Introduction to Geometry

it is a draft of lecture notes of H.M. Khudaverdian.
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Contents

1 Euclidean space

We recall important notions from linear algebra.

1.1 Vector space.

Vector space V on real numbers is a set of vectors with operations " $+$ "—addition of vector and " $\cdot$ "—multiplication of vector Lon real number (sometimes called coefficients, scalars). These operations obey the following axioms

- $\forall \mathbf{a}, \mathbf{b} \in V, \mathbf{a}+\mathbf{b} \in V$,
- $\forall \lambda \in \mathbf{R}, \forall \mathbf{a} \in V, \lambda \mathbf{a} \in V$.
- $\forall \mathbf{a}, \mathbf{b} \mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}$ (commutativity)
- $\forall \mathbf{a}, \mathbf{b}, \mathbf{c},\ \mathbf{a}+(\mathbf{b}+\mathbf{c}) = (\mathbf{a}+\mathbf{b})+\mathbf{c}$ (associativity)
- $\exists\ \mathbf{0}$ such that $\forall \mathbf{a}$, $\mathbf{a}+\mathbf{0}=\mathbf{a}$
- $\forall \mathbf{a}$ there exists a vector $-\mathbf{a}$ such that $\mathbf{a}+(-\mathbf{a})=0$.
- $\forall \lambda \in \mathbf{R}, \lambda(\mathbf{a}+\mathbf{b}) = \lambda\mathbf{a}+\lambda\mathbf{b}$
- $\forall \lambda, \mu \in \mathbf{R} (\lambda+\mu)\mathbf{a} = \lambda\mathbf{a}+\mu\mathbf{a}$

- $(\lambda\mu)\mathbf{a} = \lambda(\mu\mathbf{a})$
- $1\mathbf{a} = \mathbf{a}$

It follows from these axioms that in particularly $\mathbf{0}$ is unique and $-\mathbf{a}$ is uniquely defined by $\mathbf{a}$. (Prove it.)

Remark We denote by 0 real number 0 and *vector* $\mathbf{0}$. Sometimes we have to be careful to distinguish between zero vector $\mathbf{0}$ and number zero.

Examples of vector spaces...

1.2 Basic example of n-dimensional vector space—$\mathbf{R}^n$

A basic example of vector space (over real numbers) is a space of ordered n-tuples of real numbers.
$\mathbf{R}^2$ is a space of pairs of real numbers. $\mathbf{R}^2 = \{(x, y),\ x, y \in \mathbf{R}\}$
$\mathbf{R}^3$ is a space of triples of real numbers. $\mathbf{R}^3 = \{(x, y, z),\ x, y, z \in \mathbf{R}\}$
$\mathbf{R}^4$ is a space of quadruples of real numbers. $\mathbf{R}^4 = \{(x, y, z, t),\ x, y, z, t, \in \mathbf{R}\}$
and so on...

$\mathbf{R}^n$—is a space of n-typles of real numbers:

$$\mathbf{R}^n = \{(x^1, x^2, \dots, x^n),\ x^1, \dots, , x^n \in \mathbf{R}\} \tag{1.1}$$

If $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ are two vectors, $\mathbf{x} = (x^1, \dots, x^n)$, $\mathbf{y} = (y^1, \dots, y^n)$ then

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \dots, x_n + y_n)\,.$$

and multiplication on scalars is defined as

$$\lambda\mathbf{x} = \lambda \cdot (x^1, \dots, x^n) = (\lambda x^1, \dots, \lambda x^n)\,, \quad (\lambda \in \mathbf{R})\,.$$

$(\lambda \in \mathbf{R})$.

1.3 Linear dependence of vectors

We often consider linear combinations in vector space:

$$\sum_i \lambda_i \mathbf{x}_i = \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \dots + \lambda_m\mathbf{x}_m\,, \tag{1.2}$$

where $\lambda_1, \lambda_2, \dots, \lambda_m$ are coefficients (real numbers), $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m$ are vectors from vector space V. We say that linear combination (1.2) is *trivial* if all coefficients $\lambda_1, \lambda_2, \dots, \lambda_m$ are equal to zero.

$$\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0\,.$$

We say that linear combination (1.2) is *not trivial* if at least one of coefficients $\lambda_1, \lambda_2, \dots, \lambda_m$ is not equal to zero:

$$\lambda_1 \neq 0, \mathrm{or} \lambda_2 \neq 0, \mathrm{or} \dots \mathrm{or} \lambda_m \neq 0\,.$$

Recall definition of linearly dependent and linearly independent vectors:

Definition The vectors $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ in vector space V are *linearly dependent* if there exists a non-trivial linear combination of these vectors such that it is equal to zero.

In other words we say that the vectors $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ in vector space V are *linearly dependent* if there exist coefficients $\mu_1, \mu_2, \dots, \mu_m$ such that at least one of these coefficients is not equal to zero and

$$\mu_1 \mathbf{x}_1 + \mu_2 \mathbf{x}_2 + \cdots + \mu_m \mathbf{x}_m = 0\,. \tag{1.3}$$

Respectively vectors $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ are *linearly independent* if they are not linearly dependent. This means that an arbitrary linear combination of these vectors which is equal zero is trivial.

In other words vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_m\}$ are *linearly independent* if the condition

$$\mu_1 \mathbf{x}_1 + \mu_2 \mathbf{x}_2 + \cdots + \mu_m \mathbf{x}_m = 0$$

implies that $\mu_1 = \mu_2 = \cdots = \mu_m = 0$.

Very useful and workable

Proposition *Vectors $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m\}$ in vector space V are linearly dependent if and only if at least one of these vectors is expressed via linear combination of other vectors:*

$$\mathbf{x}_i = \sum_{j \neq i} \lambda_j \mathbf{x}_j\,.$$

Proof. If the condition (1.3) is obeyed then $x_i - \sum_{j\neq i} \lambda_j \mathbf{x}_j = 0$. This non-trivial linear combination is equal to zero. Hence vectors $\{x_1, \dots, \mathbf{x}_m\}$ are linearly dependent.

Now suppose that vectors $\{\mathbf{x}_1, \dots, \mathbf{x}_m\}$ are linearly dependent. This means that there exist coefficients $\mu_1, \mu_2, \dots, \mu_m$ such that at least one of these coefficients is not equal to zero and the sum (1.3) equals to zero. WLOG suppose that $\mu_1 \neq 0$. We see that to

$$\mathbf{x}_1 = -\frac{\mu_2}{\mu_1}\mathbf{x}_2 - \frac{\mu_3}{\mu_1}\mathbf{x}_3 - \cdots - \frac{\mu_m}{\mu_1}\mathbf{x}_m ,$$

i.e. vector $\mathbf{x}_1$ is expressed as linear combination of vectors $\{\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_m\}$ $\blacksquare$.

1.4 Dimension of vector space. Basis in vector space.

Definition Vector space V has a dimension n if there exist n linearly independent vectors in this vector space, and any $n+1$ vectors in V are linearly dependent.

In the case if in the vector space V for an arbitrary N there exist N linearly independent vectors then the space V is *infinite-dimensional*

Basis

Definition Let V be n-dimensional vector space. The ordered set $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ of n linearly independent vectors in V is called a basis (an ordered basis) of the vector space V.

Proposition 1 *Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be an arbitrary basis in n-dimensional vector space V. Then any vector $\mathbf{a} \in V$ can be expressed as a linear combination of vectors $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ in a unique way, i.e. for every vector $\mathbf{x} \in V$ there exist a set an ordered set of coefficients $\{a^1, \dots, a^n\}$ such that*

$$\mathbf{x} = x^1\mathbf{e}_1 + \cdots + x^n\mathbf{e}_n \tag{1.4}$$

and if

$$\mathbf{x} = a^1\mathbf{e}_1 + \cdots + a^n\mathbf{e}_n = b^1\mathbf{e}_1 + \cdots + b^n\mathbf{e}_n , \tag{1.5}$$

then $a^1 = b^1, a^2 = b^2, \dots, a^n = b^n$. In other words for any vector $\mathbf{x} \in V$ there exists an ordered n-tuple $(x^1, \dots, x^n)$ of coefficients such that $\mathbf{x} = \sum_{i=1}^n x^i\mathbf{e}_i$ and this n-tuple is unique.

Proof Let $\mathbf{x}$ be an arbitrary vector in vector space V. The dimension of vector space V equals to n. Hence $n+1$ vectors $(\mathbf{e}_1, \dots, \mathbf{e}_n, \mathbf{x})$ are linearly dependent: $\lambda_1\mathbf{e}_1 + \cdots + \lambda_n\mathbf{e}_n + \lambda_{n+1}\mathbf{x} = 0$ and this combination is non-trivial. If $\lambda_{n+1} = 0$ then $\lambda_1\mathbf{e}_1 + \cdots + \lambda_n\mathbf{e}_n = 0$ and this combination is non-trivial, i.e. vectors $(\mathbf{e}_1, \dots, \mathbf{e}_n$ are linearly dependent. Contradiction. Hence $\lambda_{n+1} \neq 0$, i.e. vector $\mathbf{x}$ can be expressed via vectors $(\mathbf{e}_1, \dots, \mathbf{e}_n)$: $\mathbf{x} = x^1\mathbf{e}_1 + \dots x^n\mathbf{e}_n$ where $x^i = -\frac{\lambda_i}{\lambda_{n+1}}$. We proved that any vector can be expressed via vectors of basis. Prove now uniqueness.

This expansion is unique. Indeed if (1.5) holds then $(a^1 - b^1)\mathbf{e}_1 + (a^2 - b^2)\mathbf{e}_2 + \dots + (a^n - b^n)\mathbf{e}_n = 0$. Due to linear independence of basis vectors

this means that $(a^1 - b^1) = (a^2 - b^2) = \dots = (a^n - b^n) = 0$, i.e. $a^1 = b^1, a^2 = b^2, \dots, a^n = b^n$ ■

In other words:

Basis is a set of linearly independent vectors in vector space V which span (generate) vector space V.

(Recall that we say that vector space V is *spanned* by vectors $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ (or vectors vectors $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ *span* vector space V) if any vector $\mathbf{a} \in V$ can be expresses as a linear combination of vectors $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$.

Definition Coefficients $\{a^1, \dots, a^n\}$ are called *components of the vector* $\mathbf{x}$ *in the basis* $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ or just shortly *components of the vector* $\mathbf{x}$.

Another very useful and workable statement

Proposition 2 Let $\{\mathbf{e}_1, \dots, \mathbf{e}_m\}$ be an ordered set of vectors in vector space V such that an arbitrary vector $\mathbf{x} \in V$ can be expressed as a linear combination of vectors $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ in a unique way (see (1.4) and (1.5) above). Then

- V is a finite-dimensional space of dimension m.
- $\{\mathbf{e}_1, \dots, \mathbf{e}_m\}$ is a basis in this space.

This is very practical statement: it can be often used to find a dimension of vector space.

Remark We say "a basis" not "the basis", since there are many bases in the vector space V. (See also Homework 1).

Remark Basis is a maximal set of linearly independent vectors in a linear space V.

This leads to definition of a basis in infinite-dimensional space. We have to note that in infinite-dimensional space more useful becomes the conception of *topological basis* when infinite sums are considered.

Canonical basis in $\mathbf{R}^n$

We considered above the basic example of n-dimensional vector space—a space of ordered n-tuples of real numbers: $\mathbf{R}^n = \{(x^1, x^2, \dots, x^n), x^i \in \mathbf{R}\}$

(see the subsection 1.2). What is the meaning of letter n in the definition of $\mathbf{R}^n$?

Consider vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n \in \mathbf{R}^n$:

$$\begin{array}{rl} \mathbf{e}_1 = & (1,0,0\dots,0,0) \\ \mathbf{e}_2 = & (0,1,0\dots,0,0) \\ \dots & \quad\quad\dots \\ \mathbf{e}_n = & (0,0,0\dots,0,1) \end{array} \tag{1.6}$$

Then for an arbitrary vector $\mathbf{R}^n \ni \mathbf{a} = (a^1, a^2, a^3, \dots, a^n)$

$$\mathbf{a} = (a^1, a^2, a^3, \dots, a^n) =$$

$$a^1(1,0,0\dots,0,0) + a^2(0,1,0\dots,0,0) + a^3(0,0,1,0\dots,0,0) + \dots + a^n(0,1,0\dots,0,1) =$$

$$= \sum_{i=1}^{m} a^i \mathbf{e}_i = a^i \mathbf{e}_i \qquad \text{(we will use sometimes condensed notations } \mathbf{x} = x^i \mathbf{e}_i)$$

Thus we see that for every vector $\mathbf{a} \in \mathbf{R}^n$ we have unique expansion via the vectors (1.6).

Remark One can find another basis in $\mathbf{R}^n$–just take an arbitrary ordered set of n linearly independent vectors. (See exercise 7 in Homework 1). The basis (1.6) is distinguished. Sometimes it is called *canonical basis in* $\mathbf{R}^n$.

Remark One can consider a set of ordered n-tuples in $\mathbf{R}^n$ as the set of points. Two points $\mathbf{a}, \mathbf{b} \in \mathbf{R}^n$ define a vector: if $\mathbf{a} = (a^1, \dots, a^n)$, $\mathbf{b} = (b^1, \dots, b^n)$, then the vector $\mathbf{ab}$ attached to the point $\mathbf{a}$ has coordinates $= (b^1 - a^1, b^2 - a^2 \dots, b^n - a^n)$ [1].

1.5 Scalar product. Euclidean space

In vector space one have additional structure: *scalar product of vectors.*

Definition Scalar product in a vector space V is a function $B(\mathbf{x}, \mathbf{y})$ on a pair of vectors which takes real values and satisfies the the following conditions:

$$\begin{array}{c} B(\mathbf{x}, \mathbf{y}) = B(\mathbf{y}, \mathbf{x}) \quad \text{(symmetricity condition)} \\ B(\lambda \mathbf{x} + \mu \mathbf{x}', \mathbf{y}) = \lambda B(\mathbf{x}, \mathbf{y}) + \mu B(\mathbf{x}', \mathbf{y}) \quad \text{(linearity condition)} \\ B(\mathbf{x}, \mathbf{x}) \geq 0 \ , B(\mathbf{x}, \mathbf{x}) = 0 \Leftrightarrow \mathbf{x} = 0 \ \text{(positive-definiteness condition)} \end{array} \tag{1.7}$$

[1] $\mathbf{R}^n$ considered as a set of points is called affine space

Definition Euclidean space is a vector space equipped with a scalar product.

One can easy to see that the function $B(\mathbf{x}, \mathbf{y})$ is bilinear function, i.e. it is linear function with respect to the second argument also[2]. This follows from previous axioms:

$$B(\mathbf{x}, \lambda\mathbf{y}+\mu\mathbf{y}') \underbrace{=}_{\text{symm.}} B(\lambda\mathbf{y}+\mu\mathbf{y}', \mathbf{x}) \underbrace{=}_{\text{linear.}} \lambda B(\mathbf{y}, \mathbf{x})+\mu B(\mathbf{y}', \mathbf{x}) \underbrace{=}_{\text{symm.}} \lambda B(\mathbf{x}, \mathbf{y})+\mu B(\mathbf{x}, \mathbf{y}') .$$

A bilinear function $B(\mathbf{x}, \mathbf{y})$ on pair of vectors is called sometimes *bilinear form* on vector space. Bilinear form $B(\mathbf{x}, \mathbf{y})$ which satisfies the symmetricity condition is called *symmetric bilinear form.* Scalar product is nothing but symmetric bilinear form on vectors which is positive-definite: $B(\mathbf{x}, \mathbf{x}) \geq 0$) and is non-degenerate $((\mathbf{x}, \mathbf{x}) = 0 \Rightarrow \mathbf{x} = 0$.

Example We considered the vector space $\mathbf{R}^n$, the space of n-tuples (see the subsection 1.2). One can consider the vector space $\mathbf{R}^n$ as Euclidean space provided by the scalar product

$$B(\mathbf{x}, \mathbf{y}) = x^1 y^1 + \cdots + x^n y^n \tag{1.8}$$

This scalar product sometimes is called *canonical scalar product.*

Exercise a) Check that it is indeed scalar product.

Example We consider in 2-dimensional vector space V with basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ $B(\mathbf{X}, \mathbf{Y})$ such that $B(\mathbf{e}_1, \mathbf{e}_1) = 3$, $B(\mathbf{e}_2, \mathbf{e}_2) = 5$ and $B(\mathbf{e}_1, \mathbf{e}_2) = 0$. Then for every two vectors $\mathbf{X} = x^1\mathbf{e}_1 + x^2\mathbf{e}_2$ and $\mathbf{Y} = y^1\mathbf{e}_1 + y^2\mathbf{e}_2$ we have that

$$B(\mathbf{X}, \mathbf{Y}) = (\mathbf{X}, \mathbf{Y}) = \left(x^1\mathbf{e}_1 + x^2\mathbf{e}_2, y^1\mathbf{e}_1 + y^2\mathbf{e}_2\right) =$$

$$x^1y^1(\mathbf{e}_1, \mathbf{e}_1) + x^1y^2(\mathbf{e}_1, \mathbf{e}_2) + x^2y^1(\mathbf{e}_2, \mathbf{e}_1) + x^2y^2(\mathbf{e}_2, \mathbf{e}_2) = 3x^1y^1 + 5x^2y^2 .$$

Notations!

Scalar product sometimes is called "inner" product or "dot" product. Later on we will use for scalar product $B(\mathbf{x}, \mathbf{y})$ just shorter notation $(\mathbf{x}, \mathbf{y})$ (or $\langle \mathbf{x}, \mathbf{y} \rangle$). Sometimes it is used for scalar product a notation $\mathbf{x} \cdot \mathbf{y}$. Usually this notation is reserved only for the canonical case (1.8).

Another **Counterexample** Show that operation such that $(\mathbf{e}_1, \mathbf{e}_1) = (\mathbf{e}_2, \mathbf{e}_2) = 0$ and $(\mathbf{e}_1, \mathbf{e}_2) = 1$ does not define scalar product. *Solution.* For every two vectors $\mathbf{X} = x^1\mathbf{e}_1 + x^2\mathbf{e}_2$ and $\mathbf{Y} = y^1\mathbf{e}_1 + y^2\mathbf{e}_2$ we have that

$$(\mathbf{X}, \mathbf{Y}) = \left(x^1\mathbf{e}_1 + x^2\mathbf{e}_2, y^1\mathbf{e}_1 + y^2\mathbf{e}_2\right) = x^1y^2 + x^2y^1$$

[2]Here and later we will denote scalar product $B(\mathbf{x}, \mathbf{y})$ just by $(\mathbf{x}, \mathbf{y})$. Scalar product sometimes is called inner product. Sometimes it is called dot product.

hence for vector $\mathbf{X} = (1, -1)$ $(\mathbf{X}, \mathbf{X}) = -2 < 0$. Positive-definiteness is not fulfilled.

Counterexample Show that operation $(\mathbf{X}, \mathbf{Y}) = x^1y^1 - x^2y^2$ does not define scalar product. *Solution.* Take $\mathbf{X} = (0, -1)$. Then $(\mathbf{X}, \mathbf{X}) = -1$. The condition of positive-definiteness is not fulfilled. .

(See also exercises in Homework 2)

1.6 Orthonormal basis in Euclidean space

One can see that for scalar product (1.8) and for the basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ defined by the relation (1.6) the following relations hold:

$$(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \tag{1.9}$$

Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be an ordered set of n vectors in n-dimensional Euclidean space which obeys the conditions (1.9). One can see that this ordered set is a basis [3].

Definition-Proposition The ordered set of vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ in n-dimensional Euclidean space which obey the conditions (1.9) is a basis. This basis is called *an orthonormal basis.*

One can prove that every (finite-dimensional) Euclidean space possesses orthonormal basis.

Later by default we consider only orthonormal bases in Euclidean spaces. Respectively scalar product will be defined by the formula (1.8). Indeed let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be an orthonormal basis in Euclidean space. Then for an arbitrary two vectors $\mathbf{x}, \mathbf{y}$, such that $\mathbf{x} = \sum x^i \mathbf{e}_i$, $\mathbf{y} = \sum y^j \mathbf{e}_j$ we have:

$$(\mathbf{x}, \mathbf{y}) = \left(\sum x^i \mathbf{e}_i, \sum y^j \mathbf{e}_j\right) = \sum_{i,j=1}^{n} x^i y^j (\mathbf{e}_i, \mathbf{e}_j) = \sum_{i,j=1}^{n} x^i y^j \delta_{ij} = \sum_{i=1}^{n} x^i y^i$$

We come to the canonical scalar product (1.8). Later on we usually will consider scalar product defined by the formula (1.8) ((1.6)), i.e. scalar product in orthonormal basis.

[3] Indeed prove that conditions (1.9) imply that these n vectors are linear independent. Suppose that $\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 + \cdots + \lambda_n \mathbf{e}_n = 0$. For an arbitrary i multiply the left and right hand sides of this relation on a vector $\mathbf{e}_i$. We come to condition $\lambda_i = 0$. Hence vectors $(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$ are linearly dependent.

Geometrical properties of scalar product: length of the vectors, angle between vectors

The scalar product of vector on itself defines the *length of the vector*:

$$\text{Length of the vector } \mathbf{x} = |\mathbf{x}| = \sqrt{(\mathbf{x},\mathbf{x})} = \sqrt{(x^1)^2 + \cdots + (x^n)^2} \qquad (1.10)$$

If we consider Euclidean space $\mathbf{E^n}$ as the set of points then the distance between two points $\mathbf{x}, \mathbf{y}$ is the length of corresponding vector:

$$\text{distance between points } \mathbf{x},\mathbf{y} = |\mathbf{x}-\mathbf{y}| = \sqrt{\left(y^1 - x^1\right)^2 + \cdots + \left(y^n - x^n\right)^2}$$

We recall very important formula how scalar (inner) product is related with the angle between vectors:

$$(\mathbf{x},\mathbf{y}) = x^1y^1 + x^2y^2 = |\mathbf{x}||\mathbf{y}|\cos\varphi$$

where φ is an angle between vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbf{E}^2$.

This formula is valid also in the three-dimensional case and any n-dimensional case for $n \geq 1$. It gives as a tool to calculate angle between two vectors:

$$(\mathbf{x},\mathbf{y}) = x^1y^1 + x^2y^2 + \cdots + x^ny^n = |\mathbf{x}||\mathbf{y}|\cos\varphi \qquad (1.11)$$

In particulary it follows from this formula that

angle between vectors $\mathbf{x},\mathbf{y}$ *is acute if scalar product* $(\mathbf{x},\mathbf{y})$ *is positive*
angle between vectors $\mathbf{x},\mathbf{y}$ *is obtuse if scalar product* $(\mathbf{x},\mathbf{y})$ *is negative*
vectors $\mathbf{x},\mathbf{y}$ *are perpendicular if scalar product* $(\mathbf{x},\mathbf{y})$ *is equal to zero*
(1.12)

Remark Geometrical intuition tells us that cosinus of the angle between two vectors has to be less or equal to one and it is equal to one if and only if vectors $\mathbf{x},\mathbf{y}$ are collinear. Comparing with (1.11) we come to the inequality:

$$\begin{gathered}(\mathbf{x},\mathbf{y})^2 = \left(x^1y^1 + \cdots + x^ny^n\right)^2 \leq \left((x^1)^2 + \cdots + (x^n)^2\right)\left((y^1)^2 + (\cdots + (y^n)^2\right) = (\mathbf{x},\mathbf{x})(\mathbf{y},\mathbf{y}) \\ \text{and}(\mathbf{x},\mathbf{y})^2 = (\mathbf{x},\mathbf{x})(\mathbf{y},\mathbf{y}) \quad \text{if vectors are colienar, i.e. } x^i = \lambda y^i\end{gathered} \qquad (1.13)$$

This is famous Cauchy–Buniakovsky–Schwarz inequality, one of most important inequalities in mathematics. (See for more details Homework 2)

1.7 Transition matrices. Orthogonal bases and orthogonal matrices

One can consider different bases in vector space.

Let A be $n \times n$ matrix with real entries, $A = ||a_{ij}||$, $i, j = 1, 2, \ldots, n$:

$$A = \begin{pmatrix} a_{11} & a_{12}\ldots & a_{1n} \\ a_{21} & a_{22}\ldots & a_{2n} \\ a_{31} & a_{32}\ldots & a_{3n} \\ \ldots & \ldots\ldots & \ldots \\ a_{(n-1)\,1} & a_{(n-1)2}\ldots & a_{(n-1)n} \\ a_{n\,1} & a_{n2}\ldots & a_{nn} \end{pmatrix}$$

Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be an arbitrary basis in n-dimensional vector space V.

The basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ can be considered as row of vectors, or $1 \times n$ matrix with entries–vectors.

Multiplying $1 \times n$ matrix $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ on matrix A we come to new row of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\}$ such that

$$\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}A = \tag{1.14}$$

$$\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\} \begin{pmatrix} a_{11} & a_{12}\ldots & a_{1n} \\ a_{21} & a_{22}\ldots & a_{2n} \\ a_{31} & a_{32}\ldots & a_{3n} \\ \ldots & \ldots\ldots & \ldots \\ a_{(n-1)\,1} & a_{(n-1)2}\ldots & a_{(n-1)n} \\ a_{n\,1} & a_{n2}\ldots & a_{nn} \end{pmatrix} \tag{1.15}$$

,

$$\begin{cases} \mathbf{e}'_1 = a_{11}\mathbf{e}_1 + a_{21}\mathbf{e}_2 + a_{31}\mathbf{e}_3 + \cdots + a_{(n-1)\,1}\mathbf{e}_{n-1} + a_{n\,1}\mathbf{e}_n \\ \mathbf{e}'_1 = a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2 + a_{32}\mathbf{e}_3 + \cdots + a_{(n-1)\,2}\mathbf{e}_{n-1} + a_{n\,2}\mathbf{e}_n \\ \mathbf{e}'_1 = a_{13}\mathbf{e}_1 + a_{23}\mathbf{e}_2 + a_{33}\mathbf{e}_3 + \cdots + a_{(n-1)\,3}\mathbf{e}_{n-1} + a_{n\,1}\mathbf{e}_n \\ \cdots = \ldots \cdots + \ldots \cdots + \ldots \cdots + \cdots + \ldots\ldots\ldots\ldots \\ \mathbf{e}'_n = a_{1n}\mathbf{e}_1 + a_{2n}\mathbf{e}_2 + a_{3n}\mathbf{e}_3 + \cdots + a_{(n-1)\,n}\mathbf{e}_{n-1} + a_{n\,n}\mathbf{e}_n \end{cases}$$

or shortly:

$$\mathbf{e}'_i = \sum_{k=1}^{n} \mathbf{e}_k a_{ki}\,. \tag{1.16}$$

What is the condition that the row $\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\}$ is a basis too? The row $\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\}$ is a basis if and only if vectors $(\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n)$ are linearly independent. Thus we come to

Proposition 1 *Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be a basis in n-dimensional vector space V, and let A be an $n \times n$ matrix with real entries. Then*

$$\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}A \tag{1.17}$$

is a basis if and only if the matrix A has rank n, i.e. it is non-degenerate (invertible) matrix.

