Introduction to Geometry

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Manchester, 4 February 2019

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1 Euclidean space

1.1 Recollection of vector space and Euclidean vector space

We recall here important notions from linear algebra of vector space and Euclidean vector space.

1.1.1 Vector space.

Vector space V on real numbers is a set of vectors with operations " $+$ "—addition of vector and " $\cdot$ "—multiplication of vector Lon real number (sometimes called coefficients, scalars).

[1]

Remark We denote by 0 real number 0 and *vector* $\mathbf{0}$. Sometimes we have to be careful to distinguish between zero vector $\mathbf{0}$ and number zero.

1.1.2 Basic example of (n-dimensional) vector space—$\mathbf{R}^n$

A basic example of vector space (over real numbers) is a space of ordered n-tuples of real numbers.
$\mathbf{R}^2$ is a space of pairs of real numbers. $\mathbf{R}^2 = \{(x, y),\ x, y \in \mathbf{R}\}$

[1] These operations obey the following axioms

- $\forall \mathbf{a}, \mathbf{b} \in V, \mathbf{a} + \mathbf{b} \in V$,
- $\forall \lambda \in \mathbf{R}, \forall \mathbf{a} \in V, \lambda \mathbf{a} \in V$.
- $\forall \mathbf{a}, \mathbf{b} \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ (commutativity)
- $\forall \mathbf{a}, \mathbf{b}, \mathbf{c},\ \mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$ (associativity)
- $\exists\ \mathbf{0}$ such that $\forall \mathbf{a}$, $\mathbf{a} + \mathbf{0} = \mathbf{a}$
- $\forall \mathbf{a}$ there exists a vector $-\mathbf{a}$ such that $\mathbf{a} + (-\mathbf{a}) = 0$.
- $\forall \lambda \in \mathbf{R}, \lambda(\mathbf{a} + \mathbf{b}) = \lambda \mathbf{a} + \lambda \mathbf{b}$
- $\forall \lambda, \mu \in \mathbf{R} (\lambda + \mu) \mathbf{a} = \lambda \mathbf{a} + \mu \mathbf{a}$
- $(\lambda \mu) \mathbf{a} = \lambda(\mu \mathbf{a})$
- $1 \mathbf{a} = \mathbf{a}$

$\mathbf{R}^3$ is a space of triples of real numbers. $\mathbf{R}^3 = \{(x, y, z),\ x, y, z \in \mathbf{R}\}$
$\mathbf{R}^4$ is a space of quadruples of real numbers. $\mathbf{R}^4 = \{(x, y, z, t),\ x, y, z, t, \in \mathbf{R}\}$
and so on...

$\mathbf{R}^n$—is a space of n-typles of real numbers:

$$\mathbf{R}^n = \{(x^1, x^2, \dots, x^n),\ x^1, \dots, , x^n \in \mathbf{R}\} \tag{1.1}$$

If $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ are two vectors, $\mathbf{x} = (x^1, \dots, x^n)$, $\mathbf{y} = (y^1, \dots, y^n)$ then

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \dots, x_n + y_n)\,.$$

and multiplication on scalars is defined as

$$\lambda \mathbf{x} = \lambda \cdot (x^1, \dots, x^n) = (\lambda x^1, \dots, \lambda x^n)\,, \quad (\lambda \in \mathbf{R})\,.$$

1.1.3 Linear dependence of vectors

We often consider linear combinations in vector space:

$$\sum_i \lambda_i \mathbf{x}_i = \lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \cdots + \lambda_m \mathbf{x}_m\,, \tag{1.2}$$

where $\lambda_1, \lambda_2, \dots, \lambda_m$ are coefficients (real numbers), $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m$ are vectors from vector space V. We say that linear combination (1.2) is *trivial* if all coefficients $\lambda_1, \lambda_2, \dots, \lambda_m$ are equal to zero.

$$\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0\,.$$

We say that linear combination (1.2) is *not trivial* if at least one of coefficients $\lambda_1, \lambda_2, \dots, \lambda_m$ is not equal to zero:

$$\lambda_1 \neq 0, \mathrm{or} \lambda_2 \neq 0, \mathrm{or} \dots \mathrm{or} \lambda_m \neq 0\,.$$

Recall definition of linearly dependent and linearly independent vectors:

Definition The vectors $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ in vector space V are *linearly dependent* if there exists a non-trivial linear combination of these vectors such that it is equal to zero.

