Homework 1. Solutions

1 *a) Let $x^2+y^2=R^2$ be a circle in $\mathbf{E}^2$. Write down explicitly formulae for stereographic projections with respect to the North pole (the point $(0,1)$) and South pole (the point $(0,-1)$). b) Do the same exercise for the sphere $x^2+y^2+z^2=R^2$ in $\mathbf{E}^3$. (North pole (the point $(0,0,1)$) and South pole (the point $(0,0,-1)$).*

a) Let $N=(0,R)$ be North Pole and A be a point on the circle. Let chorde NA intersects the line $y=0$ at the point P. Let B be a point on the axis OY such that BP is parallel to the axis OX. Consider similar triangles $\triangle NAB$ and $\triangle NPO$, where O is origin. Let $A=(x,y)$ and $P=(0,t)$. We have:

$$\frac{|NB|}{|NO|}=\frac{|AB|}{|OP|}\,i.e.\frac{R-y}{R}=\frac{x}{t}$$

Solving this equations we come to $t=\frac{Rx}{R-y}$. Respectively: $x=\frac{t(R-y)}{R}$, Using $x^2+y^2=\frac{t^2(R-y)^2}{R^2}+y^2=R^2$ we come to

$$\frac{t^2(R-y)}{R^2}=R+y\Rightarrow\begin{cases} x=\frac{t(R-y)}{R}=\frac{2tR^2}{R^2+t^2} \\ y=\frac{t^2-R^2}{t^2+R^2}R \end{cases}.$$

Analogously for South Pole: Let $S=(0,-R)$ be South Pole and A' a point on the circle. Let chorde SA intersects the line $y=0$ at the point P'. Consider similar triangles $\triangle SAB$ and $\triangle SP'O$. We have:

$$\frac{|SB|}{|SO|}=\frac{|AB|}{|OP'|}\,i.e.\frac{R+y}{R}=\frac{x}{t'}$$

Solving this equations we come to $t'=\frac{Rx}{R+y}$. Respectively:

$$\begin{cases} x=\frac{2t'R^2}{R^2+t'^2} \\ y=\frac{R^2-t'^2}{R^2+t'^2}R \end{cases}.$$

b) For two dimensional sphere the considerations are analogous.

Let $N=(0,0,R)$ be North Pole and A a point on the sphere with coordinates (x,y,z) Let chorde NA intersects the line $y=0$ at the point P. Let B be a point the axis OY such that BP is parallel to the axis OX. Let (u,v) be coordinates of the point P. Consider similar triangles, $\triangle NAB$ and $\triangle NPO$, where O is origin. Let $A=(x,y)$ and $P=(0,t)$. We have:

$$\frac{|NB|}{|NO|}=\frac{|AB|}{|OP|}\,i.e.\frac{R-z}{R}=\frac{x}{u}=\frac{y}{v}\,.$$

Solving this equations we come to $\begin{cases} u=\frac{Rx}{R-z} \\ v=\frac{Ry}{R-z} \end{cases}$. Respectively: $x=\frac{u(R-z)}{R}, y=\frac{v(R-z)}{R}$, Using

$$x^2+y^2+z^2=\left(\frac{u(R-z)}{R}\right)^2+\left(\frac{v(R-z)}{R}\right)^2+z^2=R^2\Rightarrow\frac{u^2(R-z)}{R^2}+\frac{v^2(R-z)}{R^2}=R-z$$

we come to

$$\begin{cases} x = \frac{2uR^2}{u^2+v^2+R^2} \\ y = \frac{2vR^2}{u^2+v^2+R^2} \\ z = \frac{u^2+v^2-R^2}{u^2+v^2+R^2}R \end{cases}$$

Analogously for stereographic projection with respect to South Pole.

