Proof of Levi-Civita Theorem.

Let connection ∇ be compatible with metric:

$$\partial_X \langle \vec{Y}, \vec{Z} \rangle = \langle \nabla_{\vec{X}} \vec{Y}, \vec{Z} \rangle + \langle \vec{Y}, \nabla_X Z \rangle$$

$$\vec{X} = \partial_i, \quad \vec{Y} = \partial_m, \quad \vec{Z} = \partial_n$$

$$\langle \vec{Y}, \vec{Z} \rangle = \langle \partial_m, \partial_n \rangle = g_{mn}, \quad \nabla_{\vec{X}} \vec{Y} = \Gamma^k_{im} \partial_k, \quad \nabla_{\vec{X}} \vec{Z} = \Gamma^k_{in} \partial_k$$

$$\partial_i g_{mn} = \langle \Gamma^k_{im} \partial_k, \partial_n \rangle + \langle \partial_m, \Gamma^r_{in} \partial_r \rangle$$

$$\partial_i g_{mn} = \underbrace{\Gamma^k_{im} g_{kn}}_{\Gamma_{imn}} + \Gamma^r_{in} g_{rm}$$

$$\partial_i g_{mn} = \Gamma_{imn} + \Gamma_{inm} \qquad \begin{pmatrix} \Gamma_{inm} = \Gamma_{nim} \\ \text{(symmetrical).} \end{pmatrix}$$

$$\underline{\Gamma_{imn}} = \partial_i g_{mn} - \Gamma_{inm} = \partial_i g_{mn} - \Gamma_{nim} =$$

$$= \partial_i g_{mn} - (\partial_n g_{im} - \Gamma_{nmi}) =$$

$$= \partial_i g_{mn} - \partial_n g_{im} + \Gamma_{mni} =$$

$$= \partial_i g_{mn} - \partial_n g_{im} + (\partial_m g_{ni} - \Gamma_{min}) =$$

$$= \partial_i g_{mn} - \partial_n g_{im} + \partial_m g_{ni} - \underline{\Gamma_{imn}}$$

$$2\Gamma_{imn} = \partial_i g_{mn} + \partial_m g_{in} - \partial_n g_{im}$$

$$\Gamma_{imn} = \frac{1}{2}\left(\frac{\partial g_{mn}}{\partial x^i} + \frac{\partial g_{in}}{\partial x^m} - \frac{\partial g_{im}}{\partial x^n}\right)$$

$$\Gamma^n_{im} = \frac{1}{2} g^{nk} \left(\frac{\partial g_{km}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^m} - \frac{\partial g_{im}}{\partial x^k}\right)$$