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Berezinians, Exterior Powers and Recurrent Sequences

To the memory of Felix Alexandrovich Berezin

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Abstract. We study power expansions of the characteristic function of a linear operator A in a $p|q$-dimensional superspace V. We show that traces of exterior powers of A satisfy universal recurrence relations of period q. 'Underlying' recurrence relations hold in the Grothendieck ring of representations of $\mathrm{GL}(V)$. They are expressed by vanishing of certain Hankel determinants of order $q+1$ in this ring, which generalizes the vanishing of sufficiently high exterior powers of an ordinary vector space. In particular, this allows to express the Berezinian of an operator as a ratio of two polynomial invariants. We analyze the Cayley–Hamilton identity in a superspace. Using the geometric meaning of the Berezinian we also give a simple formulation of the analog of Cramer's rule.

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1. Introduction

1.1. MAIN RESULTS

The Berezinian (superdeterminant) of a linear operator, discovered by Felix Alexandrovich Berezin, is a key notion of supermathematics. In this paper we show that the Berezinian possesses fundamental properties that were previously unknown. We give a new invariant formula for the Berezinian as a rational function of traces. The crucial fact is the existence of universal recurrence relations satisfied by the exterior powers in the supercase.

Our starting point is the study of the characteristic function $R_A(z)=\mathrm{Ber}(1+zA)$. The principal tool is the two power expansions of $R_A(z)$, at zero and at infinity. We also study a similar rational function taking values in the Grothendieck ring. The main results are as follows.

For an arbitrary even linear operator A in a $p|q$-dimensional superspace V consider its action in the exterior powers $\Lambda^k(V)$ and 'dual exterior powers' $\Sigma^k(V) = \operatorname{Ber} V \otimes \Lambda^{p-k} V^*$. We establish universal recurrence relations satisfied by the traces $\operatorname{Tr} \Lambda^k A$ and $\operatorname{Tr} \Sigma^k A$ [Theorem 1 and Equations (3.5) and (3.6)] and similar relations satisfied by the spaces $\Lambda^k(V)$ and $\Sigma^n(V)$ in a suitable Grothendieck ring (Theorems 4 and 5). In particular, we show that $\operatorname{Tr} \Sigma^k A$, which are rational functions of A, can be obtained from the polynomial invariants $\operatorname{Tr} \Lambda^k A$ by a sort of "analytic continuation". In this way we arrive at an invariant explicit formula for the Berezinian, $\operatorname{Ber} A$ as the ratio of two polynomial invariants

$$\operatorname{Ber} A = \frac{\operatorname{Ber}^+ A}{\operatorname{Ber}^- A} \tag{1.1}$$

where the new functions $\operatorname{Ber}^+ A$ and $\operatorname{Ber}^- A$ are Hankel determinants built of $\operatorname{Tr} \Lambda^k A$ [see Equations (5.6) and (5.7)]. One can relate them with the characters of polynomial representations corresponding to particular Young diagrams.

Besides this, we study two other related questions. We show how the minimal annihilating polynomial of a linear operator on a superspace can be obtained from its characteristic function $R_A(z)$ (an analog of the Cayley–Hamilton theorem). However, we think that in the supercase the rational characteristic function $R_A(z) = \operatorname{Ber}(1 + zA)$ is a more fundamental object than such a 'characteristic polynomial'. We also consider Cramer's rule in the supercase and give for it a geometric proof.

1.2. DISCUSSION AND BACKGROUND

Recall that the Berezinian is the analog of the determinant for the supercase, which naturally appears in integration theory involving odd variables (see [1, 3] and references therein). The main feature of $\operatorname{Ber} A$ (see (2.2)) is that it is not a polynomial in the matrix entries, but a fraction, whose numerator and denominator do not have independent invariant meaning. Exactly because $\operatorname{Ber} A$ is nonpolynomial, integration theory in the supercase is nontrivial. In particular, the straightforward generalization of the exterior algebra by standard tensor tools transferred to the $\mathbb{Z}_2$-graded situation is not sufficient, because it is not related with the Berezinian and hence with integration over supermanifolds (see, e.g., [19, 20]). The simplest objects that one has to consider besides the naive exterior powers $\Lambda^k(V)$ are the 'dual exterior powers' $\Sigma^k(V) := \operatorname{Ber} V \otimes \Lambda^{p-k} V^*$ introduced by Bernstein and Leites [5].

As we show in this paper, there are surprising "hidden relations" between the naive exterior powers $\Lambda^k(V)$ and the Berezinian. This is seen by the comparing of the two expansions of the characteristic function $R_A(z)$: the expansion at zero gives the traces in $\Lambda^k(V)$, while the expansion at infinity gives the traces in $\Sigma^k(V)$, including the Berezinian. (There is an analogy with rational numbers: the ordinary decimal expansion corresponds to an expansion near infinity, while a p-adic expansion corresponds to an expansion at zero.)

Recall that for an ordinary vector space V of dimension n all exterior powers starting from $\Lambda^{n+1}(V)$ vanish and the top exterior power $\Lambda^n(V)$ is the same as the one-dimensional space $\det V$. This gives rise to natural 'duality' isomorphisms $\det V \otimes \Lambda^{n-k}(V^*) \cong \Lambda^k(V)$. Compared with this, in the $\mathbb{Z}_2$-graded case for a vector space V of dimension $p|q$ there is an infinite sequence of the exterior powers $\Lambda^k(V)$, which does not terminate, and an infinite sequence of the 'dual' powers $\Sigma^k(V) = \operatorname{Ber} V \otimes \Lambda^{p-k} V^*$, stretching to the left, which are now essentially different from $\Lambda^k(V)$. In this paper we establish relations for the differences $\Gamma_k = \Lambda^k V - (-\Pi)^q \Sigma^{k+q} V$ in the Grothendieck ring:

$$\begin{vmatrix} \Gamma_k & \dots & \Gamma_{k+q} \\ \dots & \dots & \dots \\ \Gamma_{k+q} & \dots & \Gamma_{k+2q} \end{vmatrix} = 0$$

for all $k \in \mathbb{Z}$. (Π is the parity shift functor.) When $k \geqslant p - q + 1$ and $\Gamma_k = \Lambda^k V$ they replace the vanishing of the sufficiently high exterior powers of ordinary vector spaces. Taken in the range where both $\Lambda^k V$ and $\Sigma^k V$ are not zero, they give a proper replacement of the classical 'duality isomorphisms'.

In the last 15 years there has been an active work on noncommutative generalizations of determinants initiated by Gelfand and Retakh [9, 10] and related topics such as noncommutative Vieta formulae [6, 7] (see also a paper in this volume by the authors of [9]). Methods of [10] were applied by Bergvelt and Rabin [4] to obtain Cramer's formula for the supercase. We show in this paper how Cramer's formula can be directly deduced from the geometrical meaning of the Berezinian.

The topics of our paper are related to subtle questions concerning rational and polynomial invariants of operators in superspaces considered in the pioneer works by Berezin (see references in [3]) and Kac [11]. (Some Berezin's texts of 1975–1977 were incorporated into the English version of the posthumous book [3].) There is also a connection with in an interesting sequel of papers by Kantor and Trishin [12–14] aimed at clarifying the relations in the algebra of polynomial invariants. Part of our identities for traces $\operatorname{Tr} \Lambda^k A$ were obtained in [13, 14] by an analysis of Young diagrams.

The main new results of our paper are the recurrence relations linking $\Lambda^k V$ and $\Sigma^k V$, both for traces and in the Grothendieck ring, and the explicit invariant formula for the Berezinian as a ratio of two polynomial invariants (1.1).

1.3. NOTATION

We use standard language of superalgebra and supergeometry. Whenever it could not cause confusion, we drop the prefix 'super', writing 'spaces', 'traces', etc., instead of 'superspaces', 'supertraces', respectively. The notation Tr is used for supertrace. Parity is denoted by a tilde.

2. Expansions of the Characteristic Function

Let A be an even linear operator acting in a finite-dimensional superspace V of dimension $p|q$. Introduce the *characteristic function* of the operator A,

$$R_A(z) := \mathrm{Ber}(1 + zA)\,, \tag{2.1}$$

depending on a complex variable z. Here Ber denotes the Berezinian (superdeterminant). Recall that

$$\mathrm{Ber}\,A = \frac{\det\left(A_{00} - A_{01}A_{11}^{-1}A_{10}\right)}{\det A_{11}} \tag{2.2}$$

for an even matrix $A = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}$. In the diagonal blocks A_{00}, A_{11} the matrix entries are even and in the antidiagonal blocks A_{01}, A_{10}, odd. The Berezinian is a multiplicative function of matrices, hence it is well-defined on linear operators. In the sequel, if it cannot cause a confusion, we do not distinguish sharply operators and matrices. Matrix elements can be viewed either as belonging to a given $\mathbb{Z}_2$-graded (super)commutative ring or as free generators. This corresponds to considering an 'individual' matrix or a 'general' matrix, in the classical language. Strictly speaking one should speak about free modules over the ground ring instead of vector spaces, but we shall not push this distinction.

Consider the expansion of the rational function $R_A(z)$ at zero:

$$R_A(z) = \sum_{k=0}^{\infty} c_k(A) z^k = 1 + c_1 z + c_2 z^2 + \cdots\,. \tag{2.3}$$

In the ordinary case when $q = 0$, the function $R_A(z)$ is a polynomial and the expansion (2.3) terminates. It is well known that for a linear operator acting in a p-dimensional vector space V

$$\det(1 + zA) = 1 + c_1 z + \cdots + c_p z^p\,,$$

where $c_k(A) = \mathrm{Tr}\,\Lambda^k A$ are the traces of the action of the operator A in the exterior powers $\Lambda^k V$. In particular, $c_1(A) = \mathrm{Tr}\,A$, $c_p(A) = \det A$. For $k > p$, $c_k(A) = 0$ as $\Lambda^k V = 0$. If the odd dimension of V is not equal to zero, then $\mathrm{Ber}(1 + zA)$ is no longer a polynomial in z, but an analog of the formula above still holds:

PROPOSITION 1. *There is an infinite power expansion*

$$\mathrm{Ber}(1 + zA) = \sum_{k=0}^{\infty} c_k(A) z^k \quad \textit{where } c_k(A) = \mathrm{Tr}\,\Lambda^k A. \tag{2.4}$$

To our knowledge, this expansion was first obtained by Schmitt in [16]. It can be proved by considering diagonal matrices. Here $\Lambda^k A$ stands for the action of A in the kth exterior power of the superspace V. Recall that the exterior algebra $\Lambda(V) = \oplus \Lambda^k V$ is defined as $T(V)/\langle v \otimes u + (-1)^{\tilde{v}\tilde{u}} u \otimes v\rangle$, v, u being elements of V. Parity (the $\mathbb{Z}_2$-grading) in $\Lambda(V)$ is naturally inherited from V. There is no "top" power among $\Lambda^k V$, and the Taylor expansion (2.4) is infinite. We denote the supertrace of a supermatrix by the same symbol as the trace of an ordinary matrix. Recall that for an even supermatrix

$$\operatorname{Tr} A = \operatorname{Tr} \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} = \operatorname{Tr} A_{00} - \operatorname{Tr} A_{11} .$$

The expansion of the characteristic function at infinity leads to traces of the wedge products of the inverse matrix:

$$\operatorname{Ber}(1+zA) = \sum_{k=q-p}^{\infty} c^*_{-k}(A) z^{-k} \quad \text{where } c^*_{-k}(A) = \operatorname{Ber} A \cdot \operatorname{Tr} \Lambda^{p-q+k} A^{-1}.$$

This follows from (2.4) and the equalities $\operatorname{Ber}(1+zA) = \operatorname{Ber} A \operatorname{Ber}(A^{-1}+z) = z^{p-q} \operatorname{Ber} A \operatorname{Ber}(1+z^{-1}A^{-1})$. The geometric meaning of the coefficients is as follows. $\operatorname{Ber} A \cdot \operatorname{Tr} \Lambda^{p-k} A^{-1} = \operatorname{Tr} \Sigma^k A$ is the trace of the representation of A in the space $\Sigma^k V := \operatorname{Ber} V \otimes \Lambda^{p-k} V^*$. In the ordinary case, it would be just a "dual" description of the same $\Lambda^k V$; in the super case these two spaces are essentially different. Hence we arrive at the following proposition.

