Homework 2. Solutions

1 *Consider an upper half-plain ($y > 0$) in $\mathbf{R}^2$ equipped with Riemannian metric*

$$G = \sigma(x,y)(dx^2 + dy^2)$$

a) Show that $\sigma > 0$

b) In the case if $\sigma = \frac{1}{y^2}$ (the Lobachevsky metric) calculate the lengths of vectors $\mathbf{A} = 2\partial_x$ and $\mathbf{B} = 12\partial_x + 5\partial_y$ attached at the point $(x,y) = (1,2)$.

c) calculate the cosine of the angle between the vectors $\mathbf{A}$ and $\mathbf{B}$ and show that the answer does not depend on the choice of the function $\sigma(x,y)$.

d) Calculate the lenght of the segments $x = a + t, y = b$, and $x = a, y = b + 1$, $0 \le t \le 1$ in the case if $\sigma = \frac{1}{y^2}$ (Lobachevsky plane)

e) Suppose $\sigma(x,y) = \frac{1}{(1+x^2+y^2)^2}$. Consider two curves L_1 and L_2 in upper half-plane such that

$$L_1 = \begin{cases} x = f(t) \\ y = g(t) \end{cases}, \quad \text{and} \begin{cases} x = g(t) \\ y = f(t) \end{cases}, \qquad 0 \le t \le 1\,.$$

Show that these curves have the same length in the case if $\sigma(x,y) = \frac{1}{(1+x^2+y^2)^2}$.

a)$\sigma > 0$ since positive definiteness: e.g. $G(\mathbf{X},\mathbf{X}) = \sigma(x,y) > 0$ if $\mathbf{X} = \partial_x$.

b)

$$|\mathbf{A}| = \sqrt{G(\mathbf{A},\mathbf{A})} = \sqrt{\frac{A_x^2 + A_y^2}{y^2}} = \sqrt{\frac{2^2+0^2}{2^2}} = 1,\ |\mathbf{B}| = \sqrt{G(\mathbf{B},\mathbf{B})} = \sqrt{\frac{B_x^2 + B_y^2}{y^2}} = \sqrt{\frac{12^2+5^2}{2^2}} = \frac{13}{2}\,.$$

c)Calculate the cosine for an arbitrary σ: $\cos(\angle(\mathbf{A},\mathbf{B})) = \frac{G(\mathbf{A},\mathbf{B})}{\sqrt{G(\mathbf{A},\mathbf{A})}\sqrt{G(\mathbf{B},\mathbf{B})}} = \frac{\langle \mathbf{A},\mathbf{B}\rangle_G}{|\mathbf{A}||\mathbf{B}|} =$

$$\frac{\sigma(x,y)\left(A_xB_x + A_yB_y\right)}{\sqrt{\sigma(x,y)\left(A_x^2 + A_y^2\right)}\sqrt{\sigma(x,y)\left(B_x^2 + B_y^2\right)}} = \frac{\left(A_xB_x + A_yB_y\right)}{\sqrt{\left(A_x^2 + A_y^2\right)}\sqrt{\left(B_x^2 + B_y^2\right)}} = \frac{2\cdot 12 + 0\cdot 5}{1\cdot 2\cdot 13} = \frac{12}{13}\,.$$

d) Length of the first curve is equal to

$$\int_0^1 \sqrt{\frac{x_t^2 + y_t^2}{y^2(t)}}dt = \int_0^1 \sqrt{\frac{1+0}{b^2}}dt = \frac{1}{b}\,,$$

length of the second curve is equal to

$$\int_0^1 \sqrt{\frac{x_t^2 + y_t^2}{y^2(t)}}dt = \int_0^1 \sqrt{\frac{0+1}{(b+t)^2}}dt = \int_0^1 \frac{1}{b+t}dt = \log\left(1 + \frac{1}{b}\right)\,.$$

e) If $x \leftrightarrow y$ then metric does not change since $\sigma(x,y) = \sigma(y,x)$: $\sigma(x,y)(dx^2+dy^2) = \sigma(y,x)(dx^2+dy^2)$, and $L_1 \leftrightarrow L_2$. Hence lengths of these curves coincide.

2 *Consider the Riemannian metric on the circle of the radius R induced by the Euclidean metric on the ambient plane.*

a) Express it using polar angle as a coordinate on the circle.

b) Express the same metric using stereographic coordinate t obtained by stereographic projection of the circle on the line, passing through its centre.

