Homework 4

1 Consider parallelogram $\Pi_{\mathbf{a},\mathbf{b}}$ formed by two vectors in Euclidean space $\mathbf{E}^2$:

$$\Pi_{\mathbf{a},\mathbf{b}} = u\mathbf{a} + v\mathbf{b}\,, \quad 0 \leq u \leq 1, 0 \leq v \leq 1\,,$$

$$\mathbf{r}(u,v) = \mathbf{a}u + \mathbf{b}v = \begin{pmatrix} a_x \\ a_y \end{pmatrix} u \begin{pmatrix} b_x \\ b_y \end{pmatrix} v = \begin{cases} x = a_x u + b_x v \\ y = a_y u + b_y v \end{cases}.$$

a) Write down standard Euclidean metric $G = dx^2 + dy^2$ in coordinates (u,v).

b) Calculate the area of parallelogram $\Pi_{\mathbf{a},\mathbf{b}}$ using Riemannian volume form

c) Compare the answer with stadard formula for area of parallelogram (*See subsection 1.5.1 "Motiation. Gramm formula for volume of parallellepiped"*)

2 a) Consider the domain D on the cone $x^2 + y^2 - k^2z^2$ defined by the condition $0 < z < H$. Find an area of this domain using induced Riemannian metric. Compare with the answer when using standard formulae.

3 Find an area of the segment of the height h of the sphere of radius R (surface: $x^2 + y^2 + z^2 = R^2, \leq a \leq a + h$ for an arbitrary a: $-R \leq a \leq R - h$)

4 Find an area of 2-dimensional sphere of radius R using explicit formulae for induced Riemannian metric in stereographic coordinates.

5 Show that two spheres of different radii in Euclidean space are not isometric to each other, i.e. there is no an isometry of one sphere on another.

6 In exercise 4 of previous homework you have considered Riemannian manifolds $(\mathbf{R}^2, G^{(1)})$ and $(\mathbf{R}^2, G^{(2)})$, where

$$G^{(1)} = \frac{a(dx^2 + dy^2)}{(1 + x^2 + y^2)^2}\,, \quad \text{and} \quad G^{(2)} = \frac{4R^4(du^2 + dv^2)}{(R^2 + u^2 + v^2)^2}$$

(The second manifold is sphere of radius R without North pole in stereographic coordinates) You proved in the previous homework that in the case if $a = 4R^2$ then under isometry $\begin{cases} u = Rx \\ v = Ry \end{cases}$ these Riemannina manifolds are isometric. Using the result of previous exercise, prove now more strong statement, that in the case if the condition $a = 4R^2$ is not obeyed, then these manifolds *are not isometric.*

7 Let D be a domain in Lobachevsky plane which is lying between lines $x = a, x = -a$ and outside of the disc $x^2 + y^2 = 1$, $(0 < a < 1)$: $D = \{(x,y)\colon\ |x| < a, x^2 + y^2 > 1\}$,

a) Find the area of this domain.

b) Find the angles between lines and arc of the circle.

Lobachevsky plane, i.e. hyperbolic plane is the upper half plane with Riemannian metric $\frac{dx^2+dy^2}{y^2}$ *in Cartesian coordinates* x, y $(y > 0)$.

8* Find a volume of n-dimensional sphere of radius a. (You may use Riemannian metric in stereographic coordinates, or you may do it in other way... You just have to calculate the answer.)

Hint: One way to do it is the following. Denote by σ_n the volume of n-dimensional unit sphere embedded in Euclidean space $\mathbf{E}^{n+1}$. One can see that the volume of n-dimensional sphere of the radius R equals to $\sigma_n R^n$. We need to calculate just σ_n. Consider the following integral: $I_k = \int e^{-r^2} dx^1 dx^2 \dots dx^k$, where $r^2 = (x^1)^2 + (x^2)^2 + \dots + (x^k)^2$. One can see that on one hand $I_k = \int e^{-r^2} dx^1 dx^2 \dots dx^k = \left(\int e^{-x^2} dx\right)^n = \pi^{\frac{n}{2}}$, and on the other hand $I_k = \int e^{-r^2} dx^1 dx^2 \dots dx^k = \sigma_{k-1} \int e^{-r^2} r^{k-1} dr$. Comparing these integrals we calculate σ_n.