frac{\partial \mathbf{e}}{\partial v} = A_2(u,v)\mathbf{e} + B_2(u,v)\mathbf{f} + C_2(u,v)\mathbf{n}$$

thus

$$d\mathbf{e} = \frac{\partial \mathbf{e}}{\partial u} du + \frac{\partial \mathbf{e}}{\partial v} dv = (A_1 \mathbf{e} + B_1 \mathbf{f} + C_1 \mathbf{n}) \, du + (A_2 \mathbf{e} + B_2 \mathbf{f} + C_2 \mathbf{n}) \, dv =$$

$$= \underbrace{(A_1 du + A_2 dv)}_{M_{11}} \mathbf{e} + \underbrace{(B_1 du + B_2 dv)}_{M_{12}} \mathbf{f} + \underbrace{(C_1 du + C_2 dv)}_{M_{11}} \mathbf{e}, \qquad (4.40)$$

i.e.

$$d\mathbf{e} = M_{11}\mathbf{e} + M_{12}\mathbf{f} + M_{13}\mathbf{n},$$

where M_{11}, M_{12} and M_{13} are 1-forms on the surface M defined by the relation (4.40).

In the same way we do the expansions of vector-valued 1-forms $d\mathbf{f}$ and $d\mathbf{n}$ we come to

$$d\mathbf{e} = M_{11}\mathbf{e} + M_{12}\mathbf{f} + M_{13}\mathbf{n}$$
$$d\mathbf{f} = M_{21}\mathbf{e} + M_{22}\mathbf{f} + M_{23}\mathbf{n}$$
$$d\mathbf{n} = M_{31}\mathbf{e} + M_{32}\mathbf{f} + M_{33}\mathbf{n}$$

This equation can be rewritten in the following way:

$$d \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix} \qquad (4.41)$$

Proposition The matrix M in the equation (4.41) is antisymmetrical matrix, i.e.

$$M_{11} = M_{22} = M_{33} = 0$$
$$M_{12} = -M_{21} = a$$
$$M_{13} = -M_{31} = b$$
$$M_{23} = -M_{32} = -b$$
$$(4.42)$$

i.e.

$$d \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix}, \qquad (4.43)$$

where a, b, c are 1-forms on the surface M.

Formulae (5.5) are called *derivation formulae.*

Prove this Proposition. Recall that $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ is orthonormal basis, i.e. at every point of the surface

$$\langle \mathbf{e}, \mathbf{e} \rangle = \langle \mathbf{f}, \mathbf{f} \rangle = \langle \mathbf{n}, \mathbf{n} \rangle = 1, \text{ and } \langle \mathbf{e}, \mathbf{f} \rangle = \langle \mathbf{e}, \mathbf{n} \rangle = \langle \mathbf{f}, \mathbf{n} \rangle = 0$$

Now using (4.41) we have

$$\langle \mathbf{e}, \mathbf{e} \rangle = 1 \Rightarrow d\langle \mathbf{e}, \mathbf{e} \rangle = 0 = 2\langle \mathbf{e}, d\mathbf{e} \rangle = \langle \mathbf{e}, M_{11}\mathbf{e} + M_{12}\mathbf{f} + M_{13}\mathbf{n} \rangle =$$

$$M_{11}\langle \mathbf{e}, \mathbf{e} \rangle + M_{12}\langle \mathbf{e}, \mathbf{f} \rangle + M_{13}\langle \mathbf{e}, \mathbf{n} \rangle = M_{11} \Rightarrow M_{11} = 0 \,.$$

Analogously

$$\langle \mathbf{f}, \mathbf{f} \rangle = 1 \Rightarrow d\langle \mathbf{f}, \mathbf{f} \rangle = 0 = 2\langle \mathbf{f}, d\mathbf{f} \rangle = \langle \mathbf{f}, M_{21}\mathbf{e} + M_{22}\mathbf{f} + M_{23}\mathbf{n} \rangle = M_{22} \Rightarrow M_{22} = 0 \,,$$

$$\langle \mathbf{n}, \mathbf{n} \rangle = 1 \Rightarrow d\langle \mathbf{n}, \mathbf{n} \rangle = 0 = 2\langle \mathbf{n}, d\mathbf{n} \rangle = \langle \mathbf{n}, M_{31}\mathbf{e} + M_{32}\mathbf{f} + M_{33}\mathbf{n} \rangle = M_{33}. \Rightarrow M_{33} = 0 \,.$$

We proved already that $M_{11} = M_{22} = M_{33} = 0$. Now prove that $M_{12} = -M_{21}$, $M_{13} = -M_{31}$ and $M_{13} = -M_{31}$.

$$\langle \mathbf{e}, \mathbf{f} \rangle = 0 \Rightarrow d\langle \mathbf{e}, \mathbf{f} \rangle = 0 = \langle \mathbf{e}, d\mathbf{f} \rangle + \langle d\mathbf{e}, \mathbf{f} \rangle =$$

$$\langle \mathbf{e}, M_{21}\mathbf{e} + M_{22}\mathbf{f} + M_{23}\mathbf{n} \rangle + \langle M_{11}\mathbf{e} + M_{12}\mathbf{f} + M_{13}\mathbf{n}, \mathbf{f} \rangle = M_{21} + M_{12} = 0.$$

Analogously

$$\langle \mathbf{e}, \mathbf{n} \rangle = 0 \Rightarrow d\langle \mathbf{e}, \mathbf{n} \rangle = 0 = \langle \mathbf{e}, d\mathbf{n} \rangle + \langle d\mathbf{e}, \mathbf{n} \rangle =$$

$$\langle \mathbf{e}, M_{31}\mathbf{e} + M_{32}\mathbf{f} + M_{33}\mathbf{n} \rangle + \langle M_{11}\mathbf{e} + M_{12}\mathbf{f} + M_{13}\mathbf{n}, \mathbf{n} \rangle = M_{31} + M_{13} = 0$$

and

$$\langle \mathbf{f}, \mathbf{n} \rangle = 0 \Rightarrow d\langle \mathbf{f}, \mathbf{n} \rangle = 0 = \langle \mathbf{f}, d\mathbf{n} \rangle + \langle d\mathbf{f}, \mathbf{n} \rangle =$$

$$\langle \mathbf{f}, M_{31}\mathbf{e} + M_{32}\mathbf{f} + M_{33}\mathbf{n} \rangle + \langle M_{21}\mathbf{e} + M_{22}\mathbf{f} + M_{23}\mathbf{n}, \mathbf{n} \rangle = M_{32} + M_{23} = 0.$$

