Homework 6. Solutions

We consider here the realisation of Lobachevsky plane (hyperbolic plane) as upper half of Euclidean plane $\{(x, y)\colon y > 0\}$ *with the metric* $G = \frac{dx^2+dy^2}{y^2}$.

1 *Calculate Christoffel symbols of Levi-Civita connection on Lobachevsky plane*

Lagrangian of "free" particle on the Lobachevsky plane with metric $G = \frac{dx^2+dy^2}{y^2}$ is

$$L = \frac{1}{2}\frac{\dot{x}^2 + \dot{y}^2}{y^2}.$$

Euler-Lagrange equations are

$$\frac{\partial L}{\partial x} = 0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{d}{dt}\left(\frac{\dot{x}}{y^2}\right) = \frac{\ddot{x}}{y^2} - \frac{2\dot{x}\dot{y}}{y^3}, \text{i.e.} \quad \ddot{x} - \frac{2\dot{x}\dot{y}}{y} = 0\,,$$

$$\frac{\partial L}{\partial y} = -\frac{\dot{x}^2 + \dot{y}^2}{y^3} = \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = \frac{d}{dt}\left(\frac{\dot{y}}{y^2}\right) = \frac{\ddot{y}}{y^2} - \frac{2\dot{y}^2}{y^3}, \text{i.e.} \quad \ddot{y} + \frac{\dot{x}^2}{y} - \frac{\dot{y}^2}{y} = 0\,.$$

Comparing these equations with equations for geodesics: $\ddot{x}^i - \dot{x}^k\Gamma^i_{km}\dot{x}^m = 0$ ($i = 1, 2$, $x = x^1, y = x^2$) we come to

$$\Gamma^x_{xx} = 0, \Gamma^x_{xy} = \Gamma^x_{yx} = -\frac{1}{y},\ \Gamma^x_{yy} = 0,\ \Gamma^y_{xx} = \frac{1}{y}, \Gamma^y_{xy} = \Gamma^y_{yx} = 0, \Gamma^y_{yy} = -\frac{1}{y}\,. \blacksquare$$

2 *Show that vertical lines $x = a$ are geodesics (non-parameterised) on Lobachevsky plane.*

First solution: Consider second order differential equations defining geodesics (see the problem above) with initial conditions such that "horisontal" velocity equals to zero

$$\begin{cases} \ddot{x} - \frac{2\dot{x}\dot{y}}{y} = 0 \\ \ddot{y} + \frac{\dot{x}^2}{y} - \frac{\dot{y}^2}{y} = 0 \\ x(t)\big|_{t=t_0} = x_0, \dot{x}(t)\big|_{t=t_0} = 0 \\ y(t)\big|_{t=t_0} = y_0, \dot{y}(t)\big|_{t=t_0} = \dot{y}_0 \end{cases}$$

This equation has a solution and it is unique. One can see that if we put $x(t) \equiv 0$, i.e. curve is vertical then we come to the equation $\ddot{y} - \frac{\dot{y}^2}{y} = 0$. Solution of these equation gives curve $x = x_0$, $y = y(t)$: $\ddot{y} - \frac{\dot{y}^2}{y} = 0$. The image of this curve clearly is vertical ray $x = x_0, y > 0$.

Second solution Often it is much more practical to find integral of motions for finding geodesics than to study second order differential equations.

The Lagrangian of "free" particle $L = \frac{1}{2}\frac{\dot{x}^2+\dot{y}^2}{y^2}$ has two integrals of motions

$$p_x = \frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{y^2}, \quad \text{and } E = L = \frac{1}{2}\frac{\dot{x}^2 + \dot{y}^2}{y^2}$$

These magnitudes are preserved along geodesic. If $x(t), y(t)$ are equations of geodesics then

$$p_x(x(t), y(t)) = \frac{\dot{x}}{y^2} = C_1, \quad E(x(t), y(t)) = \frac{1}{2}\frac{\dot{x}^2 + \dot{y}^2}{y^2} = C_2.$$

are constants. These are two first order differential equations which have unique solutions defining geodesics as well as integrals of motion (constants C_1, C_2) and initial conditions are fixed. Bearing in mind the vertical rays, consider the case when $C_1 = 0$. We come to differential equations with initial conditions:

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = \pm\sqrt{2C_2y^2 - \dot{x}^2} = \pm\sqrt{2C_2}y \\ \dot{x} = 0, \dot{y} = a \end{cases}$$

The curve $x(t)=x_0$, $y=y(t)$: $\dot y=\pm\sqrt{2C_2y^2-\dot x^2}=\pm\sqrt{2C_2}y$ is the solution of this equation. The image of this curve is vertical ray. Thus we prove that vertical ray is non-parameterised geodesic.

