Riemannian Geometry

it is a draft of Lecture Notes of H.M. Khudaverdian.
Manchester, 30 April 2014

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1 Riemannian manifolds

1.1 Manifolds. Tensors. (Recalling)

I recall briefly basics of manifolds and tensor fields on manifolds.

An n-dimensional manifold is a space such that in a vicinity of any point one can consider local coordinates $\{x^1, \dots, x^n\}$ (charts). One can consider different local coordinates. If both coordinates $\{x^1, \dots, x^n\}$, $\{x^{1'}, \dots, x^{n'}\}$ are defined in a vicinity of the given point then they are related by bijective transition functions (functions defined on domains in $\mathbf{R}^n$ and taking values in $\mathbf{R}^n$).

$$\begin{cases} x^{1'} = x^{1'}(x^1, \dots, x^n) \\ x^{2'} = x^{2'}(x^1, \dots, x^n) \\ \dots \\ x^{n-1'} = x^{n-1'}(x^1, \dots, x^n) \\ x^{n'} = x^{n'}(x^1, \dots, x^n) \end{cases} \tag{1.1}$$

We say that n-dimensional manifold is *differentiable* or *smooth* if transition functions are diffeomorphisms, i.e. they are smooth and rank of Jacobian is equal to k, i.e.

$$\det \begin{pmatrix} \frac{\partial x^{1'}}{\partial x^1} & \frac{\partial x^{1'}}{\partial x^2} \dots & \frac{\partial x^{1'}}{\partial x^n} \\ \frac{\partial x^{2'}}{\partial x^1} & \frac{\partial x^{2'}}{\partial x^2} \dots & \frac{\partial x^{2'}}{\partial x^n} \\ \dots & & \\ \frac{\partial x^{n'}}{\partial x^1} & \frac{\partial x^{n'}}{\partial x^2} \dots & \frac{\partial x^{n'}}{\partial x^n} \end{pmatrix} \neq 0 \, . \tag{1.2}$$

A good example of manifold is an open domain D in n-dimensional vector space $\mathbf{R}^n$. Cartesian coordinates on $\mathbf{R}^n$ define global coordinates on D. On the other hand one can consider an arbitrary local coordinates in different domains in $\mathbf{R}^n$. E.g. one can consider polar coordinates $\{r, \varphi\}$ in a domain $D = \{x, y \colon y > 0\}$ of $\mathbf{R}^2$ defined by standard formulae:

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} , \tag{1.3}$$

$$\det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{pmatrix} = \det \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix} = r \tag{1.4}$$

or one can consider spherical coordinates $\{r, \theta, \varphi\}$ in a domain $D = \{x, y, z\colon x > 0, y > 0, z > 0\}$ of $\mathbf{R}^3$ (or in other domain of $\mathbf{R}^3$) defined by standard formulae

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases},$$

,

$$\det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{pmatrix} = \det\begin{pmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} = r^2\sin\theta \tag{1.5}$$

Choosing domain where polar (spherical) coordinates are well-defined we have to be award that coordinates have to be well-defined and transition functions (**??**) have to obey condition (**??**), i.e. they have to be diffeomorphisms. E.g. for domain D in example (1.3) Jacobian (1.4) does not vanish if and only if $r > 0$ in D.

Examples of manifolds: $\mathbf{R}^n$, Circle S^1, Sphere S^2, in general sphere S^n, torus $S^1 \times S^1$, cylinder, cone, ...

Example Consider in detail circle S^1. Suppose it is given as $x^2 + y^2 = 1$. One can consider two local polar coordinates: 1) φ, $0 < \varphi < 2\pi$ which covers all points except the point $(1, 0)$ and $\varphi'\colon -\pi < \varphi' < \pi$ which covers all points except point $(-1, 0)$. In a vicinity of any point M on the circle (except these two exceptional points) one can consider both coordinates φ and φ'

We come to another very useful coordinates on a circle using *stereographic projection*. Take north pole of the circle: the point $N = (0, 1)$. Assign to every point $M = (x, y)$ on the circle the point $(t, 0)$ on the x-axis such that the point $(t, 0)$, the point M and the north pole N are on the one line. This can be done for every point of circle except the north pole $(0, 1)$ itself. We come to stereographic projection of circle S^1 without North pole on the line $\mathbf{R}$. In the same way we can define stereographic projection of circle without south pole (the point $(0, -1)$) on the x-axis. We come to coordinate t'. One can see that these coordinates are related by the following simple formula:

$$t' = \frac{1}{t},$$

† One very important property of stereographic projetion which we do not use in this course but it is too beautiful not to mention it: under stereographic projection

all points on the circle $x^2 + y^2 = 1$ with rational coordinates x and y and only these points transform to rational points on line. Thus we come to Pythagorean triples $a^2 + b^2 = c^2$.

Tensors on Manifold

Recall briefly what are tensors on manifold. For every point $\mathbf{p}$ on manifold M one can consider tangent vector space $T_{\mathbf{p}}M$— the space of vectors tangent to the manifold at the point M.

Tangent vector $\mathbf{A}(x) = A^i(x)\frac{\partial}{\partial x^i}$. Under changing of coordinates it transforms as follows:

$$\mathbf{A} = A^i(x)\frac{\partial}{\partial x^i} = A^i(x)\frac{\partial x^{m'}(x)}{\partial x^i}\frac{\partial}{\partial x^{m'}} = A^{m'}(x'(x))\frac{\partial}{\partial x^{m'}}\,.$$

Hence

$$A^{i'}(x') = \frac{\partial x^{i'}(x)}{\partial x^i}A^i(x)\,. \tag{1.6}$$

Consider also cotangent space $T^*_{\mathbf{p}}M$ (for every point $\mathbf{p}$ on manifold M)— space of linear functions on tangent vectors, i.e. space of 1-forms which sometimes are called *covectors*.:

One-form (covector) $\omega = \omega_i(x)dx^i$ transforms as follows

$$\omega = \omega_m(x)dx^m = \omega_m\frac{\partial x^m(x')}{\partial x^{m'}}dx^{m'} = \omega_{m'}(x')dx^{m'}\,.$$

Hence

$$\omega_{m'}(x') = \frac{\partial x^m(x')}{\partial x^{m'}}\omega_m(x)\,. \tag{1.7}$$

Differential form sometimes is called *covector*.

Tensors:

One can consider *contravariant* tensors of the rank p

$$T = T^{i_1 i_2 \dots i_p}(x)\frac{\partial}{\partial x^{i_1}}\otimes\frac{\partial}{\partial x^{i_2}}\otimes\dots\otimes\frac{\partial}{\partial x^{i_k}}$$

with components $\{T^{i_1 i_2 \dots i_k}\}(x)$. Under changing of coordinates $(x^1, \dots, x^n) \to (x^{1'}, \dots, x^{n'})$ (see (1.1)) they transform as follows:

$$T^{i'_1 i'_2 \dots i'_p}(x') = \frac{\partial x^{i'_1}}{\partial x^{i_1}}\frac{\partial x^{i'_2}}{\partial x^{i_2}}\dots\frac{\partial x^{i'_p}}{\partial x^{i_p}}T^{i_1 i_2 \dots i_p}(x)\,. \tag{1.8}$$

One can consider *covariant* tensors of the rank q

$$S = S_{j_1 j_2 \dots j_q} dx^{j_1} \otimes dx^{j_2} \otimes \dots dx^{j_q}$$

with components $\{S_{j_1 j_2 \dots j_q}\}$. Under changing of coordinates $(x^1, \dots, x^n) \to (x^{1'}, \dots, x^{n'})$ they transform as follows:

$$S_{j'_1 j'_2 \dots j'_q}(x') = \frac{\partial x^{i_1}}{\partial x^{i'_1}} \frac{\partial x^{i_2}}{\partial x^{i'_2}} \dots \frac{\partial x^{i_p}}{\partial x^{i'_p}} S_{j_1 j_2 \dots j_q}(x)\,. \tag{1.9}$$

One can also consider mixed tensors:

$$Q = Q^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q} \frac{\partial}{\partial x^{i_1}} \otimes \frac{\partial}{\partial x^{i_2}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes dx^{j_2} \otimes \dots dx^{j_q}$$

with components $\{Q^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q}\}$. We call these tensors *tensors of the type* $\begin{pmatrix} p \\ q \end{pmatrix}$.

Tensors of the type $\begin{pmatrix} p \\ 0 \end{pmatrix}$ are called *contravariant tensors of the rank p*. They have p upper indices.

Tensors of the type $\begin{pmatrix} 0 \\ q \end{pmatrix}$ are called *covariant tensors of the rank q*. They have q lower indices.

Having in mind (1.6), (1.7),(1.8) and (1.9) we come to the rule of transformation for tensors which have p upper and q lover indices, tensors of type $\begin{pmatrix} p \\ q \end{pmatrix}$:

$$Q^{i'_1 i'_2 \dots i'_p}_{j'_1 j'_2 \dots j'_q}(x') = \frac{\partial x^{i'_1}}{\partial x^{i_1}} \frac{\partial x^{i'_2}}{\partial x^{i_2}} \dots \frac{\partial x^{i'_p}}{\partial x^{i_p}} \frac{\partial x^{j_1}}{\partial x^{j'_1}} \frac{\partial x^{j_2}}{\partial x^{j'_2}} \dots \frac{\partial x^{j_q}}{\partial x^{j'_q}} Q^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q}(x)\,.$$

E.g. if S_{ik} is a covariant tensor of rank 2 then

$$S_{i'k'}(x') = \frac{\partial x^i(x')}{\partial x^{i'}} \frac{\partial x^k(x')}{\partial x^{k'}} S_{ik}(x)\,. \tag{1.10}$$

If A^i_k is a tensor of rank $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (linear operator on $T_{\mathbf{p}}M$) then

$$A^{i'}_{k'}(x') = \frac{\partial x^{i'}(x)}{\partial x^i} \frac{\partial x^k(x')}{\partial x^{k'}} A^i_k(x)\,.$$

If S^m_{ik} is a tensor of the type $\begin{pmatrix}1\\2\end{pmatrix}$) then

$$S^{m'}_{i'k'} = \frac{\partial x^{m'}}{\partial x^m}\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^k}{\partial x^{k'}}S^m_{ik}(x)\,. \tag{1.11}$$

Transformations formulae (1.6)—(1.11) define vectors, covectors and in generally any tensor fields in components. E.g. covariant tensor (covariant tensor field) of the rank 2 can be defined as matrix S_{ik} (matrix valued function $S_{ik}(x)$) such that under changing of coordinates $\{x^1, x^2, \dots, x^n\} \mapsto \{x^{1'}, x^{2'}, \dots, x^{n'}\}$, (1.1) S_{ik} change by the rule (1.10).

Remark *Einstein summation rules*

In our lectures we always use so called *Einstein summation convention.* it implies that when an index occurs twice in the same expression in upper and in lower postitions, then the expression is implicitly summed over all possible values for that index. Sometimes it is called dummy indices summation rule.

1.2 Riemannian manifold— manifold equipped with Riemannian metric

Definition The Riemannian manifold is a manifold equipped with a Riemannian metric.

The Riemannian metric on the manifold M defines the length of the tangent vectors and the length of the curves.

Definition Riemannian metric G on n-dimensional manifold M^n defines for every point $\mathbf{p} \in M$ the scalar product of tangent vectors in the tangent space $T_{\mathbf{p}}M$ smoothly depending on the point $\mathbf{p}$.

It means that in every coordinate system $(x^1, \dots, x^n)$ a metric $G = g_{ik}dx^i dx^k$ is defined by a matrix valued smooth function $g_{ik}(x)$ $(i = 1, \dots, n; k = 1, \dots n)$ such that for any two vectors

$$\mathbf{A} = A^i(x)\frac{\partial}{\partial x^i},\ \mathbf{B} = B^i(x)\frac{\partial}{\partial x^i},$$

tangent to the manifold M at the point $\mathbf{p}$ with coordinates $x = (x^1, x^2, \dots, x^n)$ $(\mathbf{A}, \mathbf{B} \in T_{\mathbf{p}}M)$ the scalar product is equal to:

$$\langle \mathbf{A}, \mathbf{B}\rangle_G\big|_{\mathbf{p}} = G(\mathbf{A}, \mathbf{B})\big|_{\mathbf{p}} = A^i(x)g_{ik}(x)B^k(x) =$$

$$
\begin{pmatrix} A^1 \dots A^n \end{pmatrix} \begin{pmatrix} g_{11}(x) & \dots & g_{1n}(x) \\ \dots & \dots & \dots \\ g_{n1}(x) & \dots & g_{nn}(x) \end{pmatrix} \begin{pmatrix} B^1 \\ . \\ . \\ . \\ B^n \end{pmatrix} \tag{1.12}
$$

where

- $G(\mathbf{A}, \mathbf{B}) = G(\mathbf{B}, \mathbf{A})$, i.e. $g_{ik}(x) = g_{ki}(x)$ (symmetricity condition)
- $G(\mathbf{A}, \mathbf{A}) > 0$ if $\mathbf{A} \neq \mathbf{0}$, i.e.

  $g_{ik}(x)u^i u^k \geq 0$, $g_{ik}(x)u^i u^k = 0$ iff $u^1 = \dots = u^n = 0$ (positive-definiteness)
- $G(\mathbf{A}, \mathbf{B})\big|_{\mathbf{p}=x}$, i.e. $g_{ik}(x)$ are smooth functions.

One can say that Riemannian metric is defined by symmetric covariant smooth tensor field G of the rank 2 which defines scalar product in the tangent spaces $T_{\mathbf{p}}M$ smoothly depending on the point $\mathbf{p}$. Components of tensor field G in coordinate system are matrix valued functions $g_{ik}(x)$:

$$
G = g_{ik}(x) dx^i \otimes dx^k \,. \tag{1.13}
$$

The matrix $||g_{ik}||$ of components of the metric G we also sometimes denote by G.

Rule of transformation for entries of matrix $g_{ik}(x)$

$g_{ik}(x)$-entries of the matrix $||g_{ik}||$ are components of tensor field G in a given coordinate system.

How do these components transform under transformation of coordinates $\{x^i\} \mapsto \{x^{i'}\}$?

$$
G = g_{ik} dx^i \otimes dx^k = g_{ik} \left(\frac{\partial x^i}{\partial x^{i'}} dx^{i'} \right) \otimes \left(\frac{\partial x^k}{\partial x^{k'}} dx^{k'} \right) =
$$

$$
\frac{\partial x^i}{\partial x^{i'}} g_{ik} \frac{\partial x^k}{\partial x^{k'}} dx^{i'} \otimes dx^{k'} = g_{i'k'} dx^{i'} \otimes dx^{k'}
$$

Hence

$$
g_{i'k'} = \frac{\partial x^i}{\partial x^{i'}} g_{ik} \frac{\partial x^k}{\partial x^{k'}} \,. \tag{1.14}
$$

One can derive transformations formulae also using general formulae (1.10) for tensors.

Important remark

$$g_{ik} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k} \right\rangle . \tag{1.15}$$

Later by some abuse of notations we sometimes omit the sign of tensor product and write a metric just as

$$G = g_{ik}(x) dx^i dx^k .$$

Examples

- $\mathbf{R}^n$ with canonical coordinates $\{x^i\}$ and with metric

  $$G = (dx^1)^2 + (dx^2)^2 + \cdots + (dx^n)^2$$

  $G = ||g_{ik}|| = \text{diag}\ [1, 1, \ldots, 1]$

  Recall that this is a basis example of n-dimensional Euclidean space, where scalar product is defined by the formula:

  $$G(\mathbf{X}, \mathbf{Y}) = \langle \mathbf{X}, \mathbf{Y} \rangle = g_{ik} X^i Y^k = X^1 Y^1 + X^2 Y^2 + \cdots + X^n Y^n .$$

  In the general case if $G = ||g_{ik}||$ is an arbitrary symmetric positive-definite metric then

  $$G(\mathbf{X}, \mathbf{Y}) = \langle \mathbf{X}, \mathbf{Y} \rangle = g_{ik} X^i Y^k .$$

  One can show that there exists a new basis $\{\mathbf{e}_i\}$ such that in this basis

  $$G(\mathbf{e}_i, \mathbf{e}_k) = \delta_{ik} .$$

  This basis is called orthonormal basis. (See the Lecture notes in Geometry)

  Scalar product in vector space defines the *same* scalar product at all the points. In general case for Riemannian manifold scalar product depends on a point.

  In Riemannian manifold we consider arbitrary transformations from local coordinates to new local coordinates.

- $\mathbf{R}^2$ with polar coordinates in the domain $y > 0$ ($x = r\cos\varphi, y = r\sin\varphi$):

  $dx = \cos\varphi dr - r\sin\varphi d\varphi, dy = \sin\varphi dr + r\cos\varphi d\varphi$. In new coordinates the Riemannian metric $G = dx^2 + dy^2$ will have the following appearance:

  $$G = (dx)^2 + (dy)^2 = (\cos\varphi dr - r\sin\varphi d\varphi)^2 + (\sin\varphi dr + r\cos\varphi d\varphi)^2 = dr^2 + r^2(d\varphi)^2$$

  We see that for matrix $G = ||g_{ik}||$

  $$\underbrace{G = \begin{pmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},}_{\text{in Cartesian coordinates}} \qquad \underbrace{G = \begin{pmatrix} g_{rr} & g_{r\varphi} \\ g_{\varphi r} & g_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}}_{\text{in polar coordinates}}$$

- Circle

  Interval $[0, 2\pi)$ in the line $0 \leq x < 2\pi$ with Riemannian metric

  $$G = a^2 dx^2 \tag{1.16}$$

  Renaming $x \mapsto \varphi$ we come to habitual formula for metric for circle of the radius a: $x^2 + y^2 = a^2$ embedded in the Euclidean space $\mathbf{E}^2$:

  $$G = a^2 d\varphi^2 \qquad \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \end{cases}, 0 \leq \varphi < 2\pi, \tag{1.17}$$

- Cylinder surface

  Domain in $\mathbf{R}^2$ $D = \{(x, y) : , \ 0 \leq x < 2\pi$ with Riemannian metric

  $$G = a^2 dx^2 + dy^2 \tag{1.18}$$

  We see that renaming variables $x \mapsto \varphi$, $y \mapsto h$ we come to habitual, familiar formulae for metric in standard polar coordinates for cylinder surface of the radius a embedded in the Euclidean space $\mathbf{E}^3$:

  $$G = a^2 d\varphi^2 + dh^2 \qquad \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases}, 0 \leq \varphi < 2\pi, -\infty < h < \infty \tag{1.19}$$

- Sphere

  Domain in $\mathbf{R}^2$, $0 < x < 2\pi$, $0 < y < \pi$ with metric $G = dy^2 + \sin^2 y dx^2$ We see that renaming variables $x \mapsto \varphi$, $y \mapsto h$ we come to habitual, familiar formulae for metric in standard spherical coordinates for sphere $x^2 + y^2 + z^2 = a^2$ of the radius a embedded in the Euclidean space $\mathbf{E}^3$:

$$G = a^2 d\theta^2 + a^2 \sin^2 \theta d\varphi^2 \qquad \begin{cases} x = a \sin\theta\cos\varphi \\ y = a\sin\theta\sin\varphi z = a\cos\theta \end{cases}, 0 \leq \varphi < 2\pi, -\infty < h < \infty \tag{1.20}$$

(See examples also in the Homeworks.)

1.2.1 * Pseudoriemannian manifold

If we omit the condition of positive-definiteness for Riemannian metric we come to so called Pseudoriemannian metric. Manifold equipped with pseudoriemannian metric is called pseudoriemannian manifold. Pseudoriemannian manifolds appear in applications in the special and general relativity theory.

In pseudoriemanninan space scalar product $(\mathbf{X}, \mathbf{X})$ may take an arbitrary real values: it can be positive, negative, it can be equal to zero. Vectors $\mathbf{X}$ such that $(\mathbf{X}, \mathbf{X}) = 0$ are called null-vectors. (See the problem 6 in Homework 1).

Example Consider $n+1$-dimensional linear space $\mathbf{R}^{n+1}$ with pseudometric [1]

$$G = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - \cdots - (dx^n)^2 \,.$$

For an arbitrary vector $\mathbf{X} = (a^0, a^1, a^2, \ldots, a^n)$ scalar product $(\mathbf{X}, \mathbf{X})$ is positive if $(a^0)^2 > (a_1)^2 + (a_2)^2 + \cdots + (a_n)^2$, it is negative if $(a^0)^2 < (a_1)^2 + (a_2)^2 + \cdots + (a_n)^2$, and $\mathbf{X}$ is null-vector if $(a^0)^2 = (a_1)^2 + (a_2)^2 + \cdots + (a_n)^2$.

1.3 Scalar product, length of tangent vectors and angle between them. Length of curves

The Riemannian metric defines scalar product of tangent vectors attached at the given point. Hence it defines the length of tangent vectors and angle

[1] In the case $n = 3$ it is so called Minkovski space. The coordinate x^0 plays a role of the time: $x^0 = ct$, where c is the value of the speed of the light. Vectors $\mathbf{X}$ such that $(\mathbf{X}, \mathbf{X}) > 0$ are called time-like vectors and they called space-like vectors if $(\mathbf{X}, \mathbf{X}) < 0$

between them. If $\mathbf{X} = X^m \frac{\partial}{\partial x^m}$, $\mathbf{Y} = Y^m \frac{\partial}{\partial x^m}$ are two tangent vectors at the given point $\mathbf{p}$ of Riemannian manifold with coordinates $x^1, \dots, x^n$, then we have that lengths of these vectors equal to

$$|\mathbf{X}| = \sqrt{\langle \mathbf{X}, \mathbf{X} \rangle} = \sqrt{g_{ik}(x) X^i X^k}, \quad |\mathbf{Y}| = \sqrt{\langle \mathbf{Y}, \mathbf{Y} \rangle} = \sqrt{g_{ik}(x) Y^i Y^k}, \tag{1.21}$$

and an angle θ between these vectors is defined by the relation

$$\cos\theta = \frac{\langle \mathbf{X}, \mathbf{Y} \rangle}{|\mathbf{X}| \cdot |\mathbf{Y}|} = \frac{g_{ik} X^i Y^k}{\sqrt{g_{ik}(x) X^i X^k} \sqrt{g_{ik}(x) Y^i Y^k}} \tag{1.22}$$

Example Let M be 3-dimensional Riemannian manifold. Consider the vectors $\mathbf{X} = 2\partial_x + 2\partial_y - \partial_z$ and $\mathbf{Y} = \partial_x - 2\partial_y - 2\partial_z$ attached at the point $\mathbf{p}$ of M with local coordinates (x, y, z), where $x = y = 1, z = 0$. Find the lengths of these vectors and angle between them if the expression of Riemannian metric in these coordinates is $\frac{G = dx^2 + dy^2 + dz^2}{(1+x^2+y^2)^2}$.

We see that matrix of Riemannian $g_{ik} = \sigma(x, y, z)\delta_{ik}$, where $\sigma(x, y, z) = \frac{1}{(1+x^2+y^2+z^2)^2}$ is a scalar function, i.e. matrix $G = ||g_{ik}||$ is proportional to unity matrix. According to formulae above

$$|\mathbf{X}| = \sqrt{\langle \mathbf{X}, \mathbf{X} \rangle} = \sqrt{g_{ik}(x) X^i X^k} = \sqrt{\sigma(x, y, z)} \sqrt{X^i X^i} = 3\sqrt{\sigma(x, y, z)} = 1\,.$$

The same answer for $|\mathbf{Y}|$. The scalar product between vectors $\mathbf{X}, \mathbf{Y}$ equal to zero:

$$\langle \mathbf{X}, \mathbf{Y} \rangle = \sigma(x, y, z)\delta_{ik} X^i Y^k = 0$$

Hence these vectors are unit vectors which are orthogonal to each other. (δ_{ik} is Kronecker symbol: $\delta_{ik} = 1$ if $i = k$ and it vanishes otherwise.)

1.3.1 Length of curves

Let $\gamma\colon\ x^i = x^i(t), (i = 1, \dots, n))$ $(a \le t \le b)$ be a curve on the Riemannian manifold (M, G).

At the every point of the curve the velocity vector (tangent vector) is defined:

$$\mathbf{v}(t) = \begin{pmatrix} \dot{x}^1(t) \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}^n(t) \end{pmatrix}$$

The length of velocity vector $\mathbf{v} \in T_xM$ (vector $\mathbf{v}$ is tangent to the manifold M at the point x) equals to

$$|\mathbf{v}|_x = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle_G \big|_x} = \sqrt{g_{ik} v^i v^k}\big|_x = \sqrt{g_{ik} \frac{dx^i(t)}{dt} \frac{dx^k(t)}{dt}_x}\big|_x .$$

For an arbitrary curve its length is equal to the integral of the length of velocity vector:

$$L_\gamma = \int_a^b \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle_G \big|_{x(t)}} dt = \int_a^b \sqrt{g_{ik}(x(t)) \dot{x}^i(t) \dot{x}^k(t)} dt . \qquad (1.23)$$

Bearing in mind that metric (1.13) defines the length we often write metric in the following form

$$ds^2 = g_{ik} dx^i dx^k$$

For example consider 2-dimensional Riemannian manifold with metric

$$||g_{ik}(u,v)|| = \begin{pmatrix} g_{11}(u,v) & g_{12}(u,v) \\ g_{21}(u,v) & g_{22}(u,v) \end{pmatrix} .$$

Then

$$G = ds^2 = g_{ik} du^i dv^k = g_{11}(u,v) du^2 + 2g_{12}(u,v) du dv + g_{22}(u,v) dv^2 .$$

The length of the curve γ: $u = u(t), v = v(t)$, where $t_0 \leq t \leq t_1$ according to (1.23) is equal to $L_\gamma = \int_{t_0}^{t_1} \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \int_{t_0}^{t_1} \sqrt{g_{ik}(x) \dot{x}^i \dot{x}^k} =$

$$\int_{t_0}^{t_1} \sqrt{g_{11} (u(t), v(t)) u_t^2 + 2g_{12} (u(t), v(t)) u_t v_t + g_{22} (u(t), v(t)) v_t^2} dt . \qquad (1.24)$$

The length of curves defined by the formula(1.23) obeys the following natural conditions

- It coincides with the usual length in the Euclidean space $\mathbf{E}^n$ ($\mathbf{R}^n$ with standard metric $G = (dx^1)^2 + \cdots + (dx^n)^2$ in Cartesian coordinates). E.g. for 3-dimensional Euclidean space

$$L_\gamma = \int_a^b \sqrt{g_{ik}(x(t)) \dot{x}^i(t) \dot{x}^k(t)} dt = \int_a^b \sqrt{(\dot{x}^1(t))^2 + (\dot{x}^2(t))^2 + (\dot{x}^3(t))^2} dt$$

- It does not depend on parameterisation of the curve

$$L_\gamma = \int_a^b \sqrt{g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t)}dt = \int_{a'}^{b'} \sqrt{g_{ik}(x(\tau))\dot{x}^i(\tau)\dot{x}^k(\tau)}d\tau\,,$$

($x^i(\tau) = x^i(t(\tau))$, $a' \leq \tau \leq b'$ while $a \leq t \leq b$) since under changing of parameterisation

$$\dot{x}^i(\tau) = \frac{dx(t(\tau))}{d\tau} = \frac{dx(t(\tau))}{dt}\frac{dt}{d\tau} = \dot{x}^i(t)\frac{dt}{d\tau}\,.$$

- It *does not depend on coordinates on Riemannian manifold M*

$$L_\gamma = \int_a^b \sqrt{g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t)}dt = \int_a^b \sqrt{g_{i'k'}(x'(t))\dot{x}^{i'}(t)\dot{x}^{k'}(t)}dt\,.$$

  This immediately follows from transformation rule (1.62) for Riemannian metric:

$$g_{i'k'}\dot{x}^{i'}(t)\dot{x}^{k'}(t) = g_{ik}\left(\frac{\partial x^i}{\partial x^{i'}(t)}\dot{x}^{i'}(t)\right)\left(\frac{\partial x^k}{\partial x^{k'}}\dot{x}^{k'}(t)\right) g_{ik}\,\dot{x}^i(t)\dot{x}^k(t)\,.$$

- It is additive: length of the sum of two curves is equal to the sum of their lengths. If a curve $\gamma = \gamma_1 + \gamma$, i.e. $\gamma\colon x^i(t), a \leq t \leq b$, $\gamma_1\colon x^i(t), a \leq t \leq c$ and $\gamma_2\colon x^i(t), c \leq t \leq b$ where a point c belongs to the interval (a, b) then $L_\gamma = L_{\gamma^1} + L_{\gamma^2}$.

One can show that formula (1.23) for length is defined uniquely by these conditions. More precisely one can show under some technical conditions one may show that any local additive functional on curves which does not depend on coordinates and parameterisation, and depends on derivatives of curves of order ≤ 1 is equal to (1.23) up to a constant multiplier. To feel the taste of this statement you may do the following exercise:

Exercise Let $A = A\left(x(t), y(t), \frac{dx(t)}{dt}, \frac{dy(t)}{dt}\right)$ be a function such that an integral $L = \int A\left(x(t), y(t), \frac{dx(t)}{dt}, \frac{dy(t)}{dt}\right) dt$ over an arbitrary curve γ in $\mathbf{E}^2$ does not change under reparameterisation of this curve and under an arbitrary isometry, i.e. translation and rotation of the curve. Then one can easy show (show it!) that

$$A\left(x(t), y(t), \frac{dx(t)}{dt}, \frac{dy(t)}{dt}\right) = c\sqrt{\left(\frac{dx(t)}{dt}\right)^2 + \left(\frac{dy(t)}{dt}\right)^2}\,,$$

where c is a constant, i.e. it is a usual length up to a multiplier

1.4 Riemannian structure on the surfaces embedded in Euclidean space

In first

Let M be a surface embedded in Euclidean space. Let G be Riemannian structure on the manifold M.

Let $\mathbf{X}, \mathbf{Y}$ be two vectors tangent to the surface M at a point $\mathbf{p} \in M$. An External Observer calculate this scalar product viewing these two vectors as vectors in $\mathbf{E}^3$ attached at the point $\mathbf{p} \in \mathbf{E}^3$ using scalar product in $\mathbf{E}^3$. An Internal Observer will calculate the scalar product viewing these two vectors as vectors tangent to the surface M using the Riemannian metric G (see the formula (1.28)). Respectively

If L is a curve in M then an External Observer consider this curve as a curve in $\mathbf{E}^3$, calculate the modulus of velocity vector (speed) and the length of the curve using Euclidean scalar product of ambient space. An Internal Observer ("an ant") will define the modulus of the velocity vector and the length of the curve using Riemannian metric.

Definition Let M be a surface embedded in the Euclidean space. We say that metric G_M on the surface is induced by the Euclidean metric if the scalar product of arbitrary two vectors $\mathbf{A}, \mathbf{B} \in T_{\mathbf{p}}M$ calculated in terms of the metric G equals to Euclidean scalar product of these two vectors:

$$\langle \mathbf{A}, \mathbf{B} \rangle_{G_M} = \langle \mathbf{A}, \mathbf{B} \rangle_{G_{\text{Euclidean}}} \tag{1.25}$$

In other words we say that Riemannian metric on the embedded surface is induced by the Euclidean structure of the ambient space if External and Internal Observers come to the same results calculating scalar product of vectors tangent to the surface.

In this case modulus of velocity vector (speed) and the length of the curve is the same for External and Internal Observer.

Before going in details of this definition recall the conception of Internal and External Observers when dealing with surfaces in Euclidean space:

1.4.1 Internal and external coordinates of tangent vector

Tangent plane

Here we recall basic notions from the course of Geometry which we will need here.

Let $\mathbf{r} = \mathbf{r}(u, v)$ be parameterisation of the surface M embedded in the Euclidean space:

$$\mathbf{r}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}$$

Here as always x, y, z are Cartesian coordinates in $\mathbf{E}^3$.

Let $\mathbf{p}$ be an arbitrary point on the surface M. Consider the plane formed by the vectors which are adjusted to the point $\mathbf{p}$ and tangent to the surface M. We call this plane *plane tangent to M at the point* $\mathbf{p}$ and denote it by $T_{\mathbf{p}}M$.

For a point $\mathbf{p} \in M$ one can consider a basis in the tangent plane T_pM adjusted to the parameters u, v. Tangent basis vectors at any point (u, v) are

$$\mathbf{r}_u = \frac{\partial \mathbf{r}(u, v)}{\partial u} = \begin{pmatrix} \frac{\partial x(u,v)}{\partial u} \\ \frac{\partial y(u,v)}{\partial u} \\ \frac{\partial z(u,v)}{\partial u} \end{pmatrix} = \frac{\partial x(u, v)}{\partial u} \frac{\partial}{\partial x} + \frac{\partial y(u, v)}{\partial u} \frac{\partial}{\partial y} + \frac{\partial z(u, v)}{\partial u} \frac{\partial}{\partial z}$$

Every vector $\mathbf{X} \in T_pM$ can be expanded over this basis:

$$\mathbf{X} = X_u \mathbf{r}_u + X_v \mathbf{r}_v,$$

where X_u, X_v are coefficients, components of the vector $\mathbf{X}$.

Internal Observer views the basis vector $\mathbf{r}_u \in T_pM$, as a velocity vector for the curve $u = u_0 + t, v = v_0$, where (u_0, v_0) are coordinates of the point p. Respectively the basis vector $\mathbf{r}_v \in T_pM$ for an Internal Observer, is velocity vector for the curve $u = u_0, v = v_0 + t$, where (u_0, v_0) are coordinates of the point p.

1.4.2 Explicit formulae for induced Riemannian metric (First Quadratic form)

Now we are ready to write down the explicit formulae for the Riemannian metric on the surface induced by metric (scalar product) in ambient Euclidean space (see the Definition (1.25)).

Let $M\colon \mathbf{r} = \mathbf{r}(u, v)$ be a surface embedded in $\mathbf{E}^3$.

The formula (1.25) means that scalar products of basic vectors $\mathbf{r}_u = \partial_u, \mathbf{r}_v = \partial_v$ has to be the same calculated in the ambient space and on the surface: For example scalar product $\langle \partial_u, \partial_v \rangle_M = g_{uv}$ calculated by the Internal Observer is the same as a scalar product $\langle \mathbf{r}_u, \mathbf{r}_v \rangle_{\mathbf{E}^3}$ calculated by the External Observer, scalar product $\langle \partial_v, \partial_v \rangle_M = g_{uv}$ calculated by the Internal Observer is the same as a scalar product $\langle \mathbf{r}_v, \mathbf{r}_v \rangle_{\mathbf{E}^3}$ calculated by the External Observer and so on:

$$G = \begin{pmatrix} g_{uu} & g_{uv} \\ g_{vu} & g_{vv} \end{pmatrix} = \begin{pmatrix} \langle \partial_u, \partial_u \rangle & \langle \partial_u, \partial_v \rangle \\ \langle \partial_v, \partial_u \rangle & \langle \partial_v, \partial_v \rangle \end{pmatrix} = \begin{pmatrix} \langle \mathbf{r}_u, \mathbf{r}_u \rangle_{\mathbf{E}^3} & \langle \mathbf{r}_u, \mathbf{r}_v \rangle_{\mathbf{E}^3} \\ \langle \mathbf{r}_v, \mathbf{r}_u \rangle_{\mathbf{E}^3} & \langle \mathbf{r}_v, \mathbf{r}_v \rangle_{\mathbf{E}^3} \end{pmatrix} \tag{1.26}$$

where as usual we denote by $\langle \ , \ \rangle_{\mathbf{E}^3}$ the scalar product in the ambient Euclidean space. (Here see also the important remark (1.15))

Remark It is convenient sometimes to denote parameters (u, v) as (u^1, u^2) or u^α $(\alpha = 1, 2)$ and to write $\mathbf{r} = \mathbf{r}(u^1, u^2)$ or $\mathbf{r} = \mathbf{r}(u^\alpha)$ $(\alpha = 1, 2)$ instead $\mathbf{r} = \mathbf{r}(u, v)$

In these notations:

$$G_M = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} = \begin{pmatrix} \langle \mathbf{r}_u, \mathbf{r}_u \rangle_{\mathbf{E}^3} & \langle \mathbf{r}_u, \mathbf{r}_v \rangle_{\mathbf{E}^3} \\ \langle \mathbf{r}_u, \mathbf{r}_v \rangle_{\mathbf{E}^3} & \langle \mathbf{r}_v, \mathbf{r}_v \rangle_{\mathbf{E}^3} \end{pmatrix}, \quad g_{\alpha\beta} = \langle \mathbf{r}_\alpha, \mathbf{r}_\beta \rangle ,$$

$$G_M = g_{\alpha\beta} du^\alpha du^\beta = g_{11} du^2 + 2 g_{12} du dv + g_{22} dv^2 \tag{1.27}$$

where (,) is a scalar product in Euclidean space.

The formula (1.27) is the formula for induced Riemannian metric on the surface $\mathbf{r} = \mathbf{r}(u, v)$. It is First Quadratic Form of this surface.

If $\mathbf{X}, \mathbf{Y}$ are two tangent vectors in the tangent plane T_pC then $G(\mathbf{X}, \mathbf{Y})$ at the point p is equal to scalar product of vectors $\mathbf{X}, \mathbf{Y}$:

$$(\mathbf{X}, \mathbf{Y}) = (X^1 \mathbf{r}_1 + X^2 \mathbf{r}_2, Y^1 \mathbf{r}_1 + Y^2 \mathbf{r}_2) = \tag{1.28}$$

$$X^1 (\mathbf{r}_1, \mathbf{r}_1) Y^1 + X^1 (\mathbf{r}_1, \mathbf{r}_2) Y^2 + X^2 (\mathbf{r}_2, \mathbf{r}_1) Y^1 + X^2 (\mathbf{r}_2, \mathbf{r}_2) Y^2 =$$
$$X^\alpha (\mathbf{r}_\alpha, \mathbf{r}_\beta) Y^\beta = X^\alpha g_{\alpha\beta} Y^\beta = G(\mathbf{X}, \mathbf{Y})$$

We can come to this formula just transforming differentials. In Cartesian coordinates $\langle \mathbf{X}, \mathbf{Y} \rangle = X^1 Y^1 + X^2 Y^2 + X^3 Y^3$, i.e. the Riemannian structure of Euclidean space in Cartesian coordinates is given by

$$G_{\mathbf{E}^3} = (dx)^2 + (dy)^2 + (dz)^2 \,. \tag{1.29}$$

The condition that Riemannian metric (1.27) is induced by Euclidean scalar product means that

$$G_{\mathbf{E}^3}\big|_{\mathbf{r}=\mathbf{r}(u,v)} = \left((dx)^2+(dy)^2+(dz)^2\right)\big|_{\mathbf{r}=\mathbf{r}(u,v)} = G_M = g_{\alpha\beta}du^\alpha du^\beta \qquad (1.30)$$

i.e. $((dx)^2+(dy)^2+(dz)^2)\big|_{\mathbf{r}=\mathbf{r}(u,v)} =$

$$\left(\frac{\partial x(u,v)}{\partial u}du + \frac{\partial x(u,v)}{\partial u}du\right)^2 + \left(\frac{\partial x(u,v)}{\partial u}du + \frac{\partial x(u,v)}{\partial u}du\right)^2 + \left(\frac{\partial x(u,v)}{\partial u}du + \frac{\partial x(u,v)}{\partial u}du\right)^2 =$$

$$(x_u^2+y_u^2+z_u^2)du^2 + 2(x_ux_v+y_uy_v+z_uz_v)dudv + (x_v^2+y_v^2+z_v^2)dv^2$$

We see that

$$G_M = g_{\alpha\beta}du^\alpha du^\beta = g_{11}du^2 + 2g_{12}dudv + g_{22}dv^2, \qquad (1.31)$$

where $g_{11} = g_{uu} = (x_u^2+y_u^2+z_u^2) = \langle \mathbf{r}_u, \mathbf{r}_u\rangle_{\mathbf{E}^3}$, $g_{12} = g_{21} = g_{uv} = g_{vu} = (x_ux_v+y_uy_v+z_uz_v) = \langle \mathbf{r}_u, \mathbf{r}_v\rangle_{\mathbf{E}^3}$, $g_{22} = g_{vv} = (x_v^2+y_v^2+z_v^2) = \langle \mathbf{r}_v, \mathbf{r}_v\rangle_{\mathbf{E}^3}$. We come to same formula (1.27).

(See the examples of calculations in the next subsection.)

Remark Sometimes it is convenient to denote Cartesian coordinates of Euclidean space by x^i, $(i = 1,2,3)$. Let surface M be given in local parameterisation $x^i = x^i(u^\alpha)$. Riemannian metric of Euclidean space (1.29) has appearance

$$G_{\mathbf{E}} = dx^i\delta_{ik}dx^k\,. \qquad (1.32)$$

and induced metric (1.30) has appearance

$$G_M = dx^i\delta_{ik}dx^k\big|_{x^i=x^i(u^\alpha)} = \frac{\partial x^i(u)}{\partial u^\alpha}\delta_{ik}\frac{\partial x^k(u)}{\partial u^\beta}du^\alpha du^\beta = g_{\alpha\beta}(u)du^\alpha du^\beta \qquad (1.33)$$

Why this representation is useful? it is easy to see that formulae (1.32) ,(1.33) work for arbitrary dimensions, i.e. if we have m-dimensional manifold embedded in n-dimensional Euclidean space. We just have to suppose that in this case $i = 1,\dots,n$ and $\alpha = 1,\dots,m$; manifold is given by parameterisation $x^i = x^i(u^\alpha)$ $(\alpha = 1,\dots,m)$. Moreover in the face if manifold is embedded not in Euclidean space but in an arbitrary Riemannian space then one can see comparing formulae (1.25) and (1.32) we come to the induced metric

$$G_M = dx^i g_{ik}\left((x(u))\right)dx^k\big|_{x^i=x^i(u^\alpha)} = \frac{\partial x^i(u)}{\partial u^\alpha}g_{ik}\left((x(u))\right)\frac{\partial x^k(u)}{\partial u^\beta}du^\alpha du^\beta = g_{\alpha\beta}(x(u))du^\alpha du^\beta$$

Check explicitly again that length of the tangent vectors and curves on the surface calculating by External observer (i.e. using Euclidean metric

(1.29)) *is the same* as calculating by Internal Observer, ant (i.e. using the induced Riemannian metric (1.27), (1.31)). Let $\mathbf{X} = X^\alpha \mathbf{r}_\alpha = a\mathbf{r}_u + b\mathbf{r}_v$ be a vector tangent to the surface M. The square of the length $|\mathbf{X}|$ of this vector calculated by External observer (he calculates using the scalar product in $\mathbf{E}^3$) equals to

$$|\mathbf{X}|^2 = \langle \mathbf{X}, \mathbf{X} \rangle = \langle \mathbf{r}_u + b\mathbf{r}_v, a\mathbf{r}_u + b\mathbf{r}_v \rangle = a^2 \langle \mathbf{r}_u, \mathbf{r}_u \rangle + 2ab \langle \mathbf{r}_u, \mathbf{r}_v \rangle + b^2 \langle \mathbf{r}_v, \mathbf{r}_v \rangle \tag{1.34}$$

where $\langle\, ,\, \rangle$ is a scalar product in $\mathbf{E}^3$. The internal observer will calculate the length using Riemannian metric (1.27) (1.31):

$$G(\mathbf{X}, \mathbf{X}) = \begin{pmatrix} a, & b \end{pmatrix} \cdot \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & G_{22} \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = g_{11}a^2 + 2g_{12}ab + g_{22}b^2 \tag{1.35}$$

External observer (person living in ambient space $\mathbf{E}^3$*) calculate the length of the tangent vector using formula* (1.34)*. An ant living on the surface calculate length of this vector in internal coordinates using formula* (1.35)*. External observer deals with external coordinates of the vector, ant on the surface with internal coordinates. They come to the same answer.*

Let $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ $a \leq t \leq b$ be a curve on the surface.

