Homework 6. Solutions.

Here the full solution of exercise 4b) is added, which was absent due to Coursework

Christoffel symbols and Lagrangians

1 *Consider the Lagrangian of "free" particle $L = \frac{1}{2} g_{ik}\dot{x}^i\dot{x}^k$ for Riemannian manifold with a metric $G = g_{ik}dx^i dx^k$.*
Write down Euler-Lagrange equations of motion for this Lagrangian and compare them with differential equations for geodesics on this Riemannian manifold.

In fact show that

$$\underbrace{\frac{\partial L}{\partial x^i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}}_{\textit{Euler-Lagrange equations}} \quad \Leftrightarrow \quad \underbrace{\frac{d^2x^i}{dt^2} + \Gamma^i_{km}\dot{x}^k\dot{x}^m = 0}_{\textit{Equations for geodesics}} \ , \tag{1}$$

where

$$\Gamma^i_{km} = \frac{1}{2} g^{ij}\left(\frac{\partial g_{jk}}{\partial x^m} + \frac{\partial g_{jm}}{\partial x^k} - \frac{\partial g_{km}}{\partial x^j}\right) . \tag{2}$$

Solution: see the lecture notes.

2 *a) Write down the Lagrangian of of free particle $L = \frac{1}{2} g_{ik}\dot{x}^i\dot{x}^k$ for Euclidean plane in polar coordinates. Calculate Christoffel symbols for canonical flat connection in polar coordinates using Euler-Lagrange equations for this Lagrangian. Compare with answers which you obtained by the direct use of the formula (2). b) Do the same for cylindrical coordinates in $\mathbf{E}^3$.*

Solution. Canonical flat connection is Levi-Civita connection of Euclidean metric $G = dx^2 + dy^2$. Hence we can calculate Christoffel symbols using Lagrangian method.

Euclidean metric in polar coordinates is $dr^2 + r^2 d\varphi^2$. Hence the Lagrangian of the free particle is

$$L = \frac{\dot{r}^2 + r^2\dot{\varphi}^2}{2}$$

Euler-Lagrange equations:

1) for r:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \ddot{r} = \frac{\partial L}{\partial \varphi} = r\dot{\varphi}^2$$

i.e.

$$\ddot{r} - r\dot{\varphi}^2 = 0 \Rightarrow \Gamma^r_{rr} = \Gamma^r_{\varphi r} = \Gamma^r_{r\varphi} = 0, = \Gamma^r_{\varphi\varphi} = -r.$$

2) for φ:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = \frac{d}{dt}\left(r^2\dot{\varphi}\right) = r^2\ddot{\varphi} + 2r\dot{r}\dot{\varphi} = \frac{\partial L}{\partial \varphi} = 0,$$

i.e.

$$\ddot{\varphi} + \frac{2}{r}\dot{r}\dot{\varphi} = 0 \Rightarrow \Gamma^\varphi_{rr} = \Gamma^\varphi_{\varphi\varphi} = 0, \Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = \frac{1}{r}.$$

b) cylindrical coordinates in $\mathbf{E}^3$. Calculations almost the same as for polar coordinates in $\mathbf{E}^2$. $G = dr^2 + r^2 d\varphi^2 + dh^2$,

$$L = \frac{\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{h}^2}{2}$$

for r: $\ddot{r} - r\dot{\varphi}^2 = 0 \Rightarrow$

$$\Gamma^r_{rr} = \Gamma^r_{\varphi r} = \Gamma^r_{r\varphi} = \Gamma^r_{rh} = \Gamma^r_{hr} = \Gamma^r_{h\varphi} = \Gamma^r_{\varphi h} = \Gamma^r_{hh} = 0 \, , \Gamma^r_{\varphi\varphi} = -r.$$

for φ, $r^2\ddot{\varphi} + 2r\dot{r}\dot{\varphi} = \frac{\partial L}{\partial \varphi} = 0$, i.e. $\ddot{\varphi} + \frac{2}{r}\dot{r}\dot{\varphi} = 0 \Rightarrow$

$$\Gamma^{\varphi}_{rr} = \Gamma^{\varphi}_{rh} = \Gamma^{\varphi}_{hr} = \Gamma^{\varphi}_{\varphi\varphi} = \Gamma^{\varphi}_{\varphi h} = \Gamma^{\varphi}_{h\varphi} = \Gamma^{\varphi}_{hh} = 0,\ \Gamma^{\varphi}_{r\varphi} = \Gamma^{\varphi}_{\varphi r} = \frac{1}{r}$$

3) for h, $\ddot{h} = 0$,

$$\Gamma^{h}_{rr} = \Gamma^{h}_{r\varphi} = \Gamma^{h}_{\varphi r} = \Gamma^{h}_{rh} = \Gamma^{h}_{hr} = \Gamma^{h}_{\varphi\varphi} = \Gamma^{h}_{\varphi h} = \Gamma^{h}_{h\varphi} = \Gamma^{h}_{hh} = 0,$$

3

Calculate Christoffel symbols of Levi-Civita connection for Riemannian metric $G = adu^2 + bdv^2$. Compare with resultls of the Exercise 1b) in the Homework 5.

