Dear Geometry students. In this homework we will recall such a basic notions as determinant of linear operator, and notion of orthogonal operator.

Homework 2

1 Let A be a linear operator in 2-dimensional vector space V such that for a given basis $\{\mathbf{e}, \mathbf{f}\}$,

$$A(\mathbf{e}) = 27\mathbf{e} + 40\mathbf{f}\,, A(\mathbf{f}) = -16\mathbf{e} - \frac{71}{3}\mathbf{f}\,.$$

Write down the matrix of the operator A in this basis.

Consider the pair of vectors $\{\mathbf{e}', \mathbf{f}'\}$ such that $\mathbf{e}' = 2\mathbf{e} + 3\mathbf{f}$ and $\mathbf{f}' = 3\mathbf{e} + 5\mathbf{f}$.

Show that an ordered set of vectors $\{\mathbf{e}', \mathbf{f}'\}$ is also a basis, and find a matrix of the operator A in the new basis.

Calculate the determinant and trace of operator A (compare determinants and traces of different matrix representations of this operator.)

2 Let $\mathbf{e}, \mathbf{f}$ be orthonormal basis in Euclidean space $\mathbf{E}^2$. Consider a vector

$$\mathbf{n}_\varphi = \mathbf{e}\cos\varphi + \mathbf{f}\sin\varphi\,.$$

Let A be a linear orthogonal operator acting on the space $\mathbf{E}^2$ such that $A(\mathbf{e}) = \mathbf{n}_\varphi$.

We know that $\det A = \pm 1$ since A is orthogonal operator.

In the case if $\det A = 1$, find the image $A(\mathbf{f})$ of vector $\mathbf{f}$ and an image $A(\mathbf{x})$ of an arbitrary vector $\mathbf{x} = a\mathbf{e} + b\mathbf{f}$, write down the matrix of operator A in the basis $\mathbf{e}, \mathbf{f}$ and explain geometrical meaning of the operator A.

† How the answer will change if $\det A = -1$?

3 Let $\mathbf{e}, \mathbf{f}$ be an orthonormal basis in Euclidean space $\mathbf{E}^2$.

Consider a vector $\mathbf{N} = \mathbf{e} + \mathbf{f}$ in $\mathbf{E}^2$.

Let A be an orthogonal operator acting on the space $\mathbf{E}^2$ such that $A\mathbf{N} = \mathbf{N}$. ($\mathbf{N}$ is eigenvector of A with eigenvalue 1.) Suppose that A is not identity operator.

a) Find an action of operator A on the vector $\mathbf{R} = \mathbf{e} - \mathbf{f}$ in $\mathbf{E}^2$.

b) Write down the matrix of operator A in the basis $\mathbf{e}, \mathbf{f}$.

c) Explain geometrical meaning of the operator A.

4 Let V be a space of functions, which are solutions of differential equation

$$\frac{d^2y(x)}{dx^2} + p\frac{dy(x)}{dx} + qy(x) = 0\,, \tag{1}$$

where parameters p, q are equal to

$$p = -7\,, q = 12\,. \tag{2}$$

Show that V is 2-dimensional vector space.

Find a basis in this vector space, and write down the operator A in this basis.

Differentiation $A = \frac{d}{dx}$ is linear operator on space V which transforms every vector from V to another vector on V. Check it.

Find determinant and trace of this linear operator.

5† Solve the problem 2 in the case if parameters p, q are equal to $p = -6$, $q = 9$.