In other words we say that the vectors $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ in vector space V are *linearly dependent* if there exist coefficients $\mu_1, \mu_2, \dots, \mu_m$ such that at least one of these coefficients is not equal to zero and

$$\mu_1 \mathbf{x}_1 + \mu_2 \mathbf{x}_2 + \cdots + \mu_m \mathbf{x}_m = 0\,. \tag{1.3}$$

Respectively vectors $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ are *linearly independent* if they are not linearly dependent. This means that an arbitrary linear combination of these vectors which is equal zero is trivial.

In other words vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_m\}$ are *linearly independent* if the condition

$$\mu_1\mathbf{x}_1 + \mu_2\mathbf{x}_2 + \cdots + \mu_m\mathbf{x}_m = 0$$

implies that $\mu_1 = \mu_2 = \cdots = \mu_m = 0$.

Very useful and workable

Proposition *Vectors $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ in vector space V are linearly dependent if and only if at least one of these vectors is expressed via linear combination of other vectors:*

$$\mathbf{x}_i = \sum_{j \neq i} \lambda_j \mathbf{x}_j \,.$$

1.1.4 Dimension of vector space. Basis in vector space.

Definition Vector space V has a dimension n if there exist n linearly independent vectors in this vector space, and any $n+1$ vectors in V are linearly dependent.

In the case if in the vector space V for an arbitrary N there exist N linearly independent vectors then the space V is *infinite-dimensional*. An example of infinite-dimensional vector space is a space V of all polynomials of an arbitrary order. One can see that for an arbitrary N polynomials $\{1, x, x^2, x^3, \dots, x^N\}$ are linearly idependent. (Try to prove it!). This implies V is infinite-dimensional vector space.

Basis

Definition Let V be n-dimensional vector space. The ordered set $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ of n linearly independent vectors in V is called a basis of the vector space V.

Remark We say ‘a basis’, not ‘the basis’ since there are many bases in the vector space (see also Homeworks 1.2).

Remark Focus your attention: basis is *an ordered* set of vectors, not just a set of vectors[2].

The following Proposition is very useful:

[2]See later on orientation of vector spaces, where the ordering of vectors of basis will be highly important.

Proposition *Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be an arbitrary basis in n-dimensional vector space V. Then any vector $\mathbf{x} \in V$ can be expressed as a linear combination of vectors $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ in a unique way, i.e. for every vector $\mathbf{x} \in V$ there exists an ordered set of coefficients $\{x^1, \dots, a^n\}$ such that*

$$\mathbf{x} = x^1\mathbf{e}_1 + \cdots + x^n\mathbf{e}_n \tag{1.4}$$

and if

$$\mathbf{x} = a^1\mathbf{e}_1 + \cdots + a^n\mathbf{e}_n = b^1\mathbf{e}_1 + \cdots + b^n\mathbf{e}_n\,, \tag{1.5}$$

then $a^1 = b^1, a^2 = b^2, \dots, a^n = b^n$. In other words for any vector $\mathbf{x} \in V$ there exists an ordered n-tuple $(x^1, \dots, x^n)$ of coefficients such that $\mathbf{x} = \sum_{i=1}^n x^i\mathbf{e}_i$ and this n-tuple is unique.

In other words:

Basis is a set of linearly independent vectors in vector space V which span (generate) vector space V.

Recall that we say that vector space V is *spanned* by vectors $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ (or vectors vectors $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ *span* vector space V) if any vector $\mathbf{a} \in V$ can be expresses as a linear combination of vectors $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$.

Definition Coefficients $\{a^1, \dots, a^n\}$ are called *components of the vector* $\mathbf{x}$ *in the basis* $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ or just shortly *components of the vector* $\mathbf{x}$.

