Homework 7. Solutions

1 *Find geodesics on sphere and cylinder*

a) using straightforwardly equations for geodesics, or using the fact that for geodesic, acceleration is orthogonal to the surface.

*b *) using the fact that geodesic is shortest.*

a) Geodesics on a surface of cylinder are vertical lines, circles and helixes (see example 2 in section 3.2.1. of lecture notes)

Show here by straightforward calculations that geodesics on sphere are great circles.

The straightforward equations for geodesic: $\frac{d^2x^i}{dt^2} + \frac{dx^k}{dt}\Gamma^i_{km}\frac{dx^m}{dt} = 0$ are just equation of motion for free Lagrangian on the Riemannian surface. Hence in the case of sphere they are equations of motion of the Lagrangian of "free" particle on the sphere is $L = \frac{R^2\dot\theta^2 + R^2\sin^2\theta\dot\varphi^2}{2}$. Its equations of motion are second order differential equations

$$\begin{cases} \ddot\theta - \sin\theta\cos\theta\dot\varphi^2 = 0 \\ \ddot\varphi + 2\mathrm{cotan}\,\theta\dot\theta\dot\varphi = 0 \end{cases} \quad \text{with intitial conditions} \quad \begin{cases} \theta(t)|_{t=0} = \theta_0,\ \dot\theta(t)|_{t=0} = a \\ \varphi(t)|_{t=0} = \varphi_0,\ \dot\varphi(t)|_{t=0} = b \end{cases} \tag{1}$$

for geodesics $\theta(t), \varphi(t)$ starting at the initial point $\mathbf{p} = (\theta_0, \varphi_0)$ with initial velocity $\mathbf{v}_0 = a\frac{\partial}{\partial\theta} + b\frac{\partial}{\partial\varphi}$. (All Christoffel symbols vanish except $\Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta$, and $\Gamma^\varphi_{\varphi\theta} = \Gamma^\varphi_{\theta\varphi} = \mathrm{cotan}\,\theta$.)

This differential equation is not very easy to solve in general case. On the other hand use the fact that rotations are isometries of the sphere. Rotate the sphere in a way such that the initial point transforms to the point the point $\theta_0 = \frac{\pi}{2}, \varphi_0 = 0$ and then rotate the sphere with respect to the axis $0X$ such that θ-component of velocity becomes zero. We come to the same differential equation but with changed initial conditions:

$$\begin{cases} \ddot\theta - \sin\theta\cos\theta\dot\varphi^2 = 0 \\ \ddot\varphi + 2\mathrm{cotan}\,\theta\dot\theta\dot\varphi = 0 \end{cases} \quad \text{with intitial conditions} \quad \begin{cases} \theta(t)|_{t=0} = \frac{\pi}{2},\ \dot\theta(t)|_{t=0} = 0 \\ \varphi(t)|_{t=0} = 0,\ \dot\varphi(t)|_{t=0} = \Omega_0 \end{cases} \tag{2}$$

where we denote by Ω_0 the magnitude of initial velocity. One can easy check that the functions

$$\begin{cases} \theta(t) = \frac{\pi}{2} \\ \varphi(t) = \Omega_0 t \end{cases}$$

are the solution of the differential equations for geodesic with initial conditions (2). Hence this is geodesic passing through the point $\left(\theta_0 = \frac{\pi}{2}, \varphi = 0\right)$ with initial velocity $\Omega_0\frac{\partial}{\partial\varphi}$. We see that this geodesic is the equator of the sphere. We proved that an arbitrary geodesic after applying the suitable rotation is the great-circle—equator. On the other hand an equator is the great circle (the intersection of the sphere $x^2 + y^2 + z^2 = R^2$ with the plane $z = 0$) and the rotation transforms the equator to the another great circle. Hence all arcs of great circles are geodesics and all geodesics are arcs of great circles.

b*) See the lecture notes the section 3.4.1 ("Again on geodesics on sphere and on Lobachevsky plane".)

2 *Great circle is a geodesic. Every geodesic is a great circle.*

Are these statements correct?

Make on the base of these statements correct statements and justify them.

The correct statments are: great circles indeed are *un-parametersied* geodesics. One can consider suitable parameterisation of great circle such that it becomes geodesic (i.e. parameterised geodesics) For this prupsoe one has to consider a parameterisation such that speed is constant in this parameterisation.

