Homework 1. Solutions

1 *Let* $\mathbf{R}^2$ *be an affine space of points. Consider in* $\mathbf{R}^2$ *points* $A=(2,3)$ *and* $B=(6,6)$.

a) Find the length of the segment AB

b) Find a point C *in* $\mathbf{R}^2$ *such that vector* AC *has unit length and it is orthogonal to the vector* $\vec{AB}$

$|AB|=\sqrt{(6-2)^2+(6-3)^2}=5$.

Let point $C=(x,y)$ Consider vectors $\vec{AB}=(6-2,6-3)=(4,3)$ and $\vec{AC}=(x-2,y-3)$. Thes vectors are orthogonal:

$$(\vec{AB},\vec{AC})=\vec{AB}\cdot\vec{AC}=4(x-2)+3(y-3)=0\,.$$

Hence $x-2:y-3=-3:4$, i.e. $x-2=-3t, y-3=4t$ where t is arbitrary parameter. Since vector AC is unit vector then

$$|AC|^2=(x-2)^2+(y-3)^2=9t^2+16t^2=25t^2=1\Rightarrow t=\pm\frac{1}{5}$$

Thus $AC=\pm\left(-\frac{3}{5},\frac{4}{5}\right)$, and point C has coordinates

$$C=A+\vec{AC}=(2,3)\pm\left(-\frac{3}{5},\frac{4}{5}\right)$$

We have two solutions: $C=\left(1\frac{2}{5},3\frac{4}{5}\right)$ or $C=\left(2\frac{3}{5},2\frac{1}{5}\right)$.

2 *In affine space* $\mathbf{R}^2$ *consider points* $A=(2,1)$, $B=(2+a,1+b)$ *and* $C=(2+p,1+q)$, *where* a,b,p,q,r *are arbitrary parameters.*

Calculate the area of the triangle $\triangle ABC$ *and compare the answer with determinant of the matrix* $\begin{pmatrix} a & b\\ p & q\end{pmatrix}$.

Do it first using "brute force". Let $\varphi=\angle BAC$, let h be the length of the hight from the vertex C. Then

$$\text{Area of}\triangle ABC=\frac{1}{2}|AB||AC||\sin\angle BAC|=\frac{1}{2}\sqrt{a^2+b^2}\sqrt{p^2+q^2}\sqrt{1-\cos^2\angle BAC}=$$

$$\frac{1}{2}\sqrt{a^2+b^2}\sqrt{p^2+q^2}\sqrt{1-\left(\frac{\vec{AB}\cdot\vec{AC}}{|AB||AC|}\right)^2}=\frac{1}{2}\sqrt{a^2+b^2}\sqrt{p^2+q^2}\sqrt{1-\frac{(ap+bq)^2}{(a^2+b^2)(p^2+q^2)}}=$$

$$\frac{1}{2}\sqrt{(a^2+b^2)(p^2+q^2)-(ap+bq)^2}=$$

$$\frac{1}{2}\sqrt{(a^2q^2+b^2p^2-2abpq}=(aq-bp)^2=\frac{1}{2}|aq-bp|=\frac{1}{2}\left|\det\begin{pmatrix} a & b\\ p & q\end{pmatrix}\right|$$

Sure this beautiful property is not occasional:

$$\text{Area of}\triangle ABC = \frac{1}{2}\left|\vec{AC}\times\vec{AB}\right|$$

(We will discuss it later when will consider vector product*.)

Another

Remark Notice that during this 'not very clever' solution we come to the formula:

$$(a^2+b^2)(p^2+q^2) = (ap+bq)^2(aq-bp)^2$$

This is famous Lagrange identity. This identity implies that the set of natural numbers which are summ of two squares is closed with respect to multiplication.

3 *Let $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$ be an orthonormal basis in 3-dimensional Euclidean space $\mathbf{E}^3$.*

Let $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ be the row of three vectors. and let A be the transistion matrix from the basis $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$ to the row $\{\mathbf{a},\mathbf{b},c\}$:

$$\{\mathbf{a},\mathbf{b},c\} = \{\mathbf{e},\mathbf{f},\mathbf{g}\}A\,.$$

Consider the cases

$$\text{a) } A = \begin{pmatrix} 5 & 0 & 1 \\ 0 & 1 & 2 \\ 3 & 1 & 7 \end{pmatrix}, \qquad \text{b) } A = \frac{1}{2}\begin{pmatrix} \sqrt{3} & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -\sqrt{3} \end{pmatrix}, \qquad \text{c) } A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Show that in the case a) the row $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ is a basis. Show that this basis is not an arthonormal basis.

