Linear algebra and volume element of $G_{k,n}$

`We are working together on this text, H.Kh.` & `Thomas Honey`

§ 1 **Grassmanian**

Let $V_{k,N}$ be a space of $k\times N$ real marices.

We consider the Euclidean metric in $V_{k,N}$ induced by the norm

$$||M|| = \mathrm{Tr}\,(MM^+)\,, \qquad \text{the scalar product is}\,, \langle M,N\rangle = \mathrm{Tr}\,(MN^+)\,.$$

Let $\mathcal{V}_{k,N}$ be a subset of matrices of rank k in $V_{k,N}(M\in)$:

$$\mathcal{V}_{k,N} = \{M\colon \quad M\in V_{k,N} \text{ and } \det(MM^+)\neq 0\,.\}$$

Denote by $[M]$ the plane in $\mathbf{R}^N$ spanned by the rows of matrix M. Then we have the fibre bundle of non-degenerate rectangular $k\times N$ matrices over the Grassmanian

$$\mathcal{V}_{k,N}\stackrel{\pi}{\to}G_{k,N}\ \pi(M)=[M]=\begin{array}{c}k\text{-frames in }\mathbf{R}^N\\ \downarrow\\ k\text{-planes in }\mathbf{R}^N\end{array}.$$

One can consider $\mathcal{V}_{k,N}$ as a set of frames.

In components $[M]$ is the set of matrices $M_{ia}=\lambda_{ik}M_{ka}$.

Consider an arbitrary matrix $M\in\mathcal{V}_{k,n}$. For an arbitrary matrix N consider the matrix

$$N'_{(N.M)} = N-\lambda M$$

such that the distance between N' and M is minimal:

$$N'_{(N.M)} = N - NM^+(MM^+)^{-1}M\,.$$

We see that

$$d(N,[M]) = \min_{\lambda\in GL(k)}||N-\lambda M|| = ||N-NM^+(MM^+)^{-1}M||\,,$$

where M is an arbitrary matrix in $[M]$.

Remark Minimum value may be attained for the matrix $\lambda\notin GL(k)$. To be more precise we have to write

$$d(N,[M]) = \inf_{\lambda\in GL(k)}||N-\lambda M|| = ||N-NM^+(MM^+)^{-1}M||\,,$$

is "heavily" orthogonal to the plane $[M]$:

$$N'M=0\,.$$

This is more than just to be orthogonal: $\langle N'M \rangle = 0$.

Matrix $N' = N - NM^{+}(MM^{+})^{-1}M$ which is `heavily orthogonal` to the plane $[M]$, in particular it does not depend on the choice of the frame in the plane $[M]$:

$$N'_{N,\lambda M} = N'_{N,M}.$$

Using this condition of heavily `orthogonality` we come to

$$d(N,[M]) = ||N - NM^{+}(MM^{+})^{-1}M|| =$$

$$\sqrt{\mathrm{Tr}\left[(N - NM^{+}(MM^{+})^{-1}M)\left((N - NM^{+}(MM^{+})^{-1}M)^{+}\right)\right]} =$$

$$\sqrt{\mathrm{Tr}\left[(N - NM^{+}(MM^{+})^{-1}M)\,N^{+}\right]} = \sqrt{\mathrm{Tr}\left[N\,(\mathbf{1} - M^{+}(MM^{+})^{-1}M)\,N^{+}\right]}.$$

§ 2 **Calculation of “distance”**

Now we want to define the “distance” between arbitrary two planes $[M],[N] \in G_{k,N}$.

For arbitrary frame N in the plane $[N]$ the distance $d(N,[M])$ is well defined above. Under the changing of the frame $N \mapsto N$ the matrix which defines the distance $d(N,[M])$ is transformed in a “regular way”. Compare:

$$d(\lambda \circ N,[M]) = \sqrt{\mathrm{Tr}\left[(\lambda \circ N)\,(\mathbf{1} - M^{+}(MM^{+})^{-1}M)\,(\lambda \circ N)^{+}\right]} =$$

$$\sqrt{\mathrm{Tr}\left[\lambda^{+}\lambda \circ [N\,(\mathbf{1} - M^{+}(MM^{+})^{-1}M)\,N^{+}]\right]}$$

with $d(N,[M])$.

