Homework 3a. Solutions

1 *Let* $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$ *be a basis in* 3*-dimensional vector space* V*.*
Consider in the space V *the following ordered triples*
I)—- $\{\mathbf{e}+2f+3\mathbf{g}\,,2\mathbf{f}+\mathbf{g}\,,\mathbf{e}+2\mathbf{f}+\mathbf{g}\}$
II)—- $\{\mathbf{e}+\mathbf{f}-2\mathbf{g}\,,2\mathbf{f}+\mathbf{g}\,,\mathbf{e}+\mathbf{f}+\mathbf{g}\}$
III)—- $\{\mathbf{e}+2\mathbf{f}+4\mathbf{g}\,,\mathbf{e}+3\mathbf{f}+9\mathbf{g}\,,\mathbf{e}+4f\mathbf{f}+16\mathbf{g}\}$
Show that all these oredered triples are bases.
Show that I-st and II-nd bases have opposite orientations.
Show that II-nd and III-d bases have same orientations.
Show that I-st and III-nd bases have opposite orientations.
Calculate transition matrices T_I, T_{II} and T_{III} from intial basis $\mathbf{e},\mathbf{f},\mathbf{g}$ to the these triples:

$$\{\mathbf{e}+2f+3\mathbf{g}\,,2\mathbf{f}+\mathbf{g}\,,\mathbf{e}+2\mathbf{f}+\mathbf{g}\} = \{\mathbf{e},\mathbf{f},\mathbf{g}\}\underbrace{\begin{pmatrix}1 & 0 & 1\\ 2 & 2 & 2\\ 3 & 1 & 1\end{pmatrix}}_{T_I}, \qquad \det T_I = -4\,,$$

$$\{\mathbf{e}+f-2\mathbf{g}\,,2\mathbf{f}+\mathbf{g}\,,\mathbf{e}+\mathbf{f}+\mathbf{g}\} = \{\mathbf{e},\mathbf{f},\mathbf{g}\}\underbrace{\begin{pmatrix}1 & 0 & 1\\ 1 & 2 & 1\\ -2 & 1 & 1\end{pmatrix}}_{T_I}, \qquad \det T_I = 6\,,$$

$$\{\mathbf{e}+2f+4\mathbf{g}\,,\mathbf{e}+3\mathbf{f}+9\mathbf{g}\,,\mathbf{e}+4\mathbf{f}+16\mathbf{g}\} = \{\mathbf{e},\mathbf{f},\mathbf{g}\}\underbrace{\begin{pmatrix}1 & 1 & 1\\ 2 & 3 & 4\\ 4 & 9 & 16\end{pmatrix}}_{T_I}, \qquad \det T_I = 2\,.$$

We see that all transition matrices are not degenerate, hence all the triples are bases.

The first transition matrix has negative determinant, hence the I-st basis has orientation opposite to the orientation of the basis $\{\mathbf{e},\mathbf{f},\mathbf{g},\}$. Respectively the second transition matrix has positive determinant, hence the II-nd basis has the same orientation as the basis $\{\mathbf{e},\mathbf{f},\mathbf{g},\}$. We see that I-st and II-nd bases belong to different equivalence classes. Hence they have opposite orientations. The third transition matrix has positive determinant. Hence II-nd and III-rd bases both have the same orientation as initial basis $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$. Thus we see that II-nd and III-rd bases have the same orientation.

Finally since first basis has orientation opposite to the orientation of the basis $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$ and III-rd basis has the same orientation as the basis $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$ hence I-st and III-rd bases have opposite orientations.