Show that in the case b) the row $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ is a basis, and this is the orthonormal basis. Find in this case the transition matrix from the basis $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ to the initial orthonormal basis $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$.

Show that in the case c) the row $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ is not a basis.

a) the transition matrix A is non-degenerate, since $\det A = 5\cdot 5 - 3 = 22 \neq 0$. Hence the row $(\mathbf{a},\mathbf{b},\mathbf{c})$ is a basis. This basis is not orthonormal since the initial basis is orthonormal basis, and the transition matrix is not an orthogonal matrix (e.g. its first row has not unit length).

b) One can see directly that the transition matrix A is orthogonal matrix. Hence the row $(\mathbf{a},\mathbf{b},\mathbf{c})$ is an orthonormal basis since initial basis was orthonormal.

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* Of course this solution is evident if you know the properties of vector product (cross-product) Our aim was to do it straightforwardly

It is useful to see that

$$A=\frac{1}{2}\begin{pmatrix}\sqrt{3} & 0 & 1\\ 0 & -2 & 0\\ 1 & 0 & -\sqrt{3}\end{pmatrix}=\begin{pmatrix}\frac{\sqrt{3}}{2} & 0 & \frac{1}{2}\\ 0 & -1 & 0\\ \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2}\end{pmatrix}=\begin{pmatrix}\cos\frac{\pi}{6} & 0 & \sin\frac{\pi}{6}\\ 0 & -1 & 0\\ \sin\frac{\pi}{6} & 0 & -\cos\frac{\pi}{6}\end{pmatrix}$$

The transition matrix from the new orthonormal basis to the former basis is the matrix inverse to A, and since A is orthogonal matrix,and self-conjugate, then it is the same matrix:

$$A^{-1}=A^T=\frac{1}{2}\begin{pmatrix}\sqrt{3} & 0 & 1\\ 0 & -2 & 0\\ 1 & 0 & -\sqrt{3}\end{pmatrix}^T=\frac{1}{2}\begin{pmatrix}\sqrt{3} & 0 & 1\\ 0 & -2 & 0\\ 1 & 0 & -\sqrt{3}\end{pmatrix}=A\,.$$

c) the transition matrix A is degenerate, since

$$\det A=\det\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix}=(5\cdot 9-8\cdot 6)-2(4\cdot 9-7\cdot 6)+3(4\cdot 8-7\cdot 5)=-3-2\cdot(-6)+3\cdot(-3)=-3+12-9=$$

Hence vectors $\mathbf{a},\mathbf{b},\mathbf{c}$ are linearly dependent and the row $(\mathbf{a},\mathbf{b},\mathbf{c})$ is not a basis. (e.g. its first row has not unit length) and initial basis was orthogonal.

4 *Let P be a linear operator in 2-dimensional vector space V. Let $\{\mathbf{e}_1,\mathbf{e}_2\}$ be a basis in V such that*

$$P(\mathbf{e}_1)=7\mathbf{e}_1+9\mathbf{e}_2\,,P(\mathbf{e}_2)=2\mathbf{e}_1+3\mathbf{e}_2\,.$$

Consider in V the new bases $\{\mathbf{f}_1,\mathbf{f}_2\}$ and $\mathbf{g}_1,\mathbf{g}_2$ such that

$$\mathbf{f}_1=\frac{1}{2}\mathbf{e}_1\,,\quad \mathbf{f}_2=3\mathbf{e}_2\qquad \text{and}\quad \mathbf{g}_1=\mathbf{e}_1+\mathbf{e}_2\,,\quad \mathbf{g}_2=\mathbf{e}_2$$

($\{\mathbf{f}_1,\mathbf{f}_2\}$ and $\mathbf{g}_1,\mathbf{g}_2$ are bases since evidently vectors $\mathbf{f}_1,\mathbf{f}_2$ are linearly independent, and vectors $\mathbf{g}_1,\mathbf{g}_2$ are linearly idenpendent.)

Write down the matrices of the operator P in the basis $\{\mathbf{e}_1,\mathbf{e}_2\}$, and in the new bases $\{\mathbf{f}_1,\mathbf{f}_2\}$ and $\{\mathbf{g}_1,\mathbf{g}_2\}$.

Matrix of operator P in the basis $\{\mathbf{e}_1,\mathbf{e}_2\}$ is $\begin{pmatrix}7 & 2\\ 9 & 3\end{pmatrix}$ since $\begin{cases}\mathbf{e}_1'=P(\mathbf{e}_1)=7\mathbf{e}_1+9\mathbf{e}_2\\ \mathbf{e}_2'=P(\mathbf{e}_2)=2\mathbf{e}_1+3\mathbf{e}_2\end{cases}$. This is the transition matrix from the basis $\{e_1,\mathbf{e}_2\}$ to the row of vectors $\{\mathbf{e}_1',\mathbf{e}_2'\}$.

