## Homework 8. Solutions

**1.** *Find coordinate basis vectors, first quadratic form and unit normal vector field for the following surfaces:*

*a) sphere of the radius R:*

$$\mathbf{r}(\varphi,\theta) \qquad \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases} \qquad (0 \le \varphi < 2\pi, 0 \le \theta \le \pi)\,, \tag{1}$$

*b) cylinder*

$$\mathbf{r}(\varphi,h) \qquad \begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \\ z = h \end{cases} \qquad (0 \le \varphi < 2\pi, -\infty < h < \infty) \tag{2}$$

*c) graph of the function $z = F(x,y)$*

$$\mathbf{r}(u,,v) \qquad \begin{cases} x = u \\ y = v \\ z = F(u,v) \end{cases} \qquad (-\infty < u < \infty, -\infty < v < \infty) \tag{3}$$

*in the case if $F(u,v) = F = Au^2 + 2Buv + Cv^2$.*

*Put down the special case of saddle when $F = uv$.*

a) *sphere $x^2 + y^2 + z^2 = R^2$ (of the radius R):*

$$\mathbf{r}(\theta,\varphi) \qquad \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases} \tag{1}$$

$(0 \le \varphi < 2\pi,\ 0 \le \theta \le \pi)$,

$$\mathbf{r}_\theta = \frac{\partial \mathbf{r}(\varphi,\theta)}{\partial\theta} = \begin{pmatrix} R\cos\theta\cos\varphi \\ R\cos\theta\sin\varphi \\ -R\sin\theta \end{pmatrix},\ \mathbf{r}_\varphi = \frac{\partial \mathbf{r}(\varphi,\theta)}{\partial\varphi} = \begin{pmatrix} -R\sin\theta\sin\varphi \\ R\sin\theta\cos\varphi \\ 0 \end{pmatrix},\ \mathbf{n}(\theta,\varphi) = \frac{\mathbf{r}(\theta,\varphi)}{R} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

(Sometimes we denote by $\mathbf{r}_\theta$ by $\partial_\theta$ and $\mathbf{r}_\varphi$ by $\partial_\varphi$.)

Check that $\mathbf{n}(\theta,\varphi)$ is indeed unit normal vector (in fact this is obvious from geometric considerations):

$$(\mathbf{n},\mathbf{n}) = \sin^2\theta(\cos^2\varphi + \sin^2\varphi) + \cos^2\theta = 1\,,$$

$$(\mathbf{n},\mathbf{r}_\theta) = R\sin\theta\cos\theta(\cos^2\varphi + \sin^2\varphi) - R\sin\theta\cos\theta = 0,\ (\mathbf{n},\mathbf{r}_\varphi) = R\sin^2\theta(-\cos\varphi\sin\varphi + \cos\varphi\sin\varphi) = 0\,.$$

Unit normal vector is defined up to a sign; $-\mathbf{n}$ is unit normal vector too.

Calculate now first quadratic form. $(\mathbf{r}_\theta,\mathbf{r}_\theta) = R^2\cos^2\theta(\cos^2\varphi + \sin^2\varphi) + R^2\sin^2\theta = R^2$, $(\mathbf{r}_\theta,\mathbf{r}_\varphi) = 0$, $(\mathbf{r}_\varphi,\mathbf{r}_\varphi) = R^2\sin^2\theta(\sin^2\varphi + \cos^2\varphi) = R^2\sin^2\theta$. Thus

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_\theta,\mathbf{r}_\theta) & (\mathbf{r}_\theta,\mathbf{r}_\varphi) \\ (\mathbf{r}_\varphi,\mathbf{r}_\theta) & (\mathbf{r}_\varphi,\mathbf{r}_\varphi) \end{pmatrix} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix}$$

$$dl^2 = G_{11}d\theta^2 + 2G_{12}d\theta d\varphi + G_{22}d\varphi^2 = R^2 d\theta^2 + R^2\sin^2\theta d\varphi^2$$

The length of the curve $\mathbf{r}(t) = \mathbf{r}(\theta(t),\varphi(t))$ with $\theta = \theta(t), \varphi = \varphi(t)$, $t_1 \le t \le t_2$ is given by the integral:

$$\int_{t_1}^{t_2} |\mathbf{v}(t)|dt = \int_{t_1}^{t_2} \sqrt{G_{11}\dot\theta^2 + 2G_{12}\dot\theta\dot\varphi + G_{22}\dot\varphi^2}dt = \int_{t_1}^{t_2} \sqrt{R^2\dot\theta^2 + R^2\sin^2\theta\dot\varphi^2}dt \tag{1a}$$

*b) cylinder* $x^2 + y^2 = R^2$

$$\mathbf{r}(\varphi, h) \qquad \begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \\ z = h \end{cases} \qquad (0 \le \varphi < 2\pi, -\infty < h < \infty) \tag{2}$$

$$\mathbf{r}_\varphi\big|_{\varphi,h} = \frac{\partial \mathbf{r}(\varphi,h)}{\partial\varphi} = \begin{pmatrix} -R\sin\varphi \\ R\cos\varphi \\ 0 \end{pmatrix}, \ \mathbf{r}_h\big|_{\varphi,h} = \frac{\partial \mathbf{r}(\varphi,h)}{\partial h} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ \mathbf{n}(\varphi,h) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$$

Sometimes we denote $\mathbf{r}_\varphi$ by $\partial_\varphi$ and $\mathbf{r}_h$ by $\partial_h$.

