**Solutions of Homework 9**

*The solution of the second part of the exercise 2 and the solution of the exercise 4 (which are not compulsory) see at the end of this file.*

**1.** *Calculate the shape operator for the sphere $x^2 + y^2 + z^2 = R^2$:*

$$\mathbf{r}(\varphi, \theta) \qquad \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases} .$$

*Calculate principal curvatures, Gaussian and mean curvature for this sphere.*

*Solution* We use results of calculations of vectors $\mathbf{r}_\theta, \mathbf{r}_\varphi$ and of unit normal vector $\mathbf{n}(\theta, \varphi)$ from the exercise 1 of the Homework 8. By the definition (see lecture notes) the action of shape operator on any tangent vector $\mathbf{v}$ is given by the formula $S\mathbf{v} = -\partial_{\mathbf{v}} S$. Hence for basis vectors $\mathbf{r}_\theta = \partial_\theta, \mathbf{r}_\varphi = \partial_\varphi$ (see the calculations in the Solutions of Homework 8) we have

$$S\mathbf{r}_\theta = -\partial_\theta \mathbf{n}(\theta, \varphi) = -\partial_\theta \left(\frac{\mathbf{r}(\theta, \varphi)}{R}\right) = -\left(\frac{\partial_\theta \mathbf{r}(\theta, \varphi)}{R}\right) = -\frac{\mathbf{r}_\theta}{R}$$

and

$$S\mathbf{r}_\varphi = -\partial_\varphi \mathbf{n}(\theta, \varphi) = -\partial_\varphi \left(\frac{\mathbf{r}(\theta, \varphi)}{R}\right) = -\left(\frac{\partial_\varphi \mathbf{r}(\theta, \varphi)}{R}\right) = -\frac{\mathbf{r}_\varphi}{R}$$

We see that shape operator is equal to $S = -\frac{I}{R}$, where $I$ is an identity operator. Its matrix in the basis $\partial_\theta, \partial_\varphi$ is equal to

$$-\begin{pmatrix} \frac{1}{R} & 0 \\ 0 & \frac{1}{R} \end{pmatrix} .$$

In the case if we choose the opposite direction for unit normal vector then we will come to the same answer just with changing the signs: if $\mathbf{n} \to -\mathbf{n}$, $S \to -S$.

We see that principal curvatures, i.e. eigenvalues of shape operator are the same:

$$\lambda_1 = \lambda_2 = -\frac{1}{R}, \text{i.e. } \kappa_1 = \kappa_2 = -\frac{1}{R}$$

(if we choose the opposite sign for $\mathbf{n}$ then $\kappa_1 = \kappa_2 = \frac{1}{R}$). Thus we can calculate Gaussian and mean curvature: Gaussian curvature

$$K = \kappa_1 \cdot \kappa_2 = \det S = \frac{1}{R^2} .$$

Mean curvature

$$H = \kappa_1 + \kappa_2 = \operatorname{Tr} S = -\frac{2}{R} .$$

If we choose the opposite sign for $\mathbf{n}$ then $S \to -S$, principal curvatures change the sign, Gaussian curvature $K = \kappa_1 \cdot \kappa_2$ does not change but mean curvature $H = \kappa_1 + \kappa_2$ will change the sign: if $\mathbf{n} \to -\mathbf{n}$ then $H = \frac{2}{R}$.

**2.** *Calculate the shape operator for the cylinder $x^2 + y^2 = R^2$:*

$$\mathbf{r}(h, \varphi) \qquad \begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \\ z = h \end{cases} .$$

*Calculate principal curvatures, Gaussian and mean curvature for this cylinder.*

† *What values the normal curvature of an arbitrary curve on the cylinder of radius $R$ can take? (You may consider first horizontal circle, vertical line and helix.)*

