Examples of calculations

Example Consider dfferential equation

$$u_x^2 + u_y^2 = 1\,, \quad F(u,p,q,x,y) = p^2 + q^2 - 1\,.$$

in the space (u,p,q,x,y) $(p = u_x, q = u_y)$ consider surface

$$\Gamma\colon \quad \begin{cases} u(\xi) = \varphi(\xi) \\ p(\xi) = \sqrt{1 - \varphi'^2(\xi)} \\ q(\xi) = \varphi'(\xi) \\ x(\xi) = 0 \\ y(\xi) = \xi \end{cases}$$

This surface is generated by the function $\varphi(y)$. It belongs to the differential equation $p^2 + q^2 = 1$. Consider characteristic vector field

$$\frac{1}{2} X_F = p\partial_p + q\partial_q + \partial_u$$

This vector field emit the 1-parametric family of the curves which are solutions of characteristic equations

$$\begin{cases} \dot{u} = 1 \\ \dot{p} = 0 \\ \dot{q} = 0 \\ \dot{x} = p \\ \dot{y} = q \end{cases} \text{with boundary conditions} \begin{cases} u(\xi,\tau)\big|_{\tau=0} = \varphi(\xi) \\ p(\xi,\tau)\big|_{\tau=0} = \sqrt{1 - \varphi'^2(\xi)} \\ q(\xi,\tau)\big|_{\tau=0} = \varphi'(\xi) \\ x(\xi,\tau)\big|_{\tau=0} = 0 \\ y(\xi,\tau)\big|_{\tau=0} = \xi \end{cases}$$

$$\begin{cases} u(\xi,\tau) = \varphi(\xi) + \tau \\ p(\xi,\tau) = \sqrt{1 - \varphi'^2(\xi)} \\ q(\xi,\tau) = \varphi'(\xi) \\ x(\xi,\tau) = \tau\sqrt{1 - \varphi'^2(\xi)} \\ y(\xi,\tau) = \xi + \tau\varphi'(\xi) \end{cases}$$

It is instructive to check straightfoewardly that this is the solution: We have

$$\begin{pmatrix} \xi_x & \xi_y \\ \tau_x & \tau_y \end{pmatrix} = \left(\begin{pmatrix} x_\xi & x_\tau \\ y_\xi & y_\tau \end{pmatrix}\right)^{-1} = \frac{1}{x_\xi y_\tau - x_\tau y_\xi} \begin{pmatrix} y_\tau & -x_\tau \\ -y_\xi & x_\xi \end{pmatrix}$$