Introduction to Geometry

it is a draft of lecture notes of H.M. Khudaverdian.
Manchester, 18 May 2010

Contents

1 Euclidean space

We recall important notions from linear algebra.

1.1 Vector space.

Vector space V on real numbers is a set of vectors with operations " $+$ "—addition of vector and " $\cdot$ "—multiplication of vector Lon real number (sometimes called coefficients, scalars). These operations obey the following axioms

- $\forall \mathbf{a}, \mathbf{b} \in V, \mathbf{a} + \mathbf{b} \in V$,
- $\forall \lambda \in \mathbf{R}, \forall \mathbf{a} \in V, \lambda \mathbf{a} \in V$.
- $\forall \mathbf{a}, \mathbf{b} \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ (commutativity)
- $\forall \mathbf{a}, \mathbf{b}, \mathbf{c},\ \mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$ (associativity)
- $\exists\ \mathbf{0}$ such that $\forall \mathbf{a}\ \mathbf{a} + \mathbf{0} = \mathbf{a}$
- $\forall \mathbf{a}$ there exists a vector $-\alpha$ such that $\mathbf{a} + (-\mathbf{a}) = 0$.
- $\forall \lambda \in \mathbf{R}, \lambda(\mathbf{a} + \mathbf{b}) = \lambda \mathbf{a} + \lambda \mathbf{b}$
- $\forall \lambda, \mu \in \mathbf{R} (\lambda + \mu) \mathbf{a} = \lambda \mathbf{a} + \mu \mathbf{a}$
- $(\lambda \mu) \mathbf{a} = \lambda (\mu \mathbf{a})$
- $1 \mathbf{a} = \mathbf{a}$

It follows from these axioms that in particularly $\mathbf{0}$ is unique and $-\mathbf{a}$ is uniquely defined by $\mathbf{a}$. (Prove it.)

Examples of vector spaces...

1.2 Basic example of n-dimensional vector space—$\mathbf{R}^n$

A basic example of vector space (over real numbers) is a space of ordered n-tuples of real numbers.
$\mathbf{R}^2$ is a space of pairs of real numbers. $\mathbf{R}^2 = \{(x, y),\ x, y \in \mathbf{R}\}$
$\mathbf{R}^3$ is a space of triples of real numbers. $\mathbf{R}^3 = \{(x, y, z),\ x, y, z \in \mathbf{R}\}$
$\mathbf{R}^4$ is a space of quadruples of real numbers. $\mathbf{R}^4 = \{(x, y, z, t),\ x, y, z, t, \in \mathbf{R}\}$
and so on...

$\mathbf{R}^n$—is a space of n-typles of real numbers:

$$\mathbf{R}^n = \{(x^1, x^2, \dots, x^n),\ x^1, \dots, , x^n \in \mathbf{R}\} \tag{1.1}$$

If $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ are two vectors, $\mathbf{x} = (x^1, \dots, x^n)$, $\mathbf{y} = (y^1, \dots, y^n)$ then

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \dots, x_n + y_n)\,.$$

and multiplication on scalars is defined as

$$\lambda\mathbf{x} = \lambda \cdot (x^1, \dots, x^n) = (\lambda x^1, \dots, \lambda x^n)\,, \quad (\lambda \in \mathbf{R})\,.$$

($\lambda \in \mathbf{R}$).

1.3 Linear dependence of vectors

We often consider linear combinations in vector space:

$$\sum_i \lambda_i \mathbf{x}_i = \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \cdots + \lambda_m\mathbf{x}_m\,, \tag{1.2}$$

where $\lambda_1, \lambda_2, \dots, \lambda_m$ are coefficients (real numbers), $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m$ are vectors from vector space V.

We say that linear combination (1.2) is *trivial* if all coefficients $\lambda_1, \lambda_2, \dots, \lambda_m$ are equal to zero.

$$\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0\,.$$

We say that linear combination (1.2) is *not trivial* if at least one of coefficients $\lambda_1, \lambda_2, \dots, \lambda_m$ is not equal to zero:

$$\lambda_1 \neq 0, \mathrm{or} \lambda_2 \neq 0, \mathrm{or} \dots \mathrm{or} \lambda_m \neq 0\,.$$

Recall definition of linearly dependent and linearly independent vectors:

Definition The vectors $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ in vector space V are *linearly dependent* if there exists a non-trivial linear combination of these vectors such that it is equal to zero.

In other words we say that the vectors $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ in vector space V are *linearly dependent* if there exist coefficients $\mu_1, \mu_2, \dots, \mu_m$ such that at least one of these coefficients is not equal to zero and

$$\mu_1 \mathbf{x}_1 + \mu_2 \mathbf{x}_2 + \cdots + \mu_m \mathbf{x}_m = 0\,. \tag{1.3}$$

Respectively vectors $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ are *linearly independent* if they are not linearly dependent. This means that an arbitrary linear combination of these vectors which is equal zero is trivial.

In other words vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_m\}$ are *linearly independent* if the condition

$$\mu_1 \mathbf{x}_1 + \mu_2 \mathbf{x}_2 + \cdots + \mu_m \mathbf{x}_m = 0$$

implies that $\mu_1 = \mu_2 = \cdots = \mu_m = 0$.

Very useful and workable

Proposition *Vectors $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ in vector space V are linearly dependent if and only if at least one of these vectors is expressed via linear combination of other vectors:*

$$\mathbf{x}_i = \sum_{j \neq i} \lambda_j \mathbf{x}_j\,. \tag{1.4}$$

Proof. If the condition (1.4) is obeyed then $x_i - \sum_{j\neq i} \lambda_j \mathbf{x}_j = 0$. This non-trivial linear combination is equal to zero. Hence vectors $\{x_1, \dots, \mathbf{x}_m\}$ are linearly dependent.

Now suppose that vectors $\{\mathbf{x}_1, \dots, \mathbf{x}_m\}$ are linearly dependent. This means that there exist coefficients $\mu_1, \mu_2, \dots, \mu_m$ such that at least one of these coefficients is not equal to zero and the sum (1.3) equals to zero. WLOG suppose that $\mu_1 \neq 0$. We see that to

$$\mathbf{x}_1 = -\frac{\mu_2}{\mu_1}\mathbf{x}_2 - \frac{\mu_3}{\mu_1}\mathbf{x}_3 - \cdots - \frac{\mu_m}{\mu_1}\mathbf{x}_m\,,$$

i.e. vector $\mathbf{x}_1$ is expressed as linear combination of vectors $\{\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_m\}$ $\blacksquare$.

Formulate and give a proof of useful

Lemma Let m vectors $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ belong to the span of n vectors $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$, i.e. every vector $\mathbf{x}_i$ $(i = 1, \dots, m)$ can be expressed as a linear combination of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$:

$$\begin{cases} \mathbf{x}_1 = \lambda_1^1 \mathbf{a}_1 + \lambda_1^2 \mathbf{a}_2 + \dots + \lambda_1^n \mathbf{a}_n \\ \mathbf{x}_2 = \lambda_2^1 \mathbf{a}_1 + \lambda_2^2 \mathbf{a}_2 + \dots + \lambda_2^n \mathbf{a}_n \\ \mathbf{x}_3 = \lambda_3^1 \mathbf{a}_1 + \lambda_3^2 \mathbf{a}_2 + \dots + \lambda_3^n \mathbf{a}_n \\ \dots \\ \mathbf{x}_m = \lambda_m^1 \mathbf{a}_1 + \lambda_m^2 \mathbf{a}_2 + \dots + \lambda_m^n \mathbf{a}_n \end{cases} \tag{1.5}$$

Then vectors $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ are linearly dependent if $m > n$.

Proof

Prove using mathematical induction. If $n = 1$ then the statement is obvious: it follows from (1.5) that all vectors $\mathbf{x}_i$ are proportional to vector $\mathbf{a}_1$. Hence they are linearly dependent, since they are proportional each other.

Let a statement be true for $m = k$. Prove it for $m = k + 1$.

Consider the first equation

$$\mathbf{x}_1 = \lambda_1^1 \mathbf{a}_1 + \lambda_1^2 \mathbf{a}_2 + \dots + \lambda_1^k \mathbf{a}_k + \lambda_1^{k+1} \mathbf{a}_{k+1} \tag{1.6}$$

If vector $\mathbf{x}_1 = 0$ then nothing to prove: vectors $\{0, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_m\}$ are linearly dependent. If $\mathbf{x}_1 \neq 0$ then one of the coefficients in (1.6) is not equal to zero. WLOG suppose that $\lambda_{k+1}^1 \neq 0$. Hence $\mathbf{a}_{k+1}$ can be expressed as a linear combination of vectors $\{\mathbf{x}_1, \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k\}$:

$$\mathbf{a}_{k+1} = \mathbf{x}_1 - \frac{\lambda_1^1}{\lambda_1^{k+1}} \mathbf{a}_1 - \frac{\lambda_1^2}{\lambda_1^{k+1}} \mathbf{a}_2 - \dots - \frac{\lambda_1^k}{\lambda_1^{k+1}} \mathbf{a}_k$$

Input this expansion of $\mathbf{a}_{k+1}$ in expressions in (1.5) for vectors $\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_m$. We will see that $m - 1$ vectors

$$\mathbf{x}'_2 = \mathbf{x}_2 - \lambda_2^{k+1} \mathbf{x}_1, \mathbf{x}'_3 = \mathbf{x}_3 - \lambda_3^{k+1} \mathbf{x}_1 \dots, \mathbf{x}'_m = \mathbf{x}_m - \lambda_m^{k+1} \mathbf{x}_1 \tag{1.7}$$

are expressed as linear combinations of k vectors $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k\}$. Hence due to inductive hypothesis vectors $\mathbf{x}'_2, \mathbf{x}'_3, \dots, \mathbf{x}'_m$ are linearly dependent:

$$\mu_2 \mathbf{x}'_2 + \dots + \mu_m \mathbf{x}'_m = \mu_2 (\mathbf{x}_2 - \lambda_2^{k+1} \mathbf{x}_1) + \dots + \mu_m (\mathbf{x}_m - \lambda_m^{k+1} \mathbf{x}_1) = 0,$$

where one of coefficients $\mu_2, \dots, \mu_m$ is not equal to zero. Now it follows from (1.7) that this is non-trivial combination of vectors $\mathbf{x}_1, \dots, \mathbf{x}_m$, i.e. vectors $\mathbf{x}_1, \dots, \mathbf{x}_m$ are linearly dependent. ■

1.4 Dimension of vector space. Basis in vector space.

Definition Vector space V has a dimension n if there exist n linearly independent vectors in this vector space, and any $n+1$ vectors in V are linearly dependent.

In the case if in the vector space V there exist n linearly independent vectors for an arbitrary natural number n then the space V is *infinite-dimensional*

Basis

Recall that we say that vector space V is *spanned* by vectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ (or vectors vectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ *span* vector space V) if any vector $\mathbf{a} \in V$ can be expresses as a linear combination of vectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$.

Basis is a set of linearly independent vectors in vector space V which span (generate) vector space V. More in detail:

Definition Let V be n-dimensional vector space. The ordered set $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ of n linearly independent vectors in V is called a basis (an ordered basis) of the vector space V if these vectors are linearly independent.

Proposition 1 *Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be an arbitrary basis in n-dimensional vector space V. Then any vector $\mathbf{a} \in V$ can be expressed as a linear combination of vectors $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ in a unique way, i.e. for every vector $\mathbf{x} \in V$ there exist a set an ordered set of coefficients $\{a^1, \ldots, a^n\}$ such that*

$$\mathbf{x} = x^1\mathbf{e}_1 + \cdots + x^n\mathbf{e}_n \tag{1.8}$$

and if

$$\mathbf{x} = a^1\mathbf{e}_1 + \cdots + a^n\mathbf{e}_n = b^1\mathbf{e}_1 + \cdots + b^n\mathbf{e}_n\,, \tag{1.9}$$

then $a^1 = b^1, a^2 = b^2, \ldots, a^n = b^n$. In other words for any vector $\mathbf{x} \in V$ there exists an ordered n-tuple $(x^1, \ldots, x^n)$ of coefficients such that $\mathbf{x} = \sum_{i=1}^n x^i\mathbf{e}_i$ and this n-tuple is unique.

Proof Let $\mathbf{x}$ be an arbitrary vector in vector space V. The dimension of vector space V equals to n. Hence $n+1$ vectors $(\mathbf{e}_1, \ldots, \mathbf{e}_n, \mathbf{x})$ are linearly dependent: $\lambda_1\mathbf{e}_1 + \cdots + \lambda_n\mathbf{e}_n + \lambda_{n+1}\mathbf{x} = 0$ and this combination is non-trivial. If $\lambda_{n+1} = 0$ then $\lambda_1\mathbf{e}_1 + \cdots + \lambda_n\mathbf{e}_n = 0$ and this combination is non-trivial, i.e. vectors $(\mathbf{e}_1, \ldots, \mathbf{e}_n$ are linearly dependent. Contradiction. Hence $\lambda_{n+1} \neq 0$, i.e. vector $\mathbf{x}$ can be expressed via vectors $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$: $\mathbf{x} = x^1\mathbf{e}_1 + \ldots x^n\mathbf{e}_n$ where $x^i = -\frac{\lambda_i}{\lambda_{n+1}}$. We prove that any vector can be expressed via vectors of basis. Prove now uniqueness. This expansion is unique. Indeed if (1.9) holds then $(a^1 - b^1)\mathbf{e}_1 + (a^2 - b^2)\mathbf{e}_2 + \dots + (a^n - b^n)\mathbf{e}_n = 0$. Due to linear independence of basis vectors

this means that $(a^1 - b^1) = (a^2 - b^2) = \dots = (a^n - b^n) = 0$, i.e. $a^1 = b^1, a^2 = b^2, \dots, a^n = b^n$ ∎

Definition Coefficients $\{a^1, \dots, a^n\}$ are called *components of the vector* $\mathbf{x}$ *in the basis* $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ or just shortly *components of the vector* $\mathbf{x}$.

Another very useful and workable statement

Proposition 2 Let $\{\mathbf{e}_1, \dots, \mathbf{e}_m\}$ be an ordered set of vectors in vector space V such that an arbitrary vector $\mathbf{x} \in V$ can be expressed as a linear combination of vectors $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ in a unique way (see (1.8) and (1.9) above). Then

- V is a finite-dimensional space of dimension m.
- $\{\mathbf{e}_1, \dots, \mathbf{e}_m\}$ is a basis in this space.

This is very practical statement: it can be often used to find a dimension of vector space.

Remark We say "a basis" not "the basis", since there are many bases in the vector space V. (See below and also Homework 1).

Remark Basis is a maximal set of linearly independent vectors in a linear space V. (See exercise 5 in Homework 1)

Proof. Show first that the vectors $\{\mathbf{e}_1, \dots, \mathbf{e}_m\}$ are linearly independent. Let $\mu_1\mathbf{e}_1 + \mu_2 + \dots + \mu_m\mathbf{e}_m = 0$. This relation holds if $\mu_1 = \mu_2 = \mu_3 = \dots = \mu_m = 0$. Due to the uniqueness of the expansion applied to the vector $\mathbf{x} = 0$ we see that $\mu_1\mathbf{e}_1 + \mu_2 + \dots + \mu_m\mathbf{e}_m = 0$ implies that $\mu_1 = \mu_2 = \mu_3 = \dots = \mu_m = 0$. Hence vectors $\{\mathbf{e}_1, \dots, \mathbf{e}_m\}$ are linearly independent. We proved that $\dim V \geq m$. Consider an arbitrary $m + 1$ vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{m+1}$. For any of these vectors $\mathbf{x}_i$ vectors $\{\mathbf{x}_i, \mathbf{e}_1, \dots, \mathbf{e}_m\}$ are linearly dependent and vectors $\{\mathbf{e}_1, \dots, \mathbf{e}_m\}$ are linearly independent. Hence any of vectors $\mathbf{x}_i$ can be expressed as a linear combination of vectors $\{\mathbf{e}_1, \dots, \mathbf{e}_m\}$. Thus we proved that any $m + 1$ vectors in V belong to the space of vectors $\{\mathbf{e}_1, \dots, \mathbf{e}_m\}$. Hence according to the lemma in the subsection 1.2 of lecture notes any $m+1$ vectors in V are linearly dependent. Thus we proved that $\dim V = m$.

The ordered set $\{\mathbf{e}_1, \dots, \mathbf{e}_m\}$ is a set of m linearly independent vectors in m-dimensional vector space V. Hence $\{\mathbf{e}_1, \dots, \mathbf{e}_m\}$ is a basis.

Canonical basis in $\mathbf{R}^n$

We considered above the basic example of n-dimensional vector space—a space of ordered n-tuples of real numbers: $\mathbf{R}^n = \{(x^1, x^2, \dots, x^n),\ x, y \in \mathbf{R}\}$.

What is the meaning of letter n in the definition of $\mathbf{R}^n$?

Consider vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n \in \mathbf{R}^n$:

$$\begin{array}{rl} \mathbf{e}_1 = & (1,0,0\dots,0,0) \\ \mathbf{e}_2 = & (0,1,0\dots,0,0) \\ \dots & \quad\dots \\ \mathbf{e}_n = & (0,0,0\dots,0,1) \end{array} \tag{1.10}$$

Then for an arbitrary vector $\mathbf{R}^n \ni \mathbf{a} = (a^1, a^2, a^3, \dots, a^n)$

$$\mathbf{a} = (a^1, a^2, a^3, \dots, a^n) =$$

$$a^1(1,0,0\dots,0,0)+a^2(0,1,0\dots,0,0)+a^3(0,0,1,0\dots,0,0)+\dots+a^n(0,1,0\dots,0,1) =$$

$$= \sum_{i=1}^{m} a^i \mathbf{e}_i = a^i \mathbf{e}_i \qquad \text{(we will use sometimes condensed notations } \mathbf{x} = x^i \mathbf{e}_i\text{)}$$

Thus we see that for every vector $\mathbf{a} \in \mathbf{R}^n$ we have unique expansion via the vectors (1.10). Thus according to Proposition 2 above the dimension of the space $\mathbf{R}^n$ is equals to n and (1.10) is a basis in $\mathbf{R}^n$.

Remark One can find another basis in $\mathbf{R}^n$–just take an arbitrary ordered set of n linearly independent vectors. (See exercise 7 in Homework 1). The basis (1.10) is distinguished. Sometimes it is called *canonical basis in* $\mathbf{R}^n$.

Remark One can consider set of ordered n-tuples in $\mathbf{R}^n$ as the set of points. Two points $\mathbf{a}, \mathbf{b} \in \mathbf{R}^n$ define a vector: if $\mathbf{a} = (a^1, \dots, a^n)$, $\mathbf{b} = (b^1, \dots, b^n)$, then the vector $\mathbf{ab}$ attached to the point $\mathbf{a}$ has coordinates $= (b^1 - a^1, b^2 - a^2 \dots, b^n - a^n)$ [1].

1.5 Scalar product. Euclidean space

In vector space one have additional structure: *scalar product of vectors.*

Definition Scalar product in a vector space V is a function $B(\mathbf{x}, \mathbf{y})$ on a pair of vectors which takes real values and satisfies the the following conditions:

$$\begin{array}{c} B(\mathbf{x}, \mathbf{y}) = B(\mathbf{y}, \mathbf{x}) \quad \text{(symmetricity condition)} \\ B(\lambda\mathbf{x} + \mu\mathbf{x}', \mathbf{y}) = \lambda B(\mathbf{x}, \mathbf{y}) + \mu B(\mathbf{x}', \mathbf{y}) \quad \text{(linearity condition)} \\ (\mathbf{x}, \mathbf{x}) \geq 0 \text{ , } (\mathbf{x}, \mathbf{x}) = 0 \Leftrightarrow \mathbf{x} = 0 \text{ (positive-definiteness condition)} \end{array} \tag{1.11}$$

[1] $\mathbf{R}^n$ considered as a set of points is called affine space

Definition Euclidean space is a vector space equipped with a scalar product.

One can easy to see that the function $B(\mathbf{x}, \mathbf{y})$ is bilinear function, i.e. it is linear function with respect to the second argument also[2]. This follows from previous axioms:

$$B(\mathbf{x}, \lambda\mathbf{y}+\mu\mathbf{y}') \underbrace{=}_{\text{symm.}} B(\lambda\mathbf{y}+\mu\mathbf{y}', \mathbf{x}) \underbrace{=}_{\text{linear.}} \lambda B(\mathbf{y}, \mathbf{x})+\mu B(\mathbf{y}', \mathbf{x}) \underbrace{=}_{\text{symm.}} \lambda B(\mathbf{x}, \mathbf{y})+\mu B(\mathbf{x}, \mathbf{y}') .$$

A bilinear function $B(\mathbf{x}, \mathbf{y})$ on pair of vectors is called sometimes *bilinear form* on vector space. Bilinear form $B(\mathbf{x}, \mathbf{y})$ which satisfies the symmetricity condition is called *symmetric bilinear form.* Scalar product is nothing but symmetric bilinear form on vectors which is positive-definite: $B(\mathbf{x}, \mathbf{x}) \geq 0$) and is non-degenerate $((\mathbf{x}, \mathbf{x}) = 0 \Rightarrow \mathbf{x} = 0$.

For example one can consider $\mathbf{R}^n$ as Euclidean space provided by the scalar product

$$B(\mathbf{x}, \mathbf{y}) = x^1y^1 + \cdots + x^ny^n \tag{1.12}$$

Exercise a) Check that it is indeed scalar product.

Notations!

Scalar product sometimes is called "inner" product or "dot" product. Later on we will use for scalar product $B(\mathbf{x}, \mathbf{y})$ just shorter notation $(\mathbf{x}, \mathbf{y})$ or $\langle \mathbf{x}, \mathbf{y} \rangle$. Sometimes it is used for scalar product a notation $\mathbf{x} \cdot \mathbf{y}$. Usually this notation is reserved only for the canonical case (1.12).

b) Show that operation $(\mathbf{x}, \mathbf{y}) = x_1y_1 + x_2y_2 - x_3y_3$ *does not define scalar product* in $\mathbf{R}^3$. (See also exercises in Homework 1)

1.6 Orthonormal basis in Euclidean space

One can see that for scalar product (1.12) and for the basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ defined by the relation (1.10) the following relations hold:

$$(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \tag{1.13}$$

Definition The basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ in n-dimensional Euclidean space which obeys the conditions (1.13) is called *orthonormal basis.*

[2]Here and later we will denote scalar product $B(\mathbf{x}, \mathbf{y})$ just by $(\mathbf{x}, \mathbf{y})$. Scalar product sometimes is called inner product. Sometimes it is called dot product.

Remark Note that if for an arbitrary ordered set of n vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ in n-dimensional Euclidean space the conditions (1.13) are obeyed then this set is automatically a basis [3]. Thus we see that

The ordered set of n vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ in n-dimensional Euclidean space which obeys the conditions (1.13) *is a basis and this is an orthonormal basis.*

If we have to show that ordered set of n vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ in n-dimensional Euclidean space is an orthonormal basis it suffices to check the conditions (1.13).

One can prove that every (finite-dimensional) Euclidean space possesses orthonormal basis. Later by default we consider only orthonormal bases in Euclidean spaces. Respectively scalar product will be defined by the formula (1.12).

Indeed let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be an orthonormal basis in Euclidean space. Then for an arbitrary two vectors $\mathbf{x}, \mathbf{y}$, such that $\mathbf{x} = \sum x^i \mathbf{e}_i$, $\mathbf{y} = \sum y^j \mathbf{e}_j$ we have:

$$(\mathbf{x}, \mathbf{y}) = \left(\sum x^i \mathbf{e}_i, \sum y^j \mathbf{e}_j\right) = \sum_{i,j=1}^{n} x^i y^j (\mathbf{e}_i, \mathbf{e}_j) = \sum_{i,j=1}^{n} x^i y^j \delta_{ij} = \sum_{i=1}^{n} x^i y^i \tag{1.14}$$

We come to the scalar product (1.12). Later on we usually will consider scalar product defined by the formula (1.12) ((1.14)).

Length of the vectors, angle between vectors

The scalar product of vector on itself defines the *length of the vector*:

$$\text{Length of the vector } \mathbf{x} = |\mathbf{x}| = \sqrt{(\mathbf{x}, \mathbf{x})} = \sqrt{(x^1)^2 + \cdots + (x^n)^2} \tag{1.15}$$

If we consider Euclidean space $\mathbf{E^n}$ as the set of points then the distance between two points $\mathbf{x}, \mathbf{y}$ is the length of corresponding vector:

$$\text{distance between points } \mathbf{x}, \mathbf{y} = |\mathbf{x} - \mathbf{y}| = \sqrt{(y^1 - x^1)^2 + \cdots + (y^n - x^n)^2} \tag{1.16}$$

Geometrical properties of scalar product

---

[3] Indeed prove that conditions (1.13) imply that these n vectors are linear independent. Suppose that $\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 + \cdots + \lambda_n \mathbf{e}_n = 0$. For an arbitrary i multiply the left and right hand sides of this relation on a vector $\mathbf{e}_i$. We come to condition $\lambda_i = 0$. Hence vectors $(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$ are linearly dependent.

We recall very important formula how scalar (inner) product is related with the angle between vectors:

$$(\mathbf{x}, \mathbf{y}) = x^1 y^1 + x^2 y^2 = |\mathbf{x}||\mathbf{y}| \cos\varphi \tag{1.17}$$

where φ is an angle between vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbf{E}^2$.

This formula is valid also in the three-dimensional case and any n-dimensional case for $n \geq 1$. It gives as a tool to calculate angle between two vectors:

$$(\mathbf{x}, \mathbf{y}) = x^1 y^1 + x^2 y^2 + \cdots + x^n y^n = |\mathbf{x}||\mathbf{y}| \cos\varphi \tag{1.18}$$

In particulary it follows from this formula that

$$\begin{gathered}\textit{angle between vectors } \mathbf{x}, \mathbf{y} \textit{ is acute if scalar product } (\mathbf{x}, \mathbf{y}) \textit{ is positive}\\ \textit{angle between vectors } \mathbf{x}, \mathbf{y} \textit{ is obtuse if scalar product } (\mathbf{x}, \mathbf{y}) \textit{ is negative}\\ \textit{vectors } \mathbf{x}, \mathbf{y} \textit{ are perpendicular if scalar product } (\mathbf{x}, \mathbf{y}) \textit{ is equal to zero}\end{gathered} \tag{1.19}$$

$$|\mathbf{x}| = \sqrt{(\mathbf{x}, \mathbf{x})} \tag{1.20}$$

Remark Geometrical intuition tells us that cosinus of the angle between two vectors has to be less or equal to one and it is equal to one if and only if vectors $\mathbf{x}, \mathbf{y}$ are collinear. Comparing with (1.18) we come to the inequality:

$$\begin{gathered}(\mathbf{x}, \mathbf{y})^2 = \left(x^1 y^1 + \cdots + x^n y^n\right)^2 \leq \left((x^1)^2 + \cdots + (x^n)^2\right) \left((y^1)^2 + (\cdots + (y^n)^2\right) = (\mathbf{x}, \mathbf{x})(\mathbf{y}, \mathbf{y})\\ \text{and}(\mathbf{x}, \mathbf{y})^2 = (\mathbf{x}, \mathbf{x})(\mathbf{y}, \mathbf{y}) \quad \text{if vectors are colienar, i.e. } x^i = \lambda y^i\end{gathered} \tag{1.21}$$

This is famous Cauchy–Buniakovsky–Schwarz inequality, one of most important inequalities in mathematics. (See for more details Homework 2)

1.7 Transition matrices. Orthogonal matrices

One can consider different bases in vector space.

Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be a basis in n-dimensional vector space V.

Let A be $n \times n$ matrix with real entries:

One can consider different orthonormal bases in $\mathbf{E}^n$.

$$A = \begin{pmatrix} a_{11} & a_{12} \ldots & a_{1n} \\ a_{21} & a_{22} \ldots & a_{2n} \\ a_{31} & a_{32} \ldots & a_{3n} \\ \ldots & \ldots\ldots & \ldots \\ a_{(n-1)\,1} & a_{(n-1)2} \ldots & a_{(n-1)n} \\ a_{n\,1} & a_{n2} \ldots & a_{nn} \end{pmatrix}$$

The basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ can be considered as row of vectors, or $1 \times n$ mmatrix with entries–vectors.

Multiplying $1 \times n$ matrix $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ on matrix A we come to new row of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n\}$ such that

$$\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n\} = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}A = \tag{1.22}$$

$$\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n\} = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\} \begin{pmatrix} a_{11} & a_{12} \dots & a_{1n} \\ a_{21} & a_{22} \dots & a_{2n} \\ a_{31} & a_{32} \dots & a_{3n} \\ \dots & \dots\dots & \dots \\ a_{(n-1)\,1} & a_{(n-1)2} \cdots & a_{(n-1)n} \\ a_{n\,1} & a_{n2} \dots & a_{nn} \end{pmatrix} \tag{1.23}$$

Making matrix multiplication we come to

$$\begin{cases} \mathbf{e}'_1 = a_{11}\mathbf{e}_1 + a_{21}\mathbf{e}_2 + a_{31}\mathbf{e}_3 + \cdots + a_{(n-1)\,1}\mathbf{e}_{n-1} + a_{n\,1}\mathbf{e}_n \\ \mathbf{e}'_1 = a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2 + a_{32}\mathbf{e}_3 + \cdots + a_{(n-1)\,2}\mathbf{e}_{n-1} + a_{n\,2}\mathbf{e}_n \\ \mathbf{e}'_1 = a_{13}\mathbf{e}_1 + a_{23}\mathbf{e}_2 + a_{33}\mathbf{e}_3 + \cdots + a_{(n-1)\,3}\mathbf{e}_{n-1} + a_{n\,1}\mathbf{e}_n \\ \cdots = \dots\dots + \dots\dots + \dots\dots + \cdots + \dots\dots\dots\dots \\ \mathbf{e}'_n = a_{1n}\mathbf{e}_1 + a_{2n}\mathbf{e}_2 + a_{3n}\mathbf{e}_3 + \cdots + a_{(n-1)\,n}\mathbf{e}_{n-1} + a_{n\,n}\mathbf{e}_n \end{cases} \tag{1.24}$$

What is the condition that the row $\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n\}$ is a basis too? The row $\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n\}$ is a basis if and only if vectors $(\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n)$ are linearly independent. Thus it follows from (1.24)

We call matrix A a *transition matrix*

Proposition 1 *Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ be a basis in vector space V and let A be an $n\times n$ matrix with real entries. Then $\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n\} = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}A$ is a basis if and only if the matrix A has rank n i.e. it is non-degenerate matrix (*$\det A \neq 0$*)*

We call matrix A a *transition matrix* from a basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ to the basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n\}$,

Remark Recall that the rank of matrix A is maximal number of linearly independent rows (or columns). $n \times n$ matrix A of rank n are called non-degenerate matrix. Non-degenerate matrix = invertible matrix. Matrix is invertible if and only if its determinant is not equal to zero.

Now suppose that $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is orthonromal basis in n-dimensional Euclidean vector space. What is the condition that the row $\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n\}$ is an orthonormal basis too?

Definition We say that $n \times n$ matrix is orthogonal matrix if its product on transposed matrix is equal to unity matrix:

$$A^T A = I \tag{1.25}$$

Exercise. Prove that determinant of orthogonal matrix is equal to ± 1.

Solution $A^T A = I$. Hence $\det(A^T A) = \det A^T \det A = (\det A)^2 = \det I = 1$. Hence $\det A = \pm 1$

We see that in particular orthogonal matrix is non-degenerate and $\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n\} = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}A$ is a basis if $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is a basis and A is orthogonal matrix.

The following Proposition is valid:

Proposition 2 Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ be an orthonormal basis in n-dimensional Euclidean vector space. Then the new basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n\} = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}A$ is orthonormal basis if and only if the transition matrix is orthogonal matrix.

Proof The basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n\}$ is orthonormal means that $(\mathbf{e}'_i, \mathbf{e}'_j) = \delta_{ij}$. We have:

$$\delta_{ij} = (\mathbf{e}'_i, \mathbf{e}'_j) = \left(\sum_{m=1}^{n} \mathbf{e}_m A_{mi}, \mathbf{e}'_j = \sum_{n=1}^{n} \mathbf{e}_n A_{nj}\right) = \sum_{m,n=1}^{n} A_{mi} A_{nj} (\mathbf{e}_m, \mathbf{e}_n) =$$

$$\sum_{m,n=1}^{n} A_{mi} A_{nj} \delta_{mn} = \sum_{m=1}^{n} A_{mi} A_{mj} = \sum_{m=1}^{n} A^t_{im} A_{mj} = (A^t A)_{ij}\,, \tag{1.26}$$

Hence $(A^T A)_{ij} = \delta_{ij}$, i.e. $A^T A = I$.