Check that $\mathbf{n}(\varphi, h)$ is indeed unit normal vector:

$$(\mathbf{n},\mathbf{n}) = \cos^2\varphi + \sin^2\varphi = 1, \ (\mathbf{n},\mathbf{r}_\varphi) = R\cos\varphi\sin\varphi(-1+1) = 0, \ (\mathbf{n},\mathbf{r}_h) = 0$$

Unit normal vector is defined up to a sign; $-\mathbf{n}$ is unit normal vector too.

Calculate now first quadratic form. $(\mathbf{r}_\varphi,\mathbf{r}_\varphi) = R^2(\sin^2\varphi + \cos^2\varphi) = R^2$, $(\mathbf{r}_\varphi,\mathbf{r}_h) = 0$, $(\mathbf{r}_h,\mathbf{r}_h) = 1$. Thus

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_\varphi,\mathbf{r}_\varphi) & (\mathbf{r}_\varphi,\mathbf{r}_h) \\ (\mathbf{r}_h,\mathbf{r}_\varphi) & (\mathbf{r}_h,\mathbf{r}_h) \end{pmatrix} = \begin{pmatrix} R^2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$dl^2 = G_{11}d\varphi^2 + 2G_{12}d\varphi dh + G_{22}dh^2 = R^2 d\varphi^2 + dh^2$$

The length of the curve $\mathbf{r}(t) = \mathbf{r}(\varphi(t), h(t))$ with $\varphi = \varphi(t), h = h(t)$, $t_1 \le t \le t_2$ is given by the integral:

$$\int_{t_1}^{t_2} \sqrt{G_{11}\dot\varphi^2 + 2G_{12}\dot\varphi\dot h + G_{22}\dot h^2}dt = \int_{t_1}^{t_2} \sqrt{R^2\dot\varphi^2 + \dot h^2}dt \tag{2a}$$

*c) graph of the function* $z = F(x, y)$

$$\mathbf{r}(u, v) \qquad \begin{cases} x = u \\ y = v \\ z = F(u,v) \end{cases} \qquad (-\infty < u < \infty, -\infty < v < \infty) \tag{3}$$

in the case if $F(u, v) = Au^2 + 2Buv + Cv^2$

$$\mathbf{r}_u\big|_{u,v} = \frac{\partial \mathbf{r}(u,v)}{\partial u} = \begin{pmatrix} 1 \\ 0 \\ F_u \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2Au + 2Bv \end{pmatrix}, \ \mathbf{r}_u\big|_{u=v=0} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbf{r}_v\big|_{u,v} = \frac{\partial \mathbf{r}(u,v)}{\partial v} = \begin{pmatrix} 0 \\ 1 \\ F_v \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2Bu + 2Cv \end{pmatrix}, \ \mathbf{r}_v\big|_{u=v=0} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

$$\mathbf{n}(u, v) = \frac{1}{\sqrt{1 + F_u^2 + F_v^2}} \begin{pmatrix} -F_u \\ -F_v \\ 1 \end{pmatrix}, \ \mathbf{n}(u,v)\big|_{u=v=0} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Sometimes we denote $\mathbf{r}_u$ by $\partial_u$ and $\mathbf{r}_v$ by $\partial_v$.

Check that $\mathbf{n}(u,v)$ is indeed unit normal vector: $(\mathbf{n},\mathbf{n}) = \frac{1}{1+F_u^2+F_v^2}(F_u^2+F_v^2+1) = 1$, $(\mathbf{n},\mathbf{r}_u) = \frac{1}{\sqrt{1+F_u^2+F_v^2}}(F_u - F_u) = 0$, $(\mathbf{n},\mathbf{r}_v) = \frac{1}{\sqrt{1+F_u^2+F_v^2}}(F_v - F_v) = 0$. Calculate now first quadratic form. $(\mathbf{r}_u,\mathbf{r}_u) = 1+F_u^2$, $(\mathbf{r}_u,\mathbf{r}_v) = F_uF_v$, $(\mathbf{r}_v,\mathbf{r}_v) = 1+F_v^2$. Thus

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_u,\mathbf{r}_u) & (\mathbf{r}_u,\mathbf{r}_v) \\ (\mathbf{r}_v,\mathbf{r}_u) & (\mathbf{r}_v,\mathbf{r}_v) \end{pmatrix} = \begin{pmatrix} 1+F_u^2 & F_uF_v \\ F_uF_v & 1+F_v^2 \end{pmatrix}$$

$$dl^2 = G_{11}d\varphi^2 + 2G_{12}dd\varphi dh + G_{22}dh^2 = (1+F_u^2)du^2 + 2F_uF_vdudv + (1+F_v^2)dv^2$$

At the point $u = v = 0$, $F_u = F_v = 0$ and

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_u,\mathbf{r}_u) & (\mathbf{r}_u,\mathbf{r}_v) \\ (\mathbf{r}_v,\mathbf{r}_u) & (\mathbf{r}_v,\mathbf{r}_v) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ dl^2 = du^2 + dv^2$$