Velocity of this curve at the point $\mathbf{r}(u(t), v(t))$ is equal to

$$\mathbf{v} = \mathbf{X} = \xi \mathbf{r}_u + \eta \mathbf{r}_v \text{where } \xi = u_t, \eta = v_t : \quad \mathbf{v} = \frac{d\mathbf{r}(t)}{dt} = u_t \mathbf{r}_u + v_t \mathbf{r}_v \,.$$

The length of the curve is equal to

$$L = \int_a^b |\mathbf{v}(t)| dt = \int_a^b \sqrt{\langle \mathbf{v}(t), \mathbf{v}(t) \rangle_{\mathbf{E}^3}} dt = \int_a^b \sqrt{\langle u_t \mathbf{r}_u + v_t \mathbf{r}_v, u_t \mathbf{r}_u + v_t \mathbf{r}_v \rangle_{\mathbf{E}^3}} dt = \tag{1.36}$$

$$\int_a^b \sqrt{\langle \mathbf{r}_u, \mathbf{r}_u \rangle_{\mathbf{E}^3} u_t^2 + 2 \langle \mathbf{r}_u, \mathbf{r}_v \rangle_{\mathbf{E}^3} u_t v_t + \langle \mathbf{r}_v, \mathbf{r}_v \rangle_{\mathbf{E}^3} v_t^2} d\tau =$$

$$\int_a^b \sqrt{g_{11} u_t^2 + 2 g_{12} u_t v_t + g_{22} v_t^2} dt \tag{1.37}$$

An external observer will calculate the length of the curve using (1.36)*. An ant living on the surface calculate length of the curve using* (1.37) *using Riemannian metric on the surface. They will come to the same answer.*

1.4.3 Induced Riemannian metrics. Examples.

We consider here examples of calculating induced Riemannian metric on some quadratic surfaces in $\mathbf{E}^3$. using calculations for tangent vectors (see (1.27)) or explicitly in terms of differentials (see (1.30) and (1.31)).

First of all consider the general case when a surface M is defined by the equation $z - F(x, y) = 0$. One can consider the following parameterisation of this surface:

$$\mathbf{r}(u, v) \colon \quad \begin{cases} x = u \\ y = v \\ z = F(u, v) \end{cases} \tag{1.38}$$

Then

$$\mathbf{r}_u = \begin{pmatrix} 1 \\ 0 \\ F_u \end{pmatrix} \quad \mathbf{r}_v = \begin{pmatrix} 0 \\ 1 \\ F_v \end{pmatrix} \tag{1.39}$$

,

$$(\mathbf{r}_u, \mathbf{r}_u) = 1 + F_u^2, \quad (\mathbf{r}_u, \mathbf{r}_v) = F_u F_v, \quad (\mathbf{r}_v, \mathbf{r}_v) = 1 + F_v^2$$

and induced Riemannian metric (first quadratic form) (1.27) is equal to

$$||g_{\alpha\beta}|| = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_u, \mathbf{r}_u) & (\mathbf{r}_u, \mathbf{r}_v) \\ (\mathbf{r}_u, \mathbf{r}_v) & (\mathbf{r}_v, \mathbf{r}_v) \end{pmatrix} = \begin{pmatrix} 1 + F_u^2 & F_u F_v \\ F_u F_v & 1 + F_v^2 \end{pmatrix} \tag{1.40}$$

$$G_M = ds^2 = (1 + F_u^2) du^2 + 2 F_u F_v du dv + (1 + F_v^2) dv^2 \tag{1.41}$$

and the length of the curve $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ on C $(a \le t \le b)$ can be calculated by the formula:

$$L = \int \int_a^b \sqrt{(1 + F_u^2) u_t^2 + 2 F_u F_v u_t v_t + (1 + F_v)^2 v_t^2} dt$$

One can calculate (1.41) explicitly using (1.30):

$$G_M = \left(dx^2 + dy^2 + dz^2\right)\Big|_{x=u, y=v, z=F(u,v)} = (du)^2 + (dv)^2 + (F_u du + F_v dv)^2 =$$

$$= (1 + F_u^2) du^2 + 2 F_u F_v du dv + (1 + F_v^2) dv^2 \,. \tag{1.42}$$

Cylinder

Cylinder is given by the equation $x^2 + y^2 = a^2$. One can consider the following parameterisation of this surface:

$$\mathbf{r}(h, \varphi): \quad \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases} \tag{1.43}$$

We have $G_{cylinder} = (dx^2 + dy^2 + dz^2)\big|_{x=a\cos\varphi, y=a\sin\varphi, z=h} =$

$$= (-a\sin\varphi d\varphi)^2 + (a\cos\varphi d\varphi)^2 + dh^2 = a^2 d\varphi^2 + dh^2 \tag{1.44}$$

The same formula in terms of scalar product of tangent vectors:

$$\mathbf{r}_h = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \mathbf{r}_\varphi = \begin{pmatrix} -a\sin\varphi \\ a\cos\varphi \\ 0 \end{pmatrix} \tag{1.45}$$

,

$$(\mathbf{r}_h, \mathbf{r}_h) = 1, \quad (\mathbf{r}_h, \mathbf{r}_\varphi) = 0, \quad (\mathbf{r}_\varphi, \mathbf{r}_\varphi) = a^2$$

and

$$||g_{\alpha\beta}|| = \begin{pmatrix} (\mathbf{r}_u, \mathbf{r}_u) & (\mathbf{r}_u, \mathbf{r}_v) \\ (\mathbf{r}_u, \mathbf{r}_v) & (\mathbf{r}_v, \mathbf{r}_v) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a^2 \end{pmatrix},$$

$$G = dh^2 + a^2 d\varphi^2 \tag{1.46}$$

and the length of the curve $\mathbf{r}(t) = \mathbf{r}(h(t), \varphi(t))$ on the cylinder $(a \le t \le b)$ can be calculated by the formula:

$$L = \int_a^b \sqrt{h_t^2 + a^2\varphi_t} dt \tag{1.47}$$

Cone

Cone is given by the equation $x^2 + y^2 - k^2z^2 = 0$. One can consider the following parameterisation of this surface:

$$\mathbf{r}(h, \varphi): \quad \begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases} \tag{1.48}$$

Calculate induced Riemannian metric:

We have

$$G_{conus} = \left(dx^2 + dy^2 + dz^2\right)\big|_{x=kh\cos\varphi, y=kh\sin\varphi, z=h} =$$

$$(k\cos\varphi dh - kh\sin\varphi d\varphi)^2 + (k\sin\varphi dh + kh\cos\varphi d\varphi)^2 + dh^2$$

$$G_{conus} = k^2h^2d\varphi^2 + (1+k^2)dh^2,\ ||g_{\alpha\beta}|| = \begin{pmatrix} 1+k^2 & 0 \\ 0 & k^2h^2 \end{pmatrix} \tag{1.49}$$

The length of the curve $\mathbf{r}(t) = \mathbf{r}(h(t), \varphi(t))$ on the cone ($a \le t \le b$) can be calculated by the formula:

$$L = \int_a^b \sqrt{(1+k^2)h_t^2 + k^2h^2\varphi_t^2}dt \tag{1.50}$$

Circle

Circle of radius R is given by the equation $x^2 + y^2 = R^2$. Consider standard parameterisation φ of this surface:

$$\mathbf{r}(\varphi): \quad \begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \end{cases}$$

Calculate induced Riemannian metric (first quadratic form)

$$G_{S^2} = \left(dx^2 + dy^2\right)\big|_{x=R\cos\varphi, y=R\sin\varphi} =$$

$$(-R\sin\varphi d\varphi)^2 + (a\cos\varphi d\varphi)^2 = (R^2\cos^2\varphi + R^2\sin^2\varphi)d\varphi^2 = R^2d\varphi^2\,.$$

One can consider stereographic coordinates on the circle (see Example in the subsection 1.1) A point $x, y\colon x^2 + y^2 = R^2$ has stereographic coordinate t if points $(0,1)$ (north pole), the point (x, y) and the point $(t, 0)$ belong to the same line, i.e. $\frac{x}{t} = \frac{R-y}{R}$, i.e.

$$t = \frac{Rx}{R-y}, \qquad \begin{cases} x = \frac{2tR^2}{R^2+t^2} \\ y = \frac{t^2-R^2}{t^2+R^2}R \end{cases} \quad . \quad \text{since } x^2 + y^2 = R^2\,.$$

Induced metric in coordinate t is

$$G = (dx^2 + dy^2)\big|_{x=x(t), y=y(t)} = \left(d\left(\frac{2tR^2}{R^2+t^2}\right)\right)^2 + \left(d\left(\frac{t^2-R^2}{R^2+t^2}R\right)\right)^2 =$$

$$\left(\frac{2R^2dt}{R^2+t^2}-\frac{4t^2R^2dt}{(R^2+t^2)^2}\right)^2+\left(-\frac{4R^2tdt}{(t^2+R^2)^2}\right)^2=\frac{4R^4dt^2}{(R^2+t^2)^2}\,.$$

(See for detail Homework 2)

Remark Stereographic coordinates very often are preferable since they define birational equivalence between circle and line.

Sphere

Sphere of radius R is given by the equation $x^2+y^2+z^2=R^2$. Consider the following (standard) parameterisation of this surface:

$$\mathbf{r}(\theta,\varphi)\colon\qquad \begin{cases} x=R\sin\theta\cos\varphi \\ y=R\sin\theta\sin\varphi \\ z=R\cos\theta \end{cases} \tag{1.51}$$

Calculate induced Riemannian metric (first quadratic form)

$$G_{S^2}=\left(dx^2+dy^2+dz^2\right)\big|_{x=R\sin\theta\cos\varphi,y=R\sin\theta\sin\varphi,z=R\cos\theta}=$$

$$(R\cos\theta\cos\varphi d\theta-R\sin\theta\sin\varphi d\varphi)^2+(R\cos\theta\sin\varphi d\theta+R\sin\theta\cos\varphi d\varphi)^2+(-R\sin\theta d\theta)^2=$$

$$R^2\cos^2\theta d\theta^2+R^2\sin^2\theta d\varphi^2+R^2\sin^2\theta d\theta^2=$$

$$,\qquad =R^2d\theta^2+R^2\sin^2\theta d\varphi^2\,,\qquad ||g_{\alpha\beta}||=\begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix} \tag{1.52}$$

One comes to the same answer calculating scalar product of tangent vectors:

$$\mathbf{r}_\theta=\begin{pmatrix} R\cos\theta\cos\varphi \\ R\cos\theta\sin\varphi \\ -R\sin\theta \end{pmatrix}\qquad \mathbf{r}_\varphi=\begin{pmatrix} -R\sin\theta\sin\varphi \\ R\sin\theta\cos\varphi \\ 0 \end{pmatrix}$$

,

$$(\mathbf{r}_\theta,\mathbf{r}_\theta)=R^2,\quad (\mathbf{r}_h,\mathbf{r}_\varphi)=0,\quad (\mathbf{r}_\varphi,\mathbf{r}_\varphi)=R^2\sin^2\theta$$

and

$$||g||=\begin{pmatrix} (\mathbf{r}_u,\mathbf{r}_u) & (\mathbf{r}_u,\mathbf{r}_v) \\ (\mathbf{r}_u,\mathbf{r}_v) & (\mathbf{r}_v,\mathbf{r}_v) \end{pmatrix}=$$

$$\begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix},\quad G_{S^2}=ds^2=R^2d\theta^2+R^2\sin^2\theta d\varphi^2$$

The length of the curve $\mathbf{r}(t) = \mathbf{r}(\theta(t), \varphi(t))$ on the sphere of the radius a ($a \leq t \leq b$) can be calculated by the formula:

$$L = \int_a^b R\sqrt{\theta_t^2 + \sin^2\theta \cdot \varphi_t^2}dt \tag{1.53}$$

One can consider on sphere as well as on a circle stereographic coordinates:

$$\begin{cases} u = \frac{Rx}{R-z} \\ v = \frac{Ry}{R-z} \end{cases}, \qquad \begin{cases} x = \frac{2uR^2}{R^2+u^2+v^2} \\ y = \frac{2vR^2}{R^2+u^2+v^2} \\ z = \frac{u^2+v^2-R^2}{u^2+v^2+R^2}R \end{cases} \tag{1.54}$$

In these coordinates Riemannian metric is

$$G = (dx^2 + dy^2 + dz^2)\big|_{x=x(u,v),y=y(u,v),z=z(u,v)} =$$

$$\left(d\left(\frac{2uR^2}{R^2+u^2+v^2}\right)\right)^2 + \left(d\left(\frac{2vR^2}{R^2+u^2+v^2}\right)\right)^2 + \left(d\left(1 - \frac{2R^2}{R^2+u^2+v^2}\right)R\right)^2 =$$

$$= \frac{4R^4(du^2+dv^2)}{(R^2+u^2+v^2)^2}.$$

(see in detail Homework 2 and Solutions)

Saddle (paraboloid)[2]

Consider paraboloid $z = x^2 - y^2$. It can be rewritten as $z = axy$ and it is called sometimes "saddle" (rotation on the angle $\varphi = \pi/4$ transforms $z = x^2 - y^2$ onto $z = 2xy$.) Paraboloid and saddle they are ruled surfaces which are formed by lines. We considered this surface in the course of Geometry.

Consider the following (standard) parameterisation of this surface:

$$\mathbf{r}(u,v)\colon \quad \begin{cases} x = u \\ y = v \\ z = uv \end{cases} \tag{1.55}$$

Calculate induced metric:

$$G_{saddle} = \left(dx^2 + dy^2 + dz^2\right)\big|_{x=u\cos\varphi, y=v\sin\varphi, z=uv} = du^2 + dv^2 + (udv + vdu)^2 =$$

[2] *This example was not considered on lectures. It could be useful for learning purposes.*

$$G_{saddle} = (1+v^2)du^2 + 2uvdudv + (1+u^2)dv^2\,.$$

One-sheeted and two-sheeted hyperboloids.

These examples were mostly considered on tutorials.

Consider surface given by the equation

$$x^2 + y^2 - z^2 = c$$

If $c = 0$ it is a cone. We considered it already above.

If $c > 0$ it is one-sheeted hyperboloid—connected surface in $\mathbf{E}^3$.

If $c < 0$ it is two-sheeted hyperboloid— a surface with two sheets: upper sheet $z > 0$ and another sheet: $z < 0$.

Consider these cases separately.

1) *One-sheeted hyperboloid*: $x^2 + y^2 - z^2 = a^2$. It is ruled surface.

Exercise† Find the lines on two-sheeted hyperboloid

One-sheeted hyperboloid is given by the equation $x^2 + y^2 - z^2 = a^2$. It is convenient to choose parameterisation:

$$\mathbf{r}(\theta,\varphi)\colon \qquad \begin{cases} x = a\cosh\theta\cos\varphi \\ y = a\cosh\theta\sin\varphi \\ z = a\sinh\theta \end{cases} \tag{1.56}$$

$$x^2 + y^2 - z^2 = a^2\cosh^2\theta - a^2\sinh^2\theta = a^2\,.$$

(Compare the calculations with calculations for sphere! We changed functions $\cos, \sin$ on $\cosh, \sinh$.)

Induced Riemannian metric (first quadratic form) is equal to

$$G_{HyperbolI} = \left(dx^2 + dy^2 + dz^2\right)\big|_{x=a\cosh\theta\cos\varphi, y=a\cosh\theta\sin\varphi, z=a\sinh\theta} =$$

$$(a\sinh\theta\cos\varphi d\theta - a\cosh\theta\sin\varphi d\varphi)^2 + (a\sinh\theta\sin\varphi d\theta + a\cosh\theta\cos\varphi d\varphi)^2 + (a\cosh\theta d\theta)^2 =$$

$$a^2\sinh^2\theta d\theta^2 + a^2\cosh^2\theta d\varphi^2 + a^2\cosh^2\theta d\theta^2 =$$

$$, \qquad = a^2(1+2\sinh^2\theta)d\theta^2 + a^2\cosh^2\theta d\varphi^2\,, \qquad ||g_{\alpha\beta}|| = \begin{pmatrix} a^2(1+2\sinh^2\theta) & 0 \\ 0 & a^2\cosh^2\theta \end{pmatrix}$$

2) *Two-sheeted hyperboloid*: $z^2 - x^2 - y^2 = a^2$. It is not ruled surface!

For two-sheeted hyperboloid calculations will be very similar.

In the same way as for one-sheeted hyperboloid (see equation (1.56)) it is convenient to choose parameterisation:

$$\mathbf{r}(\theta,\varphi)\colon \qquad \begin{cases} x = a\sinh\theta\cos\varphi \\ y = a\sinh\theta\sin\varphi \\ z = a\cosh\theta \end{cases} \tag{1.57}$$

$$z^2 - x^2 - y^2 = a^2\cosh^2\theta - a^2\sinh^2\theta = a^2$$

(Compare the calculations with calculations for sphere and one-sheeted hyperboloid.

Induced Riemannian metric (first quadratic form) is equal to

$$G_{HyperbolI} = \left(dx^2 + dy^2 + dz^2\right)\big|_{x=a\sinh\theta\cos\varphi, y=a\sinh\theta\sin\varphi, z=a\cosh\theta} =$$

$$(a\cosh\theta\cos\varphi d\theta - a\sinh\theta\sin\varphi d\varphi)^2 + (a\cosh\theta\sin\varphi d\theta + a\sinh\theta\cos\varphi d\varphi)^2 + (a\sinh\theta d\theta)^2 =$$

$$a^2\cosh^2\theta d\theta^2 + a^2\sinh^2\theta d\varphi^2 + a^2\sinh^2\theta d\theta^2 =$$

$$, \qquad = a^2(1+2\sinh^2\theta)d\theta^2 + a^2\sinh^2\theta d\varphi^2\,, \qquad ||g_{\alpha\beta}|| = \begin{pmatrix} a^2(1+2\sinh^2\theta) & 0 \\ 0 & a^2\sinh^2\theta \end{pmatrix} \tag{1.58}$$

We calculated examples of induced Riemannian structure embedded in Euclidean space almost for all quadratic surfaces.

Quadratic surface is a surface defined by the equation

$$Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz + ex + fy + dz + c = 0$$

One can see that any quadratic surface by affine transformation can be transformed to one of these surfaces

- cylinder (elliptic cylinder) $x^2 + y^2 = 1$
- hyperbolic cylinder: $x^2 - y^2 = 1$)
- parabolic cylinder $z = x^2$
- paraboloid $x^2 + y^2 = z$
- hyperbolic paraboloid $x^2 - y^2 = z$
- cone $x^2 + y^2 - z^2 = 0$
- sphere $x^2 + y^2 + z^2 = 1$
- one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$
- two-sheeted hyperboloid $z^2 - x^2 - y^2 = 1$

(We exclude degenerate cases such as "point" $x^2 + y^2 + z^2 = 0$, planes, e.t.c.)

1.4.4 *Induced metric on two-sheeted hyperboloid embedded in pseudo-Euclidean space.

Consider the same two-sheeted hyperboloid $z^2 - x^2 - y^2 = 1$ embedded $\mathbf{R}^3$ (See equation (1.57). For simplicity we assume now that $a = 1$.) Now we consider the ambient space $\mathbf{R}^3$ not as Euclidean space but as *pseudo-Euclidean space*, i.e. in $\mathbf{R}^3$ instead standard scalar product

$$\langle \mathbf{X}, \mathbf{Y} \rangle = X^1Y^1 + X^2Y^2 + X^3Y^3$$

we consider pseudo-scalar product defined by bilinear form

$$\langle \mathbf{X}, \mathbf{Y} \rangle_{pseud} = X^1Y^1 + X^2Y^2 - X^3Y^3$$

The "pseudoscalar" product is bilinear, symmetric. It is defined by non-degenerate matrix. But it is not positive-definite. E.g. The "pseudo-length" of vectors $\mathbf{X} = (a\cos\varphi, a\sin\varphi, \pm a)$ is equals to zero (such vectors are called null vectors):

$$\mathbf{X} = (a\cos\varphi, a\sin\varphi, \pm a) \Rightarrow \langle \mathbf{X}, \mathbf{X} \rangle_{pseudo} = 0,$$

The corresponding pseudo-Riemannian metric is:

$$G_{pseudo} = dx^2 + dy^2 - dz^2 \tag{1.59}$$

It turns out that the following remarkable fact occurs:

Proposition *The pseudo-Riemannian metric* (1.59) *in the ambient* 3*-dimensional pseudo-Euclidean space induces Riemannian metric on two-sheeted hyperboloid* $x^2 + y^2 - z^2 = 1$.

Remark This is not the fact for one-sheeted hyperboloid (see problem 7 in Homework 2)

Show it. (See also problems 5 and 6 in Homework 2.) Repeat the calculations above for two-sheeted hyperboloid changing in the ambient space Riemannian metric $G = dx^2 + dy^2 + dz^2$ on pseudo-Riemannian $dx^2 + dy^2 - dz^2$:

Using (1.57) and (1.59) we come now to

$$G = \left(dx^2 + dy^2 - dz^2\right)\big|_{x=a\sinh\theta\cos\varphi, y=a\sinh\theta\sin\varphi, z=a\cosh\theta} =$$

$$(a\cosh\theta\cos\varphi d\theta - a\sinh\theta\sin\varphi d\varphi)^2 + (a\cosh\theta\sin\varphi d\theta + a\sinh\theta\cos\varphi d\varphi)^2 - (a\sinh\theta d\theta)^2 =$$

$$a^2\cosh^2\theta d\theta^2 + a^2\sinh^2\theta d\varphi^2 - a^2\sinh^2\theta d\theta^2$$

$$, \qquad G_L = a^2 d\theta^2 + a^2\sinh^2\theta d\varphi^2, \qquad ||g_{\alpha\beta}|| = \begin{pmatrix} 1 & 0 \\ 0 & \sinh^2\theta \end{pmatrix} \tag{1.60}$$

The two-sheeted hyperboloid equipped with this metric is called hyperbolic or Lobachevsky plane.

Now express Riemannian metric in stereographic coordinates. (We did it in detail in homework 2)

Calculations are very similar to the case of stereographic coordinates of 2-sphere $x^2 + y^2 + z^2 = 1$. (See homework 1). Centre of projection $(0, 0, -1)$: For stereographic coordinates u, v we have $\frac{u}{x} = \frac{y}{v} = \frac{1}{1+z}$. We come to

$$\begin{cases} u = \frac{x}{1+z} \\ v = \frac{y}{1+z} \end{cases} , \qquad \begin{cases} x = \frac{2u}{1-u^2-v^2} \\ y = \frac{2v}{1-u^2-v^2} \\ z = \frac{u^2+v^2+1}{1-u^2-v^2} \end{cases} \tag{4}$$

The image of upper-sheet is an open disc $u^2 + v^2 = 1$ since $u^2 + v^2 = \frac{x^2+y^2}{(1+z)^2} = \frac{z^2-1}{(1+z)^2} = \frac{z-1}{z+1}$. Since for upper sheet $z > 1$ then $0 \leq \frac{z-1}{z+1} < 1$.

$$G = (dx^2 + dy^2 - dz^2)\big|_{x=x(u,v), y=y(u,v), z=z(u,v)} = \left(d\left(\frac{2u}{1-u^2-v^2}\right)\right)^2 +$$

$$\left(d\left(\frac{2v}{1-u^2-v^2}\right)\right)^2 - \left(d\left(\frac{u^2+v^2+1}{1-u^2-v^2}\right)\right)^2 = \frac{4(du)^2 + 4(dv)^2}{(1+u^2+v^2)^2} .$$

These coordinates are very illuminating. One can show that we come to so called hyperbolic plane (see in detail Homework 2)

1.5 Isometries of Riemannian manifolds.

Let $(M_1, G_{(1)})$, $(M_2, G_{(2)})$ be two Riemannian manifolds— manifolds equipped with Riemannian metric $G_{(1)}$ and $G_{(2)}$ respectively.

Loosely speaking isometry is the diffeomorphism of Riemannian manifolds which preserves the distance.

Definition Let F be a diffeomorphims (one-one smooth map with smooth inverse) of manifold M_1 on manifold M_2.

We say that diffeomorphism F is an isometry of Riemannian manifolds $(M_1, G_{(1)})$ and $(M_2, G_{(2)})$ if it preserves the metrics, i.e. $G_{(1)}$ is pull-back of $G_{(2)}$:

$$F^* G_{(2)} = G_{(1)} . \tag{1.61}$$

In local coordinates this means the following: Let $\mathbf{p}_1$ be an arbitrary point on manifold M_1 and $\mathbf{p}_2 \in M_2$ be its image:$F(\mathbf{p}_1) = \mathbf{p}_2$. Let $\{x^i\}$ be arbitrary oordinates in a vicinity of a point $\mathbf{p}_1 \in M_1$ and $\{y^a\}$ be arbitrary coordinates in a vicinity of a point $\mathbf{p}_2 \in M_2$. Let Riemannian metrics $G_{(1)}$ on M_1 has local expression $G_{(1)} = g_{(1)ik}(x)dx^i dx^k$ in coordinates $\{x^i\}$ and respectively Riemannian metrics $G_{(2)}$ has local expression $G_{(2)} = g_{(2)ab}(y)dy^a dy^b$ in coordinates $\{y^a\}$ on M_2. Then the formula (1.61) has the following appearance in these local coordinates:

$$F^*\left(g_{(2)ab}(y)dy^a dy^b\right) = g_{(2)ab}(y)dy^a dy^b\big|_{y=y(x)} =$$

$$g_{(2)ab}(y(x))\frac{\partial y^a(x)}{\partial x^i}dx^i\frac{\partial y^b(x)}{\partial x^k}dx^k = g_{(1)ik}(x)dx^i dx^k\,, \tag{1.62}$$

i.e.

$$g_{(1)ik}(x) = \frac{\partial y^a(x)}{\partial x^i}g_{(2)ab}(y(x))\frac{\partial y^b(x)}{\partial x^k}\,, \tag{1.63}$$

where $y^a = y^a(x)$ is local expression for diffeomorphism F. We say that diffeomorphism F is *isometry* of Riemanian manifolds $(M_1, G_{(1)})$ and $M_2, G_{(2)}$.

Diffeomorphism F establishes one-one correspondence between local coordinates on manifolds M_1 and M_2. The left hand side of equation (1.62) can be considered as a local expression of metric $G_{(2)}$ in coordinates x^i on M_2 and the right hand side of this equation is local expression of metric $G_{(1)}$ in coordinates x^i on M_1. Diffeomorphism F identifies manifolds M_1 and M_2 and it can be considered as changing of coordinates.

Example Consider surface of cylinder C, $x^2 + y^2 = a^2$ in $\mathbf{E}^3$ with induced Riemannian metric $G_C = a^2 d\varphi^2 + dh^2$ (see equations (1.43) and (1.44)). If we remove the line l: $x = a, y = 0$ from the cylinder surface C we come to surface $C' = C\backslash l$. Consider a map F of this surface in Euclidean space E^2 with Cartesian coordinates u, v (with standard Euclidean metric $G_{Eucl} = du^2 + dv^2$):

$$F\colon \qquad \begin{cases} u = a\varphi \\ v = h \end{cases} \qquad 0 < \varphi < 2\pi\,. \tag{1.64}$$

One can see that F is the diffeomorphism of C' on the domain $0 < u < 2\pi a$ in $\mathbf{E}^2$ and this diffeomorphism is an isometry: it transforms the metric G_{Eucl} on Euclidean space in metric G_C on cylinder, i.e. pull-back condition (1.61) is obeyed:

$$F^* G_{Eucl} = F^*\left(du^2 + dv^2\right) = \left(du^2 + dv^2\right)\big|_{u=a\varphi, v=h} = a^2 d\varphi^2 + dh^2 = G_1\,.$$

We see that cylinder surface with removed line is isometric to domain in $\mathbf{E}^2$ and the map F establishes this isometry.

Remark Let F be diffeomorphism of manifold M_1 on a manifold M_2. Let a manifold M_2 be equipped with Riemannian metric $G_{(2)}$. Then consider the pull-back of this metric, Riemannian metric $G_{(1)} = F^*G_{(2)}$ on M_1. We see that diffeomorphism F is an isometry of Riemannian manifold $(M_1, G_{(1)})$ on Riemannian manifold $(M_2, G_{(2)})$.

1.5.1 Isometries of Riemannian manifold (on itself)

Definition Let (M, G) be a Riemannian manifold. We say that a diffeomorphism F is an isometry of Riemannian manifold on itself if it preserves the metric, i.e. $F^*G = G$. In local coordinates this means that

$$g_{ik}(x) = g_{pq}(x'(x)) \frac{\partial x^p(x')}{\partial x^i} \frac{\partial x^q(x')}{\partial x^k}, \tag{1.65}$$

where $x' = x'(x)$ is a local expression for diffeomorphism F. **Example** Let $\mathbf{E}^2$ be Euclidean plane with metric $dx^2 + dy^2$ in Cartesian coordinates x, y. Consider the transformation

$$\begin{cases} x' = p + ax + by \\ y' = q + cx + dy \end{cases}$$

is isometry if and only if the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an orthogonal matrix, i.e. if the trasformation above is combination of translation, rotation and reflection.

1.6 *Infinitesimal isometries of Riemannian manifold

Let $\mathbf{X}$ be an arbitrary vector field on Riemannian manifold M. It induces infinitesimal diffeomorphism

$$F\colon x^{i'} = x^i + \varepsilon X^i(x), \quad \text{where } \varepsilon^2 = 0\,.$$

(the condition $\varepsilon^2 = 0$ reflects the fact that we ignore terms of order ≥ 2 over ε.) Find a condition which guarantees that infinitesimal diffeomorphism is an isometry. If $x^{i'} = x^i + \varepsilon X^i(x)$, then one can see that the inverse infinitesimal diffeomorphism is defined by the equation $x^i = x^{i'} - \varepsilon X^i(x')$ and equation (1.65) implies that

$$g_{ik}(x) = g_{pq}(x'(x)) \frac{\partial x^p(x')}{\partial x^i} \frac{\partial x^q(x')}{\partial x^k} = g_{pq}(x^i + \varepsilon X^i) \left(\delta^p_i + \varepsilon \frac{pX^p(x)}{\partial x^i}\right) \left(\delta^q_k + \varepsilon \frac{pX^q(x)}{\partial x^k}\right) =$$

$$g_{ik}(x) + \varepsilon \left[X^p(x) \frac{\partial g_{ik}(x)}{\partial x^p} + g_{iq}(x) \frac{\partial X^q(x)}{\partial x^k} + g_{pk}(x) \frac{\partial X^p(x)}{\partial x^i}\,,\right]$$

Here we consider only terms of first and zero order over ε since $\varepsilon^2 = 0$ (this is related with the fact that transformation is *infinitesimal*). The last relation implies that

$$X^p(x)\frac{\partial g_{ik}(x)}{\partial x^p} + g_{iq}(x)\frac{\partial X^q(x)}{\partial x^k} + g_{pk}(x)\frac{\partial X^p(x)}{\partial x^i} = 0\,. \tag{1.66}$$

Left hand side of this relation we denote $\mathcal{L}_{\mathbf{X}}G$— Lie derivative of Riemannian metric along vector field $\mathbf{X}$. Vector field $\mathbf{X}$ induces isometry if Lie derivative of metric along this vector field vanishes. We come to

Proposition Vector field $\mathbf{X}$ on Riemannian manifold (M, G) induces infinitesimal isometry if $\mathcal{L}_{\mathbf{X}}G = 0$:

$$\mathcal{L}_{\mathbf{X}}G = X^p(x)\frac{\partial g_{ik}(x)}{\partial x^p} + g_{iq}(x)\frac{\partial X^q(x)}{\partial x^k} + g_{pk}(x)\frac{\partial X^p(x)}{\partial x^i} = 0\,. \tag{1.67}$$

Definition) We call vector field $\mathbf{X}$ *Killing vector field)* if it preserves the metric, i.e. if equation (1.67) is obeyed.

Example Consider plane (x, y) with Riemannian metric $G = \sigma(x, y)(dx^2 + dy^2)$. Find differential equation for infinitesimal isometries of this metric, i.e. write down equations (1.67) for this metric.

We have $||g_{ik}(x, y)|| = \begin{pmatrix} \sigma(x, y) & 0 \\ 0 & \sigma(x, y) \end{pmatrix}$.

Let $\mathbf{X} = A(x, y)\partial_x + B(x, y)\partial_y$. Write down equations (1.67) for components g_11, g_12, g_{21} and g_{22}: We will have the following three equations

$$\begin{cases} A(x,y)\frac{\partial \sigma}{\partial x} + B(x,y)\frac{\partial \sigma}{\partial y} + 2\frac{\partial A(x,y)}{\partial x}\sigma = 0 & \text{for component } g_{11} \\ A(x,y)\frac{\partial \sigma}{\partial x} + B(x,y)\frac{\partial \sigma}{\partial y} + 2\frac{\partial B(x,y)}{\partial y}\sigma = 0 & \text{for component } g_{22} \\ \frac{\partial B(x,y)}{\partial x} + \frac{\partial A(x,y}{\partial y} = 0 \quad \text{for components } g_{12} \text{ and } g_{21} \end{cases} \tag{1.68}$$

Practically for sphere, Lobachgevsky plane, e.t.c. it is much easier to find the Killing fields not solving these equations, but considering the usual isometries (see examples in solutions of Coursework and in the Appendix about Killing vector fields for Lobachevsky plane.))

1.6.1 Locally Euclidean Riemannian manifolds

It is useful to formulate the local isometry condition between Riemannian manifold and Euclidean space. A neighbourhood of every point of n-dimensional manifold is diffeomorphic to $\mathbf{R}^n$. Let as usual $\mathbf{E}^n$ be n-dimensional Euclidean space, i.e. $\mathbf{R}^n$ with standard Riemannian metric $G = dx^i\delta_{ik}dx^k = (dx^1)^2 + \cdots + (dx^n)^2$ in Cartesian coordinates $(x^1, \ldots, x^n)$.

Definition We say that n-dimensional Riemannian manifold (M, G) is locally isometric to Euclidean space $\mathbf{E}^n$, i.e. it is locally Euclidean Riemannian manifold, if for every point $\mathbf{p} \in M$ there exists an open neighboorhood D (domain) containing this point, $\mathbf{p} \in D$ such that D is isometric to a domain in Euclidean plane. In other words in a vicinity of every point $\mathbf{p}$ there exist local coordinates $u^1, \dots, u^n$ such that Riemannian metric G in these coordinates has an appearance

$$G = du^i \delta_{ik} du^k = (du^1)^2 + \dots + (du^n)^2 \,. \tag{1.69}$$

Consider examples.

Example Consider again cylinder surface..

We know that cylinder is not diffeomorphic to plane (there are plenty reasons for this). In the previous subsection we cutted the line from cylindre. Thus we came to surface diffeomorphic to plane. We established that this surface is isometric to Euclidean plane. (See equation (1.64) and considerations above.) Local isometry of cylinder to the Euclidean plane, i.e. the fact that it is locally Euclidean Riemannian surface immediately follows from the fact that under changing of local coordinates $u = a\varphi, v = h$ in equation (1.64), the standard Euclidean metric $du^2 + dv^2$ transforms to the metric $G_{cylinder} = a^2 d\varphi^2 + dh^2$ on cylinder.

Example Now show that cone is locally Euclidean Riemannian surface, i.e, it is locally isometric to the Euclidean plane. This means that we have to find local coordinates u, v on the cone such that in these coordinates induced metric $G|_c$ on cone would have the appearance $G|_c = du^2 + dv^2$. Recall calculations of the metric on cone in coordinates h, φ where

$$\mathbf{r}(h, \varphi)\colon \begin{cases} x = kh \cos\varphi \\ y = kh \sin\varphi \\ z = h \end{cases} ,$$

$x^2 + y^2 - k^2 z^2 = k^2 h^2 \cos^2\varphi + k^2 h^2 \sin^2\varphi - k^2 h^2 = k^2 h^2 - k^2 h^2 = 0$. We have that metric G_c on the cone in coordinates h, φ induced with the Euclidean metric $G = dx^2 + dy^2 + dz^2$ is equal to

$$G_c = \left(dx^2 + dy^2 + dz^2\right)\big|_{x = kh\cos\varphi, y = kh\sin\varphi, z = h} = (k\cos\varphi dh - kh\sin\varphi d\varphi)^2 +$$

$$(k\sin\varphi dh + kh\cos\varphi d\varphi)^2 + dh^2 = (k^2 + 1) dh^2 + k^2 h^2 d\varphi^2 \,.$$

In analogy with polar coordinates try to find new local coordinates u, v such that $\begin{cases} u = \alpha h \cos\beta\varphi \\ v = \alpha h \sin\beta\varphi \end{cases}$, where α, β are parameters. We come to $du^2 + dv^2 =$

$$(\alpha\cos\beta\varphi dh - \alpha\beta h\sin\beta\varphi d\varphi)^2 + (\alpha\sin\beta\varphi dh + \alpha\beta h\cos\beta\varphi d\varphi)^2 = \alpha^2 dh^2 + \alpha^2\beta^2 h^2 d\varphi^2 .$$

Comparing with the metric on the cone $G_c = (1+k^2)dh^2 + k^2h^2d\varphi^2$ we see that if we put $\alpha = k$ and $\beta = \frac{k}{\sqrt{1+k^2}}$ then $du^2 + dv^2 = \alpha^2 dh^2 + \alpha^2\beta^2h^2d\varphi^2 = (1+k^2)dh^2 + k^2h^2d\varphi^2$.

Thus in new local coordinates

$$\begin{cases} u = \sqrt{k^2+1}h\cos\frac{k}{\sqrt{k^2+1}}\varphi \\ v = \sqrt{k^2+1}h\sin\frac{k}{\sqrt{k^2+1}}\varphi \end{cases}$$

induced metric on the cone becomes $G|_c = du^2 + dv^2$, i.e. cone locally is isometric to the Euclidean plane ■

Of course these coordinates are local.— Cone and plane are not homeomorphic, thus they are not globally isometric.

Example and counterexample

Consider domain D in Euclidean plane with two metrics:

$$G_{(1)} = du^2 + \sin^2 v dv^2\,, \quad \text{and} \quad G_{(2)} = du^2 + \sin^2 u dv^2 \tag{1.70}$$

Thus we have two different Rimeannian manifolds $(D, G_{(1)})$ and $(D, G_{(2)})$. Metrics in (1.70) look similar. But.... It is easy to see that the first one is locally isometric to Euclidean plane, i.e. it is locally Euclidean Riemannian manifold since $\sin^2 v dv^2 = d(-\cos v)^2$: in new coordinates $u' = u, v' = \cos v$ Riemannian metric $G_{(1)}$ has appearance of standard Euclidean metric:

$$(du')^2 + (dv')^2 = (du)^2 + (d(\cos v))^2 = du^2 + \sin^2 v dv^2 = G_{(1)}\,.$$

This is not the case for second metric $G_{(2)}$. If we change notations $u \mapsto \theta$, $v \mapsto \varphi$ then $G_{(2)} = d\theta^2 + sin\theta^2 d\varphi^2$. This is local expression for Riemannian metric induced on the sphere of radius $R = 1$. Suppose that there exist coordinates $u' = u'(\theta, \varphi)$ $v' = v'(\theta, \varphi)$ such that in these coordinates metric has Eucldean appearance. This means that locally geometry of sphere is as a geometry of Euclidean plane. On the other hand we know from the course of Geometry that this is not the case: sum of angles of triangels on the sphere is not equal to π, sphere cannot be bended without shrinking. Later in this course we will return to this question....

There are plenty other examples:

2) Plane with metric $\frac{4R^4(dx^2+dy^2)}{(R^2+x^2+y^2)^2}$ is isometric to the sphere with radius R.

3) Disc with metric $\frac{du^2+dv^2}{(1-u^2-v^2)^2}$ is isometric to half plane with metric $\frac{dx^2+dy^2}{4y^2}$.

(see also exercises in Homeworks and Coursework.)

1.7 Volume element in Riemannian manifold

The volume element in n-dimensional Riemannian manifold with metric $G = g_{ik}dx^i dx^k$ is defined by the formula

$$\sqrt{\det g_{ik}}\, dx^1 dx^2 \ldots dx^n \,. \tag{1.71}$$

If D is a domain in the n-dimensional Riemannian manifold with metric $G = g_{ik}dx^i$ then its volume is equal to to the integral of volume element over this domain.

$$V(D) = \int_D \sqrt{\det g_{ik}}\, dx^1 dx^2 \ldots dx^n \,. \tag{1.72}$$

Remark Students who know the concept of exterior forms can read the volume element as n-form

$$\sqrt{\det g_{ik}}\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \,.$$

Note that in the case of $n = 1$ volume is just the length, in the case if $n = 2$ it is area.

