Introduction to Geometry

it is a draft of lecture notes of H.M. Khudaverdian.
Manchester, 18 February 2019

Contents

1 Euclidean space **1**
1.1 Recollection of vector space and Euclidean vector space 1
1.1.1 Vector space. 1
1.1.2 Basic example of (n-dimensional) vector space—$\mathbf{R}^n$ . . 1
1.1.3 Linear dependence of vectors 2
1.1.4 Dimension of vector space. Basis in vector space. 3
1.1.5 Scalar product. Euclidean space 5
1.1.6 Orthonormal basis in Euclidean space 6
1.2 Affine spaces and vector spaces 7
1.2.1 Eucldiean affine space. 9
1.3 Transition matrices. Orthogonal bases and orthogonal matrices 10
1.3.1 Bases and transition matrices 10
1.4 Orthonormal bases and orthogonal matirces 12
1.5 Linear operators. 12
1.5.1 Matrix of linear operator in a given basis 12
1.5.2 Determinant and Trace of linear operator 14
1.5.3 Orthogonal linear operators 15
1.6 Orientation in vector space 16
1.6.1 Orientation in vector space. Oriented vector space . . . 16
1.6.2 Orientation of linear operator 24
1.7 Rotations and orthogonal operators preserving orientation of $\mathbf{E}^n$ (n=2,3) 24
1.7.1 Orthogonal operators in $\mathbf{E}^2$— Rotations and reflections 25

1 Euclidean space

1.1 Recollection of vector space and Euclidean vector space

We recall here important notions from linear algebra of vector space and Euclidean vector space.

1.1.1 Vector space.

Vector space V on real numbers is a set of vectors with operations " $+$ "—addition of vector and " $\cdot$ "—multiplication of vector Lon real number (sometimes called coefficients, scalars).

[1]

Remark We denote by 0 real number 0 and *vector* $\mathbf{0}$. Sometimes we have to be careful to distinguish between zero vector $\mathbf{0}$ and number zero.

1.1.2 Basic example of (n-dimensional) vector space—$\mathbf{R}^n$

A basic example of vector space (over real numbers) is a space of ordered n-tuples of real numbers.
$\mathbf{R}^2$ is a space of pairs of real numbers. $\mathbf{R}^2 = \{(x,y),\ x,y \in \mathbf{R}\}$

[1] These operations obey the following axioms

- $\forall \mathbf{a}, \mathbf{b} \in V, \mathbf{a} + \mathbf{b} \in V$,
- $\forall \lambda \in \mathbf{R}, \forall \mathbf{a} \in V, \lambda \mathbf{a} \in V$.
- $\forall \mathbf{a}, \mathbf{b} \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ (commutativity)
- $\forall \mathbf{a}, \mathbf{b}, \mathbf{c},\ \mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$ (associativity)
- $\exists\ \mathbf{0}$ such that $\forall \mathbf{a}$, $\mathbf{a} + \mathbf{0} = \mathbf{a}$
- $\forall \mathbf{a}$ there exists a vector $-\mathbf{a}$ such that $\mathbf{a} + (-\mathbf{a}) = 0$.
- $\forall \lambda \in \mathbf{R}, \lambda(\mathbf{a} + \mathbf{b}) = \lambda \mathbf{a} + \lambda \mathbf{b}$
- $\forall \lambda, \mu \in \mathbf{R} (\lambda + \mu)\mathbf{a} = \lambda \mathbf{a} + \mu \mathbf{a}$
- $(\lambda \mu)\mathbf{a} = \lambda(\mu \mathbf{a})$
- $1\mathbf{a} = \mathbf{a}$

$\mathbf{R}^3$ is a space of triples of real numbers. $\mathbf{R}^3 = \{(x, y, z),\ x, y, z \in \mathbf{R}\}$
$\mathbf{R}^4$ is a space of quadruples of real numbers. $\mathbf{R}^4 = \{(x, y, z, t),\ x, y, z, t, \in \mathbf{R}\}$
and so on...

$\mathbf{R}^n$—is a space of n-typles of real numbers:

$$\mathbf{R}^n = \{(x^1, x^2, \dots, x^n),\ x^1, \dots, , x^n \in \mathbf{R}\} \tag{1.1}$$

If $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ are two vectors, $\mathbf{x} = (x^1, \dots, x^n)$, $\mathbf{y} = (y^1, \dots, y^n)$ then

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \dots, x_n + y_n)\,.$$

and multiplication on scalars is defined as

$$\lambda \mathbf{x} = \lambda \cdot (x^1, \dots, x^n) = (\lambda x^1, \dots, \lambda x^n)\,, \quad (\lambda \in \mathbf{R})\,.$$

1.1.3 Linear dependence of vectors

We often consider linear combinations in vector space:

$$\sum_i \lambda_i \mathbf{x}_i = \lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \cdots + \lambda_m \mathbf{x}_m\,, \tag{1.2}$$

where $\lambda_1, \lambda_2, \dots, \lambda_m$ are coefficients (real numbers), $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m$ are vectors from vector space V. We say that linear combination (1.2) is *trivial* if all coefficients $\lambda_1, \lambda_2, \dots, \lambda_m$ are equal to zero.

$$\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0\,.$$

We say that linear combination (1.2) is *not trivial* if at least one of coefficients $\lambda_1, \lambda_2, \dots, \lambda_m$ is not equal to zero:

$$\lambda_1 \neq 0, \text{or} \lambda_2 \neq 0, \text{or} \dots \text{or} \lambda_m \neq 0\,.$$

Recall definition of linearly dependent and linearly independent vectors:

Definition The vectors $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ in vector space V are *linearly dependent* if there exists a non-trivial linear combination of these vectors such that it is equal to zero.

In other words we say that the vectors $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ in vector space V are *linearly dependent* if there exist coefficients $\mu_1, \mu_2, \dots, \mu_m$ such that at least one of these coefficients is not equal to zero and

$$\mu_1 \mathbf{x}_1 + \mu_2 \mathbf{x}_2 + \cdots + \mu_m \mathbf{x}_m = 0\,. \tag{1.3}$$

Respectively vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ are *linearly independent* if they are not linearly dependent. This means that an arbitrary linear combination of these vectors which is equal zero is trivial.

In other words vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_m\}$ are *linearly independent* if the condition

$$\mu_1 \mathbf{x}_1 + \mu_2 \mathbf{x}_2 + \cdots + \mu_m \mathbf{x}_m = 0$$

implies that $\mu_1 = \mu_2 = \cdots = \mu_m = 0$.

Very useful and workable

Proposition *Vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m\}$ in vector space V are linearly dependent if and only if at least one of these vectors is expressed via linear combination of other vectors:*

$$\mathbf{x}_i = \sum_{j \neq i} \lambda_j \mathbf{x}_j \,.$$

1.1.4 Dimension of vector space. Basis in vector space.

Definition Vector space V has a dimension n if there exist n linearly independent vectors in this vector space, and any $n+1$ vectors in V are linearly dependent.

In the case if in the vector space V for an arbitrary N there exist N linearly independent vectors then the space V is *infinite-dimensional*. An example of infinite-dimensional vector space is a space V of all polynomials of an arbitrary order. One can see that for an arbitrary N polynomials $\{1, x, x^2, x^3, \ldots, x^N\}$ are linearly idependent. (Try to prove it!). This implies V is infinite-dimensional vector space.

Basis

Definition Let V be n-dimensional vector space. The ordered set $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ of n linearly independent vectors in V is called a basis of the vector space V.

Remark We say 'a basis', not 'the basis' since there are many bases in the vector space (see also Homeworks 1.2).

Remark Focus your attention: basis is *an ordered* set of vectors, not just a set of vectors[2].

The following Proposition is very useful:

[2]See later on orientation of vector spaces, where the ordering of vectors of basis will be highly important.

Proposition *Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be an arbitrary basis in n-dimensional vector space V. Then any vector $\mathbf{x} \in V$ can be expressed as a linear combination of vectors $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ in a unique way, i.e. for every vector $\mathbf{x} \in V$ there exists an ordered set of coefficients $\{x^1, \dots, a^n\}$ such that*

$$\mathbf{x} = x^1\mathbf{e}_1 + \dots + x^n\mathbf{e}_n \tag{1.4}$$

and if

$$\mathbf{x} = a^1\mathbf{e}_1 + \dots + a^n\mathbf{e}_n = b^1\mathbf{e}_1 + \dots + b^n\mathbf{e}_n\,, \tag{1.5}$$

then $a^1 = b^1, a^2 = b^2, \dots, a^n = b^n$. In other words for any vector $\mathbf{x} \in V$ there exists an ordered n-tuple $(x^1, \dots, x^n)$ of coefficients such that $\mathbf{x} = \sum_{i=1}^n x^i\mathbf{e}_i$ and this n-tuple is unique.

In other words:

Basis is a set of linearly independent vectors in vector space V which span (generate) vector space V.

Recall that we say that vector space V is *spanned* by vectors $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ (or vectors vectors $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ *span* vector space V) if any vector $\mathbf{a} \in V$ can be expresses as a linear combination of vectors $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$.

Definition Coefficients $\{a^1, \dots, a^n\}$ are called *components of the vector* $\mathbf{x}$ *in the basis* $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ or just shortly *components of the vector* $\mathbf{x}$.