*Solution*

To calculate the shape operator for the cylinder we use results of calculations of vectors $\mathbf{r}_h, \mathbf{r}_\varphi$ and of unit normal vector $\mathbf{n}(\varphi, h)$ from the exercise 1b of the Homework 8. By the definition the action of shape operator on any tangent vector $\mathbf{v}$ is given by the formula $S\mathbf{v} = -\partial_\mathbf{v} S$. Hence for basis vectors $\mathbf{r}_\varphi = \partial_\varphi, \mathbf{r}_h = \partial_h$ (see the calculations in the Solutions of Homework 8) we have

$$S\mathbf{r}_h = -\partial_h \mathbf{n}(\varphi, h) = -\partial_h \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} = 0$$

and

$$S\mathbf{r}_\varphi = -\partial_\varphi \mathbf{n}(\varphi, h) = -\partial_\varphi \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} \sin\varphi \\ -\cos\varphi \\ 0 \end{pmatrix} = -\frac{\mathbf{r}_\varphi}{R}$$

(Recall that $\mathbf{n}(h, \varphi) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$ and $\mathbf{r}_\varphi = \begin{pmatrix} -R\sin\varphi \\ R\cos\varphi \\ 0 \end{pmatrix}$ (See the solutions of Homework 8).)

For an arbitrary tangent vector $\mathbf{X} = a\mathbf{r}_h + b\mathbf{r}_\varphi$, $S\mathbf{X} = -\frac{b\mathbf{r}_\varphi}{R}$. Shape operator transforms tangent vectors to tangent vectors. Its matrix in the basis $\mathbf{r}_h, \mathbf{r}_\varphi$ equals to

$$-\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{R} \end{pmatrix}$$

In the case if we choose the opposite direction for unit normal vector then we will come to the same answer just with changing the signs: if $\mathbf{n} \to -\mathbf{n}$, $S \to -S$.

We see that principal curvatures, i.e. eigenvalues of shape operator are:

$$\kappa_1 = 0, \kappa_2 = -\frac{1}{R}$$

(if we choose the opposite sign for $\mathbf{n}$ then $\kappa_1 = \kappa_2 = \frac{1}{R}$). Thus we can calculate Gaussian and mean curvature: Gaussian curvature

$$K = \kappa_1 \cdot \kappa_2 = \det S = 0\,.$$

Mean curvature

$$H = \kappa_1 + \kappa_2 = \operatorname{Tr} S = -\frac{1}{R}\,.$$

If we choose the opposite sign for $\mathbf{n}$ then $S \to -S$, principal curvatures change the sign, Gaussian curvature $K = \kappa_1 \cdot \kappa_2$ remains the same but mean curvature $H = \kappa_1 + \kappa_2$ will change the sign: if $\mathbf{n} \to -\mathbf{n}$ then $H = \frac{1}{R}$.

*Solution of the last part † of this problem see in the end of this file*

**3.** *Calculate the shape operator for the cone $x^2 + y^2 - k^2 z^2 = 0$:*

$$\mathbf{r}(h, \varphi) \qquad \begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases} .$$

*Calculate principal curvatures, Gaussian and mean curvature for this cone.*

*Solution*

To calculate the shape operator for the cone we use the results of calculations of vectors $\mathbf{r}_h, \mathbf{r}_\varphi$ and of unit normal vector $\mathbf{n}(\varphi, h)$ from the exercise 1c of the Homework 8. Recall that for the cone cone $x^2 + y^2 - k^2 z^2 = 0$,

$$\mathbf{r}(\varphi, h) \qquad \begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases}$$

and

$$\mathbf{r}_h = \frac{\partial \mathbf{r}}{\partial h} = \begin{pmatrix} k\cos\varphi \\ k\sin\varphi \\ 1 \end{pmatrix}, \ \mathbf{r}_\varphi = \frac{\partial \mathbf{r}}{\partial \varphi} = \begin{pmatrix} -kh\sin\varphi \\ kh\cos\varphi \\ 0 \end{pmatrix}$$

and normal unit vector $\mathbf{n} = \frac{1}{\sqrt{k^2+1}} \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix}$ (See for the detail the solution of Homework 8).

