Homework 4

Often it is useful to view 3-dimensional Euclidean space $\mathbf{E}^3$ *as a space* $\mathbf{R}^3$ *with the standard Cartesian coordinates:* $\mathbf{R}^3 = \{(x, y, z), x, y, z \in \mathbf{R}\}$. *The canonical orthonormal basis* $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ *in* $\mathbf{R}^3$ *has the following geometrical meaning: The unit vector* $\mathbf{e}_x = (1, 0, 0)$ *is directed along* x*-axis, the unit vector* $\mathbf{e}_y = (0, 1, 0)$ *is directed along* y*-axis and the unit vector* $\mathbf{e}_z = (0, 0, 1)$ *is directed along* z*-axis. We suppose that an orientation in* $\mathbf{E}^3$ *is fixed by the left basis* $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$.

1 Consider an operator P on $\mathbf{E}^3$ such that P is an orthogonal operator preserving the orientation of $\mathbf{E}^3$ and

$$P(\mathbf{e}_x) = \mathbf{e}_y, P(\mathbf{e}_z) = -\mathbf{e}_z\,.$$

Find an action of the operator P on an arbitrary vector $\mathbf{x} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$.

Why P is a rotation operator? Find an angle and axis of the rotation.

2 Consider an operator P on $\mathbf{E}^3$ such that

$$P(\mathbf{e}) = \frac{2}{3}\mathbf{e} + \frac{2}{3}\mathbf{f} + \frac{1}{3}\mathbf{g}\,, P(\mathbf{f}) = -\frac{1}{3}\mathbf{e} + \frac{2}{3}\mathbf{f} - \frac{2}{3}\mathbf{g}\,, P(\mathbf{g}) = -\frac{2}{3}\mathbf{e} + \frac{1}{3}\mathbf{f} + \frac{2}{3}\mathbf{g}\,.$$

Show that this is an orthogonal operator preserving the orientation of $\mathbf{E}^3$.

Find an axis of rotation (i.e. a vector $\mathbf{N} \neq 0$ which is directed along the axis.)

(We assume that $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ is an orthonormal basis in $\mathbf{E}^3$.)

3 Consider the operator P on $\mathbf{E}^3$ such that

$$P(\mathbf{x}) = 2(\mathbf{n}, \mathbf{x})\mathbf{n} - \mathbf{x}\,,$$

where $\mathbf{n}$ is a unit vector.

Show that this is an orthogonal operator preserving orientation and find an angle of rotation and axis of rotation.

4 a) Let $\mathbf{n}$ be an arbitrary unit vector in $\mathbf{E}^3$. Consider in $\mathbf{E}^3$ an operator

$$P(\mathbf{x}) = \mathbf{n} \times \mathbf{x}\,. \tag{1}$$

Show that this is not an invertible operator in $\mathbf{E}^3$.

b) Consider a subspace $V_\mathbf{n}$, orthogonal to the vector $\mathbf{n}$. Suppose that $\mathbf{n} = \mathbf{e}_z$. In this case the subspace $V_\mathbf{n}$ is spanned by vectors $\{\mathbf{e}_y, \mathbf{e}_x\}$ (plane $z = 0$). Show that the relation (1) defines an operator on $V_{\mathbf{e}_z}$:

$$\forall \mathbf{x} \in V_{\mathbf{e}_z}\,, \quad P(\mathbf{x}) = \mathbf{e}_z \times \mathbf{x} \in V_{\mathbf{e}_z}\,. \tag{2}$$

Show that this is an invertible operator preserving orientation.

Find an angle of rotation of subspace $V_{\mathbf{e}_z}$ under the action of operator P.

c) How it will look the answers on the question above in the case if $\mathbf{n}$ is an arbitrary unit vector?

5 Students John and Sarah calculate vector product $\mathbf{a} \times \mathbf{b}$ of two vectors using two different orthonormal bases in the Euclidean space $\mathbf{E}^3$, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$. John expands the vectors with respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Sarah expands the vectors with respect to the basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$. For two arbitrary vectors $\mathbf{a}, \mathbf{b} \in \mathbf{E}^3$

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = a'_1\mathbf{e}'_1 + a'_2\mathbf{e}'_2 + a'_3\mathbf{e}'_3 \,,$$

$$\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 = b'_1\mathbf{e}'_1 + b'_2\mathbf{e}'_2 + b'_3\mathbf{e}'_3 \,.$$

John and Sarah both use the so-called "determinant" formula. Are their answers the same?

$$\mathbf{a} \times \mathbf{b} = \underbrace{\det \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}}_{\text{John's calculations}} \stackrel{?}{=} \underbrace{\det \begin{pmatrix} \mathbf{e}'_1 & \mathbf{e}'_2 & \mathbf{e}'_3 \\ a'_1 & a'_2 & a'_3 \\ b'_1 & b'_2 & b'_3 \end{pmatrix}}_{\text{Sarah's calculations}}$$

6 Calculate the area of parallelograms formed by the vectors $\mathbf{a}, \mathbf{b}$ if
a) $\mathbf{a} = (1, 2, 3), \mathbf{b} = (1, 0, 1)$;
b) $\mathbf{a} = (2, 2, 3), \mathbf{b} = (1, 1, 1)$;
c) $\mathbf{a} = (5, 8, 4), \mathbf{b} = (10, 16, 8)$.
d) $\mathbf{a} = (3, 4, 0), \mathbf{b} = (5, 17, 0)$.

7 Find a vector $\mathbf{n}$ such that the following conditions hold:
1) It has unit length
2) It is orthogonal to the vectors $\mathbf{a} = (1, 2, 3)$ and $\mathbf{b} = (1, 3, 2)$.
3) An ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{n}\}$ has an orientation opposite to the orientation of the orthonormal basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ which defines the orientation of the Euclidean space.

8 Show that for any two vectors $\mathbf{a}, \mathbf{b} \in \mathbf{E}^3$ the following identity is satisfied

$$(\mathbf{a}, \mathbf{a})(\mathbf{b}, \mathbf{b}) = (\mathbf{a}, \mathbf{b})^2 + (\mathbf{a} \times \mathbf{b}, \mathbf{a} \times \mathbf{b}) \,.$$

Write down this identity in components.
† Compare this identity with the CBS inequality . (See the problem 5 in the Homework 2).

9 In 2-dimensional Euclidean space $\mathbf{E}^2$ consider the vectors

$$\mathbf{a} = (3, 2),\ \mathbf{b} = (7, 5),\ \mathbf{c} = (17, 12)\,, \mathbf{d} = (41, 29)\,.$$

Calculate areas of the parallelograms $\Pi(\mathbf{a}, \mathbf{b}), \Pi(\mathbf{b}, \mathbf{c})$ and $\Pi(\mathbf{c}, \mathbf{d})$.

10† Do you see any relations between parallelograms in the exercise above, fractions $\frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}$ and the number... $\sqrt{2}$? Can you continue this sequence of fractions?
(*Hint: Consider the squares of these fractions.*)