Example *Canonical basis in* $\mathbf{R}^n$

We considered above the basic example of vector space—a space of ordered n-tuples of real numbers: $\mathbf{R}^n = \{(x^1, x^2, \dots, x^n), x^i \in \mathbf{R}\}$ (see (1.1)). One can see that it is n-dimensional vector space. Consider vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n \in \mathbf{R}^n$:

$$\begin{array}{rl} \mathbf{e}_1 = & (1, 0, 0 \dots, 0, 0) \\ \mathbf{e}_2 = & (0, 1, 0 \dots, 0, 0) \\ \dots & \quad\dots \\ \mathbf{e}_n = & (0, 0, 0 \dots, 0, 1) \end{array} \tag{1.6}$$

Then for an arbitrary vector $\mathbf{R}^n \ni \mathbf{a} = (a^1, a^2, a^3, \dots, a^n)$,

$$\mathbf{a} = a^1(1, 0, 0 \dots, 0, 0) + a^2(0, 1, 0 \dots, 0, 0) + a^3(0, 0, 1, 0 \dots, 0, 0) + \dots + a^n(0, 1, 0 \dots, 0, 1) =$$

$$\sum_{i=1}^m a^i\mathbf{e}_i = a^i\mathbf{e}_i \qquad \text{(we will use sometimes condensed notations } \mathbf{x} = x^i\mathbf{e}_i)$$

For every vector $\mathbf{a} \in \mathbf{R}^n$ we have unique expansion via the vectors (1.6). The set of vectors $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a basis of $\mathbf{R}^n$. The basis (1.6) is the distinguished basis. Sometimes it is called *canonical basis in* $\mathbf{R}^n$. One can find another basis in $\mathbf{R}^n$–just take an arbitrary ordered set of n linearly independent vectors. (See exercises in Homework 0).

1.1.5 Scalar product. Euclidean space

In vector space one have additional structure: *scalar product of vectors.*

Definition Scalar product in a vector space V is a function $B(\mathbf{x}, \mathbf{y})$ on a pair of vectors which takes real values and satisfies the the following conditions:

$$\begin{gathered} B(\mathbf{x}, \mathbf{y}) = B(\mathbf{y}, \mathbf{x}) \quad \text{(symmetricity condition)} \\ B(\lambda\mathbf{x} + \mu\mathbf{x}', \mathbf{y}) = \lambda B(\mathbf{x}, \mathbf{y}) + \mu B(\mathbf{x}', \mathbf{y}) \quad \text{(linearity condition)} \\ B(\mathbf{x}, \mathbf{x}) \geq 0 \text{ , } B(\mathbf{x}, \mathbf{x}) = 0 \Leftrightarrow \mathbf{x} = 0 \text{ (positive-definiteness condition)} \end{gathered} \tag{1.7}$$

Definition Euclidean space is a vector space equipped with a scalar product.

One can easy to see that the function $B(\mathbf{x}, \mathbf{y})$ is bilinear function, i.e. it is linear function with respect to the second argument also. This follows from previous axioms:

$$B(\mathbf{x}, \lambda\mathbf{y}+\mu\mathbf{y}') \underbrace{=}_{\text{symm.}} B(\lambda\mathbf{y}+\mu\mathbf{y}', \mathbf{x}) \underbrace{=}_{\text{linear.}} \lambda B(\mathbf{y}, \mathbf{x})+\mu B(\mathbf{y}', \mathbf{x}) \underbrace{=}_{\text{symm.}} \lambda B(\mathbf{x}, \mathbf{y})+\mu B(\mathbf{x}, \mathbf{y}') \,.$$

A bilinear function $B(\mathbf{x}, \mathbf{y})$ on pair of vectors is called sometimes *bilinear form* on vector space. Bilinear form $B(\mathbf{x}, \mathbf{y})$ which satisfies the symmetricity condition is called *symmetric bilinear form.* Scalar product is nothing but symmetric bilinear form on vectors which is positive-definite: $B(\mathbf{x}, \mathbf{x}) \geq 0$) and is non-degenerate $((\mathbf{x}, \mathbf{x}) = 0 \Rightarrow \mathbf{x} = 0$.

Example We considered the vector space $\mathbf{R}^n$, the space of n-tuples (see the subsection 1.2). One can consider the vector space $\mathbf{R}^n$ as Euclidean space provided by the scalar product

$$B(\mathbf{x}, \mathbf{y}) = x^1y^1 + \cdots + x^ny^n \tag{1.8}$$

This scalar product sometimes is called *canonical scalar product.*

Exercise Check that it is indeed scalar product.

Example We consider in 2-dimensional vector space V with basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ and $B(\mathbf{X}, \mathbf{Y})$ such that $B(\mathbf{e}_1, \mathbf{e}_1) = 3$, $B(\mathbf{e}_2, \mathbf{e}_2) = 5$ and $B(\mathbf{e}_1, \mathbf{e}_2) = 0$. Then for every two vectors $\mathbf{X} = x^1\mathbf{e}_1 + x^2\mathbf{e}_2$ and $\mathbf{Y} = y^1\mathbf{e}_1 + y^2\mathbf{e}_2$ we have that

$$B(\mathbf{X}, \mathbf{Y}) = (\mathbf{X}, \mathbf{Y}) = \left(x^1\mathbf{e}_1 + x^2\mathbf{e}_2, y^1\mathbf{e}_1 + y^2\mathbf{e}_2\right) =$$

$$x^1y^1(\mathbf{e}_1, \mathbf{e}_1) + x^1y^2(\mathbf{e}_1, \mathbf{e}_2) + x^2y^1(\mathbf{e}_2, \mathbf{e}_1) + x^2y^2(\mathbf{e}_2, \mathbf{e}_2) = 3x^1y^1 + 5x^2y^2 \,.$$

One can see that all axioms are obeyed.

Remark Scalar product sometimes is called "inner" product or "dot" product. Later on we will use for scalar product $B(\mathbf{x}, \mathbf{y})$ just shorter notation $(\mathbf{x}, \mathbf{y})$ (or $\langle \mathbf{x}, \mathbf{y} \rangle$). Sometimes it is used for scalar product a notation $\mathbf{x} \cdot \mathbf{y}$. Usually this notation is reserved only for the canonical case (1.8).

Counterexample Consider again 2-dimensional vector space V with basis $\{\mathbf{e}_1, \mathbf{e}_2\}$.

Show that operation such that $(\mathbf{e}_1, \mathbf{e}_1) = (\mathbf{e}_2, \mathbf{e}_2) = 0$ and $(\mathbf{e}_1, \mathbf{e}_2) = 1$ does not define scalar product. *Solution.* For every two vectors $\mathbf{X} = x^1\mathbf{e}_1 + x^2\mathbf{e}_2$ and $\mathbf{Y} = y^1\mathbf{e}_1 + y^2\mathbf{e}_2$ we have that

$$(\mathbf{X}, \mathbf{Y}) = \left(x^1\mathbf{e}_1 + x^2\mathbf{e}_2, y^1\mathbf{e}_1 + y^2\mathbf{e}_2\right) = x^1y^2 + x^2y^1$$

hence for vector $\mathbf{X} = (1, -1)$ $(\mathbf{X}, \mathbf{X}) = -2 < 0$. Positive-definiteness is not fulfilled.

1.1.6 Orthonormal basis in Euclidean space

One can see that for scalar product (1.8) and for the basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ defined by the relation (1.6) the following relations hold:

$$(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \tag{1.9}$$

Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be an ordered set of n vectors in n-dimensional Euclidean space which obeys the conditions (1.9). One can see that this ordered set is a basis [3].

[3] Indeed prove that conditions (1.9) imply that these n vectors are linear independent. Suppose that $\lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2 + \cdots + \lambda_n\mathbf{e}_n = 0$. For an arbitrary i multiply the left and right hand sides of this relation on a vector $\mathbf{e}_i$. We come to condition $\lambda_i = 0$. Hence vectors $(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$ are linearly dependent.

Definition-Proposition *The ordered set of vectors* $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ *in* n*-dimensional Euclidean space which obey the conditions* (1.9) *is a basis. This basis is called an orthonormal basis.*

One can prove that every (finite-dimensional) Euclidean space possesses orthonormal basis.

Later by default we consider only orthonormal bases in Euclidean spaces. Respectively scalar product will be defined by the formula (1.8). Indeed let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ be an orthonormal basis in Euclidean space. Then for an arbitrary two vectors $\mathbf{x}, \mathbf{y}$, such that $\mathbf{x} = \sum x^i \mathbf{e}_i$, $\mathbf{y} = \sum y^j \mathbf{e}_j$ we have:

$$(\mathbf{x}, \mathbf{y}) = \left(\sum x^i \mathbf{e}_i, \sum y^j \mathbf{e}_j\right) = \sum_{i,j=1}^{n} x^i y^j (\mathbf{e}_i, \mathbf{e}_j) = \sum_{i,j=1}^{n} x^i y^j \delta_{ij} = \sum_{i=1}^{n} x^i y^i$$

We come to the canonical scalar product (1.8). Later on we usually will consider scalar product defined by the formula (1.8) i.e. scalar product in orthonormal basis.

Remark We consider here general definition of scalar product then came to conclusion that in a special basis, (*orthonormal basis*), this is nothing but usual 'dot' product (1.8).

1.2 Affine spaces and vector spaces

`AFFINE SPACE WITH ORIGIN IS A VECTOR SPACE`

Let V be an arbitrary vector space.

Consider a set A whose elements will be called 'points' We say that A is *an affine space associated with vector space* V if the following rule is defined: to every point $P \in A$ and an arbitrary vector $\mathbf{x} \in V$ a point Q is assigned:

$$\forall P \in A\,, \quad \forall \mathbf{x} \in V\,, \quad (P, \mathbf{x}) \mapsto Q \in A \tag{1.10}$$

We denote $Q = P + \mathbf{x}$.

