MATH 61082

Three hours

THE UNIVERSITY OF MANCHESTER

RIEMANNIAN GEOMETRY

15-th May/June 2014
09:45—12:45

ANSWER ANY THREE OF QUESTIONS 1—4 AND QUESTION 5
All questions are worth 20 marks

Electronic calculators may not be used

P.T.O.

1.

(a) Explain what is meant by saying that G is a Riemannian metric on a manifold M.

Consider the upper half plane ($y > 0$) in $\mathbf{R}^2$ equipped with the Riemannian metric $G = \sigma(x,y)(dx^2 + dy^2)$.

Explain why $\sigma(x,y) > 0$.

Consider in this Riemannian manifold a curve C such that

$$C\colon \quad \begin{cases} x = 1 \\ y = a + t \end{cases} , \quad 0 \le t \le 1\,, \ (a > 0)\,.$$

Find the length of this curve in the case $\sigma(x,y) = \frac{1}{y^2}$ (the Lobachevsky metric).

[6 marks]

(b) Consider a surface (the upper sheet of a cone) in $\mathbf{E}^3$

$$\mathbf{r}(h,\varphi)\colon \quad \begin{cases} x = 2h\cos\varphi \\ y = 2h\sin\varphi \\ z = h \end{cases} , \quad h > 0, 0 \le \varphi < 2\pi\,.$$

Calculate the Riemannian metric on this surface induced by the canonical metric on Euclidean space $\mathbf{E}^3$.

Show that this surface is locally Euclidean.

[6 marks]

(c) Consider a Riemannian manifold M^n with a metric $G = g_{ik}dx^i dx^k$.
Write down the formula for the volume element on M^n (area element for $n = 2$).

Find the volume of a domain $0 < h < H$ of the cone considered in the part (**b**).

It is well-known that the metric $G = \frac{dx^2+dy^2}{y^2}$ of the Lobachevsky plane is not locally Euclidean. However show that there exist coordinates $u = u(x,y)$, $v = v(x,y)$ such that in these coordinates the area element is equal to $dudv$.

[8 marks]

P.T.O.

2.

(a) Explain what is meant by an affine connection on a manifold.

Let ∇ be an affine connection on a 2-dimensional manifold M such that in local coordinates (u, v) all Christoffel symbols vanish except $\Gamma^u_{vv} = u$ and $\Gamma^v_{uu} = v$. Calculate the vector field $\nabla_{\mathbf{X}}\mathbf{X}$, where $\mathbf{X} = \frac{\partial}{\partial u} + u\frac{\partial}{\partial v}$.

[5 marks]

(b) Explain what is meant by the induced connection on a surface in Euclidean space.

Calculate the induced connection on the cylindrical surface in $\mathbf{E}^3$

$$\mathbf{r}(h, \varphi): \quad \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases} .$$

[6 marks]

(c) Give a detailed formulation of the Levi-Civita Theorem. In particular write down the expression for the Christoffel symbols Γ^i_{km} of the Levi-Civita connection in terms of the Riemannian metric $G = g_{ik}(x)dx^i dx^k$.

Prove that the induced connection on a surface $\mathbf{r} = \mathbf{r}(u, v)$ in $\mathbf{E}^3$ is equal to the Levi-Civita connection of the Riemannnian metric induced by the canonical metric on Euclidean space $\mathbf{E}^3$.

A Riemannian metric in local coordinates u, v is equal to $G = e^{-u^2-v^2}(du^2 + dv^2)$.
Calculate the Christoffel symbols of the Levi-Civita connection at the point $u = v = 0$.

[9 marks]

P.T.O.

3.

(a) Define a geodesic on a Riemannian manifold as a parameterised curve.

Write down the differential equations for geodesics in terms of the Christoffel symbols.

Explain why the great circles are the geodesics on the sphere.

[7 marks]

(b) Explain what is meant by the Lagrangian of a "free" particle on a Riemannian manifold.

Explain what is the relation between the Lagrangian of a free particle and the differential equations for geodesics.

Calculate the Christoffel symbols on the Lobachevsky plane.
(You may use the Lagrangian of a "free" particle on this plane $L = \frac{1}{2}\frac{\dot{x}^2+\dot{y}^2}{y^2}$.)

Consider an arbitrary geodesic $\mathbf{r} = (x(t), y(t))$ on the Lobachevsky plane.
Show that the magnitude $I(t) = \frac{\dot{x}(t)}{y^2(t)}$ is preserved along the geodesic.