Remark This proof could be performed much more shortly in condensed notations. Derivation formulae (5.5) in condensed notations are

$$d\mathbf{e}_i = M_{ik}\mathbf{e}_k \tag{4.44}$$

Orthonormality condition means that $\langle \mathbf{e}_i, \mathbf{e}_k \rangle = \delta_{ik}$. Hence

$$d\langle \mathbf{e}_i, \mathbf{e}_k \rangle = 0 = \langle d\mathbf{e}_i, \mathbf{e}_k \rangle + \langle \mathbf{e}_i, d\mathbf{e}_k \rangle = \langle M_{im}\mathbf{e}_m, \mathbf{e}_k \rangle + \langle \mathbf{e}_i, M_{kn}\mathbf{e}_n \rangle = M_{ik} + M_{ki} = 0 \quad \blacksquare$$

$$\tag{4.45}$$

Much shorter, is not it?

4.4.3 Gauss condition (structure equations)

Derive the relations between 1-forms a, b and c in derivation formulae.

Recall that a, b, c are 1-forms, $\mathbf{e}, f, \mathbf{n}$ are vector valued functions (0-forms) and $d\mathbf{e}, d\mathbf{f}, d\mathbf{n}$ are vector valued 1-forms. (We use the simple identity that $ddf = 0$ and the fact that for 1-form $\omega \wedge \omega = 0$.) We have from derivation formulae (5.5) that

$$d^2\mathbf{e} = 0 = d(a\mathbf{f} + b\mathbf{n}) = da\mathbf{f} - a \wedge d\mathbf{f} + db\mathbf{n} - b \wedge d\mathbf{n} =$$

$$da\,\mathbf{f} - a \wedge (-a\mathbf{e} + c\mathbf{n}) + db\,\mathbf{n} - b \wedge (-b\mathbf{e} - c\mathbf{f}) =$$

$$(da + b \wedge c)\mathbf{f} + (a \wedge a + b \wedge b)\mathbf{e} + (db - a \wedge c)\mathbf{n} = (da + b \wedge c)\mathbf{f} + (db + c \wedge a)\mathbf{n} = 0\,.$$

We see that

$$(da + b \wedge c)\mathbf{f} + (db + c \wedge a)\mathbf{n} = 0 \tag{4.46}$$

Hence components of the left hand side equal to zero:

$$(da + b \wedge c) = 0 \; (db + c \wedge a) = 0\,. \tag{4.47}$$

Analogously

$$d^2\mathbf{f} = 0 = d(-a\mathbf{e} + c\mathbf{n}) = -da\mathbf{e} + a \wedge d\mathbf{e} + dc\mathbf{n} - c \wedge d\mathbf{n} =$$

$$-da\mathbf{e} + a \wedge (a\mathbf{f} + b\mathbf{n}) + dc\,\mathbf{n} - c \wedge (-b\mathbf{e} - c\mathbf{f}) =$$

$$(-da + c \wedge b)\mathbf{e} + (dc + a \wedge b)\mathbf{n} = 0\,.$$

Hence we come to structure equations:

$$\begin{aligned} da + b \wedge c &= 0 \\ db + c \wedge a &= 0 \\ dc + a \wedge a &= 0 \end{aligned} \tag{4.48}$$

4.4.4 Geometrical meaning of derivation formulae. Weingarten operator and second quadratic form in terms of derivation formulae.

Let M be a surface in $\mathbf{E}^3$.

Let $\mathbf{e}, \mathbf{f}, \mathbf{n}$ be three vector fields defined on the points of this surface such that they form an orthonormal basis at any point, so that the vectors $\mathbf{e}, \mathbf{f}$ are

tangent to the surface and the vector $\mathbf{n}$ is orthogonal to the surface. Note that in generally these vectors are not coordinate vectors.

Describe Riemannian geometry on the surface M in terms of this basis and derivation formulae (5.5).

Induced Riemannian metric

If G is the Riemannian metric induced on the surface M then since $\mathbf{e}, \mathbf{f}$ is orthonormal basis at every tangent space $T_\mathbf{p}M$ then

$$G(\mathbf{e}, \mathbf{e}) = G(\mathbf{f}, \mathbf{f}) = 1, \quad G(\mathbf{e}, \mathbf{f}) = G(\mathbf{f}, \mathbf{e}) = 0 \qquad (4.49)$$

The matrix of the Riemannian metric in the basis $\{\mathbf{e}, \mathbf{f}\}$ is

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b(\mathbf{e}) & b(\mathbf{f}) \\ c(\mathbf{e}) & c(\mathbf{f}) \end{pmatrix} \qquad (4.50)$$

Induced connection Let ∇ be the connection induced by the canonical flat connection on the surface M.

Then according equations (4.4) and derivation formulae (5.5) for every tangent vector $\mathbf{X}$

$$\nabla_\mathbf{X}\mathbf{e} = (\partial_\mathbf{X}\mathbf{e})_{\text{tangent}} = (d\mathbf{e}(\mathbf{X}))_{\text{tangent}} = (a(\mathbf{X})\mathbf{f} + b(\mathbf{X})\mathbf{n})_{\text{tangent}} = a(\mathbf{X})\mathbf{f}. \tag{4.51}$$

and

$$\nabla_\mathbf{X}\mathbf{f} = (\partial_\mathbf{X}\mathbf{f})_{\text{tangent}} = (d\mathbf{f}(\mathbf{X}))_{\text{tangent}} = (-a(\mathbf{X})\mathbf{e} + c(\mathbf{X})\mathbf{n})_{\text{tangent}} = -a(\mathbf{X})\mathbf{e}. \tag{4.52}$$

In particular

$$\begin{array}{ll} \nabla_\mathbf{e}\mathbf{e} = a(\mathbf{e})\mathbf{f} & \nabla_\mathbf{f}\mathbf{e} = a(\mathbf{f})\mathbf{f} \\ \nabla_\mathbf{e}\mathbf{f} = -a(\mathbf{e})\mathbf{e} & \nabla_\mathbf{f}\mathbf{f} = -a(\mathbf{f})\mathbf{e} \end{array} \qquad (4.53)$$

We know that the connection ∇ is Levi-Civita connection of the induced Riemannian metric (4.51) (see the subsection 4.2.1)[14].