The following properties must be satisfied:

- For arbitrary two vectors $\mathbf{x}, \mathbf{y} \in V$ and arbitrary point $P \in A$, $P + (\mathbf{x} + \mathbf{y}) = (P + \mathbf{x}) + \mathbf{y}$.

- For an arbitrary point $P \in A$, $P + \mathbf{0} = P$.

  (Recall that $\mathbf{0}$ is the zero vector in the vector space V.)

- For arbitrary two points $P, Q \in A$ there exists unique vector $\mathbf{y} \in V$ such that $P + \mathbf{y} = Q$.

If $P + \mathbf{x} = Q$ we often denote the vector $\mathbf{x} = Q - P = \vec{PQ}$. We say that vector $\mathbf{x} = \vec{PQ}$ starts at the point P and it ends at the point Q.

One can see that if vector $\mathbf{x} = \vec{PQ}$, then $\vec{QP} = -\mathbf{x}$; if P, Q, R are three arbitrary points then $\vec{PQ} + \vec{QR} = \vec{PR}$.

One can reconstruct vector space V in terms of an affine space A, and vice versa. Namely, let A be an affine space associated with vector space V. Choose an arbitrary point $O \in A$ as an the origin, and consider the vectors starting at the origin: We come to the vector space V:

$$V = \text{set of vectors } \vec{OQ} \text{ where } Q \text{ is an arbitrary point in } A\,,$$

which is associated with an affine space A.

Let V be an arbitrary vector space. We will define now an affine space associated with this vector space. Consider two copies of the vector space V. The elements of the *first* copy we will call "points", and the elements of the *second* copy we will call as usual "vectors":

$$\underbrace{\overbrace{\text{V}}^{\text{first copy of } V}}_{\text{elements of } V \text{ are points}} \qquad \underbrace{\overbrace{\text{V}}^{\text{second copy of } V}}_{\text{elements of } V \text{ are vectors}} \tag{1.11}$$

Let $A = \mathbf{a}$ be an arbitrary point of the affine space, (i.e. an element of the *first* copy of vector space V) and let $\mathbf{x}$ is an arbitrary vector of the vector space V (i.e. an element of the *second* copy of vector space V). We define the action (1.10) in the following way:

$$(A, \mathbf{x}) \mapsto B = A + \mathbf{x} = \mathbf{a} + \mathbf{x}\,, \quad \mathbf{x} = \vec{AB}\,.$$

The point B is the vector $\mathbf{a} + \mathbf{x} \in V$ belonging to the *first* copy of the vector space V.

We assign to two 'points' $A = \mathbf{a}, B = \mathbf{b}$ (elements of the *first* copy of vector space V) the vector $\mathbf{x} = \mathbf{b} - \mathbf{a}$ (elements of the *second* copy of vector space V).

For example vector space $\mathbf{R}^n$ of n-tuples of real numbers can be considered as a set of points. If we choose arbitrary two points $A = (a^1, a^2 \ldots, a^n)$ and

$B = (b^1, b^2, \dots, b_n)$, then these two points define a vector $\vec{AB}$ which is equal to $\vec{AB} = B - A = (b^1 - a^1, b^2 - a^2, \dots, b_n - a_n)$.

The associated with each other affine space and vector space $\mathbf{R}^n$ we will usually denote by the same letter.

1.2.1 Eucldiean affine space.

Respectively one can consider Euclidean vector space as a set of points. Let $\mathbf{E}^n$ be n-dimensional Euclidean vector space, i.e. vector space equipped with scalar product. Let $\{\mathbf{e}_i\}$ $(i = 1, \dots, n)$ be an arbitrary orthonormal basis in the vector space $\mathbf{E}^n$. Now consider this vector space as a set of points. Choose arbitrary two points (vectors of the *first* copy of the vector space $\mathbf{E}^n$), $A = a^1\mathbf{e}_1 + a^2\mathbf{e}_2 + \dots + a^n\mathbf{e}_n$ and $B = b^1\mathbf{e}_1 + b^2\mathbf{e}_2 + \dots + b^n\mathbf{e}_n$. These points define a vector $\vec{AB}$ (in the *second* copy of the vector space $\mathbf{E}^n$) which is equal to

$$\vec{AB} = B - A = (b^1 - a^1)\mathbf{e}_1 + (b^2 - a^2)\mathbf{e}_2 + \dots + (b^n - a^n)\mathbf{e}_n\,.$$

The distance between two points A, B is the length of corresponding vector $\vec{AB}$, and the length of the vector $\vec{AB}$ is defined by the scalar product:

$$\left|\vec{AB}\right| = \sqrt{\left(\vec{AB}, \vec{AB}\right)} = \sqrt{((b^1 - a^1)\,\mathbf{e}_1 + \dots + (b^n - a^n)\,\mathbf{e}_n, (b^1 - a^1)\mathbf{e}_1 + \dots + (b^n - a^n)\mathbf{e}_n)}$$

$$= \sqrt{(b^1 - a^1)^2 + \dots + (b^n - a^n)^2}\,.$$

We recall very important formula how scalar product is related with the angle between vectors: if φ is an angle between vectors $\mathbf{x}$ and $\mathbf{y}$ then

$$(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^2 + \dots + x^ny^n = |\mathbf{x}||\mathbf{y}|\cos\varphi \tag{1.12}$$

(We suppose that vectors $\mathbf{x}, \mathbf{y}$ are defined in orthonormal basis.)

In particulary it follows from this formula that

angle between vectors $\mathbf{x}, \mathbf{y}$ *is acute if scalar product* $(\mathbf{x}, \mathbf{y})$ *is positive*
angle between vectors $\mathbf{x}, \mathbf{y}$ *is obtuse if scalar product* $(\mathbf{x}, \mathbf{y})$ *is negative*
vectors $\mathbf{x}, \mathbf{y}$ *are perpendicular if scalar product* $(\mathbf{x}, \mathbf{y})$ *is equal to zero*
(1.13)

Remark The associated with each other affine space and Euclidean vector space $\mathbf{E}^n$ we will denote by the same letter.

Remark Geometrical intuition tells us that cosinus of the angle between two vectors has to be less or equal to one and it is equal to one if and only if vectors $\mathbf{x}, \mathbf{y}$ are collinear. Comparing with (1.12) we come to the inequality:

$$(\mathbf{x},\mathbf{y})^2 = \left(x^1y^1 + \cdots + x^ny^n\right)^2 \leq \left((x^1)^2 + \cdots + (x^n)^2\right)\left((y^1)^2 + (\cdots + (y^n)^2\right) = (\mathbf{x},\mathbf{x})(\mathbf{y},\mathbf{y})$$
$$\text{and}(\mathbf{x},\mathbf{y})^2 = (\mathbf{x},\mathbf{x})(\mathbf{y},\mathbf{y}) \quad \text{if vectors are colinear, i.e. } x^i = \lambda y^i \tag{1.14}$$

This is famous Cauchy–Buniakovsky–Schwarz inequality, one of most important inequalities in mathematics. (See for more details the last exercise in the Homework 0)

1.3 Transition matrices. Orthogonal bases and orthogonal matrices

1.3.1 Bases and transition matrices

One can consider different bases in vector space.

Let A be $n \times n$ matrix with real entries, $A = ||a_{ij}||$, $i, j = 1, 2, \ldots, n$:

$$A = \begin{pmatrix} a_{11} & a_{12}\ldots & a_{1n} \\ a_{21} & a_{22}\ldots & a_{2n} \\ a_{31} & a_{32}\ldots & a_{3n} \\ \ldots & \ldots\ldots & \ldots \\ a_{(n-1)\,1} & a_{(n-1)2}\ldots & a_{(n-1)n} \\ a_{n\,1} & a_{n2}\ldots & a_{nn} \end{pmatrix}$$

Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be an arbitrary basis in n-dimensional vector space V.

The basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ can be considered as row of vectors, or $1 \times n$ matrix with entries–vectors.

Multiplying $1 \times n$ matrix $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ on matrix A we come to new row of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\}$ such that

$$\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}A = \tag{1.15}$$

$$\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\} \begin{pmatrix} a_{11} & a_{12}\ldots & a_{1n} \\ a_{21} & a_{22}\ldots & a_{2n} \\ a_{31} & a_{32}\ldots & a_{3n} \\ \ldots & \ldots\ldots & \ldots \\ a_{(n-1)\,1} & a_{(n-1)2}\ldots & a_{(n-1)n} \\ a_{n\,1} & a_{n2}\ldots & a_{nn} \end{pmatrix} \tag{1.16}$$

,

$$\begin{cases} \mathbf{e}'_1 = a_{11}\mathbf{e}_1 + a_{21}\mathbf{e}_2 + a_{31}\mathbf{e}_3 + \cdots + a_{(n-1)\,1}\mathbf{e}_{n-1} + a_{n\,1}\mathbf{e}_n \\ \mathbf{e}'_1 = a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2 + a_{32}\mathbf{e}_3 + \cdots + a_{(n-1)\,2}\mathbf{e}_{n-1} + a_{n\,2}\mathbf{e}_n \\ \mathbf{e}'_1 = a_{13}\mathbf{e}_1 + a_{23}\mathbf{e}_2 + a_{33}\mathbf{e}_3 + \cdots + a_{(n-1)\,3}\mathbf{e}_{n-1} + a_{n\,1}\mathbf{e}_n \\ \cdots = \ldots\cdots + \ldots\cdots + \ldots\cdots + \cdots + \ldots\ldots\ldots\ldots \\ \mathbf{e}'_n = a_{1n}\mathbf{e}_1 + a_{2n}\mathbf{e}_2 + a_{3n}\mathbf{e}_3 + \cdots + a_{(n-1)\,n}\mathbf{e}_{n-1} + a_{n\,n}\mathbf{e}_n \end{cases}$$

or shortly:

$$\mathbf{e}'_i = \sum_{k=1}^{n} \mathbf{e}_k a_{ki}\,. \tag{1.17}$$

Definition Matrix A which transforms a basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ to the row of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\}$ (see equation (1.17)) is *transition matrix* from the basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ to the row $\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\}$.

