The solution of the problem 3c ii of the Coursework, revisited

There are errata in the solution of the second part of the question 3c) of the coursework which was distributed on 13 May 2011. I write here the corrected solution. I add also little bit more detail.

Consider a surface M in $\mathbf{E}^3$ defined by the equation $\begin{cases} x = u \\ y = v \\ z = F(u,v) \end{cases}$.

Consider a point $\mathbf{p}$ on M with coordinates $u = x_0, v = y_0$ such that (x_0, y_0) is a point of local extremum for the function F.

Calculate Christoffel symbols of Levi-Civita connection at the point $\mathbf{p}$.

Solution

WLOG we suppose that for coordinates of stationary point $x_0 = y_0 = z_0 = 0$, i,e, $F = 0$ and $F_u = F_v = 0$ at $u = v = 0$.

For the surface $\begin{cases} x = u \\ y = v \\ z = F(u,v) \end{cases}$ $G_M = \left(dx^2 + dy^2 + dz^2\right)|_{x=u,y=v,z=F(u,v)} =$

$$du^2 + dv^2 + dF^2 = du^2 + dv^2 + (F_u du + F_v dv)^2 = (1 + F_u^2)du^2 + 2F_u F_v dv + (1 + F_v^2)dv^2$$

Hence the Riemannian metric at the vicinity of the extremum point is given by quadratic form

$$G_M = \begin{pmatrix} 1 + F_u^2 & F_u F_v \\ F_v F_u & 1 + F_v^2 \end{pmatrix}.$$

Since $\mathbf{p}$ is the point of local extremum, hence it is the stationary point: the first differential vanishes: $F_u = F_v = 0$. One can see that first derivatvies of metric's components at the extremum point vanish. E.g.

$$\frac{\partial g_{12}}{\partial u}|_{u=v=0} = F_{uu} F_v|_{u=v=0} + F_u F_{vu}|_{u=v=0} = 0$$

since at the extremum point $F_{uu} = F_{uv} = F_{vv} = 0$. Hence according to Levi-Civita formula Christopher symbols vanish at the extremum point (in coordinates u, v.)

Another solution: One can see that Christoffel symbols of induced connection (which is equals to Levi-Civita) connection vanish.

Notice now that at the extremum point normal unit vector $\mathbf{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Indeed $\mathbf{r}_u = \begin{pmatrix} 1 \\ 0 \\ F_u \end{pmatrix}$, $\mathbf{r}_v = \begin{pmatrix} 0 \\ 1 \\ F_v \end{pmatrix}$, but $F_u = F_v = 0$ at the extremum point $\mathbf{p}$, hence $\mathbf{r}_u|_{\mathbf{p}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$,

$\mathbf{r}_v|_{\mathbf{p}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Hence normal unit vector at the point $\mathbf{p}$ equals to $\mathbf{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Now it is very easy to show that for induced connection (which is equal to Levi-Civita connection) Christoffel symbols vanish at the point $\mathbf{p}$. This is since $\mathbf{r}_{\alpha\beta}$ is orthogonal to the surface: At the extremum point $\mathbf{p}$

$$\nabla^M_{\partial u^\alpha} \partial_{u^\beta} = \left(\nabla^{\text{canonical flat}}_{\partial u^\alpha} \partial_{u^\beta}\right)_{\text{tangent}} = \left(\partial_{\partial u^\alpha} \partial_{u^\beta}\right)_{\text{tangent}} = (\mathbf{r}_{\alpha\beta})_{\text{tangent}} = 0$$

since $\mathbf{r}_{\alpha\beta}|_{\mathbf{p}}$ is orthogonal to the surface at the extremum point $\mathbf{p}$:

$$\mathbf{r}_{\alpha\beta}|_{\mathbf{p}} = \frac{\partial^2}{\partial u^\alpha \partial u^\beta} \begin{pmatrix} u \\ v \\ F \end{pmatrix} |_{\mathbf{p}} = \begin{pmatrix} 0 \\ 0 \\ F_{\alpha\beta} \end{pmatrix} = F_{\alpha\beta} \mathbf{n}|_{\mathbf{p}}$$

($F_{\alpha\beta} = \frac{\partial^2 F}{\partial u^\alpha \partial u^\beta}$.)

We use notation u^α ($\alpha = 1, 2$) for coordinates u, v. For example $F_{12} = \frac{\partial^2 F}{\partial u^1 \partial u^2} = \frac{\partial^2 F}{\partial u \partial v}$ and $F_{22} = \frac{\partial^2 F}{\partial u^2 \partial u^2} = \frac{\partial^2 F}{\partial v \partial v}$.