Homework 9. Solutions.

1 *Let ∇ be a connection on n-dimensional manifold M and $\{R^i{}_{rmn}\}$ be the components of the curvature tensor of a connection ∇ in local coordinates $(x^1, x^2, \dots, x^n)$.*

a) For arbitrary vector fields $\mathbf{A}, \mathbf{B}$ and $\mathbf{D}$ calculate the vector field

$$(\nabla_{\mathbf{A}}\nabla_{\mathbf{B}} - \nabla_{\mathbf{B}}\nabla_{\mathbf{A}})\,\mathbf{D} - \nabla_{\mathbf{C}}\mathbf{D}\,,$$

where the vector field $\mathbf{C}$ is a commutator of vector fields $\mathbf{A}$ and $\mathbf{B}$:

$$\mathbf{C} = C^i\frac{\partial}{\partial x^i} = [\mathbf{A}, \mathbf{B}] = \left(A^m\frac{\partial B^i(x)}{\partial x^m} - B^m\frac{\partial A^i(x)}{\partial x^m}\right)\frac{\partial}{\partial x^i}\,. \tag{1.0}$$

b) Calculate the vector field

$$(\nabla_{\mathbf{A}}\nabla_{\mathbf{B}} - \nabla_{\mathbf{B}}\nabla_{\mathbf{A}})\,\mathbf{D}$$

in the case if for vector fields $\mathbf{A}$ and $\mathbf{B}$ components A^i and B^m are constants (in the local coordinates $(x^1, \dots, x^n)$)

c) Calculate the vector field

$$(\nabla_{\mathbf{A}}\nabla_{\mathbf{B}} - \nabla_{\mathbf{B}}\nabla_{\mathbf{A}})\,\mathbf{A} - \nabla_{\mathbf{A}}\mathbf{A}$$

in the case if $\mathbf{A} = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}$, $\mathbf{B} = x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2}$.

(You have to express the answers in terms of components of the vector fields and components of the curvature tensor $R^i{}_{rmn}$.)

a) According to the definition of the curvature tensor for every vector fields $\mathbf{X} = X^m\partial_m, \mathbf{Y} = Y^m\partial_m$ and $\mathbf{Z} = Z^m\partial_m$ we have that $\mathcal{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z} =$

$$\mathcal{R}(X^m\partial_m, Y^n\partial_n)(Z^r\partial_r) = \left(\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}} - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}} - \nabla_{[\mathbf{X},\mathbf{Y}]}\right)\mathbf{Z} = Z^r R^i{}_{rmn}X^m Y^n\partial_i\,.$$

Hence for vector fields $\mathbf{A}, \mathbf{B}, \mathbf{C} = [\mathbf{A}, \mathbf{B}]$ and we have that

$$\left(\nabla_{\mathbf{A}}\nabla_{\mathbf{B}} - \nabla_{\mathbf{B}}\nabla_{\mathbf{A}} - \nabla_{[\mathbf{A},\mathbf{B}]}\right)\mathbf{D} = (\nabla_{\mathbf{A}}\nabla_{\mathbf{B}} - \nabla_{\mathbf{B}}\nabla_{\mathbf{A}})\,\mathbf{D} - \nabla_{\mathbf{C}}\mathbf{D} = \mathcal{R}(\mathbf{A}, \mathbf{B})\mathbf{D} = D^r R^i{}_{rmn}A^m B^n\partial_i\,. \tag{1.1}$$

b) in the case if in the local coordinates $(x^1, \dots, x^n)$ for vector fields $\mathbf{A}$ and $\mathbf{B}$ components A^i and B^m are constants then the commutator of these vector fields $\mathbf{C} = [\mathbf{A}, \mathbf{B}]$ vanishes: $\mathbf{C} = 0$ (see the formula (1.0)). Hence according to the formula (1.1) above we have that

$$(\nabla_{\mathbf{A}}\nabla_{\mathbf{B}} - \nabla_{\mathbf{B}}\nabla_{\mathbf{A}})\,\mathbf{D} - \nabla_{\mathbf{C}}\mathbf{D} = (\nabla_{\mathbf{A}}\nabla_{\mathbf{B}} - \nabla_{\mathbf{B}}\nabla_{\mathbf{A}})\,\mathbf{D} = \mathcal{R}(\mathbf{A}, \mathbf{B})\mathbf{D} = D^r R^i{}_{rmn}A^m B^n\partial_i\,.$$

