Homework 6

1. Calculate Levi-Civita connection of the metric $G = a(u,v)du^2 + b(u,v)dv^2$

a) in the case if functions $a(u,v)$, $b(u,v)$ are constants.

b) In general case

2. Calculate Levi-Civita connection of the metric $G = adu^2 + bdv^2$ at the point $u = v = 0$ in the case if functions $a(u,v)$, $b(u,v)$ equal to constants at the point $u = v = 0$ up to the second order:

$$a(u,v) = a_0 + \dots\,,\quad b(u,v) = b_0 + \dots$$

where dots mean the terms of the second and higher order with respect to u, v.

3. Calculate $\nabla_{\frac{\partial}{\partial u}}\left(u\frac{\partial}{\partial v}\right)$ at the point $u = v = 0$ for the Levi-Civita connection considered in the previous problem.

4. Calculate Levi-Civita connection of the Riemannian metric on the sphere in stereographic coordinates:

$$G = \frac{4R^4(du^2 + dv^2)}{(R^2 + u^2 + v^2)^2}$$

a) at the point $u = v = 0$

b)* at an arbitrary point.

5. Calculate Levi-Civita connection of Euclidean metric of a plane in

a) Cartesian coordinates

b) polar coordinates

Compare with results of previous calculations.

6 Calculate Levi-Civita connection of the Riemannian metric induced on cylinder $x^2 + y^2 = a^2$. You may use parameterisation:

$$\mathbf{r}(h,\varphi)\colon \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases}.$$

Compare with results of previous calculations (induced connection).

7. Calculate Levi-Civita connection of the Riemannian metric induced on the cone $x^2 + y^2 - k^2z^2 = 0$. You may use parameterisation:

$$\mathbf{r}(h,\varphi)\colon \begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases}.$$

Find coordinates on the cone $x^2 + y^2 - k^2z^2 = 0$ such that Christoffel symbols of Levi-Civita connection of induced metric vanish in these coordinates.

8. Calculate Levi-Civita connection of the metric $G = R^2(d\theta^2 + \sin^2\theta d\varphi^2)$ on the sphere.

Compare with results of previous calculations (induced connection).

9 Let $\mathbf{E}^2$ be the Euclidean plane with the standard Euclidean metric $G_{\text{Eucl.}} = dx^2 + dy^2$.

You know that for the Levi-Civita connection of this metric the Christoffel symbols vanish in the Cartesian coordinates x, y. (Why?)

Let ∇ be a symmetric connection on the Euclidean plane $\mathbf{E}^2$ such that its Christoffel symbols satisfy the condition $\Gamma^y_{xy} = \Gamma^y_{yx} \neq 0$.

Show that for vector fields $\mathbf{A} = \partial_x$ and $\mathbf{B} = \partial_y$, $\partial_{\mathbf{A}} \langle \mathbf{B}, \mathbf{B} \rangle \neq 2\langle \nabla_{\mathbf{A}} \mathbf{B}, \mathbf{B} \rangle$, i.e. the connection ∇ does not preserve the Euclidean scalar product $\langle \ , \ \rangle$.