NON-LINEAR HOMOMORPHISMS OF ALGEBRAS OF FUNCTIONS ARE INDUCED BY THICK MORPHISMS

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Abstract. In 2014, Voronov introduced the notion of thick morphisms of (super)manifolds as a tool for constructing L_∞-morphisms of homotopy Poisson algebras. Thick morphisms generalise ordinary smooth maps, but are not maps themselves. Nevertheless, they induce pull-backs on C^∞ functions. These pull-backs are in general non-linear maps between the algebras of functions which are so-called "non-linear homomorphisms". By definition, this means that their differentials are algebra homomorphisms in the usual sense. The following conjecture was formulated: an arbitrary non-linear homomorphism of algebras of smooth functions is generated by some thick morphism. We prove here this conjecture in the class of formal functionals. In this way, we extend the well-known result for smooth maps of manifolds and algebra homomorphisms of C^∞ functions and, more generally, provide an analog of classical "functional-algebraic duality" in the non-linear setting.

1. Introduction

A map $\varphi\colon\ M \to N$ defines the linear map

$$\varphi^*\colon\ C^\infty(N) \to C^\infty(M)\,, \tag{1}$$

which is homomorphism of algebras of functions. In 2014 Ted Voronov introduced the notion of a *thick morphism* (see [1], [2]) of manifolds, which generalises ordinary maps. A thick morphism defines a non-linear map $\Phi^*\colon\ C^\infty(N) \to C^\infty(M)$. This notion provides a natural way to construct L_∞ morphisms for homotopy Poisson algebras (see [1],[2], and [4] and also Appendix A). The notion of thick morphisms turns out to be also related with quantum mechanis and the construction of spinor representation (see [3] and [5]). The pull-back $\Phi^*\colon\ C^\infty(N) \to C^\infty(M)$ corresponding to a thick morphism is not in general a homomorphism of algebras (just because it is non-linear). However as it was proved by Voronov, the differential of this non-linear map is a usual pull-back. This motivated him to define so called *non-linear homomorphisms.*

Definition 1. (Th.Voronov, see [2]) Let $\mathbf{A}, \mathbf{B}$ be two algebras. A map L from an algebra $\mathbf{A}$ to an algebra $\mathbf{B}$ is called *a non-linear homomorphism* if at an arbitrary element of algebra $\mathbf{A}$ its derivative is a homomorphism of the algebra $\mathbf{A}$ to the algebra $\mathbf{B}$.

One can say that a thick morphism induces a non-linear homomorphism of algebras of functions in the same way as a usual morphism φ induces usual (linear) homomorphism (1). A natural question was formulated in [2]: is it true that every non-linear algebra homomorphism between algebras of smooth functions arises from a thick morphism as the pull–back? Note that the pull-backs by thick morphisms are formal mappings of algebras. Hence in the above definition of non-linear homomorphisms one can consider formal maps only. We prove here this conjecture for formal maps ("formal functionals").

The structure of the paper is as follows. We recall the construction of thick morphisms, and we define a class of *formal functionals* which are induced by thick morphisms. We recall the proof of Voronov's result that the functional induced by a thick morphism is a non-linear homomorphism (see [1] and [2] for detail). Then we show that the converse implication also holds. In Appendix A we briefly discuss the relation of thick morphisms with L_∞ morphisms of homotopy Poisson algebras. In appendix B we recall some useful polarisation formulae.

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2. Thick morphisms and non-linear functionals

Consider two manifolds M and N. We denote by x^i local coordinates on M, and we denote y^a local coordinates on N. To define the thick morphism $\Phi\colon\ M \Rrightarrow N$ we consider a function, $S = S(x,q)$, where x is the point on M and q is covector in T^*N. We supposee that $S = S(x,q)$ is a formal function, power series over q:

$$S = S(x,q) = S_0(x) + S_1^a(x)q_a + S_2^{ab}(x)q_bq_a + S_3^{cba}(x)q_cq_bq_a + \ldots =$$

$$S_0(x) + S_1^a(x)q_a + S_+(x,q)\,, \text{where } S_+(x,q) = \sum_{k=2}^{\infty} S^{a_1\ldots a_k}q_{a_1}\ldots q_{a_k}\,, \tag{2}$$

coefficients $S_k^{a_1\ldots a_k}(x)$ are usual smooth functions on x.

A formal function $S(x,q)$ is called *generating function of thick morphism.*

Remark 1. In fact $S(x,q)$ is geometrical object which transforms non- trivially under changing of local coordinates (see for detail [2]). Here and below we consider only local coordinates x^i on M and y^a on N.

To generating function $S(x,q)$ corresponds thick morphism $\Phi = \Phi_S:\ M \Rrightarrow N$ which is defined in the following way: it defines pull-back Φ_S^* such that to every smooth function $g(y) \in C^\infty(M)$ corresponds a function

$$f(x) = \Phi^*(g) = g(y) + S(x,q) - y^aq_a \tag{3}$$

where $y^a = y^a(x), q_b = q_b(x)$ are chosen in a way that

$$y^a = \frac{\partial S(x,q)}{\partial q_a}\,, \quad q_b = \frac{\partial g(y)}{\partial y^b}\,. \tag{4}$$

Remark 2. Conditions (4) imply that left hand side of equation (3) does not depend on y^a and q_b:

$$\frac{\partial}{\partial y^a}\left(g(y)+S(x,q)-y^a q_a\right)=0\,,\ \frac{\partial}{\partial q_b}\left(g(y)+S(x,q)-y^a q_a\right)=0\,.$$

In the special case if $S(x,q) = S^a(x)q_a$ $y^a = S^a(x)$ and $\Phi^* g$ is the usual pull-back corresponding to the map $y^a = S^a(x)$:

$$f(x)=\Phi^*(g)=g(y)+S(x,q)-y^a q_a=g\left(S^a(x)\right)\,, \tag{5}$$

and this pull-back corresponds to the usual morphism $y^a = S^a(x)$.

In the general case (if action $S(x,q)$ is not linear over q) maps (3) and (4) become formal maps. They become formal power series in g (see for details also equation (11) below). Namely equation (5) defines the formal functional $L(x,g)$ on $C^\infty(N)$ such that

$$L(x,g)=L_0(x,g)+L_1(x,g)+L_2(x,g)+\cdots=\sum L_k(x,g)\,,\ (g\in C^\infty(N)) \tag{6}$$

where every summand $L_k(x,g)$ takes values in smooth functions on M and it has an order k in g: $L_k(x,\lambda g)=\lambda^k L_k(x,g)$. We suppose that

$$L_k(g)=\int L(x,y_1,\dots,y_k)g(y_1)\dots g(y_k)dy_1\dots dy_k\,, \tag{7}$$

the kernel $L(x,y_1,\dots,y_n)$ of the functional $L_k(x,g)$ can be generalised functions.

Definition 2. We denote by $\mathbf{A}$ the space of all formal functionals which have appearance (6). We denote by $\mathbf{A}_k$ the subspace of functionals which have order k on g, $(k=0,1,2,\dots)$.