PROPOSITION 2. *There is an expansion at infinity*

$$\operatorname{Ber}(1+zA) = \sum_{k=q-p}^{\infty} c^*_{-k}(A) z^{-k} \quad \textit{where } c^*_{-k}(A) = \operatorname{Tr} \Sigma^{q-k} A, \tag{2.5}$$

which is a Taylor expansion when $p \leqslant q$ and a Laurent expansion when $p > q$. Here $\operatorname{Tr} \Sigma^{q-k} A = \operatorname{Ber} A \cdot \operatorname{Tr} \Lambda^{p-q+k} A^{-1}$.

Consider the coefficients $c_k(A) = \operatorname{Tr} \Lambda^k A$. They can be expressed as polynomials via $s_k(A) = \operatorname{Tr} A^k$ using the Liouville formula:

$$\operatorname{Ber}(1+zA) = e^{\operatorname{Tr} \ln(1+zA)} = \exp\left(z \operatorname{Tr} A - \frac{z^2}{2} \operatorname{Tr} A^2 + \frac{z^3}{3} \operatorname{Tr} A^3 + \cdots \right),$$

and $c_k(A) = P_k(s_0(A), \ldots, s_k(A))$, where P_k are the classical Newton's polynomials. For example, $c_0 = s_0 = 1$,

$$c_1 = s_1, \quad c_2 = \tfrac{1}{2}(s_1^2 - s_2), \quad c_3 = \tfrac{1}{6}(s_1^3 - 3 s_1 s_2 + 2 s_3),$$

etc., where $c_k = c_k(A)$, $s_k = s_k(A)$. There is a relation

$$c_{k+1} = \frac{1}{k+1}(s_1 c_k - s_2 c_{k-1} + \cdots + (-1)^k s_{k+1}). \tag{2.6}$$

These universal formulae linking $c_k(A)$ with $s_k(A)$ are true regardless whether V is a superspace or ordinary space.

For further considerations it is convenient to define the following polynomials:

$$\mathcal{H}_k(z) = z^k - c_1 z^{k-1} + c_2 z^{k-2} - \cdots + (-1)^k c_k, \tag{2.7}$$

where $k=0,1,2,\ldots$. We shall refer to them as to the *Cayley–Hamilton polynomials.* (In the classical case of an n-dimensional space, $\pm\mathcal{H}_n(z)$ is the classical characteristic polynomial $\det(A-z)$ if $c_k=c_k(A)$.) The following identities are satisfied:

$$\frac{\mathrm{d}c_{k+1}(A)}{\mathrm{d}A}=(-1)^k\mathcal{H}_k^A(A) \tag{2.8}$$

$$\frac{1}{k+1}\operatorname{Tr}\big(A\mathcal{H}_k^A(A)\big)=(-1)^k c_{k+1}(A), \tag{2.9}$$

Here $\mathcal{H}_k^A(A)$ is the value of the polynomial (2.7) where $c_k=c_k(A)$ at $z=A$. The derivative $\mathrm{d}f(A)/\mathrm{d}A$ of a scalar function of a matrix argument is defined as the matrix that satisfies

$$\left\langle\frac{\mathrm{d}f(A)}{\mathrm{d}A},B\right\rangle=\frac{\mathrm{d}}{\mathrm{d}t}f(A+tB)\big|_{t=0}$$

for an arbitrary matrix B, where the scalar product of matrices is given by $\langle A,B\rangle=\operatorname{Tr}(AB)$. Equations (2.6) and (2.8) can be deduced by differentiating the characteristic function $R_A(z)=\operatorname{Ber}(1+Az)$. Bearing in mind that $d\operatorname{Ber}A=\operatorname{Ber}A\operatorname{Tr}(A^{-1}\mathrm{d}A)$, we can come to the following identities:

$$\frac{\mathrm{d}}{\mathrm{d}z}\log R_A(z)=\operatorname{Tr}\left(A\,(1+Az)^{-1}\right)=\sum_{k=0}^{\infty}(-1)^k s_{k+1}(A)z^k,$$

$$\frac{\mathrm{d}}{\mathrm{d}A}\log R_A(z)=(1+Az)^{-1}\,z=\sum_{k=0}^{\infty}(-1)^k z^{k+1}A^k\,.$$

By writing $\mathrm{d}\log R_A(z)$ as $(R_A(z))^{-1}\mathrm{d}R_A(z)$ and comparing the power series we arrive at (2.6) and (2.8).

Unlike the polynomial functions $c_k(A)=\operatorname{Tr}\Lambda^k A$, the coefficients $c_k^*(A)=\operatorname{Tr}\Sigma^{q+k}A=\operatorname{Ber}A\cdot\operatorname{Tr}\Lambda^{p-q-k}A^{-1}$ are rational functions of the matrix entries of A. In particular,

$$c_{p-q}^*(A)=\operatorname{Tr}\Sigma^p A=\operatorname{Ber}A.$$

Our task will be to give an expression for $c_k^*(A)$ in terms of polynomial invariants of A.

3. Recurrence Relations for Traces of Exterior Powers

Recall that $\Sigma^k A$ denotes the representation of A in the space $\Sigma^k V=\operatorname{Ber}V\cdot\Lambda^{p-k}V^*$, thus $\operatorname{Tr}\Sigma^k A=\operatorname{Ber}A\cdot\operatorname{Tr}\Lambda^{p-k}A^{-1}$. By definition, $\operatorname{Tr}\Sigma^k A=0$ when $k>p$ and $\operatorname{Tr}\Lambda^k A=0$ when $k<0$. In the purely even case $q=0$ the spaces $\Lambda^k V$ and $\Sigma^k V$ are canonically isomorphic, $\operatorname{Tr}\Lambda^k A=\operatorname{Tr}\Sigma^k A$, $c_p(A)=\det A$, and $c_k(A)=0$ for $k>p$. We shall find out now what replaces these facts for a general $p|q$-dimensional superspace.

Let us analyze the expansions of the characteristic function $R_A(z)$. One can see that $R_A(z)$ is a fraction of the appearance

$$R_A(z) = \frac{P(z)}{Q(z)} = \frac{1 + a_1 z + a_2 z^2 + \cdots + a_p z^p}{1 + b_1 z + b_2 z^2 + \cdots + b_q z^q}$$

where the numerator is a polynomial of degree p and the denominator is a polynomial of degree q. (Consider the diagonal matrices.) In principle the degrees can be less than p and q, and the fraction may be reducible. However, for an operator "in a general position", this fraction is irreducible and the top coefficients a_p, b_q can be assumed to be invertible. We shall use the notation $R_A^+(z)$ and $R_A^-(z)$ for the numerator and denominator of the fraction $R_A(z)$, respectively.

From the well-known connection between rational functions and recurrent sequences (see Appendix), one can deduce the following facts:

(1) The coefficients $c_k(A) = \operatorname{Tr} \Lambda^k A$ of the expansion of $R_A(z)$ at zero (2.4) satisfy the recurrence relation of period q

$$b_0 c_{k+q} + \cdots + b_q c_k = 0 \tag{3.1}$$

for all $k > p - q$, where $b_0 = 1$. In particular, if $p < q$, then (3.1) holds for all c_k including the zero values when $p - q < k < 0$.

(2) The coefficients $c_k^*(A) = \operatorname{Tr} \Sigma^{q+k} A$ of the expansion of $R_A(z)$ at infinity (2.5) satisfy the same recurrence relation:

$$b_0 c_k^* + \cdots + b_q c_{k-q}^* = 0 \tag{3.2}$$

for all $k < 0$. In particular, if $p < q$, then the relation (3.2) holds for all c_k^* including the zero values when $p - q < k < 0$.

(3) The sequences c_k and c_k^* can be combined together into a single recurrent sequence by considering the differences

$$\gamma_k = c_k - c_k^*,$$

which satisfy the recurrence relation

$$b_0 \gamma_{k+q} + \cdots + b_q \gamma_k = 0 \tag{3.3}$$

for *all* values of $k \in \mathbb{Z}$ (notice that $c_k = 0$, $\gamma_k = -c_k^*$ for $k < 0$ and $c_k^* = 0$, $\gamma_k = c_k$ for $k > p - q$).

In particular, we have the following fundamental theorem.

THEOREM 1. *For an operator A acting in $p|q$-dimensional vector space the differences*

$$\gamma_k = c_k - c_k^* = \operatorname{Tr} \Lambda^k A - \operatorname{Tr} \Sigma^{q+k} A \tag{3.4}$$

form a recurrent sequence with period q, for all $k \in \mathbb{Z}$.

In the classical case of $q = 0$, all terms of the sequence (3.4) are zero and $\mathrm{Tr}\,\Lambda^k A = \mathrm{Tr}\,\Sigma^k A$ or $\mathrm{Tr}\,\Lambda^k A = \det A \cdot \mathrm{Tr}\,\Lambda^{p-k} A^{-1}$ for any operator A. In this case the spaces $\Lambda^k V$ and $\Sigma^k V$ are canonically isomorphic. Theorem 1 actually suggests a relation between spaces $\Lambda^k V$ and $\Sigma^{k+q} V$ for arbitrary q (see details in Section 7).

In (3.4) the terms $c_k = \mathrm{Tr}\,\Lambda^k A$ and $c_k^* = \mathrm{Tr}\,\Sigma^{q+k} A$ are together nonzero only in a finite range, for $k = 0, \ldots, p-q$ if $p > q$. Otherwise γ_k equals either $c_k(A)$ for $k \geqslant p-q+1$ or $-c_k^*(A)$ for $k \leqslant -1$. The relation (3.4) gives a tool to express terms of the two sequences $c_k^* = \mathrm{Tr}\,\Sigma^{q+k} A$ and $c_k = \mathrm{Tr}\,\Lambda^k A$ via each other. What actually happens, for large k $\gamma_k = c_k$ holds, and the sequence can be continued to the left using (3.3) to obtain $c_k^*(A)$, in particular $c_{p-q}^*(A) = \mathrm{Ber}\,A$, as

$$\mathrm{Ber}\,A = \mathrm{Tr}\,\Lambda^{p-q} A - \gamma_{p-q}\,.$$

The "continuation to the left" of $c_k(A)$ using recurrence relations corresponds to the analytic continuation to the neighborhood of infinity of the power series (2.4) representing the rational function $R_A(z)$ near zero. We give examples of calculations in the next section.

