Homework 4

1 Let P be an orthogonal operator in $\mathbf{E}^3$, and let $\mathbf{a} \neq 0$ be its eigenvector: $P\mathbf{a} = \lambda\mathbf{a}$. Show that the eigenvalue $|\lambda| = 1$.

2 In the problem 3 of Homework 3a we considered an operator P

$$P(\mathbf{e}) = \frac{2}{3}\mathbf{e} + \frac{2}{3}\mathbf{f} + \frac{1}{3}\mathbf{g}\,, P(\mathbf{f}) = -\frac{1}{3}\mathbf{e} + \frac{2}{3}\mathbf{f} - \frac{2}{3}\mathbf{g}\,, P(\mathbf{g}) = -\frac{2}{3}\mathbf{e} + \frac{1}{3}\mathbf{f} + \frac{2}{3}\mathbf{g}\,.$$

where $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ is an orthonormal basis in $\mathbf{E}^3$, We showed that it is orthogonal operator preserving orientation.

Show that this operator defines rotation, and find the axis and the angle of this rotation.

3 Consider on $\mathbf{E}^3$ following two operators:

$$P_1(\mathbf{x}) = \mathbf{x} - 2(\mathbf{n}, \mathbf{x})\mathbf{n}\,, \quad P_2(\mathbf{x}) = 2(\mathbf{n}, \mathbf{x})\mathbf{n} - \mathbf{x}\,,$$

where $\mathbf{n}$ is a unit vector.

Show that these both operators are orthogonal operators. Show that the first operator changes the orientation, and the second operator preserves orientation.

Show that the first operator is reflection operator with respect to...?

Show that the second operator is rotation operator: find the axis of the rotation and the angle of the rotation.

4 Orthogonal operator P obeys the condition

$$P \neq \mathbf{id},, \quad \text{and} \quad P^5 = \mathbf{id}\,.$$

Show that P is a rotation operator, and calculate the angle of the rotation.

5 Students John and Sarah calculate vector product $\mathbf{a} \times \mathbf{b}$ of two vectors using two different orthonormal bases in the Euclidean space $\mathbf{E}^3$, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$. John expands the vectors with respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Sarah expands the vectors with respect to the basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$. For two arbitrary vectors $\mathbf{a}, \mathbf{b} \in \mathbf{E}^3$

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = a'_1\mathbf{e}'_1 + a'_2\mathbf{e}'_2 + a'_3\mathbf{e}'_3\,, \quad \mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 = b'_1\mathbf{e}'_1 + b'_2\mathbf{e}'_2 + b'_3\mathbf{e}'_3\,.$$

John and Sarah both use the so-called "determinant" formula. Are their answers the same?

$$\mathbf{a} \times \mathbf{b} = \underbrace{\det \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}}_{\text{John's calculations}} \stackrel{?}{=} \underbrace{\det \begin{pmatrix} \mathbf{e}'_1 & \mathbf{e}'_2 & \mathbf{e}'_3 \\ a'_1 & a'_2 & a'_3 \\ b'_1 & b'_2 & b'_3 \end{pmatrix}}_{\text{Sarah's calculations}}$$

In the problems 6,7,8 and 9 we suppose that $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ *is an orthonormal basis in* $\mathbf{E}^3$ *and* $\mathbf{e}, \mathbf{f}$ *is an orthonormal basis in* $\mathbf{E}^2$.

6 Find a unit vector $\mathbf{n}$ in $\mathbf{E}^3$, such that the following conditions hold:

1) It is orthogonal to the vectors $\mathbf{a} = \mathbf{e} + 2\mathbf{f} + 3\mathbf{g}$ and $\mathbf{b} = \mathbf{e} + 3\mathbf{f} + 2\mathbf{g}$.

2) An ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{n}\}$ has an orientation opposite to the orientation of the basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$. (You have to expand vector $\mathbf{n}$ over the basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$).

7 Calculate the area of parallelograms formed by the vectors $\mathbf{a}, \mathbf{b}$ if

a) $\mathbf{a} = 2\mathbf{e} + 2\mathbf{f} + 3\mathbf{g}, \mathbf{b} = \mathbf{e} + \mathbf{f} + \mathbf{g}$;

b) $\mathbf{a} = 5\mathbf{e} + 8\mathbf{f} + 4\mathbf{g}, \mathbf{b} = 10\mathbf{e} + 16\mathbf{f} + 8\mathbf{g}$.

8 In 2-dimensional Euclidean space $\mathbf{E}^2$ consider vectors $\mathbf{a} = (3, 2)$, $\mathbf{b} = (7, 5)$, $\mathbf{c} = (17, 12)$, $\mathbf{d} = (41, 29)$. Calculate areas of the parallelograms $\Pi(\mathbf{a}, \mathbf{b}), \Pi(\mathbf{b}, \mathbf{c})$ and $\Pi(\mathbf{c}, \mathbf{d})$.

8a† Do you see any relations between parallelograms in the exercise above, fractions $\frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}$ and the number... $\sqrt{2}$? Can you continue the sequence of these fractions? (*Hint: Consider the squares of these fractions.*)

9 Let A be an operator in $\mathbf{E}^2$ such that

$$\mathbf{a} = A(\mathbf{e}) = 27\mathbf{e} + 40\mathbf{f}\,, \mathbf{b} = A(\mathbf{f}) = -16\mathbf{e} - \frac{71}{3}\mathbf{f}\,.$$

(see problem 1 in Homework 2) Compare the areas of parallelograms $\Pi(\mathbf{e}, \mathbf{f})$, $\Pi(\mathbf{a}, \mathbf{b})$ and $\det A$.

10 Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be three vectors in $\mathbf{E}^3$ such that $\mathbf{x} = \mathbf{e}+\mathbf{f}, \mathbf{y} = \mathbf{e}+5\mathbf{f}+\mathbf{g}$ and $\mathbf{z} = 2\mathbf{f}+3\mathbf{g}$, where $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ is an orthonormal basis in $\mathbf{E}^3$.

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three another vectors such that

$$\mathbf{a} = 2\mathbf{x} + 5\mathbf{y} + 7\mathbf{z} \quad \mathbf{b} = \mathbf{x} + 3\mathbf{y} + 2\mathbf{z}\,, \quad \mathbf{c} = 2\mathbf{z}\,. \qquad (*)$$

Find volume of the parallellepiped $\Pi(\mathbf{x}, \mathbf{y}, \mathbf{z})$

Find volume of the parallellepiped $\Pi(\mathbf{a}, \mathbf{b}, \mathbf{c})$

Relation (*) defines the linear operator which transforms basis $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ to the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ Calculate determinant of this linear operator.