The length of the curve $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ with $u = u(t), v = v(t)$ can be calculated by the integral:

$$\int_{t_1}^{t_2} \sqrt{G_{11}\dot{u}^2 + 2G_{12}\dot{u}\dot{v} + G_{22}\dot{v}^2}dt = \int_{t_1}^{t_2} \sqrt{(1+F_u^2)\dot{u}^2 + 2F_uF_v\dot{u}\dot{v} + (1+F_v^2)\dot{v}^2}dt \qquad (3a)$$

Special case of saddle: In the special case of saddle we just take $F = uv$ in previous formulae. In particular normal for normal unit vector we have

$$\mathbf{n}(u,v) = \frac{1}{\sqrt{1+v^2+u^2}}\begin{pmatrix} -v \\ -u \\ 1 \end{pmatrix},\ \mathbf{n}(u,v)\big|_{u=v=0} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and first quadratic form is equal to

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_u,\mathbf{r}_u) & (\mathbf{r}_u,\mathbf{r}_v) \\ (\mathbf{r}_v,\mathbf{r}_u) & (\mathbf{r}_v,\mathbf{r}_v) \end{pmatrix} = \begin{pmatrix} 1+F_u^2 & F_uF_v \\ F_uF_v & 1+F_v^2 \end{pmatrix} = \begin{pmatrix} 1+v^2 & vu \\ vu & 1+u^2 \end{pmatrix},$$

$$dl^2 = G_{11}d\varphi^2 + 2G_{12}dd\varphi dh + G_{22}dh^2 = (1+v^2)du^2 + 2uvdudv + (1+u^2)dv^2\,.$$

**2.** *Consider helix* $\mathbf{r}(t)$: $\begin{cases} x(t) = R\cos t \\ y(t) = R\sin t \\ z(t) = ct \end{cases}$.

*Show that this helix belongs to cylinder surface* $x^2 + y^2 = R^2$.

*Using first quadratic form calculate length of this curve* ($0 \le t \le t_0$). *(Compare with problem 4 from Homework 7.)*

This helix belongs to cylinder surface $x^2 + y^2 = R^2$ because $x^2 + y^2 = R^2$ on the points of the helix.

Use quadratic form which we obtained in the previous exercise (see equation (2a) and equations above in the solution of exercise 1). For the helix internal coordinates are $\varphi = \varphi(t) = t$ and $h = h(t) = ct$ ($x = R\cos\varphi, y = R\sin\varphi, z = h$)

We come to

$$L = \int_0^{t_0} \sqrt{G_{11}\dot{\varphi}^2 + 2G_{12}\dot{\varphi}\dot{h} + G_{22}\dot{h}^2}dt = \int_0^{t_0} \sqrt{R^2\dot{\varphi}^2 + \dot{h}^2}dt = \int_0^{t_0} \sqrt{R^2+c^2}dt = t\sqrt{R^2+c^2}$$

Of course the answer can be obtained without integration: speed is constant, hence

$$L = \sqrt{G_{11}\dot{\varphi}^2 + 2G_{12}\dot{\varphi}\dot{h} + G_{22}\dot{h}^2}t_0 = \sqrt{R^2\dot{\varphi}^2 + \dot{h}^2}t_0 = \sqrt{R^2+c^2}t_0\,.$$

**3.** *a) Consider on the sphere (1) the following curves:*
$C_1$: $x = R\cos t,\ y = R\sin t,\ z = 0$ *(Equator)* $(0 \le t < 2\pi)$,
$C_2$: $x = R\cos t,\ y = 0,\ z = R\sin t$ $(0 \le t < \pi)$ *("Greenwich" Meridian)*,
$C_3$: $x = R\sin\theta_0\cos t,\ y = R\sin\theta_0\sin t,\ z = R\cos\theta_0$ *(Circle of constant latitude)*
*Sketch these curves. Calculate length of these curves considering them in the ambient Euclidean space. Calculate length of these curves using first quadratic form.*

Calculate the lengths of curves $C_1, C_2$ and $C_3$ in ambient Euclidean space, i.e. from the point of view of the External Observer:

Curve $C_1$: $x = R\cos t,\ y = R\sin t,\ z = 0$ is a circle of the radius $R$. Its length is equal to $2\pi R$.

Curve $C_2$ is a semicircle of the radius $R$. Its length is equal to $\pi R$.

Curve $C_3$

$$\mathbf{r}(t):\quad \begin{cases} x(t) = R\sin\theta_0\cos\varphi(t) = R\sin\theta_0\cos t \\ y(t) = R\sin\theta_0\sin\varphi(t) = R\sin\theta_0\sin t \\ z(t) = R\cos\theta_0 \end{cases}$$

It is the circle (latitude) of the radius $R\sin\theta_0$. Its length is equal to $L = 2\pi R\sin\theta_0$.