What is the condition that the row $\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\}$ is a basis too? The row, ordered set of vectors, $\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\}$ is a basis if and only if vectors $(\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n)$ are linearly independent. Thus we come to

Proposition 1 *Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be a basis in n-dimensional vector space V, and let A be an $n \times n$ matrix with real entries. Then*

$$\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}A \tag{1.18}$$

is a basis if and only if the transition matrix A has rank n, i.e. it is non-degenerate (invertible) matrix.

Recall that $n\times$ matrix A is nondegenerate (invertible) $\Leftrightarrow$ $\det A \neq 0$.

Remark Recall that the condition that $n \times n$ matrix A is non-degenerate (has rank n) is equivalent to the condition that it is invertible matrix, or to the condition that $\det A \neq 0$.

Example let $\{\mathbf{e}_1, \mathbf{e}_2.\mathbf{e}_3\}$ be a basis in $\mathbf{R}^3$. Consider set of vectors $\{\mathbf{e}_1, 3\mathbf{e}_1 + \lambda\mathbf{e}_2, 7\mathbf{e}_1 + 5\mathbf{e}_2 + 3\mathbf{e}_3\}$, where λ is an arboitrary parameter. The transtition matrix from the basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ to the row of vectors $\{\mathbf{e}_1, 3\mathbf{e}_1 + \lambda\mathbf{e}_2, 7\mathbf{e}_1 + 5\mathbf{e}_2 + 3\mathbf{e}_3\}$ is the following:

$$\{\mathbf{e}_1, 3\mathbf{e}_1 + \lambda\mathbf{e}_2, 7\mathbf{e}_1 + 5\mathbf{e}_2 + 3\mathbf{e}_3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}A = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \begin{pmatrix} 1 & 3 & 7 \\ 0 & \lambda & 5 \\ 0 & 0 & 3 \end{pmatrix}$$

We see that $\det A = 3\lambda$. In the case if $\mathbf{l} \neq 0$ then transition matrix is non-degenerate and the row $\{\mathbf{e}_1, 3\mathbf{e}_1 + \lambda\mathbf{e}_2, 7\mathbf{e}_1 + 5\mathbf{e}_2 + 3\mathbf{e}_3\}$ is a basis.

(See another examples in the Homework)

1.4 Orthonormal bases and orthogonal matirces

Now suppose that $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is orthonoromal basis in n-dimensional Euclidean vector space. What is the condition that the new basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}A$ is an orthonormal basis too?

Definition We say that $n \times n$ matrix is orthogonal matrix if its product on transposed matrix is equal to unity matrix:

$$A^T A = I\,. \tag{1.19}$$

Exercise. Prove that determinant of orthogonal matrix is equal to ± 1:

$$A^T A = I \Rightarrow \det A = \pm 1\,. \tag{1.20}$$

Solution $A^T A = I$. Hence $\det(A^T A) = \det A^T \det A = (\det A)^2 = \det I = 1$. Hence $\det A = \pm 1$. We see that in particular orthogonal matrix is non-degenerate ($\det A \neq 0$). Hence it is a transition matrix from one basis to another. The following Proposition is valid:

Proposition 2 *Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be an orthonormal basis in n-dimensional Euclidean vector space. Then the new basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}A$ is orthonormal basis if and only if the transition matrix A is orthogonal matrix.*

Proof The basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\}$ is orthonormal means that $(\mathbf{e}'_i, \mathbf{e}'_j) = \delta_{ij}$. We have:

$$\delta_{ij} = (\mathbf{e}'_i, \mathbf{e}'_j) = \left(\sum_{m=1}^n \mathbf{e}_m A_{mi}, \mathbf{e}'_j = \sum_{n=1}^n \mathbf{e}_n A_{nj}\right) = \sum_{m,n=1}^n A_{mi} A_{nj} (\mathbf{e}_m, \mathbf{e}_n) =$$

$$\sum_{m,n=1}^n A_{mi} A_{nj} \delta_{mn} = \sum_{m=1}^n A_{mi} A_{mj} = \sum_{m=1}^n A^T_{im} A_{mj} = (A^T A)_{ij} \Rightarrow (A^T A)_{ij} = \delta_{ij}, \text{i.e. } A^T A = I\,. \tag{1.21}$$

Remark The set of orthogonal matrices form the group which is called $O(n)$. This group is a subgroup of the group $GL(n, \mathbf{R})$ of linear invertible $n \times n$ matrices with real entries.

1.5 Linear operators.

1.5.1 Matrix of linear operator in a given basis

Recall here facts about linear operators in vector space

Let P be a linear operator in vector space V:

$$P\colon V \to V, \qquad P(\lambda\mathbf{x} + \mu\mathbf{y}) = \lambda P(\mathbf{x}) + \mu P(\mathbf{y}).$$

Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be an arbitrary basis in n-dimensional vector space V. Consider the action of operator P on basis vectors: $\mathbf{e}'_i = P(\mathbf{e}_i)$. We denote by $p_{1k}, p_{2k}, \dots, p_{nk}$ coordinates of vector $\mathbf{e}'_k$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$:

$$\mathbf{e}'_i = P(\mathbf{e}_i) = \sum \mathbf{e}_k p_{ki}\,,$$

$$\begin{array}{c} \mathbf{e}'_1 = P(\mathbf{e}_1) = \mathbf{e}_1 p_{11} + \mathbf{e}_2 p_{21} + \mathbf{e}_3 p_{31} + \dots + \mathbf{e}_n p_{n1} \\ \mathbf{e}'_2 = P(\mathbf{e}_2) = \mathbf{e}_1 p_{12} + \mathbf{e}_2 p_{22} + \mathbf{e}_3 p_{32} + \dots + \mathbf{e}_n p_{n2} \\ \mathbf{e}'_3 = P(\mathbf{e}_3) = \mathbf{e}_1 p_{13} + \mathbf{e}_2 p_{23} + \mathbf{e}_3 p_{31} + \dots + \mathbf{e}_n p_{n3} \\ \dots \\ \mathbf{e}'_n = P(\mathbf{e}_n) = \mathbf{e}_1 p_{1n} + \mathbf{e}_2 p_{2n} + \mathbf{e}_3 p_{3n} + \dots + \mathbf{e}_n p_{nn} \end{array} \tag{1.22}$$

Definition-Proposition *Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ be an arbitrary basis in n-dimensional vector space V, and let P be a linear operator in V. Then matrix $P = ||p_{ik}||$ in equation (1.22) is a matrix of linear transformation P in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$. This matrix coincides with the transition matrix from the basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ to the row of vectors $\{\mathbf{e}'_1, \mathbf{e}_2, \dots, \mathbf{e}'_n\}$.*

In the case if linear operator P is non-degenerate (invertible) then vectors $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3, \dots, \mathbf{e}'_n$, form also a basis.

`Does matrix of linear operator change if we change the basis?`

See it:

Consider a new basis $\{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ in the linear space V. Let A be transition matrix from the basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ to the new basis $\{\mathbf{f}_1, \dots, \mathbf{f}_n\}$:

$$\{\mathbf{f}_1, \dots, \mathbf{f}_n\} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}A, \text{ i.e. } \mathbf{f}_i = \sum_{k=1}^{n} \mathbf{e}_k a_{ki} \tag{1.23}$$

(see equation (1.17)). Find matrix for linear operator P considered above in (1.22), in the new basis $\{\mathbf{f}_i\}$. According to the formulae (1.23) and (1.22) we have

$$\mathbf{f}'_i = P(\mathbf{f}_i) = P\left(\sum_{q=1}^{n} \mathbf{e}_q a_{qi}\right) = \sum_{q=1}^{n} a_{qi} P(\mathbf{e}_q) = \sum_{q=1}^{n} a_{qi}\left(\sum_{r=1}^{n} \mathbf{e}_r p_{rq}\right) = \sum_{q,r=1}^{n} \mathbf{e}_r p_{rq} a_{qi} =$$

$$\sum_{r=1}^{n} \mathbf{e}_r (PA)_{ri} = \sum_{r,k=1}^{n} \mathbf{f}_k (A^{-1})_{kr=1} (PA)_{ri} = \sum_{k=1}^{n} \mathbf{f}_k (A^{-1}PA)_{ki} \,.$$

We see that in the new basis $\{\mathbf{f}_i\}$ a matrix of linear operator is equal to $A^{-1}PA$.

