Algorithm "nosov" continuous fractions

Here we reproduce the algorithm "vytiagivanija nosov", which I learnt in the book of Arnold.

Let $\mathbf{e}, \mathbf{f}$ be standard basis in $\mathbf{R}^2$. $\mathbf{e} = (1,0)$, and $\mathbf{f} = (0,1)$.

It is useful also consider the module $\mathbf{Z}^2 = \mathbf{Z} \oplus \mathbf{Z}$ over vectors $\mathbf{e}, \mathbf{f}$:

$$\mathbf{Z}^2 = \{(m,n) = m\mathbf{e} + n\mathbf{f} \quad m,n \in \mathbf{Z}\} \,. \tag{1}$$

We assign to an arbitrary rational number $\frac{p}{q}$ the vector

$$\mathbf{E}\left(\frac{p}{q}\right) = q\mathbf{e} + p\mathbf{f}\,.(2a)$$

which belongs also to $\mathbf{Z} \otimes \mathbf{Z}$. (We assume that p, q are coprime)

Let α be non-negative real number and let $[a_0, a_1, \ldots,]$ be its continuous fraction, and $\frac{p_k}{q_k}$ be its k-th approximation:

$$[a_0, \ldots, a_k] = \frac{p_k}{q_k} \tag{2b}$$

(We assume that p_k, q_k are coprime)

Consider vectors $\{\mathbf{E}_{-2}, \mathbf{E}_{-1}, \mathbf{E}_0 . \mathbf{E}_1, \ldots\}$ in $\mathbf{Z}^2$ defined by real number α in the following way:

$$\mathbf{E}_{-2} = \mathbf{e}, \quad \mathbf{E}_{-1} = \mathbf{f}\,, \tag{3}$$

Proposition "vytiagivanie nosov": for arbitrary k:

$$\mathbf{E}_{k+1} = \mathbf{E}_{k-1} + a_{k+1}\mathbf{E}_k\,.(k = -1, 0, 1, 2, \ldots) \tag{Prop}$$

Proof:

Instead Proposition we will prove the statement that equation (Prop) survives up to collinearity:

for arbitrary k:

$$[\mathbf{E}_{k+1}] = [\mathbf{E}_{k-1} + a_{k+1}\mathbf{E}_k]\,, \tag{Statement}$$

where we denote by $[\mathbf{t}]$ the class of vectors collinear to the vector $\mathbf{t}$.

This statement seems to be weaker, but it implies the Proposition. Prove first that statement (*Statement*) implies Proposition (*Prop*), then we will prove statement (*Statement*)

Notice that formulae (2) define polynomials:

$$[a_0, a_1, \ldots, a_k] = \frac{p_k(a_0, a_1, \ldots, a_k)}{q_k(a_0, a_1, \ldots, a_k)} .$$

E.g.

$$p_0 = a_0, q_0 = 1, \quad q_1 = a_1, p_1 = a_0 a_1 + 1 , \quad q_2 = a_1 a_2 + 1, p_2 = a_0 a_1 a_2 + a_0 + a_2 ,$$

Equation (*Statement*) implies that

$$\mathbf{E}_{k+1} = \lambda \left(\mathbf{E}_{k-1} + a_{k+1} \mathbf{E}_k \right)$$

One can eassy to check that $\lambda = 1$ for $k = -1$. Assume by induction that this holds for $k \leq m$. Then comparing coefficients at a_{k+1} one can see that this holds for $k = m + 1$.

Now prove the statement.

Consider first small k

For $k = -1$: $\mathbf{E}_0 = q_0 \mathbf{e} + p_0 \mathbf{f} = \mathbf{e} + a_0 \mathbf{f}$ and

$$[\mathbf{E}_0] = [\mathbf{E}_{-2} + a_0 \mathbf{E}_{-1}] = [\mathbf{e} + a_0 \mathbf{f}] .$$

This is true.

For $k = 0$:

$$[\mathbf{E}_1] = [q_1 \mathbf{e} + p_1 \mathbf{f}] = \left[\mathbf{e} + \left(a_0 + \frac{1}{a_1} \right) \mathbf{f} \right] = \left[(\mathbf{e} + a_0 \mathbf{f}) + \frac{1}{a_1} \mathbf{f} \right] =$$

$$\left[\mathbf{E}_{-2} + a_0 \mathbf{E}_{-1} + \frac{1}{a_1} \mathbf{f} \right] = \left[\mathbf{E}_0 + \frac{1}{a_1} \mathbf{E}_{-1} \right] = [a_1 \mathbf{E}_0 + \mathbf{E}_{-1}] .$$

This is true.

