—-Wolf inteval and how many notes there are in octave

We know that quinte (fifth) $= \frac{3}{2}$ *and on the piano it is equal to* $2^{\frac{7}{12}} \approx \frac{3}{2}$*, If we have 12 fifths it will take 7 octaves:*

$$\left(2^{\frac{7}{12}}\right)^{12} = 2^7 \approx \left(\frac{3}{2}\right)^{12} \Leftrightarrow \frac{3^{12}}{2^{12+7}} = \frac{3^{12}}{2^{19}} = \frac{531441}{524288} \approx 1\,.$$

These are numbers which are related with so called "wolf quinte" and to openednss of circles in music*

$$C\text{-}dur \stackrel{quinte}{\to} G\text{-}dur \stackrel{quinte}{\to} D\text{-}dur \stackrel{quinte}{\to} A\text{-}dur \stackrel{quinte}{\to} M\text{-}dur \stackrel{quinte}{\to} H\text{-}dur \stackrel{quinte}{\to}$$

$$Fis\text{-}dur \stackrel{quinte}{\to} Cis\text{-}dur \stackrel{quinte}{\to} Ges\text{-}dur \stackrel{quinte}{\to} Des\text{-}dur \stackrel{quinte}{\to} As\text{-}dur \stackrel{quinte}{\to} F\text{-}dur \stackrel{quinte}{\to}$$

You return to C-dur but in the new manifestation!

In fact this is related with continuous fraction for 'quinta':

We will study this fraction and will see by the way why there are 12 notes.

Let α be a number such that

$$\alpha\colon\quad 3 = 2^\alpha$$

Then continuos fraction of α gives a good approximation to α by rational numbers:

$$\alpha = [M_1, M_2, M_3, M_4, \ldots]$$

We have $\alpha = M_1 + \ldots$, i.e.

$$2^\alpha = 2^{M_1+\ldots} = 3\,,$$

i.e.

$$2^{M_1} < 3\,,\quad 2^{M_1+1} > 3 \Rightarrow M_1 = 1\,.$$

Then $\alpha = M_1 + \frac{1}{M_2+\ldots} = 1 + \frac{1}{M_2+\ldots}$, i.e.

$$2^\alpha = 2^{1+\frac{1}{M_2+\ldots}} = 3\,,$$

i.e.

$$2^{\frac{1}{M_2+\ldots}} = \frac{3}{2} \Rightarrow \left(\frac{3}{2}\right)^{M_2+\ldots} = 2\,,$$

i.e.

$$\left(\frac{3}{2}\right)^{M_2} < 2\,,\quad \text{but}\ \left(\frac{3}{2}\right)^{M_2+1} > 2$$

* *The numbers* 3^{12} *and* 2^{19}*...Oh, sweet memory: I used very much these numbers about 25 years ago playing different games with David and Tigran. Now they appear in another manifestation: we are looking here for integers* p, q *such that* $3^p \approx 2^q$*.*

i.e.

$$3^{M_2} < 2^{M_2+1}\,, \text{ but } 3^{M_2+1} > 2^{M_2+2} \Rightarrow M_2 = 1\,.$$

Then $\alpha = M_1 + \frac{1}{M_2+\frac{1}{M_3+\dots}} = 1 + \frac{1}{1+\frac{1}{M_3+\dots}}$, i.e.

$$2^{\alpha} = 2^{1+\frac{1}{1+\frac{1}{M_3+\dots}}} = 3\,,$$

i.e.

$$2^{\frac{1}{1+\frac{1}{M_3+\dots}}} = \frac{3}{2} \Rightarrow \left(\frac{3}{2}\right)^{1+\frac{1}{M_3+\dots}} = 2\,, \Rightarrow \left(\frac{3}{2}\right)^{\frac{1}{M_3+\dots}} = \frac{4}{3}\,, \Rightarrow \frac{3}{2} = \left(\frac{4}{3}\right)^{M_3+\dots}\,,$$

i.e.

$$\left(\frac{4}{3}\right)^{M_3} < \frac{3}{2}\,, \quad \text{but } \left(\frac{4}{3}\right)^{M_3+1} > \frac{3}{2}\,,$$

i.e.

$$2^{2M_3+1} < 3^{M_3+1}\,, \quad \text{but } 2^{2M_3+3} > 3^{M_3+2} \Rightarrow M_3 = 1\,,$$

Then $\alpha = M_1 + \frac{1}{M_2+\frac{1}{M_3+\frac{1}{M_4+\dots}}} = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{M_4+\dots}}}$, i.e.

$$2^{\alpha} = 2^{1+\frac{1}{1+\frac{1}{1+\frac{1}{M_4+\dots}}}} = 3\,,$$

i.e.

