Homework 6

Christoffel symbols and Lagrangians

1 Consider the Lagrangian of a free particle $L = \frac{1}{2} g_{ik}\dot{x}^i\dot{x}^k$ for Riemannian manifold with a metric $G = g_{ik}dx^i dx^k$.
Write down the Euler-Lagrange equations of motion for this Lagrangian and compare them with differential equations for geodesics on this Riemannian manifold.

In fact show that

$$\underbrace{\frac{\partial L}{\partial x^i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}}_{\text{Euler-Lagrange equations}} \qquad \Leftrightarrow \quad \underbrace{\frac{d^2x^i}{dt^2} + \Gamma^i_{km}\dot{x}^k\dot{x}^m = 0}_{\text{Equations for geodesics}} \ , \tag{1}$$

where

$$\Gamma^i_{km} = \frac{1}{2} g^{ij}\left(\frac{\partial g_{jk}}{\partial x^m} + \frac{\partial g_{jm}}{\partial x^k} - \frac{\partial g_{km}}{\partial x^j}\right) . \tag{2}$$

2 a) Write down the Lagrangian of a free particle $L = \frac{1}{2} g_{ik}\dot{x}^i\dot{x}^k$ for Euclidean plane in polar coordinates. Calculate the Christoffel symbols for the canonical flat connection in polar coordinates using the Euler-Lagrange equations for this Lagrangian. Compare with answers which you obtained by the direct use of transformation formulae for the Christoffel symbols (see Homework 4 and lecture notes) and with answers which you obtained by direct use of the formula (2) for the Levi-Civita connection.

b) Do the same in cylindrical coordinates in $\mathbf{E}^3$: $x = r\cos\varphi, y = r\sin\varphi, z = h$.

3 Calculate the Christoffel symbols of the Levi-Civita connection for Riemannian metric $G = adu^2 + bdv^2$. Compare with results of the Exercise 1b) in the Homework 5.

4 Write down the Lagrangian of a free particle $L = \frac{1}{2} g_{ik}\dot{x}^i\dot{x}^k$ and using the Euler-Lagrange equations for this Lagrangian calculate the Christoffel symbols (the Christoffel symbols of the Levi-Civita connection) for

a) cylindrical surface of the radius a

b) for the cone $x^2 + y^2 - k^2z^2 = 0$

c) for the sphere of radius R

d) for the Lobachevsky plane

Compare with the results that you obtained using straightforwardly formula (2) or using formulae for induced connection.

5 Consider the following magnitudes:

$$\text{a)} \quad I_{\text{cylindr}}(t) = \dot{h}(t), \quad I'_{\text{cylindr}}(t) = \dot{\varphi}(t), \quad \text{for cylindre} \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases} ,$$

b) $\quad I_{\text{cone}}(t) = h^2(t)\dot{\varphi}(t), \quad$ for cone $\begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases}$,

c) $\quad I_{\text{sphere}}(t) = \sin^2\theta(t)\dot{\varphi}, \quad$ for sphere $\begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\cos\varphi \\ z = R\cos\theta \end{cases}$,

d) $\quad I_{\text{Lob.}}(t) = \dfrac{\dot{x}(t)}{y^2(t)}$, for the Lobachevsky plane (metric $G = \frac{dx^2+dy^2}{y^2}$).

Show that these magnitues are preserved along the corresponding geodesics. (You may use the Lagrangians from the previous exercise.)