Example *Canonical basis in* $\mathbf{R}^n$

We considered above the basic example of vector space—a space of ordered n-tuples of real numbers: $\mathbf{R}^n = \{(x^1, x^2, \dots, x^n), x^i \in \mathbf{R}\}$ (see (1.1)). One can see that it is n-dimensional vector space. Consider vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n \in \mathbf{R}^n$:

$$\begin{array}{rl} \mathbf{e}_1 = & (1, 0, 0 \dots, 0, 0) \\ \mathbf{e}_2 = & (0, 1, 0 \dots, 0, 0) \\ \dots & \qquad \dots \\ \mathbf{e}_n = & (0, 0, 0 \dots, 0, 1) \end{array} \tag{1.6}$$

Then for an arbitrary vector $\mathbf{R}^n \ni \mathbf{a} = (a^1, a^2, a^3, \dots, a^n)$,

$$\mathbf{a} = a^1(1, 0, 0 \dots, 0, 0) + a^2(0, 1, 0 \dots, 0, 0) + a^3(0, 0, 1, 0 \dots, 0, 0) + \cdots + a^n(0, 1, 0 \dots, 0, 1) =$$

$$\sum_{i=1}^m a^i\mathbf{e}_i = a^i\mathbf{e}_i \qquad \text{(we will use sometimes condensed notations } \mathbf{x} = x^i\mathbf{e}_i)$$

For every vector $\mathbf{a} \in \mathbf{R}^n$ we have unique expansion via the vectors (1.6). The set of vectors $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a basis of $\mathbf{R}^n$. The basis (1.6) is the distinguished basis. Sometimes it is called *canonical basis in* $\mathbf{R}^n$. One can find another basis in $\mathbf{R}^n$–just take an arbitrary ordered set of n linearly independent vectors. (See exercises in Homework 0).

1.1.5 Scalar product. Euclidean space

In vector space one have additional structure: *scalar product of vectors.*

Definition Scalar product in a vector space V is a function $B(\mathbf{x}, \mathbf{y})$ on a pair of vectors which takes real values and satisfies the the following conditions:

$$\begin{array}{c} B(\mathbf{x},\mathbf{y}) = B(\mathbf{y},\mathbf{x}) \quad \text{(symmetricity condition)} \\ B(\lambda\mathbf{x}+\mu\mathbf{x}',\mathbf{y}) = \lambda B(\mathbf{x},\mathbf{y}) + \mu B(\mathbf{x}',\mathbf{y}) \quad \text{(linearity condition)} \\ B(\mathbf{x},\mathbf{x}) \geq 0 \ , B(\mathbf{x},\mathbf{x}) = 0 \Leftrightarrow \mathbf{x} = 0 \text{ (positive-definiteness condition)} \end{array} \tag{1.7}$$

Definition Euclidean space is a vector space equipped with a scalar product.

One can easy to see that the function $B(\mathbf{x}, \mathbf{y})$ is bilinear function, i.e. it is linear function with respect to the second argument also. This follows from previous axioms:

$$B(\mathbf{x}, \lambda\mathbf{y}+\mu\mathbf{y}') \underbrace{=}_{\text{symm.}} B(\lambda\mathbf{y}+\mu\mathbf{y}', \mathbf{x}) \underbrace{=}_{\text{linear.}} \lambda B(\mathbf{y}, \mathbf{x})+\mu B(\mathbf{y}', \mathbf{x}) \underbrace{=}_{\text{symm.}} \lambda B(\mathbf{x}, \mathbf{y})+\mu B(\mathbf{x}, \mathbf{y}') \,.$$

A bilinear function $B(\mathbf{x}, \mathbf{y})$ on pair of vectors is called sometimes *bilinear form* on vector space. Bilinear form $B(\mathbf{x}, \mathbf{y})$ which satisfies the symmetricity condition is called *symmetric bilinear form.* Scalar product is nothing but symmetric bilinear form on vectors which is positive-definite: $B(\mathbf{x}, \mathbf{x}) \geq 0$) and is non-degenerate $((\mathbf{x}, \mathbf{x}) = 0 \Rightarrow \mathbf{x} = 0$.

Example We considered the vector space $\mathbf{R}^n$, the space of n-tuples (see the subsection 1.2). One can consider the vector space $\mathbf{R}^n$ as Euclidean space provided by the scalar product

$$B(\mathbf{x}, \mathbf{y}) = x^1 y^1 + \cdots + x^n y^n \tag{1.8}$$

This scalar product sometimes is called *canonical scalar product.*

Exercise Check that it is indeed scalar product.