1.7.1 Volume of parallelepiped

Use formulae (1.71),(1.72) to calculate volume of n-dimensional parallelepiped. Let $\mathbf{E}^n$ be Euclidean vector space with orthonormal basis $\{\mathbf{e}_i\}$. Let $\mathbf{a}_i$ be an arbitrary basis in this vector space (vectors $\mathbf{v}_i$ in general have not unit length and are not orthogonal to each other). Consider n-parallelepiped spanned by vectors $\{\mathbf{v}_i\}$:

$$\Pi_{\mathbf{v}_i} : \mathbf{r} = t^i \mathbf{v}_i, 0 \le t^i \le 1.$$

We know that the volume of this parallelepiped is equal to

$$Vol(\Pi_{\mathbf{v}_i}) = \det ||a_i^m|| \,, \tag{1.73}$$

where $A = ||a_i^m||$ is transition matrix, $\mathbf{v}_i = \mathbf{e}_m a_i^m$. (We know this formula t least for $n = 1$–length of interval, $n = 2$— area of parallelogram and $n = 3$ volume of paralleliped and vector product (see below))

On the other hand

$$\mathbf{r} = x^i \mathbf{e}_i = t^m \mathbf{v}_m, \text{ hence } x^i = a_m^i t^m, \text{ where } \mathbf{v}_m = \mathbf{e}_i a_m^i.$$

Let $G = (dx^1)^2 + \cdots + (dx^n)^2 = g_{ik} dt^i dt^k$ be usual Euclidean metric in new coordinates t^i. Then

$$G = (dx^1)^2 + \cdots + (dx^n)^2 = dx^i \delta_{ik} dx^k = dt^i \frac{\partial x^{i'}}{\partial t^i} \delta_{i'k'} \frac{\partial x^{k'}}{\partial t^k} dt^k.$$

Since $\frac{\partial x^{i'}}{\partial t^i} = a_i^{i'}$ then

$$g_{ik} = \sum_{i'} a_i^{i'} a_k^{i'} \Rightarrow \det g = (\det A)^2, \det g = \sqrt{\det A}.$$

and according to the formula (1.72)

$$Vol(\Pi_{\mathbf{v}_i}) = \int_{0 \le t^i \le 1} \sqrt{\det g} dt^1 dt^2 \dots dt^n = \det A\,.$$

We come to (1.73).

Perform these calculations in detail for 3-dimensional case.

Let $\mathbf{E}^3$ be 3-dimensional Euclidean space. Consider parallelepiped spanned by vectors $\{\mathbf{a}_1 \mathbf{a}_2, \mathbf{a}_3\}$:

$$\Pi_{\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3} = t^1\mathbf{a}_1 + t^2\mathbf{a}_2 + t^3\mathbf{a}_3\,, \quad 0 \le t^1, t^2, t^3 \le 1.$$

We know form standard course of Geometry (calculus, e.t.c.) that volume of parallelepiped equals to

$$Vol\Pi_{\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3} = (\mathbf{a}_1, \mathbf{a}_2 \times \mathbf{a}_3) = \begin{pmatrix} a_1^1 & a_1^2 & a_1^3 \\ a_2^1 & a_2^2 & a_2^3 \\ a_3^1 & a_3^2 & a_3^3 \end{pmatrix} = \det ||a_i^k||\,, \tag{1.74}$$

(rows of matirx are components of vectors $\mathbf{a}_i$). Vectors $\mathbf{a}_i = \mathbf{e}_k a_i^k$, where $\mathbf{e}_k$ is standard basis in $\mathbf{E}^3$: $\Pi_{\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3} \ni \mathbf{r} = t^i\mathbf{a}_i = t^i a_i^k \mathbf{e}_k = x^k \mathbf{e}_k$. Hence we come from Cartesian coordinates x^i to new coordinates t^i by changing of coordinates

$$x^k = a_i^k t^i\,.$$

The Riemannian metric in new coordinates (t^1, t^2, t^3) is equal to

$$(dx^1)^2 + (dx^2)^2 + (dx^3)^2 = dx^i \delta_{ik} \delta^k = a_m^i dt^m \delta_{ik} a_n^k dt^n,$$

i.e. in coordinates t^i $g_{mn} = a_m^i a_n^i$ and

$$\sqrt{\det g} = \sqrt{\det(a^T \cdot a)} = \det ||a_k^i||,$$

Volume of parallelpiped is equal to

$$\int_{0 \le t^i \le 1} \left(\sqrt{\det g}\right) dt^1 dt^2 dt^3 = \sqrt{\det g} \int_0^1 dt^1 \int_0^1 dt^2 \int_0^1 dt^3 = \sqrt{\det g} = \det ||a_i^k||\,.$$

1.7.2 Invariance of volume element under changing of coordinates

Prove that volume element is invariant under coordinate transformations, i.e. if $y^1, \dots, y^n$ are new coordinates: $x^1 = x^1(y^1, \dots, y^n), x^2 = x^2(y^1, \dots, y^n)...$,

$$x^i = x^i(y^p), i = 1, \dots, n \quad , p = 1, \dots, n$$

and $\tilde{g}_{pq}(y)$ matrix of the metric in new coordinates:

$$\tilde{g}_{pq}(y) = \frac{\partial x^i}{\partial y^p} g_{ik}(x(y)) \frac{\partial x^k}{\partial y^q} . \tag{1.75}$$

Then

$$\sqrt{\det g_{ik}(x)}\, dx^1 dx^2 \dots dx^n = \sqrt{\det \tilde{g}_{pq}(y)}\, dy^1 dy^2 \dots dy^n \tag{1.76}$$

This follows from (1.75). Namely

$$\sqrt{\det g_{ik}(y)}\, dy^1 dy^2 \dots dy^n = \sqrt{\det \left(\frac{\partial x^i}{\partial y^p} g_{ik}(x(y)) \frac{\partial x^k}{\partial y^q} \right)}\, dy^1 dy^2 \dots dy^n$$

Using the fact that $\det(ABC) = \det A \cdot \det B \cdot \det C$ and $\det \left(\frac{\partial x^i}{\partial y^p} \right) = \det \left(\frac{\partial x^k}{\partial y^q} \right)$[3] we see that from the formula above follows:

$$\sqrt{\det g_{ik}(y)}\, dy^1 dy^2 \dots dy^n = \sqrt{\det \left(\frac{\partial x^i}{\partial y^p} g_{ik}(x(y)) \frac{\partial x^k}{\partial y^q} \right)} dy^1 dy^2 \dots dy^n =$$

$$\sqrt{\left(\det \left(\frac{\partial x^i}{\partial y^p} \right) \right)^2} \sqrt{\det g_{ik}(x(y))} dy^1 dy^2 \dots dy^n =$$

$$\sqrt{\det g_{ik}(x(y))} \det \left(\frac{\partial x^i}{\partial y^p} \right) dy^1 dy^2 \dots dy^n = \tag{1.77}$$

Now note that

$$\det \left(\frac{\partial x^i}{\partial y^p} \right) dy^1 dy^2 \dots dy^n = dx^1 \dots dx^n$$

according to the formula for changing coordinates in n-dimensional integral [4]. Hence

$$\sqrt{\det g_{ik}(x(y))} \det \left(\frac{\partial x^i}{\partial y^p} \right) dy^1 dy^2 \dots dy^n = \sqrt{\det g_{ik}(x(y))} dx^1 dx^2 \dots dx^n \tag{1.78}$$

[3]determinant of matrix does not change if we change the matrix on the adjoint, i.e. change columns on rows.

[4]Determinant of the matrix $\left(\frac{\partial x^i}{\partial y^p} \right)$ of changing of coordinates is called sometimes Jacobian. Here we consider the case if Jacobian is positive. If Jacobian is negative then formulae above remain valid just the symbol of modulus appears.

Thus we come to (1.76).

1.7.3 Examples of calculating volume element

Consider first very simple example: Volume element of plane in Cartesian coordinates, metric $g = dx^2 + dy^2$. Volume element is equal to

$$\sqrt{\det g}dxdy = \sqrt{\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}dxdy = dxdy$$

Volume of the domain D is equal to

$$V(D) = \int_D \sqrt{\det g}dxdy = \int_D dxdy$$

If we go to polar coordinates:

$$x = r\cos\varphi, y = r\sin\varphi \tag{1.79}$$

Then we have for metric:

$$G = dr^2 + r^2 d\varphi^2$$

because

$$dx^2 + dy^2 = (dr\cos\varphi - r\sin\varphi d\varphi)^2 + (dr\sin\varphi + r\cos\varphi d\varphi)^2 = dr^2 + r^2 d\varphi^2 \tag{1.80}$$

Volume element in polar coordinates is equal to

$$\sqrt{\det g}drd\varphi = \sqrt{\det \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}}drd\varphi = drd\varphi\,.$$

Lobachesvky plane.

In coordinates x, y $(y > 0)$ metric $G = \frac{dx^2+dy^2}{y^2}$, the corresponding matrix $G = \begin{pmatrix} 1/y^2 & 0 \\ 0 & 1/y^2 \end{pmatrix}$. Volume element is equal to $\sqrt{\det g}dxdy = \frac{dxdy}{y^2}$.

Sphere in stereographic coordinates In stereographic coordinates

$$G = \frac{4R^4(du^2 + dv^2)}{(R + u^2 + v^2)^2} \tag{1.81}$$

(It is isometric to the sphere of the radius R without North pole in stereographic coordinates (see the Homeworks.))

Calculate its volume element and volume. It is easy to see that:

$$G = \begin{pmatrix} \frac{4R^4}{(R^2+u^2+v^2)^2} & 0 \\ 0 & \frac{4R^4}{(R^2+u^2+v^2)^2} \end{pmatrix} \qquad \det g = \frac{16R^8}{(R+u^2+v^2)^4} \tag{1.82}$$

and volume element is equal to $\sqrt{\det g} dudv = \frac{4R^4 dudv}{(R^2+u^2+v^2)^2}$

One can calculate volume in coordinates u, v but it is better to consider homothety $u \to Ru, v \to Rv$ and polar coordinates: $u = Rr\cos\varphi, v = Rr\sin\varphi$. Then volume form is equal to $\sqrt{\det g} dudv = \frac{4R^4 dudv}{(R^2+u^2+v^2)^2} = \frac{4R^2 rdrd\varphi}{(1+r^2)^2}$.

Now calculation of integral becomes easy:

$$V = \int \frac{4R^2 rdrd\varphi}{(1+r^2)^2} = 8\pi R^2 \int_0^\infty \frac{rdr}{(1+r^2)^2} = 4\pi R^2 \int_0^\infty \frac{du}{(1+u)^2} = 4\pi R^2 \,.$$

Segment of the sphere.

Consider sphere of the radius a in Euclidean space with standard Riemannian metric

$$a^2 d\theta^2 + a^2 \sin^2\theta d\varphi^2$$

This metric is nothing but first quadratic form on the sphere (see (1.4.3)). The volume element is

$$\sqrt{\det g} d\theta d\varphi = \sqrt{\det \begin{pmatrix} a^2 & 0 \\ 0 & a^2\sin\theta \end{pmatrix}} d\theta d\varphi = a^2 \sin\theta d\theta d\varphi$$

Now calculate the volume of the segment of the sphere between two parallel planes, i.e. domain restricted by parallels $\theta_1 \le \theta \le \theta_0$: Denote by h be the height of this segment. One can see that

$$h = a\cos\theta_0 - a\cos\theta_1 = a(\cos\theta_0 - a\cos\theta_1)$$

There is remarkable formula which express the area of segment via the height h:

$$V = \int_{\theta_1 \le \theta \le \theta_0} \left(a^2 \sin\theta\right) d\theta d\varphi = \int_{\theta^0}^{\theta^1} \left(\int_0^{2\pi} \left(a^2\sin\theta\right) d\varphi \right) d\theta =$$

$$\int_{\theta^1}^{\theta^0} 2\pi a^2 \sin\theta d\theta = 2\pi a^2(\cos\theta_0 - \cos\theta_1) = 2\pi a(a\cos\theta_0 - acos\theta_1) = 2\pi ah \tag{1.83}$$

E.g. for all the sphere $h = 2a$. We come to $S = 4\pi a^2$. It is remarkable formula: area of the segment is a polynomial function of radius of the sphere and height (Compare with formula for length of the arc of the circle)

2 Covariant differentiaion. Connection. Levi Civita Connection on Riemannian manifold

2.1 Differentiation of vector field along the vector field.— Affine connection

How to differentiate vector fields on a (smooth)manifold M?

Recall the differentiation of functions on a (smooth)manifold M.

Let $\mathbf{X} = \mathbf{X^i(x)e_i(x)} = \frac{\partial}{\partial \mathbf{x^i}}$ be a vector field on M. Recall that vector field [5] $\mathbf{X} = \mathbf{X^i e_i}$ defines at the every point x_0 an infinitesimal curve: $x^i(t) = x_0^i + tX^i$ (More exactly the equivalence class $[\gamma(t)]_\mathbf{X}$ of curves $x^i(t) = x_0^i + tX^i + \dots$).

Let f be an arbitrary (smooth) function on M and $\mathbf{X} = X^i \frac{\partial}{\partial x^i}$. Then derivative of function f along vector field $\mathbf{X} = X^i \frac{\partial}{\partial x^i}$ is equal to

$$\partial_\mathbf{X} f = \nabla_\mathbf{X} f = X^i \frac{\partial f}{\partial x^i}$$

The geometrical meaning of this definition is following: If $\mathbf{X}$ is a velocity vector of the curve $x^i(t)$ at the point $x_0^i = x^i(t)$ at the "time" $t = 0$ then the value of the derivative $\nabla_\mathbf{X} f$ at the point $x_0^i = x^i(0)$ is equal just to the derivative by t of the function $f(x^i(t))$ at the "time" $t = 0$:

$$\text{if} \quad X^i(x)\big|_{x_0=x(0)} = \frac{dx^i(t)}{dt}\big|_{t=0}, \quad \text{then} \quad \nabla_\mathbf{X} f\big|_{x^i=x^i(0)} = \frac{d}{dt} f\left(x^i(t)\right)\big|_{t=0} \tag{2.1}$$

Remark In the course of Geometry and Differentiable Manifolds the operator of taking derivation of function along the vector field was denoted by "$\partial_\mathbf{X} f$". In this course we prefer to denote it by "$\nabla_\mathbf{X} f$" to have the uniform

[5] here like always we suppose by default the summation over repeated indices. E.g.$\mathbf{X} = X^i \mathbf{e}_i$ is nothing but $\mathbf{X} = \sum_{i=1}^n X^i \mathbf{e}_i$

notation for both operators of taking derivation of functions and vector fields along the vector field.

One can see that the operation ∇_X on the space $C^\infty(M)$ (space of smooth functions on the manifold) satisfies the following conditions:

- $\nabla_{\mathbf{X}}(\lambda f + \mu g) = a\nabla_{\mathbf{X}} f + b\nabla_{\mathbf{X}} g$ where $\lambda, \mu \in \mathbf{R}$ (linearity over numbers)
- $\nabla_{h\mathbf{X}+g\mathbf{Y}}(f) = h\nabla_{\mathbf{X}}(f) + g\nabla_{\mathbf{Y}}(f)$ (linearity over the space of functions)
- $\nabla_{\mathbf{X}}(\lambda fg) = f\nabla_{\mathbf{X}}(\lambda g) + g\nabla_{\mathbf{X}}(\lambda f)$ (Leibnitz rule)

(2.2)

Remark One can prove that these properties characterize vector fields:operator on smooth functions obeying the conditions above is a vector field. (You will have a detailed analysis of this statement in the course of Differentiable Manifolds.)

How to define differentiation of vector fields along vector fields.

The formula (2.1) cannot be generalised straightforwardly because vectors at the point x_0 and $x_0 + tX$ are vectors from different vector spaces. (We cannot substract the vector from one vector space from the vector from the another vector space, because *apriori* we cannot compare vectors from different vector space. One have to define an operation of transport of vectors from the space $T_{x_0}M$ to the point $T_{x_0+tX}M$ defining the transport from the point $T_{x_0}M$ to the point $T_{x_0+tX}M$).

Try to define the operation ∇ on vector fields such that conditions (2.2) above be satisfied.

2.1.1 Definition of connection. Christoffel symbols of connection

Definition Affine connection on M is the *operation* ∇ which assigns to every vector field $\mathbf{X}$ a linear map, (but not necessarily $C(M)$-linear map!) (i.e. a map which is linear over numbers not necessarily over functions) $\nabla_{\mathbf{X}}$ on the space of vector fields on M:

$$\nabla_{\mathbf{X}}(\lambda \mathbf{Y} + \mu \mathbf{Z}) = \lambda \nabla_{\mathbf{X}} \mathbf{Y} + \mu \nabla_{\mathbf{X}} \mathbf{Z}, \qquad \text{for every } \lambda, \mu \in \mathbf{R} \tag{2.3}$$

(Compare the first condition in (2.2)).
which satisfies the following conditions:

- for arbitrary (smooth) functions f, g on M

$$\nabla_{f\mathbf{X}+\mathbf{g}\mathbf{Y}}(\mathbf{Z}) = f\nabla_{\mathbf{X}}(\mathbf{Z}) + g\nabla_{\mathbf{Y}}(\mathbf{Z}) \qquad (C(M)\text{-linearity}) \tag{2.4}$$

(compare with second condition in (2.2))

- for arbitrary function f

$$\nabla_{\mathbf{X}}(f\mathbf{Y}) = (\nabla_{\mathbf{X}}f)\,\mathbf{Y} + f\nabla_{\mathbf{X}}(\mathbf{Y}) \qquad (\text{Leibnitz rule}) \tag{2.5}$$

Recall that $\nabla_{\mathbf{X}}f$ is just usual derivative of a function f along vector field: $\nabla_{\mathbf{X}}f = \partial_{\mathbf{X}}f$.

(Compare with Leibnitz rule in (2.2)).

The operation $\nabla_{\mathbf{X}}\mathbf{Y}$ is called covariant derivative of vector field $\mathbf{Y}$ along the vector field $\mathbf{X}$.

Write down explicit formulae in a given local coordinates $\{x^i\}$ ($i = 1, 2, \dots, n$) on manifold M.

Let

$$\mathbf{X} = X^i\mathbf{e}_i = X^i\frac{\partial}{\partial x^i} \qquad \mathbf{Y} = Y^i\mathbf{e}_i = Y^i\frac{\partial}{\partial x^i}$$

The basis vector fields $\frac{\partial}{x^i}$ we denote sometimes by ∂_i sometimes by $\mathbf{e}_i$

Using properties above one can see that

$$\nabla_{\mathbf{X}}\mathbf{Y} = \nabla_{X^i\partial_i}Y^k\partial_k = X^i\left(\nabla_i\left(Y^k\partial_k\right)\right), \qquad \text{where } \nabla_i = \nabla_{\partial_i} \tag{2.6}$$

Then according to (2.4)

$$\nabla_i\left(Y^k\partial_k\right) = \nabla_i\left(Y^k\right)\partial_k + Y^k\nabla_i\partial_k$$

Decompose the vector field $\nabla_i\partial_k$ over the basis ∂_i:

$$\nabla_i\partial_k = \Gamma^m_{ik}\partial_m \tag{2.7}$$

and

$$\nabla_i\left(Y^k\partial_k\right) = \frac{\partial Y^k(x)}{\partial x^i}\partial_k + Y^k\Gamma^m_{ik}\partial_m, \tag{2.8}$$

$$\nabla_{\mathbf{X}}\mathbf{Y} = X^i\frac{\partial Y^m(x)}{\partial x^i}\partial_m + X^iY^k\Gamma^m_{ik}\partial_m, \tag{2.9}$$

In components

$$(\nabla_{\mathbf{X}}\mathbf{Y})^m = X^i \left(\frac{\partial Y^m(x)}{\partial x^i} + Y^k \Gamma^m_{ik}\right) \tag{2.10}$$

Coefficients $\{\Gamma^m_{ik}\}$ are called *Christoffel symbols* in coordinates $\{x^i\}$. These coefficients define covariant derivative—**connection**.

If operation of taking covariant derivative is given we say that the connection is given on the manifold. Later it will be explained why we us the word "connection"

We see from the formula above that to define covariant derivative of vector fields, connection, we have to define Christoffel symbols in local coordinates.

2.1.2 Transformation of Christoffel symbols for an arbitrary connection

Let ∇ be a connection on manifold M. Let $\{\Gamma^i_{km}\}$ be Christoffel symbols of this connection in given local coordinates $\{x^i\}$. Then according (2.7) and (2.8) we have

$$\nabla_{\mathbf{X}}\mathbf{Y} = X^m \frac{\partial Y^i}{\partial x^m}\frac{\partial}{\partial x^i} + X^m \Gamma^i_{mk} Y^k \frac{\partial}{\partial x^i},$$

and in particularly

$$\Gamma^i_{mk}\partial_i = \nabla_{\partial_m}\partial_k$$

Use this relation to calculate Christoffel symbols in new coordinates $x^{i'}$

$$\Gamma^{i'}_{m'k'}\partial_{i'} = \nabla_{\partial'_m}\partial_{k'}$$

We have that $\partial_{m'} = \frac{\partial}{\partial x^{m'}} = \frac{\partial x^m}{\partial x^{m'}}\frac{\partial}{\partial x^m} = \frac{\partial x^m}{\partial x^{m'}}\partial_m$. Hence due to properties (2.4), (2.5) we have

$$\Gamma^{i'}_{m'k'}\partial_{i'} = \nabla_{\partial_{m'}}\partial_{k'} = \nabla_{\partial'_m}\left(\frac{\partial x^k}{\partial x^{k'}}\partial_k\right) = \left(\frac{\partial x^k}{\partial x^{k'}}\right)\nabla_{\partial'_m}\partial_k + \frac{\partial}{\partial x^{m'}}\left(\frac{\partial x^k}{\partial x^{k'}}\right)\partial_k =$$

$$\left(\frac{\partial x^k}{\partial x^{k'}}\right)\nabla_{\frac{\partial x^m}{\partial x^{m'}}\partial_m}\partial_k + \frac{\partial^2 x^k}{\partial x^{m'}\partial x^{k'}}\partial_k = \frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\nabla_{\partial_m}\partial_k + \frac{\partial^2 x^k}{\partial x^{m'}\partial x^{k'}}\partial_k$$

$$\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\Gamma^i_{mk}\partial_i + \frac{\partial^2 x^k}{\partial x^{m'}\partial x^{k'}}\partial_k = \frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\Gamma^i_{mk}\frac{\partial x^{i'}}{\partial x^i}\partial_{i'} + \frac{\partial^2 x^k}{\partial x^{m'}\partial x^{k'}}\frac{\partial x^{i'}}{\partial x^k}\partial_{i'}$$

Comparing the first and the last term in this formula we come to the transformation law:

If $\{\Gamma^i_{km}\}$ are Christoffel symbols of the connection ∇ in local coordinates $\{x^i\}$ and $\{\Gamma^{i'}_{k'm'}\}$ are Christoffel symbols of this connection in new local coordinates $\{x^{i'}\}$ then

$$\Gamma^{i'}_{k'm'} = \frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^i}\Gamma^i_{km} + \frac{\partial^2 x^r}{\partial x^{k'}\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^r} \tag{2.11}$$

Remark Christoffel symbols do not transform as tensor. If the second term is equal to zero, i.e. transformation of coordinates are linear (see the Proposition on flat connections) then the transformation rule above is the the same as a transformation rule for tensors of the type $\begin{pmatrix}1\\2\end{pmatrix}$ (see the formula (1.1)). In general case this is not true. Christoffel symbols do not transform as tensor under arbitrary non-linear coordinate transformation: see the second term in the formula above.

Remark On the other hand note that *difference of two arbitrary connections is a tensor*. If Γ^i_{km} and $\tilde{\Gamma}^i_{km}$ are corresponding Chrstoffel symbols then it follows from (1.1)) that their difference $S^i_{km} = \Gamma^i_{km} - \tilde{\Gamma}^i_{km}$ transforms as a tensor:

$$S^{i'}_{k'm'} = \Gamma^{i'}_{k'm'} - \tilde{\Gamma}^{i'}_{k'm'} = \frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^i}\left(\Gamma^i_{km} - \tilde{\Gamma}^i_{km}\right) = \frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^i}S^i_{km}$$

(See for detail the Homework 5.)

2.1.3 Canonical flat affine connection

It follows from the properties of connection that it is suffice to define connection at vector fields which form basis at the every point using (2.7), i.e. to define Christoffel symbols of this connection.

Example Consider n-dimensional Euclidean space $\mathbf{E}^n$ with Cartesian coordinates $\{x^1, \dots, x^n\}$.

Define connection such that all Christoffel symbols are equal to zero in these Cartesian coordinates $\{x^i\}$.

$$\nabla_{\mathbf{e}_i}\mathbf{e}_k = \Gamma^m_{ik}\mathbf{e}_m = 0, \quad \Gamma^m_{ik} = 0 \tag{2.12}$$

Does this mean that Christoffel symbols are equal to zero in an arbitrary Cartesian coordinates if they equal to zero in given Cartesian coordinates?

Does this mean that Christoffel symbols of this connection equal to zero in arbitrary coordinates system?

To answer these questions note that the relations (2.12) mean that

$$\nabla_{\mathbf{X}}\mathbf{Y} = X^m \frac{\partial Y^i}{\partial x^m}\frac{\partial}{\partial x^i} \tag{2.13}$$

in coordinates $\{x^i\}$

Consider an arbitrary new coordinates $x^{i'} = x^{i'}(x^1, \dots, x^n)$. Recall the transformation rule for an arbitrary vector field (see subsection 1.1)

$$\mathbf{R} = R^m \frac{\partial}{\partial x^m} = R^m \frac{\partial x^{m'}}{\partial x^m}\frac{\partial}{\partial x^{m'}}\,, \quad \text{i.e.} R^{m'} = \frac{\partial x^{m'}}{\partial x^m} R^m\,, \text{and}\,, R^m = \frac{\partial x^m}{\partial x^{m'}} R^{m'}\,.$$

Hence we have from (2.13) that

$$\nabla_{\mathbf{X}}\mathbf{Y} = X^m \frac{\partial Y^i}{\partial x^m}\frac{\partial}{\partial x^i} = X^m \frac{\partial}{\partial x^m}\left(Y^i\right)\frac{\partial}{\partial x^i} = X^m \frac{\partial x^{m'}}{\partial x^m}\frac{\partial}{\partial x^{m'}}\left(\frac{\partial x^i}{\partial x^{i'}} Y^{i'}\right)\frac{\partial}{\partial x^i} =$$

$$X^{m'}\frac{\partial}{\partial x^{m'}}\left(\frac{\partial x^i}{\partial x^{i'}} Y^{i'}\right)\frac{\partial}{\partial x^i} = X^{m'}\frac{\partial}{\partial x^{m'}}\left(Y^{i'}\right)\frac{\partial x^i}{\partial x^{i'}}\frac{\partial}{\partial x^i} + X^{m'}\frac{\partial^2 x^i}{\partial x^{m'}\partial x^{i'}}\left(Y^{i'}\right)\frac{\partial}{\partial x^i} =$$

$$X^{m'}\frac{\partial Y^{i'}}{\partial x^{m'}}\frac{\partial}{\partial x^{i'}} + \underbrace{X^{m'}\frac{\partial^2 x^i}{\partial x^{m'}\partial x^{i'}} Y^{i'}\frac{\partial}{\partial x^i}}_{\text{an additional term}} = \tag{2.14}$$

We see that an additional term equals to zero for arbitrary vector fields $\mathbf{X}, \mathbf{Y}$ if and only if the relations between new and old coordinates are linear:

$$\frac{\partial^2 x^i}{\partial x^{m'}\partial x^{i'}} = 0, \text{ i.e. } x^i = b^i + a^i_k x^k \tag{2.15}$$

Comparing formulae (2.15) and (2.13) we come to simple but very important

Proposition *Let all Christoffel symbols of a given connection be equal to zero in a given coordinate system $\{x^i\}$. Then all Christoffel symbols of this connection are equal to zero in an arbitrary coordinate system $\{x^{i'}\}$ such that the relations between new and old coordinates are linear:*

$$x^{i'} = b^i + a^i_k x^k \tag{2.16}$$

If transformation to new coordinate system is not linear, i.e. $\frac{\partial^2 x^i}{\partial x^{m'}\partial x^{i'}} \neq 0$ then Christoffel symbols of this connection in general are not equal to zero in new coordinate system $\{x^{i'}\}$.

Definition We call connection ∇ flat if there exists coordinate system such that all Christoffel symbols of this connection are equal to zero in a given coordinate system.

In particular connection (2.12) has zero Christoffel symbols in arbitrary Cartesian coordinates.

Corollary Connection has zero Christoffel symbols in arbitrary Cartesian coordinates if it has zero Christoffel symbols in a given Cartesian coordinates.

Hence the following definition is correct:

Definition A connection on $\mathbf{E}^n$ which Christoffel symbols vanish in Cartesian coordinates is called *canonical flat connection.*

Remark Canonical flat connection in Euclidean space is uniquely defined, since Cartesian coordinates are defined globally. On the other hand on arbitrary manifold one can define flat connection locally just choosing any arbitrary *local* coordinates and define *locally flat connection* by condition that Christoffel symbols vanish in these local coordinates. This does not mean that one can define flat connection *globally.* We will study this question after learning transformation law for Christoffel symbols.

Remark One can see that flat connection is symmetric connection.

Example Consider a connection (2.12) in $\mathbf{E}^2$. It is a flat connection. Calculate Christoffel symbols of this connection in polar coordinates

$$\begin{cases} x = r\cos\varphi \\ y = y\sin\varphi \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2} \\ \varphi = \arctan\frac{y}{x} \end{cases} \tag{2.17}$$

Write down Jacobians of transformations—matrices of partial derivatives:

$$\begin{pmatrix} x_r & y_r \\ x_\varphi & y_\varphi \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -r\sin\varphi & r\cos\varphi \end{pmatrix}, \qquad \begin{pmatrix} r_x & \varphi_x \\ r_y & \varphi_y \end{pmatrix} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & -\frac{y}{x^2+y^2} \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{x}{x^2+y^2} \end{pmatrix} \tag{2.18}$$

According (2.11) and since Chrsitoffel symbols are equal to zero in Cartesian coordinates (x, y) we have

$$\Gamma^{i'}_{k'm'} = \frac{\partial^2 x^r}{\partial x^{k'}\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^r}, \tag{2.19}$$

where $(x^1, x^2) = (x, y)$ and $(x^{1'}, x^{2'}) = (r, \varphi)$. Now using (2.18) we have

$$\Gamma^r_{rr} = \frac{\partial^2 x}{\partial r\partial r}\frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial r\partial r}\frac{\partial r}{\partial y} = 0$$

$$\Gamma^r_{r\varphi} = \Gamma^r_{\varphi r} = \frac{\partial^2 x}{\partial r \partial \varphi}\frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial r \partial \varphi}\frac{\partial r}{\partial y} = -\sin\varphi\cos\varphi + \sin\varphi\cos\varphi = 0\,.$$

$$\Gamma^r_{\varphi\varphi} = \frac{\partial^2 x}{\partial \varphi \partial \varphi}\frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial r \partial \varphi}\frac{\partial r}{\partial y} = -x\frac{x}{r} - y\frac{y}{r} = -r\,.$$

$$\Gamma^\varphi_{rr} = \frac{\partial^2 x}{\partial r \partial r}\frac{\partial \varphi}{\partial x} + \frac{\partial^2 y}{\partial r \partial r}\frac{\partial \varphi}{\partial y} = 0\,.$$

$$\Gamma^\varphi_{\varphi r} = \Gamma^\varphi_{r\varphi} = \frac{\partial^2 x}{\partial r \partial \varphi}\frac{\partial \varphi}{\partial x} + \frac{\partial^2 y}{\partial r \partial \varphi}\frac{\partial \varphi}{\partial y} = -\sin\varphi\frac{-y}{r^2} + \cos\varphi\frac{x}{r^2} = \frac{1}{r}$$

$$\Gamma^\varphi_{\varphi\varphi} = \frac{\partial^2 x}{\partial \varphi \partial \varphi}\frac{\partial \varphi}{\partial x} + \frac{\partial^2 y}{\partial \varphi \partial \varphi}\frac{\partial \varphi}{\partial y} = -x\frac{-y}{r^2} - y\frac{x}{r^2} = 0\,. \tag{2.20}$$

Hence we have that the covariant derivative (2.13) in polar coordinates has the following appearance

$$\nabla_r \partial_r = \Gamma^r_{rr}\partial_r + \Gamma^\varphi_{rr}\partial_\varphi = 0\,,, \qquad \nabla_r \partial_\varphi = \Gamma^r_{r\varphi}\partial_r + \Gamma^\varphi_{r\varphi}\partial_\varphi = \frac{\partial_\varphi}{r}$$

$$\nabla_\varphi \partial_r = \Gamma^r_{\varphi r}\partial_r + \Gamma^\varphi_{\varphi r}\partial_\varphi = \frac{\partial_\varphi}{r}, \qquad \nabla_\varphi \partial_\varphi = \Gamma^r_{\varphi\varphi}\partial_r + \Gamma^\varphi_{\varphi\varphi}\partial_\varphi = -r\partial_r \tag{2.21}$$

Remark Later when we study geodesics we will learn a very quick method to calculate Christoffel symbols.

2.1.4 * Global aspects of existence of connection

We defined connection as an operation on vector fields obeying the special axioms (see the subsubsection 2.1.1). Then we showed that in a given coordinates connection is defined by Christoffel symbols. On the other hand we know that in general coordinates on manifold are not defined globally. (We had not this trouble in Euclidean space where there are globally defined Cartesian coordinates.)

- How to define connection globally using local coordinates?
- Does there exist at least one globally defined connection?
- Does there exist globally defined flat connection?

These questions are not naive questions. Answer on first and second questions is "Yes". It sounds bizzare but answer on the first question is not "Yes" [6]

[6] Topology of the manifold can be an obstruction to existence of global flat connection. E.g. it does not exist on sphere S^n if $n > 1$.

Global definition of connection

The formula (2.11) defines the transformation for Christoffer symbols if we go from one coordinates to another.

Let $\{(x^i_\alpha), U_\alpha\}$ be an atlas of charts on the manifold M.

If connection ∇ is defined on the manifold M then it defines in any chart (local coordinates) (x^i_α) Christoffer symbols which we denote by ${}_{(\alpha)}\Gamma^i_{km}$. If (x^i_α), $(x^{i'}_{(\beta)})$ are different local coordinates in a vicinity of a given point then according to (2.11)

$$ {}_{(\beta)}\Gamma^{i'}_{k'm'} = \frac{\partial x^k_{(\alpha)}}{\partial x^{k'}_{(\beta)}} \frac{\partial x^m_{(\alpha)}}{\partial x^{m'}_{(\beta)}} \frac{\partial x^{i'}_{(\beta)}}{\partial x^i_{(\alpha)}}{}_{(\beta)} \Gamma^{(\alpha)i}_{mk} + \frac{\partial^2 x^k_{(\alpha)}}{\partial x^{m'}_{(\beta)} \partial x^{k'}_{(\beta)}} \frac{\partial x^{i'}_{(\beta)}}{\partial x^k_{(\alpha)}} \tag{2.22} $$

Definition Let $\{(x^i_\alpha), U_\alpha\}$ be an atlas of charts on the manifold M

We say that the collection of Christoffel symbols $\{\Gamma^{(\alpha)i}_{km}\}$ defines globally a connection on the manifold M in this atlas if for every two local coordinates $(x^i_{(\alpha)})$, $(x^i_{(\beta)})$ from this atlas the transformation rules (2.22) are obeyed.

Using partition of unity one can prove the existence of global connection constructing it in explicit way. Let $\{(x^i_\alpha), U_\alpha\}$ $(\alpha = 1, 2, \dots, N)$ be a finite atlas on the manifold M and let $\{\rho_\alpha\}$ be a partition of unity adjusted to this atlas. Denote by ${}^{(\alpha)}\Gamma^i_{km}$ local connection defined in domain U_α such that its components in these coordinates are equal to zero. Denote by ${}^{(\alpha)}_{(\beta)}\Gamma^i_{km}$ Christoffel symbols of this local connection in coordinates $(x^i_{(\beta)})$ $({}^{(\alpha)}_{(\beta)}\Gamma^i_{km} = 0)$. Now one can define globally the connection by the formula:

$$ {}_{(\beta)}\Gamma^i_{km}(\mathbf{x}) = \sum_\alpha \rho_\alpha(\mathbf{x})\, {}^{(\alpha)}_{(\beta)}\Gamma^i_{km}(\mathbf{x}) = \sum_\alpha \rho_\alpha(\mathbf{x}) \frac{\partial x^i_{(\beta)}}{\partial x^{i'}_{(\alpha)}} \frac{\partial^2 x^{i'}_{(\alpha)}(\mathbf{x})}{\partial x^k_{(\beta)} \partial x^m_{(\beta)}}\,. \tag{2.23} $$

This connection in general is not flat connection[7]

2.2 Connection induced on the surfaces

Let M be a manifold (surface) embedded in Euclidean space[8]. Canonical flat connection on $\mathbf{E}^N$ induces the connection on surface in the following way.

Let $\mathbf{X}, \mathbf{Y}$ be tangent vector fields to the surface M and $\nabla^{\text{can.flat}}$ a canonical flat connection in $\mathbf{E}^N$. In general

$$ \mathbf{Z} = \nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y} \quad \text{is not tangent to manifold } M \tag{2.24} $$

[7]See for detail the text: "Global affine connection on manifold" " in my homepage: "www.maths.mancheser.ac.uk/khudian" in subdirectory Etudes/Geometry

[8]We know that every n-dimensional manifodl can be embedded in $2n+1$-dimensional Euclidean space

Consider its decomposition on two vector fields:

$$\mathbf{Z} = \mathbf{Z}_{tangent} + \mathbf{Z}_{\perp}, \nabla^{\text{can.flat}}_{\mathbf{X}}, \mathbf{Y} = \left(\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}\right)_{tangent} + \left(\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}\right)_{\perp}, \quad (2.25)$$

where $\mathbf{Z}_{\perp}$ is a component of vector which is orthogonal to the surface M and $\mathbf{Z}_{||}$ is a component which is tangent to the surface. Define an induced connection ∇^M on the surface M by the following formula

$$\nabla^M : \qquad \nabla^M_{\mathbf{X}}\mathbf{Y} := \left(\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}\right)_{tangent} \quad (2.26)$$

Remark One can imply this construction for an arbitrary connection in $\mathbf{E}^N$.

2.2.1 Calculation of induced connection on surfaces in $\mathbf{E}^3$.

Let $\mathbf{r} = \mathbf{r}(u, v)$ be a surface in $\mathbf{E}^3$. Let $\nabla^{\text{can.flat}}$ be a flat connection in $\mathbf{E}^3$. Then

$$\nabla^M : \qquad \nabla^M_{\mathbf{X}}\mathbf{Y} := \left(\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}\right)_{||} = \nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y} - \mathbf{n}(\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}, \mathbf{n}), \quad (2.27)$$

where $\mathbf{n}$ is normal unit vector field to M. Consider a special example

Example (Induced connection on sphere) Consider a sphere of the radius R in $\mathbf{E}^3$:

$$\mathbf{r}(\theta, \varphi) : \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases}$$

then

$$\mathbf{r}_\theta = \begin{pmatrix} R\cos\theta\cos\varphi \\ R\cos\theta\sin\varphi \\ -R\sin\theta \end{pmatrix}, \mathbf{r}_\varphi = \begin{pmatrix} -R\sin\theta\sin\varphi \\ R\sin\theta\cos\varphi \\ 0 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} sin\theta\cos\varphi \\ sin\theta\sin\varphi \\ \cos\theta \end{pmatrix},$$

where $\mathbf{r}_\theta = \frac{\partial \mathbf{r}(\theta,\varphi)}{\partial\theta}$, $\mathbf{r}_\varphi = \frac{\partial \mathbf{r}(\theta,\varphi)}{\partial\varphi}$ are basic tangent vectors and $\mathbf{n}$ is normal unit vector.

Calculate an induced connection ∇ on the sphere.

