Homework 0. Solutions of the first part of the homework

1 *Consider sets*

$$V = \{ax^2 + bx + c\,, a, b, c \in \mathbf{R}\}\,, \qquad T = \{x^2 + px + q\,, p, q \in \mathbf{R}\}$$

a) Explain why a set V is a vector space, and a set T is not a vector space (with respect to natural operations of mulitplication and addition of polynomials)

b) Explain why polynomials $1, x, x^2$ are linearly independent in V.

c) Calcualte dimension of V.

One can see that operations $+$ and $\cdot$ are well-defined: For two "vectors"—polynomials $P_1 = a_1x^2 + b_1x + c_1$ $P_2 = a_2x^2 + b_2x + c_2$

$$P_1 + P_2 = a_3x^2 + b_3x + c_3\,, \text{where } (a_3, b_3, c_3) = (a_1, b_1, c_1) + (a_2, b_2, c_2) = (a_1 + _2, b_1 + b_2, c_1 + c_2)\,,$$

$$\lambda \cdot P_1 = \lambda()a_1x^2 + b_1x + c_1) = (a, b, c)\,, \text{ where } (a, b, c) = \lambda(a_1, b_1, c_1) = (\lambda a_1, \lambda b_1, \lambda c_1)\,.$$

We see that we may identify the space V with $\mathbf{R}^3$.

On the other hand T is not vector space, since if we consider two arbitrary polynomials in T their sum does not belong T,

Now prove that polynomials (vectors) $1, x, x^2$ are linearly independent. Let $c_1, c_2, c_3 \in \mathbf{R}$ be coefficinets such that

$$c_1 \cdot 1 + c_2 \cdot x + c_3 \cdot x^2 = 0$$

i.e. polynomial $c_1 + c_2x + c_3x^2$ is identically equal to zero. In this case it is equal at zero at points $x = 0, 1, -1$:

$$P(x) = c_1 + c_2x + c_3x^2 \equiv 0 \Rightarrow \begin{cases} P(0) = c_1 = 0 \\ P(1) = c_1 + c_2 + c_3 = 0 \\ P(-1) = c_1 - c_2 + c_3 = 0 \end{cases} \Rightarrow c_1 = 0\,, c_2 = 0\,, c_3 = 0\,,$$

i.e. polynomials $1, x, x^2$ are linearly independent.

2 *Show that the vectors $\{\mathbf{a}_1, \mathbf{a}_2 \ldots, \mathbf{a}_m\}$ in vector space V are linearly dependent if at least one of these vectors is equal to zero.*

WLOG suppose that $\mathbf{a}_1 = 0$. Then

$$\lambda \mathbf{a}_1 + 0 \cdot \mathbf{a}_2 + \ldots + 0 \cdot \mathbf{a}_n = 0$$

where λ is an arbitrary non-zero real number $\lambda \neq 0$. We see that there exists a linear combinations of vectors $\{\mathbf{a}_1, \mathbf{a}_2 \ldots, \mathbf{a}_m\}$ which is equal to zero and one of the coefficients $\{\lambda, 0, \ldots, 0\}$ is not equal to zero. Hence vectors $\{\mathbf{a}_1, \mathbf{a}_2 \ldots, \mathbf{a}_m\}$ are linearly dependent.

3 *a) Show that arbitrary three vectors in* $\mathbf{R}^2$ *are linearly dependent.*
Consider the following vectors in $\mathbf{R}^2$

$$\mathbf{e}_1 = (1,0)\,, \qquad \mathbf{e}_2 = (0,1)\,, \qquad \mathbf{a} = (2,3)\,, \qquad \mathbf{b} = (3,0)\,, \tag{1}$$

b) Show that $\{\mathbf{e}_1, \mathbf{e}_2\}$ *is a basis in* $\mathbf{R}^2$.
c) Show that $\{\mathbf{a}, \mathbf{b}\}$ *is a basis in* $\mathbf{R}^2$.
d) Show that $\{\mathbf{e}_1, \mathbf{b}\}$ *is not a basis in* $\mathbf{R}^2$.

