Solutions of Homework 6

`this text is under changing. Some questions has to be omitted.`

1

Calculate the integrals of the form $\omega = xdy - ydx$ over the following three curves. Compare answers.

$$C_1\colon \mathbf{r}(t) \begin{cases} x = R\cos t \\ y = R\sin t \end{cases},\ 0 < t < \pi, \quad C_2\colon \mathbf{r}(t) \begin{cases} x = R\cos 4t \\ y = R\sin 4t \end{cases},\ 0 < t < \frac{\pi}{4}$$

$$\textit{and}\quad C_3\colon \mathbf{r}(t) \begin{cases} x = Rt \\ y = R\sqrt{1-t^2} \end{cases},\ -1 \le t \le 1.$$

We have that

$$\int_C \omega = \int_{t_1}^{t_2} \omega(\mathbf{v}(t))dt = \int_{t_1}^{t_2} (xdy - ydx)(x_t\partial_x + y_t\partial_y)dt = \int_{t_1}^{t_2} (-y(t)x_t(t) + x(t)y_t(t))dt\,,$$

where $\mathbf{v} = (x_t, y_t)$ is velocity vector: $dx(\partial_x) = dy(\partial_y) = 1$, $dx(\partial_y) = dy(\partial_x) = 0$.)

For the first curve C_1 we have $\mathbf{v}(t) = (-R\sin t, R\cos t)$ and $\int_{C_1}\omega = \int_0^\pi (xdy - ydx)(-R\sin t\partial_x + R\cos t\partial_y) =$

$$\int_0^\pi (R\cos tdy - R\sin tdx)(-R\sin t\partial_x + R\cos t\partial_y) = \int_0^\pi (R^2\cos^2 t + R^2\sin^2 t)dt = \int_0^\pi R^2 \cdot dt = \pi R^2\,.$$

For the second curve C_2 we have $\mathbf{v}(t) = (-4R\sin 4t, 4R\cos 4t)$ and $\int_{C_2}\omega = \int_0^{\frac{\pi}{4}} (xdy - ydx)(-4R\sin 4t\partial_x + 4R\cos 4t\partial_y) =$

$$\int_0^{\frac{\pi}{4}} (R\cos 4tdy - R\sin 4tdx)(-4R\sin 4t\partial_x + 4R\cos 4t\partial_y) = \int_0^{\frac{\pi}{4}} (4R^2\cos^2 4t + 4R^2\sin^2 4t)dt = \int_0^{\frac{\pi}{4}} 4R^2 \cdot dt = \pi R^2\,.$$

Answer is the same. The second curve is reparameterised first curve ($t \mapsto 4t$) and reparameterisation preserves the orientation: $(4t)' = 4 > 0$.

For the third curve C_3 we have $\mathbf{v}(t) = \left(-R, -\frac{Rt}{\sqrt{1-t^2}}\right)$ and $\omega(\mathbf{v}(t)) = (xdy - ydx)(v_x\partial_x + v + y\partial_y) =$

$$= \left(Rtdy - R\sqrt{1-t^2}dx\right)\left(R\partial_x - \frac{Rt}{\sqrt{1-t^2}}\partial_y\right) = -R^2\sqrt{1-t^2} - \frac{R^2t^2}{\sqrt{1-t^2}} = -\frac{R^2}{1-t^2}\,.$$

Hence

$$\int_{C_3}\omega = \int_0^1 \omega(\mathbf{v}(t)) = \int_0^1 \left(-\frac{R^2}{\sqrt{1-t^2}}\right)dt = -R^2\int_0^1 \frac{dt}{\sqrt{1-t^2}} = -\pi R^2\,.$$

Answer is the same up to a sign: This curve is reparameterised first curve. If we put $t = \cos\tau$ then third curve C_3 will transform to the first curve C_1. This reparameterisation changes the orientation, because $(\cos t)' = -\sin t < 0$ on the interval $(0, \pi/2)$.

Resumé: In these three examples was considered an integral over the same (non-parameteresed) half-circle. All the answers are the same up to a sign. Sign changes if reparameterisation changes an orientation. Sure if we already know the information about orientation of these curves we did not need to calcualte all the three integrals: $\int_{C_1}\omega$,$\int_{C_2}\omega$, and $\int_{C_3}\omega$. It is enough to calculate one of these integrals, e.g. the first one (the calculations are little bit simpler for first one) then use the fact that curve C_2 has the same orientation with curve C_1 and curve C_3 has opposite orientation to the curve C_1, hence

$$\int_{C_1}\omega = \pi R^2 = \int_{C_2}\omega = -\int_{C_3}\omega\,.$$

(For solutions see also lecture notes the end of subsection 2.5)

2

Consider an arc of parabola $x = 2y^2 - 1$, $0 < y < 1$.

