Calculation for Green function of disc and half-plane.

Recall that if $G_U(z_0, z)$ is Green function of domain U then the identity

$$\int_U (u\Delta v - v\Delta v) = \int_{\partial U} (u * dv - v * du) \tag{0.1}$$

implies that the function

$$u(\zeta) = \mathcal{G}_U(\zeta, z) f(z)\Omega + \int_{\partial U} *d_{(z)} G(\zeta, z)\mu\,, \quad (\Omega\text{is area form on } z\text{-plane}) \tag{0.2}$$

is the solution of boundary problem

$$\begin{cases} \Delta u = f \\ u\big|_{\partial U} = \mu \end{cases} \tag{0.3}$$

Green function for disc

Having a map

$$z \to w = \frac{z-\zeta}{1-\bar{\zeta} z}$$

which maps unic disc onto unit disc and boundary on boundary, and the point ζ on the centre we define

$$G(z_0, z) = -\frac{1}{2\pi} \log \left| \frac{z - z_0}{1 - \bar{z_0} z} \right|$$

One can see that

$$*dG_z = \Omega \rfloor \operatorname{grad} G_z = r dr \wedge d\varphi \rfloor \left(\frac{\partial G}{\partial r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial G}{\partial \varphi}\frac{\partial}{\partial \varphi} \right) = r\frac{\partial G}{\partial r}\frac{\partial}{\partial r} d\varphi - \frac{1}{r}\frac{\partial G}{\partial \varphi} dr$$

($z = x + iy = r\cos\varphi_i r\sin\varphi$, $z_0 = R\cos\theta + iR\sin\theta$) We need derivatives of Green function on boundary $r = 1$:

$$*dG_z\big||z| = 1 = -\Omega\rfloor \operatorname{grad} G_z\big||z| = 1 = -r\frac{\partial G}{\partial r}\frac{\partial}{\partial r}\big|_{r=1} d\varphi =$$

$$-r\frac{\partial}{\partial r}\left(-\frac{1}{2\pi}\log\left| r^{i\varphi} - Re^{i\theta}\right| + \frac{1}{2\pi}\log\left| 1 - Re^{-i\theta} r^{e^{i\varphi}} \right| \right)\big|_{r=1} d\varphi =$$

$$-r\frac{\partial}{\partial r}\left(-\frac{1}{2\pi}\log\sqrt{R^2 - 2Rr\cos(\theta-\varphi)r + r^2} + \frac{1}{2\pi}\log\sqrt{1 - 2Rr\cos(\theta-\varphi)r + R^2 r^2} \right)\big|_{r=1} d\varphi =$$

$$-\frac{1}{2\pi}\frac{1 - R^2}{1 - 2R\cos(\theta - \varphi) + R^2} d\varphi$$

Remark (Peut etre un problem de sign?)

This we have the solution of Dirichle problme for the circle: if W harmonic function in the disc D: $x^2+y^2<1$ such that $W_{\partial D}=\mu(\varphi)$ ($r\to 1$????) then

$$W(R,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\mu(\varphi)\frac{1-R^2}{1-2R\cos(\theta-\varphi)+R^2}d\varphi\,.$$

Look at this formula. It is useful sometimes to write it:

$$W(R,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\mu(\varphi)\frac{1-R^2}{(1-Re^{i(\theta-\varphi)})(1-Re^{-i(\theta-\varphi)})}d\varphi\,.$$

Notice also that If $\mu(\varphi)=C$ then obviously

$$W(R,\theta)=\begin{cases} C \text{ for } R<1\\ \frac{C}{2} \text{ for } R=1\end{cases}$$

if $\mu(\varphi)=e^{in\varphi}$, $n\geq 0$ then

$$W(R,\theta)=\begin{cases} z^n=R^ne^{in\varphi} \text{ for } R<1\\ \text{ for } R=1\end{cases}$$

then we come to harmonic polynomials.....

Remark *In the formula (*) there is a jump, so to be more precise we have to write:*

$$W(R,\theta)=\lim_{R\to R_-}\frac{1}{2\pi}\int_0^{2\pi}\mu(\varphi)\frac{1-R^2}{(1-Re^{i(\theta-\varphi)})(1-Re^{-i(\theta-\varphi)})}d\varphi\,.$$

sure this is important only for $R=1$:

$$W(1,\theta)=\lim_{R\to 1_-}\frac{1}{2\pi}\int_0^{2\pi}\mu(\varphi)\frac{1-R^2}{(1-Re^{i(\theta-\varphi)})(1-Re^{-i(\theta-\varphi)})}d\varphi\,.$$

Remark 2 *All this can be done much more easilly using Fourier transform : Indeed if $\mu(\varphi=e^{ik\theta})$ then $W=r^ke^{ik\varphi}$, hence*

$$W(R,\theta)=\sum\mu_kR^ke^{ik\theta}=\frac{1}{2\pi}\sum_k\left(\int_0^{2\pi}\mu(\varphi)e^{-ik\varphi}d\varphi\right)R^{|k|}e^{ik\theta}=$$