`Solution of a)`
Consider arbitrary three vectors in $\mathbf{R}^2$

$$\begin{aligned} \mathbf{x}_1 &= (a^1, a^2) \\ \mathbf{x}_2 &= (b^1, b^2) \\ \mathbf{x}_3 &= (c^1, c^2) \end{aligned}$$

If vector $\mathbf{x}_1 = (a_1, a_2) = 0$ then nothing to prove. (See exercise 2). Let $\mathbf{x}_1 \neq 0$. WLOG suppose $a_1 \neq 0$. Consider vectors

$$\begin{aligned} \mathbf{x}'_2 &= \mathbf{x}_2 - \tfrac{b_1}{a_1}\mathbf{x}_1 = (b^1, b^2) - \tfrac{b_1}{a_1}(a_1, a_2) = (0, b'_2) \\ \mathbf{x}'_3 &= \mathbf{x}_3 - \tfrac{c_1}{a_1}\mathbf{x}_1 = (c^1, c^2) - \tfrac{c_1}{a_1}(a_1, a_2) = (0, c'_2) \end{aligned}$$

We see that vectors $\mathbf{x}'_2, \mathbf{x}'_3$ are proportional—i.e. they are linearly dependent: there exist $\mu_2 \neq 0$ or $\mu_3 \neq 0$ such that $\mu_2\mathbf{x}'_2 + \mu_3\mathbf{x}'_3 = 0$ E.g. we can take $\mu_2 = c'_2$, $\mu_3 = -b'_2$ in the case if $c'_2 \neq 0$ or $b'_2 \neq 0$ (if $c'_2 = b'_2 \neq 0$ then we can take coefficients μ_1, μ_2 any real numbers.) We have:

$$0 = \mu_2\mathbf{x}'_2 + \mu_3\mathbf{x}'_3 = \mu_2\left(\mathbf{x}_3 - \frac{c_1}{a_1}\mathbf{x}_1\right) + \mu_3\left(\mathbf{x}_3 - \frac{c_1}{a_1}\mathbf{x}_1\right) = \mu_2\mathbf{x}_2 + \mu_3\mathbf{x}_3 - \left(\frac{\mu_2 b_1}{a_1} + \frac{\mu_3 c_1}{a_1}\right)\mathbf{x}_1 = 0,$$

where $\mu_2 \neq 0$ or $\mu_3 \neq 0$. Hence vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linearly dependent *. $\blacksquare$

`Solution of b)`
Vectors $\mathbf{e}_1, \mathbf{e}_2$ are linearly independent:

$$a\mathbf{e}_1 + b\mathbf{e}_2 = a(1,0) + b(0,1) = (a,b) = 0 \Rightarrow a = b = 0$$

We see that on one hand in $\mathbf{R}^3$ any trhee vectors are linearly dependent, and on the other hand there exist two linearly independent vectors. Hence dimension of $\mathbf{R}^2$ is equal to 2. Hence these two vectors $\{\mathbf{e}_1, \mathbf{e}_2\}$ form a basis

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* You may say: why so long proof? We know already that dimension of $\mathbf{R}^2$ is equal to 2 then by definition any three vectors in $\mathbf{R}^2$ have to be linear dependent. This "proof" is in fact "*circulus vicious* " since the proof of the fact that $\dim \mathbf{R}^2 = 2$ is founded on the statement of this exercise.

`Solution of c)` Vectors $\mathbf{a}, \mathbf{b}$ are also linearly independent:

$$x\mathbf{a} + y\mathbf{b} = x(2,3) + y(3,0) = (2x+3y, 3x) = 0 \Rightarrow \begin{cases} x = 0 \\ 2x + 3y = 0 \end{cases} \Rightarrow x = y = 0\,.$$

We see that two vectors $\mathbf{a}, \mathbf{b}$ are linearly independent vectors in 2-dimensional space. Hence these two vectors $\{\mathbf{a}, \mathbf{b}\}$ form a basis

`Solution of d)` Vectors $\mathbf{e}_1, \mathbf{b}$ are linearly dependent, since

$$3\mathbf{e}_1 - \mathbf{b} = 0\,.$$

Hence this is not a basis.