First calculate $\nabla_{\partial_\theta}\partial_\theta$.

$$\nabla_{\partial_\theta}\partial_\theta = \left(\frac{\partial \mathbf{r}_\theta}{\partial\theta}\right)_{tangent} = (\mathbf{r}_{\theta\theta})_{tangent}.$$

On the other hand one can see that $\mathbf{r}_{\theta\theta} = \begin{pmatrix} -Rsin\theta\cos\varphi \\ -Rsin\theta\sin\varphi \\ -R\cos\theta \end{pmatrix} = -R\mathbf{n}$ is proportional to normal vector, i.e. $(\mathbf{r}_{\theta\theta})_{tangent} = 0$. We come to

$$\nabla_{\partial_\theta}\partial_\theta = (\mathbf{r}_{\theta\theta})_{tangent} = 0 \Rightarrow \Gamma^\theta_{\theta\theta} = \Gamma^\varphi_{\theta\theta} = 0\,. \tag{2.28}$$

Now calculate $\nabla_{\partial_\theta}\partial_\varphi$ and $\nabla_{\partial_\varphi}\partial_\theta$.

$$\nabla_{\partial_\theta}\partial_\varphi = \left(\frac{\partial\mathbf{r}_\varphi}{\partial\theta}\right)_{tangent} = (\mathbf{r}_{\theta\varphi})_{tangent}\,,\ \nabla_{\partial_\varphi}\partial_\theta = \left(\frac{\partial\mathbf{r}_\theta}{\partial\varphi}\right)_{tangent} = (\mathbf{r}_{\varphi\theta})_{tangent}$$

We have

$$\nabla_{\partial_\theta}\partial_\varphi = \nabla_{\partial_\varphi}\partial_\theta = (\mathbf{r}_{\varphi\theta})_{tangent} = \begin{pmatrix} -R\cos\theta\sin\varphi \\ R\cos\theta\cos\varphi \\ 0 \end{pmatrix}_{tangent}.$$

We see that the vector $\mathbf{r}_{\varphi\theta}$ is orthogonal to $\mathbf{n}$:

$$\langle\mathbf{r}_{\varphi\theta}, \mathbf{n}\rangle = -R\cos\theta\sin\varphi\sin\theta\cos\varphi + R\cos\theta\cos\varphi\sin\theta\sin\varphi = 0.$$

Hence

$$\nabla_{\partial_\theta}\partial_\varphi = \nabla_{\partial_\varphi}\partial_\theta = (\mathbf{r}_{\varphi\theta})_{tangent} = \mathbf{r}_{\varphi\theta} = \begin{pmatrix} -R\cos\theta\sin\varphi \\ R\cos\theta\cos\varphi \\ 0 \end{pmatrix} = \operatorname{cotan}\theta\mathbf{r}_\varphi\,.$$

We come to

$$\nabla_{\partial_\theta}\partial_\varphi = \nabla_{\partial_\varphi}\partial_\theta = \operatorname{cotan}\theta\partial_\varphi \Rightarrow \Gamma^\theta_{\theta\varphi} = \Gamma^\theta_{\varphi\theta} = 0,\ \Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = \operatorname{cotan}\theta \tag{2.29}$$

Finally calculate $\nabla_{\partial_\varphi}\partial_\varphi$

$$\nabla_{\partial_\varphi}\partial_\varphi = (\mathbf{r}_{\varphi\varphi})_{tangent} = \left(\begin{pmatrix} -R\sin\theta\cos\varphi \\ -R\sin\theta\sin\varphi \\ 0 \end{pmatrix}\right)_{tangent}$$

Projecting on the tangent vectors to the sphere (see (2.27)) we have

$$\nabla_{\partial_\varphi}\partial_\varphi = (\mathbf{r}_{\varphi\varphi})_{tangent} = \mathbf{r}_{\varphi\varphi} - \mathbf{n}\langle\mathbf{n}, \mathbf{r}_{\varphi\varphi}\rangle =$$

$$\begin{pmatrix}-R\sin\theta\cos\varphi\\ -R\sin\theta\sin\varphi\\ 0\end{pmatrix}-\begin{pmatrix}\sin\theta\cos\varphi\\ \sin\theta\sin\varphi\\ \cos\theta\end{pmatrix}(-R\sin\theta\cos\varphi\sin\theta\cos\varphi-R\sin\theta\sin\varphi\sin\theta\sin\varphi)=$$

$$-\sin\theta\cos\theta\begin{pmatrix}R\cos\theta\cos\varphi\\ R\cos\theta\sin\varphi\\ -R\sin\theta\end{pmatrix}=-\sin\theta\cos\theta\mathbf{r}_\theta,$$

i.e.

$$\nabla_{\partial_\varphi}\partial_\varphi=-\sin\theta\cos\theta\mathbf{r}_\theta\Rightarrow\Gamma^\theta_{\varphi\varphi}=-\sin\theta\cos\theta,\ \Gamma^\varphi_{\varphi\varphi}=\Gamma^\varphi_{\varphi\varphi}=0\,. \tag{2.30}$$

2.3 Levi-Civita connection

2.3.1 Symmetric connection

Definition. We say that connection is symmetric if its Christoffel symbols Γ^i_{km} are symmetric with respect to lower indices

$$\Gamma^i_{km}=\Gamma^i_{mk} \tag{2.31}$$

The canonical flat connection and induced connections considered above are symmetric connections.

Invariant definition of symmetric connection

A connection ∇ is symmetric if for an arbitrary vector fields $\mathbf{X},\mathbf{Y}$

$$\nabla_{\mathbf{X}}\mathbf{Y}-\nabla_{\mathbf{Y}}\mathbf{X}-[\mathbf{X},\mathbf{Y}]=0 \tag{2.32}$$

If we apply this definition to basic fields ∂_k,∂_m which commute: $[\partial_k,\partial_m]=0$ we come to the condition

$$\nabla_{\partial_k}\partial_m-\nabla_{\partial_m}\partial_k=\Gamma^i_{mk}\partial_i-\Gamma^i_{km}\partial_i=0$$

and this is the condition (2.31).

2.3.2 Levi-Civita connection. Theorem and Explicit formulae

Let (M,G) be a Riemannian manifold.

Definition. Theorem

A symmetric connection ∇ is called Levi-Civita connection if it is compatible with metric, i.e. if it preserves the scalar product:

$$\partial_{\mathbf{X}}\langle\mathbf{Y},\mathbf{Z}\rangle=\langle\nabla_{\mathbf{X}}\mathbf{Y},\mathbf{Z}\rangle+\langle\mathbf{Y},\nabla_{\mathbf{X}}\mathbf{Z}\rangle \tag{2.33}$$

for arbitrary vector fields $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$.

There exists unique levi-Civita connection on the Riemannian manifold.

In local coordinates Christoffel symbols of Levi-Civita connection are given by the following formulae:

$$\Gamma^i_{mk} = \frac{1}{2} g^{ij} \left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^m} - \frac{\partial g_{mk}}{\partial x^j} \right) . \tag{2.34}$$

where $G = g_{ik} dx^i dx^k$ *is Riemannian metric in local coordinates and* $||g^{ik}||$ *is the matrix inverse to the matrix* $||g_{ik}||$.

Proof

Suppose that this connection exists and Γ^i_{mk} are its Christoffel symbols. Consider vector fields $\mathbf{X} = \partial_m, \mathbf{Y} = \partial_i$ and $\mathbf{Z} = \partial_k$ in (2.33). We have that

$$\partial_m g_{ik} = \langle \Gamma^r_{mi} \partial_r, \partial_k \rangle + \langle \partial_i, \Gamma^r_{mk} \partial_r \rangle = \Gamma^r_{mi} g_{rk} + g_{ir} \Gamma^r_{mk} . \tag{2.35}$$

for arbitrary indices m, i, k.

Denote by $\Gamma_{mik} = \Gamma^r_{mi} g_{rk}$ we come to

$$\partial_m g_{ik} = \Gamma_{mik} + \Gamma_{mki}, \text{ i.e.}$$

Now using the symmetricity $\Gamma_{mik} = \Gamma_{imk}$ since $\Gamma^k_{mi} = \Gamma^k_{im}$ we have

$$\Gamma_{mik} = \partial_m g_{ik} - \Gamma_{mki} = \partial_m g_{ik} - \Gamma_{kmi} = \partial_m g_{ik} - (\partial_k g_{mi} - \Gamma_{kim}) =$$

$$\partial_m g_{ik} - \partial_k g_{mi} + \Gamma_{kim} = \partial_m g_{ik} - \partial_k g_{mi} + \Gamma_{ikm} = \partial_m g_{ik} - \partial_k g_{mi} + (\partial_i g_{km} - \Gamma_{imk}) =$$

$$\partial_m g_{ik} - \partial_k g_{mi} + \partial_i g_{km} - \Gamma_{mik} .$$

Hence

$$\Gamma_{mik} = \frac{1}{2} (\partial_m g_{ik} + \partial_i g_{mk} - \partial_k g_{mi}) \Rightarrow \Gamma^k_{im} = \frac{1}{2} g^{kr} (\partial_m g_{ir} + \partial_i g_{mr} - \partial_r g_{mi}) \tag{2.36}$$

We see that if this connection exists then it is given by the formula(2.34).

On the other hand one can see that (2.34) obeys the condition (2.35). We prove the uniqueness and existence.

since $\nabla_{\partial_i} \partial_k = \Gamma^m_{ik} \partial_m$.

Consider examples.

2.3.3 Levi-Civita connection of $\mathbf{E}^n$

For Euclidean space $\mathbf{E}^n$ in standard Cartesian coordinates

$$G_{\text{Eucl}} = (dx^1)^2 + \cdots + (dx^n)^2 = \delta_{ik} dx^i dx^k$$

Components of metric are constants (they are equal to 0 or 1). Hence obviously Christoffel symbols of Levi-Civita connection in Cartesian coordinates according formula (2.34) vanish:

$$\Gamma^I_{km} = 0 \text{ in Cartesian coordiantes}$$

Recalling canonical flat connection (see 2.1.3) we come to simple but important observation:

Observation Levi-Civita connection coincides with canonical flat connection in Euclidean space $\mathbf{E}^n$. They have vanishing Cristoffel symbols in Cartesian coordinates.

2.3.4 Levi-Civita connection on 2-dimensional Riemannian manifold with metric $G = adu^2 + bdv^2$.

Example Consider 2-dimensional manifold with Riemannian metrics

$$G = a(u,v)du^2 + b(u,v)dv^2, \qquad G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} a(u,v) & 0 \\ 0 & b(u,v) \end{pmatrix}$$

Calculate Christoffel symbols of Levi Civita connection.

Using (2.36) we see that:

$$\begin{aligned}
\Gamma_{111} &= \tfrac{1}{2}\left(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}\right) = \tfrac{1}{2}\partial_1 g_{11} = & \tfrac{1}{2}a_u \\
\Gamma_{211} = \Gamma_{121} &= \tfrac{1}{2}\left(\partial_1 g_{12} + \partial_2 g_{11} - \partial_1 g_{12}\right) = \tfrac{1}{2}\partial_2 g_{11} = & \tfrac{1}{2}a_v \\
\Gamma_{221} &= \tfrac{1}{2}\left(\partial_2 g_{12} + \partial_2 g_{12} - \partial_1 g_{22}\right) = -\tfrac{1}{2}\partial_1 g_{22} = - & \tfrac{1}{2}b_u \\
\Gamma_{112} &= \tfrac{1}{2}\left(\partial_1 g_{12} + \partial_1 g_{12} - \partial_2 g_{11}\right) = -\tfrac{1}{2}\partial_2 g_{11} = - & \tfrac{1}{2}a_v \\
\Gamma_{122} = \Gamma_{212} &= \tfrac{1}{2}\left(\partial_2 g_{21} + \partial_1 g_{22} - \partial_2 g_{21}\right) = \tfrac{1}{2}\partial_1 g_{22} = & \tfrac{1}{2}b_u \\
\Gamma_{222} &= \tfrac{1}{2}\left(\partial_2 g_{22} + \partial_2 g_{22} - \partial_2 g_{22}\right) = \tfrac{1}{2}\partial_2 g_{22} = & \tfrac{1}{2}b_v
\end{aligned} \tag{2.37}$$

To calculate $\Gamma^i_{km} = g^{ir}\Gamma_{kmr}$ note that for the metric $a(u,v)du^2 + b(u,v)dv^2$

$$G^{-1} = \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{a(u,v)} & 0 \\ 0 & \frac{1}{b(u,v)} \end{pmatrix}$$

Hence

$$\begin{array}{llll} \Gamma^1_{11} = g^{11}\Gamma_{111} = \frac{a_u}{2a}, & \Gamma^1_{21} = \Gamma^1_{12} = g^{11}\Gamma_{121} = \frac{a_v}{2a}, & \Gamma^1_{22} = g^{11}\Gamma_{221} = \frac{-b_u}{2a} \\ \Gamma^2_{11} = g^{22}\Gamma_{112} = \frac{-a_v}{2b}, & \Gamma^2_{21} = \Gamma^2_{12} = g^{22}\Gamma_{122} = \frac{b_u}{2b}, & \Gamma^2_{22} = g^{22}\Gamma_{222} = \frac{b_v}{2b} \end{array} \tag{2.38}$$

2.3.5 Example of the sphere again

Calculate Levi-Civita connection on the sphere.

On the sphere first quadratic form (Riemannian metric) $G = R^2 d\theta^2 + R^2 \sin^2\theta d\varphi^2$. Hence we use calculations from the previous example with $a(\theta,\varphi) = R^2, b(\theta,\varphi) = R^2\sin^2\theta$ $(u = \theta, v = \varphi)$. Note that $a_\theta = a_\varphi = b_\varphi = 0$. Hence only non-trivial components of Γ will be:

$$\Gamma^\theta_{\varphi\varphi} = \frac{-b_\theta}{2a} = \frac{-\sin 2\theta}{2}, \qquad \left(\Gamma_{\varphi\varphi\theta} = \frac{-R^2\sin 2\theta}{2}\right), \tag{2.39}$$

$$\Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = \frac{b_\theta}{2b} = \frac{\cos\theta}{\sin\theta} \qquad \left(\Gamma_{\theta\varphi\varphi} = \frac{R^2\sin 2\theta}{2}\right) \tag{2.40}$$

All other components are equal to zero:

$$\Gamma^\theta_{\theta\theta} = \Gamma^\theta_{\theta\varphi} = \Gamma^\theta_{\varphi\theta} = \Gamma^\varphi_{\theta\theta} = \Gamma^\varphi_{\varphi\varphi} = 0$$

Remark Note that Christoffel symbols of Levi-Civita connection on the sphere coincide with Christoffel symbols of induced connection calculated in the subsection "Connection induced on surfaces". later we will understand the geometrical meaning of this fact.

2.4 Levi-Civita connection = induced connection on surfaces in $\mathbf{E}^3$

We know already that *canonical flat connection of Euclidean space is the Levi-Civita connection of the standard metric on Euclidean space.* (see section

2.3.3.) Now we show that Levi-Civita connection on surfaces in Euclidean space coincides with the connection induced on the surfaces by canonical flat connection. We perform our analysis for surfaces in $\mathbf{E}^3$.

Let $M\colon \mathbf{r} = \mathbf{r}(u,v)$ be a surface in $\mathbf{E}^3$. Let G be induced Riemannian metric on M and ∇ Levi–Civita connection of this metric.

We know that the induced connection $\nabla^{(M)}$ is defined in the following way: for arbitrary vector fields $\mathbf{X}, \mathbf{Y}$ tangent to the surface M, $\nabla^M_{\mathbf{X}}\mathbf{Y}$ equals to the projection on the tangent space of the vector field $\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}$:

$$\nabla^M_{\mathbf{X}}\mathbf{Y} = \left(\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}\right)_{\text{tangent}} ,$$

where $\nabla^{\text{can.flat}}$ is canonical flat connection in $\mathbf{E}^3$ (its Christoffel symbols vanish in Cartesian coordinates). We denote by $\mathbf{A}_{\text{tangent}}$ a projection of the vector A attached at the point of the surface on the tangent space: $\mathbf{A}_\perp = \mathbf{A} - \mathbf{n}(\mathbf{A},\mathbf{n})$, ($\mathbf{n}$ is normal unit vector field to the surface.)

Theorem *Induced connection on the surface $\mathbf{r} = \mathbf{r}(u,v)$ in $\mathbf{E}^3$ coincides with Levi-Civita connection of Riemannian metric induced by the canonical metric on Euclidean space $\mathbf{E}^3$.*

Proof

Let ∇^M be induced connection on a surface M in $\mathbf{E}^3$ given by equations $\mathbf{r} = \mathbf{r}(u,v)$. Considering this connection on the basic vectors $\mathbf{r}_h, \mathbf{r}_v$ we see that it is symmetric connection. Indeed

$$\nabla^M_{\partial_u}\partial_v = (\mathbf{r}_{uv})_{\text{tangent}} = (\mathbf{r}_{vu})_{\text{tangent}} = \nabla^M_{\partial_v}\partial_u \,. \Rightarrow \Gamma^u_{uv} = \Gamma^u_{vu}, \Gamma^v_{uv} = \Gamma^v_{vu} \,.$$

Prove that this connection preserves scalar product on M. For arbitrary tangent vector fields $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ we have

$$\partial_{\mathbf{X}}\langle \mathbf{Y}, \mathbf{Z}\rangle_{\mathbf{E}^3} = \langle \nabla^{\text{can. flat}}_{\mathbf{X}}\mathbf{Y}, \mathbf{Z}\rangle_{\mathbf{E}^3} + \langle \mathbf{Y}, \nabla^{\text{can. flat}}_{\mathbf{X}}\mathbf{Z}\rangle_{\mathbf{E}^3} \,.$$

since canonical flat connection in $\mathbf{E}^3$ preserves Euclidean metric in $\mathbf{E}^3$ (it is evident in Cartesian coordinates). Now project the equation above on the surface M. If $\mathbf{A}$ is an arbitrary vector attached to the surface and $\mathbf{A}_{\text{tangent}}$ is its projection on the tangent space to the surface, then for every tangent vector $\mathbf{B}$ scalar product $\langle \mathbf{A}, \mathbf{B}\rangle_{\mathbf{E}^3}$ equals to the scalar product $\langle \mathbf{A}_{\text{tangent}}, \mathbf{B}\rangle_{\mathbf{E}^3} = \langle \mathbf{A}_{\text{tangent}}, \mathbf{B}\rangle_M$ since vector $\mathbf{A} - \mathbf{A}_{\text{tangent}}$ is orthogonal to the surface. Hence we deduce from (2) that $\partial_{\mathbf{X}}\langle \mathbf{Y}, \mathbf{Z}\rangle_M =$

$$\langle \left(\nabla^{\text{can. flat}}_{\mathbf{X}}\mathbf{Y}\right)_{\text{tangent}}, \mathbf{Z}\rangle_{\mathbf{E}^3} + \langle \mathbf{Y}, \left(\nabla^{\text{can. flat}}_{\mathbf{X}}\mathbf{Z}\right)_{\text{tangent}}\rangle_{\mathbf{E}^3} = \langle \nabla^M_{\mathbf{X}}\mathbf{Y}, \mathbf{Z}\rangle_M + \langle \mathbf{Y}, \nabla^M_{\mathbf{X}}\mathbf{Z}\rangle_M \,.$$

We see that induced connection is symmetric connection which preserves the induced metric. Hence due to Levi-Civita Theorem it is unique and is expressed as in the formula (2.34).

Remark One can easy to reformulate and prove more general statement: Let M be a submanifold in Riemannian manifold (E, G). Then Levi-Civita connection of the metric induced on this submanifold coincides with the connection induced on the manifold by Levi-Civita connection of the metric G.

3 Parallel transport and geodesics

3.1 Parallel transport

3.1.1 Definition

Let M be a manifold equipped with affine connection ∇.

Definition Let $C\colon \mathbf{x}(t)$ with coordinates $x^i = x^i(t)$, $t_0 \le t \le t_1$ be a curve on the manifold M. Let $\mathbf{X} = \mathbf{X}(t_0)$ be an arbitrary tangent vector attached at the initial point $\mathbf{x}_0$ (with coordinates $x^i(t_0)$) of the curve C, i.e. $\mathbf{X}(t_0) \in T_{\mathbf{x}_0}M$ is a vector tangent to the manifold M at the point $\mathbf{x}_0$ with coordinates $x^i(t_0)$. (The vector $\mathbf{X}$ is not necessarily tangent to the curve C)

We say that $\mathbf{X}(t)$, $t_0 \le t \le t_1$ is a parallel transport of the vector $\mathbf{X}(t_0) \in T_{\mathbf{x}_0}M$ along the curve $C\colon x^i = x^i(t), t_0 \le t \le t_1$ if

- For an arbitrary t, $t_0 \le t \le t$, vector $\mathbf{X} = \mathbf{X}(t)$, $(\mathbf{X}(t)|_{t=t_0} = \mathbf{X}(t_0))$ is a vector attached at the point $\mathbf{x}(t)$ of the curve C, i.e. $\mathbf{X}(t)$ is a vector tangent to the manifold M at the point $\mathbf{x}(t)$ of the curve C.

- The covariant derivative of $\mathbf{X}(t)$ along the curve C equals to zero:

$$\frac{\nabla \mathbf{X}}{dt} = \nabla_{\mathbf{v}} \mathbf{X} = 0\,. \tag{3.1}$$

  In components: if $X^m(t)$ are components of the vector field $\mathbf{X}(t)$ and $v^m(t)$ are components of the velocity vector $\mathbf{v}$ of the curve C ,

$$\mathbf{X}(t) = X^m(t)\frac{\partial}{\partial x^m}|_{\mathbf{x}(t)}\,, \quad \mathbf{v} = \frac{d\mathbf{x}(t)}{dt} = \frac{dx^i}{dt}\frac{\partial}{\partial x^m}|_{\mathbf{x}(t)}$$

  then the condition (3.1) can be rewritten as

$$\frac{dX^i(t)}{dt} + v^k(t)\Gamma^i_{km}(x^i(t))X^m(t) \equiv 0\,. \tag{3.2}$$

Remark We say sometimes that $\mathbf{X}(t)$ is *covariantly constant along the curve* C If $\mathbf{X}(t)$ is parallel transport of the vector $\mathbf{X}$ along the curve C. If we consider Euclidean space with canonical flat connection then in Cartesian coordinates Christoffel symbols vanish and parallel transport is nothing but $\frac{d\mathbf{X}}{dt} = \nabla_{\mathbf{v}}\mathbf{X} = 0$, $\mathbf{X}(t)$ is constant vector.

Remark Compare this definition of parallel transport with the definition which we consider in the course of "Introduction to Geometry" where we consider parallel transport of the vector along the curve on the surface embedded in $\mathbf{E}^3$ and define parallel transport by the condition, that only orthogonal component of the vector changes during parallel transport, i.e. $\frac{d\mathbf{X}(t)}{dt}$ is a vector orthogonal to the surface (see the Exercise in the Homework 7).

3.1.2 Parallel transport is a linear map

Consider two different points $\mathbf{x}_0, \mathbf{x}_1$ on the manifold M with connection ∇. Let C be a curve $\mathbf{x}(t)$ joining these points. The parallel transport (3.1) $\mathbf{X}(t)$ defines the map between tangent vectors at the point $\mathbf{x}_0$ and tangent vectors at the point $\mathbf{x}_1$. This map depends on the curve C. Parallel transport along different curves joining the same points is in general different (if we are not in Euclidean space).

On the other hand parallel transport is a linear map of tangent spaces which *does not depend on the parameterisation of the curve* joining these points.

Proposition *Let* $\mathbf{X}(t), t_0 \leq t \leq t_1$ *be a parallel transport of the vector* $\mathbf{X}(t_0) \in T_{\mathbf{x}_0}M$ *along the curve* $C\colon \mathbf{x} = \mathbf{x}(t), t_0 \leq t \leq t_1$, *joining the points* $\mathbf{x}_0 = \mathbf{x}(t_0)$ *and* $\mathbf{x}_1 = \mathbf{x}(t_1)$. *Then the map*

$$\tau_C\colon \quad T_{\mathbf{x}_0}M \ni \mathbf{X}(t_0) \longrightarrow \mathbf{X}(t_1) \in T_{\mathbf{x}_1}M \tag{3.3}$$

is a linear map from the vector space $T_{\mathbf{x}_0}M$ *to the vector space* $T_{\mathbf{x}_1}M$ *which does not depend on the parameterisation of the curve* C.

The fact that the map (3.3) does not depend on the parameterisation follows from the differential equation (3.2) also.

Indeed let $t = t(\tau)$, $\tau_0 \leq \tau \leq \tau_1$, $t(\tau_0) = t_0, t(\tau_1) = \tau_1$ be another parameterisation of the curve C. Then multiplying the equation (3.2) on $\frac{dt}{d\tau}$ and using the fact that velocity $\mathbf{v}'(\tau) = t_\tau \mathbf{v}(t)$ we come to differential equation:

$$\frac{dX^i(t(\tau))}{d\tau} + v'^k(t(\tau))\Gamma^i_{km}(x^i(t(\tau)))X^m(t(\tau)) \equiv 0\,. \tag{3.4}$$

The functions $X^(t(\tau))$ with the same initial conditions are the solutions of this equation.

The fact that it is a linear map follows immediately from the fact that differential equations (3.2) are linear. E.g. let vector fields $\mathbf{X}(t), \mathbf{Y}(t)$ be covariantly constant along the curve C. Then since linearity $\mathbf{X}(t) + \mathbf{Y}(t)$ is a solution too.

3.1.3 Parallel transport with respect to Levi-Civita connection

We mostly consider parallel transport on Riemannian manifold. If (M, G) is Riemannian manifold we mostly consider parallel transport with respect to connection ∇ which is Levi-Civita connection of the Riemannian metric G

Proposition The scalar product of vectors preserves during parallel transport with respect to Levi-Civita connection; i.e. during parallel transport of two vectors $\mathbf{X}$ and $\mathbf{Y}$ along curve, the length of these vectors and angle between them is preserved.

Proof follows immediately from the definition (3.1) of a parallel transport and the definition (2.33) of Levi-Civita connection

$$\frac{d}{dt}\langle \mathbf{X}(t), \mathbf{X}(t)\rangle = \partial_{\mathbf{v}}\langle \mathbf{X}(t), \mathbf{X}(t)\rangle = 2\,\langle \nabla_{\mathbf{v}}\mathbf{X}(t), \mathbf{X}(t)\rangle = 2\,\langle 0, \mathbf{X}(t)\rangle = 0\,. \tag{3.5}$$

3.2 Geodesics

3.2.1 Definition. Geodesic on Riemannian manifold

Let M be manifold equipped with connection ∇.

Definition A parameterised curve $C\colon \quad x^i = x^i(t)$ is called geodesic if velocity vector $\mathbf{v}(t)\colon v^i(t) = \frac{dx^i(t)}{dt}$ is covariantly constant along this curve, i.e. it remains parallel along the curve:

$$\nabla_{\mathbf{v}}\mathbf{v} = \frac{\nabla \mathbf{v}}{dt} = \frac{dv^i(t)}{dt} + v^k(t)\Gamma^i_{km}(x(t))v^m(t) = 0, \ \ i.e. \tag{3.6}$$

$$\frac{d^2x^i(t)}{dt^2} + \frac{dx^k(t)}{dt}\Gamma^i_{km}(x(t))\frac{dx^m(t)}{dt} = 0\,. \tag{3.7}$$

These are linear second order differential equations. One can prove that this equations have solution and it is unique[9] for an arbitrary initial data ($x^i(t_0) = x^i_0, \dot{x}^i(t_0) = \dot{x}^i_0$.)

In other words the curve $C\colon x(t)$ is a geodesic if parallel transport of velocity vector to along the curve is a velocity vector at any point of the curve.

Geodesics defined with Levi-Civita connection on the Riemannian manifold is called geodesic on Riemannian manifold. We mostly consider geodesics on Riemannian manifolds.

Since velocity vector of the geodesics on Riemannian manifold at any point is a parallel transport with the Levi-Civita connection, hence due to Proposition (3.5) the length of the velocity vector remains constant:

Proposition *If* $C\colon\ \mathbf{x}(t)$ *is a geodesics on Riemannian manifold then the length of velocity vector is preserved along the geodesic.*

Proof Since the connection is Levi-Civita connection then it preserves scalar product of tangent vectors, (see (2.33)) in particularly the length of the velocity vector $\mathbf{v}$:

Example 1 *Geodesics of Euclidean space.* In Cartesian coordinates Christoffel symbols of Levi-Civita connection vanish, and differential equation (3.6), (3.7) are reduced to equation

$$\frac{d^2x^i(t)}{dt^2} = 0\,, \Rightarrow \frac{dx^i(t)}{dt} = v^i \Rightarrow x^i = x^i_0 + v^i t\,. \tag{3.8}$$

We come to straight lines.

Example 2 *Geodesics of cylindrical surface* One can see that if Riemannian metric $G = G_{ik}du^i dv^k$ have constant coefficients in coordinates u^i then Christoffel symbols of Levi-Civita connection vanish in these coordinates, (see formula (2.34)) and according to (3.8) geodesics are “straight lines” in coordinates u^i. In particular this is a case for cylinder: If surface of cylinder is given by equation $\begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases}$ then Riemannian metric is equal to

[9]this is true under additional technical conditions which we do not discuss here

$G = a^2 d\varphi^2 + dh^2$ and we come to equations:

$$\begin{cases} \frac{d^2\varphi(t)}{dt^2} = 0 \\ \frac{d^2h(t)}{dt^2} = 0 \end{cases} \Rightarrow \begin{cases} \frac{d\varphi(t)}{dt} = \Omega \\ \frac{dh(t)}{dt} = c \end{cases} \Rightarrow \begin{cases} \frac{\varphi(t)}{dt} = \varphi_0 + \Omega t \\ \frac{dh(t)}{dt} = h_0 + tc \end{cases} . \tag{3.9}$$

In general case we come to helix:

$$\begin{cases} x = a\cos\varphi(t) = a\cos(\varphi_0 + \Omega t) \\ y = a\sin\varphi(t) = a\sin(\varphi_0 + \Omega t) \\ z = h(t) = h_0 + ct \end{cases} \tag{3.10}$$

If $c = 0$ then geodesics are circles $x^2 + y^2 = a^2, z = h_0$. If angular velocity $\Omega = 0$ then geodesics are vertical lines $x = x_0, y = y_0, z = h_0 + ct$.

3.2.2 Geodesics and Lagrangians of "free" particle on Riemannian manifold.

Lagrangian and Euler-Lagrange equations

A function $L = L(x, \dot{x})$ on points and velocity vectors on manifold M is a *Lagrangian* on manifold M.

We assign to Lagrangian $L = L(x, \dot{x})$ the following second order differential equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{\partial L}{\partial x^i} \tag{3.11}$$

In detail

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{\partial^2 L}{\partial x^m \partial \dot{x}^i}\dot{x}^m + \frac{\partial^2 L}{\partial \dot{x}^m \partial \dot{x}^i}\ddot{x}^m = \frac{\partial L}{\partial x^i} . \tag{3.12}$$

These equations are called *Euler-Lagrange equations* of the Lagrangian L. We will explain later the variational origin of these equations [10].

Lagrangian of "free" particle

Let (M, G), $G = g_{ik}dx^i dx^k$ be a Riemannian manifold.

[10] To every mechanical system one can put in correspondence a Lagrangian on configuration space. The dynamics of the system is described by Euler-Lagrange equations. The advantage of Lagrangian approach is that it works in an arbitrary coordinate system: Euler-Lagrange equations are invariant with respect to changing of coordinates since they arise from variational principe.

Definition We say that *Lagrangian* $L = L(x, \dot{x})$ is the Lagrangian of a a 'free' particle on the Riemannian manifold M if

$$L = \frac{g_{ik}\dot{x}^i\dot{x}^k}{2} \tag{3.13}$$

Example "Free" particle in Euclidean space. Consider $\mathbf{E}^3$ with standard metric $G = dx^2 + dy^2 + dz^2$

$$L = \frac{g_{ik}\dot{x}^i\dot{x}^k}{2} = \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{2} \tag{3.14}$$

Note that this is the Lagrangian that describes the dynamics of a free particle.

Example A 'free' particle on a sphere.

The metric on the sphere of radius R is $G = R^2 d\theta^2 + R^2 \sin^2\theta d\varphi^2$. Respectively for the Lagrangian of "free" particle we have

$$L = \frac{g_{ik}\dot{x}^i\dot{x}^k}{2} = \frac{R^2\dot{\theta}^2 + R^2\sin^2\theta\dot{\varphi}^2}{2} \tag{3.15}$$

Equations of geodesics and Euler-Lagrange equations

Theorem. *Euler-Lagrange equations of the Lagrangian of a free particle are equivalent to the second order differential equations for geodesics.*

This Theorem makes very easy calculations for Christoffel indices.

This Theorem can be proved by direct calculations.

Calculate Euler-Lagrange equations (3.11) for the Lagrangian (3.13):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{d}{dt}\left(\frac{\partial\left(\frac{g_{mk}\dot{x}^m\dot{x}^k}{2}\right)}{\partial \dot{x}^i}\right) = \frac{d}{dt}\left(g_{ik}\dot{x}^k\right) = g_{ik}\ddot{x}^{\ k} + \frac{\partial g_{ik}}{\partial x^m}\dot{x}^m\dot{x}^k$$

and

$$\frac{\partial L}{\partial x^i} = \frac{\partial\left(\frac{g_{mk}\dot{x}^m\dot{x}^k}{2}\right)}{\partial x^i} = \frac{1}{2}\frac{\partial g_{mk}}{\partial x^i}\dot{x}^m\dot{x}^k.$$

Hence we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = g_{ik}\ddot{x}^{\ k} + \frac{\partial g_{ik}}{\partial x^m}\dot{x}^m\dot{x}^k = \frac{\partial L}{\partial x^i} = \frac{1}{2}\frac{\partial g_{mk}}{\partial x^i}\dot{x}^m\dot{x}^k\,,$$

i.e.

$$g_{ik}\ddot{x}^{\,k} + \partial_m g_{ik}\dot{x}^m\dot{x}^k = \frac{1}{2}\partial_i g_{mk}\dot{x}^m\dot{x}^k\,.$$

Note that $\partial_m g_{ik}\dot{x}^m\dot{x}^k = \frac{1}{2}\left(\partial_m g_{ik}\dot{x}^m\dot{x}^k + \partial_k g_{im}\dot{x}^m\dot{x}^k\right)$. Hence we come to equation:

$$g_{ik}\frac{d^2x^k}{dt^2} + \frac{1}{2}\left(\partial_m g_{ik} + \partial_k g_{im} - \partial_i g_{mk}\right)\dot{x}^m\dot{x}^k$$

Multiplying on the inverse matrix g^{ik} we come

$$\frac{d^2x^i}{dt^2} + \frac{1}{2}g^{ij}\left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^m} - \frac{\partial g_{mk}}{\partial x^j}\right)\frac{dx^m}{dt}\frac{dx^k}{dt} = 0\,. \tag{3.16}$$

We recognize here Christoffel symbols of Levi-Civita connection (see (2.34)) and we rewrite this equation as

$$\frac{d^2x^i}{dt^2} + \frac{dx^m}{dt}\Gamma^i_{mk}\frac{dx^k}{dt} = 0\,. \tag{3.17}$$

This is nothing but the equation (3.6).

Applications of this Theorem: calculation of Christoffel symbols of Levi-Civita connection.

3.2.3 Calculations of Christoffel symbols and geodesics using the Lagrangians of a free particle.

It turns out that equation (3.17) is the very effective tool to caluclatie Christoffel symbols of Levi-Civita connection.

Consider two examples: We calculate Levi-Civita connection on sphere in $\mathbf{E}^3$ and on Lobachevsky plane using Lagrangians and find geodesics.

1) *Sphere of the radius R in $\mathbf{E}^3$*:

Lagrangian of "free" particle on the sphere is given by (3.15):

$$L = \frac{R^2\dot{\theta}^2 + R^2\sin^2\theta\dot{\varphi}^2}{2}$$

Euler-Lagrange equations defining geodesics are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = \frac{d}{dt}\left(R^2\dot{\theta}\right) - R^2\sin\theta\cos\theta\dot{\varphi}^2 \Rightarrow \ddot{\theta} - \sin\theta\cos\theta\dot{\varphi}^2 = 0\,, \tag{3.18}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi}\right)-\frac{\partial L}{\partial \varphi}=\frac{d}{dt}\left(R^2\sin^2\theta\dot\varphi\right)=0\Rightarrow \ddot\varphi+\mathrm{cotan}\,\theta\dot\theta\dot\varphi=0\,.$$

Comparing Euler-Lagrange equations with equations for geodesic in terms of Christoffel symbols:

$$\ddot\theta+\Gamma^\theta_{\theta\theta}\dot\theta^2+2\Gamma^\theta_{\theta\varphi}\dot\theta\dot\varphi+\Gamma^\theta_{\varphi\varphi}\dot\varphi^2=0,$$

$$\ddot\varphi+\Gamma^\varphi_{\theta\theta}\dot\theta^2+2\Gamma^\varphi_{\theta\varphi}\dot\theta\dot\varphi+\Gamma^\varphi_{\varphi\varphi}\dot\varphi^2=0$$

we come to

$$\Gamma^\theta_{\theta\theta}=\Gamma^\theta_{\theta\varphi}=\Gamma^\theta_{\varphi\theta}=0\,,\Gamma^\theta_{\varphi\varphi}=-\sin\theta\cos\theta\,, \tag{3.19}$$

$$\Gamma^\varphi_{\theta\theta}=\Gamma^\varphi_{\varphi\varphi}=0,\ \Gamma^\varphi_{\theta\varphi}=\Gamma^\varphi_{\varphi\theta}=\mathrm{cotan}\,\theta\,. \tag{3.20}$$

(Compare with previous calculations for connection in subsections 2.2.1 and 2.3.4)

We know already and we will prove later in a elegant way that geodesics on the sphere are great circles. (see subsection **??** above). Consider another technically more difficult but straightforward proof of this fact. To find geodesics one have to solve second order differential equations (3.17)

One can see that the great circles: $\varphi=\varphi_0$, $\theta=\theta_0+t$ are solutions of second order differential equations (3.18) with initial conditions

$$\theta(t)\big|_{t=0}=\theta_0,\dot\theta(t)\big|_{t=0}=1,\quad \varphi(t)\big|_{t=0}=\varphi_0,\dot\theta(t)\big|_{t=0}=0\,. \tag{3.21}$$

The rotation of the sphere is isometry, which does not change Levi-Civta connection. Hence an arbitrary great circle is geodesic.

Prove that an arbitrary geodesic is an arc of great circle. Let the curve $\theta=\theta(t),\varphi=\varphi(t)$, $0\le t\le t_1$ be geodesic. Rotating the sphere we can come to the curve $\theta=\theta'(t),\varphi=\varphi'(t)$, $0\le t\le t_1$ such that velocity vector at the initial time is direccted along meridian, i.e. initial conditions are

$$\theta'(t)\big|_{t=0}=\theta_0,\dot\theta'(t)\big|_{t=0}=a,\quad \varphi'(t)\big|_{t=0}=\varphi_0,\dot\varphi'(t)\big|_{t=0}=0\,. \tag{3.22}$$

(Compare with initial conditions (3.21)) Second order differential equations with boundary conditions for coordinates and velocities at $t=0$ have unique solution. The solutions of second order differential equations (3.18) with initial conditions (3.22) is a curve $\theta'(t)=\theta_0+at$, $\varphi'(t)=\varphi_0$. It is great circle. Hence initial curve the geodesic $\theta=\theta(t),\varphi=\varphi(t)$, $0\le t\le t_1$ is an arc of great circle too.

This is another proof that geodesics are great circles.

2) *Lobachevsky plane.*

Lagrangian of "free" particle on the Lobachevsky plane with metric $G=\frac{dx^2+dy^2}{y^2}$ is

$$L=\frac{1}{2}\frac{\dot x^2+\dot y^2}{y^2}.$$

Euler-Lagrange equations are

$$\frac{\partial L}{\partial x}=0=\frac{d}{dt}\frac{\partial L}{\partial \dot x}=\frac{d}{dt}\left(\frac{\dot x}{y^2}\right)=\frac{\ddot x}{y^2}-\frac{2\dot x\dot y}{y^3},\text{i.e.}\quad \ddot x-\frac{2\dot x\dot y}{y}=0\,,$$

$$\frac{\partial L}{\partial y}=-\frac{\dot x^2+\dot y^2}{y^3}=\frac{d}{dt}\frac{\partial L}{\partial \dot y}=\frac{d}{dt}\left(\frac{\dot y}{y^2}\right)=\frac{\ddot y}{y^2}-\frac{2\dot y^2}{y^3},\text{i.e.}\quad \ddot y+\frac{\dot x^2}{y}-\frac{\dot y^2}{y}=0\,.$$

Comparing these equations with equations for geodesics: $\ddot x^i-\dot x^k\Gamma^i_{km}\dot x^m=0$ ($i=1,2,\ x=x^1, y=x^2$) we come to

$$\Gamma^x_{xx}=0, \Gamma^x_{xy}=\Gamma^x_{yx}=-\frac{1}{y},\ \Gamma^x_{yy}=0,\ \Gamma^y_{xx}=\frac{1}{y}, \Gamma^y_{xy}=\Gamma^y_{yx}=0, \Gamma^y_{yy}=-\frac{1}{y}\,.\ \blacksquare$$

In a similar way as for a sphere one can find geodesics on Lobachevsky plane. First we note that vertical rays are geodesics. Then using the inversions with centre on the absolute one can see that arcs of the circles with centre at the absolute ($y=0$) are geodesics too.

See another examples in Homework 6 and 7.

3.2.4 Magnitudes preserved along geodesics—Integrals of motion

It is very useful to find magnitudes which are preserved along geodesics, functions $F=F(x,\dot x)$ such that for geodesic $C\colon x^i=x^i(t)$ the magnitude

$$I(t)=F(x,\dot x)\big|\_x^i=x^i(t)\quad\text{is preserved along geodesic } x^i(t),\quad \frac{dI(t)}{dt}=0\,. \tag{3.23}$$

Geodesics are solutions of equations of motions for the Lagrangian of a free particle $L=\frac{g_{ik}(x)\dot x^i\dot x^k}{2}$. One can consider such functions $F=F(x,\dot x)$ for an arbitrary Lagrangian L. In this case $x^i(t)$ is a solution of the Lagrangian L.

These magnitudes which are preserved along solutions of equations of motion (in particular along geodesics in the case if L is the Lagrangian of a free particle) are called *integrals of motion* (See in detail about integrals of motion in Appendix to this lectures).

There is the following very useful criterion to find magnitudes, which are preserved on equations of motions, i.e. integrals fo motion.

Proposition *Let Lagrangian $L(x^i,\dot x^i)$ in coordinates $\{x^i\}$ does not depend, say on the coordinate x^1. $L=L(x^2,\dots,x^n,\dot x^1,\dot x^2,\dots,\dot x^n)$. Then the function*

$$F_1(x,\dot x)=\frac{\partial L}{\partial \dot x^1}$$

is integral of motion. (In the case if $L(x^i, \dot{x}^i)$ does not depend on the coordinate x^i. the function $F_i(x, \dot{x}) = \frac{\partial L}{\partial \dot{x}^i}$ will be integral of motion.)

Proof is simple. Check the condition (3.23): Euler-Lagrange equations of motion are:

$$\frac{d}{dt}\left(\frac{\partial L}{\dot{\partial} x^i}\right) - \frac{\partial L}{\partial x^i} = 0 \quad (i = 1, 2, \ldots, n)$$

In particular for first coordinate x^1, $\frac{\partial L}{\partial x^1} = 0$ and

$$\frac{d}{dt}\left(\frac{\partial L}{\dot{\partial} x^1}\right) - \frac{\partial L}{\partial x^1} = \frac{d}{dt}\left(\frac{\partial L}{\dot{\partial} x^1}\right) = 0\,,$$

i.e. the magnitude $I(t) = F(x, \dot{x})$ is preserved if $F = \frac{\partial L}{\dot{\partial} x^1}$. We see that exactly first equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\dot{\partial} x^1}\right) = \frac{d}{dt} F_1(q, \dot{q}) = 0 \quad \text{since } \frac{\partial L}{\partial x^1} = 0, .$$

(if $L(x^i, \dot{x}^i)$ does not depend on the coordinate x^i then the function $F_i(x, \dot{x}) = \frac{\partial L}{\dot{x}^1}$ is integral of motion since $i - th$ equation is exactly the condition $\dot{F}_i = 0$.)