For arbitrary functional $L(x,g)\in\mathbf{A}$ (see equation (6)) functional $L_k(x,g)$ is projection of functional $L(x,g)$ on subspace $\mathbf{A}_k$. We sometimes denote this projection by $[L(x,g)]_k$

$$L(x,g)=\sum L_k(x,g)\,,\quad L_k(x,g)=[L(x,g)]_k\,. \tag{8}$$

It is useful to denote by $\mathbf{A}_{\geq k}$ ($\mathbf{A}_{\leq k}$) the subspace of functionals which have order bigger or equal than k (less or equal than k),

$$\mathbf{A}_{\geq k}=\oplus_{i\geq k}A_a\,,\quad \mathbf{A}_{\leq k}=\oplus_{0\leq i\leq k}A_i\,, \tag{9}$$

and we say that two functionals $L_1, L_2\in\mathbf{A}$ coincide up to the order k if $L_1-L_2\in\mathbf{A}_{\geq k+1}$. We will write in this case that

$$L_1(g)=L_2(g)(\operatorname{mod}\mathbf{A}_{k+1})$$

Explain how every formal generating function $S(x,q)$, (see equation (2)) defines thick morphism Φ_S, i.e. how $S(x,q)$ defines a map $\Phi^*_S(g)$ which is a formal functional in $\mathbf{A}$. Functional $\Phi^*_{S(x,q)}(g)$ defines non-linear pull-back, assigning to every smooth function $g\in C^\infty(N)$ a formal sum of smooth functions $\left[\Phi^*_{S(x,q)}(g)\right]_k$, $(k=0,1,2,\dots)$.

$$\Phi^*_{S(x,q)}(g)=\sum\left[\Phi^*_{S(x,q)}(g)\right]_k=\left[\Phi^*_{S(x,q)}(g)\right]_0+\left[\Phi^*_{S(x,q)}(g)\right]_1+\dots\,, \tag{10}$$

where $\left[\Phi^*_{S(x,q)}(g)\right]_k$ is component of the functional $\Phi^*_{S(x,q)}(g)$ which has order k in g (see equation (8)). We will explain how to calculate this map recurrently step by step and we will write explicitly the results of calculations of its first components. (See Propositions 1 and 2)

As it was mentioned above a map $y^a = y^a(x)$ in equation (4) has to be viewed as a formal sum of smooth maps depending on g:

$$y^a(x) = y^a(x,g) = \sum y_k(x,g) = y_0^a(x) + y_1^a(x,g) + \cdots = \tag{11}$$

Here every term $y_k^a(x) = y_k^a(x,g)$ is a smooth map of order k in g:

$$y_k^a(x, \lambda g) = \lambda^k(x,g)\,.$$

We will show how to calculate map (11) step by step recurrently, and we will write the expressions for calculating first few components of this formal map (see Proposition 1 below).

One can see from equations (2) and (4) that initial term $y_0^a(x)$ in equation (11) is equal to

$$y_0^a(x) = \left[\frac{\partial S(x,q)}{\partial q_a}\right]_{q=0} = S_1^a(x)\,, \tag{12}$$

and every next term $y_{k+1}^a(x) = y_{k+1}^a(x,g)$ in (11) is expressed recurrently via previous terms $\{y_0^a(x), \ldots, y_k^a(x)\}$:

$$y_{k+1}^a = \left[\frac{\partial S(x,q)}{\partial q_a}\Big|_{q_a = \frac{\partial g(y)}{\partial y^a}}\Big|_{y^a = y^a_{\leq k}(x)}\right]_{k+1}. \tag{13}$$

Here $y^a_{\leq k}(x) = \sum_{i\leq k} y_i(x)$ according to equation (9), and $[\ ]_r$ means r-th component of the map (see expansion (8)).

We have already expression (12) for initial component $y_0(x)$ of map $y^a(x)$ in equation (11). Write down expression for next components $y_1^a(x)$ and $y_2^a(x)$ of this map. We have

$$y_1^a = \left[\frac{\partial S(x,q)}{\partial q_a}\Big|_{q_a = \frac{\partial g(y)}{\partial y^a}}\Big|_{y^a = y^a_{\leq 0}(x)}\right]_1 =$$

$$\left[\left(S^a(x) + 2S^{ab}(x)\frac{\partial g(y)}{\partial y^a}\right)_{y^a = S^a(x)}\right]_1 = 2S^{ab}(x) g_b^*(x)\,, \tag{14}$$

and

$$y_2^a = \left[\frac{\partial S(x,q)}{\partial q_a}\Big|_{q_a = \frac{\partial g(y)}{\partial y^a}}\Big|_{y^a = y^a_{\leq 1}(x)}\right]_2 =$$

$$\left[\left(S^a(x) + \sum_{j\geq 1}(j+1)S^{ab_1\ldots b_j}(x)\frac{\partial g(y)}{\partial y^{b_1}}\cdots\frac{\partial g(y)}{\partial y^{b_j}}\right)\Big|_{y^a = y_0^a(x) + y_1^a(x)}\right]_2 =$$

$$= \left[\left(S^a(x) + 2S^{ab}(x)\frac{\partial g(y)}{\partial y^b} + 3S^{abc}(x)\frac{\partial g(y)}{\partial y^b}\frac{\partial g(y)}{\partial y^c}\right)\Big|_{y^a = S_0^a(x) + 2S^{ab}g_b^*(x)}\right]_2 =$$

$$= 3S^{abc}(x) g_b^*(x) g_c^*(x) + 4S^{ab}(x) S^{cd}(x) g_{bc}^*(x) g_d^*(x) \tag{15}$$

where in equations (14) and (15) we used notations

$$g^*(x) = g\left(y^a\right)\Big|_{y^a = S_1^a(x)}\,,\quad g_a^*(x) = \frac{\partial g(y)}{\partial y^a}\Big|_{y^a = S_1^a(x)}\,,\quad g_{ab}^*(x) = \frac{\partial^2 g(y)}{\partial y^b \partial y^a}\Big|_{y^a = S_1^a(x)}\,. \tag{16}$$

Thus collecting the answers in equations (11),(12) and (13) we come to

Proposition 1. *For thick morphism $\Phi_{S(x,q)}$ formal map $y^a(x) = y^a(x,g)$ in (11) can be calculated recurrently by the equations (12), (13). In particular up to order $k \leq 2$ it is defined by the following expression: for arbitrary $g \in C^\infty(N)$,*

$$y^a(x) = y^a(x,g) = \underbrace{S_1^a(x)}_{\text{term of order } 0 \text{ in } g} +$$

$$= + \underbrace{2S_2^{ab}(x)g_b^*(x)}_{\text{term of order } 1 \text{ in } g} + \underbrace{3S^{abc}(x)g_b^*(x)g_c^*(x) + 4S^{ab}(x)S^{cd}(x)g_{bc}^*(x)g_d^*(x)}_{\text{term of order } 2 \text{ in } g} \,(\mathrm{mod}\mathbf{A}_3)\,. \quad (17)$$

