Solutions of Homework 4

1

Calculate the derivatives of the functions $f = x^2+y^2$, $g = e^{-(x^2+y^2)}$ *and* $h = q\log|r| = q\log\left(\sqrt{x^2+y^2}\right)$ *(q is a constant) along vector fields* $\mathbf{A} = x\partial_x + y\partial_y$ *and* $\mathbf{B} = x\partial_y - y\partial_x$*, i.e. calculate* $\partial_{\mathbf{A}} f, \partial_{\mathbf{A}} g, \partial_{\mathbf{A}} h$, $\partial_{\mathbf{B}} f, \partial_{\mathbf{B}} g, \partial_{\mathbf{B}} h$.

$\partial_{\mathbf{A}} f = A_x\frac{\partial f}{\partial x} + A_y\frac{\partial f}{\partial y} = x\cdot 2x + y\cdot 2y = 2(x^2+y^2)$,
$\partial_{\mathbf{A}} g = A_x\frac{\partial g}{\partial x} + A_y\frac{\partial g}{\partial y} = x\cdot 2xg - y\cdot 2yg = -2(x^2+y^2)e^{-(x^2+y^2)}$
$\partial_{\mathbf{A}} h = x\frac{\partial h}{\partial x} + y\frac{\partial h}{\partial y} = \frac{x^2 q}{x^2+y^2} + \frac{y^2 q}{x^2+y^2} = q$
$\partial_{\mathbf{B}} f = B_x\frac{\partial f}{\partial x} + B_y\frac{\partial f}{\partial y} = -y\cdot 2x + x\cdot 2y = 0$,
$\partial_{\mathbf{B}} g = -y\frac{\partial g}{\partial x} + x\frac{\partial g}{\partial y} = -y\cdot 2xg + x\cdot 2yg = 0$
$\partial_{\mathbf{B}} h = -y\frac{\partial h}{\partial x} + x\frac{\partial h}{\partial y} = \frac{-xyq}{x^2+y^2} + \frac{xyq}{x^2+y^2} = 0$

Remark We can do this exercise or using the formula for direcrtional derivative (see the Example in the end of the subsection 2.4 of lecture notes): If $\mathbf{A} = A_x\partial_x + A_y\partial_y$ then

$$\partial_A f = (A_x\partial_x + A_y\partial_y)f = A_x f_x + A_y f_y \text{ or } \partial_A f = df(\mathbf{A}) = (f_x dx + f_y dy)(\mathbf{A}) = f_x dx(\mathbf{A}) + f_y dy(\mathbf{A})$$

$$= f_x dx(A_x\partial_x + A_y\partial_y) + f_y dy(A_x\partial_x + A_y\partial_y) = f_x A_x + f_y A_y\,.$$

2

Perform the calculations of the previous exercise using polar coordinates.

For basic fields $\partial_r, \partial_\varphi$ in polar coordinates r, φ ($r = x\cos\varphi, y = r\sin\varphi$) we have that

$$\partial_r = \frac{\partial x}{\partial r}\partial_x + \frac{\partial y}{\partial r}\partial_y = \cos\varphi\partial_x + \sin\varphi\partial_y = \frac{x}{r}\partial_x + \frac{y}{r}\partial_y = \frac{x\partial_x + y\partial_y}{r} = \frac{\mathbf{A}}{r} \Rightarrow \mathbf{A} = r\partial_r$$

and

$$\partial_\varphi = \frac{\partial x}{\partial\varphi}\partial_x + \frac{\partial y}{\partial\varphi}\partial_y = -r\sin\varphi\partial_x + r\cos\varphi\partial_y = -y\partial_x + x\partial_y \Rightarrow \mathbf{B} = \partial_\varphi$$

We see that fields $\mathbf{A}, \mathbf{B}$ have very simple expression in polar coordiantes. Now calculations become almost immediate because in polar coordinates $f = r^2$, $g = e^{-r^2}$ and $h = q\log r$: $\partial_A f = r\partial_r r^2 = 2r^2$, $\partial_A g = r\partial_r e^{-r^2} = -2r^2 e^{-r^2}$, $\partial_A h = r\partial_r(q\log r) = q$. For field B it is even easier , because functions f, g, h do not depend on φ; $\partial_{\mathbf{B}} f = \partial_\varphi f = 0$. Analogously $\partial_{\mathbf{B}} g = \partial_{\mathbf{B}} h = 0$.