3 Let $\mathbf{r}=\mathbf{r}(t)$ be an arbitrary geodesic on Lobachevsky plane. Show that magnitudes $I=I=\frac{v_x^2}{y^2}$ and $E=\frac{v_x^2+v_y^2}{y^2}$ are preserved along geodesics.

The magnitudes I and E are integrals of motion. $I=\frac{v_x}{y^2}=\frac{\dot x}{y^2}=\frac{\partial L}{\partial \dot x}$. It is preserved since due to Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot x}=\frac{\partial L}{\partial x}=0$$

for the Lagrangian $L=\frac{1}{2}\frac{\dot x^2+\dot y^2}{y^2}$.

Show that $E=L=E=\frac{v_x^2+v_y^2}{y^2}=\frac{1}{2}\frac{\dot x^2+\dot y^2}{y^2}$ is preserved along geodesic. Using Euler-Lagrange equations we have

$$\frac{dE}{dt}=\frac{dL}{dt}=\frac{\partial L}{\partial x}\dot x+\frac{\partial L}{\partial \dot x}\ddot x+\frac{\partial L}{\partial y}\dot y+\frac{\partial L}{\partial \dot y}\ddot y=\frac{d}{dt}\left(\frac{\partial L}{\partial \dot x}\right)\dot x+\frac{\partial L}{\partial \dot x}\frac{d\dot x}{dt}+\frac{d}{dt}\left(\frac{\partial L}{\partial \dot y}\right)\dot y+\frac{\partial L}{\partial \dot y}\frac{d\dot y}{dt}=\frac{d}{dt}\left(\frac{\partial L}{\partial \dot x}\dot x+\frac{\partial L}{\partial \dot y}\dot y\right)$$

Notice that $E=L=\frac{1}{2}\frac{\dot x^2+\dot y^2}{y^2}$ is quadratic function over $\dot x$ and $\dot y$. Hence

$$\frac{dE}{dt}=\frac{dL}{dt}=\frac{d}{dt}\left(\frac{\partial L}{\partial \dot x}\dot x+\frac{\partial L}{\partial \dot y}\dot y\right)=\frac{d}{dt}(2L)=2\frac{dE}{dt}\Rightarrow\frac{dE}{dt}=0\,.$$

4 *Show that the following transformations are isometries of Lobachevsky plane:*

a) horizontal translation $\mathbf{r}\to\mathbf{r}+\mathbf{a}$ *where* $\mathbf{a}=(a,0)$,

b) homothety: $\mathbf{r}\to\lambda\mathbf{r}$ $(\lambda>0)$,

* *c) inversion with the centre at the points of the line* $x=0^*$:

$$\mathbf{r}\to\mathbf{a}+\frac{\mathbf{r}-\mathbf{a}}{|\mathbf{r}-\mathbf{a}|^2}\ \textit{where}\ \mathbf{a}=(a,0):\quad\begin{cases}x'=a+\frac{x-a}{(x-a)^2+y^2}\\ y'=\frac{y}{(x-a)^2+y^2}\end{cases}.$$

We have to show that Riemannian metric $G=\frac{dx^2+dy^2}{y^2}$ remains invariant under these transformations.

a) horizontal translation. If $x\to x+a, y\to y$ then dx and dy do not change. Hence G is invariant under horisontal translations.

b) Homothety. If $x\to\lambda x$, $y\to\lambda y$ where $\lambda>0$ is a constant then $\frac{dx^2+dy^2}{y^2}\to\frac{\lambda^2dx^2+\lambda^2dy^2}{\lambda^2y^2}=\frac{dx^2+dy^2}{y^2}$ does not change too.

c) * inversion with the centre at the points of the line $y=0$. Since we proved that horisontal translation is isometry it suffices to consider inversion with centre at the point $x=y=0$:

$$\mathbf{r}\to\frac{\mathbf{r}}{|\mathbf{r}|^2}:\quad\begin{cases}x'=\frac{x}{x^2+y^2}\\ y'=\frac{y}{x^2+y^2}\end{cases}$$

Now by straightforward calculations one can show that $\frac{dx^2+dy^2}{y^2}=\frac{dx'^2+dy'^2}{y'^2}$.