Every geodesic conisdered as unparametersied curve is great circle.

3 *Show that vertical lines $x = a$ are geodesics (non-parameterised) on Lobachevsky plane.*

We consider here the realisation of Lobachevsky plane (hyperbolic plane) as upper half of Euclidean plane $\{(x, y): y > 0\}$ with the metric $G = \frac{dx^2+dy^2}{y^2}$.

Consider second order differential equations defining geodesics with initial conditions such that "horisontal" velocity equals to zero: (we uses the information from Homework 6 or from Lecture notes about Christoffels for Lobachevsky plane: $\Gamma^x_{xx} = 0, \Gamma^x_{xy} = \Gamma^x_{yx} = -\frac{1}{y}$, $\Gamma^x_{yy} = 0$, $\Gamma^y_{xx} = \frac{1}{y}, \Gamma^y_{xy} = \Gamma^y_{yx} = 0, \Gamma^y_{yy} = -\frac{1}{y}$.)

$$\begin{cases} \ddot{x} - \frac{2\dot{x}\dot{y}}{y} = 0 \\ \ddot{y} + \frac{\dot{x}^2}{y} - \frac{\dot{y}^2}{y} = 0 \\ x(t)\big|_{t=t_0} = x_0, \dot{x}(t)\big|_{t=t_0} = 0 \\ y(t)\big|_{t=t_0} = y_0, \dot{y}(t)\big|_{t=t_0} = \dot{y}_0 \end{cases}$$

This equation has a solution and it is unique. One can see that if we put $x(t) \equiv 0$, i.e. curve is vertical then we come to the equation $\ddot{y} - \frac{\dot{y}^2}{y} = 0$. Solution of these equation gives curve $x = x_0$, $y = y(t)$: $\ddot{y} - \frac{\dot{y}^2}{y} = 0$. The image of this curve clearly is vertical ray $x = x_0, y > 0$.

4 *Consider a vertical ray C: $x(t) = 1, y(t) = 1 + t$, $0 \le t < \infty$ on the Lobachevsky plane.*

Find the parallel transport $\mathbf{X}(t)$ of the vector $\mathbf{X}_0 = \partial_y$ attached at the initial point $(1, 1)$ along the ray C at an arbitrary point of the ray.

Find the parallel transport $\mathbf{Y}(t)$ of the vector $\mathbf{Y}_0 = \partial_x + \partial_y$ attached at the same initial point $(1, 1)$ along the ray C at an arbitrary point of the ray. (Exam question, 2013.)

Since vertical ray is geodesic then during parallel transport vector $\mathbf{X}(t)$ remains proportional to velocity vector. Hence $\mathbf{X}(t) = k(t)\partial_y$. On the other hand during parallel transport its length is not changed, since the connection is Levi-Civita connection. i.e. scalar product

$$\langle \mathbf{X}(t), \mathbf{X}(t) \rangle = \langle k(t)\partial_y, k(t)\partial_y \rangle = \frac{k^2(t)}{(y_0 + t)^2} = \frac{k^2(t)}{(1 + t)^2} = Constant$$

At the moment $t = 0$ it is equal to $\frac{1}{1} = 1$. We have $\frac{k^2(t)}{(1+t)^2} = 1$, i.e. $k(t) = \pm(1 + t)$

Since at $t = 0$, $k = 1$ we choose sign $+$ and $k(t) = 1 + t$. We come to $\mathbf{X}(t) = (1 + t)\partial_y$

During parallel transport of two vectors along Levi-Civita connection not only their lengths but angles between them is not changed too.

Consider vector $\mathbf{Z} = \partial_x$. it is orthogonal to vector $\mathbf{X}$. Hence during parallel transport it will remain orthogonal. Hence $\mathbf{Z}(t) = k'(t)\partial_x$ since vectors ∂_x, ∂_y are orthogonal to each other at any point of the Lobachevsky plane. The length fo the vector $\mathbf{Z}(t)$ is preserved too. Hence it has to be equal always to 1 since at $t = 0$ it is equal to 1. We come to $\mathbf{Z}(t) = (1 + t)\partial_x$.