Use this Proposition to calculate components $[\Phi_S^*(g)]_k$ of functional $\Phi_S^*(g)$.
Due to definition (3) we have that

$$\Phi_S^*(g) = (g(y^a) + S(x,q) - y^a q_a)\,\big|_{y^a = \frac{\partial S(x,q)}{\partial q_a},\, q_a = \frac{\partial g(y)}{\partial y^a}} =$$

$$\left(g(y^a) + S_0(x) - \sum_{k\geq 2}(k-1)S_k^{a_1\ldots a_k}(x)\frac{\partial g(y)}{\partial y^{a_1}}\cdots\frac{\partial g(y)}{\partial y^{a_k}}\right)\big|_{y^a = y_0^a(x) + y_1^1(x) + \ldots}\,, \quad (18)$$

where $y^a = y_0^a(x) + y_1^1(x) + \ldots$ is a formal map (11). Here we used the fact that according to equations (2), (3) and (4)

$$S(x,q) - y^a q_a = S(x,q) - \frac{\partial S(x,q)}{\partial q_a}q_a = \sum_k S_k^{a_1\ldots a_k}(x)q_{a_1}\ldots q_{a_k} - \sum_k kS_k^{a_1\ldots a_k}(x)q_{a_1}\ldots q_{a_k} =$$

$$\sum_k (1-k)S_k^{a_1\ldots a_k}(x)q_{a_1}\ldots q_{a_k}\,.$$

Now using equation (18) and equation (17) in Proposition 1 write down first few components $[\Phi_S^*(g)]_k$ of non-linear functional $\Phi_S^*(g)$

$$\left[(g(y))_{y^a(x)}\right]_{\leq 3} = g\left(y_{\leq 2}^a(x)\right) = g\left(y_0(x) + y_1^a(x) + y_2^a(x)\right) =$$

$$g\left(S_1^a(x) + 2S^{ab}(x)g_b^*(x) + 4S^{ab}(x)S^{cd}(x)g_{bc}^*(x)g_d^*(x)\right) =$$

$$g^*(x) + 2S^{ab}(x)g_a^*(x)g_b^*(x) + 3S^{abc}(x)g_c^*(x)g_b^*(x)g_a^*(x) + 2S^{ab}(x)S^{cd}(x)g_{ab}^*(x)g_a^*(x)g_d^*(x)\,,$$

where we denoted by $\left[g\,(y)_{y^a(x)}\right]_{\leq 3}$ projection of functional $g \mapsto g\,(y^a\,(x))$ on $\mathbf{A}_3$. Hence it follows from equation (18) that

$$[\Phi_S^*(g)]_{\leq 3} = [\Phi_S^*(g)]_0 + [\Phi_S^*(g)]_1 + [\Phi_S^*(g)]_2 + [\Phi_S^*(g)]_3 =$$

$$S_0(x) + g\left(y_{\leq 2}^a(x)\right) - S_2^{ab}(x)\frac{\partial g(y)}{\partial y^a}\frac{\partial g(y)}{\partial y^b}\big|_{y^a = S_0^a(x) + 2S^{ab}(x)g_b^*(x)} -$$

$$-2S_3^{abc}(x)\frac{\partial g(y)}{\partial y^c}\frac{\partial g(y)}{\partial y^b}\frac{\partial g(y)}{\partial y^a}\big|_{y^a = S_0^a(x)}$$

Collecting together the terms we come to formal power sums we come to

Proposition 2. *Formal functional $\Phi_S^*(g)$ corresponding to thick morphism $\Phi_{S(x,q)}$ can be calculated recurrently by equations* (18).

In particular up to the order ≤ 3 it is defined by the following expression

$$\Phi_S^*(g) = \underbrace{S_0(x)}_{\textit{term of order } 0 \textit{ in } g} + \underbrace{g\left(S^a(x)\right)}_{\textit{term of order } 1 \textit{ in } g} + \underbrace{S^{ab}(x)g_b^*(x)g_b^*(x)}_{\textit{terms of order } 2 \textit{ in } g} +$$

$$\underbrace{S^{abc}(x)g_c^*(x)g_b^*(x)g_a^*(x) + 2S^{ac}S^{bd}(x)g_{ab}^*(x)g_d^*(x)g_c^*(x)}_{\textit{terms of order } 3 \textit{ in } g}(\mathrm{mod}\mathbf{A}_4)\,. \tag{19}$$

Thick morphisms define in general non-linear functionals $\Phi_S^*(g)$ belonging to space of formal functionals $\mathbf{A}$ (see definition of formal functionals in 2). As it was mentioned in introduction these non-linear functionals are non-linear homomorphisms. Return to definition 1 of non-linear homomorphisms formulating it for formal functionals.

Definition 3. Let $L = L(x, g)$ be formal functional in $\mathbf{A}$ (see definition 2). According to definition 1 this formal functional is *non-linear homomorphism* if its differential is usual homomorphism, i.e. for every function g there exists a map

$$y^a(x) = K^a(x, g)\,, \tag{20}$$

such that for an arbitrary function h

$$L(g + \varepsilon h) - L(g) = \varepsilon h\left(y^a(x, g)\right)\,, \quad (\varepsilon^2 = 0)\,. \tag{21}$$

The map $y^a(x, g) = K^a(x, g)$ in (20) is in general a formal map:

$$y^a(x, g) = K_0^a(x) + K_1^a(x, g) + K_2^a(x, g) + \cdots =$$

$$K_0^a(x) + \int K_1^a(x, y)g(y)dy + \int K_1^a(x, y_1, y_2)g(y_1)g(y_2)dy_1dy_2 + \ldots \tag{22}$$

Now we formulate

Theorem 1. *Let $\Phi = \Phi_S \colon M \Rrightarrow N$ be an arbitrary thick morphism. Then formal functional $\Phi_S^*(g)$ is non-linear homomorphism, i.e. for arbitrary functions g there exists a map $y^a(x) = y^a(x, g)$ such that for an arbitrary function h, ($h \in C^\infty N$)*

$$\Phi_S^*(g + \varepsilon h) - \Phi_S^*(g) = \varepsilon h\left(y^a(x, g)\right)\,, \quad \varepsilon^2 = 0\,. \tag{23}$$

This very important observation was made by Voronov in his pioneer work [1] on thick morphisms.

Example 1. For example consider pull-back

$$L(g) = \Phi_S^*(g)\,. \tag{24}$$

According to Theorem 1 this is non-linear homomorphism. One can show that the map $y^a = y^a(x, g)$ in equation (11) which we constructed above (see equations (12), (13) and equation (17) in Proposition 1) is just formal map $K^a(x, g)$ (20) for this functional. (See the proof of Theorem 1 in the next section.)

For non-linear homomorphisms we will use the notion of so called *support map*.

Definition 4. If $L(g)$ is a functional which is non-linear homomorphism then a map $K_0^a(x)$ corresponding to the functional $L(g)$, which is the zeroth part of the formal map $K^a(x)$ (see equations (20) and (22)) will be called `support map corresponding to functional` $L(g)$.

Example 2. Consider functional $L(x,g)$ corresponding to thick morphism (see equation (24) in example 1). If $S(x,q) = S_0(x) + S_1^a(x)q_a + \dots$ is generating function (2) which defines this thick morphism, then it follows from equations (11) and (12) that support map is equal to $K_0^a(x) = S_1^a(x)$ (see also equation (17).)