3

Consider in $\mathbf{E}^2$ vector fields $\mathbf{A} = x\partial_x + y\partial_y$, $\mathbf{B} = x\partial_y - y\partial_x$, $\mathbf{C} = \partial_x$, $\mathbf{D} = \partial_y$. Calculate the values of 1-forms df , dg on these vector fields if $f = (x^2+y^2)^n$ and $g = \frac{y}{x}$. For vector fields $\mathbf{A}, \mathbf{B}$ perform calculations also in polar coordinates.

We can perform calculations calculating directional derivative $\partial_{\mathbf{A}} f$ or calculating 1-form df then its value on the vector field $\mathbf{A}$ because

$$\partial_{\mathbf{A}} f = df(\mathbf{A})\,.$$

(See the Remark above or lecture notes)

Perform calculations using or first or second formula:
$df = 2n(x^2+y^2)^{n-1}(xdx + ydy)$. Hence

$$\partial_{\mathbf{A}} f = df(\mathbf{A}) = 2n(x^2+y^2)^{n-1}(xdx + ydy)(x\partial_x + y\partial_y) = 2n(x^2+y^2)^{n-1}(x^2+y^2) = 2n(x^2+y^2)^n$$

$$\partial_{\mathbf{A}} g = A_x \frac{\partial g}{\partial x} + A_y \frac{\partial g}{\partial y} = x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = x\left(\frac{-y}{x^2}\right) + y\left(\frac{1}{x}\right) = \frac{-xy}{x^2} + \frac{y}{x} = 0\,.$$

$$\partial_{\mathbf{B}} f = B_x \frac{\partial f}{\partial x} + B_y \frac{\partial f}{\partial y} = -y \cdot 2xn(x^2+y^2)^{n-1} + x \cdot 2yn(x^2+y^2)^{n-1} = 0\,.$$

$$\partial_{\mathbf{B}} g = dg(\mathbf{B}) = d\left(\frac{y}{x}\right)(\mathbf{B}) = \frac{dy(\mathbf{B})}{x} - \frac{ydx(\mathbf{B})}{x^2} = \frac{dy(x\partial_y - y\partial_x)}{x} - \frac{ydx(x\partial_y - y\partial_x)}{x^2} = 1 + \frac{y^2}{x^2}$$

$$\partial_{\mathbf{C}} f = f_x, \partial_{\mathbf{D}} f = f_y$$

We performed calculations in cartesian coordinates . Now perform calculations for vector fields $\mathbf{A}$ and $\mathbf{B}$ in polar coordinates using the fact that fields $\mathbf{A}, \mathbf{B}$ have a simple appearance in polar coordinates $\mathbf{A} = r\partial_r$ and $\mathbf{B} = \partial_\varphi$ (see the exercise above). Calculate again df, dg on $\mathbf{A}, \mathbf{B}$ in polar coordinates.

In polar coordinates $f = r^{2n}$ and $g = \tan\varphi$ and

$$df = 2nr^{2n-1}dr, \qquad dg = d\tan\varphi = \frac{d\varphi}{\cos^2\varphi}$$

Calculations become VERY TRANSPARENT:

$\partial_{\mathbf{A}} f = r\partial_r(r^{2n}) = 2nr^{2n}$ or $df(\mathbf{A}) = \partial_{\mathbf{A}} f = 2nr^{2n-1}dr(r\partial_r) = 2nr^{2n}dr$

$df(\mathbf{B}) = \partial_\varphi r^{2n} = 0$ or $df(\mathbf{B}) = \partial_{\mathbf{B}} f = 2nr^{2n-1}dr(\partial_\varphi) = 0$

$dg(\mathbf{A}) = r\partial_r(\tan\varphi) = 0$ or $dg(\mathbf{A}) = \partial_{\mathbf{A}} g = d\tan\varphi(r\partial_r) = \frac{d\varphi}{\cos^2\varphi}(r\partial_r) = 0$

$dg(\mathbf{B}) = \partial_\varphi \tan\varphi = \frac{1}{\cos^2\varphi}$ or $dg = d\tan\varphi = \frac{d\varphi}{\cos^2\varphi}(\partial_\varphi) = \frac{1}{\cos^2\varphi}$

4

Calculate the integrals of the form $\omega = \sin y\, dx$ over the following three curves. Compare answers.