Now calculate the lengths of this curve from the point of view of internal observer:

For the curve $C_1$ internal coordinates are angles $\theta, \varphi$. They are $\theta(t) = \theta_0 = \frac{\pi}{2}$ and $\varphi(t) = t$. Hence using first quadratic form (see exercise 1, equation (1a)) we see that speed is equal to

$$|\mathbf{v}(t)| = \sqrt{G_{11}\dot\theta^2 + 2G_{12}\dot\theta\dot\varphi + G_{22}\dot\varphi^2} = \sqrt{R^2\dot\theta^2 + R^2\sin^2\theta_0\dot\varphi^2} = \sqrt{R^2\sin^2\theta_0} = R\sin\theta_0 = R\,.$$

It is constant. To calculate the length we do not need to calculate integral. The length is equal to $L = |\mathbf{v}|2\pi = 2\pi R$. (Compare with an answer for external observer.)

For the curve $C_2$ internal coordinates $\theta(t) = t$ and $\varphi(t) = 0$, $0 \le t \le \pi$. Again using first quadratic form (see exercise 1, equation (1a)) we see that speed is equal to

$$|\mathbf{v}(t)| = \sqrt{G_{11}\dot\theta^2 + 2G_{12}\dot\theta\dot\varphi + G_{22}\dot\varphi^2} = \sqrt{R^2\dot\theta^2 + R^2\sin^2\theta\dot\varphi^2} = R\,.$$

It is constant. To calculate the length we do not need to calculate integral. The length is equal to $L = |\mathbf{v}|\pi = \pi R$. (Compare with an answer for external observer.)

For the curve $C_3$ internal coordinates $\theta, \varphi$ are following: $\theta(t) = \theta_0$, and $\varphi(t) = t$, $0 \le t < 2\pi$. Hence using first quadratic form (see exercise 1, equation (1a)) we see that speed is equal to

$$|\mathbf{v}(t)| = \sqrt{G_{11}\dot\theta^2 + 2G_{12}\dot\theta\dot\varphi + G_{22}\dot\varphi^2} = \sqrt{R^2\dot\theta^2 + R^2\sin^2\theta_0\dot\varphi^2} = \sqrt{R^2\sin^2\theta_0} = R\sin\theta_0\,,$$

because $\dot\theta = 0, \dot\varphi = 1$. Speed is constant. Hence to calculate the length we do not need to calculate integral. The length is equal to $L = |\mathbf{v}|2\pi = 2\pi R\sin\theta_0$. (Compare with an answer for external observer.)

**4.** *Calculate the shape operator for an arbitrary point of the sphere (1).*

*Recall the notion of normal curvature of a curve on a surface.*

*Let $C$ be an arbitrary curve on the sphere of radius $R$. Show that the normal curvature of the curve $C$ at an arbitrary point is equal to $1/R$ (up to a sign).*

We use results of calculations of vectors $\mathbf{r}_\theta, \mathbf{r}_\varphi$ and of unit normal vector $\mathbf{n}(\theta,\varphi)$ from the exercise 1a. By the definition (see lecture notes) the action of shape operator on any tangent vector $\mathbf{v}$ is given by the formula $S\mathbf{v} = -\partial_{\mathbf{v}}S$. Hence for basis vectors $\mathbf{r}_\theta = \partial_\theta, \mathbf{r}_\varphi = \partial_\varphi$

$$S\mathbf{r}_\theta = -\partial_\theta\mathbf{n}(\theta,\varphi) = -\partial_\theta\left(\frac{\mathbf{r}(\theta,\varphi)}{R}\right) = -\left(\frac{\partial_\theta\mathbf{r}(\theta,\varphi)}{R}\right) = -\frac{\mathbf{r}_\theta}{R}$$

and

$$S\mathbf{r}_\varphi = -\partial_\varphi \mathbf{n}(\theta,\varphi) = -\partial_\varphi \left(\frac{\mathbf{r}(\theta,\varphi)}{R}\right) = -\left(\frac{\partial_\varphi \mathbf{r}(\theta,\varphi)}{R}\right) = -\frac{\mathbf{r}_\varphi}{R}$$

We see that shape operator is equal to $S = -\frac{I}{R}$, where $I$ is an identity operator. Its matrix in the basis $\partial_\theta, \partial_\varphi$ is equal to

$$-\begin{pmatrix} \frac{1}{R} & 0 \\ 0 & \frac{1}{R} \end{pmatrix}.$$

In the case if we choose the opposite direction for unit normal vector then we will come to the same answer just with changing the signs: if $\mathbf{n} \to -\mathbf{n}$, $S \to -S$.

