

## Homework 1. Solutions

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- a) Show that  $(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^2 + x^3y^3$  defines a scalar product in  $\mathbf{R}^3$ .
- b) Show that  $(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^2$  does not define a scalar product in  $\mathbf{R}^3$ .
- c) Show that  $(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^2 - x^3y^3$  does not define a scalar product in  $\mathbf{R}^3$ .
- d) Show that  $(\mathbf{x}, \mathbf{y}) = x^1y^1 + 3x^2y^2 + 5x^3y^3$  defines a scalar product in  $\mathbf{R}^3$ .
- e) Show that  $(\mathbf{x}, \mathbf{y}) = x^1y^2 + x^2y^1 + x^3y^3$  does not define a scalar product in  $\mathbf{R}^3$ .

f) Find necessary and sufficient conditions for entries  $a, b, c$  of symmetrical matrix  $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$  such that the formula

$$(\mathbf{x}, \mathbf{y}) = (x^1, x^2) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = ax^1y^1 + b(x^1y^2 + x^2y^1) + cx^2y^2$$

defines scalar product in  $\mathbf{R}^2$ .

Recall that scalar product on a vector space  $V$  is a function  $B(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathbf{y})$  on a pair of vectors which takes real values and satisfies the the following conditions:

- 1)  $B(\mathbf{x}, \mathbf{y}) = B(\mathbf{y}, \mathbf{x})$  (symmetricity condition)
- 2)  $B(\lambda\mathbf{x} + \mu\mathbf{y}, \mathbf{z}) = \lambda B(\mathbf{x}, \mathbf{z}) + \mu B(\mathbf{y}, \mathbf{z})$  (linearity condition (with respect to the first argument))
- 3)  $B(\mathbf{x}, \mathbf{x}) \geq 0$ ,  $B(\mathbf{x}, \mathbf{x}) = 0 \Leftrightarrow \mathbf{x} = 0$  (positive-definiteness condition)

(The linearity condition with respect to the second argument follows from the conditions 2) and 1))

a) Check all these conditions for  $(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^2 + x^3y^3$ :

1)  $(\mathbf{y}, \mathbf{x}) = y^1x^1 + y^2x^2 + y^3x^3 = x^1y^1 + x^2y^2 + x^3y^3 = (\mathbf{x}, \mathbf{y})$ . Hence it is symmetrical.

2)  $(\lambda\mathbf{x} + \mu\mathbf{y}, \mathbf{z}) = (\lambda x^1 + \mu y^1)z^1 + (\lambda x^2 + \mu y^2)z^2 + (\lambda x^3 + \mu y^3)z^3 =$   
 $= \lambda(x^1z^1 + x^2z^2 + x^3z^3) + \mu(y^1z^1 + y^2z^2 + y^3z^3) = \lambda(\mathbf{x}, \mathbf{z}) + \mu(\mathbf{y}, \mathbf{z})$ . Hence it is linear.

3)  $(\mathbf{x}, \mathbf{x}) = (x^1)^2 + (x^2)^2 + (x^3)^2 \geq 0$ . It is non-negative. If  $\mathbf{x} = 0$  then  $(\mathbf{x}, \mathbf{x}) = 0$ . If  $(\mathbf{x}, \mathbf{x}) = (x^1)^2 + (x^2)^2 + (x^3)^2 = 0$ , then  $x^1 = x^2 = x^3 = 0$ , i.e.  $\mathbf{x} = 0$ . This we proved positive-definiteness.

All conditions are checked. Hence  $B(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^2 + x^3y^3$  is indeed a scalar product in  $\mathbf{R}^3$

**Remark** Note that  $x^1, x^2, x^3$ —are components of the vector, do not be confused with exponents!

b) Show that  $B(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^2$  does not define scalar product in  $\mathbf{R}^3$ .

To see that the formula  $(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^2$  does not define scalar product check the condition 3) of positive-definiteness:  $(\mathbf{x}, \mathbf{x}) = (x^1)^2 + (x^2)^2$  may take zero values for  $\mathbf{x} \neq 0$ . E.g. if  $\mathbf{x} = (0, 0, -1)$   $(\mathbf{x}, \mathbf{x}) = 0$ , in spite of the fact that  $\mathbf{x} \neq 0$ . The condition 3) of positive-definiteness is not satisfied. Hence it is not scalar product.

c) Show that  $B(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^2 - x^3y^3$  does not define scalar product in  $\mathbf{R}^3$ .

