From Berezinians to formal characteristic functions of maps of algebras

Hovhannes Khudaverdian and Theodore Voronov

University of Manchester, Manchester, UK

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Contents

Setup

Let A and B be commutative associative algebras with unit (say, over $\mathbb{R}$ or $\mathbb{C}$). Consider a linear map

$$\boldsymbol{\varphi}\colon\ \mathrm{A} \to \mathrm{B}\,.$$

What are “good classes” of such maps?

Example

Algebra homomorphisms: $\boldsymbol{\varphi}(1)=1$, $\boldsymbol{\varphi}(\mathrm{a}_1\mathrm{a}_2)=\boldsymbol{\varphi}(\mathrm{a}_1)\boldsymbol{\varphi}(\mathrm{a}_2)$. (The last equation is equivalent to $\boldsymbol{\varphi}(\mathrm{a}^2)=\boldsymbol{\varphi}(\mathrm{a})^2$ for all a.)

What are other interesting classes?

Algebra homomorphisms: geometric meaning

Let $A = C(X)$ and $B = \mathbb{R}$. The algebra homomorphisms $\varphi\colon C(X) \to \mathbb{R}$ correspond to the points of X:

Theorem (Gelfand–Kolmogorov, 1939)

For a compact Hausdorff topological space X, all algebra homomorphisms $C(X) \to \mathbb{R}$ are the evaluation homomorphisms $a \mapsto \mathrm{ev}_x(a) = a(x)$ (where $a \in C(X)$ and $x \in X$).

Therefore X is embedded into the linear space $C(X)^*$ as an 'algebraic variety' specified by the system of quadratic equations

$$\varphi(1) = 1$$
$$\varphi(a)^2 - \varphi(a^2) = 0$$

for $\varphi \in C(X)^*$, where a runs over all $C(X)$.

n-Homomorphisms and new developments

- Buchstaber and Rees generalized the Gelfand–Kolmogorov theorem as follows: all symmetric powers $\mathrm{Sym}^n(X)$ of the topological space X are canonically embedded into $C(X)^*$. This is based on their notion of a 'Frobenius n-homomorphism'. The embedding is given by algebraic equations (of higher degree, compared to the quadratic equations specifying the homomorphisms).
- We give an alternative approach to Buchstaber–Rees statements (more efficient) and a further generalization. Our method: a formal 'characteristic function' of a map of algebras. Source: the theory of Berezinians (superdeterminants) of linear operators acting on a superspace.

Definition

Consider an arbitrary linear map $\boldsymbol{\varphi}\colon \mathrm{A} \to \mathrm{B}$ of commutative associative algebras with unit. The (formal) characteristic function for $\boldsymbol{\varphi}$ is defined as follows:

$$\mathrm{R}(\boldsymbol{\varphi}, \mathrm{a}, \mathrm{z}) := \mathrm{e}^{\boldsymbol{\varphi} \ln(1+\mathrm{az})},$$

where $\mathrm{a} \in \mathrm{A}$ and z is a formal parameter. Note that initially $\mathrm{R}(\boldsymbol{\varphi}, \mathrm{a}, \mathrm{z})$ is just a formal power series in z:

$$\mathrm{R}(\boldsymbol{\varphi}, \mathrm{a}, \mathrm{z}) = \exp\Big(\boldsymbol{\varphi} \ln(1+\mathrm{az})\Big) = \exp\Big(\sum_{\mathrm{n}=1}^{\infty} (-1)^{\mathrm{n}} \frac{1}{\mathrm{n}} \boldsymbol{\varphi}(\mathrm{a}^{\mathrm{n}}) \mathrm{z}^{\mathrm{n}}\Big).$$

Example

Example

If φ is an algebra homomorphism, then $\mathrm{R}(\varphi, \mathrm{a}, \mathrm{z}) = 1 + \varphi(\mathrm{a})\mathrm{z}$, a linear polynomial in z.

We see that algebraic properties of the map φ are reflected in functional properties of $\mathrm{R}(\varphi, \mathrm{a}, \mathrm{z})$ w.r.t. the variable z.

Remark (Justification of the name)

Let $\varphi(\mathrm{a}) = \mathrm{tr}\,\rho(\mathrm{a})$ for a matrix representation ρ of the algebra A. Then $\mathrm{R}(\varphi, \mathrm{a}, \mathrm{z}) = \det(1 + \rho(\mathrm{a})\mathrm{z})$ is, basically, the characteristic polynomial of the operator $\rho(\mathrm{a})$.