Give examples of two different parameterisations of this curve such that these parameterisations have the opposite orientation.

Calculate the integral of the form 1-form $\omega = \sin y dx$ over this curve.

How does the answer depend on a parameterisation?

To consider a different parameterisation we may take an arbitrary number $n \neq 0$ and consider

$C_n \colon \mathbf{r}(t) \begin{cases} x = 2n^2t^2 - 1 \\ y = nt \end{cases}, \ 0 < t < 1/n,$

These two different parameterisation are related with the reparameterisation $t' = nt$. If $n > 0$, then reparameterisation preserves orientation, If $n < 0$, then reparameterisation changes orientation of the curve. For example if we take $n = 2$ then we will come to the curve

$C_2 \colon \mathbf{r}(t) \begin{cases} x = 8t^2 - 1 \\ y = 2t \end{cases}, \ 0 < t < 1/2,$, with the same orientation as initial curve and if we will take $n = -2$ we will come to the curve

$C_2' \colon \mathbf{r}(t) \begin{cases} x = 8t^2 - 1 \\ y = -2t \end{cases}, \ -\frac{1}{2} < t < 0,$,

we will come to the curve with orientation opposite to the orientation opposite to the orientation of the initial curve.

Sure we can change parametersiation in a different way. E.g. we may consider

$C_3 \colon \mathbf{r}(t) \begin{cases} x = 2\cos^2 t - 1 = \cos 2t \\ y = \cos t \end{cases}, \ 0 < t < \frac{\pi}{2}$

Curve C_3 has orientation opposite to the orientation of the curves C, C_2 and the same orienation with the curve C_2' since reparameterisation $t' = \cos t$ changes orienation ($\frac{dt'}{dt} = -\sin t < 0$ for $0 \leq t \leq \frac{\pi}{2}$).

Now we calculate integrals for all these curves. Sure we do not need to do it, it suffices to calculate the integral just for one curve, and then using orientation arguments to find integrals for other curves, but just for exercise we will do all examples.

For any curve $\mathbf{r}(t), t_1 < t < t_2$

$$\int_C \omega = \int_C \sin y dx = \int_C \sin y dx(\mathbf{v}) = \int_{t_1}^{t_2} \sin y(t) \frac{dx(t)}{dt} dt$$

where $\mathbf{v} = (x_t, y_t)$.

For the first curve C_1 $x_t = 4t$ and

$$\int_{C_1} \omega = \int_0^1 4t \sin t dt = 4(-t\cos t + \sin t)\big|_0^1 = -4\cos 1 + 4\sin 1$$

For the second curve C_2 $x_t = 16t$ and

$$\int_{C_2} \omega = \int_0^{1/2} 16t \sin 2t dt = 4(-2t\cos 2t + \sin 2t)\big|_0^{1/2} = -4\cos 1 + 4\sin 1\,.$$

Answer is the same. Non-surprising. The second curve is reparameterised first curve ($t \mapsto 2t$) and reparameterisation preserves the orientation.

For the third curve C_2' $x_t = 16t$ and

$$\int_{C_2} \omega = \int_{-1/2}^0 16t \sin(-2t) dt = -4(-2t\cos 2t + \sin 2t)\big|_{-1/2}^0 = 4\cos 1 - 4\sin 1\,.$$

Answer is the same up to a sign. Non-surprising. This curve is reparameterised first curve ($t \mapsto -2t$) and reparameterisation changes the orientation.

For the last curve $x_t = -2\sin 2t dt$ and

$$\int_{C_3} w = \int_0^{\frac{\pi}{2}} (-2\sin 2t)\sin(\cos t)dt = -4\left(\cos t\cos(\cos t) - \sin(\cos t)\right)\Big|_{-1/2}^{\pi/2} = 4\cos 1 - 4\sin 1$$

Answer is the same as for the previous curve: This curve is reparameterised first curve with opposite orientation ($t \mapsto \cos t$) and reparameterisation changes the orientation, because $(\cos t)' = -\sin t < 0$ on the interval $(0, \pi/2)$. hence the first integral is equal to the third on and it has a sign opposite to the second and first one.

Resumé: In these three examples an integral over the same (non-parameteresed) curve was considered. All the answers are the same up to a sign. Sign changes if reparameterisation changes an orientation.

