Homework 4. Solutions

1 *Consider an operator P on $\mathbf{E}^3$ such that P is an orthogonal operator preserving orientation of $\mathbf{E}^3$ and*

$$P(\mathbf{e}_x) = \mathbf{e}_y, P(\mathbf{e}_z) = -\mathbf{e}_z\,.$$

Find an action of operator P on an arbitrary vector $\mathbf{x} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$.

Why P is a rotation operator? Find an angle and axis of the rotation.

Solution. Calculate first $P(\mathbf{e}_y)$. P is orthogonal operator and unit vector $\mathbf{e}_y$ is orthogonal to vectors $\mathbf{e}_x, \mathbf{e}_z$. Hence vector $P(\mathbf{e}_y)$ is unit vector also and $P(\mathbf{e}_y)$ MUST be orthogonal to vectors $P(\mathbf{e}_x)$ and $P(\mathbf{e}_z)$:

$$(P(\mathbf{e}_y), P(\mathbf{e}_y)) = (\mathbf{e}_y, \mathbf{e}_y) = 1,\ (P(\mathbf{e}_y), P(\mathbf{e}_x)) = (\mathbf{e}_y, \mathbf{e}_x) = 0,\ (P(\mathbf{e}_y), P(\mathbf{e}_z)) = (\mathbf{e}_y, \mathbf{e}_z) = 0,$$

Since $P(\mathbf{e}_x) = \mathbf{e}_y$ and $P(\mathbf{e}_z) = -\mathbf{e}_z$ we see that vector $P(\mathbf{e}_y)$ has to be proportional to vector $\mathbf{e}_x$: $P(\mathbf{e}_y) = c\mathbf{e}_x$ with $c = \pm 1$ since the length of the vector $P(\mathbf{e}_y)$ is equal to 1. Calculate c. We already know that $c = \pm 1$. The triple $\{P(\mathbf{e}_x), P(\mathbf{e}_y), P(\mathbf{e}_z)\}$ has the same orientation as the triple $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ since operator P preserves orientation. We have

$$\{P(\mathbf{e}_x), P(\mathbf{e}_y), P(\mathbf{e}_z)\} = \{c\mathbf{e}_y, \mathbf{e}_x, -\mathbf{e}_z\} \sim \{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)\} \Rightarrow c = 1\,.$$

We see that $P(\mathbf{e}_y) = \mathbf{e}_x$. Hence for an arbitrary vector $\mathbf{x} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$

$$P(\mathbf{x}) = P(x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z) = x\mathbf{e}_y + y\mathbf{e}_x - z\mathbf{e}_z : \quad P\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ x \\ -z \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

To find an angle and axis of rotations you may see the example just at the end of the subsection 1.10 of the lecture notes, or do it again in a little bit other way: Find a vector directed along axis. This is eigenvector with eigenvalue 1:

$$P\mathbf{N} = \mathbf{N}\,, P\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ x \\ -z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \lambda \\ \lambda \\ 0 \end{pmatrix}.$$

i.e. N is proportional to the vector $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ ($\mathbf{N} = \lambda(\mathbf{e}_x + \mathbf{e}_y)$). The axis of rotation is the bisectrices of the angle between x and y axis. To find an angle of rotation we calculate $\mathrm{Tr}P = 1 + 2\cos\varphi = -1$. Hence angle of the rotation is equal to π.

We may calculate the angle of rotation in other way too: consider an arbitrary vector $\mathbf{x} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$ which is orthogonal to axis: $(\mathbf{x}, \mathbf{N}) = x + y$ (if $\mathbf{N} = \mathbf{e}_x + \mathbf{e}_y$). Hence vectors orthogonal to axis have appearance $\mathbf{x} = a(\mathbf{e}_x - \mathbf{e}_y) + b\mathbf{e}_z$ (x-component + y component equals zero.) We have that

$$\text{for } \mathbf{x} = a(\mathbf{e}_x - \mathbf{e}_y) + b\mathbf{e}_z P(\mathbf{x}) = \mathbf{x} = a(\mathbf{e}_y - \mathbf{e}_x) - b\mathbf{e}_z = -\mathbf{x}.$$

We see that any vector orthogonal to axis is multiplied on -1. Thus P is rotation on the angle π.