To see that the formula  $(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^2 - x^3y^3$  does not define scalar product check the condition 3):  $(\mathbf{x}, \mathbf{x}) = (x^1)^2 + (x^2)^2 - (x^3)^2$  may take negative values. E.g. if  $\mathbf{x} = (0, 0, -1)$   $(\mathbf{x}, \mathbf{x}) = -1 < 0$ . The condition 3) of positive-definiteness is not satisfied. Hence it is not scalar product.

d) Now show that  $(\mathbf{x}, \mathbf{y}) = x^1y^1 + 3x^2y^2 + 5x^3y^3$  is a scalar product in  $\mathbf{R}^3$ .

We need to check all the conditions above for scalar product for  $(\mathbf{x}, \mathbf{y}) = x^1y^1 + 3x^2y^2 + 5x^3y^3$ :

1)  $(\mathbf{y}, \mathbf{x}) = y^1x^1 + 3y^2x^2 + 5y^3x^3 = x^1y^1 + 3x^2y^2 + 5x^3y^3 = (\mathbf{x}, \mathbf{y})$ . Hence it is symmetrical.

2)  $(\lambda\mathbf{x} + \mu\mathbf{y}, \mathbf{z}) = (\lambda x^1 + \mu y^1)z^1 + 3(\lambda x^2 + \mu y^2)z^2 + 5(\lambda x^3 + \mu y^3)z^3 =$   
 $= \lambda(x^1z^1 + 3x^2z^2 + 5x^3z^3) + \mu(y^1z^1 + 3y^2z^2 + 5y^3z^3) = \lambda(\mathbf{x}, \mathbf{z}) + \mu(\mathbf{y}, \mathbf{z})$ . Hence it is linear.

3)  $(\mathbf{x}, \mathbf{x}) = (x^1)^2 + 3(x^2)^2 + 5(x^3)^2 \geq 0$ . It is non-negative. If  $\mathbf{x} = 0$  then obviously  $(\mathbf{x}, \mathbf{x}) = 0$ . If  $(\mathbf{x}, \mathbf{x}) = (x^1)^2 + 3(x^2)^2 + 5(x^3)^2 = 0$ , then  $x^1 = x^2 = x^3 = 0$ . Hence it is positive-definite.

All conditions are checked. Hence  $(\mathbf{x}, \mathbf{y}) = x^1y^1 + 3x^2y^2 + 5x^3y^3$  is indeed a scalar product in  $\mathbf{R}^3$

e) Show that  $B(\mathbf{x}, \mathbf{y}) = x^1y^2 + x^2y^1 + x^3y^3$  does not define scalar product in  $\mathbf{R}^3$ .

To see that the formula  $(\mathbf{x}, \mathbf{y}) = x^1y^2 + x^2y^1 + x^3y^3$  does not define scalar product check the condition 3):  $(\mathbf{x}, \mathbf{x}) = 2x^1x^2 + (x^3)^2$  may take negative values. E.g. if  $\mathbf{x} = (1, -1, 0)$   $(\mathbf{x}, \mathbf{x}) = -2 < 0$ . The condition 3) of positive-definiteness is not satisfied. Hence it is not scalar product.

f) <sup>†</sup>)

The condition of linearity and symmetricity for the bilinear form

$$B(\mathbf{x}, \mathbf{y}) = (x^1, x^2) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = ax^1y^1 + b(x^1y^2 + x^2y^1) + cx^2y^2$$

are evidently obeyed.

The general answer on this question is: symmetric matrix is positive-definite if and only if all principal minors are positive. For matrix under consideration it means that conditions  $a > 0$  and  $ac - b^2 > 0$  are necessary and sufficient conditions.

Give a proof for this special case.

Check the positive-definiteness condition.