Recall that $n\times$ matrix A is nondegenerate (invertible) $\Leftrightarrow \det A \neq 0$.

Definition Non-degenerate matrix A which transforms a basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ to another basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n\}$ (see equations (1.16)– (1.17)) is called *transition matrix* from the basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ to the basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n\}$.

Remark Recall that the condition that $n\times n$ matrix A is non-degenerate (has rank n) is equivalent to the condition that it is invertible matrix, or to the condition that $\det A \neq 0$.

Now suppose that $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is orthonoromal basis in n-dimensional Euclidean vector space. What is the condition that the new basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n\} = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}A$ is an orthonormal basis too?

Definition We say that $n\times n$ matrix is orthogonal matrix if its product on transposed matrix is equal to unity matrix:

$$A^T A = I\,. \tag{1.18}$$

Exercise. Prove that determinant of orthogonal matrix is equal to ± 1:

$$A^T A = I \Rightarrow \det A = \pm 1\,. \tag{1.19}$$

Solution $A^T A = I$. Hence $\det(A^T A) = \det A^T \det A = (\det A)^2 = \det I = 1$. Hence $\det A = \pm 1$. We see that in particular orthogonal matrix is non-degenerate ($\det A \neq 0$). Hence it is a transition matrix from one basis to another. The following Proposition is valid:

Proposition 2 Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ be an orthonormal basis in n-dimensional Euclidean vector space. Then the new basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n\} = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}A$ is orthonormal basis if and only if the transition matrix A is orthogonal matrix.

Proof The basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_n\}$ is orthonormal means that $(\mathbf{e}'_i, \mathbf{e}'_j) = \delta_{ij}$. We have:

$$\delta_{ij} = (\mathbf{e}'_i, \mathbf{e}'_j) = \left(\sum_{m=1}^n \mathbf{e}_m A_{mi}, \mathbf{e}'_j = \sum_{n=1}^n \mathbf{e}_n A_{nj}\right) = \sum_{m,n=1}^n A_{mi} A_{nj} (\mathbf{e}_m, \mathbf{e}_n) =$$

$$\sum_{m,n=1}^n A_{mi} A_{nj} \delta_{mn} = \sum_{m=1}^n A_{mi} A_{mj} = \sum_{m=1}^n A^T_{im} A_{mj} = (A^t A)_{ij}\,, \tag{1.20}$$

Hence $(A^TA)_{ij} = \delta_{ij}$, i.e. $A^TA = I$.

We know that determinant of orthogonal matrix equals to ± 1. It is very useful to consider the following groups:

- The group $O(n)$—group of orthogonal $n \times n$ matrices:

$$O(n) = \{A: \quad A^TA = I\}\,. \tag{1.21}$$

- The group $SO(n)$ special orthogonal group of $n \times n$ matrices:

$$SO(n) = \{A: \quad A^TA = I, \det A = 1\}\,. \tag{1.22}$$

1.8 Orthogonal 2×2 matrices

One can "rotate" orthonormal basis in $\mathbf{E}^2$ or consider its "reflection". What is mathematical meaning of it?

We find now orthogonal 2×2 matrices and explain that an arbitrary transition matrix from orthonormal basis to an arbitrary orthonormal basis in $\mathbf{E}^2$, is a "rotation" or "reflection" [4].

Consider 2-dimensional Euclidean space $\mathbf{E}^2$ with orthonormal basis $\{\mathbf{e}, \mathbf{f}\}$: $(\mathbf{e}, \mathbf{e}) = (\mathbf{f}, \mathbf{f}) = 1$ (i.e. $|\mathbf{e}| = |\mathbf{f}| = 1$) and $(\mathbf{e}, \mathbf{f}) = 0$—vectors $\mathbf{e}, \mathbf{f}$ have unit length and are orthogonal to each other. Let $\{\mathbf{e}', \mathbf{f}'\}$ be a new basis:

$$\{\mathbf{e}', \mathbf{f}'\} = \{\mathbf{e}, \mathbf{f}\}T = \{\mathbf{e}, \mathbf{f}\}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}, \text{ i.e. } \mathbf{e}' = \alpha\mathbf{e} + \gamma\mathbf{f},\ \mathbf{f}' = \beta\mathbf{e} + \delta\mathbf{f}$$

new basis is orthonormal basis also,

$$(\mathbf{e}', \mathbf{e}') = (\mathbf{f}', \mathbf{f}') = 1\,, \quad (\mathbf{e}', \mathbf{f}') = 0\,,$$

i.e. transition matrix is $A = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ is orthogonal matrix: according (1.18)

$$A^TA = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}^t \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}\alpha & \gamma\\ \beta & \delta\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}\alpha^2+\gamma^2 & \alpha\beta+\gamma\delta\\ \alpha\beta+\gamma\delta & \beta^2+\delta^2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

We have $\alpha^2+\gamma^2 = 1$, $\alpha\beta+\gamma\delta = 0$ and $\beta^2+\delta^2 = 1$. Hence one can choose angles $\varphi, \psi\colon 0 \leq 2\pi$ such that $\alpha = \cos\varphi,\ \gamma = \sin\varphi,\ \ \beta = \sin\psi,\ \delta = \cos\psi$. The condition $\alpha\beta + \gamma\delta =$ means that

$$\cos\varphi\sin\psi + \sin\varphi\cos\psi = \sin(\varphi+\psi) = 0$$

[4] More detailed explanation will be performed later in subsection...

Hence $\sin\varphi = -\sin\psi$, $\cos\varphi = \cos\psi$ $(\varphi+\psi=0)$ or $\sin\varphi = \sin\psi$, $\cos\varphi = -\cos\psi$ $(\varphi+\psi=\pi)$

The first case: $\sin\varphi = -\sin\psi$, $\cos\varphi = \cos\psi$,

$$A_\varphi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \quad (\det A_\varphi = 1) \tag{1.23}$$

The second case: $\sin\varphi = \sin\psi$, $\cos\varphi = -\cos\psi$,

$$\tilde{A}_\varphi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix} \quad (\det \tilde{A}_\varphi = -1) \tag{1.24}$$

Consider the first case, when a matrix A_φ is defined by the relation (1.23). In this case the new basis is:

$$(\mathbf{e}', \mathbf{f}') = (\mathbf{e}, \mathbf{f})A_\varphi = (\mathbf{e}, \mathbf{f}) \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = \begin{pmatrix} \cos\varphi\,\mathbf{e} + \sin\varphi\,\mathbf{f} \\ -\sin\varphi\,\mathbf{e} + \cos\varphi\,\mathbf{f} \end{pmatrix} \tag{1.25}$$

One can see that that new basis $\{\mathbf{e}', \mathbf{f}'\}$ is orthonormal basis too and transition matrix T_φ "rotates" the basis $(\mathbf{e}, \mathbf{f})$ on the angle φ (see Homework 1).

We call the matrix A_φ **rotation matrix**

Now consider the second case, when a matrix $\tilde{A}_\varphi$ is defined by the relation (1.24). One can see that

$$\tilde{A}_\varphi = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = A_\varphi R \tag{1.26}$$

where we denote by $R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ a transition matrix from the basis $\{\mathbf{e}, \mathbf{f}\}$ to the basis $\{\mathbf{e}, -\mathbf{f}\}$—"reflection"l.

We see that in the second case the orthogonal matrix is composition of rotation and reflection matrix: $\{\mathbf{e}, \mathbf{f}\} \xrightarrow{\tilde{A}_\varphi = A_\varphi R} \{\tilde{\mathbf{e}}, \tilde{\mathbf{f}}\}$:

$$\{\mathbf{e}, \mathbf{f}\} \xrightarrow{A_\varphi} \{\mathbf{e}' = \cos\varphi\,\mathbf{e} + \sin\varphi\mathbf{l}, \mathbf{f}, \mathbf{f}' = -\sin\varphi\,\mathbf{e} + \cos\varphi\,\mathbf{f}\} \xrightarrow{R} \{\tilde{\mathbf{e}} = \mathbf{e}', \tilde{f} = -\mathbf{f}\} \tag{1.27}$$

We come to proposition

Proposition. *Let A be an arbitrary 2×2 orthogonal matrix, i.e. $A^T A = 1$ and in particularly* $\det A = \pm 1$. *(Transition matrix transforms an orthonormal basis to an orthonormal one.)*

If $\det A = 1$ *then there exists an angle* $\varphi \in [0, 2\pi)$ *such that* $A = A_\varphi$ *is a transition matrix* (1.23) *which rotates the basis vectors on the angle* φ.

If $\det A = -1$ *then there exists an angle* $\varphi \in [0, 2\pi)$ *such that* $A = \tilde{A}_\varphi$ *is a transition matrix is a composition of rotation and reflection (see* (1.27)*).*

Remark One can show that in this case the transition matrix $\tilde{A}_\varphi$ is a reflection matrix with respect to the axis which have the angle $\frac{\varphi}{2}$ with x-axis.

Consider just examples:

$$a)\varphi = 0,\ \tilde{A}_\varphi = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{e} \\ -\mathbf{f} \end{pmatrix}$$

(reflection with respect to x-axis)

$$b)\varphi = \pi,\ \tilde{A}_\varphi = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \end{pmatrix} \mapsto \begin{pmatrix} -\mathbf{e} \\ \mathbf{f} \end{pmatrix}$$

(reflection with respect to y-axis)

$$b)\varphi = \frac{\pi}{2},\ \tilde{A}_\varphi = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{f} \\ \mathbf{e} \end{pmatrix}$$

(reflection with respect to axis $y = x$ ("swapping" of basis vectors))

1.9 Orientation in vector space

You heard words "orientation...", ""

You heard expressions like: A basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ have the same orientation as the basis $\{\mathbf{a}', \mathbf{b}', \mathbf{c}'\}$ if they both obey right hand rule or if they both obey left hand rule. In the other case we say that these bases have opposite orientation...

Try to give the exact meaning to these words.

Note that in three-dimensional Euclidean space except scalar (inner) product, one can consider another important operation: vector product. The conception of orientation is indispensable for defining this operation.

Consider the set of *all* bases in the given vector space V.

Let $(\mathbf{e}_1, \dots \mathbf{e}_n)$, $(\mathbf{e}'_1, \dots \mathbf{e}'_n)$ be two arbitrary bases in the vector space V and let T be transition matrix which transforms the basis $\{\mathbf{e}_i\}$ to the new basis $\{\mathbf{e}'_i\}$:

$$\{\mathbf{e}'_1, \dots \mathbf{e}'_n\} = \{\mathbf{e}_1, \dots \mathbf{e}_n\}T\,, \qquad (\mathbf{e}'_i = \sum_{k=1}^{n} \mathbf{e}_k t_{ki}) \tag{1.28}$$

(see also (1.15)).

Definition We say that two bases $\{\mathbf{e}_1, \dots \mathbf{e}_n\}$ and $\{\mathbf{e}'_1, \dots \mathbf{e}'_n\}$ in V have *the same orientation* if the determinant of transition matrix (1.28) from the first basis to the second one is positive: $\det T > 0$.

We say that the basis $\{\mathbf{e}_1, \dots \mathbf{e}_n\}$ has an orientation *opposite to the orientation* of the basis $\{\mathbf{e}'_1, \dots \mathbf{e}'_n\}$ (or in other words these two bases have opposite orientation) if the determinant of transition matrix from the first basis to the second one is negative: $\det T < 0$.

Remark Transition matrix from basis to basis is non-degenerate, hence its determinant cannot be equal to zero. It can be or positive or negative.

One can see that orientation establishes the equivalence relation in the set of all bases. Denote this relation by "$\sim$": $\{\mathbf{e}_1, \dots \mathbf{e}_n\} \sim \{\mathbf{e}'_1, \dots \mathbf{e}'_n\}$, if two bases $\{\mathbf{e}_1, \dots \mathbf{e}_n\}$ and $\{\mathbf{e}'_1, \dots \mathbf{e}'_n\}$ have the same orientation, i.e. $\det T > 0$ for transition matrix.

Show that "$\sim$" is an equivalence relation, i.e. this relation is reflexive, symmetric and transitive.

Check it:

- it is reflexive, i.e. for every basis $\{\mathbf{e}_1, \dots \mathbf{e}_n\}$

  $$\{\mathbf{e}_1, \dots, \mathbf{e}_n\} \sim \{\mathbf{e}_1, \dots, \mathbf{e}_n\}\,, \tag{1.29}$$

  because in this case transition matrix $T = I$ and $detI = 1 > 0$.

- it is symmetric, i.e.

  If $\{\mathbf{e}_1, \dots, \mathbf{e}_n\} \sim \{\mathbf{e}'_1, \dots, \mathbf{e}'_n\}$ then $\{\mathbf{e}'_1, \dots, \mathbf{e}'_n) \sim \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$,

  because if T is transition matrix from the first basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ to the second basis $\{\mathbf{e}'_1, \dots, \mathbf{e}'_n\}$: $\{\mathbf{e}'_1, \dots, \mathbf{e}'_n\} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}T$,

  then the transition matrix from the second basis $\{\mathbf{e}'_1, \dots, \mathbf{e}'_n\}$ to the first basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is the inverse matrix T^{-1}: $\{\mathbf{e}_1, \dots, \mathbf{e}_n\} = \{\mathbf{e}'_1, \dots, \mathbf{e}'_n\}T^{-1}$. Hence $\det T^{-1} = \frac{1}{\det T} > 0$ if $\det T > 0$.

- Is transitive, i.e. if $\{\mathbf{e}_1,\dots,\mathbf{e}_n\} \sim \{\mathbf{e}'_1,\dots,\mathbf{e}'_n\}$ and $\{\mathbf{e}'_1,\dots,\mathbf{e}'_n) \sim \{\tilde{\mathbf{e}}_1,\dots,\tilde{\mathbf{e}}_n\}$, then $\{\mathbf{e}_1,\dots,\mathbf{e}_n\} \sim \{\tilde{\mathbf{e}}_1,\dots,\tilde{\mathbf{e}}_n\}$, because if T_1 is transition matrix from the first basis $\{\mathbf{e}_1,\dots\mathbf{e}_n\}$ to the second basis $\{\mathbf{e}'_1,\dots,\mathbf{e}'_n\}$ and T_2 is transition matrix from the second basis $\{\mathbf{e}'_1,\dots,\mathbf{e}'_n\}$ to the third basis $\{\tilde{\mathbf{e}}_1,\dots,\tilde{\mathbf{e}}_n\}$ then the transition matrix T from the first basis $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ to the third basis $\{\tilde{\mathbf{e}}_1,\dots,\tilde{\mathbf{e}}_n\}$ is equal to $T = T_1T_2$. We have that $\det T = \det(T_2T_1) = \det T_2 \cdot \det T_1 > 0$ because $\det T_1 > 0$ and $\det T_2 > 0$.

Since it is equivalence relation the set of all bases is a union if disjoint equivalence classes. Two bases are in the same equivalence class if and only if they have the same orientation.

One can see that there are exactly two equivalence classes.

Proposition *Let two bases $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ and $\{\mathbf{e}'_1,\dots,\mathbf{e}'_n\}$ in vector space V have opposite orientation. Let $\{\tilde{\mathbf{e}}_1,\dots,\tilde{\mathbf{e}}_n\}$ be an arbitrary basis in V. Then the basis $\{\tilde{\mathbf{e}}_1,\dots,\tilde{\mathbf{e}}_n\}$ and the first basis $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ have the same orientation or the basis $\{\tilde{\mathbf{e}}_1,\dots,\tilde{\mathbf{e}}_n\}$ and the second basis $\{\mathbf{e}'_1,\dots,\mathbf{e}'_n\}$ have the same orientation. In other words if $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$, $\{\mathbf{e}'_1,\dots,\mathbf{e}'_n\}$ and $\{\tilde{\mathbf{e}}_1,\dots,\tilde{\mathbf{e}}_n\}$ are three bases in vector space V such that $\{\mathbf{e}_1,\dots,\mathbf{e}_n\} \not\sim \{\mathbf{e}'_1,\dots,\mathbf{e}'_n\}$ then*

$$\{\tilde{\mathbf{e}}_1,\dots,\tilde{\mathbf{e}}_n\} \sim \{\mathbf{e}_1,\dots,\mathbf{e}_n\} \text{ or } \{\tilde{\mathbf{e}}_1,\dots,\tilde{\mathbf{e}}_n\} \sim \{\mathbf{e}'_1,\dots,\mathbf{e}'_n\}\,. \tag{1.30}$$

There are two equivalence classes of bases with respect to orientation. An arbitrary basis belongs to the equivalence class of the basis $\{\mathbf{e}_1,\mathbf{e}_2\dots,\mathbf{e}_n\}$, or it belongs to the to the equivalence class of the basis $\{\mathbf{e}'_1,\mathbf{e}_2\dots,\mathbf{e}'_n\}$ (in the case if bases $\{\tilde{\mathbf{e}}'_1,\dots,\tilde{\mathbf{e}}'_n\}$, $\{\tilde{\mathbf{e}}_1,\dots,\tilde{\mathbf{e}}_n\}$ have opposite orientation).

Prove this statement. For convenience call the basis $\{\mathbf{e}_1,\mathbf{e}_2\dots,\mathbf{e}_n\}$ the "I-st basis", call the basis $\{\mathbf{e}'_1,\mathbf{e}'_2\dots,\mathbf{e}'_n\}$ the "II-nd basis" and call the basis $\{\tilde{\mathbf{e}}_1,\tilde{\mathbf{e}}_2\dots,\tilde{\mathbf{e}}_n\}$ a "III-rd basis".

Let T_0 be transition matrix from the I-st basis to the II-nd basis, let T_1 be transition matrix from the I-st basis to the III-rd basis, and let T_2 be transition matrix from the II-nd basis to the III-rd basis:

$$\{\mathbf{e}'_1,\dots,\mathbf{e}'_n\} = \{\mathbf{e}_1,\dots,\mathbf{e}_n\}T_0\,,$$

$$\{\tilde{\mathbf{e}}_1,\dots,\tilde{\mathbf{e}}_n\} = \{\mathbf{e}_1,\dots,\mathbf{e}_n\}T_1, \text{ and } \{\tilde{\mathbf{e}}_1,\dots,\tilde{\mathbf{e}}_n\} = \{\mathbf{e}'_1,\dots,\mathbf{e}'_n\}T_2, \text{ i.e.}$$

$$\{\tilde{\mathbf{e}}_1,\dots,\tilde{\mathbf{e}}_n\} = \{\mathbf{e}'_1,\dots,\mathbf{e}'_n\}T_2 = (\{\mathbf{e}_1,\dots,\mathbf{e}_n\}T_0)\,T_2 = \{\mathbf{e}_1,\dots,\mathbf{e}_n\}T_0T_2 = \{\mathbf{e}_1,\dots,\mathbf{e}_n\}T_1.$$

Hence

$$T_2 = T_0 \cdot T_1 \Rightarrow \det T_2 = \det(T_0T_1) = \det T_0 \cdot \det T_1\,.$$

On the other hand $\det T_0 < 0$ since bases $\{\mathbf{e}'_1, \dots, \mathbf{e}'_n\}$ and $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ have opposite orientation. Hence determinants of matrices T_2 and T_1 have *opposite signs*. There are just two choices. Or determinant of matrix T_2 is positive, i.e. II-nd and III-rd bases have the same orientation, or determinant of matrix T_1 is positive, i.e. I-st and III-rd bases have the same orientation. ■

Example Let $\{\mathbf{e}_1, \mathbf{e}_2 \dots, \mathbf{e}_n\}$ be an arbitrary basis in n-dimensional vector space V. Swap the vectors $\mathbf{e}_1, \mathbf{e}_2$. We come to a new basis: $\{\mathbf{e}'_1, \mathbf{e}'_2 \dots, \mathbf{e}'_n\}$

$$\mathbf{e}'_1 = \mathbf{e}_2, \mathbf{e}'_2 = \mathbf{e}_1, \text{ all other vectors are the same: } \mathbf{e}_3 = \mathbf{e}'_3, \dots, \mathbf{e}_n = \mathbf{e}'_n \tag{1.31}$$

We have:

$$\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3 \dots, \mathbf{e}'_n\} = \{\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_3, \dots, \mathbf{e}_n\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \dots, \mathbf{e}_n\} T_{\text{swap}}\,, \tag{1.32}$$

where one can easy see that the determinant for transition matrix T_{swap} is equal to -1, i.e. bases $\{\mathbf{e}_1, \mathbf{e}_2 \dots, \mathbf{e}_n\}$ and $\{\mathbf{e}_2, \mathbf{e}_1 \dots, \mathbf{e}_n\}$ have opposite orientation.

E.g. write down the transition matrix (1.32) in the case if dimension of vector space is equal to 5, $n = 5$. Then we have $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3, \mathbf{e}'_4, \mathbf{e}'_5\} = \{\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\} T$ where

$$T_{\text{swap}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (\det T_{\text{swap}} = -1)\,. \tag{1.33}$$

We see that bases $\{\mathbf{e}_1, \mathbf{e}_2 \dots, \mathbf{e}_n\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2 \dots, \mathbf{e}'_n\}$ have opposite orientation.

Hence according to Proposition above an arbitrary basis $\{\mathbf{e}'_1, \dots \mathbf{e}'_n\}$ have the same orientation as the basis $\{\mathbf{e}_1, \mathbf{e}_2 \dots, \mathbf{e}_n\}$, i.e. belongs to the equivalence class of basis $\{\mathbf{e}_1, \mathbf{e}_2 \dots, \mathbf{e}_n\}$, or it has the same orientation as the "swapped" basis $\{\mathbf{e}_2, \mathbf{e}_1 \dots, \mathbf{e}_n\}$, i.e. it belongs to the equivalence class of the "swappedd" basis $\{\mathbf{e}_2, \mathbf{e}_1 \dots, \mathbf{e}_n\}$.

The set of all bases is a union of two disjoint subsets.

Any two bases which belong to the same subset have the same orientation. Any two bases which belong to different subsets have opposite orientation.

Definition *An orientation of a vector space is an equivalence class of bases in this vector space.*

Note that fixing any basis we fix orientation, considering the subset of all bases which have the same orientation that the given basis.

There are two orientations. Every basis has the same orientation as a given basis or orientation opposite to the orientation of the given basis.

If we choose an arbitrary basis then all bases which belong to the equivalence class of this basis may be called "left" bases and all the bases which do not belong to the equivalence class of this basis may be called "right" bases

Definition *An oriented vector space is a vector space equipped with orientation.*

Consider examples.

Example (Orientation in two-dimensional space). Let $\{\mathbf{e}_x, \mathbf{e}_y\}$ be arbitrary two bases in $\mathbf{R}^2$ and let $\mathbf{a}, \mathbf{b}$ be arbitrary two vectors in $\mathbf{R}^2$. Consider an ordered pair $\{\mathbf{a}, \mathbf{b}, \}$. The transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y\}$ to the ordered pair $\{\mathbf{a}, \mathbf{b}\}$ is $T = \begin{pmatrix} a_x & b_x \\ a_y & b_y \end{pmatrix}$:

$$\{\mathbf{a}, \mathbf{b}\} = \{\mathbf{e}_x, \mathbf{e}_y\}T = \{\mathbf{e}_x, \mathbf{e}_y\} \begin{pmatrix} a_x & b_x \\ a_y & b_y \end{pmatrix}, \quad \begin{cases} \mathbf{a} = a_x \mathbf{e}_x + a_y \mathbf{e}_y \\ \mathbf{b} = b_x \mathbf{e}_x + b_y \mathbf{e}_y \end{cases}$$

One can see that the ordered pair $\{\mathbf{a}, \mathbf{b}\}$ also is a basis, (i.e. these two vectors are linearly independent in $\mathbf{R}^2$) if and only if transition matrix is not degenerate, i.e. $\det T \neq 0$. The basis $\{\mathbf{a}, \mathbf{b}\}$ has the same orientation as the basis $\{\mathbf{e}_x, \mathbf{e}_y\}$ if $\det T > 0$ and the basis $\{\mathbf{a}, \mathbf{b}\}$ has the orientation opposite to the orientation of the basis $\{\mathbf{e}_x, \mathbf{e}_y\}$ if $\det T < 0$.

Example Let $\{\mathbf{e}, \mathbf{f}\}$ be a basis in 2-dimensional vector space. Consider bases $\{\mathbf{e}, -\mathbf{f}\}$, $\{\mathbf{f}, -\mathbf{e}\}$ and $\{\mathbf{f}, \mathbf{e}\}$.

1) We come to basis $\{\mathbf{e}, -\mathbf{f}\}$ reflecting the second basis vector. Transition matrix from initial basis $\{\mathbf{e}, \mathbf{f}\}$ to the basis $\{\mathbf{e}, -\mathbf{f}\}$ is $T_{\{\mathbf{e}, -\mathbf{f}\}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Its determinant is -1. Bases $\{\mathbf{e}, \mathbf{f}\}$ and $\{\mathbf{e}, -\mathbf{f}\}$ have opposite orientation.

2) Transition matrix from initial basis $\{\mathbf{e}, \mathbf{f}\}$ to the basis $\{\mathbf{f}, -\mathbf{e}\}$ is $T_{\{\mathbf{f}, -\mathbf{e}\}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Its determinant is 1. Bases $\{\mathbf{e}, \mathbf{f}\}$ and $\{\mathbf{f}, -\mathbf{e}\}$ have same orientation. We come to basis $\{\mathbf{f}, -\mathbf{e}\}$ rotating the initial basis on the angle $\pi/2$.

3) Transition matrix from initial basis $\{\mathbf{e}, \mathbf{f}\}$ to the basis $\{\mathbf{f}, \mathbf{e}\}$ is $T_{\{\mathbf{f},\mathbf{e}\}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Its determinant is -1. Bases $\{\mathbf{e}, \mathbf{f}\}$ and $\{\mathbf{e}, -\mathbf{f}\}$ have opposite orientation.

We come to basis $\{\mathbf{f}, \mathbf{e}\}$ reflecting the initial basis.