Example We consider in 2-dimensional vector space V with basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ and $B(\mathbf{X}, \mathbf{Y})$ such that $B(\mathbf{e}_1, \mathbf{e}_1) = 3$, $B(\mathbf{e}_2, \mathbf{e}_2) = 5$ and $B(\mathbf{e}_1, \mathbf{e}_2) = 0$. Then for every two vectors $\mathbf{X} = x^1\mathbf{e}_1 + x^2\mathbf{e}_2$ and $\mathbf{Y} = y^1\mathbf{e}_1 + y^2\mathbf{e}_2$ we have that

$$B(\mathbf{X}, \mathbf{Y}) = (\mathbf{X}, \mathbf{Y}) = \left(x^1\mathbf{e}_1 + x^2\mathbf{e}_2, y^1\mathbf{e}_1 + y^2\mathbf{e}_2\right) =$$

$$x^1y^1(\mathbf{e}_1, \mathbf{e}_1) + x^1y^2(\mathbf{e}_1, \mathbf{e}_2) + x^2y^1(\mathbf{e}_2, \mathbf{e}_1) + x^2y^2(\mathbf{e}_2, \mathbf{e}_2) = 3x^1y^1 + 5x^2y^2 .$$

One can see that all axioms are obeyed.

Remark Scalar product sometimes is called "inner" product or "dot" product. Later on we will use for scalar product $B(\mathbf{x}, \mathbf{y})$ just shorter notation $(\mathbf{x}, \mathbf{y})$ (or $\langle \mathbf{x}, \mathbf{y} \rangle$). Sometimes it is used for scalar product a notation $\mathbf{x} \cdot \mathbf{y}$. Usually this notation is reserved only for the canonical case (1.8).

Counterexample Consider again 2-dimensional vector space V with basis $\{\mathbf{e}_1, \mathbf{e}_2\}$.

Show that operation such that $(\mathbf{e}_1, \mathbf{e}_1) = (\mathbf{e}_2, \mathbf{e}_2) = 0$ and $(\mathbf{e}_1, \mathbf{e}_2) = 1$ does not define scalar product. *Solution.* For every two vectors $\mathbf{X} = x^1\mathbf{e}_1 + x^2\mathbf{e}_2$ and $\mathbf{Y} = y^1\mathbf{e}_1 + y^2\mathbf{e}_2$ we have that

$$(\mathbf{X}, \mathbf{Y}) = \left(x^1\mathbf{e}_1 + x^2\mathbf{e}_2, y^1\mathbf{e}_1 + y^2\mathbf{e}_2\right) = x^1y^2 + x^2y^1$$

hence for vector $\mathbf{X} = (1, -1)$ $(\mathbf{X}, \mathbf{X}) = -2 < 0$. Positive-definiteness is not fulfilled.

1.1.6 Orthonormal basis in Euclidean space

One can see that for scalar product (1.8) and for the basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ defined by the relation (1.6) the following relations hold:

$$(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \tag{1.9}$$

Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ be an ordered set of n vectors in n-dimensional Euclidean space which obeys the conditions (1.9). One can see that this ordered set is a basis [3].

[3] Indeed prove that conditions (1.9) imply that these n vectors are linear independent. Suppose that $\lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2 + \cdots + \lambda_n\mathbf{e}_n = 0$. For an arbitrary i multiply the left and right hand sides of this relation on a vector $\mathbf{e}_i$. We come to condition $\lambda_i = 0$. Hence vectors $(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)$ are linearly dependent.

Definition-Proposition *The ordered set of vectors* $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ *in* n*-dimensional Euclidean space which obey the conditions* (1.9) *is a basis. This basis is called an orthonormal basis.*

One can prove that every (finite-dimensional) Euclidean space possesses orthonormal basis.

