Solutions of Homework 7

1 *A point moves in $\mathbf{E}^2$ along an ellipse with the law of motion $x = a\cos t$, $y = b\sin t$, $0 \le t < 2\pi$, $(0 < b < a)$. Find the velocity and acceleration vectors. Find the points of the ellipse where the angle between velocity and acceleration vectors is acute. Find the points where speed attains its maximum value.*

Calculate velocity and acceleration vectors

$$\mathbf{v} = \mathbf{r}_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} -a\sin t \\ b\cos t \end{pmatrix}, \ \mathbf{a} = \mathbf{r}_{tt} = \begin{pmatrix} x_{tt} \\ y_{tt} \end{pmatrix} = \begin{pmatrix} -a\cos t \\ -b\sin t \end{pmatrix}.$$

We see that acceleration is collinear to $\mathbf{r}$: $\mathbf{a} = -\mathbf{r}$.

The scalar product of these vectors is equal to $(\mathbf{v}, \mathbf{a}) = |\mathbf{v}||\mathbf{a}|\cos\alpha = v_x a_x + v_y a_y = (a^2 - b^2)\sin t\cos t$, where α is angle between velocity and acceleration vectors.

Speed is increasing $\Leftrightarrow$ angle α is acute $\Leftrightarrow$ $(\mathbf{v}, \mathbf{a}) > 0 \Leftrightarrow \sin t\cos t > 0 \Leftrightarrow 0 \le t \le \pi/2$ or $\pi < t < 3\pi/2$.

Speed is decreasing $\Leftrightarrow$ angle α is obtuse $\Leftrightarrow$ $(\mathbf{v}, \mathbf{a}) < 0 \Leftrightarrow \sin t\cos t < 0 \Leftrightarrow \pi/2 \le t \le \pi$ or $3\pi/2 < t < 2\pi$.

Speed is attains its maximum when $t = \frac{\pi}{2}, \frac{3\pi}{2}$ and speed attains its minimum when $t = 0, \pi$.

(At these points acceleration is orthogonal to velocity vector and scalar product is equal to zero).

2 *Find a natural parameter for the following interval of the straight line*

$$C\colon \begin{cases} x = t \\ y = 2t + 1 \end{cases}, \quad 0 < t < \infty$$

Calculate a curvature of the straight line C.

We know that natural parameter $s(t)$ measures the length of the arc of the curve between a point $\mathbf{r}(t)$ and initial point. Take a point $t = 0$: $x = 0, y = 1$ as initial point.

$$s(t) = \text{length of the interval of the line between point } (0,1) \text{ and point } (t, 2t+1)$$

If α is angle between the line and x-axis then $s(t) = t/\cos\alpha$. $\cos\alpha = \frac{1}{\sqrt{1+2^2}} = \frac{1}{\sqrt{5}}$. Hence $s(t) = t\sqrt{5}$. One comes to the same answer making straightforward integration:

$$s(t) = \int_0^t \sqrt{x_\tau^2 + y_\tau^2}\,d\tau = \int_0^t \sqrt{1 + 2^2}\,d\tau = t\sqrt{5}\,.$$

If we take another point as a initial point then natural parameter will change on a constant: E.g. if we take an initial point $(1,3)$ $(t = 1)$ then a new natural parameter:

$$s'(t) = \int_1^t \sqrt{x_\tau^2 + y_\tau^2}\,d\tau = \int_1^t \sqrt{5}\,d\tau = \sqrt{5}(t-1) = s(t) - \sqrt{5}\,.$$

Usually if a curve $\mathbf{r}(t)$ is given for parameters $t \in [t_1, t_2]$ one takes as initial a point $\mathbf{r}(t_1)$ and

$$s(t) = \int_0^t \sqrt{x_\tau^2 + y_\tau^2}\,d\tau\,.$$

The curvature of straight line equals to zero. This is evident, but one can see it from the formula $k = \frac{|\mathbf{a}_\perp|}{|v|^2}$ since for $C\colon \begin{cases} x = t \\ y = 2t + 1 \end{cases}$ velocity vector $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is constant, hence the acceleration equals to zero and the normal component of acceleration equals to zero too. One can see it also using the definition: in natural parameterisation $\begin{cases} x = \frac{s}{\sqrt{5}} \\ y = 2\frac{s}{\sqrt{5}} + 1 \end{cases}$. We see that acceleration equals to zero in natural parameterisation.