2 *Consider an operator P on $\mathbf{E}^3$ such that*

$$P(\mathbf{e}) = \frac{2}{3}\mathbf{e} + \frac{2}{3}\mathbf{f} + \frac{1}{3}\mathbf{g}\,, \quad P(\mathbf{f}) = -\frac{1}{3}\mathbf{e} + \frac{2}{3}\mathbf{f} - \frac{2}{3}\mathbf{g}\,, \quad P(\mathbf{g}) = -\frac{2}{3}\mathbf{e} + \frac{1}{3}\mathbf{f} + \frac{2}{3}\mathbf{g}\,.$$

Show that this is an orthogonal operator preserving orientation of $\mathbf{E}^3$.

Find an axis of rotation (i.e. a vector $\mathbf{N} \neq 0$ which is directed along the axis.) and an angle of rotation (up to a sign). (We assume that $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ is an orthonormal basis in $\mathbf{E}^3$.)

Solution It is easy to see that

$$
\begin{array}{rr}
(\mathbf{e}',\mathbf{e}') = (P(\mathbf{e}),P(\mathbf{e})) = & \left(\frac{2}{3}\mathbf{e}+\frac{2}{3}\mathbf{f}+\frac{1}{3}\mathbf{g}, \frac{2}{3}\mathbf{e}+\frac{2}{3}\mathbf{f}+\frac{1}{3}\mathbf{g}\right) = \quad 1,\\
(\mathbf{e}',\mathbf{f}') = (P(\mathbf{e}),P(\mathbf{f})) = & \left(\frac{2}{3}\mathbf{e}+\frac{2}{3}\mathbf{f}+\frac{1}{3}\mathbf{g}, -\frac{1}{3}\mathbf{e}+\frac{2}{3}\mathbf{f}-\frac{1}{3}\mathbf{g}\right) = \quad 0,\\
(\mathbf{e}',\mathbf{g}') = (P(\mathbf{e}),P(\mathbf{g})) = & \left(\frac{2}{3}\mathbf{e}+\frac{2}{3}\mathbf{f}+\frac{1}{3}\mathbf{g}, -\frac{2}{3}\mathbf{e}+\frac{1}{3}\mathbf{f}+\frac{2}{3}\mathbf{g}\right) = \quad 0,\\
(\mathbf{f}',\mathbf{f}') = (P(\mathbf{f}),P(\mathbf{f})) = & \left(-\frac{1}{3}\mathbf{e}+\frac{2}{3}\mathbf{f}-\frac{2}{3}\mathbf{g}, -\frac{1}{3}\mathbf{e}+\frac{2}{3}\mathbf{f}-\frac{2}{3}\mathbf{g}\right) = \quad 1,\\
(\mathbf{f}',\mathbf{g}') = (P(\mathbf{f}),P(\mathbf{g})) = & \left(-\frac{1}{3}\mathbf{e}+\frac{2}{3}\mathbf{f}-\frac{2}{3}\mathbf{g}, -\frac{2}{3}\mathbf{e}+\frac{1}{3}\mathbf{f}+\frac{2}{3}\mathbf{g}\right) = \quad 1,\\
(\mathbf{g}',\mathbf{g}') = (P(\mathbf{g}),P(\mathbf{g})) = & \left(-\frac{2}{3}\mathbf{e}+\frac{1}{3}\mathbf{f}+\frac{2}{3}\mathbf{g}, -\frac{2}{3}\mathbf{e}+\frac{1}{3}\mathbf{f}+\frac{2}{3}\mathbf{g}\right) = \quad 1
\end{array}
$$

new basis is orthonormal one. Hence P is orthogonal operator. The matrix of operator P is

$$
\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{pmatrix}
$$

Its determinant equals $\det P = 1$. Operator P preserves orientation. To find an axis we have to find eigenvector of this matrix with eigenvalue 1. Eigenvalue equals 1, since this is rotation: We have

$$
P\mathbf{N} = \mathbf{N}, \quad \frac{1}{3}\begin{pmatrix} 2 & -1 & -2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.
$$

Solving these equations we come to $x = y = -z$, i.e. $\mathbf{N}$ is proportional to the vector $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. Axis is directed along the vector $\mathbf{N} = \mathbf{e} + \mathbf{f} - \mathbf{g}$.