Later by default we consider only orthonormal bases in Euclidean spaces. Respectively scalar product will be defined by the formula (1.8). Indeed let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be an orthonormal basis in Euclidean space. Then for an arbitrary two vectors $\mathbf{x}, \mathbf{y}$, such that $\mathbf{x} = \sum x^i \mathbf{e}_i$, $\mathbf{y} = \sum y^j \mathbf{e}_j$ we have:

$$(\mathbf{x}, \mathbf{y}) = \left(\sum x^i \mathbf{e}_i, \sum y^j \mathbf{e}_j\right) = \sum_{i,j=1}^{n} x^i y^j (\mathbf{e}_i, \mathbf{e}_j) = \sum_{i,j=1}^{n} x^i y^j \delta_{ij} = \sum_{i=1}^{n} x^i y^i$$

We come to the canonical scalar product (1.8). Later on we usually will consider scalar product defined by the formula (1.8) i.e. scalar product in orthonormal basis.

Remark We consider here general definition of scalar product then came to conclusion that in a special basis, (*orthonormal basis*), this is nothing but usual 'dot' product (1.8).

1.2 Affine spaces and vector spaces

`AFFINE SPACE WITH ORIGIN IS A VECTOR SPACE`

Let V be an arbitrary vector space.

Consider a set A whose elements will be called 'points' We say that A is *an affine space associated with vector space* V if the following rule is defined: to every point $P \in A$ and an arbitrary vector $\mathbf{x} \in V$ a point Q is assigned:

$$\forall P \in A\,, \quad \forall \mathbf{x} \in V\,, \quad (P, \mathbf{x}) \mapsto Q \in A \tag{1.10}$$

We denote $Q = P + \mathbf{x}$.

The following properties must be satisfied:

- For arbitrary two vectors $\mathbf{x}, \mathbf{y} \in V$ and arbitrary point $P \in A$, $P + (\mathbf{x} + \mathbf{y}) = (P + \mathbf{x}) + \mathbf{y}$.
- For an arbitrary point $P \in A$, $P + \mathbf{0} = P$.

  (Recall that $\mathbf{0}$ is the zero vector in the vector space V.)

- For arbitrary two points $P, Q \in A$ there exists unique vector $\mathbf{y} \in V$ such that $P + \mathbf{y} = Q$.

If $P + \mathbf{x} = Q$ we often denote the vector $\mathbf{x} = Q - P = \vec{PQ}$. We say that vector $\mathbf{x} = \vec{PQ}$ starts at the point P and it ends at the point Q.

One can see that if vector $\mathbf{x} = \vec{PQ}$, then $\vec{QP} = -\mathbf{x}$; if P, Q, R are three arbitrary points then $\vec{PQ} + \vec{QR} = \vec{PR}$.

One can reconstruct vector space V in terms of an affine space A, and vice versa. Namely, let A be an affine space associated with vector space V. Choose an arbitrary point $O \in A$ as an the origin, and consider the vectors starting at the origin: We come to the vector space V:

$$V = \text{set of vectors } \vec{OQ} \text{ where } Q \text{ is an arbitrary point in } A\,,$$

which is associated with an affine space A.

Let V be an arbitrary vector space. We will define now an affine space associated with this vector space. Consider two copies of the vector space V. The elements of the *first* copy we will call "points", and the elements of the *second* copy we will call as usual "vectors":

$$\underbrace{\overbrace{\mathrm{V}}^{\text{first copy of } V}}_{\text{elements of } V \text{ are points}} \qquad \underbrace{\overbrace{\mathrm{V}}^{\text{second copy of } V}}_{\text{elements of } V \text{ are vectors}} \tag{1.11}$$

Let $A = \mathbf{a}$ be an arbitrary point of the affine space, (i.e. an element of the *first* copy of vector space V) and let $\mathbf{x}$ is an arbitrary vector of the vector space V (i.e. an element of the *second* copy of vector space V). We define the action (1.10) in the following way:

$$(A, \mathbf{x}) \mapsto B = A + \mathbf{x} = \mathbf{a} + \mathbf{x}\,, \quad \mathbf{x} = \vec{AB}\,.$$

The point B is the vector $\mathbf{a} + \mathbf{x} \in V$ belonging to the *first* copy of the vector space V.

We assign to two 'points' $A = \mathbf{a}, B = \mathbf{b}$ (elements of the *first* copy of vector space V) the vector $\mathbf{x} = \mathbf{b} - \mathbf{a}$ (elements of the *second* copy of vector space V).

