**EXAM FEEDBACK**

INTRODUCTION TO GEOMETRY. Spring 2016

MATH 20222

ANSWER **THREE** OF THE FOUR QUESTIONS
If four questions are answered credit will be given for the best three questions.
Each question is worth 20 marks.

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Electronic calculators may not be used

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**1.**

**(a)** Let $V$ be a vector space. What is meant by a scalar product on $V$?

Let $(x^1, x^2)$ be the coordinates of the vector $\mathbf{x}$ and let $(y^1, y^2)$ be the coordinates of the vector $\mathbf{y}$ in the vector space $\mathbf{R}^2$.

Show that the formula $(\mathbf{x}, \mathbf{y}) = 5x^1y^1 + 2x^2y^2$ defines a scalar product on $\mathbf{R}^2$.

Show that the formula $(\mathbf{x}, \mathbf{y}) = x^1y^2 + x^2y^1$ does not define a scalar product on $\mathbf{R}^2$.

[7 marks]

**(b)** State the Euler Theorem about rotations.

Let $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ be an orthonormal basis in $\mathbf{E}^3$.

Let $P_1$ and $P_2$ be linear operators on $\mathbf{E}^3$ such that operator $P_1$ is defined by relation

$$P_1(\mathbf{e}) = \mathbf{f}\,, \quad P_1(\mathbf{f}) = \mathbf{e}\,, \quad P_1(\mathbf{g}) = \mathbf{g}\,,$$

and for operator $P_2$, $P_2 = -P_1$, i.e. for arbitrary vector $\mathbf{x}$, $P_2(\mathbf{x}) = -P_1(\mathbf{x})$.

Explain which one of these operators is a rotation operator, and why. For rotation operator find an axis and angle of rotation.

[8 marks]

**(c)** Let $P$ be a linear operator on $\mathbf{E}^3$ such that

$$P(\mathbf{x}) = \mathbf{x} - \mathbf{n} \times (\mathbf{n} \times \mathbf{x})\,,$$

where $\mathbf{n}$ is a unit vector, and $'\times'$ is the vector product in $\mathbf{E}^3$. (We assume that $\mathbf{E}^3$ is equipped with orientation.)

Consider in $\mathbf{E}^3$ two parallelepipeds $\Pi(\mathbf{a}, \mathbf{b}, \mathbf{c})$ and $\Pi(\mathbf{a}', \mathbf{b}', \mathbf{c}')$, formed by vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, and respectively by vectors $\mathbf{a}', \mathbf{b}', \mathbf{c}'$, where

$$\mathbf{a}' = P(\mathbf{a}) , \quad \mathbf{b}' = P(\mathbf{b}) , \quad \mathbf{c}' = P(\mathbf{c}) .$$

Calculate the volume of parallelepiped $\Pi(\mathbf{a}', \mathbf{b}', \mathbf{c}')$ if the volume of parallelpiped $\Pi(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is equal to 1.

[5 marks]

## Discussion of first question

(**a**) Students have no special problems answering this question. Just answering the second part of the question you have to give an example of vector $\mathbf{x} \neq 0$ such that condition $(\mathbf{x}, \mathbf{x}) > 0$ is not obeyed. E.g. for the vector $\mathbf{x} = (0, 1)$, $(\mathbf{x}, \mathbf{x}) = 2x^1x^2 = 0$. This vector is not equal to zero, but $(\mathbf{x}, \mathbf{x}) = 0$. Hence this is not scalar product.

(**b**) Stating Euler Theorems students have no special problems to define axis, angle of rotationm but many students did not define clearly what is a rotation: it has to be clearly stated that 'every vector orthogonal to axis rotates on the angle $\varphi$ in the plane orthogonal to axis'

Answering on the question why operator is rotation operator, some students just checked the condition that operator $P$ preserves orientation, and did not check that this operator is orthogonal operator. (The fact that $\det P = 1$ does not imply that operator is orthogonal!)

Operators $P_1, P_2$ are both orthogonal operators since they transform orthonormal basis to another rorthonormal basis. This can be shown directly, and studens have no special problems to do it.