We see that bases $\{\mathbf{e}, \mathbf{f}\}$ and $\{\mathbf{f}, -\mathbf{e}\}$ have the same orientation; i.e. they belong to the same equivalenceclass. Bases $\{\mathbf{e}, -\mathbf{f}\}$ and $\{\mathbf{f}, \mathbf{e}\}$ have the same orientation too, they belong to the another equivalence class. If we say that bases $\{\mathbf{e}, \mathbf{f}\}$ and $\{\mathbf{f}, -\mathbf{e}\}$ are *left* bases then bases $\{\mathbf{e}, -\mathbf{f}\}$ and $\{\mathbf{f}, \mathbf{e}\}$ are *right* bases.

(There are plenty exercises in the Homework 3.)

Example(Orientation in three-dimensional euclidean space.) Let $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ be any basis in $\mathbf{E}^3$ and $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are arbitrary three vectors in $\mathbf{E}^3$:

$$\mathbf{a} = a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z \ \mathbf{b} = b_x\mathbf{e}_x + b_y\mathbf{e}_y + b_z\mathbf{e}_z, \ \mathbf{c} = c_x\mathbf{e}_x + c_y\mathbf{e}_y + c_z\mathbf{e}_z \,.$$

Consider ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$. The transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is $T = \begin{pmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{pmatrix}$:

$$\{\mathbf{a}, \mathbf{b}, \mathbf{c}\} = \{\mathbf{e_x}, \mathbf{e_y}, \mathbf{e_z}\}\mathbf{T} = \{\mathbf{e_x}, \mathbf{e_y}, \mathbf{e_z}\} \begin{pmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{pmatrix}$$

One can see that the ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ also is a basis, (i.e. these three vectors are linearly independent) if and only if transition matrix is not degenerate $\det T \neq 0$. The basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ has the same orientation as the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ if

$$\det T > 0 \,. \tag{1.34}$$

The basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ has the orientation opposite to the orientation of the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ if

$$\det T < 0 \,. \tag{1.35}$$

Remark Note that in the example above we considered in $\mathbf{E}^3$ *arbitrary bases* not necessarily orthonormal bases.

Relations (1.34),(1.35) define equivalence relations in the set of bases. Orientation is equivalence class of bases. There are two orientations, every basis has the same orientation as a given basis or opposite orientation.

If two bases $\{\mathbf{e}_i\}$, $\{\mathbf{e}_{i'}\}$ have the same orientation then they can be transformed to each other by continuous transformation, i.e. there exists one-parametric family of bases $\{\mathbf{e}_i(t)\}$ such that $0 \le t \le 1$ and $\{\mathbf{e}_i(t)\}|_{t=0} = \{\mathbf{e}_i\}$, $\{\mathbf{e}_i(t)\}|_{t=1} = \{\mathbf{e}_{i'}\}$. (All functions $\mathbf{e}_i(t)$ are continuous) In the case of three-dimensional space the following statement is true : *Let* $\{\mathbf{e}_i\}, \{\mathbf{e}_{i'}\}$ $(i = 1, 2, 3)$ *be two orthonormal bases in* $\mathbf{E}^3$ *which have the same orientation. Then there exists an axis* $\mathbf{n}$ *such that basis* $\{\mathbf{e}_i\}$ *transforms to the basis* $\{\mathbf{e}_{i'}\}$ *under rotation around the axis.*(This is Euler Theorem (see it later).

Exercise Show that bases $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ and $\{\mathbf{f}, \mathbf{e}, \mathbf{g}\}$ have opposite orientation but bases $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ and $\{\mathbf{f}, \mathbf{e}, -\mathbf{g}\}$ have the same orientation.

Solution. Transformation from basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ to basis $\{\mathbf{f}, \mathbf{e}, \mathbf{g}\}$ is "swapping" of vectors $((\mathbf{e}, \mathbf{f}) \mapsto (\mathbf{f}, \mathbf{e})$. This is reflection and this transformation changes orientation. One can see it using transition matrix:

$$T\colon \{\mathbf{f}, \mathbf{e}, \mathbf{g}\} = \{\mathbf{e}, \mathbf{f}, \mathbf{g}\}T = \{\mathbf{e}, \mathbf{f}, \mathbf{g}\} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \det T = -1$$

Transformation from basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ to basis $\{\mathbf{f}, \mathbf{e}, -\mathbf{g}\}$ is composition of two transformations: "swapping" of vectors $((\mathbf{e}, \mathbf{f}) \mapsto (\mathbf{f}, \mathbf{e})$ and changing direction of vector $\mathbf{g}$ $(\mathbf{g} \mapsto -\mathbf{g})$. We have two reflections:

$$\{\mathbf{e}, \mathbf{f}, \mathbf{g}\} \quad \stackrel{\text{reflection}}{\longrightarrow} \quad \{\mathbf{f}, \mathbf{e}, \mathbf{g}\} \quad \stackrel{\text{reflection}}{\longrightarrow} \quad \{\mathbf{f}, \mathbf{e}, -\mathbf{g}\}$$

Any reflection changes orientation. Two reflections preserve orinetation. One may come to this result using transition matrix:

$$T\colon \{\mathbf{f}, \mathbf{e}, -\mathbf{g}\} = \{\mathbf{e}, \mathbf{f}, \mathbf{g}\}T = \{\mathbf{e}, \mathbf{f}, \mathbf{g}\} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} . \det T = 1. \quad \text{Orientation is not changed.} \tag{1.36}$$

(See also exercises in Homework 3)

1.10 Linear operator and rotation in $\mathbf{E}^2$ and $\mathbf{E}^3$

1.10.1 Linear operators.

Recall here facts about linear operators in vector space

Let P be a linear operator in vector space V:

$$P\colon\ V \to V, \qquad P(\lambda \mathbf{x} + \mu \mathbf{y}) = \lambda P(\mathbf{x}) + \mu P(\mathbf{y}).$$

Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be an arbitrary basis in n-dimensional vector space V. Consider the action of operator P on basis vectors: $\mathbf{e}'_i = P(\mathbf{e}_i)$:

$$\begin{array}{c}
\mathbf{e}'_1 = P(\mathbf{e}_1) = \mathbf{e}_1 p_{11} + \mathbf{e}_2 p_{21} + \mathbf{e}_3 p_{31} + \dots + \mathbf{e}_n p_{n1} \\
\mathbf{e}'_2 = P(\mathbf{e}_2) = \mathbf{e}_1 p_{12} + \mathbf{e}_2 p_{22} + \mathbf{e}_3 p_{32} + \dots + \mathbf{e}_n p_{n2} \\
\mathbf{e}'_3 = P(\mathbf{e}_3) = \mathbf{e}_1 p_{13} + \mathbf{e}_2 p_{23} + \mathbf{e}_3 p_{31} + \dots + \mathbf{e}_n p_{n3} \\
\dots \\
\mathbf{e}'_n = P(\mathbf{e}_n) = \mathbf{e}_1 p_{1n} + \mathbf{e}_2 p_{2n} + \mathbf{e}_3 p_{3n} + \dots + \mathbf{e}_n p_{nn}
\end{array} \tag{1.37}$$

In the case if linear operator P is non-degenerate (invertible) then vectors $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3, \dots, \mathbf{e}'_n$, where $\mathbf{e}'_i = P(\mathbf{e}_i) = \sum \mathbf{e}_k p_{ki}$ form a basis. The matrix $P = ||p_{ik}||$ is the transition matrix from the basis $\{\mathbf{e}_i\}$ to the basis $\{\mathbf{e}'_i = P(\mathbf{e}_i)\}$. It is called *matrix of operator P in the basis $\{\mathbf{e}_i\}$.*

How matrix of linear operatot changes if we change the basis? Consider new basis $\{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ in the linear space V. Let A be transition matrix from the basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ to the new basis $\{\mathbf{f}_1, \dots, \mathbf{f}_n\}$:

$$\{\mathbf{f}_1, \dots, \mathbf{f}_n\} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\} A, \text{ i.e.} f_i = \sum_k \mathbf{e}_k a_{ki}$$

(see equation (1.16)). Then the action of operator P in the new basis is given by the formula $\mathbf{f}'_i = P(\mathbf{f}_i)$. According to the formulae (1.10.1) and (1.37) we have

$$\mathbf{f}'_i = P(\mathbf{f}_i) = P\left(\sum_q \mathbf{e}_q a_{qi}\right) = \sum_q a_{qi} \left(\sum_r \mathbf{e}_r p_{rq}\right) = \sum_{q,r} \mathbf{e}_r p_{rq} a_{qi} = \sum_r \mathbf{e}_r (PA)_{ri} =$$

$$\sum_{r,k} \mathbf{f}_k (A^{-1})_{kr} (PA)_{ri} = \sum_k \mathbf{f}_k (A^{-1}PA)_{ki}\,.$$

We see that in the new basis $\{\mathbf{f}_i\}$ a matrix of linear operator is $A^{-1}PA$:

$$\text{If } \{\mathbf{e}'_1, \dots, \mathbf{e}'_n\} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\} P, \text{ then } \{\mathbf{f}'_1, \dots, \mathbf{f}'_n\} = \{\mathbf{f}_1, \dots, \mathbf{f}_n\} A^{-1}PA, \ , \tag{1.38}$$

where A is transition matrix from the basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ to the basis $\{\mathbf{f}_1, \dots, \mathbf{f}_n\}$,

1.10.2 Determinant and Trace of linear operator

We recall the definition of determinant and explain what is the trace of linear operator,

Definition-Proposition Let P be a linear operator in vector space V and let $P_{ik} = ||p_{ik}||$ be transition matrix of this operator in an arbitrary basis in V (see construction (1.37).) Then determinant of linear operator P equals to determinant of transition matrix of this operator.

$$\det P = \det (p_{ik})$$

In the same way we define trace of operator via trace of matrix:

$$\mathrm{Tr}\, P = \mathrm{Tr}\ (||p_{ik}||) = p_{11} + p_{22} + p_{33} + \cdots + p_{nn}\,. \tag{1.39}$$

Determinant and trace of operator are well-defined. since due to (1.38) determinant and trace of transition matrice do not change if we change the basis in spite of the fact that transition matrix changes: $P \mapsto A^{-1}PA$, but

$$\det \left(A^{-1}PA\right) = \det A^{-1} \det P \det A = (\det A)^{-1} \det P \det A = \det P\,.$$

In the same way one can see that trace is invariant too:

$$\mathrm{Tr}\,(A^{-1}PA) = \sum_i (A^{-1}PA)_{ii} = \sum_{i,k,p} \left(A^{-1}\right)_{ik} p_{kp} A_{pi} = \sum_{i,k,p} A_{pi} \left(A^{-1}\right)_{ik} p_{kp} =$$

$$\sum_{p,k} \left(A \cdot A^{-1}\right)_{pk} p_{kp} A_{pi} = \sum_{p,k} \delta_{kp} p_{kp} A_{pi} = \sum_k p_{kk} = \mathrm{Tr}\, P\,.$$

Trace of linear operator is an infinitesimal version of its determinant:

$$\det(1 + tP) = 1 + t\mathrm{Tr}\, P + O(t^2)\,.$$

This is inbfinitesimal version for the followiong famous formula which relates trace and det of linear operator:

$$\det e^{tA} = e^{t\mathrm{Tr}\, A}\,. \tag{1.40}$$

where $e^P = \sum \frac{A^n}{n!}$. E.g. if $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then $e^{tA} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$, $\det e^{tA} = 1$ and $e^{t\mathrm{Tr}\, A} = e^0 = 1$.

1.10.3 Orientation of linear operator

. Let P be invertible linear operator, i.e. $\det P \neq 0$.

If a linear operator P acting on the space V has positive determinant then under the action of this operator an arbitrary basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ transforms to the new basis $\{\mathbf{e}'_1, \dots, \mathbf{e}'_n\}$ such that transition matrix from basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ to the new basis $\{\mathbf{e}'_1, \dots, \mathbf{e}'_n\}$ has positive determinant, i.e. these bases have the same orientation. Respectively if a linear operator P acting on the space V has negative determinant then under the action of this operator an arbitrary basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ transforms to the new basis $\{\mathbf{e}'_1, \dots, \mathbf{e}'_n\}$ such that transition matrix from basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ to the new basis $\{\mathbf{e}'_1, \dots, \mathbf{e}'_n\}$ has negative determinant, i.e. these bases have opposite orientation. Thus we can define does the linear operator P acting in the vector space V change an orientation or it does not change an orientation of this vector space.

Definition. Non-degenerate (invertible) linear operator P ($\det P \neq 0$) acting in vector space V preserves an orientation of the vector space V if $\det P > 0$. It changes the orientation if $\det P < 0$.

If $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is an arbitrary basis which transforms to the new basis $\{\mathbf{e}'_1, \dots, \mathbf{e}'_n\}$ under the action of nvertible operator P: $\mathbf{e}'_i = P(\mathbf{e}_i)$ then these bases have the same orientation if and only if operator P preserves an orientation, i.e. $\det P > 0$, and these bases have opposite orientation if and only if the operator P changes an orientation, i.e. $\det P < 0$.

The matrix $P = ||p_{ij}||$ is the transition matrix from the basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ to the basis $\{\mathbf{e}'_1, \dots, \mathbf{e}'_n\}$. For an arbitrary vector $\mathbf{x}$

$$\forall \mathbf{x} = \sum_{i=1}^{n} \mathbf{e}_i x^i = (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n) \cdot \begin{pmatrix} x^1 \\ x^2 \\ \dots \\ x^n \end{pmatrix}$$

$$P\mathbf{x} = (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n) \cdot P \cdot \begin{pmatrix} x^1 \\ x^2 \\ \dots \\ x^n \end{pmatrix} = \sum_{i=1}^{n} \mathbf{e}'_i x^i = \sum_{i,k=1}^{n} \mathbf{e}_k p_{ki} x^i \,.$$

If x^i are components of vector $\mathbf{x}$ at the basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ and x'^i are components of the vector $\mathbf{x}$ at the new basis $\{\mathbf{e}'_i\}$ then $x'^i = \sum_i p_{ik} x^k$.

1.10.4 Orthogonal linear operators

Now we study geometrical meaning of orthogonal linear operators in Euclidean space.

Recall that linear operator P in Euclidean space $\mathbf{E}^n$ is called orthogonal operator if it preserves scalar product:

$$(P\mathbf{x}, P\mathbf{y}) = (\mathbf{x}, \mathbf{y}), \text{ for arbitrary vectors } \mathbf{x}, \mathbf{y} \tag{1.41}$$

In particular if $\{\mathbf{e}_i\}$ is orthonormal basis in Euclidean space then due to (1.41) the new basis $\{\mathbf{e}'_i = P(\mathbf{e}_i)\}$ is orthonormal too. Thus we see that matrix of orthogonal operator P in a given orthogonal basis is orthogonal matrix:

$$P^T \cdot P = I \tag{1.42}$$

(see (1.18) in subsection 1.7). In particular we see that for orthogonal linear operator $\det P = \pm 1$ (compare with (1.19)). If $\det P = 1$ then it preserves orientation and if $\det P = -1$ then it changes orientation. Remember we already studied orthogonal matrices in 2-dimensional Euclidean space (see subsection 1.8).

1.10.5 Orthogonal operators preserving orientation of $\mathbf{E}^n$ (n=2,3)

Orthogonal operators preserving orientation in $\mathbf{E}^2$ and $\mathbf{E}^3$ are rotations. We try to explain this.

Let $\mathbf{E}^n$ be oriented vector space. Recall that oriented vector space means that it is chosen the equivalence class of bases: all bases in this class have the same orientation. We call all bases in the equivalence class defining orientation "left" bases. All "left" bases have the same orientation. To define an orientation in vector space V one may consider an arbitrary basis $\{\mathbf{e}_i^{(0)}\}$ in V and claim that this basis is "left" basis. The basis $\{\mathbf{e}_i^{(0)}\}$ defines equivalence class of "left" bases: all bases $\{\mathbf{e}_i\}$ such that $\{\mathbf{e}_i\} \sim \{\mathbf{e}_i^{(0)}$ will be called "left" bases. We can say that basis $\{\mathbf{e}_i^{(0)}\}$ defines the orientation.

Later on considering oriented vector space we usually call all bases defining the orientation (i.e. belonging to the equivalence class of bases defining orientation) "left" bases.

Now we define rotation of oriented $\mathbf{E}^2$ and oriented $\mathbf{E}^3$.

Definition Let $\mathbf{E}^2$ be an oriented Euclidean space. We say that linear operator P rotates this space on an angle "φ" if for a given "left" orthonormal basis $\{\mathbf{e}, \mathbf{f}\}$

$$\begin{cases} \mathbf{e}' = P(\mathbf{e}) = \mathbf{e}\cos\varphi + \mathbf{f}\sin\varphi \\ \mathbf{f}' = P(\mathbf{f}) = -\mathbf{e}\sin\varphi + \mathbf{f}\cos\varphi \end{cases} \quad \text{i.e.} \quad \{\mathbf{e}', \mathbf{f}'\} = \{\mathbf{e}, \mathbf{f}\} \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \tag{1.43}$$

i.e. transition matrix from basis $\{\mathbf{e}, \mathbf{f}\}$ to new basis $\{\mathbf{e}' = P(\mathbf{e}), \mathbf{f}' = P(\mathbf{f})\}$ is the rotation matrix (1.23) (see also (1.25)).

Remark One can show that the angle of rotation does not depend on the choice of "left" basis. If we will choose another left basis $\tilde{\mathbf{e}}, \tilde{\mathbf{f}}$ then the angle remains the same

Operator P rotates every vector rotates on the angle φ.

If we choose a basis with opposite orientation ("right" basis) then the angle will change: $\varphi \mapsto -\varphi$.

We see from formula (1.43) that the matrix of operator P is orthogonal matrix such that its determinant equals 1. On the other hand we proved that all orthogonal 2×2 matrices A such that $\det A = 1$ have the appearance (1.43) (see the subsection 1.8). Hence in 2-dimensional case we come to the folowing simple

Proposition Let P be an orthogonal operator in oriented 2-dimensional Euclidean space. If operator P preserves orientation ($\det P = 1$) then it is a rotation operator (1.43) on some angle φ.

The situation is little bit more tricky in 3-dimensional case.

Let $\mathbf{E}^3$ be an Euclidean vector space. (Problem of orientation we will discuss below.) Let $\mathbf{N} \neq 0$ be an arbitrary non-zero vector in $\mathbf{E}^3$. Consider the line $l_{\mathbf{N}}$, spanned by vector $\mathbf{N}$. This is *axis* directed along the vector $\mathbf{N}$. Choose a unit vector

$$\mathbf{n} = \pm \frac{\mathbf{N}}{|\mathbf{N}|} \tag{1.44}$$

Vector $\mathbf{n}$ fixes an orientation on $l_{\mathbf{N}}$. Changing $\mathbf{n} \mapsto -\mathbf{n}$ changes an orientation on opposite).

Choose an arbitrary orthonormal basis such that first vector of this basis is directed along the axis: a basis $\{\mathbf{n}, \mathbf{f}, \mathbf{g}\}$.

Definition We say that a linear operator P rotates the space $\mathbf{E}^3$ on the angle φ with respect to an axis $l_{\mathbf{N}}$ directed along a vector $\mathbf{N}$ if the following conditions are satisfied:

- $$P(\mathbf{N}) = \mathbf{N}$$

  vector $\mathbf{N}$ (and all vectors proportional to this vector) are eigenvectors of operator P with eigenvalue 1, i.e. axis remain intact

- for a basis $\{\mathbf{n}, \mathbf{f}, \mathbf{g}\}$ such that the first vector of this basis is equal to $\mathbf{n}$, ($\mathbf{n}$ is a unit vector, proportional to $\mathbf{N}$)

$$\begin{cases} \mathbf{f}' = P(\mathbf{f}) = \mathbf{f}\cos\varphi + \mathbf{g}\sin\varphi \\ \mathbf{g}' = P(\mathbf{f}) = -\mathbf{f}\sin\varphi + \mathbf{g}\cos\varphi \end{cases} \quad \text{i.e.} \quad \{\mathbf{f}', \mathbf{g}'\} = \{\mathbf{f}, \mathbf{g}\} \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}. \tag{1.45}$$

  In other words plane (subspace) orthogonal to axis rotates on the angle φ: linear operator P rotates every vector orthogonal to axis on the angle φ in the plane (subspace) orthogonal to the axis.

Linear operator P transforms the basis $\{\mathbf{n}, , \mathbf{f}, \mathbf{g}\}$ to the new basis $\{\mathbf{n}, \mathbf{f}', \mathbf{g}'\}$ $= \{\mathbf{n}, \mathbf{f}\cos\varphi + \mathbf{g}\sin\varphi, -\mathbf{f}\sin\varphi + \mathbf{g}\cos\varphi\}$. The matrix of operator P, i.e. the transition matrix from the basis $\{\mathbf{n}, , \mathbf{f}, \mathbf{g}\}$ to the basis $\{\mathbf{n}, \mathbf{f}', \mathbf{g}'\}$ is defined by the relation:

$$\{\mathbf{n}, \mathbf{f}', \mathbf{g}'\} = \{\mathbf{n}, \mathbf{f}\cos\varphi + \mathbf{g}\sin\varphi, -\mathbf{f}\sin\varphi + \mathbf{g}\cos\varphi\} = \{\mathbf{n}, , \mathbf{f}, \mathbf{g}\} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix} \tag{1.46}$$

Recalling definition (1.39) of trace of linear operator we come to the following relation

$$TrP = 1 + 2\cos\varphi \tag{1.47}$$

where φ is angle of rotation. Note that Trace of the operator does not depend on the choice of the basis. This formula express cosine of the angle of rotation in terms of operator, irrelevant of the choice of the basis.

Remark If we change orientation then $\varphi \mapsto -\varphi$. For non-oriented Euclidean space rotation is defined up to a sign[5]

Careful reader maybe already noted that even fixing the orientation of $\mathbf{E}^3$ does not fix the "sign" of the angle: If we change the orientation of the axis (changing $\mathbf{n} \mapsto -\mathbf{n}$) then changing the corresponding "left" basis will imply that $\varphi \mapsto -\varphi$. In fact angle φ is the angle of rotation of oriented plane which is orthogonal to the axis of rotation. Orientation on the plane is defined by orientation in $\mathbf{E}^3$ and orientation of the axis which is orthogonal to this plane. In the case of 3-dimensional space sign of the angle depends not only on orientation of $\mathbf{E}^3$ but on orientation of axis. In what follows we will ignore this. This means that we define rotation on the angle $\pm\varphi$ up to a sign.... Rotation is defined for operators preserving orientation. The difference between angles of rotations φ and $-\varphi$ is depending not only on orientation of $\mathbf{E}^3$ but on orientation of axis too. But we ignore this difference. Note that $\cos\varphi$ in the formula is defined up to a sign

[5]Does it recall you expressions such as "clockwise", "anticlock-wise" rotation?

Rotation operator eviently is orthogonal operator preserving orientation. Is it true converse implication? We are ready to formulate the following remarkable result.

Theorem (the Euler Theorem) *Let P be an orthogonal operator preserving an orientation of Euclidean space $\mathbf{E}^3$, i.e. operator P preserves the scalar product and orientation. Then it is a rotation operator with respect to an axis l on the angle φ. Every vector $\mathbf{N}$ directed along the axis does not change, i.e. the axis is 1-dimensional space of eigenvectors with eigenvalue 1, $P(\mathbf{N}) = \mathbf{N}$. Every vector orthogonal to axis rotates on the angle φ in the plane orthogonal to the axis,*

$$\operatorname{Tr} P = 1 + 2\cos\varphi\,.$$

The angle φ is defined up to a sign. Changing orientation of the Euclidean space and of the axis change sign of φ.

This Theorem can be restated in the following way: every orthogonal operator P preserving orientation, ($\det P \neq 0$) has an eigenvector $\mathbf{N} \neq 0$ with eigenvalue 1. This eigenvector defines the axis of rotation. In an orthonormal basis $\{\mathbf{n}, \mathbf{f}, \mathbf{g}\}$ where $\mathbf{n}$ is a unit vector along the axis, the transition matrix of operator has an appearance (1.46). Angle of rotaion can be defined via Trace of operator by formula $\operatorname{Tr} P = 1 + 2\cos\varphi$.

Remark If P is an identity operator, $P = I$ then " there is no rotation", more precisely: any line can be considered as an axis of rotation (every vector is eigenvector of identity matrix with eigenvalue 1) and angle of rotation is equal to zero. If $P \neq I$ then axis of rotation is defiend uniquely.

Proof of the Euler Theorem. The proof of the Euler Theorem has two parts. First and central part is to prove the existence of the axis. The rest is trivial: we take an arbitrary orthonormal basis $\mathbf{n}, \mathbf{f}, \mathbf{g}$ such that $\mathbf{n}$ is eigenvector and we come to relation (1.45). We expose here maybe the most beautiful proof which belongs to Coxeter.

Let P be linear orthogonal operator preserving orientation. Note that for any two not-zero distinct vectors $\mathbf{e}, \mathbf{f}$ one can consider orthogonal operator $R_{\mathbf{e},\mathbf{f}}$ which changes orientation and swaps the vectors $\mathbf{e}, \mathbf{f}$: it is reflection with respect to the plane spanned by the vectors $\mathbf{e} + \mathbf{f}$ and a vector $\mathbf{e} \times \mathbf{f}$.