One can see that any orthogonal matrix has determinant 1 or -1: $det(A^t A) = (\det A)^2 = 1 \Rightarrow \det A = \pm 1$. Hence one can consider

It is very useful to consider the following groups:

- The group $O(n)$—group of orthogonal $n \times n$ matrices:

$$O(n) = \{A: \quad A^t T = I\}\,. \tag{1.27}$$

- The group $SO(n)$ special orthogonal group of $n \times n$ matrices:

$$SO(n) = \{A: \quad A^t A = I, \}\,. \tag{1.28}$$

1.8 Orthogonal 2×2 matrices

Find orthogonal 2×2 matrices. Note that rotation of basis and reflection of basis is orthogonal transformation. We show now that an arbitrary transition matrix from orthonormal basis to an arbitrary orthonormal basis in $\mathbf{E}^2$, i.e. orthogonal 2×2 matrix is a rotation or reflection.

Consider 2-dimensional Euclidean space $\mathbf{E}^2$ with orthonormal basis $\{\mathbf{e}, \mathbf{f}\}$: $(\mathbf{e}, \mathbf{e}) = (\mathbf{f}, \mathbf{f}) = 1$ (i.e. $|\mathbf{e}| = |\mathbf{f}| = 1$) and $(\mathbf{e}, \mathbf{f}) = 0$ (i.e. vectors $\mathbf{e}, \mathbf{f}$ are orthogonal). Let $\{\mathbf{e}', \mathbf{f}'\}$ be a new basis:

$$\{\mathbf{e}', \mathbf{f}'\} = \{\mathbf{e}, \mathbf{f}\}T = \{\mathbf{e}, \mathbf{f}\} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \text{ i.e. } \mathbf{e}' = \alpha \mathbf{e} + \gamma \mathbf{f},\ \mathbf{f}' = \beta \mathbf{e} + \delta \mathbf{f}$$

new basis is orthonormal basis also,$(\mathbf{e}', \mathbf{e}') = (\mathbf{f}', \mathbf{f}') = 1$ and $(\mathbf{e}', \mathbf{f}') = 0$,i.e. transition matrix is an orthogonal matrix:

$$\begin{array}{c} 1 = (\mathbf{e}', \mathbf{e}') = (\alpha \mathbf{e} + \gamma \mathbf{f}, \alpha \mathbf{e} + \gamma \mathbf{f}) = \alpha^2 + \gamma^2 = 1 \\ 0 = (\mathbf{e}', \mathbf{f}') = (\alpha \mathbf{e} + \gamma \mathbf{f}, \beta \mathbf{e} + \delta \mathbf{f}) = \alpha\beta + \gamma\delta = 0 \\ 0 = (\mathbf{f}', \mathbf{e}') = (\beta \mathbf{e} + \delta \mathbf{f}, \alpha \mathbf{e} + \gamma \mathbf{f}) = \alpha\beta + \gamma\delta = 0 \\ 1 = (\mathbf{f}', \mathbf{f}') = (\beta \mathbf{e} + \delta \mathbf{f}, \beta \mathbf{e} + \delta \mathbf{f}) = \beta^2 + \delta^2 = 1 \end{array}$$

Or in matrix notations: The matrix $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is orthogonal matrix if and only if

$$A^t A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^t \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha^2 + \gamma^2 & \alpha\beta + \gamma\delta \\ \alpha\beta + \gamma\delta & \beta^2 + \delta^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.29}$$

We have $\alpha^2 + \gamma^2 = 1$, $\alpha\beta + \gamma\delta = 0$ and $\beta^2 + \delta^2 = 1$. Hence one can choose angles $\varphi, \psi \colon 0 \leq 2\pi$ such that $\alpha = \cos\varphi$, $\gamma = \sin\varphi$, $\beta = \sin\psi$, $\delta = \cos\psi$. The condition $\alpha\beta + \gamma\delta =$ means that

$$\cos\varphi \sin\psi + \sin\varphi \cos\psi = \sin(\varphi + \psi) = 0$$

Hence $\sin\varphi = -\sin\psi$, $\cos\varphi = \cos\psi$ $(\varphi + \psi = 0)$ or $\sin\varphi = \sin\psi$, $\cos\varphi = -\cos\psi$ $(\varphi + \psi = \pi)$

The first case: $\sin\varphi = -\sin\psi$, $\cos\varphi = \cos\psi$,

$$A_\varphi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \quad (\det T_\varphi = 1) \tag{1.30}$$

The second case: $\sin\varphi = \sin\psi$, $\cos\varphi = -\cos\psi$,

$$\tilde{A}_\varphi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix} \quad (\det \tilde{T}_\varphi = -1) \tag{1.31}$$

Consider the first case, when a matrix A_φ is defined by the relation (1.30). In this case the new basis is:

$$(\mathbf{e}', \mathbf{f}') = (\mathbf{e}, \mathbf{f})A_\varphi = (\mathbf{e}, \mathbf{f}) \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = \begin{pmatrix} \cos\varphi\,\mathbf{e} + \sin\varphi\,\mathbf{f} \\ -\sin\varphi\,\mathbf{e} + \cos\varphi\,\mathbf{f} \end{pmatrix} \tag{1.32}$$

One can see that that new basis $\{\mathbf{e}', \mathbf{f}'\}$ is orthonormal basis too and transition matrix T_φ rotates the basis $(\mathbf{e}, \mathbf{f})$ on the angle φ (see Homework 1).

We call the matrix A_φ **rotation matrix**

Now consider the second case, when a matrix $\tilde{A}_\varphi$ is defined by the relation (1.31). One can see that

$$\tilde{A}_\varphi = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = A_\varphi R \tag{1.33}$$

where we denote by $R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ a transition matrix from the basis $\{\mathbf{e}, \mathbf{f}\}$ to the basis $\{\mathbf{e}, -\mathbf{f}\}$—the reflection.

We see that in the second case the orthogonal matrix is composition of rotation and reflection matrix: $\{\mathbf{e}, \mathbf{f}\} \xrightarrow{\tilde{A}_\varphi = A_\varphi R} \{\tilde{\mathbf{e}}, \tilde{\mathbf{f}}\}$:

$$\{\mathbf{e}, \mathbf{f}\} \xrightarrow{A_\varphi} \{\mathbf{e}' = \cos\varphi\,\mathbf{e} + \sin\varphi\,\mathbf{f}, \mathbf{f}' = -\sin\varphi\,\mathbf{e} + \cos\varphi\,\mathbf{f}\} \xrightarrow{R} \{\tilde{\mathbf{e}} = \mathbf{e}', \tilde{f} = -\mathbf{f}\} \tag{1.34}$$

One can see that the transition matrix $\tilde{A}_\varphi$ is a reflection matrix with respect to the axis which have the angle $\varphi/over2$ with x-axis. We come to proposition

Proposition. *Let A be an arbitrary 2×2 orthogonal matrix, i.e. $A^t A = 1$ and in particularly* $\det A = \pm 1$. *(Transition matrix transforms an orthonormal basis to an orthonormal one.)*

If $\det A = 1$ *then there exists an angle $\varphi \in [0, 2\pi)$ such that $A = A_\varphi$ is a transition matrix* (1.30) *which rotates the basis vectors on the angle φ.*

If $\det A = -1$ *then there exists an angle $\varphi \in [0, 2\pi)$ such that $A = \tilde{A}_\varphi$ is a transition matrix is a composition of rotation and reflection (see* (1.34)*) or* *it is a reflection with respect to the axis which have the angle $\varphi/over2$ with x-axis.*

Let (x, y) be components of the vector $\mathbf{a}$ in the basis $(\mathbf{e}, \mathbf{f})$, and (x', y') be components of the vector $\mathbf{a}$ in the rotated basis $\{\mathbf{e}', \mathbf{f}'\}$.

Then it follows from (1.32) that

$$\mathbf{a} = x'\mathbf{e}' + y'\mathbf{f}' = (\mathbf{e}', \mathbf{f}') \begin{pmatrix} x' \\ y' \end{pmatrix} = (\mathbf{e}, \mathbf{f}) T_\varphi \begin{pmatrix} x' \\ y' \end{pmatrix} = (\mathbf{e}, \mathbf{f}) \begin{pmatrix} x \\ y \end{pmatrix} .$$

Hence

$$\begin{pmatrix} x \\ y \end{pmatrix} = A_\varphi \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x'\cos\varphi - y'\sin\varphi \\ x'\sin\varphi + y'\cos\varphi \end{pmatrix} \quad (1.35)$$

and respectively

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A_{-\varphi} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\varphi + y\sin\varphi \\ -x\sin\varphi + y\cos\varphi \end{pmatrix} \quad (1.36)$$

because $A_\varphi^{-1} = A_{-\varphi}$.

1.9 Orientation in vector space

In the three-dimensional Euclidean space except scalar (inner) product, one can consider another important operation: vector product. For defining this operation we need additional structure: *orientation*.

A basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ have the same orientation as the basis $\{\mathbf{a}', \mathbf{b}', \mathbf{c}'\}$ if they both obey right hand rule or if they both obey left hand rule. In the other case we say that these bases have opposite orientation.

How to make this conception more mathematical?

Consider the set of *all* bases in the given vector space V.

If $(\mathbf{e}_1, \dots \mathbf{e}_n)$, $(\mathbf{e}'_1, \dots \mathbf{e}'_n)$ are two bases then one can consider the matrix T—transition matrix which transforms the old basis to the new one (see (1.23)). The transition matrix is not degenerate, i.e. determinant of this matrix is not equal to zero.

Definition Let $\{\mathbf{e}_1, \dots \mathbf{e}_n\}, \{\mathbf{e}'_1, \dots \mathbf{e}'_n\} \in \mathbf{R}^n$ be two bases in $\mathbf{R}^n$ and T be transition matrix:

$$\{\mathbf{e}'_1, \dots \mathbf{e}'_n\} = \{\mathbf{e}_1, \dots \mathbf{e}_n\} T\,. \tag{1.37}$$

We say that these two bases have the same orientation if the determinant of transition matrix from the first basis to the second one is positive: $\det T > 0$. We say that the basis $\{\mathbf{e}'_1, \dots \mathbf{e}'_n\}$ has an orientation opposite to the orientation of the basis $\{\mathbf{e}'_1, \dots \mathbf{e}'_n\}$ (or in other words these two bases have opposite orientation) if the determinant of transition matrix from the first basis to the second one is negative: $\det T < 0$.

Remark Transition matrix from basis to basis is non-degenerate, hence its determinant cannot be equal to zero. It can be or positive or negative.

One can see that orientation establishes the equivalence relation in the set of all bases. Denote

$$\{\mathbf{e}_1, \dots \mathbf{e}_n\} \sim \{\mathbf{e}'_1, \dots \mathbf{e}'_n\}\,,$$

if two bases $\{\mathbf{e}_1, \dots \mathbf{e}_n\}$ and $\{\mathbf{e}'_1, \dots \mathbf{e}'_n\}$ have the same orientation, i.e. $\det T > 0$ for transition matrix.

Show that " $\sim$ " is an equivalence relation, i.e. this relation is reflexive, symmetric and transitive.

Check it:

- it is reflexive, i.e. for every basis $\{\mathbf{e}_1, \dots \mathbf{e}_n\}$

  $$\{\mathbf{e}_1, \dots, \mathbf{e}_n\} \sim \{\mathbf{e}_1, \dots, \mathbf{e}_n\}\,, \tag{1.38}$$

  because in this case transition matrix $T = I$ and $detI = 1 > 0$.

- it is symmetric, i.e.

  If $\{\mathbf{e}_1, \dots, \mathbf{e}_n\} \sim \{\mathbf{e}'_1, \dots, \mathbf{e}'_n\}$ then $\{\mathbf{e}'_1, \dots, \mathbf{e}'_n) \sim \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$,

  because if T is transition matrix from the first basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ to the second basis $\{\mathbf{e}'_1, \dots, \mathbf{e}'_n\}$: $\{\mathbf{e}'_1, \dots, \mathbf{e}'_n\} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}T$,

  then the transition matrix from the second basis $\{\mathbf{e}'_1, \dots, \mathbf{e}'_n\}$ to the first basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is the inverse matrix T^{-1}: $\{\mathbf{e}_1, \dots, \mathbf{e}_n\} = \{\mathbf{e}'_1, \dots, \mathbf{e}'_n\}T^{-1}$. Hence $\det T^{-1} = \frac{1}{\det T} > 0$ if $\det T > 0$.

- Is transitive, i.e. if $\{\mathbf{e}_1, \dots, \mathbf{e}_n\} \sim \{\mathbf{e}'_1, \dots, \mathbf{e}'_n\}$ and $\{\mathbf{e}'_1, \dots, \mathbf{e}'_n) \sim \{\tilde{\mathbf{e}}_1, \dots, \tilde{\mathbf{e}}_n\}$, then $\{\mathbf{e}_1, \dots, \mathbf{e}_n\} \sim \{\tilde{\mathbf{e}}_1, \dots, \tilde{\mathbf{e}}_n\}$, because if T_1 is transition matrix from the first basis $\{\mathbf{e}_1, \dots \mathbf{e}_n\}$ to the second basis $\{\mathbf{e}'_1, \dots, \mathbf{e}'_n\}$ and T_2 is transition matrix from the second basis $\{\mathbf{e}'_1, \dots, \mathbf{e}'_n\}$ to the third basis $\{\tilde{\mathbf{e}}_1, \dots, \tilde{\mathbf{e}}_n\}$ then the transition matrix T from the first basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ to the third basis $\{\tilde{\mathbf{e}}_1, \dots, \tilde{\mathbf{e}}_n\}$ is equal to $T = T_2 T_1$. We have that $\det T = \det(T_2 T_1) = \det T_2 \cdot \det T_1 > 0$ because $\det T_1 > 0$ and $\det T_2 > 0$.

Since it is equivalence relation the set of all bases is a union if disjoint equivalence classes. Two bases are in the same equivalence class if and only if they have the same orientation.

One can see that there are exactly two equivalence classes.

Indeed let $\{\mathbf{e}_1, \mathbf{e}_2 \dots, \mathbf{e}_n\}$ be an arbitrary basis in n-dimensional vector space V. Swap the vectors $\mathbf{e}_1, \mathbf{e}_2$. We come to a new basis: $\{\mathbf{e}'_1, \mathbf{e}'_2 \dots, \mathbf{e}'_n\}$

$$\mathbf{e}'_1 = \mathbf{e}_2, \mathbf{e}'_2 = \mathbf{e}_1, \ \text{all other vectors are the same: } \mathbf{e}_3 = \mathbf{e}'_3, \dots, \mathbf{e}_n = \mathbf{e}'_n \tag{1.39}$$

We have:

$$\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3 \dots, \mathbf{e}'_n\} = \{\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_3, \dots, \mathbf{e}_n\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \dots, \mathbf{e}_n\} T \,, \tag{1.40}$$

where one can easy see that the determinant for transition matrix T is equal to -1. E.g. write down the transition matrix (1.40) in the case if dimension of vector space is equal to 5, $n = 5$. Then we have $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3, \mathbf{e}'_4, \mathbf{e}'_5\} = \{\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\} T$ where

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (\det T = -1) \,. \tag{1.41}$$

We see that bases $\{\mathbf{e}_1, \mathbf{e}_2 \dots, \mathbf{e}_n\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2 \dots, \mathbf{e}'_n\}$ have opposite orientation. They does not belong to the same equivalence class.

Now consider in V an arbitrary basis $\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2 \dots, \tilde{\mathbf{e}}_n\}$. For convenience call the initial basis $\{\mathbf{e}_1, \mathbf{e}_2 \dots, \mathbf{e}_n\}$ the *first* basis, call the basis $\{\mathbf{e}'_1, \mathbf{e}'_2 \dots, \mathbf{e}'_n\}$ the *second* basis and call a new basis arbitrary basis $\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2 \dots, \tilde{\mathbf{e}}_n\}$ a "third basis". Show that this third basis and the first basis have the same orientation or the third basis and the second basis have the same orientation, i.e. the third basis belongs to the equivalence class of the first basis or it belongs to the equivalence class of the second basis. Thus we will show that there are exactly two equivalence classes.

Let T_1 be transition matrix from the basis $\{\mathbf{e}_1, \mathbf{e}_2 \ldots, \mathbf{e}_n\}$ to the basis $\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2 \ldots, \tilde{\mathbf{e}}_n\}$. Let T_2 be transition matrix from the basis $\{\mathbf{e}'_1, \mathbf{e}'_2 \ldots, \mathbf{e}'_n\}$ to the basis $\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2 \ldots, \tilde{\mathbf{e}}_n\}$. We have:

$$\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2 \ldots, \tilde{\mathbf{e}}_n\} = \{\mathbf{e}_1, \mathbf{e}_2 \ldots, \mathbf{e}_n\}T_1, \ \ \{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2 \ldots, \tilde{\mathbf{e}}_n\} = \{\mathbf{e}'_1, \mathbf{e}'_2 \ldots, \mathbf{e}'_n\}T_2$$

Recall that $\{\mathbf{e}'_1, \mathbf{e}'_2 \ldots, \mathbf{e}'_n\} = \{\mathbf{e}_1, \mathbf{e}_2 \ldots, \mathbf{e}_n\}T$.

Hence

$$\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2 \ldots, \tilde{\mathbf{e}}_n\} = \{\mathbf{e}'_1, \mathbf{e}'_2 \ldots, \mathbf{e}'_n\}T_2 = \{\mathbf{e}_1, \mathbf{e}_2 \ldots, \mathbf{e}_n\}T_1 T \ \Rightarrow T_2 = T_1 T\,. \tag{1.42}$$

We have $\det T_2 = \det(T_1 T) = \det T_1 \det T$. But $\det T < 0$ because bases $\{\mathbf{e}'_1, \mathbf{e}'_2 \ldots, \mathbf{e}'_n\}$, $\{\mathbf{e}_1, \mathbf{e}_2 \ldots, \mathbf{e}_n\}$ have opposite orientation. Hence $\det T_1$ and $\det T_2$ have opposite signs because $\det T < 0$.

We see that
$\det T_1 > 0$ and $\det T_2 < 0$ or $\det T_2 > 0$ and $\det T_1 < 0$.

If $\det T_1 > 0$ and $\det T_2 < 0$, then the bases $\{\mathbf{e}_1, \mathbf{e}_2 \ldots, \mathbf{e}_n\}$ and $\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2 \ldots, \tilde{\mathbf{e}}_n\}$ have the same orientation and the bases $\{\mathbf{e}'_1, \mathbf{e}'_2 \ldots, \mathbf{e}'_n\}$ and $\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2 \ldots, \tilde{\mathbf{e}}_n\}$ have opposite orientation.

If $\det T_2 > 0$ and $\det T_1 < 0$, then the bases $\{\mathbf{e}'_1, \mathbf{e}'_2 \ldots, \mathbf{e}'_n\}$ and $\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2 \ldots, \tilde{\mathbf{e}}_n\}$ have the same orientation and the bases $\{\mathbf{e}_1, \mathbf{e}_2 \ldots, \mathbf{e}_n\}$ and $\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2 \ldots, \tilde{\mathbf{e}}_n\}$ have opposite orientation.

In other words the basis $\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2 \ldots, \tilde{\mathbf{e}}_n\}$ belongs to the equivalence class of the basis $\{\mathbf{e}_1, \mathbf{e}_2 \ldots, \mathbf{e}_n\}$ (if $\det T_1 > 0$) or it belongs to the equivalence class of the basis $\{\mathbf{e}'_1, \mathbf{e}'_2 \ldots, \mathbf{e}'_n\}$ (if $\det T_2 > 0$).

The set of all bases is a union of two disjoint subsets.

Any two bases which belong to the same subset have the same orientation. Any two bases which belong to different subsets have opposite orientation.

Definition *An orientation of a vector space is an equivalence class of bases in this vector space.*

Note that fixing any basis we fix orientation, considering the subset of all bases which have the same orientation that the given basis.

There are two orientations. Every basis has the same orientation as a given basis or opposite orientation.

Definition *An oriented vector space is a vector space equipped with orientation.*

Consider examples.

Example (Orientation in two-dimensional space). Let $\{\mathbf{e}_x, \mathbf{e}_y\}$ be any basis in $\mathbf{R}^2$ and $\mathbf{a}, \mathbf{b}$ are arbitrary two vectors in $\mathbf{R}^2$:

$$\mathbf{a} = a_x\mathbf{e}_x + a_y\mathbf{e}_y \ \mathbf{b} = b_x\mathbf{e}_x + b_y\mathbf{e}_y,$$

Consider ordered pair $\{\mathbf{a}, \mathbf{b}, \}$. The transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y\}$ to the ordered pair $\{\mathbf{a}, \mathbf{b}\}$ is $T = \begin{pmatrix} a_x & b_x \\ a_y & b_y \end{pmatrix}$:

$$\{\mathbf{a}, \mathbf{b}\} = \{\mathbf{e}_x, \mathbf{e}_y\}T = \{\mathbf{e}_x, \mathbf{e}_y\} \begin{pmatrix} a_x & b_x \\ a_y & b_y \end{pmatrix}, \quad \begin{cases} \mathbf{a} = a_x\mathbf{e}_x + a_y\mathbf{e}_y \\ \mathbf{b} = b_x\mathbf{e}_x + b_y\mathbf{e}_y \end{cases}$$

One can see that the ordered pair $\{\mathbf{a}, \mathbf{b}\}$ also is a basis, (i.e. these two vectors are linearly independent in $\mathbf{R}^2$) if and only if transition matrix is not degenerate, i.e.

$$\det T \neq 0\,. \tag{1.43}$$

We see that the basis $\{\mathbf{a}, \mathbf{b}\}$ has the same orientation as the basis $\{\mathbf{e}_x, \mathbf{e}_y\}$ if

$$\det T > 0\,. \tag{1.44}$$

We see that the basis $\{\mathbf{a}, \mathbf{b}\}$ has the orientation opposite to the orientation of the basis $\{\mathbf{e}_x, \mathbf{e}_y\}$ if

$$\det T < 0\,. \tag{1.45}$$

Exercise Show that bases $\{\mathbf{e}_x, \mathbf{e}_y\}$ and $\{\mathbf{e}_y, \mathbf{e}_x\}$ have opposite orientation but bases $\{\mathbf{e}_x, \mathbf{e}_y\}$ and $\{-\mathbf{e}_y, \mathbf{e}_x\}$ have the same orientation.

Relations (1.44),(1.45) define equivalence relations in the set of bases. Orientation is equivalence class of bases. There are two orientations, every basis has the same orientation as a given basis or opposite orientation.

Example(Orientation in three-dimensional euclidean space.) Let $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ be any basis in $\mathbf{E}^3$ and $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are arbitrary three vectors in $\mathbf{E}^3$:

$$\mathbf{a} = a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z \ \mathbf{b} = b_x\mathbf{e}_x + b_y\mathbf{e}_y + b_z\mathbf{e}_z, \ \mathbf{c} = c_x\mathbf{e}_x + c_y\mathbf{e}_y + c_z\mathbf{e}_z\,.$$

Consider ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$. The transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is $T = \begin{pmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{pmatrix}$:

$$\{\mathbf{a}, \mathbf{b}, \mathbf{c}\} = \{\mathbf{e_x}, \mathbf{e_y}, \mathbf{e_z}\}\mathbf{T} = \{\mathbf{e_x}, \mathbf{e_y}, \mathbf{e_z}\} \begin{pmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{pmatrix}$$

One can see that the ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ also is a basis, (i.e. these three vectors are linearly independent) if and only if transition matrix is not degenerate, i.e.

$$\det T \neq 0\,. \tag{1.46}$$

We see that the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ has the same orientation as the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ if

$$\det T > 0\,. \tag{1.47}$$

We see that the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ has the orientation opposite to the orientation of the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ if

$$\det T < 0\,. \tag{1.48}$$

Exercise Show that bases $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and $\{-\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ have opposite orientation but bases $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and $\{\mathbf{e}_y, \mathbf{e}_x, -\mathbf{e}_z\}$ have the same orientation.

We say that Euclidean space is equipped with orientation if we consider in this space only orthonormal bases which have the same orientation.

Remark Note that in the example above we considered in $\mathbf{E}^3$ *arbitrary bases* not necessarily orthonormal bases.

Relations (1.47),(1.48) define equivalence relations in the set of bases. Orientation is equivalence class of bases. There are two orientations, every basis has the same orientation as a given basis or opposite orientation.

If two bases $\{\mathbf{e}_i\}$, $\{\mathbf{e}_{i'}\}$ have the same orientation then they can be transformed to each other by continuous transformation, i.e. there exist one-parametric family of bases $\{\mathbf{e}_i(t)\}$ such that $0 \leq t \leq 1$ and $\{\mathbf{e}_i(t)\}|_{t=0} = \{\mathbf{e}_i\}$, $\{\mathbf{e}_i(t)\}|_{t=1} = \{\mathbf{e}_{i'}\}$. (All functions $\mathbf{e}_i(t)$ are continuous) In the case of three-dimensional space the following statement is true (Euler Theorem): *Let* $\{\mathbf{e}_i\}, \{\mathbf{e}_{i'}\}$ $(i = 1, 2, 3)$ *be two orthonormal bases in* $\mathbf{E}^3$ *which have the same orientation. Then there exists an axis* $\mathbf{n}$ *such that basis* $\{\mathbf{e}_i\}$ *transforms to the basis* $\{\mathbf{e}_{i'}\}$ *under rotation around the axis.*

1.10 †Linear operator in $\mathbf{E}^3$ preservinig orientation is a rotation

Let P be a linear operator in vector space $\mathbf{R}^n$. Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be an arbitrary basis in $\mathbf{R}^n$. Considering the action of P on basis vectors we come to vectors $(\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n)$:

$$\mathbf{e}'_1 = P(\mathbf{e}_1), \mathbf{e}'_2 = P(\mathbf{e}_2) \ldots, \mathbf{e}'_n = P(\mathbf{e}_n) \tag{1.49}$$

If operator P is non-degenerate ($\det P \neq 0$) then ordered n-tuple $\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\}$ is a basis too.

Non-degenerate linear operator maps the basis to another basis.

Definition. Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be an arbitrary basis in $\mathbf{R}^n$. Consider the basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\}$, where $\mathbf{e}'_i = P(\mathbf{e}_i)$ We say that non-degenerate linear operator P ($\det P \neq 0$) preserves orientation if bases $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ and $\{e'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\}$, where $\mathbf{e}'_i = P(\mathbf{e}_i)$ have the same orientation. In this case $\det P > 0$.

We say that linear operator P changes orientation if bases $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\}$ have opposite orientation. In this case $\det P < 0$.

It is easy to see that this definition is correct: The property of operator P to preserve orientation does not depend on choosing a basis. If bases $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\}$, where $\mathbf{e}'_i = P(\mathbf{e}_i)$ have the same (opposite) orientation, then for an another basis $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n\}$ in $\mathbf{R}^n$, the bases $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n\}$ and $\{\mathbf{f}'_1, \mathbf{f}'_2, \ldots, \mathbf{f}'_n\}$, where $\mathbf{f}'_i = P(\mathbf{f}_i)$ have the same (opposite) orientation also.

In other words we say that non-degenerate linear operator P preserves orientation if it maps vectors of an arbitrary basis to the vectors of another basis which have the same orientation as an initial basis. We say that non-degenerate linear operator P changes orientation if it maps vectors of an arbitrary basis to the vectors of another basis which have the orientation opposite an orientation of initial basis.

Example Let $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ be an orthonormal basis in $\mathbf{E}^3$. Consider linear operator P such that

$$P(\mathbf{e}_x) = \mathbf{e}_y, P(\mathbf{e}_y) = -\mathbf{e}_x, P(\mathbf{e}_z) = \mathbf{e}_z\,.$$

This operator maps orthonormal basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the basis $\{\mathbf{e}_y, -\mathbf{e}_x, \mathbf{e}_z\}$ which is orthonormal too. Both bases have the same orientation. Hence the operator P is linear operator preserving orientation. In this case it is orthogonal operator, because it maps orthonormal basis to the orthonormal one. One can see that P is rotation operator: Under the action of operator P vectors in $\mathbf{E}^3$ rotate on the angle $\frac{\pi}{2}$ about the axis Oz. The vectors $\lambda\mathbf{e}_z$ collinear (proportional) to the vector $\mathbf{e}_z$. are eigenvectors of this operator: $P\mathbf{e}_z = \mathbf{e}_z$. The axis is a line spanned by the vector $\mathbf{e}_z$.

One can show that in Euclidean vector space $\mathbf{E}^3$ every orthogonal operator which preserves orientation is a rotation.

Theorem (Euler Theorem). Let P be an linear orthogonal operator in $\mathbf{E}^3$ preserving orientation. Then it is a rotation operator about some axis passing through the origin.

(The proof of this theorem see in the solutions of Homework 2, Exercise 7)

How to find an axis of rotation? Vectors which belong to axis (starting at origin) are *eigenvectors* of P. They all are proportional each other. Eigenvalue of these vectors is equal to 1— Rotation does not change the vectors which belong to axis. Hence

Claim To find an axis we have to find eigenvector of the operator P with eigenvalue 1.

In the example above vector $\mathbf{e}_z$ was the eigenvector of the operator P. Consider more interesting example:

Example Let $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ be an orthonormal basis in $\mathbf{E}^3$. Consider linear operator P such that

$$P(\mathbf{e}_x) = \mathbf{e}_z, P(\mathbf{e}_y) = -\mathbf{e}_y, P(\mathbf{e}_z) = \mathbf{e}_x \,. \tag{1.50}$$

This operator maps orthonormal basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the basis $\{\mathbf{e}_z, -\mathbf{e}_y, \mathbf{e}_x\}$ which is orthonormal too. Both bases have the same orientation. Hence the operator P is linear orthogonal operator preserving orientation. According to the Euler Theorem it is a rotation operator about an axis. Find this axis. Let vector $\mathbf{N}$ (starting at origin) belongs to the axis. Then

$$P\mathbf{N} = \mathbf{N} \,. \tag{1.51}$$

$\mathbf{N}$ is eigenvector of the operator P. Its eigenvalue is equal to 1. To find an axis of rotation (1.50) we have to find an eigenvector (1.51). It is easy to see that vector $\mathbf{e}_x + \mathbf{e}_z$ obeys the condition (1.51):

$$P(\mathbf{e}_x + \mathbf{e}_z) = \mathbf{e}_x + \mathbf{e}_z$$

We see that eigenvector of P is an arbitrary vector proportional (collinear) to the vector $\mathbf{e}_x + \mathbf{e}_z$. These vectors span the line $\lambda(\mathbf{e}_x + \mathbf{e}_y)$–axis of rotation. We see that the axis of rotation is the line spanned by the eigenvectors which is a bisectrix of the angle between Ox and Oz axis.

1.11 Vector product in oriented $\mathbf{E}^3$

Now we give a definition of vector product of vectors in 3-dimensional Euclidean space equipped with orientation, i.e. we fix an equivalence class of orthonormal bases with the same orientation. Recall that it suffices to fix an arbitrary orthonormal basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. Then an equivalence class is defined as the set of all orthonormal bases which have the same orientation as the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$

Let $\mathbf{E}^3$ be three-dimensional oriented Euclidean space, i.e. Euclidean space equipped with an equivalence class of bases with the same orientation. To define the orientation it suffices to consider just one orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Then the equivalence class of the bases is a set of all bases which have the same orientation as the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

By default we suppose that orthonormal basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ belongs to the equivalence class of bases defining orientation of $\mathbf{E}^3$.

Let $\mathbf{E}^3$ be three-dimensional oriented Euclidean space.

Definition Vector product $L(\mathbf{x}, \mathbf{y}) = \mathbf{x} \times \mathbf{y}$ is a function of two vectors which takes vector values such that the following conditions hold

- The vector $L(\mathbf{x}, \mathbf{y}) = \mathbf{x} \times \mathbf{y}$ is orthogonal to vector $\mathbf{x}$ and vector $\mathbf{y}$:

  $$(\mathbf{x} \times \mathbf{y}) \perp \mathbf{x}\,, \quad (\mathbf{x} \times \mathbf{y}) \perp \mathbf{y} \tag{1.52}$$

  In particular it is orthogonal to the the plane spanned by the vectors $\mathbf{x}, \mathbf{y}$ (in the case if vectors $\mathbf{x}, \mathbf{y}$ are linearly independent)

- $$\mathbf{x} \times \mathbf{y} = -\mathbf{y} \times \mathbf{x}, \qquad \text{(anticommutativity condition)} \tag{1.53}$$

- $$(\lambda\mathbf{x} + \mu\mathbf{y}) \times \mathbf{z} = \lambda(\mathbf{x} \times \mathbf{z}) + \mu(\mathbf{y} \times \mathbf{z})\,, \qquad \text{(linearity condition)} \tag{1.54}$$

- If vectors $\mathbf{x}, \mathbf{y}$ are perpendicular each other then the magnitude of the vector $\mathbf{x} \times \mathbf{y}$ is equal to the area of the rectangle formed by the vectors $\mathbf{x}$ and $\mathbf{y}$:

  $$|\mathbf{x} \times \mathbf{y}| = |\mathbf{x}| \cdot |\,\mathbf{y}|\,, \qquad \text{if } \mathbf{x} \perp \mathbf{y}\,, i.e. (\mathbf{x}, \mathbf{y}) = 0\,. \tag{1.55}$$

- If the ordered triple of the vectors $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$, where $\mathbf{z} = \mathbf{x} \times \mathbf{y}$ is a basis, then this basis and an orthonormal basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ in $\mathbf{E}^3$ have the same orientation: Vector product depends on orientation in Euclidean space.