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{k=-\infty}^{k=\infty}R^{|k|}e^{ik(\theta-\varphi)}\right)d\varphi=\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{1-Re^{i(\theta-\varphi)}}+\frac{1}{1-Re^{-i(\theta-\varphi)}}-1\right)d\varphi$$

Sure we can solve Dirichle problem just using angle function. Recall shortly the calculation of angle function

$$Angle_C(\mathbf{R})=\int_C(x-X)dy-(y-Y)dx$$

(compare with $*d\log|z| = \frac{xdy-ydx}{(x-X)^2+(y-Y)^2}$). Put $x=\cos\varphi, y=\sin\varphi$ we come to

$$L(R,\theta,\varphi)d\varphi = \frac{(x-X)dy-(y-Y)dx}{(x-X)^2+(y-Y)^2}\Big|_{X=R\cos\theta, Y=R\sin\theta, x=\cos\varphi, y=\sin\varphi} =$$

$$\frac{1-R\cos(\theta-\varphi)}{1-2R\cos(\theta-\varphi)+R^2}d\varphi$$

Now we solve Dirichle problem via integral equation.

Consider the distribution $\nu(\varphi)$ on the circle. It defines thefunction:

$$U(R,\theta) = \int_0^{2\pi} \nu(\varphi)L(R,\theta,\varphi)d\varphi = \int_0^{2\pi} \nu(\varphi)\frac{1-R\cos(\theta-\varphi)}{1-2R\cos(\theta-\varphi)+R^2}d\varphi\,.$$

This function is harmonic in the interior of the disc , in the exterior of the dis, but it has the jump:

$$U(R,\theta)\big|_{R=1-0_+} = U(R,\theta)\big|_{R=1} + \pi\nu(\theta)\,,\quad U(R,\theta)\big|_{R=1+0_+} = U(R,\theta)\big|_{R=1} - \pi\nu(\theta)\,.$$

In particular we come to the harmonic function in the disc such that its values at the boundary are equal to

$$\mu(\varphi) = U(R,\theta)\big|_{R=1-0_+} = U(R,\theta)\big|_{R=1} + \pi\nu(\theta) =$$

$$\int_0^{2\pi} \nu(\varphi)\frac{1-R\cos(\theta-\varphi)}{1-2R\cos(\theta-\varphi)+R^2}\Big|_{R=1}d\varphi + \pi\nu(\theta) = \frac{1}{2}\int_0^{2\pi}\nu(\varphi)d\varphi + \pi\nu(\theta)$$

This is linear integral equation on the function $\nu(\theta)$:

$$\mu(\theta) = \frac{1}{2}\int_0^{2\pi}\nu(\varphi)d\varphi + \pi\nu(\theta)\,.$$

Solving it we come to

$$\nu(\theta) = \frac{\mu(\theta)}{\pi} - \frac{1}{4\pi^2}\int_0^{2\pi}\mu(\varphi)d\varphi\,.$$

Hence

$$U(R,\theta) = \int_0^{2\pi}\nu(\varphi)L(R,\theta,\varphi)d\varphi =$$

$$\int_0^{2\pi}\left[\frac{\mu(\varphi)}{\pi} - \frac{1}{4\pi^2}\int_0^{2\pi}\mu(\tau)d\tau\right]L(R,\theta,\varphi)d\varphi =$$

$$\int_0^{2\pi}\mu(\varphi)\frac{L(R,\theta,\varphi)}{\pi} - \int_0^{2\pi}\frac{L(R,\theta,\varphi}{\pi}d\varphi\cdot\frac{1}{4\pi^2}\int_0^{2\pi}\mu(\tau)d\tau =$$

$$\int_0^{2\pi} \mu(\varphi) \frac{L(R,\theta,\varphi)}{\pi} - 2\pi \cdot \frac{1}{4\pi^2} \int_0^{2\pi} \mu(\varphi) d\varphi = \frac{1}{\pi} \int_0^{2\pi} \mu(\varphi) \left(L(R,\theta,\varphi) - \frac{1}{2} \right) d\varphi =$$

$$\frac{1}{\pi} \int_0^{2\pi} \mu(\varphi) \left(\frac{1 - R\cos(\theta-\varphi)}{1 - 2R\cos(\theta-\varphi) + R^2} - \frac{1}{2} \right) d\varphi = \frac{1}{2\pi} \int_0^{2\pi} \mu(\varphi) \left(\frac{1 - R^2}{1 - 2R\cos(\theta-\varphi) + R^2} - \frac{1}{2} \right) d\varphi$$

Thus we come to the solutioj of Dirichle problem.