We see that eigenvalues of shape operator are the same, i.e. principal curvatures $\kappa_+, \kappa_-$ are the same: $\kappa_+ = \kappa_- = \frac{1}{R}$ (if we choose the opposite sign for $\mathbf{n}$). We know that normal curvature $\kappa$ of an arbitrary curve on the surface takes values in the interval $(\kappa_-, \kappa_+)$:

$$\frac{1}{R} = \kappa_- \leq \kappa \leq \kappa_+ = \frac{1}{R} \Rightarrow \kappa = \frac{1}{R}$$

The fact that normal curvature (not usual curvature!) of an arbitrary curve on the surface (up to a sign) is the same can be deduced straightforwardly from the definition of normal curvature *

**5.** *a) Calculate the shape operator for an arbitrary point of the cylinder (2).*

*b) Consider on the cylinder (2) the following curves:*

$C_1$: $x = R\cos t,\ y = R\sin t,\ z = h_0$ *(circle),*

$C_2$: $x = R\cos t,\ y = R\sin t,\ z = vt$ *(helix),*

$C_3$: $x = R\cos\varphi_0,\ y = R\sin\varphi_0,\ z = t$ *(straight line).*

*Calculate the normal curvatures of these curves.*

*c) What values the normal curvature of an arbitrary curve on the cylinder of radius $R$ can take?*

a) To calculate the shape operator for the cylinder we use results of calculations vectors $\mathbf{r}_\varphi, \mathbf{r}_h$ and for unit normal vector $\mathbf{n}(\varphi, h)$ from the exercise 1b. By the definition the action of shape operator on any tangent vector $\mathbf{v}$ is given by the formula $S\mathbf{v} = -\partial_\mathbf{v} S$. Hence for basis vectors $\mathbf{r}_\varphi = \partial_\varphi, \mathbf{r}_h = \partial_h$

$$S\mathbf{r}_\varphi = -\partial_\varphi \mathbf{n}(\varphi, h) = -\partial_\varphi \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} \sin\varphi \\ -\cos\varphi \\ 0 \end{pmatrix} = -\frac{\mathbf{r}_\varphi}{R}$$

and

$$S\mathbf{r}_h = -\partial_h \mathbf{n}(\varphi, h) = -\partial_h \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

For an arbitrary tangent vector $\mathbf{X} = a\mathbf{r}_\varphi + b\mathbf{r}_h$, $S\mathbf{X} = -a/R\mathbf{r}_\varphi$. We see that shape operator transforms tangent vectors to tangent vectors. Its matrix in the basis $\mathbf{r}_\varphi, \mathbf{r}_h$ is equal to

$$-\begin{pmatrix} \frac{1}{R} & 0 \\ 0 & 0 \end{pmatrix}$$

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* Show that for an arbitrary (smooth) curve on the sphere of the radius $R$ the normal curvature is equal to $1/R$ (up to a sign) at all points of the curve. Let $\mathbf{v}(t)$ be a velocity vector, $\mathbf{a}(t)$ be an acceleration vector and $\mathbf{n}(t) = \frac{\mathbf{r}(t)}{R}$ be a unit normal vector at the point $\theta(t), \varphi(t)$ on the sphere. Note that $\frac{d\mathbf{n}(t)}{dt} = \frac{\mathbf{v}(t)}{R}$ because $\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt}$ and $\mathbf{n}(t) = \frac{\mathbf{r}(t)}{R}$. Then normal acceleration is equal to $\frac{-v^2}{R}$ since $\mathbf{a}_n = a_n\mathbf{n}$ where $\alpha_n = (\mathbf{n}(t), \mathbf{a}(t)) = \left(\mathbf{n}(t), \frac{d}{dt}\mathbf{v}(t)\right) = \frac{d}{dt}(\mathbf{n}(t), \mathbf{v}(t)) - \left(\frac{d\mathbf{n}(t)}{dt}, \mathbf{v}(t)\right) = \frac{d}{dt}(0) - \left(\frac{\mathbf{v}(t)}{R}, \mathbf{v}(t)\right) = -\frac{v^2}{R}$ and normal curvature is equal to $\kappa_n = \frac{(\mathbf{v},\mathbf{n})}{v^2} = \frac{a_n}{v^2} = -\frac{1}{R}$ We see that normal curvature (not usual curvature!) for an arbitrary curve on the sphere is always equal to $1/R$

In the case if we choose the opposite direction for unit normal vector then we will come to the same answer just with changing the signs: if $\mathbf{n} \to -\mathbf{n}$, $S \to -S$.

b) For the points of the circle $C_1$ normal curvature is equal (up to a sign) to $1/R$. Indeed consider point moving around $C_1$ with constant speed ($x = R\cos\omega t, y = R\sin\omega t, z = h_0$). Speed is equal to $v = \omega R$ and the acceleration vector $\mathbf{a} = -R\omega^2\cos\omega t\partial_x - R\omega^2\sin\omega t\partial_y$ is orthogonal to the surface: $\mathbf{a} = \omega^2 R\mathbf{n}$, where we choose unit normal vector to be $\mathbf{n} = -(x/R, y/R, 0)$ at the points $(x, y, z)$ of the cylinder ($\mathbf{n} = (-\cos\omega t, -\sin\omega t, 0)$). Normal curvature is equal to $\kappa_n = (\mathbf{a}, \mathbf{n})/(\mathbf{v}, \mathbf{v}) = \omega^2 R/\omega^2 R^2 = \frac{1}{R}$. If we choose $\mathbf{n} = +(x/R, y/R, 0) = (\cos\omega t, \sin\omega t, 0)$ then normal curvature would change a sign: $\mathbf{n} \to -\mathbf{n}$, $(\mathbf{a}, \mathbf{n}) \to -(\mathbf{a}, \mathbf{n})$ and $\kappa_{normal} \to -\kappa_{normal} = -\frac{1}{R}$.