Since $\det P_1 = 1$ and $P_2 = -P_1$ then $\det P_2 = \det(-P_1) = \det(-1) \det P_2 = -\det P_2$, because determinant of operator $-1$ in 3-dimensional space is equal to $-1$. Many students concluded that $\det P_2 = 1$ just on the base that $\det P_1 = -1$ (it changes orientation), and $P_2 = -P_1$, without even making an attempt to see that determinant of operator $-1$ in 3-dimensiona space is equal to $-1$. Note that in 2-dimensional space determinant of operator $-1$ is equal to $+1$, and $P_2 = -P_1$ *implies* that $\det P_2 = \det P_1 = 1$.

(**c**) One can see straightforwardly that vector $\mathbf{n}$ is eigenvector with eigenvalue 1, and an arbitrary vector orthogonal to the vector $\mathbf{n}$ is eigenvector with eigenvalue 2. Choosing an arbitrary basis $\mathbf{e}, \mathbf{f}, \mathbf{n}$ such that vectors $\mathbf{e}, \mathbf{f}$ are orthogonal to $\mathbf{n}$ we see that matrix of operator $P$ in this basis is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Hence $\det P = 2$. This implies that volume of parallelepiped $\Pi(\mathbf{a}', \mathbf{b}', \mathbf{c}')$ is equal to $1 \cdot 4 = 4$.

This question was considered very difficult, but it solved by much more students that it was supposed to.

**2.**

**(a)** Give the definition of a differential 1-form on $\mathbf{E}^n$.

Let $f$ be a function on $\mathbf{E}^2$ given by $f(r, \varphi) = 2r^2 \cos 2\varphi$, where $r, \varphi$ are polar coordinates on $\mathbf{E}^2$.

Calculate the 1-form $\omega = df$.

Calculate the value of the 1-form $\omega$ on the vector field $\mathbf{A} = r^2\partial_r + r\partial_\varphi$.

Express the 1-form $\omega$ in Cartesian coordinates $x, y$ ($x = r\cos\varphi, y = r\sin\varphi$).

`[6 marks]`

**(b)** Consider in $\mathbf{E}^2$ the differential form $\omega = \frac{1}{2}(xdy - ydx)$ and the circle $C$ of radius $R$ with centre at the point $A = (1, 0)$.

Choose a parameterisation of this circle and calculate the integral $\int_C \omega$.

How does your answer depend on a choice of parameterisation?

How will the answer change if instead of the circle $C$ we consider another circle of the same radius $R$ but with centre at another point?

`[8 marks]`

**(c)** Explain what is meant by saying that a differential 1-form is exact.
Show that the 1-form $\omega = xdy + ydx$ is exact.

State the theorem about the integral of an exact 1-form over a curve $C$ in $\mathbf{E}^n$.

Explain why the 1-form $\omega = \frac{1}{2}(xdy - ydx)$ considered in part b) is not an exact form.

`[6 marks]`

### Discussion of second question

(**a**) Definition of forms and calculation of $\omega(\mathbf{A})$ did not arise special problems. Just many students still use wrong notations for differentials and vector field (Differential of function is $df$, not $\partial f$, vector along $Ox$ is $\partial_x$, not $d_x$)

Some students have problems answering the last part of this question:

i) About 15-20 students instead calculating 1-form $\omega = df$ in Cartesian coordinates were trying to caclulate the *function* $\omega(A) = 4r^2(\cos 2\varphi - \sin 2\varphi)$ in Catesian coordinates

ii) About 10-15 students were trying to perform straightforward transformation of the 1-form $\omega = df = 4r\cos 2\varphi dr - 4r^2 \sin 2\varphi d\varphi$ from polar coordinates to Cartesian. These calculations are too long (for exam) and not very easy, and nobody performed them correctly. (The short way: you have to calculate first a function $f$ in Cartesian coordinates

$f = 2r^2\cos 2\varphi = 2x^2 - 2y^2$ and then take its differential.)

iii) Some students could not perform transformation of function $f$ to Cartesian coordinates because they fail to use the formula $\cos 2\varphi = 2\cos^2\varphi - 1$.