Second Quadratic form Let $A(\mathbf{X}, \mathbf{Y})$ be second quadratic form. Then according to to (4.12) and derivation formulae (5.5) we have

$$A(\mathbf{e}, \mathbf{e}) = \langle \partial_\mathbf{e}\mathbf{e}, \mathbf{n} \rangle = \langle d\mathbf{e}(\mathbf{e}), \mathbf{n} \rangle = \langle a(\mathbf{e})\mathbf{f} + b(\mathbf{e})\mathbf{n}, \mathbf{n} \rangle = b(\mathbf{e}),$$

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[14]In particular this implies that this is symmetric connection, i.e.

$$\nabla\mathbf{f}\mathbf{e} - \nabla_\mathbf{e}\mathbf{f} - [\mathbf{f}, \mathbf{e}] = a(\mathbf{f})\mathbf{f} + a(\mathbf{e})\mathbf{e} - [\mathbf{f}, \mathbf{e}] = 0. \qquad (4.54)$$

$$A(\mathbf{f}, \mathbf{e}) = \langle \partial_{\mathbf{f}} \mathbf{e}, \mathbf{n} \rangle = \langle d\mathbf{e}(\mathbf{f}), \mathbf{n} \rangle = \langle a(\mathbf{f})\mathbf{f} + b(\mathbf{f})\mathbf{n}, \mathbf{n} \rangle = b(\mathbf{f}),$$

$$A(\mathbf{e}, \mathbf{f}) = \langle \partial_{\mathbf{e}} \mathbf{f}, \mathbf{n} \rangle = \langle d\mathbf{f}(\mathbf{e}), \mathbf{n} \rangle = \langle -a(\mathbf{e})\mathbf{f} + c(\mathbf{e})\mathbf{n}, \mathbf{n} \rangle = c(\mathbf{e}),$$

$$A(\mathbf{f}, \mathbf{f}) = \langle \partial_{\mathbf{f}} \mathbf{f}, \mathbf{n} \rangle = \langle d\mathbf{f}(\mathbf{f}), \mathbf{n} \rangle = \langle -a(\mathbf{f})\mathbf{f} + c(\mathbf{f})\mathbf{n}, \mathbf{n} \rangle = c(\mathbf{f}),$$

The matrix of the second quadratic form in the basis $\{\mathbf{e}, \mathbf{f}\}$ is

$$A = \begin{pmatrix} A(\mathbf{e}, \mathbf{e}) & A(\mathbf{f}, \mathbf{e}) \\ A(\mathbf{e}, \mathbf{f}) & A(\mathbf{f}, \mathbf{f}) \end{pmatrix} = \begin{pmatrix} b(\mathbf{e}) & b(\mathbf{f}) \\ c(\mathbf{e}) & c(\mathbf{f}) \end{pmatrix} \tag{4.55}$$

This is symmetrical matrix (see the subsection 4.3.2):

$$A(\mathbf{f}, \mathbf{e}) = b(\mathbf{f}) = A(\mathbf{e}, \mathbf{f}) = c(\mathbf{e}). \tag{4.56}$$

Weingarten operator

Let S be Weingarten operator: $S\mathbf{X} = -\partial_{\mathbf{X}} \mathbf{n}$ (see the subsection 4.3.1). Then it follows from derivation formulae that

$$S\mathbf{X} = -\partial_{\mathbf{X}} \mathbf{n} = -d\mathbf{n}(\mathbf{X}) = -(-b(X)\mathbf{e} - c(\mathbf{X})\mathbf{f}) = b(\mathbf{X})\mathbf{e} + c(\mathbf{X})f$$

In particular

$$S(\mathbf{e}) = b(\mathbf{e})\mathbf{e} + c(\mathbf{e})\mathbf{f}, S(\mathbf{f}) = b(\mathbf{f})\mathbf{e} + c(\mathbf{f})\mathbf{f}$$

and the matrix of the Weingarten operator in the basis $\{\mathbf{e}, \mathbf{f}\}$ is

$$S = \begin{pmatrix} b(\mathbf{e}) & c(\mathbf{e}) \\ b(\mathbf{f}) & c(\mathbf{f}) \end{pmatrix} \tag{4.57}$$

According to the condition (4.56) the matrix S is symmetrical.

The relations $A = GS, S = G^{-1}A$ for Weingarten operator, Riemannian metric and second quadratic form are evidently obeyed for matrices of these operators in the basis $\mathbf{e}, \mathbf{f}$ where $G = 1$, $A = S$.

4.4.5 Gaussian and mean curvature in terms of derivation formulae

Now we are equipped to express Gaussian and mean curvatures (see the subsection 4.3.3) in terms of derivation formulae. Using (4.57) we have for Gaussian curvature

$$K = \det S = b(\mathbf{e})c(\mathbf{f}) - c(\mathbf{e})b(\mathbf{f}) = (b \wedge c)(\mathbf{e}, \mathbf{f}) \tag{4.58}$$

and for mean curvature

$$H = \operatorname{Tr} S = b(\mathbf{e}) + c(\mathbf{f}) \tag{4.59}$$

What next? We will study in more detail formula (4.58) later.

Now consider some examples of calculation derivation formulae, Weingarten operator, e.t..c. for some examples using derivation formulae.

4.4.6 Examples of calculations of derivation formulae for cylinder, cone and sphere

In the subsection 4.3.4 we calculated Weingarten oeprator, second quadratic form and curvatures for cylinder, cone and sphere. Now we do the same but in terms of derivation formulae.