c) In the case if $\mathbf{A} = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}$, $\mathbf{B} = x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2}$ their commutator vector field $\mathbf{C}$ equals to

$$\mathbf{AB} - \mathbf{BA} = \left(\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}\right)\left(x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2}\right) - \left(x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2}\right)\left(\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}\right)$$

$$\frac{\partial}{\partial x^1} + x^1\frac{\partial^2}{\partial x^1\partial x^1} + x^1\frac{\partial^2}{\partial x^2\partial x^1} + x^2\frac{\partial^2}{\partial x^1\partial x^2} + \frac{\partial}{\partial x^2} + x^2\frac{\partial^2}{\partial x^2\partial x^2} -$$

$$-x^1\frac{\partial^2}{\partial x^1\partial x^1} - x^2\frac{\partial^2}{\partial x^2\partial x^1} - x^1\frac{\partial^2}{\partial x^1\partial x^2} - x^2\frac{\partial^2}{\partial x^2\partial x^2} = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} = \mathbf{A}$$

We have that $[\mathbf{A}, \mathbf{B}] = \mathbf{A}$. Hence according to the formula (1.1) we have that

$$(\nabla_{\mathbf{A}}\nabla_{\mathbf{B}} - \nabla_{\mathbf{B}}\nabla_{\mathbf{A}})\,\mathbf{A} - \nabla_{\mathbf{A}}\mathbf{A} = (\nabla_{\mathbf{A}}\nabla_{\mathbf{B}} - \nabla_{\mathbf{B}}\nabla_{\mathbf{A}})\,\mathbf{A} - \nabla_{[\mathbf{A},\mathbf{B}]}\mathbf{A} = \left(\nabla_{\mathbf{A}}\nabla_{\mathbf{B}} - \nabla_{\mathbf{B}}\nabla_{\mathbf{A}} - \nabla_{[\mathbf{A},\mathbf{B}]}\right)\mathbf{A} =$$

$$\mathcal{R}(\mathbf{A}, \mathbf{B})\mathbf{A} = A^r R^i{}_{rmn}A^m B^n\partial_i$$

Since $A=\frac{\partial}{\partial x^1}+\frac{\partial}{\partial x^2}$ then $\mathcal{R}(\mathbf{A},\mathbf{B})\mathbf{A}=A^r R^i{}_{rmn}A^mB^n=A^1R^i{}_{1mn}A^mB^n\partial_i+A^2R^i{}_{2mn}A^mB^n\partial_i$ (all components A^i vanish for $i=2,3,\dots$)

Now using the antisymmetricity $\mathcal{R}^i{}_{rmn}=-R^i{}_{rnm}$ and the fact that $\mathbf{A}=\frac{\partial}{\partial x^1}+\frac{\partial}{\partial x^2}$, $x^1\frac{\partial}{\partial x^1}+x^2\frac{\partial}{\partial x^2}$ we have

$$\mathcal{R}(\mathbf{A},\mathbf{B})\mathbf{A}=A^1R^i{}_{1mn}A^mB^n\partial_i+A^2R^i{}_{2mn}A^mB^n\partial_i=A^1R^i{}_{112}(A^1B^2-A^2B^1)\partial_i+A^2R^i{}_{212}(A^1B^2-A^2B^1)\partial_i$$
$$=(R^i{}_{112}+R^i{}_{112})(x^2-x^1)\partial_i\,,$$

since $A^1=A^2=1, A^i=0$ for $i=3,4,\dots$ and $B^1=x^1, B^2=x^2$ and $B^i=0$ for $i=3,4,\dots$.

Consider a surface M in $\mathbf{E}^3$ defined by the equation

$$\begin{cases} x=u\\ y=v\\ z=F(u,v)\end{cases} \qquad (2.0)\ .$$

2 * *Calculate explicitly the component R_{1212} of the Riemannian curvature tensor at the point* $\mathbf{p}$ *with coordinates $u=v=0$ in the case if $F(u,v)=\frac{1}{2}(au^2+2buv+bv^2)$, where a,b,c are parameters.*

*Solution**

The function $F(u,v)=F(u,v)=\frac{1}{2}(au^2+2buv+bv^2)$ obeys the conditions that $F_u=F_V=0$ at the origin, the point $u=v=0$. We perform the calculation for an arbitrary function $F(u,v)$ which obeys the conditions $F_u=F_V=0$ at the origin, the point $u=v=0$, i.e. the point $u=v=0$ is stationary point for the function F. Geometrically this means that we consider the surface such that the plane OXY is tangent to the plane at the origin and the axis OZ is orthogonal to the surface at this point.