For linear recurrence relations with constant coefficients such as (3.1) or (3.3) it is possible to eliminate the coefficients to obtain the relation in a closed form, using Hankel matrices. Recall that a *Hankel matrix* has the entries $c_{ij} = c_{i+j}$. A recurrence relation for c_k of period q implies the vanishing of Hankel determinants of order $q+1$.

The statement of Theorem 1 can be reformulated in the following way: the identity

$$\begin{vmatrix} \gamma_k(A) & \ldots & \gamma_{k+q}(A) \\ \ldots & \ldots & \ldots \\ \gamma_{k+q}(A) & \ldots & \gamma_{k+2q}(A) \end{vmatrix} = 0 \tag{3.5}$$

holds for all $k \in \mathbb{Z}$.

COROLLARY. The identity

$$\begin{vmatrix} c_k(A) & \ldots & c_{k+q}(A) \\ \ldots & \ldots & \ldots \\ c_{k+q}(A) & \ldots & c_{k+2q}(A) \end{vmatrix} = 0 \tag{3.6}$$

holds for all $k > p-q$.

Remark. A system of equations for b_i similar to our (3.1) appeared previously in [2, 11] without a link with recurrence relations. Relations for $c_k = \mathrm{Tr}\,\Lambda^k A$, in particular the identity (3.6), were obtained in [13, 14] from an analysis of Young diagrams. The main difference of our work from [2, 11] and [13, 14] is in the simultaneous consideration of $c_k(A) = \mathrm{Tr}\,\Lambda^k A$ and $c_k^*(A) = \mathrm{Tr}\,\Sigma^k A$.

4. Berezinian as a Rational Function of Traces

As we established above, the coefficients $c_k(A) = \operatorname{Tr}\Lambda^k A$ for a linear operator A in a $p|q$-dimensional vector space V satisfy relations (3.1) making them a $p|q$-recurrent sequence (see Appendix for necessary notions). Basing just on this fact we will give a recurrent procedure for calculating the characteristic function $R_A(z) = \operatorname{Ber}(1+zA)$ and the Berezinian of the operator A. Then we will present a closed formula for $\operatorname{Ber} A$ using the relations (3.5) of Theorem 1.

Let $\boldsymbol{c} = \{c_n\}_{n\geqslant 0}$ be a $p|q$-recurrent sequence such that $c_0 = 1$. Denote by $R_{p|q}(z, \boldsymbol{c})$ its generating function:

$$R_{p|q}(z, \boldsymbol{c}) = \frac{1 + a_1 z + \cdots + a_p z^p}{1 + b_1 z + \cdots + b_q z^q} = 1 + c_1 z + c_2 z^2 + \cdots .$$

The fraction $R_{p|q}(z, \boldsymbol{c})$ is defined by the first $p+q$ terms $c_1, c_2, \ldots, c_{p+q}$ of the sequence $\boldsymbol{c}$:

$$R_{p|q}(z, \boldsymbol{c}) = R_{p|q}(z, c_1, \ldots, c_{p+q}) .$$

In particular, if A is a $p|q \times p|q$ matrix and $\{c_k\}$ is the sequence of the traces of exterior powers of the matrix A ($c_k = c_k(A) = \operatorname{Tr}\Lambda^k A$), then $R_{p|q}(z, \boldsymbol{c})$ coincides with the characteristic function of A:

$$R_A(z) = R_{p|q}(z, c_1(A), c_2(A), \ldots, c_{p+q}(A)) . \tag{4.1}$$

The rational functions $R_{p|q}(z, \boldsymbol{c}) = R_{p|q}(z, c_1, \ldots, c_{p+q})$ have the following properties:

(1) If $p \geqslant q$, then the sequence $\boldsymbol{c}'$ defined by $c'_k := \frac{c_{k+1}}{c_1}$ (assuming that the coefficient c_1 is invertible) is a $p-1|q$-recurrent sequence and

$$R_{p|q}(z, \boldsymbol{c}) = 1 + c_1 z\, R_{p-1|q}(z, \boldsymbol{c}'), \tag{4.2}$$

i.e.,

$$R_{p|q}(z, c_1, \ldots, c_{p+q}) = 1 + c_1 z\, R_{p-1|q}\left(z, \frac{c_2}{c_1}, \ldots, \frac{c_{p+q}}{c_1}\right) . \tag{4.3}$$

(2) The sequence $\boldsymbol{c}^{\Pi} = \{c_n^{\Pi}\}$ defined according to

$$1 + c_1^{\Pi} z + c_2^{\Pi} z^2 + \cdots = \frac{1}{1 + c_1 z + c_2 z^2 + \cdots} ,$$

for example

$$c_1^{\Pi} = -c_1,\ \ c_2^{\Pi} = -c_2 + c_1^2,\ \ c_3^{\Pi} = -c_3 + 2c_1c_2 - c_1^3,\ \ \ldots , \tag{4.4}$$

is a $q|p$-recurrent sequence, and

$$R_{p|q}\left(z, c_1, \ldots, c_{p+q}\right) = \frac{1}{R_{q|p}(z, c_1^{\Pi}, \ldots, c_{p+q}^{\Pi})} . \tag{4.5}$$

(If A is a $p|q \times p|q$ supermatrix and A^{Π} is the parity reversed $q|p \times q|p$ supermatrix, then $c_k(A^{\Pi}) = c_k(A)^{\Pi}$.)

Using these properties one can express the rational function $R_{p|q}$ corresponding to a $p|q$-recurrent sequence via the rational function $R_{0|1}$ corresponding to a $0|1$-recurrent sequence, i.e., a geometric progression. The steps are as follows. If $p<q$, we apply (4.5) to get a $p'|q'$-sequence with $p'>q'$. If $p>q$, we repeatedly apply (4.2) to decrease p.

EXAMPLE 4.1. Let A be a $p|1 \times p|1$ matrix. Then it follows from (4.2) and (4.5) that

$$\begin{aligned}
R_A(z) &= R_{p|1}\left(z, c_1(A), c_2(A), \dots, c_{p+1}(A)\right)\\
&= 1 + c_1 z\, R_{p-1|1}\left(z, \frac{c_2}{c_1}, \dots, \frac{c_{p+1}}{c_1}\right)\\
&= \dots = 1 + c_1 z + \dots + c_{p-1} z^{p-1} + c_p z^p R_{0|1}\left(z, \frac{c_{p+1}}{c_p}\right)\\
&= 1 + c_1 z + \dots + c_{p-1} z^{p-1} + \frac{c_p z^p}{1 - \frac{c_{p+1}}{c_p} z}\\
&= 1 + c_1 z + \dots + c_{p-1} z^{p-1} + \frac{c_p^2 z^p}{c_p - c_{p+1} z}
\end{aligned}$$

We can deduce from here formulae for the Berezinian. One can see from (2.5) that for a $p|q \times p|q$ matrix A

$$\operatorname{Ber} A = \lim_{z\to\infty} z^{q-p} R_A(z) \tag{4.6}$$

Let $\boldsymbol{c} = \{c_n\}$, $n \geqslant 0$, be an arbitrary $p|q$-recurrent sequence such that $c_0 = 1$ and let $R(z, \boldsymbol{c})$ be its generating function. Then mimicking (4.6) we define the *Berezinian* of this sequence $B_{p|q}(\boldsymbol{c})$ by the formula

$$B_{p|q}(\boldsymbol{c}) = \lim_{z\to\infty} z^{q-p} R_{p|q}(z, \boldsymbol{c}). \tag{4.7}$$

If $c_n = c_n(A) = \operatorname{Tr} \Lambda^k A$, then $B_{p|q}(\boldsymbol{c}) = \operatorname{Ber} A$. From (4.2) and (4.5) immediately follow relations for $B_{p|q}$:

$$B_{p|q}(\boldsymbol{c}) = B_{p|q}(c_1, \dots, c_{p+q}) = \begin{cases} c_1 B_{p-1|q}\left(\boldsymbol{c}'\right) & \text{if } p \geqslant q+1 \\ 1 + c_1 B_{p-1|q}\left(\boldsymbol{c}'\right) & \text{if } p = q \\ \frac{1}{B_{q|p}(\boldsymbol{c}^{\Pi})} & \text{if } p \leqslant q-1 \end{cases} \tag{4.8}$$

where the sequences $\boldsymbol{c}'$ and $\boldsymbol{c}^{\Pi}$ are defined as above.

Using these relations one can calculate the Berezinians of matrices in terms of traces. Note that from these recurrent relations follows that if $p>q$ then for a $p|q$-recurrent sequence $\boldsymbol{c}$, its Berezinian $B_{p|q}$ depends only on the coefficients $c_{p-q}, \dots, c_p, \dots, c_{p+q}$.

EXAMPLE 4.2. For a $1|1\times 1|1$ matrix:

$$\begin{aligned}\operatorname{Ber} A &= B_{1|1}(c_1(A), c_2(A)) = 1 + c_1 B_{0|1}\left(\frac{c_2}{c_1}\right) = 1 + \frac{c_1}{B_{1|0}\left(\left(\frac{c_2}{c_1}\right)^{\Pi}\right)}\\ &= 1 + \frac{c_1}{-\frac{c_2}{c_1}} = 1 - \frac{c_1^2}{c_2} = \frac{c_2 - c_1^2}{c_2} = \frac{\operatorname{Tr} A^2 + (\operatorname{Tr} A)^2}{\operatorname{Tr} A^2 - (\operatorname{Tr} A)^2}\end{aligned}$$

(we have applied Newton's formulae to get the last expression).

EXAMPLE 4.3. For a $p|1\times p|1$ matrix:

$$\begin{aligned}\operatorname{Ber} A &= B_{p|1}\left(c_{p-1}(A), \dots, c_p(A), \dots, c_{p+1}(A)\right) = c_{p-1} B_{1|1}\left(\frac{c_p}{c_{p-1}}, \frac{c_{p+1}}{c_{p-1}}\right)\\ &= \frac{c_{p-1}c_{p+1} - c_p^2}{c_{p+1}}.\end{aligned}$$

EXAMPLE 4.4. For a $2|2\times 2|2$ matrix:

$$\begin{aligned}\operatorname{Ber} A &= B_{2|2}\left(c_1(A), c_2(A), c_3(A), c_4(A)\right) = 1 + c_1 B_{1|2}\left(\frac{c_2}{c_1}, \frac{c_3}{c_1}, \frac{c_4}{c_1}\right)\\ &= 1 + \frac{c_1}{B_{2|1}\left(\left(\frac{c_2}{c_1}\right)^{\Pi}, \left(\frac{c_3}{c_1}\right)^{\Pi}, \left(\frac{c_4}{c_1}\right)^{\Pi}\right)}\\ &= 1 + \frac{c_1}{B_{2|1}\left(-\frac{c_2}{c_1}, -\frac{c_3}{c_1} + \left(\frac{c_2}{c_1}\right)^2, -\frac{c_4}{c_1} + 2\frac{c_2}{c_1}\frac{c_3}{c_1} - \left(\frac{c_2}{c_1}\right)^3\right)}\\ &= 1 - \frac{c_1^2}{c_2 B_{1|1}\left(\frac{c_3}{c_2} - \frac{c_2}{c_1}, \frac{c_4}{c_2} - \frac{2c_3}{c_1} + \left(\frac{c_2}{c_1}\right)^2\right)}\\ &= 1 - \frac{c_1^2}{c_2\left(1 - \frac{\left(\frac{c_3}{c_2} - \frac{c_2}{c_1}\right)^2}{\frac{c_4}{c_2} - \frac{2c_3}{c_1} + \left(\frac{c_2}{c_1}\right)^2}\right)}\end{aligned}$$

The last expression can be further simplified, and in principle one can proceed in this way to get the answer for arbitrary q, but at this point it is easier to give a general formula. It will reveal an unexpected link with classical algebraic notions.

