Homework 4

Often it is useful to view 3-dimensional Euclidean space $\mathbf{E}^3$ as a space $\mathbf{R}^3$ with the standard Cartesian coordinates: $\mathbf{R}^3 = \{(x, y, z), x, y, z \in \mathbf{R}\}$. The canonical orthonormal basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ in $\mathbf{R}^3$ has the following geometrical meaning: The unit vector $\mathbf{e}_x = (1, 0, 0)$ is directed along x-axis, the unit vector $\mathbf{e}_y = (0, 1, 0)$ is directed along y-axis and the unit vector $\mathbf{e}_z = (0, 0, 1)$ is directed along z-axis. We suppose that an orientation in $\mathbf{E}^3$ is fixed by the left basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$.

1 Consider an operator P on $\mathbf{E}^3$ such that P is an orthogonal operator preserving the orientation of $\mathbf{E}^3$ and

$$P(\mathbf{e}_x) = \mathbf{e}_y, \quad P(\mathbf{e}_z) = -\mathbf{e}_z\,.$$

Find an action of the operator P on an arbitrary vector $\mathbf{x} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$.

Why P is a rotation operator? Find an angle and axis of the rotation.

2 Consider an operator P on $\mathbf{E}^3$ such that

$$P(\mathbf{e}) = \frac{2}{3}\mathbf{e} + \frac{2}{3}\mathbf{f} + \frac{1}{3}\mathbf{g}\,, P(\mathbf{f}) = -\frac{1}{3}\mathbf{e} + \frac{2}{3}\mathbf{f} - \frac{2}{3}\mathbf{g}\,, P(\mathbf{g}) = -\frac{2}{3}\mathbf{e} + \frac{1}{3}\mathbf{f} + \frac{2}{3}\mathbf{g}\,.$$

Show that this is an orthogonal operator preserving the orientation of $\mathbf{E}^3$.

Find an axis of rotation (i.e. a vector and an angle of rotation. (We assume that $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ is an orthonormal basis in $\mathbf{E}^3$.)

3 Consider on $\mathbf{E}^3$ following two operators:

$$P_1(\mathbf{x}) = \mathbf{x} - 2(\mathbf{n}, \mathbf{x})\mathbf{n}\,, \quad P_2(\mathbf{x}) = 2(\mathbf{n}, \mathbf{x})\mathbf{n} - \mathbf{x}\,,$$

where $\mathbf{n}$ is a unit vector.

Show that these both operators are orthogonal operators. Show that first operator changes the orientation, and the second operator preserves orientation.

Show that the first operator is reflection operator with respect to....

Show that the second operator is rotation operator: find an axis of rotation and an angle of rotation.

4* Let $\mathbf{e}, \mathbf{f}$ be two distinct unit vectors. Let P be an operator such that

$$P(\mathbf{e}) = \mathbf{f}, \quad P(\mathbf{f}) = \mathbf{e}.$$

Find orthogonal operators P which obey these condition.

Hint: One of them is reflection operator, another rotation operator...

5 Let $\mathbf{n}$ be a unit vector. Consider linear operator

$$P(\mathbf{x}) = \mathbf{n} \times \mathbf{x} + (\mathbf{n}, \mathbf{x})\mathbf{n}\,.$$

Show that this is rotation operator. Find the axis and angle of rotation.

6 Students John and Sarah calculate vector product $\mathbf{a} \times \mathbf{b}$ of two vectors using two different orthonormal bases in the Euclidean space $\mathbf{E}^3$, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$. John expands the vectors with respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Sarah expands the vectors with respect to the basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$. For two arbitrary vectors $\mathbf{a}, \mathbf{b} \in \mathbf{E}^3$

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = a'_1\mathbf{e}'_1 + a'_2\mathbf{e}'_2 + a'_3\mathbf{e}'_3\,,$$

$$\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 = b'_1\mathbf{e}'_1 + b'_2\mathbf{e}'_2 + b'_3\mathbf{e}'_3\,.$$

John and Sarah both use the so-called "determinant" formula. Are their answers the same?

$$\mathbf{a} \times \mathbf{b} = \underbrace{\det \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}}_{\text{John's calculations}} \stackrel{?}{=} \underbrace{\det \begin{pmatrix} \mathbf{e}'_1 & \mathbf{e}'_2 & \mathbf{e}'_3 \\ a'_1 & a'_2 & a'_3 \\ b'_1 & b'_2 & b'_3 \end{pmatrix}}_{\text{Sarah's calculations}}$$

7 Calculate the area of parallelograms formed by the vectors $\mathbf{a}, \mathbf{b}$ if
a) $\mathbf{a} = (1, 2, 3), \mathbf{b} = (1, 0, 1)$;
b) $\mathbf{a} = (2, 2, 3), \mathbf{b} = (1, 1, 1)$;
c) $\mathbf{a} = (5, 8, 4), \mathbf{b} = (10, 16, 8)$.
d) $\mathbf{a} = (3, 4, 0), \mathbf{b} = (5, 17, 0)$.

8 Find a vector $\mathbf{n}$ such that the following conditions hold:
1) It has unit length
2) It is orthogonal to the vectors $\mathbf{a} = (1, 2, 3)$ and $\mathbf{b} = (1, 3, 2)$.
3) An ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{n}\}$ has an orientation opposite to the orientation of the orthonormal basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ which defines the orientation of the Euclidean space.

9 Show that for any two vectors $\mathbf{a}, \mathbf{b} \in \mathbf{E}^3$ the following identity is satisfied

$$(\mathbf{a}, \mathbf{a})(\mathbf{b}, \mathbf{b}) = (\mathbf{a}, \mathbf{b})^2 + (\mathbf{a} \times \mathbf{b}, \mathbf{a} \times \mathbf{b})\,.$$

Write down this identity in components.
† Compare this identity with the CBS inequality . (See the problem 5 in the Homework 2).

10 In 2-dimensional Euclidean space $\mathbf{E}^2$ consider the vectors

$$\mathbf{a} = (3, 2),\ \mathbf{b} = (7, 5),\ \mathbf{c} = (17, 12)\,, \mathbf{d} = (41, 29)\,.$$

Calculate areas of the parallelograms $\Pi(\mathbf{a}, \mathbf{b}), \Pi(\mathbf{b}, \mathbf{c})$ and $\Pi(\mathbf{c}, \mathbf{d})$.

10† Do you see any relations between parallelograms in the exercise above, fractions $\frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}$ and the number... $\sqrt{2}$? Can you continue this sequence of fractions?
(*Hint: Consider the squares of these fractions.*)