Beautiful calculations

Let $H_M = H_M(x,p)$ be Hamiltonian on T^*M and let $H_N = H_N(y,q)$ be Hamiltonian on T^*N. Let Φ^* be pull back of quantum thick morphism generated by $S(x,q)$ which intertwins these hamiltonains, i.e.

$$\int e^{\frac{i}{\hbar}\left(S(x,q)-y^i q_i\right)} H_N(\hat{y},\hat{q}) e^{\frac{i}{\hbar}g(y)} DyDq = H_M(\hat{x},\hat{p}) \int e^{\frac{i}{\hbar}\left(S(x,q)-y^i q_i\right)} e^{\frac{i}{\hbar}g(y)} DyDq \tag{1}$$

Now tend $\hbar$ to zero, and use the fact that roughly speaking

$$\lim_{\hbar\to 0}\left(H(\hat{y},\hat{q})e^{\frac{i}{\hbar}g(y)}\right) = \lim_{\hbar\to 0}\left(H\left(y,\frac{\hbar}{i}\frac{\partial}{\partial y}\right)e^{\frac{i}{\hbar}g(y)}\right) = H(y,q)\,.$$

Je ne sais pas comment ca dit plus precisement, peut etre:

$$\lim_{\hbar\to 0}\left[\left(H(\hat{y},\hat{q})e^{\frac{i}{\hbar}g(y)}\right)e^{-\frac{i}{\hbar}g(y)}\right] = \lim_{\hbar\to 0}\left[\left(H\left(y,\frac{\hbar}{i}\frac{\partial}{\partial y}\right)e^{\frac{i}{\hbar}g(y)}\right)e^{-\frac{i}{\hbar}g(y)}\right] = H(y,q)\,.$$

Now we have that if $\hbar\to 0$, then

$$\int e^{\frac{i}{\hbar}\left(S(x,q)-y^i q_i\right)} H_N(\hat{y},\hat{q}) e^{\frac{i}{\hbar}g(y)} DyDq \text{ tends to } \int e^{\frac{i}{\hbar}\left(S(x,q)-y^i q_i\right)} H_N(y,q) e^{\frac{i}{\hbar}g(y)} DyDq$$

and using the stationary phase method we drr that left hand side tends to

$$H_N\left(y^i=\frac{\partial S(x,q)}{\partial q_i},q\right) \text{ multiplied to } \int e^{\frac{i}{\hbar}\left(S(x,q)-y^i q_i+g(y)\right)} DyDq\,.$$

Respectively the right hand side tends to

$$H_M\left(x^a, p_b=\frac{\partial S(x,q)}{\partial x^b}\right) \text{multiplied to the same } \int e^{\frac{i}{\hbar}\left(S(x,q)-y^i q_i+g(y)\right)} DyDq\,.$$

Thus we come to the conclusion that the classical limit of the condition (1) is

$$H_M\left(\frac{\partial S(x,q)}{q_i},q_j\right) \equiv H_N\left(x^a,\frac{\partial S(x,q)}{x^b},\right)$$

This is nothing that the relation between Hamiltonians which are related with thick morphism.