By the definition the action of shape operator on any tangent vector $\mathbf{v}$ is given by the formula $S\mathbf{v} = -\partial_\mathbf{v} S$. Hence for basis vectors $\mathbf{r}_h = \partial_h, \mathbf{r}_\varphi = \partial_\varphi$

$$S\mathbf{r}_h = -\partial_h \mathbf{n}(\varphi, h) = -\frac{1}{\sqrt{k^2+1}} \partial_h \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix} = 0$$

and

$$S\mathbf{r}_\varphi = -\partial_\varphi \mathbf{n}(\varphi, h) = -\partial_\varphi = -\frac{1}{\sqrt{k^2+1}} \partial_\varphi \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix} = \frac{1}{\sqrt{k^2+1}} \begin{pmatrix} \sin\varphi \\ -\cos\varphi \\ 0 \end{pmatrix} =$$

$$-\frac{1}{k\sqrt{k^2+1}} \frac{\mathbf{r}_\varphi}{h} \text{ since } \mathbf{r}_\varphi = \begin{pmatrix} -kh\sin\varphi \\ kh\cos\varphi \\ 0 \end{pmatrix}.$$

We see that for an arbitrary tangent vector $\mathbf{X} = a\mathbf{r}_h + b\mathbf{r}_\varphi$,

$$S\mathbf{X} = S(a\mathbf{r}_h + b\mathbf{r}_\varphi) = -\frac{b}{kh\sqrt{k^2+1}} \mathbf{r}_\varphi .$$

. Shape operator transforms tangent vectors to tangent vectors. Its matrix in the basis $\mathbf{r}_h, \mathbf{r}_\varphi$ equals to

$$-\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{hk\sqrt{1+k^2}} \end{pmatrix}$$

In the case if we choose the opposite direction for unit normal vector then we will come to the same answer just with changing the signs: if $\mathbf{n} \to -\mathbf{n}$, $S \to -S$.

We see that principal curvatures, i.e. eigenvalues of shape operator are:

$$\kappa_1 = 0, \kappa_2 = -\frac{1}{hk\sqrt{1+k^2}}$$

(if we choose the opposite sign for $\mathbf{n}$ then $\kappa_1 = \kappa_2 = \frac{1}{hk\sqrt{1+k^2}}$). Thus we can calculate Gaussian and mean curvature: Gaussian curvature

$$K = \kappa_1 \cdot \kappa_2 = \det S = 0 .$$

Mean curvature

$$H = \kappa_1 + \kappa_2 = \operatorname{Tr} S = -\frac{1}{hk\sqrt{1+k^2}} .$$

If we choose the opposite sign for $\mathbf{n}$ then $S \to -S$, principal curvatures change the sign, Gaussian curvature $K = \kappa_+ \cdot \kappa_-$ remains the same but mean curvature $H = \kappa_+ + \kappa_-$ will change the sign: if $\mathbf{n} \to -\mathbf{n}$ then $H = \frac{1}{hk\sqrt{1+k^2}}$.

**4** † *Calculate the shape operator, Gaussian and mean curvature for the surface*

$$\mathbf{r}(u,v) \qquad \begin{cases} x = u \\ y = v \\ z = F(u,v) \end{cases} . \tag{1}$$

*at the origin, i.e. at the point $u = v = 0$ in the case if $F = Au^2 + 2Buv + Cv^2$.*

*Consider the case if $F(u, v) = uv$ (the surface is "saddle").*

*See the solution in the end of this file.*

**5** *Assume that the action of the shape operator at the tangent coordinate vectors $\mathbf{r}_u = \partial_u$, $\mathbf{r}_v = \partial_v$ at the given point $\mathbf{p}$ of the surface $\mathbf{r} = \mathbf{r}(u, v)$ is defined by the relations: $S(\partial_u) = 2\partial_u + 2\partial_v$ and $S(\partial_v) = -\partial_u + 5\partial_v$. Calculate principal curvatures, Gaussian and mean curvatures of the surface at this point.*