The integral of motion $F_i = \frac{\partial L}{\dot{\partial} x^i}$ is called sometimes *generalised momentum.*

Consider examples of calculation of preserved mangnitudes along geodesics.

Example (sphere)

Sphere of the radius R in $\mathbf{E}^3$. Riemannian metric: $G = R d\theta^2 + R^2 \sin^2\theta d\varphi^2$ and $L_{\text{free}} = \frac{1}{2}\left(R^2\dot{\theta}^2 + R^2 \sin^2\theta \dot{\varphi}^2\right)$. Lagrangian does not depend explicitly on coordinate φ. The integral of motion is

$$F = \frac{\partial L_{\text{free}}}{\partial \dot{\varphi}} = R^2 \sin^2\theta \dot{\varphi}\,.$$

It is preserved along geodesics, i.e. along great circles.

Example (cone)

Consider cone $\begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases}$. Riemannian metric:

$$G = d(kh\cos\varphi)^2 + d(kh\sin\varphi)^2 + (dh)^2 = (k^2+1)dh^2 + k^2h^2 d\varphi^2\,.$$

and free Lagrangian

$$L_{\text{free}} = \frac{(k^2+1)\dot{h}^2 + k^2 h^2 \dot{\varphi}^2}{2}\,.$$

Lagrangian does not depend explicitly on coordinate h. The integral of motion is

$$F = \frac{\partial L_{\text{free}}}{\partial \dot{\varphi}} = k^2 h^2 \dot{\varphi}.$$

It is preserved along geodesics.

Remark One has to note that for the Lagrangian of a free particle $F = L = g_{ik}\dot{x}^i\dot{x}^k$, kinetik energy, is integral of motion preserved along geodesic:it is nothing that square of the length of velocity vector which is preserved along the geodesic.

See these and other examples in Homework 6.

* *Using integral of motions to calculate geodesics*

Integrals of motions may be very useful to calculate geodesics. The equations for geodesics are second order differential equations. If we know integrals of motions they help us to solve these equations. Consider just an example.

For Lobachevsky plane the free Lagrangian $L = \frac{\dot{x}^2+\dot{y}^2}{2y^2}$. We already calculated geodesics in the subsection 3.3.4. Geodesics are solutions of second order Euler-Lagrange equations for the Lagrangian $L = \frac{\dot{x}^2+\dot{y}^2}{2y^2}$ (see the subsection 3.3.4)

$$\begin{cases} \ddot{x} - \frac{2\dot{x}\dot{y}}{y} = 0 \\ \ddot{y} + \frac{\dot{x}^2}{y} - \frac{\dot{y}^2}{y} = 0 \end{cases}$$

It is not so easy to solve these differential equations.

For Lobachevsky plane we know two integrals of motions:

$$E = L = \frac{\dot{x}^2+\dot{y}^2}{2y^2}, \quad \text{and } F = \frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{y^2}\,. \tag{3.24}$$

These both integrals are preserved in time: if $x(t), y(t)$ is geodesics then

$$\begin{cases} F = \frac{\dot{x}(t)}{y(t)^2} \\ E = \frac{\dot{x}(t)^2+\dot{y}(t)^2}{2y(t)^2} = C_2 \end{cases} \Rightarrow \begin{cases} \dot{x} = C_1 y^2 \\ \dot{y} = \pm\sqrt{2C_2 y^2 - C_1^2 y^4} \end{cases}$$

These are first order differential equations. It is much easier to solve these equations in general case than initial second order differential equations.

3.2.5 * Variational principe and Euler-Lagrange equations

Here very briefly we will explain how Euler-Lagrange equations follow from variational principe.

Let M be a manifold (not necessarily Riemannian) and $L = L(x^i, \dot{x}^i)$ be a Lagrangian on it.

Denote my $\mathcal{M}_{\mathbf{x}_1,t_1}^{\mathbf{x}_2,t_2}$ the space of curves (paths) such that they start at the point $\mathbf{x}_1$ at the "time" $t = t_1$ and end at the point $\mathbf{x}_2$ at the "time" $t = t_2$:

$$\mathcal{M}_{\mathbf{x}_1,t_1}^{\mathbf{x}_2,t_2} = \{C\colon\ \mathbf{x}(t), t_1 \le t \le t_2,\ \mathbf{x}(t_1) = \mathbf{x}_1, \mathbf{x}(t_2) = \mathbf{x}_2\}\,. \tag{3.25}$$

Consider the following functional S on the space $\mathcal{M}_{\mathbf{x}_1,t_1}^{\mathbf{x}_2,t_2}$:

$$S\left[\mathbf{x}(t)\right] = \int_{t_1}^{t_2} L\left(x^i(t), \dot{x}^i(t)\right) dt\,. \tag{3.26}$$

for every curve $\mathbf{x}(t) \in \mathcal{M}_{\mathbf{x}_1,t_1}^{\mathbf{x}_2,t_2}$.

This functional is called *action* functional.

Theorem *Let functional S attaints the minimal value on the path $\mathbf{x}_0(t) \in \mathcal{M}_{\mathbf{x}_1,t_1}^{\mathbf{x}_2,t_2}$, i.e.*

$$\forall \mathbf{x}(t) \in \mathcal{M}_{\mathbf{x}_1,t_1}^{\mathbf{x}_2,t_2} \quad S[\mathbf{x}_0(t)] \le S[\mathbf{x}(t)]\,. \tag{3.27}$$

Then the path $\mathbf{x}_0(t)$ is a solution of Euler-Lagrange equations of the Lagrangian L:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{\partial L}{\partial x^i} \text{ if } \mathbf{x}(t) = \mathbf{x}_0(t)\,. \tag{3.28}$$

Remark The path $\mathbf{x}(t)$ sometimes is called *extremal* of the action functional (3.26).

We will use this Theorem to show that the geodesics are in some sense shortest curves [11].

---

[11] The statement of this Theorem is enough for our purposes. In fact in classical mechanics another more useful statement is used: the path $\mathbf{x}_0(t)$ is a solution of Euler-Lagrange equations of the Lagrangian L if and only if it is the stationary "point" of the action functional (3.26), i.e.

$$S[\mathbf{x}_0(t) + \delta\mathbf{x}(t)] - S[\mathbf{x}_0(t) + \delta\mathbf{x}(t)] = 0(\delta\mathbf{x}(t)) \tag{3.29}$$

for an arbitrary infinitesimal variation of the path $\mathbf{x}_0(t)$: $\delta\mathbf{x}_(t_1) = \delta\mathbf{x}_(t_2) = 0$.

3.2.6 Un-parameterised geodesic

We defined a geodesic as a parameterised curve such that the velocity vector is covariantly constant along the curve.

What happens if we change the parameterisation of the curve?

Another question: Suppose a tangent vector to the curve remains tangent to the curve during parallel transport. Is it true that this curve (in a suitable parameterisation) becomes geodesic?

Definition We call un-parameterised curve geodesic if under suitable parameterisation it obeys the equation (3.6) for geodesics.

Let C–be un-parameterised geodesic. Then the following statement is valid.

Proposition *A curve C (un-parameterised) is geodesic if an only if a non-zero vector tangent to the curve remains tangent to the curve during parallel transport.*

Proof. Let $\mathbf{A}$ be tangent vector at the point $\mathbf{p} \in C$ of the curve. Parallel transport does not depend on parametersiation of the curve (see 3.1.2). Choose a suitable parameterisation $x^i = x^i(t)$ such that $x^i(t)$ obeys the equations (3.6) for geodesics, i.e. the velocity vector $\mathbf{v}(t)$ is covariantly constant along the curve: $\nabla_{\mathbf{v}}\mathbf{v} = 0$. If $\mathbf{A}(t_0) = c\mathbf{v}(t_0)$ at the given point $\mathbf{p}$ (c is a scalar coefficient) then due to linearity $\mathbf{A}(t) = c\mathbf{v}(t)$ is a parallel transport of the vector $\mathbf{A}$. The vector $\mathbf{A}(t)$ is tangent to the curve since it is proportional to velocity vector. We proved that any tangent vector remains tangent during parallel transport.

Now prove the converse: Let $\mathbf{A}(t)$ be a parallel transport of non-zero vector and it is proportional to velocity. If in a given parameterisation $\mathbf{A}(t) = c(t)\mathbf{v}(t)$ choose a reparameterisation $t = t(\tau)$ such that $\frac{dt(\tau)}{d\tau} = c(t)$. In the new parameterisation the velocity vector $\mathbf{v}'(\tau) = \frac{dt(\tau)}{d\tau}\mathbf{v}(t(\tau)) = c(t)\mathbf{v}(t) = \mathbf{A}(t(\tau))$. We come to parameterisation such that velocity vector remains covariantly constant. Thus we come to parameterised geodesic. Hence C is a geodesic.

Remark In particularly it follows from the Proposition above the following important observation:

Let C is un-parameterised geodesic, $x^i(t)$ be its arbitrary parameterisation and $\mathbf{v}(t)$ be velocity vector in this parameterisation. Then the velocity vector remains parallel to the curve since it is a tangent vector.

In spite of the fact that velocity vector is not covariantly constant along the curve, i.e. it will not remain velocity vector during parallel transport, since it will be remain tangent to the curve during parallel transport.

Remark One can see that if $x^i = x^i(t)$ is geodesic in an arbitrary parameterisation and $s = s(t)$ is a natural parameter (which defines the length of the curve) then $x^i(t(s))$ is parameterised geodesic.

3.2.7 Parallel transport of vectors along geodesics

We already now that during parallel transport along curve with respect to Levi-Civita connection scalar product of vectors, i.e. lengths of vectors and angle between them does not change (see section 3.1.3). This remark makes easy to calculate parallel transport of vectors along geodesics in Riemannian manifold. Indeed let C a geodesic (in general un-parameterised) and a vectors $\mathbf{X}(t)$ is attached to the point $\mathbf{p}_1 \in C$ on the curve C. In the special case if $\mathbf{X}$ is a tangent vector to geodesic C then during parallel transport it remains tangent, i.e. proportional to velocity vector:

$$\mathbf{X}(t) = a(t)\mathbf{v}(t)\,. \tag{3.30}$$

Here $\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt}$ and $\mathbf{r} = \mathbf{r}(t)$ is an *arbitrary parameterisation* of geodesic C. Note that in general t is not parameter such that $\mathbf{r} = \mathbf{r}(t)$ is parameterised geodesic; t is an arbitrary parameter. In the special case if t is a parameter such that $\mathbf{r} = \mathbf{r}(t)$ is parameterised geodesic then velocity vector remains velocity vector during parallel transport, i.e. $\mathbf{X}(t) = a\mathbf{v}(t)$ where a is not dependent on t.

To calculate the dependence of coefficient a on t in (3.31) we note that the length of the vector is not changed (see the section 3.1.3, i.e.

$$\langle \mathbf{X}(t), \mathbf{X}(t)\rangle = \langle a(t)\mathbf{v}(t), a(t)\mathbf{v}(t)\rangle = a^2(t)|\mathbf{v}(t)|^2 = \text{constant} \tag{3.31}$$

3.2.8 Geodesics on surfaces in $\mathbf{E}^3$

Let $M\colon \mathbf{r} = \mathbf{r}(u,v)$ be a surface in $\mathbf{E}^3$. Let G_M be induced Riemannian metric and ∇ a Levi-Civita connection on M. We consider on M Levi-Civita connection of the metric G_M.

Let C be an arbitrary geodesic and $\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt}$ the velocity vector. According to the definition of geodesic $\nabla_{\mathbf{v}}\mathbf{v} = 0$. On the other hand we know that Levi-Civita connection coincides with the connection induced on the surface by canonical flat connection in $\mathbf{E}^3$ (see the Theorem in subsection 2.4). Hence

$$\nabla_{\mathbf{v}}\mathbf{v} = 0 = \nabla^M_{\mathbf{v}}\mathbf{v} = \left(\nabla^{\text{can.flat}}_{\mathbf{v}}\mathbf{v}\right)_{\text{tangent}} \tag{3.32}$$

In Cartesian coordinates $\nabla^{\text{can.flat}}_{\mathbf{v}}\mathbf{v} = \partial_{\mathbf{v}}\mathbf{v} = \frac{d}{dt}\mathbf{v}(u(t), v(t)) = \frac{d^2\mathbf{r}(t)}{dt^2} = \mathbf{a}$.

Hence according to (3.32) the tangent component of acceleration equals to zero.

Converse if for the curve $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ the acceleration vector $\mathbf{a}(t)$ is orthogonal to the surface then due to (3.32) $\nabla_{\mathbf{v}}\mathbf{v} = 0$.

We come to very beautiful observation:

Theorem *The acceleration vector of an curve $\mathbf{r} = \mathbf{r}(u(t), v(t))$ on M is orthogonal to the surface M if and only if this curve is geodesic.*

In other words due to Newton second law particle moves along along geodesic on the surface if and only if the force is orthogonal to the surface.

One can very easy using this Proposition to calculate geodesics of cylinder and sphere.

Geodesic on the cylinder

Let $\mathbf{r}(h(t), \varphi(t))$ be a geodesic on the cylinder $\begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases}$. We have $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \begin{pmatrix} -a\dot\varphi\sin\varphi \\ a\dot\varphi\cos\varphi \\ \dot h \end{pmatrix}$ and for acceleration:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \underbrace{\begin{pmatrix} -a\ddot\varphi\sin\varphi \\ a\ddot\varphi\cos\varphi \\ \ddot h \end{pmatrix}}_{\text{tangent acceleration}} + \underbrace{\begin{pmatrix} -a\dot\varphi^2\cos\varphi \\ -a\dot\varphi^2\sin\varphi \\ 0 \end{pmatrix}}_{\text{normal acceleration}}$$

Since tangential acceleration equals to zero hence $\frac{d^2h}{dt^2} = 0$ and $h(t) = h_0 + ct$ Normal acceleration is centripetal acceleration of the rotation over circle with constant speed (projection on the plane OXY). The geodesic is helix. (Compare these calculations with calculations of geodesics of cylinder in the last example of section 3.2.1: see (3.10).)

Geodesics on sphere

Let $\mathbf{r} = \mathbf{r}(\theta(t), \varphi(t))$ be a geodesic on the sphere of the radius a: $\mathbf{r}(\theta, \varphi)$: $\begin{cases} x = a \sin\theta \cos\varphi \\ y = a \sin\theta \sin\varphi \\ z = a \cos\theta \end{cases}$

Consider the vector product of the vectors $\mathbf{r}(t)$ and velocity vector $\mathbf{v}(t)$ $\mathbf{M}(t) = \mathbf{r}(t) \times \mathbf{v}(t)$. Acceleration vector $\mathbf{a}(t)$ is proportional to the $\mathbf{r}(t)$ since due to Proposition it is orthogonal to the surface of the sphere. This implies that $\mathcal{M}(t)$ is constant vector:

$$\frac{d}{dt}\mathbf{M}(t) = \frac{d}{dt}(\mathbf{r}(t) \times \mathbf{v}(t)) = (\mathbf{v}(t) \times \mathbf{v}(t)) + (\mathbf{r}(t) \times \mathbf{a}(t)) = 0 \qquad (3.33)$$

We have $\mathbf{M}(t) = \mathbf{M}_0$. $\mathbf{r}(t)$ is orthogonal to $\mathbf{M} = \mathbf{r}(t) \times \mathbf{v}(t)$. We see that $\mathbf{r}(t)$ belongs to the sphere and to the plane orthogonal to the vector $\mathbf{M}_0 = \mathbf{r}(t) \times \mathbf{v}(t)$. The intersection of this plane with sphere is a great circle. We proved that if $\mathbf{r}(t)$ is geodesic hence it belongs to great circle (as un-parameterised curve).

The converse is evident since if particle moves along the great circle with constant velocity then obviously acceleration vector is orthogonal to the surface.

Remark The vector $\mathcal{M} = \mathbf{r}(t) \times \mathbf{v}(t)$ is the torque. The torque is integral of motion in isotropic space.—This is the core of the considerations for geodesics on the sphere.

3.2.9 * Geodesics and shortest distance.

Many of you know that geodesics are in some sense shortest curves. We will give an exact meaning to this statement and prove it using variational principe:

Let M be a Riemannian manifold.

Theorem *Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two points on M. The shortest curve which joins these points is an arc of geodesic.*

Let C be a geodesic on M and $\mathbf{x}_1 \in C$. Then for an arbitrary point $\mathbf{x}_2 \in C$ which is close to the point $\mathbf{x}_1$ the arc of geodesic joining the points $\mathbf{x}_1, \mathbf{x}_2$ is a shortest curve between these points[12] .

[12]More precisely: for every point $\mathbf{x}_1 \in C$ there exists a ball $B_\delta(\mathbf{x}_1)$ such that for an arbitrary point $\mathbf{x}_2 \in C \cap B_\delta(\mathbf{x}_1)$ the arc of geodesic joining the points $\mathbf{x}_1, \mathbf{x}_2$ is a shortest curve between these points.

This Theorem makes a bridge between two different approach to geodesic: the shortest disntance and parallel transport of velocity vector.

Sketch a proof:

Consider the following two Lagrangians: Lagrangian of a "free " particle $L_{\text{free}} = \frac{g_{ik}(x)\dot{x}^i\dot{x}^k}{2}$ and the length Lagrangian

$$L_{\text{length}}(x, \dot{x}) = \sqrt{g_{ik}(x)\dot{x}^i\dot{\mathbf{x}}^k} = \sqrt{2L_{\text{free}}}\,.$$

If $C\colon\ x^i(t), t_1 \le t \le t_2$ is a curve on M then

$$\text{Length of the curve } C =$$

$$\int_{t_1}^{t_2} L_{\text{length}}(x^i(t), \dot{x}^i(t))dt = \int_{t_1}^{t_2} \sqrt{g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t)}dt\,. \tag{3.34}$$

The proof of the Theorem follows from the following observation:

Observation Euler-Lagrange equations for the length functional (3.34) are equivalent to the Euler-Lagrange equations for action functional (3.26). This means that extremals of the length functional and action functionals coincide.

Indeed it follows from this observation and the variational principe that the shortest curves obey the Euler-Lagrange equations for the action functional. We showed before that Euler-Lagrange equations for action functional (3.26) define geodesics. Hence the shortest curves are geodesics.

One can check the observation by direct calculation: Calculate Euler-Lagrange equations for the Lagrangian $L_{\text{length}} = \sqrt{g_{ik}(x)\dot{x}^i\dot{x}^k} = \sqrt{2L_{\text{free}}}$:

$$\frac{d}{dt}\left(\frac{\partial L_{\text{length}}}{\partial \dot{x}^i}\right) - \frac{\partial L_{\text{length}}}{\partial x^i} = \frac{d}{dt}\left(\frac{1}{\sqrt{g_{ik}\dot{x}^i\dot{x}^k}} g_{ik}\dot{x}^k\right) - \frac{1}{2\sqrt{g_{ik}\dot{x}^i\dot{x}^k}}\frac{\partial g_{km}\dot{x}^k\dot{x}^m}{\partial x^i}$$

$$= \frac{d}{dt}\left(\frac{1}{L_{\text{length}}}\frac{\partial L_{\text{free}}}{\partial \dot{x}^i}\right) - \frac{1}{L_{\text{length}}}\frac{\partial L_{\text{free}}}{\partial x^i} = 0\,. \tag{3.35}$$

To facilitate calculations note that the length functional (3.34) is reparameterisation invariant. Choose the natural parameter $s(t)$ or a parameter proportional to the natural parameter on the curve $x^i(t)$. We come to $L_{\text{length}} = const$ and it follows from (3.35) that

$$\frac{d}{dt}\left(\frac{\partial L_{\text{length}}}{\partial \dot{x}^i}\right) - \frac{\partial L_{\text{length}}}{\partial x^i} = \frac{1}{L_{\text{length}}}\left(\frac{d}{dt}\left(\frac{\partial L_{\text{free}}}{\partial \dot{x}^i}\right) - \frac{\partial L_{\text{free}}}{\partial x^i}\right) = 0\,.$$

We prove that Euler-Lagrange equations for length and action Lagrangians coincide. ∎

In the Euclidean space straight lines are the shortest distances between two points. On the other hand their velocity vectors are constant. We realise now that in general Riemannian manifold the role of geodesic is twofold also: they are locally shortest and have covariantly constant velocity vectors.

3.2.10 * Again geodesics for sphere and Lobachevsky plane

The fact that geodesics are shortest gives us another tool to calculate geodesics.

Consider again examples of sphere and Lobachevsky plane and find geodesics using the fact that they are shortest. The fact that geodesics are locally the shortest curves

Consider again sphere in $\mathbf{E}^3$ with the radius R: Coordinates θ, φ, induced Riemannian metrics (first quadratic form):

$$G = R^2(d\theta^2 + \sin^2\theta d\varphi^2)\,. \tag{3.36}$$

Consider two arbitrary points A and B on the sphere. Let (θ_0, φ_0) be coordinates of the point A and (θ_1, φ_1) be coordinates of the point B

Let C_{AB} be a curve which connects these points: $C_{AB}\colon \theta(t), \varphi(t)$ such that $\theta(t_0) = \theta_0, \theta(t_1) = \theta_1$, $\varphi(t_0) = \theta_0, \theta(t_1) = \theta_1$ then:

$$L_{C_{AB}} = \int R\sqrt{\theta_t^2 + \sin^2\theta(t)\varphi_t^2}dt \tag{3.37}$$

Suppose that points A and B have the same latitude, i.e. if (θ_0, φ_0) are coordinates of the point A and (θ_1, φ_1) are coordinates of the point B then $\varphi_0 = \varphi_1$ (if it is not the fact then we can come to this condition rotating the sphere)

Now it is easy to see that an arc of meridian, the curve $\varphi = \varphi_0$ is geodesics: Indeed consider an arbitrary curve $\theta(t), \varphi(t)$ which connects the points A, B: $\theta(t_0) = \theta(t_1) = \theta_0$, $\varphi(t_0) = \varphi(t_1) = \varphi_0$. Compare its length with the length of the meridian which connects the points A, B:

$$\int_{t_0}^{t_1} R\sqrt{\theta_t^2 + \sin^2\theta\varphi_t^2}dt \geq R\int_{t_0}^{t_1}\sqrt{\theta_t^2}dt = R\int_{t_0}^{t_1}\theta_t dt = R(\theta_1 - \theta_0) \tag{3.38}$$

Thus we see that the great circle joining points A, B is the shortest. *The great circles on sphere are geodesics.* It corresponds to geometrical intuition: The geodesics on the sphere are the circles of intersection of the sphere with the plane which crosses the centre.

Geodesics on Lobachevsky plane

Riemannian metric on Lobachevsky plane:

$$G = \frac{dx^2 + dy^2}{y^2} \tag{3.39}$$

The length of the curve $\gamma\colon x = x(t, y = y(t))$ is equal to

$$L = \int \sqrt{\frac{x_t^2 + y_t^2}{y^2(t)}} dt$$

In particularly the length of the vertical interval $[1, \varepsilon]$ tends to infinity if $\varepsilon \to 0$:

$$L = \int \sqrt{\frac{x_t^2 + y_t^2}{y^2(t)}} dt = \int_\varepsilon^1 \sqrt{\frac{1}{t^2}} dt = \log \frac{1}{\varepsilon}$$

One can see that the distance from every point to the line $y = 0$ is equal to infinity. This motivates the fact that the line $y = 0$ is called *absolute*.

Consider two points $A = (x_0, y_0)$, $B = (x_1, y_1)$ on Lobachevsky plane.

It is easy to see that vertical lines are geodesics of Lobachevsky plane.

Namely let points A, B are on the ray $x = x_0$. Let C_{AB} be an arc of the ray $x = x_0$ which joins these points: $C_{AB}\colon x = x_0, y = y_0 + t$ Then it is easy to see that the length of the curve C_{AB} is less or equal than the length of the arbitrary curve $x = x(t), y = y(t)$ which joins these points: $x(t)\big|_{t=0} = x_0, y(t)\big|_{t=0} = y_0$, $x(t)\big|_{t=t_1} = x_0, y(t)\big|_{t=t_1} = y_1$:

$$\int_0^t \sqrt{\frac{x_t^2 + y_t^2}{y^2(t)}} dt \geq \int_0^t \sqrt{\frac{y_t^2}{y^2(t)}} dt = \int_{y_0}^{y_1} \frac{dt}{t} dt = \log \frac{y_1}{y_0} = \text{length of } C_{AB}$$

Hence C_{AB} is shortest. We prove that vertical rays are geodesics.

Consider now the case if $x_0 \neq x_1$. Find geodesics which connects two points A, B which are not on the same vertical ray. Consider semicircle which passes these two points such that its centre is on the absolute. We prove that it is a geodesic.

Proof Let coordinates of the centre of the circle are $(a, 0)$. Then consider polar coordinates (r, φ):

$$x = a + r\cos\varphi, y = r\sin\varphi \tag{3.40}$$

In these polar coordinates r-coordinate of the semicircle is constant.

Find Lobachevsky metric in these coordinates: $dx = -r\sin\varphi d\varphi + \cos\varphi dr$, $dy = r\cos\varphi d\varphi + \sin\varphi dr$, $dx^2 + dy^2 = dr^2 + r^2 d\varphi^2$. Hence:

$$G = \frac{dx^2 + dy^2}{y^2} = \frac{dr^2 + r^2 d\varphi^2}{r^2\sin^2\varphi} == \frac{d\varphi^2}{\sin^2\varphi} + \frac{dr^2}{r^2\sin^2\varphi} \tag{3.41}$$

We see that the length of the arbitrary curve which connects points A, B is greater or equal to the length of the arc of the circle:

$$L_{AB} = \int_{t_0}^{t_1} \sqrt{\frac{\varphi_t^2}{\sin^2\varphi} + \frac{r_t^2}{r^2\sin^2\varphi}}dt \geq \int_{t_0}^{t_1} \sqrt{\frac{\varphi_t^2}{\sin^2\varphi}}dt = \tag{3.42}$$

$$\int_{t_0}^{t_1} \frac{\varphi_t}{\sin\varphi}dt = \int_{\varphi_0}^{\varphi_1} \frac{d\varphi}{\sin\varphi} = \log\frac{\tan\varphi_1}{\tan\varphi_1}$$

The proof is finished.

4 Surfaces in $\mathbf{E}^3$

Now equipped by the knowledge of Riemannian geometry we consider surfaces in $\mathbf{E}^3$. We reconsider again conceptions of Shape (Weingarten) operator, Gaussian and mean curvatures, focusing attention on the the fact what properties are internal and what properties are external. In particular we consider again gaussian curvature and derive its internal meaning. We will consider again Theorema Egregium.

4.1 Formulation of the main result.

We formulate here three important and beautiful statements

- Theorem of parallel transport over closed curve
- Formula for Gaussian curvature in isothermal coordinates
- Theorema Egregium

Then we will consider them in detail.

4.1.1 Theorem of parallel transport over closed curve

Let M be a surface in Euclidean space $\mathbf{E}^3$. Consider a closed curve C on M, $M\colon \mathbf{r} = \mathbf{r}(u, v)$, $C\colon \mathbf{r} = \mathbf{r}(u(t, v(t)), 0 \leq t \leq t_1$, $\mathbf{x}(0) = \mathbf{x}(t_1)$. $(u(t), v(t)$ are internal coordinates of the curve C.)

Consider the parallel transport of an arbitrary tangent $\mathbf{X}$ vector along the closed curve C:

$$\mathbf{X}(t) = \underbrace{X^\alpha(t)\frac{\partial}{\partial u_\alpha}\big|_{u^\alpha(t)}}_{\text{Internal observer}} = \underbrace{X^\alpha(t)\mathbf{r}_\alpha\big|_{\mathbf{r}(u(t),v(t))}}_{\text{External observer}}, \left(\mathbf{r}_\alpha = \frac{\partial x^i}{\partial u^\alpha}\frac{\partial}{\partial x^i}\right).$$

$$\mathbf{X}(t)\colon\ \frac{\nabla \mathbf{X}(t)}{dt} = 0,\ 0 \le t \le t_1\,,$$

i.e.

$$\frac{dX^\alpha(t)}{dt} + X^\beta(t)\Gamma^\alpha_{\beta\gamma}(u(t))\frac{du^\gamma(t)}{dt} = 0,\ 0 \le t \le t_1\,, \qquad \alpha,\beta,\gamma = 1,2\,, \quad (4.1)$$

where ∇ is the connection induced on the surface M by canonical flat connection (see (2.27)), or (it is the same) the Levi-Civita connection (2.34) of the induced Riemannian metric on the surface M and $\Gamma^\alpha_{\beta\gamma}$ its Christoffel symbols:

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2}g^{\gamma\pi}\left(\frac{\partial g_{\pi\alpha}}{\partial u^\beta} + \frac{\partial g_{\pi\beta}}{\partial u^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial u^\pi}\right), \text{where}\ g_{\alpha\beta} = \langle \mathbf{r}_\alpha, \mathbf{r}_\beta\rangle = \frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^i}{\partial u^\beta} \quad (4.2)$$

are components of induced Riemannian metric $G_M = g_{\alpha\beta}du^\alpha du^\beta/$

Let $\mathbf{r}(0) = \mathbf{p}$ be a starting (and ending) point of the curve C: $\mathbf{r}(0) = \mathbf{r}(t_1) = \mathbf{p}$. The differential equation (4.1) defines the linear operator

$$R_C\colon T_\mathbf{p}M \longrightarrow T_\mathbf{p}M \quad (4.3)$$

For any vector $\mathbf{X} \in T_\mathbf{p}M$, its image the vector $R_C\mathbf{X}$ as the solution of the differential equation (4.1) with initial condition $\mathbf{X}(t)\big|_{t=0} = \mathbf{X}$.

On the other hand we know that parallel transport of the vector does not change its length (see (3.5) in the subsection 3.2.1):

$$\langle \mathbf{X}, \mathbf{X}\rangle = \langle R_C\mathbf{X}, R_C\mathbf{X}\rangle \quad (4.4)$$

We see that $\mathbf{R}_C$ is an orthogonal operator in the 2-dimensional vector space $T_\mathbf{p}M$. We know that orthogonal operator preserving orientation is the operator of rotation on some angle ϕ.

One can see that $\mathbf{R}_C$ preserves orientation [13] then the action of operator $\mathbf{R}_C$ on vectors is rotation on the angle, i.e. the result of parallel transport

[13] In our considerations we consider only the case if the closed curve C is a boundary of a compact oriented domain $D \subset M$. In this case one can see that operator R_C preserves an orientation.

along closed curve is rotation on the $\angle\phi$. This angle depends depends on the curve. The very beautiful question arises: How to calculate this angle $\Delta\Phi(C)$

Theorem *Let M be a surface in Euclidean space $\mathbf{E}^3$. Let C be a closed curve C on M such that C is a boundary of a compact oriented domain $D \subset M$. Consider the parallel transport of an arbitrary tangent vector along the closed curve C. As a result of parallel transport along this closed curve any tangent vector rotates through the angle*

$$\angle\phi = \angle(\mathbf{X}, \mathbf{R}_C\mathbf{X}) = \int_D K d\sigma\,, \tag{4.5}$$

where K is the Gaussian curvature and $d\sigma = \sqrt{\det g}dudv$ is the area element induced by the Riemannian metric on the surface M, i.e. $d\sigma = \sqrt{\det g}dudv$.

Remark One can show that the angle of rotation does not depend on initial point of the curve.

Example Consider the closed curve, "latitude" $C_{\theta_0}\colon\ \theta = \theta_0$ on the sphere of the radius R. Calculations show that

$$\angle\phi(C_{\theta_0}) = 2\pi(1 - \cos\theta_0) \tag{4.6}$$

(see also the Homework 8). On the other hand the latitude C_{θ_0} is the boundary of the segment D with area $2\pi RH$ where $H = R(1-\cos\theta_0)$. Hence

$$\angle(\mathbf{X}, \mathbf{R}_C\mathbf{X}) = \frac{2\pi RH}{R^2} = \frac{1}{R^2} \cdot \text{area of the segment} = \int_D K d\sigma$$

since Gaussian curvature is equal to $\frac{1}{R^2}$

We will present the proof of this Theorem later.

4.1.2 †Formula for Gaussian curvature in isothermal coordinates

Now we will consider one very beautiful and illuminating formula to calculate Gaussian curvature for surfaces.

Let $M\colon \mathbf{r} = \mathbf{r}(u, v)$ be a surface in $\mathbf{E}^3$.

Definition We say that coordinates (parameters) u, v are *isothermal* (or *conformal*) if the induced Riemannian metric 1.4.2 is equal to

$$G = \sigma(u, v)(du^2 + dv^2)$$

Since $\sigma > 0$ it is convenient to introduce $\sigma = e^{\Phi(u,v)}$. Then

$$G = e^{\Phi(u,v)}(du^2 + dv^2)\,. \tag{4.7}$$

One can show that locally one can always find isothermal coordinates [14]

Theorem Gaussian curvature K of the surface M is given by the formula

$$K = -\frac{1}{2}e^{-\Phi(u,v)}\Delta\Phi(u,v)\,, \qquad \text{where } \Delta = \tfrac{\partial^2}{\partial u^2} + \tfrac{\partial^2}{\partial v^2}\,. \tag{4.8}$$

We will discuss this theorem later.

4.1.3 Gauß *Theorema Egregium*

We defined Gaussian curvature in terms of Weingarten operator as a product of principal curvatures. This definition was in terms of External Observer. On the other hand the right hand side of the formula (4.8) depends on metric on the surface, i.e. Gaussian curvature maybe independently calculated by Internal Observer. We come to remarkable corollary:

Corollary *Gauß Egregium Theorema*

Gaussian curvature of the surface can be expressed in terms of induced Riemannian metric. It is invariant of isometries.

We may come to the same corollary form the Theorem about transport over closed curve. Indeed let D be a small domain around a given point $\mathbf{p}$, let C its boundary and $\angle\phi(D)$ be an angle of rotation. Denote by $S(D)$ an area of this domain. Applying the Theorem for the case when area of the domain D tends to zero we we come to the statement that

$$\text{if } S(D) \to 0 \text{ then } \angle\phi(D) = \int_D K d\sigma \to K(\mathbf{p})S(D), \text{ i.e.}$$

$$K(\mathbf{p}) = \lim_{S(D)\to 0} \frac{\angle\phi(D)}{S(D)}\,. \tag{4.9}$$

Now notice that left hand side od this equation defining Gaussian curvature $K(\mathbf{p})$ depends only on Riemannian metric on the surface C. Indeed numerator of LHS is defined by the solution of differential equation (4.1)

[14] The existence of local isothermal coordiantes is a part of famous Gauss theorem, which can be formulated in modern terms in the following way: every surface has a canonical complex structure ($z = u + iv, \bar{z} = u - iv$). We will consider this question later.

which depends on Levi-Civita connection depending on the induced Riemannian metric, and denominator is an area depending on Riemannian metric too. Thus we come again to Gauss Theorema Egregium.

In next subsections we develop the technique which itself is very interesting. One of the applications of this technique is the proof of the Theorem (4.5) and Theorem (4.8). Thus we will prove Theorema Egregium too.

Later in the fifth section we will give another proof of the Theorema Egregium.

4.2 Derivation formula

Let M be a surface embedded in Euclidean space $\mathbf{E}^3$, $M\colon \mathbf{r} = \mathbf{r}(u,v)$.

Let $\mathbf{e}, \mathbf{f}, \mathbf{n}$ be three vector fields defined on the points of this surface such that they form an orthonormal basis at any point, so that the vectors $\mathbf{e}, \mathbf{f}$ are tangent to the surface and the vector $\mathbf{n}$ is orthogonal to the surface[15]. Vector fields $\mathbf{e}, \mathbf{f}, \mathbf{n}$ are functions on the surface M:

$$\mathbf{e} = \mathbf{e}(u,v),\ \ \mathbf{f} = \mathbf{f}(u,v)\,, \mathbf{n} = \mathbf{n}(u,v)\,.$$

Consider 1-forms $d\mathbf{e}, d\mathbf{f}, d\mathbf{n}$:

$$d\mathbf{e} = \frac{\partial \mathbf{e}}{\partial u}du + \frac{\partial \mathbf{e}}{\partial v}dv,\ d\mathbf{f} = \frac{\partial \mathbf{f}}{\partial u}du + \frac{\partial \mathbf{f}}{\partial v}dv\,, d\mathbf{n} = \frac{\partial \mathbf{n}}{\partial u}du + \frac{\partial \mathbf{n}}{\partial v}dv$$

These 1-forms take values in the vectors in $\mathbf{E}^3$, i.e. they are *vector valued* 1-forms. Any vector in $\mathbf{E}^3$ attached at an arbitrary point of the surface can be expanded over the basis $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$. Thus vector valued 1-forms $d\mathbf{e}, d\mathbf{f}, d\mathbf{n}$ can be expanded in a sum of 1-forms with values in basic vectors $\mathbf{e}, \mathbf{f}, \mathbf{n}$. E.g. for $d\mathbf{e} = \frac{\partial \mathbf{e}}{\partial u}du + \frac{\partial \mathbf{e}}{\partial v}dv$ expanding vectors $\frac{\partial \mathbf{e}}{\partial u}$ and $\frac{\partial \mathbf{e}}{\partial v}$ over basis vectors we come to

$$\frac{\partial \mathbf{e}}{\partial u} = A_1(u,v)\mathbf{e} + B_1(u,v)\mathbf{f} + C_1(u,v)\mathbf{n},\ \ \frac{\partial \mathbf{e}}{\partial v} = A_2(u,v)\mathbf{e} + B_2(u,v)\mathbf{f} + C_2(u,v)\mathbf{n}$$

thus

$$d\mathbf{e} = \frac{\partial \mathbf{e}}{\partial u}du + \frac{\partial \mathbf{e}}{\partial v}dv = (A_1\mathbf{e} + B_1\mathbf{f} + C_1\mathbf{n})\,du + (A_2\mathbf{e} + B_2\mathbf{f} + C_2\mathbf{n})\,dv =$$

[15] One can say that $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ is an orthonormal basis in $T_{\mathbf{p}}\mathbf{E}^3$ at every point of surface $\mathbf{p} \in M$ such that $\{\mathbf{e}, \mathbf{f}\}$ is an orthonormal basis in $T_{\mathbf{p}}\mathbf{E}^3$ at every point of surface $\mathbf{p} \in M$.

$$= \underbrace{(A_1 du + A_2 dv)}_{M_{11}} \mathbf{e} + \underbrace{(B_1 du + B_2 dv)}_{M_{12}} \mathbf{f} + \underbrace{(C_1 du + C_2 dv)}_{M_{11}} \mathbf{e}, \tag{4.10}$$

i.e.

$$d\mathbf{e} = M_{11}\mathbf{e} + M_{12}\mathbf{f} + M_{13}\mathbf{n},$$

where M_{11}, M_{12} and M_{13} are 1-forms on the surface M defined by the relation (4.10).

In the same way we do the expansions of vector-valued 1-forms $d\mathbf{f}$ and $d\mathbf{n}$ we come to

$$\begin{array}{c} d\mathbf{e} = M_{11}\mathbf{e} + M_{12}\mathbf{f} + M_{13}\mathbf{n} \\ d\mathbf{f} = M_{21}\mathbf{e} + M_{22}\mathbf{f} + M_{23}\mathbf{n} \\ d\mathbf{n} = M_{31}\mathbf{e} + M_{32}\mathbf{f} + M_{33}\mathbf{n} \end{array}$$

This equation can be rewritten in the following way:

$$d\begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix} \tag{4.11}$$

Proposition The matrix M in the equation (4.11) *is antisymmetrical matrix, i.e.*

$$\begin{array}{c} M_{11} = M_{22} = M_{33} = 0 \\ M_{12} = -M_{21} = a \\ M_{13} = -M_{31} = b \\ M_{23} = -M_{32} = -b \end{array} \tag{4.12}$$

i.e.

$$d\begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix}, \tag{4.13}$$

where a, b, c are 1-forms on the surface M.