(**b**) Calculation of $\int_C w = \pi R^2$. No special problems. When answering question how the answer depends on parameterisation you have to give full answer: "if parameterisation changes orientation, then.... and if parameterisation does not change orientation then,...." The answer like: "the sign depends on parameterisation" were considered non-complete

If you consider circle of radius $R$ with the centre at the point $(a, b)$ then calculating the integral you will come to answer

$$\int_C w = \int_0^{2\pi} \frac{1}{2}(\cos^2 t + \sin^2 t)dt + a\int_0^{2\pi} \cos t dt + b\int_0^{2\pi} \sin t dt$$

The second and third integrals vanish since $\int_0^{2\pi} \cos t dt = \int_0^{2\pi} \sin t dt = 0$, hence the answer does not depend on the position of the centre.

Many students ignore the last part of the question: answering on the question "how integral depends on position of circle" they just answer: it does not depend. This answer was considered as an empty answer.

**c**) This question did not induce problems. Just two comments. Answering very simple question: why the form $\omega = xdy + ydx$ is exact, and giving finally the right answer, many students were analyzing the partial derivatives $A_y, B_x$ ($\omega = Adx + Bdy$): Sure the fact $A_y \neq B_x$ for the form $\omega = Adx + Bdy$ means that the form is not exact, but the fact that $A_y = B_x$ does not imply that form is exact.

The fact that $\omega = \frac{1}{2}(xdy - ydx)$ is not exact follows from the previous consideration, but one can deduce from the fact that its itegral over circle calculated in the part **b**) was not equal to zero.

## 3.

**(a)** Describe what is meant by a natural parameter on a curve in $\mathbf{E}^n$.

Find a natural parameter for the following interval of the straight line

$$\begin{cases} x = t \\ y = 1 - 6t \end{cases}, \quad 0 < t < \infty\,.$$

Let $C\colon \mathbf{r} = \mathbf{r}(t)$, $0 \le t \le 2$ be a curve in $\mathbf{E}^2$ such that at an arbitrary point of this curve the angle between the velocity and acceleration vectors is acute and, for the velocity vector $\mathbf{v}(t)$,

$$\mathbf{v}(t)\big|_{t=0} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}\,.$$

Show that the length of the curve is greater than 10. `[7 marks]`

**(b)** Give the definition of the curvature of a curve in $\mathbf{E}^n$.

Explain why the curvature of a circle of radius $R$ is equal to $k = \frac{1}{R}$.

Consider the parabola $y = ax^2 + bx + c$, $a \neq 0$.

Show that curvature attains a maximum at the vertex of the parabola, and calculate the curvature of the parabola at this point.

`[7 marks]`

**(c)** Write down the formula expressing the curvature of a curve in $\mathbf{E}^n$ in terms of the area of the parallelogram $\Pi(\mathbf{v}, \mathbf{a})$ formed by velocity and acceleration vectors $\mathbf{v}(t)$, $\mathbf{a}(t)$.

A particle moves along a closed curve $C$ in $\mathbf{E}^n$ such that at the given point $A$ of the curve the speed attains the minimum value $v_{\text{min}} = 10$.
Calculate the length of the acceleration vector of the particle at the point $A$ if the curvature of the curve at the point $A$ is equal to $k = 1$.

`[6 marks]`

### Discussion of third question

(**a**) Definition of natural parameter and its calculation for straight line was alright.

The last part of the question: Since the angle is acute the length of the arc

$$L = \int_0^2 |\mathbf{v}(t)|dt \geq \int_0^2 |\mathbf{v}_0|dt = 2|\mathbf{v}_0| = 2\sqrt{3^2 + 4^2} = 10.$$

That is all. Surprisingly many students calculated initial velocity but they could not perfom this deduction. Many students confused between conception of velocity and speed. The sentence "*velocity* $\mathbf{v}$ *is increasing*" is not correct, velocity is a vector.

(**b**) Some students instead defining curvature (length of the acceleration vector in natural parameterisation) were giving different formulae for curvature.