3 *Let C: $\mathbf{r}=\mathbf{r}(t)$, $0 \le t \le 2$ be a curve in $\mathbf{E}^2$ such that at an arbitrary point of this curve the velocity and acceleration vectors $\mathbf{v}(t)$ and $\mathbf{a}(t)$ are orthogonal to each other and*

$$\mathbf{v}(t)\big|_{t=0} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} .$$

Find the length of this curve.

The speed is constant because the acceleration is orthogonal to the velocity vector: $\frac{d}{dt}|\mathbf{v}|^2 = \frac{d}{dt}(\mathbf{v},\mathbf{v}) = 2(\mathbf{v},\mathbf{a}) = 0$. Hence the length of the curve is equal to $|v|t = \sqrt{3^2+4^2}\cdot 2 = 10$.

4 *Consider the following curve (a helix):*

$$\mathbf{r}(t)\colon \quad \begin{cases} x(t) = R\cos\Omega t \\ y(t) = R\sin\Omega t \\ z(t) = ct \end{cases} .$$

Find velocity and acceleration vector of this curve.

Find the curvature of this helix

Calculate velocity and acceleration vectors:

$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \begin{pmatrix} -R\Omega\sin t \\ R\Omega\cos t \\ c \end{pmatrix}, |\mathbf{v}| = \sqrt{R^2\Omega^2+c^2}\,, \mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \frac{d^2\mathbf{r}(t)}{dt^2} = \begin{pmatrix} -\Omega^2 R\cos t \\ -\Omega^2 R\sin t \\ 0 \end{pmatrix}, |\mathbf{a}| = R\,.$$

The scalar (inner) product of velocity and acceleration vectors is equal to zero: $(\mathbf{v}(t),\mathbf{a}(t)) = 0$, i.e. these vectors are orthogonal. Hence the projection of acceleration vector on velocity vector (tangential vector to the curve) is equal to zero. Thus tangential acceleration is equal to zero. (Note that speed $|\mathbf{v}|$ is constant. This also implies that tangential acceleration is equal to zero.)

One can see that helix belongs to the surface of cylinder $x^2+y^2=R^2$ and acceleration is orthogonal to surface of the cylinder.

In this exercise we have to calculate curvature of the curve in three-dimensional Euclidean space. So we need to use the formula

$$k(t) = \frac{\text{Area of parallelogram formed by vectors } \mathbf{v},\mathbf{a}}{|\mathbf{v}|^3} = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$$

We already calculated velocity and acceleration vectors for helix (see exercise 3)

We already noticed that acceleration is orthogonal to velocity vector, since their scalar product equals to zero. Hence

$$|\mathbf{v}\times\mathbf{a}| = |\mathbf{v}|\cdot|\mathbf{a}| = \Omega^2 R\sqrt{\Omega^2R^2+c^2}\,.$$

and curvature is equal to

$$k = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{|\mathbf{v}||\mathbf{a}|}{|\mathbf{v}|^3} = \frac{|\mathbf{a}|}{|\mathbf{v}|^2} = \frac{\Omega^2 R}{\Omega^2R^2+c^2} \qquad (*).$$

Another solution we could calculate curvature using the formula $k = \frac{|\mathbf{a}_\perp|}{|\mathbf{v}|^2}$. We already know that tangential acceleration is equal to zero, hence $\mathbf{a} = \mathbf{a}_{norm}$ and

$$k = \frac{|\mathbf{a}_\perp|}{|\mathbf{v}|^2} = k = \frac{|\mathbf{a}|}{|\mathbf{v}|^2}$$

We come to the formula (*).