Let $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ be an arbitrary orthonormal basis in $\mathbf{E}^3$ and let $\mathbf{e}', \mathbf{f}', \mathbf{g}'$ be image of this basis under operator P

$$P(\mathbf{e}) = \mathbf{e}',\ P(\mathbf{f}) = \mathbf{f}'\ P(\mathbf{g}) = \mathbf{g}'\,.$$

If $\mathbf{e} = \mathbf{e}'$ nothing to prove ($\mathbf{e}$ is eigenvector with eigenvalue 1). If this is not the case, apply reflection operator $R_{\mathbf{e},\mathbf{e}'}$ to the initial basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ we come to the orthonormal basis $\{\mathbf{e}', \tilde{f}, \tilde{g}\}$, Then applying reflection operator $R_{\tilde{f},\mathbf{f}'}$ to this basis we come to the basis $\mathbf{e}', \mathbf{f}', \tilde{\tilde{g}}$. The third vector has no choice it has to be equal to $\mathbf{g}'$ since in the case if it is equal to $-\mathbf{g}'$ orientation is opposite. Hence we see that operator P is the product of two reflections operators. Consider the line l, intersection of these planes, we come to eigenvectors with eigenvalue 1. ∎

There are many other proofs, for example:

Another proof: Any non-degenerate 3×3 matrix has at least one eigenvector $\mathbf{x}$: $P\mathbf{x} = \lambda\mathbf{x}$, since cubic equation $\det(P - \lambda I) = 0$ has at lest one real root. Since P is orthogonal operator, then $\lambda = \pm 1$. If $\lambda = 1$, then $\mathbf{x}$ defines the axis. If $\lambda = -1$, $P\mathbf{x} = -\mathbf{x}$, then eigenvector with eigenvalue 1 belongs to the plane orthogonal to $\mathbf{x}$. ∎

Example Consider linear operator P such that for orthonormal basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$

$$P(\mathbf{e}_x) = \mathbf{e}_y, P(\mathbf{e}_y) = \mathbf{e}_x, P(\mathbf{e}_z) = -\mathbf{e}_z \tag{1.48}$$

This is obviously orthogonal operator since it transforms orthogonal basis to orthogonal one. This operator swaps first two vectors and reflects the third one. It preserves orientation: matrix of operator in the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$, i.e. the transition matrix from the basis $\{\mathbf{e}_x, , \mathbf{e}_y, \mathbf{e}_z\}$ to the basis $\{P(\mathbf{e}_x), P(\mathbf{e}_y), P(\mathbf{e}_z)\}$ is defined by the relation:

$$\{P(\mathbf{e}_x), P(\mathbf{e}_y), P(\mathbf{e}_z)\} = \{\mathbf{e}_y, \mathbf{e}_x, -\mathbf{e}_z\} = \{\mathbf{e}_x, , \mathbf{e}_y, \mathbf{e}_z\} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$\det P = 1$. This operator preserves orientation. Hence by Euler Theorem it is a rotation. Find first axis of rotation. It is easy to see from (1.48) that $\mathbf{N} = \lambda(\mathbf{e}_x + \mathbf{e}_y)$ is eigenvector with eigenvalue 1:

$$P(\mathbf{N}) = P(\mathbf{e}_x + \mathbf{e}_y) = \mathbf{e}_y + \mathbf{e}_x = \mathbf{N}\,.$$

Hence axis of rotation is directed along the vector $\mathbf{e}_x = \mathbf{e}_y$. $\operatorname{Tr} P = 1 + 2\cos\varphi = 0$. hence angle of rotation $\varphi = \pm\frac{\pi}{2}$.

One can calculate explicitly angle of rotation: Consider orthonormal basis $\{\mathbf{n}, f, \mathbf{g}\}$ adjusted to the axis $(\mathbf{n}||\mathbf{N})$. We have that $\mathbf{n} = \frac{\mathbf{e}_x + \mathbf{e}_y}{\sqrt{2}}$ since $\mathbf{n}$ is proportional to $\mathbf{N}$ and it is unit vector. Choose $\mathbf{f} = \frac{-\mathbf{e}_x + \mathbf{e}_y}{\sqrt{2}}$ and $\mathbf{g} = \mathbf{e}_z$. Then it is easy to see that

$$\{\mathbf{n}, \mathbf{f}, \mathbf{g}\} = \left\{\frac{\mathbf{e}_x + \mathbf{e}_y}{\sqrt{2}}, \frac{-\mathbf{e}_x + \mathbf{e}_y}{\sqrt{2}}, \mathbf{g}\right\}$$

is orthonormal basis.Using (1.48)one can see that

$$P(\mathbf{n}) = P\left(\frac{\mathbf{e}_x + \mathbf{e}_y}{\sqrt{2}}\right) = \frac{\mathbf{e}_y + \mathbf{e}_x}{\sqrt{2}} = \mathbf{n}\,,$$

$$P(\mathbf{f}) = P\left(\frac{-\mathbf{e}_x + \mathbf{e}_y}{\sqrt{2}}\right) = \frac{-\mathbf{e}_y + \mathbf{e}_x}{\sqrt{2}} = -\mathbf{f}, \quad P(\mathbf{g}) = -\mathbf{g}$$

We see that

$$\{\mathbf{n},\mathbf{f},\mathbf{g}\}\xrightarrow{P}\{\mathbf{n},-\mathbf{f},-\mathbf{g}\}\,.$$

Comparing with (1.45) and (1.46) we see that the operator P is rotation of $\mathbf{E}^3$ on the angle π with respect to the axis directed along the vector $\mathbf{e}_x+\mathbf{e}_y$.

1.11 Vector product in oriented $\mathbf{E}^3$

Now we give a definition of vector product of vectors in 3-dimensional Euclidean space equipped with orientation.

Let $\mathbf{E}^3$ be three-dimensional oriented Euclidean space, i.e. Euclidean space equipped with an equivalence class of bases with the same orientation. To define the orientation it suffices to consider just one orthonormal basis $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$ which is claimed to be left basis. Then the equivalence class of the left bases is a set of all bases which have the same orientation as the basis $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$.

Definition Vector product $L(\mathbf{x},\mathbf{y})=\mathbf{x}\times\mathbf{y}$ is a function of two vectors which takes vector values such that the following axioms (conditions) hold

- The vector $L(\mathbf{x},\mathbf{y})=\mathbf{x}\times\mathbf{y}$ is orthogonal to vector $\mathbf{x}$ and vector $\mathbf{y}$:

  $$(\mathbf{x}\times\mathbf{y})\perp\mathbf{x}\,,\quad(\mathbf{x}\times\mathbf{y})\perp\mathbf{y}\tag{1.49}$$

  In particular it is orthogonal to the the plane spanned by the vectors $\mathbf{x},\mathbf{y}$ (in the case if vectors $\mathbf{x},\mathbf{y}$ are linearly independent)

- $$\mathbf{x}\times\mathbf{y}=-\mathbf{y}\times\mathbf{x},\qquad\text{(anticommutativity condition)}\tag{1.50}$$

- $$(\lambda\mathbf{x}+\mu\mathbf{y})\times\mathbf{z}=\lambda(\mathbf{x}\times\mathbf{z})+\mu(\mathbf{y}\times\mathbf{z})\,,\qquad\text{(linearity condition)}\tag{1.51}$$

- If vectors $\mathbf{x},\mathbf{y}$ are perpendicular each other then the magnitude of the vector $\mathbf{x}\times\mathbf{y}$ is equal to the area of the rectangle formed by the vectors $\mathbf{x}$ and $\mathbf{y}$:

  $$|\mathbf{x}\times\mathbf{y}|=|\mathbf{x}|\cdot|\,\mathbf{y}|\,,\qquad\text{if }\mathbf{x}\perp\mathbf{y}\,,i.e.(\mathbf{x},\mathbf{y})=0\,.\tag{1.52}$$

- If the ordered triple of the vectors $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$, where $\mathbf{z} = \mathbf{x} \times \mathbf{y}$ is a basis, then this basis and an orthonormal basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ defining orientation of $\mathbf{E}^3$ have the same orientation:

$$\{\mathbf{x}, \mathbf{y}, \mathbf{z}\} = \{\mathbf{e}, \mathbf{f}, \mathbf{g}\}T, \text{ where for transition matrix } T, \det T > 0. \tag{1.53}$$

Vector product depends on orientation in Euclidean space.

Comments on conditions (axioms) (1.49)—(1.53):

1. The condition (1.51) of linearity of vector product with respect to the first argument and the condition (1.50) of anticommutativity imply that vector product is an operation which is linear with respect to the second argument too. Show it:

$$\mathbf{z} \times (\lambda\mathbf{x} + \mu\mathbf{y}) = -(\lambda\mathbf{x} + \mu\mathbf{y}) \times \mathbf{z} = -\lambda(\mathbf{x} \times \mathbf{z}) - \mu(\mathbf{y} \times \mathbf{z}) = \lambda(\mathbf{z} \times \mathbf{x}) + \mu(\mathbf{z} \times \mathbf{y})\,.$$

Hence vector product is bilinear operation. Comparing with scalar product we see that vector product is bilinear anticommutative (antisymmetric) operation which takes vector values, while scalar product is bilinear symmetric operation which takes real values.

2. The condition of anticommutativity immediately implies that vector product of two colinear (proportional) vectors $\mathbf{x}, \mathbf{y}$ ($\mathbf{y} = \lambda\mathbf{x}$) is equal to zero. It follows from linearity and anticommuativity conditions. Show it: Indeed

$$\mathbf{x} \times \mathbf{y} = \mathbf{x} \times (\lambda\mathbf{x}) = \lambda(\mathbf{x} \times \mathbf{x}) = -\lambda(\mathbf{x} \times \mathbf{x}) = -\mathbf{x} \times (\lambda\mathbf{x}) = -\mathbf{x} \times \mathbf{y}. \tag{1.54}$$

Hence $\mathbf{x} \times \mathbf{y} = 0$, if $\mathbf{y} = \lambda\mathbf{x}$ $\blacksquare$.

3. It is very important to emphasize again that vector product depends on orientation. According the condition (1.53) if $\mathbf{z} = \mathbf{x} \times \mathbf{y}$ and we change the orientation of Euclidean space, then $\mathbf{z} \to -\mathbf{z}$ since the basis $\{\mathbf{x}, \mathbf{y}, -\mathbf{z}\}$ as an orientation opposite to the orientation of the basis $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$.

You may ask a question: Does this operation (taking the vector product) which obeys all the conditions (axioms) (1.49)—(1.53) exist? And if it exists is it unique? We will show that the vector product is well-defined by the axioms (1.49)—(1.53), i.e. there exists an operation $\mathbf{x} \times y$ which obeys the axioms (1.49)—(1.53) and these axioms define the operation uniquely.

We will assume first that there exists an operation $L(\mathbf{x}, \mathbf{y}) = \mathbf{x} \times \mathbf{y}$ which obeys all the axioms (1.49)—(1.53). Under this assumption we will construct explicitly this operation (if it exists!). We will see that the operation that we constructed indeed obeys all the axioms (1.49)—(1.53).

Let $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ be an *arbitrary* left orthonormal basis of oriented Euclidean space $\mathbf{E}^3$, i.e. a basis which belongs to the equivalence class of the basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ defining orientation of $\mathbf{E}^3$. Then it follows from the considerations above for vector product that

$$\begin{array}{ccc} \mathbf{e}_x \times \mathbf{e}_x = 0, & \mathbf{e}_x \times \mathbf{e}_y = \mathbf{e}_z, & \mathbf{e}_x \times \mathbf{e}_z = -\mathbf{e}_y \\ \mathbf{e}_y \times \mathbf{e}_x = -\mathbf{e}_z, & \mathbf{e}_y \times \mathbf{e}_y = 0, & \mathbf{e}_y \times \mathbf{e}_z = \mathbf{e}_x \\ \mathbf{e}_z \times \mathbf{e}_x = \mathbf{e}_y, & \mathbf{e}_z \times \mathbf{e}_y = -\mathbf{e}_x, & \mathbf{e}_z \times \mathbf{e}_z = 0 \end{array} \tag{1.55}$$

E.g. $\mathbf{e}_x \times \mathbf{e}_x = 0$, because of (1.50), $\mathbf{e}_x \times \mathbf{e}_y$ is equal to $\mathbf{e}_z$ or to $-\mathbf{e}_z$ according to (1.52), and according to orientation arguments (1.53) $\mathbf{e}_x \times \mathbf{e}_y = \mathbf{e}_z$.

Now it follows from linearity and (1.55) that for two arbitrary vectors $\mathbf{a} = a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z$, $\mathbf{b} = b_x\mathbf{e}_x + b_y\mathbf{e}_y + b_z\mathbf{e}_z$

$$\mathbf{a}\times\mathbf{b} = (a_x\mathbf{e}_x+a_y\mathbf{e}_y+a_z\mathbf{e}_z)\times(b_x\mathbf{e}_x+b_y\mathbf{e}_y+b_z\mathbf{e}_z) = a_xb_y\mathbf{e}_x\times\mathbf{e}_y+a_xb_z\mathbf{e}_x\times\mathbf{e}_z+$$

$$a_yb_x\mathbf{e}_y \times \mathbf{e}_x + a_yb_z\mathbf{e}_y \times \mathbf{e}_z + a_zb_x\mathbf{e}_z \times \mathbf{e}_x + a_zb_y\mathbf{e}_z \times \mathbf{e}_y =$$

$$(a_yb_z - a_zb_y)\mathbf{e}_x + (a_zb_x - a_xb_z)\mathbf{e}_y + (a_xb_y - a_yb_x)\mathbf{e}_z\,. \tag{1.56}$$

It is convenient to represent this formula in the following very familiar way:

$$L(\mathbf{a}, \mathbf{b}) = \mathbf{a} \times \mathbf{b} = \det \begin{pmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{pmatrix} \tag{1.57}$$

We see that the operation $L(\mathbf{x}, \mathbf{y}) = \mathbf{x} \times \mathbf{y}$ which obeys all the axioms (1.49)—(1.53), if it exists, has an appearance (1.57), where $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ is an arbitrary orthonormal basis (with rightly chosen orientation). On the other hand using the properties of determinant and the fact that vectors are orthogonal if and only if their scalar product equals to zero one can easy see that the vector product defined by this formula indeed obeys all the conditions (1.49)—(1.53).

Thus we proved that the vector product is well-defined by the axioms (1.49)—(1.53) and it is given by the formula (1.57) in an arbitrary orthonormal basis (with rightly chosen orientation).

Remark In the formula above we have chosen an arbitrary orthonormal basis which belongs to the equivalence class of bases defining the orientation. What will happen if we choose instead the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ an arbitrary orthonormal basis $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$. We see that such that answer does not change if both bases $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ have the same orientation, Formulae (1.55) are valid for an arbitrary orthonormal basis which have the same orientation as the orthonormal basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$.— In oriented Euclidean space $\mathbf{E}^3$ we may take an arbitrary basis from the equivalence class of bases defining orientation. On the other hand if we will consider the basis with opposite orientation then according to the axiom (1.53) vector product will change the sign. (See also the question 6 in Homework 4)

1.11.1 Vector product—area of parallelogram

The following Proposition states that vector product can be considered as area of parallelogram:

Proposition 2 *The modulus of the vector* $\mathbf{z} = \mathbf{x} \times \mathbf{y}$ *is equal to the area of parallelogram formed by the vectors* $\mathbf{x}$ *and* $\mathbf{y}$.:

$$S(\mathbf{x}, \mathbf{y}) = |\mathbf{x} \times \mathbf{y}| \,, \tag{1.58}$$

where we denote by $S(\mathbf{x}, \mathbf{y})$ the area of parallelogram formed by the vectors $\mathbf{x}, \mathbf{y}$.

Proof: Consider the expansion $\mathbf{y} = \mathbf{y}_{||} + \mathbf{y}_{\perp}$, where the vector $\mathbf{y}_{\perp}$ is orthogonal to the vector $\mathbf{x}$ and the vector $\mathbf{y}_{||}$ is parallel to to vector $\mathbf{x}$. The area of the parallelogram formed by vectors $\mathbf{x}$ and $\mathbf{y}$ is equal to the product of the length of of the vector $\mathbf{x}$ on the height. The height is equal to the length of the vector $\mathbf{y}_{\perp}$. We have $S(\mathbf{x}, \mathbf{y}) = |\mathbf{x}||\mathbf{y}_{\perp}|$. On the other $\mathbf{z} = \mathbf{x} \times \mathbf{y} = \mathbf{x} \times (\mathbf{y}_{||} + \mathbf{y}_{\perp}) = \mathbf{x} \times \mathbf{y}_{||} + \mathbf{x} \times \mathbf{y}_{\perp}$. But $\mathbf{x} \times \mathbf{y}_{||} = 0$, because these vectors are colinear. Hence $\mathbf{z} = \mathbf{x} \times \mathbf{y}_{\perp}$ and $|\mathbf{z}| = |\mathbf{x}||\mathbf{y}_{\perp}| = S(\mathbf{x}, \mathbf{y})$ because vectors $\mathbf{x}, \mathbf{y}_{\perp}$ are orthogonal to each other.

This Proposition is very important to understand the meaning of vector product. Shortly speaking *vector product of two vectors is a vector which is orthogonal to the plane spanned by these vectors, such that its magnitude is equal to the area of the parallelogram formed by these vectors. The direction is defined by orientation.*

Remark It is useful sometimes to consider area of parallelogram not as a positive number but as an real number positive or negative (see the next subsubsection.)

It is not worthless to recall the formula which we know from the school that area of parallelogram formed by vectors $\mathbf{x}, \mathbf{y}$ equals to the product of the base on the height. Hence

$$|\mathbf{x} \times \mathbf{y}| = |\mathbf{x}| \cdot |\mathbf{y}| |\sin\theta| , \tag{1.59}$$

where θ is an angle between vectors $\mathbf{x}, \mathbf{y}$.

Finally I would like again to stress:

Vector product of two vectors is equal to zero if these vectors are colinear (parallel). Scalar product of two vectors is equal to zero if these vector are orthogonal.

Exercise†Show that the vector product obeys to the following identity:

$$((\mathbf{a} \times \mathbf{b}) \times \mathbf{c}) + ((\mathbf{b} \times \mathbf{c}) \times \mathbf{a}) + ((\mathbf{c} \times \mathbf{a}) \times \mathbf{b}) = 0 . \quad \text{(Jacoby identity)} \tag{1.60}$$

This identity is related with the fact that heights of the triangle intersect in the one point.

Exercise† Show that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a}, \mathbf{c}) - \mathbf{c}(\mathbf{a}, \mathbf{b})$.

1.11.2 Area of parallelogram in $\mathbf{E}^2$ and determinant of 2×2 matrices

.

Let $\mathbf{a}, \mathbf{b}$ be two vectors in 2-dimensional vector space $\mathbf{E}^2$.

One can consider $\mathbf{E}^2$ as a plane in 3-dimensional Euclidean space $\mathbf{E}^3$. Let $\mathbf{n}$ be a unit vector in $\mathbf{E}^3$ which is orthogonal to $\mathbf{E}^2$. One can see $\mathbf{a} \times \mathbf{b}$ is proportional to the normal vector $\mathbf{n}$ to the plane $\mathbf{E}^2$:

$$\mathbf{a} \times \mathbf{b} = A(\mathbf{a}, \mathbf{b})\mathbf{n} \tag{1.61}$$

and area of parallelogram equals to the modulus of the coefficient $A(\mathbf{c}, \mathbf{b})$:

$$S(\mathbf{a}, \mathbf{b}) = |\mathbf{a} \times \mathbf{b}| = |A(\mathbf{a}, \mathbf{b})| \tag{1.62}$$

The normal unit vector $\mathbf{n}$ and coefficient $A(\mathbf{a}, \mathbf{b})$ are defined up to a sign: $\mathbf{n} \to -\mathbf{n}$, $A \to -A$. On the other hand the vector product $\mathbf{a} \times \mathbf{b}$ is defined up to a sign too: vector product depends on orientation. The answer for $\mathbf{a} \times \mathbf{b}$ is not changed if we perform calculations for vector product in an arbitrary basis $\{\mathbf{e}'_x, \mathbf{e}'_y, \mathbf{e}'_z\}$ which have the same orientation as the the basis $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ and $\mathbf{a} \times \mathbf{b} \mapsto\to -\mathbf{a} \times \mathbf{b}$. If we consider an arbitrary basis $\{\mathbf{e}'_x, \mathbf{e}'_y, \mathbf{e}'_z\}$ which have the orientation opposite to the orientation of the basis $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ (e.g. the basis $\{\mathbf{e}, \mathbf{f}, -\mathbf{n}\}$) then $A(\mathbf{a}, \mathbf{b}) \to$

$-A(\mathbf{a},\mathbf{b})$. The magnitude $A(\mathbf{a},\mathbf{b})$ is so called algebraic area of parallelogram. It can positive and negative.

If (a_1,a_2), (b_1,b_2) are coordinates of the vectors $\mathbf{a},\mathbf{b}$ in the basis $\{\mathbf{e},\mathbf{f}\}$: $\mathbf{a}=a_1\mathbf{e}+a_2\mathbf{f}$, $\mathbf{b}=b_1\mathbf{e}+b_2\mathbf{f}$ and according to (1.57)

$$\mathbf{a}\times\mathbf{b}=\det\begin{pmatrix}\mathbf{e} & \mathbf{f} & \mathbf{n}\\ a_1 & a_2 & 0\\ b_1 & b_2 & 0\end{pmatrix}=\mathbf{n}\det\begin{pmatrix}a_x & a_y\\ b_x & b_y\end{pmatrix} \tag{1.63}$$

Hence

$$A(\mathbf{a},\mathbf{b})=\det\det\begin{pmatrix}a_x & a_y\\ b_x & b_y\end{pmatrix}\ \text{ and } S(\mathbf{a},\mathbf{b})=\left|\det\begin{pmatrix}a_x & a_y\\ b_x & b_y\end{pmatrix}\right|. \tag{1.64}$$

We come to very beautiful formulae for relation between determinant of 2×2 matrix, area of parallelogram and vector product.

One can deduce this relation in other way:

Let $\mathbf{E}^2$ be a 2-dimensional Euclidean space. The function $A(\mathbf{a},\mathbf{b})$ defined by the relation (1.64) obeys the following conditions:

- It is anticommutative:
$$A(\mathbf{a},\mathbf{b})=-A(\mathbf{a},\mathbf{b}) \tag{1.65}$$
- It is bilinear
$$A(\lambda\mathbf{a}+\mu\mathbf{b},\mathbf{c})=\lambda A(\mathbf{a},\mathbf{c})+\mu A(\mathbf{b},\mathbf{c});\ A(\mathbf{c},\lambda\mathbf{a}+\mu\mathbf{b})=\lambda A(\mathbf{c},\mathbf{a})+\mu A(\mathbf{c},\mathbf{b}). \tag{1.66}$$
- and it obeys normalisation condition:
$$A(\mathbf{e},\mathbf{f})=\pm 1 \tag{1.67}$$
for an arbitrary orthonormal basis.
(Compare with conditions (1.49)—(1.53).)

One can see that these conditions define uniquely $A(\mathbf{a},\mathbf{b})$ and these are the conditions which define the determinant of the 2×2 matrix.

1.11.3 Volume of parallelepiped

The vector product of two vectors is related with area of parallelogram. What about a volume of parallelepiped formed by three vectors $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$?

Consider parallelepiped formed by vectors $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$. The parallelogram formed by vectors $\mathbf{b},\mathbf{c}$ is considered as a base of this parallelepiped.

Let θ be an angle between height and vector $\mathbf{a}$. It is just the angle between the vector $\mathbf{b} \times \mathbf{c}$ and the vector $\mathbf{a}$. Then the volume is equal to the length of the height multiplied on the area of the parallelogram, $V = Sh = S|\mathbf{a}|\cos\theta$, i.e. volume is equal to scalar product of the vectors $\mathbf{a}$ on the vector product of vectors $\mathbf{b}$ and $\mathbf{c}$:

$$V(\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}) = (\mathbf{a}, \mathbf{b} \times \mathbf{c}) = \left(a_x \mathbf{e}_x + a_y \mathbf{e}_y + a_z \mathbf{e}_z, \det \begin{pmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{pmatrix} \right)$$

$$= (a_x \mathbf{e}_x + a_y \mathbf{e}_y + a_z \mathbf{e}_z, (b_y c_z - b_z c_y)\mathbf{e}_x + (b_z c_x - b_x c_z)\mathbf{e}_y + (b_x c_y - b_y c_x)\mathbf{e}_z) =$$

$$a_x(b_y c_z - b_z c_y) + a_y(b_z c_x - b_x c_z) + a_z(b_x c_y - b_y c_x) = \begin{pmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{pmatrix}$$

We come to beautiful and useful formula:

$$(\mathbf{a}, [\mathbf{b} \times \mathbf{c}]) = \begin{pmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{pmatrix} . \tag{1.68}$$

Remark The volume of the parallelepiped if considered as a positive number equals to the modulus of the number $(\mathbf{a}, [\mathbf{b} \times \mathbf{c}])$. On the other hand often it is very useful to consider the volume as a real number (it could be positive and negative).

Exercise Consider the function $F(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{a}, \mathbf{b} \times \mathbf{c})$.

1. Show that $F(\mathbf{a}, \mathbf{b}, \mathbf{c}) = 0$ if and only if vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are linear dependent.

2. Show that for an arbitrary vector $\mathbf{a}$, $F(\mathbf{a}, \mathbf{a}, \mathbf{c}) = 0$.

3. Show that for arbitrary vectors $\mathbf{a}, \mathbf{b}$, $F(\mathbf{a}, \mathbf{b}, \mathbf{c}) = -F(\mathbf{a}, \mathbf{b}, \mathbf{c})$. Can you deduce 3) from the 2)?

2 Differential forms

2.1 Tangent vectors, curves, velocity vectors on the curve

Tangent vector is a vector $\mathbf{v}$ applied at the given point $\mathbf{p} \in \mathbf{E}^n$.

The set of all tangent vectors at the given point $\mathbf{p}$ is a vector space. It is called tangent space of $\mathbf{E}^3$ at the point $\mathbf{p}$ and it is denoted $T_{\mathbf{p}}(\mathbf{E}^n)$.

One can consider *vector field* on $\mathbf{E}^n$, i.e.a function which assigns to every point $\mathbf{p}$ vector $\mathbf{v}(\mathbf{p}) \in T_{\mathbf{p}}(\mathbf{E}^n)$.

It is instructive to study the conception of tangent vectors and vector fields on the curves and surfaces embedded in $\mathbf{E}^n$. We begin with curves.

A curve in $\mathbf{E}^n$ with parameter $t \in (a,b)$ is a continuous map

$$C\colon \qquad (a,b) \to \mathbf{E}^n \qquad \mathbf{r}(t) = (x^1(t), \dots, x^n(t)), \quad a < t < b \tag{2.1}$$

For example consider in $\mathbf{E}^2$ the curve

$$C\colon \qquad (0, 2\pi) \to \mathbf{E}^2 \qquad \mathbf{r}(t) = (R\cos t, R\sin t),\ 0 \leq t < 2\pi\,.$$

The image of this curve is the circle of the radius R. It can be defined by the equation:

$$x^2 + y^2 = R^2\,.$$

To distinguish between curve and its image we say that curve C in (2.1) is *parameterised* curve or *path*. We will call the image of the curve *unparameterised curve* (see for details the next subsection). It is very useful to think about parameter t as a "time" and consider parameterised curve like *point moving along a curve*. Unparameterised curve is the trajectory of the moving point. The using of word "curve" without adjective "parameterised" or "nonparameterised" sometimes is ambiguous.

Vectors tangent to curve—velocity vector

Let $\mathbf{r}(t) \quad \mathbf{r} = \mathbf{r}(t)$ be a curve in $\mathbf{E}^n$.