Let $S=(0,0,-R)$ be South Pole and A a point on the sphere with coordinates (x,y,z) Let chorde NA intersects the line $y=0$ at the point P' with coordinates (u',v'). Then we come to

$$\begin{cases} u = \frac{Rx}{R-z} \\ v = \frac{Ry}{R-z} \end{cases}, \qquad \begin{cases} x = \frac{2uR^2}{u^2+v^2+R^2} \\ y = \frac{2vR^2}{u^2+v^2+R^2} \\ z = \frac{u^2+v^2-R^2}{u^2+v^2+R^2}R \end{cases}.$$

2 *Consider in* $\mathbf{E}^n$ *the transformation:*

$$\varphi(\mathbf{r}) = \frac{\mathbf{r}}{|\mathbf{r}|^2} \tag{1}$$

inversion with the centre at origin. (Strictly speaking this transformation is defined on $\mathbf{E}^n \backslash 0$*).*

Analyze its geometrical meaning.

Show that this trasformation is an involution: $\varphi(\varphi(\mathbf{r})) = \mathbf{r}$.

Let $\mathbf{a}$ *be an arbitrary vector attached at the arbitrary point* $\mathbf{r}$ *of* $\mathbf{E}^n$ $(\mathbf{r} \neq 0)$*. Let* $\mathbf{a}'$ *be a vector attached at the point* $\mathbf{r}' = \varphi(\mathbf{r})$*, such that* $\mathbf{a}' = \varphi_* \mathbf{a}$ *is the image of vector* $\mathbf{a}$ *under inversion (1), Show that*

$$\mathbf{a}' = \frac{\mathbf{a}r^2 - 2\mathbf{r}(\mathbf{r},\mathbf{a})}{|\mathbf{r}|^4}, \tag{2}$$

where $(\ ,\ )$ *is the scalar product in Euclidean space* $\mathbf{E}^n$*.*

Using this equation show that inversion preserves angles between vectors.

Find the image of the hyperplane $x^n = a$ *under the inversion (1).* (*$(x^1, \dots, x^n)$ are standard Cartesian coordinates.*) [1].

For simplicity you may consider just cases $n = 2, 3$

Solution

The involution property follows from geometrical picture. Points $\mathbf{r}, \mathbf{r}' = \varphi(\mathbf{r})$ and $\tilde{\mathbf{r}} = \varphi(\mathbf{r}') = \varphi(\varphi((\mathbf{r}))$ all belong to the same ray. On the other hand $|\mathbf{r}'||\mathbf{r}| = |\tilde{\mathbf{r}}||\mathbf{r}'| = 1$, hence $|\tilde{r}| = |\mathbf{r}|$. Hence $\tilde{\mathbf{r}} = \mathbf{r}$.

[1] *Show that this is a sphere of radius* $\frac{1}{2|a|}$ *with the centre at the point* $x^1 = \ldots = x^{n-1} = 0, x^n = \frac{1}{2a}$.

If a point $\mathbf{r}$ with coordinates x^i tansformes under the inversion to the point $\mathbf{r}'$ with coordinates $y^i = y^i(x)$ then the tangent vector $\mathbf{a}$ at this point transforms to the tangent vector $\mathbf{a}'$ at the point $\mathbf{r}'$ such that

$$(a')^i = \frac{\partial y^i(x)}{\partial x^m} a^m$$

For inversion (1) we have $y^i(x) = \frac{x^i}{r^2} = \frac{x^i}{(x^1)^2+\ldots+(x^n)^2}$ hence

$$\frac{\partial y^i}{\partial x^m} = \frac{\delta^i_m}{r^2} - \frac{2x^i x^m}{r^4} = \frac{r^2\delta^i_m - 2x^i x^m}{r^4},$$

hence

$$(a')^i = \frac{\partial y^i(x)}{\partial x^m} a^m = \left(\frac{r^2\delta^i_m - 2x^i x^m}{r^4}\right) a^m = \frac{r^2 a^i - 2x^i(x^1a^1 + \ldots + x^n a^n}{r^4},$$

i.e.

$$\mathbf{a}' = \frac{r^2\mathbf{a} - 2\mathbf{r}(\mathbf{a},\mathbf{r})}{r^4}. \quad \text{where } r^2 = |\mathbf{r}|^2 = (\mathbf{r},\mathbf{r}).$$