The condition of linearity of vector product with respect to the first argument (1.54) and the condition (1.53) of anticommutativity imply that vector product is an operation which is linear with respect to the second argument too. Show it:

$$\mathbf{z} \times (\lambda\mathbf{x} + \mu\mathbf{y}) = -(\lambda\mathbf{x} + \mu\mathbf{y}) \times \mathbf{z} = -\lambda(\mathbf{x} \times \mathbf{z}) - \mu(\mathbf{y} \times \mathbf{z}) = \lambda(\mathbf{z} \times \mathbf{x}) + \mu(\mathbf{z} \times \mathbf{y})\,.$$

Hence vector product is bilinear operation. Comparing with scalar product we see that vector product is bilinear anticommutative (antisymmetric) operation which takes vector values, while scalar product is bilinear symmetric operation which takes real values.

Remark You may ask: Does this operation exist? In other words is vector product well-defined. Yes, and we will show it (see below Proposition 2).

Exercise Vector product of two colinear vectors $\mathbf{x}, \mathbf{y}$ ($\mathbf{y} = \lambda\mathbf{x}$) is equal to zero. Show it: It follows from linearity and antisymmetricity conditions. Indeed

$$\mathbf{x} \times \mathbf{y} = \mathbf{x} \times (\lambda\mathbf{x}) = \lambda(\mathbf{x} \times \mathbf{x}) = -\lambda(\mathbf{x} \times \mathbf{x}) = -\mathbf{x} \times (\lambda\mathbf{x}) = -\mathbf{x} \times \mathbf{y}. \quad (1.56)$$

Hence $\mathbf{x} \times \mathbf{y} = 0$, if $\mathbf{y} = \lambda\mathbf{x}$.

Now it is a time for explicit formulae for calculations for vector product.

Let $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ be an *arbitrary* orthonormal basis of oriented Euclidean space $\mathbf{E}^3$ which belongs to the equivalence class of bases defining orientation. Then it follows from the considerations above for vector product that

$$\begin{matrix} \mathbf{e}_x \times \mathbf{e}_x = 0, & \mathbf{e}_x \times \mathbf{e}_y = \mathbf{e}_z, & \mathbf{e}_x \times \mathbf{e}_z = -\mathbf{e}_y \\ \mathbf{e}_y \times \mathbf{e}_x = -\mathbf{e}_z, & \mathbf{e}_y \times \mathbf{e}_y = 0, & \mathbf{e}_y \times \mathbf{e}_z = \mathbf{e}_x \\ \mathbf{e}_z \times \mathbf{e}_x = \mathbf{e}_y, & \mathbf{e}_z \times \mathbf{e}_y = -\mathbf{e}_x, & \mathbf{e}_z \times \mathbf{e}_z = 0 \end{matrix} \quad (1.57)$$

E.g. $\mathbf{e}_x \times \mathbf{e}_x = 0$, because of (1.56), $\mathbf{e}_x \times \mathbf{e}_y$ is equal to $\mathbf{e}_z$ or to $-\mathbf{e}_z$ according to (1.55), and according to orientation arguments $\mathbf{e}_x \times \mathbf{e}_y = \mathbf{e}_z$.

Remark Formulae above are valid for an arbitrary orthonormal basis which have the same orientation as the orthonormal basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$.— In oriented Euclidean space $\mathbf{E}^3$ we may take an arbitrary basis from the equivalence class of bases defining orientation.

Now it follows from linearity and (1.57) that for two arbitrary vectors $\mathbf{a} = a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z$, $\mathbf{b} = b_x\mathbf{e}_x + b_y\mathbf{e}_y + b_z\mathbf{e}_z$

$$\mathbf{a} \times \mathbf{b} = (a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z) \times (b_x\mathbf{e}_x + b_y\mathbf{e}_y + b_z\mathbf{e}_z) = a_xb_y\mathbf{e}_x \times \mathbf{e}_y + a_xb_z\mathbf{e}_x \times \mathbf{e}_z +$$

$$a_yb_x\mathbf{e}_y \times \mathbf{e}_x + a_yb_z\mathbf{e}_y \times \mathbf{e}_z + a_zb_x\mathbf{e}_z \times \mathbf{e}_x + a_zb_y\mathbf{e}_z \times \mathbf{e}_y =$$

$$(a_yb_z - a_zb_y)\mathbf{e}_x + (a_zb_x - a_xb_z)\mathbf{e}_y + (a_xb_y - a_yb_x)\mathbf{e}_z \,. \quad (1.58)$$

It is convenient to represent this formula in the following way:

$$L(\mathbf{a}, \mathbf{b}) = \mathbf{a} \times \mathbf{b} = \det \begin{pmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{pmatrix} \quad (1.59)$$

Note that in the formula above we have chosen an arbitrary orthonormal basis which belongs to the equivalence class of bases defining the orientation. We can choose instead the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ an arbitrary basis $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ such that both bases $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ have the same orientation.

Two questions arise.

- Does the formula (1.59) in fact defines the function $L(\mathbf{a}, \mathbf{b})$ obeying axioms of the vector product in the Definition above?
- Is it unique?

One can easy see to answer "Yes" on the first question.
Indeed by the properties of determinant $\mathbf{a} \times b$ defined by (1.59) is

- $\mathbf{a} \times \mathbf{b}$ is orthogonal to both vectors $\mathbf{a}$ and $\mathbf{b}$ This is easy to check by direct calculation of scalar products $(\mathbf{a} \times \mathbf{b}, \alpha) = (\mathbf{a} \times \mathbf{b}, \alpha) = 0$.
- linear with respect to vectors $\mathbf{a}$ and $\mathbf{b}$
- anticommutative
- $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|$ if these vectors are orthogonal. One can check it by direct calculations
- The orientation of the ordered pair $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is the same as the orientation of the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. This can be checked by direct calculation too.

It is a time to formulate and prove the uniqueness.

Lemma *An arbitrary two vectors* $\mathbf{x}, \mathbf{y}$ *in* $\mathbf{E}^3$ *are linear independent if and only if three vectors* $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ *where* $\mathbf{z} = \mathbf{x} \times \mathbf{y}$ *are linear independent, i.e. the ordered set of vectors* $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ *is a basis*

(When considering $\mathbf{z} = \mathbf{x} \times \mathbf{y}$ we assume that an orinetation in $\mathbf{E}^3$ is chosen.)

Proof. If vectors $\mathbf{x}, \mathbf{y}$ are linearly dependent, then due to anticommutativity condition (1.53) $\mathbf{x} \times \mathbf{y} = 0$, hence vectors $\mathbf{x}, \mathbf{y}$ and $\mathbf{z} = \mathbf{x} \times \mathbf{y}$ are linear dependent and $\{\mathbf{x}, \mathbf{y}, \mathbf{x} \times \mathbf{y}\}$ is not a basis.

Suppose now that vectors $\mathbf{x}$ and $\mathbf{y}$ are linearly independent. Show first that that $\mathbf{z} = \mathbf{x} \times \mathbf{y} \neq 0$. Consider the expansion

$$\mathbf{y} = \frac{(\mathbf{x}, \mathbf{y})}{(\mathbf{x}, \mathbf{x})}\mathbf{x} + \mathbf{b}'_{\perp} .$$

One can easy see that vector $\mathbf{b}'_\perp$ is perpendicular to the vector $\mathbf{x}$: $(\mathbf{b}'_\perp, \mathbf{y}) = \left(\mathbf{y} - \frac{(\mathbf{x},\mathbf{y})}{(\mathbf{x},\mathbf{x})}\mathbf{x}, \mathbf{x}\right) = 0$. Hence due to definition of vector product (see the conditions (1.54), (1.53) and (1.55) above) we have $\mathbf{z} = \mathbf{x} \times \mathbf{y} = \mathbf{x} \times \mathbf{y}'_\perp = |\mathbf{x}| \cdot |\mathbf{y}'_\perp|$. On the other hand $\mathbf{y}'_\perp \neq 0$. (If $\mathbf{y}'_\perp \neq 0$ then $\mathbf{y} = \frac{(\mathbf{x},\mathbf{y})}{(\mathbf{x},\mathbf{x})}\mathbf{x}$, but vectors $\mathbf{x}, \mathbf{y}$ are linearly independent.) Hence $\mathbf{z} = \mathbf{x} \times \mathbf{y} = \mathbf{x} \times \mathbf{y}'_\perp \neq 0$.

Now it is easy to see that vectors $\mathbf{x}, \mathbf{y}\mathbf{z}$ are linear independent (and hence form a basis.) Indeed let $\lambda\mathbf{x} + \mu\mathbf{y} + \tau\mathbf{z} = 0$. Multiplying this relation on the vector $\mathbf{z}$ we come to

$$0 = (\lambda\mathbf{x} + \mu y + \tau z, \mathbf{z}) = \tau(\mathbf{z}, \mathbf{z}) = 0, \text{since } \mathbf{z} \perp \mathbf{x} \text{ and } \mathbf{z} \perp \mathbf{y}$$

Hence $\tau = 0$ since $\mathbf{z} \neq 0$. Hence $\lambda\mathbf{x} + \mu y = 0$, i.e. $\lambda = \mu = 0$. We proved that three vectors are linear independent. ∎

Proposition 2 *The vector product* $\mathbf{a} \times \mathbf{b} = L(\mathbf{a}, \mathbf{b})$ *is well-defined.*

The formula (1.59) *defines a unique function* $L(\mathbf{a}, \mathbf{b})$ *such that it obeys all axioms of vector product independently of the choice of orthonormal basis with the given orientation.*

Proof Consider another orthonormal basis $\{\mathbf{f}_x, \mathbf{f}_y, \mathbf{f}_z\}$ such that bases $\{\mathbf{f}_x, \mathbf{f}_y, \mathbf{f}_z\}$ and $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ have the same orientation.

Perform calculation (1.59) in this new basis: Let

$$\mathbf{a} = a'_x\mathbf{f}_x + a'_y\mathbf{f}_y + a'_z\mathbf{f}_z, \mathbf{b} = b'_x\mathbf{f}_x + b'_y\mathbf{f}_y + b'_z\mathbf{f}_z$$

and

$$L'(\mathbf{a}, \mathbf{b}) = \mathbf{c}' = \det \begin{pmatrix} \mathbf{f}_x & \mathbf{f}_y & \mathbf{f}_z \\ a'_x & a'_y & a'_z \\ b'_x & b'_y & b'_z \end{pmatrix} \tag{1.60}$$

Show that $\mathbf{c} = L(\mathbf{a}, \mathbf{b})$ obtained via the formula (1.59) coincides with $\mathbf{c}' = L'(\mathbf{a}, \mathbf{b})$ obtained via the formula (1.60).

If vectors $\mathbf{a}, \mathbf{b}$ are linearly dependent, then nothing to prove: Both formulae (1.59) and (1.60) lead to the same answer: 0.

If vectors $\mathbf{a}, \mathbf{b}$ are linearly independent, then it follows from the Lemma above that vectors $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ in $\mathbf{E}^3$ are linearly independent. Hence $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is a basis in $\mathbf{E}^3$. Consider an expansion of vector $\mathbf{c}' = L'(\mathbf{a}, \mathbf{b})$ with respect to this basis: $\mathbf{c}' = \lambda\alpha + \mu\mathbf{b} + \tau\mathbf{c}$. Prove that $\lambda = \mu = 0$. If $\lambda \neq 0$ or $\mu \neq 0$ we come to the vector $\mathbf{x} = \mathbf{c}' - \tau\mathbf{a}$ such that $\mathbf{x} = \mathbf{c}' = \lambda\mathbf{a} + \mu\mathbf{b} \neq 0$ and $\mathbf{x} \perp \mathbf{b}\,\mathbf{c}$. Contradiction.

Hence $\lambda = \mu = 0$. We come to $\mathbf{c}' = \tau\mathbf{c}$. Hence $\tau = \pm 1$. The orientation arguments lead to the fact that $\tau = 1$ $\blacksquare$

Exercise Given two not-collinear vectors $\mathbf{a}, \mathbf{b}$ find a vector $\mathbf{n}$ such that the vector $\mathbf{n}$ is orthogonal to vectors $\mathbf{a}, \mathbf{b}$, and it has a unit length.

Solution: Consider the vector $\mathbf{N} = \mathbf{a} \times \mathbf{b}$, then the vector $\mathbf{n} = \pm\frac{\mathbf{N}}{|\mathbf{N}|}$ obeys all the conditions.

Remark The ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{N}\}$ as well as the ordered triple $\{\mathbf{N}, \mathbf{a}, \mathbf{b}\}$ have the same orientation as a basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ of $\mathbf{E}^3$, which defines the orinetation. The ordered triples $\{\mathbf{a}, \mathbf{b}, -\mathbf{N}\}$, $\{\mathbf{a}, \mathbf{N}, \mathbf{b}\}$, $\{-\mathbf{N}, \mathbf{a}, \mathbf{b}\}$, $\{\mathbf{a}, \mathbf{N}, \mathbf{b}\}$ have orientation opposite to the orientation of the orthonormal basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$.

The following Proposition states that vector product can be considered as area of parallelogram:

Proposition 2 *The modulus of the vector $\mathbf{z} = \mathbf{x} \times \mathbf{y}$ is equal to the area of parallelogram formed by the vectors $\mathbf{x}$ and $\mathbf{y}$.*

Proof: Consider the expansion $\mathbf{y} = \mathbf{y}_{||} + \mathbf{y}_{\perp}$, where the vector $\mathbf{y}_{\perp}$ is orthogonal to the vector $\mathbf{x}$ and the vector $\mathbf{y}_{||}$ is parallel to to vector $\mathbf{x}$. The area of the parallelogram formed by vectors $\mathbf{x}$ and $\mathbf{y}$ is equal to the product of the length of of the vector $\mathbf{x}$ on the height. The height is equal to the length of the vector $\mathbf{y}_{\perp}$. We have $S = |\mathbf{x}||\mathbf{y}_{\perp}|$. On the other $\mathbf{z} = \mathbf{x} \times \mathbf{y} = \mathbf{x} \times (\mathbf{y}_{||} + \mathbf{y}_{\perp})$ $= \mathbf{x} \times \mathbf{y}_{||} + \mathbf{x} \times \mathbf{y}_{\perp}$. But $\mathbf{x} \times \mathbf{y}_{||} = 0$, because these vectors are colinear. Hence $\mathbf{z} = \mathbf{x} \times \mathbf{y}_{\perp})$ and $|\mathbf{z}| = |\mathbf{x}||\mathbf{y}_{\perp}| = S$ because vectors $\mathbf{x}, \mathbf{y}_{\perp}$ are orthogonal to each other.

This Proposition is very important to understand the meaning of vector product. Shortly speaking vector product of two vectors is a vectors which is orthogonal to the plane spanned by these vectors, such that its magnitude is equal to the area of the parallelogram formed by these vectors. The direction is defined by orientation.

It is not worthless to recall the formula whcih we know from the school that area of parallelogram formed by vectors $\mathbf{x}, \mathbf{y}$ equals to the product of the base on the height. Hence

$$|\mathbf{x} \times \mathbf{y}| = |\mathbf{x}| \cdot |\mathbf{y}| |\sin\theta| \,, \tag{1.61}$$

where θ is an angle between vectors $\mathbf{x}, \mathbf{y}$.

Finally I would like again to stress:

Vector product of two vectors is equal to zero if these vectors are colinear (parallel). Scalar product of two vectors is equal to zero if these vector are orthogonal.

Exercise† Show that the vector product obeys to the following identity:

$$((\mathbf{a}\times\mathbf{b})\times\mathbf{c}) + ((\mathbf{b}\times\mathbf{c})\times\mathbf{a}) + ((\mathbf{c}\times\mathbf{a})\times\mathbf{b}) \quad \text{(Jacoby identity)} \tag{1.62}$$

This identity is related with the fact that heights of the triangle intersect in the one point.

Exercise† Show that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a},\mathbf{c}) - \mathbf{c}(\mathbf{a},\mathbf{b})$.

1.11.1 Volume of parallelepiped

The vector product of two vectors is related with area of parallelogram. What about a volume of parallelepiped formed by three vectors $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$?

Consider parallelepiped formed by vectors $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$. The parallelogram formed by vectors $\mathbf{b},\mathbf{c}$ is considered as a base of this parallelepiped.

Let θ be an angle between height and vector $\mathbf{a}$. It is just the angle between the vector $\mathbf{b}\times\mathbf{c}$ and the vector $\mathbf{a}$. Then the volume is equal to the length of the height multiplied on the area of the parallelogram, $V = Sh = S|\mathbf{a}|\cos\theta$, i.e. volume is equal to scalar product of the vectors $\mathbf{a}$ on the vector product of vectors $\mathbf{b}$ and $\mathbf{c}$:

$$V(\{\mathbf{a},\mathbf{b},\mathbf{c}\}) = (\mathbf{a},[\mathbf{b}\times\mathbf{c}]) = \left(a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z, \det\begin{pmatrix}\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z\\ b_x & b_y & b_z\\ c_x & c_y & c_z\end{pmatrix}\right) \tag{1.63}$$

$$= (a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z, (b_yc_z - b_zc_y)\mathbf{e}_x + (b_zc_x - b_xc_z)\mathbf{e}_y + (b_xc_y - b_yc_x)\mathbf{e}_z) =$$

$$a_x(b_yc_z - b_zc_y) + a_y(b_zc_x - b_xc_z) + a_z(b_xc_y - b_yc_x) = \begin{pmatrix}a_x & a_y & a_z\\ b_x & b_y & b_z\\ c_x & c_y & c_z\end{pmatrix}$$

We come to beautiful and useful formula:

$$(\mathbf{a},[\mathbf{b}\times\mathbf{c}]) = \begin{pmatrix}a_x & a_y & a_z\\ b_x & b_y & b_z\\ c_x & c_y & c_z\end{pmatrix}. \tag{1.64}$$

Remark The volume of the parallelepiped if considered as a positive number equals to the modulus of the number $(\mathbf{a},[\mathbf{b}\times\mathbf{c}])$. On the other hand often it is very useful to consider the volume as a real number (it could be positive and negative).

2 Differential forms in $\mathbf{E}^2$ and $\mathbf{E}^3$

2.1 Tangent vectors, curves, velocity vectors on the curve

Tangent vector is a vector $\mathbf{v}$ applied at the given point $\mathbf{p} \in \mathbf{E}^3$.

The set of all tangent vectors at the given point $\mathbf{p}$ is a vector space. It is called tangent space of $\mathbf{E}^3$ at the point $\mathbf{p}$ and it is denoted $T_{\mathbf{p}}(\mathbf{E}^3)$.

One can consider *vector field* on $\mathbf{E}^3$, i.e.a function which assigns to every point $\mathbf{p}$ vector $\mathbf{v}(\mathbf{p}) \in T_{\mathbf{p}}(\mathbf{R}^3)$.

It is instructive to study the conception of tangent vectors and vector fields on the curves and surfaces embedded in $\mathbf{E}^3$. We begin with curves.

A curve in $\mathbf{E}^n$ with parameter $t \in (a, b)$ is a continuous map

$$C\colon \qquad (a,b) \to \mathbf{E}^n \qquad \mathbf{r}(t) = (x^1(t), \ldots, x^n(t)), \quad a < t < b \tag{2.1}$$

For example consider in $\mathbf{E}^2$ the curve

$$C\colon \qquad (0, 2\pi) \to \mathbf{E}^2 \qquad \mathbf{r}(t) = (R\cos t, R\sin t),\ 0 \leq t < 2\pi \tag{2.2}$$

The image of this curve is the circle of the radius R. It can be defined by the equation:

$$x^2 + y^2 = R^2 \tag{2.3}$$

To distinguish between curve and its image we say that curve C in (2.1) is *parameterised* curve or *path*. We will call the image of the curve *unparameterised curve* (see for details the next subsection). It is very useful to think about parameter t as a "time" and consider parameterised curve like *point moving along a curve.* Unparameterised curve is the trajectory of the moving point. The using of word "curve" without adjective "parameterised" or "nonparameterised" sometimes is ambiguous.

Vectors tangent to curve—velocity vector

Let $\mathbf{r}(t) \quad \mathbf{r} = \mathbf{r}(t)$ be a curve in $\mathbf{E}^n$.

Velocity $\mathbf{v}(t)$ it is the vector

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \big(\dot{x}^1(t), \ldots, \ldots \dot{x}^n(t)\big) = \big(v^1(t), \ldots, v^n(t)\big) \tag{2.4}$$

in $\mathbf{E}^n$. Velocity vector is *tangent vector to the curve.*

Let $C\colon \mathbf{r} = \mathbf{r}(t)$ be a curve and $\mathbf{r}_0 = \mathbf{r}(t_0)$ any given point on it. Then the set of all vectors tangent to the curve at the point $\mathbf{r}_0 = \mathbf{r}(t_0)$ is one-dimensional vector space $T_{\mathbf{r}_0}C$. It is linear subspace in vector space $T_{\mathbf{r}_0}C$. The points of the tangent space $T_{\mathbf{r}_0}C$ are the points of tangent line.

In the next section we will return to curves and consider them in more details.

Remark We consider only smooth and regular curves. Curve $\mathbf{r}(t) = (x^1(t), \dots, x^n(t))$ is called smooth if all functions $x^i(t)$, $(i = 1, 2, \dots, n)$ are smooth functions (Function is called smooth if it has derivatives of arbitrary order.) Curve $\mathbf{r}(t)$ is called regular if velocity vector $\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt}$ is not equal to zero at all t. By default we consider simple curves, i.e. curves which have no intersection points.

2.2 Reparameterisation

One can move along trajectory with different velocities, i.e. one can consider different parameterisation. E.g. consider

$$C_1\colon \quad \begin{cases} x(t) = t \\ y(t) = t^2 \end{cases} \quad 0 < t < 1 \qquad C_2\colon \quad \begin{cases} x(t) = \sin t \\ y(t) = \sin^2 t \end{cases} \quad 0 < t < \frac{\pi}{2} \tag{2.5}$$

Images of these two parameterised curves are the same. In both cases point moves along a piece of the same parabola but with different velocities.

Definition

Two smooth curves $C_1\colon \quad \mathbf{r}_1(t)\colon (a_1, b_1) \to \mathbf{E}^n$ and $C_2\colon \quad \mathbf{r}_2(\tau)\colon (a_2, b_2) \to \mathbf{E}^n$ are called equivalent if there exists reparameterisation map:

$$t(\tau)\colon (a_2, b_2) \to (a_1, b_1),$$

such that

$$r_2(\tau) = r_1(t(\tau)) \tag{2.6}$$

Reparameterisation $t(\tau)$ is diffeomorphism, i.e. function $t(\tau)$ has derivatives of all orders and first derivative $t'(\tau)$ is not equal to zero.

E.g. curves in (2.5) are equivalent because a map $\varphi(t) = \sin t$ transforms first curve to the second.

Equivalence class of equivalent parameterised curves is called non-parameterised curve.

It is useful sometimes to distinguish curves in the same equivalence class which differ by orientation.

Definition Let curves C_1, C_2 be two equivalent curves. We say that they have same orientation (parameterisations $\mathbf{r}_1(t$ and $\mathbf{r}_(\tau)$ have the same orientation) if reparameterisation $t = t(\tau)$ has positive derivative, $t'(\tau) > 0$. We say that they have opposite orientation (parameterisations $\mathbf{r}_1(t$ and $\mathbf{r}_(\tau)$ have the opposite orientation) if reparameterisation $t = t(\tau)$ has negative derivative, $t'(\tau) < 0$.

Changing orientation means changing the direction of "walking" around the curve.

Equivalence class of equivalent curves splits on two subclasses with respect to orientation.

Non-formally: Two curves are equivalent curves (belong to the same equivalence class) if these parameterised curves (paths) have the same images. Two equivalent curves have the same image. They define the same set of points in $\mathbf{E}^n$. Different parameters correspond to moving along curve with different velocity. Two equivalent curves have different orientation If two parameterisations correspond to moving along the curve in different directions then these parameterisations define opposite orientation.

What happens with velocity vector if we change parameterisation? It changes its value, but it can change its direction only on opposite (If these parameterisations have opposite orientation of the curve):

$$\mathbf{v}(\tau) = \frac{d\mathbf{r}_2(\tau)}{d\tau} = \frac{d\mathbf{r}(t(\tau))}{d\tau} = \frac{dt(\tau)}{d\tau} \cdot \frac{d\mathbf{r}(t)}{dt}\Big|_{t=t(\tau)} \tag{2.7}$$

Or shortly: $\mathbf{v}(\tau)\big|_\tau = t_\tau(\tau)\mathbf{v}(t)\big|_{t=t(\tau)}$

We see that velocity vector is multiplied on the coefficient (depending on the point of the curve), i.e. velocity vectors for different parameterisations are collinear vectors.

(We call two vectors $\mathbf{a}, \mathbf{b}$ collinear, if they are proportional each other, i,e, if $\mathbf{a} = \lambda\mathbf{b}$.)

Example Consider following curves in $\mathbf{E}^2$:

$$C_1\colon \quad \begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases} ,0 < \theta < \pi, \qquad C_2\colon \quad \begin{cases} x = u \\ y = \sqrt{1-u^2} \end{cases} ,-1 < u < 1, \tag{2.8}$$

$$\begin{cases} x = \tan t \\ y = \frac{\sqrt{\cos 2t}}{\cos t} \end{cases} \quad , -\frac{\pi}{4} < t < \frac{\pi}{4} \tag{2.9}$$

These three parameterised curves,(paths) define the same non-parameterised curve: the upper piece of the circle: $x^2 + y^2 = 1, y > 0$. The reparameterisation $u(\theta) = \cos\theta$ transforms the second curve to the first one.

The reparameterisation $u(\theta) = \cos\theta$ transforms the second curve to the first one.

The reparameterisation $u(\theta) = \tan t$ transforms the second curve to the third one one: $\frac{\sqrt{\cos 2t}}{\cos t} = \frac{\sqrt{\cos^2 t - \sin^2 t}}{\cos t} = \sqrt{1 - \tan^2 t}$.

Curves C_1, C_2 have opposite orientation because $u'(\theta) < 0$. Curves C_2, C_3 have the same orientation, because $u'(t) > 0$. Curves C_1 and C_2 have opposite orientations too (Why?).

In the first case point moves with constant pace $|\mathbf{v}(\theta)| = 1$ anti clock-wise "from right to left" from the point $A = (1, 0)$ to the point $B = (-1, 0)$. In the second case pace is not constant, but $v_x = 1$ is constant. Point moves clock-wise "from left to right", from the point $B = (-1, 0)$ to the point $A = (1, 0)$. In the third case point also moves clock-wise "from the left to right".

There are other examples in the Homeworks.

2.3 0-forms and 1-forms

Most of considerations of this and next subsections can be considered only for $\mathbf{E}^2$ or $\mathbf{E}^3$. Compulsory material for differential forms is only for $\mathbf{E}^2$, $\mathbf{E}^3$.

0-form on $\mathbf{E}^n$ it is just function on $\mathbf{E}^n$ (all functions under consideration are differentiable)

Now we define 1-forms.

Definition Differential 1-form ω on $\mathbf{E}^n$ is a function on tangent vectors of $\mathbf{E}^n$, such that it is linear at each point:

$$\omega(\mathbf{r}, \lambda\mathbf{v}_1 + \mu\mathbf{v}_2) = \lambda\omega(\mathbf{r}, \mathbf{v}_1) + \mu\omega(\mathbf{r}, \mathbf{v}_2)\,. \tag{2.10}$$

Here $\mathbf{v}_1, \mathbf{v}_2$ are vectors tangent to $\mathbf{E}^n$ at the point $\mathbf{r}$, ($\mathbf{v}_1, \mathbf{v}_2 \in T_x\mathbf{E}^n$) (We recall that vector tangent at the point $\mathbf{r}$ means vector attached at the point $\mathbf{r}$). We suppose that ω is smooth function on points $\mathbf{r}$.

If $\mathbf{X}(\mathbf{r})$ is vector field and ω-1-form then evaluating ω on $\mathbf{X}(\mathbf{r})$ we come to the function $w(\mathbf{r}, \mathbf{X}(\mathbf{r}))$ on $\mathbf{E}^3$.

Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be a basis in $\mathbf{E}^n$ and $(x^1, \dots, x^n)$ corresponding coordinates: an arbitrary point with coordinates $(x^1, \dots, x^n)$ is assigned to the vector $\mathbf{r} = x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + \dots x^n\mathbf{e}_n$ starting at the origin.

Translating basis vectors $\mathbf{e}_i$ $(i = 1, \dots, n)$ from the origin to other points of $\mathbf{E}^n$ we come to vector field which we also denote $\mathbf{e}_i$ $(i = 1, \dots, n)$. The value of vector field $\mathbf{e}_i$ at the point $(x^1, \dots, x^n)$ is the vector $\mathbf{e}_i$ attached at this point (tangent to this point).

Let ω be an 1-form on $\mathbf{E}^n$. Consider an arbitrary vector field $\mathbf{A}(\mathbf{r}) = \mathbf{A}(x^1, \dots, x^n)$:

$$\mathbf{A}(\mathbf{r}) = A^1(\mathbf{r})\mathbf{e}_1 + \dots + A^n(\mathbf{r})\mathbf{e}_n = \sum_{i=1}^{n} A^i(\mathbf{r})\mathbf{e}_i \tag{2.11}$$

Then by linearity

$$\omega(\mathbf{r}, \mathbf{A}(\mathbf{r})) = \omega(\mathbf{r}, A^1(\mathbf{r})\mathbf{e}_1 + \dots + A^n(\mathbf{r})\mathbf{e}_n) = A^1\omega(\mathbf{r}, \mathbf{e}_1) + \dots + A^n\omega(\mathbf{r}, \mathbf{e}_n)$$

Consider *basic* differential forms $dx^1, dx^2, \dots, dx^n$ such that

$$dx^i(\mathbf{e}_j) = \delta^i_j = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} . \tag{2.12}$$

Then it is easy to see that

$$dx^1(\mathbf{A}) = \mathbf{A}^1, dx^2(\mathbf{A}) = \mathbf{A}^2, ..., \text{i.e.} dx^i(\mathbf{A}) = A^i$$

Hence

$$\omega(\mathbf{r}, \mathbf{A}(\mathbf{r})) = \left(\omega_1(\mathbf{r})dx^1 + \omega_2(\mathbf{r})dx^2 + \dots + \omega_n(\mathbf{r})dx^n\right)(\mathbf{A}(\mathbf{r}))$$

where components $\omega_i(\mathbf{r}) = \omega(\mathbf{r}, \mathbf{e}_i)$.

In the same way as an arbitrary vector field on $\mathbf{E}^n$ can be expanded over the basis $\{\mathbf{e}_i\}$ (see (2.11)), an arbitrary differential 1-form ω can be expanded over the basis forms(2.12)

$$\omega = \omega_1(x^1, \dots, x^n)dx^1 + \omega_2(x^1, \dots, x^n)x^2 + \dots + \omega_n(x^1, \dots, x^n)dx^n .$$

Example Consider in $\mathbf{E}^3$ a basis $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ and corresponding coordinates (x, y, z). Then

$$\begin{array}{c} dx(\mathbf{e}_x) = 1, dx(\mathbf{e}_y) = 0, dx(\mathbf{e}_z) = 0 \\ dy(\mathbf{e}_x) = 0, dy(\mathbf{e}_y) = 1, dy(\mathbf{e}_z) = 0 \\ dz(\mathbf{e}_x) = 0, dz(\mathbf{e}_y) = 0, dz(\mathbf{e}_z) = 1 \end{array} \tag{2.13}$$

The value of a differential 1-form $\omega = a(x,y,z)dx + b(x,y,z)dy + c(x,y,z)dz$ on vector field $\mathbf{X} = A(x,y,z)\mathbf{e}_x + B(x,y,z)\mathbf{e}_y + C(x,y,z)\mathbf{e}_z$ is equal to

$$\omega(\mathbf{r},\mathbf{X}) = a(x,y,z)dx(\mathbf{X}) + b(x,y,z)dx(\mathbf{X}) + c(x,y,z)dx(\mathbf{X}) =$$

$$a(x,y,z)A(x,y,z) + b(x,y,z)B(x,y,z) + c(x,y,z)C(x,y,z)$$

It is very useful to introduce new notation for vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$. To introduce this new notation we need a short recalling of derivational derivative of functions.

