I know well angle function and its relation with Green function:

In fact Green function generalises this concept.

recall sketchly : Let $G_U == G(z_0, z)$ be a Green function of the domain U, i.e. the function such that

1) $G \approx \log(z - z_0)$ in a vicnity of the point z_0,

2) it is harmonic elswhere in U except a point z_0

3) it vanishes at boundary ∂U

Using the identity:

$$\int_U u\Delta v - \int_U v\Delta u = \int_{\partial U} u * dv - \int_{\partial V} v * du = \qquad (identity)$$

we solve the boundary problem

$$\begin{cases} \Delta F = f \\ F\big|_{\partial U} = \mu \end{cases}, \qquad F = \int_U G \circ f + \int_{\partial U} *dG\mu$$

where $\circ$ is convolution. (this is called Dirichle problem if $f \equiv 0$.)

On the other hand one can come to the Green function G_U trhough

1) Green function $G_\infty(z_0, z) = -\frac{1}{2\pi}\log(z - z_0)$

2) and the "angle function" defined on the boundary solving (with Arsenin) the integral equation. Recall that angle function defines for every curve the function on plane which is equal to the angle that we look at this curve *.

Remark *We wrote identity (ident.) in a way to emphasize as much as possible the relation between 'angle' function and Green function. This identity is written in standard way as following:*

$$\int_U u\Delta v - \int_U v\Delta u = \oint_{\partial U} u\frac{\partial v}{\partial n}d\mathbf{S} - \oint_{\partial V} v\frac{\partial u}{\partial n}d\mathbf{S} = \qquad (identity')$$

(see any book). In fact for every vector field $\mathbf{A}$

$$\underbrace{\int}_{C} \mathbf{A}d\mathbf{s}_{\textit{flux of } \mathbf{A} \textit{ trhough the surface } C} = \int \Omega\rfloor\mathbf{A}\,,$$

where Ω *is volume* 2*-form, and*

$$\frac{\partial f}{\partial n} = \operatorname{grad} f$$

and $*$ *is Hodge operation:*

$$*df = \Omega\rfloor\operatorname{grad} f$$

* another name: double layer potential

Notice that for harmonic function u, $d^{-1}(*du)$ is conjugate function:

$$\Delta u(x,y)=0 \Leftrightarrow u(x,y)+id^{-1}(*du(x,y))=F(z) \textit{ is holomorphoc function}$$

, e.g.

$$d^{-1}(*d\log\sqrt{x^2+y^2})=\mathrm{arctan}\frac{y}{x}, \log x^2+y^2+i\mathrm{arctan}\frac{y}{x}=\log(x+iy)$$

it is how angle function relates with normal derivative of Green function.

Thus on the space $C(\partial U)$ of functions on boundary we have to linear operators:
First operator:

$$C(\partial U)\ni\nu\mapsto\mu\in C(\partial U):\quad s(t)=\int_{\partial U}L(t,t')\nu(t')dt'$$

Second operator

$$C(\partial U)\ni\nu\mapsto W\in C(U):\quad W(\mathbf{r})=\int_U L(\mathbf{r},t')\nu(t')dt'$$

On the other hand we know that for the angle function L the difference of values of these operators on the boundary ∂U is equal to $\pi\nu(t)$:

$$C(\partial U)\ni\nu\mapsto\left(\lim_{\mathbf{r}\to t} int_U L(\mathbf{r},t')\nu(t')dt'\right)-\int_{\partial U}L(t,t')\nu(t')dt'=\pi\nu(t)$$

Thus to solve the Dirichle problem, i.e. to reconstruct harmonic function W by its value ν at the boundary $\partial U_{0\_}$ we first solving integral equation find a function ν such that

$$\pi\nu+\int_{\partial U}L(t,t')\nu(t')dt'=\mu(t)$$

then we reconstruct W in terms of ν.

Example

Tro to recontruct harmonic function W in the disc $x^2+y^2<1$ using different methods.
First: Green function:

$$W=\int_{x^2+y^2=1}*dG$$

but here we need to know $G=G(z_0,0)$ We already know the conformal map whoch transforms circle to circle, and we know the Green fucntion for $z_0=0$, hence

$$G(z_0,z)=-\frac{1}{2\pi}\log\left|\frac{z-z_0}{1-\bar{z_0}z}\right|$$

Another way to calculate: