Solutions of problem 1 of Homework 6

In the file of Solutions of Homework 1 I put the function g equal to $e^{-(x^2+y^2)}$ instead putting it $g = y^2 - x^2$. Here I rewrite again the solutions of problems 1 and 2 putting $g = y^2 - x^2$.

1

Calculate the derivatives of the functions $f = x^2 + y^2$, $g = y^2 - x^2$ and $h = q \log |r| = q \log \left(\sqrt{x^2 + y^2}\right)$ (q is a constant) along vector fields $\mathbf{A} = x\partial_x + y\partial_y$ and $\mathbf{B} = x\partial_y - y\partial_x$,

a) calculating directional derivatives $\partial_{\mathbf{A}} f, \partial_{\mathbf{A}} g, \partial_{\mathbf{A}} h$, $\partial_{\mathbf{B}} f, \partial_{\mathbf{B}} g, \partial_{\mathbf{B}} h$

b) calculating $df(\mathbf{A}), dg(\mathbf{A}), dh(\mathbf{A}), df(\mathbf{B}), dg(\mathbf{B}), dh(\mathbf{B})$.

We can do this exercise or using the formula for directional derivative or using the 1-form, differential of function: $\partial_{\mathbf{A}} f = df(\mathbf{A})$.

a) First do using directional derivatives:

$\partial_{\mathbf{A}} f = A_x \frac{\partial f}{\partial x} + A_y \frac{\partial f}{\partial y} = x \cdot 2x + y \cdot 2y = 2(x^2 + y^2)$,

$\partial_{\mathbf{A}} g = A_x \frac{\partial g}{\partial x} + A_y \frac{\partial g}{\partial y} = x \cdot (-2x) + y \cdot 2y = 2(y^2 - x^2)$,

$\partial_{\mathbf{A}} h = x \frac{\partial h}{\partial x} + y \frac{\partial h}{\partial y} = \frac{x^2 q}{x^2+y^2} + \frac{y^2 q}{x^2+y^2} = q$

$\partial_{\mathbf{B}} f = B_x \frac{\partial f}{\partial x} + B_y \frac{\partial f}{\partial y} = -y \cdot 2x + x \cdot 2y = 0$,

$\partial_{\mathbf{B}} g = -y \frac{\partial g}{\partial x} + x \frac{\partial g}{\partial y} = -y \cdot (-2x) + x \cdot 2y = 4xy$

$\partial_{\mathbf{B}} h = -y \frac{\partial h}{\partial x} + x \frac{\partial h}{\partial y} = \frac{-xyq}{x^2+y^2} + \frac{xyq}{x^2+y^2} = 0$

b) Now calculate using 1-form using the fact that $\partial_{\mathbf{A}} f = df(\mathbf{A})$:

We have that $df = d(x^2 + y^2) = 2xdx + 2ydy$, $dg = d\left(y^2 - x^2\right) = g_x dx + g_y dy = (2ydy - 2xdx)$, $dh = d\left(q \log \sqrt{x^2 + y^2}\right) = h_x dx + h_y dy = \frac{qxdx+qydy}{x^2+y^2}$.

Hence

$\partial_{\mathbf{A}} f = df(\mathbf{A}) = (2xdx + 2ydy)(x\partial_x + y\partial_y) = 2x^2 dx(\partial_x) + 2y^2 dy(\partial_y) = 2x^2 + 2y^2$,

$\partial_{\mathbf{A}} g = dg(\mathbf{A}) = (2ydy - 2xdx)((x\partial_x + y\partial_y)) = 2ydy(y\partial_y) - 2xdx(x\partial_x) = 2y^2 - 2x^2$.

$\partial_{\mathbf{A}} h = dh(\mathbf{A}) = \frac{qxdx+qydy}{x^2+y^2} (x\partial_x + y\partial_y) = \frac{qxdx(x\partial_x)+qydy(y\partial_y)}{x^2+y^2} = \frac{qx^2+qy^2}{x^2+y^2} = q$

$\partial_{\mathbf{B}} f = df(\mathbf{A}) = (2xdx + 2ydy)(-y\partial_x + x\partial_y) = -2xydx(\partial_x) + 2xydy(\partial_y) = 0$,

$\partial_{\mathbf{B}} g = dg(\mathbf{A}) = (2ydy - 2xdx)((x\partial_y - y\partial_x)) = 2ydy(x\partial_y) - 2xdx(-y\partial_x) = 2xy + 2xy = 4xy$.

$\partial_{\mathbf{A}} h = dh(\mathbf{A}) = \frac{qxdx+qydy}{x^2+y^2} (-y\partial_x + x\partial_y) = \frac{qxdx(-y\partial_x)+qydy(x\partial_y)}{x^2+y^2} = \frac{-qxy+qxy}{x^2+y^2} = 0$.

2

Perform the calculations of the previous exercise using polar coordinates.

For basic fields $\partial_r, \partial_\varphi$ in polar coordinates r, φ ($r = x \cos\varphi, y = r \sin\varphi$) we have that

$$\partial_r = \frac{\partial x}{\partial r}\partial_x + \frac{\partial y}{\partial r}\partial_y = \cos\varphi \partial_x + \sin\varphi \partial_y = \frac{x}{r}\partial_x + \frac{y}{r}\partial_y = \frac{x\partial_x + y\partial_y}{r} = \frac{\mathbf{A}}{r} \Rightarrow \mathbf{A} = r\partial_r$$

and

$$\partial_\varphi = \frac{\partial x}{\partial \varphi}\partial_x + \frac{\partial y}{\partial \varphi}\partial_y = -r\sin\varphi \partial_x + r\cos\varphi \partial_y = -y\partial_x + x\partial_y \Rightarrow \mathbf{B} = \partial_\varphi$$

We see that fields $\mathbf{A}, \mathbf{B}$ have very simple expression in polar coordinates. Now calculations become almost immediate because in polar coordinates $f = x^2 + y^2 = r^2$, $g = y^2 - x^2 = r^2(\sin^2\varphi - \cos^\varphi) = -r^2 \cos 2\varphi$ and $h = q \log r$ and

$$\partial_A f = r\partial_r r^2 = 2r^2 = 2(x^2 + y^2)\,,$$

$$\partial_A g = r\partial_r(-r^2 \cos 2\varphi) = -2r^2 \cos 2\varphi = 2(y^2 - x^2), \;\; \partial_A h = r\partial_r(q \log r) = q\,.$$

For field B we have that: $\partial_{\mathbf{B}} = \partial_\varphi$, hence

$$\partial_{\mathbf{B}} f = \partial_{\mathbf{B}} g = \partial_{\mathbf{B}} h = 0\,.$$

since the functions f and h do not depend on φ. For the function $g = y^2 - x^2 = -r^2 \cos 2\varphi$ we have:

$$\partial_{\mathbf{B}} g = \partial_\varphi(-r^2 \cos 2\varphi) = 2r^2 \sin 2\varphi = 4r^2 \sin\varphi \cos\varphi = 4r^2 \left(\frac{y}{r}\right)\left(\frac{x}{r}\right) = 4xy.$$