5 *Calculate the curvature of the parabola $x = t, y = mt^2$ $(m > 0)$ at an arbitrary point. Let s be a natural parameter on this parabola. Show that the integral $\int_0^\infty k(s)ds = \int_0^\infty k(t)|\mathbf{v}(t)|dt$ and calculate this integral.*

Sure it is not practical to use the definition of curvature for calculations. It is much more practical to use the formula for curvature in arbitrary parameterisation:

$$k(t) = \frac{|x_t y_{tt} - y_t x_{tt}|}{(x_t^2 + y_t^2)^{3/2}} = \frac{|1 \cdot 2m - 2mt \cdot 0|}{(1^2 + (2mt)^2)^{3/2}} = \frac{2m}{(1 + 4m^2t^2)^{3/2}}, \quad (m > 0)\,.$$

We see that the curvature at the point (t, mt^2) is equal to $k(t) = \frac{2m}{(1+m^2t^2)^{3/2}}$ $(a > 0)$.

(Curvature is positive by definition. If $m < 0$, then $k(t) = \frac{-2m}{(1+4m^2t^2)^{3/2}}$).

To show that $\int k(s)ds = \int_0^\infty k(t)|\mathbf{v}(t)|dt$, where s is natural parameter, use the fact that $\frac{ds(t)}{dt} = |\mathbf{v}(t)|$. Hence

$$\int k(s)ds = \int k(s(t))\frac{ds(t)}{dt}dt = \int k(t)|\mathbf{v}(t)|dt$$

To calculate the integral $\int_0^\infty k(t)|\mathbf{v}(t)|$ use the results of the previous exercise:

$$\int_0^\infty k(t)|\mathbf{v}(t)| = \int_0^\infty \frac{|x_t t y_t - y_{tt} x_t|}{(x_t^2 + y_t^2)^{\frac{3}{2}}} \sqrt{x_t^2 + y_t^2} dt =$$

$$\int_0^\infty \frac{|x_t t y_t - y_{tt} x_t|}{x_t^2 + y_t^2} dt = \int_0^\infty \frac{2m}{(1 + 4m^2t^2)} dt = \int_0^\infty \frac{du}{(1 + u^2)} du = \arctan u|_0^\infty = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

Another solution:

In fact answer does depend only on "boundaries" of the curve: One can see that

$$k(t)|\mathbf{v}(t)| = \frac{d}{dt}\varphi(t),$$

where $\varphi(t)$ is the angle between the velocity vector and a given direction . One can see this also by straightforward calculation:

$$\pm k(t)|\mathbf{v}(t)| = \frac{x_t y_{tt} - x_{tt} y_t}{x_t^2 + y_t^2} = \frac{d}{dt} \arctan \frac{y_t}{x_t}$$

Hence $\int k(s)ds = \varphi|_0^{+\infty} = \pi/2$. (see in detail appendix to lecture notes)

6 Consider the parabola

$$\mathbf{r}(t)\colon \begin{cases} x = v_x t \\ y = v_y t - \frac{gt^2}{2} \end{cases}.$$

(It is path of the point moving under the gravity force with initial velocity $\mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$.) Calculate the curvature at the vertex of this parabola.

To calculate the curvature one has to perform the same calculations as in the exercise 5:

$$k(t) = \frac{|x_t y_{tt} - y_t x_{tt}|}{(x_t^2 + y_t^2)^{3/2}} = \frac{|v_x \cdot (-g)|}{((v_x^2 + (v_y - gt)^2))^{3/2}}$$

In the vertex of this parabola vertical component of velocity is equal to zero. Hence curvature at the vertex is equal to

$$k = \frac{|v_x \cdot (-g)|}{\left((v_x^2 + (v_y - gt)^2)\right)^{3/2}}|_{v_y = gt} = \frac{g}{v_x^2}\,.$$

Another solution: Curvature at any point equals to the ratio of normal acceleration to the square of the velocity: $k = \frac{|\mathbf{a}_\perp|}{v^2}$. The normal acceleration at the vertex equals to g. Hence $k = \frac{g}{v_x^2}$.

