# Introduction to Geometry

it is a draft of lecture notes of H.M. Khudaverdian.
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# Contents

# 1 Euclidean space

We recall important notions from linear algebra.

## 1.1 Vector space.

Vector space $V$ on real numbers is a set of vectors with operations " + "—addition of vector and " $\cdot$ "—multiplication of vector Lon real number (sometimes called coefficients, scalars). These operations obey the following axioms

- $\forall \mathbf{a}, \mathbf{b} \in V, \mathbf{a} + \mathbf{b} \in V$,
- $\forall \lambda \in \mathbf{R}, \forall \mathbf{a} \in V, \lambda \mathbf{a} \in V$.
- $\forall \mathbf{a}, \mathbf{b} \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ (commutativity)
- $\forall \mathbf{a}, \mathbf{b}, \mathbf{c},\ \mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$ (associativity)
- $\exists\ \mathbf{0}$ such that $\forall \mathbf{a}$, $\mathbf{a} + \mathbf{0} = \mathbf{a}$
- $\forall \mathbf{a}$ there exists a vector $-\mathbf{a}$ such that $\mathbf{a} + (-\mathbf{a}) = 0$.
- $\forall \lambda \in \mathbf{R}, \lambda(\mathbf{a} + \mathbf{b}) = \lambda \mathbf{a} + \lambda \mathbf{b}$
- $\forall \lambda, \mu \in \mathbf{R} (\lambda + \mu) \mathbf{a} = \lambda \mathbf{a} + \mu \mathbf{a}$
- $(\lambda \mu) \mathbf{a} = \lambda (\mu \mathbf{a})$
- $1 \mathbf{a} = \mathbf{a}$

It follows from these axioms that in particularly $\mathbf{0}$ is unique and $-\mathbf{a}$ is uniquely defined by $\mathbf{a}$. (Prove it.)

**Remark** We denote by 0 real number 0 and *vector* $\mathbf{0}$. Sometimes we have to be careful to distinguish between zero vector $\mathbf{0}$ and number zero.

Examples of vector spaces... Consider now just one non-trivial example: a space of polynomials of order $\leq 2$:

$$V = \left\{ax^2 + bx + c, a, b, c \in \mathbf{R}\right\}.$$

It is easy to see that polynomials are 'vectors' with respect to operation of addition and multiplication on numbers.

Consider **conterexample**: a space of polynomials of order 2 such that leading coefficient is equal to 1:

$$V = \{x^2 + bx + c, a, b, c \in \mathbf{R}\} .$$

This is not vcto space: why? since the for any two polynomials $f, g$ from thsi space the polynomials $f - g$, $f + g$ *does not belong to this space.*

## 1.2 Basic example of ($n$-dimensional) vector space—$\mathbf{R}^n$

A basic example of vector space (over real numbers) is a space of ordered $n$-tuples of real numbers.
$\mathbf{R}^2$ is a space of pairs of real numbers. $\mathbf{R}^2 = \{(x, y),\ x, y \in \mathbf{R}\}$
$\mathbf{R}^3$ is a space of triples of real numbers. $\mathbf{R}^3 = \{(x, y, z),\ x, y, z \in \mathbf{R}\}$
$\mathbf{R}^4$ is a space of quadruples of real numbers. $\mathbf{R}^4 = \{(x, y, z, t),\ x, y, z, t, \in \mathbf{R}\}$
and so on...

$\mathbf{R}^n$—is a space of $n$-typles of real numbers:

$$\mathbf{R}^n = \{(x^1, x^2, \dots, x^n),\ x^1, \dots, , x^n \in \mathbf{R}\} \tag{1.1}$$

If $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ are two vectors, $\mathbf{x} = (x^1, \dots, x^n)$, $\mathbf{y} = (y^1, \dots, y^n)$ then

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \dots, x_n + y_n) .$$

and multiplication on scalars is defined as

$$\lambda\mathbf{x} = \lambda \cdot (x^1, \dots, x^n) = (\lambda x^1, \dots, \lambda x^n) , \quad (\lambda \in \mathbf{R}) .$$

$(\lambda \in \mathbf{R})$.

**Remark** Why $\mathbf{R}^n$ is $n$-dimensional vector space? We see it later in the subsection 1.5

## 1.3 Affine spaces and vector spaces

Let $V$ be a vector space. A set $A$ whose elements will be called 'points' is an affine space associated with a vector space $V$ if the following rule is defined: to every point $P \in A$ and an arbitrary vector $\mathbf{x} \in V$ a point $Q$ is assigned: $(P, \mathbf{x}) \mapsto Q$. We denote $Q = P + \mathbf{x}$.

The following properties must be satisfied:

- For arbitrary two vectors $\mathbf{x}, \mathbf{y} \in V$ and arbitrary point $P \in A$, $P + (\mathbf{x} + \mathbf{y}) = (P + \mathbf{x}) + \mathbf{y}$.

- For an arbitrary point $P \in A$, $P + \mathbf{0} = P$.

  For arbitrary two points $P, Q \in A$ there exists unique vector $\mathbf{y} \in V$ such that $P + \mathbf{y} = Q$.

If $P + \mathbf{x} = Q$ we often denote the vector $\mathbf{x} = Q - P = \vec{PQ}$. We say that vector $\mathbf{x} = \vec{PQ}$ starts at the point $P$ and it ends at the point $Q$.

One can see that if vector $\mathbf{x} = \vec{PQ}$, then $\vec{QP} = -x$; if $P, Q, R$ are three arbitrary points then $\vec{PQ} + \vec{QR} = \vec{PR}$.

*Examples of affine space.*

Every vector space can be considered as an affine space in the following way. We define affine space $A$ as a same set as vector space $V$, but we consider vectors of $V$ as points of this affine space. If $A = \mathbf{a}$ is an arbitrary point of the affine space, and $\mathbf{b}$ is an arbitrary vector of vector space $V$, then $A + \mathbf{b}$ is equal to the vector $\mathbf{a} + \mathbf{b}$. We assign to two 'points' $A = \mathbf{a}, B = \mathbf{b}$ the vector $\mathbf{x} = \mathbf{b} - \mathbf{a}$.

