Homework 7. Solutions.

1 *Let (M,G) be a Riemannian manifold. Let C be a curve on M starting at the point $\mathbf{p}_1$ and ending at the point $\mathbf{p}_2$.*

Define an operator $P_C\colon T_{\mathbf{p}_1}M \to T_{\mathbf{p}_2}M$.

Explain why the parallel transport P_C is a linear orthogonal operator.

Let the points $\mathbf{p}_1$ and $\mathbf{p}_2$ coincide, so that C is a closed curve.

Let $\mathbf{a}$ be a vector attached at the point $\mathbf{p}_1$, and $\mathbf{b} = P_C(\mathbf{a})$.

Consider operator P_C^2. Suppose that $P_C(\mathbf{a}) = \mathbf{b}$ and $P_C^2(\mathbf{a}) = -\mathbf{a}$. Show that vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal to each other.

Solution Definition and porperties of the parallel transport see in Lecture Notes.

Since parallel transport is orthogonal operators, hence it preserves the scalar product, i.e. $<\mathbf{a},\mathbf{b}>=<P_C\mathbf{a}, P_C\mathbf{b}>=<\mathbf{b},-\mathbf{a}>=-<\mathbf{a},\mathbf{b}>$, i.e. $<\mathbf{a},\mathbf{b}>=0$, i.e. these vectors are orthogonal to each other.

2 *Consider $\mathbf{R}^2$ equipped with Riemannian metric $G = \sigma(x,y)(dx^2+dy^2)$.*

Consider in this Riemannian manifold upper half-circle equipped with two different parameterisations

$$C_1\colon \begin{cases} x = R\cos t \\ y = R\sin t \end{cases}, \quad 0 \le t \le \pi\,, \qquad C_2\colon \begin{cases} x = R\cos 2t \\ y = R\sin 2t \end{cases}, \quad 0 \le t \le \frac{\pi}{2}$$

Write down explicitly equations of motion defining parallel transport for the curve C_1.'

Show explictily that operator of parallel transport does not change if we change C_1 on C_2.

Solution Let $\mathbf{X}(t) = (X^1(t), X^2(t))$ be parallel transport along the circle of the initial vector $\mathbf{X}$. We have the differential equations

$$\frac{dX^\alpha(t)}{dt} + \frac{dx^\alpha(t)}{dt}\Gamma^\alpha_{\beta\gamma}(x(t))X^\gamma(t) = 0\,. \tag{1}$$

with initial conditions $\mathbf{X}^\alpha(t) = \mathbf{X}_0^\alpha$ $(\alpha,\beta,\gamma = 1,2)$ In components:

$$\frac{dx^1(t)}{dt} + \frac{dx(t)}{dt}\gamma^1_{1\gamma}(x(t,y(t)))x^\gamma(t) + \frac{dy(t)}{dt}\gamma^1_{2\gamma}(x(t,y(t)))X^\gamma(t) =$$

$$\frac{dX^1(t)}{dt} - \cos t\Gamma^1_{1\gamma}(x(t),y(t)))X^\gamma(t) + \sin t\Gamma^1_{2\gamma}(x(t),y(t))X^\gamma(t) = 0$$

and for $X^2(t)$:

$$\frac{dX^2(t)}{dt} - \cos t\Gamma^2_{1\gamma}(x(t),y(t)))X^\gamma(t) + \sin t\Gamma^2_{2\gamma}(x(t),y(t))X^\gamma(t) = 0\,.$$

here Christoffel symbols can be calculated from Levi-Civita formula:

$$\Gamma^y_{xy} = \Gamma^y_{yx} = -\Gamma^x_{xx} = -\Gamma^x_{yy} = \frac{1}{2\sigma}\frac{\partial\sigma}{\partial x}\,, \Gamma^x_{yx} = \Gamma^x_{xy} = -\Gamma^y_{xx} = -\Gamma^y_{yy} = \frac{1}{2\sigma}\frac{\partial\sigma}{\partial y},,$$

$(x \leftrightarrow 1,\ y \leftrightarrow 2)$.