The answer in fact immediately follows from considerations of classical mechanics: If curvature in the vertex is equal to k then radius of the circle which has second order touching is equal to $R = \frac{1}{k}$ and centripetal acceleration is equal to $a = \frac{v_x^2}{R}$. On the other hand $a = g$. Hence $R = \frac{v_x^2}{g}$ and $k = \frac{g}{v_x^2}$.

Remark Note that $v_x = \sqrt{\frac{g}{k}} = \sqrt{Rg}$. if we take $R \approx 6400km$ (radius of the Earth) then $v_x \approx 8km\,\mathrm{sec}$— if a point has this velocity then it will become satellite of the Earth (we ignore resistance of atmosphere).

7 *Consider the ellipse $x = a\cos t, y = b\sin t$ $(a, b > 0,\ 0 \leq t < 2\pi)$ in $\mathbf{E}^2$. Calculate the curvature $k(t)$ at an arbitrary point of this ellipse.*

Find the radius of a circle which has second order touching with the ellipse at the point $(0, b)$.

† *Calculate $\int k(s)ds$ over the ellipse where s is a natural parameter.*

For the ellipse $\mathbf{r}(t)$: $x = a\cos t, y = b\sin t$ velocity vector $\mathbf{v}(t) = (-a\sin t, b\cos t)$, acceleration vector $\mathbf{a}(t) = (-a\cos t, -b\sin t)$ and for curvature

$$k(t) = \frac{|v_x a_y - v_y a_x|}{(v_x^2 + v_y^2)^{3/2}} = \frac{ab\sin^2 t + ab\cos^2 t}{(a^2\sin^2 t + b^2\cos^2 t)^{\frac{3}{2}}} = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{\frac{3}{2}}}\,.$$

The value of parameter t at the point $(0, b)$ is $t = \frac{\pi}{2}$. The curvature of the ellipse at the point $(0, b)$ is equal to $k(t)|_{t=\frac{\pi}{2}} = \frac{ab}{(a^2)^{\frac{3}{2}}} = \frac{b}{a^2}$. The circle has the same curvature $k = \frac{1}{R}$. Hence its radius is equal to $\frac{a^2}{b}$.

† As it follows from the previous exercise $\int k(s)ds = \int k(t)|\mathbf{v}(t)|dt$. One can calculate this integral using explicit formulae for curvature and velocity. On the other hand we already know that

$$\int_C k(s)ds = \int_C k(t)|v(t)|dt = \int_C \frac{d}{dt}\arctan\frac{y_t}{x_t} = \Delta\varphi = 2\pi.$$

8 *Calculate the curvature of the following curve (latitude on the sphere)*

$$\begin{cases} x = R\sin\theta_0\cos\varphi(t) \\ y = R\sin\theta_0\sin\varphi(t) \\ z = R\cos\theta_0 \end{cases},\ \text{where } \varphi(t) = t, 0 \leq t < 2\pi\,.$$

The curve under consideration is the circle of the radius $r = R\sin\theta_0$. Hence its curvature equals to $k = \frac{1}{R\sin\theta_0}$.

9† *Show that the curvature of an arbitrary curve on the sphere of the radius R is greater or equal to $\frac{1}{R}$.*

Let $\mathbf{r}(s)$ be a curve on the sphere of the radius R in natural parameterisat ion. We have that the curve is on the sphere. Hence $\langle \mathbf{r}(s), \mathbf{r}(s)\rangle = R^2$. Differentiate it by s we come to $\left\langle \frac{d\mathbf{r}(s)}{ds}, \mathbf{r}(s)\right\rangle = 0$. Differentiate it again over s we come to

$$0 = \frac{d}{ds}\left(\left\langle \frac{d\mathbf{r}(s)}{ds}, \mathbf{r}(s)\right\rangle\right) = \left\langle \frac{d^2\mathbf{r}(s)}{ds^2}, \mathbf{r}(s)\right\rangle + \left\langle \frac{d\mathbf{r}(s)}{ds}, \frac{d\mathbf{r}(s)}{ds}\right\rangle = kR\cos\Psi + 1 = 0 \Rightarrow k \geq \frac{1}{R}\,.$$

(Here Ψ is the angle between acceleration vector and the vector $\mathbf{r}$.)