5. Berezinian and Resultant

Let A be an even linear operator in a $p|q$-dimensional superspace. Consider the relation (3.5) of Theorem 1 for $k = p - q$. Recall that $\gamma_{p-q} = c_{p-q} - c^*_{p-q}$, $\gamma_k = c_k$ for $k \geqslant p - q + 1$ and $c^*_{p-q} = \operatorname{Ber} A$. Hence we have the following equalities:

$$0 = \begin{vmatrix} \gamma_{p-q} & \dots & \gamma_p \\ \dots & \dots & \dots \\ \gamma_p & \dots & \gamma_{p+q} \end{vmatrix} = \begin{vmatrix} c_{p-q} - \operatorname{Ber} A & \dots & c_p \\ \dots & \dots & \dots \\ c_p & \dots & c_{p+q} \end{vmatrix}$$
$$= \begin{vmatrix} c_{p-q} & \dots & c_p \\ \dots & \dots & \dots \\ c_p & \dots & c_{p+q} \end{vmatrix} - \operatorname{Ber} A \begin{vmatrix} c_{p-q+2} & \dots & c_{p+1} \\ \dots & \dots & \dots \\ c_{p+1} & \dots & c_{p+q} \end{vmatrix}$$

We arrive at the formula

$$\operatorname{Ber} A = \frac{\begin{vmatrix} c_{p-q} & \dots & c_p \\ \dots & \dots & \dots \\ c_p & \dots & c_{p+q} \end{vmatrix}}{\begin{vmatrix} c_{p-q+2} & \dots & c_{p+1} \\ \dots & \dots & \dots \\ c_{p+1} & \dots & c_{p+q} \end{vmatrix}} = \frac{|c_{p-q} \dots c_p|_{q+1}}{|c_{p-q+2} \dots c_{p+1}|_q}, \tag{5.1}$$

where we used a short notation for Hankel determinants with subscripts denoting their orders. Here as always $c_k = 0$ for $k \leqslant -1$ and $c_0 = 1$.

Let us make an important observation. The Hankel determinants appearing in the numerator and denominator of Equation (5.1) are nothing but the traces of the representations in the subspaces of tensors corresponding to certain Young diagrams. Indeed, denote by $D = D_{[\lambda_1,\dots,\lambda_s]}$ the Young diagram with s columns, such that the ith column contains λ_i cells, $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_s$. Let V_D be an invariant subspace in the tensor power $V^{\otimes N}$, $N = \lambda_1 + \dots + \lambda_s$, corresponding to the Young diagram $D = D_{[\lambda_1,\dots,\lambda_s]}$, and A_D be the representation of A in V_D. Then the *Schur–Weyl formula* (see [21]) tells that the trace of A_D is expressed via the traces $c_k(A) = \operatorname{Tr} \Lambda^k A$ as the determinant of the $s \times s$ matrix with the entries $a_{ij} = c_{\lambda_i + j - i}(A) = \operatorname{Tr} \Lambda^{\lambda_i + j - i} A$. This formula remains valid in the supercase. Let $D(r, s)$ be the rectangular Young diagram with r rows and s columns. So $D(r, s) = D_{[\lambda_1,\dots,\lambda_s]}$ with $\lambda_i = r$ for all i. For $D = D(r, s)$ the 'Schur determinant' $\operatorname{Tr} A_D$ is equal to the Hankel determinant $|c_{r-s+1} \dots c_r|_s$ with the inverted order of rows. In other words, Hankel determinants appearing in this paper can be interpreted as characters of tensor representations corresponding to rectangular Young diagrams.

The formulae obtained above deserve to be called a theorem.

THEOREM 2. *The Berezinian of a linear operator A in a $p|q$-dimensional space is equal to the ratio of the traces of the representations in the invariant subspaces of tensors corresponding to the rectangular Young diagrams $D(p, q+1)$ and $D(p+1, q)$*

$$\operatorname{Ber} A = \frac{|\operatorname{Tr} \Lambda^{p-q} A \dots \operatorname{Tr} \Lambda^p A|_{q+1}}{|\operatorname{Tr} \Lambda^{p-q+2} A \dots \operatorname{Tr} \Lambda^{p+1} A|_q} = (-1)^q \frac{\operatorname{Tr} A_{D(p,q+1)}}{\operatorname{Tr} A_{D(p+1,q)}}, \tag{5.2}$$

the sign coming from a change of order of the rows in the determinant.

EXAMPLE 5.1. For a $2|3 \times 2|3$ matrix we have

$$\operatorname{Ber} A = \frac{\begin{vmatrix} 0 & 1 & c_1 & c_2 \\ 1 & c_1 & c_2 & c_3 \\ c_1 & c_2 & c_3 & c_4 \\ c_2 & c_3 & c_4 & c_5 \end{vmatrix}}{\begin{vmatrix} c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \\ c_3 & c_4 & c_5 \end{vmatrix}} = -\frac{\operatorname{Tr} A_{D(2,4)}}{\operatorname{Tr} A_{D(3,3)}}.$$

Remark. In the classical case $q = 0$ when $c_k(A)$ are the elementary symmetric functions of the eigenvalues of A, Schur's determinants corresponding to Young diagrams, when written as functions of eigenvalues, are special symmetric functions known as *Schur functions* (see [15]); in the supercase the same Schur determinants when expressed via the eigenvalues are no longer classical symmetric Schur functions but are certain combinations of functions separately symmetric in the 'bosonic' and 'fermionic' eigenvalues.

What is the meaning – as polynomial invariants of A – of the determinants appearing as the numerator and denominator in Equation (5.2)?

DEFINITION. Define the following functions of A:

$$\operatorname{Ber}^+ A := \lambda_1 \cdots \lambda_p \prod_{i,\alpha} (\lambda_i - \mu_\alpha), \tag{5.3}$$

$$\operatorname{Ber}^- A := \mu_1 \cdots \mu_q \prod_{i,\alpha} (\lambda_i - \mu_\alpha). \tag{5.4}$$

We assume for a moment that A can be diagonalized and λ_i, μ_α, $i = 1, \dots, p$, $\alpha = 1, \dots, q$ stand for its eigenvalues. So

$$\operatorname{Ber} A = \frac{\lambda_1 \cdots \lambda_p}{\mu_1 \cdots \mu_q} = \frac{\operatorname{Ber}^+ A}{\operatorname{Ber}^- A}.$$

We shall see that $\operatorname{Ber}^\pm A$ make sense for all A. Denote the product $\prod_{i,\alpha}(\lambda_i - \mu_\alpha)$ by R or $R(A)$. If $R_A^+(z)$ and $R_A^-(z)$ stand for the numerator and denominator of the characteristic function $R_A(z)$, respectively, then it is easy to check that R is the classical Silvester's resultant for the polynomials $R_A^+(z)$ and $R_A^-(z)$, $R = \operatorname{Res}(R_A^-(z), R_A^+(z))$.

PROPOSITION 3. *The resultant of $R_A^+(z)$ and $R_A^-(z)$ is a polynomial in the matrix entries of A and can be expressed by the following formula:*

$$\begin{aligned} R = \operatorname{Res}(R_A^-(z), R_A^+(z)) &= \prod_{i,\alpha} (\lambda_i - \mu_\alpha) \\ &= (-1)^{q(q-1)/2} |c_{p-q+1} \dots c_p|_q = \operatorname{Tr} A_{D(p,q)}. \end{aligned} \tag{5.5}$$

Proof. The Hankel determinant in the r.h.s. of (5.5) vanishes when $\lambda_i = \mu_\alpha$ for any pair i, α. This follows from our recurrence relation (3.6) applied a $(p-1|q-1)$-dimensional space. Hence $|c_{p-q+1} \dots c_p|_q$ is divisible by the resultant. As polynomials in λ_i, μ_α they have the same degree pq, hence they must coincide up to a numerical factor, which can be checked, for example, by setting all $\mu_\alpha = 0$. □

The statement of Proposition 3 can be extracted from Berezin's paper [2].

THEOREM 3. *The functions* $\mathrm{Ber}^+ A$ *and* $\mathrm{Ber}^- A$ *are polynomial invariants of* A. *The following equalities hold:*

$$\mathrm{Ber}^+ A = \lambda_1 \cdots \lambda_p \prod_{i,\alpha} (\lambda_i - \mu_\alpha) = |c_{p-q} \dots c_p|_{q+1} \tag{5.6}$$

$$\mathrm{Ber}^- A = \mu_1 \cdots \mu_q \prod_{i,\alpha} (\lambda_i - \mu_\alpha) = |c_{p-q+2} \dots c_{p+1}|_q \,, \tag{5.7}$$

i.e., $\mathrm{Ber}^+ A$ *and* $\mathrm{Ber}^- A$ *give exactly the numerator and denominator of the expression for* $\mathrm{Ber}\, A$ *in formula* (5.2).

Proof. Indeed, $\lambda_1 \cdots \lambda_p$ and $\mu_1 \cdots \mu_q$ are equal, respectively, to the coefficients a_p and b_q in $R_A^+(z)$ and $R_A^-(z)$. In general, all the coefficients a_i, b_k can be obtained from c_k, $k = 1, \dots, p+q$, by solving simultaneous equations, with the determinant of the system being exactly R. Therefore, all coefficients a_i, b_k have the appearance of a polynomial in c_k divided by the same denominator $R = \pm |c_{p-q+1} \dots c_p|_q = \mathrm{Tr}\, A_{D(p,q)}$. Equations (5.6) and (5.7) follow by a direct application of Cramer's rule. (In particular, this yields another proof of the expression for the Berezinian (5.2).) □

From the proof follows that the polynomials $R^+(z)$ and $R^-(z)$ are well-defined if the resultant $R = |c_{p-q+1} \dots c_p|_q$ is invertible.

Notice that the numerator and denominator of the standard definition of the Berezinian given by fraction (2.2) are noninvariant and nonpolynomial functions of the matrix. The products $\lambda_1 \cdots \lambda_p$ and $\mu_1 \cdots \mu_q$ are invariant, but nonpolynomial. The functions $\mathrm{Ber}^\pm A$ are polynomial invariants, and they are the "minimally possible" modifications of the products of eigenvalues with this property.

We have four remarkable Hankel (or Schur) determinants in this paper: $\mathrm{Tr}\, A_{D(p,q)}$, $\mathrm{Tr}\, A_{D(p+1,q)}$, $\mathrm{Tr}\, A_{D(p,q+1)}$ and $\mathrm{Tr}\, A_{D(p+1,q+1)}$; the first being the resultant R, the last giving the identity (3.6) of the smallest degree, and the two others arising in the formula for the Berezinian (5.2).

