Symmetries

§1Symmetries of triangles, quadrates, ...

We consider symmetries of polygons (triangles,....)

Symmetry, it is a transformation of polygon such that the new polygon coincides with former one.

1 Let ABC be an isoscales triangle ($AC = CB$).

We have transformation (reflection) $R = \begin{pmatrix} AB \\ BA \end{pmatrix}$. $R^2 = I$:

$$\begin{pmatrix} AB \\ BA \end{pmatrix} \circ \begin{pmatrix} AB \\ BA \end{pmatrix} = \begin{pmatrix} AB \\ AB \end{pmatrix} = I$$

We come to *group* of transformations which possesses two elements $\{I, R\}$. Multiplication table is:

$$I \circ I = I, I \circ R = R, R \circ R = I$$

We will denote this group by D_2.

2 Let ABC be an equilateral triangle ($AB = BC = AC$).

We have much more symmetries:

There are three reflections

$$R_{AB}\colon\ R_{AB} = \begin{pmatrix} AB \\ BA \end{pmatrix}, R_{AB}\colon\ R_{BC} = \begin{pmatrix} BC \\ CB \end{pmatrix}\ R_{AC}\colon\ R_{AC} = \begin{pmatrix} CA \\ BA \end{pmatrix}.$$

All reflections of course obey the law: $R \circ R = I$

But there are also another symmetry transformations– rotations. Consider rotation S: $\begin{pmatrix} ABC \\ BCA \end{pmatrix}$. It is anticlock-wise rotation on the angle 120°. One can see that $S \circ S$ is anticlock-wise rotation on the angle 240°, or clock-wise rotation on the angle 120°:

$$S^2 = I = \begin{pmatrix} ABC \\ BCA \end{pmatrix} \circ \begin{pmatrix} ABC \\ BCA \end{pmatrix} = \begin{pmatrix} ABC \\ CAB \end{pmatrix}$$

So we have 6 symmetries (including identical) $\{I, S, S^2, R_{AB}, R_{AC}, R_{BC}\}$:

$$I = \begin{pmatrix} ABC \\ ABC \end{pmatrix}, S = \begin{pmatrix} ABC \\ BCA \end{pmatrix}, S^2 = \begin{pmatrix} ABC \\ CAB \end{pmatrix},$$

$$R_{AB} = \begin{pmatrix} ABC \\ BAC \end{pmatrix}, R_{AC} = \begin{pmatrix} ABC \\ CBA \end{pmatrix}, R_{BC} = \begin{pmatrix} ABC \\ ACBC \end{pmatrix}$$

Look on multiplication table. Now multiplication table is not so simple. In particular $R_{AB} \circ S = \begin{pmatrix} ABC \\ BAC \end{pmatrix} \circ \begin{pmatrix} ABC \\ BCA \end{pmatrix} = \begin{pmatrix} ABC \\ CBA \end{pmatrix} = R_{AC}$, but $S \circ R_{AB} = \begin{pmatrix} ABC \\ BCA \end{pmatrix} \circ \begin{pmatrix} ABC \\ BAC \end{pmatrix} = \begin{pmatrix} ABC \\ ACB \end{pmatrix} = R_{BC}$. We see that now multiplication is not commutative:

$$R_{AB} \circ S \neq S \circ R_{AB} .$$

We denote the group by D_3.

Exercise 1 (for sixthformers)

Calculate multiplication table.

Exercise 2 Consider subsets $\{I, S, S^2\}$, $\{I, R_{AB}\}$, $\{S, R_{AB}\}$. Analyse which sets are subgroups?

Exercise 3 Describe all subgroups in D_3.

The group D_3 of symmetries of triangle possesses 6 elements: Note that it is just group of *all permutations* of three elements (group S_3), say $\{A, B, C\}$ or say $\{pen, chair, table\}$....

Exercise 4 Consider quadrat: quadrat $ABCD$. Find all symmetry transformations.

Exercise Find all symmetry transformations.

Exercise 5 How many permutations one can do if we have four elements $\{A, B, C, D\}$ or say $\{pen, pencil, chair, table\}$

Exercise Is it TRUE or FALSE: Any permutation of letters $\{A, B, C, D\}$ is a symmetry transformation.

§2 **Applications of symmetries to polynomial equations.**

Consider now the problem from algebra.

Let x_1, x_2 be roots of quadratic polynomial $x^2 + px + q = 0$.

By the reasons which will be clear later denote one root by A and the second root by the letter B Consider the following expressions:

a) $A - B$, $A + B$, $A^2 + B^2$, $A^3 - B^3$, $AB + B$, $AB + A + B$

Apply to these expressions symmetry group D_2 of isoscales.

We see that some of these expressions are D_2-invariant, some not.

Theorem (Viète Theorem). $A + B = -p$, $AB = q$. This you know well. But Viète Theorem is equivalent to another theorem

Theorem $'$ An arbitrary polynomial on A, B (recall that A, B are roots of quadratic equation $x^2 + px + q$) which is D_2-invariant can be expressed as a polynomial on p, q

You see the relation of group D_2 of isoscales with the problem of calculation expressions formed with the roots of quadratic polynomial:

D_2-symmetric expressions is easy to calculate!

Exercises (for sixthformers)

1. Let A, B be roots of polynomial $x^2 - x + 1$.

a) Calculate $a_n = A^n + B^n$ for $n = 1, 2, 3, \ldots$

b) Calculate A^2+B^2 for the polynomial x^2+x+10. (The answer will be negative number. Explain it)

c) something of this type

d) *Formula for roots via Viète Theorem*

Calculate $A-B=\pm\sqrt{(A-B)^2}=\pm\sqrt{(A+B)^2-4AB}$

Note that

$$A,B=\frac{(A+B)\pm(A-B)}{2}=\frac{(A+B)\pm\sqrt{(A+B)^2-4AB}}{2}=\dots$$

We come to the formula for roots of quadratic polynomial.

Above we see that D_2-invariant expressions of roots A,B of quadratic polynomial x^2+px+q is easy to calculate, i.e. you do not need to find roots themself.

What about... D_3-invariant expressions of roots A,B,C of cubic polynomial x^3+px^2+qx+r

This question is much more tricker? Not only you do not know how to calculate rooots of the cubic polynomial, but people who know it realise that formulae are much more complicated and out of practical use.

It turns out that in spite of the fact that we cannot calculate roots but Vi‘ete Theorem still works:

Let x_1,x_2,x_3 be roots. As before we denote them by A,B,C Then comparing polynomials x^3+px^2+qx+r

We come to

$$A+B+C=-p, AB+AC+BC=q, ABC=-r$$

Note that expressions in left hand side are D_3-invariant.

We come to

Theorem 2 (Viète Theorem). $A+B=-p$, $AB=q$. This you know well. But Viète Theorem is equivalent to another theorem

Theorem $2'$ An arbitrary polynomial on A,B,C (recall that A,B,C are roots of cubic equation $x^3+px^2+qx+r=0$) which is D_3-invariant can be expressed as a polynomial on p,q

Exercises Consider expressions $A^2+B^2-C^2$, $A^2+B^2+C^2$... $A^2B+B^2C+C^2A$, $A^2B+B^2C+C^2A+AB^2+BC^2+CA^2$ where A,B,C are roots of polynomial $x^3-x-1=0$

where A,B,C are roots of Find D_3-invariant expresisions

Calculate D_3-invariant expression.

What next.

One can try to consider group D_4. It is related with quadratic equation.

May be you heard the name of Evarist Galois...

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