Cylinder

We have to define three vector fields $\mathbf{e}, \mathbf{f}, \mathbf{n}$ on the points of the cylinder

$$\mathbf{r}(h, \varphi)\colon \quad \begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \\ z = h \end{cases} \quad \text{such that they form an orthonormal basis at any}$$

point, so that the vectors $\mathbf{e}, \mathbf{f}$ are tangent to the surface and the vector $\mathbf{n}$ is orthogonal to the surface.

We already calculated coordinate vector fields $\mathbf{r}_h, \mathbf{r}_\varphi$ and normal unit vector field $\mathbf{n}$ (see (4.17)):

$$\mathbf{r}_h = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{r}_\varphi = \begin{pmatrix} -R\sin\varphi \\ R\cos\varphi \\ 0 \end{pmatrix}, \quad \mathbf{n} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}. \tag{4.60}$$

Vectors $\mathbf{r}_h, \mathbf{r}_\varphi$ and $\mathbf{n}$ are orthogonal to each other but not all of them have unit length. One can choose

$$\mathbf{e} = \mathbf{r}_h = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{f} = \frac{\mathbf{r}_\varphi}{R} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} \tag{4.61}$$

These vectors form an orthonormal basis and $\mathbf{e}, \mathbf{f}$ form an orthonormal basis in tangent space.

Derive for this basis derivation formulae (5.5). For vector fields $\mathbf{e}, \mathbf{f}, \mathbf{n}$ in (4.61) we have

$$d\mathbf{e} = 0, d\mathbf{f} = d \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} -\cos\varphi \\ -\sin\varphi \\ 0 \end{pmatrix} d\varphi = -\mathbf{n}d\varphi,$$

$$d\mathbf{n} = d \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} = \mathbf{f}d\varphi,$$

i.e.

$$d \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -d\varphi \\ 0 & d\varphi & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix}, \qquad (4.62)$$

i.e. in derivation formulae $a = b = 0$, $c = -d\varphi$.

The matrix of Weingarten operator in the basis $\{\mathbf{e}, \mathbf{f}\}$ is

$$S = \begin{pmatrix} b(\mathbf{e}) & c(\mathbf{e}) \\ b(\mathbf{f}) & c(\mathbf{f}) \end{pmatrix} = \begin{pmatrix} 0 & -d\varphi(\mathbf{e}) \\ 0 & -d\varphi(\mathbf{f}) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{R} \end{pmatrix}$$

According to (4.58) and (4.59) Gaussian curvature $K = b(\mathbf{e})c(\mathbf{f}) - b(\mathbf{e})c(\mathbf{f}) = 0$ and mean curvature

$$H = b(\mathbf{e}) + c(\mathbf{f}) = -d\varphi(\mathbf{f}) = -d\varphi\left(\frac{\mathbf{r}_\varphi}{R}\right) = -\frac{1}{R}$$

(Compare with calculations in the subsection 4.3.4)

Cone
For cone:

$$\mathbf{r}(h, \varphi): \quad \begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases},$$

$$\mathbf{r}_h = \begin{pmatrix} k\cos\varphi \\ k\sin\varphi \\ 1 \end{pmatrix}, \quad \mathbf{r}_\varphi = \begin{pmatrix} -kh\sin\varphi \\ kh\cos\varphi \\ 0 \end{pmatrix}, \quad \mathbf{n} = \frac{1}{\sqrt{1+k^2}} \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix}$$

Tangent vectors $\mathbf{r}_h, \mathbf{r}_\varphi$ are orthogonal to each other. The length of the vector $\mathbf{r}_h$ equals to $\sqrt{1+k^2}$ and the length of the vector $\mathbf{r}_\varphi$ equals to kh. Hence we

can choose orthonormal basis $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ such that vectors $\mathbf{e}, \mathbf{f}$ are unit vectors in the directions of the vectors $\mathbf{r}_h, \mathbf{r}_\varphi$:

$$\mathbf{e} = \frac{\mathbf{r}_h}{\sqrt{1+k^2}} = \frac{1}{\sqrt{1+k^2}} \begin{pmatrix} k\cos\varphi \\ k\sin\varphi \\ 1 \end{pmatrix}, \ \mathbf{f} = \frac{\mathbf{r}_\varphi}{hk} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}, \ \mathbf{n} = \frac{1}{\sqrt{1+k^2}} \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix}$$

Calculate $d\mathbf{e}, d\mathbf{f}$ and $d\mathbf{n}$:

$$d\mathbf{e} = d\begin{pmatrix} k\cos\varphi \\ k\sin\varphi \\ 1 \end{pmatrix} = \frac{kd\varphi}{\sqrt{1+k^2}} \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} = \frac{kd\varphi}{\sqrt{1+k^2}}\mathbf{f},$$

$$d\mathbf{f} = d\begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} -\cos\varphi \\ -\sin\varphi \\ 0 \end{pmatrix} d\varphi =$$

$$\frac{-k}{1+k^2} \begin{pmatrix} k\cos\varphi \\ k\sin\varphi \\ 1 \end{pmatrix} d\varphi - \frac{d\varphi}{1+k^2} \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix} = \frac{-kd\varphi}{\sqrt{1+k^2}}\mathbf{e} - \frac{d\varphi}{\sqrt{1+k^2}}\mathbf{n},$$

and

$$d\mathbf{n} = \frac{1}{\sqrt{1+k^2}}d\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix} = \frac{d\varphi}{\sqrt{1+k^2}} \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}.$$

We come to

$$d\begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 & \frac{kd\varphi}{\sqrt{1+k^2}} & 0 \\ -\frac{kd\varphi}{\sqrt{1+k^2}} & 0 & \frac{-d\varphi}{\sqrt{1+k^2}} \\ 0 & \frac{d\varphi}{\sqrt{1+k^2}} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix},$$

$$(4.63)$$

i.e. in derivation formulae for 1-forms $a = \frac{kd\varphi}{\sqrt{1+k^2}}$, $b = 0$ and and $c = -\frac{d\varphi}{\sqrt{1+k^2}}$.