[8 marks]

(c) Consider a plane $\mathbf{R}^2$ equipped with the Riemannian metric $G = \frac{4R^4(du^2+dv^2)}{(R^2+u^2+v^2)^2}$.
We know that it is isometric to the sphere of radius R in $\mathbf{E}^3$ (without the North pole) in stereographic coordinates $u = \frac{Rx}{R-z}$, $v = \frac{Ry}{R-z}$, $(x^2 + y^2 + z^2 = R^2)$.

Consider the parallel transport of the vector $\mathbf{A} = \partial_u$ attached at the point $u = R, v = 0$ along the circle $u^2 + v^2 = R^2$ with respect to this Riemannian metric.
Show that during the parallel transport along this circle it will always be orthogonal to this circle.

[5 marks]

P.T.O.

4.

(a) Consider the sphere of radius R in Euclidean space $\mathbf{E}^3$

$$\mathbf{r}(\theta,\varphi)\colon \quad \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases} .$$

Let $\mathbf{e}, \mathbf{f}$ be unit vectors in the directions of the vectors $\mathbf{r}_\theta = \frac{\partial \mathbf{r}}{\partial\theta}$ and $\mathbf{r}_\varphi = \frac{\partial \mathbf{r}}{\partial\varphi}$ and $\mathbf{n}$ be a unit normal vector to the sphere.
Express these vectors explicitly.

For the obtained orthonormal basis $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ calculate the 1-forms a, b and c in the derivation formula

$$d\begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix} .$$

Deduce from these calculations the mean curvature and the Gaussian curvature of the sphere.

[7 marks]

(b) Give a definition of the curvature tensor for a manifold equipped with a connection.

On two-dimensional Riemannian manifold with coordinates x^1, x^2 consider the vector fields $\mathbf{A} = \frac{\partial}{\partial x^1}$, $\mathbf{B} = \frac{\partial}{\partial x^2}$, $\mathbf{X} = (1+x^1x^2)\frac{\partial}{\partial x^2}$, and the vector field $\mathbf{Y} = (\nabla_\mathbf{A}\nabla_\mathbf{B} - \nabla_\mathbf{B}\nabla_\mathbf{A})\,\mathbf{X}$, where ∇ is a connection.
Calculate the value of the field $\mathbf{Y}$ at the point $x^1 = x^2 = 0$ if the curvature tensor of the connection ∇ is such that $R^1{}_{212} = 1$ and $R^2{}_{212} = 0$ at this point.

[5 marks]

(c) State the relation between the Riemann curvature tensor of the Levi-Civita connection of a surface in $\mathbf{E}^3$ and its Gaussian curvature K.

Explain why the sphere is not a locally Euclidean Riemannian manifold.

On the sphere of radius R give an example of local coordinates in the vicinity of an arbitrary point $\mathbf{p}$ such that in these coordinates standard Riemannian metric of the sphere is equal to $du^2 + dv^2$ at this point $\mathbf{p}$.

[8 marks]

P.T.O.

The following question is compulsory.

5.

(a) Explain what is meant by saying that F is an isometry between two Riemannian manifolds.

Consider the plane $\mathbf{R}^2$ with coordinates (x, y) and with the Riemannian metric

$$G_{(1)} = \frac{a(dx^2 + dy^2)}{(1 + x^2 + y^2)^2}, \quad (a > 0),$$

and the sphere of radius R (without the North pole) with standard metric G which in stereographic coordinates (u, v) has the appearance

$$G_{(2)} = \frac{4R^4(du^2 + dv^2)}{(R^2 + u^2 + v^2)^2}.$$

Find an isometry F: $\begin{cases} u = u(x, y) \\ v = v(x, y) \end{cases}$ between these two Riemannian manifolds in the case of $a = 4R^2$.

Explain why, in the case where $a \neq 4R^2$, there is no isometry between these Riemannian manifolds.

[10 marks]

(b) Describe all infinitesimal isometries (Killing vector fields) of the Lobachevsky plane (the upper half plane $(y > 0)$ with the metric $G = \frac{dx^2+dy^2}{y^2}$), and deduce equations for the geodesics.

You may use the fact that translations $\begin{cases} x' = x + a \\ y' = y \end{cases}$, homotheties $\begin{cases} x' = \lambda x \\ y' = \lambda y \end{cases}$, $(\lambda > 0)$ and inversion $\begin{cases} x' = \frac{x}{x^2+y^2} \\ y' = \frac{y}{x^2+y^2} \end{cases}$ are isometries of the Lobachevsky plane.

[10 marks]

END OF EXAMINATION PAPER