Definition 5. Let L be an arbitrary functional in $\mathbf{A}$,

$$L(x,g) = \sum_k L_k(x,g)\,, \text{ where } L_k(x,g) = [L(x,g)]_k \in \mathbf{A}_k$$

(see equations (6) and (8)). Taking the values of this functional on linear functions $y = y^a l_a$ we assign to this functional, *formal function*

$$S_L(x,q) = L(x,g)\big|_{g=y^a q_a} = S_0(x) + \sum_k S_k^{a_1\dots a_k}(x) q_{a_1}\dots q_{a_k}\,, \tag{25}$$

where tensors $\{S_k^{a_1\dots a_k}(x)\}$ can be expressed through polarised form of functionals L_k (see equations (55) and (56) in Appendix B):

$$S_k^{a_1\dots a_k}(x) = L_k^{\text{polaris.}}\left(x, y^{a_1}, \dots, y^{a_k}\right)\,,$$

where $\{y^a\}$ are coordinates on N. E.g.

$$S_L^{ab}(x) = L_2^{\text{polaris.}}(x, y^a, y^b) = \frac{1}{2}\left(L_2\left(y^a + y^b\right) - L_2\left(y^a\right) - L_2\left(y^a\right)\right)\,.$$

We say that $S_L(x,q)$ is *formal function associated with functional L.*

Let $S = S(x,q)$ be an arbitrary formal generating function (2). Let Φ_S be a thick morphism defined by this generating function, and let $L(x,g)$ be a formal functional, $L(x,g) \in \mathbf{A}$, which defines pull-back of functions produced by this thick morphism: $L(x,g) = \Phi^*_{S(x,q)}(g)$. Then one can see that formal generating function associated with functional $L(x,g) = \Phi^*_{S(x,q)}(g)$ coincides with formal generating function $S(x,q)$:

$$L(x,g) = \Phi^*_{S(x,q)}(g) \Rightarrow S_L(x,q) \equiv S(x,q)\,. \tag{26}$$

Indeed in the case if function $g = y^a l_a$ is linear then calculations of pull-back $\Phi^*_S(g)$ by formulae (3) and (4) become evident. Indeed in this case we immediately come to equation (26) since according to equations (3) and (4)

$$f(x) = g(y) + S(x,q) - y^a q_a = S(x,l)$$

because for linear function $g(y) = y^a q_a$.

It turns out that converse implication is also valid for non-linear homomorphisms.

Theorem 2. *Let $L = L(x,g) \in \mathbf{A}$ be an arbitrary non-linear homomorphism, and let $S(x,q)$ be an action associated to it. Then*

$$L(g) = \Phi^*_S(g)\,.$$

This is main result of this paper.

3. Proof of the Theorems

We recall here the proof of Theorem 1 and give a proof of Theorem 2.

3.1. **Proof of Theorem 1.** Check straightforwardly that a formal map $y^a(x,g)$ constructed in Proposition 1 (see equations (12), (13) and equation (17) in Proposition 1) is just a map corresponding to function g i.e. equation

$$\Phi_S^*(g+\varepsilon h) - \Phi_S^*(g) = \varepsilon h(y(x,g))\,, (\varepsilon^2 = 0) \tag{27}$$

is satisfied. (See also example 1.)

Using definition (3) we see that in (27)

$$\Phi_S^*(g+\varepsilon h) - \Phi_S^*(g) =$$
$$\left[(g(y)+\varepsilon h(y))\big|_{y^a=y^a(x,g+\varepsilon h)} + S(x,q)\big|_{q_a=q_a(x,g+\varepsilon h)} - y^a q_a\big|_{y^a=y^a(x,g+\varepsilon h), q_a=q_a(x,g+\varepsilon h)}\right] -$$
$$\left[(g(y))\big|_{y^a=y^a(x,g)} + S(x,q)\big|_{q_a=q_a(x,g)} - y^a q_a\big|_{y^a=y^a(x,g), q_a=q_a(x,g)}\right] \tag{28}$$

Here we introduced notation

$$q_a(x,g) = \frac{\partial g(y)}{\partial y^a}\big|_{y^a=y^a(x,g)}\,.$$

To see that right hand sides of equations (27) and (28) coincide we note that in equation (28) the following relations hold

$$(g(y)+\varepsilon h(y))\big|_{y^a=y^a(x,g+\varepsilon h)} - g(y)\big|_{y^a=y^a(x,g)} = \varepsilon \frac{\partial g(y)}{\partial y^a}\big|_{y^a=y^a(x,g)} t^a = \varepsilon q_a(x,g) t^a\,, \tag{29}$$

$$S(x,q)\big|_{q_a=q_a(x,g+\varepsilon h)} - S(x,q)\big|_{q_a=q_a(x,g)} = \varepsilon \frac{\partial S(x,q)}{\partial q_a}\big|_{y^a=y^a(x,g)} r_a = \varepsilon y^a(x,g) r_a(x,g;h)\,, \tag{30}$$

and

$$y^a q_a\big|_{y^a=y^a(x,g+\varepsilon h), q_a=q_a(x,g)+\varepsilon h)} - y^a q_a\big|_{y^a=y^a(x,g,q_a=q_a(x,g)} = \varepsilon t^a q_a(x,g) + \varepsilon y^a(x,g) r_a \tag{31}$$

In equations (29), (30) and (31) we used notations t^a, r_b such that

$$y^a(x,g+\varepsilon h) - y^a(x,g) = \varepsilon t^a \text{ and } q_a(x,g+\varepsilon h) - q_a(x,g) = \varepsilon r_a\,.$$

Comparing right hand sides of equations (29), (30) and (31) we come to conclusion that equation (27) is obeyed. ■

3.2. **Proof of Theorem 2.** To prove Theorem 2 we will formulate two lemmas.

Lemma 1. *Let $L = L(x,g) = \sum_{k\geq 0} L_k(x,g)$ be an arbitrary functional in $\mathbf{A}$ which is non-linear homomorphism (see definition3). Let $S_0(x)$ be a function which is equal to value of this functional on function $g = 0$*

$$S_0(x) = L(x,g)\big|_{g=0}\,, \tag{32}$$

we will call sometimes this function an affine component of functional L.