On the other hand if $A$ is an affine space with associated vector space $V$, then choose an arbitrary point $O \in A$ and consider the vectors starting at the at the origin. We come to the vector space $V$.

One can say that vector space is an affine space with fixed origin.

For example vector space $\mathbf{R}^2$ of pairs of real numbers can be considered as a set of points. If we choose arbitrary two points $A = (a^1, a^2), B = (b^1, b^2)$, then the vector $\vec{AB} = B - A = (b^1 - a^1, b^2 - a^2)$.

## 1.4 Linear dependence of vectors

We often consider linear combinations in vector space:

$$\sum_i \lambda_i \mathbf{x}_i = \lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \cdots + \lambda_m \mathbf{x}_m \,, \tag{1.2}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are coefficients (real numbers), $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ are vectors from vector space $V$. We say that linear combination (1.2) is *trivial* if all coefficients $\lambda_1, \lambda_2, \ldots, \lambda_m$ are equal to zero.

$$\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0 \,.$$

We say that linear combination (1.2) is *not trivial* if at least one of coefficients $\lambda_1, \lambda_2, \ldots, \lambda_m$ is not equal to zero:

$$\lambda_1 \neq 0, \mathrm{or} \lambda_2 \neq 0, \mathrm{or} \ldots \mathrm{or} \lambda_m \neq 0 \,.$$

Recall definition of linearly dependent and linearly independent vectors:

**Definition** The vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ in vector space $V$ are *linearly dependent* if there exists a non-trivial linear combination of these vectors such that it is equal to zero.

In other words we say that the vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ in vector space $V$ are *linearly dependent* if there exist coefficients $\mu_1, \mu_2, \ldots, \mu_m$ such that at least one of these coefficients is not equal to zero and

$$\mu_1 \mathbf{x}_1 + \mu_2 \mathbf{x}_2 + \cdots + \mu_m \mathbf{x}_m = 0 \,. \tag{1.3}$$

Respectively vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ are *linearly independent* if they are not linearly dependent. This means that an arbitrary linear combination of these vectors which is equal zero is trivial.

In other words vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_m\}$ are *linearly independent* if the condition

$$\mu_1 \mathbf{x}_1 + \mu_2 \mathbf{x}_2 + \cdots + \mu_m \mathbf{x}_m = 0$$

implies that $\mu_1 = \mu_2 = \cdots = \mu_m = 0$.

Very useful and workable

**Proposition** *Vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ in vector space $V$ are linearly dependent if and only if at least one of these vectors is expressed via linear combination of other vectors:*

$$\mathbf{x}_i = \sum_{j \neq i} \lambda_j \mathbf{x}_j \,.$$

*Proof.* If the condition (1.4) is obeyed then $x_i - \sum_{j \neq i} \lambda_j \mathbf{x}_j = 0$. This non-trivial linear combination is equal to zero. Hence vectors $\{x_1, \ldots, \mathbf{x}_m\}$ are linearly dependent.

Now suppose that vectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$ are linearly dependent. This means that there exist coefficients $\mu_1, \mu_2, \ldots, \mu_m$ such that at least one of these coefficients is not equal to zero and the sum (1.3) equals to zero. WLOG suppose that $\mu_1 \neq 0$. We see that to

$$\mathbf{x}_1 = -\frac{\mu_2}{\mu_1} \mathbf{x}_2 - \frac{\mu_3}{\mu_1} \mathbf{x}_3 - \cdots - \frac{\mu_m}{\mu_1} \mathbf{x}_m \,,$$

i.e. vector $\mathbf{x}_1$ is expressed as linear combination of vectors $\{\mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_m\}$ ■.

## 1.5 Dimension of vector space. Basis in vector space.

**Definition** Vector space $V$ has a dimension $n$ if there exist $n$ linearly independent vectors in this vector space, and any $n+1$ vectors in $V$ are linearly dependent.

In the case if in the vector space $V$ for an arbitrary $N$ there exist $N$ linearly independent vectors then the space $V$ is *infinite-dimensional.* An example of infinite-dimensional vector space is a space $V$ of all polynomials of an arbitrary order. One can see that for an arbitrary $N$ polynomials

$$\{1, x, x^2, x^3, \dots, x^N\}$$

are linearly idependent. (Try to prove it!). This implies $V$ is infinite-dimensional vector space.

*Basis*

**Definition** Let $V$ be $n$-dimensional vector space. The ordered set $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ of $n$ linearly independent vectors in $V$ is called a basis of the vector space $V$.

**Remark** We say 'a basis', not 'the basis' since there are many bases in the vector space (see also Homeworks 1.2).

**Remark** Focus your attention: basis is *an ordered* set of vectors, not just a set of vectors[1].

**Proposition** *Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be an arbitrary basis in $n$-dimensional vector space $V$. Then any vector $\mathbf{x} \in V$ can be expressed as a linear combination of vectors $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ in a unique way, i.e. for every vector $\mathbf{x} \in V$ there exists an ordered set of coefficients $\{x^1, \dots, a^n\}$ such that*

$$\mathbf{x} = x^1\mathbf{e}_1 + \dots + x^n\mathbf{e}_n \tag{1.4}$$

*and if*

$$\mathbf{x} = a^1\mathbf{e}_1 + \dots + a^n\mathbf{e}_n = b^1\mathbf{e}_1 + \dots + b^n\mathbf{e}_n\,, \tag{1.5}$$

*then $a^1 = b^1, a^2 = b^2, \dots, a^n = b^n$. In other words for any vector $\mathbf{x} \in V$ there exists an ordered $n$-tuple $(x^1, \dots, x^n)$ of coefficients such that $\mathbf{x} = \sum_{i=1}^n x^i\mathbf{e}_i$ and this $n$-tuple is unique.*