The matrix of Weingarten operator in the basis $\{\mathbf{e}, \mathbf{f}\}$ is

$$S = \begin{pmatrix} b(\mathbf{e}) & c(\mathbf{e}) \\ b(\mathbf{f}) & c(\mathbf{f}) \end{pmatrix} = S = \begin{pmatrix} 0 & \frac{-d\varphi(\mathbf{e})}{\sqrt{1+k^2}} \\ 0 & \frac{-d\varphi(\mathbf{f})}{\sqrt{1+k^2}} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{-1}{kh\sqrt{1+k^2}} \end{pmatrix}.$$

since $d\varphi(\mathbf{f}) = d\varphi\left(\frac{\mathbf{r}_\varphi}{kh}\right) = \frac{1}{kh}d\varphi(\partial_\varphi) = \frac{1}{kh}$.

According to (4.58), (4.59) Gaussian curvature

$$K = b(\mathbf{e})c(\mathbf{f}) - b(\mathbf{e})c(\mathbf{f}) = 0$$

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and mean curvature

$$H = b(\mathbf{e}) + c(\mathbf{f}) = -d\varphi(\mathbf{f}) = -d\varphi\left(\frac{\mathbf{r}_\varphi}{R}\right) = -\frac{1}{R}$$

(Compare with calculations in the subsection 4.3.4)

Sphere

For sphere

$$\mathbf{r}(\theta, \varphi): \quad \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases} \qquad (4.64)$$

$$\mathbf{r}_\theta(\theta, \varphi) = \frac{\partial\mathbf{r}}{\partial\theta} = \begin{pmatrix} R\cos\theta\cos\varphi \\ R\cos\theta\sin\varphi \\ -R\sin\theta \end{pmatrix}, \quad \mathbf{r}_\varphi(\theta, \varphi) = \frac{\partial\mathbf{r}}{\partial\varphi} = \begin{pmatrix} -R\sin\theta\sin\varphi \\ R\sin\theta\cos\varphi \\ 0 \end{pmatrix},$$

$$\mathbf{n}(\theta, \varphi) = \frac{\mathbf{r}}{R} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}.$$

Tangent vectors $\mathbf{r}_\theta, \mathbf{r}_\varphi$ are orthogonal to each other. The length of the vector $\mathbf{r}_\theta$ equals to R and the length of the vector $\mathbf{r}_\varphi$ equals to $R\sin\theta$. Hence we can choose orthonormal basis $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ such that vectors $\mathbf{e}, \mathbf{f}$ are unit vectors in the directions of the vectors $\mathbf{r}_\theta, \mathbf{r}_\varphi$:

$$\mathbf{e}(\theta, \varphi) = \frac{\mathbf{r}_\theta}{R} = \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix}, \mathbf{f}(\theta, \varphi) = \frac{\mathbf{r}_\varphi}{R\sin\theta} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}, \mathbf{n}(\theta, \varphi) = \frac{\mathbf{r}}{R} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}.$$

Calculate $d\mathbf{e}, d\mathbf{f}$ and $d\mathbf{n}$:

$$d\mathbf{e} = d\begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} =$$

$$\begin{pmatrix} -\cos\theta\sin\varphi \\ \cos\theta\cos\varphi \\ 0 \end{pmatrix} d\varphi - \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} = \cos\theta d\varphi\mathbf{f} - d\theta\mathbf{n},$$

$$d\mathbf{f} = d\begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} = -\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} d\varphi =$$

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$$-\cos\theta d\varphi \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} - \sin\theta d\varphi \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} = -\cos\theta d\varphi\mathbf{e} - \sin\theta d\varphi\mathbf{n}\,,$$

$$d\mathbf{n} = d\begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} d\theta + \begin{pmatrix} -\sin\theta\sin\varphi \\ \sin\theta\cos\varphi \\ 0 \end{pmatrix} d\varphi$$

$$= d\theta\mathbf{e} + \sin\theta d\varphi\mathbf{f}\,.$$

i.e.

$$d\begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 & \cos\theta d\varphi & -d\theta \\ -\cos\theta d\varphi & 0 & -\sin\theta d\varphi \\ d\theta & \sin\theta d\varphi & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix}\,,$$
$$(4.65)$$

i.e. in derivation formulae $a = \cos\theta d\varphi$, $b = -d\theta$, $c = -\sin\theta d\varphi$.

The matrix of Weingarten operator in the basis $\{\mathbf{e}, \mathbf{f}\}$ is

$$S = \begin{pmatrix} b(\mathbf{e}) & c(\mathbf{e}) \\ b(\mathbf{f}) & c(\mathbf{f}) \end{pmatrix} = \begin{pmatrix} -d\theta(\mathbf{e}) & -\sin\theta d\varphi(\mathbf{e}) \\ -d\theta(\mathbf{f}) & -\sin\theta d\varphi(\mathbf{f}) \end{pmatrix} = \begin{pmatrix} \frac{-1}{R} & 0 \\ 0 & -\frac{1}{R} \end{pmatrix}$$

since $d\theta(\mathbf{e}) = d\theta\left(\frac{\partial_\theta}{R}\right) = \frac{1}{R}d\theta(\partial_\theta) = \frac{1}{R}$, $d\varphi(\mathbf{e}) = d\varphi\left(\frac{\partial_\theta}{R}\right) = \frac{1}{R}d\varphi(\partial_\theta) = 0$.

According to (4.58) and(4.59) Gaussian curvature

$$K = b(\mathbf{e})c(\mathbf{f}) - b(\mathbf{e})c(\mathbf{f}) = \frac{1}{R^2}$$

and mean curvature

$$H = b(\mathbf{e}) + c(\mathbf{f}) = -\frac{2}{R}$$

Notice that for calculation of Weingarten operator and curvatures we used only 1-forms b and c, i.e. the derivation equation for $d\mathbf{n}$, ($d\mathbf{n} = d\theta\mathbf{e} + \sin\theta d\varphi\mathbf{f}$). (Compare with calculations in the subsection 4.3.4)

Mean curvature is define up to a sign. If we change $\mathbf{n} \to -\mathbf{n}$ mean curvature $H \to \frac{1}{R}$ and Gaussian curvature will not change.

We see that for the sphere Gaussian curvature is not equal to zero, whilst for cylinder and cone Gaussian curvature equals to zero.