Riemannian metric of Euclidean space is $G = dx^2 + dy^2$.

a) using the angle: In this case parametric equation of circle is $\begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \end{cases}$. Then

$$G = (dx^2 + dy^2)\big|_{x=R\cos\varphi, y=R\sin\varphi} = (d\cos\varphi)^2 + (d\sin\varphi)^2 = R^2 d\varphi^2 \,.$$

b) In stereographic coordinate using (1) and the fact that

$$y = R\frac{t^2 - R^2}{t^2 + R^2} = R\left(1 - \frac{2R^2}{t^2 + R^2}\right)$$

we have that

$$G = (dx^2 + dy^2)\big|_{x=x(t), y=y(t)} = \left(d\left(\frac{2tR^2}{R^2 + t^2}\right)\right)^2 + \left(d\left(\frac{t^2 - R^2}{R^2 + t^2}R\right)\right)^2 =$$

$$\left(\frac{2R^2 dt}{R^2 + t^2} - \frac{4t^2R^2 dt}{(R^2 + t^2)^2}\right)^2 + \left(-\frac{4R^2 t dt}{(t^2 + R^2)^2}\right)^2 == \frac{4R^4 dt^2}{(R^2 + t^2)^2} \blacksquare$$

Another solution Using the fact that stereographic projection is restriction of inversion with the radius $R\sqrt{2}$ we come to the same formula (see in details lecture notes).

3 *Consider the Riemannian metric on the sphere of the radius R induced by the Euclidean metric on the ambient 3-dimensional space.*

a) Express it using spherical coordinates on the sphere.

b) Express the same metric using stereographic coordinates u, v obtained by stereographic projection of the sphere on the plane, passing through its centre.

Solution

Riemannian metric of Euclidean space is $G = dx^2 + dy^2 + dz^2$.

a) using the spherical coordinates: In this case parametric equation of sphere is $\begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases}$. Then

$$G = (dx^2 + dy^2 + dz^2)\big|_{x=R\sin\theta\cos\varphi, y=R\sin\theta\sin\varphi, z=R\cos\theta} = R^2\left((d\sin\theta\cos\varphi)\right)^2 + R^2\left((d\sin\theta\sin\varphi)\right)^2 + R^2\left((d\cos\theta)\right)^2 =$$

$$R^2(\cos\theta\cos\varphi d\theta - \sin\theta\sin\varphi d\varphi)^2 + R^2(\cos\theta\sin\varphi d\theta + \sin\theta\cos\varphi d\varphi)^2 + R^2(-\sin\theta d\theta)^2 = R^2 d\theta^2 + R^2\sin^2\theta d\varphi^2 \,. \blacksquare$$

b) in stereographic coordinates using (2) we have $G = (dx^2 + dy^2 + dz^2)\big|_{x=x(u,v), y=y(u,v), z=z(u,v)} =$

$$\left(d\left(\frac{2uR^2}{R^2 + u^2 + v^2}\right)\right)^2 + \left(d\left(\frac{2vR^2}{R^2 + u^2 + v^2}\right)\right)^2 + \left(d\left(1 - \frac{2R^2}{R^2 + u^2 + v^2}\right)R\right)^2 =$$

$$R^4\left(\frac{2du}{R^2 + u^2 + v^2} - \frac{2u(2udu + 2vdv)}{(R^2 + u^2 + v^2)^2}\right)^2 + R^4\left(\frac{2dv}{R^2 + u^2 + v^2} - \frac{2v(2udu + 2vdv)}{(R^2 + u^2 + v^2)^2}\right)^2 + \frac{16R^6(udu + vdv)^2}{(R^2 + u^2 + v^2)^4} =$$

$$\frac{4R^4}{(R^2 + u^2 + v^2)^2}\left[\left(du - \frac{2u(udu + vdv)}{R^2 + u^2 + v^2}\right)^2 + \left(dv - \frac{2v(udu + vdv)}{R^2 + u^2 + v^2}\right)^2 + \frac{4R^2(udu + vdv)^2}{(R^2 + u^2 + v^2)^2}\right] =$$

$$\frac{4R^4(du^2 + dv^2)}{(R^2 + u^2 + v^2)^2} \blacksquare$$

Another solution One can avoid this straightforward long caluclations, just noting that stereographic projection is the restriction of inversion, of radius $\sqrt{2}R$. This immediately implies the answer. (See in deails lecture notes.)