For  $\mathbf{x} = (1, 0)$   $B(\mathbf{x}, \mathbf{x}) = a$ . Hence  $a > 0$  is necessary condition. Now consider

$$B(\mathbf{x}, \mathbf{x}) = a(x^1)^2 + 2bx^1x^2 + c(x^2)^2 = \frac{(ax^1 + bx^2)^2 + (ac - b^2)(x^2)^2}{a} \geq 0 \Leftrightarrow ac - b^2 \geq 0$$

We see that  $B(\mathbf{x}, \mathbf{x}) > 0$  for all  $\mathbf{x} \neq 0$  iff  $a > 0$  and  $(ac - b^2) > 0$ .

The positive-definiteness condition is in fact the condition that discriminant of quadratic polynomial  $at^2 + 2bt + c$  is non-positive, i.e. it does not take negative values (if  $a > 0$ ).

**2 a)** Let  $\mathbf{e}, \mathbf{f}$  and  $\mathbf{g}$  be three vectors in 3-dimensional Euclidean space  $\mathbf{E}^3$  such that all these vectors have unit length and they are pairwise orthogonal. Show explicitly that the ordered set of these vectors  $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$  is a basis.

b) Let  $\mathbf{a}, \mathbf{b}$  and  $\mathbf{c}$  be three vectors in 3-dimensional Euclidean space  $\mathbf{E}^3$  such that vectors  $\mathbf{a}$  and  $\mathbf{b}$  have unit length, and are orthogonal to each other and vector  $\mathbf{c}$  has length  $\sqrt{3}$  and it forms an angle  $\varphi = \arccos \frac{1}{\sqrt{3}}$  with vectors  $\mathbf{a}$  and  $\mathbf{b}$ .

Show that an ordered set  $\{\mathbf{a}, \mathbf{b}, \mathbf{c} - \mathbf{a} - \mathbf{b}\}$  of vectors is an orthonormal basis in  $\mathbf{E}^3$ .

a) The space is 3-dimensional. Hence to show that  $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$  is a basis it suffices to show that vectors  $(\mathbf{e}, \mathbf{f}, \mathbf{g})$  are linearly independent. Suppose  $c_1\mathbf{e} + c_2\mathbf{f} + c_3\mathbf{g} = 0$ . Take scalar product of this equation on the vector  $\mathbf{e}$ . Since vectors  $\mathbf{e}, \mathbf{f}$  and  $\mathbf{g}$  have unit length and they are pairwise orthogonal then

$$(c_1\mathbf{e} + c_2\mathbf{f} + c_3\mathbf{g}, \mathbf{e}) = c_1(\mathbf{e}, \mathbf{e}) + c_2(\mathbf{f}, \mathbf{e}) + c_3(\mathbf{g}, \mathbf{e}) = c_1 \cdot 1 + c_2 \cdot 0 + c_3 \cdot 0 = c_1 = 0.$$

In the same way we prove that  $c_2 = c_3 = 0$ . Hence vectors  $(\mathbf{e}, \mathbf{f}, \mathbf{g})$  are linearly independent.

b) Since vectors  $\mathbf{a}$  and  $\mathbf{b}$  have unit length and they are orthogonal to each other then  $(\mathbf{a}, \mathbf{a}) = (\mathbf{b}, \mathbf{b}) = 1$  and  $(\mathbf{a}, \mathbf{b}) = 0$ . Since angle  $\varphi$  between vectors  $\mathbf{a}$  and  $\mathbf{c}$  equals to  $\arccos \frac{1}{\sqrt{3}}$  and length of vector  $\mathbf{c}$  equals to  $\sqrt{3}$  then

$$(\mathbf{a}, \mathbf{c}) = |\mathbf{a}||\mathbf{c}| \cos \varphi = 1 \cdot \sqrt{3} \cdot \frac{1}{\sqrt{3}} = 1.$$