Consider now a point moving around the curve $C_2$ (helix) with constant speed: $x = R\cos\omega t$, $y = R\sin\omega t$, $z = vt$. Then velocity vector is equal to $\mathbf{v} = -R\omega\sin\omega t\partial_x + R\omega\cos\omega t\partial_y + v\partial_z$ and $\mathbf{a} = -R\omega^2\cos\omega t\partial_x - R\omega^2\sin\omega t\partial_y$ (it is normal (centripetal) acceleration ). We see that $v = \sqrt{\omega^2 R^2 + v^2}$, $\mathbf{a}_n = (\mathbf{a}, \mathbf{n}) = \omega^2 R$ (we choose $\mathbf{n} = -(x/R, y/R, 0)$ at the point $(x, y, z)$ of the cylinder) and normal curvature is equal to

$$\kappa_n = \frac{\omega^2 R}{\omega^2 R^2 + v^2}$$

(In the case if we change $\mathbf{n} \to -\mathbf{n}$ then normal curvature will change a sign too) One can see that

$$0 < \kappa_{normal} = \frac{\omega^2 R}{\omega^2 R^2 + v^2} \leq \frac{\omega^2 R}{\omega^2 R^2} = \frac{1}{R}$$

On the cylinder normal curvature varies from zero (for straight line) to $1/R$: $0 \leq \kappa_{normal} \leq \frac{1}{R}$

For the straight line $C_3$ normal curvature obviously is equal to 0: e.g. if particle moves on $C_3$ with constant speed, then its acceleration is equal to zero.

c) Now consider an arbitrary curve on the cylinder. We know that normal curvature takes values in the interval between principal curvatures. According to calculation of shape operator we see that principal curvatures for cylinder are equal to

$$\kappa_- = 0, \kappa_+ = \frac{1}{R}.$$

Hence for an arbitrary curve on the cylinder (2) normal curvature takes values in the interval $\left[0, \frac{1}{R}\right]$. Of course this can be shown straightforwardly**.

** Show straightforwardly that for an arbitrary curve on the cylinder (2) normal curvature takes values in the interval $\left[0, \frac{1}{R}\right]$. For convenience we choose normal unit vector attached at the point $(x, y, z)$ of the cylinder $\mathbf{n} = -(x/R, y/R, 0)$. (In this case normal curvature of circle and helix were positive). Consider an arbitrary smooth curve $\mathbf{r} = \mathbf{r}(\varphi(t), h(t))$ on the cylinder (2) and take any point $P = \mathbf{r} = (x, y, z)$ on it ($x^2 + y^2 + z^2 = R^2$). Let $\mathbf{v}$ be velocity vector at this point, $\mathbf{v} = (v_x, v_y, v_z)$. Consider expansion of the vector $\mathbf{v}$ on horizontal and vertical components. $\mathbf{v} = \mathbf{v}_{horisont} + \mathbf{v}_{vertical}$, $\mathbf{v}_{horisontal} = v_x\partial_x + v_y\partial_y$, $\mathbf{v}_{vertical} = v_z\partial_z$. Normal (centripetal) acceleration $\mathbf{a}_n$ is equal to $\mathbf{a}_n = -a_n\mathbf{n}$, where $a_n = \frac{|\mathbf{v}_{horisontal}|^2}{R} = \frac{v_x^2 + v_y^2}{R}$. We have for normal curvature:

$$\kappa_{normal} = \frac{\frac{v^2_{horisontlal}}{R}}{(\mathbf{v}, \mathbf{v})} = \frac{v^2_{horisontlal}}{R(v^2_{horisontlal} + v^2_{vertical})} = \frac{v_x^2 + v_y^2}{R(v_x^2 + v_y^2 + v_z^2)} \leq \frac{1}{R}\,.$$

We see that normal curvature could be less or equal than $1/R$ (normal curvature of the circle) and bigger or equal than 0 (normal curvature for straight line) (See also the example in Lecture notes.)

**Remark** Note that one can consider on the cylinder $x^2+y^2=R^2$ a circle of very small radius $r$. The curvature (usual curvature which we studied before) of this circle will be equal to $1/r$. We see that usual curvature of curve can be very big, but normal curvature cannot be bigger than $1/R$.

**6.** *Calculate the shape operator for the surface (3) at the point $u=v=0$.*
*Put down the shape operator for this surface at the point $u=v=0$ in the special case $F=uv$ (a "saddle")*

We use results of calculations vectors $\mathbf{r}_u, \mathbf{r}_v$ and for unit normal vector $\mathbf{n}(u,v)$ from the exercise 1c For basic vectors $\mathbf{r}_u=\partial_u, \mathbf{r}_v=\partial_v$ we have $S\mathbf{r}_u=-\partial_u\left(\mathbf{n}(u,v)\right)_{|u=v=0}=$

$$-\partial_u\left(\frac{1}{\sqrt{1+F_u^2+F_v^2}}\begin{pmatrix}-F_u\\-F_v\\1\end{pmatrix}\right)_{|u=v=0}=\left(\frac{1}{\sqrt{1+F_u^2+F_v^2}}\right)_{|u=v=0}\begin{pmatrix}F_{uu}\\F_{uv}\\1\end{pmatrix}_{|u=v=0}=\begin{pmatrix}2A\\2B\\0\end{pmatrix}=2A\mathbf{r}_u+2B\mathbf{r}_v$$