4.4.7 Proof of the Theorem of parallel transport along closed curve.

We are ready now to prove the Theorem. Recall that the Theorem states following:

If C is a closed curve on a surface M such that C is a boundary of a compact oriented domain $D \subset M$, then during the parallel transport of an arbitrary tangent vector along the closed curve C the vector rotates through the angle

$$\Delta\Phi = \angle\left(\mathbf{X}, \mathbf{R}_C \mathbf{X}\right) = \int_D K d\sigma\,, \tag{4.66}$$

where K is the Gaussian curvature and $d\sigma = \sqrt{\det g}\,dudv$ is the area element induced by the Riemannian metric on the surface M, i.e. $d\sigma = \sqrt{\det g}\,dudv$. (see (4.37).

Recall that for derivation formulae (5.5) we obtained structure equations

$$\begin{aligned} da + b \wedge c &= 0 \\ db + c \wedge a &= 0 \\ dc + a \wedge a &= 0 \end{aligned} \tag{4.67}$$

We need to use only one of these equations, the equation

$$da + b \wedge c = 0\,. \tag{4.68}$$

This condition sometimes is called *Gauß condition*.

Let as always $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ be an orthonormal basis in $T_\mathbf{p}\mathbf{E}^3$ at every point of surface $\mathbf{p} \in M$ such that $\{\mathbf{e}, \mathbf{f}\}$ is an orthonormal basis in $T_\mathbf{p}\mathbf{E}^3$ at every point of surface $\mathbf{p} \in M$. Then the Gauß condition (4.68) and equation (4.58) mean that for Gaussian curvature on the surface M can be expressed through the 2-form da and base vectors $\{\mathbf{e}, \mathbf{f}\}$:

$$K = b \wedge c(\mathbf{e}, \mathbf{f}) = -da(\mathbf{e}, \mathbf{f}) \tag{4.69}$$

We use this formula to prove the Theorem.

Now calculate the parallel transport of an arbitrary tangent vector over the closed curve C on the surface M.

Let $\mathbf{r} = \mathbf{r}(u, v) = \mathbf{r}(u\alpha)$ $(\alpha = 1, 2,\ (u, v) = (u^1, v^1))$ be an equation of the surface M.

Let $u^\alpha = u^\alpha(t)$ $(\alpha = 1, 2)$ be the equation of the curve C. Let $\mathbf{X}(t)$ be the parallel transport of vector field along the closed curve C, i.e. $\mathbf{X}(t)$ is

tangent to the surface M at the point $u(t)$ of the curve C and vector field $\mathbf{X}(t)$ is covariantly constant along the curve:

$$\frac{\nabla \mathbf{X}(t)}{dt} = 0$$

To write this equation in components we usually expanded the vector field in the coordinate basis $\{\mathbf{r}_u = \partial_u, \mathbf{r}_v = \partial_v\}$ and used Christoffel symbols of the connection $\Gamma^\alpha_{\beta\gamma}: \nabla_\beta \partial_\gamma = \Gamma^\alpha_{\beta\gamma} \partial_\alpha$.

Now we will do it in different way: *instead coordinate basis* $\{\mathbf{r}_u = \partial_u, \mathbf{r}_v = \partial_v\}$ *we will use the basis* $\{\mathbf{e}, \mathbf{f}\}$. In the subsection 3.4.4 we obtained that the connection ∇ has the following appearance in this basis

$$\nabla_{\mathbf{v}} \mathbf{e} = a(\mathbf{v})\mathbf{f}, \ , \nabla_{\mathbf{v}} \mathbf{f} = -a(\mathbf{v})\mathbf{e} \tag{4.70}$$

(see the equations (4.51) and (4.52))

Let

$$\mathbf{X} = \mathbf{X}(u(t)) = X^1(t)\mathbf{e}(u(t)) + X^2(t)\mathbf{f}(u(t))$$

Lbe an expansion of tangent vector field $\mathbf{X}(t)$ over basis $\{\mathbf{e}, \mathbf{f}\}$. Let $\mathbf{v}$ be velocity vector of the curve C. Then the equation of parallel transport $\frac{\nabla \mathbf{X}(t)}{dt} = 0$ will have the following appearance:

$$\frac{\nabla \mathbf{X}(t)}{dt} = 0 = \nabla_{\mathbf{v}} \left(X^1(t)\mathbf{e}(u(t)) + X^2(t)\mathbf{f}(u(t)) \right) =$$

$$\frac{dX^1(t)}{dt}\mathbf{e}(u(t)) + X^1(t)\nabla_{\mathbf{v}}\mathbf{e}(u(t)) + \frac{dX^2(t)}{dt}\mathbf{f}(u(t)) + X^2(t)\nabla_{\mathbf{v}}\mathbf{f}(u(t)) =$$

$$\frac{dX^1(t)}{dt}\mathbf{e}(u(t)) + X^1(t)a(\mathbf{v})\mathbf{f}(u(t)) + \frac{dX^2(t)}{dt}\mathbf{f}(u(t)) - X^2(t)a(\mathbf{v})\mathbf{e}(u(t)) =$$

$$\left(\frac{dX^1(t)}{dt} - X^2(t)a(\mathbf{v}) \right) \mathbf{e}(u(t)) + \left(\frac{dX^2(t)}{dt} + X^1(t)a(\mathbf{v}) \right) \mathbf{f}(u(t)) = 0.$$

Thus we come to equation:

$$\begin{cases} \dot{X}^1(t) - a(\mathbf{v}(t))X^2 = 0 \\ \dot{X}^2(t) + a(\mathbf{v}(t))X^1 = 0 \end{cases}$$

There are many ways to solve this equation. It is very convenient to consider complex variable

$$Z(t) = X^1(t) + iX^2(t)$$

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We see that

$$\dot{Z}(t) = \dot{X}^1(t) + i\dot{X}^2(t) = a(\mathbf{v}(t))X^2 - ia(\mathbf{v}(t)X^1 = -ia(\mathbf{v})Z(t),$$

i.e.