Vectors—directional derivatives of functions

Let $\mathbf{R}$ be a vector in $\mathbf{E}^n$ tangent to the point $\mathbf{r} = \mathbf{r}_0$ (attached at a point $\mathbf{r} = \mathbf{r}_0$). Define the operation of derivative of an arbitrary (differentiable) function at the point $\mathbf{r}_0$ along the vector $\mathbf{R}$— directional derivative of function f along the vector $\mathbf{R}$

Definition

Let $\mathbf{r}(t)$ be a curve such that

- $\mathbf{r}(t)\big|_{t=0} = \mathbf{r}_0$
- Velocity vector of the curve at the point $\mathbf{r}_0$ is equal to $\mathbf{R}$: $\frac{d\mathbf{r}(t)}{dt}\big|_{t=0} = \mathbf{R}$

Then directional derivative of function f with respect to the vector $\mathbf{R}$ at the point $\mathbf{r}_0$ $\partial_{\mathbf{R}} f\big|_{\mathbf{r}_0}$ is defined by the relation

$$\partial_{\mathbf{R}} f\big|_{\mathbf{r}_0} = \frac{d}{dt}\left(f\left(\mathbf{r}(t)\right)\right)\big|_{t=0}. \tag{2.14}$$

Using chain rule one come from this definition to the following important formula for the directional derivative:

$$\text{If } \mathbf{R} = \sum_{i=1}^{n} R^i \mathbf{e}_i \text{ then } \partial_{\mathbf{R}} f\big|_{\mathbf{r}_0} = \sum_{i=1}^{n} R^i \frac{\partial}{\partial x^i} f(x^1,\dots,x^n)\big|_{\mathbf{r}=\mathbf{r}_0} \tag{2.15}$$

It follows form this formula that

One can assign to every vector $\mathbf{R} = \sum_{i=1}^n R^i \mathbf{e}_i$ *the operation* $\partial_{\mathbf{R}} = R^1 \frac{\partial}{\partial x^1} + R^2 \frac{\partial}{\partial x^2} + \cdots + R^n \frac{\partial}{\partial x^n}$ *of taking directional derivative:*

$$\mathbf{R} = \sum_{i=1}^n R^i \mathbf{e}_i \mapsto \partial_{\mathbf{R}} = \sum_{i=1}^n R^i \frac{\partial}{\partial x^i} \tag{2.16}$$

Vector $\mathbf{e}_x$ we will denote sometimes by ∂_x, vectors $\mathbf{e}_y, \mathbf{e}_z$ by ∂_y, ∂_z respectively. The symbols $\partial_x, \partial_y, \partial_z$ correspond to partial derivative with respect to coordinate x or y or z . Later we see that these new notations are very illuminating when we deal with arbitrary coordinates, such as polar coordinates or spherical coordinates, The conception of orthonormal basis is ill-defined in arbitrary coordinates, but one can still consider the corresponding partial derivatives. Vector fields $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ (or in new notation $\partial_x, \partial_y, \partial_z$) can be considered as a basis[4] in the space of all vector fields on $\mathbf{E}^3$.

An arbitrary vector field (2.11) can be rewritten in the following way:

$$\mathbf{A}(\mathbf{r}) = A^1(\mathbf{r})\mathbf{e}_1 + \cdots + A^n(\mathbf{r})\mathbf{e}_n = A^1(\mathbf{r})\frac{\partial}{\partial x^1} + A^2(\mathbf{r})\frac{\partial}{\partial x^2} + \cdots + A^n(\mathbf{r})\frac{\partial}{\partial x^n} \tag{2.17}$$

Differential on 0*-forms*

Now we introduce very important operation: Differential d which acts on 0-forms and transforms them to 1 forms.

$$\boxed{\begin{matrix}\text{Differential} \\ \text{0-forms}\end{matrix}} \xrightarrow{d} \boxed{\begin{matrix}\text{Differential} \\ \text{1-forms}\end{matrix}}$$

Later we will learn how differential acts on 1-forms transforming them to 2-forms.

Definition Let $f = f(x)$-be 0-form, i.e. function on $\mathbf{E}^n$. Then

$$df = \sum_{i=1}^n \frac{\partial f(x^1, \ldots, x^n)}{\partial x^i} dx^i \,. \tag{2.18}$$

The value of 1-form df on an arbitrary vector field (2.17) is equal to

$$df(\mathbf{A}) = \sum_{i=1}^n \frac{\partial f(x^1, \ldots, x^n)}{x^i} dx^i(\mathbf{A}) = \sum_{i=1}^n \frac{\partial f(x^1, \ldots, x^n)}{x^i} A^i = \partial_{\mathbf{A}} f \tag{2.19}$$

[4]Coefficients of expansion are functions, elements of algebra of functions, not numbers ,elements of field. To be more careful, these vector fields are basis of the *module* of vector fields on $\mathbf{E}^3$

We see that *value of differential of* 0*-form* f *on an arbitrary vector field* $\mathbf{A}$ *is equal to directional derivative of function* f *with respect to the vector* $\mathbf{A}$.

The formula (2.19) defines df in invariant way without using coordinate expansions. Later we check straightforwardly the coordinate-invariance of the definition (2.18).

Exercise Check that

$$dx^i(\mathbf{A}) = \partial_{\mathbf{A}} x^i \tag{2.20}$$

Example If $f = f(x,y)$ is a function $(0 - form)$ on $\mathbf{E}^2$ then

$$df = \frac{\partial f(x,y)}{\partial x}dx + \frac{\partial f(x,y)}{\partial y}dy$$

and for an arbitrary vector field $\mathbf{A} = \mathbf{A} = A_x\mathbf{e}_x + \mathbf{A}_y\mathbf{e}_y = A_x(x,y)\partial_x + A_y(x,y)\partial_y$

$$df(\mathbf{A}) = \frac{\partial f(x,y)}{\partial x}dx(\mathbf{A}) + A_y(x,y)\frac{\partial f(x,y)}{\partial y}dy(\mathbf{A}) =$$

$$\mathbf{A}_x(x,y)\frac{\partial f(x,y)}{\partial x} + A_y(x,y)\frac{\partial f(x,y)}{\partial y} = \partial_{\mathbf{A}} f\,.$$

Example Find the value of 1-form $\omega = df$ on the vector field $\mathbf{A} = x\partial_x + y\partial_y$ if $f = \sin(x^2+y^2)$.

$\omega(\mathbf{A}) = df(\mathbf{A})$. One can calculate it using formula (2.18) or using formula (2.19).

Solution (using (2.18)*)*:

$$\omega = df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 2x\cos(x^2+y^2)dx + 2y\cos(x^2+y^2)dy\,.$$

$$\omega(\mathbf{A}) = 2x\cos(x^2+y^2)dx(\mathbf{A}) + 2y\cos(x^2+y^2)dy(\mathbf{A}) =$$

$$2x\cos(x^2+y^2)A_x + 2y\cos(x^2+y^2)dA_y = 2(x^2+y^2)\cos(x^2+y^2)\,,$$

Another solution (using (2.19)*)*

$$df(\mathbf{A}) = \partial_{\mathbf{A}} f = A_x\frac{\partial f}{\partial x} + A_x\frac{\partial f}{\partial y} = 2(x^2+y^2)\cos(x^2+y^2)\,.$$

See other examples in Homeworks.

2.4 Differential 1-form in arbitrary coordinates

Why differential forms? Why so strange notations for vector fields.

Now we see that working with differential forms we in fact do not care about what coordinates we work in.

One of advantages of the technique of differential form is that calculations are the same in arbitrary coordinates.

Consider an arbitrary (local) coordinates $u^1, \dots, u^n$ on $\mathbf{E}^n$: $u^i = u^i(x^1, \dots, x^n)$, $i = 1, \dots, n$. Show first that

$$du^i = \sum_{k=1}^{n} \frac{\partial u^i(x^1, \dots, x^n)}{\partial x^k} dx^k \,. \tag{2.21}$$

It is enough to check it on basic fields:

$$du^i\left(\frac{\partial}{\partial x^m}\right) = \partial_{\left(\frac{\partial}{\partial x^m}\right)} u^i = \frac{\partial u^i(x^1, \dots, x^n)}{x^m} = \sum_{k=1}^{n} \frac{\partial u^i(x^1, \dots, x^n)}{\partial x^k} dx^k \left(\left(\frac{\partial}{\partial x^m}\right)\right) = \left(\frac{\partial}{\partial x^m}\right) .$$

because (see (2.12)):

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \,. \tag{2.22}$$

(We rewrite the formula (2.12) using new notations ∂_i instead $\mathbf{e}_i$). In the previous formula (2.12) we considered *cartesian* coordinates.

Show that the formula above is valid in an *arbitrary coordinates*.

One can see using chain rule that

$$\frac{\partial}{\partial u^i} = \frac{\partial x^1}{\partial u^i}\frac{\partial}{\partial x^1} + \frac{\partial x^2}{\partial u^i}\frac{\partial}{\partial x^2} + \dots + \frac{\partial x^n}{\partial u^i}\frac{\partial}{\partial x^n} = \sum_{k=1}^{n} \frac{\partial x^k}{\partial u^i}\frac{\partial}{\partial x^k} \tag{2.23}$$

Calculate the value of differential form du^i on vector field $\frac{\partial}{\partial u^j}$ using (2.21) and (2.23):

$$du^i\left(\frac{\partial}{\partial u^j}\right) = \sum_{k=1}^{n} \frac{\partial u^i(x^1, \dots, x^n)}{\partial x^k} dx^k \left(\sum_{r=1}^{n} \frac{\partial x^r}{\partial u^j}\frac{\partial}{\partial x^r}\right) = \tag{2.24}$$

$$\sum_{k,r=1}^{n} \frac{\partial u^i(x^1, \dots, x^n)}{\partial x^k} \frac{\partial x^r(u^1, \dots, u^n)}{\partial u^j} dx^k \left(\frac{\partial}{\partial x^r}\right) =$$

$$\sum_{k,r=1}^{n} \frac{\partial u^i(x^1, \dots, x^n)}{\partial x^k} \frac{\partial x^r(u^1, \dots, u^n)}{\partial u^j} \delta^k_r = \sum_{k=1}^{n} \frac{\partial x^k}{\partial u^j}\frac{\partial u^i}{\partial x^k} = \delta^j_i$$

We come to

$$du^i\left(\frac{\partial}{\partial u^j}\right) = \delta^i_j = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \,. \tag{2.25}$$

We see that formula (2.22) has the same appearance in arbitrary coordinates. In other words it is invariant with respect to an arbitrary transformation of coordinates.

Exercise Check straightforwardly the invariance of the definition (2.18). In coordinates $(u^1, \dots, u^n)$

Solution We have to show that the formula (2.18) does not changed under changing of coordinates $u^i = u^i(x^1, \dots, x^n)$.

$$df = \sum_{i=1}^{n} \frac{\partial f(x^1, \dots, x^n)}{\partial x^i} dx^i = \sum_{i=1,k}^{n} \frac{\partial f(x^1, \dots, x^n)}{\partial x^i} \frac{\partial x^i}{\partial u^k} du^k == \sum_{i=1}^{n} \frac{\partial f}{\partial u^k} du^k \,,$$

because $\sum_{i=1}^n \frac{\partial f(x^1,\dots,x^n)}{\partial x^i} \frac{\partial x^i}{\partial u^k} = \frac{\partial f}{\partial u^k}$

Example

Consider more in detail $\mathbf{E}^2$. (For $\mathbf{E}^3$ considerations are the same, just calculations little bit more complicated) Let u, v be an arbitrary coordinates in $\mathbf{E}^2$, $u = u(x, y), v = v(x, y)$.

$$du = \frac{\partial u(x,y)}{\partial x} dx + \frac{\partial u(x,y)}{\partial y} dy, \qquad dv = \frac{\partial v(x,y)}{\partial x} dx + \frac{\partial v(x,y)}{\partial y} dy \tag{2.26}$$

and

$$\partial_u = \frac{\partial x(u,v)}{\partial u} \partial_x + \frac{\partial y(u,v)}{\partial u} \partial_y, \;\; \partial_v = \frac{\partial x(u,v)}{\partial v} \partial_x + \frac{\partial y(u,v)}{\partial v} \partial_y \tag{2.27}$$

(As always sometimes we use notation ∂_u instead $\frac{\partial}{\partial u}$, ∂_x instead $\frac{\partial}{\partial x}$ e.t.c.) Then

$$\begin{gathered} du(\partial_u) = 1, du(\partial_v) = 0 \\ dv(\partial_w) = 0, dv(\partial_v) = 1 \end{gathered} \tag{2.28}$$

This follows from the general formula but it is good exercise to repeat the previous calculations for this case:

$$du(\partial_u) = \left(\frac{\partial u(x,y)}{\partial x} dx + \frac{\partial u(x,y)}{\partial y} dy \right) \left(\frac{\partial x(u,v)}{\partial u} \partial_x + \frac{\partial y(u,v)}{\partial u} \partial_y \right) =$$

$$\frac{\partial u(x,y)}{\partial x} \frac{\partial x(u,v)}{\partial u} + \frac{\partial u(x,y)}{\partial y} \frac{\partial y(u,v)}{\partial u} = \frac{\partial x(u,v)}{\partial u} \frac{\partial u(x,y)}{\partial x} + \frac{\partial y(u,v)}{\partial u} \frac{\partial u(x,y)}{\partial y} = 1$$

We just apply chain rule to the function $u = u(x, y) = u(x(u, v), y(u, v))$:

Analogously

$$du(\partial_v) = \left(\frac{\partial u(x,y)}{\partial x} dx + \frac{\partial u(x,y)}{\partial y} dy \right) \left(\frac{\partial x(u,v)}{\partial v} \partial_x + \frac{\partial y(u,v)}{\partial v} \partial_y \right)$$

$$\frac{\partial u(x,y)}{\partial x} \frac{\partial x(u,v)}{\partial v} + \frac{\partial u(x,y)}{\partial y} \frac{\partial y(u,v)}{\partial v} = \frac{\partial x(u,v)}{\partial v} \frac{\partial u(x,y)}{\partial x} + \frac{\partial y(u,v)}{\partial v} \frac{\partial u(x,y)}{\partial y} = 0$$

The same calculations for dv.

Example (Polar coordinates) Consider polar coordinates in $\mathbf{E}^2$:

$$\begin{cases} x(r,\varphi) = r\cos\varphi \\ y(r,\varphi) = r\sin\varphi \end{cases} \qquad (0 \le \varphi < 2\pi, 0 \le r < \infty),$$

Respectively

$$\begin{cases} r(x,y) = \sqrt{x^2+y^2} \\ \varphi = \arctan\frac{y}{x} \end{cases} . \tag{2.29}$$

We have that for basic 1-forms

$$dr = r_x dx + r_y dy = \frac{x}{\sqrt{x^2+y^2}}dx + \frac{y}{\sqrt{x^2+y^2}}dy = \frac{xdx+ydy}{r}$$

and

$$d\varphi = \varphi_x dx + \varphi_y dy = \frac{-ydx}{x^2+y^2} + \frac{xdy}{x^2+y^2}dx = \frac{xdy-ydx}{r^2}$$

Respectively

$$dx = x_r dr + x_\varphi d\varphi = \cos\varphi dr - r\sin\varphi d\varphi$$

and

$$dy = y_r dr + y_\varphi d\varphi = \sin\varphi dr + r\cos\varphi d\varphi \tag{2.30}$$

For basic vector field

$$\partial_r = \frac{\partial x}{\partial r}\partial_x + \frac{\partial y}{\partial r}\partial_y = \cos\varphi\partial_x + \sin\varphi\partial_y = \frac{x\partial_x + y\partial_y}{r},$$

$$\partial_\varphi = \frac{\partial x}{\partial\varphi}\partial_x + \frac{\partial y}{\partial\varphi}\partial_y = -r\sin\varphi\partial_x + r\cos\varphi\partial_y = x\partial_y - y\partial_x,$$

respectively

$$\partial_x = \frac{\partial r}{\partial x}\partial_r + \frac{\partial\varphi}{\partial x}\partial_\varphi = \frac{x}{r}\partial_r - \frac{y}{r^2}\partial_\varphi$$

and

$$\partial_y = \frac{\partial r}{\partial y}\partial_r + \frac{\partial\varphi}{\partial y}\partial_\varphi = \frac{y}{r}\partial_r + \frac{x}{r^2}\partial_\varphi \tag{2.31}$$

Example Calculate the value of forms $\omega_1 = xdx + ydy$, $\omega_2 = xdy - ydx$ on vector fields $\mathbf{A} = x\partial_x + y\partial_y$, $\mathbf{B} = x\partial_y - y\partial_x$. Perform calculations in cartesian and in polar coordinates.

In cartesian coordinates:

$$\omega_1(\mathbf{A}) = xdx(x\partial_x+y\partial_y)+ydy(x\partial_x+y\partial_y) = x^2+y^2,\ \omega_1(\mathbf{B}) = xdx(\mathbf{B})+ydy(\mathbf{B}) = 0,$$

$$\omega_2(\mathbf{A}) = xdy(\mathbf{A}) - ydx(\mathbf{A}) = 0,\ \omega_2(\mathbf{B}) = xdy(\mathbf{B}) - ydx(\mathbf{B}) = x^2 + y^2\,.$$

Now perform calculations in polar coordinates. According to calculations in previous example we have that

$$\omega_1 = xdx + ydy = rdr,\ \omega_2 = xdy - ydx = r^2 d\varphi$$

and

$$\mathbf{A} = x\partial_x + y\partial_y = r\partial_r,\ \mathbf{B} = x\partial_y - y\partial_x = \partial_\varphi$$

Hence $\omega_1(\mathbf{A}) = rdr(\mathbf{A}) = r^2, \omega_1(\mathbf{B} = rdr(\partial_\varphi) = 0,$

$$\omega_2(\mathbf{A}) = r^2 d\varphi(r\partial_r) = 0, \quad \omega_2(\mathbf{B}) = r^2 d\varphi(\partial_\varphi) = r^2$$

Example Calculate the value of form $\omega = \frac{xdy-ydx}{x^2+y^2}$ on the vector field $\mathbf{A} = \partial_\varphi$. We have to transform form from cartesian coordinates to polar or vector field from polar to cartesian.

$$\frac{xdy - ydx}{x^2 + y^2} = d\varphi, \quad \omega(\mathbf{A}) = d\varphi(\partial_\varphi) = 1$$

or

$$\partial_\varphi = x\partial_y - y\partial_x,\ \omega(\mathbf{A}) = \frac{xdy(x\partial_y - y\partial_x) - ydx(x\partial_y - y\partial_x)}{x^2 + y^2} = 1\,.$$

2.5 Integration of differential 1-forms over curves

Let $\omega = \omega_1(x^1, \dots, x^n)dx^1 + \dots + \omega_1(x^1, \dots, x^n)dx^n = \sum_{i=1}^n \omega_i dx^i$ be an arbitrary 1-form in $\mathbf{E}^n$

and $C\colon \mathbf{r} = \mathbf{r}(t), t_1 \le t \le t_2$ be an arbitrary smooth curve in $\mathbf{E}^n$.

One can consider the value of one form ω on the velocity vector field $\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt}$ of the curve:

$$\omega(\mathbf{v}(t)) = \sum_{i=1}^n \omega_i \left(x^1(t), \dots, x^n(t)\right) dx^i(\mathbf{v}(t)\right) = \sum_{i=1}^n \omega_i \left(x^1(t), \dots, x^n(t)\right) \frac{dx^i(t)}{dt}$$

We define now integral of 1-form ω over the curve C.

Definition The integral of the form $\omega = \omega_1(x^1,\dots,x^n)dx^1 + \dots + \omega_n(x^1,\dots,x^n)dx^n$ over the curve C: $\quad \mathbf{r} = \mathbf{r}(t) \quad t_1 \le t \le t_2$ is equal to the integral of the function $\omega(\mathbf{v}(t))$ over the interval $t_1 \le t \le t_2$:

$$\int_C \omega = \int_{t_1}^{t_2} \omega(\mathbf{v}(t))dt = \int_{t_1}^{t_2} \left(\sum_{i=1}^{n} \omega_i \left(x^1(t),\dots,x^n(t)\right) \frac{dx^i(t)}{dt} \right) dt \,. \quad (2.32)$$

Proposition *The integral $\int_C \omega$ does not depend on the choice of coordinates on $\mathbf{E}^n$. It does not depend (up to a sign) on parameterisation of the curve: if C: $\quad \mathbf{r} = \mathbf{r}(t) \quad t_1 \le t \le t_2$ is a curve and $t = t(\tau)$ is reparameterisation, i.e. new curve C': $\quad \mathbf{r}'(\tau) = \mathbf{r}(t(\tau)) \quad \tau_1 \le \tau \le \tau_2$, then $\int_C \omega = \pm \int_C' \omega$:*

$$\int_C \omega = \int_C' \omega, \quad \textit{if orientaion is not changed, i.e. if } t'(\tau) > 0 \qquad (2.33)$$

and

$$\int_C \omega = -\int_C' \omega, \quad \textit{if orientaion is changed, i.e. if } t'(\tau) < 0 \qquad (2.34)$$

If reparameterisation changes the orientation then starting point of the curve becomes the ending point and vice versa.

Proof of the Proposition Show that integral does not depend (up to a sign) on the parameterisation of the curve. Let $t(\tau)$ $(\tau_1 \le t \le \tau_2)$ be reparameterisation. We come to the new curve C': $\mathbf{r}'(\tau) = \mathbf{r}(t(\tau))$. Note that the new velocity vector $\mathbf{v}'(\tau) = \frac{dr(t(\tau))}{d\tau} = t'(\tau)\mathbf{v}(t(\tau))$. Hence $\omega(\mathbf{v}'(\tau)) = w(\mathbf{v}(t(\tau)))t'(\tau)$. For the new curve C'

$$\int_{C'} \omega = \int_{\tau_1}^{\tau_2} \omega(\mathbf{v}'(\tau))d\tau = \int_{\tau_1}^{\tau_2} \omega(\mathbf{v}(t(\tau))) \frac{dt(\tau)}{d\tau} d\tau = \int_{t(\tau_1)}^{t(\tau_2)} \omega(\mathbf{v}(t))dt$$

$t(\tau_1) = t_1$, $t(\tau_2) = t_2$ if reparameterisation does not change orientation and $t(\tau_1) = t_2$, $t(\tau_2) = t_1$ if reparameterisation changes orientation.

Hence $\int_{C'} w = \int_{t_1}^{t_2)} \omega(\mathbf{v}(t))dt = \int_C \omega$ if orientation is not changed and $\int_{C'} w = \int_{t_2}^{t_1)} \omega(\mathbf{v}(t))dt = -\int_{t_1}^{t_2)} \omega(\mathbf{v}(t))dt = -\int_C \omega$ is orientation is changed.

Example
Let
$$\omega = \omega_1(x,y)dx + \omega_2(x,y)dy$$
be 1-form in $\mathbf{E}^2$ (x,y–are usual cartesian coordinates). Let $C\colon \quad \mathbf{r} = \mathbf{r}(t) \quad \begin{cases} x = x(t) \\ y = y(t) \end{cases}$, $t_1 \le t \le t_2$ be a curve in $\mathbf{E}^2$.

Consider velocity vector field of this curve
$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} = \begin{pmatrix} x_t(t) \\ y_t(t) \end{pmatrix} = x_t\partial_x + y_t\partial_y \tag{2.35}$$
($x_t = \frac{dx(t)}{dt}$, $y_t = \frac{dy(t)}{dt}$).

One can consider the value of one form ω on the velocity vector field $\mathbf{v}(t)$ of the curve: $\omega(\mathbf{v}) = \omega_1 dx(\mathbf{v}) + \omega_2 dy(\mathbf{v}) =$
$$\omega_1(x(t),y(t))x_t(t) + \omega_2(x(t),y(t))y_t(t)\,.$$
The integral of the form $\omega = \omega_1(x,y)dx + \omega_2(x,y)dy$ over the curve $C\colon \quad \mathbf{r} = \mathbf{r}(t) \quad t_1 \le t \le t_2$ is equal to the integral of the function $\omega(\mathbf{v}(t))$ over the interval $t_1 \le t \le t_2$:
$$\int_C \omega = \int_{t_1}^{t_2} \omega(\mathbf{v}(t))dt = \int_{t_1}^{t_2} \left(\omega_1(x(t),y(t))\frac{dx(t)}{dt} + \omega_2(x(t),y(t))\frac{dy(t)}{dt}\right) dt\,. \tag{2.36}$$

Example Consider an integral of the form $\omega = 3dy + 3y^2dx$ over the curve $C\colon \mathbf{r}(t) \begin{cases} x = \cos t \\ y = \sin t \end{cases}$, $0 \le t \le \pi/2$. (C is the arc of the circle $x^2 + y^2 = 1$ defined by conditions $x,y \ge 0$).

Velocity vector $\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} = \begin{pmatrix} x_t(t) \\ y_t(t) \end{pmatrix} = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$. The value of the form on velocity vector is equal to
$$\omega(\mathbf{v}(t)) = 3y^2(t)v_x(t) + 3v_y(t) = 3\sin^2 t(-\sin t) + 3\cos t = 3\cos t - 3\sin^3 t$$
and
$$\int_C \omega = \int_0^{\frac{\pi}{2}} w(\mathbf{v}(t))dt = \int_0^{\frac{\pi}{2}} (3\cos t - 3\sin^3 t)dt = 3\left(\sin t + \cos t - \frac{\cos^3 t}{3}\right)\Big|_0^{\frac{\pi}{2}}$$

For another examples see Homeworks and Coursework.

Now consider integrals of the same form ω over three curves which differ by reparameterisation.

Example Consider 1-form $\omega = xdy - ydx$ and curve C—- upper half of the circle $x^2+y^2=R^2$, $(y \geq 0)$.

We have the image of the curve not the parameterised curve. Consider different parameterisations of this curve:

$$\mathbf{r}_1(t)\colon \begin{cases} x = R\cos t \\ y = R\sin t \end{cases}, 0 \leq t \leq \pi\,,\ \mathbf{r}_2(t)\colon \begin{cases} x = R\cos \Omega t \\ y = R\sin \Omega t \end{cases}, 0 \leq t \leq \frac{\pi}{\Omega}\,, (\Omega > 0)$$

and

$$\mathbf{r}_3(t)\colon \begin{cases} x = t \\ y = \sqrt{R^2-t^2} \end{cases}, 0 \leq t \leq R,, \tag{2.37}$$

All these curves are the same image. If $\Omega = 1$ the second curve coincides with the first one. First and second curve have the same orientation (reparameterisation $t \mapsto \Omega t$) The third curve have orientation opposite to first and second (reparameterisation $t \mapsto \cos t$, the derivative $\frac{d\cos t}{dt} < 0$).

Calculate integrals $\int_{C_1}\omega$, $\int_{C_2}\omega$, $\int_{C_3}\omega$

$$\int_{C_1}\omega = \int_0^\pi (xy_t - yx_t)dt = \int_0^\pi (R^2\cos^2 t + R^2\sin^2 t)dt = \pi R^2$$

$$\int_{C_2}\omega = \int_0^{\frac{\pi}{\Omega}} (xy_t - yx_t)dt = \int_0^\pi (R^2\Omega\cos^2 \Omega t + R^2\Omega\sin^2 \Omega t)dt = \pi R^2\,.$$

These answers coincide: both parameterisation have the same orientation. Note that these integrals is much nicer to calculate in polar coordinates: Recall that in polar coordinates

$$\omega = xdy - ydx = r\cos\varphi d(r\cos\varphi) - r\sin\varphi d(r\cos\varphi) = r^2 d\varphi$$

Hence

$$\int_{C_1}\omega = \int_0^\pi (r^2\varphi_t)dt = \pi R^2,\ \int_{C_2}\omega = \int_0^{\pi/\Omega} (r^2\varphi_t)dt = \pi R^2\,.$$

For the third parameterisation:

$$\int_{C_3}\omega = \int_0^R (xy_t - yx_t)dt = \int_0^1 \left(t\left(\frac{-t}{\sqrt{R^2-t^2}}\right) - \sqrt{R^2-t^2}\right)dt =$$

$$-R^2 \int_0^R \frac{dt}{\sqrt{R^2 - t^2}} = -R^2 \int_0^1 \frac{du}{\sqrt{1-u^2}} = -\pi R^2$$

We see that the sign is changed.

For other examples see Homeworks.

2.6 Integral over curve of exact form

1-form ω is called exact if there exists a function f such that $\omega = df$.

Theorem
Let ω be an exact 1-form in $\mathbf{E}^n$, $\omega = df$.
Then the integral of this form over an arbitrary curve C: $\mathbf{r} = \mathbf{r}(t)$ $t_1 \leq t \leq t_2$ is defined by the ending and starting point of the curve:

$$\int_C \omega = f\big|_{\partial C} = f(\mathbf{r}_2) - f(\mathbf{r}_1)\,, \quad \mathbf{r}_1 = \mathbf{r}(t_1), \mathbf{r}_2 = \mathbf{r}(t_2)\,. \tag{2.38}$$

Proof: $\int_C df = \int_{t_1}^{t_2} df(\mathbf{v}(t)) = \int_{t_1}^{t_2} \frac{d}{dt} f(\mathbf{r}(t))dt = f(\mathbf{r}(t))|_{t_1}^{t_2}$.

Example Calculate an integral of the form $\omega = 3x^2(1+y)dx + x^3dy$ over the arc of the semicircle $x^2 + y^2 = 1, y \geq 0$.

One can calculate the integral naively using just the formula (2.36): Choose a parameterisation of C,e.g., $x = \cos t, y = \sin t$, then $\mathbf{v}(t) = -\sin t\partial_x + \cos t\partial_x$ and $\omega(\mathbf{v}(t)) = (3x^2(1+y)dx + x^3dy)(-\sin t\partial_x + \cos t\partial_y) = -3\cos^2 t(1+\sin t)\sin t + \cos^3 t \cdot \cos t$ and

$$\int_C \omega = \int_0^\pi (-3\cos^2 t \sin t - 3\cos^2 t \sin^2 t + \cos^4 t)dt = ...$$

Calculations are little bit long.

But for the form $\omega = 3x^2(1+y)dx + x^3dy$ one can calculate the integral in much more efficient way noting that it is an exact form:

$$\omega = 3x^2(1+y)dx + x^3dy = d\left(x^3(1+y)\right) \tag{2.39}$$

Hence it follows from the Theorem that

$$\int_C \omega = f(\mathbf{r}(\pi)) - f(\mathbf{r}(0)) = x^3(1+y)\big|_{x=1,y=0}^{x=-1,y=0} = -2 \tag{2.40}$$

Remark If we change the orientation of curve then the starting point becomes the ending point and the ending point becomes the starting point.— The integral changes the sign in accordance with general statement, that integral of 1-form over parameterised curve is defined up to reparameterisation.

Corollary *The integral of an exact form over an arbitrary closed curve is equal to zero.*

Proof. According to the Theorem $\int_C \omega = \int_C df = f\big|_{\partial C} = 0$, because the starting and ending points of closed curve coincide.

Example. Show that the integral of 1-form $\omega = x^5 dy + 5x^4 y dx$ over the ellipse $x^2 + \frac{y^2}{9} = 1$.

The form $\omega = x^5 dy + 5x^4 y dx$ is exact form because $\omega = x^5 dy + 5x^4 y dx = d(x^5 y)$. Hence the integral over ellipse is equal to zero, because it is a closed curve.

2.7 Differential 2-forms in $\mathbf{E}^2$

We considered detailed definition of 1-forms. Now we give some formal approach to describe 2-forms.