3

Calculate the integral of the form $\omega = xdy$ over the following curves
a) closed curve $x^2 + y^2 = 12y$.
b) arc of the ellipse $x^2 + y^2/9 = 1$ defined by the condition $y \geq 0$.

a) Consider closed curve $x^2 + y^2 = 12y$. We have

$$0 = x^2 + y^2 - 12y = x^2 + (y-6)^2 - 36\,.$$

That is this curve is a circle of the radius 6 with a centre at the point $(0, 6)$. The parametric equation of this circle is

$$\begin{cases} x = 6\cos t \\ y = 6 + 6\sin t \end{cases}, \quad 0 \leq t \leq 2\pi\,.$$

We have that

$$\mathbf{v} = \begin{pmatrix} -6\sin t \\ 6\cos t \end{pmatrix} \text{ and } \quad \omega(\mathbf{v}) = xdy(v_x\partial_x + v_y\partial_y) = xv_y = 6x(t)\cdot 6\cos t = 36\cos^2 t\,,$$

$$\int_C \omega = \int_0^{2\pi} \omega(\mathbf{v}(t))dt = \int_0^{2\pi} 36\cos^2 t\, dt = 36\cdot\frac{2\pi}{2} = 36\pi\,.$$

So for an arbitrary parameterisation answer will be $\pm 36\pi$. (36π if orientation is the same and -36π if opposite) E.g. if we change parameterisation above on the parameterisation $\tau = -t$ then integral will change a sign, since this reparameterisation changes the orientation of the circle.

b) For the the arc of the ellipse $x^2 + y^2/9 = 1, y \geq 0$ choose a parameterisation: $\begin{cases} x = \cos t \\ y = 3\sin t \end{cases}$, $0 \leq t \leq \pi$. Then $\mathbf{v} = (-\sin t, 3\cos t)$ and

$$\int_C \omega = \int_0^{\pi} \omega(\mathbf{v})dt = \int_0^{\pi} x(t)y_t dt = \int_0^{\pi 2} 3\cos t\cos t dt = \int_0^{\pi} 3\cos^2 t dt = 3\pi/2$$

So for an arbitrary parameterisation answer will be $\pm 3\pi/2$, sign is depending on orientation of parameterisation. E.g. if we change parameterisation above on the parameterisation $\tau = -t$ then integral will change a sign, since this reparameterisation changes the orientation of the ellipse.

Exact forms

4

Calculate the integral $\int_C \omega$ where $\omega = xdx + ydy$ and C is

a) the straight line segment $x = t, y = 1 - t$, $0 \le t \le 1$
b) the segment of parabola $x = t$, $y = 1 - t^n$, $0 \le t \le 1$, $n = 2, 3, 4, \dots$
c) **an arbitrary** *curve starting at the point $(0, 1)$ and ending at the point $((1, 0)$.*

For any of these curves we can perform calculations naively just using definition of integral
E.g. for the curve a)

$$\int_C \omega = \int_0^1 (x(t)x_t + y(t)y_t)dt = \int_0^1 (t + (1-t)(-1))dt = \int_0^1 (2t-1)dt = 0\,,$$

for the curve b) if $n = 2$

$$\int_C \omega = \int_0^1 (x(t)x_t + y(t)y_t)dt = \int_0^1 (x(t)x_t + y(t)y_t)dt = \int_0^1 (t + (1-t^2)(-2t))dt = \int_0^1 (2t^3 - 3t^2)dt = 0\,,$$

for the curve b) in general case:

$$\int_C \omega = \int_0^1 (x(t)x_t + y(t)y_t)dt = \int_0^1 (x(t)x_t + y(t)y_t)dt =$$

$$\int_0^1 (t + (1-t^n)(-nt^{n-1}))dt = \int_0^1 (t - nt^{n-1} + nt^{2n-1})dt = 0\,.$$

But there is another nice way to calculate these integrals. We immediately come to these results in a clear and elegant way if we use the fact that $\omega = xdx + ydy$ is an **exact form**, i.e. $\omega = df$ where $f = \frac{x^2+y^2}{2}$. Indeed using Theorem we see that for an arbitrary curve starting at the point $A = (0, 1)$ and ending at the point $B = (1, 0)$

$$\int_C \omega = \int_C df = f(x, y)|_A^B = f(1, 0) - f(0, 1) = 0\,.$$

5

Show that the form 1-form $\omega = 3x^2ydx + x^3dy$ is an exact 1-form. Calculate integral of this form over the curves considered in exercises 2) and 3)

One can see that $\omega = 3xydx + x^3dy = d(x^3y)$ $(d(x^3y) = \frac{\partial(x^3y)}{\partial x}dx + \frac{\partial(x^3y)}{\partial y}dy = 3x^2ydx + x^3dy.)$
It is an exact form.