Properties of characteristic function

- Exponential property:
  $R(\varphi_1+\varphi_2,a,z)=R(\varphi_1,a,z)R(\varphi_2,a,z)$.
- Explicit power expansion at zero:

$$R(\varphi,a,z)=1+\psi_1(\varphi,a)z+\psi_2(\varphi,a)z^2+\dots$$

where $\psi_k(\varphi,a)=P_k(s_1,\dots,s_k)$ with $s_k=\varphi(a^k)$ and

$$P_k(s_1,\dots,s_k)=\frac{1}{k!}\begin{vmatrix} s_1 & 1 & 0 & \dots & 0 \\ s_2 & s_1 & 2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ s_{k-1} & s_{k-2} & s_{k-3} & \dots & k-1 \\ s_k & s_{k-1} & s_{k-2} & \dots & s_1 \end{vmatrix}$$

(the classical Newton polynomials giving expression of elementary symmetric functions via sums of powers).

Expansion at infinity and φ-Berezinian

Suppose now that $\mathrm{R}(\varphi,\mathrm{a},\mathrm{z})$ extends to a genuine function of z regarded, say, as a complex variable. Consider its behavior at infinity. By a formal transformation, $\mathrm{R}(\varphi,\mathrm{a},\mathrm{z})=\mathrm{z}^{\varphi(1)}\mathrm{e}^{\varphi\ln\mathrm{a}}\mathrm{e}^{\varphi\ln(1+\mathrm{a}^{-1}\mathrm{z}^{-1})}$. In particular, for $\mathrm{a}=1$ we have $\mathrm{R}(\varphi,1,\mathrm{z})=(1+\mathrm{z})^{\varphi(1)}$. Hence $\varphi(1)=\chi\in\mathbb{Z}$ is an integer, which is the order of the pole at infinity. Hence we have the expansion $\mathrm{R}(\varphi,\mathrm{a},\mathrm{z})=\sum_{\mathrm{k}\leqslant\chi}\psi^*_{\mathrm{k}}(\varphi,\mathrm{a})\mathrm{z}^{\mathrm{k}}$ at infinity, where $\psi^*_{\mathrm{k}}(\varphi,\mathrm{a}):=\mathrm{e}^{\varphi\ln\mathrm{a}}\psi_{\chi-\mathrm{k}}(\varphi,\mathrm{a}^{-1})$. Denote the leading term of the expansion as

$$\mathrm{ber}(\varphi,\mathrm{a}):=\mathrm{e}^{\varphi\ln\mathrm{a}}$$

and call it, the φ-Berezinian of $\mathrm{a}\in\mathrm{A}$.

Theorem (The multiplicativity of φ-Berezinian)

$$\mathrm{ber}(\varphi,\mathrm{a}_1\mathrm{a}_2)=\mathrm{ber}(\varphi,\mathrm{a}_1)\,\mathrm{ber}(\varphi,\mathrm{a}_2)$$

n-Homomorphisms: definition

Let the characteristic function $R(\varphi, a, z)$ be a polynomial for all a. In particular it follows that the integer $\chi = \varphi(1)$ must be positive; denote it $n \in \mathbb{N}$. Hence n is the degree of $R(\varphi, a, z)$ for all a. So $\psi_k(\varphi, a) = 0$ for all $k \geqslant n+1$ and all $a \in A$.

Definition

A linear map $\varphi\colon A \to B$ satisfying $\varphi(1) = n \in \mathbb{N}$ and $\psi_k(\varphi, a) = 0$ for all $k \geqslant n+1$ and all $a \in A$ is called an n-homomorphism.

For $n = 1$ it is a usual algebra homomorphism.

Example

For $n = 2$, a 2-homomorphism satisfies $\varphi(1) = 2$ and

$$\begin{vmatrix} \varphi(a) & 1 & 0 \\ \varphi(a^2) & \varphi(a) & 2 \\ \varphi(a^3) & \varphi(a^2) & \varphi(a) \end{vmatrix} = 0\,.$$

n-Homomorphisms: properties

- The sum of an n-homomorphism and an m-homomorphism is an $(n+m)$-homomorphism.
  (Follows immediately from the exponential property of the characteristic function.)
- The composition of n-homomorphism and an m-homomorphism is an nm-homomorphism.
  (Indeed, consider
  $R(\varphi_1 \circ \varphi_2, a, z) = e^{\varphi_1 \varphi_2 \ln(1+az)} = e^{\varphi_1 \ln R(\varphi_2, a, z)} = \mathrm{ber}(\varphi_1, R(\varphi_2, a, z))$.
  Since we know that $R(\varphi_2, a, z)$ is a polynomial in z of degree at most m, and the φ_1-Berezinian $\mathrm{ber}(\varphi_1, b)$ is a polynomial in $b \in B$ of degree n, we conclude that $R(\varphi_1 \circ \varphi_2, a, z)$ has degree at most nm in z.)

