Contact vector field

Let J^1M be a space of first jets of functions on manifold M. Cooridnates on J^1M are (p_i, q^j, u), where q^j are coordinates on M. Jet of every function $u = u(x)$ has coordinates $\left(p_i = \frac{\partial u}{\partial x q^i}, q^i, u\right)$.

Consider $\mathcal{C}$, the Cartan distribution of $2n$-dimensional planes in J^1M defined by the form $\omega = p_i dq^i - du$

$$\mathcal{C}_{\mathbf{p}} \subset T_{\mathbf{p}} J^1 M = \{T_{\mathbf{p}}(J^1M) \ni \mathbf{X}\colon \ \omega(\mathbf{X}) = 0\}\,,$$

Vector field

$$M^i \frac{\partial}{\partial q^i} + N_i \frac{\partial}{\partial p_i} + A \frac{\partial}{\partial u} \text{ belongs to Cartan distribution } \mathcal{C} \text{ if } A = p_i M^i\,.$$

$\mathcal{C}$ is non-integrable distribution.

Consider differential equation,

$$\mathcal{E}\colon F(p, q, u) = 0\,.$$

Differential equation is sumbmanifold of codimension 1.

The Cartan distribution $\mathcal{C}$ of hyperplanes on J^1M defines distribution $\mathcal{C}(\mathcal{E})$ in $T\mathcal{E}$:

$$\mathcal{C}(E) = \mathcal{C} \cap T\mathcal{E}\,.$$

$$\mathbf{X} = M^i \frac{\partial}{\partial q^i} + N_i \frac{\partial}{\partial p_i} + A \frac{\partial}{\partial u} \in \mathcal{C}(\mathcal{E}) \text{ if } A = p_i M^i \& \left(M^i \frac{\partial}{\partial q^i} + N_i \frac{\partial}{\partial p_i} + A \frac{\partial}{\partial u}\right) F(p, q, u)\big|_{F=0} = 0\,.$$

Definition 1 The vector field $\mathbf{K}$ in $2n+1$ is *is an infinitesiaml symmetry* of differential equation $\mathcal{E} = 0$ if it belongs to $\mathcal{C}(\mathcal{E})$:

$$\mathcal{L}_{\mathbf{X}} \mathcal{C}(\mathcal{E}) = 0 \tag{2a}$$

In what follows we consider here mostly an empty differential equation. (We focus the attention on the equation in the next file tomorrow.)

Definition 2 The vector field $\mathbf{K}$ in $2n + 1$ is called *contact vector field* if it is an infinitesimal symmetry of empty differential equation, i.e. if it preserves the Cartan distribution $\mathcal{C}$

$$\mathcal{L}_{\mathbf{X}} \mathcal{C} = 0 \tag{2b}$$

Theorem *There is one-one corrspondence between functions on M and contact vector fields:*

$$C^\infty(M) \ni F = F(p_i, q^j, u) \leftrightarrow \mathbf{X}_F$$

such that

$$F=\omega(\mathbf{X}_F)\,, \text{ and } \quad \mathbf{X}_F=\frac{\partial F}{\partial p_m}\frac{\partial}{\partial q^m}-\left(\frac{\partial F}{\partial q^m}+p_m\frac{\partial F}{\partial u}\right)\frac{\partial}{\partial p_m}+\left(p_m\frac{\partial F}{\partial p_m}-F\right)\frac{\partial}{\partial u}$$

The proof of the Theorem follows from the

Lemma If $\mathbf{X}$ is contact vector field and $\omega(\mathbf{X})\equiv 0$ then $\mathbf{X}\equiv 0$.

This lemma implies that for every function F there exists at most one contact vector field $\mathbf{X}_F$ such that $\omega(\mathbf{X}_F)=F$.