Velocity $\mathbf{v}(t)$ it is the vector

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \big(\dot{x}^1(t), \dots, \dots \dot{x}^n(t)\big) = \big(v^1(t), \dots, v^n(t)\big)$$

in $\mathbf{E}^n$. Velocity vector is *tangent vector to the curve.*

Let $C\colon \mathbf{r} = \mathbf{r}(t)$ be a curve and $\mathbf{r}_0 = \mathbf{r}(t_0)$ any given point on it. Then the set of all vectors tangent to the curve at the point $\mathbf{r}_0 = \mathbf{r}(t_0)$ is one-dimensional vector space $T_{\mathbf{r}_0}C$. It is linear subspace in vector space $T_{\mathbf{r}_0}C$. The points of the tangent space $T_{\mathbf{r}_0}C$ are the points of tangent line.

In the next section we will return to curves and consider them in more details.

Remark We consider by default only *smooth, regular* and *simple* curves. Curve $\mathbf{r}(t) = (x^1(t), \dots, x^n(t))$ is called smooth if all functions $x^i(t)$, $(i = 1, 2, \dots, n)$ are smooth functions (Function is called smooth if it has derivatives of arbitrary order.) Curve $\mathbf{r}(t)$ is called regular if velocity vector $\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt}$ is not equal to zero at all t. Simple curves are curves which have no intersection points.

2.2 Reparameterisation

One can move along trajectory with different velocities, i.e. one can consider different parameterisation. E.g. consider

$$
C_1\colon \quad \begin{cases} x(t) = t \\ y(t) = t^2 \end{cases} \quad 0 < t < 1\,, \qquad C_2\colon \quad \begin{cases} x(t) = \sin t \\ y(t) = \sin^2 t \end{cases} \quad 0 < t < \frac{\pi}{2}
$$

Images of these two parameterised curves are the same. In both cases point moves along a piece of the same parabola but with different velocities.

Definition

Two smooth curves $C_1\colon \quad \mathbf{r}_1(t)\colon (a_1, b_1) \to \mathbf{E}^n$ and $C_2\colon \quad \mathbf{r}_2(\tau)\colon (a_2, b_2) \to \mathbf{E}^n$ are called equivalent if there exists reparameterisation map:

$$
t(\tau)\colon (a_2, b_2) \to (a_1, b_1),
$$

such that

$$
r_2(\tau) = r_1(t(\tau)) \tag{2.2}
$$

Reparameterisation $t(\tau)$ is diffeomorphism, i.e. function $t(\tau)$ has derivatives of all orders and first derivative $t'(\tau)$ is not equal to zero.

E.g. curves in (2.2) are equivalent because a map $\varphi(t) = \sin t$ transforms first curve to the second.

Equivalence class of equivalent parameterised curves is called non-parameterised curve.

It is useful sometimes to distinguish curves in the same equivalence class which differ by orientation.

Definition Let curves C_1, C_2 be two equivalent curves. We say that they have same orientation (parameterisations $\mathbf{r}_1(t$ and $\mathbf{r}_(\tau)$ have the same orientation) if reparameterisation $t = t(\tau)$ has positive derivative, $t'(\tau) > 0$. We say that they have opposite orientation (parameterisations $\mathbf{r}_1(t$ and $\mathbf{r}_(\tau)$ have the opposite orientation) if reparameterisation $t = t(\tau)$ has negative derivative, $t'(\tau) < 0$.

Changing orientation means changing the direction of "walking" around the curve.

Equivalence class of equivalent curves splits on two subclasses with respect to orientation.

Non-formally: Two curves are equivalent curves (belong to the same equivalence class) if these parameterised curves (paths) have the same images. Two equivalent curves have the same image. They define the same set of points in $\mathbf{E}^n$. Different parameters correspond to moving along curve with different velocity. Two equivalent curves have opposite orientation If two parameterisations correspond to moving along the curve in different directions then these parameterisations define opposite orientation.

What happens with velocity vector if we change parameterisation? It changes its value, but it can change its direction only on opposite (If these parameterisations have opposite orientation of the curve):

$$\mathbf{v}(\tau) = \frac{d\mathbf{r}_2(\tau)}{d\tau} = \frac{d\mathbf{r}(t(\tau))}{d\tau} = \frac{dt(\tau)}{d\tau} \cdot \frac{d\mathbf{r}(t)}{dt}\Big|_{t=t(\tau)} \tag{2.3}$$

Or shortly: $\mathbf{v}(\tau)\big|_\tau = t_\tau(\tau)\mathbf{v}(t)\big|_{t=t(\tau)}$

We see that velocity vector is multiplied on the coefficient (depending on the point of the curve), i.e. velocity vectors for different parameterisations are collinear vectors.
(We call two vectors $\mathbf{a}, \mathbf{b}$ collinear, if they are proportional each other, i,e, if $\mathbf{a} = \lambda\mathbf{b}$.)

Example Consider following curves in $\mathbf{E}^2$:

$$C_1\colon \quad \begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases} ,0 < \theta < \pi, \qquad C_2\colon \quad \begin{cases} x = u \\ y = \sqrt{1-u^2} \end{cases} ,-1 < u < 1,$$

$$\begin{cases} x = \tan t \\ y = \frac{\sqrt{\cos 2t}}{\cos t} \end{cases} ,-\frac{\pi}{4} < t < \frac{\pi}{4} \tag{2.4}$$

These three parameterised curves,(paths) define the same non-parameterised curve: the upper piece of the circle: $x^2 + y^2 = 1, y > 0$. The reparameterisation $u(\theta) = \cos\theta$ transforms the second curve to the first one.

The reparameterisation $u(\theta) = \cos\theta$ transforms the second curve to the first one.

The reparameterisation $u(\theta) = \tan t$ transforms the second curve to the third one one: $\frac{\sqrt{\cos 2t}}{\cos t} = \frac{\sqrt{\cos^2 t - \sin^2 t}}{\cos t} = \sqrt{1-\tan^2 t}$.

Curves C_1, C_2 have opposite orientation because $u'(\theta) < 0$. Curves C_2, C_3 have the same orientation, because $u'(t) > 0$. Curves C_1 and C_2 have opposite orientations too (Why?).

In the first case point moves with constant pace $|\mathbf{v}(\theta)| = 1$ anti clock-wise "from right to left" from the point $A = (1, 0)$ to the point $B = (-1, 0)$. In the second case pace is not constant, but $v_x = 1$ is constant. Point moves clock-wise "from left to right", from the point $B = (-1, 0)$ to the point $A = (1, 0)$. In the third case point also moves clock-wise "from the left to right".

There are other examples in the Homeworks.

2.3 0-forms and 1-forms

Most of considerations of this and next subsections can be considered only for $\mathbf{E}^2$ or $\mathbf{E}^3$. All examples for differential forms is only for $\mathbf{E}^2$, $\mathbf{E}^3$.

0-form on $\mathbf{E}^n$ it is just function on $\mathbf{E}^n$ (all functions under consideration are differentiable)

Now we define 1-forms.

Definition Differential 1-form ω on $\mathbf{E}^n$ is a function on tangent vectors of $\mathbf{E}^n$, such that it is linear at each point:

$$\omega(\mathbf{r}, \lambda\mathbf{v}_1 + \mu\mathbf{v}_2) = \lambda\omega(\mathbf{r}, \mathbf{v}_1) + \mu\omega(\mathbf{r}, \mathbf{v}_2) \,. \tag{2.5}$$

Here $\mathbf{v}_1, \mathbf{v}_2$ are vectors tangent to $\mathbf{E}^n$ at the point $\mathbf{r}$, ($\mathbf{v}_1, \mathbf{v}_2 \in T_x\mathbf{E}^n$) (We recall that vector tangent at the point $\mathbf{r}$ means vector attached at the point $\mathbf{r}$). We suppose that ω is smooth function on points $\mathbf{r}$.

If $\mathbf{X}(\mathbf{r})$ is vector field and ω-1-form then evaluating ω on $\mathbf{X}(\mathbf{r})$ we come to the function $w(\mathbf{r}, \mathbf{X}(\mathbf{r}))$ on $\mathbf{E}^3$.

Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be a basis in $\mathbf{E}^n$ and $(x^1, \ldots, x^n)$ corresponding coordinates: an arbitrary point with coordinates $(x^1, \ldots, x^n)$ is assigned to the vector $\mathbf{r} = x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + \ldots x^n\mathbf{e}_n$ starting at the origin.

Translating basis vectors $\mathbf{e}_i$ $(i = 1, \ldots, n)$ from the origin to other points of $\mathbf{E}^n$ we come to vector field which we also denote $\mathbf{e}_i$ $(i = 1, \ldots, n)$. The value of vector field $\mathbf{e}_i$ at the point $(x^1, \ldots, x^n)$ is the vector $\mathbf{e}_i$ attached at this point (tangent to this point).

Let ω be an 1-form on $\mathbf{E}^n$. Consider an arbitrary vector field $\mathbf{A}(\mathbf{r}) = \mathbf{A}(x^1, \dots, x^n)$:

$$\mathbf{A}(\mathbf{r}) = A^1(\mathbf{r})\mathbf{e}_1 + \cdots + A^n(\mathbf{r})\mathbf{e}_n = \sum_{i=1}^{n} A^i(\mathbf{r})\mathbf{e}_i$$

Then by linearity

$$\omega(\mathbf{r}, \mathbf{A}(\mathbf{r})) = \omega(\mathbf{r}, A^1(\mathbf{r})\mathbf{e}_1 + \cdots + A^n(\mathbf{r})\mathbf{e}_n) = A^1\omega(\mathbf{r}, \mathbf{e}_1) + \cdots + A^n\omega(\mathbf{r}, \mathbf{e}_n)$$

Consider *basic* differential forms $dx^1, dx^2, \dots, dx^n$ such that

$$dx^i(\mathbf{e}_j) = \delta^i_j = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} . \tag{2.6}$$

Then it is easy to see that

$$dx^1(\mathbf{A}) = \mathbf{A}^1, dx^2(\mathbf{A}) = \mathbf{A}^2, ..., \text{i.e.} dx^i(\mathbf{A}) = A^i$$

Hence

$$\omega(\mathbf{r}, \mathbf{A}(\mathbf{r})) = \left(\omega_1(\mathbf{r})dx^1 + \omega_2(\mathbf{r})dx^2 + \cdots + \omega_n(\mathbf{r})dx^n\right)(\mathbf{A}(\mathbf{r}))$$

where components $\omega_i(\mathbf{r}) = \omega(\mathbf{r}, \mathbf{e}_i)$.

In the same way as an arbitrary vector field on $\mathbf{E}^n$ can be expanded over the basis $\{\mathbf{e}_i\}$ (see (2.3)), an arbitrary differential 1-form ω can be expanded over the basis forms(2.3)

$$\omega = \omega_1(x^1, \dots, x^n)dx^1 + \omega_2(x^1, \dots, x^n)dx^2 + \cdots + \omega_n(x^1, \dots, x^n)dx^n .$$

Example Consider in $\mathbf{E}^3$ a basis $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ and corresponding coordinates (x, y, z). Then

$$\begin{array}{l} dx(\mathbf{e}_x) = 1, dx(\mathbf{e}_y) = 0, dx(\mathbf{e}_z) = 0 \\ dy(\mathbf{e}_x) = 0, dy(\mathbf{e}_y) = 1, dy(\mathbf{e}_z) = 0 \\ dz(\mathbf{e}_x) = 0, dz(\mathbf{e}_y) = 0, dz(\mathbf{e}_z) = 1 \end{array} \tag{2.7}$$

The value of a differential 1-form $\omega = a(x, y, z)dx + b(x, y, z)dy + c(x, y, z)dz$ on vector field $\mathbf{X} = A(x, y, z)\mathbf{e}_x + B(x, y, z)\mathbf{e}_y + C(x, y, z)\mathbf{e}_z$ is equal to

$$\omega(\mathbf{r}, \mathbf{X}) = a(x, y, z)dx(\mathbf{X}) + b(x, y, z)dx(\mathbf{X}) + c(x, y, z)dx(\mathbf{X}) =$$

$$a(x,y,z)A(x,y,z)+b(x,y,z)B(x,y,z)+c(x,y,z)C(x,y,z)$$

It is very useful (see below) introduce for basic vectors new notations:

$$\mathbf{e}_i \mapsto \frac{\partial}{\partial x^i} \text{ for basic vectors } \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z \text{ in } \mathbf{E}^3\, \mathbf{e}_x \mapsto \frac{\partial}{\partial x} \ \mathbf{e}_y \mapsto \frac{\partial}{\partial y} \ \mathbf{e}_z \mapsto \frac{\partial}{\partial z}\,. \tag{2.8}$$

In these new notations the formula (2.3) looks like

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \quad .$$

and the formula (2.7) looks like

$$\begin{array}{c} dx\left(\frac{\partial}{\partial x}\right) = 1, dx\left(\frac{\partial}{\partial y}\right) = 0, dx\left(\frac{\partial}{\partial z}\right) = 0 \\ dy\left(\frac{\partial}{\partial x}\right) = 0, dy\left(\frac{\partial}{\partial y}\right) = 1, dy\left(\frac{\partial}{\partial z}\right) = 0 \\ dz\left(\frac{\partial}{\partial x}\right) = 0, dz\left(\frac{\partial}{\partial y}\right) = 0, dz\left(\frac{\partial}{\partial z}\right) = 1 \end{array}$$

It is very useful to introduce new notation for vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$.

In the next subsection we will consider the directional derivative of function along vector fields. The directional derivative will justify our new notations (2.8).

2.3.1 Vectors—directional derivatives of functions

Let $\mathbf{R}$ be a vector in $\mathbf{E}^n$ tangent to the point $\mathbf{r} = \mathbf{r}_0$ (attached at a point $\mathbf{r} = \mathbf{r}_0$). Define the operation of derivative of an arbitrary (differentiable) function at the point $\mathbf{r}_0$ along the vector $\mathbf{R}$— directional derivative of function f along the vector $\mathbf{R}$

Definition
Let $\mathbf{r}(t)$ be a curve such that

- $\mathbf{r}(t)\big|_{t=0} = \mathbf{r}_0$
- Velocity vector of the curve at the point $\mathbf{r}_0$ is equal to $\mathbf{R}$: $\frac{d\mathbf{r}(t)}{dt}\big|_{t=0} = \mathbf{R}$

Then directional derivative of function f with respect to the vector $\mathbf{R}$ at the point $\mathbf{r}_0$ $\partial_{\mathbf{R}} f\big|_{\mathbf{r}_0}$ is defined by the relation

$$\partial_{\mathbf{R}} f\big|_{\mathbf{r}_0} = \frac{d}{dt}\left(f\left(\mathbf{r}(t)\right)\right)\big|_{t=0}. \tag{2.9}$$

Using chain rule one come from this definition to the following important formula for the directional derivative:

$$\text{If } \mathbf{R} = \sum_{i=1}^{n} R^i \mathbf{e}_i \text{ then } \partial_{\mathbf{R}} f\big|_{\mathbf{r}_0} = \sum_{i=1}^{n} R^i \frac{\partial}{\partial x^i} f(x^1, \dots, x^n)\big|_{\mathbf{r}=\mathbf{r}_0} \tag{2.10}$$

It follows form this formula that

One can assign to every vector $\mathbf{R} = \sum_{i=1}^{n} R^i \mathbf{e}_i$ *the operation* $\partial_{\mathbf{R}} = R^1 \frac{\partial}{\partial x^1} + R^2 \frac{\partial}{\partial x^2} + \cdots + R^n \frac{\partial}{\partial x^n}$ *of taking directional derivative:*

$$\mathbf{R} = \sum_{i=1}^{n} R^i \mathbf{e}_i \mapsto \partial_{\mathbf{R}} = \sum_{i=1}^{n} R^i \frac{\partial}{\partial x^i} \tag{2.11}$$

Thus we come to notations (2.8). The symbols $\partial_x, \partial_y, \partial_z$ correspond to partial derivative with respect to coordinate x or y or z . Later we see that these new notations are very illuminating when we deal with arbitrary coordinates, such as polar coordinates or spherical coordinates, The conception of orthonormal basis is ill-defined in arbitrary coordinates, but one can still consider the corresponding partial derivatives. Vector fields $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ (or in new notation $\partial_x, \partial_y, \partial_z$) can be considered as a basis[6] in the space of all vector fields on $\mathbf{E}^3$.

An arbitrary vector field (2.3) can be rewritten in the following way:

$$\mathbf{A}(\mathbf{r}) = A^1(\mathbf{r})\mathbf{e}_1 + \cdots + A^n(\mathbf{r})\mathbf{e}_n = A^1(\mathbf{r})\frac{\partial}{\partial x^1} + A^2(\mathbf{r})\frac{\partial}{\partial x^2} + \cdots + A^n(\mathbf{r})\frac{\partial}{\partial x^n} \tag{2.12}$$

Differential on 0*-forms*

Now we introduce very important operation: Differential d which acts on 0-forms and transforms them to 1 forms.

$$\boxed{\begin{array}{l}\text{Differential}\\ \text{0-forms}\end{array}} \xrightarrow{d} \boxed{\begin{array}{l}\text{Differential}\\ \text{1-forms}\end{array}}$$

[6]Coefficients of expansion are functions, elements of algebra of functions, not numbers ,elements of field. To be more careful, these vector fields are basis of the *module* of vector fields on $\mathbf{E}^3$

Later we will learn how differential acts on 1-forms transforming them to 2-forms.

Definition Let $f = f(x)$-be 0-form, i.e. function on $\mathbf{E}^n$. Then

$$df = \sum_{i=1}^{n} \frac{\partial f(x^1, \dots, x^n)}{\partial x^i} dx^i \,. \tag{2.13}$$

The value of 1-form df on an arbitrary vector field (2.12) is equal to

$$df(\mathbf{A}) = \sum_{i=1}^{n} \frac{\partial f(x^1, \dots, x^n)}{x^i} dx^i(\mathbf{A}) = \sum_{i=1}^{n} \frac{\partial f(x^1, \dots, x^n)}{x^i} A^i = \partial_{\mathbf{A}} f \tag{2.14}$$

We see that *value of differential of* 0*-form* f *on an arbitrary vector field* $\mathbf{A}$ *is equal to directional derivative of function* f *with respect to the vector* $\mathbf{A}$.

The formula (2.14) defines df in invariant way without using coordinate expansions. Later we check straightforwardly the coordinate-invariance of the definition (2.13).

Exercise Check that

$$dx^i(\mathbf{A}) = \partial_{\mathbf{A}} x^i \tag{2.15}$$

Example If $f = f(x, y)$ is a function $(0 - form)$ on $\mathbf{E}^2$ then

$$df = \frac{\partial f(x, y)}{\partial x} dx + \frac{\partial f(x, y)}{\partial y} dy$$

and for an arbitrary vector field $\mathbf{A} = \mathbf{A} = A_x \mathbf{e}_x + \mathbf{A}_y \mathbf{e}_y = A_x(x, y)\partial_x + A_y(x, y)\partial_y$

$$df(\mathbf{A}) = \frac{\partial f(x, y)}{\partial x} dx(\mathbf{A}) + A_y(x, y) \frac{\partial f(x, y)}{\partial y} dy(\mathbf{A}) =$$

$$\mathbf{A}_x(x, y) \frac{\partial f(x, y)}{\partial x} + A_y(x, y) \frac{\partial f(x, y)}{\partial y} = \partial_{\mathbf{A}} f \,.$$

Example Find the value of 1-form $\omega = df$ on the vector field $\mathbf{A} = x\partial_x + y\partial_y$ if $f = \sin(x^2 + y^2)$.

$\omega(\mathbf{A}) = df(\mathbf{A})$. One can calculate it using formula (2.13) or using formula (2.14).

Solution (using (2.13)*)*:

$$\omega = df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 2x \cos(x^2 + y^2) dx + 2y \cos(x^2 + y^2) dy \,.$$

$$\omega(\mathbf{A}) = 2x\cos(x^2+y^2)dx(\mathbf{A}) + 2y\cos(x^2+y^2)dy(\mathbf{A}) =$$
$$2x\cos(x^2+y^2)A_x + 2y\cos(x^2+y^2)dA_y = 2(x^2+y^2)\cos(x^2+y^2)\,,$$

Another solution (using (2.14)*)*

$$df(\mathbf{A}) = \partial_{\mathbf{A}} f = A_x \frac{\partial f}{\partial x} + A_x \frac{\partial f}{\partial y} = 2(x^2+y^2)\cos(x^2+y^2)\,.$$

See other examples in Homeworks.

2.4 Differential 1-form in arbitrary coordinates

Why differential forms? Why so strange notations for vector fields.

If we use the technique of differential forms we in fact do not care about what coordinates we work in: calculations are the same in arbitrary coordinates.

2.4.1 Calculations in arbitrary coordinates *

Consider an arbitrary (local) coordinates $u^1, \dots, u^n$ on $\mathbf{E}^n$: $u^i = u^i(x^1, \dots, x^n)$, $i = 1, \dots, n$. Show first that

$$du^i = \sum_{k=1}^n \frac{\partial u^i(x^1, \dots, x^n)}{\partial x^k} dx^k\,. \tag{2.16}$$

It is enough to check it on basic fields:

$$du^i\left(\frac{\partial}{\partial x^m}\right) = \partial_{\left(\frac{\partial}{\partial x^m}\right)} u^i = \frac{\partial u^i(x^1, \dots, x^n)}{x^m} = \sum_{k=1}^n \frac{\partial u^i(x^1, \dots, x^n)}{\partial x^k} dx^k\left(\left(\frac{\partial}{\partial x^m}\right)\right) = \left(\frac{\partial}{\partial x^m}\right).$$

because (see (2.3)):

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}\,. \tag{2.17}$$

(We rewrite the formula (2.3) using new notations ∂_i instead $\mathbf{e}_i$). In the previous formula (2.3) we considered *cartesian* coordinates.

Show that the formula above is valid in an *arbitrary coordinates.*

One can see using chain rule that

$$\frac{\partial}{\partial u^i} = \frac{\partial x^1}{\partial u^i}\frac{\partial}{\partial x^1} + \frac{\partial x^2}{\partial u^i}\frac{\partial}{\partial x^2} + \dots + \frac{\partial x^n}{\partial u^i}\frac{\partial}{\partial x^n} = \sum_{k=1}^n \frac{\partial x^k}{\partial u^i}\frac{\partial}{\partial x^k} \tag{2.18}$$

Calculate the value of differential form du^i on vector field $\frac{\partial}{\partial u^j}$ using (2.16) and (2.18):

$$du^i\left(\frac{\partial}{\partial u^j}\right) = \sum_{k=1}^n \frac{\partial u^i(x^1, \dots, x^n)}{\partial x^k} dx^k \left(\sum_{r=1}^n \frac{\partial x^r}{\partial u^j}\frac{\partial}{\partial x^r}\right) = \tag{2.19}$$

$$\sum_{k,r=1}^{n} \frac{\partial u^i(x^1,\dots,x^n)}{\partial x^k}\frac{\partial x^r(u^1,\dots,u^n)}{\partial u^j} dx^k\left(\frac{\partial}{\partial x^r}\right) =$$

$$\sum_{k,r=1}^{n} \frac{\partial u^i(x^1,\dots,x^n)}{\partial x^k}\frac{\partial x^r(u^1,\dots,u^n)}{\partial u^j}\delta^k_r = \sum_{k=1}^{n}\frac{\partial x^k}{\partial u^j}\frac{\partial u^i}{\partial x^k} = \delta^j_i$$

We come to

$$du^i\left(\frac{\partial}{\partial u^j}\right) = \delta^i_j = \begin{cases} 1 \text{ if } i=j \\ 0 \text{ if } i\neq j \end{cases} . \tag{2.20}$$

We see that formula (2.17) has the same appearance in arbitrary coordinates. In other words it is invariant with respect to an arbitrary transformation of coordinates.

Exercise Check straightforwardly the invariance of the definition (2.13). In coordinates $(u^1,\dots,u^n)$

Solution We have to show that the formula (2.13) does not changed under changing of coordinates $u^i = u^i(x^1,\dots,x^n)$.

$$df = \sum_{i=1}^{n}\frac{\partial f(x^1,\dots,x^n)}{\partial x^i}dx^i = \sum_{i=1,k}^{n}\frac{\partial f(x^1,\dots,x^n)}{\partial x^i}\frac{\partial x^i}{\partial u^k}du^k == \sum_{i=1}^{n}\frac{\partial f}{\partial u^k}du^k\,,$$

because $\sum_{i=1}^{n}\frac{\partial f(x^1,\dots,x^n)}{\partial x^i}\frac{\partial x^i}{\partial u^k} = \frac{\partial f}{\partial u^k}$

Example

Consider more in detail $\mathbf{E}^2$. (For $\mathbf{E}^3$ considerations are the same, just calculations little bit more complicated) Let u, v be an arbitrary coordinates in $\mathbf{E}^2$, $u = u(x,y), v = v(x,y)$.

$$du = \frac{\partial u(x,y)}{\partial x}dx + \frac{\partial u(x,y)}{\partial y}dy, \qquad dv = \frac{\partial v(x,y)}{\partial x}dx + \frac{\partial v(x,y)}{\partial y}dy \tag{2.21}$$

and

$$\partial_u = \frac{\partial x(u,v)}{\partial u}\partial_x + \frac{\partial y(u,v)}{\partial u}\partial_y,\ \ \partial_v = \frac{\partial x(u,v)}{\partial v}\partial_x + \frac{\partial y(u,v)}{\partial v}\partial_y \tag{2.22}$$

(As always sometimes we use notation ∂_u instead $\frac{\partial}{\partial u}$, ∂_x instead $\frac{\partial}{\partial x}$ e.t.c.) Then

$$\begin{gathered} du(\partial_u) = 1, du(\partial_v) = 0 \\ dv(\partial_w) = 0, dv(\partial_v) = 1 \end{gathered} \tag{2.23}$$

This follows from the general formula but it is good exercise to repeat the previous calculations for this case:

$$du(\partial_u) = \left(\frac{\partial u(x,y)}{\partial x}dx + \frac{\partial u(x,y)}{\partial y}dy\right)\left(\frac{\partial x(u,v)}{\partial u}\partial_x + \frac{\partial y(u,v)}{\partial u}\partial_y\right) =$$

$$\frac{\partial u(x,y)}{\partial x}\frac{\partial x(u,v)}{\partial u} + \frac{\partial u(x,y)}{\partial y}\frac{\partial y(u,v)}{\partial u} = \frac{\partial x(u,v)}{\partial u}\frac{\partial u(x,y)}{\partial x} + \frac{\partial y(u,v)}{\partial u}\frac{\partial u(x,y)}{\partial y} = 1$$

We just apply chain rule to the function $u = u(x, y) = u(x(u, v), y(u, v))$:
Analogously

$$du(\partial_v) = \left(\frac{\partial u(x,y)}{\partial x}dx + \frac{\partial u(x,y)}{\partial y}dy\right)\left(\frac{\partial x(u,v)}{\partial v}\partial_x + \frac{\partial y(u,v)}{\partial v}\partial_y\right)$$

$$\frac{\partial u(x,y)}{\partial x}\frac{\partial x(u,v)}{\partial v} + \frac{\partial u(x,y)}{\partial y}\frac{\partial y(u,v)}{\partial v} = \frac{\partial x(u,v)}{\partial v}\frac{\partial u(x,y)}{\partial x} + \frac{\partial y(u,v)}{\partial v}\frac{\partial u(x,y)}{\partial y} = 0$$

The same calculations for dv.