Lagrangian of free particle for this metric is

$$L = \frac{a(u,v)\dot{u}^2 + b(u,v)\dot{v}^2}{2}\,.$$

Euler-lagrange equations

for u:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}}\right) = \frac{d}{dt}(a\dot{u}) = a_u\dot{u}^2 + a_v\dot{v}\dot{u} + a\ddot{u} = \frac{\partial L}{\partial u} = \frac{a_u\dot{u}^2 + b_u\dot{v}^2}{2}$$

hence

$$\ddot{u} + \frac{1}{2}\frac{a_u}{a}\dot{u}^2 + \frac{a_v}{a}\dot{v}\dot{u} - \frac{1}{2}\frac{b_u}{a}\dot{u}^2$$

Comparing with equation

$$\ddot{u} + \Gamma^u_{uu}\dot{u}\dot{u} + \Gamma^u_{uv}\dot{u}\dot{v} + \Gamma^u_{vu}\dot{v}\dot{u} + \Gamma^u_{vv}\dot{v}\dot{v}\ddot{u} + \Gamma^u_{uu}\dot{u}\dot{u} + 2\Gamma^u_{uv}\dot{u}\dot{v} + \Gamma^u_{vv}\dot{v}\dot{v} = 0$$

we see that

$$\Gamma^u_{uu} = \frac{1}{2}\frac{a_u}{a}\,, \Gamma^u_{uv} = \Gamma^u_{vu} = \frac{1}{2}\frac{a_v}{a}, \Gamma^u_{vv} = -\frac{1}{2}\frac{b_u}{a}\,,$$

Analogously v:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{v}}\right) = \frac{d}{dt}(b\dot{v}) = b_v\dot{v}^2 + b_u\dot{u}\dot{v} + b\ddot{b} = \frac{\partial L}{\partial v} = \frac{a_v\dot{u}^2 + b_v\dot{v}^2}{2}$$

hence

$$\ddot{v} + \frac{1}{2}\frac{b_v}{b}\dot{b}^2 + \frac{b_u}{b}\dot{u}\dot{v} - \frac{1}{2}\frac{a_v}{b}\dot{v}^2 \Rightarrow \Gamma^v_{vv} = \frac{1}{2}\frac{b_v}{b}\,, \Gamma^v_{vu} = \Gamma^v_{uv} = \frac{1}{2}\frac{b_u}{b}, \Gamma^v_{uu} = -\frac{1}{2}\frac{a_v}{b}\,.$$

4

Write down the Lagrangian of free particle $L = \frac{1}{2}g_{ik}\dot{x}^i\dot{x}^k$ and using Euler-Lagrange equations for this Lagrangian calculate Christoffel symbols (Christoffel symbols of Levi-Civita connection) for

a) cylindrical surface of the radius R

b) for the cone $x^2 + y^2 - k^2z^2 = 0$

c) for the sphere of radius R

d) for Lobachevsky plane

Compare with the results that you obtained using straightforwardly the formula (1) or using formulae for induced connection.

Solution.

For cylindrical surface of the radius a: $x^2 + y^2 = a^2$ $\mathbf{r}(h,\varphi) = \begin{cases} x = a\cos\ \varphi \\ y = a\sin\varphi \\ z = h \end{cases}$ we have that induced metric is $G = dh^2 + a^2 d\varphi^2$ and the Lagrangian of free particle is

$$L = \frac{a^2\dot\varphi^2 + \dot h^2}{2}$$

for φ, Euler-Lagrange equations of motion:

$$\frac{\partial L}{\partial \varphi} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi}\right)\ . \quad \frac{\partial L}{\partial \varphi} = 0,\ \frac{\partial L}{\partial \dot\varphi} = a^2\dot\varphi$$

hence

$$\frac{d}{dt}\left(a^2\dot\varphi\right) = a^2\ddot\varphi = 0\,, \ddot\varphi = 0.$$

