Algorithm "nosov" continuous fractions

Here we reproduce the algorithm, which I learnt in the book of Arnold.

Let α be non-negative real number and let $[a_0, a_1, \dots,]$ be its continuous fraction.

Let $\mathbf{e}, \mathbf{f}$ be standard basis in $\mathbf{R}^2$.

Assign to an arbitrary rational number $\frac{p}{q}$ the vector $\mathbf{E}\left(\frac{p}{q}\right) = p\mathbf{e} + q\mathbf{f}$. We assume that p, q are coprime:

Consider vectors $\{\mathbf{E}_{-2}, \mathbf{E}_{-1}, \mathbf{E}_0 . \mathbf{E}_1, \dots\}$ such that

$$\mathbf{E}_{-2} = \mathbf{e}, \quad \mathbf{E}_{-1} = \mathbf{f}\,,$$

and

$$\mathbf{E}_k = \mathbf{E}\left(\frac{p_k}{q_k}\right).$$

We see that

$$\frac{p_0}{q_0} = a_0 = [1 : a_0] = \mathbf{e} + a_0\mathbf{f}\,. \tag{basic}$$

Thus

$$\mathbf{E}_0 \colon , \mathbf{E}_0 = \mathbf{E}_{-2} + a_0\mathbf{E}_{-1}\,,$$

Proposition vytiagivanie nosov: for arbitrary k:

$$\mathbf{E}_{k+1} = \mathbf{E}_{k-1} + a_k\mathbf{E}_k\,.$$

Proof:

For the continuous fraction $\alpha = [a_0, a_1, a_2, dots]$ denote by

$$a'_k = a_k + \frac{1}{a_{k+1}}$$

First check the Proposition for $k = 0$. We have

$$\frac{p_1}{q_1} = a_0 + \frac{1}{a_1} = a'_0 = \left(1 + \frac{1}{a_1}\right).$$

Now using "projectivisation":

$$[p : q] = [\lambda p : \lambda q]\,, \quad \mathbf{E} + \lambda\mathbf{F} = \lambda\left(\mathbf{E} + \frac{1}{\lambda}\mathbf{F}\right) = \mathbf{E} + \frac{1}{\lambda}\mathbf{F}$$

we will come to

$$\mathbf{E}_1 = \mathbf{e} + a_1\mathbf{f} = \mathbf{E}_{-2} + a_0'\mathbf{E}_{-1} = \mathbf{E}_{-2} + \left(a_0 + \frac{1}{a_1}\right)\mathbf{E}_{-1} = \mathbf{E}_{-2} + a_0\mathbf{E}_{-1} + \frac{1}{a_1}\mathbf{E}_{-1} = \mathbf{E}_0 + \frac{1}{a_1}\mathbf{E}_{-1}$$

$$(\text{multiplying the last expression on } a_1 \text{ we come to}) = a_1\mathbf{E}_0 + \mathbf{E}_{-1}\,,$$

i.e.

$$\mathbf{E}_1 = \mathbf{E}_{-1} + a_1\mathbf{E}_0\,.$$

$$\mathbf{E}_2\colon\,,\mathbf{E}_2 = \mathbf{E}_0 + a_2\mathbf{E}_1\,,$$

$$\mathbf{E}_3\colon\,,\mathbf{E}_3 = \mathbf{E}_1 + a_3\mathbf{E}_2\,,$$

and so on:

$$\mathbf{E}_k\colon\,,\mathbf{E}_k = \mathbf{E}_{k-2} + a_k\mathbf{E}_{k-1}\,,$$

We see that

$$\mathbf{E}_{-2} = (1,0)\,,\quad \mathbf{E}_{-1} = (0,1)\,,$$

$$\mathbf{E}_0 = \mathbf{E}_{-2} + a_0\mathbf{E}_{-1} = (1,0) + a_0(0,1) = (1,a_0)\,,$$

$$\mathbf{E}_1 = \mathbf{E}_{-1} + a_1\mathbf{E}_0 = (0,1) + a_1(1,a_0) = (a_1, 1 + a_1a_0)\,,$$

$$\mathbf{E}_2 = \mathbf{E}_0 + a_2\mathbf{E}_1 = (1,a_0) + a_2(a_1, 1 + a_1a_0) =$$

$$\mathbf{E}_3\colon\,,\mathbf{E}_3 = \mathbf{E}_1 + a_3\mathbf{E}_2\,,$$

and so on:

$$\mathbf{E}_k\colon\,,\mathbf{E}_k = \mathbf{E}_{k-2} + a_k\mathbf{E}_{k-1}\,,$$