$$\frac{dZ(t)}{dt} = -ia(\mathbf{v}(t))Z(t) \tag{4.71}$$

The solution of this equation is:

$$Z(t) = Z(0)e^{-i\int_0^t a(\mathbf{v}(\tau))d\tau} \tag{4.72}$$

Calculate $\int_0^{t_1} a(\mathbf{v}(\tau))d\tau$ for closed curve $u(0) = u(t_1)$. Due to Stokes Theorem:

$$\int_0^{t_1} a(\mathbf{v}(t))dt = \int_C a = \int_D da$$

Hence using Gauss condition (4.68) we see that

$$\int_0^{t_1} a(\mathbf{v}(t))dt = \int_C a = \int_D da = -\int_D b \wedge c$$

Claim

$$\int_D b \wedge c = -\int_D da = \int K d\sigma \,. \tag{4.73}$$

Theorem follows from this claim:

$$Z(t_1) = Z(0)e^{-i\int_C a} = Z(0)e^{i\int_D b \wedge C} \tag{4.74}$$

Denote the integral $i\int_D b \wedge C$ by $\Delta\Phi$: $\Delta\Phi = i\int_D b \wedge C$. We have

$$Z(t_1) = X^1(t_1) + iX^2(t_1) = \left(X^1(0) + iX^2(0)\right)e^{i\Delta\Phi} = \tag{4.75}$$

It remains to prove the claim. The induced volume form $d\sigma$ is 2-form. Its value on two orthogonal unit vector $\mathbf{e}, \mathbf{f}$ equals to 1:

$$d\sigma(\mathbf{e}, \mathbf{f}) = 1 \tag{4.76}$$

(In coordinates u, v volume form $d\sigma = \sqrt{\det g}\,du \wedge dv$).

The value of the form $b \wedge c$ on vectors $\{\mathbf{e}, \mathbf{f}\}$ equals to Gaussian curvature according to (4.69). We see that

$$b \wedge c(\mathbf{e}, \mathbf{f}) = -da(\mathbf{e}, \mathbf{f}) = K d\sigma(\mathbf{e}, \mathbf{f})$$

Hence 2-forms $b \wedge c$, $-da$ and volume form $d\sigma$ coincide. Thus we prove (4.73).

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5 Curvtature tensor

5.1 Definition

Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ be an arbitrary vector fields on the manifold equipped with affine connection ∇.

Consider the following operation which assings to the vector fields $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$ the new vector field.

$$\mathcal{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z} = \left(\nabla_{\mathbf{X}} \nabla_{\mathbf{Y}} - \nabla_{\mathbf{Y}} \nabla_{\mathbf{X}} - \nabla_{[\mathbf{X}, \mathbf{Y}]} \right) \mathbf{Z} \tag{5.1}$$

This operation is obviously linear over the scalar coefficients and it is $C^{\infty}(M)$ with respect to vector fields $\mathbf{X}, \mathbf{Y}\, \mathbf{Z}$, i.e. for an arbitrary functions f, g, h

$$\mathcal{R}(f\mathbf{X}, g\mathbf{Y})(h\mathbf{Z}) = fgh\mathcal{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z}, \tag{5.2}$$

i.e. it defines the tensor field of the type $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$: If $\mathbf{X} = X^i \partial_i, \mathbf{X} = X^i \partial_i, \mathbf{X} = X^i \partial_i$ then according to (5.2)

$$\mathcal{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z} = \mathcal{R}(X^m \partial_m, Y^n \partial_n)(Z^r \partial_r) = Z^r R^i_{rmn} X^m Y^n$$

where we denote by R^i_{rmn} the components of the tensor $\mathcal{R}$ in the coordinate basis ∂_i

$$R^i_{rmn} \partial_i = \mathcal{R}(\partial_m, \partial_n)\partial_r \tag{5.3}$$

Express components of the curvature tensor in terms of Christoffel symbols of the connection. If $\nabla_m \partial_n = \Gamma^r_{mn} \partial_r$ then according to the (5.1) we have:

$$R^i_{rmn} \partial_i = \mathcal{R}(\partial_m, \partial_n)\partial_r = \nabla_{\partial_m} \nabla_{\partial_n} \partial_r - \nabla_{\partial_n} \nabla_{\partial_m} \partial_r =$$

$$\nabla_{\partial_m} \left(\Gamma^p_{nr} \partial_p \right) - \nabla_{\partial_n} \left(\Gamma^p_{mr} \partial_p \right) =$$

$$\partial_m \Gamma^i_{nr} + \Gamma^i_{mp} \Gamma^p_{nr} - (m \leftrightarrow n) \tag{5.4}$$

The proof of the property (5.2) can be given just by straightforward calculations: Consider e.g. the case $f = g = 1$, then

$$\mathcal{R}(\mathbf{X}, \mathbf{Y})(h\mathbf{Z}) = \nabla_{\mathbf{X}} \nabla_{\mathbf{Y}}(h\mathbf{Z}) - \nabla_{\mathbf{Y}} \nabla_{\mathbf{X}}(h\mathbf{Z}) - \nabla_{[\mathbf{X}, \mathbf{Y}]}(h\mathbf{Z}) =$$

$$\nabla_{\mathbf{X}} \left(\partial_{\mathbf{Y}} h\mathbf{Z} + h\nabla_{\mathbf{Y}}\mathbf{Z} \right) - \nabla_{\mathbf{Y}} \left(\partial_{\mathbf{X}} h\mathbf{Z} + h\nabla_{\mathbf{X}}\mathbf{Z} \right) - \partial_{[\mathbf{X}, \mathbf{Y}]} h\mathbf{Z} - h\nabla_{[\mathbf{X}, \mathbf{Y}]}\mathbf{Z} =$$

$$\partial_{\mathbf{X}} \partial_{\mathbf{Y}} h\mathbf{Z} + \partial_{\mathbf{Y}} h\nabla_{\mathbf{X}}\mathbf{Z} + \partial_{\mathbf{X}} h\nabla_{\mathbf{Y}}\mathbf{Z} + h\nabla_{\mathbf{X}} \nabla_{\mathbf{Y}}\mathbf{Z} -$$

$$\partial_{\mathbf{Y}}\partial_{\mathbf{X}}h\mathbf{Z} - \partial_{\mathbf{X}}h\nabla_{\mathbf{Y}}\mathbf{Z} - \partial_{\mathbf{Y}}h\nabla_{\mathbf{X}}\mathbf{Z} + h\nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}\mathbf{Z}-$$
$$\partial_{[\mathbf{X},\mathbf{Y}]}h\mathbf{Z} - h\nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z} =$$
$$h\left[\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{Z} - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}\mathbf{Z}) - \nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z}\right] + \left[\partial_{\mathbf{X}}\partial_{\mathbf{Y}}h - \partial_{\mathbf{Y}}\partial_{\mathbf{X}}h - \partial_{[\mathbf{X},\mathbf{Y}]h}\right]\mathbf{Z} =$$
$$h\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{Z} - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}\mathbf{Z}) - \nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z} = h\mathcal{R}(\mathbf{X},\mathbf{Y})\mathbf{Z},$$

since $\partial_{\mathbf{X}}\partial_{\mathbf{Y}}h - \partial_{\mathbf{Y}}\partial_{\mathbf{X}}h - \partial_{[\mathbf{X},\mathbf{Y}]}h = 0$.

