22 December 2018

Taylor identity

I know this identity "hundred years". Karabegov's proof makes me to re- alise that this is really effective.

Let $f = f(x)$ be a smooth function. Then integrating by parts we come to

$$f(x) = f(0) + \int_0^x \frac{df(t))}{dt} dt = f(0) + \underbrace{\int_0^x \frac{df(t))}{dt} \cdot 1 dt}_{\mathrm{I}} =$$

$$f(0) + \int_0^x \frac{d}{dt} f(t) \frac{d}{dt}\,(t-x)\,dt = f(0) + \frac{d}{dt} f(t)(t-x)\Big|_0^x - \int_0^x \frac{d^2}{dt^2} f(t)\,(t-x)\,dt =$$

$$f(0) + f'(0)x + \underbrace{\int_0^x \frac{d^2}{dt^2} f(t)\,(x-t)\,dt}_{\mathrm{II}} =$$

$$f(0) + f'(0)x - \frac{1}{2}\int_0^x \frac{d^2}{dt^2} f(t) \frac{d}{dt}\left((t-x)^2\right) dt =$$

$$f(0) + f'(0)x - \frac{1}{2}\frac{d^2}{dt^2} f(t)\,(t-x)^2\,\Big|_0^x + \frac{1}{2}\int_0^x \frac{d^3}{dt^3} f(t)\,(t-x)^2\,dt =$$

$$f(0) + f'(0)x + f^{''}(0)\frac{x^2}{2} + \frac{1}{2}\underbrace{\int_0^x \frac{d^3}{dt^3} f(t)\,(x-t)^2\,dt}_{\mathrm{III}} =$$

$$f(0) + f'(0)x + f^{''}(0)\frac{x^2}{2} + \frac{1}{6}\int_0^x \frac{d^3}{dt^3} f(t) \frac{d}{dt}\left((x-t)^3\right) dt =$$

$$f(0) + f'(0)x + f^{''}(0)\frac{x^2}{2} + \frac{1}{6}\frac{d^3}{dt^3} f(t)\,(t-x)^3\,\Big|_0^x - \frac{1}{6}\int_0^x \frac{d^4}{dt^4} f(t)\,(x-t)^3\,dt =$$

$$f(0) + f'(0)x + f^{''}(0)\frac{x^2}{2} + f^{'''}(0)\frac{x^3}{6} + \frac{1}{6}\underbrace{\int_0^x \frac{d^4}{dt^4} f(t)\,(x-t)^3\,dt}_{\mathrm{IV}} =$$

$$f(0) + f'(0)x + f^{''}(0)\frac{x^2}{2} + f^{'''}(0)\frac{x^3}{6} - \frac{1}{24}\int_0^x \frac{d^4}{dt^4} f(t) \frac{d}{dt}\left((t-x)^4\right) dt =$$

$$f(0) + f'(0)x + f^{''}(0)\frac{x^2}{2} + f^{'''}(0)\frac{x^3}{6} - \frac{1}{24}\frac{d^4}{dt^4} f(t)\,(t-x)^4\,\Big|_0^x + \frac{1}{24}\int_0^x \frac{d^5}{dt^5} f(t)\,(t-x)^4\,dt =$$

$$f(0) + f'(0)x + f^{''}(0)\frac{x^2}{2} + f^{'''}(0)\frac{x^3}{6} + f^{''''}(0)\frac{x^4}{24} + \frac{1}{24}\underbrace{\int_0^x \frac{d^5}{dt^5} f(t)\,(x-t)^4\,dt}_{\mathrm{IV}} =$$

and so on:

$$\ldots = \sum_{k=1}^{n} f^{(k)}(0)\frac{x^k}{k!} + \frac{1}{n!}\int_0^x \frac{d^{n+1}}{dt^{n+1}} f(t)(x-t)^n dt$$

This identity is very useful to prove the Sasha Karabegov' question (see Etudes,/Algebra/taylor4.tex)∎

Appendix 1

`Here I reproduce the Karabegov's proof of the Theorem (3)..`

Shortly speaking his proof is the following: if $f(x_0)=0$ then for all $x \leq x_0$, $f(x)=0$ also since $f'(x) \geq 0$ for $x \leq x_0$. Thus all derivatives of the smooth function f vanish at the point x_0. Hence it follows from the identity (1) that for all x and for all n,

$$f(x) = \frac{1}{n!}\int_{x_0}^x \frac{d^{n+1}}{dt^{n+1}} f(t)(x-t)^n dt\,. \tag{A1}$$

for an arbitary n. Hence for every x_1 and for every n

$$\int_{x_0}^{x_1} f(x)dx = \int_{x_0}^{x_1} dx \left(\frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n dt\right) =$$

$$\frac{1}{n!}\int_{x_0}^{x_1} dt \left(\int_t^x f^{(n+1)}(t)(x-t)^n dx\right) = \frac{1}{(n+1)!}\int_{x_0}^{x_1} f^{(n+1)}(t)(x_1-t)^{n+1}\,. \tag{A2}$$

Choose an arbitrary $x_1 > x_0$. Then $x_1 - t < x_1 - x_0$. Thus it follows from equations (A2) and (A1) that

$$\text{for } x_1 > x_0 \quad \int_{x_0}^{x_1} f(x)dx = \frac{1}{(n+1)!}\int_{x_0}^{x_1} f^{(n+1)}(t)(x_1-t)^{n+1} \leq$$

$$\frac{x_1-x_0}{n}\left(\frac{1}{n!}\int_{x_0}^{x_1} f^{(n+1)}(t)(x_1-t)^n\right) = \frac{x_1-x_0}{n} f(x_1) \Rightarrow f(x_1)=0 \ \blacksquare$$

since this unequality holds or arbitrary n.