We are ready to define the “distance” between two planes,

$$d\left([N],[M]\right) = \sqrt{\mathrm{Tr}\left(\left(N'_{(N,M)}N'^{+}_{(N,M)}\right)\left(NN^{+}\right)^{-1}\right)} =$$

$$\sqrt{\mathrm{Tr}\left[\left(N\,(\mathbf{1} - M^{+}(MM^{+})^{-1}M)\,N^{+}\right)\left(NN^{+}\right)^{-1}\right]} = \sqrt{\mathrm{Tr}\left[\mathbf{1} - NM^{+}(MM^{+})^{-1}MN^{+}(NN^{+})^{-1}\right]}$$

Is it good???

It is almost evident that

1) it is well-defined function:

$$d([\lambda_1 M],[\lambda_2 N]) = d([M],[N])$$

2) it is symmetric

$$d([M],[N]) = d([N],[M])$$

One can prove that it is positive definite. I believe (????) that triangle law is obeyed.....

To see the geometrical meaning consider for these planes orthonormal bases: i.e. M, N are such that $MM^+ = NN^+ = 1$, in these bases the functionhas very elegant expression:

$$d(N, M) = \sqrt{\mathrm{Tr}\ [\mathbf{1} - NM^+MN^+]}\,,$$

it is useful to consider rows of M as vectors $\{\mathbf{m_i}\}$ and rows of N as $\{\mathbf{n_i}\}$. They both form orthonormal bases and

$$d(N, M) = \sqrt{\mathrm{Tr}\ [\mathbf{1} - NM^+MN^+]} = \sqrt{\langle \mathbf{n}_i, \mathbf{n}_j\rangle\langle \mathbf{m}_j, \mathbf{m}_i\rangle - \langle \mathbf{n}_i, \mathbf{m}_j\rangle\langle \mathbf{m}_j, \mathbf{n}_i\rangle} = \sqrt{k - \langle \mathbf{n}_i, \mathbf{m}_j\rangle\langle \mathbf{m}_j, \mathbf{n}_i$$

Remark if it is indeed positive, then it is the version of Cauchy-Bunyakovski inequality.....???!. (see the blog for January 2019)

§ 3 Calculation of metric

We still do not know is it a distance, but we can consider its infinitesimal version: $N = N = \delta n$. We come to bilinear form on tangent vectors, and we will see that it iwill be positive definite, e.t.c., thus we will define the metric.

Let

$$N = M = \delta m, N_{ia} = M + \delta m_{ia}$$

It is convenient to consider the square of distance

$$ds^2 = d^2([N], [M]) = d([M + \delta m], [M]) =$$

$$\mathrm{Tr}\ \left[(M + \delta m)\left(\mathbf{1} - M^+(MM^+)^{-1}M\right)(M^+ + \delta m^+)\left[(M + \delta m)(M^+ + \delta m^+)\right]^{-1}\right].$$

One can see that

$$(M + \delta m)\left(\mathbf{1} - M^+(MM^+)^{-1}M\right)(M^+ + \delta m^+) = \delta m\left(\mathbf{1} - M^+(MM^+)^{-1}M\right)\delta m^+\,,$$

hence

$$ds^2 = d^2([N], [M]) = d([M + \delta m], [M]) =$$

$$\mathrm{Tr}\ \left[(M + \delta m)\left(\mathbf{1} - M^+(MM^+)^{-1}M\right)(M^+ + \delta m^+)\left[(M + \delta m)(M^+ + \delta m^+)\right]^{-1}\right] =$$

$$\mathrm{Tr}\ \left[\delta m\left(\mathbf{1} - M^+(MM^+)^{-1}M\right)\delta m^+\left[(M + \delta m)(M^+ + \delta m^+)\right]^{-1}\right].$$

For metric we can ignore infinitesiamls of order ≥ 3. We come to

Proposition Metric on tangent vectors is defined by

$$ds^2 = G = \mathrm{Tr}\ \left[\delta m\left(\mathbf{1} - M^+(MM^+)^{-1}M\right)\delta m^+\left[MM^+\right]^{-1}\right].$$

One has to prove thqt this is positive-definite. (We will see it doing straightforward calcilations.)