To show that inversion does not change the angles between tangent vectors, calculate the scalar products. Let $\mathbf{a}, \mathbf{b}$ be two arbitrary vectors attached at arbitrary point $\mathbf{r}$ ($\mathbf{r} \neq 0$). Let $\mathbf{a}', \mathbf{b}'$ be vectors attached at the point $\mathbf{r}' = \varphi(\mathbf{r})$, such that vector $\mathbf{a}'$ is the image of the vector $\mathbf{a}$ under inversion φ, and respectively vector $\mathbf{b}$ is the image of the vector $\mathbf{b}$ under inversion φ. Then according to previous calculations we have

$$(\mathbf{a}', \mathbf{b}') = \left(\frac{r^2\mathbf{a} - 2\mathbf{r}(\mathbf{a},\mathbf{r})}{r^4}, \frac{r^2\mathbf{b} - 2\mathbf{r}(\mathbf{b},\mathbf{r})}{r^4}\right) = \frac{r^4(\mathbf{a},\mathbf{b}) - 4r^2(\mathbf{a},r)(\mathbf{b},\mathbf{r}) + 4r^2(\mathbf{a},\mathbf{r})(\mathbf{b},\mathbf{r})}{r^8} = \frac{\mathbf{a},\mathbf{b}}{r^4}$$

We see that inversion does not change the scalar product of vectors up to a factor depending on the point.

Now find the image of hyperplane. Its image has to be sphere which passes via origin, centre of inversion. Show it. Let $A = (u^1, u^2, \ldots, u^{n-1}, a)$ be an arbitrary point of the plane $x^n = a$. Image of this point will be the point A' with coordinates $(v^1, \ldots, v^n, b)$ such that

$$v^i = \frac{u^i}{a^2 + r^2}, b = \frac{a}{a^2 + r^2}, \quad \text{where } r^2 = (u^1)^2 + \ldots + (u^n)^2.$$

Perform calculations which check that point $(v^1, \ldots, v^{n-1}, b)$ belongs to the sphere

$$\sum_i v^i v^i + \left(b - \frac{1}{2a}\right)^2 = \frac{\sum_i u^i u^i}{(a^2+r^2)^2} + b^2 - \frac{b}{a} + \frac{1}{4a^2} = \frac{\sum_i u^i u^i}{(a^2+r^2)^2} + \frac{a^2}{a^2+r^2} - \frac{1}{a^2+r^2} + \frac{1}{4a^2} = \frac{1}{4a^2}.$$

3* *Consider in* $\mathbf{E}^3$ *the transformation:*

$$\varphi_N(\mathbf{r}) = \mathbf{N} + \frac{C\mathbf{r}}{|\mathbf{r} - \mathbf{N}|^2}, \quad C \neq 0, \tag{2}$$

where $\mathbf{N}$ *is an arbitrary vector.*

(Stricly speaking this transformation is defined on $\mathbf{E}^n \backslash N$.

Analyze geometrical meaning of this transformation: in particular analyze the relation of this transformation with the transformation (2).

Show that this transformation preserves angles between vectors.

Find a transformation inverse to transformation (2).

Find an image of a plane under the transformation (2).

The transformation (2) is compostions of four transformations

i) translation $\mathbf{r} \to \mathbf{r} - \mathbf{N}$

ii) inversion $\mathbf{r} \to \frac{\mathbf{r}}{r^2}$

iii) dilation $\mathbf{r}\ C\mathbf{r}$

iy) translation $\mathbf{r} + \mathbf{N}$

All these transformations preserve angles between tangent vectors. For translations, dilations this is evident, and for inversion it was proved in the previous exercie. Hence the composition of these transformations preserve also angles.

We see that transformation (2) is the composition

$$T_{\mathbf{N}} \circ D_C \circ \text{inversion} \circ T_{-\mathbf{N}}\,,$$

where $T_{\mathbf{a}}: \mathbf{r} \to \mathbf{r} + \mathbf{a}$ is a translation on the vector $\mathbf{a}$, and D_C: $\mathbf{r} \to C\mathbf{r}$ is dilation.