Trace of the operator P is equal to $\operatorname{Tr} P = \frac{1}{3}(2+2+2) = 2 = 1+2\cos\varphi$. Hence $\cos\varphi = \frac{1}{2}$, i.e. $\varphi = \pm\frac{\pi}{3}$.

3 *Consider on* $\mathbf{E}^3$ *following two operators:*

$$
P_1(\mathbf{x}) = \mathbf{x} - 2(\mathbf{n},\mathbf{x})\mathbf{n}, \quad P_2(\mathbf{x}) = 2(\mathbf{n},\mathbf{x})\mathbf{n} - \mathbf{x},
$$

where $\mathbf{n}$ *is a unit vector.*

Show that these both operators are orthogonal operators. Show that first operator changes the orientation, and the second operator preserves orientation.

Show that the first operator is reflection operator with respect to....

Show that the second operator is rotation operator: find an axis of rotation and an angle of rotation.

One can see that for operator P_1

$$
P_1(\mathbf{n}) = \mathbf{n} - 2(\mathbf{n},\mathbf{n})\mathbf{n} = -\mathbf{n}
$$

since $(\mathbf{n},\mathbf{n}) = 1$, and for an arbitrary vector $\mathbf{y}$ belonging to the plane $\alpha_{\mathbf{n}}$ which is orthogonal to the vector $\mathbf{n}$

$$
P_1(\mathbf{y}) = \mathbf{y} - 2(\mathbf{n},\mathbf{y})\mathbf{n} = \mathbf{y}
$$

since $(\mathbf{n},\mathbf{y}) = 0$. We see that operator P is identical on the plane orthogonal to $\mathbf{n}$ and $\mathbf{n} \mapsto -\mathbf{n}$. Hence P_1 is orthogonal operator and it is reflection operator with respect to the plane $\alpha_{\mathbf{n}}$.

Operator $P_2 = -P_1$, Operator P_1 changes orientation, hence operator P_2 preserves orientation: $\det P_2 = -\det P_1 > 0$. Hence it is orthogonal operator, preserving orientation i.e. rotation operator. One can see that

$$
P_2(\mathbf{n}) = -P_1(\mathbf{n}) = \mathbf{n}
$$

Axis of this operator is directed along vector $\mathbf{n}$. Arbitrary vector $\mathbf{y}$ belonging to the plane α_y is multiplied on -1 (is eigenvector with eigenvalue -1):

$$
P_2(\mathbf{y}) = -P_1(\mathbf{y}) = -\mathbf{y}
$$

This means that plane $\alpha_{\mathbf{n}}$ rotates on the angel π.

Remark One can do it using brute force: calculate the matrix of operator and convince that its determinant equals 1. But calculations in general case are long and boring: Matrix of operator P_2 is

$$\begin{pmatrix} 2n_x^2 - 1 & n_x n_y & n_x n_z \\ n_y n_x & 2n_y^2 - 1 & n_y n_z \\ n_z n_x & n_z n_y & 2n_z^2 - 1 \end{pmatrix}$$

, where $\mathbf{n} = (n_x, n_y, n_z)$!?.). Vector $\mathbf{n} = (n_x, n_y, n_z)$ is eigenvector with eigenvalue 1 and Trace of this matrix is equal to 1. Thus we will come to the same answer, but claculations are really much more complicated...

4* *Let* $\mathbf{e}, \mathbf{f}$ *be two distinct unit vectors. Let* P *be an operator such that*

$$P(\mathbf{e}) = \mathbf{f}, \quad P(\mathbf{f}) = \mathbf{e}.$$

Find orthogonal operators P *which obey these condition.*

Hint: One of them is reflection operator, another rotation operator...