To wrk with this formula go to local affine coordinates:

$$M_{ia}\colon M_{ij}=\delta_{ij}\,,\,, M_{ia}=(\delta_{ij},W_{i\alpha})\,,\quad \alpha=k+1,\dots,n$$

We have

$$MM^{+}=\mathbf{1}+WW^{+}\delta m_{ia}=(0,\delta m_{i\alpha})\,,$$

and metric has the following expression in these coordinates:

$$ds^2=G=\mathrm{Tr}\,\left[\delta m\left(\mathbf{1}-W^{+}(\mathbf{1}+WW^{+})^{-1}W\right)\delta m^{+}\left[\mathbf{1}+WW^{+}\right]^{-1}\right]=$$

$$\delta m_{ia}\left[\delta_{ab}-(W^{+}(\mathbf{1}+WW^{+})^{-1}W)_{ab}\right]\delta m_{kb}\left[\mathbf{1}+WW^{+}\right]^{-1}_{ki}$$

§ 4 **Calculation of the determinant of the metric**

Calculate the determinant of the metric. We have (see the last formula above) that

$$ds^2=\delta mG\delta m=\delta m_{i\alpha}G_{ij;\alpha\beta}\delta m_{j\beta}\,,$$

where

$$G=K\otimes L=\left(\left[\mathbf{1}+WW^{+}\right]^{-1}\right)^{+}\otimes\left[\mathbf{1}-(W^{+}(\mathbf{1}+WW^{+})^{-1}W)\right]\,,$$

i.e.

$$G_{ij;ab}=K_{ij}L_{ab}\,,\quad K_{ij}=\left[\mathbf{1}+WW^{+}\right]^{-1}_{ji}\,,\,L_{ab}=\left[\delta_{\alpha\beta}-(W^{+}(\mathbf{1}+WW^{+})^{-1}W)\right]_{\alpha\beta}\,,$$

$(i,j=1,\dots,k,\ \alpha,\beta=k+1,\dots,n-k)$.

We have that

$$\det G=(\det K)^{n-k}(\det L)^{k}=\frac{1}{(\det(\mathbf{1}+WW^{+}))^{n-k}}(\det L)^{k}\,.$$

For operator L one can see that

$$\det L=\frac{1}{(\det(\mathbf{1}+WW^{+}))}\,.$$

This can be done using the elementary linear algebra *. Hence

$$\det G=\left(\frac{1}{\det(\mathbf{1}+WW^{+})}\right)^{n}\,.$$

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* Indeed consider

$$L_{\alpha\beta}=\delta_{\alpha\beta}-\left(W^{+}(\mathbf{1}+WW^{+})^{-1}W\right)_{\alpha\beta}$$

§ 5 Formula for volume of the Grassmanian

Now we have that

$$\text{Volume of } G_{k,N} = \int \sqrt{\det G} \prod_{i,\alpha} dW_{i\alpha} = \int \frac{\prod_{i,\alpha} dW_{i\alpha}}{(\det(1+WW^+))^{\frac{N}{2}}} \,.$$

Use the formula $\int e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$ we come to

$$\text{Volume of } G_{k,N} = \int \frac{\prod_{i,\alpha} dW_{i\alpha}}{(\det(1+WW^+))^{\frac{n}{2}}} \,. = \int \prod_{i=1,\alpha=k+1}^{k,N} dW_{i\alpha} \left(\frac{1}{\prod_k (1+\lambda_k)^{\frac{n}{2}}} \right) =$$

$$\text{Volume of } G_{1,N} = \int \frac{\prod_{i,\alpha} dW_{i\alpha}}{(\det(1+WW^+))^{\frac{n}{2}}} \,. = \int \prod_{i=1,\alpha=k+1}^{k,N} dW_{i\alpha} \left(\frac{1}{\prod_k (1+\lambda_k)^{\frac{n}{2}}} \right) =$$

Matrix W defines the operator which maps $\mathbf{R}^k$ to $\mathbf{R}^{n-k}$ Notice that arbitrary vector which is orthogonal to the image of this operator: $\mathbf{t}\colon W_{i\alpha} t_\alpha = 0$, we have that $L(\mathbf{t}) = \mathbf{t}$, i.e. it is the eigenvector of operator L with eigenvalue 1. On the other hand for arbitrary vector which belongs to the image of this operator $\mathbf{l}$: $\quad l_\alpha = l_k W_{k\alpha}$ (linear compbination of rows) we have that

$$L\mathbf{l}_\alpha = \left(\delta_{\alpha\beta} - \left(W^+ \left(\mathbf{1} + WW^+\right)^{-1} W \right)_{\alpha\beta} \right) l_k W_{k\beta} = l_k W_{k\alpha} -$$

$$-W_{i\alpha} \left(\mathbf{1} + WW^+\right)^{-1}_{ij} W_{j\beta} l_k W_{k\beta} l_k W_{k\alpha} = -W_{i\alpha} \left(\left(\mathbf{1} + WW^+\right)^{-1} WW^+ \right)_{ik} l_k$$

i.e.