Remark. As a by-product of Proposition 3 we have the following formula for the resultant of two polynomials:

$$\mathrm{Res}(Q, P) = \begin{vmatrix} c_{p-q+1} & \dots & c_p \\ \dots & \dots & \dots \\ c_p & \dots & c_{p+q-1} \end{vmatrix} \tag{5.8}$$

where $P(z)=a_p z^p+\cdots+1$, $Q(z)=b_q z^q+\cdots+1$, and the coefficients $c_k=c_k(Q,P)$ are defined as follows:

$$c_k(Q,P)=\sum_{i+j=k} a_i\tau_j(-1)^j \tag{5.9}$$

where τ_j are the complete symmetric functions of the roots of Q. The r.h.s. of (5.8) can be interpreted as the (super)trace $\pm\operatorname{Tr} A_{D(p,q)}$, where A is an operator in a $p|q$-dimensional space associated with the pair of polynomials P, Q such that $R_A(z)=P(z)/Q(z)$.

6. Cayley–Hamilton Identity

In the previous section we obtained explicit formulae expressing the Berezinian as a rational function of traces. The Berezinian is an example of a rational invariant on supermatrices. Let us briefly review general facts concerning such functions. This will be applied to the analysis of an analog of the Cayley–Hamilton theorem.

In the classical case rational invariant functions $F(A)$ on $p\times p$ matrices, $F(A)=F(C^{-1}AC)$, are in a 1-1 correspondence with rational symmetric functions $f(\lambda_1,\dots,\lambda_p)$ of p variables, the eigenvalues of A. The same is true for polynomial functions, due to the fundamental theorem on symmetric functions and to the fact that the elementary symmetric polynomials $\sigma_k(\lambda)$ (or the power sums $s_k(\lambda)$) are restrictions of the polynomial functions of matrices $\operatorname{Tr}\Lambda^k A$ (resp., $\operatorname{Tr} A^k$).

This is not the case for $p|q\times p|q$ matrices, where there is a wide gap between rational and polynomial invariants.

Every rational invariant function $F(A)$ of $p|q\times p|q$ matrices, such that $F(A)=F(C^{-1}AC)$ for every even invertible matrix C, defines a function $f(\lambda_1,\dots,\lambda_p,\mu_1,\dots,\mu_q)$ of the eigenvalues of A, with λ_i corresponding to even eigenvectors and μ_α to odd eigenvectors, symmetric separately in the variables $\lambda_1,\dots,\lambda_p$ and $\mu_1,\dots,\mu_q$. (Even and odd eigenvectors cannot be permuted by a similarity transformation.)

PROPOSITION 4. *Every rational $S_p\times S_q$-invariant function of λ_i,μ_α can be expressed as a rational function of the polynomials $c_1,\dots,c_{p+q}$ or $s_1,\dots,s_{p+q}$, where $c_k(\lambda,\mu)=\operatorname{Tr}\Lambda^k A$, $s_k(\lambda,\mu)=\operatorname{Tr} A^k$.*

EXAMPLE 6.1. Consider the $S_1\times S_1$-invariant polynomial $f(\lambda,\mu)=\lambda+\mu$. We have

$$\lambda+\mu=\frac{\lambda^2-\mu^2}{\lambda-\mu}=\frac{s_2}{s_1}=\frac{c_1^2-c_2}{c_1}, \tag{6.1}$$

and it is a rational invariant function on $1|1\times 1|1$ matrices.

Proposition 4 (Berezin [2,3, p. 315], Kac [11) immediately follows from considerations of the previous section. Indeed, all $S_p\times S_q$-invariant functions of λ_i,μ_α

are expressed via the elementary symmetric functions of λ_i and μ_α, i.e., the coefficients a_k, b_k of the numerator and denominator of the characteristic function $R_A(z)$, which are rational functions of $c_1, \dots, c_{p+q}$. Moreover, *for $S_p \times S_q$-invariant polynomials $f(\lambda, \mu)$ the corresponding rational invariant functions $F(A)$ can be written as fractions with the numerator being a polynomial invariant function of A and the denominator being a power of the resultant R.*

In Example 6.1 we see that $S_p \times S_q$-invariant polynomials of the eigenvalues do not necessarily extend to polynomial invariants of matrices. The following non-trivial statement holds.

PROPOSITION 5 [2,3, p. 294 17,18]. *For a $S_p \times S_q$-invariant polynomial $f(\lambda, \mu)$ three conditions are equivalent:*

(a) *the equations*

$$\left(\frac{\partial f}{\partial \lambda_i} + \frac{\partial f}{\partial \mu_\alpha}\right)\Big|_{\lambda_i = \mu_\alpha} = 0 \tag{6.2}$$

are satisfied;

(b) *$f(\lambda, \mu)$ extends to a polynomial invariant on matrices;*

(c) *$f(\lambda, \mu)$ can be expressed as a polynomial of a finite number of functions $c_k(\lambda, \mu)$* (or $s_k(\lambda, \mu)$).

The implication (c)⇒(b) is obvious, the implication (b)⇒(a) can be deduced from the invariance condition, the implication (a)⇒(c) is the key nontrivial part.

EXAMPLE 6.2. The $S_1 \times S_1$-invariant polynomial $f(\lambda, \mu) = \mu^N(\lambda - \mu)$ satisfies (6.2) and is in fact equal to the polynomial $(-1)^N c_{N+1}(A)$. It cannot be expressed as a polynomial in $c_1, \dots, c_k$ if $k \leqslant N$. On the other hand, in full accordance with Proposition 4, we can express it rationally via c_1, c_2: $\mu^N(\lambda - \mu) = (-1)^N c_{N+1}(A) = c_2^N / c_1^{N-1}$.

Example 6.2 demonstrates that, differently from the classical case, the algebra of polynomial invariants on supermatrices is not finitely generated (no a priori number of c_k is sufficient) and is not free (the generators c_k, $k = 1, 2, \dots$ satisfy an infinite number of relations (3.6)).

Remark. It would be interesting to describe the class of rational $S_p \times S_q$-invariant functions of λ_i, μ_α that obey Equation (6.2). For example, the characteristic function $R_A(z)$ and the Berezinian $\operatorname{Ber} A$ belong to this class. Hence it contains products of polynomial invariants with arbitrary powers of the Berezinian.

An important fact immediately follows from Proposition 5 and will be useful for the analysis of the Cayley–Hamilton theorem. If $f(\lambda, \mu)$ is an arbitrary $S_p \times S_q$-invariant polynomial, then the product Rf extends to a polynomial invariant. Here $R = \prod_{i,\alpha}(\lambda_i - \mu_\alpha)$ is the resultant considered above. This statement can be found in Berezin [3].

Now let us turn to the Cayley–Hamilton theorem. Let A be an operator on a $p|q$-dimensional space. Clearly A annihilates the polynomial $\mathcal{P}_A(z) = \prod(\lambda_i - z)(\mu_\alpha - z)$, where λ_i, μ_α stand for the eigenvalues of A as above. For a generic operator A, every polynomial annihilating A is divisible by $\mathcal{P}_A(z)$, exactly as in the classical case. Hence, the polynomial $\mathcal{P}_A(z)$ is a minimal polynomial for generic operators. 'Generic' here means that all the differences of the eigenvalues, $\lambda_i - \lambda_j$, $\lambda_i - \mu_\alpha$, $\mu_\alpha - \mu_\beta$, are invertible. In particular, $R_A^{\pm}(z)$ make sense and $R = \operatorname{Res}(R_A^-, R_A^+)$ is invertible. This *'naive' characteristic polynomial* can be expressed as

$$\begin{aligned}\mathcal{P}_A(z) &= (-z)^{p+q} R_A^+\Big(-\frac{1}{z}\Big) R_A^-\Big(-\frac{1}{z}\Big)\\ &= \big(a_p - a_{p-1}z + \cdots + (-1)^p z^p\big)\big(b_q - b_{q-1}z + \cdots + (-1)^q z^q\big)\end{aligned} \tag{6.3}$$

or

$$\mathcal{P}_A(z) = R^{-2}\,\mathrm{Ber}^+(A-z)\,\mathrm{Ber}^-(A-z). \tag{6.4}$$

where we used our notions Ber^+ and Ber^-. The coefficients of $\mathcal{P}_A(z)$ are rational (not polynomial) invariant functions of A. As it follows from the fact mentioned above, the denominators of the coefficients of $\mathcal{P}_A(z)$ are equal to R and the product $\mathrm{Ber}^+(A-z)\,\mathrm{Ber}^-(A-z)$ is divisible by R. Define the polynomial $\tilde{\mathcal{P}}_A(z)$ by the equality

$$\tilde{\mathcal{P}}_A(z) = R\,\mathcal{P}_A(z) \tag{6.5}$$

for generic operators. Its coefficients are polynomial invariants of A. Getting rid of the denominator R makes $\tilde{\mathcal{P}}_A(z)$ an annihilating polynomial for arbitrary operators, not necessarily generic. Notice that both polynomials $\mathcal{P}_A(z)$ and $\tilde{\mathcal{P}}_A(z)$ are reducible over the ring of rational invariants of A (and $R\tilde{\mathcal{P}}_A(z)$, over the ring of polynomial invariants).

EXAMPLE 6.3. Consider a linear operator A in a $p|1$-dimensional vector space V. Using the formula for $R_A(z)$ from Example 4.1 we obtain that

$$\begin{aligned}\mathcal{P}_A(z) = (-1)^{p+1}\bigg(&z^p - \frac{c_1c_p - c_{p+1}}{c_p}\,z^{p-1} + \frac{c_2c_p - c_1c_{p+1}}{c_p}\,z^{p-2} -\\ &- \cdots + (-1)^p\,\frac{c_pc_p - c_{p-1}c_{p+1}}{c_p}\bigg)\bigg(z + \frac{c_{p+1}}{c_p}\bigg)\end{aligned}$$

and after simplification using the identity $c_pc_{p+2} - c_{p+1}^2 = 0$ we get

$$\mathcal{P}_A(z) = \sum_{k=0}^{p+1}(-1)^{p+1-k}\,\frac{c_kc_p - 2c_{k-1}c_{p+1} + c_{k-2}c_{p+2}}{c_p}\,z^{p+1-k} \tag{6.6}$$

(here $c_p = R$). Hence

$$\tilde{\mathcal{P}}_A(z) = \sum_{k=0}^{p+1} (-1)^{p+1-k} \left(c_k c_p - 2c_{k-1}c_{p+1} + c_{k-2}c_{p+2}\right) z^{p+1-k}\,. \tag{6.7}$$

One can come to the characteristic polynomial $\tilde{\mathcal{P}}_A(z)$ by differentiating the recurrence relations of Theorem 1. In this way the coefficients of $\tilde{\mathcal{P}}_A(z)$ will be explicitly expressed in terms of $c_k(A)$. Recall that in the classical case when A is a linear operator on a p-dimensional vector space differentiating the identity $c_{p+1}(A) \equiv 0$ leads to the classical Cayley–Hamilton theorem.