Remark *Sure using angle function we can reconstruct not only* $*dG$ *but the Green fucntion also. We take the ordinary Green function,* $G_\infty(z_0, z)$ *and using Dirichle problem solution we will find harmonic function* W *such that* $G_\infty + W$ *becomes the Green function: this harmonic function is the soluton of Dirichle problem* $W\big|_{boundary} = -G_\infty\big|_{boundary}$

Upper-half plane: Green function, Dirichle problem

G We know that $w = \frac{iz+1}{z+i}$ maps $\mathbf{H} \leftrightarrow D$ (in fact this is not arbitrary: see the blog on 10-th August)

taking composition we come to the map:

$$w = w(\xi, z) = \frac{\frac{iz+1}{z+i} - \frac{i\zeta+1}{\zeta+i}}{1 - \frac{-i\bar{\zeta}+1}{\bar{\zeta}-i} \frac{iz+1}{z+i}}$$

maps upper half plane H on the disc $x^2 + y^2 < 1$ and a point ζ on the centre.

Hence Green function for half-plane is

$$G(z_0, z) = \log |w(z_0, z)| \tag{3.1}$$

Remark this is very stupid way to calculate Green function. Much easier another way: see the blog on 10 August!

It seems that here it is much easier way to calculate Dirichle problem solution straightforwardly, using angle function. We will do it, but in tomorrow file you will see that it is much easier to calculate straightforwardly Green function using reciprocity method.

Indeed due to

$$A_C(\mathbf{R}) = \int_C \frac{(x-X)dy - (y-Y)dx}{(x-X)^2 + (y-Y)^2}$$

Put $x = t, y = 0$ we come to

$$L(\mathbf{R}, t)dt = \frac{Y dt}{(t-X)^2 + Y^2} = d\left(\arctan \frac{t-X}{y}\right)$$

The potential of double layer with density $\nu(t)$ (compare with disc) is equal to

$$U(X,Y) = \int_{-\infty}^{\infty} \nu(t) \frac{Y dt}{(t-X)^2 + Y^2} eqno(2.1)$$

in the same way as for the circle

$$U(X, 0_+) = \pi\nu(x) + U(X, 0),$$

but for the boundary of half plane this function vanishes: the calculations are simpler, we do not need to solve the integral equation. The function (2.1) is the solution of Dirichle problem:

$$\begin{cases} \Delta W = 0 \\ W\big| = \mu(x) \end{cases} \Rightarrow W(X, Y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \mu(t) \frac{Y dt}{(t-X)^2 + Y^2} \tag{2.2}$$

Remark *One can see straightforwardly that the function*

$$F(X) = \lim_{Y \to 0} \frac{Y dt}{(t-X)^2 + Y^2} = \pi\delta(t-x)\,.$$

Indeed $F(X) = 0$ for all $X \neq t$ and $\int F(X) dX = \pi$

This implies the boundary condition

Calculate Green function using solution of Dirichle problem: Let

$$G_{\mathbf{H}}(z_0, z) = G_{\text{classic}}(z_0, z) + W =$$

$$-\frac{1}{2\pi}\log|z - z_0| + W = -\frac{1}{2\pi}\log\sqrt{(x-X)^2 + (y-Y)^2} + W(x, y)$$

The condition that $G_{\mathbf{H}}$ vanishes at absolute implies that

$$W(x, y)\big|_{y=0} = \frac{1}{2\pi}\log\sqrt{(x-X)^2 + (y-Y)^2}\big|_{y=0} = \frac{1}{2\pi}\log\sqrt{(x-X)^2 + Y^2}$$

thus we have that in Dirichle problem (2.2) $\mu(t) = \frac{1}{2\pi}\log\sqrt{(t-X)^2 + Y^2}$, thus

$$W(x, y) = \frac{1}{2\pi^2}\int_{-\infty}^{\infty}\log\sqrt{(t-X)^2 + Y^2}\frac{y dt}{(t-x)^2 + y^2}$$

and

$$G_{\mathbf{H}}(z_0, z) = -\frac{1}{2\pi}\log\sqrt{(x-X)^2 + (y-Y)^2} + \frac{1}{2\pi^2}\int_{-\infty}^{\infty}\log\sqrt{(t-X)^2 + Y^2}\frac{y dt}{(t-x)^2 + y^2}\,.$$

It seems that calculation of the integral here are not an easy task. But instead, note that the Green function can be immediately obtained using symmetry arguments:

$$G_{\mathbf{H}}(z_0, z) = -\frac{1}{2\pi}\left(\log|z - z_0| - \log|z - \bar{z}_0|\right) = -\frac{1}{2\pi}\log\sqrt{(x-X)^2 + (y-Y)^2} + \frac{1}{2\pi}\log\sqrt{(x-X)^2 +$$

Thus:

$$\int_{-\infty}^{\infty}\log\sqrt{(t-X)^2 + Y^2}\frac{y dt}{(t-x)^2 + y^2} = \ldots\log|z - \bar{z}_0| = \ldots\log\sqrt{(x-X)^2 + (y+Y)^2} +$$

Beautiful, is not it???