Composition of morphisms

Let $S_1 = S_1(x,p)$ be action of thick morphism $M \to N$ and $S_2(y,t)$ be the action of thick morphism from $N \to R$, and let $S_{13} = S_{13}(x,t)$ be the action of composition of these morphisms. Let h be a function on R, $g(y) = \Phi^*_{23} h$ and $f(x) = \Phi^*_{12} g$ and

$$e^{\frac{i}{\hbar} f(x)} = \int e^{\frac{i}{\hbar}(S_{13}(x,t) - zt + h(z))} DtDz = \int e^{\frac{i}{\hbar}(S_{12}(x,q) - yq + g(y))} DqDy\,,$$

and

$$e^{\frac{i}{\hbar} g(y)} = \int e^{\frac{i}{\hbar}(S_{23}(y,t) - zt + h(z))} DtDz =$$

hence

$$e^{\frac{i}{\hbar} f(x)} = \int e^{\frac{i}{\hbar}(S_{13}(x,t) - zt + h(z))} DtDz = \int e^{\frac{i}{\hbar}(S_{12}(x,q) - yq + g(y))} DqDy =$$

$$\int e^{\frac{i}{\hbar}(S_{12}(x,q) - yq + S_{23}(y,t) - zt + h(z))} DtDzDqDy\,,$$

Thus we see that

$$e^{\frac{i}{\hbar} S_{13}(x,t} = \int e^{\frac{i}{\hbar}(S_{12}(x,q) + S_{23}(y,t) - yq} DyDq\,, \tag{1}$$

i.e. for classical case

$$S_{13}(x,t) = S_{12}(x,q) + S_{23}(y,t) - yq\,.$$

and it does not depend on y, q, i.e.

$$\frac{\partial S_{12}(x,q)}{\partial q} - y = \frac{\partial S_{23}(y,t)}{\partial y} - q = 0\,.$$

For quantum morphisms we have to consider the integral (1).

Consider the special cases

1) $S_{23}(y,t) = g(y)t$, i.e. the second morphism is usual one.

In this case

$$S_{13}(x,t) = S_{12}(x,y) + g(y)q - yq$$

and one can see that

$$\Phi^*_{13}(z) = \Phi^*_{12} g$$

2) First and second morphisms are quadratic. Then

$$e^{\frac{i}{\hbar} S_{13}(x,t} = \int e^{\frac{i}{\hbar}(S_{12}(x,q) + S_{23}(y,t) - yq} DyDq =$$

$$\int e^{\frac{i}{\hbar}\left(\frac{1}{2} x^a a_{ab} x^b + x^a \mathcal{A}_a^i q_i + \frac{1}{2} A^{ij} q_i q_j + \frac{1}{2} y^i b_{ij} y^j + y^i \mathcal{B}_i^\alpha t_\alpha + \frac{1}{2} B^{\alpha\beta} t_\alpha t_\beta - y^i q_i\right)} DyDq\,, \tag{3}$$

Classicaly

$$\begin{cases} y^i = \mathcal{A}^i_a x^a + A^{ij} q_j \\ q_i = \mathcal{B}^\alpha_i t_\alpha + b_{ij} y^j \end{cases} \Rightarrow \begin{cases} y = (1 - Ab)^{-1}(\mathcal{A}x + A\mathcal{B}t) \\ q = (1 - bA)^{-1}(\mathcal{B}t + b\mathcal{A}x) \end{cases}$$

$$S_{13}(x,t) = S_{12}(x,q) + S_{23}(y,t) - yq \quad (y = \dots, q = \dots)\,.$$

For quantum case we have to calculate (3). Rewrite it (we change little bit notations)

$$e^{\frac{i}{\hbar}S_{13}(x,t} = \int e^{\frac{i}{\hbar}(S_{12}(x,q)+S_{23}(y,t)-yq} DyDq =$$

$$\int e^{\frac{i}{\hbar}\left(\frac{1}{2}x^a a_{ab} x^b + x^a \mathcal{A}^i_a q_i + \frac{1}{2}A^{ij} q_i q_j + \frac{1}{2} y^i b_{ij} y^j + y^i \mathcal{B}^\alpha_i t_\alpha + \frac{1}{2} B^{\alpha\beta} t_\alpha t_\beta - y^i q_i\right)} DyDq =$$

$$\int \exp \frac{i}{\hbar}\left(\frac{1}{2}x^a a_{ab} x^b + x^a \mathcal{A}^i_a \tilde{q}_i + \frac{1}{2}A^{ij}\tilde{q}_i\tilde{q}_j + \frac{1}{2}\tilde{y^i} b_{ij}\tilde{y^j} + \tilde{y^i}\mathcal{B}^\alpha_i t_\alpha + \frac{1}{2}B^{\alpha\beta}t_\alpha t_\beta - \tilde{y^i}\tilde{q}_i\right) D\tilde{y}D\tilde{q} =$$

Now we change $\tilde{q} = q + \beta$, $\tilde{y} = y + \alpha$

$$\int DqDp \exp\frac{i}{\hbar}\,($$

$$\frac{1}{2}xax + x\mathcal{A}(q+\beta) + \frac{1}{2}A(q+\beta)(q+\beta) + \frac{1}{2}(y+\alpha)b(y+\alpha) + (y+\alpha)\mathcal{B}t + \frac{1}{2}Btt - (y+\alpha)(q+\beta)\Big)$$

$$= \int DqDp \exp\frac{i}{\hbar}\,($$

Now we choose α and β such that

$$\begin{cases} \mathcal{A}x + A\beta = \alpha \\ v\alpha + \mathcal{B}t = \beta \end{cases} \Rightarrow \begin{cases} \alpha = (1 - Ab)^{-1}(\mathcal{A}x + A\mathcal{B}t) \\ \beta = (1 - bA)^{-1}(\mathcal{B}t + b\mathcal{A}x) \end{cases}$$

In this way we will eliminate all the terms which are linear by q and y. We will come to

$$= \int DqDp \exp\frac{i}{\hbar}\,($$

$$\frac{1}{2}x^a a_{ab} x^b + \frac{1}{2}A^{ij} q_i q_j + \frac{1}{2} b_{ij} y^i y^j - y^i q_i + \frac{1}{2}B^{\alpha\beta} t_\alpha t_\beta +$$

$$+\mathcal{A}^i_a x^a \beta_i + \frac{1}{2}A^{ij}\beta_i\beta_j + \frac{1}{2}b_{ij}\alpha^i\alpha^j + \mathcal{B}^\alpha_i \alpha^i t_\alpha - \alpha^i\beta_i\Big) =$$

$$\frac{C}{\sqrt{\det\left(\frac{1}{2}\begin{pmatrix} b_{ij} & -1 \\ -1 & A^{ij}\end{pmatrix}\right)}} \exp\frac{i}{\hbar}\left(\frac{1}{2}x^a a_{ab}x^b + \frac{1}{2}B^{\alpha\beta}t_\alpha t_\beta +\right)$$

$$\exp\frac{i}{\hbar}\left(\mathcal{A}^i_a x^a\beta_i + \frac{1}{2}A^{ij}\beta_i\beta_j + \frac{1}{2}b_{ij}\alpha^i\alpha^j + \mathcal{B}^\alpha_i\alpha^i t_\alpha - \alpha^i\beta_i\right)\,,$$

where α^i and β_j are defined above.