Homework 2. Solutions

1

a) Show that $(\mathbf{x},\mathbf{y}) = x^1y^1 + x^2y^2 + x^3y^3$ *defines a scalar product in* $\mathbf{R}^3$.

b) Show that $\langle\mathbf{x},\mathbf{y}\rangle = x^1y^1 + x^2y^2$ *does not define a scalar product in* $\mathbf{R}^3$.

c) Show that $(\mathbf{x},\mathbf{y}) = x^1y^1 + x^2y^2 - x^3y^3$ *does not define a scalar product in* $\mathbf{R}^3$.

d) Show that $(\mathbf{x},\mathbf{y}) = x^1y^1 + 3x^2y^2 + 5x^3y^3$ *defines a scalar product in* $\mathbf{R}^3$.

e†*) Find necessary and sufficient conditions for entries* a, b, c *of symmetrical matrix* $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ *such that the formula*

$$(\mathbf{x},\mathbf{y}) = (\, x^1, x^2 \,) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = ax^1y^1 + b(x^1y^2 + x^2y^1) + cx^2y^2$$

defines scalar product in $\mathbf{R}^2$.

Recall that scalar product on a vector space V is a function $B(\mathbf{x},\mathbf{y}) = (\mathbf{x},\mathbf{y})$ on a pair of vectors which takes real values and satisfies the the following conditions:

1) $B(\mathbf{x},\mathbf{y}) = B(\mathbf{y},\mathbf{x})$ (symmetricity condition)

2) $B(\lambda\mathbf{x} + \mu\mathbf{y}, \mathbf{z}) = \lambda B(\mathbf{x},\mathbf{z}) + \mu B(\mathbf{y},\mathbf{z})$ (linearity condition (with respect to the first argument))

3) $B(\mathbf{x},\mathbf{x}) \geq 0$, $B(\mathbf{x},\mathbf{x}) = 0 \Leftrightarrow \mathbf{x} = 0$ (positive-definiteness condition)

(The linearity condition with respect to the second argument follows from the conditions 2) and 1))

Check all these conditions for $(\mathbf{x},\mathbf{y}) = x^1y^1 + x^2y^2 + x^3y^3$:

1) $(\mathbf{y},\mathbf{x}) = y^1x^1 + y^2x^2 + y^3x^3 = x^1y^1 + x^2y^2 + x^3y^3 = (\mathbf{x},\mathbf{y})$. Hence it is symmetrical.

2) $(\lambda\mathbf{x} + \mu\mathbf{y}, \mathbf{z}) = (\lambda x^1 + \mu y^1)z^1 + (\lambda x^2 + \mu y^2)z^2 + (\lambda x^3 + \mu y^3)z^3 =$
$= \lambda(x^1z^1 + x^2z^2 + x^3z^3) + \mu(y^1z^1 + y^2z^2 + y^3z^3) = \lambda(\mathbf{x},\mathbf{y}) + \mu(\mathbf{y},\mathbf{z})$. Hence it is linear.

3) $(\mathbf{x},\mathbf{x}) = (x^1)^2 + (x^2)^2 + (x^3)^2 \geq 0$. It is non-negative. If $\mathbf{x} = 0$ then $(\mathbf{x},\mathbf{x}) = 0$. If $(\mathbf{x},\mathbf{x}) = (x^1)^2 + (x^2)^2 + (x^3)^2 = 0$, then $x^1 = x^2 = x^3 = 0$, i.e. $\mathbf{x} = 0$. This we proved positive-definiteness.

All conditions are checked. Hence $B(\mathbf{x},\mathbf{y}) = x^1y^1 + x^2y^2 + x^3y^3$ is indeed a scalar product in $\mathbf{R}^3$

Remark Note that x^1, x^2, x^3—are components of the vector, do not be confused with exponents!

Show that $B(\mathbf{x},\mathbf{y}) = x^1y^1 + x^2y^2$ *does not define scalar product* in $\mathbf{R}^3$.

To see that the formula $(\mathbf{x},\mathbf{y}) = x^1y^1 + x^2y^2$ *does not* define scalar product check the condition 3) of positive-definiteness: $(\mathbf{x},\mathbf{x}) = (x^1)^2 + (x^2)^2$ may take zero values for $\mathbf{x} \neq 0$. E.g. if $\mathbf{x} = (0, 0, -1)$ $(\mathbf{x},\mathbf{x}) = 0$, in spite of the fact that $\mathbf{x} \neq 0$. The condition 3) of positive-definiteness is not satisfied. Hence it is not scalar product.

Show that $B(\mathbf{x},\mathbf{y}) = x^1y^1 + x^2y^2 - x^3y^3$ *does not define scalar product* in $\mathbf{R}^3$.