$$(L\mathbf{l})_\alpha = \tilde{l}_k W_{k\alpha} \,, \text{ where } \tilde{l}_k = l_k - \left(\left(\mathbf{1} + WW^+\right)^{-1} \left(WW^+\right) \right)_{kn} l_n \,.$$

This means that $\det L$ is equal to the product of 1 (the determinant of this operator restricted on vectors orthogonal to the image of W) on the determinant of the operator $\mathbf{1} - \left(\left(\mathbf{1} + WW^+\right)^{-1} \left(WW^+\right) \right)$. Hence we see that

$$\det L = 1 \cdot \det \left(\mathbf{1} - \left(\left(\mathbf{1} + WW^+\right)^{-1} \left(WW^+\right) \right) \right) = \frac{1}{\det((\mathbf{1} + WW^+)}$$

The last relation follows from the fact that in the case if the operator WW^+ has diagonal representation, $WW^+ = \mathrm{diag}[\lambda_1, \dots, \lambda_n]$ then

$$\det L = \det \left(\mathbf{1} - \left(\left(\mathbf{1} + WW^+\right)^{-1} \left(WW^+\right) \right) \right) = \prod_{i=1}^{n} \left(1 - \frac{\lambda_1}{1+\lambda_i} \right) = \frac{1}{\prod_{i=1}^{n} (1+\lambda_i)}$$

$$= \frac{1}{\pi^{\frac{N}{2}}} \int \prod_{i,\alpha} dW_{i\alpha} \left(\int dz_1 dz_2 \dots dz_k \prod_{r=1}^{k} e^{-(1+\lambda_r) z_r^2} \right)^N =$$

$$\frac{1}{\pi^{\frac{N}{2}}} \int \prod_{i=1,\alpha=k+1}^{k,N} dW_{i\alpha} \left(\int \prod_{r=1}^{k} dz_r e^{-(\delta_{ij} + W_{i\alpha} W_{j\alpha}) z_i z_j} \right)^N =$$

$$\frac{1}{\pi^{\frac{N}{2}}} \int \prod_{i=1,\alpha=k+1}^{k,N} dW_{i\alpha} \prod_{r=1,b=1}^{k,N} dz_{rb} e^{-(\delta_{ij} + W_{i\alpha} W_{j\alpha}) z_{ib} z_{jb}}$$

$$= \frac{1}{\pi^{\frac{N}{2}}} \int \prod_{i,\alpha} dW_{i\alpha} \left(\int dz_1 dz_2 \dots dz_k \prod_{r=1}^{k} e^{-(1+\lambda_r) z_r^2} \right)^N =$$

$$\frac{1}{\pi^{\frac{N}{2}}} \int \prod_{i=1,\alpha=k+1}^{k,N} dW_{i\alpha} \left(\int \prod_{r=1}^{k} dz_r e^{-(\delta_{ij} + W_{i\alpha} W_{j\alpha}) z_i z_j} \right)^N =$$

$$\frac{1}{\pi^{\frac{N}{2}}} \int \prod_{i=1,\alpha=k+1}^{k,N} dW_{i\alpha} \prod_{r=1,b=1}^{k,N} dz_{rb} e^{-(\delta_{ij} + W_{i\alpha} W_{j\alpha}) z_{ib} z_{jb}}$$

$$= \frac{1}{\pi^{\frac{N}{2}}} \int \prod_{r=1,b=1}^{k,N} dz_{rb} \prod_{i=1,\alpha=k+1}^{k,N} dW_{i\alpha} e^{-(\delta_{ij} + W_{i\alpha} W_{j\alpha}) z_{ib} z_{jb}} =$$

$$\frac{1}{\pi^{\frac{N}{2}}} \int e^{-z_{ib} z_{ib}} \left(\frac{\pi}{\det [z_{ib} z_{jb}]} \right)^{\frac{k}{2}} \prod_{r=1,b=1}^{k,N} dz_{rb} = \frac{1}{\pi^{\frac{N-k}{2}}} \int \frac{e^{-z_{ib} z_{ib}}}{(\det [z_{ib} z_{jb}])^{\frac{k}{2}}} \prod_{r=1,b=1}^{k,N} dz_{rb} \,.$$