(These results were originally obtained by Buchstaber and Rees, by a much harder method.)

Typical example of an n-homomorphism

Example

Let $\varphi_i\colon A \to B$ be algebra homomorphisms for all $i = 1 \ldots n$. Then

$$\varphi = \varphi_1 + \ldots + \varphi_n$$

is an n-homomorphism.

Frobenius recursion

The following construction can be traced back to Frobenius. For a given linear map $\varphi\colon A \to B$, define maps $\Phi_n\colon A \times \ldots \times A \to B$ by induction: $\Phi_1(a) = \varphi(a)$ and

$$\begin{aligned}\Phi_{k+1}(a_1, \ldots, a_{k+1}) &= \varphi(a_1)\Phi_k(a_2, \ldots, a_{k+1}) \\ &\quad - \Phi_k(a_1 a_2, \ldots, a_{k+1}) - \ldots - \Phi_k(a_2, \ldots, a_1 a_{k+1}).\end{aligned}$$

One can show by induction that the multilinear functions Φ_n are symmetric in their arguments. It follows that it is sufficient to consider them on the diagonal. Again, by induction,

$$\Phi_k(a, \ldots, a) = k!\,\psi_k(\varphi, a).$$

It follows that if $\psi_n(\varphi, a) = 0$ for all a, then $\psi_k(\varphi, a) = 0$ for all a and $k \geqslant n$.

Buchstaber–Rees theorem: statement

Examples of n-homomorphisms date back to Frobenius's works on matrix representations of finite groups. This theory was revived recently by Buchstaber and Rees motivated by studies of multi-valued groups. Their main algebraic result is the following.

Theorem (Buchstaber–Rees, 2002)

There is a one-to-one correspondence between the n-homomorphisms $\mathrm{A} \to \mathrm{B}$ and the algebra homomorphisms $\mathrm{S}^{\mathrm{n}}\mathrm{A} \to \mathrm{B}$.

Here $\mathrm{S}^{\mathrm{n}}\mathrm{A} \subset \mathrm{A}^{\otimes \mathrm{n}}$ is the symmetric power of A considered as a subalgebra of the tensor power $\mathrm{A}^{\otimes \mathrm{n}}$.

Geometric meaning

Geometrically the statement of Buchstaber and Rees gives a canonical embedding of the symmetric power $\mathrm{Sym}^n(X) = X^n/S_n$ of a topological space X into $C(X)^*$ by a system of algebraic equations.

Example

Let $n = 2$. The embedding $\mathrm{Sym}^2(X) \to C(X)^*$ is given by the formulas

$$[x_1, x_2] \mapsto \boldsymbol{\varphi} = \mathrm{ev}_{[x_1,x_2]} \quad \text{where} \quad \mathrm{ev}_{[x_1,x_2]}(a) = a(x_1) + a(x_2)\,.$$

The equations for a linear functional $\boldsymbol{\varphi}\colon\ C(X) \to \mathbb{R}$ are

$$\boldsymbol{\varphi}(1) = 2 \quad \text{and} \quad \begin{vmatrix} \boldsymbol{\varphi}(a) & 1 & 0 \\ \boldsymbol{\varphi}(a^2) & \boldsymbol{\varphi}(a) & 2 \\ \boldsymbol{\varphi}(a^3) & \boldsymbol{\varphi}(a^2) & \boldsymbol{\varphi}(a) \end{vmatrix} = 0 \quad \text{for all } a \in C(X)\,.$$

A simple proof

Using the characteristic functions, the main theorem of Buchstaber and Rees can now be easily obtained as follows. The key is to construct a homomorphism $S^nA \to B$ from an n-homomorphism $A \to B$. Set it to $\frac{1}{n!}\Phi_n(\varphi, a_1, \ldots, a_n)$. It is a linear map $S^nA \to B$. The most difficult part is to establish that it is an algebra homomorphism, i.e., multiplicative. Since the elements $a \otimes \ldots \otimes a$ span S^nA, it is sufficient to check for them. But on the diagonal we have $\frac{1}{n!}\Phi_n(a, \ldots, a) = \psi_n(\varphi, a) = \mathrm{ber}(\varphi, a)$ and we simply apply the multiplicativity of φ-Berezinian.

p|q-Homomorphisms: definition

Suppose now the characteristic function $\mathrm{R}(\varphi, \mathrm{a}, \mathrm{z})$ is not a polynomial, but a rational function. We arrive at a further generalization of ring homomorphisms.