2 *Consider an operator* P *on* $\mathbf{E}^3$ *such that* P *is an orthogonal operator preserving the orientation of* $\mathbf{E}^3$ *and*

$$P(\mathbf{e}_x) = \mathbf{e}_y, P(\mathbf{e}_z) = -\mathbf{e}_z\,.$$

Find an action of the operator P *on an arbitrary vector* $\mathbf{x} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$*.*

Why P *is a rotation operator? Find an angle and axis of the rotation.*

(We assume that $\{\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z\}$ *is an orthonormal basis.)*

Solution. Calculate first $P(\mathbf{e}_y)$. P is orthogonal operator and unit vector $\mathbf{e}_y$ is orthogonal to vectors $\mathbf{e}_x,\mathbf{e}_z$. Hence vector $P(\mathbf{e}_y)$ is unit vector also and $P(\mathbf{e}_y)$ MUST be orthogonal to vectors $P(\mathbf{e}_x)$ and $P(\mathbf{e}_z)$:

$$(P(\mathbf{e}_y),P(\mathbf{e}_y)) = (\mathbf{e}_y,\mathbf{e}_y) = 1,\ (P(\mathbf{e}_y),P(\mathbf{e}_x)) = (\mathbf{e}_y,\mathbf{e}_x) = 0,\ (P(\mathbf{e}_y),P(\mathbf{e}_z)) = (\mathbf{e}_y,\mathbf{e}_z) = 0,$$

Since $P(\mathbf{e}_x) = \mathbf{e}_y$ and $P(\mathbf{e}_z) = -\mathbf{e}_z$ we see that vector $P(\mathbf{e}_y)$ has to be proportional to vector $\mathbf{e}_x$: $P(\mathbf{e}_y) = c\mathbf{e}_x$ with $c = \pm 1$ since the length of the vector $P(\mathbf{e}_y)$ is equal to 1. Calculate c. We already know that $c = \pm 1$. The triple $\{P(\mathbf{e}_x),P(\mathbf{e}_y),P(\mathbf{e}_z)\}$ has the same orientation as the triple $\{\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z\}$ since operator P preserves orientation. We have

$$\{P(\mathbf{e}_x),P(\mathbf{e}_y),P(\mathbf{e}_z)\} = \{c\mathbf{e}_y,\mathbf{e}_x,-\mathbf{e}_z\} \sim \{\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z)\} \Rightarrow c = 1\,.$$

We see that $P(\mathbf{e}_y) = \mathbf{e}_x$. Hence for an arbitrary vector $\mathbf{x} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$

$$P(\mathbf{x}) = P(x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z) = x\mathbf{e}_y + y\mathbf{e}_x - z\mathbf{e}_z : \quad P\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ x \\ -z \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

In particular $P(\mathbf{e}_x + \mathbf{e}_y) = \mathbf{e}_x + \mathbf{e}_y$. Hence $\mathbf{N} = \mathbf{e}_x + \mathbf{e}_y$ is an eigenvector with eigenvalue $\lambda = 1$. Axis of rotation is directed along this vector. One can come to this answer another way doing 'matrix calculus':

$$P\mathbf{N} = \mathbf{N}\,, P\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ x \\ -z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \lambda \\ \lambda \\ 0 \end{pmatrix}.$$

i.e. N is proportional to the vector $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $(\mathbf{N} = c(\mathbf{e}_x + \mathbf{e}_y))$.

The axis of rotation is the bisectrices of the angle between $\mathbf{e}_x$ and $\mathbf{e}_y$ axis.

To find an angle of rotation we calculate $\mathrm{Tr}P = 1 + 2\cos\varphi = -1$. Hence angle of the rotation is equal to π.

We may calculate the angle of rotation in other way too: consider an arbitrary vector $\mathbf{x} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$ which is orthogonal to axis: $(\mathbf{x}, \mathbf{N}) = x + y$ (if $\mathbf{N} = \mathbf{e}_x + \mathbf{e}_y$). Hence vectors orthogonal to axis have appearance $\mathbf{x} = a(\mathbf{e}_x - \mathbf{e}_y) + b\mathbf{e}_z$ (x-component + y component equals zero.) We have that

$$\text{for } \mathbf{x} = a(\mathbf{e}_x - \mathbf{e}_y) + b\mathbf{e}_z P(\mathbf{x}) = \mathbf{x} = a(\mathbf{e}_y - \mathbf{e}_x) - b\mathbf{e}_z = -\mathbf{x}.$$

We see that any vector orthogonal to axis is multiplied on -1. Thus P is rotation on the angle π. (See also the end of the section 1.11 in lecture notes)