For example vector space $\mathbf{R}^n$ of n-tuples of real numbers can be considered as a set of points. If we choose arbitrary two points $A = (a^1, a^2 \ldots, a^n)$ and

$B=(b^1,b^2,\dots,b_n)$, then these two points define a vector $\vec{AB}$ which is equal to $\vec{AB}=B-A=(b^1-a^1,b^2-a^2,\dots,b_n-a_n)$.

The associated with each other affine space and vector space $\mathbf{R}^n$ we will usually denote by the same letter.

1.2.1 Eucldiean affine space.

Respectively one can consider Euclidean vector space as a set of points. Let $\mathbf{E}^n$ be n-dimensional Euclidean vector space, i.e. vector space equipped with scalar product. Let $\{\mathbf{e}_i\}$ $(i=1,\dots,n)$ be an arbitrary orthonormal basis in the vector space $\mathbf{E}^n$. Now consider this vector space as a set of points. Choose arbitrary two points (vectors of the *first* copy of the vector space $\mathbf{E}^n$), $A=a^1\mathbf{e}_1+a^2\mathbf{e}_2+\cdots+a^n\mathbf{e}_n$ and $B=b^1\mathbf{e}_1+b^2\mathbf{e}_2+\cdots+b^n\mathbf{e}_n$. These points define a vector $\vec{AB}$ (in the *second* copy of the vector space $\mathbf{E}^n$) which is equal to

$$\vec{AB}=B-A=(b^1-a^1)\mathbf{e}_1+(b^2-a^2)\mathbf{e}_2+\cdots+(b^n-a^n)\mathbf{e}_n\,.$$

The distance between two points A,B is the length of corresponding vector $\vec{AB}$, and the length of the vector $\vec{AB}$ is defined by the scalar product:

$$\left|\vec{AB}\right|=\sqrt{\left(\vec{AB},\vec{AB}\right)}=\sqrt{((b^1-a^1)\,\mathbf{e}_1+\cdots+(b^n-a^n)\,\mathbf{e}_n,(b^1-a^1)\mathbf{e}_1+\cdots+(b^n-a^n)\mathbf{e}_n)}$$

$$=\sqrt{(b^1-a^1)^2+\cdots+(b^n-a^n)^2}\,.$$

We recall very important formula how scalar product is related with the angle between vectors: if φ is an angle between vectors $\mathbf{x}$ and $\mathbf{y}$ then

$$(\mathbf{x},\mathbf{y})=x^1y^1+x^2y^2+\cdots+x^ny^n=|\mathbf{x}||\mathbf{y}|\cos\varphi \qquad (1.12)$$

(We suppose that vectors $\mathbf{x},\mathbf{y}$ are defined in orthonormal basis.)

In particulary it follows from this formula that

angle between vectors $\mathbf{x},\mathbf{y}$ *is acute if scalar product* $(\mathbf{x},\mathbf{y})$ *is positive*
angle between vectors $\mathbf{x},\mathbf{y}$ *is obtuse if scalar product* $(\mathbf{x},\mathbf{y})$ *is negative*
vectors $\mathbf{x},\mathbf{y}$ *are perpendicular if scalar product* $(\mathbf{x},\mathbf{y})$ *is equal to zero*
(1.13)

Remark The associated with each other affine space and Euclidean vector space $\mathbf{E}^n$ we will denote by the same letter.

Remark Geometrical intuition tells us that cosinus of the angle between two vectors has to be less or equal to one and it is equal to one if and only if vectors $\mathbf{x}, \mathbf{y}$ are collinear. Comparing with (1.12) we come to the inequality:

$$
\begin{gathered}
(\mathbf{x},\mathbf{y})^2 = \left(x^1y^1 + \cdots + x^ny^n\right)^2 \leq \left((x^1)^2 + \cdots + (x^n)^2\right)\left((y^1)^2 + (\cdots + (y^n)^2\right) = (\mathbf{x},\mathbf{x})(\mathbf{y},\mathbf{y}) \\
\text{and}(\mathbf{x},\mathbf{y})^2 = (\mathbf{x},\mathbf{x})(\mathbf{y},\mathbf{y}) \quad \text{if vectors are colinear, i.e. } x^i = \lambda y^i
\end{gathered} \tag{1.14}
$$

This is famous Cauchy–Buniakovsky–Schwarz inequality, one of most important inequalities in mathematics. (See for more details the last exercise in the Homework 0)

1.3 Transition matrices. Orthogonal bases and orthogonal matrices

1.3.1 Bases and transition matrices

One can consider different bases in vector space.