C_1: $\mathbf{r}(t)$ $\begin{cases} x = 2t^2 - 1 \\ y = t \end{cases}$, $0 < t < 1$, $\qquad$ C_2: $\mathbf{r}(t)$ $\begin{cases} x = 8t^2 - 1 \\ y = 2t \end{cases}$, $0 < t < 1/2$,

C_3: $\mathbf{r}(t)$ $\begin{cases} x = \cos 2t \\ y = \cos t \end{cases}$, $0 < t < \frac{\pi}{2}$

For any curve $\mathbf{r}(t), t_1 < t < t_2$

$$\int_C \omega = \int_C \sin y dx = \int_C \sin y dx(\mathbf{v}) = \int_{t_1}^{t_2} \sin y(t) \frac{dx(t)}{dt} dt$$

where $\mathbf{v} = (x_t, y_t)$.

For the first curve $x_t = 4t$ and

$$\int_{C_1} \omega = \int_0^1 4t \sin t dt = 4(-t\cos t + \sin t)\big|_0^1 = -4\cos 1 + 4\sin 1$$

For the second curve $x_t = 16t$ and

$$\int_{C_2} \omega = \int_0^{1/2} 16t \sin 2t dt = 4(-2t\cos 2t + \sin 2t)\big|_0^{1/2} = -4\cos 1 + 4\sin 1$$

Answer is the same. Non-surprising. The second curve is reparameterised first curve ($t \mapsto 2t$) and reparameterisation preserves the orientation.

For the third curve $x_t = -2\sin 2t dt$ and

$$\int_{C_3} w = \int_0^{\frac{\pi}{2}} (-2\sin 2t)\sin(\cos t)dt = -4\left(\cos t\cos(\cos t) - \sin(\cos t)\right)\big|_0^{\pi/2} = 4\cos 1 - 4\sin 1$$

Answer is the same up to a sign: This curve is reparameterised first curve ($t \mapsto \cos t$) and reparameterisation changes the orientation, because $(\cos t)' = -\sin t < 0$ on the interval $(0, \pi/2)$.

Resumé: In these three examples an integral over the same (non-parameteresed) curve was considered. All the answers are the same up to a sign. Sign changes if reparameterisation changes an orientation.

5

Calculate the integral of the form $\omega = e^{-y}dx + \sin x dy$ over the segment of straight line which connects the points $A = (1,1)$, $B = (2,3)$. At what extent an answer depends on a chosen parameterisation?

Choose any parameterisation of this segment, e.g. $x = 1 + t, y = 1 + 2t$, $0 \le t \le 1$. Then $\mathbf{v} = (v_x, v_y) = (1,2)$ ($x_t = 1, y_t = 2$) and

$$\int_C e^{-y}dx + \sin x dy = \int_0^1 \left(e^{-(1+2t)}x_t + \sin(1+t)y_t\right)dt = \int_0^1 \left(e^{-(1+2t)} + 2\sin(1+t)\right)dt\,.$$

What happens if we choose another parameterisation, i.e. consider reparameterisation $t = t(\tau)$. Answer remains the same if the reparameterisation does not change orientation i.e. $t'(\tau) > 0$. This means that starting and ending points of the curve remain the same.

Answer is multiplied on -1 in other case: if the reparameterisation changes orientation i.e. $t'(\tau) < 0$.

6

Calculate the integral of the form $\omega = xdy$ over the upper arc of the unit circle starting at the point $A = (1,0)$ and ending at the point $(0,1)$.

Calculate the integral of the form $w = xdy$ over arc of the unit circle starting at the point $A = (1,0)$ and ending at the point $(0,1)$.

The equation of the arc is $x^2 + y^2 = 1, x \ge 0, y \ge 0$. We know that answer up to a sign does not depend on parameterisation. Choose an arbitrary parameterisation, e.g.

