Orthogonal transfromations of $\mathbf{E}^2$ from the point of view of $\mathbf{E}^3$

Let A be an orthogonal transformation of $\mathbf{E}^2$ changing its orientation, $\det A = -1$.

We know that A has two eignevectors $\tilde{\mathbf{e}}, \tilde{f}$ with eignevalues $1, -1$. One can see it easily from three-dimensional point of view:

Consider natural embedding of $\mathbf{E}^2$ (with basis $\{\mathbf{e}, \mathbf{f}\}$) in $\mathbf{E}^3$, and consider the operator $\hat{A}$ such that $\hat{A}(\mathbf{g}) = -\mathbf{g}$.

Operator $\hat{A}$ is orthogonal operator and it preserves orientation. Hence it has axis $\mathbf{N}$: $A(\mathbf{N}) = \mathbf{N}$. It is easy to see that axis $\mathbf{N}$ is orthogonal to vector $\mathbf{g}$

$$(\mathbf{N}, \mathbf{g}) = (\hat{A}(\mathbf{N}), -\hat{A}(\mathbf{g})) = -(\mathbf{N}, \mathbf{g}) \Leftarrow (\mathbf{N}.\mathbf{g}) = 0\,.$$

i.e. $\mathbf{N}$ belongs to $\mathbf{E}^2$, thus we see that $A(\tilde{\mathbf{e}}) = \tilde{\mathbf{e}}$ for $\mathbf{e} = \frac{\mathbf{N}}{|\mathbf{N}|}$. If $\tilde{\mathbf{f}}$ is a vector orthogonal to $\mathbf{N}$ then $\hat{A}(\mathbf{N}) = A(\mathbf{N})$ is also orthogonal to $\mathbf{N} = A(\mathbf{N})$, but A is not identical operator, hence hence $A(\tilde{\mathbf{f}}) = -\tilde{\mathbf{f}}$.