Differential forms on $\mathbf{E}^2$ is an expression obtained by adding and multiplying functions and differentials dx, dy. These operations obey usual associativity and distributivity laws but multiplications is not moreover of one-forms on each other is *anticommutative*:

$$\omega \wedge \omega' = -\omega' \wedge \omega \quad \text{if } \omega, \omega' \text{ are 1-forms} \tag{2.41}$$

In particular

$$dx \wedge dy = -dy \wedge dx, dx \wedge dx = 0, dy \wedge dy = 0 \tag{2.42}$$

Example If $\omega = xdy + zdx$ and $\rho = dz + ydx$ then

$$\omega \wedge \rho = (xdy + zdx) \wedge (dz + ydx) = xdy \wedge dz + zdx \wedge dz + xydy \wedge dx$$

and

$$\rho \wedge \omega = (dz + ydx) \wedge (xdy + zdx) = xdz \wedge dy + zdz \wedge dx + xydx \wedge dy = -\omega \wedge \rho$$

Changing of coordinates. If $\omega = a(x,y)dx \wedge dy$ be two form and $x = x(u,v), y = y(u,v)$ new coordinates then $dx = x_u du + x_v dv$, $dy = y_u du + y_v dv$ ($x_u = \frac{\partial x(u,v)}{\partial u}$, $x_v = \frac{\partial x(u,v)}{\partial v}$, $y_u = \frac{\partial y(u,v)}{\partial u}$, $y_v = \frac{\partial y(u,v)}{\partial v}$). and

$$a(x,y)dx \wedge dy = a\left(x(u,v), y(u,v)\right)(x_u du + x_v dv) \wedge (y_u du + y_v dv) = \tag{2.43}$$

$$a\left(x(u,v), y(u,v)\right)(x_u du + x_v dv)(x_u y_v du \wedge dv + x_v y_u dv \wedge du) =$$

$$a\left(x(u,v), y(u,v)\right)(x_u y_v - x_v y_u) du \wedge dv$$

Example Let $\omega = dx \wedge dy$ then in polar coordinates $x = r\cos\varphi, y = r\sin\varphi$

$$dx \wedge dy = (\cos\varphi dr - r\sin\varphi d\varphi) \wedge (\sin\varphi dr + r\cos\varphi d\varphi) = r dr \wedge d\varphi \tag{2.44}$$

2.8 0-forms (functions) $\xrightarrow{d}$ 1-forms $\xrightarrow{d}$ 2-forms

We introduced differential d of functions (0-forms) which transform them to 1-form. It obeys the following condition:

- d: is linear operator: $d(\lambda f + \mu g) = \lambda df + \mu dg$
- $d(fg) = df \cdot g + f \cdot dg$

Now we introduce differential on 1-forms such that

- d: is linear operator on 1-forms also
- $d(fw) = df \wedge w + f dw$
- $ddf = 0$

Remark Sometimes differential d is called *exterior differential.*

Perform calculations using this definition and (2.41):

$$d\omega = d(\omega_1 dx + \omega_2 dy) = dw_1 \wedge dx + dw_2 \wedge dy = \left(\frac{\partial\omega_1(x,y)}{\partial x}dx + \frac{\partial\omega_1(x,y)}{\partial y}dy\right) \wedge dx +$$

$$\left(\frac{\partial\omega_2(x,y)}{\partial x}dx + \frac{\partial\omega_2(x,y)}{\partial y}dy\right) \wedge dy = \left(\frac{\partial\omega_2(x,y)}{\partial x} - \frac{\partial\omega_1(x,y)}{\partial y}\right) dx \wedge dy$$

Example Consider 1-form $\omega = xdy$. Then $d\omega = d(xdy) = dx \wedge dy$.

2.9 †Exact and closed forms

We know that it is very easy to integrate exact 1-forms over curves (see the subsection "Integral over curve of exact form")

How to know is the 1-form exact or no?

Definition We say that one form ω is *closed* if two form $d\omega$ is equal to zero.

Example One-form $xdy + ydx$ is closed because $d(xdy + ydx) = 0$.

(See other examples in the Homeworks.)

It is evident that exact 1-form is closed:

$$\omega = d\rho \Rightarrow d\omega = d(d\rho) = d \circ d\rho = 0 \tag{2.45}$$

We see that the condition that form is closed is necessary condition that form is exact.

So if $d\omega \neq 0$, i.e. the form is not closed, then it is not exact.

Is this condition sufficient? Is it true that a closed form is exact?

In general the answer is: *No.*

E.g. we considered differential 2-form

$$\omega = \frac{xdy - ydx}{x^2 + y^2} \tag{2.46}$$

defined in $\mathbf{E}^2 \backslash 0$. It is closed, but it is not exact (See non-compulsory exercises 11,12,13 in the Homework 6).

How to recognize for 1-form ω is it exact or no?

Inverse statement (Poincaré lemma) is true if 1-form is well-defined in $\mathbf{E}^2$:

A closed 1-form ω in $\mathbf{E}^n$ is exact if it is well-defined at all points of $\mathbf{E}^n$, i.e. if it is differentiable function at all points of $\mathbf{E}^n$.

Sketch a proof for 1-form in $\mathbf{E}^2$: if ω is defined in whole $\mathbf{E}^2$ then consider the function

$$F(\mathbf{r}) = \int_{C_{\mathbf{r}}} \omega \tag{2.47}$$

where we denote by $C_{\mathbf{r}}$ an arbitrary curve which starts at origin and ends at the point $\mathbf{r}$. It is easy to see that the integral is well-defined and one can prove that $\omega = df$.

The explicit formula for the function (2.47) is the following: If $\omega = a(x, y)dx + b(x, y)dy$ then $F(x, y) = \int_0^1 (a(tx, ty)x + b(tx, ty)y)\, dt$.

Exercise Check by straightforward calculation that $\omega = dF$ (See exercise 14 in Homework 6).

2.10 †Integration of two-forms. Area of the domain

We know that 1-form is a linear function on tangent vectors. If $\mathbf{A}, \mathbf{B}$ are two vectors attached at the point $\mathbf{r}_0$, i.e. tangent to this point and ω, ρ are two 1-forms then one defines the value of $\omega \wedge \rho$ on $\mathbf{A}, B$ by the formula

$$\omega \wedge \rho(\mathbf{A}, \mathbf{B}) = \omega(\mathbf{A})\rho(B) - \omega(B)\rho(A) \tag{2.48}$$

We come to bilinear anisymmetric function on tangent vectors. If $\sigma = a(x, y)dx \wedge dy$ is an arbitrary two form then this form defines bilinear form on pair of tangent vectors: $\sigma(\mathbf{A}, \mathbf{B}) =$

$$a(x, y)dx \wedge dy(A, B) = a(x, y)\left(dx(\mathbf{A})dy(\mathbf{B}) - dx(\mathbf{B})dy(\mathbf{A})\right) = a(x, y)(A_x B_y - A_y B_y) \tag{2.49}$$

One can see that in the case if $a = 1$ then right hand side of this formula is nothing but the area of parallelogram spanned by the vectors $\mathbf{A}, \mathbf{B}$.

This leads to the conception of integral of form over domain.

Let $\omega = a(x)dx \wedge dy$ be a two form and D be a domain in $\mathbf{E}^2$. Then by definition

$$\int_D \omega = \int_D a(x, y)dxdy \tag{2.50}$$

If $\omega = dx \wedge dy$ then

$$\int_D w = \int_D (x, y)dxdy = \text{Area of the domain } D \tag{2.51}$$

The advantage of these formulae is that we do not care about coordinates[5]

Example Let D be a domain defined by the conditions

$$\begin{cases} x^2 + y^2 \leq 1 \\ y \geq 0 \end{cases} \tag{2.53}$$

Calculate $\int_D dx \wedge dy$.

$\int_D dx \wedge dy = \int_D dxdy =$ area of the $D = \frac{\pi}{2}$.

If we consider polar coordinates then according (2.44)

$$dx \wedge dy = rdr \wedge d\varphi$$

Hence $\int_D dx \wedge dy = \int_D rdr \wedge d\varphi = \int_D rdrd\varphi = \int_0^1 \left(\int_0^\pi d\varphi\right) rdr = \pi \int_0^1 rdr = \pi/2$.

---

[5]If we consider changing of coordinates then jacobian appears: If u, v are new coordinates, $x = x(u, v)$, $y = y(u, v)$ are new coordinates then

$$\int a(x, y)dxdy = \int a(x(u, v), y(u, v)) \det \begin{pmatrix} x_u & x_v \\ x_u & x_v \end{pmatrix} dudv \tag{2.52}$$

In formula(5.9) it appears under as a part of coefficient of differential form.

Another example
Example Let D be a domain in $\mathbf{E}^2$ defined by the conditions

$$\begin{cases} \frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} \leq 1 \\ y \geq 0 \end{cases} \tag{2.54}$$

D is domain restricted by upper half of the ellipse and x-axis. Ellipse has the centre at the point $(c, 0)$. Its area is equal to $S = \int_D dx \wedge dy$. Consider new variables x', y': $x = c + ax', y = by'$. In new variables domain D becomes the domain from the previous example:

$$\frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} = {x'}^2 + {y'}^2$$

and $dx \wedge dy = abdx' \wedge dy'$. Hence

$$S = \int_{\frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} \leq 1, y \geq 0} dx \wedge dy = ab \int_{{x'}^2 + {y'}^2 \leq 1, y' \geq 0} dx' \wedge dy' = \frac{\pi ab}{2} \tag{2.55}$$

Theorem 2 (Green formula) Let ω be 2-form such that $\omega = d\omega'$ and D be a domain– interior of the closed curve C. Then

$$\int_D w = \int_C w' \tag{2.56}$$

3 Curves in $\mathbf{E}^3$. Curvature

3.1 Curves. Velocity and acceleration vectors

We already study velocity vector of curves. Consider now acceleration vector $\mathbf{a} = \frac{d^2\mathbf{r}(t)}{dt^2}$. For curve in $\mathbf{r} = \mathbf{r}(t)$ $\mathbf{E}^n$ we have

$$\mathbf{v} = \frac{d\mathbf{r}(t)}{dt}, \; v^i = \frac{dx^i(t)}{dt}, \; (i = 1, 2, \ldots, n) \; ,$$

and

$$\mathbf{a} = \frac{d\mathbf{v}(t)}{dt} = \frac{d^2\mathbf{r}(t)}{dt^2}, a^i = \frac{d^2x^i(t)}{dt^2}, (i = 1, 2, \ldots, n) \; . \tag{3.1}$$

Velocity vector $\mathbf{v}(t)$ is tangent to the curve. In general acceleration vector is not tangent to the curve. One can consider decomposition of acceleration vector $\mathbf{a}$ on tangential and normal component:

$$\mathbf{a} = \mathbf{a}_{tangent} + \mathbf{a}_{\perp}, \tag{3.2}$$

where $\mathbf{a}_{tangent}$ is the vector tangent to the curve (collinear to velocity vector) and $\mathbf{a}_\perp$ is orthogonal to the tangent vector (orthogonal to the velocity vector). The vector $\mathbf{a}_\perp$ is called normal acceleration vector of the curve [6].

Example Consider a curve

$$C\colon \quad \begin{cases} x = R\cos\Omega t \\ y = R\sin\Omega t \end{cases}, \tag{3.3}$$

If we consider parameter t as a time then we have the point which moves over circle of the radius R with angular velocity Ω. We see that

$$\mathbf{v} = \begin{pmatrix} -R\Omega\sin\Omega t \\ R\Omega\cos\Omega t \end{pmatrix}, \mathbf{a} = -\begin{pmatrix} R\Omega^2\cos\Omega t \\ R\Omega^2\sin\Omega t \end{pmatrix} = -\Omega^2\mathbf{r}(t)$$

Speed is constant: $|\mathbf{v}| = R\Omega$. Acceleration is perpendicular to the velocity. (It is just *centripetal acceleration*.)

What happens if speed is increasing, or decreasing, i.e. if angular velocity is not constant? One can see that in this case tangential acceleration is not equal to zero, i.e. the velocity and acceleration are not orthogonal to each other.

Analyze the meaning of an angle between velocity and acceleration vectors for an arbitrary parameterised curve $\mathbf{r} = \mathbf{r}(t)$. For this purpose consider the equation for speed: $|\mathbf{v}|^2 = (\mathbf{v}, \mathbf{v})$ and differentiate it:

$$\frac{d|\mathbf{v}|^2}{dt} = \frac{d}{dt}(\mathbf{v}(t), \mathbf{v}(t)) = 2(\mathbf{v}(t), \mathbf{a}(t)) = |\mathbf{v}(t)||\mathbf{a}(t)|\cos\theta(t) = (\mathbf{v}(t), \mathbf{a}_{tangent}(t)) \tag{3.4}$$

where θ is an angle between velocity vector and acceleration vector.

We formulate the following

Proposition

Suppose that parameter t is just time. We see from this formula that if point moves along the curve $\mathbf{r}(t)$ then

- speed is increasing in time if and only if the angle between velocity and acceleration vector is acute, i.e. tangential acceleration has the same direction as a velocity vector:

$$\frac{d|\mathbf{v}|^2}{dt} > 0 \Leftrightarrow (\mathbf{v}, \mathbf{a}) > 0 \Leftrightarrow \cos\theta > 0 \Leftrightarrow \mathbf{a}_{tang} = \lambda\mathbf{v} \text{ with } \lambda > 0\,. \tag{3.5}$$

[6] Component of acceleration orthogonal to the velocity vector sometimes is called also *centripetal acceleration*

- speed is decreasing in time if and only if the angle between velocity and acceleration vector is obtuse, i.e. tangential acceleration has the direction opposite to the direction of a velocity vector.

$$\frac{d|\mathbf{v}|^2}{dt} < 0 \Leftrightarrow (\mathbf{v}, \mathbf{a}) < 0 \Leftrightarrow \cos\theta < 0 \Leftrightarrow \mathbf{a}_{tang} = \lambda \mathbf{v} \text{ with } \lambda < 0\,. \tag{3.6}$$

- speed is constant in time if and only if the velocity and acceleration vectors are orthogonal to each other, i..e. tangential acceleration is equal to zero.

$$\frac{d|\mathbf{v}|^2}{dt} = 0 \Leftrightarrow (\mathbf{v}, \mathbf{a}) = 0 \Leftrightarrow \cos\theta = 0 \Leftrightarrow \mathbf{a}_{tang} = 0\,. \tag{3.7}$$

Example Consider the curve $\mathbf{r}(t)\colon \begin{cases} x(t) = v_x t \\ y(t) = v_y t - \frac{gt^2}{2} \end{cases}$ It is path of the point moving under the gravity force with initial velocity $\mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$. One can see that the curve is parabola: $y = \left(\frac{v_y}{v_x}\right) x - \left(\frac{gv_y^2}{v_x^2}\right) x^2$. We have that $\mathbf{v}(t) = \begin{pmatrix} v_x \\ v_y - gt \end{pmatrix}$ and acceleration vector $\mathbf{a} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$. Suppose that $v_y > 0$. $(\mathbf{v}, \mathbf{a}) = -g(v_y - gt)$. Then at the highest point (vertex of the parabola) $(t = v_y/g)$ acceleration is orthogonal to the velocity. For $t < v_y/g$ angle between acceleration and velocity vectors is obtuse. Speed is decreasing. For $t > v_y/g$ angle between acceleration and velocity vectors is acute. Speed is increasing.

3.2 Behaviour of acceleration vector under reparameterisation

How acceleration vector changes under changing of parameterisation of the curve?

Let $C\colon \quad \mathbf{r} = \mathbf{r}(t), t_1 \le t \le t_2$ be a curve and $t = t(\tau)$ reparametrisation of this curve. We know that for new parameterised curve $C'\colon \quad \mathbf{r}'(\tau) = \mathbf{r}(t(\tau)), \tau_1 \le \tau \le \tau_2$ velocity vector $\mathbf{v}'(\tau)$ is collinear to the velocity vector $\mathbf{v}(t)$ (see (2.7)):

$$\mathbf{v}'(\tau) = \frac{d\mathbf{r}'(\tau)}{d\tau} = \frac{d\mathbf{r}(t(\tau))}{d\tau} = \frac{dt(\tau)}{d\tau}\frac{d\mathbf{r}(t(\tau))}{dt} = t_\tau \mathbf{v}(t(\tau))$$

Taking second derivative we see that for acceleration vector:

$$\mathbf{a}'(\tau) = \frac{d^2\mathbf{r}'(\tau)}{d\tau^2} = \frac{d\mathbf{v}'(\tau)}{d\tau} = \frac{d}{d\tau}\left(t_\tau \mathbf{v}(t(\tau))\right) = t_{\tau\tau}\mathbf{v}(t(\tau)) + t_\tau^2 \mathbf{a}(t(\tau)) \quad (3.8)$$

Under reparameterisation acceleration vector in general changes its direction: new acceleration vector becomes linear combination of old velocity and acceleration vectors: direction of acceleration vector does not remain unchanged [7].

We observed this phenomenon already when we considered the moving along the curve with different velocities (see (3.5), (3.6) and (3.7)).

We know that acceleration vector can be decomposed on tangential and normal components (see (3.2)). Study how tangential and normal components change under reparameterisation.

Decompose left and right hand sides of the equation (3.8) on tangential and orthogonal components:

$$\mathbf{a}'(\tau)_{tangent} + \mathbf{a}'(\tau)_\perp = t_{\tau\tau}\mathbf{v}(t) + t_\tau^2\left(\mathbf{a}(t)_{tangent} + \mathbf{a}(t)_\perp\right)$$

Then comparing tangential and orthogonal components we see that new tangential acceleration is equal to

$$\mathbf{a}'(\tau)_{tangent} = t_{\tau\tau}\mathbf{v}(t) + t_\tau^2\mathbf{a}(t)_{tangent} \quad (3.9)$$

and normal acceleration is equal to

$$\mathbf{a}'(\tau)_\perp = t_\tau^2\mathbf{a}(t)_\perp \quad (3.10)$$

The magnitude of normal (centripetal) acceleration under changing of parameterisation is multiplied on the t_τ^2. Now recall that magnitude of velocity vector under reparameterisation is multiplied on t_τ. We come to very interesting and important observation:

Observation

The magnitude $\frac{|\mathbf{a}_\perp|}{|\mathbf{v}^2|}$ remains unchanged under reparameterisation. (3.11)

We come to the expression which is independent of parameterisation: it must have deep mechanical and geometrical meaning. We see later that it is nothing but curvature.

[7] The plane spanned by velocity and acceleration vectors remains unchanged.(This plane is called osculating plane.)

3.3 Length of the curve

If $\mathbf{r}(t), a \leq t \leq b$ is a parameterisation of the curve L and $\mathbf{v}(t)$ velocity vector then length of the curve is equal to the integral of of $|\mathbf{v}(t)|$ over curve:

$$\text{Length of the curve } L = \int_a^b |\mathbf{v}(t)|dt = \tag{3.12}$$

$$\int_a^b \sqrt{\left(\frac{dx^1(t)}{dt}\right)^2 + \left(\frac{dx^2(t)}{dt}\right)^2 + \dots + \left(\frac{dx^n(t)}{dt}\right)^2} dt\,.$$

Note that formula above is *reparameterisation* invariant. The length of the image of the curve does not depend on parameterisation. This corresponds to our intuition.

Proof Consider curve $\mathbf{r}_1 = \mathbf{r}_1(t)$, $a_1 \leq t \leq b_1$. Let $t = t(\tau)$, $a_2 < \tau < b_2$ be another parameterisation of the curve $\mathbf{r} = \mathbf{r}(t)$, In other words we have two different parameterised curves $\mathbf{r}_1 = \mathbf{r}_1(t), a_1 \leq t \leq b_1$ and $\mathbf{r}_2 = \mathbf{r}_1(t(\tau))$, $a_2 \leq \tau \leq \mathbf{b}_2$ such that their images coincide (See (2.6)). Then under reparameterisation velocity vector is multiplied on t_τ

$$\mathbf{v}_2(\tau) = \frac{d\mathbf{r}_2}{d\tau} = \frac{dt}{d\tau}\frac{d\mathbf{r}_1}{dt} = t_\tau(\tau)\mathbf{v}_1(t(\tau))$$

Hence

$$L_1 = \int_{a_1}^{b_1} |\mathbf{v}_1(t)|dt = \int_{a_2}^{b_2} |\mathbf{v}_1(t)|\frac{dt(\tau)}{d\tau}d\tau = \int_{a_2}^{b_2} |t_\tau \mathbf{v}_1(t)|d\tau = \int_{a_2}^{b_2} |\mathbf{v}_2(\tau)|d\tau = L_2\,, \tag{3.13}$$

i.e. length of the curve does not change under reparameterisation.

If C: $\quad \mathbf{r} = \mathbf{r}(t)$ $t_1 \leq t \leq t_2$ is a curve in $\mathbf{E}^2$ then its length is equal to

$$L_C = \int_{t_1}^{t_2} |\mathbf{v}(t)|dt = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx(t)}{dt}\right)^2 + \left(\frac{dy(t)}{dt}\right)^2} dt \tag{3.14}$$

3.4 Natural parameterisation of the curves

Non-parameterised curve can be parameterised in many different ways.

Is there any distinguished parameterisation? Yes, it is.

Definition A natural parameter $s = s(t)$ on the curve $\mathbf{r} = r(t)$ is a parameter which defines the length of the arc of the curve between initial point $\mathbf{r}(t_1)$ and the point $\mathbf{r}(t)$.

If a natural parameter s is chosen we say that a curve $\mathbf{r} = \mathbf{r}(s)$ is given in natural parameterisation.

Write down explicit formulae for natural parameter.

Let $C : \mathbf{r}(t), a < t < b$ be a curve in $\mathbf{E}^n$. As always we suppose that it is smooth and regular curve: (i.e. $\mathbf{r}(t)$ has derivatives of arbitrary order, and velocity vector $\mathbf{v} \neq 0$.

Then it follows from (3.12) that

$$s(t) = \{\text{length of the arc of the curve between points } \mathbf{r}(a) \text{ and } \mathbf{r}(t)\} \tag{3.15}$$

$$= \int_a^t |\mathbf{v}(t')| dt' =$$

$$= \int_a^t \sqrt{\left(\frac{dx^1(t')}{dt'}\right)^2 + \left(\frac{dx^2(t')}{dt'}\right)^2 + \cdots + \left(\frac{dx^n(t')}{dt'}\right)^2} dt' \,. \tag{3.16}$$

(As always we suppose that it is smooth and regular curve: (i.e. $\mathbf{r}(t)$ has derivatives of arbitrary order, and velocity vector $\mathbf{v} \neq 0$.)

Example Consider circle: $x = R\cos t, y = R\sin t$ in $\mathbf{E}^2$. Then we come to the obvious answer

$$s(t) = \{\text{length of the arc of the circle between points } \mathbf{r}(0) \text{ and } \mathbf{r}(t)\} = Rt =$$

$$\int_0^t \sqrt{\left(\frac{dx(t')}{dt'}\right)^2 + \left(\frac{dy(t')}{dt'}\right)^2} dt' = \int_0^t \sqrt{R^2 \sin^2 t' + R^2 \cos^2 t'} dt' = \int_a^t R dt' = Rt$$

$s = Rt$. Hence in natural parameterisation $x = R\cos\frac{s}{R}$, $y = R\sin\frac{s}{R}$.

Remark If we change an initial point then a natural parameter changes on a constant.

For example if we choose as a initial point for the circle above a point $\mathbf{r}(t_1)$ for $t_1 = -\frac{\pi}{2}$, then the length of the arc between points $\mathbf{r}(-\frac{\pi}{2})$ and $\mathbf{r}(0)$ is equal to $R\frac{\pi}{2}$ and

$$s'(t) = s(t) + R\frac{\pi}{2} \,.$$

Another

Example Consider arc of the parabola $x = t, y = t^2, 0 < t < 1$:

$$s(t) = \{\text{length of the arc of the curve for parameter less or equal to } t\} = \tag{3.17}$$

$$\int_0^t \sqrt{\left(\frac{dx(\tau)}{d\tau}\right)^2 + \left(\frac{dy(\tau)}{d\tau}\right)^2} d\tau =$$

$$\int_0^t \sqrt{1+4\tau^2} d\tau = \frac{t\sqrt{1+4t^2}}{2} + \frac{1}{4}\log\left(2t + \sqrt{1+4t^2}\right)$$

The first example was very simple. The second is harder to calculate [8]. In general case natural parameter is not so easy to calculate. But its notion is very important for studying properties of curves.

Natural parameterisation is distinguished. Later we will often use the following very important property of natural parameterisation:

Proposition *If a curve is given in natural parameterisation then*

- *the speed is equal to* 1

$$(\mathbf{v}(s), \mathbf{v}(s)) \equiv 1, \quad \text{i.e. } |\mathbf{v}(s)| \equiv 1\,, \tag{3.18}$$

- *acceleration is orthogonal to velocity, i.e. tangential acceleration is equal to zero:*

$$(\mathbf{v}(s), \mathbf{a}(s)) = 0\,, \quad \text{i.e. } \mathbf{a}_{tangent} = 0\,. \tag{3.19}$$

Proof: For an arbitrary parameterisation $|\mathbf{v}(t)| = \frac{dL(t)}{dt}$, where $L(t)$ is a length of the curve. In the case of natural parameter $L(s) = s$, i.e. $|\mathbf{v}(t)| = \frac{dL(t)}{dt} = 1$. We come to the first relation.

The second relation means that value of the speed does not change (see (3.4) and (3.7)).

---

[8]Denote by $I = \int_0^t \sqrt{1+4\tau^2} d\tau$. Then integrating by parts we come to:

$$I = t\sqrt{1+4t^2} - \int \frac{4\tau^2}{\sqrt{1+4\tau^2}} d\tau = t\sqrt{1+4t^2} - I + \int \frac{1}{\sqrt{1+4\tau^2}} d\tau\,.$$

Hence

$$I = \frac{t\sqrt{1+4t^2}}{2} + \frac{1}{2}\int \frac{1}{\sqrt{1+4\tau^2}} d\tau\,.$$

and we come to the answer.

3.5 Curvature. Curvature of curves in $\mathbf{E}^2$

How to find invariants of non-parameterised curve, i.e. magnitudes which depend on the points of non-parameterised curve but which do not depend on parameterisation?

Answer at the first sight looks very simple: Consider the distinguished natural parameterisation $\mathbf{r} = \mathbf{r}(s)$ of the curve. Then arbitrary functions on $x^i(s)$ and its derivatives do not depend on parameterisation. But the problem is that it is not easy to calculate natural parameter explicitly (See e.g. calculations of natural parameter for parabola in the previous subsection). So it is preferable to know how to construct these magnitudes in arbitrary parameterisation, i.e. construct functions $f(\frac{dx^i}{dt}, \frac{d^2x^i}{dt^2}, \dots)$ such that they *do not depend on parameterisation.*

We define now curvature. First formulate reasonable conditions on curvature:

- it has to be a function of the points of the curve
- it does not depend on parameterisation
- curvature of the line must be equal to zero
- curvature of the circle with radius R must be equal to $1/R$

We first give definition of curvature in natural parameterisation. Then study how to calculate it for a curve in an arbitrary parameterisation.

For a given non-parameterised curve consider natural parameterisation $\mathbf{r} = \mathbf{r}(s)$. We know already that velocity vector has length 1 and acceleration vector is orthogonal to curve in natural parameterisation (see (3.18) and (3.19)). It is just normal (centripetal) acceleration.

Definition. The curvature of the curve in a given point is equal to the modulus (length) of acceleration vector (normal acceleration) in natural parameterisation. Namely, let $\mathbf{r}(s)$ be natural parameterisation of this curve. Then curvature at every point $\mathbf{r}(s)$ of the curve is equal to the length of acceleration vector:

$$k = |\mathbf{a}(s)|, \qquad \mathbf{a}(s) = \frac{d^2\mathbf{r}(s)}{ds^2} \tag{3.20}$$

First check that it corresponds to our intuition (see reasonable conditions above)

It does not depend on parameterisation by definition.

It is evident that for the line in normal parameterisation $x^i(s) = x_0^i + b^i s$ ($\sum b^i b^i = 1$) the acceleration is equal to zero.

Now check that the formula (3.20) gives a natural answer for circle. For circle of radius R in natural parameterisation

$$\mathbf{r} = \mathbf{r}(s) = (x(s), y(s)), \quad \text{where} \quad x(s) = R\cos\frac{s}{R}, \quad y(s) = R\sin\frac{s}{R}$$

(length of the arc of the angle θ of the circle is equal to $s = R\theta$.) Then

$$\mathbf{a}(s) = \frac{d\mathbf{r}^2(s)}{ds^2} = \left(-\frac{1}{R}\cos\frac{s}{R}, -\frac{1}{R}\sin\frac{s}{R}\right)$$

and for curvature

$$k = |\mathbf{a}(s)| = \frac{1}{R} \tag{3.21}$$

we come to the answer which agrees with our intuition.

Geometrical meaning of curvature: One can see from this example that $\frac{1}{k}$ is *just a radius of the circle which has second order touching to curve.*(See the subsection "Second order contact" (this is not compulsory))

3.6 Curvature of curve in an arbitrary parameterisation.

Let curve be given in an arbitrary parameterisation. How to calculate curvature. One way is to go to natural parameterisation. But in general it is very difficult (see the example of parabola in the subsection "Natural parameterisation").

We do it in another more elegant way.

Proposition *Curvature of the curve in terms of an arbitrary parameterisation* $\mathbf{r} = \mathbf{r}(t)$ *is given by the formula:*

$$k = \frac{|\mathbf{a}_\perp(t)|}{|\mathbf{v}(t)|^2} = \frac{\textit{Area of parallelogram spanned by the vectors } \mathbf{a}, \mathbf{v}}{|\mathbf{v}|^3}, \tag{3.22}$$

where $\mathbf{v}(t) = d\mathbf{r}(t)/dt$ *is velocity vector and* $\mathbf{a}_\perp(t)$ *is normal acceleration.*

Proof of the Proposition

Prove first that $k = \frac{|\mathbf{a}_\perp(t)|}{|\mathbf{v}(t)|^2}$. Note that in natural parameterisation speed is equal to 1 and acceleration is orthogonal to curve: $\mathbf{a} = \mathbf{a}_\perp$, $|\mathbf{v}| = 1$ (see

(3.18), (3.19)). Hence in natural parameterisation the ration $\frac{|\mathbf{a}_\perp|}{|\mathbf{v}|^2}$ is equal just to modulus of acceleration vector, i.e. to the curvature (3.20). On the other hand according to the observation (3.11) (see the end of the subsection "Velocity and acceleration vectors") the ratio $\frac{|\mathbf{a}_\perp|}{|\mathbf{v}|^2} = \frac{|\mathbf{a}_\perp|}{(\mathbf{v},\mathbf{v})}$ *does not* depend on parameterisation. Hence curvature is defined by the formula $k = \frac{|\mathbf{a}_\perp(t)|}{|\mathbf{v}(t)|^2}$ in an arbitrary parameterisation.

Advantage of the formula $k = \frac{|\mathbf{a}_\perp(t)|}{|\mathbf{v}(t)|^2}$ is that it is given in an arbitrary parameterisation. Disadvantage of this formula is that we still do not know how to calculate $\mathbf{a}_\perp(t)$. Do now the next step: Note that

$$\frac{|\mathbf{a}_\perp(t)|}{|\mathbf{v}|^2} = \frac{|\mathbf{a}_\perp(t)| \cdot |\mathbf{v}|}{|\mathbf{v}|^3} = \frac{\text{Area of parallelogram spanned by the vectors } \mathbf{a}, \mathbf{v}}{|\mathbf{v}|^3} \tag{3.23}$$

The last formula is practical. We already know how to calculate area of parallelogram spanned by the vectors $\mathbf{a}, \mathbf{v}$. In particularly it is easy to do for $\mathbf{E}^3$ and $\mathbf{E}^2$ (general case $n > 3$ we consider later.). In $\mathbf{E}^3$ it is just given by vector product: $S = |\mathbf{v} \times \mathbf{a}|$. In $\mathbf{E}^2$, $S = |v_x a_y - v_y a_x|$ because $\mathbf{v} \times \mathbf{a} = (v_x \mathbf{e}_x + v_y \mathbf{e}_y) \times (a_x \mathbf{e}_x + a_y \mathbf{e}_y) = (v_x a_y - v_y a_x)\mathbf{e}_z$ if curve is in $\mathbf{E}^2$.

Hence for $\mathbf{E}^3$ formula for curvature is:

$$k = \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{|\mathbf{v}(t)|^3}, \qquad \text{(if curve is in } \mathbf{E}^3) \tag{3.24}$$

In $\mathbf{E}^2$ formula for curvature is

$$k = \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{|\mathbf{v}(t)|^3} = \frac{|v_x a_y - v_y a_x|}{(v_x^2 + v_y^2)^{\frac{3}{2}}} \qquad \text{(if curve is in } \mathbf{E}^2) \tag{3.25}$$

or more explicit formula:

$$k = \frac{|x_t y_{tt} - y_t x_{tt}|}{(x_t^2 + y_t^2)^{\frac{3}{2}}} \qquad \text{(if curve is in } \mathbf{E}^2) \tag{3.26}$$

This is workable formula.

In general case if curve is in $\mathbf{E}^n$ then to calculate the area S of parallelogram note that $S = |\mathbf{v}||\mathbf{a}||\sin\theta|$ where $|\mathbf{v}||\mathbf{a}|\cos\theta = (\mathbf{v}, \mathbf{a})$. Hence $S = |\mathbf{v}||\mathbf{a}|\sqrt{1-\cos^2\theta} = \sqrt{\mathbf{v}^2\mathbf{a}^2 - (\mathbf{v}\cdot\mathbf{a})^2}$ and curvature is equal to

$$k = \frac{\text{Area of parallelogram formed by the vectors } \mathbf{v} \text{ and } \mathbf{a}}{\text{Cube of the speed}} = \frac{\sqrt{\mathbf{v}^2\mathbf{a}^2 - (\mathbf{v}\cdot\mathbf{a})^2}}{|\mathbf{v}|^3} \tag{3.27}$$

Remark 1. Of course one can come to formulae (3.27), (3.24) and (3.25) by "brute force" making straightforward attack. Instead considering explicitly natural parameterisation of the curve we just try to rewrite the formula in definition (3.20) in arbitrary parameterisation using chain rule. The calculations are not transparent. Try to do it.