Notice that vector $\mathbf{e}+\mathbf{f}$ and $\mathbf{e}-\mathbf{f}$ are orthogonal to each other. Consider the unit vector $\mathbf{n}_{(1)}$ which is proportional to the vector $\mathbf{e}+\mathbf{f}$ which is directed along the bisectrices, the unit vector $\mathbf{n}_{(2)}$ which is proportional to the vector $\mathbf{e}-\mathbf{f}$, and finally the unit vector $\mathbf{n}_{(3)}$ which is proportional to the vector $\mathbf{e}\times\mathbf{f}$. It is evident that these vector form an orthonormal basis. Since $P(\mathbf{e}) = \mathbf{f}$ and $P(\mathbf{f}) = \mathbf{e}$ hence $P(\mathbf{n}_{(1)}) = \mathbf{n}_{(1)}$ and $P_{(2)} = -\mathbf{n}_{(2)}$. Hence $P(\mathbf{n}_{(3)}) = \pm\mathbf{n}_{(3)}$. If $P(\mathbf{n}_{(3)}) = \mathbf{n}_{(3)}$ then orthogonal operator P is reflection operator. If $P(\mathbf{n}_{(3)}) = -\mathbf{n}_{(3)}$ it is rotation operator along the axis $\mathbf{n}_{(1)}$ on the angle π.

5 *Let* $\mathbf{n}$ *be a unit vector. Consider linear operator*

$$P(\mathbf{x}) = \mathbf{n}\times\mathbf{x} + (\mathbf{n},\mathbf{x})\mathbf{n}\,.$$

Show that this is rotation operator. Find the axis and angle of rotation.

Choose an orthonormal basis $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$ such that vector $\mathbf{g} = \mathbf{n}$ and vectors $\mathbf{e}$ and $\mathbf{f}$ are orthornomal to the vector $\mathbf{n}$. (Suppose that orientation this basis defines the orientation) We have

$$P(\mathbf{e}) = \mathbf{n}\times\mathbf{e}+(\mathbf{n},\mathbf{e})\mathbf{n} = \mathbf{n}\times\mathbf{e} = \mathbf{f}, \quad P(\mathbf{f}) = \mathbf{n}\times\mathbf{f}+(\mathbf{n},\mathbf{f})\mathbf{n} = \mathbf{n}\times\mathbf{f} = -\mathbf{e}, \quad \mathbf{n}\times\mathbf{g}+(\mathbf{n},\mathbf{g})\mathbf{n} = \mathbf{n}\times\mathbf{n}+(\mathbf{n},\mathbf{n})\mathbf{n} = \mathbf{n}$$

We see that P transforms orthonromal basis $\{\mathbf{e},\mathbf{f},\mathbf{g},\}$ to orthonormal basis $\{\mathbf{f},-\mathbf{e},\mathbf{n}\}$. Hence this is rotation around axis directed along the vector $\mathbf{n}$ on the angle $\partial\backslash 2$. Changing the orientation of the basis $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$ as well changing the orientation of the axis will change the sign of the angle

6 Students John and Sarah calculate vector product $\mathbf{a}\times\mathbf{b}$ of two vectors using two different orthonormal bases in the Euclidean space $\mathbf{E}^3$, $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ and $\{\mathbf{e}'_1,\mathbf{e}'_2,\mathbf{e}'_3\}$. John expands the vectors with respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$. Sarah expands the vectors with respect to the basis $\{\mathbf{e}'_1,\mathbf{e}'_2,\mathbf{e}'_3\}$. For two arbitrary vectors $\mathbf{a},\mathbf{b}\in\mathbf{E}^3$

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = a'_1\mathbf{e}'_1 + a'_2\mathbf{e}'_2 + a'_3\mathbf{e}'_3\,,$$

$$\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 = b'_1\mathbf{e}'_1 + b'_2\mathbf{e}'_2 + b'_3\mathbf{e}'_3\,.$$

John and Sarah both use so called "determinant" formula. Are their answers the same?

