Again about strings on the circle

This is third attempt. We have to calculate eigenvalues of the potential energy, which is described by the circulant $N+1\times N+1$ matrix matrix (see the previous blogs)

$$U = M_{ik}x^i x^k, M = \begin{pmatrix} 2 & -1 & 0 & 0 & 0\dots & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0\dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0\dots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1\dots & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2\dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots\dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & -1\dots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0\dots & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & 0\dots & 0 & -1 & 2 \end{pmatrix},$$

This matrix is symmetric, however caclualte its complex eigenvalues:

There are $N+1$ eigenvectors over $\mathbf{C}$. They are:

$$\mathbf{f}_i = \begin{pmatrix} 1 \\ \varepsilon_i \\ \varepsilon_i^2 \\ \dots\varepsilon_i^N \end{pmatrix}, \quad i = 0, 1, \dots N,$$

where ε_i is a root of unity:

$$\varepsilon_k = \exp\left(\frac{2\pi k i}{N+1}\right), \quad (i = \sqrt{-1}).$$

We see that

$$M\mathbf{f}_i = (2 - \varepsilon_i - \varepsilon_i^N)\mathbf{f}_i$$

Notice that eigenvalues are conjugate to each other.