Consider examples of calculating curvature for curves in $\mathbf{E}^2$.

Example. Consider circle of the radius R, $x^2 + y^2 = R^2$. Take any parameterisation, e.g. $x = R\cos t, y = R\sin t$. Then $\mathbf{v} = (-R\cos t, R\sin t)$, $\mathbf{a} = (-R\sin t, -R\cos t)$. Applying the formula (3.26) we come to

$$k = \frac{|x_t y_{tt} - y_t x_{tt}|}{(x_t^2 + y_t^2)^{\frac{3}{2}}} = \frac{|R^2\cos^2 t + R^2\sin^2 t|}{(R^2\cos^2 t + R^2\sin^2 t)^{\frac{3}{2}}} = \frac{R^2}{R^3} = \frac{1}{R}$$

Example Consider ellipse

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2}$$

. Choose a parameterisation, e.g. $\mathbf{r}(t)\colon \begin{cases} x = x_0 + a\cos t \\ y = y_0 + b\sin t \end{cases}$, $0 \le t < 2\pi$.

Then $\mathbf{v}(t) = \begin{pmatrix} -a\sin t \\ b\cos t \end{pmatrix}$, $\mathbf{a}(t) = \begin{pmatrix} -a\cos t \\ -b\sin t \end{pmatrix}$ and for curvature we have

$$k = \frac{|x_t y_{tt} - y_t x_{tt}|}{(x_t^2 + y_t^2)^{\frac{3}{2}}} = k = \frac{|ab\sin^2 t + ab\cos^2 t|}{(a^2\sin^2 t + b^2\cos^2 t)^{\frac{3}{2}}} = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{\frac{3}{2}}}\,. \tag{3.28}$$

Exercise Calculate curvatures at the points $(x_0 \pm a, y_0)$ and $(x_0, y_0 \pm b)$.

See other examples in Homework 7.

Example For any function $f = f(x)$ one can consider its graph as not-parameterised curve C_f. Calculate curvature of the curve C_f at any point $(x, f(x))$.

One can choose parameteristaion: $\mathbf{r}(t)\colon \begin{cases} x = t \\ y = f(t) \end{cases}$.

Then $\mathbf{v}(t) = \begin{pmatrix} 1 \\ f'(t) \end{pmatrix}$, $\mathbf{a}(t) = \begin{pmatrix} 0 \\ f''(t) \end{pmatrix}$ and we have for the curvature that

$$k = \frac{|x_t y_{tt} - y_t x_{tt}|}{(x_t^2 + y_t^2)^{\frac{3}{2}}} = k = \frac{|f''(t)|}{(1 + f'(t)^2)^{\frac{3}{2}}} \tag{3.29}$$

4 Surfaces in $\mathbf{E}^3$. Curvatures and Shape operator.

In this section we study surfaces in $\mathbf{E}^3$. One can define surfaces by equation $F(x, y, z) = 0$ or by parametric equation

$$\mathbf{r}(u, v)\colon \begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases}, \tag{4.1}$$

Example the equation $x^2 + y^2 = R^2$ defines cylinder (cylindrical surface). z-axis is the axis of this cylinder, R is radius of this cylinder. One can define this cylinder by the parametric equation

$$\mathbf{r}(\varphi, h)\colon \begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \\ z = h \end{cases}, \tag{4.2}$$

where φ is the angle $0 \leq \varphi < 2\pi$ and $-\infty < h < \infty$ takes arbitrary real values.

Example sphere $x^2 + y^2 + z^2 = R^2$:

$$\mathbf{r}(\theta, \varphi)\colon \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases}, 0 \leq \theta \leq \pi,\ 0 \leq \varphi \leq 2\pi \tag{4.3}$$

Example cone $k^2x^2 + k^2y^2 - z^2 = 0$:

$$\mathbf{r}(h, \varphi)\colon \begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases}, -infty < h < \infty\ \ 0 \leq \varphi \leq 2\pi \tag{4.4}$$

Example graph of the surface $z = F(x, y)$:

$$\mathbf{r}(u, v)\colon \begin{cases} x = u \\ y = v \\ z = F(u, v) \end{cases}, -\infty < u < \infty,\ -\infty < v < \infty \tag{4.5}$$

It is interesting to consider this example when $F = uv$ we come to the surface, *saddle*:

$$\mathbf{r}(u,v)\colon \begin{cases} x = u \\ y = v \\ z = uv \end{cases}, -\infty < u < \infty,\ -\infty < v < \infty \qquad (4.6)$$

4.1 Coordinate basis, tangent plane to the surface.

Coordinate basis vectors are $\mathbf{r}_u = \partial_u, \mathbf{r}_v = \partial_v$. At the any point $\mathbf{p}$, $\mathbf{p} = \mathbf{r}(u,v)$ these vectors span the plane, (two-dimensional linear space) $T_\mathbf{p}M$ in three dimensional vector space $T_\mathbf{p}E^3$.

$$T_\mathbf{p}M = \{\lambda\mathbf{r}_u + \mu\mathbf{r}_v, \lambda, \mu \in \mathbf{R}\}, \qquad T_\mathbf{p}\,\text{subspace in}\,T_\mathbf{p}E^3 \qquad (4.7)$$

E.g. consider the point $\mathbf{p} = (R, 0, 0)$ on the cylinder (4.2). Then $\mathbf{p} = \mathbf{r}(\varphi, h)$ for $\varphi = 0, h = 0$. Coordinate basis vectors are

$$\mathbf{r}_\varphi = \begin{pmatrix} -R\sin\varphi \\ R\cos\varphi \\ 0 \end{pmatrix}, \qquad \mathbf{r}_h = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad (4.8)$$

or in other notations

$$\mathbf{r}_\varphi = -R\sin\varphi\partial_x + R\cos\varphi\partial_y, \qquad \mathbf{r}_h = \partial_z \qquad (4.9)$$

At the point $\mathbf{p} = (R, 0, 0)$ they are are equal to the vectors ∂_y and ∂_z respectively attached at this point. Tangent plane at the point $\mathbf{p}$ is the plane passing through the point $\mathbf{p}$ spanned by the vectors ∂_y and ∂_z.

4.2 Curves on surfaces. Length of the curve. Internal and external point of the view. First Quadratic Form

Let $M\colon \mathbf{r} = \mathbf{r}(u,v)$ be a surface and C curve on this surface, i.e. $C\colon \mathbf{r}(t) = \mathbf{r}(u(t, v(t)))$.

Consider an arbitrary point $\mathbf{p} = \mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ at this curve.

- $T_\mathbf{p}E^3$—three-dimensional tangent space to the point $\mathbf{p}$,

- $T_{\mathbf{p}}M$—two dimensional linear space tangent to the surface at the point $\mathbf{p}$, spanned by the tangent vectors ∂_u, ∂_v
- $T_{\mathbf{p}}M$—one dimensional linear space tangent to the curve at the point $\mathbf{p}$ spanned by the velocity vector $\mathbf{v}(t)$.

$$\mathbf{v}(t) = \frac{d\mathbf{r}(u(t), v(t))}{dt} = u_t \frac{\partial \mathbf{r}}{\partial u} + v_t \frac{\partial \mathbf{r}}{\partial v} = u_t \mathbf{r}_u + v_t \mathbf{r}_v \tag{4.10}$$

These tangent spaces form flag of subspaces $T_{\mathbf{p}}C < T_{\mathbf{p}}M < T_{\mathbf{p}}E^3$.

How to calculate the length of the arc of the curve:

$$C\colon \mathbf{r}(t) = \mathbf{r}(u(t, v(t))) = \begin{cases} x = x(u(t), v(t)) \\ y = y(u(t), v(t)) \\ z = z(u(t), v(t)) \end{cases} \qquad t_1 \leq t_2.$$

External and internal observer do it in different ways. External observer just looks at the curve as the curve in ambient space. He uses the formula (3.12):

$$L = \text{Length of the curve } L = \int_a^b |\mathbf{v}(t)| dt = \int_a^b \sqrt{\left(\frac{dx(t)}{dt}\right)^2 + \left(\frac{dy(t)}{dt}\right)^2 + \left(\frac{dz(t)}{dt}\right)^2} dt\,. \tag{4.11}$$

What about internal observer?

Internal observer will perform calculations in coordinates u, v. We have $|\mathbf{v}(t)| = \sqrt{(\mathbf{v}, \mathbf{v})}$. We have

$$\mathbf{v} = \frac{d\mathbf{r}(t)}{dt} = \frac{d\mathbf{r}(u(t), v(t))}{dt} = \dot{u}\frac{\partial r(u, v)}{\partial u} + \dot{v}\frac{\partial r(u, v)}{\partial v} = \dot{u}\mathbf{r}_u\,.$$

Hence the scalar product

$$(\mathbf{v}, \mathbf{v}) = (u_t\mathbf{r}_u + v_t\mathbf{r}_v, u_t\mathbf{r}_u + v_t\mathbf{r}_v) = u_t^2(\mathbf{r}_u, \mathbf{r}_u) + 2u_tv_t(\mathbf{r}_u, \mathbf{r}_v) + v_t^2(v_t, v_t)\,.$$

To understand how internal observer can calculate the length of the curve we have to introduce

$$G_{uu} = (\mathbf{r}_u, \mathbf{r}_u),\ \ G_{uv} = (\mathbf{r}_u, \mathbf{r}_v)\ \ G_{vu} = (\mathbf{r}_v, \mathbf{r}_u),\ \ \Gamma_{vv} = (\mathbf{r}_v, \mathbf{r}_v) \tag{4.12}$$

Of course $G_{uv} = G_{vu}$. We see that internal observer calculates the length of the curve using time derivatives u_t, v_t of internal coordinates u, v and coefficients (4.12):

$$(\mathbf{v}, \mathbf{v}) = u_t^2(\mathbf{r}_u, \mathbf{r}_u) + 2u_tv_t(\mathbf{r}_u, \mathbf{r}_v) + v_t^2(v_t, v_t) = G_{11}u_t^2 + 2G_{12}u_tv_t + G_{22}v_t^2. \tag{4.13}$$

We come to conception of *first quadratic form.*

Definition First quadratic form defines length of the tangent vector to the surface in internal coordinates and length of the curves on the surface.

The first quadratic form at the point $\mathbf{r} = \mathbf{r}(u, v)$ is defined by symmetric matrix:

$$\begin{pmatrix} G_{uu} & G_{uv} \\ G_{vu} & G_{vv} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_u, \mathbf{r}_u) & (\mathbf{r}_u, \mathbf{r}_v) \\ (\mathbf{r}_u, \mathbf{r}_v) & (\mathbf{r}_v, \mathbf{r}_v) \end{pmatrix}, \tag{4.14}$$

where (,) is a scalar product.

E.g. calculate the first quadratic form for the cylinder (4.2). Using (4.8), (4.9) we come to

$$\begin{pmatrix} G_{hh} & G_{h\varphi} \\ G_{\varphi h} & G_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_h, \mathbf{r}_h) & (\mathbf{r}_h, \mathbf{r}_\varphi) \\ (\mathbf{r}_\varphi, \mathbf{r}_h) & (\mathbf{r}_\varphi, \mathbf{r}_\varphi) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & R^2 \end{pmatrix} \tag{4.15}$$

(See this example and other examples in Homework 5)

Let $\mathbf{X} = a\mathbf{r}_u + b\mathbf{r}_v$ be a vector tangent to the surface M at the point $\mathbf{r}(u, v)$. Then the length of this vector is defined by the scalar product $(\mathbf{X}, \mathbf{X})$:

$$|\mathbf{X}|^2 = (\mathbf{X}, \mathbf{X}) = (a\mathbf{r}_u + b\mathbf{r}_v, a\mathbf{r}_u + b\mathbf{r}_v) = a^2(\mathbf{r}_u, \mathbf{r}_u) + 2ab(\mathbf{r}_u, \mathbf{r}_v) + b^2(\mathbf{r}_v, \mathbf{r}_v) \tag{4.16}$$

It is just equal to the value of the first quadratic form on this tangent vector:

$$(\mathbf{X}, \mathbf{X}) = G(\mathbf{X}, \mathbf{X}) = \begin{pmatrix} a, & b \end{pmatrix} \cdot \begin{pmatrix} G_{uu} & G_{uv} \\ G_{vu} & G_{vv} \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = G_{uu}a^2 + 2G_{uv}ab + G_{vv}b^2 \tag{4.17}$$

External observer (person living in ambient space $\mathbf{E}^3$) calculate the length of the tangent vector using formula (4.16). An ant living on the surface (internal observer) calculate length of this vector in internal coordinates using formula (4.17). External observer deals with external coordinates of the vector, ant on the surface with internal coordinates.

If $\mathbf{X}, \mathbf{Y}$ are two tangent vectors in the tangent plane T_pC then $G(\mathbf{X}, \mathbf{Y})$ at the point p is equal to scalar product of vectors $\mathbf{X}, \mathbf{Y}$: $(\mathbf{X}, \mathbf{Y}) = (X^1\mathbf{r}_1 + X^2\mathbf{r}_2, Y^1\mathbf{r}_1 + Y^2\mathbf{r}_2) = X^1(\mathbf{r}_1, \mathbf{r}_1)Y^1 + X^1(\mathbf{r}_1, \mathbf{r}_2)Y^2 + X^2(\mathbf{r}_2, \mathbf{r}_1)Y^1 + X^2(\mathbf{r}_2, \mathbf{r}_2)Y^2 = X^\alpha(\mathbf{r}_\alpha, \mathbf{r}_\beta)Y^\beta = X^\alpha G_{\alpha\beta} Y^\beta = G(\mathbf{X}, \mathbf{Y})$. We identify quadratic forms and corresponding symmetric bilinear forms. Bilinear symmetric form $B(\mathbf{X}, \mathbf{Y}) = B(\mathbf{Y}, \mathbf{X})$ defines quadratic form $Q(\mathbf{X}) = B(\mathbf{X}, \mathbf{X})$. Quadratic form satisfies the condition $Q(\lambda\mathbf{X}) = \lambda^2 Q(\mathbf{X})$ and so called parallelogram condition

$$Q(\mathbf{X} + \mathbf{Y}) + Q(\mathbf{X} - \mathbf{Y}) = 2Q(\mathbf{X}) + 2Q(\mathbf{Y}) \tag{4.18}$$

First quadratic form and length of the curve

Let $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ $a \leq t \leq b$ be a curve on the surface.

The first quadratic form measures the length of velocity vector at every point of this curve. Write down again the formula for length of the curve in internal coordinates using First Quadratic form (compare with (4.13)).

Velocity of this curve at the point $\mathbf{r}(u(t), v(t))$ is equal to $\mathbf{v} = \frac{d\mathbf{r}(t)}{dt} = u_t\mathbf{r}_u + v_t\mathbf{r}_v$. The length of the curve is equal to

$$L = \int_a^b |\mathbf{v}(t)|dt = \int_a^b \sqrt{(\mathbf{v}(t), \mathbf{v}(t))}dt = \int_a^b \sqrt{(u_t\mathbf{r}_u + v_t\mathbf{r}_v, u_t\mathbf{r}_u + v_t\mathbf{r}_v)}dt = \tag{4.19}$$

$$\int_a^b \sqrt{(\mathbf{r}_u, \mathbf{r}_u)u_t^2 + 2(\mathbf{r}_u, \mathbf{r}_v)u_t v_t + (\mathbf{r}_v, \mathbf{r}_v)v_t^2}d\tau =$$

$$\int_a^b \sqrt{G_{11}u_t^2 + 2G_{12}u_t v_t + G_{22}v_t^2}dt\,. \tag{4.20}$$

An external observer will calculate the length of the curve using (4.16). An ant living on the surface calculate length of the curve via first quadratic form using (4.20): first quadratic form defines Riemannian metric on the surface:

$$ds^2 = G_{11}du^2 + 2G_{12}dudv + G_{22}dv^2 \tag{4.21}$$

Example Consider the curve

$$\mathbf{r}(t) \quad \begin{cases} x = R\cos t \\ y = R\sin t \\ z = vt \end{cases} \quad , \quad 0 \leq t \leq 1$$

on the cylinder (4.2) (helix). The coordinates of this curve on the cylinder (internal coordinates) are

$$\begin{cases} \varphi(t) = t \\ h(t) = vt \end{cases} \,.$$

To calculate the length of this curve the external observer will perform the calculations

$$L = \int_0^1 \sqrt{x_t^2 + y_t^2 + z_t^2}dt = \int_0^1 \sqrt{R^2\sin^2 t + R^2\cos^2 t + v^2}dt = \int_0^1 \sqrt{R^2 + v^2}dt = \sqrt{R^2 + v^2}.$$

An internal observer ("ant") uses quadratic form (4.15) and perform the following calculations:

$$L = \int_0^1 \sqrt{G_{11}\varphi_t^2 + 2G_{12}\varphi_t h_t + G_{22}h_t^2}dt = \int_0^1 \sqrt{R^2\varphi_t^2 + h_t^2}dt = \int_0^1 \sqrt{R^2+v^2}dt = \sqrt{R^2+v^2}\,.$$

The answer will be the same. (See this and other examples in Homework 8).

4.3 Unit normal vector to surface

We define unit normal vector field for surfaces in $\mathbf{E}^3$.

Consider vector field defined on the points of surface.

Definition Let $M\colon\ \mathbf{r} = \mathbf{r}(u,v)$ be a surface in $\mathbf{E}^3$. We say that vector $\mathbf{n}(u,v)$ is *normal unit vector* at the point $\mathbf{p} = \mathbf{r}(u,v)$ of the surface M if it has unit length $|\mathbf{n}| = 1$, and it is orthogonal to the surface, i.e. it is orthogonal to the tangent plane $T_{\mathbf{p}}M$. This means that it is orthogonal to any tangent vector $\xi \in T_pM$, i.e. it is orthogonal to the coordinate vectors $\mathbf{r}_u = \partial_u$, $\mathbf{r}_v = \partial_v$ at the point $\mathbf{p}$.

$$\mathbf{n}\colon (\mathbf{n}, \mathbf{r}_u) = (\mathbf{n}, \mathbf{r}_v) = 0, (\mathbf{n}, \mathbf{n}) = 1\,. \tag{4.22}$$

Write down this equation in components:

If surface is given by equation $\mathbf{r}(u,v)\colon \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$ then

$$\mathbf{r}_u = \begin{pmatrix} x_u \\ y_u \\ z_u \end{pmatrix},\ \mathbf{r}_v = \begin{pmatrix} x_v \\ y_v \\ z_v \end{pmatrix},$$

and $\mathbf{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$ is unit normal vector. Then writing the previous conditions in components we come to

$$(\mathbf{n}, \mathbf{r}_u) = n_x x_u + n_y y_u + n_z z_v = 0,\ (\mathbf{n}, \mathbf{r}_v) = n_x x_v + n_y y_v + n_z z_v = 0,\ (\mathbf{n}, \mathbf{n}) = n_x^2 + n_y^2 + n_z^2 = 1$$

Normal unit vector is defined up to a sign. At any point there are two normal unit vetors: the transformation $\mathbf{n} \to -\mathbf{n}$ transforms normal unit vector to normal unit vector.

Vector field defined at the points of the surface is called normal unit vector field if any vector is normal unit vector.

In simple cases one can guess how to find unit normal vector field using geometrical intuition and just check that conditions above are satisfied. E.g. for sphere (4.4) $\mathbf{r}$ is orthogonal to the surface, hence

$$\mathbf{n}(\theta,\varphi) = \frac{\mathbf{r}(\theta,\varphi)}{R} = \pm \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

For cylinder (4.2) it is easy to see that at any point (φ, h) (4.2), $\mathbf{r}\colon x = R\cos\varphi, y = R\sin\varphi, z = h$, a normal unit vector is equal to

$$\mathbf{n}(\varphi,h) = \pm \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} \tag{4.23}$$

Indeed it is easy to see that the conditions (4.22) are satisfied.

In general case one can define $\mathbf{n}(u,v)$ in two steps using vector product formula:

$$\mathbf{n}(u,v) = \frac{\mathbf{N}(u,v)}{|\mathbf{N}(u,v)|} \quad \text{where} \quad \mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v \tag{4.24}$$

Indeed by definition of vector product vector field $\mathbf{N}(u,v)$ is orthogonal to $\mathbf{r}_u$ and $\mathbf{r}_v$, i.e. it is orthogonal to the surface. Dividing $\mathbf{N}$ on the length we come to unit normal vector field $\mathbf{n}(u,v)$ at the point $\mathbf{r}(u,v)$. (See other examples of calculating normal unit vector in the Homework 8)

4.4 † Curves on surfaces—normal acceleration and normal curvature

We know already how to measure the length of the curve belonging to the given surface. What about curvature? Answering this question we will be able to study curvature of the surface.

Before we have to introduce normal acceleration and normal curvature for curves on the surfaces.

We know that acceleration vector $\mathbf{a}$ in general is not tangent to the curve. Recall that when studying curvature we consider decomposition of acceleration vector on tangential component and the component which is perpendicular to velocity vector: $\mathbf{a} = \mathbf{a}_{tang} + \mathbf{a}_{\perp}$. The curvature of curve is nothing but the magnitude of normal acceleration $\mathbf{a}_{\perp}$ of particle which moves along the curve with unit speed: $k = \frac{|\mathbf{a}_{\perp}|}{|\mathbf{v}|}$.

Now we consider *normal acceleration of the curve on the surface.*

Let $M\colon\ \mathbf{r} = \mathbf{r}(u, v)$ be a surface and $C\colon u = u(t), v = v(t)$, i.e. $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$, be curve on the surface M. Consider an arbitrary point $\mathbf{p} = \mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ on this curve and velocity and acceleration vectors $\mathbf{v} = \frac{d\mathbf{r}(t)}{dt}$, $\mathbf{a} = \frac{d^2\mathbf{r}(t)}{dt^2}$ at this point.

Definition The component of acceleration vector of the curve on the surface orthogonal to the surface is called a normal acceleration of curve on the surface. If $\mathbf{a}$ is acceleration vector then

$$\mathbf{a} = \mathbf{a}_{||} + \mathbf{a}_n, \ , \tag{4.25}$$

where the vector $\mathbf{a}_{||}$ is tangent to the surface and the vector $\mathbf{a}_n$ is orthogonal (perpendicular) to the surface. Calculate vector $\mathbf{a}_n$.

If $\mathbf{n}$ is a normal unit vector to the surface, then vector $\mathbf{a}_n$ is collinear (proportional) to the vector $\mathbf{n}$ and vector $\mathbf{a}_{||}$ is orthogonal to this vector:

$$\mathbf{a}_n = a_n \mathbf{n},\ (\mathbf{n}, \mathbf{a}_{||}) = 0\,.$$

Take a scalar product of left and right hand sides of the formula (4.25) on the vector $\mathbf{n}$. We come to:

$$(\mathbf{n}, \mathbf{a}) = (\mathbf{n}, \mathbf{a}_{||} + \mathbf{a}_n) = (\mathbf{n}, \mathbf{a}_{||}) + (\mathbf{n}, \mathbf{a}_n) = 0 + a_n(\mathbf{n}, \mathbf{n}) = a_n.$$

Hence we come to

$$\mathbf{a} = a_n \mathbf{n} = (\mathbf{n}, \mathbf{a})\mathbf{n}\,. \tag{4.26}$$

Avoid confusion! The normal acceleration vector $\mathbf{a}_n$ of the curve on the surface is orthogonal to the surface. The normal acceleration vector of the curve in $\mathbf{E}^3$ $\mathbf{a}_\perp$ is orthogonla to the velocity vector of the curve.

Now we are ready give a definition of normal curvature of the curve on the surface.

Definition Let C be a curve on the surface M. Let $\mathbf{v}$, $\mathbf{a}$ be velocity and acceleration vectors at the given point of this curve and $\mathbf{n}$ be normal unit vector at this point. Then

$$\kappa_n = \frac{a_n}{|\mathbf{v}|^2} = \frac{(\mathbf{n}, \mathbf{a})}{(\mathbf{v}, \mathbf{v})} \tag{4.27}$$

is called *normal curvature of the curve* C on the surface M at the point $\mathbf{p}$. Or in other words

$$|\kappa_n| = \frac{|\mathbf{a}_n|}{(\mathbf{v}, \mathbf{v})}, \tag{4.28}$$

i.e. up to a sign normal curvature is equal to modulus of normal acceleration divided on the square of speed (Compare with formula (3.23) for usual curvature.)

Remark Avoid confusion: We know that usual curvature k of the curve is defined by the formula $k = \frac{|\mathbf{a}_\perp|}{|\mathbf{v}|^2}$, where $\mathbf{a}_\perp$ is a magnitude of the acceleration vector orthogonal to the curve (see the formula (3.23)). Normal curvature of the curve on the surface is defined by the analogous formula bunt in terms of normal acceleration $\mathbf{a}_n$ which is orthogonal to the surface, not to the curve!

In fact one can see that $|\mathbf{a}_\perp| \leq |\mathbf{a}_n|$, i.e. modulus of the normal curvature is less or equal to the usual curvature of the curve. (See in details the Appendix "Relations between usual curvature, normal curvature and geodesic curvature")

4.5 Shape operator on the surface

Let $M\colon \mathbf{r} = \mathbf{r}(u,v)$ be a surface and $\mathbf{L}(u,v)$ be an arbitrary (not necessarily unit normal) vector field at the points of the surface M. We define at every point $\mathbf{p} = \mathbf{r}(u,v)$ a linear operator K_L acting on the vectors tangent to the surface M such that its value is equal to the derivative of vector field $\mathbf{L}(u,v)$ along vector $\boldsymbol{\xi}$

$$K_{\mathbf{L}}\colon\ \boldsymbol{\xi} \in T_{\mathbf{p}}M \mapsto K_{\mathbf{L}}(\boldsymbol{\xi}) = \partial_{\boldsymbol{\xi}} L = \xi_u \frac{\partial \mathbf{L}(u,v)}{\partial u} + \xi_v \frac{\partial \mathbf{L}(u,v)}{\partial v}\,, \tag{4.29}$$

ξ_u, ξ_v are components of vector $\boldsymbol{\xi}$

$$\boldsymbol{\xi} = \xi_u \mathbf{r}_u + \xi_v \mathbf{r}_v \tag{4.30}$$

The vector $K_{\mathbf{L}}\boldsymbol{\xi} \in T_{\mathbf{p}}\mathbf{E}^3$ in general is not a vector tangent to the surface C and $K_{\mathbf{L}}$ is linear operator from the space $T_{\mathbf{p}}M$ in the space $T_{\mathbf{p}}\mathbf{E}^3$ of all vectors in $\mathbf{E}^3$ attached at the point $\mathbf{p}$

It turns out that in the case if vector field $\mathbf{L}(u,v)$ is *a unit normal vector field* then operator $K_{\mathbf{L}}$ takes values in vectors tangent to M and it is very important geometric properties.

Definition-Proposition Let $\mathbf{n}(u,v)$ be a unit normal vector field to the surface M. Then operator

$$S\colon = L_{-\mathbf{n}}\colon\ S(\boldsymbol{\xi}) = \partial_{\boldsymbol{\xi}}(-\mathbf{n}) = -\xi_u \frac{\partial \mathbf{n}(u,v)}{\partial u} - \xi_v \frac{\partial \mathbf{n}(u,v)}{\partial v} \tag{4.31}$$

maps tangent vectors to the tangent vectors:

$$S\colon T_{\mathbf{p}}M \to T_{\mathbf{p}}M \text{ for every } \boldsymbol{\xi} \in T_{\mathbf{p}}M, \qquad S(\boldsymbol{\xi}) \in T_{\mathbf{p}}M \tag{4.32}$$

This operator is called *shape operator*.

Remark The sign " $-$ " seems to be senseless: if $\mathbf{n}$ is unit normal vector field then $-\mathbf{n}$ is normal vector field too. Later we will see why it is convenient (see the proof of the Proposition below).

Show that the property (4.32) is indeed obeyed, i.e. vector $\boldsymbol{\xi}' = S(\boldsymbol{\xi})$ is tangent to surface. Consider derivative of scalar product $(\mathbf{n}, \mathbf{n})$ with respect to the vector field $\boldsymbol{\xi}$. We have that $(\mathbf{n}, \mathbf{n}) = 1$. Hence

$$\partial_{\boldsymbol{\xi}}(\mathbf{n}, \mathbf{n}) = 0 = \partial_{\boldsymbol{\xi}}(\mathbf{n}, \mathbf{n}) = (\partial_{\boldsymbol{\xi}}\mathbf{n}, \mathbf{n}) + (\mathbf{n}, \partial_{\boldsymbol{\xi}}\mathbf{n}) = 2(\partial_{\boldsymbol{\xi}}\mathbf{n}, \mathbf{n})\,.$$

Hence $(\partial_{\boldsymbol{\xi}}\mathbf{n}, \mathbf{n}) = -(S(\boldsymbol{\xi}), \mathbf{n}) = -(\boldsymbol{\xi}', \mathbf{n}) = 0$, i.e. vector $\partial_{\boldsymbol{\xi}}\mathbf{n} = -\boldsymbol{\xi}'$ is orthogonal to the vector $\mathbf{n}$. This means that vector $\boldsymbol{\xi}'$ is tangent to the surface.

Examples of shape operator see in the Homework 9.

We show now that normal acceleration of a curve on the surface and normal curvature are expressed in terms of shape operator.

Let $C\colon \mathbf{r}(t)$ be a curve on the surface M, $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$. Let $\mathbf{v} = \mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt}$, $\mathbf{a} = \mathbf{a}(t) = \frac{d^2 r(t)}{dt^2}$ be velocity and acceleration vectors respectively. Recall that

$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \dot{x}\mathbf{e}_x + \dot{y}\mathbf{e}_y + \dot{z}\mathbf{e}_z = \frac{d\mathbf{r}(u(t), v(t))}{dt} = \dot{u}\mathbf{r}_u + \dot{v}\mathbf{r}_v \tag{4.33}$$

be velocity vector; $\dot{u}, \dot{v}$ are internal components of the velocity vector with respect to the basis $\{\mathbf{r}_u = \partial_u, \mathbf{r}_v = \partial_v\}$ and $\dot{x}, \dot{y}, \dot{z}$, are external components velocity vectors with respect to the basis $\{\mathbf{e}_x = \partial_x, \mathbf{e}_y = \partial_y, \mathbf{e}_z = \partial_z\}$. As always we denote by $\mathbf{n}$ normal unit vector.

Proposition *The normal acceleration at an arbitrary point $\mathbf{p} = \mathbf{r}(u(t_0), v(t_0))$ of the curve C on the surface M is defined by the scalar product of the velocity vector $\mathbf{v}$ of the curve at the point $\mathbf{p}$ on the value of the shape operator on the velocity vector:*

$$\mathbf{a}_n = a_n\mathbf{n} = (\mathbf{v}, S\mathbf{v})\,\mathbf{n} \tag{4.34}$$

and normal curvature (4.27) *is equal to*

$$\kappa_n = \frac{(\mathbf{n}, \mathbf{a})}{(\mathbf{v}, \mathbf{v})} = \frac{(\mathbf{v}, S\mathbf{v})}{(\mathbf{v}, \mathbf{v})} \tag{4.35}$$

Proof of the Proposition. According to (5.32) we have

$$\mathbf{a}_n = (\mathbf{n}, \mathbf{a})\mathbf{n} = \mathbf{n}\left(\mathbf{n}, \frac{d}{dt}\mathbf{v}(t)\right)\mathbf{n} = \mathbf{n}\frac{d}{dt}\left(\mathbf{n}, \mathbf{v}(t)\right) - \mathbf{n}\left(\frac{d}{dt}\mathbf{n}(u(t), v(t)), \mathbf{v}(t)\right)$$

$$= 0 + \left(-\partial_{\mathbf{v}}\mathbf{n}, \mathbf{v}\right)\mathbf{n} = (S\mathbf{v}, \mathbf{v})\mathbf{n}$$

This proves Proposition.

4.6 Principal curvatures, Gaussian and mean curvatures and shape operator

Now we introduce on surfaces, principal curvatures, Gaussian curvature and mean curvature.

Let $\mathbf{p}$ be an arbitrary point of the surface M and S be shape operator at this point. S is symmetric operator: $(S\mathbf{a}, \mathbf{b}) = (\mathbf{b}, S\mathbf{a})$. Consider eigenvalues λ_1, λ_2 and eigenvectors $\mathbf{l}_1, \mathbf{l}_2$ of the shape operator S

$$\mathbf{l}_1, \mathbf{l}_2 \in T_{\mathbf{p}}M, \qquad S\,\mathbf{l}_1 = \kappa_1\mathbf{l}_1, \qquad S\,\mathbf{l}_2 = \kappa_2\mathbf{l}_2, \tag{4.36}$$

Definition Eigenvalues of shape operator λ_1, λ_2 are called *principal curvatures*:

$$\lambda_1 = \kappa_1, \;\; \lambda_2 = \kappa_2$$

Eigenvectors $\mathbf{l}_1, \mathbf{l}_2$ define the two directions such that curves directed along these vectors have normal curvature equal to the principal curvatures κ_+, κ_-.