Let a map $K_0^a(x)$ be a support map corresponding to this functional (see definition 3). Then

$$L(g) = S_0(x) + g(K_0^a(x)) \;(\mathrm{mod}\mathbf{A}_2)$$

Lemma 2. *Let $L(x,g)$ and $\widetilde{L}(x,g)$ be two functionals on $\mathbf{A}$ which both are non-linear homomorphisms, and which coincide up to the order $k-1$ ($k \geq 2$):*

$$\begin{gathered}\widetilde{L}(g)=\sum_i \widetilde{L}_i(x,g)\,, \quad \widetilde{L}_i(x,g)\in A_i\\ L(g)=\sum_i L_i(x,g)\,, \quad L_i(x,g)\in A_i\\ \widetilde{L}_j=L_j \text{ for } j\leq k-1\end{gathered}$$

Then the difference of these functionals in the order k is given by k-linear functional $T_k(x,\partial g)\in A_k$:

$$\widetilde{L}_k(x,g)-L_k(x,g)=T_k(\partial g)$$

where

$$\mathbf{A}_k \ni T_k(\partial g)=T^{a_1\dots a_k}(x)g^*_{a_1}(x)\dots g^*_{a_k} \quad \text{and } g^*_a(x)=\frac{\partial g(y)}{\partial y}\Big|_{y^a=K^a(x)}\,, \tag{33}$$

$K^a_0(x)$ is a support map 4 which is the same for both these functionals, and tensor $T^{a_1\dots a_k}$ is defined by equation

$$T^{a_1\dots a_k}(x)=\widetilde{L}^{\text{polaris.}}_k(x,y^{a_1},\dots,y^{a_k})-L^{\text{polaris.}}_k(x,y^{a_1},\dots,y^{a_k}) \tag{34}$$

where $\widetilde{L}_k(x,g)$ and $L_k(x,g)$ are the terms of order k in the expansion (6) *of functionals $\widetilde{L}(x,g)$ and $L(x,g)$), and respectively $\widetilde{L}^{\text{polaris.}}_k(x,g_1,\dots,g_k)$ is polarised form of functional $\widetilde{L}_k(x,g)$, and $L^{\text{polaris.}}_k(x,g_1,\dots,g_k)$ is polarised form of functional $L_k(x,g)$ (see equation* (57) *in definition 6 in Appendix B).*

Prove Theorem 2 using these lemmas.

Let $L=L(g)$ be a functional in $\mathbf{A}$ which is non-linear homomorphism, i.e, condition (21) (see definition 3) holds for this functional, and

$$L(x,g)=L_0(x,g)+L_1(x,g)+\dots+L_k(x,g)+\dots\,,$$

where every functional $L_r(x,g)$ has order r in g: $L_r\in A_r$.

Consider an action $S(x,q)$ associated with this functional (see equation (25) in definition 5).

Consider the sequence of thick morphisms $\{\Phi_k\}$ ($k=0,1,2,\dots$) such that the thick morphism Φ_k is generated by the action

$$\mathbf{S}_k(x,q)=S_0(x)+S^a_1(x)q_a+S^{ab}_2(x)q_aq_b+\dots+S^{a_1\dots a_k}_k(x)q_{a_1}\dots q_{a_k}\,,$$

and respectively the sequence $\{\Phi^*_k(g)=\Phi^*_{\mathbf{S}_k}(g)\}$ of functionals, generated by these thick morphisms.

Prove that for every k, non-linear homomorphism $L(g)$ coincides up to terms of order k in g with functional $\Phi^*_{\mathbf{S}_k}$:

$$L(g)=\Phi^*_k(g)(\operatorname{mod}\mathbf{A}_{k+1})\,. \tag{35}$$

This will be the proof of Theorem 2.

Remark 3. Thick morphisms $\{\Phi_k\}$ can be viewed as a sequense of morphisms tending to morphims Φ_S.

We prove equation (35) by induction. If $k=1$ then $\mathbf{S}_1(x)=S_0(x)+S^a_1(x)q_a$ and

$$\Phi^*_1(g)=S_0(x)+g(S^a_1(x))=L(g)(\operatorname{mod}\mathbf{A}_2)\,.$$

due to Lemma 1. Thus equation (35) is obeyed if $k=1$. Now suppose that equation (35) is obeyed for $k=m$, $m\geq 1$. Prove it for $k=m+1$. Denote by

$$\widetilde{L}(g)=\Phi^*_m(g)\,. \tag{36}$$

Due to Theorem 1 this functional is also non-linear homomorphism. Both functionals are non-linear homomorphisms and by inductive hypothesis functionals $L(g)$ and $\tilde{L}(g)$ coincide up to the order m. Hence lemma 2 implies that there exists tensor $T^{a_1\dots a_{m+1}}(x)$ such that

$$L(g)=\widetilde{L}(g)+T_{m+1}(\partial g)=\Phi^*_{\mathbf{S}_m}(g)+T_{m+1}(\partial g)\,(\mathrm{mod}\mathbf{A}_{m+2}), \tag{37}$$

where

$$T_{m+1}(\partial g)=T^{a_1\dots a_{m+1}}(x)g^*_{a_1}(x)\dots g^*_{a_{m+1}}(x)\,,\left(g^*_a(x)=\frac{\partial g(y)}{\partial y^a}\Big|_{y^a=S^a_1(x)}\right)\,,$$

and tensor $T^{a_1\dots a_{m+1}}(x)$ according to equation (34) is defined by equation

$$T^{a_1,\dots,a_{m+1}}_{m+1}=L^{\text{polaris.}}_{m+1}\left(x,y^{a_1},\dots,y^{a_{m+1}}\right)-\widetilde{L}^{\text{polaris.}}_{m+1}\left(x,y^{a_1},\dots,y^{a_{m+1}}\right)\,, \tag{38}$$

where $L^{\text{polaris.}}_{m+1}$ is polarised form of functional $L_{m+1}(g)$ which contains terms of order $m+1$ of functional $L(g)$. Respectively functional $\widetilde{L}^{\text{polaris.}}_{m+1}$ is polarised form of functional $\widetilde{L}_{m+1}(g)$ which contains terms of order $m+1$ of functional $\widetilde{L}(g)=\Phi_{\mathbf{S}_m}(g)$. It is easy to see that functional $\widetilde{L}^{\text{polaris.}}_{m+1}$ is vanished on arbitrary linear functions:

$$\widetilde{L}\left(x,l_1,\dots,l_{m+1}\right)=0\,,\quad \text{if functions } l_i \text{ are linear: } l_i=y^al_{ai},\ i=1,\dots,m+1\,.: \tag{39}$$

Indeed functional $\widetilde{L}(g)=\Phi^*_{\mathbf{S}_m}(g)$ is assigned to the action $\mathbf{S}_m(x,q)$ which is a polynomial of order $\leq m$, hence due to equation (26) it vanishes for arbitrary linear function $g=y^al_a$, hence polarised form vanishes also on linear functions (see equation (55) in Appendix B). Thus we come to condition (39). This condition means that in particular

$$\widetilde{L}^{\text{polaris.}}_{m+1}\left(x,y^{a_1},\dots,y^{a_{m+1}}\right)=0\,,\quad \text{for } \widetilde{L}(g)=\Phi^*_{m+1}(g)\,,$$

hence we come to conclusion that tensor $T^{a_1\dots a_{m+1}}(x)$ in equation (38) is equal to $S^{a_1\dots a_{m+1}}(x)$.