If A is an even linear operator in a $p|q$-dimensional vector space, then the traces of its exterior powers obey relations (3.6) for all $k > p-q$. For $k = p-q+1$ we have

$$\begin{vmatrix} c_{p-q+1}(A) & \dots & c_{p+1}(A) \\ \dots & \dots & \dots \\ c_{p+1}(A) & \dots & c_{p+q+1}(A) \end{vmatrix} = |c_{p-q+1}(A)\dots c_{p+1}(A)|_{q+1} = 0\,. \tag{6.8}$$

This is a scalar equation valid for any even matrix in a $p|q$-dimensional space. By differentiating it one obtains a matrix identity. Denote by F_r the partial derivative of the Hankel determinant:

$$F_r = \frac{\partial}{\partial c_r}\,|c_{p-q+1}\dots c_{p+1}|_{q+1}\,, \tag{6.9}$$

and by F_r^A its value when $c_k = c_k(A)$. Define the polynomial $\tilde{\tilde{\mathcal{P}}}(z)$ in z of degree $p+q$, with coefficients polynomially depending on c_k:

$$\tilde{\tilde{\mathcal{P}}}(z) := \sum_{r=p-q}^{p+q} (-1)^r F_{r+1}\mathcal{H}_r(z)\,, \tag{6.10}$$

where $\mathcal{H}_r(z)$ are the 'Cayley–Hamilton polynomials' (2.7). We shall write $\tilde{\tilde{\mathcal{P}}}(z) = \tilde{\tilde{\mathcal{P}}}_A(z)$ if $c_k = c_k(A)$.

PROPOSITION 6. *The polynomial $\tilde{\tilde{\mathcal{P}}}_A(z)$ is an annihilating polynomial for any operator A in a $p|q$-dimensional space, and it coincides with $\tilde{\mathcal{P}}_A(z) = R\,\mathcal{P}_A(z)$ introduced above.*

Proof. By differentiating (6.8) and applying (2.8), we get the equality

$$\sum_{r=p-q+1}^{p+q+1} (-1)^{r-1} F_r^A \mathcal{H}_{r-1}^A(A) = 0 \tag{6.11}$$

or $\tilde{\tilde{\mathcal{P}}}_A(A) = 0$. Now, for generic matrices, $\mathcal{P}_A(z)$ is a minimal polynomial. Hence the annihilating polynomial $\tilde{\tilde{\mathcal{P}}}_A(z)$ is divisible by $\mathcal{P}_A(z)$ and $\tilde{\tilde{\mathcal{P}}}_A(z) = c\cdot\mathcal{P}_A(z)$, where c is a constant, as both polynomials are of the same degree. Check that $c = R$. Compare the top coefficient in $\tilde{\tilde{\mathcal{P}}}_A(z)$, which is $(-1)^{p+q}F_{p+q+1}$, with that of $\mathcal{P}_A(z)$, which is $(-1)^{p+q}$.

$$F_{p+q+1} = \frac{\partial}{\partial c_{p+q+1}} \begin{vmatrix} c_{p-q+1} & \dots & c_{p+1} \\ \dots & \dots & \dots \\ c_{p+1} & \dots & c_{p+q+1} \end{vmatrix} = \begin{vmatrix} c_{p-q+1} & \dots & c_p \\ \dots & \dots & \dots \\ c_p & \dots & c_{p+q-1} \end{vmatrix} = R\,.$$

Hence $c = R$ and $\tilde{\tilde{\mathcal{P}}}_A(z) = \tilde{\mathcal{P}}_A(z)$.

EXAMPLE 6.4. Let us make a calculation for $p|1 \times p|1$ matrices. We have the identity $c_p c_{p+2} - c_{p+1}^2 \equiv 0$. By differentiating we get $F_p = c_{p+2}$, $F_{p+1} = -2c_{p+1}$, $F_{p+2} = c_p$. Thus

$$\tilde{\mathcal{P}}_A(z) = (-1)^{p-1} F_p \mathcal{H}_{p-1}(z) + (-1)^p F_{p+1} \mathcal{H}_p(z) + (-1)^{p+1} F_{p+2} \mathcal{H}_{p+1}(z).$$

After substituting the expressions (2.7) for $\mathcal{H}_r(z)$ and collecting similar terms we immediately arrive at the polynomial coinciding with (6.7).

Cayley–Hamilton type identities such as (6.11) were obtained from relations for traces in [14] by means of some formal differential calculus. As we see here, the annihilating polynomial obtained by differentiation coincides with the naive polynomial $\mathcal{P}_A(z)$ up to a factor R.

7. Recurrence Relations in the Grothendieck Ring

Recurrence relations for the traces of exterior powers of an operator A in a $p|q$-dimensional superspace hold good for any operator, their form being independent of the operator. Such universal relations for traces suggest the existence of underlying relations for the spaces themselves, such as in the case of $q = 0$ the equality $\Lambda^k V = 0$ when $k > p$. We shall deduce these relations now.

First of all, let us explain in which sense we may speak about recurrence relations for vector spaces. They hold in a suitable Grothendieck ring. One can consider the Grothendieck ring of the category of all finite-dimensional vector superspaces (i.e., $\mathbb{Z}_2$-graded vector spaces). This ring is isomorphic to $\mathbb{Z}[\Pi]/\langle \Pi^2 - 1 \rangle$, which is the ring where dimensions of superspaces take values. An equality in this ring means just the equality of dimensions. Alternatively, one can fix a superspace V and consider the Grothendieck ring of the category of all finite-dimensional superspaces with an action of the supergroup $\mathrm{GL}(V)$, i.e., the Grothendieck ring of the finite-dimensional representations of $\mathrm{GL}(V)$. Equality of two “natural” vector spaces like spaces of tensors over V in this ring should mean the existence of an isomorphism commuting with the action of $\mathrm{GL}(V)$.

As a starting point we use the following relation, which holds for any superspace V:

$$\Lambda_z(V) \cdot S_{-z}(V) = 1, \tag{7.1a}$$

which one might prefer to rewrite as

$$\Lambda_z(V) \cdot \Lambda_{-z\Pi}(\Pi V) = 1 \tag{7.1b}$$

(for a proof it is sufficient to consider one-dimensional spaces). Here $\Lambda_z(V) = \sum z^k \Lambda^k V = 1 + zV + z^2\Lambda^2 V + \cdots$, etc. These are power series in either of the Grothendieck rings described above. We do not distinguish in notation a vector space and its class in the Grothendieck ring. Notice that the unity 1 is the class of the main field. Equalities (7.1) hold in both senses. For example, expanding in z one gets $V - V = 0$, $S^2V + \Lambda^2 V - V \otimes V = 0$, etc.

Now, for a superspace V we have $V = V_0 \oplus V_1$ where V_0 is purely even and V_1 is purely odd. We can rewrite this as $V = U \oplus \Pi W$ where both U, W are purely even vector spaces. It follows that $\Lambda_z(V) = \Lambda_z(U)\Lambda_z(\Pi W)$, therefore by (7.1b)

$$\Lambda_z(V) = \frac{\Lambda_z(U)}{\Lambda_{-z\Pi}(W)} = \frac{1 + zU + z^2\Lambda^2 U + \cdots + z^p \Lambda^p U}{1 - z\Pi W + z^2\Lambda^2 W - \cdots + (-z)^q \Pi^q \Lambda^q W}\,. \tag{7.2}$$

Note that though U and W with their exterior powers do not belong to the ring of representations of $\mathrm{GL}(V)$, they can be thought of as ideal elements that can be adjoined to it, or, which is the same, as elements of the representation ring of the block-diagonal subgroup $\mathrm{GL}(U) \times \mathrm{GL}(W) \subset \mathrm{GL}(V)$. We see that the power series $\Lambda_z(V)$ represents a rational function with the numerator of degree p and denominator of degree q. Denote it by $R_V(z)$; it replaces the characteristic function $R_A(z) = \mathrm{Ber}(1 + zA)$ of our previous analysis. $R_A(z)$ can be viewed as the character of $R_V(z)$, for the ring of representations of $\mathrm{GL}(V)$.

We can apply to $R_V(z)$ the same reasoning as to $R_A(z)$ above and conclude that the exterior powers $\Lambda^k V$ for a $p|q$-dimensional vector space V satisfy a recurrence relation of period q

$$b_0\Lambda^{k+q}V + \cdots + b_q\Lambda^k V = 0 \tag{7.3}$$

for all $k \geqslant p - q + 1$. Here $b_i = (-\Pi)^i \Lambda^i W$. Evidently, in the classical case of $q = 0$ this reduces to $\Lambda^k V = 0$ for $k \geqslant p + 1$. The relations for $c_k(A) = \mathrm{Tr}\,\Lambda^k A$ then follow from (7.3).

As in Section 3, it is possible to eliminate the coefficients $b_i = (-\Pi)^i \Lambda^i W$ from the recurrence relations (7.3) and express them in a closed form using Hankel determinants. We arrive at the following theorem.

THEOREM 4. *For an arbitrary $p|q$-dimensional vector space V the following Hankel determinants vanish:*

$$\begin{vmatrix} \Lambda^k V & \dots & \Lambda^{k+q} V \\ \dots & \dots & \dots \\ \Lambda^{k+q} V & \dots & \Lambda^{k+2q} V \end{vmatrix} = 0 \tag{7.4}$$

for all $k \geqslant p - q + 1$.

Notice that the expression of the recurrence relation for $\Lambda^k V$ in the form of Hankel's determinant has an advantage of not using the elements that are not in the ring of representations of $\mathrm{GL}(V)$.

EXAMPLE 7.1. Let $\dim V = p|1$. Then (7.4) gives the relation

$$\begin{vmatrix} \Lambda^k V & \Lambda^{k+1} V \\ \Lambda^{k+1} V & \Lambda^{k+2} V \end{vmatrix} = 0\,, \tag{7.5}$$

i.e., $\Lambda^k V \cdot \Lambda^{k+2} V = (\Lambda^{k+1} V)^2$ (product means tensor product) for $k \geqslant p$. This can be seen directly as follows. $V = U \oplus \Pi W$ where $\dim U = p$, $\dim W = 1$. Hence $\Lambda^k V = \oplus_{i+j=k} \Lambda^i U \otimes \Pi^j S^j W$. Note that $S^j W = W^j$. Thus for $k \geqslant p$ we have $\Lambda^k V = \oplus_{i=0}^{p} \Lambda^i U \otimes (\Pi W)^{k-i}$, therefore $\Lambda^{k+1} V = \Lambda^k V \otimes \Pi W$ (a geometric progression). Obviously, by tensor multiplying $\Lambda^k V$ and $\Lambda^{k+2} V$ we get the isomorphisms $\Lambda^k V \otimes \Lambda^{k+2} V = \Lambda^k V \otimes \Lambda^{k+1} V \otimes \Pi W = \Lambda^{k+1} V \otimes \Lambda^{k+1} V$, which is exactly the relation (7.5).

Let us obtain the expansion at infinity for the rational function $R_V(z)$. For this, we shall rearrange the numerator and denominator in (7.2). Since $\Lambda^i(U) = \det U \otimes \Lambda^{p-i}(U^*)$ and $\Lambda^j(W) = \det W \otimes \Lambda^{q-j}(W^*)$, we have

$$\begin{aligned}
R_V(z) &= \frac{\det U}{\det W}\, \frac{\Lambda^p(U^*) + z\Lambda^{p-1}(U^*) + \cdots + z^p}{\Lambda^q(W^*) - z\Pi\Lambda^{q-1}(W^*) + \cdots + (-z)^q \Pi^q} \\
&= \operatorname{Ber} V\, (-\Pi)^q\, z^{p-q}\, \frac{1 + z^{-1} U^* + z^{-2}\Lambda^2 U^* + \cdots + z^{-p}\Lambda^p U^*}{1 - z^{-1}\Pi W^* + z^{-2}\Lambda^2 W^* - \cdots + (-z)^{-q}\Pi^q \Lambda^q W^*} \\
&= \operatorname{Ber} V\, (-\Pi)^q\, z^{p-q}\, \frac{\Lambda_{\frac{1}{z}}(U^*)}{\Lambda_{-\frac{1}{z}\Pi}(W^*)} = \operatorname{Ber} V\, (-\Pi)^q\, z^{p-q}\, \Lambda_{\frac{1}{z}}(V^*) \\
&= (-\Pi)^q\, z^{p-q}\, \operatorname{Ber} V \sum_{k \leqslant 0} z^k \Lambda^k(V^*) = (-\Pi)^q \sum_{k \leqslant 0} z^{k+p-q}\, \Sigma^{p-k}(V) \\
&= (-\Pi)^q \sum_{k \leqslant p-q} z^k\, \Sigma^{k+q}(V)\,.
\end{aligned}$$

Hence the rational function $R_V(z)$ taking values in a Grothendieck ring has the following expansions:

$$R_V(z) = \sum_{k \geqslant 0} z^k\, \Lambda^k(V), \quad \text{(at zero)} \tag{7.6}$$

$$= \sum_{k \leqslant p-q} z^k\, (-\Pi)^q\, \Sigma^{k+q}(V), \quad \text{(at infinity)} \tag{7.7}$$

In the same way as in Section 4 we arrive at the following theorem.

