One fact from linear algbera of Lagrangian surfaces

Today is 85 years to Albert Solomonovitch Schwarz...

Let V be finite-dimensional symplectic vector space, i.e. V is equipped with non-degenerate antisymmetric bilinear form $\langle u, v\rangle = \omega(u, v)$:

$$\langle u, v\rangle = -\langle v, u\rangle\,, \quad \langle \mathbf{x}, V\rangle = 0 \Rightarrow \mathbf{x} = 0\,. \tag{1a}$$

Let X be a Lagrangian plane in finite-dimensional symplectiv vector space V, i.e. scalar product vanishes on X

$$\text{for arbitrary } \mathbf{x}, \mathbf{x}' \text{ in } X \quad \langle \mathbf{x}, \mathbf{x}'\rangle = 0\,, \tag{1b}$$

and this cannot be enlarged, i.e. an arbitrary vector $\mathbf{y}$ which is orthogonal to X belongs to X:

$$\text{if } \langle \mathbf{y}, \mathbf{x}\rangle = 0 \text{ for every } \mathbf{x} \in X \text{ then } y \in X\,. \tag{1c}$$

In particular this means that Lagrangian plane is half-dimensional in symplectic vector space: $\dim X = \frac{\dim V}{2} = n$.

We want to study Lagrangian planes which are transversal to X. Denote a space of such Lagrangian planes $\mathcal{L}_X$:

$$\mathcal{L}_X = \{Y\colon\ Y \text{ is Lagrangian plane in } V \text{ and } X \cap Y = 0\}\,. \tag{2a}$$

Theorem

i) The set $\mathcal{L}_X$ of Lagrangian planes which are transversal to Lagrangian plane X, X is in one-one correspondence with a set of , linear operators P on V, such that

$$(Pu, v) + (u, PV) = (u, v)\,, \tag{3a}$$

and

$$\ker P = X\,. \tag{3b}$$

Namely every Lagrangian plane Y which is transversal to the plane X defines the following operator $P = P_Y$:

$$\forall u \in V\,, \quad P_Y(u) = P_Y(\mathbf{x} + \mathbf{y}) = \mathbf{y}\,, \tag{4a}$$

where $u = \mathbf{x} + \mathbf{y}$ is expansion of a vector over transversal Lagrangian planes X and Y, $\mathbf{x} \in X$, $\mathbf{y} \in Y$. Operator $P = P_Y$ in (4a) evidently obeys conditions (3a) and (3b).

Conversely every operator P which obeys equations (3a) and (3b) defines Lagrangian plane

$$Y = Y_P = \operatorname{Im} P\,, \tag{4b}$$

which is transversal to plane X.

The correspondence

Transversal Lagrangian planes $\leftrightarrow$ linear operator P obeying (3a) and (3b)

is reciprocal:

$$P_{Y_P} = P\,, Y_{P_Y} = Y\,.$$

ii) The set of $\mathcal{L}_X$ of Lagrangian planes which is transversal to the plane given Lagrangian plane X is an affine space which is associated with the vector space of symmetric bilinear forms on the factor space $V\backslash X$. In particular

$$\dim \mathcal{L}_X = \frac{nn+1}{2}\,, \quad (\dim V = 2n)\,. \tag{3c}$$

Remark Note that for Lagrangian plane X condition (3b) may be weaken (is equivalent) to condition $P\big|X = 0$, i.e. $X \subseteq \ker P$.

Indeed suppose that X is Lagrangian and operator P vanishes on X. Show that this implies that $X = \ker P$. Let $\mathbf{y}$ be a vector such that $\mathbf{y} \in \ker P$, then we have that for every $\mathbf{x} \in X$

$$\langle P(\mathbf{y}), \mathbf{x}\rangle + \langle \mathbf{y}, P(\mathbf{x})\rangle = \langle \mathbf{y}, \mathbf{x}\rangle \neq 0 \Rightarrow \text{hence due to (1c) } \mathbf{y} \in X\,. \tag{5}$$

This proves that $X = \ker P$.

Now prove (4b), that $Y = \operatorname{Im} P$ is Lagrangian plane transversal to X if conditions (3a,3b) are obeyed.