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases}, \quad 0 \le t \le \frac{\pi}{2}\,.$$

Then $\mathbf{v} = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$ and

$$\int_C w = \int_0^{\pi/2} w(\mathbf{v})dt = \int_0^{\pi/2} x(t)y_t dt = \int_0^{\pi/2} \cos t\cos t dt = \int_0^{\pi/2} \cos^2 t dt = \pi/4$$

So for an arbitrary parameterisation answer will be $\pm\pi/2$. ($\pi/2$ if orientation is the same and $= \pi/2$ if opposite)

7

Solve the previous problem for the arc of the ellipse $x^2 + y^2/9 = 1$ defined by the condition $y \ge 0$.

The equation of the arc is $x^2 + y^2/9 = 1, y \ge 0$. We know that answer up to a sign does not depend on parameterisation so choose any, e.g. $x = \cos t, y = 3\sin t$, $0 \le t \le \pi$. Then $\mathbf{v} = (-\sin t, 3\cos t)$ and

$$\int_C w = \int_0^{\pi} w(\mathbf{v})dt = \int_0^{\pi} x(t)y_t dt = \int_0^{\pi 2} 3\cos t\cos t dt = \int_0^{\pi} 3\cos^2 t dt = 3\pi/2$$

So for an arbitrary parameterisation answer will be $\pm 3\pi/2$, sign depending of parameterisation.

8

Calculate the integral $\int_C \omega$ *where* $\omega = xdx + ydy$ *and* C *is*
a) the straight line segment $x = t, y = 1 - t$, $0 \le t \le 1$
b) the segment of parabola $x = t$, $y = 1 - t^n$, $0 \le t \le 1$, $n = 2, 3, 4, \ldots$
c) the segment of the sinusoid $x = t$, $y = \cos \frac{\pi}{2} t$, $0 \le t \le 1$
d) **an arbitrary** *curve starting at the point* $(0, 1)$ *and ending at the point* $((1, 0)$.

For any of these curves we can perform calculations naively just using definition of integral
E.g. for the curve a)

$$\int_C w = \int_0^1 (x(t)x_t + y(t)y_t)dt = \int_0^1 (t + (1-t)(-1))dt = \int_0^1 (2t - 1)dt = 0\,,$$

for the curve b) if $n = 2$

$$\int_C w = \int_0^1 (x(t)x_t + y(t)y_t)dt = \int_0^1 (x(t)x_t + y(t)y_t)dt = \int_0^1 (t + (1-t^2)(-2t))dt = \int_0^1 (2t^3 - 3t^2)dt = 0\,,$$

for the curve b) in general case:

$$\int_C w = \int_0^1 (x(t)x_t + y(t)y_t)dt = \int_0^1 (x(t)x_t + y(t)y_t)dt =$$

$$\int_0^1 (t + (1-t^n)(-nt^{n-1}))dt = \int_0^1 (t - nt^{n-1} + nt^{2n-1})dt = 0\,.$$

But we immediately come to these results in a clear and elegant way if we use the fact that $w = xdx + ydy$ is an **exact form**, i.e. $w = df$ where $f = \frac{x^2+y^2}{2}$. Indeed using Theorem we see that for an arbitrary curve starting at the point $A = (0, 1)$ and ending at the point $B = (1, 0)$

$$\int_C w = \int_C df = f(x, y)|_A^B = f(1, 0) = f(0, 1) = 0\,.$$

Hints to exercises 9 and 10.

(Detailed solutions will be on web *after deadline for coursework*)

9 *Calculate the integral of the form* $\omega = \frac{xdy - ydx}{x^2+y^2}$ *over the curves* $a), b), c)$ *from the previous exercise.*

10* *What values can take the integral* $\int_C \omega$ *if* C *is an arbitrary curve starting at the point* $(0, 1)$ *and ending at the point* $((1, 0)$ *and* $\omega = \frac{xdy - ydx}{x^2+y^2}$.

One can show (do it!) that the form $\omega = \frac{xdy - ydx}{X^2+y^2} = d\varphi$ in polar coordinates.

Using this remark one can calculate the integral over an arbitrary curve starting at the point $A = (0, 1)$ and ending at the point $B = (1, 0)$ such that this curve **does not pass** through the point $(0, 0)$, where the angle φ is not defined. You have to study how the angle φ changes when you pass from the initial point to the ending point of the curve.

Remark Please, note that this form strictly speaking is not exact, because it is not defined for all points (it is not defined at origin)