2.4.2 Calculations in polar coordinates

Example (Polar coordinates) Consider polar coordinates in $\mathbf{E}^2$:

$$\begin{cases} x(r,\varphi) = r\cos\varphi \\ y(r,\varphi) = r\sin\varphi \end{cases} \qquad (0 \le \varphi < 2\pi, 0 \le r < \infty),$$

Respectively

$$\begin{cases} r(x,y) = \sqrt{x^2+y^2} \\ \varphi = \arctan\frac{y}{x} \end{cases} . \tag{2.24}$$

We have that for basic 1-forms

$$dr = r_x dx + r_y dy = \frac{x}{\sqrt{x^2+y^2}}dx + \frac{y}{\sqrt{x^2+y^2}}dy = \frac{xdx + ydy}{r} \tag{2.25}$$

and

$$d\varphi = \varphi_x dx + \varphi_y dy = \frac{-ydx}{x^2+y^2} + \frac{xdy}{x^2+y^2}dx = \frac{xdy - ydx}{r^2} \tag{2.26}$$

Respectively

$$dx = x_r dr + x_\varphi d\varphi = \cos\varphi dr - r\sin\varphi d\varphi$$

and

$$dy = y_r dr + y_\varphi d\varphi = \sin\varphi dr + r\cos\varphi d\varphi \tag{2.27}$$

For basic vector fields

$$\partial_r = \frac{\partial x}{\partial r}\partial_x + \frac{\partial y}{\partial r}\partial_y = \cos\varphi\partial_x + \sin\varphi\partial_y = \frac{x\partial_x + y\partial_y}{r},$$

$$\partial_\varphi = \frac{\partial x}{\partial\varphi}\partial_x + \frac{\partial y}{\partial\varphi}\partial_y = -r\sin\varphi\partial_x + r\cos\varphi\partial_y = x\partial_y - y\partial_x, \tag{2.28}$$

respectively

$$\partial_x = \frac{\partial r}{\partial x}\partial_r + \frac{\partial\varphi}{\partial x}\partial_\varphi = \frac{x}{r}\partial_r - \frac{y}{r^2}\partial_\varphi$$

and

$$\partial_y = \frac{\partial r}{\partial y}\partial_r + \frac{\partial\varphi}{\partial y}\partial_\varphi = \frac{y}{r}\partial_r + \frac{x}{r^2}\partial_\varphi \tag{2.29}$$

Example Calculate the value of forms $\omega = xdx + ydy$ and $\sigma = xdy - ydx$ on vector fields $\mathbf{A} = x\partial_x + y\partial_y$, $\mathbf{B} = x\partial_y - y\partial_x$. Perform calculations in Cartesian and in polar coordinates.

In Cartesian coordinates:

$$\omega(\mathbf{A}) = xdx(x\partial_x + y\partial_y) + ydy(x\partial_x + y\partial_y) = x^2 + y^2,\ \omega(\mathbf{B}) = xdx(\mathbf{B}) + ydy(\mathbf{B}) = 0,$$

$$\sigma(\mathbf{A}) = xdy(\mathbf{A}) - ydx(\mathbf{A}) = 0,\ \sigma(\mathbf{B}) = xdy(\mathbf{B}) - ydx(\mathbf{B}) = x^2 + y^2\,.$$

Now perform calculations in polar coordinates. According to relation (2.25)

$$\omega = xdx + ydy = rdr,\ \sigma = xdy - ydx = r^2 d\varphi$$

and according to relations (2.28) and (2.29)

$$\mathbf{A} = x\partial_x + y\partial_y = r\partial_r,\ \mathbf{B} = x\partial_y - y\partial_x = \partial_\varphi$$

Hence $\omega(\mathbf{A}) = rdr(\mathbf{A}) = r^2 = x^2 + y^2, \omega(\mathbf{B} = rdr(\partial_\varphi) = 0$,

$$\sigma(\mathbf{A}) = r^2 d\varphi(r\partial_r) = 0, \quad \sigma(\mathbf{B}) = r^2 d\varphi(\partial_\varphi) = r^2 = x^2 + y^2\,.$$

Answers coincide.

Example. Let $F = x^4 - y^4$ and vector field $\mathbf{A} = r\partial_r$. Calculate $\partial_{\mathbf{A}} F$. We have to transform form from Cartesian coordinates to polar or vector field from polar to Cartesian.

In Cartesian coordinates: $\mathbf{A} = r\frac{\partial}{\partial r} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$. Hence $\partial_{\mathbf{A}} F = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)(x^4 - y^4) = 4x^4 - 4y^4$. Or using the fact that $\partial_{\mathbf{A}} F = dF$ we have that $\partial_{\mathbf{A}}(x^4 - y^4) = d(x^4 - y^4)(\mathbf{A}) =$

$$(4x^3dx - 4y^3dy)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) = 4x^3dx\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) - 4y^3dy\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) = 4x^4 - 4y^4\,.$$

In polar coordinates $F = x^4 - y^4 = (x^2 - y^2)(x^2 + y^2) = r^2(r^2\cos^\varphi - r^2\sin^2\varphi) = r^4\cos 2\varphi$. $\partial_{\mathbf{A}} F = r\frac{\partial}{\partial r}(r^4\cos 2\varphi) = 4r^4\cos 2\varphi = 4(x^4 - y^4)$. Or using 1-forms:

$$\partial_{\mathbf{A}} F = dF(\mathbf{A}) = d(r^4\cos 2\varphi)(\mathbf{A}) = (4r^3\cos 2\varphi - 2r^4\sin 2\varphi d\varphi)(r\partial_r) = 4r^4\cos 2\varphi\,.$$

Example Calculate the value of form $\omega = \frac{xdy-ydx}{x^2+y^2}$ on the vector field $\mathbf{A} = \partial_\varphi$. We have to transform form from Cartesian coordinates to polar or vector field from polar to Cartesian.

$$\frac{xdy - ydx}{x^2+y^2} = d\varphi, \quad \omega(\mathbf{A}) = d\varphi(\partial_\varphi) = 1$$

or

$$\partial_\varphi = x\partial_y - y\partial_x,\ \omega(\mathbf{A}) = \frac{xdy(x\partial_y - y\partial_x) - ydx(x\partial_y - y\partial_x)}{x^2+y^2} = 1\,.$$

2.5 Integration of differential 1-forms over curves

Let $\omega = \omega_1(x^1,\dots,x^n)dx^1 + \cdots + \omega_1(x^1,\dots,x^n)dx^n = \sum_{i=1}^n \omega_i dx^i$ be an arbitrary 1-form in $\mathbf{E}^n$

and $C\colon \mathbf{r} = \mathbf{r}(t), t_1 \le t \le t_2$ be an arbitrary smooth curve in $\mathbf{E}^n$.

One can consider the value of one form ω on the velocity vector field $\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt}$ of the curve:

$$\omega(\mathbf{v}(t)) = \sum_{i=1}^n \omega_i\left(x^1(t),\dots,x^n(t)) dx^i(\mathbf{v}(t)\right) = \sum_{i=1}^n \omega_i\left(x^1(t),\dots,x^n(t)\right)\frac{dx^i(t)}{dt}$$

We define now integral of 1-form ω over the curve C.

Definition The integral of the form $\omega = \omega_1(x^1,\dots,x^n)dx^1 + \cdots + \omega_n(x^1,\dots,x^n)dx^n$ over the curve $C\colon \quad \mathbf{r} = \mathbf{r}(t) \quad t_1 \le t \le t_2$ is equal to the integral of the function $\omega(\mathbf{v}(t))$ over the interval $t_1 \le t \le t_2$:

$$\int_C \omega = \int_{t_1}^{t_2} \omega(\mathbf{v}(t))dt = \int_{t_1}^{t_2} \left(\sum_{i=1}^n \omega_i\left(x^1(t),\dots,x^n(t)\right)\frac{dx^i(t)}{dt}\right) dt\,. \quad (2.30)$$

Proposition *The integral $\int_C \omega$ does not depend on the choice of coordinates on $\mathbf{E}^n$. It does not depend (up to a sign) on parameterisation of the curve: if $C\colon \quad \mathbf{r} = \mathbf{r}(t) \quad t_1 \le t \le t_2$ is a curve and $t = t(\tau)$ is an arbitrary reparameterisation, i.e. new curve $C'\colon \quad \mathbf{r}'(\tau) = \mathbf{r}(t(\tau)) \quad \tau_1 \le \tau \le \tau_2$, then $\int_C \omega = \pm \int_C' \omega$:*

$$\int_C \omega = \int_{C'} \omega, \quad \textit{if orientaion is not changed, i.e. if } t'(\tau) > 0$$

and

$$\int_C \omega = -\int_{C'} \omega, \quad \textit{if orientaion is changed, i.e. if } t'(\tau) < 0$$

If reparameterisation changes the orientation then starting point of the curve becomes the ending point and vice versa.

Proof of the Proposition Show that integral does not depend (up to a sign) on the parameterisation of the curve. Let $t(\tau)$ ($\tau_1 \leq t \leq \tau_2$) be reparameterisation. We come to the new curve C': $\mathbf{r}'(\tau) = \mathbf{r}(t(\tau))$. Note that the new velocity vector $\mathbf{v}'(\tau) = \frac{dr(t(\tau))}{d\tau} = t'(\tau)\mathbf{v}(t(\tau))$. Hence $\omega(\mathbf{v}'(\tau)) = w(\mathbf{v}(t(\tau)))t'(\tau)$. For the new curve C'

$$\int_{C'} \omega = \int_{\tau_1}^{\tau_2} \omega(\mathbf{v}'(\tau))d\tau = \int_{\tau_1}^{\tau_2} \omega(\mathbf{v}(t(\tau))\frac{dt(\tau)}{d\tau}d\tau = \int_{t(\tau_1)}^{t(\tau_2)} \omega(\mathbf{v}(t))dt$$

$t(\tau_1) = t_1$, $t(\tau_2) = t_2$ if reparameterisation does not change orientation and $t(\tau_1) = t_2$, $t(\tau_2) = t_1$ if reparameterisation changes orientation.

Hence $\int_{C'} w = \int_{t_1}^{t_2)} \omega(\mathbf{v}(t))dt = \int_C \omega$ if orientation is not changed and $\int_{C'} w = \int_{t_2}^{t_1)} \omega(\mathbf{v}(t))dt = -\int_{t_1}^{t_2)} \omega(\mathbf{v}(t))dt = -\int_C \omega$ is orientation is changed.

Example

Let

$$\omega = a(x,y)dx + b(x,y)dy$$

be 1-form in $\mathbf{E}^2$ (x, y–are usual Cartesian coordinates). Let C: $\mathbf{r} = \mathbf{r}(t)$ $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$, $t_1 \leq t \leq t_2$ be a curve in $\mathbf{E}^2$.

Consider velocity vector field of this curve

$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} = \begin{pmatrix} x_t(t) \\ y_t(t) \end{pmatrix} = x_t\partial_x + y_t\partial_y \tag{2.31}$$

($x_t = \frac{dx(t)}{dt}$, $y_t = \frac{dy(t)}{dt}$).

One can consider the value of one form ω on the velocity vector field $\mathbf{v}(t)$ of the curve: $\omega(\mathbf{v}) = a(x(t), y(t))dx(\mathbf{v}) + b(x(t), y(t))dy(\mathbf{v}) =$

$$a(x(t), y(t))x_t(t) + b(x(t), y(t))y_t(t)\,.$$

The integral of the form $\omega = a(x,y)dx + b(x,y)dy$ over the curve C: $\mathbf{r} = \mathbf{r}(t)$ $t_1 \leq t \leq t_2$ is equal to the integral of the function $\omega(\mathbf{v}(t))$ over the interval $t_1 \leq t \leq t_2$:

$$\int_C \omega = \int_{t_1}^{t_2} \omega(\mathbf{v}(t))dt = \int_{t_1}^{t_2} \left(a(x(t), y(t))\frac{dx(t)}{dt} + b(x(t), y(t))\frac{dy(t)}{dt} \right) dt\,. \tag{2.32}$$

Example Consider an integral of the form $\omega = 3dy + 3y^2dx$ over the curve $C\colon \mathbf{r}(t) \begin{cases} x = \cos t \\ y = \sin t \end{cases}, 0 \leq t \leq \pi/2$. ($C$ is the arc of the circle $x^2 + y^2 = 1$ defined by conditions $x, y \geq 0$).

Velocity vector $\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \begin{pmatrix} v_x(t) \\ v_y(t) \end{pmatrix} = \begin{pmatrix} x_t(t) \\ y_t(t) \end{pmatrix} = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$. The value of the form on velocity vector is equal to

$$\omega(\mathbf{v}(t)) = 3y^2(t)v_x(t) + 3v_y(t) = 3\sin^2 t(-\sin t) + 3\cos t = 3\cos t - 3\sin^3 t$$

and

$$\int_C \omega = \int_0^{\frac{\pi}{2}} w(\mathbf{v}(t))dt = \int_0^{\frac{\pi}{2}} (3\cos t - 3\sin^3 t)dt = 3\left(\sin t + \cos t - \frac{\cos^3 t}{3}\right)\Big|_0^{\frac{\pi}{2}}$$

Now consider integrals of the same form ω over three curves which differ by reparameterisation.

Example Consider 1-form $\omega = xdy - ydx$ and curve C—- upper half of the circle $x^2 + y^2 = R^2$, $(y \geq 0)$.

We have the image of the curve not the parameterised curve. Consider different parameterisations of this curve:

$$\mathbf{r}_1(t)\colon \begin{cases} x = R\cos t \\ y = R\sin t \end{cases}, 0 \leq t \leq \pi\,, \ \mathbf{r}_2(t)\colon \begin{cases} x = R\cos\Omega t \\ y = R\sin\Omega t \end{cases}, 0 \leq t \leq \frac{\pi}{\Omega}\,, (\Omega > 0)$$

and

$$\mathbf{r}_3(t)\colon \begin{cases} x = t \\ y = \sqrt{R^2 - t^2} \end{cases}, -R \leq t \leq R,, \tag{2.33}$$

All these curves are the same image. If $\Omega = 1$ the second curve coincides with the first one. First and second curve have the same orientation (reparameterisation $t \mapsto \Omega t$) The third curve have orientation opposite to first and second (reparameterisation $t \mapsto \cos t$, the derivative $\frac{d\cos t}{dt} < 0$).

Calculate integrals $\int_{C_1} \omega$, $\int_{C_2} \omega$, $\int_{C_3} \omega$

$$\int_{C_1} \omega = \int_0^{\pi} (xy_t - yx_t)dt = \int_0^{\pi} (R^2\cos^2 t + R^2\sin^2 t)dt = \pi R^2$$

$$\int_{C_2} \omega = \int_0^{\frac{\pi}{\Omega}} (xy_t - yx_t)dt = \int_0^{\pi} (R^2\Omega \cos^2 \Omega t + R^2\Omega \sin^2 \Omega t)dt = \pi R^2 .$$

These answers coincide: both parameterisation have the same orientation.

For the third parameterisation:

$$\int_{C_3} \omega = \int_0^R (xy_t - yx_t)dt = \int_0^R \left(t \left(\frac{-t}{\sqrt{R^2 - t^2}} \right) - \sqrt{R^2 - t^2} \right) dt =$$

$$-R^2 \int_0^R \frac{dt}{\sqrt{R^2 - t^2}} = -R^2 \int_0^1 \frac{du}{\sqrt{1 - u^2}} = -\pi R^2$$

We see that the sign is changed.

Finally consider the integral of the form $\omega = xdy - ydx$ over the semicircle in polar coordinates instead Cartesian coordinates, We have that in polar coordinates semicircle is $\begin{cases} r(t) = R \\ \varphi(t) = t \end{cases}$, $0 \le t \le \pi$. The form $\omega = xdy - ydx = r\cos\varphi d(r\sin\varphi) - r\sin\varphi d(r\sin\varphi) = r^2 d\varphi$ and $\mathbf{v}(t) = (r_t, \varphi_t) = (0,1)$, i.e. $\mathbf{v}(t) = \partial_\varphi$. We have that $\omega(\mathbf{v}(t)) = r(t)^2 d\varphi(\partial_\varphi) = R^2$. Hence $\int_C \omega = \int_0^\pi R^2 dt = \pi R^2$. Answer is the same: The value of integral does not change if we change coordinates in the plane.

For other examples see Homeworks.

2.6 Integral over curve of exact form

1-form ω is called exact if there exists a function f such that $\omega = df$.

Theorem

Let ω be an exact 1-form in $\mathbf{E}^n$, $\omega = df$.

Then the integral of this form over an arbitrary curve C: $\mathbf{r} = \mathbf{r}(t)$ $t_1 \le t \le t_2$ is equal to the difference of the values of the function f at starting and ending points of the curve C:

$$\int_C \omega = f\big|_{\partial C} = f(\mathbf{r}_2) - f(\mathbf{r}_1) , \quad \mathbf{r}_1 = \mathbf{r}(t_1), \mathbf{r}_2 = \mathbf{r}(t_2) . \tag{2.34}$$

Proof: $\int_C df = \int_{t_1}^{t_2} df(\mathbf{v}(t)) = \int_{t_1}^{t_2} \frac{d}{dt} f(\mathbf{r}(t))dt = f(\mathbf{r}(t))|_{t_1}^{t_2}$.

Example Calculate an integral of the form $\omega = 3x^2(1+y)dx + x^3 dy$ over the arc of the semicircle $x^2 + y^2 = 1, y \ge 0$.

One can calculate the integral naively using just the formula (2.32): Choose a parameterisation of C,e.g., $x = \cos t, y = \sin t$, then $\mathbf{v}(t) = -\sin t\partial_x + \cos t\partial_x$ and $\omega(\mathbf{v}(t)) = (3x^2(1+y)dx + x^3dy)(-\sin t\partial_x + \cos t\partial_y) = -3\cos^2 t(1+\sin t)\sin t + \cos^3 t \cdot \cos t$ and

$$\int_C \omega = \int_0^\pi (-3\cos^2 t \sin t - 3\cos^2 t \sin^2 t + \cos^4 t)dt = ...$$

Calculations are little bit long.

But for the form $\omega = 3x^2(1+y)dx + x^3dy$ one can calculate the integral in a much more efficient way noting that it is an exact form:

$$\omega = 3x^2(1+y)dx + x^3dy = d\left(x^3(1+y)\right) \tag{2.35}$$

Hence it follows from the Theorem that

$$\int_C \omega = f(\mathbf{r}(\pi)) - f(\mathbf{r}(0)) = x^3(1+y)\Big|_{x=1,y=0}^{x=-1,y=0} = -2 \tag{2.36}$$

Remark If we change the orientation of curve then the starting point becomes the ending point and the ending point becomes the starting point.— The integral changes the sign in accordance with general statement, that integral of 1-form over parameterised curve is defined up to reparameterisation.

Corollary *The integral of an exact form over an arbitrary closed curve is equal to zero.*

Proof. According to the Theorem $\int_C \omega = \int_C df = f\big|_{\partial C} = 0$, because the starting and ending points of closed curve coincide.

Example. Calculate the integral of 1-form $\omega = x^5dy + 5x^4ydx$ over the ellipse $x^2 + \frac{y^2}{9} = 1$.

The form $\omega = x^5dy + 5x^4ydx$ is exact form because $\omega = x^5dy + 5x^4ydx = d(x^5y)$. Hence the integral over ellipse is equal to zero, because it is a closed curve.

Remark The remarkable Theorem and Corollary of this section works only for exact forms. Of course not any form is an exact form (see exercises in Homeworks and subsection 2.9 below) E.g. 1-form $xdy - ydx$ is not an exact form[7].

[7] if $xdy - ydx = df = f_xdx + f_ydy$, then $f_y = x$ and $f_x = -y$. We see that on one hand $f_{xy} = (f_x)_y = -1$ and on the other hand $f_{yx} = (f_y)_x = 1$. Contradiction.

2.7 † Differential 2-forms in $\mathbf{E}^2$

We considered detailed definition of 1-forms. Now we give some formal approach to describe 2-forms.

Differential forms on $\mathbf{E}^2$ is an expression obtained by adding and multiplying functions and differentials dx, dy. These operations obey usual associativity and distributivity laws but multiplications is not moreover of one-forms on each other is *anticommutative*:

$$\omega \wedge \omega' = -\omega' \wedge \omega \quad \text{if } \omega, \omega' \text{ are 1-forms} \tag{2.37}$$

In particular

$$dx \wedge dy = -dy \wedge dx, dx \wedge dx = 0, dy \wedge dy = 0 \tag{2.38}$$

Example If $\omega = xdy + zdx$ and $\rho = dz + ydx$ then

$$\omega \wedge \rho = (xdy + zdx) \wedge (dz + ydx) = xdy \wedge dz + zdx \wedge dz + xydy \wedge dx$$

and

$$\rho \wedge \omega = (dz + ydx) \wedge (xdy + zdx) = xdz \wedge dy + zdz \wedge dx + xydx \wedge dy = -\omega \wedge \rho$$

Changing of coordinates. If $\omega = a(x, y)dx \wedge dy$ be two form and $x = x(u, v), y = y(u, v)$ new coordinates then $dx = x_u du + x_v dv$, $dy = y_u du + y_v dv$ ($x_u = \frac{\partial x(u,v)}{\partial u}$, $x_v = \frac{\partial x(u,v)}{\partial v}$, $y_u = \frac{\partial y(u,v)}{\partial u}$, $y_v = \frac{\partial y(u,v)}{\partial v}$). and

$$a(x, y)dx \wedge dy = a\left(x(u, v), y(u, v)\right)(x_u du + x_v dv) \wedge (y_u du + y_v dv) = \tag{2.39}$$

$$a\left(x(u, v), y(u, v)\right)(x_u du + x_v dv)\left(x_u y_v du \wedge dv + x_v y_u dv \wedge du\right) =$$

$$a\left(x(u, v), y(u, v)\right)(x_u y_v - x_v y_u) du \wedge dv$$

Example Let $\omega = dx \wedge dy$ then in polar coordinates $x = r\cos\varphi, y = r\sin\varphi$

$$dx \wedge dy = (\cos\varphi dr - r\sin\varphi d\varphi) \wedge (\sin\varphi dr + r\cos\varphi d\varphi) = rdr \wedge d\varphi \tag{2.40}$$

2.8 † 0-forms (functions) $\xrightarrow{d}$ 1-forms $\xrightarrow{d}$ 2-forms

We introduced differential d of functions (0-forms) which transform them to 1-form. It obeys the following condition:

- d: is linear operator: $d(\lambda f + \mu g) = \lambda df + \mu dg$
- $d(fg) = df \cdot g + f \cdot dg$

Now we introduce differential on 1-forms such that

- d: is linear operator on 1-forms also
- $d(fw) = df \wedge w + fdw$

- $ddf = 0$

Remark Sometimes differential d is called *exterior differential.*

Perform calculations using this definition and (2.37):

$$d\omega = d(\omega_1 dx + \omega_2 dy) = dw_1 \wedge dx + dw_2 \wedge dy = \left(\frac{\partial\omega_1(x,y)}{\partial x}dx + \frac{\partial\omega_1(x,y)}{\partial y}dy\right) \wedge dx +$$

$$\left(\frac{\partial\omega_2(x,y)}{\partial x}dx + \frac{\partial\omega_2(x,y)}{\partial y}dy\right) \wedge dy = \left(\frac{\partial\omega_2(x,y)}{\partial x} - \frac{\partial\omega_1(x,y)}{\partial y}\right) dx \wedge dy$$

Example Consider 1-form $\omega = xdy$. Then $d\omega = d(xdy) = dx \wedge dy$.

2.9 †Exact and closed forms

We know that it is very easy to integrate exact 1-forms over curves (see the subsection "Integral over curve of exact form")

How to know is the 1-form exact or no?

Definition We say that one form ω is *closed* if two form $d\omega$ is equal to zero.

Example One-form $xdy + ydx$ is closed because $d(xdy + ydx) = 0$.

It is evident that exact 1-form is closed:

$$\omega = d\rho \Rightarrow d\omega = d(d\rho) = d \circ d\rho = 0 \tag{2.41}$$

We see that the condition that form is closed is necessary condition that form is exact.

So if $d\omega \neq 0$, i.e. the form is not closed, then it is not exact.

Is this condition sufficient? Is it true that a closed form is exact?

In general the answer is: *No.*

E.g. we considered differential 2-form

$$\omega = \frac{xdy - ydx}{x^2 + y^2} \tag{2.42}$$

defined in $\mathbf{E}^2\backslash 0$. It is closed, but it is not exact (See non-compulsory exercises 11,12,13 in the Homework 6).

How to recognize for 1-form ω is it exact or no?

Inverse statement (Poincaré lemma) is true if 1-form is well-defined in $\mathbf{E}^2$:

A closed 1-form ω in $\mathbf{E}^n$ is exact if it is well-defined at all points of $\mathbf{E}^n$, i.e. if it is differentiable function at all points of $\mathbf{E}^n$.

Sketch a proof for 1-form in $\mathbf{E}^2$: if ω is defined in whole $\mathbf{E}^2$ then consider the function

$$F(\mathbf{r}) = \int_{C_\mathbf{r}} \omega \tag{2.43}$$

where we denote by $C_{\mathbf{r}}$ an arbitrary curve which starts at origin and ends at the point $\mathbf{r}$. It is easy to see that the integral is well-defined and one can prove that $\omega = df$.

The explicit formula for the function (2.43) is the following: If $\omega = a(x,y)dx + b(x,y)dy$ then $F(x,y) = \int_0^1 (a(tx,ty)x + b(tx,ty)y)\,dt$.

Exercise Check by straightforward calculation that $\omega = dF$ (See exercise 14 in Homework 6).