First of all recall the expression for Riemannian metric for the surface (2.0):

$$G=\begin{pmatrix}1+F_u^2 & F_uF_v\\ F_vF_u & 1+F_v^2\end{pmatrix} \qquad (2.1)$$

(see the file "The solution of the problem 5 in the coursework, revisited.")

Note that Riemannian metric g_{ik} in (2.1) is defined by unity matrix $g=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $g_{uu}=g_{vv}=1$, $g_{uv}=g_{vu}=0$ at the point $u=v=0$ since $\mathbf{p}$ *is stationary point ($F_u=F_v=0$ at the point* $\mathbf{p}$*): $G=\begin{pmatrix}1+F_u^2 & F_uF_v\\ F_vF_u & 1+F_v^2\end{pmatrix}\big|_{\mathbf{p}}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$. Hence the components of the tensor $R^i{}_{kmn}$ and $R_{ikmn}=g_{ij}R^j{}_{kmn}$ at the point* $\mathbf{p}$ *are the same.*

The components of $R^i{}_{kmn}$ are defined by the formula

$$R^i{}_{kmn}=\partial_m\Gamma^i_{nk}+\Gamma^i_{mp}\Gamma^p_{nk}-\partial_n\Gamma^i_{mk}-\Gamma^i_{np}\Gamma^p_{mk} \qquad (2.2)$$

Notice that at the point $\mathbf{p}$ *not only the matrix of the metric g_{ik} equals to unity matrix $g=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, but more: Christoffel symbols vanish at this point in coordinates u,v since the derivatives of metric at this point vanish. (Why they vanish: see in detail the file "The solution of the problem 5 in the coursework, revisited.")*

Hence to calculate $R^i{}_{kmn}$ at the point $\mathbf{p}$ *one can consider more simple formula:*

$$R^i{}_{kmn}|_{\mathbf{p}}=\partial_m\Gamma^i_{nk}|_{\mathbf{p}}-\partial_n\Gamma^i_{mk}|_{\mathbf{p}} \qquad (2.3)$$

Try to calculate in a more "economical" way. Due to Levi-Civita formula

$$\Gamma^i_{mk}=\frac{1}{2}g^{ij}\left(\frac{\partial g_{jm}}{\partial x^k}+\frac{\partial g_{jk}}{\partial x^m}-\frac{\partial g_{mk}}{\partial x^j}\right) \qquad (2.4)$$

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* *The solution below is almost the same as the proof of the Proposition about the relation between Gaussian and scalar curvature for surfaces in $\mathbf{E}^3$ (see the section 5.)*

Since metric g_{ik} equals to unity matrix $g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ at the point $\mathbf{p}$ hence g^{ij} is unity matrix also:

$$g^{ik}|_{\mathbf{p}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \delta^{ik} \tag{2.5}$$

(We denote δ^{ik} the unity matrix: all diagonal components equal to 1, all other components equal to zero. (It is so called Kronecker symbols)) Moreover we know also that all the first derivatives of the metric vanish at the point $\mathbf{p}$:

$$\frac{\partial g_{ik}}{\partial x^m}|_{\mathbf{p}} = 0\,. \tag{2.6}$$

Hence it follows from the formulae (2.5) and (2.6) that for an arbitrary indices i, j, k, m, n

$$\frac{\partial}{\partial x^i}\left(g^{km}\frac{\partial g_{pr}}{\partial x^j}\right)|_{\mathbf{p}} = \frac{\partial g^{km}}{\partial x^i}|_{\mathbf{p}}\frac{\partial g_{pr}}{\partial x^j}|_{\mathbf{p}} + g^{km}|_{\mathbf{p}}\frac{\partial^2 g_{pr}}{\partial x^i \partial x^j}|_{\mathbf{p}} = \delta^{km}\frac{\partial^2 g_{pr}}{\partial x^i \partial x^j}|_{\mathbf{p}}$$