Proposition *Let P be a linear operator acting in n-dimensional vector space V. Let $\{\mathbf{e}_i\}$ and $\{\mathbf{f}_j\}$ be two arbitrary bases in V. Let $P = ||p_{ik}||$ be a matrix of the operator P in the basis $\{\mathbf{e}_i\}$, and let $P' = ||p'_{ik}||$ be a matrix of the operator P in the basis $\{\mathbf{f}_j\}$:*

$$\text{basis } \{\mathbf{e}_i\} \text{ in } V ----- ||p_{ik}|| \text{matrix of operator } P \text{ in the basis } \{\mathbf{e}_i\}$$

$$\text{basis } \{\mathbf{f}_i\} \text{ in } V ----- ||p'_{ik}|| \text{matrix of operator } P \text{ in the basis } \{\mathbf{f}_i\}$$

Then

$$p'_{ik} = (A^{-1} \circ P \circ A)_{ik} = \sum_{m,r=1}^{n} a_{im} p_{mr} a_{rk} \,. \tag{1.24}$$

Remark Let a matrix $||p_{ij}||$ be a matrix of linear operator P in the basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. Then for an arbitrary vector $\mathbf{x}$

$$\forall \mathbf{x} = \sum_{i=1}^{n} \mathbf{e}_i x^i = (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n) \cdot \begin{pmatrix} x^1 \\ x^2 \\ \dots \\ x^n \end{pmatrix} \text{, then}$$

$$P(\mathbf{x}) = (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n) \cdot P \cdot \begin{pmatrix} x^1 \\ x^2 \\ \dots \\ x^n \end{pmatrix} = \sum_{i=1}^{n} \mathbf{e}'_i y^i = \sum_{i,k=1}^{n} \mathbf{e}_k p_{ki} x^i \,.$$

If x^i are components of vector $\mathbf{x}$ at the basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ and x'^i are components of the vector $\mathbf{x}$ at the new basis $\{\mathbf{e}'_i\}$ then $x'^i = \sum_{k=1}^{n} p_{ik} x^k$.

1.5.2 Determinant and Trace of linear operator

We recall the definition of determinant and explain what is the trace of linear operator,

Definition-Proposition *Let P be a linear operator in vector space V, let $\{\mathbf{e}_i\}$ be an arbitrary basis in V, and let $||p_{ik}||$ be a matrix of operator P*

in this basis. Then we define determiant of linear operator as a determinant of its matrix:

$$\det P = \det\left(||p_{ik}||\right) \,,$$

and in the same way we define we define trace of operator via trace of matrix:

$$\mathrm{Tr}\, P = \mathrm{Tr}\,\left(||p_{ik}||\right) = p_{11} + p_{22} + p_{33} + \cdots + p_{nn}\,. \tag{1.25}$$

Determinant and trace of operator are well-defined. since due to the proposition above (see equation (1.24)), determinant and trace of transition matrice do not change if we change the basis in spite of the fact that transition matrix changes: $P \mapsto A^{-1}PA$, but

$$\det\left(A^{-1}PA\right) = \det A^{-1} \det P \det A = (\det A)^{-1} \det P \det A = \det P \,,$$

and

$$\mathrm{Tr}\,(A^{-1}PA) = \sum_i (A^{-1}PA)_{ii} = \sum_{i,k,p} \left(A^{-1}\right)_{ik} p_{kp} A_{pi} = \sum_{i,k,p} A_{pi} \left(A^{-1}\right)_{ik} p_{kp} =$$

$$\sum_{p,k} \left(A \cdot A^{-1}\right)_{pk} p_{kp} = \sum_{p,k} \delta_{kp} p_{kp} = \sum_k p_{kk} = \mathrm{Tr}\, P \,.$$

Trace of linear operator is an infinitesimal version of its determinant:

$$\det(1 + tP) = 1 + t\mathrm{Tr}\, P + O(t^2) \,.$$

This is infinitesimal version for the followiong famous formula which relates trace and det of linear operator:

$$\det e^{tA} = e^{t\mathrm{Tr}\, A} \,. \tag{1.26}$$

where $e^{tA} = \sum \frac{t^n A^n}{n!}$. E.g. if $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then $e^{tA} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$, $\det e^{tA} = 1$ and $e^{t\mathrm{Tr}\, A} = e^0 = 1$.

1.5.3 Orthogonal linear operators

Now two words on orthogonal linear operators in Euclidean space.

Recall that linear operator P in Euclidean space $\mathbf{E}^n$ is called orthogonal operator if it preserves scalar product:

$$(P\mathbf{x}, P\mathbf{y}) = (\mathbf{x}, \mathbf{y}), \text{ for arbitrary vectors } \mathbf{x}, \mathbf{y} \tag{1.27}$$

In particular if $\{\mathbf{e}_i\}$ is orthonormal basis in Euclidean space then due to (1.27) the new basis $\{\mathbf{e}'_i = P(\mathbf{e}_i)\}$ is orthonormal too. Thus we see that matrix of orthogonal operator P in a given orthogonal basis is orthogonal matrix:

$$P^T \cdot P = I \tag{1.28}$$

(see (1.19) in subsection 1.7). In particular we see that for orthogonal linear operator $\det P = \pm 1$ (compare with (1.20)).

1.6 Orientation in vector space

You have heard a words 'orientation', you have heard expressions like:

A basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ *have the same orientation as the basis* $\{\mathbf{a}', \mathbf{b}', \mathbf{c}'\}$ *if they both obey right hand rule or if they both obey left hand rule. In the other case we say that these bases have opposite orientation...*

When you look in the mirror you know that 'left' is changing on the 'right'

Try to give the exact meaning to these expressions.

1.6.1 Orientation in vector space. Oriented vector space

Consider the set of *all* bases in the given vector space V.

Let $(\mathbf{e}_1, \ldots \mathbf{e}_n)$, $(\mathbf{e}'_1, \ldots \mathbf{e}'_n)$ be two arbitrary bases in the vector space V and let T be transition matrix which transforms the basis $\{\mathbf{e}_i\}$ to the new basis $\{\mathbf{e}'_i\}$:

$$\{\mathbf{e}'_1, \ldots \mathbf{e}'_n\} = \{\mathbf{e}_1, \ldots \mathbf{e}_n\} T \,, \qquad (\mathbf{e}'_i = \sum_{k=1}^{n} \mathbf{e}_k t_{ki}) \tag{1.29}$$

(see also (1.16)).

Definition We say that two bases $\{\mathbf{e}_1, \ldots \mathbf{e}_n\}$ and $\{\mathbf{e}'_1, \ldots \mathbf{e}'_n\}$ in V have *the same orientation* if the determinant of transition matrix (1.29) from the first basis to the second one is positive: $\det T > 0$.

We say that the basis $\{\mathbf{e}_1, \ldots \mathbf{e}_n\}$ has an orientation *opposite to the orientation* of the basis $\{\mathbf{e}'_1, \ldots \mathbf{e}'_n\}$ (or in other words these two bases have opposite orientation) if the determinant of transition matrix from the first basis to the second one is negative: $\det T < 0$.

Remark Transition matrix from basis to basis is non-degenerate, hence its determinant cannot be equal to zero. It can be or positive or negative.

Consider examples.

First the simplest example.

Example 0 Consider a line $\mathbf{R} = \mathbf{R}^1$ as 1-dimensional vector space with an origin at the point 0. Consider on $\mathbf{R}^1$ vectors

$$\mathbf{e} = (2)\,, \qquad \mathbf{e}' = (-8)\,, \qquad \tilde{\mathbf{e}} = (10)\,.$$

Vector $\mathbf{e}$ is a basis of $\mathbf{R}$, as well as vector $\mathbf{e}'$ is a basis, and vector $\tilde{\mathbf{e}}$ is a basis also. (Since space is 1-dimensional every non-zero vector is a basis!)

The basis $\{\mathbf{e}\}$ and the basis $\{\tilde{\mathbf{e}}\}$ have the same orientation since $\tilde{\mathbf{e}} = 5\cdot\mathbf{e}$: transition matrix is 1×1 matrix, the determinant of transition matrix is equal to 5 and $5 > 0$.

Respectively the basis $\{\mathbf{e}\}$ and the basis $\{\mathbf{e}'\}$ have the opposite orientation since $\mathbf{e}' = -4\cdot\mathbf{e}$: determinant of transition matrix is equal to -4 and $-4 < 0$.

Now example of 2-dimensional space:

Example 1 Consider two dimensional vector space $\mathbf{R}^2$ with a canonical basis

$$\{\mathbf{e}_1, \mathbf{e}_2\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

Consider in $\mathbf{R}^2$ another basis

$$\{\mathbf{e}'_1, \mathbf{e}'_2\} = \left\{ \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

One can see that $\{\mathbf{e}'_1, \mathbf{e}'_2\} = \{-2\mathbf{e}_1, \mathbf{e}_2\}$, transition matrix $T = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$, and $\det T = -2 < 0$, i.e. bases $\{\mathbf{e}'_1, \mathbf{e}'_2\}$ and $\{-2\mathbf{e}_1, \mathbf{e}_2\}$ have opposite orientation.

One can see that orientation establishes the equivalence relation in the set of all bases. Show it. We say that $\{\mathbf{e}_1, \dots \mathbf{e}_n\} \sim \{\mathbf{e}'_1, \dots \mathbf{e}'_n\}$, if two bases $\{\mathbf{e}_1, \dots \mathbf{e}_n\}$ and $\{\mathbf{e}'_1, \dots \mathbf{e}'_n\}$ have the same orientation, i.e. $\det T > 0$ for transition matrix.

Proposition *Relation "$\sim$" is an equivalence relation, i.e. this relation is reflexive, symmetric and transitive.*

Prove it:

- `Proof of reflexivity`

it is reflexive, i.e. for every basis $\{\mathbf{e}_1, \dots \mathbf{e}_n\}$

$$\{\mathbf{e}_1, \dots, \mathbf{e}_n\} \sim \{\mathbf{e}_1, \dots, \mathbf{e}_n\}\,, \tag{1.30}$$

because in this case transition matrix $T = I$ and $detI = 1 > 0$.