2.10 †Integration of two-forms. Area of the domain

We know that 1-form is a linear function on tangent vectors. If $\mathbf{A}, \mathbf{B}$ are two vectors attached at the point $\mathbf{r}_0$, i.e. tangent to this point and ω, ρ are two 1-forms then one defines the value of $\omega \wedge \rho$ on $\mathbf{A}, B$ by the formula

$$\omega \wedge \rho(\mathbf{A}, \mathbf{B}) = \omega(\mathbf{A})\rho(B) - \omega(B)\rho(A) \tag{2.44}$$

We come to bilinear anisymmetric function on tangent vectors. If $\sigma = a(x,y)dx \wedge dy$ is an arbitrary two form then this form defines bilinear form on pair of tangent vectors: $\sigma(\mathbf{A}, \mathbf{B}) =$

$$a(x,y)dx \wedge dy(A,B) = a(x,y)\left(dx(\mathbf{A})dy(\mathbf{B}) - dx(\mathbf{B})dy(\mathbf{A})\right) = a(x,y)(A_xB_y - A_yB_y) \tag{2.45}$$

One can see that in the case if $a = 1$ then right hand side of this formula is nothing but the area of parallelogram spanned by the vectors $\mathbf{A}, \mathbf{B}$.

This leads to the conception of integral of form over domain.

Let $\omega = a(x)dx \wedge dy$ be a two form and D be a domain in $\mathbf{E}^2$. Then by definition

$$\int_D \omega = \int_D a(x,y)dxdy \tag{2.46}$$

If $\omega = dx \wedge dy$ then

$$\int_D w = \int_D (x,y)dxdy = \text{Area of the domain } D \tag{2.47}$$

The advantage of these formulae is that we do not care about coordinates[8]

Example Let D be a domain defined by the conditions

$$\begin{cases} x^2 + y^2 \le 1 \\ y \ge 0 \end{cases} \tag{2.49}$$

[8]If we consider changing of coordinates then jacobian appears: If u, v are new coordinates, $x = x(u,v)$, $y = y(u,v)$ are new coordinates then

$$\int a(x,y)dxdy = \int a(x(u,v), y(u,v)) \det \begin{pmatrix} x_u & x_v \\ x_u & x_v \end{pmatrix} dudv \tag{2.48}$$

In formula(5.9) it appears under as a part of coefficient of differential form.

Calculate $\int_D dx \wedge dy$.

$\int_D dx \wedge dy = \int_D dxdy =$ area of the $D = \frac{\pi}{2}$.

If we consider polar coordinates then according (2.40)

$$dx \wedge dy = rdr \wedge d\varphi$$

Hence $\int_D dx \wedge dy = \int_D rdr \wedge d\varphi = \int_D rdrd\varphi = \int_0^1 \left(\int_0^\pi d\varphi\right) rdr = \pi \int_0^1 rdr = \pi/2$.

Another example

Example Let D be a domain in $\mathbf{E}^2$ defined by the conditions

$$\begin{cases} \frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} \leq 1 \\ y \geq 0 \end{cases} \tag{2.50}$$

D is domain restricted by upper half of the ellipse and x-axis. Ellipse has the centre at the point $(c, 0)$. Its area is equal to $S = \int_D dx \wedge dy$. Consider new variables x', y': $x = c + ax', y = by'$. In new variables domain D becomes the domain from the previous example:

$$\frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} = {x'}^2 + {y'}^2$$

and $dx \wedge dy = abdx' \wedge dy'$. Hence

$$S = \int_{\frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} \leq 1, y \geq 0} dx \wedge dy = ab \int_{x'^2 + y'^2 \leq 1, y' \geq 0} dx' \wedge dy' = \frac{\pi ab}{2} \tag{2.51}$$

Theorem 2 (Green formula) Let ω be 2-form such that $\omega = d\omega'$ and D be a domain–interior of the closed curve C. Then

$$\int_D \omega = \int_C \omega' \tag{2.52}$$

3 Curves in Euclidean space. Curvature

3.1 Curves. Velocity and acceleration vectors

We already study velocity vector of curves. Consider now acceleration vector $\mathbf{a} = \frac{d^2\mathbf{r}(t)}{dt^2}$. For curve $\mathbf{r} = \mathbf{r}(t)$ in $\mathbf{E}^n$ we have

$$\mathbf{v} = \frac{d\mathbf{r}(t)}{dt}, \ v^i = \frac{dx^i(t)}{dt}, \ (i = 1, 2, \dots, n) \ ,$$

and

$$\mathbf{a} = \frac{d\mathbf{v}(t)}{dt} = \frac{d^2\mathbf{r}(t)}{dt^2}, a^i = \frac{d^2x^i(t)}{dt^2}, (i = 1, 2, \dots, n) \ . \tag{3.1}$$

Velocity vector $\mathbf{v}(t)$ is tangent to the curve. In general acceleration vector is not tangent to the curve. One can consider decomposition of acceleration vector $\mathbf{a}$ on tangential and normal component:

$$\mathbf{a} = \mathbf{a}_{tangent} + \mathbf{a}_{\perp}, \tag{3.2}$$

where $\mathbf{a}_{tangent}$ is the vector tangent to the curve (collinear to velocity vector) and $\mathbf{a}_{\perp}$ is orthogonal to the tangent vector (orthogonal to the velocity vector). The vector $\mathbf{a}_{\perp}$ is called normal acceleration vector of the curve [9].

Example Consider a curve

$$C\colon \quad \begin{cases} x = R\cos\Omega t \\ y = R\sin\Omega t \end{cases}, \tag{3.3}$$

If we consider parameter t as a time then we have the point which moves over circle of the radius R with angular velocity Ω. We see that

$$\mathbf{v} = \begin{pmatrix} -R\Omega\sin\Omega t \\ R\Omega\cos\Omega t \end{pmatrix}, \mathbf{a} = -\begin{pmatrix} R\Omega^2\cos\Omega t \\ R\Omega^2\sin\Omega t \end{pmatrix} = -\Omega^2\mathbf{r}(t)$$

Speed is constant: $|\mathbf{v}| = R\Omega$. Acceleration is perpendicular to the velocity. (It is just *centripetal acceleration*.)

What happens if speed is increasing, or decreasing, i.e. if angular velocity is not constant? One can see that in this case tangential acceleration is not equal to zero, i.e. the velocity and acceleration are not orthogonal to each other.

Analyze the meaning of an angle between velocity and acceleration vectors for an arbitrary parameterised curve $\mathbf{r} = \mathbf{r}(t)$. For this purpose consider the equation for speed: $|\mathbf{v}|^2 = (\mathbf{v}, \mathbf{v})$ and differentiate it:

$$\frac{d|\mathbf{v}|^2}{dt} = \frac{d}{dt}(\mathbf{v}(t), \mathbf{v}(t)) = 2(\mathbf{v}(t), \mathbf{a}(t)) = 2|\mathbf{v}(t)||\mathbf{a}(t)|\cos\theta(t), \tag{3.4}$$

where θ is an angle between velocity vector and acceleration vector.

We formulate the following

Proposition

Suppose that parameter t is just time. We see from this formula that if point moves along the curve $\mathbf{r}(t)$ then

[9] Component of acceleration orthogonal to the velocity vector sometimes is called also *centripetal acceleration*

- speed is increasing in time if and only if the angle between velocity and acceleration vector is acute, i.e. tangential acceleration has the same direction as a velocity vector:

$$\frac{d|\mathbf{v}|^2}{dt} > 0 \Leftrightarrow (\mathbf{v}, \mathbf{a}) > 0 \Leftrightarrow \cos\theta > 0 \Leftrightarrow \mathbf{a}_{tang} = \lambda\mathbf{v} \text{ with } \lambda > 0\,. \quad (3.5)$$

- speed is decreasing in time if and only if the angle between velocity and acceleration vector is obtuse, i.e. tangential acceleration has the direction opposite to the direction of a velocity vector.

$$\frac{d|\mathbf{v}|^2}{dt} < 0 \Leftrightarrow (\mathbf{v}, \mathbf{a}) < 0 \Leftrightarrow \cos\theta < 0 \Leftrightarrow \mathbf{a}_{tang} = \lambda\mathbf{v} \text{ with } \lambda < 0\,. \quad (3.6)$$

- speed is constant in time if and only if the velocity and acceleration vectors are orthogonal to each other, i..e. tangential acceleration is equal to zero.

$$\frac{d|\mathbf{v}|^2}{dt} = 0 \Leftrightarrow (\mathbf{v}, \mathbf{a}) = 0 \Leftrightarrow \cos\theta = 0 \Leftrightarrow \mathbf{a}_{tang} = 0\,. \quad (3.7)$$

Example Consider the curve $\mathbf{r}(t)$: $\begin{cases} x(t) = v_x t \\ y(t) = v_y t - \frac{gt^2}{2} \end{cases}$ It is path of the point moving under the gravity force with initial velocity $\mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$. One can see that the curve is parabola: $y = \left(\frac{v_y}{v_x}\right) x - \left(\frac{gv_y^2}{v_x^2}\right) x^2$. We have that $\mathbf{v}(t) = \begin{pmatrix} v_x \\ v_y - gt \end{pmatrix}$ and acceleration vector $\mathbf{a} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$. Suppose that $v_y > 0$. $(\mathbf{v}, \mathbf{a}) = -g(v_y - gt)$. Then at the highest point (vertex of the parabola) ($t = v_y/g$) acceleration is orthogonal to the velocity. For $t < v_y/g$ angle between acceleration and velocity vectors is obtuse. Speed is decreasing. For $t > v_y/g$ angle between acceleration and velocity vectors is acute. Speed is increasing.

3.2 Behaviour of acceleration vector under reparameterisation

How acceleration vector changes under changing of parameterisation of the curve?

Let $C\colon \quad \mathbf{r} = \mathbf{r}(t), t_1 \leq t \leq t_2$ be a curve and $t = t(\tau)$ reparametrisation of this curve. We know that for new parameterised curve $C'\colon \quad \mathbf{r}'(\tau) = \mathbf{r}(t(\tau)), \tau_1 \leq \tau \leq \tau_2$ velocity vector $\mathbf{v}'(\tau)$ is collinear to the velocity vector $\mathbf{v}(t)$ (see (2.3)):

$$\mathbf{v}'(\tau) = \frac{d\mathbf{r}'(\tau)}{d\tau} = \frac{d\mathbf{r}(t(\tau))}{d\tau} = \frac{dt(\tau)}{d\tau}\frac{d\mathbf{r}(t(\tau))}{dt} = t_\tau \mathbf{v}(t(\tau))$$

Taking second derivative we see that for acceleration vector:

$$\mathbf{a}'(\tau) = \frac{d^2\mathbf{r}'(\tau)}{d\tau^2} = \frac{d\mathbf{v}'(\tau)}{d\tau} = \frac{d}{d\tau}\left(t_\tau \mathbf{v}(t(\tau))\right) = t_{\tau\tau}\mathbf{v}(t(\tau)) + t_\tau^2 \mathbf{a}(t(\tau)) \quad (3.8)$$

Under reparameterisation acceleration vector in general changes its direction: new acceleration vector becomes linear combination of old velocity and acceleration vectors: direction of acceleration vector does not remain unchanged [10].

We know that acceleration vector can be decomposed on tangential and normal components (see (3.2)). Study how tangential and normal components change under reparameterisation.

Decompose left and right hand sides of the equation (3.8) on tangential and orthogonal components:

$$\mathbf{a}'(\tau)_{tangent} + \mathbf{a}'(\tau)_\perp = t_{\tau\tau}\mathbf{v}(t) + t_\tau^2\left(\mathbf{a}(t)_{tangent} + \mathbf{a}(t)_\perp\right)$$

Then comparing tangential and orthogonal components we see that new tangential acceleration is equal to

$$\mathbf{a}'(\tau)_{tangent} = t_{\tau\tau}\mathbf{v}(t) + t_\tau^2 \mathbf{a}(t)_{tangent} \quad (3.9)$$

and normal acceleration is equal to

$$\mathbf{a}'(\tau)_\perp = t_\tau^2 \mathbf{a}(t)_\perp \quad (3.10)$$

The magnitude of normal (centripetal) acceleration under changing of parameterisation is multiplied on the t_τ^2. Now recall that magnitude of velocity vector under reparameterisation is multiplied on t_τ. We come to very interesting and important observation:

[10] The plane spanned by velocity and acceleration vectors remains unchanged.(This plane is called osculating plane.)

Observation

$$\text{The magnitude } \frac{|\mathbf{a}_\perp|}{|\mathbf{v}^2|} \text{ remains unchanged under reparameterisation.} \quad (3.11)$$

We come to the expression which is independent of parameterisation: it must have deep mechanical and geometrical meaning. We see later that it is nothing but curvature.

3.3 Length of the curve

If $\mathbf{r}(t), a \le t \le b$ is a parameterisation of the curve L and $\mathbf{v}(t)$ velocity vector then length of the curve is equal to the integral of of $|\mathbf{v}(t)|$ over curve:

$$\text{Length of the curve } L = \int_a^b |\mathbf{v}(t)|dt = \quad (3.12)$$

$$\int_a^b \sqrt{\left(\frac{dx^1(t)}{dt}\right)^2 + \left(\frac{dx^2(t)}{dt}\right)^2 + \cdots + \left(\frac{dx^n(t)}{dt}\right)^2} dt\,.$$

Note that formula above is *reparameterisation* invariant. The length of the image of the curve does not depend on parameterisation. This corresponds to our intuition.

Proof Consider curve $\mathbf{r}_1 = \mathbf{r}_1(t)$, $a_1 \le t \le b_1$. Let $t = t(\tau)$, $a_2 < \tau < b_2$ be another parameterisation of the curve $\mathbf{r} = \mathbf{r}(t)$, In other words we have two different parameterised curves $\mathbf{r}_1 = \mathbf{r}_1(t), a_1 \le t \le b_1$ and $\mathbf{r}_2 = \mathbf{r}_1(t(\tau))$, $a_2 \le \tau \le \mathbf{b}_2$ such that their images coincide (See (2.2)). Then under reparameterisation velocity vector is multiplied on t_τ

$$\mathbf{v}_2(\tau) = \frac{d\mathbf{r}_2}{d\tau} = \frac{dt}{d\tau}\frac{d\mathbf{r}_1}{dt} = t_\tau(\tau)\mathbf{v}_1(t(\tau))$$

Hence

$$L_1 = \int_{a_1}^{b_1} |\mathbf{v}_1(t)|dt = \int_{a_2}^{b_2} |\mathbf{v}_1(t)|\frac{dt(\tau)}{d\tau}d\tau = \int_{a_2}^{b_2} |t_\tau \mathbf{v}_1(t)|d\tau = \int_{a_2}^{b_2} |\mathbf{v}_2(\tau)|d\tau = L_2\,, \quad (3.13)$$

i.e. length of the curve does not change under reparameterisation.

If C: $\mathbf{r} = \mathbf{r}(t)$ $t_1 \le t \le t_2$ is a curve in $\mathbf{E}^2$ then its length is equal to

$$L_C = \int_{t_1}^{t_2} |\mathbf{v}(t)|dt = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx(t)}{dt}\right)^2 + \left(\frac{dy(t)}{dt}\right)^2} dt \quad (3.14)$$

3.4 Natural parameterisation of the curves

Non-parameterised curve can be parameterised in many different ways.

Is there any distinguished parameterisation? Yes, it is.

Definition A natural parameter $s = s(t)$ on the curve $\mathbf{r} = r(t)$ is a parameter which defines the length of the arc of the curve between initial point $\mathbf{r}(t_1)$ and the point $\mathbf{r}(t)$.

If a natural parameter s is chosen we say that a curve $\mathbf{r} = \mathbf{r}(s)$ is given in natural parameterisation.

Write down explicit formulae for natural parameter.

Let $C : \mathbf{r}(t), a < t < b$ be a curve in $\mathbf{E}^n$. As always we suppose that it is smooth and regular curve: (i.e. $\mathbf{r}(t)$ has derivatives of arbitrary order, and velocity vector $\mathbf{v} \neq 0$.

Then it follows from (3.12) that

$$s(t) = \{\text{length of the arc of the curve between points } \mathbf{r}(a) \text{ and } \mathbf{r}(t)\} \quad (3.15)$$

$$= \int_a^t |\mathbf{v}(t')|dt' =$$

$$= \int_a^t \sqrt{\left(\frac{dx^1(t')}{dt'}\right)^2 + \left(\frac{dx^2(t')}{dt'}\right)^2 + \cdots + \left(\frac{dx^n(t')}{dt'}\right)^2} dt' \,. \quad (3.16)$$

(As always we suppose that it is smooth and regular curve: (i.e. $\mathbf{r}(t)$ has derivatives of arbitrary order, and velocity vector $\mathbf{v} \neq 0$.)

Example Consider circle: $x = R\cos t, y = R\sin t$ in $\mathbf{E}^2$. Then we come to the obvious answer

$$s(t) = \{\text{length of the arc of the circle between points } \mathbf{r}(0) \text{ and } \mathbf{r}(t)\} = Rt =$$

$$\int_0^t \sqrt{\left(\frac{dx(t')}{dt'}\right)^2 + \left(\frac{dy(t')}{dt'}\right)^2} dt' = \int_0^t \sqrt{R^2 \sin^2 t' + R^2 \cos^2 t'} dt' = \int_a^t R dt' = Rt$$

$s = Rt$. Hence in natural parameterisation $x = R\cos\frac{s}{R}$, $y = R\sin\frac{s}{R}$.

Remark If we change an initial point then a natural parameter changes on a constant.

For example if we choose as a initial point for the circle above a point $\mathbf{r}(t_1)$ for $t_1 = -\frac{\pi}{2}$, then the length of the arc between points $\mathbf{r}(-\frac{\pi}{2})$ and $\mathbf{r}(0)$ is equal to $R\frac{\pi}{2}$ and

$$s'(t) = s(t) + R\frac{\pi}{2}\,.$$

Another
Example Consider arc of the parabola $x = t, y = t^2, 0 < t < 1$:

$$s(t) = \{\text{length of the arc of the curve for parameter less or equal to } t\} = \tag{3.17}$$

$$\int_0^t \sqrt{\left(\frac{dx(\tau)}{d\tau}\right)^2 + \left(\frac{dy(\tau)}{d\tau}\right)^2} d\tau =$$

$$\int_0^t \sqrt{1+4\tau^2} d\tau = \frac{t\sqrt{1+4t^2}}{2} + \frac{1}{4}\log\left(2t + \sqrt{1+4t^2}\right)$$

The first example was very simple. The second is harder to calculate [11]. In general case natural parameter is not so easy to calculate. But its notion is very important for studying properties of curves.

Natural parameterisation is distinguished. Later we will often use the following very important property of natural parameterisation:

Proposition *If a curve is given in natural parameterisation then*

- *the speed is equal to* 1

$$(\mathbf{v}(s), \mathbf{v}(s)) \equiv 1, \quad \text{i.e. } |\mathbf{v}(s)| \equiv 1\,, \tag{3.18}$$

- *acceleration is orthogonal to velocity, i.e. tangential acceleration is equal to zero:*

$$(\mathbf{v}(s), \mathbf{a}(s)) = 0\,, \quad \text{i.e. } \mathbf{a}_{tangent} = 0\,. \tag{3.19}$$

Proof: For an arbitrary parameterisation $|\mathbf{v}(t)| = \frac{dL(t)}{dt}$, where $L(t)$ is a length of the curve. In the case of natural parameter $L(s) = s$, i.e. $|\mathbf{v}(t)| = \frac{dL(t)}{dt} = 1$. We come to the first relation.

The second relation means that value of the speed does not change (see (3.4) and (3.7)).

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[11] Denote by $I = \int_0^t \sqrt{1+4\tau^2} d\tau$. Then integrating by parts we come to:

$$I = t\sqrt{1+4t^2} - \int \frac{4\tau^2}{\sqrt{1+4\tau^2}} d\tau = t\sqrt{1+4t^2} - I + \int \frac{1}{\sqrt{1+4\tau^2}} d\tau\,.$$

Hence

$$I = \frac{t\sqrt{1+4t^2}}{2} + \frac{1}{2}\int \frac{1}{\sqrt{1+4\tau^2}} d\tau\,.$$

and we come to the answer.

3.5 Curvature. Curvature of curves in $\mathbf{E}^2$ and $\mathbf{E}^3$

How to find invariants of non-parameterised curve, i.e. magnitudes which depend on the points of non-parameterised curve but which do not depend on parameterisation?

Answer at the first sight looks very simple: Consider the distinguished natural parameterisation $\mathbf{r} = \mathbf{r}(s)$ of the curve. Then arbitrary functions on $x^i(s)$ and its derivatives do not depend on parameterisation. But the problem is that it is not easy to calculate natural parameter explicitly (See e.g. calculations of natural parameter for parabola in the previous subsection). So it is preferable to know how to construct these magnitudes in arbitrary parameterisation, i.e. construct functions $f(\frac{dx^i}{dt}, \frac{d^2x^i}{dt^2}, \dots)$ such that they *do not depend on parameterisation.*

We define now curvature. First formulate reasonable conditions on curvature:

- it has to be a function of the points of the curve
- it does not depend on parameterisation
- curvature of the line must be equal to zero
- curvature of the circle with radius R must be equal to $1/R$

We first give definition of curvature in natural parameterisation. Then study how to calculate it for a curve in an arbitrary parameterisation.

For a given non-parameterised curve consider natural parameterisation $\mathbf{r} = \mathbf{r}(s)$. We know already that velocity vector has length 1 and acceleration vector is orthogonal to curve in natural parameterisation (see (3.18) and (3.19)). It is just normal (centripetal) acceleration.

Definition. The curvature of the curve in a given point is equal to the modulus (length) of acceleration vector (normal acceleration) in natural parameterisation. Namely, let $\mathbf{r}(s)$ be natural parameterisation of this curve. Then curvature at every point $\mathbf{r}(s)$ of the curve is equal to the length of acceleration vector:

$$k = |\mathbf{a}(s)|, \qquad \mathbf{a}(s) = \frac{d^2\mathbf{r}(s)}{ds^2} \tag{3.20}$$

First check that it corresponds to our intuition (see reasonable conditions above)

It does not depend on parameterisation by definition.

It is evident that for the line in normal parameterisation $x^i(s) = x^i_0 + b^i s$ ($\sum b^i b^i = 1$) the acceleration is equal to zero.

Now check that the formula (3.20) gives a natural answer for circle. For circle of radius R in natural parameterisation

$$\mathbf{r} = \mathbf{r}(s) = (x(s), y(s)), \quad \text{where} \quad x(s) = R\cos\frac{s}{R}, \quad y(s) = R\sin\frac{s}{R}$$

(length of the arc of the angle θ of the circle is equal to $s = R\theta$.) Then

$$\mathbf{a}(s) = \frac{d\mathbf{r}^2(s)}{ds^2} = \left(-\frac{1}{R}\cos\frac{s}{R}, -\frac{1}{R}\sin\frac{s}{R}\right)$$

and for curvature

$$k = |\mathbf{a}(s)| = \frac{1}{R} \tag{3.21}$$

we come to the answer which agrees with our intuition.

Geometrical meaning of curvature: One can see from this example that $\frac{1}{k}$ is *just a radius of the circle which has second order touching to curve.*(See the subsection "Second order contact" (this is not compulsory))

3.6 Curvature of curve in an arbitrary parameterisation.

Let curve be given in an arbitrary parameterisation. How to calculate curvature. One way is to go to natural parameterisation. But in general it is very difficult (see the example of parabola in the subsection "Natural parameterisation").

We do it in another more elegant way.

Proposition *Curvature of the curve in terms of an arbitrary parameterisation* $\mathbf{r} = \mathbf{r}(t)$ *is given by the formula:*

$$k = \frac{|\mathbf{a}_\perp(t)|}{|\mathbf{v}(t)|^2} = \frac{\textit{Area of parallelogram } \Pi_{\mathbf{v},\mathbf{a}} \textit{ formed by the vectors } \mathbf{a}, \mathbf{v}}{|\mathbf{v}|^3}, \tag{3.22}$$

where $\mathbf{v}(t) = d\mathbf{r}(t)/dt$ *is velocity vector and* $\mathbf{a}_\perp(t)$ *is normal acceleration.*

Proof of the Proposition

Prove first that $k = \frac{|\mathbf{a}_\perp(t)|}{|\mathbf{v}(t)|^2}$. Note that in natural parameterisation speed is equal to 1 and acceleration is orthogonal to curve: $\mathbf{a} = \mathbf{a}_\perp$, $|\mathbf{v}| = 1$ (see

(3.18), (3.19)). Hence in natural parameterisation the ration $\frac{|\mathbf{a}_\perp|}{|\mathbf{v}|^2}$ is equal just to modulus of acceleration vector, i.e. to the curvature (3.20). On the other hand according to the observation (3.11) (see the end of the subsection "Velocity and acceleration vectors") the ratio $\frac{|\mathbf{a}_\perp|}{|\mathbf{v}|^2} = \frac{|\mathbf{a}_\perp|}{(\mathbf{v},\mathbf{v})}$ *does not* depend on parameterisation. Hence curvature is defined by the formula $k = \frac{|\mathbf{a}_\perp(t)|}{|\mathbf{v}(t)|^2}$ in an arbitrary parameterisation.

Advantage of the formula $k = \frac{|\mathbf{a}_\perp(t)|}{|\mathbf{v}(t)|^2}$ is that it is given in an arbitrary parameterisation. Disadvantage of this formula is that we still do not know how to calculate $\mathbf{a}_\perp(t)$. Do the next step. Note that

$$\frac{|\mathbf{a}_\perp(t)|}{|\mathbf{v}|^2} = \frac{|\mathbf{a}_\perp(t)| \cdot |\mathbf{v}|}{|\mathbf{v}|^3} =$$

$$\frac{|\mathbf{a}_\perp(t)|}{|\mathbf{v}|^2} = \frac{|\mathbf{a}_\perp(t)| \cdot |\mathbf{v}|}{|\mathbf{v}|^3} = \frac{\text{Area of parallelogram } \Pi_{\mathbf{v},\mathbf{a}} \text{ formed by the vectors } \mathbf{a},\mathbf{v}}{|\mathbf{v}|^3}. \tag{3.23}$$

Thus we proved formula (3.22). We express the curvature in terms of area of the parallelogram $\Pi_{\mathbf{v},\mathbf{a}}$ in an arbitrary parameterisation. We have that under an arbitrary change of parameterisation $t = t(\tau)$

$$\begin{aligned} \mathbf{v} &\mapsto t_\tau \mathbf{v} \\ \mathbf{a}_\perp &\mapsto t_\tau^2 \mathbf{a}_\perp \\ \text{Area of parallelogram}\Pi_{\mathbf{v},\mathbf{a}} &\mapsto t_\tau^3 \text{Area of parallelogram}\Pi_{\mathbf{v},\mathbf{a}} \end{aligned} \tag{3.24}$$

Numerator and denominator of the fraction, which is in the RHS of the equation (3.23) are multiplied on t_τ^3. The fraction, i.e. curvature does not change.