We see that

$$L(g)=\Phi^*_m(g)+S_{m+1}(\partial g)\ (\mathrm{mod}\mathbf{A}_{m+2})\,. \tag{40}$$

On the other hand up to the terms of order $m+1$, right hand sight of this equation is equal to Φ^*_{m+1}:

$$\Phi^*_{m+1}(g)=\Phi^*_m(g)+S_{m+1}(\partial g)(\mathrm{mod}\mathbf{A}_{m+2})\,. \tag{41}$$

One can see it straightforwardly using equation (3) or it is much easier to check equation taking differential of this equation. Namely taking differential of equation (41) and using equations (13) and (27) we come to equation

$$h\left(y^a_{m+1}(x,g)\right)=h\left(y^a_m(x,g)\right)+S^{aa_1\dots a_m}_{m+1}g^*_{a_1}\dots g^*_{a_m}(\mathrm{mod}\mathbf{A}_{m+1})\,,$$

where $y^a_{\mathbf{S}_k}(x,g$ is a map $y^a(x,g)$ corresponding to thick morphism $\Phi_{\mathbf{S}_k}$ ($\Phi^*_{\mathbf{S}_k}(g+\varepsilon h)-\Phi^*_{\mathbf{S}_k}(g)=h\left(y^a_{\mathbf{S}_{m+1}}(x,g)\right)h$). Comparing left hand sides of equations (40) and (41) we see that equation (35) holds for $k=m+1$. This ends the proof.

∎.

It remians to prove lemmas.

4. Proofs of lemmas

4.1. **Proof of the Lemma 1.** Let $L = L(x, g)$ be a functional in $\mathbf{A}$ which is non-linear homomorphism.

$$L(x, g) = L_0(x) + L_1(x, g) + \cdots = L_0(x) + L_1(x, g) (\mathrm{mod}\, \mathbf{A}_2) \tag{42}$$

If we put $g = 0$ we come to $L_0(x) = S_0(x) = L(g)\big|_{g=0}$.

Differentiate equation (42). Using equation (22) we come to

$$L(x, g+\varepsilon h) - L(x, g) = \varepsilon h \left(y^a(x, g)\right) = \varepsilon h \left(K_0^a(x) + K_1^a(x, g) + \dots\right) = \varepsilon h \left(K_0^a(x)\right) \, (\mathrm{mod}\mathbf{A}_1)$$

This is true for arbitrary smooth function h. This implies that $L_1(x, g) = g\left(K_0^a(x)\right)$. Hence

$$L(g) = L_0(g) + L_1(g) \, (\mathrm{mod}\mathbf{A}_2) = S_0(x) + g\left(K_0^a(x)\right) \, (\mathrm{mod}\mathbf{A}_2) =$$

$$S_0(x) + \int K(x, y) g(y) dy + \text{terms of order} \geq 2 \text{ in } g\,, \quad \text{with } K(x, y) = \delta(y^a - K_0^a(x))\,.$$

First lemma is proved.

4.2. **Proof of lemma 2.** Let functionals $L(g)$ and $\widetilde{L}(g)$ both be functionals which are non-linear homomorphisms (see definition 3). Suppose these functionals coincide up to the order $k-1$ ($k = 2, 3, \dots$). According to expansion (6) this means that difference of these functionals is a functional $T_k(g)$ of order k

$$\widetilde{L}(g) - L(g) = T_k(x, g) \in \mathbf{A}_{k+1} \quad \text{i.e. } \widetilde{L}(g) - L(g) - T_k(g) = 0 (\mathrm{mod}\mathbf{A}_{k+1})\,, \tag{43}$$

where

$$T_k(x, g) = \int T(x, y_1, \dots, y_k) g(y_1) \dots g(y_k) dy_1 \dots dy_k\,. \tag{44}$$

Take the differential of equation (43). We come to

$$\left(\widetilde{L}(g + \varepsilon h) - \widetilde{L}(g)\right) - \left(L(g + \varepsilon h) - L(g)\right) = \varepsilon h \left(\widetilde{y^a}(x, g)\right) - \varepsilon h \left(y^a(x, g)\right) =$$

$$= T_k(x, g + \varepsilon h) - T_k(g) + \text{terms of order} \geq k \text{ in } g =$$

$$= \varepsilon k T_k^{\text{polaris.}} \left(h, \underbrace{g, \dots, g}_{k-1 \text{ times}} \right) + \text{terms of order} \geq k \text{ in } g = \tag{45}$$

Here $T_k^{\text{polaris.}} = T_k(x, g_1, g_2, \dots, g_k)$ is the polarisation of the form $T_k(x, g)$ (see equation (55) in definition 6). Recall that if function $T(x, y_1, y_2, \dots, y_k)$ which correspond to functional $T_k(g)$ in equation (44) is symmetric function on variables $y_1, \dots y_k$ then (see equation (7))

$$T_k^{\text{polaris.}}(x, g_1, \dots, g_k) = \int T(x, y_1, y_2, \dots, y_k) g_1(y_1) g_2(y_2) \dots g_k(y_k) dy_1 dy_2 \dots dy_k\,.$$

Formal maps $y^a(x, g)$ corresponding to differential $dL(g) = L(g+\varepsilon h) - L(g)$ of functional $L(g)$ and $\widetilde{y}^a(x, g)$ corresponding to differential $d\widetilde{L}(g) = \widetilde{L}(g+\varepsilon h) - \widetilde{L}(g)$ of functional $\widetilde{L}(g)$ according to equation (22) are given by formal power series

$$y^a(x, g) = K_0^a(x) + K_1^a(x, g) + \cdots + K_{k-2}^a(x, g) + K_{k-1}^a(x, g) + \text{terms of order} \geq k \text{ in } g$$

and

$$\widetilde{y}^a(x, g) = \widetilde{K}_0^a(x) + \widetilde{K}_1^a(x, g) + \cdots + \widetilde{K}_{k-2}^a(x, g) + \widetilde{K}_{k-1}^a(x, g) + \text{terms of order} \geq k \text{ in } g\,. \tag{46}$$

Recall that here $K^a_r(x,g)$ and $\widetilde{K}^a_r(x,g)$ are maps of order r in g:

$$K^a_r(x,g)=\int K(x,y_1,\dots,y_r)g(y_1)\dots g(y_r)dy_1\dots dy_r\,.$$

Since functionals $L(g)$ and $\widetilde{L}(g)$ coincide up to the order $k-1$, their differentials coincide up to the order $k-2$. Hence it follows from equation (45) that in equation (46) all the maps K^a_r coincide with maps $\widetilde{K}^a_r$ for $r=0,1,2,\dots,k-2$

$$K^a_0(x)=\widetilde{K}^a_0(x)\,,\dots\,,K^a_{k-2}(x,g)=\widetilde{K}^a_{k-2}(x,g)\,,$$

and it is the difference between maps $\widetilde{K}_{k-1}$ and K_{k-1} which produces the functional $T_k(x,g)$.