- `Proof of simmetricity`

Prove, that relation "$\sim$" is symmetric, i.e. If $\{\mathbf{e}_1,\dots,\mathbf{e}_n\} \sim \{\mathbf{e}'_1,\dots,\mathbf{e}'_n\}$ then $\{\mathbf{e}'_1,\dots,\mathbf{e}'_n) \sim \{\mathbf{e}_1,\dots,\mathbf{e}_n\}$.

Let T be a transition matrix from the first basis $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ to the second basis $\{\mathbf{e}'_1,\dots,\mathbf{e}'_n\}$: $\{\mathbf{e}'_1,\dots,\mathbf{e}'_n\} = \{\mathbf{e}_1,\dots,\mathbf{e}_n\}T$, and $\det T > 0$ since $\{\mathbf{e}_1,\dots,\mathbf{e}_n\} \sim \{\mathbf{e}'_1,\dots,\mathbf{e}'_n\}$. Then the transition matrix from the second basis $\{\mathbf{e}'_1,\dots,\mathbf{e}'_n\}$ to the first basis $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ is the inverse matrix T^{-1}: $\{\mathbf{e}_1,\dots,\mathbf{e}_n\} = \{\mathbf{e}'_1,\dots,\mathbf{e}'_n\}T^{-1}$. Hence $\det T^{-1} = \frac{1}{\det T} > 0$ since $\det T > 0$. Hence $\{\mathbf{e}'_1,\dots,\mathbf{e}'_n) \sim \{\mathbf{e}_1,\dots,\mathbf{e}_n\}$. Symmetricity is proved.

- `Proof of transitivity`

We have to prove that if $\{\mathbf{e}_1,\dots,\mathbf{e}_n\} \sim \{\mathbf{e}'_1,\dots,\mathbf{e}'_n\}$ and $\{\mathbf{e}'_1,\dots,\mathbf{e}'_n) \sim \{\tilde{\mathbf{e}}_1,\dots,\tilde{\mathbf{e}}_n\}$, then $\{\mathbf{e}_1,\dots,\mathbf{e}_n\} \sim \{\tilde{\mathbf{e}}_1,\dots,\tilde{\mathbf{e}}_n\}$.

Do it in detail.

Formulate the following statement:

Proposition-Lemma *Let $\{\mathbf{e}_i\}$, $\{\mathbf{e}'_i\}$ and $\{\tilde{\mathbf{e}}_i\}$ be arbitrary three bases in the vector space V. For convenience call a basis $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ the 'I-st' basis, call a basis $\{\mathbf{e}'_1,\dots,\mathbf{e}'_n\}$ the 'II-nd' basis and call a basis $\{\tilde{\mathbf{e}}_1,\dots,\tilde{\mathbf{e}}_n\}$ the 'III-rd' basis.*

Let $T^{(12)}$ be a transition matrix from the I-st basis to the II-nd basis; let $T^{(13)}$ be a transition matrix from the I-st basis to the III-rd basis, and let $T^{(23)}$ be a transition matrix from the II-nd basis to the III-rd basis:

$$\begin{aligned} \{\mathbf{e}'_1,\dots,\mathbf{e}'_n\} &= \{\mathbf{e}_1,\dots,\mathbf{e}_n\}T^{(12)} \\ \{\tilde{\mathbf{e}}_1,\dots,\tilde{\mathbf{e}}_n\} &= \{\mathbf{e}_1,\dots,\mathbf{e}_n\}T^{(13)} \\ \{\tilde{\mathbf{e}}_1,\dots,\tilde{\mathbf{e}}_n\} &= \{\mathbf{e}'_1,\dots,\mathbf{e}'_n\}T^{(23)}\,. \end{aligned} \tag{1.31}$$

Then

$$\underbrace{T^{(13)}}_{\textit{I-st} \to \textit{III-rd}} = \underbrace{T^{(12)}}_{\textit{I-st} \to \textit{II-nd}} \circ \underbrace{T^{(23)}}_{\textit{II-nd} \to \textit{II-rd}} \Rightarrow$$

$$\det T^{(13)} = \det(T^{(12)} \circ T^{(23)}) = \det T^{(12)} \cdot \det T^{(23)}\,. \tag{1.32}$$

Transitivity immediately follows from this statement: if I-st $\sim$ II and II-nd $\sim$ III-rd, then determinants of matrices $T^{(12)}$ and $T^{(23)}$ are positive. Hence according to relation (1.32) $\det T^{(13)}$ is positive too, i.e. I-st $\sim$ III-rd.

It remains to prove equation (1.32). This equation follows from equation (1.31): $\{\tilde{\mathbf{e}}_1,\dots,\tilde{\mathbf{e}}_n\} = \{\mathbf{e}'_1,\dots,\mathbf{e}'_n\}T^{(23)} =$

$$\left(\{\mathbf{e}_1,\dots,\mathbf{e}_n\}T^{(12)}\right)T^{(23)} = \{\mathbf{e}_1,\dots,\mathbf{e}_n\}T^{(12)}\circ T^{(23)} = \{\mathbf{e}_1,\dots,\mathbf{e}_n\}T^{(13)}\,.$$

Thus we proved that relation $\sim$ is equivalence realtion.

Since it is equivalence relation the set of all bases is a union if disjoint equivalence classes. Two bases are in the same equivalence class if and only if they have the same orientation.

How many equivalence classes exist? One, two or more?

Show first that there are at least two equivalence classes.

Example Let $\{\mathbf{e}_1,\mathbf{e}_2\dots,\mathbf{e}_n\}$ be an arbitrary basis in n-dimensional vector space V. Swap the vectors $\mathbf{e}_1,\mathbf{e}_2$. We come to a new basis: $\{\mathbf{e}'_1,\mathbf{e}'_2\dots,\mathbf{e}'_n\}$

$$\mathbf{e}'_1=\mathbf{e}_2,\mathbf{e}'_2=\mathbf{e}_1,\ \text{ all other vectors are the same: } \mathbf{e}_3=\mathbf{e}'_3,\dots,\mathbf{e}_n=\mathbf{e}'_n \tag{1.33}$$

We have:

$$\{\mathbf{e}'_1,\mathbf{e}'_2,\mathbf{e}'_3\dots,\mathbf{e}'_n\} = \{\mathbf{e}_2,\mathbf{e}_1,\mathbf{e}_3,\dots,\mathbf{e}_n\} = \{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\dots,\mathbf{e}_n\}T_{\text{swap}}\,, \tag{1.34}$$

where one can easy see that the determinant for transition matrix T^{swap} is equal to -1, i.e. bases $\{\mathbf{e}_1,\mathbf{e}_2\dots,\mathbf{e}_n\}$ and $\{\mathbf{e}_2,\mathbf{e}_1\dots,\mathbf{e}_n\}$ have opposite orientation.

E.g. write down the transition matrix (1.34) in the case if dimension of vector space is equal to 5, $n=5$. Then we have $\{\mathbf{e}'_1,\mathbf{e}'_2,\mathbf{e}'_3,\mathbf{e}'_4,\mathbf{e}'_5\} = \{\mathbf{e}_2,\mathbf{e}_1,\mathbf{e}_3,\mathbf{e}_4,\mathbf{e}_5\} = \{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4,\mathbf{e}_5\}T$ where

$$T_{\text{swap}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (\det T_{\text{swap}} = -1)\,. \tag{1.35}$$

We see that bases $\{\mathbf{e}_1,\mathbf{e}_2\dots,\mathbf{e}_n\}$ and $\{\mathbf{e}'_1,\mathbf{e}'_2\dots,\mathbf{e}'_n\}$ have opposite orientation.

We see that there are at least two equivalence classes.

One can see that there are *exactly two equivalence classes.*

Proposition *Let two bases* $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ *and* $\{\mathbf{e}'_1,\dots,\mathbf{e}'_n\}$ *in vector space* V *have opposite orientation. Let* $\{\tilde{\mathbf{e}}_1,\dots,\tilde{\mathbf{e}}_n\}$ *be an arbitrary basis in* V.

Then the basis $\{\tilde{\mathbf{e}}_1, \ldots, \tilde{\mathbf{e}}_n\}$ *and the first basis* $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ *have the same orientation or the basis* $\{\tilde{\mathbf{e}}_1, \ldots, \tilde{\mathbf{e}}_n\}$ *and the second basis* $\{\mathbf{e}'_1, \ldots, \mathbf{e}'_n\}$ *have the same orientation.*

In other words if bases $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, $\{\mathbf{e}'_1, \ldots, \mathbf{e}'_n\}$ *and* $\{\tilde{\mathbf{e}}_1, \ldots, \tilde{\mathbf{e}}_n\}$ *are three bases in vector space* V *such that* $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\} \not\sim \{\mathbf{e}'_1, \ldots, \mathbf{e}'_n\}$ *then*

$$\{\tilde{\mathbf{e}}_1, \ldots, \tilde{\mathbf{e}}_n\} \sim \{\mathbf{e}_1, \ldots, \mathbf{e}_n\} \text{ or } \{\tilde{\mathbf{e}}_1, \ldots, \tilde{\mathbf{e}}_n\} \sim \{\mathbf{e}'_1, \ldots, \mathbf{e}'_n\}. \tag{1.36}$$

There are two equivalence classes of bases with respect to orientation.

In the case if bases $\{\tilde{\mathbf{e}}'_1, \ldots, \tilde{\mathbf{e}}'_n\}$, $\{\tilde{\mathbf{e}}_1, \ldots, \tilde{\mathbf{e}}_n\}$ *have opposite orientation, then an arbitrary basis belongs to the equivalence class of the basis* $\{\mathbf{e}_1, \mathbf{e}_2 \ldots, \mathbf{e}_n\}$, *or it belongs to the to the equivalence class of the basis* $\{\mathbf{e}'_1, \mathbf{e}'_2 \ldots, \mathbf{e}'_n\}$.