These directions are called principal directions

Remark As it was noted above normal unit vector as well as a shape operator are defined up to a sign. Hence principal curvatures, i.e. eigenvalues of shape operator are defined up to a sign too:

$$\mathbf{n} \to -\mathbf{n}, \text{then } S \to -S, \;\text{ then } (\kappa_1, \kappa_2) \to (-\kappa_1, -\kappa_2) \tag{4.37}$$

Remark. Principal directions are well-defined in the case if principal curvatures (eigenvalues of shape operator) are different: $\lambda_1 = \kappa_1 \neq \kappa_2 = \lambda_2$. In the case if eigenvalues $\lambda_1 = \lambda_2 = \lambda$ then $S = \lambda E$ is proportional to unity operator. In this case all vectors are eigenvectors, i.e. all directions are principal directions. (This happens for the shape operator of the sphere: see the Homework 9.)

Remark Do shape operator have always two eigenvectors? Yes, in fact one can prove that it is symmetrical operator: $\langle S\mathbf{a}, \mathbf{b}\rangle = \langle S\mathbf{b}, \mathbf{a}\rangle$ for arbitrary two vectors $\mathbf{a}, \mathbf{b}$, hence it has two eigenvectors. This implies that principal directions are orthogonal to each other. Indeed one can see that $\lambda_2(\mathbf{l}_2, \mathbf{l}_1) = (S\mathbf{l}_2, \mathbf{l}_1) = (\mathbf{l}_2, S\mathbf{l}_1) = \lambda_1(\mathbf{l}_2, \mathbf{l}_1)$. It follows from this relation that eigenvectors are orthogonal $((\mathbf{l}_-, \mathbf{l}_+) = 0)$ if $\lambda_- \neq \lambda_+$ If $\lambda_- = \lambda_+$ then all vectors are eigenvectors. One can choose in this case $\mathbf{l}_-$, $\mathbf{l}_+$ to be orthogonal.

Definition

- *Gaussian curvature* K of the surface M at a point $\mathbf{p}$ is equal to the product of principal curvatures.

$$K = \kappa_1 \kappa_2 \tag{4.38}$$

- *Mean curvature* K of the surface M at a point S is equal to the arithmetic mean of the principal curvatures:

$$H = \kappa_1 + \kappa_2 \tag{4.39}$$

Recall that the product of eigenvalues of a linear operator is determinant of this operator, and the sum of eigenvalues of linear operator is *trace* of this operator. Thus we immediately come to the useful formulae for calculating Gaussian and mean curvatures:

Proposition Let S be a shape operator at the point $\mathbf{p}$ on the surface M. Then

- Gaussian curvature K of the surface M at the point $\mathbf{p}$ is equal to the determinant of the shape operator:

$$K = \kappa_1 \kappa_2 = \det S \tag{4.40}$$

- Mean curvature H of the surface M at the point $\mathbf{p}$ is equal to the half of the trace of the shape operator S:

$$H = \kappa_1 + \kappa_2 = \operatorname{Tr} S \tag{4.41}$$

E.g. if in a given basis a shape operator is given by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$K = \det S = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc, \quad H = \frac{1}{2} \operatorname{Tr} S = \frac{1}{2} \operatorname{Tr} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d \tag{4.42}$$

(See also example 5 in the Homework 9)

4.7 †Principal curvatures and normal curvature

In this subsection we principal curvatures, eigenvectors of the shape operator by κ_-, κ_+ and respectively eigenvectors by $\mathbf{l}_-, \mathbf{l}_+$.

One can consider different curves passing through an arbitrary point $\mathbf{p}$ on the surface M. We know that if $\mathbf{v}$ velocity vector of the curve then normal curvature is equal to $\kappa_n = \frac{(S\mathbf{v},\mathbf{v})}{(\mathbf{v},\mathbf{v})}$ (see (4.35)). What are the relations between normal curvature of curves and principal curvature? The following Proposition establishes these relations.

Proposition

Let κ_-, κ_+, $\kappa_- \leq \kappa_+$ be principal curvatures of the surface M at the point $\mathbf{p}$ (eigenvalues of shape operator S at the point $\mathbf{p}$).

Then normal curvature κ of an arbitrary curve on the surface M at the point $\mathbf{p}$ takes values in the interval (κ_-, κ_+):

$$\kappa_- \leq k_n \leq \kappa_+ \tag{4.43}$$

Example E.g. consider cylinder surface of the radius R. One can calculate that principal curvatures are equal to $\kappa_- = 0, \kappa_+ = \frac{1}{R}$ (see Homework 8). Then for an arbitrary curve on the surface normal curvature κ_n takes values in the interval $(0, \frac{1}{R})$ (up to a sign). (See Homework 8 and appendix "Normal curvature of curves on cylinder surface")

Proof of Proposition: If velocity vector $\mathbf{v}$ of curve is collinear to the eigenvector $\mathbf{l}_+$, $\mathbf{v} = \lambda\mathbf{l}_+$ then normal curvature of the curve C at the point $\mathbf{p}$ according to (4.35) is equal to

$$\kappa_n = \frac{(\mathbf{v}, S\mathbf{v})}{(\mathbf{v},\mathbf{v})} = \frac{(\lambda\mathbf{l}_+, S\lambda\mathbf{l}_+)}{(\lambda\mathbf{l}_+, \lambda\mathbf{l}_+)} = \frac{\lambda^2\,(\mathbf{l}_+, \kappa_+\mathbf{l}_+)}{\lambda^2(\mathbf{l}_+, \mathbf{l}_+)} = \frac{\kappa_+\,(\mathbf{l}_+, \mathbf{l}_+)}{(\mathbf{l}_+, \mathbf{l}_+)} = \kappa_+ \,.$$

Analogously if velocity vector $\mathbf{v}$ is collinear to the eigenvector $\mathbf{l}_-$ then normal curvature of the curve C at the point $\mathbf{p}$ is equal to $\kappa_n = \frac{(\mathbf{v},S\mathbf{v})}{(\mathbf{v},\mathbf{v})} = \frac{(\mathbf{l}_-,S\mathbf{l}_-)}{(\mathbf{l}_-,\mathbf{l}_-)} = \frac{(\mathbf{l}_-,\kappa_-\mathbf{l}_-)}{(\mathbf{l}_-,\mathbf{l}_-)} = \kappa_-$.

In the general case if $\mathbf{v} = v_+\mathbf{l}_+ + v_-\mathbf{l}_-$ is expansion of velocity vector with respect to the basis of eigenvectors then we have for normal curvature

$$k_n = \frac{(\mathbf{v}, S\mathbf{v})}{(\mathbf{v},\mathbf{v})} = \frac{(v_+\mathbf{l}_+ + v_-\mathbf{l}_-, \lambda_+ v_+\mathbf{l}_+ + \lambda_- v_-\mathbf{l}_-)}{(v_+\mathbf{l}_+ + v_-\mathbf{l}_-, v_+\mathbf{l}_+ + v_-\mathbf{l}_-)} = \frac{\kappa_+ v_+^2 + \kappa_- v_-^2}{v_+^2 + v_-^2} \,. \tag{4.44}$$

Hence we come to the conclusion that

$$\kappa_- \leq \kappa_{normal} = \frac{\kappa_+ v_+^2 + \kappa_- v_-^2}{v +^2 + v_-^2} \leq \kappa_+ \tag{4.45}$$

Thus we prove that normal curvature varies in the interval (λ_-, λ_+).

Now remember the definition of principal curvatures from the subsection 4.4: we see that λ_-, λ_+ are just principal curvatures.

Summarize all the relations between normal curvature, shape operator and Gaussian and mean curvature.

- *Principal curvatures* κ_-, κ_+ of the surface M at the given point $\mathbf{p}$ are eigenvalues of shape operator S acting at the tangent space $T_{\mathbf{p}}M$ (κ_-, κ_+). Corresponding eigenvectors $\mathbf{l}_+, \mathbf{l}_-$ define directions which are called *principal directions.* Principal directions are orthogonal or can be chosen to be orthogonal if $\kappa_- = \kappa_+$. The normal curvature κ_n for an arbitrary curve on the surface M at the point $\mathbf{p}$ varies in the interval (κ_-, κ_+):
$$\kappa_- \leq \kappa_n \leq \kappa_+ \tag{4.46}$$

- *Gaussian curvature* K of the surface M at a point S is equal to the product of principal curvatures, i.e. determinant of shape operator S:
$$K = \kappa_+ \cdot \kappa_- = \det S \tag{4.47}$$

- *Mean curvature* H of the surface M at a point S is equal to the half-sum of the principal curvatures, i.e. half of the trace of shape operator S:
$$H = \kappa_+ + \kappa_- = \mathrm{Tr}\ S \tag{4.48}$$

(See also example 3 in the Homework 9)

5 †Appendices

5.1 Formulae for vector fields and differentials in cylindrical and spherical coordinates

Cylindrical and spherical coordinates

- Cylindrical coordinates in $\mathbf{E}^3$
$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = h \end{cases} \qquad (0 \leq \varphi < 2\pi, 0 \leq r < \infty) \tag{5.1}$$

- Spherical coordinates in $\mathbf{E}^3$
$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\cos\varphi \\ z = r\cos\theta \end{cases} \quad (0 \leq \varphi < 2\pi, 0 \leq r < \infty) - - - \text{cylindrical coordiantes in } \mathbf{E}^3 \tag{5.2}$$

Example (Basis vectors and forms for cylindrical coordinates)

Consider cylindrical coordinates in $\mathbf{E}^3$: $u = r, v = \varphi, w = h$. Then calculating partial derivatives we come to

$$\begin{cases} \partial_r = \frac{\partial x}{\partial r}\partial_x + \frac{\partial y}{\partial r}\partial_y + \frac{\partial z}{\partial r}\partial_z = \cos\varphi\partial_x + \sin\varphi\partial_y \\ \\ \partial_\varphi = \frac{\partial x}{\partial \varphi}\partial_x + \frac{\partial y}{\partial \varphi}\partial_y + \frac{\partial z}{\partial \varphi}\partial_z = -r\sin\varphi\partial_x + r\cos\varphi\partial_y \\ \\ \partial_h = \frac{\partial x}{\partial h}\partial_x + \frac{\partial y}{\partial h}\partial_y + \frac{\partial z}{\partial h}\partial_z = \partial_z \end{cases} \tag{5.3}$$

Basic forms are $dr, d\varphi, dh$ and

$$\begin{aligned} &dr(\partial_r) = 1, dr(\partial_\varphi) = 0, dr(\partial_h) = 0 \\ &d\varphi(\partial_r) = 0, d\varphi(\partial_\varphi) = 1, d\varphi(\partial_h) = 0 \\ &dh(\partial_r) = 0, dh(\partial_\varphi) = 0, dh(\partial_h) = 1 \end{aligned} \tag{5.4}$$

Example (Basis vectors for spheric coordinates)

Consider spheric coordinates in $\mathbf{E}^3$: $u = r, v = \theta, w = \varphi$. Then calculating partial derivatives we come to

$$\begin{cases} \partial_r = \frac{\partial x}{\partial r}\partial_x + \frac{\partial y}{\partial r}\partial_y + \frac{\partial z}{\partial r}\partial_z = \sin\theta\cos\varphi\partial_x + \sin\theta\sin\varphi\partial_y + \cos\theta\partial_z \\ \\ \partial_\theta = \frac{\partial x}{\partial \theta}\partial_x + \frac{\partial y}{\partial \theta}\partial_y + \frac{\partial z}{\partial \theta}\partial_z = r\cos\theta\cos\varphi\partial_x + r\cos\theta\sin\varphi\partial_y - r\sin\theta\partial_z \\ \\ \partial_\varphi = \frac{\partial x}{\partial \varphi}\partial_x + \frac{\partial y}{\partial \varphi}\partial_y + \frac{\partial z}{\partial \varphi}\partial_z = -r\cos\theta\sin\varphi\partial_x + r\sin\theta\cos\varphi\partial_y \end{cases} \tag{5.5}$$

Basic forms are $dr, d\theta, d\varphi$ and

$$\begin{aligned} &dr(\partial_r) = 1, dr(\partial_\theta) = 0, dr(\partial_\varphi) = 0 \\ &d\theta(\partial_r) = 0, d\theta(\partial_\theta) = 1, d\theta(\partial_\varphi) = 0 \\ &d\varphi(\partial_r) = 0, d\varphi(\partial_\theta) = 0, d\varphi(\partial_\varphi) = 1 \end{aligned} \tag{5.6}$$

We know that 1-form is a linear function on tangent vectors. If $\mathbf{A}, \mathbf{B}$ are two vectors attached at the point $\mathbf{r}_0$, i.e. tangent to this point and ω, ρ are two 1-forms then one defines the value of $\omega \wedge \rho$ on $\mathbf{A}, B$ by the formula

$$\omega \wedge \rho(\mathbf{A}, \mathbf{B}) = \omega(\mathbf{A})\rho(B) - \omega(B)\rho(A) \tag{5.7}$$

We come to bilinear anisymmetric function on tangent vectors. If $\sigma = a(x,y)dx \wedge dy$ is an arbitrary two form then this form defines bilinear form on pair of tangent vectors: $\sigma(\mathbf{A}, \mathbf{B}) =$

$$a(x,y)dx \wedge dy(A,B) = a(x,y)\left(dx(\mathbf{A})dy(\mathbf{B}) - dx(\mathbf{B})dy(\mathbf{A})\right) = a(x,y)(A_x B_y - A_y B_y) \tag{5.8}$$

One can see that in the case if $a = 1$ then right hand side of this formula is nothing but the area of parallelogram spanned by the vectors $\mathbf{A}, \mathbf{B}$.

This leads to the conception of integral of form over domain.

Let $\omega = a(x)dx \wedge dy$ be a two form and D be a domain in $\mathbf{E}^2$. Then by definition

$$\int_D \omega = \int_D a(x, y)dxdy \tag{5.9}$$

If $\omega = dx \wedge dy$ then

$$\int_D w = \int_D (x, y)dxdy = \text{Area of the domain } D \tag{5.10}$$

The advantage of these formulae is that we do not care about coordinates[9]

Example Let D be a domain defined by the conditions

$$\begin{cases} x^2 + y^2 \leq 1 \\ y \geq 0 \end{cases} \tag{5.12}$$

Calculate $\int_D dx \wedge dy$.

$\int_D dx \wedge dy = \int_D dxdy =$ area of the $D = \frac{\pi}{2}$.

If we consider polar coordinates then according (2.44)

$$dx \wedge dy = rdr \wedge d\varphi$$

Hence $\int_D dx \wedge dy = \int_D rdr \wedge d\varphi = \int_D rdrd\varphi = \int_0^1 \left(\int_0^\pi d\varphi\right) rdr = \pi \int_0^1 rdr = \pi/2$.

Another example

Example Let D be a domain in $\mathbf{E}^2$ defined by the conditions

$$\begin{cases} \frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} \leq 1 \\ y \geq 0 \end{cases} \tag{5.13}$$

D is domain restricted by upper half of the ellipse and x-axis. Ellipse has the centre at the point $(c, 0)$. Its area is equal to $S = \int_D dx \wedge dy$. Consider new variables x', y': $x = c + ax', y = by'$. In new variables domain D becomes the domain from the previous example:

$$\frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} = x'^2 + y'^2$$

---

[9]If we consider changing of coordinates then jacobian appears: If u, v are new coordinates, $x = x(u, v)$, $y = y(u, v)$ are new coordinates then

$$\int a(x, y)dxdy = \int a(x(u, v), y(u, v)) \det \begin{pmatrix} x_u & x_v \\ x_u & x_v \end{pmatrix} dudv \tag{5.11}$$

In formula(5.9) it appears under as a part of coefficient of differential form.

and $dx \wedge dy = ab dx' \wedge dy'$. Hence

$$S = \int_{\frac{(x-c)^2}{a^2}+\frac{y^2}{b^2}\leq 1, y\geq 0} dx \wedge dy = ab \int_{x'^2+y'^2\leq 1, y'\geq 0} dx' \wedge dy' = \frac{\pi ab}{2} \tag{5.14}$$

Theorem 2 (Green formula) Let ω be 2-form such that $\omega = d\omega'$ and D be a domain–interior of the closed curve C. Then

$$\int_D w = \int_C w' \tag{5.15}$$

5.2 Curvature and second order contact (touching) of curves

Let C_1, C_2 be two curves in $\mathbf{E}^2$. For simplicity we here consider only curves in $\mathbf{E}^2$.

Definition Two non-parameterised curves C_1, C_2 have second order contact (touching) at the point $\mathbf{r}_0$ if

- They coincide at the point $\mathbf{r}_0$
- they have the same tangent line at this point
- they have the same curvature at the point $\mathbf{r}_0$

If $\mathbf{r}_1(t), \mathbf{r}_2(t)$ are an arbitrary parameterisations of these curves such that $\mathbf{r}_1(t_0) = \mathbf{r}_2(t_0) = \mathbf{r}_0$ then the condition that they have the same tangent line means that velocity vectors $\mathbf{v}_1(t), \mathbf{v}_2(t)$ are collinear at the point t_0.

(As always we assume that curves under considerations are smooth and regular, i.e. $x(t), y(t)$ are smooth functions and velocity vector $\mathbf{v}(t) \neq 0$.)

Example Consider two curves C_f, C_g—graphs of the functions f_1, f_2. Recall that curvature of the graph of the function f at the point $(x, y = f(x))$ is equal to (see (3.29))

$$k(x) = \frac{f''(x)}{(1 + f'(x))^{\frac{3}{2}}} \tag{5.16}$$

Then condition of the second order touching at the point $\mathbf{r}_0 = (x_0, y_0)$ means that

$$\begin{cases} \text{They coincide at the point } \mathbf{r}_0\text{: } f(x_0) = g(x_0) \\ \text{They have the same tangent line at this point: } f'(x_0) = g'(x_0) \\ \text{They have the same curvature at the point } \mathbf{r}_0\text{:} \frac{f''(x_0)}{(1+f'(x_0))^{\frac{3}{2}}} = \frac{g''(x_0)}{(1+g'(x_0))^{\frac{3}{2}}}, \text{ i.e. } f''(x_0) = g''(x_0) \end{cases}$$

We see that second order touching means that difference of the functions in vicinity of the point x_0 is of order $o((x - x_0)^2)$. Indeed due to Taylor formula

$$\begin{array}{c} f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \dots \\ g(x) = g(x_0) + g'(x_0)(x - x_0) + \frac{1}{2} g''(x_0)(x - x_0)^2 + \dots \end{array} \tag{5.17}$$

where we denote by dots terms which are $o(x-x_0)^2$. (They say that $f(x) = o(x-x_0)^n$ if $\lim_{x\to x_0} \frac{f(x)}{(x-x_0)^n} = 0$).

Hence

$$f(x) - g(x) = o(x-x_0)^2 \tag{5.18}$$

because $f(x_0) = g(x_0), f(x_0) = g(x_0), f'(x_0) = g'(x_0)$ and $f''(x_0) = g''(x_0)$

In general case if two curves have second order contact then in the vicinity of the contact point one can consider these curves as a graphs of the functions $y = f(x)$ (or $x = f(y)$).

To clarify geometrical meaning of second order touching consider the case where one of the curves is a circle. Then second order touching means that curvature of one of these curves is equal to $1/R$, where R is a radius of the circle.

We see that to calculate the radius of the circle which has the second order touching with the given curve at the given point we have to calculate the curvature of this curve at this given point.

Example. Let C_1 be parabola $y = ax^2$ and C_2 be a circle. Suppose these curves have second order contact at the vertex of the parabola: point $0, 0$.

Calculate the curvature of the parabola at the vertex. Curvature at the vertex is equal to $k(t)|_{t=0} = 2a$ (see Homework). Hence the radius of the circle which has second order touching is equal to

$$R = \frac{1}{2a}.$$

To find equation of this circle note that the circle which has second order touching to parabola at the vertex passes trough the vertex (point $(0,0)$) and is tangent to x-axis. The radius of this circle is equal to $R = \frac{1}{2a}$. Hence equation of the circle is

$$(x - R^2) + y^2 = 0, \text{ where } R = \frac{1}{2a}$$

One comes to the same answer by the following detailed analysis:

Consider equation of a circle: $(x-x_0)^2 + (y-y_0)^2 = R^2$. The condition that curves coincide at the point $(0,0)$ means that $x_0^2 + y_0^2 = R^2$. x-axis is tangent to parabola at the vertex. Hence it is tangent to the circle too. Hence $y_0^2 = R^2$ and $x_0 = 0$. We see that an equation of the circle is $x^2 + (y-R)^2 = R^2$. The circle $x^2 + (y-R)^2 = x^2 + y^2 - 2yR = 0$ in the vicinity of the point $(0,0)$ can be considered as a graph of the function $y = R - \sqrt{R^2 - x^2}$.. The condition that functions $y = ax^2$ and $y = R - \sqrt{R^2 - x^2}$ have second order contact means that

$$R - \sqrt{R^2 - x^2} = ax^2 + \text{terms of the order less that } x^2 \,.$$

But

$$R - \sqrt{R^2 - x^2} = R - R\sqrt{1 - \frac{x^2}{2R^2}} = R - R\left(1 - \frac{x^2}{2R^2} + o(x^2)\right) = \frac{x^2}{2R} + o(x^2)\,.$$

Comparing we see that $a = \frac{1}{2R}$ and $\frac{1}{R} = 2a$. But curvature of the parabola at the vertex is equal to $k = 2a$ (if $a > 0$). We see that $k = \frac{1}{R}$.

5.3 Integral of curvature over planar curve.

We consider here the following problem: Let $C = \mathbf{r}(t)$ be a planar curve, i.e. a curve in $\mathbf{E}^2$.

Let $\mathbf{n}(\mathbf{r}(t)$ be a unit normal vector field to the curve, i.e. $\mathbf{n}$ is orthogonal to the curve (velocity vector) and it has unit length.

E.g. if $\mathbf{r}(t) : x(t) = R\cos t, y(t) = R\sin t$, then $\mathbf{n}(\mathbf{r}(t)) = \begin{pmatrix}\cos t\\ \sin t\end{pmatrix}$

If point moves along the curve $\mathbf{r}(t), t_1 \leq t \leq t_2$ then velocity vector and vector field $\mathbf{n}(t)$ rotate on the same angle. It turns out that this angle is expressed via integral of curvature over the curve...

Try to analyze the situation:

Proposition Let C: $\mathbf{r}(t)$ be a curve in $\mathbf{E}^2$, $\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt}$, velocity vector, $k(\mathbf{r}(t))$— curvature and $\mathbf{n}(t)$ unit normal vector field. Denote by $\varphi(t)$ the angle between normal vector $\mathbf{n}(t)$ and x-axis.

Then

$$\frac{d\mathbf{n}(t)}{dt} = \pm k(\mathbf{r}(t))\mathbf{v}(t) \tag{5.19}$$

$$\frac{d\varphi(t)}{dt} = \pm k(\mathbf{r}(t))|\mathbf{v}(t)| \tag{5.20}$$

(Sign depends on the orientation of the pair of vectors $(\mathbf{v}, \mathbf{n})$)

Note that the second statement of the Proposition has a clear geometrical meaning: If C is a circle of the radius R then RHS of (5.20) is equal to $\frac{v}{R}$. It is just angular velocity $d\varphi/dt$.

To prove this Proposition note that $(\mathbf{n}, \mathbf{n}) = 1$. Hence

$$0 = \frac{d}{dt}(\mathbf{n}(t), \mathbf{n}(t)) = 2\left(\frac{d\mathbf{n}(t)}{dt}, \mathbf{n}(t)\right),$$

i.e. vector $\frac{d\mathbf{n}(t)}{dt}$ is orthogonal to the vector $\mathbf{n}$. This means that $\frac{d\mathbf{n}(t)}{dt}$ is collinear to $\mathbf{v}(t)$, because curve is planar. We have $\frac{d\mathbf{n}(t)}{dt} = \kappa(\mathbf{r}(t))\mathbf{v}(t)$ where κ is a coefficient. Show that the coefficient κ is just equal to curvature k (up to a sign). Clearly $(\mathbf{n}, \mathbf{v}) = 0$ because these vectors are orthogonal. Hence

$$0 = \frac{d}{dt}(\mathbf{n}(t), \mathbf{v}(t)) = \left(\frac{d\mathbf{n}(t)}{dt}, \mathbf{v}(t)\right) + \left(\mathbf{n}(t), \frac{d\mathbf{v}(t)}{dt}\right) =$$

$$(\kappa(\mathbf{r}(t))\mathbf{v}(t), \mathbf{v}(t)) + (\mathbf{n}(t), \mathbf{a}(t)) = \kappa(\mathbf{r}(t))|\mathbf{v}(t)|^2 + (\mathbf{n}, \mathbf{a}_\perp),$$

because $(\mathbf{n}(t), \mathbf{a}(t)) = (\mathbf{n}, \mathbf{a}_\perp)$. But $(\mathbf{n}, \mathbf{a}_\perp)$ is just centripetal acceleration: $(\mathbf{n}, \mathbf{a}_\perp) = \pm|\mathbf{a}_\perp|$ and curvature is equal to $|\mathbf{a}_\perp|/|\mathbf{v}|^2$. Hence we come to $\kappa(\mathbf{r}(t)) = \pm\frac{|\mathbf{a}_\perp|}{|\mathbf{v}|^2} = \pm k$. Thus we prove (5.19).

To prove (5.20) consider expansion of vectors $\mathbf{n}(t), \mathbf{v}(t)$ over basis vectors ∂_x, ∂_y. We see that

$$\mathbf{n}(t) = \cos\varphi(t)\partial_x + \sin\varphi(t)\partial_y \text{ and } \mathbf{v}(t) = |\mathbf{v}(t)|\left(-\sin\varphi(t)\partial_x + \cos\varphi(t)\partial_y\right) \tag{5.21}$$

Differentiating $\mathbf{n}(t)$ by t we come to $\frac{d\mathbf{n}(t)}{dt} = \frac{d\varphi(t)}{dt}\left(-\sin\varphi(t)\partial_x + \cos\varphi(t)\partial_y\right) = \frac{d\varphi(t)}{dt}\frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$. Comparing this equation with equation (5.19) we come to (5.20).

The appearance of sign factor in previous formulae related with the fact that normal vector field is defined up to a sign factor $\mathbf{n} \to -\mathbf{n}$.

It is useful to write formulae (5.19), (5.20) in explicit way. Let $\mathbf{r}(t)\colon x(t), y(t)$ be a parameterisation of the curve. Then $\mathbf{v}(t) = \begin{pmatrix} x_t \\ y_t \end{pmatrix}$ velocity vector. One can define normal vector field as

$$\mathbf{n}(t) = \frac{1}{\sqrt{x_t^2 + y_t^2}}\begin{pmatrix} -y_t \\ x_t \end{pmatrix} \tag{5.22}$$

or changing the sign as

$$\mathbf{n}(t) = \frac{1}{\sqrt{x_t^2 + y_t^2}}\begin{pmatrix} y_t \\ -x_t \end{pmatrix} \tag{5.23}$$

If we consider (5.22) for normal vector field then

$$\frac{d\mathbf{n}(t)}{dt} = \frac{x_{tt}y_t - y_{tt}x_t}{(x_t^2 + y_t^2)^{\frac{3}{2}}}\begin{pmatrix} x_t \\ y_t \end{pmatrix} \tag{5.24}$$

Recalling that $k = \frac{|x_{tt}y_t - y_{tt}|x_t}{(x_t^2+y_t^2)^{\frac{3}{2}}}$ we come to (5.19). For the angle we have

$$\frac{d\varphi}{dt} = \frac{x_t y_{tt} - y_t x_{tt}}{(x_t^2 + y_t^2)^{\frac{3}{2}}}\sqrt{x_t^2 + y_t^2} = \frac{x_t y_{tt} - y_t x_{tt}}{(x_t^2 + y_t^2)} \tag{5.25}$$

This follows from the considerations above but it can be also calculated straightforwardly.

Remark Note that last two formulae do not possess indefenity in sign.

This Proposition has very important application. Consider just two examples:

Consider upper half part of the ellipse $x^2/a^2 + y^2/b^2 = 1, y \geq 0$. We already know that curvature at the point $x = a\cos t, y = b\cos t$ of the ellipse is equal to

$$k = \frac{ab}{(a^2\sin t^2 t + b^2\cos^2 t)^{3/2}}$$

and speed is equal to $\sqrt{a^2\sin^2 t + b^2\cos^2 t}$ Apply formula (5.20) of Proposition. The curvature is not equal to zero at all the point. Hence the sign in the (5.20) is the same for all the points, i.e.

$$\pi = \int_0^\pi d\varphi(t)dt = \pm\int_0^\pi k(\mathbf{r}(t))|\mathbf{v}(t)| = \tag{5.26}$$

$$\int_0^\pi \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}\sqrt{a^2\sin^2 t + b^2\cos^2 t}\,dt = \int_0^\pi \frac{abdt}{a^2\sin^2 t + b^2\cos^2 t}\,.$$

We calculated this integral using geometrical considerations: left hand side represents the angle of rotation of normal unit vector and this angle is equal to π. Try to calculate the last integral straightforwardly: it is not easy exercise in calculus.

Another example: Let $\mathbf{r} = \mathbf{r}(t), x = x(t), y = y(t), t_1 \leq t \leq t_2$ be a closed curve in $\mathbf{E}^2$ ($\mathbf{r}(t_1) = \mathbf{r}(t_2)$.) We suppose that it possesses self-intersections points. We cannot us a formula (5.20) for integration because in general curvature may vanish at some points, but we still can use the formula (5.25). The rotation of the angle φ is equal to $2\pi n$, (n-is called winding number of the curve). Hence according to (5.25) see that

$$\int_{t_1}^{t_2} \frac{x_t y_{tt} - x_t y_{tt}}{x_t^2 + y_t^2} = 2\pi n$$

or

$$\frac{1}{2\pi} \int_{t_1}^{t_2} \frac{x_t y_{tt} - x_t y_{tt}}{x_t^2 + y_t^2} = n \tag{5.27}$$

The integrand us equal to the curvature multiplied by the speed (up to a sign). Left hand side is integral of continuous function divided by transcendent number π. The geometry tells us that the answer must to be equal to integer number.

5.4 Relations between usual curvature normal curvature and geodesic curvature.

Consider at any point $\mathbf{p}$ of the curve the following basis $\{\mathbf{v}, \mathbf{f}, \mathbf{n}\}$, where

- velocity vector $\mathbf{v}$ is tangent to the curve
- the vector $\mathbf{f}$ is the vector tangent to the surface but orthogonal to the vector $\mathbf{v}$.
- $\mathbf{n}$ is the unit normal vector to the surface, i.e.it is orthogonal to vectors $\mathbf{v}$ and $\mathbf{f}$.

Decompose acceleration vector over three directions, i.e. over three one-dimensional spaces spanned by vectors $\mathbf{v}, \mathbf{f}$ and $\mathbf{n}$:

$$\mathbf{a} = \mathbf{a}_{\text{orthogonal to surface}} + \mathbf{a}_{\text{tang.to surf. and orthog. to curve}} + \mathbf{a}_{\text{tangent to curve}} \tag{5.28}$$

The vector $\mathbf{a}_{\text{orthogonal to surface}}$ which is collinear to normal unit vector $\mathbf{n}$, will be called *vector of normal acceleration of the curve on the surface*. We denote it by $\mathbf{a}_n$.

The vector $\mathbf{a}_{\text{tang.to surf. and orthog. to curve}}$, collinear to unit vector $\mathbf{f}_C$ will be called *vector of geodesic acceleration* . We denote it by $\mathbf{a}_{geod}$.

The vector $\mathbf{a}_{\text{tangent to curve}}$, collinear to velocity vector $\mathbf{v}$, is just *vector of tangential acceleration* . We denote it $\mathbf{a}_{tang}$. We can rewrite (5.28) as

$$\mathbf{a} = \mathbf{a}_n + \mathbf{a}_{geod} + \mathbf{a}_{tang} \tag{5.29}$$

Study the expansion (5.29). Both vectors $\mathbf{a}_n$ and $\mathbf{a}_{geod}$ are orthogonal to the curve. The vector $\mathbf{a}_{geod}$ is orthogonal to the curve but it is tangent to the surface. The vector $\mathbf{a}_n$ is orthogonal not only to the curve. It is orthogonal to the surface.

The vector $\mathbf{a}_{geod} + \mathbf{a}_n = \mathbf{a}_\perp$ is orthogonal to the curve. It is the vector of normal acceleration of the curve.

Remark Please note that when we consider the curves on the surface it could arise the confusion between the vector $\mathbf{a}_n$—normal acceleration of the curve on the surface and the vector $\mathbf{a}_\perp$ of normal acceleration of the curve (see (3.2)).