Now by linearity of parallel transport $\mathbf{Y}(t) = \mathbf{X}(t) + \mathbf{Z}(t) = (1 + t)(\partial_x + \partial_y)$.

5 *Find a parameterisation of vertical lines in the Lobachevsky plane such that they become parameterised geodesics.*

We know also that vertical line is geodesic. Let $x = x_0, y = f(t)$ be right parameterisation, i.e. parameterisation such that veclotiy vector remains velocity vector during parallel transport. Velocity vector $\mathbf{v}(t) = \begin{pmatrix} 0 \\ f_t \end{pmatrix}$. Its length is equal to $\sqrt{\frac{x_t^2+y_t^2}{y^2}} = \sqrt{\frac{0+f_t^2}{f^2}} = \frac{f_t}{f}$ and it has remain the same. Hence $\frac{f_t}{f} = c$, i.e. $f(t) = Ae^{ct}$. We see that $x = x_0, y = ae^{ct}$ is parameterised geodesic. (One can see that differential equation of geodesics are obeyed (see the exercise 2)).

6 *Consider the plane $\mathbf{R}^2$ with Cartesian coordinates and with Riemannian metric*

$$G = \frac{4R^2(du^2 + dv^2)}{(R^2 + u^2 + v^2)^2)}.$$

Show that all lines passing through the origin ($u = v = 0$) and only these lines are geodesics of the Levi-Civita connection of this metric.

Give examples of other geodesics.

† *Find all geodesics of this metric.*

(You may use the fact that this Rimeannian manifold is isometric to the sphere without North pole.)

$\mathbf{R}^2$ with this Riemannian metric is isometric to the sphere of raidus R without North Pole. One can say in other way: u, v are stereographic coordiantes on the sphere and the metric G is just the metric of the sphere.

The straigt lines passing thorug origin is $u = kv$ or $v = ku$, is the image of great cirle which passes through North Pole: it is evident form the definition, but one can see it also from the formulae for stereographic coordiantes:

$$u = \frac{Rx}{R-z}, \quad v = \frac{Rx}{R-z}$$

We have $\frac{u}{v} = k$, hence $\frac{x}{y=k}$. Hence the line $u = kv$ is the image of the curve $x = ky$ on the sphere. This curve is intersection of the plane $x = ky$ with sphere, it is a great circle.

Now we know that all geodesics are images of great circles under stereographic projection. E.g. if we take an equatior on the sphere: $x^2 + y^2 = R^2$, $z = 0$, then we come to $u = x, v = y$, i.e. $u^2 + v^2 = R^2$ is aq geodesic.

Remark Other circles with centre at origin will not be geodesics, since they are image of the circles which are intersection of the plane $z = a \neq 0$ with a sphere, and this i snto a great circle.

† Find all geodesics. (For simplicity we consider only the case $R = 1$.) Any great circle on sphere is the intersection of the plane passing thorugh origin and the sphere. Hence it is given ny equations $\begin{cases} Ax + By + Cz = 0 \\ x^2 + y^2 + z^2 = 1 \end{cases}$ Using formulae for stereographic projections we see that in coordiantes u, v these equations will be rewritten in the following way:

$$A\frac{2u}{1+u^2+v^2} + B\frac{2v}{1+u^2+v^2} + C\frac{u^2+v^2-1}{1+u^2+v^2} = 0\,,$$

i.e. they are lines passing through origin if $C = 0$ and in the case if $C \neq 0$ we come to circles such that

$$(u+a)^2 + (v+b)^2 = 1 + a^2 + b^2\,.$$

, we see that geodesics are lines passing through origin and circles such that $R = \sqrt{1+d^2}$, where d is a distance between origin and the centre of the circle and R is radiis of the circle.

7 *Find parallel transport of vector $\frac{\partial}{\partial y}$ attached attached at the point $(0,1)$ of Lobachevsky plane along curve C: $x = t, y = \sqrt{1-t^2}$, $0 \leq t < 1$.*

(You may use the facts about geodesics in Lobachevsky plane.)