Formulae (5.13) are called *derivation formula.*

Prove this Proposition. Recall that $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ is orthonormal basis, i.e. at every point of the surface

$$\langle \mathbf{e}, \mathbf{e} \rangle = \langle \mathbf{f}, \mathbf{f} \rangle = \langle \mathbf{n}, \mathbf{n} \rangle = 1, \text{ and } \langle \mathbf{e}, \mathbf{f} \rangle = \langle \mathbf{e}, \mathbf{n} \rangle = \langle \mathbf{f}, \mathbf{n} \rangle = 0$$

Now using (4.11) we have

$$\langle \mathbf{e}, \mathbf{e} \rangle = 1 \Rightarrow d\langle \mathbf{e}, \mathbf{e} \rangle = 0 = 2\langle \mathbf{e}, d\mathbf{e} \rangle = \langle \mathbf{e}, M_{11}\mathbf{e} + M_{12}\mathbf{f} + M_{13}\mathbf{n} \rangle =$$

$$M_{11}\langle \mathbf{e}, \mathbf{e}\rangle + M_{12}\langle \mathbf{e}, \mathbf{f}\rangle + M_{13}\langle \mathbf{e}, \mathbf{n}\rangle = M_{11} \Rightarrow M_{11} = 0\,.$$

Analogously

$$\langle \mathbf{f}, \mathbf{f}\rangle = 1 \Rightarrow d\langle \mathbf{f}, \mathbf{f}\rangle = 0 = 2\langle \mathbf{f}, d\mathbf{f}\rangle = \langle \mathbf{f}, M_{21}\mathbf{e}+M_{22}\mathbf{f}+M_{23}\mathbf{n}\rangle = M_{22} \Rightarrow M_{22} = 0\,,$$

$$\langle \mathbf{n}, \mathbf{n}\rangle = 1 \Rightarrow d\langle \mathbf{n}, \mathbf{n}\rangle = 0 = 2\langle \mathbf{n}, d\mathbf{n}\rangle = \langle \mathbf{n}, M_{31}\mathbf{e}+M_{32}\mathbf{f}+M_{33}\mathbf{n}\rangle = M_{33}. \Rightarrow M_{33} = 0\,.$$

We proved already that $M_{11} = M_{22} = M_{33} = 0$. Now prove that $M_{12} = -M_{21}$, $M_{13} = -M_{31}$ and $M_{13} = -M_{31}$.

$$\langle \mathbf{e}, \mathbf{f}\rangle = 0 \Rightarrow d\langle \mathbf{e}, \mathbf{f}\rangle = 0 = \langle \mathbf{e}, d\mathbf{f}\rangle + \langle d\mathbf{e}, \mathbf{f}\rangle =$$

$$\langle \mathbf{e}, M_{21}\mathbf{e} + M_{22}\mathbf{f} + M_{23}\mathbf{n}\rangle + \langle M_{11}\mathbf{e} + M_{12}\mathbf{f} + M_{13}\mathbf{n}, \mathbf{f}\rangle = M_{21} + M_{12} = 0.$$

Analogously

$$\langle \mathbf{e}, \mathbf{n}\rangle = 0 \Rightarrow d\langle \mathbf{e}, \mathbf{n}\rangle = 0 = \langle \mathbf{e}, d\mathbf{n}\rangle + \langle d\mathbf{e}, \mathbf{n}\rangle =$$

$$\langle \mathbf{e}, M_{31}\mathbf{e} + M_{32}\mathbf{f} + M_{33}\mathbf{n}\rangle + \langle M_{11}\mathbf{e} + M_{12}\mathbf{f} + M_{13}\mathbf{n}, \mathbf{n}\rangle = M_{31} + M_{13} = 0$$

and

$$\langle \mathbf{f}, \mathbf{n}\rangle = 0 \Rightarrow d\langle \mathbf{f}, \mathbf{n}\rangle = 0 = \langle \mathbf{f}, d\mathbf{n}\rangle + \langle d\mathbf{f}, \mathbf{n}\rangle =$$

$$\langle \mathbf{f}, M_{31}\mathbf{e} + M_{32}\mathbf{f} + M_{33}\mathbf{n}\rangle + \langle M_{21}\mathbf{e} + M_{22}\mathbf{f} + M_{23}\mathbf{n}, \mathbf{n}\rangle = M_{32} + M_{23} = 0.$$

Remark This proof could be performed much more shortly in condensed notations. Derivation formula (5.13) in condensed notations are

$$d\mathbf{e}_i = M_{ik}\mathbf{e}_k \tag{4.14}$$

Orthonormality condition means that $\langle \mathbf{e}_i, \mathbf{e}_k\rangle = \delta_{ik}$. Hence

$$d\langle \mathbf{e}_i, \mathbf{e}_k\rangle = 0 = \langle d\mathbf{e}_i, \mathbf{e}_k\rangle + \langle \mathbf{e}_i, d\mathbf{e}_k\rangle = \langle M_{im}\mathbf{e}_m, \mathbf{e}_k\rangle + \langle \mathbf{e}_i, M_{kn}\mathbf{e}_n\rangle = M_{ik} + M_{ki} = 0 \quad \blacksquare \tag{4.15}$$

Much shorter, is not it?

4.2.1 Gauss condition (structure equations)

Derive the relations between 1-forms a, b and c in derivation formula.

Recall that a, b, c are 1-forms, $\mathbf{e}, f, \mathbf{n}$ are vector valued functions (0-forms) and $d\mathbf{e}, d\mathbf{f}, d\mathbf{n}$ are vector valued 1-forms. (We use the simple identity that

$ddf = 0$ and the fact that for 1-form $\omega \wedge \omega = 0$.) We have from derivation formula (5.13) that

$$d^2\mathbf{e} = 0 = d(a\mathbf{f} + b\mathbf{n}) = da\mathbf{f} - a \wedge d\mathbf{f} + db\mathbf{n} - b \wedge d\mathbf{n} =$$

$$da\,\mathbf{f} - a \wedge (-a\mathbf{e} + c\mathbf{n}) + db\,\mathbf{n} - b \wedge (-b\mathbf{e} - c\mathbf{f}) =$$

$$(da+b\wedge c)\mathbf{f}+(a\wedge a+b\wedge b)\mathbf{e}+(db-a\wedge c)\mathbf{n} = (da+b\wedge c)\mathbf{f}+(db+c\wedge a)\mathbf{n} = 0\,.$$

We see that

$$(da + b \wedge c)\mathbf{f} + (db + c \wedge a)\mathbf{n} = 0 \tag{4.16}$$

Hence components of the left hand side equal to zero:

$$(da + b \wedge c) = 0 \ (db + c \wedge a) = 0\,. \tag{4.17}$$

Analogously

$$d^2\mathbf{f} = 0 = d(-a\mathbf{e} + c\mathbf{n}) = -da\mathbf{e} + a \wedge d\mathbf{e} + dc\mathbf{n} - c \wedge d\mathbf{n} =$$

$$-da\mathbf{e} + a \wedge (a\mathbf{f} + b\mathbf{n}) + dc\,\mathbf{n} - c \wedge (-b\mathbf{e} - c\mathbf{f}) =$$

$$(-da + c \wedge b)\mathbf{e} + (dc + a \wedge b)\mathbf{n} = 0\,.$$

Hence we come to structure equations:

$$\begin{aligned} da + b \wedge c = 0 \\ db + c \wedge a = 0 \\ dc + a \wedge a = 0 \end{aligned} \tag{4.18}$$

4.3 Geometrical meaning of derivation formula. Weingarten operator (and second quadratic form)* in terms of derivation formula.

Let M be a surface in $\mathbf{E}^3$.

Let $\mathbf{e}, \mathbf{f}, \mathbf{n}$ be three vector fields defined on the points of this surface such that they form an orthonormal basis at any point, so that the vectors $\mathbf{e}, \mathbf{f}$ are tangent to the surface and the vector $\mathbf{n}$ is orthogonal to the surface. Note that in generally these vectors are not coordinate vectors.

Describe Riemannian geometry on the surface M in terms of this basis and derivation formula (5.13).

Induced Riemannian metric

If G is the Riemannian metric induced on the surface M then since $\mathbf{e}, \mathbf{f}$ is orthonormal basis at every tangent space $T_{\mathbf{p}}M$ then

$$G(\mathbf{e},\mathbf{e}) = G(\mathbf{f},\mathbf{f}) = 1, \;\; G(\mathbf{e},\mathbf{f}) = G(\mathbf{f},\mathbf{e}) = 0 \tag{4.19}$$

The matrix of the Riemannian metric in the basis $\{\mathbf{e},\mathbf{f}\}$ is

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.20}$$

Induced connection Let ∇ be the connection induced by the canonical flat connection on the surface M.

Then according equations (**??**) and derivation formula (5.13) for every tangent vector $\mathbf{X}$

$$\nabla_{\mathbf{X}}\mathbf{e} = (\partial_{\mathbf{X}}\mathbf{e})_{\text{tangent}} = (d\mathbf{e}(\mathbf{X}))_{\text{tangent}} = (a(\mathbf{X})\mathbf{f} + b(\mathbf{X})\mathbf{n})_{\text{tangent}} = a(\mathbf{X})\mathbf{f}\,. \tag{4.21}$$

and

$$\nabla_{\mathbf{X}}\mathbf{f} = (\partial_{\mathbf{X}}\mathbf{f})_{\text{tangent}} = (d\mathbf{f}(\mathbf{X}))_{\text{tangent}} = (-a(\mathbf{X})\mathbf{e} + c(\mathbf{X})\mathbf{n})_{\text{tangent}} = -a(\mathbf{X})\mathbf{e}\,. \tag{4.22}$$

In particular

$$\begin{array}{cc} \nabla_{\mathbf{e}}\mathbf{e} = a(\mathbf{e})\mathbf{f} & \nabla_{\mathbf{f}}\mathbf{e} = a(\mathbf{f})\mathbf{f} \\ \nabla_{\mathbf{e}}\mathbf{f} = -a(\mathbf{e})\mathbf{e} & \nabla_{\mathbf{f}}\mathbf{f} = -a(\mathbf{f})\mathbf{e} \end{array} \tag{4.23}$$

We know that the connection ∇ is Levi-Civita connection of the induced Riemannian metric (4.21) (see the subsection 4.2.1)[16].

Second Quadratic form Second quadratic form is a bilinear symmetric function $A(\mathbf{X},\mathbf{Y})$ on tangent vectors which is well-defined by the condition $A(\mathbf{X},\mathbf{Y})\mathbf{n} = (\partial_{\mathbf{X}}\mathbf{Y})_{\text{orthogonal}}$ (see e.g. subsection 6.4 in Appendices.)

Let $A(\mathbf{X},\mathbf{Y})$ be second quadratic form. Then according to derivation formula (5.13) we have

$$A(\mathbf{e},\mathbf{e}) = \langle \partial_{\mathbf{e}}\mathbf{e}, \mathbf{n}\rangle = \langle d\mathbf{e}(\mathbf{e}), \mathbf{n}\rangle = \langle a(\mathbf{e})\mathbf{f} + b(\mathbf{e})\mathbf{n}, \mathbf{n}\rangle = b(\mathbf{e})\,,$$

$$A(\mathbf{f},\mathbf{e}) = \langle \partial_{\mathbf{f}}\mathbf{e}, \mathbf{n}\rangle = \langle d\mathbf{e}(\mathbf{f}), \mathbf{n}\rangle = \langle a(\mathbf{f})\mathbf{f} + b(\mathbf{f})\mathbf{n}, \mathbf{n}\rangle = b(\mathbf{f})\,,$$

$$A(\mathbf{e},\mathbf{f}) = \langle \partial_{\mathbf{e}}\mathbf{f}, \mathbf{n}\rangle = \langle d\mathbf{f}(\mathbf{e}), \mathbf{n}\rangle = \langle -a(\mathbf{e})\mathbf{f} + c(\mathbf{e})\mathbf{n}, \mathbf{n}\rangle = c(\mathbf{e})\,,$$

[16] In particular this implies that this is symmetric connection, i.e.

$$\nabla\mathbf{f}\mathbf{e} - \nabla_{\mathbf{e}}\mathbf{f} - [\mathbf{f},\mathbf{e}] = a(\mathbf{f})\mathbf{f} + a(\mathbf{e})\mathbf{e} - [\mathbf{f},\mathbf{e}] = 0\,. \tag{4.24}$$

$$A(\mathbf{f},\mathbf{f}) = \langle \partial_{\mathbf{f}} \mathbf{f}, \mathbf{n} \rangle = \langle d\mathbf{f}(\mathbf{f}), \mathbf{n} \rangle = \langle -a(\mathbf{f})\mathbf{f} + c(\mathbf{f})\mathbf{n}, \mathbf{n} \rangle = c(\mathbf{f}),$$

The matrix of the second quadratic form in the basis $\{\mathbf{e},\mathbf{f}\}$ is

$$A = \begin{pmatrix} A(\mathbf{e},\mathbf{e}) & A(\mathbf{f},\mathbf{e}) \\ A(\mathbf{e},\mathbf{f}) & A(\mathbf{f},\mathbf{f}) \end{pmatrix} = \begin{pmatrix} b(\mathbf{e}) & b(\mathbf{f}) \\ c(\mathbf{e}) & c(\mathbf{f}) \end{pmatrix} \tag{4.25}$$

This is symmetrical matrix (see the subsection 4.3.2):

$$A(\mathbf{f},\mathbf{e}) = b(\mathbf{f}) = A(\mathbf{e},\mathbf{f}) = c(\mathbf{e}). \tag{4.26}$$

Weingarten operator

Let S be Weingarten operator: $S\mathbf{X} = -\partial_{\mathbf{X}}\mathbf{n}$ (see the subsection 6.4 in Appendix, or Geometry lectures). Then it follows from derivation formula that

$$S\mathbf{X} = -\partial_{\mathbf{X}}\mathbf{n} = -d\mathbf{n}(\mathbf{X}) = -\left(-b(X)\mathbf{e} - c(\mathbf{X})\mathbf{f}\right) = b(\mathbf{X})\mathbf{e} + c(\mathbf{X})f$$

In particular

$$S(\mathbf{e}) = b(\mathbf{e})\mathbf{e} + c(\mathbf{e})\mathbf{f}\,, S(\mathbf{f}) = b(\mathbf{f})\mathbf{e} + c(\mathbf{f})\mathbf{f}$$

and the matrix of the Weingarten operator in the basis $\{\mathbf{e},\mathbf{f}\}$ is

$$S = \begin{pmatrix} b(\mathbf{e}) & c(\mathbf{e}) \\ b(\mathbf{f}) & c(\mathbf{f}) \end{pmatrix} \tag{4.27}$$

Remark According to the condition (4.26) the matrix S is symmetrical. The relations $A = GS, S = G^{-1}A$ for Weingarten operator, Riemannian metric and second quadratic form are evidently obeyed for matrices of these operators in the basis $\mathbf{e},\mathbf{f}$ where $G = 1$, $A = S$.

4.3.1 Gaussian and mean curvature in terms of derivation formula

Now we are equipped to express Gaussian and mean curvatures in terms of derivation formula. Using (4.27) we have for Gaussian curvature

$$K = \det S = b(\mathbf{e})c(\mathbf{f}) - c(\mathbf{e})b(\mathbf{f}) = (b \wedge c)(\mathbf{e},\mathbf{f}) \tag{4.28}$$

and for mean curvature

$$H = \operatorname{Tr} S = b(\mathbf{e}) + c(\mathbf{f}) \tag{4.29}$$

What next? We will study in more detail formula (4.28) later.

Now consider some examples of calculation of Weingarten operator, e.t..c. for using derivation formula.

4.4 Examples of calculations of derivation formulae and curvatures for cylinder, cone and sphere

Last year we calculated Weingarten oeprator, second quadratic form and curvatures for cylinder, cone and sphere (see also the subsection 6.4 in Appendices.). Now we do the same but in terms of derivation formula.

Cylinder

We have to define three vector fields $\mathbf{e}, \mathbf{f}, \mathbf{n}$ on the points of the cylinder surface $x^2 + y^2 = a^2$:

$$\mathbf{r}(h, \varphi): \quad \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases} \tag{4.30}$$

such that they form an orthonormal basis at any point, so that the vectors $\mathbf{e}, \mathbf{f}$ are tangent to the surface and the vector $\mathbf{n}$ is orthogonal to the surface. We calculated many times coordinate vector fields $\mathbf{r}_h, \mathbf{r}_\varphi$ and normal unit vector field:

$$\mathbf{r}_h = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{r}_\varphi = \begin{pmatrix} -a\sin\varphi \\ a\cos\varphi \\ 0 \end{pmatrix}, \quad \mathbf{n} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}. \tag{4.31}$$

Vectors $\mathbf{r}_h, \mathbf{r}_\varphi$ and $\mathbf{n}$ are orthogonal to each other but not all of them have unit length. One can choose

$$\mathbf{e} = \mathbf{r}_h = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{f} = \frac{\mathbf{r}_\varphi}{a} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}, \mathbf{n} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} \tag{4.32}$$

These vectors form an orthonormal basis and $\mathbf{e}, \mathbf{f}$ form an orthonormal basis in tangent space.

Derive for this basis derivation formula (5.13). For vector fields $\mathbf{e}, \mathbf{f}, \mathbf{n}$ in (4.32) we have

$$d\mathbf{e} = 0, d\mathbf{f} = d\begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} -\cos\varphi \\ -\sin\varphi \\ 0 \end{pmatrix} d\varphi = -\mathbf{n}d\varphi,$$

$$d\mathbf{n} = d\begin{pmatrix}\cos\varphi\\ \sin\varphi\\ 0\end{pmatrix} = \begin{pmatrix}-\sin\varphi\\ \cos\varphi\\ 0\end{pmatrix} = \mathbf{f}d\varphi,$$

i.e.

$$d\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\\ \mathbf{n}\end{pmatrix} = \begin{pmatrix}0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{pmatrix}\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\\ \mathbf{n}\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -d\varphi\\ 0 & d\varphi & 0\end{pmatrix}\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\\ \mathbf{n}\end{pmatrix}, \tag{4.33}$$

i.e. in derivation formula 1-forms a, b vanish $a = b = 0$ and $c = -d\varphi$.

The matrix of Weingarten operator in the basis $\{\mathbf{e}, \mathbf{f}\}$ is

$$S = \begin{pmatrix}b(\mathbf{e}) & c(\mathbf{e})\\ b(\mathbf{f}) & c(\mathbf{f})\end{pmatrix} = \begin{pmatrix}0 & -d\varphi(\mathbf{e})\\ 0 & -d\varphi(\mathbf{f})\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & -\frac{1}{R}\end{pmatrix}$$

According to (4.28) and(4.29) Gaussian curvature $K = b(\mathbf{e})c(\mathbf{f}) - b(\mathbf{e})c(\mathbf{f}) = 0$ and mean curvature

$$H = b(\mathbf{e}) + c(\mathbf{f}) = -d\varphi(\mathbf{f}) = -d\varphi\left(\frac{\mathbf{r}_\varphi}{R}\right) = -\frac{1}{a}$$

Remark We denote by the same letter a the radius of the cylinder surface (4.30) and 1-form a in derivation formula. I hope that this will not lead to the confusion. (May be it is better to denote the radius of the cylindrical surface by the letter R.)

Cone

For cone:

$$\mathbf{r}(h, \varphi)\colon \quad \begin{cases} x = kh\cos\varphi\\ y = kh\sin\varphi\\ z = h\end{cases},$$

$$\mathbf{r}_h = \begin{pmatrix}k\cos\varphi\\ k\sin\varphi\\ 1\end{pmatrix}, \quad \mathbf{r}_\varphi = \begin{pmatrix}-kh\sin\varphi\\ kh\cos\varphi\\ 0\end{pmatrix}, \quad \mathbf{n} = \frac{1}{\sqrt{1+k^2}}\begin{pmatrix}\cos\varphi\\ \sin\varphi\\ -k\end{pmatrix}$$

Tangent vectors $\mathbf{r}_h, \mathbf{r}_\varphi$ are orthogonal to each other. The length of the vector $\mathbf{r}_h$ equals to $\sqrt{1+k^2}$ and the length of the vector $\mathbf{r}_\varphi$ equals to kh. Hence we can choose orthonormal basis $\{\mathbf{e},\mathbf{f},\mathbf{n}\}$ such that vectors $\mathbf{e},\mathbf{f}$ are unit vectors in the directions of the vectors $\mathbf{r}_h,\mathbf{r}_\varphi$:

$$\mathbf{e}=\frac{\mathbf{r}_h}{\sqrt{1+k^2}}=\frac{1}{\sqrt{1+k^2}}\begin{pmatrix}k\cos\varphi\\ k\sin\varphi\\ 1\end{pmatrix},\ \mathbf{f}=\frac{\mathbf{r}_\varphi}{hk}=\begin{pmatrix}-\sin\varphi\\ \cos\varphi\\ 0\end{pmatrix},\ \mathbf{n}=\frac{1}{\sqrt{1+k^2}}\begin{pmatrix}\cos\varphi\\ \sin\varphi\\ -k\end{pmatrix}$$

Calculate $d\mathbf{e}, d\mathbf{f}$ and $d\mathbf{n}$:

$$d\mathbf{e}=d\begin{pmatrix}k\cos\varphi\\ k\sin\varphi\\ 1\end{pmatrix}=\frac{kd\varphi}{\sqrt{1+k^2}}\begin{pmatrix}-\sin\varphi\\ \cos\varphi\\ 0\end{pmatrix}=\frac{kd\varphi}{\sqrt{1+k^2}}\mathbf{f},$$

$$d\mathbf{f}=d\begin{pmatrix}-\sin\varphi\\ \cos\varphi\\ 0\end{pmatrix}=\begin{pmatrix}-\cos\varphi\\ -\sin\varphi\\ 0\end{pmatrix}d\varphi=$$

$$\frac{-k}{1+k^2}\begin{pmatrix}k\cos\varphi\\ k\sin\varphi\\ 1\end{pmatrix}d\varphi-\frac{d\varphi}{1+k^2}\begin{pmatrix}\cos\varphi\\ \sin\varphi\\ -k\end{pmatrix}=\frac{-kd\varphi}{\sqrt{1+k^2}}\mathbf{e}-\frac{d\varphi}{\sqrt{1+k^2}}\mathbf{n}\,,$$

and

$$d\mathbf{n}=\frac{1}{\sqrt{1+k^2}}d\begin{pmatrix}\cos\varphi\\ \sin\varphi\\ -k\end{pmatrix}=\frac{d\varphi}{\sqrt{1+k^2}}\begin{pmatrix}-\sin\varphi\\ \cos\varphi\\ 0\end{pmatrix}.$$

We come to

$$d\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\\ \mathbf{n}\end{pmatrix}=\begin{pmatrix}0&a&b\\ -a&0&c\\ -b&-c&0\end{pmatrix}\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\\ \mathbf{n}\end{pmatrix}=\begin{pmatrix}0&\frac{kd\varphi}{\sqrt{1+k^2}}&0\\ -\frac{kd\varphi}{\sqrt{1+k^2}}&0&\frac{-d\varphi}{\sqrt{1+k^2}}\\ 0&\frac{d\varphi}{\sqrt{1+k^2}}&0\end{pmatrix}\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\\ \mathbf{n}\end{pmatrix}, \tag{4.34}$$

i.e. in derivation formula for 1-forms $a=\frac{kd\varphi}{\sqrt{1+k^2}}$, $b=0$ and and $c=-\frac{-d\varphi}{\sqrt{1+k^2}}$.

Remark Note that calculation of $d\mathbf{f}$ are little bit hard. On the other hand the answer for $d\mathbf{f}$ follows from answers for $d\mathbf{e}$ and $d\mathbf{n}$ since the matrix in (4.34) is antisymmetric. So we can omit the straightforward calculations of $d\mathbf{f}$.

The matrix of Weingarten operator in the basis $\{\mathbf{e},\mathbf{f}\}$ is

$$S=\begin{pmatrix}b(\mathbf{e})&c(\mathbf{e})\\ b(\mathbf{f})&c(\mathbf{f})\end{pmatrix}=S=\begin{pmatrix}0&\frac{-d\varphi(\mathbf{e})}{\sqrt{1+k^2}}\\ 0&\frac{-d\varphi(\mathbf{f})}{\sqrt{1+k^2}}\end{pmatrix}=\begin{pmatrix}0&0\\ 0&\frac{-1}{kh\sqrt{1+k^2}}\end{pmatrix}.$$

since $d\varphi(\mathbf{f}) = d\varphi\left(\frac{\mathbf{r}_\varphi}{kh}\right) = \frac{1}{kh} d\varphi(\partial_\varphi) = \frac{1}{kh}$.

According to (4.28), (4.29) Gaussian curvature

$$K = b(\mathbf{e})c(\mathbf{f}) - b(\mathbf{e})c(\mathbf{f}) = 0$$

and mean curvature

$$H = b(\mathbf{e}) + c(\mathbf{f}) = -d\varphi(\mathbf{f}) = -d\varphi\left(\frac{\mathbf{r}_\varphi}{R}\right) = -\frac{1}{kh\sqrt{1+k^2}}\,.$$

Sphere

For sphere

$$\mathbf{r}(\theta,\varphi)\colon \qquad \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases} \tag{4.35}$$

$$\mathbf{r}_\theta(\theta,\varphi) = \frac{\partial \mathbf{r}}{\partial\theta} = \begin{pmatrix} R\cos\theta\cos\varphi \\ R\cos\theta\sin\varphi \\ -R\sin\theta \end{pmatrix}, \quad \mathbf{r}_\varphi(\theta,\varphi) = \frac{\partial \mathbf{r}}{\partial\varphi} = \begin{pmatrix} -R\sin\theta\sin\varphi \\ R\sin\theta\cos\varphi \\ 0 \end{pmatrix},$$

$$\mathbf{n}(\theta,\varphi) = \frac{\mathbf{r}}{R} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}.$$

Tangent vectors $\mathbf{r}_\theta, \mathbf{r}_\varphi$ are orthogonal to each other. The length of the vector $\mathbf{r}_\theta$ equals to R and the length of the vector $\mathbf{r}_\varphi$ equals to $R\sin\theta$. Hence we can choose orthonormal basis $\{\mathbf{e},\mathbf{f},\mathbf{n}\}$ such that vectors $\mathbf{e},\mathbf{f}$ are unit vectors in the directions of the vectors $\mathbf{r}_\theta, \mathbf{r}_\varphi$:

$$\mathbf{e}(\theta,\varphi) = \frac{\mathbf{r}_\theta}{R} = \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix}, \mathbf{f}(\theta,\varphi) = \frac{\mathbf{r}_\varphi}{R\sin\theta} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}, \mathbf{n}(\theta,\varphi) = \frac{\mathbf{r}}{R} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}.$$

Calculate $d\mathbf{e}, d\mathbf{f}$ and $d\mathbf{n}$:

$$d\mathbf{e} = d\begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} =$$

$$\begin{pmatrix} -\cos\theta\sin\varphi \\ \cos\theta\cos\varphi \\ 0 \end{pmatrix} d\varphi - \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} d\theta = \cos\theta d\varphi\mathbf{f} - d\theta\mathbf{n},$$

$$d\mathbf{f} = d\begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} = -\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} d\varphi =$$

$$-\cos\theta d\varphi \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} - \sin\theta d\varphi \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} = -\cos\theta d\varphi\mathbf{e} - \sin\theta d\varphi\mathbf{n}\,,$$

$$d\mathbf{n} = d\begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} d\theta + \begin{pmatrix} -\sin\theta\sin\varphi \\ \sin\theta\cos\varphi \\ 0 \end{pmatrix} d\varphi$$

$$= d\theta\mathbf{e} + \sin\theta d\varphi\mathbf{f}\,.$$

i.e.

$$d\begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 & \cos\theta d\varphi & -d\theta \\ -\cos\theta d\varphi & 0 & -\sin\theta d\varphi \\ d\theta & \sin\theta d\varphi & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix}, \tag{4.36}$$

i.e. in derivation formula $a = \cos\theta d\varphi$, $b = -d\theta$, $c = -\sin\theta d\varphi$.

Remark The same remark as for cone: equipped by the properties of derivation formula we do not need to calculate $d\mathbf{f}$. The calculation of $d\mathbf{e}$ and $d\mathbf{n}$ and the property that the matrix in derivation formula is antisymmetric gives us the answer for $d\mathbf{f}$.

The matrix of Weingarten operator in the basis $\{\mathbf{e}, \mathbf{f}\}$ is

$$S = \begin{pmatrix} b(\mathbf{e}) & c(\mathbf{e}) \\ b(\mathbf{f}) & c(\mathbf{f}) \end{pmatrix} = \begin{pmatrix} -d\theta(\mathbf{e}) & -\sin\theta d\varphi(\mathbf{e}) \\ -d\theta(\mathbf{f}) & -\sin\theta d\varphi(\mathbf{f}) \end{pmatrix} = \begin{pmatrix} \frac{-1}{R} & 0 \\ 0 & -\frac{1}{R} \end{pmatrix}$$

since $d\theta(\mathbf{e}) = d\theta\left(\frac{\partial_\theta}{R}\right) = \frac{1}{R}d\theta(\partial_\theta) = \frac{1}{R}$, $d\varphi(\mathbf{e}) = d\varphi\left(\frac{\partial_\theta}{R}\right) = \frac{1}{R}d\varphi(\partial_\theta) = 0$.

According to (4.28) and(4.29) Gaussian curvature

$$K = b(\mathbf{e})c(\mathbf{f}) - b(\mathbf{e})c(\mathbf{f}) = \frac{1}{R^2}$$

and mean curvature

$$H = b(\mathbf{e}) + c(\mathbf{f}) = -\frac{2}{R}$$

Notice that for calculation of Weingarten operator and curvatures we used only 1-forms b and c, i.e. the derivation equation for $d\mathbf{n}$, ($d\mathbf{n} = d\theta\mathbf{e} + \sin\theta d\varphi\mathbf{f}$).

Mean curvature is define up to a sign. If we change $\mathbf{n} \to -\mathbf{n}$ mean curvature $H \to \frac{1}{R}$ and Gaussian curvature will not change.

We see that for the sphere Gaussian curvature is not equal to zero, whilst for cylinder and cone Gaussian curvature equals to zero.

4.5 †Proof of the Theorem of parallel transport along closed curve.

We are ready now to prove the Theorem. Recall that the Theorem states following:

If C is a closed curve on a surface M such that C is a boundary of a compact oriented domain $D \subset M$, then during the parallel transport of an arbitrary tangent vector along the closed curve C the vector rotates through the angle

$$\Delta\Phi = \angle\left(\mathbf{X}, \mathbf{R}_C\mathbf{X}\right) = \int_D K d\sigma\,, \tag{4.37}$$

where K is the Gaussian curvature and $d\sigma = \sqrt{\det g} dudv$ is the area element induced by the Riemannian metric on the surface M, i.e. $d\sigma = \sqrt{\det g} dudv$.

(see (4.5).

Recall that for derivation formula (5.13) we obtained structure equations

$$\begin{aligned} da + b \wedge c &= 0 \\ db + c \wedge a &= 0 \\ dc + a \wedge a &= 0 \end{aligned} \tag{4.38}$$

We need to use only one of these equations, the equation

$$da + b \wedge c = 0\,. \tag{4.39}$$

This condition sometimes is called *Gauß condition.*

Let as always $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ be an orthonormal basis in $T_\mathbf{p}\mathbf{E}^3$ at every point of surface $\mathbf{p} \in M$ such that $\{\mathbf{e}, \mathbf{f}\}$ is an orthonormal basis in $T_\mathbf{p}M$ at every point of surface $\mathbf{p} \in M$. Then the Gauß condition (4.39) and equation (4.28) mean that for Gaussian curvature on the surface M can be expressed through the 2-form da and base vectors $\{\mathbf{e}, \mathbf{f}\}$:

$$K = b \wedge c(\mathbf{e}, \mathbf{f}) = -da(\mathbf{e}, \mathbf{f}) \tag{4.40}$$

We use this formula to prove the Theorem.

Now calculate the parallel transport of an arbitrary tangent vector over the closed curve C on the surface M.

Let $\mathbf{r} = \mathbf{r}(u, v) = \mathbf{r}(u\alpha)$ ($\alpha = 1, 2$, $(u, v) = (u^1, v^1)$) be an equation of the surface M.

Let $u^\alpha = u^\alpha(t)$ ($\alpha = 1, 2$) be the equation of the curve C. Let $\mathbf{X}(t)$ be the parallel transport of vector field along the closed curve C, i.e. $\mathbf{X}(t)$ is tangent to the surface M at the point $u(t)$ of the curve C and vector field $\mathbf{X}(t)$ is covariantly constant along the curve:

$$\frac{\nabla \mathbf{X}(t)}{dt} = 0$$

To write this equation in components we usually expanded the vector field in the coordinate basis $\{\mathbf{r}_u = \partial_u, \mathbf{r}_v = \partial_v\}$ and used Christoffel symbols of the connection $\Gamma^\alpha_{\beta\gamma}$: $\nabla_\beta \partial_\gamma = \Gamma^\alpha_{\beta\gamma}\partial_\alpha$.

Now we will do it in different way: *instead coordinate basis* $\{\mathbf{r}_u = \partial_u, \mathbf{r}_v = \partial_v\}$ *we will use the basis* $\{\mathbf{e}, \mathbf{f}\}$. In the subsection 3.4.4 we obtained that the connection ∇ has the following appearance in this basis

$$\nabla_{\mathbf{v}}\mathbf{e} = a(\mathbf{v})\mathbf{f},\ , \nabla_{\mathbf{v}}\mathbf{f} = -a(\mathbf{v})\mathbf{e} \tag{4.41}$$

(see the equations (4.21) and (4.22))

Let

$$\mathbf{X} = \mathbf{X}(u(t)) = X^1(t)\mathbf{e}(u(t)) + X^2(t)\mathbf{f}(u(t))$$

Lbe an expansion of tangent vector field $\mathbf{X}(t)$ over basis $\{\mathbf{e}, \mathbf{f}\}$. Let $\mathbf{v}$ be velocity vector of the curve C. Then the equation of parallel transport $\frac{\nabla \mathbf{X}(t)}{dt} = 0$ will have the following appearance:

$$\frac{\nabla \mathbf{X}(t)}{dt} = 0 = \nabla_{\mathbf{v}}\left(X^1(t)\mathbf{e}(u(t)) + X^2(t)\mathbf{f}(u(t))\right) =$$
$$\frac{dX^1(t)}{dt}\mathbf{e}(u(t)) + X^1(t)\nabla_{\mathbf{v}}\mathbf{e}(u(t)) + \frac{dX^2(t)}{dt}\mathbf{f}(u(t)) + X^2(t)\nabla_{\mathbf{v}}\mathbf{f}(u(t)) =$$
$$\frac{dX^1(t)}{dt}\mathbf{e}(u(t)) + X^1(t)a(\mathbf{v})\mathbf{f}(u(t)) + \frac{dX^2(t)}{dt}\mathbf{f}(u(t)) - X^2(t)a(\mathbf{v})\mathbf{e}(u(t)) =$$
$$\left(\frac{dX^1(t)}{dt} - X^2(t)a(\mathbf{v})\right)\mathbf{e}(u(t)) + \left(\frac{dX^2(t)}{dt} + X^1(t)a(\mathbf{v})\right)\mathbf{f}(u(t)) = 0.$$

Thus we come to equation:

$$\begin{cases} \dot{X}^1(t) - a(\mathbf{v}(t))X^2 = 0 \\ \dot{X}^2(t) + a(\mathbf{v}(t))X^1 = 0 \end{cases}$$

There are many ways to solve this equation. It is very convenient to consider complex variable

$$Z(t) = X^1(t) + iX^2(t)$$

We see that

$$\dot{Z}(t) = \dot{X}^1(t) + i\dot{X}^2(t) = a(\mathbf{v}(t))X^2 - ia(\mathbf{v}(t)X^1 = -ia(\mathbf{v})Z(t),$$

i.e.

$$\frac{dZ(t)}{dt} = -ia(\mathbf{v}(t))Z(t) \tag{4.42}$$

The solution of this equation is:

$$Z(t) = Z(0)e^{-i\int_0^t a(\mathbf{v}(\tau))d\tau} \tag{4.43}$$

Calculate $\int_0^{t_1} a(\mathbf{v}(\tau))d\tau$ for closed curve $u(0) = u(t_1)$. Due to Stokes Theorem:

$$\int_0^{t_1} a(\mathbf{v}(t))dt = \int_C a = \int_D da$$

Hence using Gauss condition (4.39) we see that

$$\int_0^{t_1} a(\mathbf{v}(t))dt = \int_C a = \int_D da = -\int_D b \wedge c$$

Claim

$$\int_D b \wedge c = -\int_D da = \int K d\sigma\,. \qquad (4.44)$$

Theorem follows from this claim:

$$Z(t_1) = Z(0)e^{-i\int_C a} = Z(0)e^{i\int_D b\wedge C} \qquad (4.45)$$

Denote the integral $i\int_D b \wedge C$ by $\Delta\Phi$: $\Delta\Phi = i\int_D b \wedge C$. We have

$$Z(t_1) = X^1(t_1) + iX^2(t_1) = \left(X^1(0) + iX^2(0)\right)e^{i\Delta\Phi} = \qquad (4.46)$$

It remains to prove the claim. The induced volume form $d\sigma$ is 2-form. Its value on two orthogonal unit vector $\mathbf{e}, \mathbf{f}$ equals to 1:

$$d\sigma(\mathbf{e}, \mathbf{f}) = 1 \qquad (4.47)$$

(In coordinates u, v volume form $d\sigma = \sqrt{\det g} du \wedge dv$).

The value of the form $b \wedge c$ on vectors $\{\mathbf{e}, \mathbf{f}\}$ equals to Gaussian curvature according to (4.40). We see that

$$b \wedge c(\mathbf{e}, \mathbf{f}) = -da(\mathbf{e}, \mathbf{f}) = K d\sigma(\mathbf{e}, \mathbf{f})$$

Hence 2-forms $b \wedge c$, $-da$ and volume form $d\sigma$ coincide. Thus we prove (4.44).

4.6 † Gaussian curvature in isothermal coordinates again

We return to isothermal coordinates and formula (4.8) for Gaussian curvature considered in (4.1.2).

Theorem. For surface M in $\mathbf{E}^3$

- in a vicinity of an arbitrary point there exist isothermal coordinates i.e. coordinates such that induced metric $G = e^\Phi(du^2 + dv^2) = e^\Phi dz d\bar{z}$.
- If (u, v) and (u', v') are two arbitrary isothermal coordinates then the function $z = f(w)$ is holomorphic function or anti-holomorphic function,

We denote $u + iv = z, u - iv = \bar{z}$. Recall that if $z = u + iv$ then

$$F_z = \frac{\partial F}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)F, \quad \textit{and } F_{\bar{z}} = \frac{\partial F}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)F \quad (4.48)$$

Function $F = f + ig$ is holomorphic $\Leftrightarrow F_{\bar{z}} = 0 \Leftrightarrow (f_u + ig_u) + i(f_v + ig_v) = 0$ $\Leftrightarrow f_u = g_v$ and $f_v = -g_v$ (Cauchy Riemann conditions). Function $F = f+ig$ is anti-holomorphic $\Leftrightarrow F_z = 0 \Leftrightarrow (f_u + ig_u) - i(f_v - ig_v) = 0$. E.g. $F = z^2 = (u+iv)^2 = u^2 - v^2 + 2iuv$ is holomorphic function, $F = e^{\bar{z}} = e^{u-iv} = e^u(\cos v - i \sin v)$ is anti-holomorphic function.

Remark Metric $G = e^{\Phi}(du^2 + dv^2)$ which is proportional to Euclidean metric up to a scalar function e^{Φ} is called conformal metric.

The Theorem immediately implies the following important

Corollary Two-dimensional surface in $\mathbf{E}^3$ has canonical complex structure, i.e. on can consider an atlas of local complex charts such that transition funtctions are analytic.

Idea of Proof of first part of Theorem

Let (ξ, η) be arbitrary parameters of surface and $G = Ad\xi^2 + 2Bd\xi d\eta + Cd\eta^2$ ($g_{11} = A, g_{12} = g_{21} = B$, $g_{22} = C$). The positive-definiteness of the metric implies that $G = \omega\overline{\omega}$ where $\omega = df + idg$ is 1-form. Use the fact that an arbitrary 1-form up to a mulitplier function is an exact form: $\omega = \lambda dF$. We come to isothermal coordinates: $G = \lambda\bar{\lambda} dF d\bar{F}$.

Illustrate this idea on the example: Let $G = dx^2 + f^2(x)dy^2$ be a metric on a domain of Riemannian manifold (e.g. for sphere $x = \theta, y = \varphi, f(x) = \sin^2 x$, for cone $x = h, y = \varphi, f(x) = x$). Then $G = (dx + ifdy)()dx - if(x)dy)$. For 1-form $\omega = dx + if(x)dy$ we have that $dx + if(x)dy = f(x)\left(dG(x) + idy\right) = f(x)d\left(L(x) + iy\right)$, where $L(x)$ is antiderivative of a function $\frac{1}{f}$ and $dx^2 + f^2(x)dy^2 = f^2(x)d\left(L(x) + iy\right)d\left(G(x) - iy\right) = e^{\Phi}(du^2 + dv^2)$, where $e^{\Phi} = f^2(x)$, $u = L(x), v = y$ [17]

Proof of second part of Theorem. To prove it we just perform straightforward calculation. Let $G = e^{\Phi}(du^2 + dv^2) = e^{\Phi} dz d\bar{z}$ in local coordinates $z = u + iv$, and in new local coordinates $w = u' + iv'$ $G = e^{\Phi'}(du'^2 + dv'^2) = e^{\Phi'} dw d\bar{w}$, where Let $w = F(z)$. Then

$$G = e^{\Phi} dz d\overline{z} = e^{\Phi}\left(F_w dw + F_{\bar{w}} d\bar{w}\right)\left(\overline{F_w dw + F_{\bar{w}} d\bar{w}}\right) =$$

$$e^{\Phi}\left(F_w \overline{F_{\bar{w}}} dw^2 + \left(|F_w|^2 + |F_{\bar{w}}|^2|\right) dw d\bar{w} + F_{\bar{w}} \overline{F_w} d\bar{w}^2\right) \qquad (4.49)$$

The condition that new coordinates are isothermal too means that $F_w \overline{F_{\bar{w}}} = 0$, i.e. $F_w = 0$, i.e. F is anti-holomorphic function or $F_{\bar{w}} = 0$, i.e. F is holomorphic function (see 4.48.)

Now return to calculation (4.8) of Gaussian curvature. Let (u, v) be local isothermal coordinates, and metric $G = e^{\Phi}(du^2 + dv^2)$. Consider vectors

$$\mathbf{e} = e^{-\frac{\Phi}{2}} \frac{\partial}{\partial u}, \quad \mathbf{f} = e^{-\frac{\Phi}{2}} \frac{\partial}{\partial v}, \quad \mathbf{n} = \mathbf{e} \times \mathbf{f}.$$

[17] in general case we use essentially the condition of analiticity. This proof was done by Gauss. The general smooth case was proved only in the beginning of XX century.

