On one subset in $SO(3)$ and Euler Theorem

This text was written around 2006. Korchagina had a beautiful talk with very facinating title "On the second generation of proof of Theorem on finite simple groups". Sasha Borovik did some interesting remarks about status of this Theorem and about relation between $\mathbf{R}P^3$ and $SO(3)$.

around 2006 *The group $S0(3)$ has fantom memories*
of lost operations
(...initiated by remark of Sasha Borovik) $\blacksquare$

It is well-well-known that $SO(3) \approx \mathbf{R}P^3$, but in fact one can say much more: $S0(3)$ knows about structure of projective space. In particular the subspace $\mathbf{R}P^2 \subset \mathbf{R}P^3$ can be canonically embedded in $SO(3)$. It is the following subset in $SO(3)$ (not subgroup!):

$$L = \{A\colon\ A \in SO(3),\, \det(1+A) = 0\}\,. \tag{1}$$

Geometrically it means following: $A \in L$ means that it is orthogonal transformations which are rotation around axis on the angle π. It will be $\mathbf{R}P^2$. This is not hard. But the point is that this appears as a special case when proving Euler Theorem in algebraic way.

Recall first the most beautiful proof of Euler Theorem (Coxeter Proof) Let $A \in SO(3)$, $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ be an orthogonal basis and $\mathbf{e}' = A\mathbf{e}, \mathbf{f}' = A\mathbf{e}, \mathbf{g}' = A\mathbf{g}$ be the new basis. Consider the reflection O_1 which transforms $\mathbf{e}$ to $\mathbf{e}'$; the invariant plane of this reflection is spanned by vectors $\mathbf{e} + \mathbf{e}'$ and $\mathbf{e} \times \mathbf{e}'$. Then consider the reflection O_2 which transforms $\mathbf{f}$ to $\mathbf{f}'$. The vector $\mathbf{e}'$ belongs to invariant plane of this reflection. Under composition of these two reflections f transforms to f' too. Indeed in other case we consider third reflecti on with respect to the plane $\mathbf{e}', \mathbf{f}'$ but composition of three reflections has determinant -1). Hence $A = O_2 O_1$. The intersection of invariant planes of these reflection is an axis. $\blacksquare$

Second (algebraic) proof Let $A \in SO(3)$. Note first that if there exists eigenvector $\mathbf{n}$ with eigenvalue 1 then everything is proved: restriction of transformation A on the plane orthogonal to the vector $\mathbf{n}$ A is nothing but orthogonal rotation. Hence A is rotation around axis $\mathbf{n}$. Hence it remains to prove that such an eigenvector $\mathbf{n}$ exists. Consider characteristic polynomial $P(z) = \det(z - A)$. It is cubic polynomial. Let λ_0 be its root and $\mathbf{n}$ be corresponding eigenvector. $\lambda = \pm 1$ since A preserves scalar product. If $A \notin L$ then $\lambda_0 \neq -1$ hence $\lambda_0 = 1$ and everything is proved. Suppose $\lambda_0 = -1$,i.e. $A \in L$. Consider a plane α, which is orthogonal to the eigenvector $\mathbf{n}$. Restriction of A on the plane α is orthogonal transformation with determinant -1. Hence restriction of A on the plane α is a reflection with respect to a line $l \in \alpha$ (A on α has eigenvectors $\mathbf{f}, \mathbf{g}$ with eigenvalues $1, -1$ respectively, i.e. l is directed along vector $\mathbf{f}$.) We see that in this special case A is a rotation around axis $\mathbf{f}$ on the angle π. $\blacksquare$