Proof of the statement immediately follows from statement (1.32).

In the same way like in statement (1.32) we call a basis $\{\mathbf{e}_1, \mathbf{e}_2 \ldots, \mathbf{e}_n\}$ the "I-st basis", a basis $\{\mathbf{e}'_1, \mathbf{e}'_2 \ldots, \mathbf{e}'_n\}$ the "II-nd basis" and a basis $\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2 \ldots, \tilde{\mathbf{e}}_n\}$ the "III-rd basis". We have to prove that the third basis has the same orientation as the first basis or it has the same orientation as the second basis.

Suppose the third basis has not the same orientation as the first basis, then for the transition matrix $T^{(13)}$ (see equation (1.31)) $\det T^{(13)} < 0$. On the other hand $\det T^{(12)} < 0$ also since the first and second bases have opposite orientation. Hence it follows from equation (1.32) that $\det T^{(23)} < 0$, thus second and third bases have opposite orientation. ■

In the example considered above (see (1.33)) an arbitrary basis $\{\mathbf{e}'_1, \ldots \mathbf{e}'_n\}$ have the same orientation as the basis $\{\mathbf{e}_1, \mathbf{e}_2 \ldots, \mathbf{e}_n\}$, i.e. belongs to the equivalence class of basis $\{\mathbf{e}_1, \mathbf{e}_2 \ldots, \mathbf{e}_n\}$, or it has the same orientation as the "swapped" basis $\{\mathbf{e}_2, \mathbf{e}_1 \ldots, \mathbf{e}_n\}$, i.e. it belongs to the equivalence class of the "swappedd" basis $\{\mathbf{e}_2, \mathbf{e}_1 \ldots, \mathbf{e}_n\}$.

The set of all bases is a union of two disjoint subsets.

Any two bases which belong to the same subset have the same orientation. Any two bases which belong to different subsets have opposite orientation.

Definition *An orientation of a vector space is an equivalence class of bases in this vector space.*

Note that fixing any basis we fix orientation, considering the subset of all bases which have the same orientation that the given basis.

There are two orientations. Every basis has the same orientation as a given basis or orientation opposite to the orientation of the given basis.

We choose an arbitrary basis, and call it 'left' basis. Then all bases which belong to the equivalence class of this basis may be called "left" bases and all the bases which do not belong to the equivalence class of this basis may be called "right" bases

Sure we could call this arbitrary basis "right" basis, (or any other `term`, this is just problem of consensus), then all the bases belonging to the equivalence class of this basis woudl be called by the same `term`.

Definition *An oriented vector space is a vector space equipped with orientation.*

Consider examples.

Example (Orientation in two-dimensional space). Let $\{\mathbf{e}_x, \mathbf{e}_y\}$ be arbitrary two bases in $\mathbf{R}^2$ and let $\mathbf{a}, \mathbf{b}$ be arbitrary two vectors in $\mathbf{R}^2$. Consider an ordered pair $\{\mathbf{a}, \mathbf{b}, \}$. The transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y\}$ to the ordered pair $\{\mathbf{a}, \mathbf{b}\}$ is $T = \begin{pmatrix} a_x & b_x \\ a_y & b_y \end{pmatrix}$:

$$\{\mathbf{a}, \mathbf{b}\} = \{\mathbf{e}_x, \mathbf{e}_y\}T = \{\mathbf{e}_x, \mathbf{e}_y\} \begin{pmatrix} a_x & b_x \\ a_y & b_y \end{pmatrix}, \qquad \begin{cases} \mathbf{a} = a_x \mathbf{e}_x + a_y \mathbf{e}_y \\ \mathbf{b} = b_x \mathbf{e}_x + b_y \mathbf{e}_y \end{cases}$$

One can see that the ordered pair $\{\mathbf{a}, \mathbf{b}\}$ also is a basis, (i.e. these two vectors are linearly independent in $\mathbf{R}^2$) if and only if transition matrix is not degenerate, i.e. $\det T \neq 0$. The basis $\{\mathbf{a}, \mathbf{b}\}$ has the same orientation as the basis $\{\mathbf{e}_x, \mathbf{e}_y\}$ if $\det T > 0$ and the basis $\{\mathbf{a}, \mathbf{b}\}$ has the orientation opposite to the orientation of the basis $\{\mathbf{e}_x, \mathbf{e}_y\}$ if $\det T < 0$.

If we call the basis $\{\mathbf{e}_x, \mathbf{e}_y\}$ `left` basis then the basis $\{\mathbf{a}, \mathbf{b}\}$ will be called also `left` basis in the case if $\det T > 0$, and the basis $\{\mathbf{a}, \mathbf{b}\}$ will be called `right` basis in the case if $\det T < 0$; respectively if we call the basis $\{\mathbf{e}_x, \mathbf{e}_y\}$ `right` basis then the basis $\{\mathbf{a}, \mathbf{b}\}$ will be called also `right` basis in the case if $\det T > 0$, and the basis $\{\mathbf{a}, \mathbf{b}\}$ will be called `left` basis in the case if $\det T < 0$.

Example Let $\{\mathbf{e}, \mathbf{f}\}$ be a basis in 2-dimensional vector space. Consider bases $\{\mathbf{e}, -\mathbf{f}\}$, $\{\mathbf{f}, -\mathbf{e}\}$ and $\{\mathbf{f}, \mathbf{e}\}$.

1) We come to basis $\{\mathbf{e}, -\mathbf{f}\}$ reflecting the second basis vector. Transition matrix from initial basis $\{\mathbf{e}, \mathbf{f}\}$ to the basis $\{\mathbf{e}, -\mathbf{f}\}$ is $T_{\{\mathbf{e}, -\mathbf{f}\}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Its determinant is -1. Bases $\{\mathbf{e}, \mathbf{f}\}$ and $\{\mathbf{e}, -\mathbf{f}\}$ have opposite orientation. If $\{\mathbf{e}, \mathbf{f}\}$ is `left` basis then $\{\mathbf{e}, -\mathbf{f}\}$ is `right basis` and vice versa.

2) Transition matrix from initial basis $\{\mathbf{e},\mathbf{f}\}$ to the basis $\{\mathbf{f},-\mathbf{e}\}$ is $T_{\{\mathbf{f},-\mathbf{e}\}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Its determinant is 1. Bases $\{\mathbf{e},\mathbf{f}\}$ and $\{\mathbf{f},-\mathbf{e}\}$ have same orientation. They both are `left` bases or they both are `right` bases. Note that we come to basis $\{\mathbf{f},-\mathbf{e}\}$ *rotating* the initial basis (on the angle $\pi/2$).

3) Transition matrix from initial basis $\{\mathbf{e},\mathbf{f}\}$ to the basis $\{\mathbf{f},\mathbf{e}\}$ is $T_{\{\mathbf{f},\mathbf{e}\}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Its determinant is -1. Bases $\{\mathbf{e},\mathbf{f}\}$ and $\{\mathbf{e},-\mathbf{f}\}$ have opposite orientation. Basis $\{\mathbf{e},-\mathbf{f}\}$ is `right` basis in the case if basis $\{\mathbf{e},\mathbf{f}\}$ is `left` basis, and vice versa, Basis $\{\mathbf{e},-\mathbf{f}\}$ is `left` basis in the case if basis $\{\mathbf{e},\mathbf{f}\}$ is `right` basis.

Notice that we come to basis $\{\mathbf{f},\mathbf{e}\}$ *reflecting* the initial basis.

(There are plenty exercises in the Homework 2.)

Example(Orientation in three-dimensional euclidean space.) Let $\{\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z\}$ be any basis in $\mathbf{E}^3$ and $\mathbf{a},\mathbf{b},\mathbf{c}$ are arbitrary three vectors in $\mathbf{E}^3$:

$$\mathbf{a} = a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z \ \ \mathbf{b} = b_x\mathbf{e}_x + b_y\mathbf{e}_y + b_z\mathbf{e}_z,\ \ \mathbf{c} = c_x\mathbf{e}_x + c_y\mathbf{e}_y + c_z\mathbf{e}_z\,.$$

Consider ordered triple $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$. The transition matrix from the basis $\{\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z\}$ to the ordered triple $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ is $T = \begin{pmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{pmatrix}$:

$$\{\mathbf{a},\mathbf{b},\mathbf{c}\} = \{\mathbf{e_x},\mathbf{e_y},\mathbf{e_z}\}\mathbf{T} = \{\mathbf{e_x},\mathbf{e_y},\mathbf{e_z}\}\begin{pmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{pmatrix}$$

One can see that the ordered triple $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ also is a basis, (i.e. these three vectors are linearly independent) if and only if transition matrix is not degenerate $\det T \neq 0$. The basis $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ has the same orientation as the basis $\{\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z\}$ if

$$\det T > 0\,. \tag{1.37}$$

The basis $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ has the orientation opposite to the orientation of the basis $\{\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z\}$ if

$$\det T < 0\,. \tag{1.38}$$

The usage of words "left" "right" is defined as always: if basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ is `left` basis, then basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is also `left` if determinant of transition matrix is positive, and basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is `right` if determinant of transition matrix is negative, and vice versa: if basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ is `right` basis, then basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is also `right` if determinant of transition matrix is positive, and basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is `left` if determinant of transition matrix is negative.

Remark Note that in the example above we considered in $\mathbf{E}^3$ *arbitrary bases* not necessarily orthonormal bases.