*Solution*

We see that the matrix of the shape operator in the basis $\partial_u, \partial_v$ is equal to

$$S = \begin{pmatrix} 2 & -1 \\ 2 & 5 \end{pmatrix}$$

Hence Gaussian curvature $K = \det S = 12$ and mean curvature $H = \operatorname{Tr} S = 7$. To calculate principal curvatures $k_1, k_2$ note that

$$\begin{cases} k_1 + k_2 = H = 7 \\ k_1 \cdot k_2 = K = 12 \end{cases}$$

Hence $k_1 = 3, k_2 = 4$; $\kappa_1, k_2$ are eigenvalues of the shape operator.

**6** † *Let $A$ and $B$ be two given points on the sphere with radius $R$ with spherical coordinates $\{\theta_A, \varphi_A\}$ and $\{\theta_B, \varphi_B\}$.*

*$a^\dagger$) Show that the shortest curve joining these points is the arc of the great circle.*

*$b^\dagger$) Find the length of this curve.*

This problem is the same as the last question of the last problem in the Homework 8 (see the solutions of the Homework 8).

Solution of the last part † of the problem 2 (not compulsory)

† For the points of the circle $C_1$ normal curvature is equal (up to a sign) to $1/R$. Indeed consider point moving around $C_1$ with constant speed ($x = R\cos\omega t, y = R\sin\omega t, z = h_0$). Speed is equal to $v = \omega R$ and the acceleration vector $\mathbf{a} = -R\omega^2\cos\omega t\partial_x - R\omega^2\sin\omega t\partial_y$ is orthogonal to the surface: $\mathbf{a} = \omega^2 R\mathbf{n}$, where we choose unit normal vector to be $\mathbf{n} = -(x/R, y/R, 0)$ at the points $(x, y, z)$ of the cylinder ($\mathbf{n} = (-\cos\omega t, -\sin\omega t, 0)$). Normal curvature is equal to $\kappa_n = (\mathbf{a}, \mathbf{n})/(\mathbf{v}, \mathbf{v}) = \omega^2 R/\omega^2 R^2 = \frac{1}{R}$. If we choose $\mathbf{n} = +(x/R, y/R, 0) = (\cos\omega t, \sin\omega t, 0)$ then normal curvature would change a sign: $\mathbf{n} \to -\mathbf{n}$, $(\mathbf{a}, \mathbf{n}) \to -(\mathbf{a}, \mathbf{n})$ and $\kappa_{normal} \to -\kappa_{normal} = -\frac{1}{R}$.

†Consider now a point moving around the curve $C_2$ (helix) with constant speed: $x = R\cos\omega t$, $y = R\sin\omega t$, $z = vt$. Then velocity vector is equal to $\mathbf{v} = -R\omega\sin\omega t\partial_x + R\omega\cos\omega t\partial_y + v\partial_z$ and $\mathbf{a} = -R\omega^2\cos\omega t\partial_x - R\omega^2\sin\omega t\partial_y$ (it is normal (centripetal) acceleration ). We see that $v = \sqrt{\omega^2 R^2 + v^2}$, $\mathbf{a}_n = (\mathbf{a}, \mathbf{n}) = \omega^2 R$ (we choose $\mathbf{n} = -(x/R, y/R, 0)$ at the point $(x, y, z)$ of the cylinder) and normal curvature is equal to

$$\kappa_n = \frac{\omega^2 R}{\omega^2 R^2 + v^2}$$

(In the case if we change $\mathbf{n} \to -\mathbf{n}$ then normal curvature will change a sign too) One can see that

$$0 < \kappa_{normal} = \frac{\omega^2 R}{\omega^2 R^2 + v^2} \leq \frac{\omega^2 R}{\omega^2 R^2} = \frac{1}{R}$$

On the cylinder normal curvature varies from zero (for straight line) to $1/R$: $0 \leq \kappa_{normal} \leq \frac{1}{R}$

For the straight line $C_3$ normal curvature obviously is equal to 0: e.g. if particle moves on $C_3$ with constant speed, then its acceleration is equal to zero.