Analogously  $(\mathbf{b}, \mathbf{c}) = 1$  too. Hence scalar product of vector  $\mathbf{c} - \mathbf{a} - \mathbf{b}$  with vector  $\mathbf{a}$  equals to  $(\mathbf{c} - \mathbf{a} - \mathbf{b}, \mathbf{a}) = 1 - 1 - 0 = 0$ , i.e. vector  $\mathbf{c} - \mathbf{a} - \mathbf{b}$  is orthogonal to the vector  $\mathbf{a}$ . In the same way we prove that vector  $\mathbf{c} - \mathbf{a} - \mathbf{b}$  is orthogonal to the vector  $\mathbf{b}$ . Hence we proved that all vectors  $\mathbf{a}, \mathbf{b}$  and  $\mathbf{c} - \mathbf{a} - \mathbf{b}$  are pairwise

orthogonal to each other. To see that  $\{\mathbf{a}, \mathbf{b}, \mathbf{c} - \mathbf{a} - \mathbf{b}\}$  is orthonormal basis it remains to prove that vector  $\mathbf{c} - \mathbf{a} - \mathbf{b}$  is unit vector. This is the fact since

$$(\mathbf{c} - \mathbf{a} - \mathbf{b}, \mathbf{c} - \mathbf{a} - \mathbf{b}) = (\mathbf{c}, \mathbf{c}) + (\mathbf{a}, \mathbf{a}) + (\mathbf{b}, \mathbf{b}) - 2(\mathbf{c}, \mathbf{a}) - 2(\mathbf{c}, \mathbf{b}) + 2(\mathbf{a}, \mathbf{b}) = \sqrt{3} \cdot \sqrt{3} + 1 + 1 - 2 \cdot 1 - 2 \cdot 1 = 0. \blacksquare$$

**3 a)** Show explicitly that matrix  $A_\varphi = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}$  is an orthogonal matrix.

b) Show explicitly that under the transformation  $(\mathbf{e}'_1, \mathbf{e}'_2) = (\mathbf{e}_1, \mathbf{e}_2) A_\varphi$  an orthonormal basis transforms to an orthonormal one.

c) Show that for orthogonal matrix  $A_\varphi$  the following relations are satisfied:

$$A_\varphi^{-1} = A_\varphi^T = A_{-\varphi}, \quad A_{\varphi+\theta} = A_\varphi \cdot A_\theta.$$

a) Check straightforwardly that  $A_\varphi^T \cdot A = I$  (this is definition of orthogonal matrix):

$$A_\varphi^T \cdot A = \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix} \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} = \begin{pmatrix} \cos^2 \varphi + \sin^2 \varphi & -\cos \varphi \sin \varphi + \sin \varphi \cos \varphi \\ -\sin \varphi \cos \varphi + \cos \varphi \sin \varphi & \sin^2 \varphi + \cos^2 \varphi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \blacksquare$$

b) We have to check that scalar products  $(\mathbf{e}'_1, \mathbf{e}'_1) = (\mathbf{e}'_2, \mathbf{e}'_2) = 1$  and  $(\mathbf{e}'_1, \mathbf{e}'_2) = 0$ . Calculate.

$$\begin{aligned} (\mathbf{e}'_1, \mathbf{e}'_1) &= (\cos \varphi \mathbf{e}_1 + \sin \varphi \mathbf{e}_2, \cos \varphi \mathbf{e}_1 + \sin \varphi \mathbf{e}_2) = \cos^2 \varphi (\mathbf{e}_1, \mathbf{e}_1) + 2 \cos \varphi \sin \varphi (\mathbf{e}_1, \mathbf{e}_2) + \sin^2 \varphi (\mathbf{e}_2, \mathbf{e}_2) = \\ &= \cos^2 \varphi \cdot 1 + 2 \cos \varphi \sin \varphi \cdot 0 + \sin^2 \varphi \cdot 1 = 1. \end{aligned}$$

$$(\mathbf{e}'_2, \mathbf{e}'_2) = (-\sin \varphi \mathbf{e}_1 + \cos \varphi \mathbf{e}_2, -\sin \varphi \mathbf{e}_1 + \cos \varphi \mathbf{e}_2) = \sin^2 \varphi (\mathbf{e}_1, \mathbf{e}_1) - 2 \cos \varphi \sin \varphi (\mathbf{e}_1, \mathbf{e}_2) + \cos^2 \varphi (\mathbf{e}_2, \mathbf{e}_2) = 1,$$

and

$$(\mathbf{e}'_1, \mathbf{e}'_2) = (\cos \varphi \mathbf{e}_1 + \sin \varphi \mathbf{e}_2, -\sin \varphi \mathbf{e}_1 + \cos \varphi \mathbf{e}_2) = -\cos \varphi \sin \varphi (\mathbf{e}_1, \mathbf{e}_1) + (\cos^2 \varphi - \sin^2 \varphi) (\mathbf{e}_1, \mathbf{e}_2) + \sin \varphi \cos \varphi (\mathbf{e}_2, \mathbf{e}_2) = 0. \blacksquare$$