I would like to emphasize again:

relations (1.37),(1.38) *define equivalence relations in the set of bases. Orientation is equivalence class of bases. There are two orientations, every basis has the same orientation as a given basis or opposite orientation.*

If two bases $\{\mathbf{e}_i\}$, $\{\mathbf{e}_{i'}\}$ have the same orientation then they can be transformed to each other by continuous transformation, i.e. there exists one-parametric family of bases $\{\mathbf{e}_i(t)\}$ such that $0 \le t \le 1$ and $\{\mathbf{e}_i(t)\}|_{t=0} = \{\mathbf{e}_i\}$, $\{\mathbf{e}_i(t)\}|_{t=1} = \{\mathbf{e}_{i'}\}$. (All functions $\mathbf{e}_i(t)$ are continuous) In the case of three-dimensional space the following statement is true : *Let* $\{\mathbf{e}_i\}, \{\mathbf{e}_{i'}\}$ $(i = 1, 2, 3)$ *be two orthonormal bases in* $\mathbf{E}^3$ *which have the same orientation. Then there exists an axis* $\mathbf{n}$ *such that basis* $\{\mathbf{e}_i\}$ *transforms to the basis* $\{\mathbf{e}_{i'}\}$ *under rotation around the axis.*(This is Euler Theorem (see it later).

Exercise Show that bases $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ and $\{\mathbf{f}, \mathbf{e}, \mathbf{g}\}$ have opposite orientation but bases $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ and $\{\mathbf{f}, \mathbf{e}, -\mathbf{g}\}$ have the same orientation.

Solution. Transformation from basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ to basis $\{\mathbf{f}, \mathbf{e}, \mathbf{g}\}$ is "swapping" of vectors $((\mathbf{e}, \mathbf{f}) \mapsto (\mathbf{f}, \mathbf{e})$. This is reflection and this transformation changes orientation. One can see it using transition matrix:

$$T\colon \{\mathbf{f}, \mathbf{e}, \mathbf{g}\} = \{\mathbf{e}, \mathbf{f}, \mathbf{g}\}T = \{\mathbf{e}, \mathbf{f}, \mathbf{g}\} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \det T = -1$$

Transformation from basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ to basis $\{\mathbf{f}, \mathbf{e}, -\mathbf{g}\}$ is composition of two transformations: "swapping" of vectors $((\mathbf{e}, \mathbf{f}) \mapsto (\mathbf{f}, \mathbf{e})$ and changing direction of vector $\mathbf{g}$ $(\mathbf{g} \mapsto -\mathbf{g})$. We have two reflections:

$$\{\mathbf{e}, \mathbf{f}, \mathbf{g}\} \quad \overset{\text{reflection}}{\longrightarrow} \quad \{\mathbf{f}, \mathbf{e}, \mathbf{g}\} \quad \overset{\text{reflection}}{\longrightarrow} \quad \{\mathbf{f}, \mathbf{e}, -\mathbf{g}\}$$

Any reflection changes orientation. Two reflections preserve orinetation. One may come to this result using transition matrix:

$$T\colon \{\mathbf{f}, \mathbf{e}, -\mathbf{g}\} = \{\mathbf{e}, \mathbf{f}, \mathbf{g}\}T = \{\mathbf{e}, \mathbf{f}, \mathbf{g}\} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} . \det T = 1. \quad \text{Orientation is not changed.} \tag{1.39}$$

(See also exercises in Homework 2)

1.6.2 Orientation of linear operator

. Let P be a linear operator acting in vector space V.

Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be an arbitrary basis in V. Linear operator P transforms this basis to another basis $\{\mathbf{e}'_1, \dots, \mathbf{e}'_n\}$ in the case if $\det P \neq 0$. Bearing in mind that determinant of transition matrix from basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ to the basis $\{\mathbf{e}'_1, \dots, \mathbf{e}'_n\}$ is a matrix of operator P in the basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ we see that these both bases

$$\{\mathbf{e}_1, \dots, \mathbf{e}_n\}, \quad \{\mathbf{e}'_1, \dots, \mathbf{e}'_n\}, \mathbf{e}'_i = P(\mathbf{e}_i)$$

have the same orientation if and only if $\det P > 0$ and they have opposite orientation if and only if $\det P < 0$.

In the case if $\det P = 0$, P is not invertible matrix, and it does not transform bases to bases.

If a linear operator P acting on the space V has positive determinant then under the action of this operator an arbitrary basis transforms to the basis with the same orientation. Respectively if a linear operator P acting on the space V has negative determinant then under the action of this operator an arbitrary basis transforms to the new basis which has opposite orientation.

Definition. Non-degenerate (invertible) linear operator P ($\det P \neq 0$) acting in vector space V preserves an orientation of the vector space V if $\det P > 0$. It changes the orientation if $\det P < 0$.

1.7 Rotations and orthogonal operators preserving orientation of $\mathbf{E}^n$ (n=2,3)

Recall the notion of orthogonal operator (see 1.5.3). We study here orthogonal operators in $\mathbf{E}^2$ and $\mathbf{E}^3$. In particular we will show that orthogonal operators preserving orientations define rotations.

1.7.1 Orthogonal operators in $\mathbf{E}^2$— Rotations and reflections

We show that an orthogonal operator in $\mathbf{E}^2$ 'rotates the space' or makes a 'reflection'.

Let A be an orthogonal operator acting in Euclidean space $\mathbf{E}^2$: $(A\mathbf{x}, A\mathbf{y}) = (\mathbf{x}, \mathbf{y})$. Let $\{\mathbf{e}, \mathbf{f}\}$ be an orthonormal basis in 2-dimensional Euclidean space $\mathbf{E}^2$: $(\mathbf{e}, \mathbf{e}) = (\mathbf{f}, \mathbf{f}) = 1$ (i.e. $|\mathbf{e}| = |\mathbf{f}| = 1$) and $(\mathbf{e}, \mathbf{f}) = 0$–vectors $\mathbf{e}, \mathbf{f}$ have unit length and are orthogonal to each other.

Consider a new basis $\{\mathbf{e}', \mathbf{f}'\}$, an image of basis $\mathbf{e}, \mathbf{f}$ under action of A: $\mathbf{e}' = A(\mathbf{e})$, $\mathbf{f}' = A(\mathbf{f})$. Let $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ be matrix of operator A in the basis $\mathbf{e}, \mathbf{f}$, (see equation (1.22) and defintion after this equation):

$$\{\mathbf{e}', \mathbf{f}'\} = \{\mathbf{e}, \mathbf{f}\}A = \{\mathbf{e}, \mathbf{f}\} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \text{ i.e. } \mathbf{e}' = \alpha\mathbf{e} + \gamma\mathbf{f},\ \mathbf{f}' = \beta\mathbf{e} + \delta\mathbf{f}$$

New basis is orthonormal basis also, $(\mathbf{e}', \mathbf{e}') = (\mathbf{f}', \mathbf{f}') = 1\,, \quad (\mathbf{e}', \mathbf{f}') = 0$.

Operator A is orthogonal operator, and its matrix is orthogonal matrix:

$$A^T A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^t \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha^2 + \gamma^2 & \alpha\beta + \gamma\delta \\ \alpha\beta + \gamma\delta & \beta^2 + \delta^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.40}$$

Remark With some abuse of notation, (if it is not a reason of confusion) we sometimes use the same letter for linear operator and the matrix of this operator in orthonormal basis.

Remark Note that condition (1.40) implies that $\det A = \pm 1$.

We have $\alpha^2 + \gamma^2 = 1$, $\alpha\beta + \gamma\delta = 0$ and $\beta^2 + \delta^2 = 1$.

Hence one can choose angles φ, ψ: $0 \leq 2\pi$ such that $\alpha = \cos\varphi$, $\gamma = \sin\varphi$, $\beta = \cos\psi$, $\delta = \sin\psi$. The condition $\alpha\beta + \gamma\delta =$ means that

$$\cos\varphi\cos\psi + \sin\varphi\sin\psi = \cos(\varphi - \psi) = 0$$

We have

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \cos\varphi & \cos\psi \\ \sin\varphi & \sin\psi \end{pmatrix}, \quad \text{with } \cos(\varphi - \psi) = 0\,.$$

Condition $\cos(\varphi - \psi) = 0$ means $\psi - \varphi = \frac{\pi}{2} + \pi k$ $(k = 0, \pm 1, \pm 2, \dots)$

We have

$$\begin{cases} \text{I-st case } \psi = \varphi + \frac{\pi}{2} + \pi m \ (m = 0, \pm 2, \pm 4 \dots), \text{ hence } \cos\psi = -\sin\varphi,\ \sin\psi = \cos\varphi \\ \text{II-nd case } \psi = \varphi + \frac{\pi}{2} + \pi k \ (m = \pm 1, \pm 3 \dots), \text{ hence } \cos\psi = \sin\varphi,\ \sin\psi = -\cos\varphi \end{cases}$$

In the I-st case $\cos\psi = -\sin\varphi$, $\sin\psi = \cos\psi$, and

$$A_\varphi = \begin{pmatrix} \cos\varphi & \cos\psi \\ \sin\varphi & \sin\psi \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}, \quad \det A_\varphi = 1\,. \tag{1.41}$$

i.e. operator A preserves orientation.

In the II-nd case $\cos\psi = \sin\varphi$, $\sin\psi = -\cos\psi$, and

$$\tilde{A}_\varphi = \begin{pmatrix} \cos\varphi & \cos\psi \\ \sin\varphi & \sin\psi \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}, \quad \det A_\varphi = -1\,. \tag{1.42}$$

i.e. operator A changes orientation.