Introduction to Geometry (20222)

2019

COURSEWORK

This assignment counts for 20% of your marks.

It will take about 8 hours of work

Solutions are due by 28 March, 3pm

Write solutions in the provided spaces.

STUDENT'S NAME:

Academic Advisor (Tutor):

1

a) Let (x^1, x^2) be coordinates of the vector $\mathbf{x}$, and (y^1, y^2) be coordinates of the vector $\mathbf{y}$ in $\mathbf{R}^2$.

Consider the formula

$$(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^2 + kx^1y^2 + kx^2y^1 \, ,$$

where k is a real parameter. Show that this formula defines a scalar product in $\mathbf{R}^2$ in the case if $|k| < 1$.

Give an example of orthonormal basis for this scalar product.

Explain why this formula does not define a scalar product on $\mathbf{R}^2$ in the case if $|k| \geq 1$.

[2 marks]

b) Consider the matrix $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Calculate the matrix A^9 in the case if $\theta = \frac{\pi}{27}$.

Calculate the matrix A^{2019} in the case if $\theta = \frac{\pi}{6}$.

[1 marks]

c) In Euclidean space $\mathbf{E}^3$ consider the following linear operator

$$A(\mathbf{x}) = \mathbf{x} + (\mathbf{a}, \mathbf{x})\mathbf{a}$$

where the vector $\mathbf{a} = 3\mathbf{e} + 4\mathbf{f} + 12\mathbf{g}$. Here $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ is an orthonormal basis in $\mathbf{E}^3$.

Calculate the trace and determinant of the operator A.

[2 marks]

d) Let $\{\mathbf{e}, \mathbf{f}\}$ be an orthonormal basis of Euclidean space $\mathbf{E}^2$. Consider a linear operator P such that $\mathbf{a} = P(\mathbf{e}) = 91\mathbf{e} + 50\mathbf{f}$, $\mathbf{b} = P(\mathbf{f}) = 20\mathbf{e} + 11\mathbf{f}$.

Calculate determinant of the operator P.

Show that P is not an orthogonal operator.

Does this operator preserve an orientation of $\mathbf{E}^2$? Justify you ranswer.

Consider the parallelogram $\Pi_{\mathbf{a},\mathbf{b}}$ formed by the vectors $\mathbf{a}$ and $\mathbf{b}$ attached at the origin. Find the area of this parallelogram.

Show that the vertices of the parallelogram $\Pi_{\mathbf{a},\mathbf{b}}$ are the only points of $\Pi_{\mathbf{a},\mathbf{b}}$, whose coordinates are both integers.

[3 marks]

2

We consider in this question 3-dimensional Euclidean space $\mathbf{E}^3$. *We suppose that* $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$ *is an orthonormal basis in this space.*

a) Let P be a linear orthogonal operator acting in $\mathbf{E}^3$ such that its matrix in the basis $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$ has the following appearance

$$P = \frac{1}{7}\begin{pmatrix} 3 & * & 6 \\ -6 & -3 & 2 \\ 2 & -6 & * \end{pmatrix}.$$

Find the entries of the matrix denoted by *.

Show that the operator P preserves orientation.

We know that due to the Euler Theorem the linear operator P considered above is a rotation operator. Find the axis and the angle of this rotation.

[3 marks]

b) Let P_1 be a rotation operator on the angle θ around the axis directed along the vector $\mathbf{g}$, and P_2 be a rotation operator on the same angle θ around the axis directed along the vector $\mathbf{e}$:

$$\{\mathbf{e},\ \mathbf{f},\ \mathbf{g}\}\xrightarrow{P_1}\{\cos\theta\mathbf{e}+\sin\theta\mathbf{f},\ -\sin\theta\mathbf{e}+\cos\theta\mathbf{f},\ \mathbf{g}\},$$

$$\{\mathbf{e},\ \mathbf{f},\ \mathbf{g}\}\xrightarrow{P_2}\{\mathbf{e},\ \cos\theta\mathbf{f}+\sin\theta\mathbf{g},\ -\sin\theta\mathbf{f}+\cos\theta\mathbf{g}\}.$$

Show that the operator $P = P_1 \circ P_2$ is also a rotation operator. Find the axis of rotation and the angle $\Phi = \Phi(\theta)$ of rotation for the operator P.

Calculate the angle $\Phi = \Phi(\theta)$ in the case $\theta = \frac{\pi}{2}$.

Show that in the case if θ is small, then $\Phi(\theta) \approx \sqrt{2}\theta$, i.e.

$$\lim_{\theta\to 0}\frac{\Phi(\theta)}{\theta} = \sqrt{2}.$$

[5 marks]

3

a) Consider the curve $\mathbf{r}(t)$: $\begin{cases} x = Rt \\ y = R\sqrt{1-t^2} \end{cases}$, $\quad 0 \leq t \leq 1$.

Draw the image of this curve.

Give an example of a parameterisation of this curve with opposite orientation.

[1 marks]

b) Let f be a function in $\mathbf{E}^2$ given by $f = r^2 \cos 2\varphi$, where r, φ are polar coordinates in $\mathbf{E}^2$ ($x = r\cos\varphi, y = r\sin\varphi$). Consider vector fields which are given in Cartesian coordinates by $\mathbf{A} = x\partial_x + y\partial_y$, $\mathbf{B} = x\partial_y - y\partial_x$.

Calculate $\partial_{\mathbf{A}} f$, $\partial_{\mathbf{B}} f$.

Let g be a function on $\mathbf{E}^2$ such that differential form $\omega = df$ vanishes at the vector field $\mathbf{B} = x\partial_y - y\partial_x$: $\omega(\mathbf{B}) \equiv 0$. Find a function g if it is known that

$$g(x,y)\big|_{y=0} = x^6 \, .$$

[3 marks]