Homework 5

1 Consider the following curves:

$C_1\colon \mathbf{r}(t)\ \begin{cases} x = t \\ y = 2t^2 - 1 \end{cases},\ 0 < t < 1,\qquad C_2\colon \mathbf{r}(t)\ \begin{cases} x = t \\ y = 2t^2 - 1 \end{cases},\ -1 < t < 1,$

$C_3\colon \mathbf{r}(t)\ \begin{cases} x = 2t \\ y = 8t^2 - 1 \end{cases},\ 0 < t < \frac{1}{2},\qquad C_4\colon \mathbf{r}(t)\ \begin{cases} x = \cos t \\ y = \cos 2t \end{cases},\ 0 < t < \frac{\pi}{2},$

$C_5\colon \mathbf{r}(t)\ \begin{cases} x = t \\ y = 2t - 1 \end{cases},\ 0 < t < 1,\qquad C_6\colon \mathbf{r}(t)\ \begin{cases} x = 1 - t \\ y = 1 - 2t \end{cases},\ 0 < t < 1,$

$C_7\colon \mathbf{r}(t)\ \begin{cases} x = \sin^2 t \\ y = -\cos 2t \end{cases},\ 0 < t < \frac{\pi}{2},\qquad C_8\colon \mathbf{r}(t)\ \begin{cases} x = t \\ y = \sqrt{1 - t^2} \end{cases},\ -1 < t < 1,$

$C_9\colon \mathbf{r}(t)\ \begin{cases} x = \cos t \\ y = \sin t \end{cases},\ 0 < t < \pi,\qquad C_{10}\colon \mathbf{r}(t)\ \begin{cases} x = \cos 2t \\ y = \sin 2t \end{cases},\ 0 < t < \frac{\pi}{2},$

$C_{11}\colon \mathbf{r}(t)\ \begin{cases} x = \cos t \\ y = \sin t \end{cases},\ 0 < t < 2\pi,\qquad C_{12}\colon \mathbf{r}(t)\ \begin{cases} x = a\cos t \\ y = b\sin t \end{cases},\ 0 < t < 2\pi \text{ (ellipse)},$

Draw the images of these curves.

Write down their velocity vectors.

Indicate parameterised curves which have the same image (equivalent curves).

In each equivalence class of parameterised curves indicate curves with same and opposite orientations.

2 Consider differential forms $\omega = xdy - ydx$, $\sigma = xdx + ydy$ and vector fields $\mathbf{A} = x\partial_x + y\partial_y$, $\mathbf{B} = x\partial_y - y\partial_x$.

a) Calculate $\omega(\mathbf{A}), \omega(\mathbf{B})$, $\sigma(\mathbf{A}), \sigma(\mathbf{B})$.

3 Consider a function $f = x^3 - y^3$.

Calculate the value of 1-form $\omega = df$ on the vector field $\mathbf{B} = x\partial_y - y\partial_x$.

4 Calculate the derivatives of the functions $f = x^2 + y^2$, $g = y^2 - x^2$ and $h = q\log|r| = q\log\left(\sqrt{x^2 + y^2}\right)$ (q is a constant) along vector fields $\mathbf{A} = x\partial_x + y\partial_y$ and $\mathbf{B} = x\partial_y - y\partial_x$

a) calculating directional derivatives $\partial_{\mathbf{A}}f, \partial_{\mathbf{A}}g, \partial_{\mathbf{A}}h$, $\partial_{\mathbf{B}}f, \partial_{\mathbf{B}}g, \partial_{\mathbf{B}}h$,

b) calculating $df(\mathbf{A}), dg(\mathbf{A}), dh(\mathbf{A}), df(\mathbf{B}), dg(\mathbf{B}), dh(\mathbf{B})$.

5 Let f be a function on $\mathbf{E}^2$ given by $f(r, \varphi) = r^3\cos 3\varphi$, where r, φ are polar coordinates in $\mathbf{E}^2$.

Calculate the 1-form $\omega = df$.

Calculate the value of the 1-form $\omega = df$ on the vector field $\mathbf{X} = r\partial_r + \partial_\varphi$.

Express the 1-form ω in Cartesian coordinates x, y.

(You may use the fact that $\cos 3\varphi = 4\cos^3\varphi - 3\cos\varphi$.)

6 Show that 1-form $\omega = xdy + ydx$ is exact.

Show that 1-form $\omega = \sin y dx + x \cos y dy$ is exact.
Show that 1-form $\omega = x^3 dy$ is not an exact 1=form.
(We call 1*-form* ω *exact if there exists a function* F *such that* $\omega = dF$*.)*