Using the fact that inversion is involution, and the decomposition abive, we see that inverse of (2) is a trasformation

$$T_{\mathbf{N}} \circ D_{\frac{1}{C}} \circ \text{inversion} \circ T_{-\mathbf{N}}\,,$$

Indeed:

$$\left(T_{\mathbf{N}} \circ \text{inversion} \circ D_{\frac{1}{C}} \circ T_{-\mathbf{N}}\right) \circ (T_{\mathbf{N}} \circ D_C \circ \text{inversion} \circ T_{-\mathbf{N}}\,,) = T_{\mathbf{N}} \circ \text{inversion} \circ D_{\frac{1}{C}} \circ T_{-\mathbf{N}} \circ T_{\mathbf{N}} \circ D_C \circ \text{inve}$$

$$T_{\mathbf{N}} \circ \text{inversion} \circ D_{\frac{1}{C}} \circ D_C \circ \text{inversion} \circ T_{-\mathbf{N}} = T_{\mathbf{N}} \circ \text{inversion} \circ \text{inversion} \circ T_{-\mathbf{N}} = T_{\mathbf{N}} \circ T_{-\mathbf{N}} = \mathbf{id}\,.$$

We proved that the transformation inverse to transformation (2) is the trasnformation

$$T_{\mathbf{N}} \circ \text{inversion} \circ D_{\frac{1}{C}} \circ T_{-\mathbf{N}}: \quad \mathbf{r}' = \frac{\frac{\mathbf{r}-\mathbf{N}}{C}}{\left|\frac{\mathbf{r}-\mathbf{N}}{C}\right|^2} + \mathbf{N} = \frac{C(\mathbf{r} - \mathbf{N})}{|\mathbf{r} - \mathbf{N}|^2} + \mathbf{N} =$$

if $C > 0$. We see that transformation (2) is an involution if $C > 0$, and it changes the sign in the case if $C < 0$. Sure this can be checked straightforwardly.

Transformation (2) is an inversion with centre at the point $\mathbf{N}$. the radius of the sphere of inversion is equal to $\sqrt{C}$, $(C > 0)$—th points which are on the distance $\sqrt{C}$ of the centre stay intact under the inversion.

4* *Let φ be stereographic projection of unit sphere $x^2+y^2+z^2=1$ in $\mathbf{E}^3$ on the plane $z=0$ with respect to the North pole $N=(0,0,1)$. Find a transformation (2) of $\mathbf{E}^3$ such that its restriction on the sphere is this stereographic projection.*

Consider the inversion (2) with centre at the point $N=(0,0,1)$ (North pole of the sphere) with the radius of inversion $\sqrt{2}$:

$$\begin{cases} x \mapsto x' = \frac{2x}{x^2+y^2+(z-1)^2} \\ y \mapsto y' = \frac{2y}{x^2+y^2+(z-1)^2} \\ z \mapsto z' = 1+\frac{2(z-1)}{x^2+y^2+(z-1)^2} \end{cases}$$

ne can see that this is inversion (2), the points of the equator $x^2+y^2=1, z=0$ of the sphere S^2 remian fixed, the North pole of the sphere 'goes to infinity'. Hence the image of this sphere under the inversion is the plane $z=0$. It follows from the formula above that if $x^2+y^2+z^2=1$ (point is on the sphere) then its image has coordinates

$$\begin{array}{c} x \mapsto u = \frac{2x}{x^2+y^2+(z-1)^2} = \frac{2x}{x^2+y^2+z^2-2z+1} = \frac{x}{1-z} \\ y \mapsto v = \frac{2y}{x^2+y^2+(z-1)^2} = \frac{2y}{x^2+y^2+z^2-2z+1} = \frac{y}{1-z} \\ z \mapsto 1+\frac{2(z-1)}{x^2+y^2+(z-1)^2} = 0 \end{array}$$