It is evident that $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ form orthonoromal basis:

$$\langle \mathbf{e}, \mathbf{e} \rangle = 1\,, \langle \mathbf{e}, \mathbf{f} \rangle = 0\,, \langle \mathbf{e}, \mathbf{n} \rangle = 0,\ \langle \mathbf{f}, \mathbf{f} \rangle = 1\, \langle \mathbf{f}, \mathbf{n} \rangle = 0\,, \langle \mathbf{n}, \mathbf{n} \rangle = 1\,.$$

Consider derivation formula (5.13) for this basis:

$$d \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix}\,, \tag{4.50}$$

To calculate Gaussian curvature we need to calculate 1-form a in this equations since accordind equations (4.28) and (4.18) $K = b \wedge c(\mathbf{e}, \mathbf{f})$ and $da + b \wedge c = 0$, i.e. $K = -da(\mathbf{e}, \mathbf{f})$. Now calculate 1-form a. We have

$$d\mathbf{e} = d\left(e^{-\frac{\Phi}{2}} \mathbf{r}_u\right) = a\mathbf{f} + b\mathbf{n}\,.$$

Taking scalar product of this equation of $\mathbf{f}$ we come to

$$a = \langle d\mathbf{e}, \mathbf{f} \rangle = \left\langle d\left(e^{-\frac{\Phi}{2}} \mathbf{r}_u\right), e^{-\frac{\Phi}{2}} \mathbf{r}_v \right\rangle. \tag{4.51}$$

Calculate it. Since $\langle \mathbf{r}_u, \mathbf{r}_v \rangle = 0$ then

$$\left\langle d\left(e^{-\frac{\Phi}{2}} \mathbf{r}_u\right), e^{-\frac{\Phi}{2}} \mathbf{r}_v \right\rangle = e^{-\Phi} \langle d\mathbf{r}_u, \mathbf{r}_v \rangle = e^{-\Phi} \langle \mathbf{r}_{uu}, \mathbf{r}_v \rangle du + e^{-\Phi} \langle \mathbf{r}_{uv}, \mathbf{r}_v \rangle dv\,.$$

Now using the fact that $\langle \mathbf{r}_v, \mathbf{r}_v \rangle = \langle \mathbf{r}_u, \mathbf{r}_u \rangle = e^{\Phi}$ and $\langle \mathbf{r}_u, \mathbf{r}_v \rangle = 0$ calculate $\langle \mathbf{r}_{uv}, \mathbf{r}_v \rangle$ and $\langle \mathbf{r}_{uu}, \mathbf{r}_v \rangle$. We have

$$\langle \mathbf{r}_{uv}, \mathbf{r}_v \rangle = \frac{1}{2} \frac{\partial}{\partial u} \langle \mathbf{r}_v, \mathbf{r}_v \rangle = \frac{1}{2} \frac{\partial}{\partial u} \left(e^{\Phi}\right) = \frac{1}{2} \Phi_u e^{\Phi}$$

and

$$\langle \mathbf{r}_{uu}, \mathbf{r}_v \rangle = \frac{\partial}{\partial u} \langle \mathbf{r}_u, \mathbf{r}_v \rangle - \langle \mathbf{r}_u, \mathbf{r}_{uv} \rangle = 0 - \langle \mathbf{r}_u, \mathbf{r}_{uv} \rangle = -\frac{1}{2} \frac{\partial}{\partial v} \langle \mathbf{r}_v, \mathbf{r}_v \rangle = -\frac{1}{2} \frac{\partial}{\partial v} \left(e^{\Phi}\right) = -\frac{1}{2} \Phi_v e^{\Phi}$$

Hence we see that 1-form a in (4.51) is equal to

$$a = \langle d\mathbf{e}, \mathbf{f} \rangle = \left\langle d\left(e^{-\frac{\Phi}{2}} \mathbf{r}_u\right), \left(e^{-\frac{\Phi}{2}} \mathbf{r}_v\right) \right\rangle = e^{-\Phi} \langle \mathbf{r}_{uu}, \mathbf{r}_v \rangle du + e^{-\Phi} \langle \mathbf{r}_{uv}, \mathbf{r}_v \rangle dv = \frac{1}{2} (\Phi_u dv - \Phi_v du)\,, \tag{4.52}$$

and 2-form

$$da = d\left(\frac{1}{2}(\Phi_u dv - \Phi_v du)\right) = \frac{1}{2}(\Phi_{uu} du\wedge dv - \Phi_{vv} dv\wedge du) = \frac{1}{2}\left(\Phi_{uu} + \Phi_{vv}\right) dv\wedge du\,.$$

Now using Gauss formula (4.18) and (4.28) we come to

$$K = b\wedge c(\mathbf{e},\mathbf{f}) = -da(\mathbf{e},\mathbf{f}) = -\frac{1}{2}\left(\Phi_{uu} + \Phi_{vv}\right) du\wedge dv\left(e^{-\frac{\Phi}{2}}\frac{\partial}{\partial u}, e^{-\frac{\Phi}{2}}\frac{\partial}{\partial v}\right) =$$

$$-\frac{e^{-\frac{\Phi}{2}}}{2}\left(\Phi_{uu} + \Phi_{vv}\right) du\wedge dv\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = -\frac{e^{-\Phi}}{2}\left(\Phi_{uu} + \Phi_{vv}\right) = -\frac{e^{-\Phi}}{2}\Delta\Phi,$$

where Laplacian $\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$.

Write down the formula in holomorphic coordinates: $z = u+iv, \bar z = u-iv$. We have that

$$K = -\frac{e^{-\Phi}}{2}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\Phi = -\frac{e^{-\Phi}}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right)\Phi = -2e^{-\Phi}\frac{\partial^2\Phi}{\partial\bar z\partial z}\,, \tag{4.53}$$

(for definition of $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar z}$ see (4.48)). This expression is sometimes very convenient for calculations.

Example Consider sphere of radius 1 in stereographic coordinates. Then $G = \frac{4(du^2+dv^2 0}{(1+u^2+v^2)^2)}$. In complex coordinates $G = \frac{4dzd\bar z}{(1+z\bar z)^2} = e^\Phi dzd\bar z$ with $e^\Phi = \frac{4}{(1+z\bar z)^2}$, i.e. $\Phi = \log 4 - 2\log(1+z\bar z)$. We see that $\Phi_z = -\frac{2}{1+z\bar z}$ and $\Phi_{z\bar z} = -\frac{2}{(1+z\bar z)^2}$, i.e. $K = -2e^{-\Phi}\Phi_{\bar z z} = 1$.

Exercise Let $z = f(w)$ be an holomorphic changing of complex coordinates. Due to Theorem new coordinates u', v' ($w = u' + iv'$, $z = u + iv$) are isothermal coordinates too: If

$$G = e^\Phi(du^2 + dv^2) = e^\Phi dzd\bar z = e^{\Phi'} dwd\bar w = e^{\Phi'}(du'^2 + dv'^2)\,.$$

It is very illuminating to check straightforwardly that calculating of Gaussian curvature in new coordinates we will come to the same answer. Do it. According to (4.49) we see that $e^\Phi dzd\bar z = e^\Phi f_w\overline{f_w}dwd\bar w$, i.e. $\Phi' = \Phi + \log f_w + \log\overline{f_w}$. Hence

$$\frac{\partial^2\Phi'}{\partial\bar w\partial w} = \frac{\partial^2\Phi}{\partial\bar w\partial w} + \frac{\partial^2}{\partial\bar w\partial w}\left(\log f_w + \log\overline{f_w}\right)\,.$$

Notice that the function $\log f_w$ is holomorphic function $\Leftrightarrow \frac{\partial}{\partial\bar w}\log f_w = 0$ and the function $\log\overline{f_w}$ is anti-holomorphic function $\Leftrightarrow \frac{\partial}{\partial w}\log\overline{f_w} = 0$ too. Hence

$$\frac{\partial^2}{\partial\bar w\partial w}\left(\log f_w + \log\overline{f_w}\right) = 0\,.$$

This implies that

$$\frac{\partial^2\Phi'}{\partial\bar{w}\partial w} = \frac{\partial^2\Phi}{\partial\bar{w}\partial w}\,.$$

Again using the fact that functions $z = f(w)$ and $z_w = f_w$ are holomorphic functions we see that

$$\frac{\partial^2\Phi'}{\partial\bar{w}\partial w} = \frac{\partial^2\Phi}{\partial\bar{w}\partial w} = \frac{\partial}{\partial\bar{w}}\left(\Phi_z f_w\right) = \frac{\partial\Phi_z}{\partial\bar{w}} f_w = \Phi_{\bar{z}z} f_w \overline{f_w}.$$

Finally we come to

$$K = -2e^{-\Phi'}\frac{\partial^2\Phi'}{\partial\bar{w}\partial w} = -2e^{-\Phi-\log f_w-\log\overline{f_w}}\Phi_{\bar{z}z} f_w\overline{f_w} = -2e^{-\Phi}\frac{\partial^2\Phi}{\partial\bar{z}\partial z}\,.$$

Thus we check by straightforward calculations that Gaussian curvature remains the same.In these calculations In these calculations we used intensively properties (4.48) of holomorphic and anti-holomorphic functions.

5 Curvature tensor

5.1 Curvature tensor for connection

Definition-Proposition Let manifold M be equipped with connection ∇. Consider the following operation which assigns to arbitrary vector fields $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$ on M the new vector field:

$$\mathcal{R}(\mathbf{X},\mathbf{Y})\mathbf{Z} = \left(\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}} - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}} - \nabla_{[\mathbf{X},\mathbf{Y}]}\right)\mathbf{Z} \tag{5.1}$$

This operation is obviously linear over the scalar coefficients.

One can show that this operation is $C^\infty(M)$-linear with respect to vector fields $\mathbf{X}, \mathbf{Y}\,\mathbf{Z}$, i.e. for an arbitrary functions f, g, h

$$\mathcal{R}(f\mathbf{X}, g\mathbf{Y})(h\mathbf{Z}) = fgh\mathcal{R}(\mathbf{X},\mathbf{Y})\mathbf{Z}\,. \tag{5.2}$$

This means that the operation defines the tensor field of the type $\begin{pmatrix}1\\3\end{pmatrix}$: If $\mathbf{X} = X^i\partial_i, \mathbf{X} = X^i\partial_i, \mathbf{X} = X^i\partial_i$ then according to (5.2)

$$\mathcal{R}(\mathbf{X},\mathbf{Y})\mathbf{Z} = \mathcal{R}(X^m\partial_m, Y^n\partial_n)(Z^r\partial_r) = Z^r R^i_{rmn} X^m Y^n$$

where we denote by R^i_{rmn} the components of the tensor $\mathcal{R}$ in the coordinate basis ∂_i

$$R^i{}_{rmn}\partial_i = \mathcal{R}(\partial_m, \partial_n)\partial_r \tag{5.3}$$

This tensor is called *curvature tensor of the connection* ∇.

Express components of the curvature tensor in terms of Christoffel symbols of the connection. If $\nabla_m \partial_n = \Gamma^r_{mn}\partial_r$ then according to the (5.1) we have:

$$\begin{gathered} R^i{}_{rmn}\partial_i = \mathcal{R}(\partial_m, \partial_n)\partial_r = \nabla_{\partial_m}\nabla_{\partial_n}\partial_r - \nabla_{\partial_n}\nabla_{\partial_m}\partial_r, \\ R^i{}_{rmn} = \nabla_{\partial_m}\left(\Gamma^p_{nr}\partial_p\right) - \nabla_{\partial_n}\left(\Gamma^p_{mr}\partial_p\right) = \\ \partial_m \Gamma^i_{nr} + \Gamma^i_{mp}\Gamma^p_{nr} - \partial_n \Gamma^i_{mr} - \Gamma^i_{np}\Gamma^p_{mr}\,. \end{gathered} \tag{5.4}$$

The proof of the property (5.2) can be given just by straightforward calculations: Consider e.g. the case $f = g = 1$, then

$$\begin{gathered} \mathcal{R}(\mathbf{X},\mathbf{Y})(h\mathbf{Z}) = \nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}(h\mathbf{Z}) - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}(h\mathbf{Z}) - \nabla_{[\mathbf{X},\mathbf{Y}]}(h\mathbf{Z}) = \\ \nabla_{\mathbf{X}}\left(\partial_{\mathbf{Y}}h\mathbf{Z} + h\nabla_{\mathbf{Y}}\mathbf{Z}\right) - \nabla_{\mathbf{Y}}\left(\partial_{\mathbf{X}}h\mathbf{Z} + h\nabla_{\mathbf{X}}\mathbf{Z}\right) - \partial_{[\mathbf{X},\mathbf{Y}]}h\mathbf{Z} - h\nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z} = \\ \partial_{\mathbf{X}}\partial_{\mathbf{Y}}h\mathbf{Z} + \partial_{\mathbf{Y}}h\nabla_{\mathbf{X}}\mathbf{Z} + \partial_{\mathbf{X}}h\nabla_{\mathbf{Y}}\mathbf{Z} + h\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{Z} - \\ \partial_{\mathbf{Y}}\partial_{\mathbf{X}}h\mathbf{Z} - \partial_{\mathbf{X}}h\nabla_{\mathbf{Y}}\mathbf{Z} - \partial_{\mathbf{Y}}h\nabla_{\mathbf{X}}\mathbf{Z} + h\nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}\mathbf{Z} - \\ \partial_{[\mathbf{X},\mathbf{Y}]}h\mathbf{Z} - h\nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z} = \\ h\left[\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{Z} - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}\mathbf{Z}) - \nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z}\right] + \left[\partial_{\mathbf{X}}\partial_{\mathbf{Y}}h - \partial_{\mathbf{Y}}\partial_{\mathbf{X}}h - \partial_{[\mathbf{X},\mathbf{Y}]h}\right]\mathbf{Z} = \\ h\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{Z} - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}\mathbf{Z}) - \nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z} = h\mathcal{R}(\mathbf{X},\mathbf{Y})\mathbf{Z}\,, \end{gathered}$$

since $\partial_{\mathbf{X}}\partial_{\mathbf{Y}}h - \partial_{\mathbf{Y}}\partial_{\mathbf{X}}h - \partial_{[\mathbf{X},\mathbf{Y}]}h = 0$.

5.1.1 Properties of curvature tensor

Tensor $R^i{}_{kmn}$ is expressed trough derivatives of Christoffel symbols. In spite this fact it is is much more "pleasant" object than Christoffel symbols, since the latter is not the tensor.

It follows from the definition that the tensor R^i_{kmn} is antisymmetrical with respect to indices m, n:

$$R^i_{kmn} = -R^i_{knm}\,. \tag{5.5}$$

One can prove that for symmetric connection this tensor obeys the following identities:

$$R^i{}_{kmn} + R^i_{mnk} + R^i_{nkm} = 0\,, \tag{5.6}$$

The curvature tensor corresponding to Levi-Civita connection obeys also another identities too (see the next subsection.)

We know well that If Christoffel symbols vanish in a vicinity of a given point **p** in some chosen coordinate system then in general Christoffel symbols do not vanish in arbitrary coordinate systems. E.g. Christoffel symbols of canonical flat connection in $\mathbf{E}^2$ vanish in Cartesian coordinates but do not vanish in polar coordinates. This unpleasant property of Christoffel symbols is due to the fact that Christoffel symbols do not form a tensor.

In particular if a tensor vanishes in some coordinate system, then it vanishes in arbitrary coordinate system too. This implies very simple but important

Proposition *If curvature tensor $R^i{}_{kmn}$ vanishes in some coordinate system, then it vanishes in arbitrary coordinate systems.*

We see that if Christoffel symbols vanish in a vicinity of a given point **p** in some chosen coordinate system then its Riemannian curvature tensor vanishes in a vicinity of the point **p** (see the formula (5.4)) and hence it vanishes locally (in a vicinity of point **p**) in arbitrary coordinate system.

In fact one can prove

Theorem If a connection is symmetric then curvature tensor vanishes in a vicinity of a point if and only there exist local Cartesian coordiantes,n a vicinity of this point i.e. coordinates in which Christoffel symbol of connection vanish.

5.2 Riemann curvature tensor of Riemannian manifolds.

Let M be Riemannian manifold equipped with Riemannian metric G

In this section we will consider curvature tensor of Levi-Civita connection ∇ of Riemannian metric G.

The curvature tensor for Levi-Civita connection will be called later Riemann curvature tensor, or Riemann tensor.

Using Riemannian metric one can consider Riemann tensor with all low indices

$$R_{ikmn} = g_{ij} R^j{}_{kmn} \tag{5.7}$$

Due to identities (5.5) and (5.6) for curvature tensor Riemann tensor obeys the following identities:

$$R_{ikmn} = -R_{iknm} \ , \quad R_{ikmn} + R_{imnk} + R_{inkm} = 0 \tag{5.8}$$

Riemann curvature tensor which is curvature tensor for Levi-Civita connection obeys also the following identities:

$$R_{ikmn} = -R_{kimn}\ , \quad R_{ikmn} = R_{mnki}\,. \tag{5.9}$$

These condition lead to the fact that for 2-dimensional Riemannian manifold the Riemann curvature tensor of Levi-Civita connection has essentially only one non-vanishing component: all components vanish or equal to component R_{1212}up to a sign. Indeed consider for 2-dimensional Riemannian manifold Riemann tensor R_{ikmn}, where $i,k,m,n = 1,2$. Since antisymmetricity with respect to third and fourth indices ($R_{ikmn} = -R_{iknm}$), $R_{ik11} = R_{ik22} = 0$ and $R_{ik12} = -R_{ik21}$. The same for first and second indices: since antisymmetricity with respect to the the first and second indices ($R_{12mn} = -R_{21mn}$), $R_{11mn} = R_{22mn} = 0$ and $R_{12mn} = -R_{ik21}$. If we denote $R_{1212} = a$ then

$$R_{1212} = R_{2121} = a, R_{1221} = R_{2112} = -a$$

and all other components vanish.

For Riemann tensor one can consider Ricci tensor,

$$R_{mn} = R^i_{min} \tag{5.10}$$

which is symmetrical tensor: $R_{mn} = R_{nm}$.

One can consider scalar curvature:

$$R = R^i_{kin} g^{kn} = g^{kn} R_{kn} \tag{5.11}$$

where g^{kn} is Riemannian metric with indices above (the matrix $||g^{ik}||$ is inverse to the matrix $||g_{il}||$).

Ricci tensor and scalar curvature play essential role for formulation of Einstein gravity equations. In particular the space without matter the Einstein equations have the following form:

$$R_{ik} - \frac{1}{R} g_{ik} = 0\,. \tag{5.12}$$

5.3 *Curvature of surfaces in $\mathbf{E}^3$.. *Theorema Egregium* again

Express Riemannian curvature of surfaces in $\mathbf{E}^3$ in terms of derivation formula (5.13).

Consider derivation formula for the orthonormal basis $\{\mathbf{e}, \mathbf{f}, \mathbf{n}, \}$ adjusted to the surface M:

$$d\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\\ \mathbf{n}\end{pmatrix} = \begin{pmatrix}0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{pmatrix}\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\\ \mathbf{n}\end{pmatrix}, \tag{5.13}$$

where as usual $\mathbf{e}, \mathbf{f}, \mathbf{n}$ vector fields of unit length which are orthogonal to each other and $\mathbf{n}$ is orthogonal to the surface M. As we know the induced connection on the surface M is defined by the formula (4.21) and (4.22):

$$\nabla_{\mathbf{Y}}\mathbf{e} = (d\mathbf{e}(\mathbf{Y}))_{\text{tangent}} = a(\mathbf{X})\mathbf{f}\,, \nabla_{\mathbf{Y}}\mathbf{f} = (d\mathbf{f}(\mathbf{Y}))_{\text{tangent}} = -a(\mathbf{X})\mathbf{e}\,, \tag{5.14}$$

According to the definition of curvature calculate

$$R(\mathbf{e}, \mathbf{f})\mathbf{e} = \nabla_{\mathbf{e}}\nabla_{\mathbf{f}}\mathbf{e} - \nabla_{\mathbf{f}}\nabla_{\mathbf{e}}\mathbf{e} - \nabla_{[\mathbf{e},\mathbf{f}]}\mathbf{e}\,.$$

Using these formulae one can calculate straightforwardly that for surfaces in $\mathbf{E}^3$ Gaussian curvature is equal to half of the scalar curvature:

$$K = \frac{R}{2} \tag{5.15}$$

Detailed calculations are following:

Note that since the induced connection is symmetrical connection then:

$$\nabla_{\mathbf{e}}\mathbf{f} - \nabla_{\mathbf{f}}\mathbf{e} - [\mathbf{e}, \mathbf{f}] = 0\,.$$

hence due to (5.14)

$$[\mathbf{e}, \mathbf{f}] = \nabla_{\mathbf{e}}\mathbf{f} - \nabla_{\mathbf{f}}\mathbf{e} = -a(\mathbf{e})\mathbf{e} - a(\mathbf{f})\mathbf{f}$$

Thus we see that $R(\mathbf{e}, \mathbf{f})\mathbf{e} =$

$$\nabla_{\mathbf{e}}\nabla_{\mathbf{f}}\mathbf{e} - \nabla_{\mathbf{f}}\nabla_{\mathbf{e}}\mathbf{e} - \nabla_{[\mathbf{e},\mathbf{f}]}\mathbf{e} = \nabla_{\mathbf{e}}\left(a(\mathbf{f})\mathbf{f}\right) - \nabla_{\mathbf{f}}\left(a(\mathbf{e})\mathbf{f}\right) + \nabla_{a(\mathbf{e})\mathbf{e}+a(\mathbf{f})\mathbf{f}}\mathbf{e} =$$

$$\partial_{\mathbf{e}}a(\mathbf{f})\mathbf{f} + a(\mathbf{f})\nabla_{\mathbf{e}}\mathbf{f} - \partial_{\mathbf{f}}a(\mathbf{e})\mathbf{f} - a(\mathbf{e})\nabla_{\mathbf{f}}\mathbf{f} + a(\mathbf{e})\nabla_{\mathbf{e}}\mathbf{e} + a(\mathbf{f})\nabla_{\mathbf{f}}\mathbf{e} =$$

$$\partial_{\mathbf{e}}a(\mathbf{f})\mathbf{f} - a(\mathbf{f})a(\mathbf{e})\mathbf{e} - \partial_{\mathbf{f}}a(\mathbf{e})\mathbf{f} + a(\mathbf{e})a(\mathbf{f})\mathbf{e} + a(\mathbf{e})a(\mathbf{e})\mathbf{f} + a(\mathbf{f})a(\mathbf{f})\mathbf{f} =$$

$$[\partial_{\mathbf{e}}a(\mathbf{f})\mathbf{f} - \partial_{\mathbf{f}}a(\mathbf{e})\mathbf{f} - a\left[-a(\mathbf{e})\mathbf{e} - a(\mathbf{f})\mathbf{f}\right]]\,\mathbf{f} =$$

$$= [\partial_{\mathbf{e}}a(\mathbf{f})\mathbf{f} - \partial_{\mathbf{f}}a(\mathbf{e})\mathbf{f} - a\left([\mathbf{e}, \mathbf{f}]\right)]\,\mathbf{f} = da(\mathbf{e}, \mathbf{f})\mathbf{f}\,.$$

Recall that we established in 4.40 that for Gaussian curvature K

$$K = b \wedge c(\mathbf{e}, \mathbf{f}) = -da(\mathbf{e}, \mathbf{f})$$

Hence we come to the relation:

$$R(\mathbf{e},\mathbf{f})\mathbf{e} = da(\mathbf{e},\mathbf{f}) = -K\mathbf{f}\,.$$

This means that

$$R^2_{112} = -K$$

(in the basis $\mathbf{e},\mathbf{f}$), i.e. the scalar curvature

$$R = 2R_{1212} = 2K$$

Thus we come to equation (5.15).

Equation (5.15) is the fundamental relation which claims that the Gaussian curvature (the magnitude defined in terms of External observer) equals to the scalar curvature (up to a coefdficient), the magnitude defined in terms of Internal Observer. This gives us another proof of Theorema Egregium.

Below we will give the proof of Theorema Egregium by straightforward calculations.

5.4 Relation between Gaussian curvature and Riemann curvature tensor and *Theorema Egregium*

Let M be a surface in $\mathbf{E}^3$ and $R^i{}_{kmp}$ be Riemann tensor, Riemann curvature tensor of Levi-Civita connection. Recall that this means that $R^i{}_{kmp}$ is curvature tensor of the connection ∇, which is Levi-Civita connection of the Riemannian metric $g_{\alpha\beta}$ induced on the surface M by standard Euclidean metric $dx^2 + dy^2 + dz^2$. Recall that Riemann curvature tensor is expressed via Christoffel symbols of connection by the formula

$$R^i{}_{kmn} = \partial_m \Gamma^i_{nk} + \Gamma^i_{mp}\Gamma^p_{nk} - \partial_n \Gamma^i_{mk} - \Gamma^i_{np}\Gamma^p_{mk} \tag{5.16}$$

(see he formula (5.4)) where Christoffel symbols of Levi-Civita connection are defined by the formula

$$\Gamma^i_{mk} = \frac{1}{2} g^{ij} \left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^m} - \frac{\partial g_{mk}}{\partial x^j} \right) \tag{5.17}$$

(see Levi-Civita Theorem)

Recall that scalar curvature R of Riemann tensor equals to $R = R^i_{kim} g^{km}$, where g^{km} is Riemannian metric tensor with upper indices (matrix $||g^{ik}||$ is inverse to the matrix $||g_{ik}||$). Note that as it was mentioned before the formula for scalar curvature becomes very simple in two-dimensional case (see formulae (5.8) and (5.8) above). Using these formulae calculate Ricci tensor and scalar curvature R. Let $R_{1212} = a$. We know that all other components of Riemann tensor equal to zero or equal to $\pm a$ (see (5.8) and (5.8)). One can show that scalar curvature R can be expressed via the component $R_{1212} = a$ by the formula

$$R = \frac{2R_{1212}}{\det g} \tag{5.18}$$

where $\det g = \det g_{ik} = g_{11}g_{22} = g_{12}^2$.

Show it. Using identities (5.8) and (5.8) we see that

$$R_{11} = R^i{}_{1i1} = R^2{}_{121} = g^{22}R_{2121} + g^{21}R_{1121} = g^{22}R_{1212} = g^{22}a \tag{5.19}$$

$$R_{22} = R^i{}_{2i2} = R^1{}_{212} = g^{11}R_{1212} + g^{12}R_{2221} = g^{11}R_{1212} = g^{11}a \tag{5.20}$$

$$R_{12} = R_{21} = R^i{}_{1i2} = R^1{}_{112} = g^{12}R_{2112} = -g^{12}R_{1212} = -g^{12}a \tag{5.21}$$

Thus using the formula for inverse 2×2 matrix we come to the relation

$$R_{ik} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} = \begin{pmatrix} g^{22}a & -g^{12}a \\ -g^{21}a & g^{11}a \end{pmatrix} = \frac{1}{\det g} \begin{pmatrix} g_{11}a & g_{12}a \\ g_{21}a & g_{11}a \end{pmatrix} ,$$

i.e. for 2-dimensional Riemannian manifold

$$R_{ik} = \frac{1}{\det g} R_{1212} g_{ik} \,, \tag{5.22}$$

Hence for scalar curvatre of 2-dimension Riemannian manifold

$$R = R^i{}_{kim} g^{km} = R_{km} g^{km} = \frac{1}{\det g} R_{1212} g_{ik} g^{ik} = \frac{2}{\det g} R_{1212} \,. \tag{5.23}$$

Note that the relations (5.22) and (5.23) imply thatt

$$R_{ik} = \frac{1}{2} R g_{ik} \,. \tag{5.24}$$

One can say that gravity equation for $n = 2$ are trivial. The mathematical meaning of this formula is the following: equation (5.24) means that variation of functional $S = \int R\sqrt{\det g} d\sigma$ vanishes and this is one of corollaries of Gauss-Bonet Theorem (see later).

Formula (5.18) expresses scalar curvature for surface in terms of non-trivial component R_{1212}. Using considerations of subsection above (where we considered the relation between Gaussian curvature and scalar curvature) we come to

Proposition *For an arbitrary point of the surface M*

$$K = \frac{R}{2} = \frac{R_{1212}}{\det g}\,. \tag{5.25}$$

where $R = R^i_{kim} g^{km}$ is scalar curvature and K is Gaussian curvature.

We know also that for surface M the scalar curvature R is expressed via Riemann curvature tensor by the formula (5.18). Hence if we know the Gaussian curvature then we know all components of Riemann curvature tensor (since all components vanish or equal to $\pm a$.). This is nothing but Theorema Egregium! Indeed Theorema Egregium (see beginning of the section 4) immediately follows from this Proposition. Indeed according to the formulae (5.16) and (5.17) the left hand side of the relation $R = 2K$ depends only on induced Riemannian metric. Hence Gaussian curvature K depends only on induced Riemannian metric.

Example Let $M = S^2$ be sphere of radius R in $\mathbf{E}^3$. Show that one cannot find local coordinates u, v on the sphere such that induced Riemannian metric equals to $du^2 + dv^2$ in these coordinates.

This immediately follows from the Proposition. Indeed suppose there exist local coordinates u, v on the sphere such that induced Riemannian metric equals to $du^2 + dv^2$, i.e. Riemannian metric is given by unity matrix. Then according to the formulae for Levi-Civita connection, the Christoffel symbols equal to zero in these coordinates. Hence Riemann curvature tensor equals to zero, and scalar curvature too. Due to Proposition this is in contradiction with the fact that Gaussian curvature of the sphere equals to $\frac{1}{R^2}$.

(The straightforward proof of this proposition see in the Appendix 6.2.6)

5.5 †Gauss Bonnet Theorem

Consider the integral of curvature over whole closed surface M. According to the Theorem above the answer has to be equal to 0 (modulo 2π), i.e. $2\pi N$ where N is an integer, because this integral is a limit when we consider very small curve. We come to the formula:

$$\int_D K d\sigma = 2\pi N$$

(Compare this formula with formula (4.5)).

What is the value of integer N?

We present now one remarkable Theorem which answers this question and prove this Theorem using the formula (4.5).

Let M be a closed orientable surface.[18] All these surfaces can be classified up to a diffeomorphism. Namely arbitrary closed oriented surface M is diffeomorphic either to sphere (zero holes), or torus (one hole), or pretzel (two holes),... "Number k" of holes is intuitively evident characteristic of the surface. It is related with very important characteristic—Euler characteristic $\chi(M)$ by the following formula:

$$\chi(M) = 2(1 - g(M)), \quad \text{where } g \text{ is number of holes} \tag{5.26}$$

Remark What we have called here "holes" in a surface is often referred to as "handles" attached o the sphere, so that the sphere itself does not have any handles, the torus has one handle, the pretzel has two handles and so on. The number of handles is also called genus.

Euler characteristic appears in many different way. The simplest appearance is the following:

Consider on the surface M an arbitrary set of points (vertices) connected with edges (graph on the surface) such that surface is divided on polygons with (curvilinear sides)—plaquets. ("Map of world")

Denote by P number of plaquets (countries of the map)

Denote by E number of edges (boundaries between countries)

Denote by V number of vertices.

Then it turns out that

$$P - E + V = \chi(M) \tag{5.27}$$

It does not depend on the graph, it depends only on how much holes has surface.

E.g. for every graph on M, $P - E + V = 2$ if M is diffeomorphic to sphere. For every graph on M $P - E + V = 0$ if M is diffeomorphic to torus.

Now we formulate Gauß-Bonnet Theorem.

Let M be closed oriented surface in $\mathbf{E}^3$.

[18]Closed means compact surface without boundaries. Intuitively orientability means that one can define out and inner side of the surface. In terms of normal vectors orientability means that one can define the continuous field of normal vectors at all the points of M. The direction of normal vectors at any point defines outward direction. Orientable surface is called oriented if the direction of normal vector is chosen.

Let $K(p)$ be Gaussian curvature at any point p of this surface.

Theorem (Gauß -Bonnet) The integral of Gaussian curvature over the closed compact oriented surface M is equal to 2π multiplied by the Euler characteristic of the surface M

$$\frac{1}{2\pi}\int_M K d\sigma = \chi(M) = 2(1 - \text{number of holes}) \tag{5.28}$$

In particular for the surface M diffeomorphic to the sphere $\kappa(M) = 2$, for the surface diffeomorphic to the torus it is equal to 0.

The value of the integral does not change under continuous deformations of surface! It is integer number (up to the factor π) which characterises topology of the surface.

E.g. consider surface M which is diffeomorphic to the sphere. If it is sphere of the radius R then curvature is equal to $\frac{1}{R^2}$, area of the sphere is equal to $4\pi R^2$ and left hand side is equal to $\frac{4\pi}{2\pi} = 2$.

If surface M is an arbitrary surface diffeomorphic to M then metrics and curvature depend from point to the point, Gauß -Bonnet states that integral nevertheless remains unchanged.

Very simple but impressive corollary:

Let M be surface diffeomorphic to sphere in $\mathbf{E}^3$. Then there exists at least one point where Gaussian curvature is positive.

Proof: Suppose it is not right. Then $\int_M K\sqrt{\det g} dudv \leq 0$. On the other hand according to the Theorem it is equal to 4π. Contradiction.

Proof of Gauß-Bonet Theorem

Consider triangulation of the surface M. Suppose M is covered by N triangles. Then number of edges will be $3N/over2$. If V number of vertices then according to Euler Theorem

$$N - \frac{3N}{2} + V = V - \frac{N}{2} = \chi(M).$$

Calculate the sum of the angles of all triangles. On the one hand it is equal to $2\pi V$. On the other hand according the formula (4.5) it is equal to

$$\sum_{i=1}^{N}\left(\pi + \int_{\triangle_i} K d\sigma\right) = \pi N + \sum_{i=1}^{N}\left(\int_{\triangle_i} K d\sigma\right) = N\pi + \int_M K d\sigma$$

We see that $2\pi V = N\pi + \int_M K d\sigma$, i.e.

$$\int_M K d\sigma = \pi\left(2V - \frac{N}{2}\right) = 2\pi\chi(M) \blacksquare$$

6 Appendices

6.1 *Integrals of motions and geodesics.

We see how useful in Riemannian geometry to use the Lagrangian approach.

To solve and study solutions of Lagrangian equations (in particular geodesics which are solutions of Euler-Lagrange equations for Lagrangian of free particle) it is very useful to use integrals of motion

6.1.1 *Integral of motion for arbitrary Lagrangian $L(x, \dot{x})$

Let $L = L(x, \dot{x})$ be a Lagrangian, the function of point and velocity vectors on manifold M (the function on tangent bundle TM).

Definition We say that the function $F = F(q, \dot{q})$ on TM is *integral of motion* for Lagrangian $L = L(x, \dot{x})$ if for any curve $q = q(t)$ which is the solution of Euler-Lagrange equations of motions the magnitude $I(t) = F(x(t), \dot{x}(t))$ is preserved along this curve:

$$F(x(t), \dot{x}(t)) = \mathit{const} \text{ if } x(t) \text{ is a solution of Euler-Lagrange equations(3.11).} \tag{6.1}$$

In other words

$$\frac{d}{dt}\left(F(x(t), \dot{x}(t))\right) = 0 \text{ if } x^i(t)\colon \ \frac{d}{dt}\left(\frac{\partial L}{\dot{\partial} x^i}\right) - \frac{\partial L}{\partial x^i} = 0\,. \tag{6.2}$$

6.1.2 *Basic examples of Integrals of motion: Generalised momentum and Energy

Let $L(x^i, \dot{x}^i)$ does not depend on the coordinate x^1. $L = L(x^2, \dots, x^n, \dot{x}^1, \dot{x}^2, \dots, \dot{x}^n)$. Then the function

$$F_1(x, \dot{x}) = \frac{\partial L}{\partial \dot{x}^1}$$

is integral of motion. (In the case if $L(x^i, \dot{x}^i)$ does not depend on the coordinate x^i. the function $F_i(x, \dot{x}) = \frac{\partial L}{\partial \dot{x}^i}$ will be integral of motion.)

Proof is simple. Check the condition (6.2): Euler-Lagrange equations of motion are:

$$\frac{d}{dt}\left(\frac{\partial L}{\dot{\partial} x^i}\right) - \frac{\partial L}{\partial x^i} = 0 \quad (i = 1, 2, \dots, n)$$

We see that exactly first equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\dot{\partial} x^1}\right) = \frac{d}{dt} F_1(q, \dot{q}) = 0 \quad \text{since } \tfrac{\partial L}{\partial x^1} = 0,.$$

(if $L(x^i, \dot{x}^i)$ does not depend on the coordinate x^i then the function $F_i(x, \dot{x}) = \frac{\partial L}{\dot{x}^1}$ is integral of motion since $i-th$ equation is exactly the condition $\dot{F}_i = 0$.)

The integral of motion $F_i = \frac{\partial L}{\dot{\partial} x^i}$ is called sometimes *generalised momentum.*

Another very important example of integral of motion is: energy.

$$E(x,\dot{x}) = \dot{x}^i \frac{\partial L}{\partial \dot{x}^i} - L\,. \tag{6.3}$$

One can check by direct calculation that it is indeed integral of motion. Using Euler Lagrange equations $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i}$ we have:

$$\frac{d}{dt}E(x(t),\dot{x}(t)) = \frac{d}{dt}\left(\dot{x}^i\frac{\partial L}{\partial \dot{x}^i} - L\right) = \frac{\partial L}{\partial \dot{x}^i}\frac{d\dot{x}^i}{dt} + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right)\dot{x}^i - \frac{dL}{dt} =$$

$$\frac{\partial L}{\partial \dot{x}^i}\frac{d\dot{x}^i}{dt} + \frac{\partial L}{\partial x^i}\frac{dx^i}{dt} - \frac{dL(x,\dot{x})}{dt} = \frac{dL(x,\dot{x})}{dt} - \frac{dL(x,\dot{x})}{dt} = 0\,.$$

6.1.3 *Integrals of motion for geodesics

Apply the integral of motions for studying geodesics.

The Lagrangian of "free" particle $L_{\text{free}} = \frac{g_{ik}(x)\dot{x}^i\dot{x}^k}{2}$. For Lagrangian of free particle solution of Euler-Lagrange equations of motions are geodesics.

If $F = F(x,\dot{x})$ is the integral of motion of the free Lagrangian $L_{\text{free}} = \frac{g_{ik}(x)\dot{x}^i\dot{x}^k}{2}$ then the condition (6.1) means that the magnitude $I(t) = F(x^i(t),\dot{x}^i(t))$ is preserved along the geodesics:

$$I(t) = F(x^i(t),\dot{x}^i(t)) = const, \text{i.e. } \frac{d}{dt}I(t) = 0 \text{ if } x^i(t) \text{ is geodesic.} \tag{6.4}$$

Consider examples of integrals of motion for free Lagrangian, i.e. magnitudes which preserve along the geodesics:

Example 1 Note that for an arbitrary "free" Lagrangian Energy integral (6.3) is an integral of motion:

$$E = \dot{x}^i\frac{\partial L}{\partial \dot{x}^i} - L = \dot{x}^i\frac{\partial\left(\frac{g_{pq}(x)\dot{x}^p\dot{x}^q}{2}\right)}{\partial \dot{x}^i} - \frac{g_{ik}(x)\dot{x}^i\dot{x}^k}{2} =$$

$$\dot{x}^i g_{iq}(x)\dot{x}^q - \frac{g_{ik}(x)\dot{x}^i\dot{x}^k}{2} = \frac{g_{ik}(x)\dot{x}^i\dot{x}^k}{2}\,. \tag{6.5}$$

This is an integral of motion for an arbitrary Riemannian metric. It is preserved on an arbitrary geodesic

$$\frac{dE(t)}{dt} = \frac{d}{dt}\left(\frac{1}{2}g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t)\right) = 0\,.$$

In fact we already know this integral of motion: Energy (6.5) is proportional to the square of the length of velocity vector:

$$|\mathbf{v}| = \sqrt{g_{ik}(x)\dot{x}^i\dot{x}^k} = \sqrt{2E}\,. \tag{6.6}$$

We already proved that velocity vector is preserved along the geodesic (see the Proposition in the subsection 3.2.1 and its proof (**??**).)

Example 2 Consider Riemannian metric $G = adu^2 + bdv^2$ (see also calculations in subsection 2.3.3) in the case if $a = a(u)$, $b = b(u)$, i.e. coefficients do not depend on the second coordinate v:

$$G = a(u)du^2 + b(u)dv^2,\ L_{\text{free}} = \frac{1}{2}\left(a(u)\dot{u}^2 + b(u)\dot{v}^2\right) \tag{6.7}$$

We see that Lagrangian does not depend on the second coordinate v hence the magnitude

$$F = \frac{\partial L_{\text{free}}}{\partial \dot{v}} = b(u)\dot{v} \tag{6.8}$$

is preserved along geodesic. It is integral of motion because Euler-Lagrange equation for coordinate v is

$$\frac{d}{dt}\frac{\partial L_{\text{free}}}{\partial \dot{v}} - \frac{\partial L_{\text{free}}}{\partial v} = \frac{d}{dt}\frac{\partial L_{\text{free}}}{\partial \dot{v}} = \frac{d}{dt}F = 0 \ \text{ since } \tfrac{\partial L_{\text{free}}}{\partial v} = 0..$$

In fact all revolution surfaces which we consider here (cylinder, cone, sphere,...) have Riemannian metric of this type. Indeed consider further examples.

Example (sphere)

Sphere of the radius R in $\mathbf{E}^3$. Riemannian metric: $G = Rd\theta^2 + R^2\sin^2\theta d\varphi^2$ and $L_{\text{free}} = \frac{1}{2}\left(R^2\dot{\theta}^2 + R^2\sin^2\theta\dot{\varphi}^2\right)$ It is the case (6.7) for $u = \theta, v = \varphi$, $b(u) = R^2\sin^2\theta$ The integral of motion is

$$F = \frac{\partial L_{\text{free}}}{\partial \dot{\varphi}} = R^2\sin^2\theta\dot{\varphi}$$

Example (cone)

Consider cone $\begin{cases} x = ah\cos\varphi \\ y = ah\sin\varphi \\ z = bh \end{cases}$. Riemannian metric:

$$G = d(ah\cos\varphi)^2 + d(ah\sin\varphi)^2 + (dbh)^2 = (a^2+b^2)dh^2 + a^2h^2d\varphi^2\,.$$

and free Lagrangian

$$L_{\text{free}} = \frac{(a^2+b^2)\dot{h}^2 + a^2h^2\dot{\varphi}^2}{2}\,.$$

The integral of motion is

$$F = \frac{\partial L_{\text{free}}}{\partial \dot{\varphi}} = a^2h^2\dot{\varphi}.$$

Example (general surface of revolution)

Consider a surface of revolution in $\mathbf{E}^3$:

$$\mathbf{r}(h,\varphi)\colon \begin{cases} x = f(h)\cos\varphi \\ y = f(h)\sin\varphi \\ z = h \end{cases} \qquad (f(h) > 0) \tag{6.9}$$

(In the case $f(h) = R$ it is cylinder, in the case $f(h) = kh$ it is a cone, in the case $f(h) = \sqrt{R^2 - h^2}$ it is a sphere, in the case $f(h) = \sqrt{R^2 + h^2}$ it is one-sheeted hyperboloid, in the case $z = \cos h$ it is catenoid,...)