*Proof* Let $\mathbf{x}$ be an arbitrary vector in vector space $V$. The dimension of vector space $V$ equals to $n$. Hence $n+1$ vectors $(\mathbf{e}_1, \dots, \mathbf{e}_n, \mathbf{x})$ are linearly

[1] See later on orientation of vector spaces, where the ordering of vectors of basis will be highly important.

dependent: $\lambda_1\mathbf{e}_1+\cdots+\lambda_n\mathbf{e}_n+\lambda_{n+1}\mathbf{x}=0$ and this combination is non-trivial. If $\lambda_{n+1}=0$ then $\lambda_1\mathbf{e}_1+\cdots+\lambda_n\mathbf{e}_n=0$ and this combination is non-trivial, i.e. vectors $(\mathbf{e}_1,\ldots,\mathbf{e}_n$ are linearly dependent. Contradiction. Hence $\lambda_{n+1}\neq 0$, i.e. vector $\mathbf{x}$ can be expressed via vectors $(\mathbf{e}_1,\ldots,\mathbf{e}_n)$: $\mathbf{x}=x^1\mathbf{e}_1+\ldots x^n\mathbf{e}_n$ where $x^i=-\frac{\lambda_i}{\lambda_{n+1}}$. We proved that any vector can be expressed via vectors of basis. Prove now the uniqueness of this expansion. Namely, if (1.5) holds then $(a^1-b^1)\mathbf{e}_1+(a^2-b^2)\mathbf{e}_2+\cdots+(a^n-b^n)\mathbf{e}_n=0$. Due to linear independence of basis vectors this means that $(a^1-b^1)=(a^2-b^2)=\cdots=(a^n-b^n)=0$, i.e. $a^1=b^1, a^2=b^2,\ldots,a^n=b^n$ ■

In other words:

**Basis is a set of linearly independent vectors in vector space $V$ which span (generate) vector space $V$.**

(Recall that we say that vector space $V$ is *spanned* by vectors $\{\mathbf{x}_1,\ldots,\mathbf{x}_n\}$ (or vectors vectors $\{\mathbf{x}_1,\ldots,\mathbf{x}_n\}$ *span* vector space $V$ ) if any vector $\mathbf{a}\in V$ can be expresses as a linear combination of vectors $\{\mathbf{x}_1,\ldots,\mathbf{x}_n\}$.

**Definition** Coefficients $\{a^1,\ldots,a^n\}$ are called *components of the vector* $\mathbf{x}$ *in the basis* $\{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$ or just shortly *components of the vector* $\mathbf{x}$.

**Remark** Basis is a maximal set of linearly independent vectors in a linear space $V$.

This leads to definition of a basis in infinite-dimensional space. We have to note that in infinite-dimensional space more useful becomes the conception of *topological basis* when infinite sums are considered.

*Canonical basis in* $\mathbf{R}^n$

We considered above the basic example of $n$-dimensional vector space—a space of ordered $n$-tuples of real numbers: $\mathbf{R}^n=\{(x^1,x^2,\ldots,x^n), x^i\in\mathbf{R}\}$ (see the subsection 1.2). What is the meaning of letter '$n$' in the definition of $\mathbf{R}^n$?

Consider vectors $\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_n\in\mathbf{R}^n$:

$$\begin{array}{rl}\mathbf{e}_1= & (1,0,0\ldots,0,0)\\ \mathbf{e}_2= & (0,1,0\ldots,0,0)\\ \ldots & \quad\ldots\\ \mathbf{e}_n= & (0,0,0\ldots,0,1)\end{array}\tag{1.6}$$

Then for an arbitrary vector $\mathbf{R}^n\ni\mathbf{a}=(a^1,a^2,a^3,\ldots,a^n)$

$$\mathbf{a}=(a^1,a^2,a^3,\ldots,a^n)=$$

$a^1(1,0,0\dots,0,0)+a^2(0,1,0\dots,0,0)+a^3(0,0,1,0\dots,0,0)+\dots+a^n(0,1,0\dots,0,1) =$

$$= \sum_{i=1}^{m} a^i \mathbf{e}_i = a^i \mathbf{e}_i \qquad \text{(we will use sometimes condensed notations } \mathbf{x} = x^i \mathbf{e}_i\text{)}$$

Thus we see that for every vector $\mathbf{a} \in \mathbf{R}^n$ we have unique expansion via the vectors (1.6).

The basis (1.6) is the distinguished basis. Sometimes it is called *canonical basis in* $\mathbf{R}^n$. One can find another basis in $\mathbf{R}^n$–just take an arbitrary ordered set of $n$ linearly independent vectors. (See exercises 7 and 8 in Homework 1).

## 1.6 Scalar product. Euclidean space

In vector space one have additional structure: *scalar product of vectors.*

**Definition** Scalar product in a vector space $V$ is a function $B(\mathbf{x}, \mathbf{y})$ on a pair of vectors which takes real values and satisfies the the following conditions:

$$\begin{array}{c} B(\mathbf{x}, \mathbf{y}) = B(\mathbf{y}, \mathbf{x}) \quad \text{(symmetricity condition)} \\ B(\lambda\mathbf{x} + \mu\mathbf{x}', \mathbf{y}) = \lambda B(\mathbf{x}, \mathbf{y}) + \mu B(\mathbf{x}', \mathbf{y}) \quad \text{(linearity condition)} \\ B(\mathbf{x}, \mathbf{x}) \geq 0 \ , B(\mathbf{x}, \mathbf{x}) = 0 \Leftrightarrow \mathbf{x} = 0 \ \text{(positive-definiteness condition)} \end{array} \tag{1.7}$$

**Definition** Euclidean space is a vector space equipped with a scalar product.

One can easy to see that the function $B(\mathbf{x}, \mathbf{y})$ is bilinear function, i.e. it is linear function with respect to the second argument also[2]. This follows from previous axioms:

$$B(\mathbf{x}, \lambda\mathbf{y}+\mu\mathbf{y}') \underbrace{=}_{\text{symm.}} B(\lambda\mathbf{y}+\mu\mathbf{y}', \mathbf{x}) \underbrace{=}_{\text{linear.}} \lambda B(\mathbf{y}, \mathbf{x})+\mu B(\mathbf{y}', \mathbf{x}) \underbrace{=}_{\text{symm.}} \lambda B(\mathbf{x}, \mathbf{y})+\mu B(\mathbf{x}, \mathbf{y}') \,.$$

