One combinatorial lemma

Let M be $n \times n$ matrix such that all diagonal entries are 0 and all other entries are ± 1:

$$M_{ij} = \begin{cases} 0 \text{ if } i = j \\ \pm 1 \text{ if } i \not j \end{cases}$$

Then for characteristic polynomial

$$\det(\lambda + M) = \sum_{k=0}^{n} a_k \lambda^{n-k} = \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} \ldots + a_{n-1} \lambda + a_n \tag{2}$$

the following relation holds:

$$a_k = C_n^k S_k = C_n^k (k+1) \ (mod 2) \, .$$

Corollary For characteristic polynomial (2)

$$C_n^k \text{ is odd and } k \text{ is even} \Rightarrow a_k \neq 0 \, .$$