On the other hand the vecot field (3)

i) is defined for an arbitrary smooth function F

ii) it evidently obyes the condition $\omega(\mathbf{X}_F)=F$

iii) is contact vector field

Conditions ii) and iii) hold evidently. may be checked by direct calculations:

$$\omega\left(\mathbf{X}_F\right)=\left(p_m dq^m-du\right)\left(\frac{\partial F}{\partial p_m}\frac{\partial}{\partial q^m}-\left(\frac{\partial F}{\partial q^m}+p_m\frac{\partial F}{\partial u}\right)\frac{\partial}{\partial p_m}+\left(p_m\frac{\partial F}{\partial p_m}-F\right)\frac{\partial}{\partial u}\right)=$$

$$=p_m\frac{\partial F}{\partial p_m}-\left(p_m\frac{\partial F}{\partial p_m}-F\right)=F$$

and

$$\mathcal{L}_{\mathbf{X}_F}=d\left(\omega\rfloor\mathbf{X}_F\right)+d\omega\left(\rfloor\mathbf{X}_F\right)=d\left(\omega\left(\mathbf{X}_F\right)\right)+dp_m\wedge dq^m\left(\rfloor\mathbf{X}_F\right)=$$

$$=dF-\frac{\partial F}{\partial p_m}dp_m-\left(\frac{\partial F}{\partial q^m}+p_m\frac{\partial F}{\partial u}\right)dq^m=\frac{\partial F}{\partial u}\left(du-p_m dq^m\right)=F_u\omega\,,$$

i.e. $\mathbf{X}_F$ preserves the Cartan distribution $\mathcal{C}$.

It remians to prove the lemma.

Suppose that the vector field $\mathbf{X}=M^i\frac{\partial}{\partial q^i}+N_i\frac{\partial}{\partial p_i}+A\frac{\partial}{\partial u}$ is contact vector field

$$\mathcal{L}_{\mathbf{X}}\omega=\lambda\omega\,, \tag{3a}$$

and

$$\omega(\mathbf{X})=p_i M^i-A=0\,. \tag{3b}$$

Condition (3a) means that

$$\mathcal{L}_{\mathbf{X}}=d\left(\omega\rfloor\mathbf{X}\right)+d\omega\left(\rfloor\mathbf{X}\right)=d\left(\omega\left(\mathbf{X}\right)\right)+dp_m\wedge dq^m\left(\rfloor\mathbf{X}\right)=$$

$$0-M^m dp_m-N_m dq^m=\lambda(pdq^m-du)\,.$$

Thus $\lambda\equiv 0$, and $M^m\equiv 0$, $N_m\equiv 0$ and due to equation (3b), $A\equiv 0$. Hence $\mathbf{X}\equiv 0$ $\blacksquare$.

Jacobi, Moyal and Poisson brackets

The bijection between algebra of smooth functions and algebra of contact vector fields (with respect to commutator) defines brackets on functions

i) Jacobi bracket:

$$\{F,H\}\colon ,\mathbf{X}_{\{F,G\}} = [\mathbf{X}_F,\mathbf{X}_G]\ .$$

We have

$$\mathbf{X}_F = \frac{\partial F}{\partial p_m}\frac{\partial}{\partial q^m} - \left(\frac{\partial F}{\partial q^m} + p_m\frac{\partial F}{\partial u}\right)\frac{\partial}{\partial p_m} + \left(p_m\frac{\partial F}{\partial p_m} - F\right)\frac{\partial}{\partial u} = F^m\partial_m - (F_m + p_m F_u)\,\partial^m + (p_m F^m -$$

where we denote

$$F_m = \frac{\partial F}{\partial q^m},\ F^m = \frac{\partial F}{\partial p_m}\ ,\ \partial_m = \frac{\partial}{\partial q^m},\ \partial^m = \frac{\partial}{\partial p_m}, i\,F_u = \frac{\partial F}{\partial u},\ \text{and}\ \partial_u = \frac{\partial}{\partial u}\ .$$

Thus $[\mathbf{X}_F,\mathbf{X}_G] =$

$$\big[F^m\partial_m - (F_m + p_m F_u)\,\partial^m + (p_m F^m - F)\,\partial_u, G^k\partial_k - (G_k + p_k G_u)\,\partial^k + \big(p_k G^k - G\big)\,\partial_u\big] =$$

$$\big(F^m G^k_m - (F_m + p_m F_u)\,G^{mk} + (p_m F^m - F)\,G^k_u\big)\,\partial_k +$$

$$(-F^m\,(G_{mk} + p_k G_{um}) + (F_m + p_m F_u)\,(G^m_k + \delta^m_k G_u + p_k G^m_u) - (p_m F^m - F)\,(G_{ku} + p_k G_{uu}))\,\partial^k$$