5.2 Curvature of surfaces in $\mathbf{E}^3$.. *Theorema Egregium egain*

Express Riemannian curvature of surfaces in $\mathbf{E}^3$ in terms of derivation formulae (5.5).

Consider derivation formulae for the orthonormal basis $\{\mathbf{e}, \mathbf{f}, \mathbf{n}, \}$ adjusted to the surface M:

$$d\begin{pmatrix}\mathbf{e}\\\mathbf{f}\\\mathbf{n}\end{pmatrix} = \begin{pmatrix} 0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{pmatrix}\begin{pmatrix}\mathbf{e}\\\mathbf{f}\\\mathbf{n}\end{pmatrix}, \tag{5.5}$$

where as usual $\mathbf{e}, \mathbf{f}, \mathbf{n}$ vector fields of unit length which are orthogonal to each other and $\mathbf{n}$ is orthogonal to the surface M. As we know the induced connection on the surface M is defined by the formulae (4.51) and (4.52):

$$\nabla_{\mathbf{Y}}\mathbf{e} = (d\mathbf{e}(\mathbf{Y}))_{\text{tangent}} = a(\mathbf{X})\mathbf{f}, \nabla_{\mathbf{Y}}\mathbf{f} = (d\mathbf{f}(\mathbf{Y}))_{\text{tangent}} = -a(\mathbf{X})\mathbf{e}, \tag{5.6}$$

According to the definition of curvature calculate

$$R(\mathbf{e},\mathbf{f})\mathbf{e} = \nabla_{\mathbf{e}}\nabla_{\mathbf{f}}\mathbf{e} - \nabla_{\mathbf{f}}\nabla_{\mathbf{e}}\mathbf{e} - \nabla_{[\mathbf{e},\mathbf{f}]}\mathbf{e}.$$

Note that since the induced connection is symmetrical connection then:

$$\nabla_{\mathbf{e}}\mathbf{f} - \nabla_{\mathbf{f}}\mathbf{e} - [\mathbf{e},\mathbf{f}] = 0. \tag{5.7}$$

hence due to (5.6)

$$[\mathbf{e},\mathbf{f}] = \nabla_{\mathbf{e}}\mathbf{f} - \nabla_{\mathbf{f}}\mathbf{e} = a(\mathbf{e})\mathbf{e} + a(\mathbf{f})\mathbf{f} \tag{5.8}$$

Thus we see that $R(\mathbf{e},\mathbf{f})\mathbf{e} =$

$$\nabla_{\mathbf{e}}\nabla_{\mathbf{f}}\mathbf{e} - \nabla_{\mathbf{f}}\nabla_{\mathbf{e}}\mathbf{e} - \nabla_{[\mathbf{e},\mathbf{f}]}\mathbf{e} = \nabla_{\mathbf{e}}\left(a(\mathbf{f})\mathbf{f}\right) - \nabla_{\mathbf{f}}\left(a(\mathbf{e})\mathbf{f}\right) - \nabla_{a(\mathbf{e})\mathbf{e}+a(\mathbf{f})\mathbf{f}}\mathbf{e} =$$

$$\partial_{\mathbf{e}} a(\mathbf{f})\mathbf{f} + a(\mathbf{f})\nabla_{\mathbf{e}}\mathbf{f} - \partial_{\mathbf{f}} a(\mathbf{e})\mathbf{f} - a(\mathbf{e})\nabla_{\mathbf{f}}\mathbf{f} - a(\mathbf{e})\nabla_{\mathbf{e}}\mathbf{e} - a(\mathbf{f})\nabla_{\mathbf{f}}\mathbf{e} =$$

$$\partial_{\mathbf{e}} a(\mathbf{f})\mathbf{f} - a(\mathbf{f})a(\mathbf{e})\mathbf{e} - \partial_{\mathbf{f}} a(\mathbf{e})\mathbf{f} + a(\mathbf{e})a(\mathbf{f})\mathbf{e} - a(\mathbf{e})a(\mathbf{e})\mathbf{f} - a(\mathbf{f})a(\mathbf{f})\mathbf{f} =$$

$$[\partial_{\mathbf{e}} a(\mathbf{f})\mathbf{f} - \partial_{\mathbf{f}} a(\mathbf{e})\mathbf{f} - a\left((\mathbf{e})\mathbf{e} - a(\mathbf{f})\mathbf{f}\right)]\mathbf{f} = da(\mathbf{e}, \mathbf{f})\mathbf{f}\,.$$

Recall that we established in 4.69 that for Gaussian curvature K

$$K = b \wedge c(\mathbf{e}, \mathbf{f}) = -da(\mathbf{e}, \mathbf{f})$$

Hence we come to the relation:

$$R(\mathbf{e}, \mathbf{f})\mathbf{e} = da(\mathbf{e}, \mathbf{f}) = -K\mathbf{f}\,. \tag{5.9}$$

This means that

$$R^2_{112} = -K$$

(in the basis $\mathbf{e}, \mathbf{f}$), i.e. the scalar curvature

$$R = 2R_{1212} = 2K$$

We come to fundamental relation which claims that the Gaussian curvature (the magnitude defined in terms of External observer) equals to the scalar curvature, the magnitude defined in terms of Internal observer. This gives us the straightforward proof of Theorema Egregium.