$$\mathbf{a}\times\mathbf{b} = \underbrace{\det\begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3\end{pmatrix}}_{\text{John's calculations}} \stackrel{?}{=} \underbrace{\det\begin{pmatrix}\mathbf{e}'_1 & \mathbf{e}'_2 & \mathbf{e}'_3\\ a'_1 & a'_2 & a'_3 \\ b'_1 & b'_2 & b'_3\end{pmatrix}}_{\text{Sarah's calculations}}$$

Solution: In the case if bases $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ and $\{\mathbf{e}'_1,\mathbf{e}'_2,\mathbf{e}'_3\}$ have the same orientation, then answer will be the same. If bases $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ and $\{\mathbf{e}'_1,\mathbf{e}'_2,e'_3\}$ have opposite orientation then the answer of John will differ from the answer of Sarah by sign. Explain why.

Let third student, say David enters, the "game". David knows that formulae of John and Sarah both obey to axioms defining vector product (see the lecture notes). Without paying attention on formulae of John and Sarah he just uses the axioms defining vector product: He will consider the direction orthogonal to the plane spanned by vectors $\mathbf{a}, \mathbf{b}$ and take the vector such that its length equals the area of parallelogram. One thing that David also have to do it is to choose the direction of this vector. It is here where the question of orientation of bases becomes crucial.

Suppose David uses an orthonoromal basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ defining the orientation, which has the same orientation as the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ which John uses.

According of the fifth axiom he chooses the direction of the vector $\mathbf{c}$ in a such way that bases $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ and $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ have the same orientation.

Now the answer is clear: if bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ (of John) and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ (of Sarah) have the same orientation then all three bases of David, John and Sarah will have the same orientation, hence all three answers will coincide: all bases $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ (calculation of vector product), $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ (David's basis) $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ (John's basis) and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ (Sarah's basis) have the same orientation.

If bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ (of John) and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ (of Sarah) have opposite orientation then answer of David will coincide with answer of John and it will have the opposite sign with answer of Sarah:

Indeed in this case the bases $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ (David's basis) $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ (John's basis) will have the same orientation, hence the bases $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ (of Sarah) will have opposite orientation. Hence calculations of vector product in the basis which Sarah is using lead to the answer $-\mathbf{c}$: in this case the bases $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ (of Sarah) will have the same orientation. *

7 *Calculate the area of parallelograms formed by the vectors* $\mathbf{a}, \mathbf{b}$ *if*

a) $\mathbf{a} = (1, 2, 3), \mathbf{b} = (1, 0, 1)$;
b) $\mathbf{a} = (2, 2, 3), \mathbf{b} = (1, 1, 1)$;
c) $\mathbf{a} = (5, 8, 4), \mathbf{b} = (10, 16, 8)$.
d) $\mathbf{a} = (3, 4, 0), \mathbf{b} = (5, 17, 0)$.

Solution

Area of parallelogram formed by the vectors $\mathbf{a}, \mathbf{b}$ is equal to the length of the vector $\mathbf{c} = \mathbf{a} \times \mathbf{b}$.

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = (a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z) \times (b_x\mathbf{e}_x + b_y\mathbf{e}_y + b_z\mathbf{e}_z) = a_xb_x\mathbf{e}_x \times \mathbf{e}_x + a_xb_y\mathbf{e}_x \times \mathbf{e}_y + a_xb_z\mathbf{e}_x \times \mathbf{e}_z +$$

$$a_yb_x\mathbf{e}_y \times \mathbf{e}_x + a_yb_y\mathbf{e}_y \times \mathbf{e}_y + a_yb_z\mathbf{e}_y \times \mathbf{e}_z + a_zb_x\mathbf{e}_z \times \mathbf{e}_x + a_zb_y\mathbf{e}_z \times \mathbf{e}_y + a_zb_z\mathbf{e}_z \times \mathbf{e}_z =$$

$$(a_xb_y - a_yb_x)\mathbf{e}_z + (a_yb_z - a_zb_y)\mathbf{e}_x + (a_zb_x - a_xb_z)\mathbf{e}_y$$