THEOREM 5. *The sequence in the Grothendieck ring*

$$\Gamma_k = \Lambda^k V - (-\Pi)^q \Sigma^{k+q} V \tag{7.8}$$

for all $k \in \mathbb{Z}$ is a recurrent sequence of period q.

It very well fits with the equality $\Lambda^k V = \Sigma^k V$ of the classical case of $q = 0$, i.e., $\Lambda^k V = \det V \otimes \Lambda^{p-k} V^*$, which is a canonical isomorphism compatible with the action of $\mathrm{GL}(V)$. Theorem 5 implies the vanishing of the Hankel determinants of order $q+1$ made of the elements Γ_k.

EXAMPLE 7.2. Consider V where $\dim V = 1|1$. Then $\Lambda^k(V) = 0$ for $k < 0$, $\dim \Lambda^0(V) = 1$, $\dim \Lambda^k(V) = 1 + \Pi$ for $k \geqslant 1$. In the same way $\dim \Sigma^{k+1}(V) = 1 + \Pi$ for $k \leqslant -1$, $\dim \Sigma^1(V) = 1$, $\dim \Sigma^{k+1}(V) = 0$ for $k > 0$. It follows that $\dim \Lambda^k V - (-\Pi) \dim \Sigma^{k+1} V = 1 + \Pi$ for all $k \in \mathbb{Z}$, which is a geometric progression with ratio Π infinite in both directions. This verifies the statement of Theorem 5 for V at the level of dimensions.

EXAMPLE 7.3 (Continuation of Examples 7.1 and 7.2). For a superspace V such that $\dim V = 1|1$ we shall show explicitly an isomorphism $\varphi\colon \Lambda^k V \otimes \Lambda^{k+2} V \to \Lambda^{k+1} V \otimes \Lambda^{k+1} V$ commuting with the action of $\mathrm{GL}(V)$. Let $e \in V_0$, $\varepsilon \in V_1$ be a basis of V. Then $E_k = \underbrace{\varepsilon \wedge \cdots \wedge \varepsilon}_{k}$ and $F_k = e \wedge \underbrace{\varepsilon \wedge \cdots \wedge \varepsilon}_{k-1}$ is a basis in $\Lambda^k V$ for $k \geqslant 1$. The desired isomorphism φ is as follows: $\varphi(E_k \otimes E_{k+2}) = \alpha\, E_{k+1} \otimes E_{k+1}$, $\varphi(E_k \otimes F_{k+2}) = \frac{1}{2}\left(-\alpha + \frac{k}{k+1}\,\beta\right) E_{k+1} \otimes F_{k+1} + (-1)^k \frac{1}{2}\left(\alpha + \frac{k}{k+1}\,\beta\right) F_{k+1} \otimes E_{k+1}$, $\varphi(F_k \otimes E_{k+2}) = (-1)^k \frac{1}{2}\left(\alpha + \frac{k+2}{k+1}\,\beta\right) E_{k+1} \otimes F_{k+1} + \frac{1}{2}\left(\alpha - \frac{k+2}{k+1}\,\beta\right) F_{k+1} \otimes E_{k+1}$, and $\varphi(F_k \otimes F_{k+2}) = \beta\, F_{k+1} \otimes F_{k+1}$, where α, β are arbitrary nonzero parameters. In particular, notice that φ is not unique.

8. Cramer's Rule in Supermathematics

In this section we formulate Cramer's rule in supermathematics basing on the geometrical meaning of the Berezinian. Earlier such a generalization was obtained by Bergveldt and Rabin in [4], who used the 'hard tools' of the Gelfand–Retakh quasi-determinants theory [9, 10]. Our approach does not use anything but the main properties of the Berezinian.

Let us first formulate the ordinary Cramer's rule geometrically. Let A be a linear operator in an n-dimensional vector space V. Consider a linear equation

$$A(\boldsymbol{x}) = \boldsymbol{y}\,.$$

Here $\boldsymbol{x}, \boldsymbol{y}$ are vectors in V. For any volume form $\boldsymbol{\rho}$ on V and arbitrary vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{n-1}$ we obviously have

$$\boldsymbol{\rho}(A(\boldsymbol{x}), A(\boldsymbol{v}_1), \dots, A(\boldsymbol{v}_{n-1})) = \det A \cdot \boldsymbol{\rho}(\boldsymbol{x}, \boldsymbol{v}_1, \dots, \boldsymbol{v}_{n-1}).$$

Considering this equation for different vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{n-1}$ we can express $\boldsymbol{x}$ via $\boldsymbol{y} = A(\boldsymbol{x})$. Namely, let $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ be an arbitrary basis in V. Take as $\boldsymbol{\rho}$ the coordinate volume form, i.e., $\boldsymbol{\rho}(\boldsymbol{e}_1, \dots, \boldsymbol{e}_n) = 1$. Then for the kth coordinate of $\boldsymbol{x}$ we have $x^k = \boldsymbol{\rho}(\boldsymbol{e}_1, \dots, \boldsymbol{x}, \dots, \boldsymbol{e}_n)$ ($\boldsymbol{x}$ stands at the kth place), hence

$$x^k = \frac{1}{\det A}\, \boldsymbol{\rho}(A(\boldsymbol{e}_1), \dots, \boldsymbol{y}, \dots, A(\boldsymbol{e}_n)) = \frac{1}{\det A} \begin{vmatrix} a_1{}^1 & \dots & a_1{}^n \\ \dots & \dots & \dots \\ y^1 & \dots & y^n \\ \dots & \dots & \dots \\ a_n{}^1 & \dots & a_n{}^n \end{vmatrix},$$

where at the r.h.s. the coordinates of $\boldsymbol{y}$ replace the kth row of the matrix of the operator A. This is exactly Cramer's rule. Here we use row vectors rather than columns because it will be more convenient in the supercase.

These considerations can be generalized to the supercase as follows.

Let V be a $p|q$-dimensional space. Consider a volume form $\boldsymbol{\rho}$. Recall that in the supercase a volume form is defined as a function on bases such that a change of basis is equivalent to the multiplying by the Berezinian of the transition matrix. Let $\boldsymbol{e}_1, \dots, \boldsymbol{e}_{p+q}$, where $\boldsymbol{e}_1, \dots, \boldsymbol{e}_p$ are even vectors and $\boldsymbol{e}_{p+1}, \dots, \boldsymbol{e}_{p+q}$ are odd, be a basis in V. For another basis $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{p+q}$, the coordinate volume form associated with $\boldsymbol{e}_1, \dots, \boldsymbol{e}_{p+q}$, takes the value

$$\operatorname{Ber} \begin{pmatrix} v_1{}^1 & \dots & v_1{}^{p+q} \\ \dots & \dots & \dots \\ v_{p+q}{}^1 & \dots & v_{p+q}{}^{p+q} \end{pmatrix}.$$

Here $\boldsymbol{v}_i = v_i{}^j \boldsymbol{e}_j$. It follows that a volume form is linear in the first p arguments and hence can be extended by linearity to arbitrary vectors (the last q arguments must remain linearly independent odd vectors!). In particular, it is possible to insert an odd vector into one of the first p "even" positions.

As above, for any volume form $\boldsymbol{\rho}$ on V and vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{p+q-1}$ of the appropriate parity we have

$$\boldsymbol{\rho}(A(\boldsymbol{v}_1), \dots, A(\boldsymbol{x}), \dots, A(\boldsymbol{v}_{p+q-1})) = \operatorname{Ber} A \cdot \boldsymbol{\rho}(\boldsymbol{v}_1, \dots, \boldsymbol{x}, \dots, \boldsymbol{v}_{p+q-1}),$$

where the vector $\boldsymbol{x}$ stands at the one of the first p "even" places. A is assumed to be an even invertible operator. This leads to the solution of a linear equation

$$A(\boldsymbol{x}) = \boldsymbol{y} \tag{8.1}$$

in the superspace V as follows. Take as $\boldsymbol{\rho}$ the coordinate volume form associated with a basis $\boldsymbol{e}_1, \dots, \boldsymbol{e}_{p+q}$. Then $\boldsymbol{\rho}(\boldsymbol{e}_1, \dots, \boldsymbol{x}, \dots, \boldsymbol{e}_{p+q}) = x^k$, if $k = 1, \dots, p$. Hence the formula for the first p coordinates of $\boldsymbol{x}$ corresponding to the even basis vectors is exactly the same as in the classical case. For $k = 1, \dots, p$

$$x^k = \frac{1}{\operatorname{Ber} A}\, \boldsymbol{\rho}(A(\boldsymbol{e}_1), \dots, \boldsymbol{y}, \dots, A(\boldsymbol{e}_{p+q})) = \frac{1}{\operatorname{Ber} A}\, \Delta_k(A, \boldsymbol{y}), \tag{8.2}$$

where

$$\Delta_k(A, \boldsymbol{y}) = \operatorname{Ber} \begin{pmatrix} a_1{}^1 & \dots & a_1{}^{p+q} \\ \dots & \dots & \dots \\ y^1 & \dots & y^{p+q} \\ \dots & \dots & \dots \\ a_n{}^1 & \dots & a_n{}^n \end{pmatrix}, \tag{8.3}$$

($\boldsymbol{y}$ inserted at the kth "even" position).

