Homework 5

1. Calculate Levi-Civita connection of the metric $G = a(u,v)du^2 + b(u,v)dv^2$

a) in the case if functions $a(u,v)$, $b(u,v)$ are constants.

b*) In general case

2. Calculate Levi-Civita connection of the metric $G = adu^2 + bdv^2$ at the point $u = v = 0$ in the case if functions $a(u,v)$, $b(u,v)$ equal to constants at the point $u = v = 0$ up to the second order:

$$a(u,v) = a_0 + \dots\,, \quad b(u,v) = b_0 + \dots$$

where dots mean the terms of the second and higher order with respect to u, v.

3. Let ∇ be a symmetric connection in $\mathbf{E}^3$ such that in Cartesian coordinates x, y, z, $\Gamma^x_{yz} = \Gamma^x_{zy} = 1$ and all other components vanish. Show explicitly that this connection is not Levi-Civita connection of standard Euclidean metric $G_{\text{Eucl}} = dx^2 + dy^2 + dz^2$, i.e. G_{Eucl} is not preserved with respect to this connection. (You have to show an example of vector fields $\mathbf{A}, \mathbf{B}, \mathbf{C}$ such that $\partial_{\mathbf{A}}\langle \mathbf{B}, \mathbf{C}\rangle \neq \langle \nabla_{\mathbf{A}}\mathbf{B}, \mathbf{C}\rangle + \langle \mathbf{B}, \nabla_{\mathbf{A}}\mathbf{C}\rangle$.)

4. Calculate Levi-Civita connection of the Riemannian metric on the sphere in stereographic coordinates:

$$G = \frac{4R^4(du^2 + dv^2)}{(R^2 + u^2 + v^2)^2}$$

a) at the point $u = v = 0$

b)* at an arbitrary point.

5 *. Calculate Levi-Civita connection of the Riemannian metric $e^{\Phi(u,v)}(du^2 + dv^2)$

6. Calculate Levi-Civita connection of Euclidean metric of a plane in

a) Cartesian coordinates

b) polar coordinates

Compare with results of previous calculations.

7. Calculate Levi-Civita connection of the Riemannian metric induced on

a) the surface of a cylinder $x^2 + y^2 = a^2$ (Compare the answer with exercise 6a.)

b) the sphere of radius R (in spherical coordinates) (Compare the answer with exercise 4)

c) the cone $x^2 + y^2 - k^2z^2 = 0$. You may use parameterisation:

$$\mathbf{r}(h,\varphi)\colon \begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases}.$$

8. Find coordinates on the surface of cylindre $x^2 + y^2 = a^2$ and on the cone $x^2 + y^2 - k^2z^2 = 0$ such that Christoffel symbols of Levi-Civita connection of induced metric vanish in these coordinates. Is it possible to do this on a sphere?