Riemannian Geometry (31082, 41082, 61082)

2013

COURSEWORK

Starred questions (in slanted font) are for the 15 credit version

This assignment counts for 20% of your marks.

Solutions are due by 19-th April

Write solutions in the provided spaces.

STUDENTS'S NAME:

(a) Explain why the positive-definiteness of a Riemannian metric implies its non-degeneracy.

You know that Riemannian metric on the sphere of radius R in the stereographic coordinates is expressed by the formula

$$G_{\text{stereogr.}} = \frac{4R^4(du^2+dv^2)}{(R^2+u^2+v^2)^2}\,. \tag{1}$$

(See Homework 2, or Lecture Notes.) Show that this metric is preserved under rotation $\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$.

* *Give an example of non-linear transformation of coordinates u, v such that it preserves metric (1).*
(Hint: You may find this transformation considering transformations of the sphere.)

Consider the plane $\mathbf{R}^2$ with standard coordinates (x, y) equipped with the Riemannian metric $G = \frac{4R^2(dx^2+dy^2)}{(1+x^2+y^2)^2}$. Using the formula (1) show that this Riemannian manifold is isometric to the Riemannian sphere of the radius R without North pole, i.e. find diffeomorphism $\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$ such that under this diffeomorphism metric (1) transforms to the metric G.

(b) Find the length of the line $v = au$ in $\mathbf{R}^2$ with respect to the metric (1).

Explain why the length of this curve does not depend on a.

* *Consider the line l: $u\cos\tau + v\sin\tau = \rho$ in $\mathbf{R}^2$, where τ, ρ are parameters. It is easy to see that this line is on the distance ρ from the origin with respect to the standard Euclidean metric. Calculate the length of this line with respect to the metric (1), and explain why it does not depend on the parameter τ.*
(Hint: To facilitate calculations you may first show that the length does not depend on parameter τ, then calculated it for special choice of τ, e.g. $\tau = 0$ or $\tau = \pi/2$.)

* *Let C be a curve on the sphere of the radius R, such that the line l considered above is its image under stereographic projection. Describe this curve.*

(c) Find the length of the shortest curve on the cone $x^2 + y^2 - k^2z^2 = 0$, joining the points $A = (kh, 0, h)$ and $B = (-kh, 0, h)$.

2

(a) Consider a Riemannian manifold M with the metric $G = g_{ik}dx^i dx^k$. Write down the formula for the volume element on M.

Write down the volume form on the sphere of radius R in $\mathbf{E}^3$ in spherical coordinates θ, φ.

Find local coordinates u, v such that the volume element on the sphere in new coordinates equals $du \wedge dv$.

Express the Riemannian metric on the sphere in these coordinates.

(b) Calculate the total area of $\mathbf{R}^2$ equipped with the Riemannian metric

$$G = (1 + u^2 + v^2)e^{-u^2-v^2}\left(du^2 + dv^2\right) .$$

* *Calculate the volume of 3-dimensional sphere of radius R. (You may use the formula for Riemannian metric in stereographic coordinates from Homework 2).*

Hint: It could be useful to use spheric coordinates when calculating the integral.

(c) Evaluate the area of the part of the sphere of radius $R = 1$ between the planes given by equations $2x + 2y + z = 1$ and $2x + 2y + z = 2$.

3

(a) Explain shortly what is meant by Christoffel symbols and write down their transformation law.

Let ∇ be an affine connection on the 2-dimensional manifold M such that in local coordinates (u, v), $\nabla_{\frac{\partial}{\partial u}} \left(u^2 \frac{\partial}{\partial v}\right) = 3u \frac{\partial}{\partial v} + u \frac{\partial}{\partial u}$.

Calculate the Christoffel symbols Γ^u_{uv} and Γ^v_{uv} of this connection.

Let ∇ be an arbitrary connection on a manifold M. Show that

$$\cos F \nabla_{\mathbf{A}}(\sin F\mathbf{B}) - \sin F\nabla_{\mathbf{A}}(\cos F\mathbf{B}) = (\partial_{\mathbf{A}} F)\mathbf{B}\,,$$

where F is an arbitrary function.

* *Let $\Gamma^{i(1)}_{km}$ be the Christoffel symbols of a connection $\nabla^{(1)}$ and $\Gamma^{i(2)}_{km}$ be the Christoffel symbols of a connection $\nabla^{(2)}$. Show that the linear combinations $f\Gamma^{i(1)}_{km} + g\Gamma^{i(2)}_{km}$, (where f and g are some functions) are Christoffel symbols for some connection if $f + g \equiv 1$.*

Explain why $\frac{1}{2}\Gamma^{i(1)}_{km} + \frac{1}{3}\Gamma^{i(2)}_{km}$ is not a connection.