To obtain the last q coordinates of $\boldsymbol{x}$ corresponding to the odd basis vectors $\boldsymbol{e}_{p+1},\dots,\boldsymbol{e}_{p+q}$, consider the space ΠV with reversed parity. Let $\boldsymbol{\rho}^{\Pi}$ be the coordinate volume form on ΠV corresponding to the basis $\boldsymbol{e}_{p+1}\Pi,\dots,\boldsymbol{e}_{p+q}\Pi,\boldsymbol{e}_1\Pi,\dots,\boldsymbol{e}_p\Pi$. Now we have

$$\boldsymbol{\rho}^{\Pi}\big(\boldsymbol{e}_{p+1}\Pi,\dots,\boldsymbol{x}\Pi,\dots,\boldsymbol{e}_{p+q}\Pi,\boldsymbol{e}_1\Pi,\dots,\boldsymbol{e}_p\Pi\big)=x^k$$

for $k=p+1,\dots,p+q$. Introducing the notation

$$\boldsymbol{\rho}^*(\boldsymbol{v}_1,\dots,\boldsymbol{v}_{p+q}):=\boldsymbol{\rho}^{\Pi}(\boldsymbol{v}_{p+1}\Pi,\dots,\boldsymbol{v}_{p+q}\Pi,\boldsymbol{v}_1\Pi,\dots,\boldsymbol{v}_p\Pi)$$

and

$$\operatorname{Ber}^* M:=\operatorname{Ber} M^{\Pi} \tag{8.4}$$

for a matrix M, we can rewrite this as $x^k=\boldsymbol{\rho}^*(\boldsymbol{e}_1,\dots,\boldsymbol{x},\dots,\boldsymbol{e}_{p+q})$, $k=p+1,\dots,p+q$. Hence for $k=p+1,\dots,p+q$

$$x^k=\frac{1}{\operatorname{Ber}^* A}\,\boldsymbol{\rho}^*(A(\boldsymbol{e}_1),\dots,\boldsymbol{y},\dots,A(\boldsymbol{e}_{p+q}))=\frac{1}{\operatorname{Ber}^* A}\,\Delta^*_k(A,\boldsymbol{y})\,, \tag{8.5}$$

where

$$\Delta^*_k(A,\boldsymbol{y})=\operatorname{Ber}^*\begin{pmatrix} {a_1}^1 & \dots & {a_1}^{p+q}\\ \dots & \dots & \dots\\ y^1 & \dots & y^{p+q}\\ \dots & \dots & \dots\\ {a_n}^1 & \dots & {a_n}^n \end{pmatrix}, \tag{8.6}$$

($\boldsymbol{y}$ inserted at the kth "odd" position). Formulae (8.2)–(8.6) give a complete solution of the Equation (8.1). Recall that the matrix of a linear operator is defined by the formula $A(\boldsymbol{e}_i)={a_i}^j\boldsymbol{e}_j$. Hence $A(\boldsymbol{x})=A(x^i\boldsymbol{e}_i)=x^i{a_i}^j\,\boldsymbol{e}_j$ if A is even.

Remark. For even invertible matrices the function Ber^* is the same as Ber^{-1}. However, for matrices that are not invertible, Ber^* can make sense, taking a nonzero nilpotent value, while Ber and Ber^{-1} are not defined.

The "super" Cramer formulae (8.2)–(8.6) motivate the following definition. Let $D_{ij}(A)$ denote the matrix obtained from an even matrix A by replacing all elements in the i-th row by zeros except for the j-th element replaced by 1. Notice that $D_{ij}(A)$ may be odd depending on positions of the indices i,j.

DEFINITION. The (i,j)*th cofactor* or *adjunct* of an even $p|q\times p|q$ matrix A is

$$(\operatorname{adj} A)_{ij}:=\begin{cases}\operatorname{Ber} D_{ij}(A) & \text{when } i=1,\dots,p\\ \operatorname{Ber}^* D_{ij}(A) & \text{when } i=p+1,\dots,p+q\end{cases} \tag{8.7}$$

In the previous notation, $(\operatorname{adj} A)_{ij} = \Delta_i(A, \boldsymbol{e}_j)$ for $i = 1, \dots, p$ and $(\operatorname{adj} A)_{ij} = \Delta_i^*(A, \boldsymbol{e}_j)$ for $i = p+1, \dots, p+q$. Notice that this notion is not symmetrical w.r.t. rows and columns. We have the following formulae for the entries of the inverse matrix:

$$(A^{-1})_{ij} = \begin{cases} \dfrac{(\operatorname{adj} A)_{ji}}{\operatorname{Ber} A} & \text{when } j = 1, \dots, p \\ \dfrac{(\operatorname{adj} A)_{ji}}{\operatorname{Ber}^* A} & \text{when } j = p+1, \dots, p+q \end{cases} \tag{8.8}$$

EXAMPLE 8.1. Consider a $1|1 \times 1|1$ even matrix

$$A = \begin{pmatrix} a_1{}^1 & a_1{}^2 \\ a_2{}^1 & a_2{}^2 \end{pmatrix} = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}.$$

Then by Equation (8.7) we get

$$(\operatorname{adj} A)_{11} = \operatorname{Ber}\begin{pmatrix} 1 & 0 \\ \gamma & d \end{pmatrix} = \frac{1}{d}, \ (\operatorname{adj} A)_{12} = \operatorname{Ber}\begin{pmatrix} 0 & 1 \\ \gamma & d \end{pmatrix} = -\frac{\gamma}{d},$$
$$(\operatorname{adj} A)_{21} = \operatorname{Ber}^*\begin{pmatrix} a & \beta \\ 1 & 0 \end{pmatrix} = \operatorname{Ber}\begin{pmatrix} 0 & 1 \\ \beta & a \end{pmatrix} = -\frac{\beta}{a^2},$$
$$(\operatorname{adj} A)_{22} = \operatorname{Ber}^*\begin{pmatrix} a & \beta \\ 0 & 1 \end{pmatrix} = \operatorname{Ber}\begin{pmatrix} 1 & 0 \\ \beta & a \end{pmatrix} = \frac{1}{a}.$$

Thus for the transpose adjunct matrix we have:

$$B = \begin{pmatrix} \frac{1}{d} & -\frac{\beta}{a^2} \\ -\frac{\gamma}{d^2} & \frac{1}{a} \end{pmatrix},$$

and

$$AB = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} \begin{pmatrix} \frac{1}{d} & -\frac{\beta}{a^2} \\ -\frac{\gamma}{d^2} & \frac{1}{a} \end{pmatrix} = \begin{pmatrix} \frac{a}{d} - \frac{\beta\gamma}{d^2} & 0 \\ 0 & \frac{d}{a} - \frac{\gamma\beta}{a^2} \end{pmatrix} = \begin{pmatrix} \operatorname{Ber} A & 0 \\ 0 & \operatorname{Ber}^* A \end{pmatrix},$$

as expected.

Remark. A different approach to Cramer's rule was suggested in [12 . Instead of solutions of $A(\boldsymbol{x}) = \boldsymbol{y}$, they introduced '$\lambda$-solutions' satisfying $A(\boldsymbol{x}) = \lambda \cdot \boldsymbol{y}$, with λ being one of certain polynomial 'relative determinants' of A defined in [12]. Such approach allows to avoid division and use only polynomial expressions.

Appendix A. Elementary Properties of Recurrent Sequences

Here we summarize the relations between recurrent sequences and rational functions used in the main text. Let

$$R(z) = \frac{a_0 + a_1 z + \cdots + a_p z^p}{b_0 + b_1 z + \cdots + b_q z^q} \tag{A.1}$$

be a rational function. We assume that the numerator has degree p and the denominator degree q. The coefficients can be in an arbitrary commutative ring with unit. Consider formal power expansions of the fraction (A.1) at zero and at infinity. Let $R(z)=\sum_{k\geqslant 0} c_k z^k$ near zero and $R(z)=\sum_{k\leqslant p-q} c^*_k z^k$ near infinity. It is convenient to assume that the coefficients such as a_k, b_k, c_k, etc., are defined for all values of $k\in\mathbb{Z}$ but may be zero for some k. Hence we have the equalities $a_n=\sum_{i=0}^q b_i c_{n-i}$ and $a_n=\sum_{i=0}^q b_i c^*_{n-i}$ for all n, where $c_k=0$ for $k<0$ and $c^*_k=0$ for $k>p-q$. Taking into account that $a_n=0$ for $n>p$ or $n<0$, we obtain that

$$\sum_{i=0}^{q} b_i c_{k+q-i}=0 \tag{A.2}$$

for all $k>p-q$, and that

$$\sum_{i=0}^{q} b_i c^*_{k-i}=0 \tag{A.3}$$

for all $k<0$. For arbitrary $k\in\mathbb{Z}$, we obtain that

$$\sum_{i=0}^{q} b_i \gamma_{n-i}=0, \tag{A.4}$$

where $\gamma_k = c_k - c^*_k$ (by taking the difference of $a_n = \sum_{i=0}^q b_i c^*_{n-i}$ and $a_n=\sum_{i=0}^q b_i c_{n-i}$).

We say that a sequence $\{c_k\}_{k\in\mathbb{Z}}$ is *right* if $c_k=0$ for $k<0$. We call a right sequence $\{c_k\}$ a *$p|q$-recurrent sequence* or, shortly, a *$p|q$-sequence*, if the elements c_k satisfy a recurrence relation of the form (A.2) for all $k>p-q$. The coefficients c_k of the power expansion at zero of the fraction (A.1) make a $p|q$-recurrent sequence. (The coefficients of the expansion of (A.1) at infinity also make a $p|q$-sequence after the re-indexing that makes them a right sequence, $c'_k:=c^*_{p-q-k}$.) The fraction (A.1) is classically referred to as the *generating function* or the *symbol* of the recurrent sequence $\{c_k\}$.

For a sequence $\{c_k\}_{k\geqslant 0}$ to be a $p|q$-sequence means, if $p\geqslant q$, that it satisfies a recurrence relation of period q except for the $p-q+1$ initial terms $c_0,\dots,c_{p-q}$, and if $p<q$, that it satisfies a recurrence relation of period q for all terms c_k, $k\geqslant 0$, and can be extended to the left by extra $q-p-1$ zero terms so that the relation still holds.

Hence we have the following picture for the coefficients of the expansions of the rational function (A.1). The coefficients of the expansions at zero and at infinity satisfy the same recurrence relations of period q. If $p<q$, the coefficients c_k and c^*_k can be nonzero only in the disjoint ranges $k\geqslant 0$ and $k\leqslant p-q$, respectively. The recurrence relation holds for all terms. If $p\geqslant q$ (that is, when the fraction is improper), the coefficients c_k and c^*_k can be simultaneously nonzero in the finite range $0\leqslant k\leqslant p-q$. Separate recurrence relations break down in this range. However, in all cases the sequence $\gamma_k=c_k-c^*_k$, infinite in both directions and which

coincides with either c_k or $-c_k^*$ 'almost everywhere', satisfies the recurrence relation for all $k \in \mathbb{Z}$.

If a sequence $\{c_k\}$ is given, one can consider the associated infinite Hankel matrix with the entries $a_{ij} = c_{i+j}$. Let $\{c_k\}$ satisfy a recurrence relation of the form (A.2) for all $k \geqslant N$. Assume that b_0 is invertible. Then the infinite vector $\boldsymbol{c}_{N+q} = \{c_{q+k}\}_{k \geqslant N}$ is a linear combination of the vectors $\boldsymbol{c}_N = \{c_k\}_{k \geqslant N}, \dots,$ $\boldsymbol{c}_{N+q-1} = \{c_{q+k-1}\}_{k \geqslant N}$. In particular it implies the vanishing of the Hankel minors of order $q+1$:

$$\begin{vmatrix} c_k & \dots & c_{k+q} \\ \dots & \dots & \dots \\ c_{k+q} & \dots & c_{k+2q} \end{vmatrix} = 0$$

where $k \geqslant N$. It is a classical fact noticed by Kronecker that a power series represents a rational function if and only if the corresponding infinite Hankel matrix has finite rank. There is a vast literature devoted to theoretical and practical aspects of recurrent sequences and Hankel matrices. We presented the material in the form convenient for our purposes. Note that the classical expositions, see [8], make use of the expansion of a rational function at one point (infinity), while our main results are based on comparing two such expansions, at zero and at infinity.

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