It follows from the following statement.

$$\ker P = \operatorname{Im}(P - I)\,. \tag{6}$$

Indeed condition $X = \ker P$ in (3b) implies that $\dim \operatorname{Im} P = n$. we have that $\dim \ker P = \dim (P - I) = n$ also. According equation (6) planes $\operatorname{Im} P$ and $\operatorname{Im}(P - I) = \ker P$ intersect by zero vector. Hence we see that planes $\ker P = X$ and $\operatorname{Im} P = Y$ are transversal. Thus Y is transversal to X. It remains to prove that every two vectors in $Y = \operatorname{Im} P$ are orthogonal to each other:

$$\langle \mathbf{y}, \mathbf{y}'\rangle = \langle P\mathbf{y}, \mathbf{y}'\rangle\langle \mathbf{y}, P\mathbf{y}'\rangle = 2\langle \mathbf{y}, \mathbf{y}'\rangle \Rightarrow \langle \mathbf{y}, \mathbf{y}'\rangle = 0\,.$$

It remains to prove equation (6) Sure it can be proved using results of previous blog on 22-nd June [1)], however do it independently: For every $\mathbf{x}\ker P = X$,

$$\mathbf{x} = P(-\mathbf{x}) - I(-\mathbf{x}),$$

hence $\ker P \subseteq \mathrm{Im}\,(P-I)$. Now prove the converse implication. Let $\mathbf{y} = Pu - u \in \mathrm{Im}\,(P_I)$, then due to (3a) for an arbitrary $\mathbf{x} \in X$ we have that

$$\langle \mathbf{y}, \mathbf{x} \rangle = \langle Pu - u, \mathbf{x} \rangle = \langle u, P(\mathbf{x}) \rangle = 0 \Rightarrow \mathbf{y} \in X = \ker P$$

Thus we proved that $Y = \mathrm{Im}\,P$ is Lagrangian plane transversal to the Lagrangian plane X.

Now prove the second part of Theorem.

Operator P obeying conditions (3a) can be codified by symmetric bilinear form

$$Q(u,v) = \langle P(u), v \rangle - \frac{1}{2}\langle u, v \rangle$$

Condition (3b) means that this form vanishes for $u \in X$. If we choose an arbitrary 'point' in $\mathcal{L}_X$ a Lagrangian plane $\mathbf{Y}$ then we come to:

$$\text{for } u = \mathbf{x} + \mathbf{y},\ v = \mathbf{x}' + \mathbf{y}',\ Q(\mathbf{x}, \mathbf{x}') = 0\,, Q(\mathbf{x}, \mathbf{y}') = \frac{1}{2}\langle \mathbf{y}', \mathbf{x} \rangle\,, Q(\mathbf{y}, \mathbf{x}') = \frac{1}{2}\langle \mathbf{y}, \mathbf{x}' \rangle\,,$$

where $\mathbf{x} + \mathbf{y}$ is an expansion over planes X, Y. Thus we see that operator P obeying to equations (3a) and (3b) is codified by bilinear form on Y.

Example Consider $V = T^*\mathbf{R}^n$ with basis ∂_i, ∂^j:

$$\partial_i = \frac{\partial}{\partial q^i}\,, \partial^j = \frac{\partial}{\partial p_j}\,, i, j = 1, \dots, n\,.$$

and with scalar product

$$\langle \partial^i, \partial^j = 0 \rangle\,, \langle \partial^i, \partial_j = \delta^i_j \rangle\,, \langle \partial_i, \partial^j = -\delta^j_i \rangle\,, \langle \partial_i, \partial_j = -0 \rangle\,.$$

and consider X is Lagrangian plane spanned by $\{\partial_i\}$ (coordinates), and transversal plane Y spanned by vectors $\{\partial^j\}$. Any symmetric bilinear form Q^{mn} codifies operator P such that

$$P(\partial_i) = 0\,, P(\partial^k) = \partial^k + Q^{km}\partial_m$$

Let Q_{mn} be

[1)] Note that in the case if only condition (3a) is obeyed, then operator $A = P - \frac{1}{2}I$ belongs to $sp(2)$ and we come to the fact that

$$\ker P = (\mathrm{Im}\ (P-I))^{\perp}$$

(see blog on 22-nd June of this month.)