$$2^{\frac{1}{1+\frac{1}{1+\frac{1}{M_4+\dots}}}} = \frac{3}{2} \Rightarrow \left(\frac{3}{2}\right)^{1+\frac{1}{1+\frac{1}{M_4+\dots}}} = 2\,, \Rightarrow \left(\frac{3}{2}\right)^{\frac{1}{1+\frac{1}{M_4+\dots}}} = \frac{4}{3}\,, \Rightarrow \frac{3}{2} = \left(\frac{4}{3}\right)^{1+\frac{1}{M_4+\dots}}\,,$$

i.e.

$$\left(\frac{4}{3}\right)^{\frac{1}{M_4+\dots}} = \frac{3}{2}\cdot\frac{3}{4} \Rightarrow \left(\frac{9}{8}\right)^{M_4+\dots} = \frac{4}{3}\,,$$

i.e.

$$\left(\frac{9}{8}\right)^{M_4} < \frac{4}{3}\,, \quad \text{but } \left(\frac{9}{8}\right)^{M_4+1} > \frac{4}{3}\,,$$

i.e.

$$3^{2M_4+1} < 2^{3M_4+2}\,, \quad \text{but } 3^{2M_4+3} > 2^{3M_4+5} \Rightarrow M_4 = 2\,,$$

Indeed if $M_4 = 2$ then $3^5 = 243 < 2^8 = 256$, but if $M_4 = 3$ then $3^7 = 729 \times 3 = 2187 > 2^11 = 1024 \times 2 = 2048$.

Continue: $\alpha = M_1 + \frac{1}{M_2+\frac{1}{M_3+\frac{1}{M_4+\frac{1}{M_5+\dots}}}} = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{2+\frac{1}{M_5+\dots}}}}$, i.e.

$$2^{\alpha} = 2^{1+\frac{1}{1+\frac{1}{1+\frac{1}{2+\frac{1}{M_5+\dots}}}}} = 3\,,$$

i.e.

$$2^{\frac{1}{1+\frac{1}{1+\frac{1}{2+\frac{1}{M_5+\dots}}}}} = \frac{3}{2} \Rightarrow \left(\frac{3}{2}\right)^{1+\frac{1}{1+\frac{1}{2+\frac{1}{M_5+\dots}}}} = 2\,, \Rightarrow \left(\frac{3}{2}\right)^{\frac{1}{1+\frac{1}{2+\frac{1}{M_5+\dots}}}} = \frac{4}{3}\,, \Rightarrow \frac{3}{2} = \left(\frac{4}{3}\right)^{1+\frac{1}{2+\frac{1}{M_5+\dots}}}\,,$$

i.e.

$$\left(\frac{4}{3}\right)^{\frac{1}{2+\frac{1}{M_5+\dots}}} = \frac{3}{2}\cdot\frac{3}{4} \Rightarrow \left(\frac{9}{8}\right)^{2+\frac{1}{M_5+\dots}} = \frac{4}{3}\,, \Rightarrow \left(\frac{9}{8}\right)^{\frac{1}{M_5+\dots}} = \frac{4}{3}\cdot\frac{64}{81} \Rightarrow \left(\frac{256}{243}\right)^{M_5+\dots} = \frac{9}{8}$$

i.e.

$$\left(\frac{256}{243}\right)^{M_5} < \frac{9}{8}\,, \quad \text{but} \quad \left(\frac{256}{243}\right)^{M_5+1} > \frac{9}{8}\,,$$

i.e.

$$2^{8M_5+3} < 3^{5M_5+2}\,, \quad \text{but } 2^{8M_5+11} > 3^{5M_5+7} \Rightarrow M_5 = 2\,,$$

Indeed if $M_5 = 2$ then $2^{19} = 524288 < 3^{12} = 531441$ (these are famous phone numbers!!!!) , but if $M_5 = 3$ then

$$2^{27} = 134217728 > 3^{17} = 129140163$$

Calculate next? Let us try:
Repeat recurrently:
We already have:
$\alpha = M_1 + \frac{1}{M_2+\frac{1}{M_3+\frac{1}{M_4+\frac{1}{M_5+\frac{1}{M_6+\dots}}}}} = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{2+\frac{1}{2+\frac{1}{M_6+\dots}}}}}$, ‘ i.e.

$$2^{\alpha} = 2^{1+\frac{1}{1+\frac{1}{1+\frac{1}{2+\frac{1}{2+\frac{1}{M_6+\dots}}}}}} = 3\,,$$

i.e.