c) We have that  $A_\varphi = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}$ . Then calculate inverse matrix  $A_\varphi^{-1}$ . One can see that  $A_\varphi^T = A_\varphi^{-1} = \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix}$ , because  $A_\varphi^T A_\varphi = I$  (see equation (1) above). On the other hand  $\cos \varphi = \cos(-\varphi)$  and  $\sin \varphi = -\sin(-\varphi)$ . Hence

$$A_\varphi^T = A_\varphi^{-1} = \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix} = \begin{pmatrix} \cos(-\varphi) & -\sin(-\varphi) \\ \sin(-\varphi) & \cos(-\varphi) \end{pmatrix} = A_{-\varphi}.$$

Now prove that  $A_{\varphi+\theta} = A_\varphi \cdot A_\theta$ :

$$\begin{aligned} A_\varphi \cdot A_\theta &= \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} (\cos \varphi \cos \theta - \sin \varphi \sin \theta) & -(\cos \varphi \sin \theta + \sin \varphi \cos \theta) \\ (\cos \varphi \sin \theta + \sin \varphi \cos \theta) & (\cos \varphi \cos \theta - \sin \varphi \sin \theta) \end{pmatrix} = \\ &= \begin{pmatrix} \cos(\varphi + \theta) & -\sin(\varphi + \theta) \\ \sin(\varphi + \theta) & \cos(\varphi + \theta) \end{pmatrix} = A_{\varphi+\theta} \end{aligned}$$

**Remark** Geometrical meaning of this relation is that composition of “rotations” on angle  $\varphi$  and  $\theta$  is “rotation” on angle  $\varphi + \theta$ .

**4** Let  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  be an orthonormal basis of Euclidean space  $\mathbf{E}^3$ . Consider the ordered set of vectors  $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$  which is expressed via basis  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  as in the exercise 8 of the Homework 1.

- a)  $\mathbf{e}'_1 = \mathbf{e}_2, \mathbf{e}'_2 = \mathbf{e}_1, \mathbf{e}'_3 = \mathbf{e}_3$ ;  
b)  $\mathbf{e}'_1 = \mathbf{e}_1, \mathbf{e}'_2 = \mathbf{e}_1 + 3\mathbf{e}_3, \mathbf{e}'_3 = \mathbf{e}_3$ ;  
c)  $\mathbf{e}'_1 = \mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}'_2 = 3\mathbf{e}_1 - 3\mathbf{e}_2, \mathbf{e}'_3 = \mathbf{e}_3$ ;  
d)  $\mathbf{e}'_1 = \mathbf{e}_2, \mathbf{e}'_2 = \mathbf{e}_1, \mathbf{e}'_3 = \mathbf{e}_1 + \mathbf{e}_2 + \lambda\mathbf{e}_3$  (where  $\lambda$  is an arbitrary coefficient)?

Write down explicitly transition matrix from the basis  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  to the ordered set of the vectors  $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ . What is the rank of this matrix? Is this matrix orthogonal?

Find out is the ordered set of vectors  $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$  a basis in  $\mathbf{E}^3$ . Is this basis an orthonormal basis of  $\mathbf{E}^3$ ? (you have to consider all cases a), b) c) and d)).

Case a) The ordered set  $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\} = \{\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_3\}$  is evidently orthonormal basis. Transition matrix  $T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ,  $(\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3) = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)T$ . This matrix is non-degenerate, its rank is equal to 3 ( $\det T = 1 \neq 0$ ). It is orthogonal because both bases are orthonormal.

Case b) The ordered set  $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\} = \{\mathbf{e}_1, \mathbf{e}_1 + 3\mathbf{e}_3, \mathbf{e}_3\}$  is not a basis because vectors are linear dependent:  $\mathbf{e}'_1 - \mathbf{e}'_2 + 3\mathbf{e}'_3 = 0$ . Transition matrix  $T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 1 \end{pmatrix}$ ,  $(\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3) = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)T$ . This matrix is degenerate, its rank  $\leq 2$ . One can see it noting that rows are linear dependent or noting that  $\det T = 0$ . Vectors  $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$  are linear dependent. On the other hand vectors  $\{\mathbf{e}'_1, \mathbf{e}'_2\}$  are linear independent. Hence rank of the matrix  $T$  is equal to 2. This matrix is not orthogonal.