For the surface of revolution (6.9)

$$G = d(f(h)\cos\varphi)^2 + d(f(h)\sin\varphi)^2 + (dh)^2 = (f'(h)\cos\varphi dh - f(h)\sin\varphi d\varphi)^2 +$$

$$(f'(h)\sin\varphi dh + f(h)\cos\varphi d\varphi)^2 + dh^2 = (1 + f'^2(h))dh^2 + f^2(h)d\varphi^2 \,.$$

The "free" Lagrangian of the surface of revolution is

$$L_{\text{free}} = \frac{\left(1 + f'^2(h)\right)\dot{h}^2 + f^2(h)\dot{\varphi}^2}{2} \,.$$

and the integral of motion is

$$F = \frac{\partial L_{\text{free}}}{\partial \dot{\varphi}} = f^2(h)\dot{\varphi}.$$

6.1.4 *Using integral of motions to calculate geodesics

Integrals of motions may be very useful to calculate geodesics. The equations for geodesics are second order differential equations. If we know integrals of motions they help us to solve these equations. Consider just an example.

For Lobachevsky plane the free Lagrangian $L = \frac{\dot{x}^2 + \dot{y}^2}{2y^2}$. We already calculated geodesics in the subsection 3.3.4. Geodesics are solutions of second order Euler-Lagrange equations for the Lagrangian $L = \frac{\dot{x}^2 + \dot{y}^2}{2y^2}$ (see the subsection 3.3.4)

$$\begin{cases} \ddot{x} - \frac{2\dot{x}\dot{y}}{y} = 0 \\ \ddot{y} + \frac{\dot{x}^2}{y} - \frac{\dot{y}^2}{y} = 0 \end{cases}$$

It is not so easy to solve these differential equations.

For Lobachevsky plane we know two integrals of motions:

$$E = L = \frac{\dot{x}^2 + \dot{y}^2}{2y^2}, \quad \text{and } F = \frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{y^2} \,. \tag{6.10}$$

These both integrals preserve in time: if $x(t), y(t)$ is geodesics then

$$\begin{cases} F = \frac{\dot{x}(t)}{y(t)^2} \\ E = \frac{\dot{x}(t)^2 + \dot{y}(t)^2}{2y(t)^2} = C_2 \end{cases} \Rightarrow \begin{cases} \dot{x} = C_1 y^2 \\ \dot{y} = \pm\sqrt{2C_2 y^2 - C_1^2 y^4} \end{cases}$$

These are first order differential equations. It is much easier to solve these equations in general case than initial second order differential equations.

6.1.5 *Killing vectors of Lobachevsky plane and geodesics

Killing vector field of Rimeannina manifold (M, G) is an infinitesimal isometry of the Rimeannian metric G: under infinitesimal transform $x \to x + \varepsilon \mathbf{K}, x^i \to x^i + \varepsilon K^I(x)$ ($\varepsilon^2 = 0$) metric does not change:

$$g_{ik}(x)dx^i dx^k = g_{ik}(x^r + \varepsilon K^i)(dx^i + \partial_m K^i dx^m)(dx^k + \partial_n K^k dx^n) \,. \tag{1}$$

Expanding this formula by ε and using the fact that $\varepsilon^2 = 0$ we come to

$$K^i \partial_i g_{km} + \partial_k K^r g_{rm} + \partial_m K^r g_{rk} = 0 \,, \tag{1a}$$

(i.e. Lie derivative $\mathcal{L}_{\mathbf{K}} G = 0$.)

Examples: Killings of plane, sphere, cylindre, Lobachevsky plane.....

Theorem Let V be a vector space of all Killing vector fields of Riemannian manifold M. Then the dimension of V is less or equal than $\frac{n(n+1}{2)}$.

It means that for surfaces the number of independent Killing vector fields is less or equal to 3.

One can prove that it is only for plane, sphere and Lobachevsky plane that number of independent Killing vector fiels is equal to 3.

We calculate here Killing vector fields for Lobachevsky plane and use them for finding geodesics.

Theorem Let $\mathbf{K}$ be Killing tor field on Riemannian manifold (M, G) , and $L = \frac{g_{kp}\dot{x}^k \dot{x}^p}{2}$ Lagragian of 'free' particle on M. We know that geodesics are solutions of its equations of motions.

The magnitude

$$I = I_{\mathbf{K}} = K^i(x) \frac{\partial L(x, \dot{x})}{\partial \dot{x}^i}$$

is an integral of motion, i.e. it is preserved along geodesics.

The proof of the Theorem is obvious. The condition that $\mathbf{K}$ is Killing vector field means that

$$L(x^i + \varepsilon K^i, \dot{x}^i + \varepsilon \dot{K}^i) \,, \tag{2}$$

i.e.

$$K^i(x) \frac{\partial L}{\partial x^i} + \frac{dK^i}{dt} \frac{\partial L}{\partial \dot{x}^i} = 0 \,. \tag{2a}$$

Hence

$$\frac{d}{dt} I_{\mathbf{K}} = \frac{d}{dt} \left(K^i(x) \frac{\partial L(x, \dot{x})}{\partial \dot{x}^i} \right) = \frac{dK^i}{dt} \frac{\partial L(x, \dot{x})}{\partial \dot{x}^i} + K^i \frac{d}{dt} \left(\frac{\partial L(x, \dot{x})}{\partial \dot{x}^i} \right) =$$

$$\underbrace{K^i(x) \frac{\partial L}{\partial x^i} + \frac{dK^i}{dt} \frac{\partial L}{\partial \dot{x}^i}}_{\text{condition that } \mathbf{K} \text{ is Killing}} + K^i \underbrace{\left(\frac{\partial L(x, \dot{x})}{\partial x^i} - \frac{\partial L(x, \dot{x})}{\partial \dot{x}^i} \right)}_{\text{equations of motion}} = 0 \,.$$

Use this Theorem to find geodesics.

First find Killing vector fields, i.e. infinitesimal isometries.

Since the dimension is equal 2, the dimension of space of Killing vector fields is ≤ 3. We will find three independent Killing vector fields.

There are two evident Killing vectors: Metric $G = \frac{dx^2+dy^2}{y^2}$ is evidently invariant with respect to translations $x \to x + a$ and homothety: $\begin{cases} x \to \lambda x \\ y \mapsto \lambda y \end{cases}$: $\frac{d(\lambda x)^2+d(\lambda y)^2}{(\lambda y)^2} = \frac{dx^2+dy^2}{y^2}$.

Infinitesimal translation is $x' = x + \varepsilon, y' = y$, the vector field $D_1 = \frac{\partial}{\partial x}$. Infinitesimal homothety is $x' = x + \varepsilon x, y' = y + \varepsilon y$, the vector field $D_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$.

Now most interesting: find the third Killing vector field. Use the fact that inversion $\mathbf{O}\colon (x, y) \mapsto \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$ preserves the metric. Consider the infinitesimal transformation $L_\varepsilon = \mathbf{O} \circ T_\varepsilon \circ \mathbf{O}$ ($L_0 = \mathbf{id}$):

$$L_\varepsilon\colon \quad \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \frac{x}{x^2+y^2} \\ \frac{y}{x^2+y^2} \end{pmatrix} \to \begin{pmatrix} \frac{x}{x^2+y^2} + \varepsilon \\ \frac{y}{x^2+y^2} \end{pmatrix} \mapsto \begin{pmatrix} \frac{\frac{x}{x^2+y^2}+\varepsilon}{\left(\frac{x}{x^2+y^2}+\varepsilon\right)^2+\left(\frac{y}{x^2+y^2}\right)^2} \\ \frac{\frac{y}{x^2+y^2}}{\left(\frac{x}{x^2+y^2}+\varepsilon\right)^2+\left(\frac{y}{x^2+y^2}\right)^2} \end{pmatrix}.$$

6.2 Induced metric on surfaces.

Recall here again induced metric (see for detail subsection 1.4)

If surface $M\colon\ \mathbf{r} = \mathbf{r}(u, v)$is embedded in $\mathbf{E}^3$ then induced Riemannian metric G_M is defined by the formulae

$$\langle \mathbf{X}, \mathbf{Y} \rangle = G_M(\mathbf{X}, \mathbf{Y}) = G(\mathbf{X}, \mathbf{Y}) , \tag{6.11}$$

where G is Euclidean metric in $\mathbf{E}^3$:

$$G_M = dx^2 + dy^2 + dz^2\big|_{\mathbf{r}=\mathbf{r}(u,v)} = \sum_{i=1}^{3}(dx^i)^2\big|_{\mathbf{r}=\mathbf{r}(u,v)} = \sum_{i=1}^{3}\left(\frac{\partial x^i}{\partial u^\alpha}du^\alpha\right)^2$$

$$= \frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^i}{\partial u^\beta}du^\alpha du^\beta ,$$

i.e.

$$G_M = g_{\alpha\beta}du^\alpha, \text{ where } g_{\alpha\beta} = \frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^i}{\partial u^\beta}du^\alpha du^\beta .$$

We use notations x, y, z or x^i ($i = 1, 2, 3$) for Cartesian coordinates in $\mathbf{E}^3$, u, v or u^α ($\alpha = 1, 2$) for coordinates on the surface. We usually omit summation symbol over dummy indices. For coordinate tangent vectors

$$\underbrace{\frac{\partial}{\partial u_\alpha}}_{\text{Internal observer}} = \underbrace{\mathbf{r}_\alpha = \frac{\partial x^i}{\partial u^\alpha}\frac{\partial}{\partial x^i}}_{\text{External observer}}$$

We have already plenty examples in the subsection 1.4. In particular for scalar product

$$g_{\alpha\beta} = \left\langle \frac{\partial}{u_\alpha}, \frac{\partial}{u_\beta} \right\rangle = x^i\alpha x^i\beta\,.\langle \mathbf{r}_\alpha, \mathbf{r}_\beta \rangle\,. \tag{6.12}$$

6.2.1 Recalling Weingarten operator

Continue to play with formulae [19].

Recall the Weingarten (shape) operator which acts on tangent vectors:

$$S\mathbf{X} = -\partial_{\mathbf{X}}\mathbf{n}\,, \tag{6.13}$$

where we denote by $\mathbf{n}$-unit normal vector field at the points of the surface M: $\langle \mathbf{n}, \mathbf{r}_\alpha \rangle = 0$, $\langle \mathbf{n}, \mathbf{r}_\alpha \rangle = 1$.

Now we realise that the derivative $\partial_{\mathbf{X}}\mathbf{R}$ of vector field with respect to another vector field is not a well-defined object: we need a connection. The formula $\partial_{\mathbf{X}}\mathbf{R}$ in Cartesian coordinates, is nothing but the derivative with respect to flat canonical connection: If we work only in Cartesian coordinates we do not need to distinguish between $\partial_{\mathbf{X}}\mathbf{R}$ and $\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{R}$. Sometimes with some abuse of notations we will use $\partial_{\mathbf{X}}\mathbf{R}$ instead $\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{R}$, but never forget: this can be done only in Cartesian coordinates where Christoffel symbols of flat canonical connection vanish:

$$\partial_{\mathbf{X}}\mathbf{R} = \nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{R} \quad \text{in Cartesian coordinates}\,.$$

So the rigorous definition of Weingarten operator is

$$S\mathbf{X} = -\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{n}\,, \tag{6.14}$$

but we often use the former one (equation (6.16)) just remembering that this can be done only in Cartesian coordinates.

Recall that the fact that Weingarten operator S maps tangent vectors to tangent vectors follows from the property: $\langle \mathbf{n}, \mathbf{X} \rangle = 0 \Rightarrow \mathbf{X}$ is tangent to the surface.

Indeed:

$$0 = \partial_{\mathbf{X}}\langle \mathbf{n}, \mathbf{n} \rangle = 2\langle \partial_{\mathbf{X}}\mathbf{n}, \mathbf{n} \rangle = -2\langle S\mathbf{X}, \mathbf{n} \rangle = 0 \Rightarrow S\mathbf{X} \text{ is tangent to the surface}$$

Recall also that normal unit vector is defined up to a sign, $\mathbf{n} \to -\mathbf{n}$. On the other hand if $\mathbf{n}$ is chosen then $\mathbf{S}$ is defined uniquely.

We use notations x, y, z or x^i $(i = 1, 2, 3)$ for Cartesian coordinates in $\mathbf{E}^3$, u, v or u^α $(\alpha = 1, 2)$ for coordinates on the surface. We usually omit summation symbol over dummy indices. For coordinate tangent vectors

$$\underbrace{\frac{\partial}{\partial u_\alpha}}_{\text{Internal observer}} = \underbrace{\mathbf{r}_\alpha = \frac{\partial x^i}{\partial u^\alpha}\frac{\partial}{\partial x^i}}_{\text{External observer}}$$

We have already plenty examples in the subsection 1.4. In particular for scalar product

$$g_{\alpha\beta} = \left\langle \frac{\partial}{u_\alpha}, \frac{\partial}{u_\beta} \right\rangle = x^i\alpha x^i\beta\,.\langle \mathbf{r}_\alpha, \mathbf{r}_\beta \rangle\,. \tag{6.15}$$

[19] In some sense differential geometry it is when we write down the formulae expressing the geometrical facts, differentiate these formulae then reveal the geometrical meaning of the new obtained formulae e.t.c.

6.2.2 Recalling Weingarten operator

Continue to play with formulae [20].

Recall the Weingarten (shape) operator which acts on tangent vectors:

$$S\mathbf{X} = -\partial_{\mathbf{X}}\mathbf{n}\,, \tag{6.16}$$

where we denote by $\mathbf{n}$-unit normal vector field at the points of the surface M: $\langle \mathbf{n}, \mathbf{r}_\alpha \rangle = 0$, $\langle \mathbf{n}, \mathbf{r}_\alpha \rangle = 1$.

Now we realise that the derivative $\partial_{\mathbf{X}}\mathbf{R}$ of vector field with respect to another vector field is not a well-defined object: we need a connection. The formula $\partial_{\mathbf{X}}\mathbf{R}$ in Cartesian coordinates, is nothing but the derivative with respect to flat canonical connection: If we work only in Cartesian coordinates we do not need to distinguish between $\partial_{\mathbf{X}}\mathbf{R}$ and $\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{R}$. Sometimes with some abuse of notations we will use $\partial_{\mathbf{X}}\mathbf{R}$ instead $\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{R}$, but never forget: this can be done only in Cartesian coordinates where Christoffel symbols of flat canonical connection vanish:

$$\partial_{\mathbf{X}}\mathbf{R} = \nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{R} \quad \text{in Cartesian coordinates}\,.$$

So the rigorous definition of Weingarten operator is

$$S\mathbf{X} = -\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{n}\,, \tag{6.17}$$

but we often use the former one (equation (6.16)) just remembering that this can be done only in Cartesian coordinates.

Recall that the fact that Weingarten operator S maps tangent vectors to tangent vectors follows from the property: $\langle \mathbf{n}, \mathbf{X} \rangle = 0 \Rightarrow \mathbf{X}$ is tangent to the surface.

Indeed:

$$0 = \partial_{\mathbf{X}}\langle \mathbf{n}, \mathbf{n} \rangle = 2\langle \partial_{\mathbf{X}}\mathbf{n}, \mathbf{n} \rangle = -2\langle S\mathbf{X}, \mathbf{n} \rangle = 0 \Rightarrow S\mathbf{X} \text{ is tangent to the surface}$$

Recall also that normal unit vector is defined up to a sign, $\mathbf{n} \to -\mathbf{n}$. On the other hand if $\mathbf{n}$ is chosen then $\mathbf{S}$ is defined uniquely.

6.2.3 Second quadratic form

We define now the new object: *second quadratic form*

Definition. For two tangent vectors $\mathbf{X}, \mathbf{Y}$ $A(\mathbf{X}, \mathbf{Y})$ is defined such that

$$\left(\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}\right)_\perp = A(\mathbf{X}, \mathbf{Y})\mathbf{n} \tag{6.18}$$

i.e. we take orthogonal component of the derivative of $\mathbf{Y}$ with respect to $\mathbf{X}$.

[20] In some sense differential geometry it is when we write down the formulae expressing the geometrical facts, differentiate these formulae then reveal the geometrical meaning of the new obtained formulae e.t.c.

This definition seems to be very vague: to evaluate covariant derivative we have to consider not a vector $\mathbf{Y}$ at a given point but the vector field. In fact one can see that $A(\mathbf{X},\mathbf{Y})$ does depend only on the value of $\mathbf{Y}$ at the given point.

Indeed it follows from the definition of second quadratic form and from the properties of Weingarten operator that

$$A(\mathbf{X},\mathbf{Y}) = \left\langle \left(\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}\right)_{\perp}, \mathbf{n}\right\rangle = \left\langle \nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}, \mathbf{n}\right\rangle =$$
$$\partial_{\mathbf{X}}\langle \mathbf{Y}, \mathbf{n}\rangle - \left\langle \mathbf{Y}, \nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{n}\right\rangle = \langle S(\mathbf{X}), \mathbf{Y}\rangle \tag{6.19}$$

We proved that second quadratic form depends only on value of vector field $\mathbf{Y}$ at the given poit and we established the relation between second quadratic form and Weingarten operator.

Proposition *The second quadratic form $A(\mathbf{X},\mathbf{Y})$ is symmetric bilinear form on tangent vectors $\mathbf{X},\mathbf{Y}$ in a given point.*

$$A\colon T_{\mathbf{p}}M \otimes T_{\mathbf{p}}M \to \mathbf{R}, \quad A(\mathbf{X},\mathbf{Y}) = A(\mathbf{Y},\mathbf{X}) = \langle S\mathbf{X}, \mathbf{Y}\rangle\,. \tag{6.20}$$

In components

$$A = A_{\alpha\beta}du^\alpha du^\beta = \langle \mathbf{r}_{\alpha\beta}, \mathbf{n}\rangle = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} n^i\,. \tag{6.21}$$

and

$$S^\alpha_\beta = g^{\alpha\pi}A_{\pi\beta} = g^{\alpha\pi}x^i_{\pi\beta}n^i\,, \tag{6.22}$$

i.e.

$$A = GS, S = G^{-1}A\,.$$

Remark The normal unit vector field is defined up to a sign.

6.2.4 Gaussian and mean curvatures

Recall that Gaussian curvature

$$K = \det S$$

and mean curvature

$$H = \operatorname{Tr} S$$

It is easy to see that for Gaussian curvature

$$K = \det S = \det(G^{-1}A) = \frac{\det A}{\det G}.$$

We know already the geometrical meaning of Gaussian and mean curvatures from the point of view of the External Observer:

Gaussian curvature K equals to the product of principal curvatures, and mean curvartures equals to the sum of principal curvatures.

Now we ask a principal question: what bout internal observer, "aunt" living on the surface?

We will show that Gaussian curvature can be expressed in terms of induced Riemannian metric, i.e. it is an internal characteristic of the surface, invariant of isometries.

It is not the case with mean curvature: cylinder is isometric to the plane but it have non-zero mean curvature.

6.2.5 Examples of calculation of Weingarten operator, Second quadratic forms, curvatures for cylinder, cone and sphere.

Cylinder

We already calculated induced Riemannian metric on the cylinder (see (1.46)).

Cylinder is given by the equation $x^2 + y^2 = R^2$. One can consider the following parameterisation of this surface:

$$\mathbf{r}(h,\varphi)\colon \quad \begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \\ z = h \end{cases}, \quad \mathbf{r}_h = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{r}_\varphi = \begin{pmatrix} -R\sin\varphi \\ R\cos\varphi \\ 0 \end{pmatrix}, \tag{6.23}$$

$$G_{cylinder} = \left(dx^2 + dy^2 + dz^2\right)\big|_{x=a\cos\varphi, y=a\sin\varphi, z=h} =$$

$$= (-a\sin\varphi d\varphi)^2 + (a\cos\varphi d\varphi)^2 + dh^2 = a^2 d\varphi^2 + dh^2\,, \quad ||g_{\alpha\beta}|| = \begin{pmatrix} 1 & 0 \\ 0 & R^2 \end{pmatrix}.$$

Normal unit vector $\mathbf{n} = \pm \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$. Choose $\mathbf{n} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$. Weingarten operator

$$S\partial_h = -\nabla^{\text{can.flat}}_{\mathbf{r}_h}\mathbf{n} = -\partial_{\mathbf{r}_h}\mathbf{n} = -\partial_h \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} = 0\,,$$

$$S\partial_\varphi = -\nabla^{\text{can.flat}}_{\mathbf{r}_\varphi}\mathbf{n} = -\partial_{\mathbf{r}_\varphi}\mathbf{n} = -\partial_\varphi \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} \sin\varphi \\ -\cos\varphi \\ 0 \end{pmatrix} = -\frac{\partial_\varphi}{R}.$$

$$S\begin{pmatrix} \partial_h \\ \partial_\varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{-\partial_\varphi}{R} \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 \\ 0 & \frac{-1}{R} \end{pmatrix}. \tag{6.24}$$

Calculate second quadratic form: $\mathbf{r}_{hh} = \partial_h \mathbf{r}_h = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{r}_{h\varphi} = \mathbf{r}_{\varphi h} =$

$$\partial_h \begin{pmatrix} -R\sin\varphi \\ R\cos\varphi \\ 0 \end{pmatrix} = 0,\ \mathbf{r}_{\varphi\varphi} = \partial_\varphi \begin{pmatrix} -R\sin\varphi \\ R\cos\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} -R\cos\varphi \\ -R\sin\varphi \\ 0 \end{pmatrix} = -R\mathbf{n}\,.$$

We have

$$A_{\alpha\beta} = \langle \mathbf{r}_{\alpha\beta}, \mathbf{n}\rangle,\ A = \begin{pmatrix} \langle \mathbf{r}_{hh}, \mathbf{n}\rangle & \langle \mathbf{r}_{h\varphi}, \mathbf{n}\rangle \\ \langle \mathbf{r}_{\varphi h}, \mathbf{n}\rangle & \langle \mathbf{r}_{\varphi\varphi}, \mathbf{n}\rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -R \end{pmatrix}, \tag{6.25}$$

$$A = GS = \begin{pmatrix} 1 & 0 \\ 0 & R^2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{R} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -R \end{pmatrix},$$

For Gaussian and mean curvatures we have

$$K = \det S = \frac{\det A}{\det G} = \det \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{R} \end{pmatrix} = 0\,, \tag{6.26}$$

and mean curvature

$$H = \operatorname{Tr} S = \operatorname{Tr} \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{R} \end{pmatrix} = -\frac{1}{R}\,. \tag{6.27}$$

Mean curvature is define up to a sign. If we change $\mathbf{n} \to -\mathbf{n}$ mean curvature $H \to \frac{1}{R}$ and Gaussian curvature will not change.

Cone

We already calculated induced Riemannian metric on the cone (see (**??**).

Cone is given by the equation $x^2 + y^2 - k^2 z^2 = 0$. One can consider the following parameterisation of this surface:

$$\mathbf{r}(h,\varphi)\colon \quad \begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases}, \quad \mathbf{r}_h = \begin{pmatrix} k\cos\varphi \\ k\sin\varphi \\ 1 \end{pmatrix}, \quad \mathbf{r}_\varphi = \begin{pmatrix} -kh\sin\varphi \\ kh\cos\varphi \\ 0 \end{pmatrix}, \tag{6.28}$$

$$G_{cone} = \left(dx^2 + dy^2 + dz^2\right)\big|_{x=kh\cos\varphi, y=kh\sin\varphi, z=h} =$$

$$= (-kh\sin\varphi d\varphi + k\cos\varphi dh)^2 + (kh\cos\varphi d\varphi + k\sin\varphi dh)^2 + dh^2 =$$

$$k^2h^2 d\varphi^2 + (k^2+1)dh^2, \quad ||g_{\alpha\beta}|| = \begin{pmatrix} k^2+1 & 0 \\ 0 & k^2h^2 \end{pmatrix}.$$

One can see that $\mathbf{N} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix}$ is orthogonal to the surface: $\mathbf{N} \perp \mathbf{r}_h, \mathbf{r}_\varphi$. Hence normal unit vector $\mathbf{n} = \pm\frac{1}{\sqrt{1+k^2}} \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix}$. Choose $\mathbf{n} = \frac{1}{\sqrt{1+k^2}} \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix}$. Weingarten operator

$$S\partial_h = -\nabla^{\text{can.flat}}_{\mathbf{r}_h}\mathbf{n} = -\partial_{\mathbf{r}_h}\mathbf{n} = -\partial_h \left(\frac{1}{\sqrt{1+k^2}} \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix} \right) = 0\,,$$

$$S\partial_\varphi = -\nabla^{\text{can.flat}}_{\mathbf{r}_\varphi}\mathbf{n} = -\partial_{\mathbf{r}_\varphi}\mathbf{n} = -\partial_\varphi \left(\frac{1}{\sqrt{1+k^2}} \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix} \right) =$$

$$\frac{1}{\sqrt{1+k^2}} \begin{pmatrix} \sin\varphi \\ -\cos\varphi \\ 0 \end{pmatrix} = -\frac{\partial_\varphi}{kh\sqrt{k^2+1}}.$$

$$S\begin{pmatrix} \partial_h \\ \partial_\varphi \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{\partial_\varphi}{kh\sqrt{k^2+1}} \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 \\ 0 & \frac{-1}{kh\sqrt{k^2+1}} \end{pmatrix}. \tag{6.29}$$

Calculate second quadratic form: $\mathbf{r}_{hh} = \partial_h \mathbf{r}_h = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{r}_{h\varphi} = \mathbf{r}_{\varphi h} =$

$$\partial_h \begin{pmatrix} -kh\sin\varphi \\ kh\cos\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} -k\sin\varphi \\ k\cos\varphi \\ 0 \end{pmatrix},\ \mathbf{r}_{\varphi\varphi} = \partial_\varphi \begin{pmatrix} -kh\sin\varphi \\ kh\cos\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} -kh\cos\varphi \\ -kh\sin\varphi \\ 0 \end{pmatrix}.$$

We have

$$A_{\alpha\beta} = \langle \mathbf{r}_{\alpha\beta}, \mathbf{n} \rangle,\ A = \begin{pmatrix} \langle \mathbf{r}_{hh}, \mathbf{n} \rangle & \langle \mathbf{r}_{h\varphi}, \mathbf{n} \rangle \\ \langle \mathbf{r}_{\varphi h}, \mathbf{n} \rangle & \langle \mathbf{r}_{\varphi\varphi}, \mathbf{n} \rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{kh}{\sqrt{1+k^2}} \end{pmatrix}, \tag{6.30}$$

$$A = GS = \begin{pmatrix} k^2+1 & 0 \\ 0 & k^2h^2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \frac{-1}{kh\sqrt{k^2+1}} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{-kh}{\sqrt{k^2+1}} \end{pmatrix},$$

For Gaussian and mean curvatures we have

$$K = \det S = \frac{\det A}{\det G} = \det \begin{pmatrix} 0 & 0 \\ 0 & \frac{-1}{kh\sqrt{k^2+1}} \end{pmatrix} = 0\,, \tag{6.31}$$

and mean curvature

$$H = \operatorname{Tr} S = \operatorname{Tr} \begin{pmatrix} 0 & 0 \\ 0 & \frac{-1}{kh\sqrt{k^2+1}} \end{pmatrix} = \frac{-1}{kh\sqrt{k^2+1}}\,. \tag{6.32}$$

Mean curvature is define up to a sign. If we change $\mathbf{n} \to -\mathbf{n}$ mean curvature $H \to \frac{1}{R}$ and Gaussian curvature will not change.

Sphere

Sphere is given by the equation $x^2 + y^2 + z^2 = a^2$. Consider the parameterisation of sphere in spherical coordinates

$$\mathbf{r}(\theta, \varphi)\colon \quad \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases} \tag{6.33}$$

We already calculated induced Riemannian metric on the sphere (see (6.2.5)). Recall that

$$\mathbf{r}_\theta = \begin{pmatrix} R\cos\theta\cos\varphi \\ R\cos\theta\sin\varphi \\ -R\sin\theta \end{pmatrix}, \quad \mathbf{r}_\varphi = \begin{pmatrix} -R\sin\theta\sin\varphi \\ R\sin\theta\cos\varphi \\ 0 \end{pmatrix}$$

and

$$G_{S^2} = \left(dx^2 + dy^2 + dz^2\right)\big|_{x=R\sin\theta\cos\varphi, y=R\sin\theta\sin\varphi, z=R\cos\theta} =$$

$$(R\cos\theta\cos\varphi d\theta - R\sin\theta\sin\varphi d\varphi)^2 + (R\cos\theta\sin\varphi d\theta + R\sin\theta\cos\varphi d\varphi)^2 +$$

$$(-R\sin\theta d\theta)^2 = R^2\cos^2\theta d\theta^2 + R^2\sin^2\theta d\varphi^2 + R^2\sin^2\theta d\theta^2 =$$

$$= R^2 d\theta^2 + R^2 \sin^2\theta d\varphi^2\,, \qquad ||g_{\alpha\beta}|| = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix}\,.$$

For the sphere $\mathbf{r}(\theta,\varphi)$ is orthogonal to the surface. Hence normal unit vector $\mathbf{n}(\theta,\varphi) = \pm\frac{\mathbf{r}(\theta,\varphi)}{R} = \pm\begin{pmatrix}\sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta\end{pmatrix}$. Choose $\mathbf{n} = \frac{\mathbf{r}}{R} = \begin{pmatrix}\sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta\end{pmatrix}$. Weingarten operator

$$S\partial_\theta = -\nabla^{\text{can.flat}}_{\mathbf{r}_\theta}\mathbf{n} = -\partial_\theta\mathbf{n} = -\partial_\theta\left(\frac{\mathbf{r}}{R}\right) = -\frac{\mathbf{r}_\theta}{R}\,,$$

$$S\partial_\varphi = -\nabla^{\text{can.flat}}_{\mathbf{r}_\varphi}\mathbf{n} = -\partial_\varphi\mathbf{n} = -\partial_\varphi\left(\frac{\mathbf{r}}{R}\right) = -\frac{\mathbf{r}_\varphi}{R}\,.$$

$$S\begin{pmatrix}\partial_\theta \\ \partial_\varphi\end{pmatrix} = \begin{pmatrix}-\frac{\partial_\theta}{R} \\ -\frac{\partial_\varphi}{R}\end{pmatrix}, \quad S = -\begin{pmatrix}\frac{1}{R} & 0 \\ 0 & \frac{1}{R}\end{pmatrix}\,. \tag{6.34}$$

For second quadratic form: $\mathbf{r}_{\theta\theta} = \partial_\theta\mathbf{r}_\theta = \begin{pmatrix}-R\sin\theta\cos\varphi \\ -R\sin\theta\sin\varphi \\ -R\cos\theta\end{pmatrix}$, $\mathbf{r}_{\theta\varphi} = \mathbf{r}_{\varphi\theta} =$

$\partial_\theta\begin{pmatrix}-R\sin\theta\sin\varphi \\ R\sin\theta\cos\varphi \\ 0\end{pmatrix} = \begin{pmatrix}-R\cos\theta\sin\varphi \\ R\cos\theta\cos\varphi \\ 0\end{pmatrix}$, $\mathbf{r}_{\varphi\varphi} = \partial_\varphi\begin{pmatrix}-R\sin\theta\sin\varphi \\ R\sin\theta\cos\varphi \\ 0\end{pmatrix} = \begin{pmatrix}-R\sin\theta\cos\varphi \\ -R\sin\theta\sin\varphi \\ 0\end{pmatrix}$.

We have

$$A_{\alpha\beta} = \langle\mathbf{r}_{\alpha\beta},\mathbf{n}\rangle,\ A = \begin{pmatrix}\langle\mathbf{r}_{\theta\theta},\mathbf{n}\rangle & \langle\mathbf{r}_{\theta\varphi},\mathbf{n}\rangle \\ \langle\mathbf{r}_{\varphi\theta},\mathbf{n}\rangle & \langle\mathbf{r}_{\varphi\varphi},\mathbf{n}\rangle\end{pmatrix} = \begin{pmatrix}-R & 0 \\ 0 & -R\sin^2\theta\end{pmatrix}, \tag{6.35}$$

$$A = GS = \begin{pmatrix}R^2 & 0 \\ 0 & R^2\sin^2\theta\end{pmatrix}\begin{pmatrix}\frac{-1}{R} & 0 \\ 0 & \frac{-1}{R}\end{pmatrix} = -R\begin{pmatrix}1 & 0 \\ 0 & \sin^2\theta\end{pmatrix},$$

For Gaussian and mean curvatures we have

$$K = \det S = \frac{\det A}{\det G} = \det\begin{pmatrix}-\frac{1}{R} & 0 \\ 0 & -\frac{1}{R}\end{pmatrix} = \frac{1}{R^2}\,, \tag{6.36}$$

and mean curvature

$$H = \operatorname{Tr} S = \operatorname{Tr}\begin{pmatrix}-\frac{1}{R} & 0 \\ 0 & -\frac{1}{R}\end{pmatrix} = -\frac{2}{R}\,, \tag{6.37}$$

Mean curvature is define up to a sign. If we change $\mathbf{n}\to-\mathbf{n}$ mean curvature $H\to\frac{1}{R}$ and Gaussian curvature will not change.

We see that for the sphere Gaussian curvature is not equal to zero, whilst for cylinder and cone Gaussian curvature equals to zero.

6.2.6 Straightforward proof of the Proposition (5.25)

We prove this fact by direct calculations. The plan of calculations is following:

Let M be a surface in $\mathbf{E}^3$ For an arbitrary point $\mathbf{p}$ of the surface M we consider Cartesian coordinates x, y, z such that orgin coincides with the point $\mathbf{p}$ and coordinate plane OXY is the palne attached at the surface at the point $\mathbf{p}$ and the axis OZ is orthogonal to the surface. In these coordinates calculations become easy. The surface M in these Cartesian coordinates can be expressed by the equation

$$\begin{cases} x = u \\ y = v \\ z = F(u,v) \end{cases} \tag{6.38}$$

where $F(u,v)$ has local extremum at the point $u = v = 0$[21].

Calculate explicitly Gaussian curvature at the point $\mathbf{p}$. In a vicinity of the point $\mathbf{p}$ basis vector $\mathbf{r}_u = \begin{pmatrix} 1 \\ 0 \\ F_u \end{pmatrix}$, basis vector $\mathbf{r}_v = \begin{pmatrix} 0 \\ 1 \\ F_v \end{pmatrix}$. One can see that

$$\mathbf{n}(u,v) = \frac{1}{1 + F_u^2 + F_v^2} \begin{pmatrix} -F_u \\ -F_v \\ 1 \end{pmatrix} \tag{6.39}$$

is unit normal vector. In particular in the origin at the point $\mathbf{p}$, we have

$$\mathbf{r}_u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{r}_v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{ and } \mathbf{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{6.40}$$

Now calculate shape operator. We need this operator only at the origin, hence during calculations of all derivatives we have to note that we need the final result only at the point $u = v = 0$. This will essentially simplify calculations since at the point $\mathbf{p}$ derivatives F_u, F_v equal to zero.

$$S\mathbf{r}_u = -\partial_u \mathbf{n}(u,v)|_{\mathbf{p}} = -\frac{\partial}{\partial u}\left(\frac{1}{1 + F_u^2 + F_v^2}\begin{pmatrix} -F_u \\ -F_v \\ 1 \end{pmatrix}\right)|_{u=v=0} =$$

$$\begin{pmatrix} F_{uu} \\ -F_{vu} \\ 1 \end{pmatrix}|_{u=v=0} = F_{uu}\mathbf{r}_u + F_{uv}\mathbf{r}_v$$

and

$$S\mathbf{r}_v = -\partial_v \mathbf{n}(u,v)|_{\mathbf{p}} = -\frac{\partial}{\partial v}\left(\frac{1}{1 + F_u^2 + F_v^2}\begin{pmatrix} -F_u \\ -F_v \\ 1 \end{pmatrix}\right)|_{u=v=0} =$$

[21]more in detail this is stationary point. It is local extremum if quadratic form corresponding to second differential is positive or negative definite, i.e. $F_{uu}F_{vv} - F_{uv}^2$ keeps the sign. This means that the Gaussian curvature is not negative

$$\begin{pmatrix} F_{uv} \\ -F_{vv} \\ 1 \end{pmatrix}|_{u=v=0} = F_{uv}\mathbf{r}_u + F_{vv}\mathbf{r}_v$$

since $F_u = F_v = 0$. We see that at the origin the Shape (Weingarten) operator equals to

$$S = \begin{pmatrix} F_{uu} & F_{uv} \\ F_{vu} & F_{vv} \end{pmatrix} \tag{6.41}$$

(all the derivatives at the origin).

Gaussian curvature at the point $\mathbf{p}$ equals to

$$K = \det S = F_{uu}F_{vv} - F_{uv}^2 \,. \tag{6.42}$$

(all the derivatives at the origin).

Now it is time to calculate the Riemann curvature tensor at the origin.

First of all recall the expression for Riemannian metric for the surface M in a vicinity of origin is

$$G = \begin{pmatrix} \langle \mathbf{r}_u, \mathbf{r}_u \rangle & \langle \mathbf{r}_u, \mathbf{r}_v \rangle \\ \langle \mathbf{r}_v, \mathbf{r}_u \rangle & \langle \mathbf{r}_v, \mathbf{r}_v \rangle \end{pmatrix} = \begin{pmatrix} 1 + F_u^2 & F_uF_v \\ F_vF_u & 1 + F_v^2 \end{pmatrix} \,. \tag{6.43}$$

This immediately follows from the expression for basic vectors $\mathbf{r}_u, \mathbf{r}_v$

Note that Riemannian metric g_{ik} at the point $u = v = 0$ is defined by unity matrix $g_{uu} = g_{vv} = 1$, $g_{uv} = g_{vu} = 0$ since $\mathbf{p}$ is extremum point: $G = \begin{pmatrix} 1 + F_u^2 & F_uF_v \\ F_vF_u & 1 + F_v^2 \end{pmatrix}|_{\mathbf{p}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ since $\mathbf{p}$ is stationary point, extremum ($F_u = F_v = 0$). Hence the components of the tensor $R^i{}_{kmn}$ and $R_{ikmn} = g_{ij}R^j{}_{kmn}$ at the point $\mathbf{p}$ are the same.

Recall that the components of $R^i{}_{kmn}$ are defined by the formula

$$R^i{}_{kmn} = \partial_m \Gamma^i_{nk} + \Gamma^i_{mp}\Gamma^p_{nk} - \partial_n \Gamma^i_{mk} - \Gamma^i_{np}\Gamma^p_{mk} \,.$$

Notice that at the point $\mathbf{p}$ not only the matrix of the metric g_{ik} equals to unity matrix, but more: Christoffel symbols vanish at this point in coordinates u, v since the derivatives of metric at this point vanish. (Why they vanish: this immediately follows from Levi-Civita formula applied to the metric (6.43), see also in detail the file "The solution of the problem 5 in the coursework, revisited"). Hence to calculate R^i_{kmn} at the point $\mathbf{p}$ one can consider more simple formula:

$$R^i{}_{kmn}|_{\mathbf{p}} = \partial_m \Gamma^i_{nk}|_{\mathbf{p}} - \partial_n \Gamma^i_{mk}|_{\mathbf{p}}$$

Try to calculate in a more "economical" way. Due to Levi-Civita formula

$$\Gamma^i_{mk} = \frac{1}{2} g^{ij} \left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^m} - \frac{\partial g_{mk}}{\partial x^j} \right)$$

Since metric g_{ik} equals to unity matrix $g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ at the point $\mathbf{p}$ hence g^{ij} is unity matrix also:

$$g^{ik}|_{\mathbf{p}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \delta^{ik} \,.$$

(We denote δ^{ik} the unity matrix: all diagonal components equal to 1, all other components equal to zero. (It is so called Kronecker symbols)) Moreover we know also that all the first derivatives of the metric vanish at the point $\mathbf{p}$:

$$\frac{\partial g_{ik}}{\partial x^m}|_{\mathbf{p}} = 0\,.$$

Hence it follows from the formulae above that for an arbitrary indices i, j, k, m, n

$$\frac{\partial}{\partial x^i}\left(g^{km}\frac{\partial g_{pr}}{\partial x^j}\right)\big|_{\mathbf{p}} = \frac{\partial g^{km}}{\partial x^i}\big|_{\mathbf{p}}\frac{\partial g_{pr}}{\partial x^j}\big|_{\mathbf{p}} + g^{km}\big|_{\mathbf{p}}\frac{\partial^2 g_{pr}}{\partial x^i\partial x^j}\big|_{\mathbf{p}} = \delta^{km}\frac{\partial^2 g_{pr}}{\partial x^i\partial x^j}\big|_{\mathbf{p}}\,.$$

Now using the Levi-Civita formula for the Christoffel symbols of connection:

$$\Gamma^i_{mk} = \frac{1}{2}g^{ij}\left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^m} - \frac{\partial g_{mk}}{\partial x^j}\right)$$

we come to $\partial_n\Gamma^i_{mk}|_{\mathbf{p}} = \frac{\partial}{\partial x^n}\left(\frac{1}{2}g^{ij}\left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^m} - \frac{\partial g_{mk}}{\partial x^j}\right)\right)\big|_{\mathbf{p}} =$

$$\frac{1}{2}\delta^{ij}\left(\frac{\partial^2 g_{jm}}{\partial x^n\partial x^k} + \frac{\partial^2 g_{jk}}{\partial x^n\partial x^m} - \frac{\partial^2 g_{mk}}{\partial x^n\partial x^j}\right)\big|_{\mathbf{p}}\,. \tag{6.44}$$

Now using this formula we are ready to calculate Riemann curvature tensor $R^i{}_{kmn}$. Remember that it is enough to calculate $R^1{}_{212}$ and $R^1{}_{212} = R_{1212}$ at the point $\mathbf{p}$ since $g_{ik} = \delta_{ik}$ at the point $\mathbf{p}$. We have that at $\mathbf{p}$ $R^1{}_{212}|_{\mathbf{p}} = \partial_1\Gamma^1_{22}|_{\mathbf{p}} - \partial_2\Gamma^1_{12}|_{\mathbf{p}}$