Remark

In the case of n-dimensional sphere S^n of radius R in $(n+1)$-dimensional Euclidean space $\mathbf{E}^{n+1}$ (it can be defined by the equation $(x^1)^2+\ldots+(x^{n+1})^2 = R^2$ in Cartesian coordinates $x^1,\ldots,x^n,x^{n+1}$) Riemannian metric on this sphere induced by the Euclidean metric in the ambient space in stereographic coordinates has following appearance:

$$G = \left((dx^1)^2 + \ldots + (dx^{n+1})^2\right)\big|_{x^\mu = x^i(u^i)} = \left(\sum_{j=1}^n \left(d\left(\frac{2R^2u^j}{R^2+\sum_{i=1}^n (u^i)^2}\right)\right)\right)^2 + \left(d\left(R\frac{\sum_{i=1}^n (u^i)^2 - R^2}{R^2+\sum_{i=1}^n (u^i)^2}\right)\right)^2 =$$

$$= \frac{4R^4\sum_{i=1}^n (du^i)^2}{(R^2+\sum_{i=1}^n (u^i)^2)^2}$$

4 *Consider the surface L which is the upper sheet of two-sheeted hyperboloid in $\mathbf{R}^3$:*

$$L: \qquad z^2 - x^2 - y^2 = 1, \quad z > 0\,.$$

a) Find parametric equation of the surface L using hyperbolic functions $\cosh, \sinh$ *following an analogy with spherical coordinates on the sphere.*

(The surface L sometimes is called pseudo-sphere.)

b) Consider the stereographic projection of the surface L on the plane OXY, i.e. the central projection on the plane $z = 0$ with the centre at the point $(0, 0, -1)$.

Show that the image of projection of the surface L is the open disc $x^2 + y^2 < 1$ in the plane OXY.

a) Parametric equation is $\begin{cases} x = \sinh\theta\cos\varphi \\ y = \sinh\theta\sin\varphi \\ z = \cosh\theta\theta\cos\varphi \end{cases}$ We see that the condition $z^2 - x^2 - y^2 = 1$ is fulfilled. (Compare with equation of sphere in spheric coordinates.)

b) Calculations are very similar to the case of stereographic coordinates for 2-sphere $x^2 + y^2 + z^2 = 1$ of the radius $R = 1$. Stereographic coordinates u, v. Centre of projection $(0, 0, -1)$: We have $\frac{u}{x} = \frac{v}{y} = \frac{1}{1+z}$. Hence $\begin{cases} u = \frac{x}{1+z} \\ v = \frac{y}{1+z} \end{cases}$. Since $x = u(1+z), y = v(1+z)$ then $z^2 - 1 = x^2 + y^2$ and $z^2 - 1 = (u^2+v^2)(1+z)^2$, i.e. $z = \frac{1+u^2+v^2}{1-u^2-v^2}$. We come to

$$\begin{cases} u = \frac{x}{1+z} \\ v = \frac{y}{1+z} \end{cases}, \qquad \begin{cases} x = \frac{2u}{1-u^2-v^2} \\ y = \frac{2v}{1-u^2-v^2} \\ z = \frac{u^2+v^2+1}{1-u^2-v^2} \end{cases}, \quad u^2+v^2 < 1\,. \tag{4}$$

The image of upper-sheet is an open disc $u^2 + v^2 < 1$ since $u^2 + v^2 = \frac{x^2+y^2}{(1+z)^2} = \frac{z^2-1}{(1+z)^2} = \frac{z-1}{z+1}$. Since for upper sheet $z > 1$ then $0 \le \frac{z-1}{z+1} < 1$.

* *Consider the pseudo-Riemannian, pseudo-Euclidean metric on $\mathbf{R}^3$ given by the formula*

$$ds^2 = dx^2 + dy^2 - dz^2\,.$$

Calculate the induced metric on the surface L considered in the Exercise 4, and show that it is a Riemannian metric (it is positive-definite).

Perform calculations in spherical-like coordinates (see Exercise 4a) above) and in stereographic coordinates (see exercise 4b) above)

In stereographic coordinates we come to realisation of Lobachevsky plane on the disc in $\mathbf{E}^2$. It is so called Poincare model of Lobachevsky geometry.

Solution. The calculations will be very similar to the calculations performed in the exercise 3 above. Just we need consider $\cosh\theta, \sinh\theta$ instead $\cos\theta, \sin\theta$ and and sometimes changes the signs.