For $k = 1$:

$$[\mathbf{E}_2] = [q_2 \mathbf{e} + p_2 \mathbf{f}] = \left[\mathbf{e} + \left(a_0 + \frac{1}{a_1 + \frac{1}{a_2}} \right) \mathbf{f} \right] = \left[(\mathbf{e} + a_0 \mathbf{f}) + \frac{1}{a_1 + \frac{1}{a_2}} \mathbf{f} \right] =$$

$$\left[\mathbf{E}_{-2} + a_0 \mathbf{E}_{-1} + \frac{1}{a_1 + \frac{1}{a_2}} \mathbf{E}_{-1} \right] = \left[\mathbf{E}_0 + \frac{1}{a_1 + \frac{1}{a_2}} \mathbf{E}_{-1} \right] = \left[\left(a_1 + \frac{1}{a_2} \right) \mathbf{E}_0 + \mathbf{E}_{-1} \right] =$$

$$\left[\mathbf{E}_{-1} + a_1\mathbf{E}_0 + \frac{1}{a_2}\mathbf{E}_0\right] = \left[\mathbf{E}_1 + \frac{1}{a_2}\mathbf{E}_0\right] = [a_2\mathbf{E}_1 + \mathbf{E}_0]\ .$$

This is true.

and so on:

$$[\mathbf{E}_m] = [q_m\mathbf{e} + p_m\mathbf{f}] = [\mathbf{e} + [a_0, \ldots, a_m]\,\mathbf{f}] = \left[\mathbf{e} + \left(a_0 + \frac{1}{[a_1, \ldots, a_m]}\right)\mathbf{f}\right] =$$

$$\left[(\mathbf{e} + a_0\mathbf{f}) + \frac{1}{[a_1, \ldots, a_m]}\mathbf{f}\right] = \left[\mathbf{E}_0 + \frac{1}{[a_1, \ldots, a_m]}\mathbf{E}_{-1}\right] = [[a_1, \ldots, a_m]\,\mathbf{E}_0 + \mathbf{E}_{-1}] =$$

$$\left[\left(a_1 + \frac{1}{[a_2, \ldots, a_m]}\right)\mathbf{E}_0 + \mathbf{E}_{-1}\right] = \left[a_1\mathbf{E}_0 + \mathbf{E}_{-1} + \frac{1}{[a_2, \ldots, a_m]}\mathbf{E}_0\right] =$$

$$\left[\mathbf{E}_1 + \frac{1}{[a_2, \ldots, a_m]}\mathbf{E}_0\right] = [[a_2, \ldots, a_m]\,\mathbf{E}_1 + \mathbf{E}_0] =$$

$$\left[\left(a_2 + \frac{1}{[a_3, \ldots, a_m]}\right)\mathbf{E}_1 + \mathbf{E}_0\right] = \left[a_2\mathbf{E}_1 + \mathbf{E}_0 + \frac{1}{[a_3, \ldots, a_m]}\mathbf{E}_1\right] =$$

$$\left[\mathbf{E}_2 + \frac{1}{[a_3, \ldots, a_m]}\mathbf{E}_1\right] = [[a_3, \ldots, a_m]\,\mathbf{E}_2 + \mathbf{E}_1] =$$

$$\left[\left(a_3 + \frac{1}{[a_4, \ldots, a_m]}\right)\mathbf{E}_2 + \mathbf{E}_1\right] = \left[a_3\mathbf{E}_2 + \mathbf{E}_1 + \frac{1}{[a_4, \ldots, a_m]}\mathbf{E}_2\right] =$$

$$\left[\mathbf{E}_3 + \frac{1}{[a_4, \ldots, a_m]}\mathbf{E}_2\right] = [[a_4, \ldots, a_m]\,\mathbf{E}_3 + \mathbf{E}_2] =$$

$$\left[\left(a_4 + \frac{1}{[a_5, \ldots, a_m]}\right)\mathbf{E}_3 + \mathbf{E}_2\right] = \left[a_4\mathbf{E}_3 + \mathbf{E}_2 + \frac{1}{[a_5, \ldots, a_m]}\mathbf{E}_3\right] =$$

$$\left[\mathbf{E}_4 + \frac{1}{[a_5, \ldots, a_m]}\mathbf{E}_3\right] = [[a_5, \ldots, a_m]\,\mathbf{E}_4 + \mathbf{E}_3] = \ldots$$

$$\left[\mathbf{E}_{m-2} + \frac{1}{[a_{m-1}, a_m]}\mathbf{E}_{m-3}\right] = [[a_{m-1}, a_m]\,\mathbf{E}_{m-2} + \mathbf{E}_{m-3}] =$$

$$\left[\left(a_{m-1} + \frac{1}{a_m}\right)\mathbf{E}_{m-2} + \mathbf{E}_{m-3}\right] = \left[a_{m-1}\mathbf{E}_{m-2} + \mathbf{E}_{m-3} + \frac{1}{a_m}\mathbf{E}_{m-2}\right] =$$

$$\left[\mathbf{E}_{m-1} + \frac{1}{a_m}\mathbf{E}_{m-2}\right] = [a_m\mathbf{E}_{m-1} + \mathbf{E}_{m-2}]\ .$$