Now for the new bases we have

$$\begin{cases}\mathbf{f}_1=\frac{\mathbf{e}_1}{2}\\ \mathbf{f}_2=3\mathbf{e}_2\end{cases}\Leftrightarrow\begin{cases}\mathbf{e}_1=2\mathbf{f}_1\\ \mathbf{e}_2=\frac{1}{3}\mathbf{f}_2\end{cases},\quad \text{and respectively}\quad \begin{cases}\mathbf{g}_1=\mathbf{e}_1+\mathbf{e}_2\\ \mathbf{g}_2=\mathbf{e}_2\end{cases}\Leftrightarrow\begin{cases}\mathbf{e}_1=\mathbf{g}_1-\mathbf{g}_2\\ \mathbf{e}_2=\mathbf{g}_2\end{cases}.$$

Hence

$$P(\mathbf{f}_1)=P\left(\frac{\mathbf{e}_1}{2}\right)=\frac{1}{2}P(\mathbf{e}_1)=\frac{7}{2}\mathbf{e}_1+\frac{9}{2}\mathbf{e}_2=7\mathbf{f}_1+\frac{3}{2}\mathbf{f}_2\,,$$

$$P(\mathbf{f}_2) = P(3\mathbf{e}_1) = 3P(\mathbf{e}_2) = 6\mathbf{e}_1 + 9\mathbf{e}_2 = 12\mathbf{f}_1 + 3\mathbf{f}_2\,,$$

i.e. the matrix of operator P in the basis $\{\mathbf{f}_1, \mathbf{f}_2\}$ is the matrix $\begin{pmatrix} 7 & 12 \\ \frac{3}{2} & 3 \end{pmatrix}$,

and respectively

$$P(\mathbf{g}_1) = P(\mathbf{e}_1 + \mathbf{e}_2) = (7\mathbf{e}_1 + 9\mathbf{e}_2) + (2\mathbf{e}_1 + 3\mathbf{e}_2) = 9\mathbf{e}_1 + 12\mathbf{e}_2 = 9(\mathbf{g}_1 - \mathbf{g}_2) + 12\mathbf{g}_2 = 9\mathbf{g}_1 + 3\mathbf{g}_2$$

$$P(\mathbf{g}_2) = P(\mathbf{e}_2) = 2\mathbf{e}_1 + 3\mathbf{e}_2 = 2(\mathbf{g}_1 - \mathbf{g}_2) + 3\mathbf{g}_2 = 2\mathbf{g}_1 + \mathbf{g}_2\,.$$

i.e. the matrix of operator P in the basis $\{\mathbf{g}_1, \mathbf{g}_2\}$ is the matrix $\begin{pmatrix} 9 & 3 \\ 2 & 1 \end{pmatrix}$.

5 *Let A be a linear operator in 2-dimensional vector space V such that for a given basis $\{\mathbf{e}, \mathbf{f}\}$,*

$$A(\mathbf{e}) = 27\mathbf{e} + 40\mathbf{f}\,, A(\mathbf{f}) = -16\mathbf{e} - \frac{71}{3}\mathbf{f}\,.$$

Write down the matrix of the operator A in this basis.

Consider the pair of vectors $\{\mathbf{e}', \mathbf{f}'\}$ such that $\mathbf{e}' = 2\mathbf{e} + 3\mathbf{f}$ and $\mathbf{f}' = 3\mathbf{e} + 5\mathbf{f}$.

Show that these vectors arre eigenvectors of linear operator A.

Show that an ordered set of vectors $\{\mathbf{e}', \mathbf{f}'\}$ is also a basis, and find a matrix of the operator A in the new basis.

Calculate the determinant and trace of operator A (compare determinants and traces of different matrix representations of this operator.)

We have that for operator A,

$$\{\mathbf{e}', \mathbf{f}'\} = \{\mathbf{e}, \mathbf{f}\} \begin{pmatrix} 27 & -16 \\ 40 & -\frac{71}{3} \end{pmatrix}$$

Hence matrix of operator A in the basis $\{\mathbf{e}, \mathbf{f}\}$ is $\begin{pmatrix} 27 & -16 \\ 40 & -\frac{71}{3} \end{pmatrix}$.