A bilinear function $B(\mathbf{x}, \mathbf{y})$ on pair of vectors is called sometimes *bilinear form* on vector space. Bilinear form $B(\mathbf{x}, \mathbf{y})$ which satisfies the symmetricity condition is called *symmetric bilinear form.* Scalar product is nothing but symmetric bilinear form on vectors which is positive-definite: $B(\mathbf{x}, \mathbf{x}) \geq 0$) and is non-degenerate $((\mathbf{x}, \mathbf{x}) = 0 \Rightarrow \mathbf{x} = 0$.

[2]Here and later we will denote scalar product $B(\mathbf{x}, \mathbf{y})$ just by $(\mathbf{x}, \mathbf{y})$. Scalar product sometimes is called inner product. Sometimes it is called dot product.

**Example** We considered the vector space $\mathbf{R}^n$, the space of $n$-tuples (see the subsection 1.2). One can consider the vector space $\mathbf{R}^n$ as Euclidean space provided by the scalar product

$$B(\mathbf{x}, \mathbf{y}) = x^1 y^1 + \cdots + x^n y^n \tag{1.8}$$

This scalar product sometimes is called *canonical scalar product.*

**Exercise** Check that it is indeed scalar product.

**Example** We consider in 2-dimensional vector space $V$ with basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ and $B(\mathbf{X}, \mathbf{Y})$ such that $B(\mathbf{e}_1, \mathbf{e}_1) = 3$, $B(\mathbf{e}_2, \mathbf{e}_2) = 5$ and $B(\mathbf{e}_1, \mathbf{e}_2) = 0$. Then for every two vectors $\mathbf{X} = x^1\mathbf{e}_1 + x^2\mathbf{e}_2$ and $\mathbf{Y} = y^1\mathbf{e}_1 + y^2\mathbf{e}_2$ we have that

$$B(\mathbf{X}, \mathbf{Y}) = (\mathbf{X}, \mathbf{Y}) = \left(x^1\mathbf{e}_1 + x^2\mathbf{e}_2, y^1\mathbf{e}_1 + y^2\mathbf{e}_2\right) =$$

$$x^1y^1(\mathbf{e}_1, \mathbf{e}_1) + x^1y^2(\mathbf{e}_1, \mathbf{e}_2) + x^2y^1(\mathbf{e}_2, \mathbf{e}_1) + x^2y^2(\mathbf{e}_2, \mathbf{e}_2) = 3x^1y^1 + 5x^2y^2 \,.$$

One can see that all axioms are obeyed.

*Notations!*

Scalar product sometimes is called "inner" product or "dot" product. Later on we will use for scalar product $B(\mathbf{x}, \mathbf{y})$ just shorter notation $(\mathbf{x}, \mathbf{y})$ (or $\langle \mathbf{x}, \mathbf{y} \rangle$). Sometimes it is used for scalar product a notation $\mathbf{x} \cdot \mathbf{y}$. Usually this notation is reserved only for the canonical case (1.8).

**Counterexample** Consider again 2-dimensional vector space $V$ with basis $\{\mathbf{e}_1, \mathbf{e}_2\}$.

Show that operation such that $(\mathbf{e}_1, \mathbf{e}_1) = (\mathbf{e}_2, \mathbf{e}_2) = 0$ and $(\mathbf{e}_1, \mathbf{e}_2) = 1$ does not define scalar product. *Solution.* For every two vectors $\mathbf{X} = x^1\mathbf{e}_1 + x^2\mathbf{e}_2$ and $\mathbf{Y} = y^1\mathbf{e}_1 + y^2\mathbf{e}_2$ we have that

$$(\mathbf{X}, \mathbf{Y}) = \left(x^1\mathbf{e}_1 + x^2\mathbf{e}_2, y^1\mathbf{e}_1 + y^2\mathbf{e}_2\right) = x^1y^2 + x^2y^1$$

hence for vector $\mathbf{X} = (1, -1)$ $(\mathbf{X}, \mathbf{X}) = -2 < 0$. Positive-definiteness is not fulfilled.

Another **Counterexample** Show that operation $(\mathbf{X}, \mathbf{Y}) = x^1y^1 - x^2y^2$ does not define scalar product. *Solution.* Take $\mathbf{X} = (0, -1)$. Then $(\mathbf{X}, \mathbf{X}) = -1$. The condition of positive-definiteness is not fulfilled. (See also exercises in Homework 2.)

## 1.7 Orthonormal basis in Euclidean space

One can see that for scalar product (1.8) and for the basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ defined by the relation (1.6) the following relations hold:

$$(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \tag{1.9}$$

Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ be an ordered set of $n$ vectors in $n$-dimensional Euclidean space which obeys the conditions (1.9). One can see that this ordered set is a basis [3].

**Definition-Proposition** The ordered set of vectors $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ in $n$-dimensional Euclidean space which obey the conditions (1.9) is a basis. This basis is called *an orthonormal basis.*

One can prove that every (finite-dimensional) Euclidean space possesses orthonormal basis.

Later by default we consider only orthonormal bases in Euclidean spaces. Respectively scalar product will be defined by the formula (1.8). Indeed let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ be an orthonormal basis in Euclidean space. Then for an arbitrary two vectors $\mathbf{x}, \mathbf{y}$, such that $\mathbf{x} = \sum x^i \mathbf{e}_i$, $\mathbf{y} = \sum y^j \mathbf{e}_j$ we have:

$$(\mathbf{x}, \mathbf{y}) = \left(\sum x^i \mathbf{e}_i, \sum y^j \mathbf{e}_j\right) = \sum_{i,j=1}^{n} x^i y^j (\mathbf{e}_i, \mathbf{e}_j) = \sum_{i,j=1}^{n} x^i y^j \delta_{ij} = \sum_{i=1}^{n} x^i y^i$$

We come to the canonical scalar product (1.8). Later on we usually will consider scalar product defined by the formula (1.8) i.e. scalar product in orthonormal basis.