Let functions $X^\alpha(t)$ are solutions of differential equations (1). Consider an arbitrary reparameterisation: $t = t(\tau)$ preserving orientation, Then initial conditions will not change, and consider the functions:

$X^(\tau) = X^\alpha(t(\tau))$, then equation (1) implies:

$$\frac{dX^\alpha(\tau)}{d\tau} + \frac{dx^\alpha(\tau)}{d\tau}\Gamma^\alpha_{\beta\gamma}(x(\tau))X^\gamma(\tau) = \frac{dt}{d\tau}\left(\frac{dX^\alpha(t)}{dt} + \frac{dx^\alpha(t)}{dt}\Gamma^\alpha_{\beta\gamma}(x(t))X^\gamma(t)\right) = 0\,.$$

i.e. it is reparameterisation invariant if $t'(\tau) > 0$.

3 *Consider the Lagrangian of "free" particle $L = \frac{1}{2} g_{ik} \dot{x}^i \dot{x}^k$ for Riemannian manifold with a metric $G = g_{ik} dx^i dx^k$.*
Write down Euler-Lagrange equations of motion for this Lagrangian and compare them with differential equations for geodesics on this Riemannian manifold.

In fact show that

$$\underbrace{\frac{\partial L}{\partial x^i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}}_{\text{Euler-Lagrange equations}} \qquad \Leftrightarrow \quad \underbrace{\frac{d^2 x^i}{dt^2} + \Gamma^i_{km} \dot{x}^k \dot{x}^m = 0}_{\text{Equations for geodesics}} \ , \tag{1}$$

where

$$\Gamma^i_{km} = \frac{1}{2} g^{ij} \left(\frac{\partial g_{jk}}{\partial x^m} + \frac{\partial g_{jm}}{\partial x^k} - \frac{\partial g_{km}}{\partial x^j} \right) . \tag{2}$$

Solution: see the lecture notes.

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Write down the Lagrangian of free particle $L = \frac{1}{2} g_{ik} \dot{x}^i \dot{x}^k$ and using Euler-Lagrange equations for this Lagrangian calculate Christoffel symbols (Christoffel symbols of Levi-Civita connection) for

a) cylindrical surface of the radius R

b) for the cone $x^2 + y^2 - k^2 z^2 = 0$

c) for the sphere of radius R

d) for Lobachevsky plane

Compare with the results that you obtained using straightforwardly the formula (1) or using formulae for induced connection.

Solution.

a) *For the sphere:*

Riemannian metric on sphere in spherical coordinates is $G = R^2 d\theta^2 + R^2 \sin^2 \theta d\varphi^2$. Hence the Lagrangian of the free particle is

$$L = \frac{R^2 \dot{\theta}^2 + R^2 \sin^2 \theta \dot{\varphi}^2}{2}$$

Euler-Lagrange equations for θ:

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) , \quad \frac{\partial L}{\partial \theta} = R^2 \sin\theta \cos\theta \dot{\varphi}^2, \ \frac{\partial L}{\partial \dot{\theta}} = R^2 \dot{\theta}$$

Hence

$$\frac{d}{dt}\left(R^2 \dot{\theta}\right) = R^2 \sin\theta\cos\theta\dot{\varphi}^2, R^2 \ddot{\theta} = R^2 \sin\theta\cos\theta\dot{\varphi}^2,$$

hence

$$\ddot{\theta} - \sin\theta\cos\theta\dot{\varphi}^2 = 0 \, .$$

Comparing with equation for geodesic

$$\ddot{\theta} + \Gamma^\theta_{\theta\theta}\dot{\theta}\dot{\theta} + \Gamma^\theta_{\theta\varphi}\dot{\theta}\dot{\varphi} + \Gamma^\theta_{\varphi\theta}\dot{\varphi}\dot{\theta} + \Gamma^\theta_{\varphi\varphi}\dot{\varphi}\dot{\varphi} = \ddot{\theta} + \Gamma^\theta_{\theta\theta}\dot{\theta}\dot{\theta} + 2\Gamma^\theta_{\theta\varphi}\dot{\theta}\dot{\varphi} + \Gamma^\theta_{\varphi\varphi}\dot{\varphi}\dot{\varphi} = 0$$

we see that

$$\Gamma^\theta_{\theta\theta} = \Gamma^\theta_{\theta\varphi} = \Gamma^\theta_{\varphi\theta} = 0, \ \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta$$