Last part of this question was to show that curvature is maximal in the vertex and calculate the curvature at this point. The shortest solution: parabola $y = ax^2 + bx + c$ by translations in axis $OX$ and $OY$ can be transformed to parabola $y = ax^2$. Translations does not change curvature. Curvature of parabola $y = ax^2$ can be easily calculated and we see that the curvature $k = 2|a|$ is maximum at the vertex.

Many students who solved or partially solved this problem, instead did detailed calculations for parabola $\begin{cases} x = t \\ y = at^2 + bt + c \end{cases}$ then analysing the function $k(t)$ they found its maximum,.... Many of these students overhelmed these cacluclatons, many did mistakes and were confused in calculations. A common mistake was to write the final answer $k = 2a$ instead $k = 2|a|$.

(**c**) Many students trying to answer the question about the length of acceleration vector did not give explanation why acceleration is equal to normal acceleration (this is because the speed is minimal) Then one can use formula for paralllelogram, or the formula $k = \frac{|a_n|}{v^2}$.

## 4.

**(a)** Consider the sphere of radius $R$ in $\mathbf{E}^3$.

$$\mathbf{r}(\theta, \varphi): \quad \begin{cases} x = R \sin\theta \cos\varphi \\ y = R \sin\theta \sin\varphi \\ z = R \cos\theta \end{cases} .$$

Calculate the first quadratic form of this sphere.

Find the length of the curve $\varphi(t) = t$, $\theta(t) = \theta_0$, $0 \le t \le \frac{\pi}{4}$ on this sphere.

`[6 marks]`

**(b)** Explain what is meant by the shape operator for a surface $\mathbf{r} = \mathbf{r}(u, v)$ in $\mathbf{E}^3$ by defining its action on an arbitrary tangent vector to the surface.

Explain why the value of the shape operator is also a tangent vector to the surface.

`[6 marks]`

**(c)** Write down explicitly the action of the shape operator on tangent vectors for the sphere of radius $R$ in $\mathbf{E}^3$.

Calculate the Gaussian and the mean curvatures of the sphere of radius $R$.

Explain why the mean curvature is only defined up to a sign.

Is the Gaussian curvature only defined up to a sign? (Justify your answer.)

Let $D$ be a domain on a sphere of radius $R$ such that its area is equal to one fifth of the area of the sphere. Calculate the integral of the Gaussian curvature of the sphere over the domain $D$.

*(You may use the fact that the area of sphere of radius $R$ equals $4\pi R^2$.)*

`[8 marks]`

### Discussion of fourth question

General remark: This question as usual was not the most popular question: Only about 20-30 students (25 %) were answering this question. I would like to tell that the average mark of answers on this question was high.

(**a**) No problem with definition of quadratic form,but when calculating the length of the curve, half of students (who were trying to do it) came to answers which were against common sence: e.g. something like "length is proprotional to $R^2$" (lenght has to be proportional to $R$). Another typical mistake: writing for the length of the curve that $L = \int_0^{\frac{\pi}{4}} R \sin\theta_0 dt$ they have just to calculate

$$L = \int_0^{\frac{\pi}{4}} R \sin\theta_0 dt = \frac{\pi}{4} \sin\theta_0 \,.$$

This is simple and true, but some students doing this last step, instead doing this simple and right thing decided to caclulate the integral over variable $\theta$ ($\int_0^{\frac{\pi}{4}} R \sin\theta d\theta = -\cos\theta\Big|_0^{\frac{\pi}{2}}$. This has nothing to do with the question..

(**b**)No problem in this question.

(**c**) Students who answered this question answered it surprisingly good, except its last and easy part: to calculate the integral of curvature along the domain in the sphere.

*Solution*: Gaussian Curvature is constant. Hence integral of curvature over any domain is equal to the area of this domain multiplied on the curvature $= \frac{4\pi R^2}{5} \cdot K = \frac{4\pi R^2}{5} \cdot \frac{1}{R^2} = \frac{4\pi}{5}$.

**END OF EXAMINATION PAPER**