**Remark** We consider here general definition of scalar product then came to conclusion that in a special basis, (*orthonormal basis*), this is nothing but usual 'dot' product (1.8).

*Geometrical properties of scalar product: length of the vectors, angle between vectors*

The scalar product of vector on itself defines the *length of the vector*:

$$\text{Length of the vector } \mathbf{x} = |\mathbf{x}| = \sqrt{(\mathbf{x}, \mathbf{x})} = \sqrt{(x^1)^2 + \cdots + (x^n)^2} \tag{1.10}$$

[3] Indeed prove that conditions (1.9) imply that these $n$ vectors are linear independent. Suppose that $\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 + \cdots + \lambda_n \mathbf{e}_n = 0$. For an arbitrary $i$ multiply the left and right hand sides of this relation on a vector $\mathbf{e}_i$. We come to condition $\lambda_i = 0$. Hence vectors $(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)$ are linearly dependent.

If we consider Euclidean space $\mathbf{E^n}$ as the set of points (affine space) then the distance between two points $\mathbf{x}, \mathbf{y}$ is the length of corresponding vector:

$$\text{distance between points } \mathbf{x}, \mathbf{y} = |\mathbf{x} - \mathbf{y}| = \sqrt{(y^1 - x^1)^2 + \cdots + (y^n - x^n)^2}$$

We recall very important formula how scalar (inner) product is related with the angle between vectors:

$$(\mathbf{x}, \mathbf{y}) = x^1 y^1 + x^2 y^2 = |\mathbf{x}||\mathbf{y}| \cos \varphi$$

where $\varphi$ is an angle between vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbf{E}^2$.

This formula is valid also in the three-dimensional case and any $n$-dimensional case for $n \geq 1$. It gives as a tool to calculate angle between two vectors:

$$(\mathbf{x}, \mathbf{y}) = x^1 y^1 + x^2 y^2 + \cdots + x^n y^n = |\mathbf{x}||\mathbf{y}| \cos \varphi \tag{1.11}$$

In particulary it follows from this formula that

$$\begin{gathered}
\textit{angle between vectors } \mathbf{x}, \mathbf{y} \textit{ is acute if scalar product } (\mathbf{x}, \mathbf{y}) \textit{ is positive} \\
\textit{angle between vectors } \mathbf{x}, \mathbf{y} \textit{ is obtuse if scalar product } (\mathbf{x}, \mathbf{y}) \textit{ is negative} \\
\textit{vectors } \mathbf{x}, \mathbf{y} \textit{ are perpendicular if scalar product } (\mathbf{x}, \mathbf{y}) \textit{ is equal to zero}
\end{gathered} \tag{1.12}$$

**Remark** Geometrical intuition tells us that cosinus of the angle between two vectors has to be less or equal to one and it is equal to one if and only if vectors $\mathbf{x}, \mathbf{y}$ are collinear. Comparing with (1.11) we come to the inequality:

$$\begin{gathered}
(\mathbf{x}, \mathbf{y})^2 = \left(x^1 y^1 + \cdots + x^n y^n\right)^2 \leq \left((x^1)^2 + \cdots + (x^n)^2\right) \left((y^1)^2 + (\cdots + (y^n)^2\right) = (\mathbf{x}, \mathbf{x})(\mathbf{y}, \mathbf{y}) \\
\text{and} (\mathbf{x}, \mathbf{y})^2 = (\mathbf{x}, \mathbf{x})(\mathbf{y}, \mathbf{y}) \quad \text{if vectors are colienar, i.e. } x^i = \lambda y^i
\end{gathered} \tag{1.13}$$

This is famous Cauchy–Buniakovsky–Schwarz inequality, one of most important inequalities in mathematics. (See for more details Homework 2)

## 1.8 Transition matrices. Orthogonal bases and orthogonal matrices

One can consider different bases in vector space.

Let $A$ be $n \times n$ matrix with real entries, $A = ||a_{ij}||$, $i, j = 1, 2, \ldots, n$:

$$A = \begin{pmatrix}
a_{11} & a_{12} \ldots & a_{1n} \\
a_{21} & a_{22} \ldots & a_{2n} \\
a_{31} & a_{32} \ldots & a_{3n} \\
\ldots & \ldots\ldots & \ldots \\
a_{(n-1)\,1} & a_{(n-1)2} \ldots & a_{(n-1)n} \\
a_{n\,1} & a_{n2} \ldots & a_{nn}
\end{pmatrix}$$

Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ be an arbitrary basis in $n$-dimensional vector space $V$.

The basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ can be considered as row of vectors, or $1 \times n$ matrix with entries–vectors.

Multiplying $1 \times n$ matrix $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ on matrix $A$ we come to new row of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n\}$ such that

$$\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n\} = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\} A = \tag{1.14}$$

$$\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n\} = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\} \begin{pmatrix} a_{11} & a_{12} \dots & a_{1n} \\ a_{21} & a_{22} \dots & a_{2n} \\ a_{31} & a_{32} \dots & a_{3n} \\ \dots & \dots\dots & \dots \\ a_{(n-1)\,1} & a_{(n-1)2} \dots & a_{(n-1)n} \\ a_{n\,1} & a_{n2} \dots & a_{nn} \end{pmatrix} \tag{1.15}$$