and $S\mathbf{r}_v=-\partial_v\left(\mathbf{n}(u,v)\right)_{|u=v=0}=$

$$-\partial_v\left(\frac{1}{\sqrt{1+F_u^2+F_v^2}}\begin{pmatrix}-F_u\\-F_v\\1\end{pmatrix}\right)_{|u=v=0}=\left(\frac{1}{\sqrt{1+F_u^2+F_v^2}}\right)_{|u=v=0}\begin{pmatrix}F_{vu}\\F_{vv}\\1\end{pmatrix}_{|u=v=0}=\begin{pmatrix}2B\\2C\\0\end{pmatrix}=2B\mathbf{r}_u+2C\mathbf{r}_v$$

The matrix of the shape operator in the basis $\mathbf{r}_u, \mathbf{r}_v$ is $\begin{pmatrix}2A & 2B\\2B & 2C\end{pmatrix}$. In the case of saddle $F=uv$, i.e. $A=C=0, B=\frac{1}{2}$. The shape operator at the point $u=v=0$ equals to

$$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$$

**7** Calculate principal curvatures, Gaussian and mean curvature
a) at the points of the sphere of radius $R$
b) at the points of cylinder surface of radius $R$
c) at the point $u=v=0$ of the surface (3).
d) at the point $u=v=0$ of the saddle.

a) We obtained in the exercise 4 that for sphere of radius $R$ shape operator is the following: $S=-\begin{pmatrix}\frac{1}{R} & 0\\0 & \frac{1}{R}\end{pmatrix}$. Hence principal curvatures (eigenvalues of the shape operator)are $\kappa_-=\kappa_+=\frac{1}{R}$ (up to a sign)

Gaussian curvature $K=k_-\cdot k_-=\det S=\frac{1}{R^2}$ and mean curvature $H=\frac{k_-+k_+}{2}=\frac{1}{2}\mathrm{Tr}\,S=-\frac{1}{R}$. If we change $\mathbf{n}\to-\mathbf{n}$ principal curvatures and mean curvature will change the sign but Gaussian curvature remains intact.

b) We obtained in the exercise 5a that for the cylinder shape operator is equal to $S=-\begin{pmatrix}\frac{1}{R} & 0\\0 & 0\end{pmatrix}$. Gaussian curvature $K=k_-\cdot k_+=\det S=0$ and mean curvature $H=\frac{1}{2}(k_-+k_+)=\frac{1}{2}\mathrm{Tr}\,S=-\frac{1}{R}$. We have that $k_-\cdot k_+=0$ and $k_-+k_+=\frac{1}{R}$. Principal curvatures (eigenvalues of operator $S$) equal to $k_-=0$, $k_+=-\frac{1}{R}$.

Answers are the same up to a sign: we choose different directions for $\mathbf{n}$ in Exercises 5 and 7: If we change $\mathbf{n}\to-\mathbf{n}$ principal curvatures and mean curvature will change the sign but Gaussian curvature remains intact.

We already calculated shape operator at the point $u=v=0$ in the exercise 6: its matrix an the basis $\partial_u, \partial_v$ is equal to $S=\begin{pmatrix}2A & 2B\\2B & 2C\end{pmatrix}$. Gaussian curvature $K=k_-\cdot k_+=\det S=4AC-4B^2$ and mean curvature $H=\frac{1}{2}(k_-+k_+)=\frac{1}{2}\mathrm{Tr}\,S=A+C$.

d) In the case of saddle $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. (see exercise 6) Gaussian curvature $K = k_- \cdot k_+ = \det S = 4AC - 4B^2 = -1$ and mean curvature $H = \frac{1}{2}(k_- + k_+) = \frac{1}{2}\mathrm{Tr}\, S = A + C = 0^{***}$.

**8** *Assume that the action of the shape operator at the tangent coordinate vectors $\partial_u, \partial_v$ at the given point* $\mathbf{p}$ *of the surface $\mathbf{r} = \mathbf{r}(u, v)$ is defined by the relations: $S(\partial_u) = 2\partial_u + 2\partial_v$ and $S(\partial_v) = -\partial_u + 5\partial_v$. Calculate principal curvatures, Gaussian and mean curvatures of the surface at this point.*

We see that the matrix of the shape operator in the basis $\partial_u, \partial_v$ is equal to

$$S = \begin{pmatrix} 2 & -1 \\ 2 & 5 \end{pmatrix}$$

Hence Gaussian curvature $K = \det S = 12$ and mean curvature $H = \frac{1}{2}\mathrm{Tr}\, S = 7/2$ To calculate principal curvatures $k_-, k_+$ note that

$$\begin{cases} k_- + k_+ = 2H = 7 \\ k_- \cdot k+ = K = 12 \end{cases}$$

Hence $k_- = 3, k_+ = 4$; $\kappa_-, k_+$ are eigenvalues of the shape operator.