Hence all Christoffel symbols $\Gamma^\varphi_{\varphi\varphi}, \Gamma^\varphi_{\varphi h}, \Gamma^\varphi_{h\varphi}$ vanish:

$$\Gamma^\varphi_{\varphi\varphi} = 0, \Gamma^\varphi_{\varphi h} = \Gamma^\varphi_{h\varphi} = 0$$

for h, we have the same. Euler-Lagrange equations of motion::

$$\frac{\partial L}{\partial h} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot h}\right)\ . \quad \frac{\partial L}{\partial h} = 0,\ \frac{\partial L}{\partial \dot h} = \dot h$$

hence

$$\frac{d}{dt}\left(\dot h\right) = \ddot h = 0\,,$$

Hence all Christoffel symbols $\Gamma^h_{\varphi\varphi}, \Gamma^h_{\varphi h}, \Gamma^h_{h\varphi}$ vanish:

$$\Gamma^h_{\varphi\varphi} = 0, \Gamma^h_{\varphi h} = \Gamma^h_{h\varphi} = 0$$

We see that on cylindrical surface in coordinates h, φ all Christoffel symbols vanish: this is not surprising, since Riemannian metric $dh^2 + a^2 d\varphi^2$ has constant coefficients.

For cone: $x^2 + y^2 - k^2z^2$ *we have* $\mathbf{r}(h,\varphi) = \begin{cases} x = kh\cos\ \varphi \\ y = kh\sin\varphi \\ z = h \end{cases}$ *and induced metric is* $G = (k^2+1)dh^2 + k^2h^2d\varphi^2$ *(see previous Homeworks) and the Lagrangian of free particle is*

$$L = \frac{k^2h^2\dot\varphi^2 + (k^2+1)^2\dot h^2}{2}$$

for φ, Euler-Lagrange equations of motion:

The full solution will appear after coursework.

Here it is

For cone: $x^2 + y^2 = k^2z^2$ we have $\mathbf{r}(h,\varphi) = \begin{cases} x = kh\cos\ \varphi \\ y = kh\sin\varphi \\ z = h \end{cases}$

$$\frac{\partial}{\partial h} = \mathbf{r}_h = \begin{pmatrix} k\cos\varphi \\ k\sin\varphi \\ 1 \end{pmatrix},\ \frac{\partial}{\partial \varphi} = \mathbf{r}_\varphi = \begin{pmatrix} -kh\sin\varphi \\ kh\cos\varphi \\ 0 \end{pmatrix},\ \mathbf{n} = \frac{1}{\sqrt{1+k^2}}\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix}$$

We have $\mathbf{r}_{hh} = 0$, hence $\nabla_{\partial_h}\partial_h = 0$. i.e. $\Gamma^h_{hh} = \Gamma^\varphi_{hh} = 0$.

We have that $\mathbf{r}_{h\varphi} = \mathbf{r}_{\varphi h} = \begin{pmatrix} -k\sin\varphi \\ k\cos\varphi \\ 0 \end{pmatrix} = \frac{\mathbf{r}_\varphi}{h}$, i.e. $\nabla_{\partial_h}\partial_\varphi = \nabla_{\partial_\varphi}\partial_h = \frac{\mathbf{r}_\varphi}{h}$:

$$\Gamma^\varphi_{h\varphi} = \Gamma^\varphi_{\varphi,h} = \frac{1}{h}\,, \quad \Gamma^h_{h\varphi} = \Gamma^h_{\varphi h}.$$

Now calculate $\mathbf{r}_{\varphi\varphi}$: $\mathbf{r}_{\varphi\varphi} = \begin{pmatrix} -kh\cos\varphi \\ -kh\sin\varphi \\ 0 \end{pmatrix}$. This vector is neither tangent to the cone nor orthogonal to the cone: $0 \neq \langle \mathbf{r}_{\varphi\varphi}, \mathbf{n}\rangle = -\frac{kh}{\sqrt{1+k^2}}$. Hence we have to consider its decomposition:

$$\mathbf{r}_{\varphi\varphi} = \underbrace{\mathbf{r}_{\varphi\varphi} - \langle \mathbf{r}_{\varphi\varphi}, \mathbf{n}\rangle \mathbf{n}}_{\text{tangent component}} + \underbrace{\langle \mathbf{r}_{\varphi\varphi}, \mathbf{n}\rangle \mathbf{n}}_{\text{orthogonal component}}$$