To see that the formula $(\mathbf{x},\mathbf{y}) = x^1y^1 + x^2y^2 - x^3y^3$ *does not* define scalar product check the condition 3): $(\mathbf{x},\mathbf{x}) = (x^1)^2 + (x^2)^2 - (x^3)^2$ may take negative values. E.g. if $\mathbf{x} = (0, 0, -1)$ $(\mathbf{x},\mathbf{x}) = -1 < 0$. The condition 3) of positive-definiteness is not satisfied. Hence it is not scalar product.

Now show that $(\mathbf{x},\mathbf{y}) = x^1y^1 + 3x^2y^2 + 5x^3y^3$ is a scalar product in $\mathbf{R}^3$.

We need to check all the conditions above for scalar product for $(\mathbf{x},\mathbf{y}) = x^1y^1 + 3x^2y^2 + 5x^3y^3$:

1) $(\mathbf{y},\mathbf{x}) = y^1x^1 + 3y^2x^2 + 5y^3x^3 = x^1y^1 + 3x^2y^2 + 5x^3y^3 = (\mathbf{x},\mathbf{y})$. Hence it is symmetrical.

2) $(\lambda\mathbf{x} + \mu\mathbf{y}, \mathbf{z}) = (\lambda x^1 + \mu y^1)z^1 + 3(\lambda x^2 + \mu y^2)z^2 + 5(\lambda x^3 + \mu y^3)z^3 =$
$= \lambda(x^1z^1 + 3x^2z^2 + 5x^3z^3) + \mu(y^1z^1 + 3y^2z^2 + 5y^3z^3) = \lambda(\mathbf{x},\mathbf{y}) + \mu(\mathbf{y},\mathbf{z})$. Hence it is linear.

3) $(\mathbf{x},\mathbf{x}) = (x^1)^2 + 3(x^2)^2 + 5(x^3)^2 \geq 0$. It is non-negative. If $\mathbf{x} = 0$ then obviously $(\mathbf{x},\mathbf{x}) = 0$. If $(\mathbf{x},\mathbf{x}) = (x^1)^2 + 3(x^2)^2 + 5(x^3)^2 = 0$, then $x^1 = x^2 = x^3 = 0$. Hence it is positive-definite.

All conditions are checked. Hence $(\mathbf{x},\mathbf{y}) = x^1y^1 + 3x^2y^2 + 5x^3y^3$ is indeed a scalar product in $\mathbf{R}^3$

e†)

The condition of linearity and symmetricity for the bilinear form

$$B(\mathbf{x},\mathbf{y}) = (\, x^1, x^2\,) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = ax^1y^1 + b(x^1y^2 + x^2y^1) + cx^2y^2$$

are evidently obeyed.

The general answer on this question is: symmetric matrix is positive-definite if and only if all principal minors are positive. For matrix under consideration it means that conditions $a > 0$ and $ac - b^2 > 0$ are necessary and sufficient conditions.

Give a proof for this special case.

Check the positive-definiteness condition.

For $\mathbf{x} = (1,0)$ $B(\mathbf{x},\mathbf{x}) = a$. Hence $a > 0$ is necessary condition. Now consider

$$B(\mathbf{x},\mathbf{x}) = a(x^1)^2 + 2bx^1x^2 + c(x^2)^2 = \frac{(ax^1 + bx^2)^2 + (ac - b^2)(x^2)^2}{a} \geq 0 \Leftrightarrow ac - b^2 \geq 0$$

We see that $B(\mathbf{x},\mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$ iff $a > 0$ and $(ac - b^2) > 0$.

2 *The matrix* $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ *obeys the conditions* $A^T A = I$ *. Show that*

a) $\det A = \pm 1$

b) if $\det A = 1$ *then there exists an angle* $\varphi : 0 \leq \varphi < 2\pi$ *such that* $A = A_\varphi$ *where*

$$A_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \textit{ (rotation matrix)}$$

c) if $\det A = -1$ *then then there exists an angle* $\varphi : 0 \leq \varphi < 2\pi$ *such that* $A = A_\varphi R$*, where* $R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *(a reflection matrix).*

The answers on this exercise see in lecture notes.

