Homework 5

1 Consider the following curves:

$C_1\colon \mathbf{r}(t)\ \begin{cases} x = t \\ y = 2t^2 - 1 \end{cases},\ 0 < t < 1,$ $\qquad C_2\colon \mathbf{r}(t)\ \begin{cases} x = t \\ y = 2t^2 - 1 \end{cases},\ -1 < t < 1,$

$C_3\colon \mathbf{r}(t)\ \begin{cases} x = 2t \\ y = 8t^2 - 1 \end{cases},\ 0 < t < \frac{1}{2},$ $\qquad C_4\colon \mathbf{r}(t)\ \begin{cases} x = \cos t \\ y = \cos 2t \end{cases},\ 0 < t < \frac{\pi}{2},$

$C_5\colon \mathbf{r}(t)\ \begin{cases} x = t \\ y = 2t - 1 \end{cases},\ 0 < t < 1,$ $\qquad C_6\colon \mathbf{r}(t)\ \begin{cases} x = 1 - t \\ y = 1 - 2t \end{cases},\ 0 < t < 1,$

$C_7\colon \mathbf{r}(t)\ \begin{cases} x = \sin^2 t \\ y = -\cos 2t \end{cases},\ 0 < t < \frac{\pi}{2},$ $\qquad C_8\colon \mathbf{r}(t)\ \begin{cases} x = t \\ y = \sqrt{1 - t^2} \end{cases},\ -1 < t < 1,$

$C_9\colon \mathbf{r}(t)\ \begin{cases} x = \cos t \\ y = \sin t \end{cases},\ 0 < t < \pi,$ $\qquad C_{10}\colon \mathbf{r}(t)\ \begin{cases} x = \cos 2t \\ y = \sin 2t \end{cases},\ 0 < t < \frac{\pi}{2},$

$C_{11}\colon \mathbf{r}(t)\ \begin{cases} x = \cos t \\ y = \sin t \end{cases},\ 0 < t < 2\pi,$ $\qquad C_{12}\colon \mathbf{r}(t)\ \begin{cases} x = a\cos t \\ y = b\sin t \end{cases},\ 0 < t < 2\pi$ (ellipse),

Draw the images of these curves.

Write down their velocity vectors.

Indicate parameterised curves which have the same image (equivalent curves).

In each equivalence class of parameterised curves indicate curves with same and opposite orientations.

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2 Consider the following curve (helix): $\mathbf{r}(t)\colon\ \begin{cases} x(t) = R\cos\Omega t \\ y(t) = R\sin\Omega t \\ z(t) = ct \end{cases},\qquad 0 \le t \le t_0.$

Show that the image of this curve belongs to the surface of cylinder $x^2 + y^2 = R^2$.

Find the velocity vector of this curve.

Find the length of this curve.

Finish the following sentence:

After developing the surface of cylinder to the plane the curve will develop to the...

3 Consider differential forms $\omega = xdy - ydx$, $\sigma = xdx + ydy$ and vector fields $\mathbf{A} = x\partial_x + y\partial_y$, $\mathbf{B} = x\partial_y - y\partial_x$.

a) Calculate $\omega(\mathbf{A}), \omega(\mathbf{B})$, $\sigma(\mathbf{A}), \sigma(\mathbf{B})$.

b) Calculate differential forms ω and σ in polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$.

4 Consider differential forms $\omega = xdy - ydx$ and $\sigma = xdx + ydy + zdz$ in $\mathbf{E}^3$. Calculate $\omega(\mathbf{v})$ and $\sigma(\mathbf{v})$ on the velocity vectors of helix considered in question 3).

5 Calculate the derivatives of the functions $f = x^2 + y^2$, $g = y^2 - x^2$ and $h = q\log|r| = q\log\left(\sqrt{x^2 + y^2}\right)$ (q is a constant) along vector fields $\mathbf{A} = x\partial_x + y\partial_y$ and $\mathbf{B} = x\partial_y - y\partial_x$

a) calculating directional derivatives $\partial_{\mathbf{A}} f, \partial_{\mathbf{A}} g, \partial_{\mathbf{A}} h$, $\partial_{\mathbf{B}} f, \partial_{\mathbf{B}} g, \partial_{\mathbf{B}} h$

b) calculating $df(\mathbf{A}), dg(\mathbf{A}), dh(\mathbf{A}), df(\mathbf{B}), dg(\mathbf{B}), dh(\mathbf{B})$.

6 Consider a function $f = x^4 - y^4$.
Calculate the value of 1-form $\omega = df$ on the vector field $\mathbf{B} = x\partial_y - y\partial_x$.

7 Let f be a function on $\mathbf{E}^2$ given by $f(r, \varphi) = r^3 \cos 3\varphi$, where r, φ are polar coordinates in $\mathbf{E}^2$.
Calculate the 1-form $\omega = df$.
Calculate the value of the 1-form $\omega = df$ on the vector field $\mathbf{X} = r\partial_r + \partial_\varphi$.
Express the 1-form ω in Cartesian coordinates x, y[1)]

8 Consider a function $f = x^4 - y^4$.
Calculate the value of 1-form $\omega = df$ on the vector field $\mathbf{B} = x\partial_y - y\partial_x$.
Express this 1-form ω in polar coordinates r, φ ($x = r\cos\varphi, y = r\sin\varphi$).
Show that $\mathbf{B} = \frac{\partial}{\partial\varphi}$ in polar coordinates.

We call 1-form ω exact if there exists a function F such that $\omega = dF$
9 Show that 1-form $\omega = xdy + ydx$ is exact.
Show that 1-form $\omega = \sin y dx + x \cos y dy$ is exact.

[1)] You may use the fact that $\cos 3\varphi = 4\cos^3\varphi - 3\cos\varphi$.