Integral of this exact form over the circle $x^2 + y^2 = 12y$ (exercise 2a) equals to zero, since it is closed curve: starting and ending points coincide.

Integral of this exact form over the arc of the ellipse $x^2 + y/9 = 1(exercise 2b), y \ge 0$ and the integral over arc of the unit circle $x^2 + y^2 = 1, y > 0$ both are equal zero in spite of the fact that these curves are not closed. The reason is that the function $f = x^3y$ $(df = \omega)$ vanishes at starting and ending points of these curves.

The integral of this form over arc of the unit circle starting at the point $A = (4, 0)$ and ending at the point $(2, 0)$ (see the exercise 3) is equal to $\int_C \omega = f|_B^A = f(1, 0) = f(0, 1) = 0$ because $f = x^2y$ and $f(1, 0) = f(0, 1) = 0$. Answer is equal to zero. Hence it does not depend on orientation of the curve.

6.

Consider the following differential 1-forms in $\mathbf{E}^2$:
a) xdx , b) xdy c) $xdx + ydy$, d)$xdy + ydx$, e) $xdy - ydx$
f) $x^4dy + 4x^3ydx$
a) Show that 1-forms a), c), d) and f) are exact forms

b) Why 1-forms b) and e) are not exact?

a) It is an exact form since $xdx = df$ where $f = \frac{x^2}{2} + c$, where c is a constant.

b) Suppose $\omega = xdy$ is an exact form: $\omega = df = f_x dx + f_y dy$. Hence $f_x = 0, f_y = x$. We see that $f_{xy} = \frac{\partial}{\partial x} f_y = 1$. On the other hand $f_{yx} = \frac{\partial}{\partial y} f_x = f_{xy} = 0$. Contradiction.

Another solution; There is another way to show why $\omega = xdy$ is not an exact form. We already calculated that the integral of the form $\omega = xdy$ over the closed circle $x^2 + y^2 = 12y$ is equal to $36\pi \neq 0$. (see the exercise 4) and its solution above)Hence ω is not exact, since the integral of an exact form over an arbitrary closed curve is equal to zero.

c) It is an exact form since $xdx + ydy = d\left(\frac{x^2+y^2}{2} + c\right)$, ($c$ is a constant).

d)It is an exact form since $xdy + ydx = d(xy + c)$,where c is a constant.

e)Suppose $\omega = xdy - ydx$ is an exact form: $\omega = df = f_x dx + f_y dy$. Hence $f_x = -y, f_y = x$. We see that $f_{xy} = 1$. On the other hand $f_{yx} = f_{xy} = -1$. Contradiction.

f) It is an exact form since $x^4 dy + 4x^3 ydx = d(x^4 y + c)$, where c is a constant.

All the exercises below are not compulsory

7†

Consider one-form

$$\omega = \frac{xdy - ydx}{x^2 + y^2} \tag{1}$$

This form is defined in $\mathbf{E}^2 \setminus 0$.

Calculate differential of this form.

Write down this form in polar coordinates

Find a function f *such that* $\omega = df$.

Is this function defined in the same domain as ω?

First calculate differential in cartesian coordinates with "brute force"

$$d\omega = d\left(\frac{xdy - ydx}{x^2 + y^2}\right) = \frac{d(xdy - ydx)}{x^2 + y^2} - (xdy - ydx) \wedge d\left(\frac{1}{x^2 + y^2}\right) = \frac{2dx \wedge dy}{x^2 + y^2} +$$

$$\frac{(xdy - ydx) \wedge d(x^2 + y^2)}{(x^2 + y^2)^2} = \frac{2dx \wedge dy}{x^2 + y^2} + \frac{(xdy - ydx) \wedge (2xdx + 2ydy)}{(x^2 + y^2)^2} =$$

$$\frac{2dx \wedge dy}{x^2 + y^2} + \frac{2x^2 dy \wedge dx + 2y^2 dy \wedge dx}{(x^2 + y^2)^2} = 0\,.$$

Much more illuminating to write down this form in polar coordinates then calculate its differential. We know already that $xdy - ydx = r^2 d\varphi$. Indeed

$dx = d(r\cos\varphi) = \cos\varphi dr - r\sin\varphi d\varphi = \frac{x}{r} dr - yd\varphi$ and $dy = d(r\sin\varphi) = \sin\varphi dr + r\cos\varphi d\varphi = \frac{y}{r} dr + xd\varphi$. Hence

$$xdy - ydx = x\left(\frac{y}{r} dr + xd\varphi\right) - y\left(\frac{x}{r} dr - yd\varphi\right) = (x^2 + y^2)d\varphi \text{ and } \frac{xdy - ydx}{x^2 + y^2} = d\varphi$$

Hence the form is closed.