† Now consider an arbitrary curve on the cylinder. We know that normal curvature takes values in the interval between principal curvatures. According to calculation of shape operator we see that principal curvatures for cylinder are equal to

$$\kappa_- = 0, \kappa_+ = \frac{1}{R}.$$

Hence for an arbitrary curve on the cylinder (2) normal curvature takes values in the interval $\left[0, \frac{1}{R}\right]$. Of course this can be shown straightforwardly**.

**Remark** Note that one can consider on the cylinder $x^2 + y^2 = R^2$ a circle of very small radius $r$. The curvature (usual curvature which we studied before) of this circle will be equal to $1/r$. We see that usual curvature of curve can be very big, but normal curvature cannot be bigger than $1/R$.

Solution of the exercise 4 (not compulsory)

*Solution*

By the definition the action of shape operator on any tangent vector $\mathbf{v}$ is given by the formula $S\mathbf{v} = -\partial_{\mathbf{v}} S$. Hence for basis vectors $\mathbf{r}_u = \partial_u$ and $\mathbf{r}_v = \partial_v$ we have

$$S\mathbf{r}_u = -\frac{\partial \mathbf{n}(u,v)}{\partial u}, S\mathbf{r}_v = -\frac{\partial \mathbf{n}(u,v)}{\partial u}$$

To calculate these vectors in the origin (at the point $u = v = 0$) we need to know the value of normal unit vector field $\mathbf{n}(u,v)$ in the vicinity of the origin. Recall that the normal unit vector field $\mathbf{n}(u,v)$ equals to

$$\mathbf{n}(u,v) = \frac{1}{\sqrt{1 + F_u^2 + F_v^2}} \begin{pmatrix} -F_u \\ -F_v \\ 1 \end{pmatrix}$$

(see the Solution of Homework 8). Hence

$$S\mathbf{r}_u = -\frac{\partial \mathbf{n}(u,v)}{\partial u} = -\frac{\partial}{\partial u} \left( \frac{1}{\sqrt{1 + F_u^2 + F_v^2}} \begin{pmatrix} -F_u \\ -F_v \\ 1 \end{pmatrix} \right),$$

** Show straightforwardly that for an arbitrary curve on the cylinder (2) normal curvature takes values in the interval $\left[0, \frac{1}{R}\right]$. For convenience we choose normal unit vector attached at the point $(x, y, z)$ of the cylinder $\mathbf{n} = -(x/R, y/R, 0)$. (In this case normal curvature of circle and helix were positive). Consider an arbitrary smooth curve $\mathbf{r} = \mathbf{r}(\varphi(t), h(t))$ on the cylinder (2) and take any point $P = \mathbf{r} = (x, y, z)$ on it $(x^2 + y^2 + z^2 = R^2)$. Let $\mathbf{v}$ be velocity vector at this point, $\mathbf{v} = (v_x, v_y, v_z)$. Consider expansion of the vector $\mathbf{v}$ on horizontal and vertical components. $\mathbf{v} = \mathbf{v}_{horisont} + \mathbf{v}_{vertical}$, $\mathbf{v}_{horisontal} = v_x \partial_x + v_y \partial_y$, $\mathbf{v}_{vertical} = v_z \partial_z$. Normal (centripetal) acceleration $\mathbf{a}_n$ is equal to $\mathbf{a}_n = -a_n \mathbf{n}$, where $a_n = \frac{|\mathbf{v}_{horisontal}|^2}{R} = \frac{v_x^2 + v_y^2}{R}$. We have for normal curvature:

$$\kappa_{normal} = \frac{\frac{v^2_{horisontlal}}{R}}{(\mathbf{v}, \mathbf{v})} = \frac{v^2_{horisontlal}}{R(v^2_{horisontlal} + v^2_{vertical})} = \frac{v_x^2 + v_y^2}{R(v_x^2 + v_y^2 + v_z^2)} \le \frac{1}{R}.$$

We see that normal curvature could be less or equal than $1/R$ (normal curvature of the circle) and bigger or equal than 0 (normal curvature for straight line) (See also the example in Lecture notes.)