Notice that C is the arc of the geodesic– C: $\begin{cases} x = t \\ y = \sqrt{1-t^2} \end{cases}$, $0 < t < 1$ is upper semicircle with center on the absolute $y = 0$. The vector $\mathbf{Y} = \frac{\partial}{\partial y}$ is not tangent to half-circle circle C at the point $(0,1)$. Another vector, vector $\mathbf{X} = \frac{\partial}{\partial x}$ at the $(0,1)$ is tangent to C. Since C is geodesics, vector $\mathbf{X}$ will remain tangent to the circle during parallel transport. Hence during parallel transport it will be proportional to the tangent vector; the vector $\mathbf{Y}$ initially is orthogonal to the vector $\mathbf{X}$ (orthogonality in Euclidean and Lobachevsky metric is the same: since both metrics are proportional to $dx^2 + dy^2$). Hence during parallel transport vector $\mathbf{Y}(t)$ will be always orthogonal to the vector $X(t)$ since parapllel transport preserves orthogonality (vanishing of scalar product). Hence we see that up to coefficient vector fields $\mathbf{X}(t)$ and vector fields $\mathbf{Y}(t)$ are equal to:

$$\mathbf{X}(t) \sim \begin{pmatrix} \sqrt{1-t^2} \\ -t \end{pmatrix} = a(t)\left(\sqrt{1-t^2}\frac{\partial}{\partial x} - t\frac{\partial}{\partial y}\right), \mathbf{Y}(t) \sim \begin{pmatrix} t \\ \sqrt{1-t^2} \end{pmatrix} = b(t)\left(t\frac{\partial}{\partial x} + \sqrt{1-t^2}\frac{\partial}{\partial y}\right),$$

where $b(t)|_{t=0} = 1$. Now note that preservation of scalar product means that length does not change:

$$\langle \mathbf{Y}(t), \mathbf{Y}(t) \rangle = b^2(t) \frac{t^2 + (\sqrt{1-t^2})^2}{y^2} = \frac{b^2(t)}{1-t^2} = consta$$

Hence $b(t) = \sqrt{1-t^2}$ (since $b(t)|_{t=0} = 1$).

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8* *Let $\mathbf{X}(t)$ be parallel transport of the vector $\mathbf{X}$ along the curve on the surface M embedded in $\mathbf{E}^3$, i.e. $\nabla_{\mathbf{v}} \mathbf{X} = 0$, where $\mathbf{v}$ is a velocity vector of the curve C and ∇ Levi-Civita connection of the metric induced on the surface. Compare the condition $\nabla_{\mathbf{v}} \mathbf{X} = 0$ (this is condition of parallel transport for internal observer) with the condition that for the vector $\mathbf{X}(t)$, the derivative $\frac{d\mathbf{X}(t)}{dt}$ is orthogonal to the surface (this is condition of parallel transport for external observer)[2).*

Do these two conditions coincide, i.e. do they imply the same parallel transport?

We know that Levi-Civita connection on surfaces coicides with induced conenction. We have by definition of induced connection that

$$\nabla_{\mathbf{v}} \mathbf{X} = \left(\nabla_{\mathbf{v}}^{\text{can. flat}} \mathbf{X} \right) = \left(\partial_{\mathbf{v}} \mathbf{X} \right)_{\text{tangent}} = \left(\frac{d\mathbf{X}(t)}{dt} \right)_{\text{tangent}}$$

Hence $\nabla_{\mathbf{v}} \mathbf{X} = 0$ if and only if the derivative $\frac{d\mathbf{X}(t)}{dt}$ is orthogonal to the surface.

* **Remark** Let us recall how to show that upper arcs of semicircles $(x-a)^2 + y^2 = R^2, y > 0$ are (non-parametersied) geodesics.

You may do this exercise solving explicitly differential equations for geodesics, but it is much more nice to use inversion (Mobius) transformation: Consider the inversion of the Lobachevsky plane with the centre at the point $x = a - R, y = 0$ (see the exercise above). This inversion does not change Riemannian metric, it is isometry. Isometry transforms geodesics to geodesics. On the other hand it transforms the semicircle $(x-a)^2 + y^2 = R^2, y > 0$ to the vertical ray $x = a - R + \frac{1}{2R}, y > 0$. This can be checked directly. On the other hand the vertical ray is geodesic. Hence the initial curve was the geodesic too.

2) *We defined parallel transport in Geometry course using this condition*