(b) Consider a cone $\mathbf{r}(h, \varphi)$: $\begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases}$ in $\mathbf{E}^3$.

Calculate the induced connection on the cone (the connection induced by canonical flat connection in the ambient Euclidean space: $\nabla_{\mathbf{X}}\mathbf{Y} = \left(\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}\right)_{\text{tangent}}$.)

Calculate the Riemannian metric on the cone induced by the canonical metric in ambient Euclidean space $\mathbf{E}^3$ and calculate explicitly the Levi-Civita connection of this metric using the Levi-Civita Theorem or using equations of motions of Lagrangian of the "free" particle on the cone.

(c) Calculate Levi-Civita connection of the Riemannian metric on the sphere in stereographic coordinates:

$$G = \frac{4R^4(du^2 + dv^2)}{(R^2 + u^2 + v^2)^2}$$

at the point $u = v = 0$

* *at an arbitrary point.*

* *(**c**) Consider a surface M in $\mathbf{E}^3$ defined by the equation* $\begin{cases} x = u \\ y = v \\ z = F(u, v) \end{cases}$.

Consider a point $\mathbf{p}$ *on* M *with coordinates* $u = x_0, v = y_0$ *such that* (x_0, y_0) *is a point of local extremum for the function* F.

Calculate Christoffel symbols of Levi-Civita connection at the point $\mathbf{p}$.

4

(a) * *Suppose a curve C: $x^i = x^i(t)$ is a geodesic on a Riemannian manifold in an arbitrary parameterisation. Show that the velocity vector of this curve remains tangent during parallel transport.*

Consider upper-half circle C: $x(t) = t, y(t) = \sqrt{1-t^2}$, $-1 < t < 1$, on the Lobachevsky plane. Consider the vectors $\mathbf{X}_0 = \frac{\sqrt{3}}{2}\partial_x + \frac{1}{2}\partial_y$ and $\mathbf{Y}_0 = -\frac{1}{2}\partial_x + \frac{\sqrt{3}}{2}\partial_y$ attached at the point $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ of C.

Show that the vector $\mathbf{X}_0$ is tangent to C and find the parallel transports $\mathbf{X}(t), \mathbf{Y}(t)$ of the vectors $\mathbf{X}_0, \mathbf{Y}_0$ along the curve C.

(You may use the fact that the curve C is a geodesic in the Lobachevsky plane.)

(**b**) Find the length of the shortest curve on the sphere of the radius R which joins points A and B on the sphere with spherical coordinates in the case if $\theta_A = \theta_B = \theta_0, \varphi_A = 0, \varphi_B = \pi$

* *in the case if $\theta_A = \theta_B = \theta_0, \varphi_A = 0, \varphi_B = \frac{\pi}{2}$.*

(**c**) On the sphere $x^2 + y^2 + z^2 = R^2$ in $\mathbf{E}^3$ consider points $A = (0, 0, R)$, $B = (R, 0, 0)$ and $C = (R\cos\varphi, R\sin\varphi, 0)$, $(0 \leq \varphi \leq \pi)$. Consider the isosceles triangle ABC. (Sides of this triangle are the shortest curves joining these points.) Show that:

$$KS(\triangle ABC) = \alpha + \beta + \gamma - \pi, \tag{1}$$

where K is the Gaussian curvature of the sphere, $S(\triangle ABC)$ is the area of the triangle $\triangle ABC$ and α, β, γ are angles of this triangle.

(d) Consider the points $B = (a, \sqrt{1-a^2})$, $C = (-a, \sqrt{1-a^2})$ on the Lobachevsky plane realised as half-plane. Consider vertical lines $x = \pm a$ and the half-circle $x^2 + y^2 = 1$ which pass through the points B and C.

Find the angles β, γ between these lines and the circle.

* *Show that the distance between points $B'_h = (a, h)$ and $C'_h = (-a, h)$ on vertical lines tends to zero if $h \to \infty$.*

* *Find the area $S(D)$ of the domain D delimited by the vertical lines and above half-circle and show that $KS(D) = \beta + \gamma - \pi$, where $K = -1$ is Gaussian curvature of Lobachevsky plane. (Compare with the formula (1).)*

(*You may use the fact that Lobachevsky metric of half-plane is $G = \frac{dx^2+dy^2}{y^2}$.)*