3 *Show that for matrix* A_φ *defined in the previous exercise the following relations are satisfied:*

$$A_\varphi^{-1} = A_\varphi^T = A_{-\varphi}\,, \qquad A_{\varphi+\theta} = A_\varphi \cdot A_\theta\ .$$

We know (see lecture notes) that $A_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$. Then calculate inverse matrix A_φ^{-1}. One can see that $A_\varphi^T = A_\varphi^{-1} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$, because $A_\varphi^T A_\varphi = I$. On the other hand $\cos\varphi = \cos(-\varphi)$ and $\sin\varphi = -\sin(-\varphi)$. Hence

$$A_\varphi^T = A_\varphi^{-1} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} = \begin{pmatrix} \cos(-\varphi) & -\sin(-\varphi) \\ \sin(-\varphi) & \cos(-\varphi) \end{pmatrix} = A_{-\varphi}\,.$$

Now prove that $A_{\varphi+\theta} = A_\varphi \cdot A_\theta$:

$$A_\varphi \cdot A_\theta = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} (\cos\varphi\cos\theta - \sin\varphi\sin\theta) & -(\cos\varphi\sin\theta + \sin\varphi\cos\theta) \\ (\cos\varphi\sin\theta + \sin\varphi\cos\theta) & (\cos\varphi\cos\theta - \sin\varphi\sin\theta) \end{pmatrix} =$$

$$\begin{pmatrix} \cos(\varphi+\theta) & -\sin(\varphi+\theta) \\ \sin(\varphi+\theta) & \cos(\varphi+\theta) \end{pmatrix} = A_{\varphi+\theta}$$

4 *Show that under the transformation* $(\mathbf{e}'_1, \mathbf{e}'_2) = (\mathbf{e}_1, \mathbf{e}_2)\, A_\varphi$ *an orthonormal basis transforms to an orthonormal one.*

How coordinates of vectors change if we rotate the orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2)$ on the angle $\varphi = \frac{\pi}{3}$ anticlockwise?

We have to check that scalar products $(\mathbf{e}'_1, \mathbf{e}'_1) = (\mathbf{e}'_2, \mathbf{e}'_2) = 1$ and $(\mathbf{e}'_1, \mathbf{e}'_2) = 0$. Calculations show that $(\mathbf{e}'_1, \mathbf{e}'_1) = (\cos\varphi\mathbf{e}_1 + \sin\varphi\mathbf{e}_2, \cos\varphi\mathbf{e}_1 + \sin\varphi\mathbf{e}_2) = \cos^2\varphi(\mathbf{e}_1, \mathbf{e}_1) + 2\cos\varphi\sin\varphi(\mathbf{e}_1, \mathbf{e}_2) + \sin^2\varphi(\mathbf{e}_2, \mathbf{e}_2) = 1$, $(\mathbf{e}'_2, \mathbf{e}'_2) = (-\sin\varphi\mathbf{e}_1 + \cos\varphi\mathbf{e}_2, -\sin\varphi\mathbf{e}_1 + \cos\varphi\mathbf{e}_2) = 1$, $(\mathbf{e}'_1, \mathbf{e}'_2) = (\cos\varphi\mathbf{e}_1 + \sin\varphi\mathbf{e}_2, -\sin\varphi\mathbf{e}_1 + \cos\varphi\mathbf{e}_2) = 0$.

Now answer the second question.

If $\mathbf{a} = x\mathbf{e}_x + y\mathbf{e}_2 = x'\mathbf{e}'_x + y'\mathbf{e}'_2$ and $A_\varphi = A_{\frac{\pi}{3}} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ is the matrix of bases transformation then we have:

$$\mathbf{a} = (\mathbf{e}_x, \mathbf{e}_y)\begin{pmatrix} x \\ y \end{pmatrix} = (\mathbf{e}'_x, \mathbf{e}'_y)\begin{pmatrix} x' \\ y' \end{pmatrix} = (\mathbf{e}_x, \mathbf{e}_y)A_{\frac{\pi}{3}}\begin{pmatrix} x' \\ y' \end{pmatrix} = (\mathbf{e}_x, \mathbf{e}_y)\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

Hence

$$\begin{pmatrix} x \\ y \end{pmatrix} = A_{\frac{\pi}{3}}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

and

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = T^{-1}_{\frac{\pi}{3}}\begin{pmatrix} x \\ y \end{pmatrix} = A_{-\frac{\pi}{3}}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

5 *Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal basis of Euclidean space $\mathbf{E}^3$. Consider the ordered set of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ which is expressed via basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as in the exercise 6 of the Homework 1.*

Write down explicitly transition matrix from the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to the ordered set of the vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$. What is the rank of this matrix? Is this matrix orthogonal?

Find out is the ordered set of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ a basis in $\mathbf{E}^3$. Is this basis an orthonormal basis of $\mathbf{E}^3$?

(you have to consider all cases a),b) c) and d)).

Case a) The ordered set $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\} = \{\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_3\}$ is evidently orthonormal basis. Transition matrix $T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $(\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3) = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)T$. This matrix is non-degenerate, its rank is equal to 3 ($\det T = 1 \neq 0$). It is orthogonal because both bases are orthonormal.