For the form $\omega = \frac{xdy - ydx}{x^2+y^2}$ one can consider the function $f = \varphi = \arctan\frac{y}{x}$, such that $\omega = df$, but the function f is not well-defined on whole $\mathbf{E}^2$. It is well-defined e.g. we remove the ray $(-\infty, 0]$.

Note that ω is defined in $\mathbf{E}^2 \setminus 0$, but f is defined on $\mathbf{E}^2 \setminus (-\infty, 0]$.

On the other hand it is well defined in any domain where we can define one-valued continuous function $f = \varphi$, i.e. the domain does not contain a loop which rotates around origin. (The function $f = \varphi$ is multi-valued function in the domain $\mathbf{R}^2 \setminus 0$ which contains loops rotating around origin). E.g. one can see that for an arbitrary convex domain which does not contain the origin, or for an arbitrary domain which does not contain a ray $[-\infty, 0]$ a function $f = \varphi$ is well defined one-valued function.

8†

Calculate the integral of the form $\omega = \frac{xdy - ydx}{x^2+y^2}$ *over the curves*

a) circle $x^2 + y^2 = 1$

b) circle $(x-3)^2 + y^2 = 1$

c) ellipse $\frac{x^2}{9} + \frac{x^2}{16} = 1$

As it follows from the previous exercise answer equals to $\pm 2\pi$ for the first curve and third curves and it is equal to zero for the second curve.

9†

What values can take the integral $\int_C \omega$ *if* C *is an arbitrary curve starting at the point* $(0,1)$ *and ending at the point* $((1,0)$ *and* $\omega = \frac{xdy - ydx}{x^2+y^2}$.

Answer is the same as in previous exercise: if the curve **does not pass the origin** then the integral is well-defined, It is equal $\frac{\pi}{2} + 2\pi n$ if starting point of the curve is $(1,0)$ and ending point is

The integer n depends on the curve.

Remark Please, note that the form $\omega = \frac{xdy - ydx}{x^2+y^2}$ strictly speaking is not exact, because it is not defined for all points (it is not defined at origin) and moreover its "antiderivative" $f = \varphi$ $(\omega = df)$ is not well-defined function.

In the next exercise we show that for 1-forms which are defined in the whole $\mathbf{E}^2$ the exactness coincide with closeness.

10†

Let $\omega = a(x,y)dx + b(x,y)dy$ *be a closed form in* $\mathbf{E}^2$, $d\omega = 0$.

Consider the function

$$f(x,y) = x \int_0^1 a(tx,ty)dt + y \int_0^1 b(tx,ty)dt \tag{2}$$

Show that

$$\omega = df\,.$$

This proves that an arbitrary closed form in $\mathbf{E}^2$ *is an exact form. (Converse implication is always true.)*

Why we cannot apply the formula (2) to the form ω *defined by the expression (1)?*

Perform the calculations: $df = f_x d + f_y dy$.

$$f_x = \int_0^1 a(tx,ty)dt + x \int_0^1 a_x(tx,ty)tdt + y \int_0^1 b_x(tx,ty)tdt\,.$$

and

$$f_y = \int_0^1 b(tx,ty)dt + x \int_0^1 a_y(tx,ty)tdt + y \int_0^1 b_y(tx,ty)tdt\,.$$

On the other hand $d\omega = d(adx + bdy) = (b_x - a_y)dx \wedge dy = 0$. Hence $b_x = a_y$ and

$$f_x = \int_0^1 a(tx,ty)dt + x \int_0^1 a_x(tx,ty)tdt + y \int_0^1 a_y(tx,ty)tdt = \int_0^1 \left(\frac{d}{dt}\left(ta(tx,ty)\right)\right) = ta(tx,ty)\Big|_0^1 = a(x,y)\,,$$

because

$$\frac{d}{dt}\left(ta(tx,ty)\right) = a(tx,ty) + xta_x(tx,ty) + yta_y(tx,ty).$$

Analogously

$$f_y = \int_0^1 b(tx,ty)dt + x \int_0^1 b_x(tx,ty)tdt + y \int_0^1 b_y(tx,ty)tdt = \int_0^1 \left(\frac{d}{dt}\left(tb(tx,ty)\right)\right) = tb(tx,ty)\Big|_0^1 = b(x,y)\,,$$

We see that $f_x = a(x,y)$ and $f_y = b(x,y)$, i.e. $df = adx + bdy$ ■