Stirling formula

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it is more than 40 years ago that I learned the Stirling formula using the stationary phase method. It is today that I realised, that I did a mistake in calculations.

Stirling formula is an excellent example of applying stationary phase method: Consider

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} t dt\,, \quad n! = \Gamma(n+1)\,.$$

We have

$$N! = \Gamma(N+1) = \int_0^\infty \exp[-t + N\log t] dt = N\int_0^\infty \exp[-Nx + N\log Nx] dx =$$

$$N^{N+1}\int_0^\infty \exp F(x) dx\,, \quad \text{where } F(x) = N\log x - Nx \tag{1}$$

Then consider Taylor series expansion of the function $F(x)$ in the vicinity of the stationary point $x_0 = 1$

$$F(x) = F(x_0) + F'(x_0)(x-x_0) + \frac{1}{2}F^{''}(x_0)(x-x_0)^2 + \frac{1}{6}F^{'''}(x_0)(x-x_0)^3 + \frac{1}{24}F^{''''}(x_0)(x-x_0)^4 + \ldots =$$

$$-N - \frac{N}{2}(x-1)^2 + \frac{N}{3}(x-1)^3 - \frac{N}{4}(x-1)^4 + \ldots =$$

and

$$N! = N^{N+1}\int_0^\infty \exp[F(t)] dt = N^{N+1}\int_0^\infty \exp\left[-N - \frac{N}{2}(x-1)^2 + \frac{N}{3}(x-1)^3 - \frac{N}{4}(x-1)^4 + \ldots\right] d$$

`Zeroth approximmation`

$$N! = N^{N+1}\int_0^\infty \exp\left[-N + \ldots\right] dx \approx N\left(\frac{N}{e}\right)^N\,,$$

`First approximmation`

$$N! = N^{N+1}\int_0^\infty \exp\left[-N - \frac{N}{2}(x-1)^2 + \ldots\right] dx =$$

$$N\left(\frac{N}{e}\right)^N \int_{-1}^\infty e^{-\frac{Ny^2}{2}} dy = N\left(\frac{N}{e}\right)^N \frac{1}{\sqrt{N}}\int_{-\sqrt{N}}^\infty e^{-\frac{u^2}{2}} du \approx$$

$$N\left(\frac{N}{e}\right)^N \frac{1}{\sqrt{N}}\int_{-\infty}^\infty e^{-\frac{u^2}{2}} du = \sqrt{2\pi N}\left(\frac{N}{e}\right)^N\,.$$

`Second apprximation`

$$N! = N^{N+1}\int_0^\infty \exp[F(t)]dt = N^{N+1}\int_0^\infty \exp\left[-N - \frac{N}{2}(x-1)^2 + \frac{N}{3}(x-1)^3 - \dots\right] dx =$$

$$N\left(\frac{N}{e}\right)^N \int_0^\infty \exp\left[-\frac{N}{2}(x-1)^2 + \frac{N}{3}(x-1)^3 + \dots\right] =$$

$$N\left(\frac{N}{e}\right)^N \frac{1}{\sqrt{N}}\int_{-\sqrt{N}}^\infty e^{-\frac{u^2}{2}} \exp\left[\frac{1}{3\sqrt{N}}u^3 + \dots\right] du \approx \sqrt{2\pi N}\left(\frac{N}{e}\right)^N ,$$

since $\int_{-\infty}^\infty e^{-\frac{u^2}{2}} u^3 du = 0$.

`Third apprximation`

$$N! = N^{N+1}\int_0^\infty \exp[F(t)]dt = N^{N+1}\int_0^\infty \exp\left[-N - \frac{N}{2}(x-1)^2 + \frac{N}{3}(x-1)^3 - \frac{N}{4}(x-1)^4 + \dots\right] d$$

$$N\left(\frac{N}{e}\right)^N \int_0^\infty \exp\left[-\frac{N}{2}(x-1)^2 + \frac{N}{3}(x-1)^3 - \frac{N}{4}(x-1)^4 + \dots\right] dx =$$

$$N\left(\frac{N}{e}\right)^N \frac{1}{\sqrt{N}}\int_{-\sqrt{N}}^\infty e^{-\frac{u^2}{2}} \exp\left[\frac{1}{3\sqrt{N}}u^3 - \frac{1}{4N}u^4 + \dots\right] du$$

$$\approx \sqrt{N}\left(\frac{N}{e}\right)^N \int_{-\infty}^\infty e^{-\frac{u^2}{2}} \left[1 + \frac{1}{3\sqrt{N}}u^3 - \frac{1}{4N}u^4 + \dots\right] du \approx$$

$$\sqrt{N}\left(\frac{N}{e}\right)^N \left[\sqrt{2\pi} + \int_{-\infty}^\infty e^{-\frac{u^2}{2}} \left(\frac{1}{3\sqrt{N}}u^3 - \frac{1}{4N}u^4\right) du\right] =$$

$$\sqrt{2\pi N}\left(\frac{N}{e}\right)^N \left[1 - \frac{1}{4N}\int_{-\infty}^\infty u^4 e^{-\frac{u^2}{2}} du\right] =$$