Hence we have

$$\nabla_\varphi \partial_\varphi = (\mathbf{r}_{\varphi\varphi})_{tangent} = \mathbf{r}_{\varphi\varphi} - \langle \mathbf{r}_{\varphi\varphi}, \mathbf{n}\rangle \mathbf{n} = \mathbf{r}_{\varphi\varphi} + \frac{kh}{\sqrt{1+k^2}}\mathbf{n} =$$

$$\begin{pmatrix} -kh\cos\varphi \\ -kh\sin\varphi \\ 0 \end{pmatrix} + \frac{kh}{1+k^2}\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix} = -\frac{hk^2}{1+k^2}\begin{pmatrix} k\cos\varphi \\ k\sin\varphi \\ 1 \end{pmatrix} = -\frac{hk^2}{1+k^2}\mathbf{r}_h,$$

i.e.

$$\Gamma^h_{\varphi\varphi} = -\frac{hk^2}{1+k^2}\,,\ \Gamma^\varphi_{\varphi\varphi} = 0\,.$$

Calculations for Levi-Civita connection using metric see in Lecture Notes.

Calculate Christoffel symbols using equations of motion of Lagrangian of free particle:

For cone: $x^2 + y^2 - k^2 z^2$ we have $\mathbf{r}(h,\varphi) = \begin{cases} x = kh\cos\ \varphi \\ y = kh\sin\varphi \\ z = h \end{cases}$ and induced metric is $G = (k^2+1)dh^2 + k^2h^2 d\varphi^2$ (see previous Homeworks) and the Lagrangian of free particle is

$$L = \frac{k^2h^2\dot\varphi^2 + (k^2+1)^2\dot h^2}{2}$$

for φ, Euler-Lagrange equations of motion:

$$\frac{\partial L}{\partial \varphi} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi}\right)\,, \quad \frac{\partial L}{\partial \varphi} = 0,\ \frac{\partial L}{\partial \dot\varphi} = k^2h^2\dot\varphi$$

hence

$$\frac{d}{dt}\left(k^2h^2\dot\varphi\right) = k^2h^2\ddot\varphi + 2k^2h\dot h\dot\varphi = 0\,,$$

i.e.

$$\ddot\varphi + \frac{2}{h}\dot h\dot\varphi = 0\,,$$

i.e. for Christoffel symbols $\Gamma^\varphi_{\cdots}$ we have:

$$\Gamma^\varphi_{\varphi h} = \Gamma^\varphi_{h\varphi} = \frac{1}{h}$$

and $\Gamma^\varphi_{\varphi\varphi}, \Gamma^\varphi_{hh}$ vanish.

For h, we have Euler-Lagrange equations of motion::

$$\frac{\partial L}{\partial h} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot h}\right)\,. \quad \frac{\partial L}{\partial h} = k^2h\dot\varphi^2,\ \frac{\partial L}{\partial \dot h} = (k^2+1)\dot h,$$

hence

$$\frac{d}{dt}\left((k^2+1)\dot{h}\right) = (k^2+1)\ddot{h} = k^2 h\dot{\varphi}^2\,, \text{i.e.}\quad \ddot{h} = \frac{k^2 h}{k^2+1}\dot{\varphi}^2$$

Hence we have that $\Gamma^h_{\varphi\varphi} = -\frac{k^2h}{k^2+1}$ and $\Gamma^h_{\varphi h} = \Gamma^h_{h\varphi} = \Gamma^h_{hh} = 0$.

c) *For the sphere:*

Riemannian metric on sphere in spherical coordinates is $G = R^2 d\theta^2 + R^2\sin^2\theta d\varphi^2$. Hence the Lagrangian of the free particle is

$$L = \frac{R^2\dot{\theta}^2 + R^2\sin^2\theta\dot{\varphi}^2}{2}$$

Euler-Lagrange equations for θ:

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)\,,\quad \frac{\partial L}{\partial \theta} = R^2\sin\theta\cos\theta\dot{\varphi}^2,\ \frac{\partial L}{\partial \dot{\theta}} = R^2\dot{\theta}$$