When we decompose in (5.29) the acceleration vector $\mathbf{a}$ in the sum of three vectors $\mathbf{a}_n, \mathbf{a}_{geod}$ and $\mathbf{a}_{tang}$ then the vector $\mathbf{a}_n$, *the normal acceleration of the curve on the surface* is orthogonal to the surface not only to the curve. The vector

$$\mathbf{a}_\perp = \mathbf{a}_n + \mathbf{a}_{geod},$$

is orthogonal only to the curve and in general it is not orthogonal to the surface (if $\mathbf{a}_{geod} \neq 0$). It is the normal acceleration of the curve. It depends only on the curve. The normal acceleration $\mathbf{a}_n$ of the curve on the surface which is orthogonal to the surface depends on the surface where the curve lies.

We know that the curvature of the curve is equal to the magnitude of normal acceleration of the curve divided on the square of the speed (see (3.22)). We have:

$$\text{curvature of the curve } k = \frac{|\mathbf{a}_\perp|}{|\mathbf{v}|^2} = \frac{|\mathbf{a}_n + \mathbf{a}_{geod}|}{|\mathbf{v}|^2}.$$

The vectors $\mathbf{a}_n$ and $\mathbf{a}_{geod}$ transform under reparameterisation in the same way as a vector $\mathbf{a}_\perp$ (see (3.11)). If $t \to t(\tau)$ then

$$\mathbf{a}'_\perp(\tau) = t_\tau^2 \mathbf{a}_\perp \text{ and } \mathbf{a}'_n(\tau) = t_\tau^2 \mathbf{a}_n(t), \quad \mathbf{a}'_{geod}(\tau) = t_\tau^2 \mathbf{a}_{geod}(t) \tag{5.30}$$

where $\mathbf{a}'(\tau) = \frac{d^2}{d\tau^2}\mathbf{r}(t(\tau)) = t_\tau^2 \mathbf{a} + t_{\tau\tau}\mathbf{v}$ (see (3.9), (3.10), (3.8)). Hence the magnitudes

$$\frac{|\mathbf{a}_{geod}|}{|\mathbf{v}|^2} \text{ and } \frac{|\mathbf{a}_n|}{|\mathbf{v}|^2} \tag{5.31}$$

are reparameterisation invariant as well as magnitude $k = \frac{|\mathbf{a}_\perp|}{|\mathbf{v}|^2} = \frac{|\mathbf{a}_n + \mathbf{a}_{geod}|}{|\mathbf{v}|^2}$.

Multiply left and right hand sides of the equation (5.29) on unit normal vector $\mathbf{n}$. Then $(\mathbf{a}_{tang}, \mathbf{n}) = (\mathbf{a}_{geod}, \mathbf{n}) = 0$ because vectors $\mathbf{a}_{geod}$ and $\mathbf{a}_{tang}$ are orthogonal to the vector $\mathbf{n}$. We come to the relation

$$\mathbf{a}_n = (\mathbf{n}, \mathbf{a})\mathbf{n} \text{ and}, |\mathbf{a}_n| = |(\mathbf{a}, \mathbf{n})|. \tag{5.32}$$

Or in other words scalar product $(\mathbf{n}, \mathbf{a})$ is equal to $|\mathbf{a}_n|$ (up to a sign).

Compare the formula

$$\kappa_n = \frac{(\mathbf{n}, \mathbf{a})}{(\mathbf{v}, \mathbf{v})} \tag{5.33}$$

(see (4.27)) for normal curvature with the formula

$$k = \frac{|\mathbf{a}_\perp|}{(\mathbf{v}, \mathbf{v})}$$

for usual curvature (see (3.22)).

It follows from (5.30), (5.31) and (4.27) (or (5.33)) that for any curve on the surface the modulus of the normal curvature is less or equal than usual curvature.

$$|\kappa_n| \leq k \tag{5.34}$$

Indeed we have for usual curvature

$$k = \frac{|\mathbf{a}_\perp|}{|\mathbf{v}|^2} = \frac{|\mathbf{a}_{geod} + \mathbf{a}_{normal}|}{|\mathbf{v}|^2} = \sqrt{\frac{\mathbf{a}^2_{geod} + \mathbf{a}^2_{norm}}{|\mathbf{v}|^2}} \geq \frac{|\mathbf{a}_{normal}|}{|\mathbf{v}|^2} = |\kappa_n| \tag{5.35}$$

Normal curvature is a positive or negative real number. (Usual curvature is non-negative real number). Normal curvature changes a sign if $\mathbf{n} \to -\mathbf{n}$.

Remark We obtained in (5.31) that the magnitude $\frac{|\mathbf{a}_{geod}|}{|\mathbf{v}|^2}$ is reparameterisation invariant. It defines so called *geodesic curvature* $\kappa_{geod} = \frac{|\mathbf{a}_{geod}|}{|\mathbf{v}|^2}$. We see that usual curvature k, normal curvature κ and geodesic curvature κ_{geod} are related by the formula

$$k^2 = \kappa^2_{geod} + \kappa^2_{normal} \tag{5.36}$$

5.5 Normal curvature of curves on cylinder surface.

Example Consider an arbitrary curve $C\colon h = h(t), \varphi = \varphi(t)$ on the cylinder

$$\mathbf{r}(\varphi, h)\colon \begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \\ z = h \end{cases}$$

Pick any point $\mathbf{p}$ on this curve and find normal acceleration vector at this point of this curve.

Without loss of generality suppose that point $\mathbf{p}$ is just a point $(R, 0, 0)$. Note that vector $\mathbf{e}_x$ attached at the point $(R, 0, 0)$ is unit vector orthogonal to the surface of cylinder, i.e. $\mathbf{e}_x = -\mathbf{n}$ at the point $\mathbf{p} = (R, 0, 0)$.

RemarkUnit vector, as well as normal curvature is defined up to a sign. It is convenient for us to choose $\mathbf{n} = -\mathbf{e}_x$, not $\mathbf{n} = \mathbf{e}_x$.

Vectors $\mathbf{e}_y, \mathbf{e}_z$ are tangent to the surface of cylinder. At the point $\mathbf{p} = (R, 0, 0)$ $\varphi = 0, h = 0$.

We have

$$\mathbf{v} = \frac{d\mathbf{r}(t)}{dt} = \frac{dx(t)}{dt}\mathbf{e}_x + \frac{dy(t)}{dt}\mathbf{e}_y + \frac{dz(t)}{dt}\mathbf{e}_z =$$

$$R\frac{d\cos\varphi(t)}{dt}\mathbf{e}_x + R\frac{d\sin\varphi(t)}{dt}\mathbf{e}_y + \frac{dh(t)}{dt}\mathbf{e}_z = -R\sin\varphi\dot\varphi\mathbf{e}_x + R\cos\varphi\dot\varphi\mathbf{e}_y + \dot h\mathbf{e}_z$$

$$\text{Thus } \mathbf{v} = R\dot\varphi\mathbf{e}_y + \dot h\mathbf{e}_z \text{ at the point } \mathbf{p} = (R,0,0)\,. \tag{5.37}$$

For acceleration vector

$$\mathbf{a} = \frac{d^2\mathbf{r}(t)}{dt^2} = \frac{d^2x(t)}{dt^2}\mathbf{e}_x + \frac{d^2y(t)}{dt^2}\mathbf{e}_y + \frac{d^2z(t)}{dt^2}\mathbf{e}_z = R\frac{d^2\cos\varphi(t)}{dt^2}\mathbf{e}_x + R\frac{d^2\sin\varphi(t)}{dt^2}\mathbf{e}_y + \frac{d^2h(t)}{dt^2}\mathbf{e}_z =$$

$$R\left(-(\dot\varphi)^2\cos\varphi - \ddot\varphi\sin\varphi\right)\mathbf{e}_x + R\left(-(\dot\varphi)^2\sin\varphi + \ddot\varphi\cos\varphi\right)\mathbf{e}_y + \ddot h\mathbf{e}_z = \ddot\varphi R\mathbf{e}_y + \ddot h\mathbf{e}_z - (\dot\varphi)^2R\mathbf{e}_x$$

at the point $\mathbf{p} = (R,0,0)$ where $\cos\varphi = 0, \sin\varphi = 1$. We see that

$$\mathbf{a} = \underbrace{\ddot\varphi R\mathbf{e}_y + \ddot h\mathbf{e}_z}_{\text{tangent to the surface}} - \underbrace{(\dot\varphi)^2R\mathbf{e}_x}_{\text{normal to the surface}} \tag{5.38}$$

We see that $\mathbf{a}_n = (\dot\varphi)^2R\mathbf{e}_x$. Comparing with velocity vector (5.37) we see that

$$\mathbf{a}_n = \frac{\mathbf{v}^2_{horizontal}}{R}\mathbf{n} \tag{5.39}$$

We see that for any curve on the cylinder $x^2 + y^2 = R^2$ the normal curvature $\frac{(\mathbf{a}_n,\mathbf{n})}{|\mathbf{v}|^2}$ (see (4.27)) is equal to

$$\frac{(\mathbf{a}_n,\mathbf{n})}{|\mathbf{v}|^2} = \frac{R\dot\varphi^2}{R^2\dot\varphi^2 + \dot h^2} \tag{5.40}$$

and it obeys relations

$$0 \le \kappa_{normal} \le \frac{1}{R}$$

depending of the curve. E.g. if the curve on the cylinder is a straight line $x = x_0, y = y_0, z = t$ then $\mathbf{a} = 0$ and normal curvature of this curve is equal to the naught as well as usual curvature.

If the curve is circle $x = R\cos t, y = R\sin t, z = z_0$ then normal curvature of this curve as well as usual curvature is equal to $\frac{1}{R}$.

Remark Very important conclusion from this example is
normal curvature of the cylinder of the radius R takes values in the interval $\left(0, \frac{1}{R}\right)$. *It cannot be greater than* $\frac{1}{R}$
Note that we can consider on cylinder very curly curve of very big curvature. The normal curvature at the points of this curve will be still less than $\frac{1}{R}$.

At any point of the surface normal curvature in general depends on the curve but it takes values in the restricted interval.

E.g. for the sphere of radius R one can see that normal curvature at any point is equal to $\frac{1}{R}$ independent of curve. In spite of this fact the usual curvature of curve can be very big [10]. If we consider the circle of very small radius r on the sphere then its usual curvature is equal to $k = \frac{1}{r}$ and $k \to \infty$ if $r \to 0$ So we see that one can define curvature of surface in terms of normal curvature.

[10] It is the geodesic curvature of the curves which characterises its curvature with respect to the curve. The relation between usual geodesic and normal curvature is given by the formula (5.36).

5.6 Concept of parallel transport

Parallel transport of the vectors is one of the fundamental concept of differential geometry. Here we just give some preliminary ideas and formulate the concept of parallel transport for surfaces embedded in Euclidean space.

Let M be a surface $\mathbf{r} = \mathbf{r}(u, v)$ in $\mathbf{E}^3$ and $C\colon \mathbf{r}(t) = \mathbf{r}(u(t), v(t))$, $t_1 \leq t \leq t_2$ be a curve on this surface.

Let $\mathbf{X}_1$ be a vector tangent to the surface at the initial point $p = \mathbf{r}(t_1)$ of the curve $\mathbf{r}(t)$ on the surface: $\mathbf{X}_1 \in T_pM$. Note that $\mathbf{X}_1$ is a vector tangent to the surface, not necessarily to the curve. We define now parallel transport of the vector along the curve C:

Definition Let $\mathbf{X}(t)$ be a family of vectors depending on the parameter t ($t_1 \leq t \leq t_2$) such that following conditions hold

- For every $t \in [t_1, t_2]$ vector $\mathbf{X}(t)$ is a vector tangent to the surface M at the point $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ of the curve C.
- $\mathbf{X}(t) = \mathbf{X}_1$ for $t = t_1$
- $\frac{d\mathbf{X}(t)}{dt}$ is orthogonal to the surface, i.e.

  $$\frac{d\mathbf{X}(t)}{dt} \text{ is collinear to the normal vector } \mathbf{n}(t), \quad \frac{d\mathbf{X}(t)}{dt} = \lambda(t)\mathbf{n}(t) \tag{5.41}$$

  Recall that normal vector $\mathbf{n}(t)$ is a vector attached to the point $\mathbf{r}(t)$ of the curve $C\colon \mathbf{r}(t)$. This vector is orthogonal to the surface M.

  The condition (5.41) means that only orthogonal component of vector field $\mathbf{X}(t)$ can be changed.

We say that a family $\mathbf{X}(t)$ is a parallel transport of the vector $\mathbf{X}_1$ along a curve $C\colon \mathbf{r}(t)$ on the surface M. The final vector $\mathbf{X}_2 = \mathbf{X}(t_2)$ is the image of the vector $\mathbf{X}_1$ under the parallel transport along the curve C.

Using the relation (5.41) it is easy to see that the scalar product of two vectors remains invariant under parallel transport. In particularly it means that length of the vector does not change. If $\mathbf{X}(t)$, $\mathbf{Y}(t)$ are parallel transports of vectors $\mathbf{X}_1, \mathbf{Y}_1$ then

$$\frac{d}{dt}\left(\mathbf{X}(t), \mathbf{Y}(t)\right) = \left(\frac{d\mathbf{X}(t)}{dt}, \mathbf{Y}(t)\right) + \left(\mathbf{X}(t), \frac{d\mathbf{Y}(t)}{dt}\right) = 0$$

because vector $\frac{d\mathbf{X}(t)}{dt}$ is orthogonal to the vector $\mathbf{Y}(t)$ and vector $\frac{d\mathbf{Y}(t)}{dt}$ is orthogonal to the vector $\mathbf{X}(t)$. In particularly length does not change:

$$\frac{d}{dt}|\mathbf{X}(t)|^2 = \frac{d}{dt}(\mathbf{X}(t), \mathbf{X}(t)) = 2\left(\frac{d\mathbf{X}(t)}{dt}, \mathbf{X}(t)\right) = 2(\lambda(t)\mathbf{n}(t), \mathbf{X}(t)) = 0 \tag{5.42}$$

Remark The relation (5.41) shows how the surface is engaged in the parallel transport. Note that it is non-sense to put the right hand side of the equation (5.41) equal to zero: In general a tangent vector ceased to be tangent to the surface if it is not changed! (E.g. consider the vector which transports along the great circle on the sphere)

We consider an example of parallel transport of vectors along meridians in the sphere and equator and more interesting examples and very important Theorems in the last Appendices.

5.7 Parallel transport of vectors tangent to the sphere.

1. In the case if surface is a plane then everything is easy. If vector $\mathbf{X}_1$ is tangent to the plane at the given point, it is tangent at all the points. Vector does not change under parallel transport $\mathbf{X}(t) \equiv \mathbf{X}$.

Consider a case of parallel transport along curves on the sphere.

Consider on the sphere $x^2 + y^2 + z^2 = a^2$ (a is a radius) tangent vectors:

$$\mathbf{r}_\theta = \begin{pmatrix} a\cos\theta\cos\varphi \\ a\cos\theta\sin\varphi \\ -a\sin\theta \end{pmatrix} \quad \mathbf{r}_\varphi = \begin{pmatrix} -a\sin\theta\sin\varphi \\ a\sin\theta\cos\varphi \\ 0 \end{pmatrix} \tag{5.43}$$

attached at the point $\mathbf{r}(\theta,\varphi) = \begin{pmatrix} a\sin\theta\cos\varphi \\ a\sin\theta\sin\varphi \\ a\cos\theta \end{pmatrix}$. One can see that

$$(\mathbf{r}_\theta, \mathbf{r}_\theta) = a, \quad (\mathbf{r}_h, \mathbf{r}_\varphi) = 0, \quad (\mathbf{r}_\varphi, \mathbf{r}_\varphi) = a^2\sin^2\theta$$

It is convenient to introduce vectors which are parallel to these vectors but have unit length:

$$\mathbf{e}_\theta = \frac{\mathbf{r}_\theta}{a}, \quad \mathbf{e}_\varphi = \frac{\mathbf{r}_\varphi}{a\sin\theta} \qquad (\mathbf{e}_\theta,\mathbf{e}_\theta) = 1, (\mathbf{e}_\theta,\mathbf{e}_\varphi) = 0, (\mathbf{e}_\varphi,\mathbf{e}_\varphi) = 1\,. \tag{5.44}$$

How these vectors change if we move along parallel (i.e. what is the value of $\frac{\partial \mathbf{e}_\theta}{\partial\varphi}, \frac{\partial \mathbf{e}_\varphi}{\partial\varphi}$); how these vectors change if we move along meridians (i.e. what is the value of $\frac{\partial \mathbf{e}_\theta}{\partial\theta}, \frac{\partial \mathbf{e}_\varphi}{\partial\theta}$). First of all recall that unit normal vector to the sphere at the point θ,φ is equal to $\frac{\mathbf{r}(\theta,\varphi)}{a}$:

$$\mathbf{n}(\theta,\varphi) = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

Now calculate:

$$\frac{\partial \mathbf{e}_\theta}{\partial\theta} = \frac{\partial}{\partial\theta}\begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} = \begin{pmatrix} -\sin\theta\cos\varphi \\ -\sin\theta\sin\varphi \\ -\cos\theta \end{pmatrix} = -\mathbf{n} \tag{5.45}$$

,

$$\frac{\partial \mathbf{e}_\theta}{\partial\varphi} = \frac{\partial}{\partial\varphi}\begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} = \begin{pmatrix} -\cos\theta\sin\varphi \\ \cos\theta\cos\varphi \\ 0 \end{pmatrix} = \cos\theta\mathbf{e}_\varphi, \tag{5.46}$$

,

$$\frac{\partial \mathbf{e}_\varphi}{\partial\theta} = \frac{\partial}{\partial\theta}\begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} -\cos\theta\sin\varphi \\ \cos\theta\cos\varphi \\ 0 \end{pmatrix} = 0, \tag{5.47}$$

$$\frac{\partial \mathbf{e}_\varphi}{\partial\varphi} = \frac{\partial}{\partial\varphi}\begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} -\cos\varphi \\ -\sin\varphi \\ 0 \end{pmatrix} = -\sin\theta\mathbf{n} - \cos\theta\mathbf{e}_\theta, \tag{5.48}$$

Some of these formulaes are intuitively evident: For example formula (5.45) which means that family of the vectors $\mathbf{e}_\theta(\theta)$ is just parallel transport along meridian, because its derivation is equal to $-\mathbf{n}$.

Another intuitively evident example: consider the meridian $\theta(t) = t$, $\varphi(t) = \varphi_0$, $0 \le t \le \pi$. It is easy to see that the vector field

$$\mathbf{X}(t) = \mathbf{e}_\theta(\theta(t), \varphi_0) = \begin{pmatrix} \cos\theta(t)\cos\varphi_0 \\ \cos\theta(t)\sin\varphi_0 \\ -\sin\theta(t) \end{pmatrix}$$

attached at the point $(\theta(t), \varphi_0)$ is a parallel transport because for family of vectors $\mathbf{X}(t)$ all the conditions of parallel transport are satisfied. In particular according to (5.45)

$$\frac{d\mathbf{X}(t)}{dt} = \frac{d\theta(t)}{dt}\frac{\partial}{\partial\theta}\begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} = -\mathbf{n}(\theta(t), \varphi_0)$$

Now consider an example which is intuitively not-evident.

Example. Calculate parallel transport of the vector $\mathbf{e}_\varphi$ along the parallel. On the sphere of the radius a consider the parallel

$$\theta(t) = \theta_0, \varphi(t) = t, \quad 0 \le t \le 2\pi \tag{5.49}$$

In cartesian coordinates equation of parallel will be:

$$\mathbf{r}(t) = \begin{pmatrix} a\sin\theta(t)\cos\varphi(t) \\ a\sin\theta(t)\sin\varphi(t) \\ -a\cos\theta(t) \end{pmatrix} = \begin{pmatrix} a\sin\theta_0\cos t \\ a\sin\theta(t)\sin t \\ -a\cos\theta_0 \end{pmatrix}, \quad 0 \le t \le 2\pi \tag{5.50}$$

It is easy to see that the family of the vectors $\mathbf{e}_\varphi(\theta_0, \varphi(t))$ on parallel, is not parallel transport! because $\frac{d\mathbf{e}_\varphi(\theta_0,\varphi(t))}{dt} = \frac{d\mathbf{e}_\varphi(\theta_0,\varphi)}{d\varphi}$ is not equal to zero (see (5.48) above). Let a family of vectors $\mathbf{X}(t)$ be a parallel transport of the vector $\mathbf{e}_\varphi$ along the parallel (5.49): $\mathbf{X}(t) = a(t)\mathbf{e}_\theta(t) + b(t)\mathbf{e}_\varphi(t)$ where $a(t), b(t)$ are components of the tangent vector $\mathbf{X}(t)$ with respect to the basis $\mathbf{e}_\theta, \mathbf{e}_\varphi$ at the point $\theta = \theta_0, \varphi = t$ on the sphere. Initial conditions for coefficients are $a(t)|_{t=0} = 0, b(t)|_{t=0} = 1$ According to the definition of parallel transport and formulae (5.45)—(5.48) we have:

$$\frac{d\mathbf{X}(t)}{dt} = \frac{d\left(a(t)\mathbf{e}_\theta(t) + b(t)\mathbf{e}_\varphi(t)\right)}{dt} = \left(\frac{da(t)}{dt}\right)\mathbf{e}_\theta + a(t)\cos\theta_0\mathbf{e}_\varphi + \frac{db(t)}{dt}\mathbf{e}_\varphi +$$

$$b(t)\left(-\sin\theta_0\mathbf{n} - \cos\theta\mathbf{e}_\theta\right) =$$

$$= \left(\frac{da(t)}{dt} - b(t)\cos\theta_0\right)\mathbf{e}_\theta + \left(\frac{db(t)}{dt} + a(t)\cos\theta_0\right)\mathbf{e}_\varphi - b(t)\sin\theta_0\mathbf{n} \tag{5.51}$$

Under parallel transport only orthogonal component of the vector changes. Hence we come to differential equations

$$\begin{cases} \frac{da(t)}{dt} - wb(t) = 0 \\ \frac{db(t)}{dt} + wa(t) \end{cases} \qquad a(0) = 0, b(0) = 0, w = \cos\theta_0 \tag{5.52}$$

The solution of these equations is $a(t) = \sin wt, b(t) = \cos wt$. We come to the following answer: parallel transport along parallel $\theta = \theta_0$ of the initial vector $\mathbf{e}_\varphi$ is the family

$$\mathbf{X}(t) = \sin wt\,\mathbf{e}_\theta + \cos wt\,\mathbf{e}_\varphi, w = \cos\theta_0 \tag{5.53}$$

During traveling along the parallel $\theta = \theta_0$ the $\mathbf{e}_\theta$ component becomes non-zero At the end of the traveling the initial vector $\mathbf{X}(t)|_{t=0} = \mathbf{e}_\varphi$ becomes $\mathbf{X}(t)|_{t=2\pi} = \sin 2\pi w \mathbf{e}_\theta + \cos 2\pi w \mathbf{e}_\varphi$: **the vector $\mathbf{e}_\varphi$ after woldtrip traveling along the parallel $\theta = \theta_0$ transforms to the vector $\sin(2\pi\cos\theta_0)\mathbf{e}_\theta + \cos(2\pi\cos\theta_0)\mathbf{e}_\varphi$. In particularly this means that the vector $\mathbf{e}_\varphi$ after parallel transport will rotate on the angle**

$$\text{angle of rotation} = 2\pi\cos\theta_0$$

Compare the angle of rotation with the area of the segment of the sphere above the parallel $\theta = \theta_0$. According to the formula (**??**) area of this segment is equal to $S = 2\pi a h = 2\pi a^2(1-\cos\theta_0)$. On the other hand Gaussian curvature of the sphere is equal to $\frac{1}{a^2}$. Hence we see that up to the sign angle of rotation is equal to area of the seqment divided on the Gaussian curvature:

$$\Delta\varphi = \pm\frac{S}{K} = \pm 2\pi\cos\theta_0 \tag{5.54}$$

5.8 Parallel transport along a closed curve on arbitrary surface.

The formula above for the parallel transport along parallel on the sphere keeps in the general case.

Theorem Let M be a surface in $\mathbf{E}^3$. Let $\mathbf{r}(t)\colon \mathbf{r}(t), t_1 \le t \le t_2, \mathbf{r}(t_1) = \mathbf{r}(t_2)$ be a closed curve on the surface M such that it is a boundary of domain D of the surface M. (We suppose that the domain D is bounded an orientate.) Let $\mathbf{X}(t)$ be a parallel transport of the arbitrary tangent vector along this closed curve. Consider initial and final vectors $\mathbf{X}(t_1), \mathbf{X}(t_2)$. They have the same length according to (5.42).

Theorem *The angle $\Delta\varphi$ between these vectors is equal to the integral of Gaussian curvature over the domain D:*

$$\Delta\varphi = \pm\int_D K d\sigma \tag{5.55}$$

where we denote by $d\sigma$ the element of the area of surface of M.

The calculations above for traveling along the parallel are just example of this Theorem. The integral of Gaussian curvature over the domain above parallel $\theta = \theta_0$ is equal to $K\cdot 2\pi a(1-\cos\theta_0) = \frac{1}{a^2}\cdot 2\pi a^2(1-\cos\theta_0) = 2\pi(1-\cos\theta_0)$. This is equal to the angle of rotation $2\pi\cos\theta_0$ (up to a sign and modulo 2π). Another simple

Example. Consider on the sphere $x^2+y^2+z^2 = a^2$ points $A = (0,0,1)$, $B = (1,0,0)$ and $C = (0,1,0)$. Consider arcs of great circles which connect these points. Consider the vector $\mathbf{e}_x$ attached at the point A. This vector is tangent to the sphere. It is easy to see that under parallel transport along the arc AB it will transform at the point B to the vector $-\mathbf{e}_z$. The vector $-\mathbf{e}_z$ under parallel transport along the arc BC will remain the same vector $-\mathbf{e}_z$. And finally under parallel transport along the arc CA the vector $-\mathbf{e}_z$ will transform at the point A to the vector $-\mathbf{e}_y$. We see that under traveling along the curvilinear triangle ABC vector $\mathbf{e}_x$ becomes the vector $-\mathbf{e}_y$, i.e. it rotates on the angle $\frac{\pi}{2}$. It is just the integral of the curvature $\frac{1}{a^2}$ over the triangle ABC: $K \cdot S = \frac{1}{a^2} \cdot \frac{4\pi a^2}{8} = \frac{\pi}{2}$.

We know that for plnar triangles sum of the angles is equal to π. It turns out that

Corollary Let ABC be a triangle on the surface formed by geodesics. Then

$$\angle A + \angle B + \angle C = \pi + \int_{\triangle ABC} K ds \tag{5.56}$$

The Gaussian curvature measures the difference of π and sum of angles.

The corollary evidently follows form the Theorem. It is of great importance: It gives us tool to measure curvature. (See the tale about ant.)

5.9 Gauss Bonnet Theorem

Consider the integral of curvature over whole closed surface M. According to the Theorem above the answer has to be equal to 0 (modulo 2π), i.e. $2\pi N$ where N is an integer, because this integral is a limit when we consider very small curve. We come to the formula:

$$\int_D K d\sigma = 2\pi N$$

(Compare this formula with formula (5.27)).

What is the value of integer N?

We present now one remarkable Theorem which answers this question and prove this Theorem using the formula (5.56).

Let M be a closed orientable surface.[11] All these surfaces can be classified up to a diffeomorphism. Namely arbitrary closed oriented surface M is diffeomorphic either to sphere (zero holes), or torus (one hole), or pretzel (two holes),... "Number k" of holes is intuitively evident characteristic of the surface. It is related with very important characteristic—Euler characteristic $\chi(M)$ by the following formula:

$$\chi(M) = 2(1 - g(M)), \quad \text{where } g \text{ is number of holes} \tag{5.57}$$

Remark What we have called here "holes" in a surface is often referred to as "handles" attached o the sphere, so that the sphere itself does not have any handles, the torus has one handle, the pretzel has two handles and so on. The number of handles is also called genus.

Euler characteristic appears in many different way. The simplest appearance is the following:

---

[11] Closed means compact surface without boundaries. Intuitively orientability means that one can define out and inner side of the surface. In terms of normal vectors orientability means that one can define the continuous field of normal vectors at all the points of M. The direction of normal vectors at any point defines outward direction. Orientable surface is called oriented if the direction of normal vector is chosen.

Consider on the surface M an arbitrary set of points (vertices) connected with edges (graph on the surface) such that surface is divided on polygons with (curvilinear sides)—plaquets. ("Map of world")

Denote by P number of plaquets (countries of the map)

Denote by E number of edges (boundaries between countries)

Denote by V number of vertices.

Then it turns out that

$$P - E + V = \chi(M) \tag{5.58}$$

It does not depend on the graph, it depends only on how much holes has surface.

E.g. for every graph on M, $P - E + V = 2$ if M is diffeomorphic to sphere. For every graph on M $P - E + V = 0$ if M is diffeomorphic to torus.

Now we formulate Gauß-Bonnet Theorem.

Let M be closed oriented surface in $\mathbf{E}^3$.

Let $K(p)$ be Gaussian curvature at any point p of this surface.

Recall that sign of Gaussian curvature does not depend on the orientation. If we change direction of normal vector $\mathbf{n} \to -\mathbf{n}$ then both principal curvatures change the sign and Gaussian curvature $K = \det A / \det G$ does not change the sign [12].

Theorem (Gauß -Bonnet) The integral of Gaussian curvature over the closed compact oriented surface M is equal to 2π multiplied by the Euler characteristic of the surface M

$$\frac{1}{2\pi} \int_M K d\sigma = \chi(M) = 2(1 - \text{number of holes}) \tag{5.59}$$

In particular for the surface M diffeomorphic to the sphere $\kappa(M) = 2$, for the surface diffeomorphic to the torus it is equal to 0.

The value of the integral does not change under continuous deformations of surface! It is integer number (up to the factor π) which characterises topology of the surface.

E.g. consider surface M which is diffeomorphic to the sphere. If it is sphere of the radius R then curvature is equal to $\frac{1}{R^2}$, area of the sphere is equal to $4\pi R^2$ and left hand side is equal to $\frac{4\pi}{2\pi} = 2$.

If surface M is an arbitrary surface diffeomorphic to M then metrics and curvature depend from point to the point, Gauß-Bonnet states that integral nevertheless remains unchanged.

Very simple but impressive corollary:

Let M be surface diffeomorphic to sphere in $\mathbf{E}^3$. Then there exists at least one point where Gaussian curvature is positive.

---

[12] For an arbitrary point p of the surface M one can always choose cartesian coordinates (x, y, z) such that surface in a vicinity of this spoint is defined by the equation $z = ax^2 + bx^2 + \dots$, where dots means terms of the order higher than 2. Then Gaussian curvature at this point will be equal to ab. If a, b have the same sign then a surfaces looks as paraboloid in the vicinity of the point p. If If a, b have different signs then a surfaces looks as saddle in the vicinity of the point p. Gaussian curvature is positive if $ab > 0$ (case of paraboloid) and negative if $ab < 0$ saddle

Proof: Suppose it is not right. Then $\int_M K\sqrt{\det g}dudv \leq 0$. On the other hand according to the Theorem it is equal to 4π. Contradiction.

In the first section in the subsection "Integrals of curvature along the plane curve" we proved that the integral of curvature over closed convex curve is equal to 2π. This Theorem seems to be "ancestor" of Gauß-Bonnet Theorem[13].

Proof of Gauß-Bonet Theorem

Consider triangulation of the surface M. Suppose M is covered by N triangles. Then number of edges will be $3N/over2$. If V number of vertices then according to Euler Theorem

$$N - \frac{3N}{2} + V = V - \frac{N}{2} = \chi(M).$$

Calculate the sum of the angles of all triangles. On the one hand it is equal to $2\pi V$. On the other hand according the formula (5.56) it is equal to

$$\sum_{i=1}^{N}\left(\pi + \int_{\triangle_i} K d\sigma\right) = \pi N + \sum_{i=1}^{N}\left(\int_{\triangle_i} K d\sigma\right) = N\pi + \int_M K d\sigma$$

We see that $2\pi V = N\pi + \int_M K d\sigma$, i.e.

$$\int_M K d\sigma = \pi\left(2V - \frac{N}{2}\right) = 2\pi\chi(M) \blacksquare$$

5.10 Theorema Egregium

The Gaussian curvature of surface is defined by first quadratic form.

If two surfaces have the same quadratic form then they have the same Gaussian curvature.

This Theorem explains why sphere cannot be transformed to the plane without deformation.

(Cylinder and cone can be transformed to the plane, their Gaussian curvature equals to zero.)

[13]Note that there is a following deep difference: Gaussian curvature is internal property of the surface: it does not depend on isometries of surface. Curvature of curve depends on the position of the curve in ambient space.