Homework 10 Solutions.

1 *On the sphere $x^2+y^2+z^2=R^2$ in $\mathbf{E}^3$ consider a circle C which is the intersection of the sphere with the plane $z=R-h$, $0<h<R$*

Let $\mathbf{X}$ be an arbitrary vector tangent to the sphere at a point of C.

Find the angle between $\mathbf{X}$ and the result of parallel transport of $\mathbf{X}$ along C.

The circle C is a boundary of the sphere segment of the height H. The area of this domain is equal to $2\pi Rh$. The Gaussian curvature of sphere iis equal to $\frac{K=1}{R^2}$. Hence due to Theorem we see that vector $\mathbf{X}$ through parallel transport rotates on the angle $KS=\frac{2\pi R}{h}$.

2 *On two-dimensional Riemannian manifold with coordinates x^1, x^2 consider the vector fields $\mathbf{A}=\frac{\partial}{\partial x^1}$, $\mathbf{B}=\frac{\partial}{\partial x^2}$, $\mathbf{X}=(1+x^1x^2)\frac{\partial}{\partial x^2}$, and the vector field $\mathbf{Y}=(\nabla_\mathbf{A}\nabla_\mathbf{B}-\nabla_\mathbf{B}\nabla_\mathbf{A})\,\mathbf{X}$, where ∇ is a connection. Calculate the value of the field $\mathbf{Y}$ at the point $x^1=x^2=0$ if the curvature tensor of the connection ∇ is such that $R^1{}_{212}=1$ and $R^2{}_{212}=0$ at this point.*

The commutator of fields $\mathbf{A}$ and $\mathbf{B}$ vanishes. Hence by definition of curvature tensor

$$\mathbf{Y}=(\nabla_\mathbf{A}\nabla_\mathbf{B}-\nabla_\mathbf{B}\nabla_\mathbf{A})\mathbf{X}=(=\mathcal{R}(\mathbf{A},\mathbf{B})\mathbf{X}=R^i_{kmn}A^mB^n\partial_i=R^i_{212}(1+x^1x^2)\partial_i$$

At the point $x^1=x^2=0$, $\mathbf{Y}=R^i{}_{212}\partial_i=R^1{}_{212}\partial_{x^1}+R^2{}_{212}\partial_{x^2}=\partial_{x^1}$.

3 *Write down components of curvature tensor in terms of Christoffel symbols.*

$$R^i_{kmn}\partial_i=\mathcal{R}(\partial_m,\partial_n)\partial_k=\nabla_m\left(\nabla_n\partial_k\right)-(m\leftrightarrow n)=\nabla_m(\Gamma^r_{nk}\partial_r)-(m\leftrightarrow n)=\left(\partial_m\Gamma^i_{nk}+\Gamma^i_{mr}\Gamma^r_{nk}\right)\partial_i-($$

i.e.

$$R^i_{kmn}=\partial_m\Gamma^i_{nk}+\Gamma^i_{mr}\Gamma^r_{nk}-\partial_n\Gamma^i_{mk}-\Gamma^i_{nr}\Gamma^r_{mk}\,.$$

(See also lecture notes).

4 *For every of the statements below prove it or show that it is wrong considering counterexample.*

a) If there exist coordinates u,v such that Riemannian metric G at the given point $\mathbf{p}$ is equal to $G=du^2+dv^2$ in these coordinates, then curvature of Levi-Civita connection at the point $\mathbf{p}$ vanishes.

b) If all derivatives of components of Riemannian metric in coordinates u,v vanish at the given point with coordinates (u_0,v_0):*

$$\frac{\partial g_{ik}(u,v)}{\partial u}\Big|_{u=u_0,v=v_0}=\frac{\partial g_{ik}(u,v)}{\partial v}\Big|_{u=u_0,v=v_0}=0\,,$$

then curvature of Levi-Civita connection at this point vanishes

c) If all first and second derivatives of components of Riemannian metric

$$\frac{\partial g_{ik}(u,v)}{\partial u},\ \frac{\partial g_{ik}(u,v)}{\partial v},\ \frac{\partial^2 g_{ik}(u,v)}{\partial u^2},\ \frac{\partial^2 g_{ik}(u,v)}{\partial u\partial v},\ \frac{\partial^2 g_{ik}(u,v)}{\partial v^2}\,,$$

vanish at the given point then curvature of Levi-Civita connection vanishes at this point.

First and second statements are wrong. The thrid statement is true.

Counterexample to the first statement: Consider on the unit sphere metric $G = d\theta^2 + \sin^2\theta d\varphi^2$.

At the points of equator (but not in their neighborhood!!!!) this metric is Euclidean and first derivatives of components vanish, but curvature is not vanished.

Counterexample to the second statement is again the sphere.

Consider sphere of the radius R: $\mathbf{r} = \mathbf{r}(u,v)$: $\begin{cases} x = u cry = v \\ z = F(u,v) = \sqrt{R^2-u^2-v^2} \end{cases}$. One can see that at the point $u = v = 0$ the induced Riemannian metric is equal to $du^2 + dv^2$ and Christoffel symbols vanish since it is the extremum of the function F. Indeed

$$G = (dx^2 + dy^2 + dz^2)\big|_M = du^2 + dv^2 + (F_u du + F_v dv)^2\,,$$

and at the point $u = v = 0$ $F_u = F_v = 0$ (see also the exercise in the Coursework).

On the other hand the Gaussian curvature is equal to $K = \frac{1}{R^2}$ and curvature of Levi-Civita connection does not vanish. At the point $u = v = 0$ we have

$$K = \frac{R}{2} = \frac{R_{1212}}{\det G} = R_{1212}\,.$$

since $G = du^2 + dv^2$ at this point. ere: $F = \sqrt{R^2 - u^2 - v^2}$.