Hence

$$\frac{d}{dt}\left(R^2\dot{\theta}\right) = R^2\sin\theta\cos\theta\dot{\varphi}^2, R^2\ddot{\theta} = R^2\sin\theta\cos\theta\dot{\varphi}^2,$$

hence

$$\ddot{\theta} - \sin\theta\cos\theta\dot{\varphi}^2 = 0\,.$$

Comparing with equation for geodesic

$$\ddot{\theta} + \Gamma^\theta_{\theta\theta}\dot{\theta}\dot{\theta} + \Gamma^\theta_{\theta\varphi}\dot{\theta}\dot{\varphi} + \Gamma^\theta_{\varphi\theta}\dot{\varphi}\dot{\theta} + \Gamma^\theta_{\varphi\varphi}\dot{\varphi}\dot{\varphi} = \ddot{\theta} + \Gamma^\theta_{\theta\theta}\dot{\theta}\dot{\theta} + 2\Gamma^\theta_{\theta\varphi}\dot{\theta}\dot{\varphi} + \Gamma^\theta_{\varphi\varphi}\dot{\varphi}\dot{\varphi} = 0$$

we see that

$$\Gamma^\theta_{\theta\theta} = \Gamma^\theta_{\theta\varphi} = \Gamma^\theta_{\varphi\theta} = 0,\ \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta$$

Analogously Euler-Lagrange equations for φ:

$$\frac{\partial L}{\partial \varphi} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right)\,,\quad \frac{\partial L}{\partial \varphi} = 0,\ \frac{\partial L}{\partial \dot{\varphi}} = R^2\sin^2\theta\dot{\varphi}\,.$$

Hence

$$\frac{d}{dt}\left(R^2\sin^2\theta\dot{\varphi}\right) = 0,\ R^2\sin^2\theta\ddot{\varphi} + 2R^2\sin\theta\cos\theta\dot{\theta}\dot{\varphi} = 0,$$

hence

$$\ddot{\theta} + \text{cotan}\,\theta\dot{\theta}\dot{\varphi} = 0,$$

Comparing with equation for geodesic

$$\ddot{\varphi} + \Gamma^\varphi_{\theta\theta}\dot{\theta}\dot{\theta} + \Gamma^\varphi_{\theta\varphi}\dot{\theta}\dot{\varphi} + \Gamma^\varphi_{\varphi\theta}\dot{\varphi}\dot{\theta} + \Gamma^\varphi_{\varphi\varphi}\dot{\varphi}\dot{\varphi} = \ddot{\theta} + \Gamma^\varphi_{\theta\theta}\dot{\theta}\dot{\theta} + 2\Gamma^\varphi_{\theta\varphi}\dot{\theta}\dot{\varphi} + \Gamma^\varphi_{\varphi\varphi}\dot{\varphi}\dot{\varphi} = 0$$

we see that

$$\Gamma^\varphi_{\theta\theta} = \Gamma^\varphi_{\varphi\varphi} = 0\,, \Gamma^\varphi_{\varphi\theta} = \Gamma^\varphi_{\theta\varphi} = \text{cotan}\,\theta\,.$$

d) *For Lobachevsky plane*:

Lagrangian of "free" particle on the Lobachevsky plane with metric $G = \frac{dx^2+dy^2}{y^2}$ is

$$L = \frac{1}{2}\frac{\dot{x}^2+\dot{y}^2}{y^2}.$$

Euler-Lagrange equations are

$$\frac{\partial L}{\partial x} = 0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{d}{dt}\left(\frac{\dot{x}}{y^2}\right) = \frac{\ddot{x}}{y^2} - \frac{2\dot{x}\dot{y}}{y^3}, \text{i.e.} \quad \ddot{x} - \frac{2\dot{x}\dot{y}}{y} = 0\,,$$

$$\frac{\partial L}{\partial y} = -\frac{\dot{x}^2+\dot{y}^2}{y^3} = \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = \frac{d}{dt}\left(\frac{\dot{y}}{y^2}\right) = \frac{\ddot{y}}{y^2} - \frac{2\dot{y}^2}{y^3}, \text{i.e.} \quad \ddot{y} + \frac{\dot{x}^2}{y} - \frac{\dot{y}^2}{y} = 0\,.$$

Comparing these equations with equations for geodesics: $\ddot{x}^i - \dot{x}^k\Gamma^i_{km}\dot{x}^m = 0$ ($i = 1, 2$, $x = x^1, y = x^2$) we come to

$$\Gamma^x_{xx} = 0, \Gamma^x_{xy} = \Gamma^x_{yx} = -\frac{1}{y}, \; \Gamma^x_{yy} = 0, \; \Gamma^y_{xx} = \frac{1}{y}, \Gamma^y_{xy} = \Gamma^y_{yx} = 0, \Gamma^y_{yy} = -\frac{1}{y}\,. \; \blacksquare$$

The answers are the same as calculated with other methods. We see that Lagrangians give us the nice and quick way to calculate Christoffel symbols.