First of all consider spherical-like coordinates:

Equation of two-sheeted hyperboloid is $\begin{cases} x = \sinh\theta\cos\varphi \\ y = \sinh\theta\sin\varphi \\ z = \cosh\theta \end{cases}$. Then

$$G = (dx^2+dy^2-dz^2)\big|_{x=\sinh\theta\cos\varphi, y=\sinh\theta\sin\varphi, z=\cosh\theta} = ((d\sinh\theta\cos\varphi))^2+((d\sinh\theta\sin\varphi))^2-((d\cosh\theta))^2 =$$

$$(\cosh\theta\cos\varphi d\theta - \sinh\theta\sin\varphi d\varphi)^2 + (\cosh\theta\sin\varphi d\theta + \sinh\theta\cos\varphi d\varphi)^2 + (\sinh\theta d\theta)^2 = d\theta^2 + \sinh^2\theta d\varphi^2 .$$

matrix of Riemannian metric is $G = \begin{pmatrix} 1 & 0 \\ 0 & \sinh^2\theta \end{pmatrix}$. In the same way as for sphere these coordinates are well-defined in all points except $z = \pm 1$, where $\sin^2\theta = 0$.

Now express Riemannian metric in stereographic coordinates (4):

$$G = (dx^2+dy^2-dz^2)\big|_{x=x(u,v), y=y(u,v), z=z(u,v)} = \left(d\left(\frac{2u}{1-u^2-v^2}\right)\right)^2 + \left(d\left(\frac{2v}{1-u^2-v^2}\right)\right)^2 - \left(d\left(\frac{2}{1-u^2-v^2}-1\right)\right)^2$$

$$= \left(\frac{2du}{1-u^2-v^2} + \frac{2u(2udu+2vdv)}{(1-u^2-v^2)^2}\right)^2 + \left(\frac{2dv}{1-u^2-v^2} + \frac{2v(2udu+2vdv)}{(1-u^2-v^2)^2}\right)^2 - \frac{16(udu+vdv)^2}{(1-u^2-v^2)^4} =$$

$$= \frac{4(du)^2+4(dv)^2}{(1-u^2-v^2)^2} .$$

(Compare with calculations for sphere $x^2+y^2+z^2=1$).

Resume: We come to Riemannian metric on the surface L induced by pseudo-Riemannian metric in ambient space.

Remark The surface L sometimes is called pseudo-sphere. The Riemannian metric on this surface sometimes is called Lobachevsky (hyperbolic) metric. The surface L with this metric realises Lobachevsky (hyperbolic) geometry, where Euclid‘s 5-th Axiom fails. This Riemannian manifold (manifold+Riemannian metric) is called Lobachevsky (hyperbolic) plane. In stereographic coordinates Lobachevsky plane is realised as an open disc $u^2+v^2<1$ in $\mathbf{E}^2$. It is so called Poincare model of Lobachevsky geometry. In the exercise 8 below we will consider realisation of Lobachevsky plane as upper half-plane.

6* *In the exercises 5 and 6 we showed that pseudo-Euclidean metric (1) in $\mathbf{R}^3$ induces Riemanian metric on two-sheeted hyperboloid $z^2-x^2-y^2=1$. Show that it is not true for one-sheeted hyperboloid: metric on one-sheeted hyperboloid $x^2+y^2-z^2=1$ in $\mathbf{R}^3$ is not Riemannian if it is induced with the pseudo-Euclidean metric (1).*

Solution. One can perform straightforward calculations in spherical-like coordinates: Equation of one-sheeted hyperboloid is $\begin{cases} x = \cosh\theta\cos\varphi \\ y = \cosh\theta\sin\varphi \\ z = \sinh\theta \end{cases}$. Then

$$G = (dx^2+dy^2-dz^2)\big|_{x=\cosh\theta\cos\varphi, y=\cosh\theta\sin\varphi, z=\sinh\theta} = ((d\cosh\theta\cos\varphi))^2+((d\cosh\theta\sin\varphi))^2-((d\sinh\theta))^2 =$$

$$(\sinh\theta\cos\varphi d\theta - \cosh\theta\sin\varphi d\varphi)^2 + (\sinh\theta\sin\varphi d\theta + \cosh\theta\cos\varphi d\varphi)^2 - (\cosh\theta d\theta)^2 = -d\theta^2 + \cosh^2\theta d\varphi^2 .$$

matrix is $G = \begin{pmatrix} -1 & 0 \\ 0 & \cosh^2\theta \end{pmatrix}$. The condition of positive-definiteness is not fulfilled. This is not Riemannian metric.