,

$$\begin{cases} \mathbf{e}'_1 = a_{11}\mathbf{e}_1 + a_{21}\mathbf{e}_2 + a_{31}\mathbf{e}_3 + \dots + a_{(n-1)\,1}\mathbf{e}_{n-1} + a_{n\,1}\mathbf{e}_n \\ \mathbf{e}'_1 = a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2 + a_{32}\mathbf{e}_3 + \dots + a_{(n-1)\,2}\mathbf{e}_{n-1} + a_{n\,2}\mathbf{e}_n \\ \mathbf{e}'_1 = a_{13}\mathbf{e}_1 + a_{23}\mathbf{e}_2 + a_{33}\mathbf{e}_3 + \dots + a_{(n-1)\,3}\mathbf{e}_{n-1} + a_{n\,1}\mathbf{e}_n \\ \dots = \dots\dots + \dots\dots + \dots\dots + \dots + \dots\dots\dots\dots \\ \mathbf{e}'_n = a_{1n}\mathbf{e}_1 + a_{2n}\mathbf{e}_2 + a_{3n}\mathbf{e}_3 + \dots + a_{(n-1)\,n}\mathbf{e}_{n-1} + a_{n\,n}\mathbf{e}_n \end{cases}$$

or shortly:

$$\mathbf{e}'_i = \sum_{k=1}^{n} \mathbf{e}_k a_{ki}\,. \tag{1.16}$$

**Definition** Matrix $A$ which transforms a basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ to the row of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n\}$ (see equation (1.16)) is *transition matrix* from the basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ to the row $\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n\}$.

What is the condition that the row $\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n\}$ is a basis too? The row, ordered set of vectors, $\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n\}$ is a basis if and only if vectors $(\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n)$ are linearly independent. Thus we come to

**Proposition 1** *Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ be a basis in $n$-dimensional vector space $V$, and let $A$ be an $n \times n$ matrix with real entries. Then*

$$\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n\} = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\} A \tag{1.17}$$

*is a basis if and only if the transition matrix $A$ has rank $n$, i.e. it is non-degenerate (invertible) matrix.*

Recall that $n\times$ matrix $A$ is nondegenerate (invertible) $\Leftrightarrow \det A \neq 0$.

**Remark** Recall that the condition that $n \times n$ matrix $A$ is non-degenerate (has rank $n$) is equivalent to the condition that it is invertible matrix, or to the condition that $\det A \neq 0$.

Now suppose that $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is orthonoromal basis in $n$-dimensional Euclidean vector space. What is the condition that the new basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n\} = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}A$ is an orthonormal basis too?

**Definition** We say that $n \times n$ matrix is orthogonal matrix if its product on transposed matrix is equal to unity matrix:

$$A^T A = I \,. \tag{1.18}$$

**Exercise**. Prove that determinant of orthogonal matrix is equal to $\pm 1$:

$$A^T A = I \Rightarrow \det A = \pm 1 \,. \tag{1.19}$$

*Solution* $A^T A = I$. Hence $\det(A^T A) = \det A^T \det A = (\det A)^2 = \det I = 1$. Hence $\det A = \pm 1$. We see that in particular orthogonal matrix is non-degenerate ($\det A \neq 0$). Hence it is a transition matrix from one basis to another. The following Proposition is valid:

**Proposition 2** Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ be an orthonormal basis in $n$-dimensional Euclidean vector space. Then the new basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n\} = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}A$ is orthonormal basis if and only if the transition matrix $A$ is orthogonal matrix.

*Proof* The basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n\}$ is orthonormal means that $(\mathbf{e}'_i, \mathbf{e}'_j) = \delta_{ij}$. We have:

$$\delta_{ij} = (\mathbf{e}'_i, \mathbf{e}'_j) = \left(\sum_{m=1}^{n} \mathbf{e}_m A_{mi}, \mathbf{e}'_j = \sum_{n=1}^{n} \mathbf{e}_n A_{nj}\right) = \sum_{m,n=1}^{n} A_{mi} A_{nj} (\mathbf{e}_m, \mathbf{e}_n) =$$

$$\sum_{m,n=1}^{n} A_{mi} A_{nj} \delta_{mn} = \sum_{m=1}^{n} A_{mi} A_{mj} = \sum_{m=1}^{n} A^T_{im} A_{mj} = (A^T A)_{ij} \,, \tag{1.20}$$

Hence $(A^T A)_{ij} = \delta_{ij}$, i.e. $A^T A = I$.

We know that determinant of orthogonal matrix equals to $\pm 1$. It is very useful to consider the following groups:

- The group $O(n)$—group of orthogonal $n \times n$ matrices:

$$O(n) = \{A \colon \quad A^T A = I\} \,. \tag{1.21}$$

- The group $SO(n)$ special orthogonal group of $n \times n$ matrices:

$$SO(n) = \{A \colon \quad A^T A = I, \det A = 1\} \,. \tag{1.22}$$

## 1.9 Linear operators.

### 1.9.1 Matrix of linear operator in a given basis

Recall here facts about linear operators in vector space

Let $P$ be a linear operator in vector space $V$:

$$P\colon\ V \to V, \qquad P(\lambda \mathbf{x} + \mu \mathbf{y}) = \lambda P(\mathbf{x}) + \mu P(\mathbf{y}).$$

Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be an arbitrary basis in $n$-dimensional vector space $V$. Consider the action of operator $P$ on basis vectors: $\mathbf{e}'_i = P(\mathbf{e}_i)$:

$$\begin{array}{c}
\mathbf{e}'_1 = P(\mathbf{e}_1) = \mathbf{e}_1 p_{11} + \mathbf{e}_2 p_{21} + \mathbf{e}_3 p_{31} + \dots + \mathbf{e}_n p_{n1} \\
\mathbf{e}'_2 = P(\mathbf{e}_2) = \mathbf{e}_1 p_{12} + \mathbf{e}_2 p_{22} + \mathbf{e}_3 p_{32} + \dots + \mathbf{e}_n p_{n2} \\
\mathbf{e}'_3 = P(\mathbf{e}_3) = \mathbf{e}_1 p_{13} + \mathbf{e}_2 p_{23} + \mathbf{e}_3 p_{31} + \dots + \mathbf{e}_n p_{n3} \\
\dots \\
\mathbf{e}'_n = P(\mathbf{e}_n) = \mathbf{e}_1 p_{1n} + \mathbf{e}_2 p_{2n} + \mathbf{e}_3 p_{3n} + \dots + \mathbf{e}_n p_{nn}
\end{array} \tag{1.23}$$