$$|\mathbf{c}| = \sqrt{(a_yb_z - a_zb_y)^2 + (a_zb_x - a_xb_z)^2 + (a_xb_y - a_yb_x)^2}$$

a) $S = |\mathbf{a} \times \mathbf{b}| = |-2\mathbf{e}_z + 2\mathbf{e}_x + 2\mathbf{e}_y|$, $S = \sqrt{4+4+4} = 2\sqrt{3}$.
b) $S = |\mathbf{a} \times \mathbf{b}|$. $\mathbf{a} \times \mathbf{b} = -\mathbf{e}_x + \mathbf{e}_y$, $S = \sqrt{1+1} = \sqrt{2}$
c) Vectors $\mathbf{a} = (5, 8, 4), \mathbf{b} = (10, 16, 8)$ are collinear, hence $\mathbf{a} \times \mathbf{b} = 0$, $S = 0$.
d) $S = |\mathbf{a} \times \mathbf{b}|$. $\mathbf{a} \times \mathbf{b} = 31\mathbf{e}_z$, $S = 31$

8 *Find a vector* $\mathbf{n}$ *such that the following conditions hold:*
1) It has a unit length
2) It is orthogonal to the vectors $\mathbf{a} = (1, 2, 3)$ *and* $\mathbf{b} = (1, 3, 2)$.
3) An ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{n}\}$ *has an orientation opposite to the orientation of the orthonormal basis* $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ *which defines the orientation of the Euclidean space.*

Solution: Consider a vector $\mathbf{N} = \mathbf{a} \times \mathbf{b}$ and a vector $\frac{\mathbf{N}}{|\mathbf{N}|}$. The vector $\mathbf{N}$ is orthogonal to vectors $\mathbf{a}, \mathbf{b}$ (vector product) and a vector $\frac{\mathbf{N}}{|\mathbf{N}|}$ is a unit vector. It remains to solve the problem of orientation. Both vectors $\pm\frac{\mathbf{N}}{|\mathbf{N}|}$ are unit vectors which are orthogonal to vectors $\mathbf{a}, \mathbf{b}$. On the other hand the ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{N}\}$ is a basis and this basis has the same orientation as a basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. This follows from the

* In the case if one of vectors equals zero and vectors do not span plane then on can see that all three students John, Sarah and David will come to the answer: zero.

axioms defining the vector product and the fact that vectors $\mathbf{N} \neq 0$, i.e. the ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{N}\}$ is a basis. Hence the ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{n}\}$ where $\mathbf{n} = -\frac{\mathbf{N}}{|\mathbf{N}|}$ has an orientation opposite to the orientation of the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$.

The vector

$$\mathbf{n} = -\frac{\mathbf{N}}{|\mathbf{N}|} = -\frac{(\mathbf{e}_x + 2\mathbf{e}_y + 3\mathbf{e}_z) \times (\mathbf{e}_x + 3\mathbf{e}_y + 2\mathbf{e}_z)}{|\mathbf{N}|} = \frac{5\mathbf{e}_x - \mathbf{e}_y - \mathbf{e}_z}{3\sqrt{3}}.$$

9 *Show that for any two vectors $\mathbf{a}, \mathbf{b} \in \mathbf{E}^3$ the following identity is satisfied*

$$(\mathbf{a}, \mathbf{a})(\mathbf{b}, \mathbf{b}) = (\mathbf{a}, \mathbf{b})^2 + (\mathbf{a} \times \mathbf{b}, \mathbf{a} \times \mathbf{b}).$$

Write down this identity in components.
Compare this identity with CBS inequality . See the problem 7 in the Homework 2.