$$2^{\frac{1}{1+\frac{1}{1+\frac{1}{2+\frac{1}{2+\frac{1}{M_6+\dots}}}}}} = \frac{3}{2} \Rightarrow \left(\frac{3}{2}\right)^{1+\frac{1}{1+\frac{1}{2+\frac{1}{2+\frac{1}{M_6+\dots}}}}} = 2\,, \Rightarrow \left(\frac{3}{2}\right)^{\frac{1}{1+\frac{1}{2+\frac{1}{2+\frac{1}{M_6+\dots}}}}} = \frac{4}{3}\,, \Rightarrow$$

$$\frac{3}{2} = \left(\frac{4}{3}\right)^{1+\frac{1}{2+\frac{1}{2+\frac{1}{M_6+\dots}}}}\,,$$

i.e.

$$\left(\frac{4}{3}\right)^{\frac{1}{2+\frac{1}{2+\frac{1}{M_6+\dots}}}} = \frac{3}{2}\cdot\frac{3}{4} \Rightarrow \left(\frac{9}{8}\right)^{2+\frac{1}{2+\frac{1}{M_6+\dots}}} = \frac{4}{3}\,, \Rightarrow$$

$$\left(\frac{9}{8}\right)^{\frac{1}{2+\frac{1}{M_6+\dots}}} = \frac{4}{3}\cdot\frac{64}{81} \Rightarrow \left(\frac{256}{243}\right)^{2+\frac{1}{M_6+\dots}} = \frac{9}{8}\,, \Rightarrow \left(\frac{256}{243}\right)^{\frac{1}{M_6+\dots}} = \frac{9}{8}\cdot\left(\frac{243}{256}\right)^{2} \Rightarrow$$

$$\frac{256}{243}=\left(\frac{9\cdot 243^2}{8\cdot 256^2}\right)^{M_6+\dots}=\left(\frac{3^{12}}{2^{19}}\right)^{M_6+\dots}$$

i.e.

$$\left(\frac{3^{12}}{2^{19}}\right)^{M_6}<\frac{256}{243}\quad\text{but}\quad\left(\frac{3^{12}}{2^{19}}\right)^{M_6+1}>\frac{256}{243}$$

i.e.

$$3^{12M_6+5}<2^{19M_6+8}\quad\text{but } 3^{12M_6+17}>2^{19M_6+27}\Rightarrow M_6=3\,,$$

Indeed if $M_6=3$ then

$$\frac{3^{12M_6+5}}{2^{19M_6+8}}=\frac{3^{41}}{2^{65}}\approx 0.9886\quad\text{and}\quad\frac{3^{12(M_6+1)+5}}{2^{19(M_6+1)+8}}=\frac{3^{53}}{2^{84}}\approx 1.00201$$

This we can continue.....

The rules for the fraction:

$$\begin{gathered}3^{M_2}<2^{M_2}\\ 2^{2M_3+1}<3M_3+1\\ 3^{2M_4+1}<2^{3M_4+2}\\ 2^{8M_5+3}<35M_5+2\\ 3^{12M_6+5}<2^{19M_6+8}\\ \dots\end{gathered}$$

Aproximations:

$$\log_2 3=[1,1,1,2,2,3\dots]=1+\frac{1}{1+\frac{1}{1+\frac{1}{2+\frac{1}{2+\frac{1}{3+\dots}}}}}$$

1) $\alpha=\log_2 3\approx 1$
2) $\alpha=\log_2 3\approx 1+\frac{1}{1}=2$
3) $\alpha=\log_2 3\approx 1+\frac{1}{1+\frac{1}{1}}=\frac{3}{2}$
There are two notes: C and G.

4) $\alpha=\log_2 3\approx 1+\frac{1}{1+\frac{1}{1+\frac{1}{2}}}=\frac{8}{5}$
There are 5 notes in octave, the third is quinte.

$$2^{\frac{3}{5}}\approx 1.5157\approx\frac{3}{2}\,.$$

5) $\alpha=\log_2 3\approx 1+\frac{1}{1+\frac{1}{1+\frac{1}{2+\frac{1}{2}}}}=\frac{19}{12}$
there are 12 notes in octave, the seventh is quinte:

$$2^{\frac{7}{12}}\approx 1.498307...\approx\frac{3}{2}\,,$$

6) $\alpha = \log_2 3 \approx 1 + \frac{1}{1+\frac{1}{1+\frac{1}{2+\frac{1}{2+\frac{1}{3}}}}} = \frac{65}{41}$

there are 41 notes in octave, the twenty-forth is quinte?

$$2^{\frac{24}{41}} \approx 1.5004194... \approx \frac{3}{2}\,.$$