**Definition** Let $\{\mathbf{e}_i\}$ be a basis. Then the transition matrix $p_{ik}$ defined by relation (1.23) is called *matrix of operator $P$ in the basis* $\{\mathbf{e}_i\}$.

$$\mathbf{e}'_i = P(\mathbf{e}_i) = \sum \mathbf{e}_k p_{ki}\,.$$

In the case if linear operator $P$ is non-degenerate (invertible) then vectors $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3, \dots, \mathbf{e}'_n$, form a basis. The matrix $P = ||p_{ik}||$ is the transition matrix from the basis $\{\mathbf{e}_i\}$ to the basis $\{\mathbf{e}'_i = P(\mathbf{e}_i)\}$.

How matrix of linear operatot changes if we change the basis? Consider a new basis $\{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ in the linear space $V$. Let $A$ be transition matrix from the basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ to the new basis $\{\mathbf{f}_1, \dots, \mathbf{f}_n\}$:

$$\{\mathbf{f}_1, \dots, \mathbf{f}_n\} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}A, \text{ i.e.} f_i = \sum_k \mathbf{e}_k a_{ki}$$

(see equation (1.16)). Then the action of operator $P$ in the new basis is given by the formula $\mathbf{f}'_i = P(\mathbf{f}_i)$. According to the formulae (1.9.1) and (1.23) we have

$$\mathbf{f}'_i = P(\mathbf{f}_i) = P\left(\sum_q \mathbf{e}_q a_{qi}\right) = \sum_q a_{qi}\left(\sum_r \mathbf{e}_r p_{rq}\right) = \sum_{q,r} \mathbf{e}_r p_{rq} a_{qi} = \sum_r \mathbf{e}_r (PA)_{ri} =$$

$$\sum_{r,k} \mathbf{f}_k (A^{-1})_{kr} (PA)_{ri} = \sum_k \mathbf{f}_k (A^{-1}PA)_{ki}\,.$$

We see that in the new basis $\{\mathbf{f}_i\}$ a matrix of linear operator is $A^{-1}PA$:

$$\text{If } \{\mathbf{e}'_1, \dots, \mathbf{e}'_n\} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}P, \text{ then } \{\mathbf{f}'_1, \dots, \mathbf{f}'_n\} = \{\mathbf{f}_1, \dots, \mathbf{f}_n\}A^{-1}PA, \ , \tag{1.24}$$

where $A$ is transition matrix from the basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ to the basis $\{\mathbf{f}_1, \dots, \mathbf{f}_n\}$,

Consider simple but important example.

**Example** Let $P$ be a linear operator in 2-dimensional vector space $V$ such that in a basis $\mathbf{e}_1, \mathbf{e}_2$ it is given by the following relation:

$$P(\mathbf{e}) = 2\mathbf{e}\,, \quad P(\mathbf{e}_2) = \mathbf{e}_2\,.$$

Then the matrix of operator $P$ in this basis is obviously

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.25}$$

Now consider another basis, $\{\mathbf{f}_1, \mathbf{f}_2\}$ in the space $V$:

$$\begin{cases} \mathbf{f}_1 = 7\mathbf{e}_1 + 5\mathbf{e}_2 \\ \mathbf{f}_2 = 4\mathbf{e}_1 + 3\mathbf{e}_2 \end{cases}, \qquad \text{respectively} \qquad \begin{cases} \mathbf{e}_1 = 3\mathbf{f}_1 - 5\mathbf{f}_2 \\ \mathbf{e}_2 = -4\mathbf{f}_1 + 7\mathbf{f}_2 \end{cases}. \tag{1.26}$$

Calculate matrix of the operator $P$ on this new basis:

$$P(\mathbf{f}_1) = P(7\mathbf{e}_1 + 5\mathbf{e}_2) = 14\mathbf{e}_1 + 5\mathbf{e}_2 = 14(3\mathbf{f}_1 - 5\mathbf{f}_2) + 5(-4\mathbf{f}_1 + 7\mathbf{f}_2) = 22\mathbf{f}_1 - 35\mathbf{f}_2\,,$$

$$P(\mathbf{f}_2) = P(4\mathbf{e}_1 + 3\mathbf{e}_2) = 8\mathbf{e}_1 + 3\mathbf{e}_2 = 8(3\mathbf{f}_1 - 5\mathbf{f}_2) + 3(-4\mathbf{f}_1 + 7\mathbf{f}_2) = 12\mathbf{f}_1 - 19\mathbf{f}_2\,.$$

Hence the matrix of operator $P$ in the basis $\{\mathbf{f}_1, \mathbf{f}_2\}$ is matrix

$$\begin{pmatrix} 22 & 12 \\ -35 & -19 \end{pmatrix}. \tag{1.27}$$

matrices (1.25) and (1.27) are different matrices which are represented the same linear operator $P$ in different bases.

One can see that

### 1.9.2 Determinant and Trace of linear operator

We recall the definition of determinant and explain what is the trace of linear operator,

**Definition-Proposition** Let $P$ be a linear operator in vector space $V$ and let $P_{ik} = ||p_{ik}||$ be transition matrix of this operator in an arbitrary basis in $V$ (see construction (1.23).) Then determinant of linear operator $P$ equals to determinant of transition matrix of this operator.

$$\det P = \det (p_{ik})$$

In the same way we define trace of operator via trace of matrix:

$$\mathrm{Tr}\, P = \mathrm{Tr}\ (||p_{ik}||) = p_{11} + p_{22} + p_{33} + \cdots + p_{nn}\,. \tag{1.28}$$

Determinant and trace of operator are well-defined. since due to (1.24) determinant and trace of transition matrice do not change if we change the basis in spite of the fact that transition matrix changes: $P \mapsto A^{-1}PA$, but

$$\det\left(A^{-1}PA\right) = \det A^{-1} \det P \det A = (\det A)^{-1} \det P \det A = \det P\,.$$

