Homework 1

1 Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a basis of the vector space V. Let $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m\}$ be an ordered set of an arbitrary m vectors in this vector space.

Show that the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m\}$ is linear dependent if $m \geq 4$.

Show that the ordered set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is a basis of V if and only if these three vectors are linear independent.

Show that the ordered set of vectors $\{\mathbf{a}_1, \mathbf{a}_2\}$ is not a basis of V.

Show that the ordered set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is not a basis of V in the case if $\mathbf{a_3} = 0$.

2 Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a basis of the vector space V.

Show that an arbitrary basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_m\}$ also possesses three vectors, i.e. if the ordered sets of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}_m\}$ in this vector space is also a basis, then $m = 3$.

3 Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a basis of the vector space V.

Is a set of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ a basis of V in the case if

a) $\mathbf{e}'_1 = \mathbf{e}_2$, $\mathbf{e}'_2 = \mathbf{e}_1$, $\mathbf{e}'_3 = \mathbf{e}_3$;

b) $\mathbf{e}'_1 = \mathbf{e}_1$, $\mathbf{e}'_2 = \mathbf{e}_1 + 3\mathbf{e}_3$, $\mathbf{e}'_3 = \mathbf{e}_3$;

c) $\mathbf{e}'_1 = \mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{e}'_2 = 3\mathbf{e}_1 - 3\mathbf{e}_2$, $\mathbf{e}'_3 = \mathbf{e}_3$;

d) $\mathbf{e}'_1 = \mathbf{e}_2$, $\mathbf{e}'_2 = \mathbf{e}_1$, $\mathbf{e}'_3 = \mathbf{e}_1 + \mathbf{e}_2 + \lambda\mathbf{e}_3$ (where λ is an arbitrary coefficient)?

4 Show that $(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^2 + x^3y^3$ is a scalar product in $\mathbf{R}^3$.

Show that $(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^2$ *does not define scalar product* in $\mathbf{R}^3$.

Show that $(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^2 - x^3y^3$ *does not define scalar product* in $\mathbf{R}^3$.

Show that $(\mathbf{x}, \mathbf{y}) = x^1y^1 + 3x^2y^2 + 5x^3y^3$ is a scalar product in $\mathbf{R}^3$.

5 The matrix $T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ obeys the conditions $T^tT = I$. Show that

a) $\det T = \pm 1$

b) if $\det T = 1$ then there exists an angle $\varphi : 0 \leq \varphi < 2\pi$ such that $T = T_\varphi$ where

$$T_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \text{ (rotation matrix)}$$

c) if $\det T = -1$ then then there exists an angle $\varphi : 0 \leq \varphi < 2\pi$ such that $T = T_\varphi R$, where $R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (a reflection matrix).

6 Show that for matrix T_φ defined in the previous exercise the following relations are satisfied:

$$T_\varphi^{-1} = T_\varphi^t = T_{-\varphi}\,, \qquad T_{\varphi+\theta} = T_\varphi \cdot T_\theta\ .$$

7 Show that under the transformation $(\mathbf{e}'_1, \mathbf{e}'_2) = (\mathbf{e}_1, \mathbf{e}_2)\, T_\varphi$ an orthonormal basis transforms to an orthonormal one.

How coordinates of vectors change if we rotate the orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2)$ on the angle $\varphi = \frac{\pi}{3}$ anticlockwise?

8 Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal basis of Euclidean space $\mathbf{E}^3$. Consider the ordered set of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ which is expressed via basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as in the exercise 3.

Find out is the ordered set of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ an orthonormal basis of $\mathbf{E}^3$.

Write down explicitly transition matrix from the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to the ordered set of the vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$.

What is the rank of this matrix?

Is this matrix orthogonal?

(you have to consider all cases a),b) c) and d)).

9†. Show that an arbitrary orthogonal transformation of two-dimensional Euclidean space can be considered as a composition of reflections.

10† Prove the Cauchy–Bunyakovsky–Schwarz inequality

$$(\mathbf{x}, \mathbf{y})^2 \leq (\mathbf{x}, \mathbf{x})(\mathbf{y}, \mathbf{y})\,,$$

where $\mathbf{x}, \mathbf{y}$ are arbitrary two vectors and $(\ ,\ )$ is a scalar product in Euclidean space.

Hint: For any two given vectors $\mathbf{x}, \mathbf{y}$ *consider the quadratic polynomial* $At^2 + 2Bt + C$ *where* $A = (\mathbf{x}, \mathbf{x})$, $B = (\mathbf{x}, \mathbf{y})$, $C = (\mathbf{y}, \mathbf{y})^2$. *Show that this polynomial has at most one real root and consider its discriminant.*