Definition

We call a linear map $\varphi\colon \mathrm{A} \to \mathrm{B}$ a p|q-homomorphism if $\mathrm{R}(\varphi, \mathrm{a}, \mathrm{z})$ can be written as the ratio of polynomials of degrees p and q.

We have $\chi = \varphi(1) = \mathrm{p} - \mathrm{q}$ for p|q-homomorphisms.

p|q-Homomorphisms: examples

Examples

The negative $-\varphi$ of a ring homomorphism φ is a 0|1-homomorphism.
The difference $\varphi_{(\mathrm{p})} - \varphi_{(\mathrm{q})}$ of a p-homomorphism $\varphi_{(\mathrm{p})}$ and a q-homomorphism $\varphi_{(\mathrm{q})}$ is a p|q-homomorphism.
In particular, a linear combination of algebra homomorphisms

$$\varphi = \varphi_1 + \ldots + \varphi_{\mathrm{p}} - \varphi_{\mathrm{p}+1} - \ldots - \varphi_{\mathrm{p}+\mathrm{q}}$$

is a p|q-homomorphism.

(It all follows from the exponential property of the characteristic function.)

Algebraic equations for p|q-homomorphisms

The condition that $\boldsymbol{\varphi}\colon \mathrm{A}\to\mathrm{B}$ is a p|q-homomorphism can be expressed by the equations

$$\boldsymbol{\varphi}(1)=\mathrm{p}-\mathrm{q} \quad \text{and} \quad \begin{vmatrix} \psi_{\mathrm{k}}(\boldsymbol{\varphi},\mathrm{a}) & \dots & \psi_{\mathrm{k+q}}(\boldsymbol{\varphi},\mathrm{a}) \\ \dots & \dots & \dots \\ \psi_{\mathrm{k+q}}(\boldsymbol{\varphi},\mathrm{a}) & \dots & \psi_{\mathrm{k+2q}}(\boldsymbol{\varphi},\mathrm{a}) \end{vmatrix}=0 \qquad (1)$$

(the Hankel determinant), for all $\mathrm{k}\geqslant \mathrm{p}-\mathrm{q}+1$ and all $\mathrm{a}\in\mathrm{A}$.

‘Generalized symmetric powers’ for algebras and spaces

What is the geometrical meaning of this notion?
Consider a topological space X. We define its $\mathrm{p}|\mathrm{q}$-th symmetric power $\mathrm{Sym}^{\mathrm{p}|\mathrm{q}}(\mathrm{X})$ as the identification space of $\mathrm{X}^{\mathrm{p}+\mathrm{q}}$ with respect to the action of $\mathrm{S}_\mathrm{p}\times\mathrm{S}_\mathrm{q}$ and the relations

$$(x_1,\ldots,x_{p-1},y,x_{p+1}\ldots,x_{p+q-1},y)\sim(x_1,\ldots,x_{p-1},z,x_{p+1}\ldots,x_{p+q-1},z).$$

The algebraic analog of $\mathrm{Sym}^{\mathrm{p}|\mathrm{q}}(\mathrm{X})$ is the $\mathrm{p}|\mathrm{q}$-th symmetric power $\mathrm{S}^{\mathrm{p}|\mathrm{q}}\mathrm{A}$ of a commutative associative algebra with unit A. We define $\mathrm{S}^{\mathrm{p}|\mathrm{q}}\mathrm{A}$ as the subalgebra $\mu^{-1}\left(\mathrm{S}^{\mathrm{p}-1}\mathrm{A}\otimes\mathrm{S}^{\mathrm{q}-1}\mathrm{A}\right)$ in $\mathrm{S}^\mathrm{p}\mathrm{A}\otimes\mathrm{S}^\mathrm{q}\mathrm{A}$ where $\mu\colon\ \mathrm{S}^\mathrm{p}\mathrm{A}\otimes\mathrm{S}^\mathrm{q}\mathrm{A}\to\mathrm{S}^{\mathrm{p}-1}\mathrm{A}\otimes\mathrm{S}^{\mathrm{q}-1}\mathrm{A}\otimes\mathrm{A}$ is the multiplication of the last arguments.