$$\big(F^m\,\big(p_k G^k_m - G_m\big) - (F_m + p_m F_u)\,\big(\delta^m_k G^k + p_k G^{km} - G^m\big) + (p_m F^m - F)\,\big(p_k G^k_u - G_u\big)\big)\,\partial_u -$$

$$(F \leftrightarrow G) =$$

$$\big(F^m G^k_m - F_m G^{mk} - (F \leftrightarrow G)\big)\,\partial_k - ((F^m G_{mk} - F_m G^m_k) - (F \leftrightarrow G))\,\partial^k + \qquad (*)$$

$$\big(\big(-p_m F_u G^{mk} + p_m F^m G^k_u - F G^k_u\big) - (F \leftrightarrow G)\big)\,\partial_k$$

$$((-p_m F^m G_{ku} + F G_{ku} + p_m F_u G^m_k) - (F \leftrightarrow G))\,\partial^k +$$

$$((-p_k F^m G_{um} - p_m p_k F^m G_{uu} + F p_k G_{uu}) - ((F \leftrightarrow G))\,\partial^k$$

$$((\delta^m_k F_m G_u + p_k F_u G_u + p_k F_m G^m_u + p_m p_k F_u G^m_u) - ((F \leftrightarrow G))\,\partial^k + \mathbf{K}_{F,G} - \mathbf{K}_{F,G}$$

where $\mathbf{K}_{F,G} =$

$$\big(F^m\,\big(p_k G^k_m - G_m\big) - (F_m + p_m F_u)\,\big(\delta^m_k G^k + p_k G^{km} - G^m\big) + (p_m F^m - F)\,\big(p_k G^k_u - G_u\big)\big)\,\partial_u\ . \qquad (***)$$

Introduce the field

$$P(F,G) = \frac{\partial F}{\partial p_m}\frac{\partial G}{\partial q^m} - \frac{\partial G}{\partial p_m}\frac{\partial F}{\partial q^m} = \{F,G\}_P\ .$$

(Later we see why) Then the contact vector field corresponding to this function is equal to

$$\mathbf{X}_{\{F,G\}} = \frac{\partial}{\partial p_k}\left(F^m G_m - G^m F_m\right)\frac{\partial}{\partial q^k} - \frac{\partial}{\partial q^k}\left(F^m G_m - G^m F_m\right)\frac{\partial}{\partial p_k} - \qquad (**)$$

Notice also that the vector field (***) is equal to

$$\mathbf{K}_{F,G} = [p_k F^m G^k_m - F^m G_m - p_k F_m G^{km} - p_m p_k G^{km} F_u + p_m p_k F^m G^k_u -$$

$$-p_k F G^k_u - p_m F^m G_u + F G_u - (F \leftrightarrow G)]\partial_u =$$

$$\left[p_k \frac{\partial}{\partial p^k}\{F,G\} - \{F,G\}\right] + (FG_u - GF_u) - p_m\,(FG_u - GF_u)^m +$$

$$p_m p_k\,(F^m G_u - F_u G^m)^k$$

and it follows from the previous calculations that

$$[\mathbf{X}_F, \mathbf{X}_G] - \mathbf{X}_{\{F,G\}} =$$

since the line (*) is equa just to (**)

$$\left(\left(-p_m F_u G^{mk} + p_m F^m G^k_u - F G^k_u\right) - (F \leftrightarrow G)\right)\partial_k$$

$$\left(\left(-p_m F^m G_{ku} + F G_{ku} + p_m F_u G^m_k\right) - (F \leftrightarrow G)\right)\partial^k +$$

$$\left(\left(-p_k F^m G_{um} - p_m p_k F^m G_{uu} + F p_k G_{uu}\right) - ((F \leftrightarrow G)\right)\partial^k$$

$$\left(\left(\delta^m_k F_m G_u + p_k F_u G_u + p_k F_m G^m_u + p_m p_k F_u G^m_u\right) - ((F \leftrightarrow G)\right)\partial^k + \mathbf{K}_{F,G} - \mathbf{K}_{F,G}$$

$$\frac{\partial}{\partial p^k}\left(\frac{\partial F}{\partial p_m}\frac{\partial G}{\partial q^m} - \frac{\partial G}{\partial p_m}\frac{\partial F}{\partial q^m}\right)\frac{\partial}{\partial q^k}$$