Rewrite equation (45) projecting all terms on subspace A_{k-1}. We come to

$$\left[\left(\widetilde{L}(g+\varepsilon h)-\widetilde{L}(g)\right)\right]_{k-1}-\left[L(g+\varepsilon h)-L(g)\right]_{k-1}=\varepsilon\left[h\left(\widetilde{y^a}(x,g)\right)-\varepsilon h\left(y^a(x,g)\right)\right]_{k-1}=$$

$$\frac{\partial h}{\partial y^a}\big|_{y^a=K^a_0(x)}\left[\widetilde{K}^a_{k-1}(x,g)-K^a_{k-1}(x,g)\right]=\frac{\partial h}{\partial y^a}\big|_{y^a=K^a_0(x)}P^a_{k-1}(x,g)=$$

$$=T_k(x,g+\varepsilon h)-T_k(g)=$$

$$=\varepsilon k T_k^{\text{polaris.}}\left(x,h,\underbrace{g,\dots,g}_{k-1\text{ times}}\right)\,.$$

where we denote by $P^a_{k-1}(x,g)$ the difference between maps $\widetilde{K}^a_{k-1}(x,g)$ and $K^a_{k-1}(x,g)$

$$P^a_{k-1}(x,g)=\widetilde{K}^a_{k-1}(x,g)-K^a_{k-1}(x,g)=\int P^a_{k-1}(x,y_1,\dots,y_{k-1})g(y_1)\dots g(y_{k-1})dy_1\dots dy_k\,.$$

The map $P^a_{k-1}(x,g)$ has order $n-1$ over g. Consider polarisation $P^{a\,\text{polaris.}}_{k-1}(x,_1,\dots,g_{k-1})$ (55) of this map. Equation (4.2) implies

$$\frac{\partial h}{\partial y^a}\big|_{y^a=K^a_0(x)}P^{a\,\text{polaris.}}_{k-1}(x_1,\dots,g_{k-1})=\varepsilon k T_k^{\text{polaris.}}\left(x,h,g_1,\dots,g_{k-1}\right)\big|_{g_1=\dots=g_{k-1}=g}\,.$$

Thus we come to equation

$$T_k^{\text{polaris.}}\left(x,g_1,\dots,g_k\right)=\frac{1}{k}\frac{\partial g_1}{\partial y^a}\big|_{y^a=K^a_0(x)}P^{a\,\text{polaris.}}_{k-1}(x,g_2,\dots,g_k)\,,\tag{47}$$

where $g_1,\dots,g_k$ are arbitrary functions and left hand side of this equation is symmetric with respect to transposition of functions $\{g_1,\dots,g_k\}$. It follows from equation (47) that

$$P^{a\,\text{polaris.}}_{k-1}(x,g_2,\dots,g_k)\,,=kT_k^{\text{polaris.}}\left(x,y^a,g_2,\dots,g_k\right)$$

hence

$$T_k^{\text{polaris.}}\left(x,g_1,\dots,g_k\right)=\frac{\partial g_1}{\partial y^a}\big|_{y^a=K^a_0(x)}T_k^{\text{polaris.}}\left(x,y^a,g_2,\dots,g_k\right)\,.\tag{48}$$

Equation (48) and symmetricity of functional $T_k(x,g_1,\dots,g_k)$ imply that

$$T_k^{\text{polaris.}}\left(x,g_1,g_2,\dots,g_k\right)=\frac{\partial g_1}{\partial y^a}\big|_{y^a=K^a_0(x)}T_k^{\text{polaris.}}\left(x,y^a,g_2,\dots,g_k\right)=$$

$$T_k^{\text{polaris.}}\left(x,g_2,g_1,\dots,g_k\right)=\frac{\partial g_2}{\partial y^a}\big|_{y^a=K^a_0(x)}T_k^{\text{polaris.}}\left(x,y^a,g_1,\dots,g_k\right)=\dots=$$

$$\frac{\partial g_1}{\partial y^{a_1}}\Big|_{y^{a_1}=K_0^{a_1}(x)}\dots\frac{\partial g_k}{\partial y^{a_k}}\Big|_{y^{a_k}=K_0^{a_k}(x)}T_k^{\text{polaris.}}\left(x,y^{a_1},\dots,y^{a_k}\right)=$$

$$=g^*_{a_1}(x)\dots g^*_{a_k}(x)T^{a_1\dots a_k}(x)\,, \tag{49}$$

where

$$T^{a_1\dots a_k}(x)=T_k\left(x,y^{a_1},\dots,y^{a_k}\right)=\text{and } g^*_a(x)=\frac{\partial g}{\partial y^a}\Big|_{y^a=K_0^a(x)}\,.$$

Now returning to equation (43) and comparing it with formulation of lemma 2 we come to proof of lemma 2:

$$\widetilde{L}_k(g)-L_k(g)=T_k(x,g_1,\dots,g_k)\big|_{g_1=\dots=g_k=g}=T_k(\partial g)\,.$$

5. Appendix A. Thick morphisms and L_∞ maps

We briefly here discuss why thick morphisms is an adequate tool to describe L_∞-morphisms of homotopy Poisson algebras (see [1] and [2] for detail). For this purpose we need to consider thick morphisms of supermanifolds. However we can catch some improtant features considering just usual manifolds. We first consider thick morphisms for usual manifolds, and show that in this case thick morphisms describe morphisms of algebras of functions on these manifolds which are provided with multilinear symmetric brackets. It turns out that if we consider supermanifold, then under some assumptions these algebras become homotopy Poisson algebras.

Let M be an arbitrary manifold, and $H=H(x,p)$ be a function (Hamiltonian) on cotangent bundle T^*M. This Hamiltonian H defines the series of symmetric brackets on M via canonical symplectic structure on T^*M

$$\langle\emptyset\rangle_H\,,\langle f_1\rangle_H\,,\langle f_1,f_2\rangle_H\,,\langle f_1,f_2,f_3\rangle_H\,,\dots\langle f_1,f_2,\dots,f_k\rangle_H\,,$$

where

$$\langle\emptyset\rangle_H=H(x,p)\big|_{p=0}=H_0(x)$$

$$\langle f_1\rangle_H=(H,f_1)\big|_{p=0}=H_1^a(x)\frac{\partial f_1(x)}{\partial x^a}\,,$$

$$\langle f_1,f_2\rangle_H=((H,f_1),f_2)\big|_{p=0}=H_1^{ab}(x)\frac{\partial f_1(x)}{\partial x^a}\frac{\partial f_2(x)}{\partial x^a}\,,$$

and so on:

$$\langle f_1,f_2,\dots,f_k\rangle_H=\underbrace{(\dots(}_{k\text{ times}}H,f_1),f_2)\dots f_k)\big|_{p=0}=H_k^{a_1\dots a_k}(x)\frac{\partial f_1(x)}{\partial x^{a_1}}\dots\frac{\partial f_k(x)}{\partial x^{a_k}}\,. \tag{50}$$

Here $(\_,\_)$ is Poisson bracket on T^*M corresponding to canonical symplectic structure:

$$(f(x,p),g(x,p))=\frac{\partial f(x,p)}{\partial p_a}\frac{\partial g(x,p)}{\partial x^a}-\frac{\partial g(x,p)}{\partial p_a}\frac{\partial f(x,p)}{\partial x^a}\,. \tag{51}$$

We suppose that Hamiltonian $H=H(x,p)$ is a *formal Hamiltonian*, i.e. formal function, power series over p:

$$H=H(x,p)=H_0(x)+H_1^a(x)p_a+H_2^{ab}(x)p_bp_a+H_3^{abc}(x)p_cp_bp_a+\dots$$

where all coefficients are smooth functions on x.