Solution
Let θ be an angle between vectors $\mathbf{a}, \mathbf{b}$. Then

$$(\mathbf{a}, \mathbf{a})(\mathbf{b}, \mathbf{b}) = |\mathbf{a}|^2|\mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2(\cos^2\theta + \sin^2\theta) = \underbrace{|\mathbf{a}|^2|\mathbf{b}|^2\cos^2\theta}_{\text{scalar product}} + \underbrace{|\mathbf{a}|^2|\mathbf{b}|^2\sin^2\theta}_{\text{Area of parallelogram = vector product}}$$

$$= (\mathbf{a}, \mathbf{b})^2 + |\mathbf{a} \times \mathbf{b}|^2 \tag{10.1}$$

In components:

$$(a_x^2 + a_y^2 + a_z^2)(b_x^2 + b_y^2 + b_z^2) = (a_x b_x + a_y b_y + a_z b_z)^2 + (a_x b_y - a_y b_x)^2 + (a_y b_z - a_z b_y)^2 + (a_z b_x - a_x b_z)^2 \tag{10.2}$$

Notice that for $n = 2, 3$ this identity is more strong statement than CBS inequality: $(a_x^2 + a_y^2 + a_z^2)(b_x^2 + b_y^2 + b_z^2) \geq (a_x b_x + a_y b_y + a_z b_z)^2$. CBS inequality $(\mathbf{a}, \mathbf{a})(\mathbf{b}, \mathbf{b}) \geq |\mathbf{a}|^2|\mathbf{b}|^2$ follows from the identity (1.10).

The proof of the identity (10.2) becomes more complicated if we use only algebraical methods.

10 *In 2-dimensional Euclidean space $\mathbf{E}^2$ consider vectors $\mathbf{a} = (3, 2)$, $\mathbf{b} = (7, 5)$, $\mathbf{c} = (17, 12)$, $\mathbf{d} = (41, 29)$. Calculate areas of the parallelograms $\Pi(\mathbf{a}, \mathbf{b}), \Pi(\mathbf{b}, \mathbf{c})$ and $\Pi(\mathbf{c}, \mathbf{d})$.*

Solution a) $\mathbf{A}(\mathbf{a}, \mathbf{b}) = \left|\det\begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}\right| = 1.$

b) $\mathbf{A}(\mathbf{b}, \mathbf{c}) = \left|\det\begin{pmatrix} 7 & 5 \\ 17 & 12 \end{pmatrix}\right| = 84 - 85 = -1.$

c) $\mathbf{A}(\mathbf{c}, \mathbf{d}) = \left|\det\begin{pmatrix} 17 & 12 \\ 41 & 29 \end{pmatrix}\right| = 2871 - 2871 = 1.$

$A(\mathbf{x}, \mathbf{y})$ is algebraic area of parallelogram formed by vectors $\mathbf{x}, \mathbf{y}$. It is equal to area $S(\mathbf{x}, \mathbf{y})$ with positive sign if the triple $\{\mathbf{x}, \mathbf{y}, \mathbf{n}\}$ has "left" orientation and it is equal to $-S(\mathbf{x}, \mathbf{y})$ if the triple $\{\mathbf{x}, \mathbf{y}, \mathbf{n}\}$ has "right" orientation.

10† *Do you see any relations between parallelograms in the exercise above, fractions $\frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}$ and the number... $\sqrt{2}$? Can you continue the sequence of these fractions?*
(Hint: Consider the squares of these fractions.)

One can consider continued fraction $\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}}$, Consider approximations: $a_0 = 1$, $a_1 = 1 + \frac{1}{2} = \frac{3}{2}$, $a_2 = 1 + \frac{1}{2 + \frac{1}{2}} = \frac{7}{5}$, and so on we come to the sequence of fractions:

$$a_k = \frac{p_k}{q_k} \text{ where } p_0 = q_0 = 1, q_{k+1} = p_k + q_k, p_k = 2q_k + p_k.$$

One can see that $\left|\frac{p_k}{q_k} - \frac{p_{k+1}}{q_{k+1}}\right| = \frac{1}{q_k q_{k+1}}$ which is just another manifestation of the fact that the area of the parallelogram formed by the vectors $\mathbf{a} = (p_k, q_k), \mathbf{b} = p_{k+1}, q_{k+1}$ equals 1. Vectors $\mathbf{a} = (p_k, q_k), \mathbf{b} = p_{k+1}, q_{k+1}$ form the parallelograms which become longer and longer but all have the same area.