On the other hand if $F = Au^2 + 2Buv + Cv^2$ then it is easy to see that $\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{1+F_u^2+F_v^2}}\right) = 0$ at the point $u = v = 0$. Hence at the origin

$$S\mathbf{r}_u = -\frac{\partial \mathbf{n}(u,v)}{\partial u} = -\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{1+F_u^2+F_v^2}}\begin{pmatrix}-F_u\\-F_v\\1\end{pmatrix}\right) =$$

$$-\frac{1}{\sqrt{1+F_u^2+F_v^2}}\frac{\partial}{\partial u}\left(\begin{pmatrix}-F_u\\-F_v\\1\end{pmatrix}\right) = \begin{pmatrix}F_{uu}\\F_{vu}\\0\end{pmatrix}$$

since $\frac{1}{\sqrt{1+F_u^2+F_v^2}} = 0$ at the origin.

Analogously we come to

$$S\mathbf{r}_v = -\frac{\partial \mathbf{n}(u,v)}{\partial v} = -\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{1+F_u^2+F_v^2}}\begin{pmatrix}-F_u\\-F_v\\1\end{pmatrix}\right) = \begin{pmatrix}F_{uv}\\-F_{vv}\\0\end{pmatrix}$$

Now recall that tangent vectors $\mathbf{r}_u, \mathbf{r}_v$ are the following

$$\mathbf{r}_u = \frac{\partial \mathbf{r}(u,v)}{\partial u} = \begin{pmatrix}1\\0\\F_u\end{pmatrix},\ \mathbf{r}_v = \frac{\partial \mathbf{r}(u,v)}{\partial v} = \begin{pmatrix}0\\1\\F_v\end{pmatrix}$$

(See the exercise 1d) in the Homework 8 and the solution of this exercise in the Solutions of Homework 8) hence at the origin (at the point $u = v = 0$) for $F = Au^2 + 2Buv + Cv^2$

$$\mathbf{r}_u\big|_{u=v=0} == \begin{pmatrix}1\\0\\0\end{pmatrix}, \mathbf{r}_v\big|_{u=v=0} == \begin{pmatrix}0\\1\\0\end{pmatrix}$$

Hence we have that at the origin

$$S\mathbf{r}_u = -\frac{\partial \mathbf{n}(u,v)}{\partial u} = -\begin{pmatrix}F_{uu}\\F_{vu}\\0\end{pmatrix} = F_{uu}\mathbf{r}_u + F_{uv}\mathbf{r}_v$$

$$S\mathbf{r}_v = -\frac{\partial \mathbf{n}(u,v)}{\partial v} = -\begin{pmatrix}F_{uv}\\F_{vv}\\0\end{pmatrix} = F_{uv}\mathbf{r}_u + F_{vv}\mathbf{r}_v$$

i.e. matrix of the shape operator $S$ is

$$S = \begin{pmatrix}F_{uu} & F_{uv}\\F_{vu} & F_{vv}\end{pmatrix} = \begin{pmatrix}2A & 2B\\2B & 2C\end{pmatrix}.$$

Gaussian curvature at the origin equals to

$$K = \det S = F_{uu}F_{vv} - F_{uv}^2 = 4AC - 4B^2\,..$$

Mean curvature

$$H = \operatorname{Tr} S = F_{uu} + F_{vv} = 2A + 2C\,.$$

For the special case of the saddle shape operator

$$S = \begin{pmatrix}F_{uu}F_{uv}\\F_{vu}F_{vv}\end{pmatrix} = \begin{pmatrix}01\\10\end{pmatrix}$$

Gaussian curvature at the origin equals to

$$K = \det S = F_{uu}F_{vv} - F_{uv}^2 = -1\,.$$

Mean curvature

$$H = \operatorname{Tr} S = F_{uu} + F_{vv} = 0\,.$$