In particular it follows from this formula that

$$\partial_n \Gamma^i_{mk}|_{\mathbf{p}} = \frac{\partial}{\partial x^n}\left(\frac{1}{2}g^{ij}\left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^m} - \frac{\partial g_{mk}}{\partial x^j}\right)\right)|_{\mathbf{p}} = \frac{1}{2}\delta^{ij}\left(\frac{\partial^2 g_{jm}}{\partial x^n \partial x^k} + \frac{\partial^2 g_{jk}}{\partial x^n \partial x^m} - \frac{\partial^2 g_{mk}}{\partial x^n \partial x^j}\right)|_{\mathbf{p}} \tag{2.7}$$

Now we are ready to calculate $\partial_n \Gamma^i_{mk}|_{\mathbf{p}}$ using the last formula (2.7) and the formula (2.1) for the metric. Remember that we want to calculate $R^1{}_{212}$ which is equal at the point $\mathbf{p}$ according to (2.3) to

$$R^1{}_{212}|_{\mathbf{p}} = \partial_1 \Gamma^1_{22}|_{\mathbf{p}} - \partial_2 \Gamma^1_{12}|_{\mathbf{p}} \tag{2.3}$$

and according to (2.7)

$$\partial_1 \Gamma^1_{22} = \frac{1}{2}\delta^{1j}\left(\frac{\partial^2 g_{j2}}{\partial x^1 \partial x^2} + \frac{\partial^2 g_{j2}}{\partial x^1 \partial x^2} - \frac{\partial^2 g_{22}}{\partial x^1 \partial x^j}\right)|_{\mathbf{p}} = \frac{1}{2}\left(\frac{\partial^2 g_{12}}{\partial x^1 \partial x^2} + \frac{\partial^2 g_{12}}{\partial x^1 \partial x^2} - \frac{\partial^2 g_{22}}{\partial x^1 \partial x^1}\right)|_{\mathbf{p}} = \frac{\partial^2 g_{12}}{\partial x^1 \partial x^2} - \frac{1}{2}\frac{\partial^2 g_{22}}{\partial x^1 \partial x^1}|_{\mathbf{p}},$$

$$\partial_2 \Gamma^1_{12} = \frac{1}{2}\delta^{1j}\left(\frac{\partial^2 g_{j1}}{\partial x^2 \partial x^2} + \frac{\partial^2 g_{j2}}{\partial x^2 \partial x^1} - \frac{\partial^2 g_{12}}{\partial x^2 \partial x^j}\right)|_{\mathbf{p}} = \frac{1}{2}\left(\frac{\partial^2 g_{11}}{\partial x^2 \partial x^2} + \frac{\partial^2 g_{12}}{\partial x^2 \partial x^1} - \frac{\partial^2 g_{12}}{\partial x^2 \partial x^1}\right)|_{\mathbf{p}} = \frac{1}{2}\frac{\partial^2 g_{11}}{\partial x^2 \partial x^2}|_{\mathbf{p}},$$

Hence

$$R_{1212}|_{\mathbf{p}} = R^1{}_{212}|_{\mathbf{p}} = \partial_1 \Gamma^1_{22}|_{\mathbf{p}} - \partial_2 \Gamma^1_{12}|_{\mathbf{p}} = \left(\frac{\partial^2 g_{12}}{\partial x^1 \partial x^2} - \frac{1}{2}\frac{\partial^2 g_{22}}{\partial x^1 \partial x^1} - \frac{1}{2}\frac{\partial^2 g_{11}}{\partial x^2 \partial x^2}\right)|_{\mathbf{p}}$$

Now using (2.1) calculate second derivatives $\frac{\partial^2 g_{12}}{\partial x^1 \partial x^2}$, $\frac{\partial^2 g_{11}}{\partial x^2 \partial x^2}$ and $\frac{\partial^2 g_{22}}{\partial x^1 \partial x^1}$:

$$\frac{\partial^2 g_{12}}{\partial x^1 \partial x^2}|_{\mathbf{p}} = \frac{\partial^2 (F_u F_v)}{\partial u \partial v}\Big|_{\mathbf{p}} = \frac{\partial}{\partial u}\left(F_u F_{vv} + F_{uv} F_v\right)|_{\mathbf{p}} = \left(F_{uu} F_{vv} + F_{uv}^2\right)|_{\mathbf{p}}\,.$$

since $F_u = F_v = 0$ at the point $\mathbf{p}$ ($x^1 = u, x^2 = v$). Analogoulsy

$$\frac{\partial^2 g_{11}}{\partial x^2 \partial x^2}|_{\mathbf{p}} = \frac{\partial^2 (1 + F_u^2)}{\partial v \partial v}\Big|_{\mathbf{p}} = \frac{\partial}{\partial v}\left(2F_u F_{uv}\right)|_{\mathbf{p}} = 2F_{uv}^2|_{\mathbf{p}}$$

and

$$\frac{\partial^2 g_{22}}{\partial x^1 \partial x^1}|_{\mathbf{p}} = \frac{\partial^2 (1 + F_v^2)}{\partial u \partial u}\Big|_{\mathbf{p}} = \frac{\partial}{\partial u}\left(2F_v F_{uv}\right)|_{\mathbf{p}} = 2F_{uv}^2|_{\mathbf{p}}\,,$$