Vectors $\mathbf{e}', \mathbf{f}'$ are linearly independent. Indeed

$$0 = c_1\mathbf{e}' + c_2\mathbf{f}' = c_1(2\mathbf{e} + 3\mathbf{f}) + c_2(3\mathbf{e} + 5\mathbf{f}) = (2c_1 + 3c_2)\mathbf{e} + (3c_1 + 5c_2)\mathbf{f} = 0\,.$$

Hence $2c_1 + 3c_2 = 0, 3c_1 + 5c_2 = 0$, i.e. $c_1 = c_2 = 0$.

Hence $\{\mathbf{e}', \mathbf{f}'\}$ is a basis also.

Sure the fact that $\{e', \mathbf{f}'\}$ is alos the basis follows from the fact that the matrix is non degenerate: $\det T = 1$.

We have that

$$\begin{cases} \mathbf{e}' = 2\mathbf{e} + 3\mathbf{f} \\ \mathbf{f}' = 3\mathbf{e} + 5\mathbf{f} \end{cases} \qquad \text{hence} \qquad \begin{cases} \mathbf{e} = 5\mathbf{e} - 3\mathbf{f} \\ \mathbf{f} = -3\mathbf{e} + 2\mathbf{f} \end{cases}$$

We have that for basis

$$A(\mathbf{e}') = A(2\mathbf{e}+3\mathbf{f}) = 2(27\mathbf{e}+40\mathbf{f}) + 3\left(-16\mathbf{e} - \frac{71}{3}\mathbf{f}\right) = 6\mathbf{e}+9\mathbf{f} = 3\mathbf{e}',$$

$$A(\mathbf{f}') = A(3\mathbf{e}+5\mathbf{f}) = 3(27\mathbf{e}+40\mathbf{f}) + 5\left(-16\mathbf{e} - \frac{71}{3}\mathbf{f}\right) = \mathbf{e}+\frac{5}{3}\mathbf{f} = \frac{1}{3}\mathbf{f}'.$$

These linear independent vectors are *eigenvectors* of the operator A.

We see that the matrix of operator A in the new basis is $\begin{pmatrix} 3 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$. To calculate trace and determinant of operator A it is convenient to use the representation of this operator by matirx in the second basis, on the other hand it is good to double check the answer in both bases:

$$\det A = \det \begin{pmatrix} 3 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} = 3\cdot\frac{1}{3} = \det \begin{pmatrix} 27 & -16 \\ 40 & -\frac{71}{3} \end{pmatrix} = 27\cdot\left(-\frac{71}{3}\right) - 40\cdot(-16) = -639+640 = 1,$$

$$\operatorname{Tr} A = \operatorname{Tr} \begin{pmatrix} 3 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} = 3+\frac{1}{3} = \operatorname{Tr} \begin{pmatrix} 27 & -16 \\ 40 & -\frac{71}{3} \end{pmatrix} = 27 - \frac{71}{3} = \frac{10}{3},$$

6† *Let V be a space of functions, which are solutions of differential equation*

$$\frac{d^2y(x)}{dx^2} + p\frac{dy(x)}{dx} + qy(x) = 0, \qquad (1)$$

where parameters p, q are equal to $p=-7$ and $q=12$.

Show that V is 2-dimensional vector space.

Find a basis in this vector space, and write down the operator A in this basis.

Show that the differentiation $A = \frac{d}{dx}$ is a linear operator on the space V which transforms every vector from V to another vector on V.

Find determinant and trace of this linear operator.

This is linear differential equation. Linear combination of solutions is a solution. Hence space of solutions is a vector space.

If $y(x)$ is a solution of differential equation (1), then obviously $Ay(x) = \frac{d}{dx}y(x)$ is a solution also. Hence A is an operator on space of solutions.

One can see that an arbitrary solution of this equation is

$$y(x) = c_1 e^{3x} + c_2 e^{4x},$$

where functions e^{3x}, e^{4x} are eigenvectors of the operator A with eigenvalues 3 and 4 respectively. Space of solutions is a span of eignevectors e^{3x}, e^{4x}.

These vectors (functions) are eigenvectors, and they form a basis $\{\mathbf{e}, \mathbf{f}\}$ in the vector space V, $\mathbf{e} = e^{3x}$, $\mathbf{f} = e^{4x}$.

$$A(\mathbf{e}) = \frac{d}{dx}e^{3x} = 3e^{3x} = 3\mathbf{e} \quad A(\mathbf{f}) = \frac{d}{dx}e^{4x} = 4e^{3x} = 4\mathbf{f}$$

matrix of the operator A in this basis is $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$. We have that $\det A = 12$ and $\operatorname{Tr} A = 7$.