**9** *Consider on the sphere (1) the points $A = (1, 0, 0)$, $B = (0, 1, 0)$ and $C = (0, 0, 1)$ and arcs of great circles $AB$, $BC$ and $CA$.*
*Find the image $\mathbf{A}_2$ of the vector $\mathbf{A}_1$ under parallel transport along the closed curve $ABC$.*

Do it in three steps.

First find the image of the vector $\partial_z$ under the parallel transport along the curve $AB$. One can see that parallel transport of the vector $\mathbf{A}_1 = \partial_z$ along the arc $AB$: $\mathbf{r}(t) = \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix}$ is the same vector $\partial_z$. Indeed this vector field is tangent to the sphere at the points of this curve, its derivative evidently equal to zero

Now find the image of the vector $\mathbf{A}_2 = \partial_z$ under the parallel transport along the curve $BC$. The vector $\partial_z$ could not remain the same, because the vector $\partial_z$ is not tangent to the sphere at the points of the curve $BC$. One can see that parallel transport of the vector $\mathbf{A}_2 = \partial_z$ along the arc $BC$:

$$\mathbf{r}(t) = \begin{pmatrix} \sin\theta(t)\cos\varphi(t) \\ \sin\theta(t)\sin\varphi(t) \\ \cos\theta(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \cos t \\ \sin t \end{pmatrix} \quad 0 \le t \le \frac{\pi}{2}$$

is the vector field

$$\mathbf{A}(t) = \begin{pmatrix} 0 \\ -\sin t \\ \cos t \end{pmatrix} = -\sin t\partial_y + \cos t\partial_z$$

Note that unit normal vector field on the curve is $\mathbf{n}(t) = \frac{\mathbf{r}(t)}{R} = \begin{pmatrix} 0 \\ \cos t \\ \sin t \end{pmatrix}$. It is easy to check that all conditions of parallel transport are obeyed:

1) $\mathbf{A}(t)$ is attached at the point $\mathbf{r}(t)$ and it is tangent to the sphere because $(\mathbf{A}(t), \mathbf{n}(t)) = 0$.
2) $\frac{d\mathbf{A}(t)}{dt}$ is collinear to the normal vector because $\frac{d\mathbf{A}(t)}{dt} = -\cos t\partial_y - \sin t\partial_z = -\mathbf{n}(t)$
3) at $t = 0$, $\mathbf{A}(t)$ coincides with the initial vector $\partial_z$.

$^{***}$ We see that the gaussian curvature at the point $u = v = 0$ of the saddle is negative.

Hence we see that image of the vector $\partial_z$ under the parallel transport along the curve $BC$ is the vector $\mathbf{A}_2 = \mathbf{A}(t)|_{t=\frac{\pi}{2}} = -\partial_y$ attached at the point $C$, North pole.

Now find the image of the vector $\mathbf{A}_3 = -\partial_y$ under the parallel transport along the arc $CA$.

One can see that during parallel transport of the vector $\mathbf{A}_2 = -\partial_y = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$ along the arc $CA$: $\mathbf{r}(t) = \begin{pmatrix} \sin t \\ 0 \\ \cos t \end{pmatrix}$, $0 \leq t \leq \frac{\pi}{2}$ it remains tangent to this arc: $(\mathbf{A}_2, \mathbf{n}(t)) = 0$, $(\mathbf{n}(t) = \mathbf{r}(t))$. Hence we see that image of the vector $\mathbf{A}_2 = -\partial_y$ under the parallel transport along the curve $CA$ is the same vector $\mathbf{A}_2 = -\partial_y$ attached at the point $A$.

Summarizing we see that the image of the initial vector $\mathbf{A}_1 = \partial_z$ under the parallel transport along the closed spherical triangle $ABC$ is the vector $\mathbf{A}_2 = -\partial_y$ attached at the same point $A$. The vector rotates on the angle $\frac{\pi}{2}$.

$^\dagger$**10** *Show that there are two straight lines which pass through the point* $(3, 4, 12)$ *on the saddle* $z = xy$ *and lie on this saddle.*

*Show that this is true for an arbitrary point of the saddle.*

Let $\mathbf{r}_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ be an arbitrary point on the saddle: $z_0 = x_0 y_0$.

Consider the following two lines: the line

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{a}, \text{ where } \mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ y_0 \end{pmatrix}, \text{ i.e. } \begin{cases} x = x_0 + t \\ y = y_0 \\ z = z_0 + y_0 t \end{cases}$$

and the line

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{b}, \text{ where } \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ x_0 \end{pmatrix}, \text{ i.e. } \begin{cases} x = x_0 + t \\ y = y_0 \\ z = z_0 + y_0 t \end{cases}$$

It is easy to check that these both lines belong to the saddle: $xy = (x_0 + t)y_0 = z_0 + ty_0 = z$ and $xy = x_0(y_0 + t) = z_0 + x_0 t$.

One the other hand it is easy to see that it is all: If $\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$ is an arbitrary straight line on the saddle passing through the point $(x_0, y_0, z_0)$, $x_0 y_0 = z_0$ then

$$xy = (x_0 + at)(y_0 + bt) = z = z_0 + ct \text{ for all } t$$

Hence $ab = 0$. Thus $a = 0$ or $b = 0$. We see that through the an arbitrary point on the saddle pass exactly two straight lines.