Analogously Euler-Lagrange equations for φ:

$$\frac{\partial L}{\partial \varphi} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi}\right), \quad \frac{\partial L}{\partial \varphi} = 0, \ \frac{\partial L}{\partial \dot\varphi} = R^2 \sin^2\theta \dot\varphi \,.$$

Hence

$$\frac{d}{dt}\left(R^2 \sin^2\theta \dot\varphi\right) = 0, \ R^2 \sin^2\theta \ddot\varphi + 2R^2 \sin\theta\cos\theta\dot\theta\dot\varphi = 0,$$

hence

$$\ddot\theta + \mathrm{cotan}\, \theta\dot\theta\dot\varphi = 0,$$

Comparing with equation for geodesic

$$\ddot\varphi + \Gamma^\varphi_{\theta\theta}\dot\theta\dot\theta + \Gamma^\varphi_{\theta\varphi}\dot\theta\dot\varphi + \Gamma^\varphi_{\varphi\theta}\dot\varphi\dot\theta + \Gamma^\varphi_{\varphi\varphi}\dot\varphi\dot\varphi = \ddot\theta + \Gamma^\varphi_{\theta\theta}\dot\theta\dot\theta + 2\Gamma^\varphi_{\theta\varphi}\dot\theta\dot\varphi + \Gamma^\varphi_{\varphi\varphi}\dot\varphi\dot\varphi = 0$$

we see that

$$\Gamma^\varphi_{\theta\theta} = \Gamma^\varphi_{\varphi\varphi} = 0\,, \Gamma^\varphi_{\varphi\theta} = \Gamma^\varphi_{\theta\varphi} = \mathrm{cotan}\,\theta\,.$$

b) *For Lobachevsky plane*:
Lagrangian of "free" particle on the Lobachevsky plane with metric $G = \frac{dx^2+dy^2}{y^2}$ is

$$L = \frac{1}{2}\frac{\dot x^2 + \dot y^2}{y^2}.$$

Euler-Lagrange equations are

$$\frac{\partial L}{\partial x} = 0 = \frac{d}{dt}\frac{\partial L}{\partial \dot x} = \frac{d}{dt}\left(\frac{\dot x}{y^2}\right) = \frac{\ddot x}{y^2} - \frac{2\dot x\dot y}{y^3}, \text{i.e.} \quad \ddot x - \frac{2\dot x \dot y}{y} = 0\,,$$

$$\frac{\partial L}{\partial y} = -\frac{\dot x^2 + \dot y^2}{y^3} = \frac{d}{dt}\frac{\partial L}{\partial \dot y} = \frac{d}{dt}\left(\frac{\dot y}{y^2}\right) = \frac{\ddot y}{y^2} - \frac{2\dot y^2}{y^3}, \text{i.e.} \quad \ddot y + \frac{\dot x^2}{y} - \frac{\dot y^2}{y} = 0\,.$$

Comparing these equations with equations for geodesics: $\ddot x^i - \dot x^k \Gamma^i_{km}\dot x^m = 0$ ($i = 1, 2$, $x = x^1, y = x^2$) we come to

$$\Gamma^x_{xx} = 0, \Gamma^x_{xy} = \Gamma^x_{yx} = -\frac{1}{y}, \ \Gamma^x_{yy} = 0, \ \Gamma^y_{xx} = \frac{1}{y}, \Gamma^y_{xy} = \Gamma^y_{yx} = 0, \Gamma^y_{yy} = -\frac{1}{y}\,. \ \blacksquare$$

The answers are the same as calculated with other methods. We see that Lagrangians give us the nice and quick way to calculate Christoffel symbols.

5 *Find geodesics on cylinder*
a) using straightforwardly equations for geodesics,
b) using the fact that geodesic is shortest.

6* *Find geodesics on sphere*
a) using straightforwardly equations for geodesics,
b) using the fact that geodesic is shortest.
Show here by straightforward calculations that geodesics on sphere are great circles.