Case b) The ordered set $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\} = \{\mathbf{e}_1, \mathbf{e}_1 + 3\mathbf{e}_3, \mathbf{e}_3\}$ is not a basis because vectors are linear dependent: $\mathbf{e}'_1 - \mathbf{e}'_2 + 3t\mathbf{e}'_3 = 0$. Transition matrix $T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 1 \end{pmatrix}$, $(\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3) = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)T$. This matrix is degenerate, its rank ≤ 2. One can see it noting that rows are linear dependent or noting that $\det T = 0$. Vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ are linear dependent. On the other hand vectors $\{\mathbf{e}'_1, \mathbf{e}'_2\}$ are linear independent. Hence rank of the matrix T is equal to 2. This matrix is not orthogonal.

Case c) The ordered set $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\} = \{\mathbf{e}_1 - \mathbf{e}_2, 3\mathbf{e}_1 - 3\mathbf{e}_2, \mathbf{e}_3\}$ is not a basis because vectors are linear dependent: $3\mathbf{e}'_1 - \mathbf{e}'_2 = 0$.

One can see it also studying the transition matrix. Transition matrix $T = \begin{pmatrix} 1 & 3 & 0 \\ -1 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $(\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3) = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)T$. This matrix is degenerate, its rank ≤ 2, $\det T = 0$. On the other hand second and third row of this matrix are linear dependent. Hence rank of the matrix T is equal to 2. This matrix is not orthogonal.

Case d)

The transition matrix from the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to the ordered triple $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\} = \{\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2 + \lambda\mathbf{e}_3\}$ is $T = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & \lambda \end{pmatrix}$, $(\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3) = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)T$

I-st case. $\lambda \neq 0$. The ordered set $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ is a basis because vectors are linear independent (see the exercise 3), This basis is not orthogonal, because the length vector is not equal to 1 ($(\mathbf{e}'_3, \mathbf{e}'_3) = |\mathbf{e}'_3|^2 = 2 + \lambda^2$). This matrix is not orthogonal, because the new basis is not orthonormal.

II-nd case $\lambda = 0$. The ordered set $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ is not a basis because vectors are linear independent: $\mathbf{e}'_1 + \mathbf{e}'_2 - \mathbf{e}'_3 = 0$. The transition matrix T has rank less or equal to 2, because vectors are linear dependent. On the other hand vectors $\mathbf{e}'_1, \mathbf{e}'_2$ are linear independent. Hence the rank of the matrix is equal to 2.

6†. *Show that an arbitrary orthogonal transformation of two-dimensional Euclidean space can be considered as a composition of reflections.*

If the determinant of orthogonal transformation is equal to -1 then

$$T = \tilde{T}_\varphi = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}$$

One can see that this is reflection with respect to the axis which has an angle $\varphi/2$ with Ox axis.

If the determinant of orthogonal transformation is equal to 1 then

$$T = T_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \tilde{T}_\varphi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

i.e. rotation on the angle φ is a composition of two reflections.

7† Prove the Cauchy–Bunyakovsky–Schwarz inequality

$$(\mathbf{x}, \mathbf{y})^2 \leq (\mathbf{x}, \mathbf{x})(\mathbf{y}, \mathbf{y}),$$

where $\mathbf{x}, \mathbf{y}$ are arbitrary two vectors and $(\ ,\ )$ is a scalar product in Euclidean space.

Hint: For any two given vectors $\mathbf{x}, \mathbf{y}$ consider the quadratic polynomial $At^2 + 2Bt + C$ where $A = (\mathbf{x}, \mathbf{x})$, $B = (\mathbf{x}, \mathbf{y})$, $C = (\mathbf{y}, \mathbf{y})$. Show that this polynomial has at most one real root and consider its discriminant.

Consider quadratic polynomial $P(t) = \sum_{i=1}^n (tx^i + y^i)^2 = At^2 + 2Bt + C$, where $A = \sum_{i=1}^n (x^i)^2 = (\mathbf{x}, \mathbf{x})$, $B = \sum_{i=1}^n (x^i y^i) = (\mathbf{x}, \mathbf{y})$, $C = \sum_{i=1}^n (y^i)^2 = (\mathbf{y}, \mathbf{y})$. We see that equation $P(t) = 0$ has at most one root (and this is the case if only vector $\mathbf{x}$ is collinear to the vector $\mathbf{y}$). This means that discriminant of this equation is less or equal to zero. But discriminant of this equation is equal to $4B^2 - 4AC$. Hence $B^2 \leq AC$. It is just CBS inequality. ($(\mathbf{x}, \mathbf{y})^2 = (\mathbf{x}, \mathbf{x})(\mathbf{y}, \mathbf{y})$, i.e. discriminant is equal to zero $\Leftrightarrow$ vectors $\mathbf{x}$, $\mathbf{y}$ are colinear.