Remark 4. All these formulae are written in local coordinates (x^a, p_b) in T^*M corresponding to local coordinates x^a on M (if $x^{a'}$ are new local coordinates on M, then new local coordinates $(x^{a'}, p_{b'})$) are

$$x^{a'} = x^{a'}(x)\,, p_{b'} = \frac{\partial x^b(x')}{\partial x_{b'}} p_b\,. \tag{52}$$

Notice that every Hamiltonian $H(x,p)$ defines vector field

$$X_H = \int H\left(f\,(x)\,, \frac{\partial f\,(x)}{\partial x}\right) dx$$

on the space of function. Vector field X_H assigns to every function $f \in C^\infty(M)$ infinitesimal curve

$$f + \varepsilon X_H = f(x) + \varepsilon H\left(f\,(x)\,, \frac{\partial f\,(x)}{\partial x}\right)\,, \quad (\varepsilon^2 = 0)\,. \tag{53}$$

Now consider two manifolds M and N. Let $H_M(x,p)$ be formal Hamiltonian on M, and let $H_N(y,q)$ be formal Hamiltonian on N. Hamiltonian $H_M(x,p)$ induces on M the sequence of multilienar symmetric brackets $\{\langle f_1, \dots, f_p\rangle_M\}$ on functions on M, and respectively Hamiltonian $H_N(y,q)$ induces on N the sequence of multilinear symmetric brackets $\{\langle g_1, \dots, g_q\rangle_M\}$ on functions on N $(p, q = 0, 1, 2, 3, \dots)$.

We say that formal functional $L(g)$ is morphism of multilinear symmetric brackets on N to multilinear symmetric brackets on M if vector fields X_{H_M} and X_{H_N} are connected by functional $L(g)$, i.e. according to formulae (53)

$$L\,(g + \varepsilon X_N) = L\,(g) + \varepsilon X_M\,.$$

Consider thick morphism $\Phi_S \colon M \Rrightarrow N$ generated by $S(x,q)$ and consider formal functional $\Phi_S^*(g)$ on $C^\infty(N)$ defined by this thick morphism (see equations (2)—(19) and remark **??**).

We say that Hamiltonians H_M and H_N are S-related if

$$H_M\left(x, \frac{\partial S\,(x,q)}{\partial x}\right) \equiv H_N\left(\frac{\partial S\,(x,q)}{\partial q}, q\right)$$

The following remarkable theorem takes place:

Theorem 3. *(Voronov, 2014) If Hamiltonians H_M and H_N are S-related, then formal functional $L(g)$ defined by thick morphism Φ_S, $L(g) = \Phi_S^*(g)$ defines morphisms of multilinear brackets*
$\{\langle f_1, \dots, f_p\rangle_M\}$ *and* $\{\langle g_1, \dots, g_q\rangle_M\}$ $\{\langle g_1, \dots, g_q\rangle_M\}$. *In other words thick morphism connects these brackets.*

Now consider the case of supermanifolds.

In this case all the constructions above will remain the same, just in some formulae will appear a sign factor. (See [1] and [2] for detail). In particular arbitrary Hamiltonian $H = H(x,p)$ which is a function on cotangent bundle T^*M to supermanifold M will define the collection of symmetric brackets like in the case (50). On the other hand if Hamiltonian H_M is *odd* and Hamiltonian H_M obeys condition

$$(H_M, H_M) \equiv 0\,, \tag{54}$$

then these brackets will become *homotopy Poisson brackets*. This is famous construction of homotopy Poisson brackets derived by odd Hamiltonian H_M which obeys so called master-equation (54) (see for detail [4]).

6. Appendix B. Polarisation of functionals

It is useful to consider polarised form of formal functionals.

Definition 6. Let $L_k(x,g)$ be formal functional of order k, $L_k(x,g) \in \mathbf{A}_k$ (See for definition 2.) Polarisation of functional $L_k(x,g)$ is the functional $L_k^{\text{polaris.}}(x,g_1,\dots,g_k)$ which linearly depends on k functions $g_1,\dots,g_k$ such that for every function g

$$L_k(x,g) = L_k^{\text{polaris.}}(x,g_1,\dots,g_k)\big|_{g_1=g_2=\dots=g_k=g} . \tag{55}$$

Using elementary combinatoric one can express polarised form $L_k^{\text{polaris.}}(x,g_1,\dots,g_k)$ explicitly in terms of functional $L_k(x,g)$, $(L_k \in A_k)$:

$$L_k^{\text{polaris.}}(x,g_1,\dots,g_k) = \frac{1}{k!}\sum (-1)^{k-n} L_k\left(x, g_{i_1}+\dots+g_{i_n}\right) , \tag{56}$$

where summation goes over all non-empty subsets of the set $\{g_1,\dots,g_k\}$. E.g. if $L=L_3$ then

$$L^{\text{polaris.}}(x,g_1,g_2,g_r) = \frac{1}{6}\left(L_3\left(x,g_1+g_2+g_3\right) - L_3\left(x,g_1+g_2\right) - L_3\left(x,g_1+g_3\right) - L_3\left(x,g_2+g_3\right)\right.$$
$$\left. + L_3\left(x,g_1\right) + L_3\left(x,g_2\right) + L_3\left(x,g_3\right)\right) .$$

If functional $L_r(x,g)$ is expressed through (generalised) functions $L(x,y_1,\dots,y_r)$ (see equation (7)) such that it is symmetric with respect to coordinates $y_1,\dots,y_r$ then

$$L^{\text{polaris.}}(g_1,\dots,g_r) = \int L(x,y_1,\dots,y_r) g_1(y_1)\dots g(y_r) dy_1 \dots dy_r . \tag{57}$$

It is useful also to note that if $L(x,g) = L_0(x) + L_1(x,g) + \dots + L_n(x,g)$ then for every k: $k=0,1,\dots,n$

$$L_k^{\text{polaris.}}(x,g_1,\dots,g_k) = \frac{1}{k!}\sum (-1)^{k-n} L\left(x, g_{i_1}+\dots+g_{i_n}\right) , \tag{58}$$

where summation goes over all subsets of the set $\{g_1,\dots,g_k\}$ including empty subset. (For empty subset $L(x,\emptyset) = L_0(x)$.)

References

[1] Th.Th. Voronov "Nonlinear pullbacks" of functions and L_∞-morphisms for homotopy Poisson structures. J. Geom. Phys. 111 (2017), 94-110. arXiv:1409.6475

[2] Th.Th. Voronov Microformal geometry and homotopy algebras. Proc. Steklov Inst. Math. 302 (2018), 88-129. arXiv:1411.6720

[3] Th. Th. Voronov. Thick morphisms of supermanifolds and oscillatory integral operators. *Russian Math. Surveys*, 71(4):784–786, 2016.

[4] H.M.Khudaverdian, Th.Th.Voronov. Thick morphisms, higher Koszul brackets, and L_∞-algebroids. arXiv:1808.10049

[5] H.M.Khudaverdian, Th,Th,Voronov. Thick morphisms of supermanifolds, quantum mechanics and spinor representation. J. Geom. Phys. 113 (2019), DOI: 10.1016/j.geomphys.2019.103540, arXiv:1909.00290

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