Homework 5

Christoffel symbols and Lagrangians

1 Consider the Lagrangian of "free" particle $L = \frac{1}{2} g_{ik} \dot{x}^i \dot{x}^k$ for Riemannian manifold with a metric $G = g_{ik} dx^i dx^k$.
Write down Euler-Lagrange equations of motion for this Lagrangian and compare them with differential equations for geodesics on this Riemannian manifold.

In fact show that

$$\underbrace{\frac{\partial L}{\partial x^i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}}_{\text{Euler-Lagrange equations}} \qquad \Leftrightarrow \qquad \underbrace{\frac{d^2 x^i}{dt^2} = \Gamma^i_{km} \dot{x}^k \dot{x}^m}_{\text{Equations for geodesics}} \quad , \tag{1}$$

where

$$\Gamma^i_{km} = \frac{1}{2} g^{ij} \left(\frac{\partial g_{jk}}{\partial x^m} + \frac{\partial g_{jm}}{\partial x^k} - \frac{\partial g_{km}}{\partial x^j} \right) . \tag{2}$$

2 a) Write down the Lagrangian of of "free" particle $L = \frac{1}{2} g_{ik} \dot{x}^i \dot{x}^k$ for Euclidean plane in polar coordinates. Calculate Christoffel symbols for canonical flat connection in polar coordinates using Euler-Lagrange equations for this Lagrangian. Compare with answers which you obtained by the direct use of the formula (2).

b) Do the same in cylindrical coordinates in $\mathbf{E}^3$: $x = r\cos\varphi, y = r\sin\varphi, z = h$.

3 Write down the Lagrangian of of "free" particle $L = \frac{1}{2} g_{ik} \dot{x}^i \dot{x}^k$ for the sphere of radius R in $\mathbf{E}^3$ in spherical coordinates. Calculate Christoffel symbols of Levi-Civita connection on the sphere in spherical coordinates using Euler-Lagrange equations for this Lagrangian. (The induced Riemannian metric on the sphere equals $G = R^2 d\theta^2 + R^2 \sin^2\theta d\varphi^2$.)

4 Calculate Christoffel symbols of Levi-Civita connection for Riemannian metric $G = adu^2 + bdv^2$.