To avoid the straightforward calculations consider coordinates r,φ: $x=r\cos\varphi, y=r\sin\varphi$ then

$$\frac{dx^2+dy^2}{y^2}=\frac{dr^2+r^2d\varphi^2}{r^2\sin^2\varphi}=\frac{1}{\sin^2\varphi}\frac{dr^2}{r^2}+\frac{d\varphi^2}{\sin^2\varphi}=\frac{1}{\sin^2\varphi}(d\log r)^2+\frac{d\varphi^2}{\sin^2\varphi}\,.$$

Under transformation of inversion in these coordinates φ does not change, $r\to\frac{1}{r}$, $\log r\to-\log\frac{1}{r}$. It is evident that in coordinates $u=\log r,\varphi$ metric does not change. Hence inversion (with centre at the point

* *This line is called absolute.*

$x = y = 0$) is isometry. Hence an inversion with a centre at the arbitrary point $(a, 0)$ and with an arbitrary radius is isometry, since horisontal translation and homothety are isometries.

5* Show that upper arcs of semicircles $(x - a)^2 + y^2 = R^2, y > 0$ are (non-parametersied) geodesics.

(*You can do this exercise solving explicitly differential equations for geodesics, or using integrals of motion obtained in exercise 3, or (and this is most beautiful) use inversion transformation and the results of exercise 2.)*

Consider the inversion of the Lobachevsky plane with the centre at the point $x = a - R, y = 0$ (see the exercise above). This inversion does not change Riemannian metric, it is isometry. Isometry transforms geodesics to geodesics. On the other hand it transforms the semicircle $(x - a)^2 + y^2 = R^2, y > 0$ to the vertical ray $x = a - R + \frac{1}{2R}, y > 0$. This can be checked directly. On the other hand the vertical ray is geodesic. Hence the initial curve was the geodesic too.

6 *Show that parallel transport along the given curve does not depend on parameterisation of the curve.*

Let $\mathbf{X}(t)$ be a parallel transport along the curve $x^i(t)$, i.e.

$$\frac{dX^i(t)}{dt} + \dot{x}^m(t)\Gamma^i_{mk}(x(t))X^k(t) = 0.$$

Consider reparameterisation: $x'^i(\tau) = x^i(t(\tau))$ and prove that

$$\frac{dX^i(t(\tau))}{d\tau} + \dot{x}'^m(\tau)\Gamma^i_{mk}(x(t(\tau)))X^k(t(\tau)) = 0. \tag{$*$}$$

Using chain rule we have

$$\frac{dX^i(t(\tau))}{d\tau} + \dot{x}'^m(\tau)\Gamma^i_{mk}(x(t(\tau)))X^k(t(\tau)) = \frac{dt}{d\tau}\frac{dX^i(t)}{dt} + \frac{dt}{d\tau}\dot{x}^m(t)\Gamma^i_{mk}(x(t))X^k(t) =$$

$$\frac{dt}{d\tau}\left(\frac{dX^i(t)}{dt} + \dot{x}^m(t)\Gamma^i_{mk}(x(t))X^k(t)\right) = 0. \tag{$**$}$$

We see that equation (*) implies equation (**), i.e. vector field $\mathbf{X}(t(\tau))$ is parallel transport over the reparameterised curve $\mathbf{x}'(\tau) = \mathbf{x}(t(\tau))$, that is the value of vector field at the same points of the image of the curve remains the same.

7 *Consider the vector* $\mathbf{X} = -\frac{\partial}{\partial x}$ *attached at the point (0,a) of the Lobachevsky plane and the arc of circle* C: $\begin{cases} x(t) = a\cos t \\ y(t) = a\sin t \end{cases}, \frac{\pi}{2} \le t \le \frac{3\pi}{4}$ *in the Lobachevsky plane. Find parallel transport of the vector* $\mathbf{X}$ *along the curve* C.

(You may use the fact that C *is an arc of geodesic (non-parameterised) geodesic.)*

We know that the the curve C is the arc of geodesic on the Lobachevsky plane. Hence during parallel transport tangent vector remains tangent to the curve. It means that parallel transport $\mathbf{X}(t)$ is proportional to the velocity vector: $\mathbf{X}(t) = k(t)\mathbf{v}(t)$.