Another solution Consider the vectors $\mathbf{e} = \frac{\partial}{\partial y}$ and $\mathbf{f} = \frac{\partial}{\partial z}$ attached at the point $(1,0,0)$. One can see that these vectors are tangent to the hyperboloid, but they have the "length" of different sign. (One of these vectors is space-like vector, another time like vector.) We have pseudoriemannian metric at the tangent space spanned by these two vectors.

7* In the exercise 6 we realised Lobachevsky plane as a disc $u^2+v^2<1$. Find new coordinates x, y such that in these coordinates Lobachevsky plane (hyperbolic plane) can be considered as an upper half plane $\{(x,y)\colon\ y>0\}$ in $\mathbf{E}^2$ and write down explicitly Riemannian metric in these coordinates.

Hint: *You may use complex coordinates:*

$$z = x+iy, \bar z = x-iy, \omega = u+iv, \bar w = u-iv$$

and consider a holomorphic transformation:

$$\omega = \frac{1+iz}{1-iz} \Leftrightarrow z = i\frac{1-\omega}{1+\omega},$$

which transforms the open disc $w\bar w < 1$ onto the upper plane $\mathbf{Im} z > 0$.

Solution.

Recall that in the previous exercise we calculated expression for Lobachevsky metric in stereographic coordinates $u, v, u^2+v^2<1$. We come to the answer: $G = \frac{4du^2+4dv^2}{(1-u^2-v^2)^2}$ (see exercise 6). (It was realisation of Lobachevsky plane on the Euclidean disc, so called Poincare model of Lobachevsky (hyperbolic) geometry.)

In complex coordinates $\omega = u+iv, \bar\omega = u-iv$ the metric $G = \frac{4du^2+4dv^2}{(1-u^2-v^2)^2}$ obtained in the exercise 6 can be rewritten $G = \frac{4dwd\bar w^2}{(1-w\bar w)}$. Indeed

$$G = \frac{4d\omega d\bar\omega}{(1-w\bar w)^2} = G = \frac{4d(u+iv)d(u-iv)}{(1-(u+iv)(u-iv))^2} = \frac{4du^2+4dv^2}{(1-u^2-v^2)^2}.$$

Now consider Mobius transformation $\omega = \frac{1=iz}{1-iz}$, which transforms the disc, interior of circle $\omega\bar\omega = 1$ onto upper half plane $Imz > 0$. One can see that

$$\omega = \frac{1+iz}{1-iz}, \qquad z = i\frac{1-\omega}{1+\omega}$$

(Can you find all Mobius transformations which transform upper half plane to the interior of unit circle?.)

Now calculate G in coordinates $z, \bar z$. i.e. in coordinates (x, y):

$$G = \frac{4du^2+4dv^2}{(1-u^2-v^2)^2} = \frac{4dwd\bar w}{(1-w\bar w)^2}$$

We have

$$d\omega = d\left(\frac{1+iz}{1-iz}\right) = \frac{2idz}{(1-iz)^2},\ d\bar\omega = \frac{-2id\bar z}{(1+i\bar z)^2},$$
$$1-\omega\bar\omega = 1-\frac{1+iz}{1-iz}\frac{1-i\bar z}{1+i\bar z} = \frac{2i(\bar z - z)}{(1-iz)(1+i\bar z)}$$

Hence

$$G = \frac{4d\omega d\bar\omega}{(1-\omega\bar\omega)^2} = \frac{4\left(\frac{2idz}{(1-iz)^2}\right)\left(\frac{-2id\bar z}{(1+i\bar z)^2}\right)}{\frac{-4(\bar z - z)^2}{(1-iz)^2(1+i\bar z)^2}} = \frac{-4dd\bar z}{(\bar z - z)^2} = \frac{dx^2+dy^2}{y^2},$$

since $z = x + iy$ and $\bar{z} - z = -2iy$.

We come to the very useful and nice interpretation of hyperbolic geometry: upper half plane in $\mathbf{E}^2$ with metric $G = \frac{dx^2+dy^2}{y^2}$. Later by default we will call "Lobachevsky (hyperbolic) plane" the realisation of Lobachevsky plane as an half-upper plane in $\mathbf{E}^2$ with these coordinates x, y $(y > 0)$ with metric $G = \frac{dx^2+dy^2}{y^2}$.

Remark What will happen if we consider another Mobius transformation of disc $\omega\bar{\omega} < 1$ onto plane $\operatorname{Im} z > 0$?