If at the given point first and second derivatves of metric vanish then due to Levi-Civita formula Christoffely symbols and their first derivatives vanish. This imply that cuirvature vanishes too.

5 *State the relation between the Riemann curvature tensor of the Levi-Civita connection of a surface in* $\mathbf{E}^3$ *and its Gaussian curvature* K*. Deduce the Theorema Egregium from this relation.*

Let M *be a surface* $\mathbf{r} = \mathbf{r}(u,v)$ *in* $\mathbf{E}^3$*, such that at the given point* $\mathbf{p}$ *Gaussian curvature* $K = 1$*, and the induced Riemannian metric is equal to* $G = du^2 + dv^2$ *at this point.*

Calculate all components of the Riemannian curvature tensor R_{ikmn} *in coordinates* u, v *at the point* $\mathbf{p}$.

Show that induced Riemannian metric cannot be equal identically to $du^2 + dv^2$ *in a vicinity of the point* $\mathbf{p}$.

Let M be two-dimensional surface in $\mathbf{E}^3$. Let G be induced Riemannian metric on this surface. Let R be scalar curvature defined by the Riemann curvature tensor of Levi-Civita connection on the surface. Let K be Gaussian curvature. Then

$$\frac{R}{2} = \frac{R_{1212}}{g_{11}g_{22} - g_{12}^2} = K$$

In the right hand side of this formula stands Gaussian curvature. It is defined by the way of how surface is embedded in $\mathbf{E}^3$. External observer calculates shape operator and obtains Gaussian curvature. In the left hand side of this formula stands Riemannian scalar curvature. It is defined by the metric on the surface. Internal observer takes the metric induced on the surface, calculates connection, then curvature tensor $R^i{}_{rmn}$ and takes scalar curvature

$$R = g^{ip}g^{kq}R_{ikpq} = 2R_{1212}(g^{11}g^{22} - g^{12}g^{21}) = \frac{2R_{1212}}{\det g},$$

and then Gaussian curvature. The internal observer and external observer come to the same answer for the Gaussian curvature. This is the statement of *Theorema Egregium.*

calculate components of the Riemann curvature tensor.

It follows from this formula that $R_{1212} = K$ since $\det G\big|_{\mathbf{p}} = 1$.

Using symmetry property of Remann curvature tensor (see the solution of the next exercise) we come to

$$R_{1212} = -R_{1221} = -R_{2112} = R_{2121} = K \det g = 1$$

and all other components are equal to zero.

The fact that Gaussian curvature is not equal to zero, means that metric cannot be equal identically to $du^2 + dv^2$ in the vicinity of the point $\mathbf{p}$. Indeed if so, then first and second derivatives of metric vanish at the point $\mathbf{p}$, i.e. Riemann curvature tensor vanishes at this point. hence $K = 0$ due to the relation (*). This is contradiction.

6 *Using relation between Gaussian curvature and Riemann curvature tensor for Levi-Civita connection, write down all components $\{R_{ikmn}\}$ of Riemann curvature tensor for sphere of radius R in spehrical coordinates.*

Using symmetry properties $R_{ikmn} = -R_{kimn} = -R_{iknm}$ we have that

$$R_{1111} = R_{1112} = R_{1121} = R_{1211} = R_{2111} = R_{2222} = R_{2212} = R_{2221} = R_{1222} = R_{2122} = R_{1122} = R_{2211}$$

It remains to calculate $R_{1212}, R_{1221}, R_{2112}, R_{2121}$ We have that

$$\frac{R_{1212}}{\det g} = K = \frac{1}{\rho^2},.$$

Since $G = \rho^2(d\theta^2 + \sin^2\theta d\varphi^2)$, $\det g = \rho^4 \sin^2\theta$, we have that

$$R_{1212} = -R_{1221} = -R_{2112} = R_{2121} = K \det g = \rho^2 \sin^2\theta\,.$$

Remark What about to calculate R_{1212} straightforwardly without using the relation with Gaussian curvature? This can be done with very simple calculations at the points where Christoffel symbols vanish. We know that since $G = \rho^2(d\theta^2 + \sin^2\theta d\varphi^2)$ then

$$\Gamma^\theta_{\varphi\varphi} = \Gamma^1_{22} = -\sin\theta\cos\theta\,, \quad \Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = \Gamma^2_{12} = \Gamma^2_{21} = \mathrm{cotan}\,\theta\,,$$

and all other components vanish, i.e. at the points where $\theta = \frac{\pi}{2}$ we can directly caclulate curvature using the defnition of curvature

$$R^r_{212}\big|_{\theta=\frac{\pi}{2}} = \partial_1 \Gamma^r_{21} - \partial_2 \Gamma^r_{11} \,.$$

Hence

$$R^1{}_{212}\big|_{\theta=\frac{\pi}{2}} = \partial_\theta \Gamma^\theta_{\varphi\varphi}\big|_{\theta=\frac{\pi}{2}} = \partial_\theta(-\sin\theta\cos\theta)\big|_{\theta=\frac{\pi}{2}} = 1\,,$$

$$R^2{}_{121}\big|_{\theta=\frac{\pi}{2}} = -\partial_\theta \Gamma^\varphi_{\varphi\theta}\big|_{\theta=\frac{\pi}{2}} = -\partial_\theta\left(\cos\theta\sin\theta\right)\big|_{\theta=\frac{\pi}{2}} = 1\,,$$

i.e. at the points $\theta = \frac{\pi}{2}$ $R1212 = \frac{1}{\rho^2}$, and $R_{11} = R_{22} = 1$, and $R = \frac{2}{\rho^2}$.