To find inverse formulae (from plane (u,v) to the sphere) we just use the fact that inversion (2) is the involution (see the previous exercise):

$$\begin{cases} x \mapsto x' = \frac{2x}{x^2+y^2+(z-1)^2} \\ y \mapsto y' = \frac{2y}{x^2+y^2+(z-1)^2} \\ z \mapsto z' = 1+\frac{2(z-1)}{x^2+y^2+(z-1)^2} \end{cases} \Leftrightarrow \begin{cases} x' \mapsto x = \frac{2x'}{x'^2+y'^2+(z'-1)^2} \\ y' \mapsto y = \frac{2y'}{x'^2+y'^2+(z'-1)^2} \\ z' \mapsto z = 1+\frac{2(z'-1)}{x'^2+y'^2+(z'-1)^2} \end{cases}$$

In particular if $u=x', v=y'$ are stereographic coordinates then $z'=0$ and we come to

$$\begin{cases} u = x' \mapsto x = \frac{2x'}{x'^2+y'^2+(z'-1)^2} = \frac{2u}{u^2+v^2+1} \\ y' \mapsto y = \frac{2y'}{x'^2+y'^2+(z'-1)^2} = \frac{2v}{1+u^2+v^2} \\ z' = 0 \mapsto z = 1+\frac{2(z'-1)}{x'^2+y'^2+(z'-1)^2} = 1+\frac{-2}{u^2+v^2+1} = \frac{u^2+v^2-1}{u^2+v^2+1} \end{cases}$$

(Compare these calculations with calculations performed in the exercise 1)

5* *Using the result of previous exercise explain why stereographic projection of unit sphere $x^2+y^2+z^2=1$ establsihes bijection between points with rational coordinates on the unit sphere with points with rational coordinates on the plane $z=0$.*

THis is because the inversion which leads to the stereographic projection evidently is birational map (see above)

6 *Show that the condition of non-degeneracy for a symmetric matrix $||g_{ik}||$ follows from the condition that this matrix is positive-definite.*

Solution Suppose $\det g = 0$, i.e. g is degenerate matrix (rows and columns of the matrix are linear dependent). Then there exists non-zero vector $\mathbf{x} = (x^1, x^2)$ such that $g_{ik}x^k = 0$, hence $g_{ik}x^i x^k = 0$ for $\mathbf{x} \neq 0$. Contradiction to the condition of positive-definiteness.

7 *Let (u, v) be local coordinates on 2-dimensional Riemannian manifold M. Let Riemannian metric be given in these local coordinates by the matrix*

$$||g_{ik}|| = \begin{pmatrix} A(u,v) & B(u,v) \\ C(u,v) & D(u,v) \end{pmatrix}, \tag{2}$$

where $A(u,v), B(u,v), C(u,v), D(u,v)$ are smooth functions. Show that the following conditions are fulfilled:

a) $B(u,v) = C(u,v)$,

b) $A(u,v)D(u,v) - B(u,v)C(u,v) \neq 0$,

c) $A(u,v) > 0$,

d) $A(u,v)D(u,v) - B(u,v)C(u,v) > 0$.*

e) Show that conditions a), c) and d) are necessary and sufficient conditions for matrix $||g_{ik}||$ to define locally a Riemannian metric.*

Solution

Consider Riemannian scalar product $G(\mathbf{X}, \mathbf{Y}) = g_{ik}X^iY^k$.

a) The condition that $G(\mathbf{X}, \mathbf{Y}) = G(\mathbf{Y}, \mathbf{X})$ means that $g_{ik} = g_{ki}$, i.e. $B(u, v = C(u,v))$.

b) $\det G = A(u,v)D(u,v) - B(u,v)C(u,v) = AD - B^2 \neq 0$ *since it is non-degenerate (see the solution of exercise 1)*

c) Consider quadratic form $G(\mathbf{x}, \mathbf{x}) = g_{ik}x^i x^k = Ax^2 + 2Bxy + Dy^2$. (We already know that $B = C$) Positive -definiteness means that $G(\mathbf{x}, \mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$. In particular if we put $\mathbf{x} = (1, 0)$ we come to $G(\mathbf{x}, \mathbf{x}) = A > 0$. Thus $A > 0$.