3 *Consider an operator P on $\mathbf{E}^3$ such that*

$$P(\mathbf{e}) = \frac{2}{3}\mathbf{e} + \frac{2}{3}\mathbf{f} + \frac{1}{3}\mathbf{g}\,, P(\mathbf{f}) = -\frac{1}{3}\mathbf{e} + \frac{2}{3}\mathbf{f} - \frac{2}{3}\mathbf{g}\,, P(\mathbf{g}) = -\frac{2}{3}\mathbf{e} + \frac{1}{3}\mathbf{f} + \frac{2}{3}\mathbf{g}\,.$$

Show that this is an orthogonal operator preserving the orientation of $\mathbf{E}^3$.

Find eigenvectors of this operator.

(We assume that $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ is an orthonormal basis in $\mathbf{E}^3$.)

Solution It is easy to see that

$$\begin{array}{rcr} (\mathbf{e}', \mathbf{e}') = (P(\mathbf{e}), P(\mathbf{e})) = & \left(\frac{2}{3}\mathbf{e} + \frac{2}{3}\mathbf{f} + \frac{1}{3}\mathbf{g}, \frac{2}{3}\mathbf{e} + \frac{2}{3}\mathbf{f} + \frac{1}{3}\mathbf{g}\right) = & 1, \\ (\mathbf{e}', \mathbf{f}') = (P(\mathbf{e}), P(\mathbf{f})) = & \left(\frac{2}{3}\mathbf{e} + \frac{2}{3}\mathbf{f} + \frac{1}{3}\mathbf{g}, -\frac{1}{3}\mathbf{e} + \frac{2}{3}\mathbf{f} - \frac{2}{3}\mathbf{g}\right) = & 0, \\ (\mathbf{e}', \mathbf{g}') = (P(\mathbf{e}), P(\mathbf{g})) = & \left(\frac{2}{3}\mathbf{e} + \frac{2}{3}\mathbf{f} + \frac{1}{3}\mathbf{g}, -\frac{2}{3}\mathbf{e} + \frac{1}{3}\mathbf{f} + \frac{2}{3}\mathbf{g}\right) = & 0, \\ (\mathbf{f}', \mathbf{f}') = (P(\mathbf{f}), P(\mathbf{f})) = & \left(-\frac{1}{3}\mathbf{e} + \frac{2}{3}\mathbf{f} - \frac{2}{3}\mathbf{g}, -\frac{1}{3}\mathbf{e} + \frac{2}{3}\mathbf{f} - \frac{2}{3}\mathbf{g}\right) = & 1, \\ (\mathbf{f}', \mathbf{g}') = (P(\mathbf{f}), P(\mathbf{g})) = & \left(-\frac{1}{3}\mathbf{e} + \frac{2}{3}\mathbf{f} - \frac{2}{3}\mathbf{g}, -\frac{2}{3}\mathbf{e} + \frac{1}{3}\mathbf{f} + \frac{2}{3}\mathbf{g}\right) = & 0, \\ (\mathbf{g}', \mathbf{g}') = (P(\mathbf{g}), P(\mathbf{g})) = & \left(-\frac{2}{3}\mathbf{e} + \frac{1}{3}\mathbf{f} + \frac{2}{3}\mathbf{g}, -\frac{2}{3}\mathbf{e} + \frac{1}{3}\mathbf{f} + \frac{2}{3}\mathbf{g}\right) = & 1 \end{array}$$

new basis is orthonormal one. Hence P is orthogonal operator. The matrix of operator P is

$$\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{pmatrix}$$

Its determinant equals $\det P = 1$. Operator P preserves orientation. To find an axis we have to find eigenvector of this matrix with eigenvalue 1. Eigenvalue equals 1, since this is rotation: We have

$$P\mathbf{N} = \mathbf{N}, \quad \frac{1}{3}\begin{pmatrix} 2 & -1 & -2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Solving these equations we come to $x = y = -z$, i.e. $\mathbf{N}$ is proportional to the vector $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.