Riemannian Geometry (31082, 41082, 61082)

2017

COURSEWORK

Starred questions are for the 15 credit version

This assignment counts for 20% of your marks.

Solutions are due by 30 March 3pm

Write solutions in the provided spaces.

STUDENT'S NAME:

1

(a) Consider a surface C, the upper sheet of the cone

$$\begin{cases} x = 5h\cos\varphi \\ y = 5h\sin\varphi \\ z = 12h \end{cases}, \qquad 0 \le \varphi < 2\pi\,,\ h > 0\,.$$

Find the induced Riemannian metric on this surface and show that this surface is locally Euclidean. Explain why it is not isometric to Euclidean surface.

(b) Consider the stereographic projection of the sphere $x^2 + y^2 + z^2 = R^2$ in $\mathbf{E}^3$ with respect to the North pole, the point $N = (0, 0, R)$, on the plane $\mathbf{R}^2$.

Let C_1, C_2 be two circles on the sphere such that the circle C_1 is the intersection of this sphere with the plane $x + y = 0$, and C_2 is the intersection of the sphere with the plane $x + z = 0$. Let C_1' and C_2' be the images of the curves C_1 and C_2 respectively under the stereographic projection.

How do the curves C_1', C_2' look? Find the lengths of these curves with respect to the Riemannian metric

$$G_{\text{stereogr.}} = \frac{4R^4(dx^2 + dy^2)}{(R^2 + x^2 + y^2)^2}\,, \tag{1}$$

on the plane $z = 0$ which is induced from the metric of the sphere by the stereographic projection.

2

Consider the upper half-plane $(y > 0)$ with metric $G = \frac{dx^2+dy^2}{y^2}$ (Lobachevsky plane).

(a) Calculate the length of the vertical segment starting at the point $A = (0, a)$ and ending at the point $B = (0, b)$.

(b) Calculate the length of the horizontal segment starting at the point $C = (-\cos\varphi, \sin\varphi)$ and ending at the point $D = (\cos\varphi, \sin\varphi)$ $(0 < \varphi < \frac{\pi}{2})$. (The chord CD of the upper-half circle $x^2 + y^2 = 1$, $y > 0$.)

Calculate also the length of the arc of the unit circle connecting these points. Compare the lengths of these two curves.

* Show explicitly that the length of every curve between points A and B is greater or equal than the length of this vertical segment.

Show that translations $\begin{cases} x' = x + a \\ y' = y \end{cases}$ and homotheties $\begin{cases} x' = \lambda x \\ y' = \lambda y \end{cases}$ $(\lambda > 0)$ are isometries of the Lobachevsky plane.

* Show that inversion $\begin{cases} x' = \frac{x}{x^2+y^2} \\ y' = \frac{y}{x^2+y^2} \end{cases}$ is also isometry of the Lobachevsky plane.

* Find infinitesimal isometries (Killing vector fields) corresponding to translations and homotheties.

3

(a) Write down the volume form on the sphere of radius R in $\mathbf{E}^3$ in spherical coordinates θ, φ.

(b) Find local coordinates u, v such that the volume element on the sphere in new coordinates equals $dudv$.

Express the Riemannian metric on the sphere in these coordinates.

(c) Evaluate the area of the part of the sphere of radius $R = 1$ between the planes given by equations $2x + 2y + z = 1$ and $2x + 2y + z = 2$.

(d) Consider the plane $\mathbf{R}^2$ with standard coordinates (x, y) equipped with Riemannian metric

$$G = (1 + x^2 + y^2)e^{-x^2-y^2} \left(dx^2 + dy^2\right) .$$

Calculate the total area of this plane.

4

(a) Let ∇ be an affine connection on the 2-dimensional manifold M such that in local coordinates (u,v), $\nabla_{\frac{\partial}{\partial u}}\left(u^2\frac{\partial}{\partial v}\right) = 3u\frac{\partial}{\partial v} + u\frac{\partial}{\partial u}$.

Calculate the Christoffel symbols Γ^u_{uv} and Γ^v_{uv} of this connection.

(b) Let ∇ be an arbitrary connection on a manifold M. Show that

$$\cos F\nabla_{\mathbf{A}}(\sin F\mathbf{B}) - \sin F\nabla_{\mathbf{A}}(\cos F\mathbf{B}) = (\partial_{\mathbf{A}}F)\mathbf{B}\,,$$

where F is an arbitrary function.

* Let $\Gamma^{i(1)}_{km}$ be the Christoffel symbols of a connection $\nabla^{(1)}$ and $\Gamma^{i(2)}_{km}$ be the Christoffel symbols of a connection $\nabla^{(2)}$. Show that the linear combinations $f\Gamma^{i(1)}_{km} + g\Gamma^{i(2)}_{km}$, (where f and g are some functions) are Christoffel symbols for some connection if $f+g\equiv 1$.

* Explain why $\frac{1}{2}\Gamma^{i(1)}_{km} + \frac{1}{3}\Gamma^{i(2)}_{km}$ is not a connection.

5

(a) Calculate the induced connection on the sphere of radius R [1]. Perform calculations in spherical coordinates.

Calculate explicitly the Christoffel symbols of the Levi-Civita connection in spherical coordinates. (Metric is induced by the Euclidean metric in $\mathbf{E}^3$).

Compare answers.

(b) Calculate the Levi-Civita connection of the Riemannian metric on the sphere in stereographic coordinates:

$$G = \frac{4R^4(du^2 + dv^2)}{(R^2 + u^2 + v^2)^2}$$

at the point $u = v = 0$.

(c)* Consider a surface M in $\mathbf{E}^3$ defined by the equation $\begin{cases} x = u \\ y = v \\ z = F(u, v) \end{cases}$.

Consider a point $\mathbf{p}$ on M with coordinates $u = x_0, v = y_0$ such that (x_0, y_0) is a point of local extremum for the function F.

Calculate the Christoffel symbols of the Levi-Civita connection at the point $\mathbf{p}$ (in coordinates u, v).

[1] The connection induced by canonical flat connection in the ambient Euclidean space: $\nabla_{\mathbf{X}}\mathbf{Y} = \left(\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}\right)_{\text{tangent}}$ ..