Facts about continuous fractions

Fact 1

Let α be a real number, and let $\alpha = [a_0, a_1, a_2, \ldots]$ be a continuous fraction of this munber:

$$a_0 = E(\alpha)\,, \quad a_1 = E\left(\frac{1}{\alpha - a_0}\right)\,, \quad a_2 = E\left(\frac{1}{\frac{1}{\alpha - a_0} - a_1}\right)\,,$$

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}}\,,$$

where $E(x)$, `Entier of` x:

$$E(x) \text{ is an integer } n \text{ such that } n \le x < n + 1\,.$$

Later we will suppose that a number α is positive and irrational number, thus all a_i are positive integers.

Let (p_k, q_k) be a pair of coprime integeres such that the rational number p_k/q_k is the k-th approximation of the number α:

$$\frac{p_k}{q_k} = [a_0, \ldots, a_k]\,, (p_k, q_k) = 1\,.$$

It is evident that

$$\frac{p_k}{q_k} = a_0 + \frac{r}{s} \text{ where } \frac{r}{s} = [a_1, \ldots, a_k]\,, \tag{1.1}$$

let

In the case if the rational number $[a_1,]\frac{r}{s}$

$$\frac{p_k}{q_k} = a_0 + \frac{r}{s}$$

Proposition *Let α be positive irrational number and $\alpha = [a_0, a_1, \ldots, a_n, \ldots]$.*

Consider

$$\frac{p_k}{q_k} = [a_0, \ldots, a_k]\,, \quad k = 0, 1, 2, 3,$$

be rational number. We suppose that p_k, q_k be comprime.

Then for an arbitrary $k = 0, 1, \ldots,$

1) ‘

$$\frac{p_{2k}}{q_{2k}} < \alpha < \frac{p_{2k+1}}{q_{2k+1}}\,,$$

and

$$\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k} = \frac{1}{q_{k+1}q_k}. \qquad Prop2$$

The first statement is evident, the second statement can be easily proved by induction. For $k = 0$ it is true.

Let $\left\{\frac{p_k}{q_k}\right\}$ be series of approximation of real number $\alpha = [a_0, a_1, \dots, a_n, \dots]$ by rational numbers, and let $\left\{\frac{s_k}{t_k}\right\}$ be series of approximation of real number $\alpha' = [a_1, \dots, a_n, \dots]$ by rational numbers:

$$\frac{p_k}{q_k} = [a_0, \dots, a_k]\,, k = 0, 1, 2, \dots, \qquad \frac{s_k}{t_k} = [a'_0, \dots, a'_k] = [a_1, \dots, a_{k+1}]\,.$$

Then suppose that equation (2) is already proved for $k \le m$ in equation (Prop2). Then we have that

$$\frac{p_k}{q_k}\Big|_{k=m+1} = a_0 + \frac{1}{\frac{s_k}{t_k}\Big|_{k=m}}\,,$$

i.e.

$$\frac{p_{m+1}}{q_{m+1}} = a_0 + \frac{1}{\frac{s_m}{t_m}\Big|_{k=m}} = \frac{a_0 s_m + t_m}{s_m}\,,$$

and

$$\frac{p_{m+1}}{q_{m+1}} - \frac{p_m}{q_m} = \left(a_0 + \frac{1}{\frac{s_m}{t_m}}\right) - \left(a_0 + \frac{1}{\frac{s_{m-1}}{t_{m-1}}}\right) = \frac{1}{\frac{s_m}{t_m}} - \frac{1}{\frac{s_{m-1}}{t_{m-1}}}\,.$$

Due to the inductive hypothese the right hand side is equal to

$$\frac{\pm 1}{s_m s_{m-1}} = \frac{\pm 1}{q_{m+1} q_m}\,.$$

During the School in Ratmino Sasha Vweselov explained me that this property can be formlised in the following way:

Theorem (`F.Klein`)

Convex span and continuous fraction

Let α be a number. Consider on the lattice $Z \times Z$ two sets

$$\Pi_- = \{p, q \in \mathbf{Z}\colon \frac{p}{q} < \alpha\}\,, \quad \Pi_+ = \{p, q \in \mathbf{Z}\colon \frac{p}{q} > \alpha\}\,.$$

Let such that its continuus fraction is equal to $\alpha = [a_0, a_1, a_2, \dots]$.

Consider the points A_k corresponding to approximmation of α by continuous fraction of $\alpha = [a_0, \ldots, a_n, \ldots]$:

$$A_k = (q_k, p_k)\,, \text{where} \frac{p_k}{q_k} = [a_0, \ldots, a_k]$$

Consider two polygonal chaines $L_-(\alpha)$ and $L_+(\alpha)$:

$$L_- = \cup\,(A_{2k}A_{2k+2})\ , \quad L_+ = \cup\,(A_{2k+1}A_{2k+3})\ ,$$

These chaines define the convex spans $\hat{P}_-$, $\hat{P}_+$ of the sets P_- and P_+:

$$\hat{P}_- = \text{union of trapezoids } B_{2k}A_{2k}A_{2k+2}B_{2k+2}\ ,$$

$$\hat{P}_+ = \text{union of trapezoids } B_{2k+1}A_{2k+1}A_{2k+3}B_{2k+3}\ ,$$

where $B_{2k} = (q_{2k}, 0)$ are the points on the axis OX and $B_{2k+1} = q_{2k+1}, \infty$ are the points at infinity.