We have finally that

$$R_{1212}|_{\mathbf{p}} = R^1{}_{212}|_{\mathbf{p}} = \partial_1 \Gamma^1_{22}|_{\mathbf{p}} - \partial_2 \Gamma^1_{12}|_{\mathbf{p}} = \left(\frac{\partial^2 g_{12}}{\partial x^1 \partial x^2} - \frac{1}{2}\frac{\partial^2 g_{22}}{\partial x^1 \partial x^1} - \frac{1}{2}\frac{\partial^2 g_{11}}{\partial x^2 \partial x^2}\right)|_{\mathbf{p}} = F_{uu} F_{vv} - F_{uv}^2\,.$$

This statement is so good that it is worth to write it as an

Proposition *Consider a surface M in $\mathbf{E}^3$ defined by the equation*

$$\begin{cases} x = u \\ y = v \\ z = F(u,v) \end{cases} .$$

Suppose at the origin, the point $\mathbf{p}$ *with coordinates* $u = v = 0$ $F_u = F_v = 0.$, *i.e. normal unit vector is colinear to Oz axis. Then at the point* $\mathbf{p}$

$$R^1{}_{212}|_{\mathbf{p}} = R_{1212}|_{\mathbf{p}} = F_{uu}F_{vv} - F_{uv}^2|_{\mathbf{p}} = \det \begin{pmatrix} F_{uu} & F_{uv} \\ F_{uv} & F_{vv} \end{pmatrix} |_{\mathbf{p}} \tag{2.4}$$

One can see that the matrix above is nothing but shape operator, and we almost proved the relation between Gaussian and scalar curvature (see the lecture notes.)

Now very last and simplest step: in the case if $F(u,v) = \frac{1}{2}(au^2 + 2buv + cv^2)$ we see that

$$R^1{}_{212}|_{\mathbf{p}} = R_{1212}|_{\mathbf{p}} = F_{uu}F_{vv} - F_{uv}^2|_{\mathbf{p}} = (ac - b^2) .$$

3 * *Consider a point* $\mathbf{p}$ *on the surface M with coordinates $u = x_0, v = y_0$ such that (x_0, y_0) is a point of local extremum for the function F.*

Using the results of previous exercise calculate the component R_{1212} of the Riemannian curvature tensor at the point $\mathbf{p}$. *In this case the equation of the surface can be written as*

$$\begin{cases} x = x_0 + u \\ y = y_0 + v \\ z = F(x_0 + u, y_0 + v) \end{cases} .$$

Consider $z = z_0 = F(x_0, y_0)$ and the function $\tilde{F}(u,v) = F(x_0 + u, y_0 + v) - z_0$ we come to the equation

$$\begin{cases} x = x_0 + u \\ y = y_0 + v \\ z = z_0 + \tilde{F}(u,v) \end{cases}$$

Considering new Cartesian coordinates $x' = x - x_0, y' = y - y_0, z' = z - z_0$ we come to the equation

$$\begin{cases} x' = u \\ y' = v \\ z' = \tilde{F}(u,v) \end{cases}$$

in the vicinity of the extremum point with coordinates $u = v = 0$. This is just the case of Porposition that was studied above (see (2.5) Hence in the extremum point

$$R^1{}_{212}|_{\mathbf{p}} = \tilde{F}_{uu}\tilde{F}_{vv} - \tilde{F}_{uv}^2|_{u=v=0} = F_{uu}F_{vv} - F_{uv}^2|_{u=x_0, v=y_0} .$$

4 † *Using the results of the calculations in the previous exercise calculate the Riemannian curvature tensor at the arbitrary point of the surface (1).*

For an arbitrary point one can always consider adjusted Cartesian coordinates $x', y'.z'$ such that the conditions of Proposition will be fulfilled.