5 *Consider the following magnitudes:*

$$\text{a)} \quad I_{\text{cylindr}}(t) = \dot{h}(t), \quad I'_{\text{cylindr}}(t) = \dot{\varphi}(t), \quad \textit{for cylindre} \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases},$$

$$\text{b)} \quad I_{\text{cone}}(t) = h^2(t)\dot{\varphi}(t), \quad \textit{for cone} \begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases},$$

$$\text{c)} \quad I_{\text{sphere}}(t) = \sin^2\theta(t)\dot{\varphi}, \quad \textit{for sphere} \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\cos\varphi \\ z = R\cos\theta \end{cases},$$

$$\text{d)} \quad I_{\text{Lob.}}(t) = \frac{\dot{x}(t)}{y^2(t)}\,, \textit{for the Lobachevsky plane (metric } G = \tfrac{dx^2+dy^2}{y^2}\textit{)}.$$

Show that these magnitues are preserved along the corresponding geodesics. (You may use the Lagrangians from the previous exercise.)

a) Let $C\colon h(t), \varphi(t)$ be an arbitrary geodesic on cylindre.

The Lagrangian of a free prticle on cylindre is $L = \frac{a^2\dot{\varphi}^2+\dot{h}^2}{2}$.

The Lagrangian L does not depend explicitly on h hence the magnitude $\dot{h} = \frac{\partial L}{\partial \dot{h}}$ is preserved on geodesic. In detail:

$$\frac{dI}{dt} = \frac{d}{dt}\left(\dot{h}\right)\big|_{\text{on equat. of motion}} = 0$$

since on equation of motion

$$\frac{d}{dt}\left(\dot{h}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{h}}\right) = \frac{\partial L}{\partial h} = 0\,.$$

We see that $\dot{h}$ is preserved on the geodesic.

The Lagrangian L does not depend explicitly on φ also, and the magnitude $a\dot{h} = \frac{\partial L}{\partial \dot{\varphi}}$ is preserved along geodesic. In detail:

$$\frac{dI}{dt} = \frac{d}{dt}\left(a\dot{\varphi}\right)\big|_{\text{on equat. of motion}} = 0$$

since on equation of motion

$$\frac{d}{dt}\left(a\dot{\varphi}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = \frac{\partial L}{\partial \varphi} = 0\,.$$

'We see that $a\dot{\varphi}$ is preserved on the geodesic, hence $\dot{\varphi}$ is preserved also.

b) Let $C\colon h(t), \varphi(t)$ be an arbitrary geodesic on the cone $x = kh\cos\varphi, y = kh\sin\varphi, z = h$.

The Lagrangian of a free prticle on the cone is $L = \frac{(k^2+1)\dot{h}^2 + k^2\dot{\varphi}^2}{2}$.

The Lagrangian L does not depend explicitly on h and the magnitude $k^2h^2\dot{\varphi} = \frac{\partial L}{\partial \dot{\varphi}}$ is an integral of motion and it is preserving along geodesics. In detail:

$$\frac{dI}{dt} = \frac{d}{dt}\left(k^2h^2\dot{\varphi}\right)\big|_{\text{on equat. of motion}} = 0$$

since on equation of motion

$$\frac{d}{dt}\left(k^2h^2\dot{\varphi}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = \frac{\partial L}{\partial \varphi} = 0\,.$$

We see that the magnitude $k^2h^2\dot{\varphi}$ is preserved on the geodesic hence $I(t) = h^2(t)\varphi(t)$ is preserved also.

c) Let $C\colon x(t), y(t)$ be an arbitrary geodesic on the Lobachevsky plane.

The Lagrangian of a free prticle on the Lobachevsky plane is is $L = \frac{\dot{x}^2+\dot{y}^2}{2y^2}$.

The Lagrangian L does not depend explicitly on x and the magnitude $I = \frac{\dot{x}}{y^2} = \frac{\partial L}{\partial \dot{x}}$ is an integral of motion and it is preserving along geodesics. In detail:

$$\frac{dI}{dt} = \frac{d}{dt}\left(\frac{\dot{x}}{y^2}\right)\big|_{\text{on equat. of motion}} = 0$$

since on equation of motion

$$\frac{d}{dt}\left(\frac{\dot{x}}{y^2}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} = 0\,.$$

We see that the magnitude $I = \frac{\dot{x}}{y^2}$ is preserved along the geodesic on Lobachevsky plane.