d) Consider quadratic form $G(\mathbf{x}, \mathbf{x}) = g_{ik}x^i x^k = Ax^2 + 2Bxy + Dy^2$. We have an identity

$$G(\mathbf{x}, \mathbf{x}) = g_{ik}x^i x^k = Ax^2 + 2Bxy + Dy^2 = \frac{(Ax + By)^2 + (AD - B^2)y^2}{A}. \tag{1}$$

*We already know that $A > 0$ (take $\mathbf{x} = (x, 0)$). Now take $\mathbf{x} = (x, y)$: $Ax + By = 0$ (e.g. $\mathbf{x}$=(-B,A)) we come to $G(\mathbf{x}, \mathbf{x}) = \frac{(AD-B^2)y^2}{A} > 0$. Hence $(AD - B^2) = \det G > 0$ *.*

* *This special trick works good for dimension is $n = 2$. We could notice that A and $AD - B^2$ are principal main minors of the matrix G. In the general case (if G is $n \times n$ symmetric matrix) using triangular transformations one can show that quadratic form*

e) it follows from condition a) that matrix (1) is symmetric. It follows from conditionis (c) and (d) and equation (2) that $G(\mathbf{x},\mathbf{x}) > 0$ for any non-zero vector $\mathbf{x}$.

8 *Consider two-dimensional Riemannian manifold with Euclidean metric $G = dx^2 + dy^2$. How this metric will transform under arbitrary linear transformation $\begin{cases} x = ax' + by' + e \\ y = cx' + dy' + f \end{cases}$?*

Solution: Perform straightforward calculations: $dx = adx' + bdy'$ and $dy = cdx' + dy'$. Hence

$$G = dx^2+dy^2 = (adx'+bdy')^2+(cdx'+dy')^2 = (a^2+c^2)(dx')^2+2(ab+cd)dx'dy'+(b^2+d^2)(dy')^2 \,.$$

In coordinates (x,y) $||g_{ik}|| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and in coordinates (x',y') $||g'_{ik}|| = \begin{pmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{pmatrix}$.

9 *Consider domain in two-dimensional Riemannian manifold with Riemannian metric $G = du^2 + 2bdudv + dv^2$ in local coordinates u, v, where b is a constant.*

Show that $b^2 < 1$

Solution: Matrix of the metric G $\begin{pmatrix} 1 & b \\ b & 1 \end{pmatrix}$ is positive definite, hence $\det g = 1 - b^2 > 0$, i.e. $b^2 < 1$.

Another solution: for any non-zero vector $\mathbf{x}$, $G(\mathbf{x},\mathbf{x}) > 0$. Consider $\mathbf{x} = (t,1)$. Then for an arbitrary t $(t,1) \neq 0$ and $G(\mathbf{x},\mathbf{x}) = t^2 + 2bt + 1 > 0$. Hence polynomial $t^2 + 2bt + 1$ has no real roots, i.e. $b^2 < 1$.

One can see that the condition $b^2 < 1$ is not only necessary but it is sufficient condition for G to be a metric.

10 * *Show that $G = dx^2 + dy^2 + cdz^2$ in $\mathbf{R}^3$ defines Riemannian metric iff $c > 0$.*

* *Find null-vectors of pesudo-Riemannian metric G if $c < 0$.*

It is symmetric matrix which is positive definite iff $c > 0$. If $c < 0$ condition of positive-definiteness is failed, but matrix G is still non-degenerate. Null-vector $\mathbf{X} = (x,y,z)$: $G_{ik}X^iX^k = x^2 + y^2 + cz^2 = 0$, i.e. vector $\mathbf{X}$ belongs to the cone $x^2 + y^2 - cz^2 = 0$.

$A(\mathbf{X},\mathbf{X}) = a_{ik}x^ix^k$ (and respectively) is positive-definite if and only if all the leading principal minors Δ_k are positive (leading Principal minor Δ_k of the matrix A is a determinant of the matrix formed by first k columns and first k rows of the matrix A.). In this case matrix G_{ik} of bilinear form is transformed to unity matrix.