On the other hand during parallel transport with respect to Levi-Civita connection the vector does not change its length, since Levi-Civita connection preserves the scalar product. Hence

$$\langle \mathbf{X}(t), \mathbf{X}(t)\rangle = \langle k(t)\mathbf{v}(t), k(t)\mathbf{v}(t)\rangle = k^2(t)\langle \mathbf{v}(t), \mathbf{v}(t)\rangle = const$$

We have that velocity vector $\mathbf{v}(t) = -a\sin t\partial_x + a\cos t\partial_y$ and

$$\langle \mathbf{v}(t), \mathbf{v}(t)\rangle = g_{xx}v_x^2 + 2g_{xy}v_xv_y + g_{yy}v_y^2 = \frac{v_x^2 + v_y^2}{y^2} = \frac{a^2\sin^2 t + a^2\cos^2 t}{a^2\sin^2 t} = \frac{1}{sin^2 t}.$$

We have $k^2(t)\langle \mathbf{v}(t), \mathbf{v}(t)\rangle = \frac{k^2(t)}{\sin^2 t} = const$. Hence $k(t) = const \sin t$. On the other hand at the $t = \frac{\pi}{2}$ $\mathbf{X} = -\frac{\partial}{\partial x}$, i.e. $k(t)|_{t=\frac{\pi}{2}} = 1$. Hence $k(t) = \sin t$ and

$$\mathbf{X}(t) = \sin t \mathbf{v}(t) = -a \sin^2 t \partial_x + a \cos t \sin t \partial_y, \ \frac{\pi}{2} \le t \le \frac{3\pi}{4}.$$

8 *Find geodesics on the sphere in Euclidean space.*

There are many many solutions. Give just one. The Lagrangian of "free" particle on the sphere is $L = \dot\theta^2 + \sin^2\theta \dot\varphi^2$. Its equations of motion:

$$\begin{cases} \frac{d}{dt}(\sin^2\theta\dot\varphi) \Rightarrow \ddot\varphi + 2\mathrm{cotan}\,\theta\dot\theta\dot\varphi = 0 \\ \ddot\theta - \sin\theta\cos\theta\dot\varphi^2 = 0 \end{cases}$$

One can see that the arc of great circle the curve C_0: $\varphi = 0$, $\theta = t$ obeys these equations. It is a geodesic. The isometry preserves the metric and Levi-Civita connection. Hence it preserves the geodesics. Take an arbitrary geodesic. By rotation which is an isometry one can transform this curve to the curve such that it coincides with the great circle C_0 at a point and the velocity vector is the same. Hence it coincides with the curve C_0. Hence all arcs of great circles are geodesics and all geodesics are arcs of great circles.

Another solution is based on the fact that locally geodesic is shortest...

Another solution is based on the fact that geodesic acceleration for geodesic equals to zero...

9 *Consider a sphere of the radius R in $\mathbf{E}^3$ and an arbitrary vector $\mathbf{X}$ attached at the point (θ_0, φ_0) and tangent to this sphere.*

What will be the result of parallel transport of the vector $\mathbf{X}$ along the following closed curves on the sphere

a) C_1: $\varphi(t) = \varphi_0, \varphi + \pi$ (two meridians), *b) C_2: $\theta(t) = \theta_0$ (latitude.)*

(θ, φ are spherical coordinates.)

It follows from the previous exercise (or we can do it in many other ways (see the Lecture notes)) that Levi-Civita connection ∇ on the sphere is

$$\begin{cases} \nabla_{\partial_\theta}\partial_\theta = 0, \Gamma^\theta_{\theta\theta} = 0 \\ \nabla_{\partial_\theta}\partial_\varphi = \nabla_{\partial_\varphi}\partial_\theta = \mathrm{cotan}\,\theta\partial_\varphi, \Gamma^\theta_{\theta\varphi} = \Gamma^\theta_{\varphi\theta} = 0, \Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = \mathrm{cotan}\,\theta \\ \nabla_{\partial_\varphi}\partial_\varphi = -\sin\theta\cos\theta\partial_\theta, \Gamma^\varphi_{\varphi\varphi} = 0, \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta \end{cases}$$