Invariants of GL(p|q)

Example

For $A = \mathbb{C}[x]$, the algebra $S^{p|q}A$ is the algebra of all polynomial invariants of $p|q$ by $p|q$ matrices.

(This is a non-trivial statement essentially due to Berezin.)

Embedding of $\mathrm{Sym}^{p|q}(X)$ into $C(X)^*$

Example

An element $x = [x_1, \ldots, x_{p+q}] \in \mathrm{Sym}^{p|q}(X)$ defines the $p|q$-homomorphism $\mathrm{ev}_x \colon C(X) \to \mathbb{R}$:

$$a \mapsto a(x_1) + \ldots + a(x_p) - \ldots - a(x_{p+q}) .$$

This gives a natural map $\mathrm{Sym}^{p|q}(X) \to A^*$, where $A = C(X)$, which generalizes the Gelfand–Kolmogorov and Buchstaber–Rees maps (in fact, an embedding). The image of $\mathrm{Sym}^{p|q}(X)$ in A^* satisfies equations (1) above for $\boldsymbol{\varphi} \in A^*$. It is a system of polynomial equations for 'coordinates' of a linear map $\boldsymbol{\varphi} \in A^*$.

A conjectured statement is that these equations give precisely the image of $\mathrm{Sym}^{p|q}(X)$.

Definition of Berezinian

For an even invertible $p|q \times p|q$ matrix, $A = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}$, the Berezinian or superdeterminant is defined by

$$\operatorname{Ber} A = \frac{\det\left(A_{00} - A_{01}A_{11}^{-1}A_{10}\right)}{\det A_{11}}$$

(a rational expression!). It is related with the supertrace $\operatorname{str} A = \operatorname{tr} A_{00} - \operatorname{tr} A_{11}$ by the Liouville relation

$$e^{\operatorname{str} A} = \operatorname{Ber} e^{A}\,.$$

The main property of Berezinian is its multiplicativity:

$$\operatorname{Ber}(AB) = \operatorname{Ber} A \cdot \operatorname{Ber} B\,.$$

Exterior powers: recurrence relations

In the ordinary case $q=0$, $\mathrm{Ber}=\det$ and it is given by the action on the top exterior power of a vector space. In the super case, the sequence $\Lambda^k(V)$ is infinite to the right.

Theorem (Kh.–V., 2003)

If $\dim V = p|q$, then the exterior powers $\Lambda^k(V)$ satisfy recurrence relations with $q+1$ terms in an appropriate Grothendieck ring for $k\geqslant p-q+1$. For any linear operator A on V there are ‘universal recurrence relations’ for the traces $\mathrm{str}\,\Lambda^k(A)$. This can be expressed by the equations

$$\begin{vmatrix} c_k & \dots & c_{k+q} \\ \dots & \dots & \dots \\ c_{k+q} & \dots & c_{k+2q} \end{vmatrix} = 0$$

for $k\geqslant p-q+1$. Here c_k are either $\mathrm{str}\,\Lambda^k(A)$ or $\Lambda^k V$.

A new formula for Berezinian

Theorem (Kh.–V., 2003)

For $\dim V = p|q$, the Berezinian of a linear operator can be expressed as the following ratio of polynomial invariants:

$$\operatorname{Ber} A = \frac{\begin{vmatrix} c_{p-q} & \dots & c_p \\ \dots & \dots & \dots \\ c_p & \dots & c_{p+q} \end{vmatrix}}{\begin{vmatrix} c_{p-q+2} & \dots & c_{p+1} \\ \dots & \dots & \dots \\ c_{p+1} & \dots & c_{p+q} \end{vmatrix}} = \frac{|c_{p-q} \dots c_p|_{q+1}}{|c_{p-q+2} \dots c_{p+1}|_q},$$

where $c_k = \operatorname{str} \Lambda^k(A)$.

Characteristic function of a linear operator

A crucial tool for obtaining these and other results: the rational characteristic function of a linear operator

$$R_A(z) := \mathrm{Ber}(1 + zA),$$

for which we consider expansions at <u>zero</u> and at <u>infinity</u>.

Conclusion

IDEAS MOTIVATED BY SUPER GEOMETRY ARE USEFUL EVERYWHERE!

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