Let A be $n \times n$ matrix with real entries, $A = ||a_{ij}||$, $i, j = 1, 2, \ldots, n$:

$$
A = \begin{pmatrix}
a_{11} & a_{12} \ldots & a_{1n} \\
a_{21} & a_{22} \ldots & a_{2n} \\
a_{31} & a_{32} \ldots & a_{3n} \\
\ldots & \ldots\ldots & \ldots \\
a_{(n-1)\,1} & a_{(n-1)2} \ldots & a_{(n-1)n} \\
a_{n\,1} & a_{n2} \ldots & a_{nn}
\end{pmatrix}
$$

Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be an arbitrary basis in n-dimensional vector space V.

The basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ can be considered as row of vectors, or $1 \times n$ matrix with entries–vectors.

Multiplying $1 \times n$ matrix $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ on matrix A we come to new row of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\}$ such that

$$
\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}A = \tag{1.15}
$$

$$
\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}
\begin{pmatrix}
a_{11} & a_{12} \ldots & a_{1n} \\
a_{21} & a_{22} \ldots & a_{2n} \\
a_{31} & a_{32} \ldots & a_{3n} \\
\ldots & \ldots\ldots & \ldots \\
a_{(n-1)\,1} & a_{(n-1)2} \ldots & a_{(n-1)n} \\
a_{n\,1} & a_{n2} \ldots & a_{nn}
\end{pmatrix} \tag{1.16}
$$

,

$$\begin{cases} \mathbf{e}'_1 = a_{11}\mathbf{e}_1 + a_{21}\mathbf{e}_2 + a_{31}\mathbf{e}_3 + \cdots + a_{(n-1)\,1}\mathbf{e}_{n-1} + a_{n\,1}\mathbf{e}_n \\ \mathbf{e}'_1 = a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2 + a_{32}\mathbf{e}_3 + \cdots + a_{(n-1)\,2}\mathbf{e}_{n-1} + a_{n\,2}\mathbf{e}_n \\ \mathbf{e}'_1 = a_{13}\mathbf{e}_1 + a_{23}\mathbf{e}_2 + a_{33}\mathbf{e}_3 + \cdots + a_{(n-1)\,3}\mathbf{e}_{n-1} + a_{n\,1}\mathbf{e}_n \\ \cdots = \ldots \cdots + \ldots \cdots + \ldots \cdots + \cdots + \ldots\ldots\ldots\ldots \\ \mathbf{e}'_n = a_{1n}\mathbf{e}_1 + a_{2n}\mathbf{e}_2 + a_{3n}\mathbf{e}_3 + \cdots + a_{(n-1)\,n}\mathbf{e}_{n-1} + a_{n\,n}\mathbf{e}_n \end{cases}$$

or shortly:

$$\mathbf{e}'_i = \sum_{k=1}^{n} \mathbf{e}_k a_{ki}\,. \tag{1.17}$$

Definition Matrix A which transforms a basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ to the row of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\}$ (see equation (1.17)) is *transition matrix* from the basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ to the row $\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\}$.

What is the condition that the row $\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\}$ is a basis too? The row, ordered set of vectors, $\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\}$ is a basis if and only if vectors $(\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n)$ are linearly independent. Thus we come to

Proposition 1 *Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be a basis in n-dimensional vector space V, and let A be an $n \times n$ matrix with real entries. Then*

$$\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}A \tag{1.18}$$

is a basis if and only if the transition matrix A has rank n, i.e. it is non-degenerate (invertible) matrix.

Recall that $n\times$ matrix A is nondegenerate (invertible) $\Leftrightarrow \det A \neq 0$.

Remark Recall that the condition that $n \times n$ matrix A is non-degenerate (has rank n) is equivalent to the condition that it is invertible matrix, or to the condition that $\det A \neq 0$.