Example. Volume of $G_{1,N} = \mathbf{R}P^{N-1}$

$$\text{Volume of } G_{1,N} = \int \frac{dw_1 \dots dw_{N-1}}{(1 + w_1^2 + \dots + w_{N-1}^2))^{\frac{N}{2}}} =$$

$$\frac{1}{\pi^{\frac{N}{2}}} \int dw_1 \dots dw_{N-1} dz_1 \dots dz_N e^{-\left(1+w_1^2+\dots+w_{n-1}^2\right)\left(z_1^2+\dots+z_N^2\right)} =$$

$$\frac{1}{\pi^{\frac{N}{2}}} \int \left(dw_1 \dots dw_{N-1} e^{-\left(1+w_1^2+\dots+w_{n-1}^2\right)\left(z_1^2+\dots+z_N^2\right)} \right) dz_1 \dots dz_N =$$

$$\frac{1}{\pi^{\frac{N}{2}}} \pi^{\frac{N-1}{2}} \int \frac{e^{-\left(z_1^2+\dots+z_N^2\right)}}{\left(z_1^2 + \dots + z_N^2\right)^{\frac{N-1}{2}}} dz_1 \dots dz_N \,.$$

First calculate explicitly the second integral (this is much easier to do):
We have:

$$\text{Volume of } G_{k,N} = \frac{1}{\sqrt{\pi}} \int dw_1 \dots dw_{N-1} dz_1 \dots dz_N e^{-\left(1+w_1^2+\dots+w_{n-1}^2\right)\left(z_1^2+\dots+z_N^2\right)} =$$

$$\frac{1}{\sqrt{\pi}}\int\frac{e^{-\left(z_1^2+\ldots+z_N^2\right)}}{(z_1^2+\ldots+z_N^2)^{\frac{N-1}{2}}}dz_1\ldots dz_N=\frac{1}{\sqrt{\pi}}\int_0^\infty\frac{e^{-r^2}}{r^{N-1}}\sigma_{N-1}r^{N-1}dr=$$

$$\sigma_{N-1}\frac{1}{\sqrt{\pi}}\int_0^\infty e^{-r^2}dr=\frac{\sigma_{N-1}}{2}$$

We come to answer which is not etonnant:

$$\text{Volume of } \mathbf{R}P^n=\frac{\text{volume of } S^n \text{ in } \mathbf{E}^{n+1}}{2}\,,\qquad \left(]RP^n=S^n\backslash\frac{Z}{2Z}\right)$$

(Here we introduced $r^2=z_1^2+\ldots+z_N^2$ and $\sigma_k=$ areaq of unit k-sphere (in $\mathbf{E}^{k+1}$).
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Now caclulate explicitly the first integral (and see that the answer is the same?)

$$\text{Volume of } G_{k,N}=\int\frac{dw_1\ldots dw_{N-1}}{(1+w_1^2+\ldots+w_{N-1}^2))^{\frac{N}{2}}}=\int\frac{\sigma_{N-2}r^{N-2}dr}{(1+r^2)^{\frac{N}{2}}}=$$

$$\sigma_{N-2}\int_0^\infty\frac{u^{\frac{N-2}{2}}}{(1+u)^{\frac{N}{2}}}\frac{du}{2\sqrt{u}}=$$

To calculate this integral we use the fact that

$$F(x,y)=\int_0^\infty\frac{u^x}{(1+u)^y}=B(x,y-x-1)=$$

One can easy check this formula using substitution $t=\frac{u}{1+u}$**.

$$\sigma_{N-2}\int_0^\infty\frac{u^{\frac{N-2}{2}}}{(1+u)^{\frac{N}{2}}}\frac{du}{2\sqrt{u}}=$$

** Indeed we see that $u=\frac{t}{1-t}$, $1+u=\frac{1}{1-t}$, $du=\frac{dt}{(1-t)^2}$ and

$$F(x,y)=\int_0^\infty\frac{u^x}{(1+u)^y}du=$$

$$\int_0^1\left(\frac{t}{1-t}\right)^x(1-t)^y\frac{dt}{(1-t)^2}=\int_0^1 t^x(1-t)^{y-x-2}=B(x+1,y-x-1)=\frac{\Gamma(x+1)\Gamma(y-x-1)}{\Gamma(y)}.\ \blacksquare$$