The straightforward equations for geodesic: $\frac{d^2x^i}{dt^2} + \frac{dx^k}{dt}\Gamma^i_{km}\frac{dx^m}{dt} = 0$ are just equation of motion for free Lagrangian on the Riemannian surface. Hence in the case of sphere they are equations of motion of the

Lagrangian of "free" particle on the sphere is $L = \frac{R^2\dot\theta^2+R^2\sin^2\theta\dot\varphi^2}{2}$. Its equations of motion are second order differential equations

$$\begin{cases}\ddot\theta - \sin\theta\cos\theta\dot\varphi^2 = 0\\ \ddot\varphi + 2\text{cotan}\,\theta\dot\theta\dot\varphi = 0\end{cases}\quad \text{with intitial conditions}\quad \begin{cases}\theta(t)|_{t=0} = \theta_0,\ \dot\theta(t)|_{t=0} = a\\ \varphi(t)|_{t=0} = \varphi_0,\ \dot\varphi(t)|_{t=0} = b\end{cases}\tag{1}$$

for geodesics $\theta(t), \varphi(t)$ starting at the initial point $\mathbf{p} = (\theta_0, \varphi_0)$ with initial velocity $\mathbf{v}_0 = a\frac{\partial}{\partial\theta} + b\frac{\partial}{\partial\varphi}$. (All Christoffel symbols vanish except $\Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta$, and $\Gamma^\varphi_{\varphi\theta} = \Gamma^\varphi_{\theta\varphi} = \text{cotan}\,\theta$.)

This differential equation is not very easy to solve in general case. On the other hand use the fact that rotations are isometries of the sphere. Rotate the sphere in a way such that the initial point transforms to the point the point $\theta_0 = \frac{\pi}{2}, \varphi_0 = 0$ and then rotate the sphere with respect to the axis $0X$ such that θ-component of velocity becomes zero. We come to the same differential equation but with changed initial conditions:

$$\begin{cases}\ddot\theta - \sin\theta\cos\theta\dot\varphi^2 = 0\\ \ddot\varphi + 2\text{cotan}\,\theta\dot\theta\dot\varphi = 0\end{cases}\quad \text{with intitial conditions}\quad \begin{cases}\theta(t)|_{t=0} = \frac{\pi}{2},\ \dot\theta(t)|_{t=0} = 0\\ \varphi(t)|_{t=0} = 0,\ \dot\varphi(t)|_{t=0} = \Omega_0\end{cases}\tag{2}$$

where we denote by Ω_0 the magnitude of initial velocity. One can easy check that the functions

$$\begin{cases}\theta(t) = \frac{\pi}{2}\\ \varphi(t) = \Omega_0 t\end{cases}$$

are the solution of the differential equations for geodesic with initial conditions (2). Hence this is geodesic passing through the point $\left(\theta_0 = \frac{\pi}{2}, \varphi = 0\right)$ with initial velocity $\Omega_0\frac{\partial}{\partial\varphi}$. We see that this geodesic is the equator of the sphere. We proved that an arbitrary geodesic after applying the suitable rotation is the great-circle—equator. On the other hand an equator is the great circle (the intersection of the sphere $x^2+y^2+z^2 = R^2$ with the plane $z = 0$) and the rotation transforms the equator to the another great circle. Hence all arcs of great circles are geodesics and all geodesics are arcs of great circles.

To show that great circles arcs are shortest, one can first to look for two points which belong to the same meridian. It is easy to see that $\theta = consta$ is the shortest. Then the symmetry conisderations (rotations) imply the same answer for arbitrary two points.

7 *Great circle is a geodesic.*
Every geodesic is a great circle.
Are these statements correct?
Make on the base of these statements correct statements and justify them.

The correct statments are: great circles indeed are *un-parametersied* geodesics. One can consider suitable parameterisation of great circle such that it becomes geodesic (i.e. parameterised geodesics) For this purpose one has to consider a parameterisation such that speed is constant in this parameterisation.

Every geodesic conisdered as unparametersied curve is great circle.

(See in detail lecture notes.)

8 *On the unit sphere $x^2+y^2+z^2 = 1$ in $\mathbf{E}^3$ consider the curve C defined by the equaton $\cos\theta - \sin\theta\sin\varphi = 0$ in spherical coordinates.*

Show that in the process of parallel transport along the curve C an arbitrary tangent vector to the curve remains tangent to the curve. Notice that $\cos\theta - \sin\theta\sin\varphi = x - z\big|_{x^2+y^2+z^2=1}$, i.e. the curve C is the intersection of the plane $x - z$ which goes through origin with the sphere. This means that C is great circle. Hence tangent vector remains tangent (and keeping its length) due to parallel transport.