Case c) The ordered set  $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\} = \{\mathbf{e}_1 - \mathbf{e}_2, 3\mathbf{e}_1 - 3\mathbf{e}_2, \mathbf{e}_3\}$  is not a basis because vectors are linear dependent:  $3\mathbf{e}'_1 - \mathbf{e}'_2 = 0$ .

One can see it also studying the transition matrix. Transition matrix  $T = \begin{pmatrix} 1 & 3 & 0 \\ -1 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ,  $(\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3) = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)T$ . This matrix is degenerate,  $\det T = 0$ . (Its rank  $\leq 2$ . On the other hand second and third row of this matrix are linear independent. Hence rank of the matrix  $T$  is equal to 2). This matrix is not orthogonal.

Case d)

The transition matrix from the basis  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  to the ordered triple  $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\} = \{\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2 + \lambda\mathbf{e}_3\}$  is  $T = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & \lambda \end{pmatrix}$ ,  $(\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3) = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)T$

I-st case.  $\lambda \neq 0$ . The ordered set  $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$  is a basis because vectors are linear independent (see the exercise 3), This basis is not orthogonal, because the length of vector  $\mathbf{e}'_3$  is not equal to 1 ( $(\mathbf{e}'_3, \mathbf{e}'_3) = |\mathbf{e}'_3|^2 = 2 + \lambda^2$ ). This matrix is not orthogonal, because the new basis is not orthonormal.

II-nd case  $\lambda = 0$ . The ordered set  $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$  is not a basis because vectors are linear dependent:  $\mathbf{e}'_1 + \mathbf{e}'_2 - \mathbf{e}'_3 = 0$ . The transition matrix  $T$  has rank less or equal to 2, because vectors are linear dependent. On the other hand vectors  $\mathbf{e}'_1, \mathbf{e}'_2$  are linear independent. Hence the rank of the matrix is equal to 2.

5<sup>†</sup> Prove the Cauchy–Bunyakovsky–Schwarz inequality

$$(\mathbf{x}, \mathbf{y})^2 \leq (\mathbf{x}, \mathbf{x})(\mathbf{y}, \mathbf{y}),$$

where  $\mathbf{x}, \mathbf{y}$  are arbitrary two vectors and  $(\ , \ )$  is a scalar product in Euclidean space.

Hint: For any two given vectors  $\mathbf{x}, \mathbf{y}$  consider the quadratic polynomial  $At^2 + 2Bt + C$  where  $A = (\mathbf{x}, \mathbf{x})$ ,  $B = (\mathbf{x}, \mathbf{y})$ ,  $C = (\mathbf{y}, \mathbf{y})$ . Show that this polynomial has at most one real root and consider its discriminant.

Consider quadratic polynomial  $P(t) = \sum_{i=1}^n (tx^i + y^i)^2 = At^2 + 2Bt + C$ , where  $A = \sum_{i=1}^n (x^i)^2 = (\mathbf{x}, \mathbf{x})$ ,  $B = \sum_{i=1}^n (x^i y^i) = (\mathbf{x}, \mathbf{y})$ ,  $C = \sum_{i=1}^n (y^i)^2 = (\mathbf{y}, \mathbf{y})$ . We see that equation  $P(t) = 0$  has at most one root (and this is the case if only vector  $\mathbf{x}$  is collinear to the vector  $\mathbf{y}$ ). This means that discriminant of this equation is less or equal to zero. But discriminant of this equation is equal to  $4B^2 - 4AC$ . Hence  $B^2 \leq AC$ . It is just CBS inequality.  $((\mathbf{x}, \mathbf{y})^2 = (\mathbf{x}, \mathbf{x})(\mathbf{y}, \mathbf{y}))$ , i.e. discriminant is equal to zero  $\Leftrightarrow$  vectors  $\mathbf{x}, \mathbf{y}$  are colinear.