If $\mathbf{X}(t) = a(t)\partial_\theta|_t + b(t)\partial_\varphi|_t$ is the parallel transport along the curve $\theta = \theta(t), \varphi = \varphi(t)$ then the equations of parallel transport

$$\frac{\nabla \mathbf{X}(t)}{dt} = 0, \frac{dX^i(t)}{dt} + \frac{dx^m(t)}{dt}\Gamma^i_{mk}X^k = 0$$

will have on the sphere the following appearance: $\mathbf{X}(t) = a(t)\partial_\theta|_t + b(t)\partial_\varphi|_t$

$$\begin{cases} \frac{da(t)}{dt} + \dot\varphi\Gamma^\theta_{\varphi\varphi}b(t) = \frac{da(t)}{dt} - \sin\theta\cos\theta\dot\varphi b(t) = 0 \\ \frac{db(t)}{dt} + \dot\varphi\Gamma^\varphi_{\varphi\theta}a(t) + \dot\theta\Gamma^\varphi_{\theta\varphi}b(t) = \frac{db(t)}{dt} + \mathrm{cotan}\,\theta(\dot\varphi a(t) + \dot\theta b(t)) = 0 \end{cases}$$

In the of case parallel transport along the great circle $\theta = t, \varphi = 0$, $\dot\varphi = 0, \dot\theta = 1$ hence

$$\begin{cases} \frac{da(t)}{dt} - \sin\theta\cos\theta\dot\varphi b(t) = \frac{da(t)}{dt} = 0 \\ \frac{db(t)}{dt} + \mathrm{cotan}\,\theta(\dot\varphi a(t) + \dot\theta b(t)) = \frac{db(t)}{dt} + b(t))\mathrm{cotan}\,t = 0. \end{cases}$$

We see that $a(t) = a_0$ and $b(t) = \frac{b_0}{\sin t}$.

Note that this can be easily deduced from the fact of preservation of the length.

The component $a(t)$ does not change in time (This was obvious without any formulae). On the other hand length does not change. Hence $b(t) \sim \frac{1}{\sin t}$ since the length of the vector ∂_φ equals to $\sin\theta$.

In particular the result of parallel transport along the closed curve C_1 we will come to the same vector.

The second case is little bit more contr-intuitive.

In the of case parallel transport along the latitude $\theta(t) = \theta_0, \varphi = t$ we have $\dot\theta = 0$, $\dot\varphi = 1$ and

$$\begin{cases} \frac{da(t)}{dt} - \sin\theta\cos\theta\dot\varphi b(t) = \frac{da(t)}{dt} - \sin\theta_0\cos\theta_0 b(t) = 0 \\ \frac{db(t)}{dt} + \operatorname{cotan}\theta(\dot\varphi a(t) + \dot\theta b(t)) = \frac{db(t)}{dt} + \operatorname{cotan}\theta_0 a(t) = 0. \end{cases}$$

Here we have two first order differential equations. Introduce $b'(t) = \sin\theta_0 b(t)$. We have

$$\begin{cases} \frac{da(t)}{dt} - \cos\theta_0 b'(t) = 0 \\ \frac{db'(t)}{dt} + \cos\theta_0 a(t) = 0 \end{cases}, \quad \text{i.e.} \ \frac{d}{dt}\begin{pmatrix} a(t) \\ b'(t) \end{pmatrix} = \cos\theta_0 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a(t) \\ b'(t) \end{pmatrix}$$

or in other way

$$\frac{d}{dt}(a + ib') = -i\cos\theta_0(a + ib')$$

The solution is

$$a(t) + ib'(t) = \mathbf{e}^{-it\cos\theta_0}(a_0 + ib'_0) = (\cos(t\cos\theta_0) - i\sin(t\cos\theta_0))(a_0 + ib'_0)$$

In particular as a result of parallel transport along the closed latitude

$$a(t) + ib'(t)\big|_{t=2\pi} = \mathbf{e}^{-2\pi i\cos\theta_0}(a_0 + ib'_0)$$

that is the vector rotates on the angle $\Phi = 2\pi\cos\theta_0$.

Remark Notice that Φ is just equals to the area of domain of the sphere over the latitude $(2\pi R^2\cos\theta_0)$ divided on the Gaussian curvature $\frac{1}{R^2}$.

Later we will learn that in general the vector $\mathbf{X}$ rotates on the angle $\Phi = \int_M K d\sigma$ if the closed curve C is a boundary of the domain M.