In the example above (see equations (1.25) and (1.27)) we have different matrices which represent the same but one operator $P$ in different bases, but according to(1.26)

$$\begin{pmatrix} 22 & 12 \\ -35 & -19 \end{pmatrix} = \begin{pmatrix} 7 & 4 \\ 5 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 7 & 4 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ -5 & 7 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 7 & 4 \\ 5 & 3 \end{pmatrix},$$

and

$$\det P = \det \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = 2 \cdot 1 = \det \begin{pmatrix} 22 & 12 \\ -35 & -19 \end{pmatrix} = 22 \cdot (-19) - (-35) \cdot 12 = 2$$

$$\mathrm{Tr}\, P = \mathrm{Tr} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = 2 + 1 = \mathrm{Tr} \begin{pmatrix} 22 & 12 \\ -35 & -19 \end{pmatrix} = 22 - 19 = 3$$

In the same way one can see that trace is invariant too:

$$\mathrm{Tr}\,(A^{-1}PA) = \sum_i (A^{-1}PA)_{ii} = \sum_{i,k,p} \left(A^{-1}\right)_{ik} p_{kp} = \sum_{i,k,p} A_{pi} \left(A^{-1}\right)_{ik} p_{kp} =$$

$$\sum_{p,k} \left(A \cdot A^{-1}\right)_{pk} p_{kp} = \sum_{p,k} \delta_{kp} p_{kp} = \sum_k p_{kk} = \mathrm{Tr}\, P\,.$$

Trace of linear operator is an infinitesimal version of its determinant:

$$\det(1 + tP) = 1 + t\mathrm{Tr}\, P + O(t^2)\,.$$

This is infinitesimal version for the followiong famous formula which relates trace and det of linear operator:

$$\det e^{tA} = e^{t\mathrm{Tr}\, A}\,. \tag{1.29}$$

where $e^{tA} = \sum \frac{t^n A^n}{n!}$. E.g. if $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then $e^{tA} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$, $\det e^{tA} = 1$ and $e^{t\mathrm{Tr}\, A} = e^0 = 1$.

### 1.9.3 Orthogonal linear operators

Now we study geometrical meaning of orthogonal linear operators in Euclidean space.

Recall that linear operator $P$ in Euclidean space $\mathbf{E}^n$ is called orthogonal operator if it preserves scalar product:

$$(P\mathbf{x}, P\mathbf{y}) = (\mathbf{x}, \mathbf{y}), \text{ for arbitrary vectors } \mathbf{x}, \mathbf{y} \tag{1.30}$$

In particular if $\{\mathbf{e}_i\}$ is orthonormal basis in Euclidean space then due to (1.30) the new basis $\{\mathbf{e}'_i = P(\mathbf{e}_i)\}$ is orthonormal too. Thus we see that matrix of orthogonal operator $P$ in a given orthogonal basis is orthogonal matrix:

$$P^T \cdot P = I \tag{1.31}$$

(see (1.18) in subsection 1.7). In particular we see that for orthogonal linear operator $\det P = \pm 1$ (compare with (1.19)).

## 1.10 Orthogonal operators in $\mathbf{E}^2$—Rotations and reflections

We show that an orthogonal operator 'rotates the space' or makes a 'reflection'.

Let $A$ be an arothogonal operator acting in Euclidean space $\mathbf{E}^2$: $(A\mathbf{x}, A\mathbf{y}) = (\mathbf{x}, \mathbf{y})$. Let $\{\mathbf{e}, \mathbf{f}\}$ be an orthonormal basis in 2-dimensional Euclidean space $\mathbf{E}^2$: $(\mathbf{e}, \mathbf{e}) = (\mathbf{f}, \mathbf{f}) = 1$ (i.e. $|\mathbf{e}| = |\mathbf{f}| = 1$) and $(\mathbf{e}, \mathbf{f}) = 0$–vectors $\mathbf{e}, \mathbf{f}$ have unit length and are orthogonal to each other.

Consider a new basis $\{\mathbf{e}', \mathbf{f}'\}$, an image of basis $\mathbf{e}, \mathbf{f}$ under action of $A$: $\mathbf{e}' = A(\mathbf{e})$, $\mathbf{f}' = A(\mathbf{f})$. Let $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ be matrix of operator $A$ in the basis $\mathbf{e}, \mathbf{f}$, (see equation (**??**) and defintion after this equation):

$$\{\mathbf{e}', \mathbf{f}'\} = \{\mathbf{e}, \mathbf{f}\}A = \{\mathbf{e}, \mathbf{f}\} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \text{ i.e. } \mathbf{e}' = \alpha\mathbf{e} + \gamma\mathbf{f},\ \mathbf{f}' = \beta\mathbf{e} + \delta\mathbf{f}$$

New basis is orthonormal basis also, $(\mathbf{e}', \mathbf{e}') = (\mathbf{f}', \mathbf{f}') = 1\,, \quad (\mathbf{e}', \mathbf{f}') = 0$.

Operator $A$ is orthogonal operator, and its matrix is orthogonal matrix:

$$A^T A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^t \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha^2 + \gamma^2 & \alpha\beta + \gamma\delta \\ \alpha\beta + \gamma\delta & \beta^2 + \delta^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

**Remark** With some abuse of notation, (if it is not a reason of confusion) we sometimes use the same letter for linear operator and the matrix of this operator in orthonormal basis.

We have $\alpha^2 + \gamma^2 = 1$, $\alpha\beta + \gamma\delta = 0$ and $\beta^2 + \delta^2 = 1$.

It can be shown easily that the last equation implies that matrix of operator $A$ has the following appearance:

$$A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi \cos\varphi & \end{pmatrix} ----\text{rotation on anlge } \varphi$$

or

$$A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi \cos\varphi & \end{pmatrix} ----\text{reflection on anlge ...}$$