We know how to calculate area of parallelogram spanned by the vectors $\mathbf{a}, \mathbf{v}$. In particularly it is easy to do for $\mathbf{E}^3$ and $\mathbf{E}^2$, where this is just the magnitude of vector product (see the formulae for vector product in the subsections 1.11.1 and 1.11. 2):

$$k = \frac{\text{Area of parallelogram } \Pi_{\mathbf{v},\mathbf{a}} \text{ formed by the vectors } \mathbf{a},\mathbf{v}}{|\mathbf{v}|^3} = \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{|\mathbf{v}(t)|^3}, \tag{3.25}$$

if curve is in $\mathbf{E}^3$.

In the case if curve is in $\mathbf{E}^2$ then formula for curvature is

$$k = \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{|\mathbf{v}(t)|^3} = \frac{|v_x a_y - v_y a_x|}{(v_x^2 + v_y^2)^{\frac{3}{2}}} = \frac{|v_x a_y - v_y a_x|}{(v_x^2 + v_y^2)^{\frac{3}{2}}} =$$

$$= \frac{|x_t y_{tt} - y_t x_{tt}|}{(x_t^2 + y_t^2)^{\frac{3}{2}}} \qquad \text{(if curve is in } \mathbf{E}^2) \tag{3.26}$$

This is workable formula.

In general case if curve is in $\mathbf{E}^n$ then to calculate the area S of parallelogram note that $S = |\mathbf{v}||\mathbf{a}||\sin\theta|$ where $|\mathbf{v}||\mathbf{a}|\cos\theta = (\mathbf{v}, \mathbf{a})$. Hence $S = |\mathbf{v}||\mathbf{a}|\sqrt{1-\cos^2\theta} = \sqrt{\mathbf{v}^2\mathbf{a}^2 - (\mathbf{v}\cdot\mathbf{a})^2}$ and curvature is equal to

$$k = \frac{\text{Area of parallelogram formed by the vectors } \mathbf{v} \text{ and } \mathbf{a}}{\text{Cube of the speed}} = \frac{\sqrt{\mathbf{v}^2\mathbf{a}^2 - (\mathbf{v}\cdot\mathbf{a})^2}}{|\mathbf{v}|^3} \tag{3.27}$$

Remark . Of course one can come to formulae (3.27), (3.25) and (3.6) by "brute force" making straightforward attack. Instead considering explicitly natural parameterisation of the curve we just try to rewrite the formula in definition (3.20) in arbitrary parameterisation using chain rule. The calculations are not transparent. Try to do it.

Consider examples of calculating curvature for curves in $\mathbf{E}^2$ and $\mathbf{E}^3$.

Example Consider a curve C_f: $\mathbf{r}(t)$: $\begin{cases} x = t \\ y = f(t) \end{cases}$ (It is parameterisationn of graph of the function $f = f(x)$). Calculate curvature of this curve. We see that $\mathbf{v}(t) = \begin{pmatrix} 1 \\ f'(t) \end{pmatrix}$, $\mathbf{a}(t) = \begin{pmatrix} 0 \\ f''(t) \end{pmatrix}$ and we have for the curvature that

$$k = \frac{|x_t y_{tt} - y_t x_{tt}|}{(x_t^2 + y_t^2)^{\frac{3}{2}}} = k = \frac{|f''(t)|}{(1 + f'(t)^2)^{\frac{3}{2}}} \tag{3.28}$$

Example. Consider circle of the radius R, $x^2 + y^2 = R^2$. Take any parameterisation, e.g. $x = R\cos t, y = R\sin t$. Then $\mathbf{v} = (-R\cos t, R\sin t)$, $\mathbf{a} = (-R\sin t, -R\cos t)$. Applying the formula (3.26) we come to

$$k = \frac{|x_t y_{tt} - y_t x_{tt}|}{(x_t^2 + y_t^2)^{\frac{3}{2}}} = \frac{|R^2\cos^2 t + R^2\sin^2 t|}{(R^2\cos^2 t + R^2\sin^2 t)^{\frac{3}{2}}} = \frac{R^2}{R^3} = \frac{1}{R}$$

:

Example Consider ellipse $\mathbf{r}(t)$: $\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$, $0 \leq t < 2\pi$. Then $\mathbf{v}(t) = \begin{pmatrix} -a\sin t \\ b\cos t \end{pmatrix}$, $\mathbf{a}(t) = \begin{pmatrix} -a\cos t \\ -b\sin t \end{pmatrix}$ and for curvature we have

$$k = \frac{|x_t y_{tt} - y_t x_{tt}|}{(x_t^2 + y_t^2)^{\frac{3}{2}}} = k = \frac{|ab\sin^2 t + ab\cos^2 t|}{(a^2\sin^2 t + b^2\cos^2 t)^{\frac{3}{2}}} = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{\frac{3}{2}}} \,. \tag{3.29}$$

In particular we see that at the points $(\pm a, 0)$ $(t = 0, \pi)$ curvature is equal to $k = \frac{ab}{b^3} = \frac{a}{b^2}$ and at the points $(0, \pm b)$ $(t = \pm\frac{\pi}{2})$ curvature is equal to $k = \frac{ab}{a^3} = \frac{b}{a^2}$.

Example Consider helix

$$\mathbf{r}(t)\colon \quad \begin{cases} x = R\cos t \\ y = R\sin t \\ z = ct \end{cases} \tag{3.30}$$

We see that velocity and acceleration vectors are equal to

$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \begin{pmatrix} -R\sin t \\ R\cos t \\ c \end{pmatrix}, \quad \mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \begin{pmatrix} -R\cos t \\ R\sin t \\ 0 \end{pmatrix}$$

One can calculate curvature using the formula (3.25):

$$k = \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{|\mathbf{v}(t)|^3} = \frac{\left|\det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -R\sin t & R\cos t & c \\ -R\cos t & -R\sin t & 0 \end{pmatrix}\right|}{(R^2+c^2)^{\frac{3}{2}}} =$$

$$\frac{\sqrt{c^2R^2+R^4}}{\left(\sqrt{R^2+c^2}\right)^3} = \frac{R\sqrt{c^2+R^2}}{\left(\sqrt{R^2+c^2}\right)^3} = \frac{R}{R^2+c^2}\,.$$

Speed is constant: $|\mathbf{v}(t)| = \sqrt{R^2\Omega^2 + c^2t^2}$. Velocity vector is orthogonal to acceleration vector. This is evident without calculations since speed is constant. (Acceleration vector is orthogonal not only to the velocity vector but to an arbitrary vector at the surface of the cylinder $x^2 + y^2 = R^2$ since it is orthogonal to vertical vectors and to velocity vector.)

How to calculate curvature. We may use the formula (3.25), but we may do it in a more nice way. Indeed acceleration is orthogonal to velocity. Hence

$$k = \frac{|\mathbf{a}_\perp|}{\mathbf{v}^2|} = \frac{|\mathbf{a}|}{\mathbf{v}^2} = \frac{R\Omega^2}{R^2\Omega^2 + c^2t^2}$$

since $|\mathbf{a}| = R\Omega^2$ Notice that in this formula curvature tends to $\frac{1}{R}$ if $c \to 0$ (in this case helix tends to the circle) and curvature k tends to 0 if $\Omega \to 0$ (in this case) helix tends to straight line.

It is useful to consider helix in parameterisation (3.30) as a "point" moves with constant angular velocity Ω along the circle of radius R and it moves in along z axis with constant velocity c. We see that velocity and acceleration vectors are equal to

See also examples in Homework 8.

4 Surfaces in $\mathbf{E}^3$. Curvatures and Shape operator.

In this section we study surfaces in $\mathbf{E}^3$. One can define surfaces by equation $F(x, y, z) = 0$ or by parametric equation

$$\mathbf{r}(u, v)\colon \begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases}, \tag{4.1}$$

Example the equation $x^2+y^2 = R^2$ defines cylinder (cylindrical surface). z-axis is the axis of this cylinder, R is radius of this cylinder. One can define this cylinder by the parametric equation

$$\mathbf{r}(\varphi, h)\colon \begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \\ z = h \end{cases}, \tag{4.2}$$

where φ is the angle $0 \leq \varphi < 2\pi$ and $-\infty < h < \infty$ takes arbitrary real values.

Example sphere $x^2 + y^2 + z^2 = R^2$:

$$\mathbf{r}(\theta, \varphi)\colon \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases}, 0 \leq \theta \leq \pi,\ 0 \leq \varphi \leq 2\pi \tag{4.3}$$

Example cone $k^2x^2 + k^2y^2 - z^2 = 0$:

$$\mathbf{r}(h, \varphi)\colon \begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases}, -\infty < h < \infty\ \ 0 \leq \varphi \leq 2\pi \tag{4.4}$$

Example saddle $z - axy = 0$:

$$\mathbf{r}(h,\varphi)\colon \begin{cases} x = u\cos\varphi \\ y = v\sin\varphi \\ z = auv \end{cases}, -\infty < u, v < \infty \tag{4.5}$$

Example graph of the surface $z = F(x,y)$:

$$\mathbf{r}(u,v)\colon \begin{cases} x = u \\ y = v \\ z = F(u,v) \end{cases}, -\infty < u < \infty,\ -\infty < v < \infty \tag{4.6}$$

Consider an example when $F = u^2 - v^2$ we come to the surface:

$$\mathbf{r}(u,v)\colon \begin{cases} x = u \\ y = v \\ z = u^2 - v^2 \end{cases}, -\infty < u < \infty,\ -\infty < v < \infty \tag{4.7}$$

Do you see that it is a saddle? Yes it is. Rotate the space on the angle $\frac{\pi}{4}$ with respect to z axis: $x \to \frac{x-y}{\sqrt{2}}$, $y \to \frac{x+y}{\sqrt{2}}$. Then $2xy \to x^2 - y^2$. We see that surface (4.5) with parameter $a = 2$ is the surface (4.7) after rotation on the angle $\frac{\pi}{4}$.

4.1 Coordinate basis, tangent plane to the surface.

Coordinate basis vectors are $\mathbf{r}_u = \partial_u, \mathbf{r}_v = \partial_v$. At the any point $\mathbf{p}$, $\mathbf{p} = \mathbf{r}(u,v)$ these vectors span the plane, (two-dimensional linear space) $T_{\mathbf{p}}M$ in three dimensional vector space $T_{\mathbf{p}}E^3$.

$$T_{\mathbf{p}}M = \{\lambda\mathbf{r}_u + \mu\mathbf{r}_v, \lambda, \mu \in \mathbf{R}\}, \qquad T_{\mathbf{p}}\,\text{subspace in}\,T_{\mathbf{p}}E^3 \tag{4.8}$$

E.g. consider the point $\mathbf{p} = (R,0,0)$ on the cylinder (4.2). Then $\mathbf{p} = \mathbf{r}(\varphi,h)$ for $\varphi = 0, h = 0$. Coordinate basis vectors are

$$\mathbf{r}_\varphi = \begin{pmatrix} -R\sin\varphi \\ R\cos\varphi \\ 0 \end{pmatrix}, \qquad \mathbf{r}_h = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{4.9}$$

or in other notations

$$\mathbf{r}_\varphi = -R\sin\varphi\partial_x + R\cos\varphi\partial_y, \qquad \mathbf{r}_h = \partial_z \tag{4.10}$$

At the point $\mathbf{p} = (R, 0, 0)$ they are are equal to the vectors ∂_y and ∂_z respectively attached at this point. Tangent plane at the point $\mathbf{p}$ is the plane passing through the point $\mathbf{p}$ spanned by the vectors ∂_y and ∂_z.

4.2 Curves on surfaces. Length of the curve. Internal and external point of the view. First Quadratic Form

Let $M\colon \mathbf{r} = \mathbf{r}(u, v)$ be a surface and C curve on this surface, i.e. $C\colon \mathbf{r}(t) = \mathbf{r}(u(t, v(t)))$.

Consider an arbitrary point $\mathbf{p} = \mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ at this curve.

- $T_\mathbf{p}E^3$—three-dimensional tangent space to the point $\mathbf{p}$,
- $T_\mathbf{p}M$—two dimensional linear space tangent to the surface at the point $\mathbf{p}$, spanned by the tangent vectors ∂_u, ∂_v
- $T_\mathbf{p}M$—one dimensional linear space tangent to the curve at the point $\mathbf{p}$ spanned by the velocity vector $\mathbf{v}(t)$.

$$\mathbf{v}(t) = \frac{d\mathbf{r}(u(t), v(t))}{dt} = u_t\frac{\partial \mathbf{r}}{\partial u} + v_t\frac{\partial \mathbf{r}}{\partial v} = u_t\mathbf{r}_u + v_t\mathbf{r}_v \tag{4.11}$$

These tangent spaces form flag of subspaces $T_\mathbf{p}C < T_\mathbf{p}M < T_\mathbf{p}E^3$.

How to calculate the length of the arc of the curve:

$$C\colon \mathbf{r}(t) = \mathbf{r}(u(t, v(t))) = \begin{cases} x = x(u(t), v(t)) \\ y = y(u(t), v(t)) \\ z = z(u(t), v(t)) \end{cases} \qquad t_1 \leq t_2.$$

External and internal observer do it in different ways. External observer just looks at the curve as the curve in ambient space. He uses the formula (3.12):

$$L = \text{Length of the curve } L = \int_a^b |\mathbf{v}(t)| dt = \int_a^b \sqrt{\left(\frac{dx(t)}{dt}\right)^2 + \left(\frac{dy(t)}{dt}\right)^2 + \left(\frac{dz(t)}{dt}\right)^2} dt\,. \tag{4.12}$$

What about internal observer?

Internal observer will perform calculations in coordinates u, v. We have $|\mathbf{v}(t)| = \sqrt{(\mathbf{v}, \mathbf{v})}$. We have

$$\mathbf{v} = \frac{d\mathbf{r}(t)}{dt} = \frac{d\mathbf{r}(u(t), v(t))}{dt} = \dot{u}\frac{\partial r(u,v)}{\partial u} + \dot{v}\frac{\partial r(u,v)}{\partial v} = \dot{u}\mathbf{r}_u \,.$$

Hence the scalar product

$$(\mathbf{v}, \mathbf{v}) = (u_t\mathbf{r}_u + v_t\mathbf{r}_v, u_t\mathbf{r}_u + v_t\mathbf{r}_v) = u_t^2(\mathbf{r}_u, \mathbf{r}_u) + 2u_tv_t(\mathbf{r}_u, \mathbf{r}_v) + v_t^2(v_t, v_t)\,.$$

To understand how internal observer can calculate the length of the curve we have to introduce

$$G_{uu} = (\mathbf{r}_u, \mathbf{r}_u),\ \ G_{uv} = (\mathbf{r}_u, \mathbf{r}_v)\ \ G_{vu} = (\mathbf{r}_v, \mathbf{r}_u),\ \ \Gamma_{vv} = (\mathbf{r}_v, \mathbf{r}_v) \tag{4.13}$$

Of course $G_{uv} = G_{vu}$. We see that internal observer calculates the length of the curve using time derivatives u_t, v_t of internal coordinates u, v and coefficients (4.13):

$$(\mathbf{v}, \mathbf{v}) = u_t^2(\mathbf{r}_u, \mathbf{r}_u) + 2u_tv_t(\mathbf{r}_u, \mathbf{r}_v) + v_t^2(v_t, v_t) = G_{11}u_t^2 + 2G_{12}u_tv_t + G_{22}v_t^2. \tag{4.14}$$

We come to conception of *first quadratic form.*

Definition First quadratic form defines length of the tangent vector to the surface in internal coordinates and length of the curves on the surface.

The first quadratic form at the point $\mathbf{r} = \mathbf{r}(u, v)$ is defined by symmetric matrix:

$$\begin{pmatrix} G_{uu} & G_{uv} \\ G_{vu} & G_{vv} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_u, \mathbf{r}_u) & (\mathbf{r}_u, \mathbf{r}_v) \\ (\mathbf{r}_u, \mathbf{r}_v) & (\mathbf{r}_v, \mathbf{r}_v) \end{pmatrix}, \tag{4.15}$$

where (,) is a scalar product.

E.g. calculate the first quadratic form for the cylinder (4.2). Using (4.9), (4.10) we come to

$$\begin{pmatrix} G_{hh} & G_{h\varphi} \\ G_{\varphi h} & G_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_h, \mathbf{r}_h) & (\mathbf{r}_h, \mathbf{r}_\varphi) \\ (\mathbf{r}_\varphi, \mathbf{r}_h) & (\mathbf{r}_\varphi, \mathbf{r}_\varphi) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & R^2 \end{pmatrix} \tag{4.16}$$

(See this example and other examples in Homework 5)

Let $\mathbf{X} = a\mathbf{r}_u + b\mathbf{r}_v$ be a vector tangent to the surface M at the point $\mathbf{r}(u, v)$. Then the length of this vector is defined by the scalar product $(\mathbf{X}, \mathbf{X})$:

$$|\mathbf{X}|^2 = (\mathbf{X}, \mathbf{X}) = (a\mathbf{r}_u + b\mathbf{r}_v, a\mathbf{r}_u + b\mathbf{r}_v) = a^2(\mathbf{r}_u, \mathbf{r}_u) + 2ab(\mathbf{r}_u, \mathbf{r}_v) + b^2(\mathbf{r}_v, \mathbf{r}_v) \tag{4.17}$$

It is just equal to the value of the first quadratic form on this tangent vector:

$$(\mathbf{X},\mathbf{X}) = G(\mathbf{X},\mathbf{X}) = \begin{pmatrix} a, & b \end{pmatrix} \cdot \begin{pmatrix} G_{uu} & G_{uv} \\ G_{vu} & G_{vv} \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = G_{uu}a^2 + 2G_{uv}ab + G_{vv}b^2 \tag{4.18}$$

External observer (person living in ambient space $\mathbf{E}^3$) calculate the length of the tangent vector using formula (4.17). An ant living on the surface (internal observer) calculate length of this vector in internal coordinates using formula (4.18). External observer deals with external coordinates of the vector, ant on the surface with internal coordinates.

If $\mathbf{X}, \mathbf{Y}$ are two tangent vectors in the tangent plane T_pC then $G(\mathbf{X}, \mathbf{Y})$ at the point p is equal to scalar product of vectors $\mathbf{X}, \mathbf{Y}$: $(\mathbf{X}, \mathbf{Y}) = (X^1\mathbf{r}_1 + X^2\mathbf{r}_2, Y^1\mathbf{r}_1 + Y^2\mathbf{r}_2) = X^1(\mathbf{r}_1, \mathbf{r}_1)Y^1 + X^1(\mathbf{r}_1, \mathbf{r}_2)Y^2 + X^2(\mathbf{r}_2, \mathbf{r}_1)Y^1 + X^2(\mathbf{r}_2, \mathbf{r}_2)Y^2 = X^\alpha(\mathbf{r}_\alpha, \mathbf{r}_\beta)Y^\beta = X^\alpha G_{\alpha\beta}Y^\beta = G(\mathbf{X}, \mathbf{Y})$. We identify quadratic forms and corresponding symmetric bilinear forms. Bilinear symmetric form $B(\mathbf{X}, \mathbf{Y}) = B(\mathbf{Y}, \mathbf{X})$ defines quadratic form $Q(\mathbf{X}) = B(\mathbf{X}, \mathbf{X})$. Quadratic form satisfies the condition $Q(\lambda\mathbf{X}) = \lambda^2 Q(\mathbf{X})$ and so called parallelogram condition

$$Q(\mathbf{X}+\mathbf{Y}) + Q(\mathbf{X}-\mathbf{Y}) = 2Q(\mathbf{X}) + 2Q(\mathbf{Y}) \tag{4.19}$$

First quadratic form and length of the curve

Let $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ $a \le t \le b$ be a curve on the surface.

The first quadratic form measures the length of velocity vector at every point of this curve. Write down again the formula for length of the curve in internal coordinates using First Quadratic form (compare with (4.14)).

Velocity of this curve at the point $\mathbf{r}(u(t), v(t))$ is equal to $\mathbf{v} = \frac{d\mathbf{r}(t)}{dt} = u_t\mathbf{r}_u + v_t\mathbf{r}_v$. The length of the curve is equal to

$$L = \int_a^b |\mathbf{v}(t)|dt = \int_a^b \sqrt{(\mathbf{v}(t), \mathbf{v}(t))}dt = \int_a^b \sqrt{(u_t\mathbf{r}_u + v_t\mathbf{r}_v, u_t\mathbf{r}_u + v_t\mathbf{r}_v)}dt = \tag{4.20}$$

$$\int_a^b \sqrt{(\mathbf{r}_u, \mathbf{r}_u)u_t^2 + 2(\mathbf{r}_u, \mathbf{r}_v)u_t v_t + (\mathbf{r}_v, \mathbf{r}_v)v_t^2}d\tau =$$

$$\int_a^b \sqrt{G_{11}u_t^2 + 2G_{12}u_t v_t + G_{22}v_t^2}dt\,. \tag{4.21}$$

An external observer will calculate the length of the curve using (4.17). An ant living on the surface calculate length of the curve via first quadratic form using (4.21): first quadratic form defines Riemannian metric on the surface:

$$ds^2 = G_{11}du^2 + 2G_{12}dudv + G_{22}dv^2 \tag{4.22}$$

Example Consider the curve

$$\mathbf{r}(t) \quad \begin{cases} x = R\cos t \\ y = R\sin t \\ z = vt \end{cases}, \quad 0 \le t \le 1$$

on the cylinder (4.2) (helix). The coordinates of this curve on the cylinder (internal coordinates) are

$$\begin{cases} \varphi(t) = t \\ h(t) = vt \end{cases}.$$

To calculate the length of this curve the external observer will perform the calculations

$$L = \int_0^1 \sqrt{x_t^2 + y_t^2 + z_t^2}dt = \int_0^1 \sqrt{R^2\sin^2 t + R^2\cos^2 t + v^2}dt = \int_0^1 \sqrt{R^2 + v^2}dt = \sqrt{R^2 + v^2}.$$

An internal observer ("ant") uses quadratic form (4.16) and perform the following calculations:

$$L = \int_0^1 \sqrt{G_{11}\varphi_t^2 + 2G_{12}\varphi_t h_t + G_{22}h_t^2}dt = \int_0^1 \sqrt{R^2\varphi_t^2 + h_t^2}dt = \int_0^1 \sqrt{R^2 + v^2}dt = \sqrt{R^2 + v^2}\,.$$

The answer will be the same. (See this and other examples in Homework 8).

4.3 Unit normal vector to surface

We define unit normal vector field for surfaces in $\mathbf{E}^3$.

Consider vector field defined on the points of surface.

Definition Let $M\colon\ \mathbf{r} = \mathbf{r}(u, v)$ be a surface in $\mathbf{E}^3$. We say that vector $\mathbf{n}(u, v)$ is *normal unit vector* at the point $\mathbf{p} = \mathbf{r}(u, v)$ of the surface M if it has unit length $|\mathbf{n}| = 1$, and it is orthogonal to the surface, i.e. it is orthogonal to the tangent plane $T_{\mathbf{p}}M$. This means that it is orthogonal to any tangent vector $\xi \in T_pM$, i.e. it is orthogonal to the coordinate vectors $\mathbf{r}_u = \partial_u$, $\mathbf{r}_v = \partial_v$ at the point $\mathbf{p}$.

$$\mathbf{n}\colon (\mathbf{n}, \mathbf{r}_u) = (\mathbf{n}, \mathbf{r}_v) = 0, (\mathbf{n}, \mathbf{n}) = 1\,. \tag{4.23}$$

Write down this equation in components:

If surface is given by equation $\mathbf{r}(u,v)$: $\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$ then

$$\mathbf{r}_u = \begin{pmatrix} x_u \\ y_u \\ z_u \end{pmatrix}, \ \mathbf{r}_v = \begin{pmatrix} x_v \\ y_v \\ z_v \end{pmatrix},$$

and $\mathbf{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$ is unit normal vector. Then writing the previous conditions in components we come to

$$(\mathbf{n}, \mathbf{r}_u) = n_x x_u + n_y y_u + n_z z_v = 0, \ (\mathbf{n}, \mathbf{r}_v) = n_x x_v + n_y y_v + n_z z_v = 0, \ (\mathbf{n}, \mathbf{n}) = n_x^2 + n_y^2 + n_z^2 = 1$$

Normal unit vector is defined up to a sign. At any point there are two normal unit vetors: the transformation $\mathbf{n} \to -\mathbf{n}$ transforms normal unit vector to normal unit vector.

Vector field defined at the points of the surface is called normal unit vector field if any vector is normal unit vector.

In simple cases one can guess how to find unit normal vector field using geometrical intuition and just check that conditions above are satisfied. E.g. for sphere (4.4) $\mathbf{r}$ is orthogonal to the surface, hence

$$\mathbf{n}(\theta, \varphi) = \frac{\mathbf{r}(\theta, \varphi)}{R} = \pm \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

For cylinder (4.2) it is easy to see that at any point (φ, h) (4.2), $\mathbf{r}$: $x = R\cos\varphi, y = R\sin\varphi, z = h$, a normal unit vector is equal to

$$\mathbf{n}(\varphi, h) = \pm \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} \tag{4.24}$$

Indeed it is easy to see that the conditions (4.23) are satisfied.

In general case one can define $\mathbf{n}(u,v)$ in two steps using vector product formula:

$$\mathbf{n}(u,v) = \frac{\mathbf{N}(u,v)}{|\mathbf{N}(u,v)|} \quad \text{where} \quad \mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v \tag{4.25}$$

Indeed by definition of vector product vector field $\mathbf{N}(u,v)$ is orthogonal to $\mathbf{r}_u$ and $\mathbf{r}_v$, i.e. it is orthogonal to the surface. Dividing $\mathbf{N}$ on the length we come to unit normal vector field $\mathbf{n}(u,v)$ at the point $\mathbf{r}(u,v)$. (See other examples of calculating normal unit vector in the Homework 9)

4.4 † Curves on surfaces—normal acceleration and normal curvature

We know already how to measure the length of the curve belonging to the given surface. What about curvature? Answering this question we will be able to study curvature of the surface.

Before we have to introduce normal acceleration and normal curvature for curves on the surfaces.

We know that acceleration vector $\mathbf{a}$ in general is not tangent to the curve. Recall that when studying curvature we consider decomposition of acceleration vector on tangential component and the component which is perpendicular to velocity vector: $\mathbf{a} = \mathbf{a}_{tang} + \mathbf{a}_{\perp}$. The curvature of curve is nothing but the magnitude of normal acceleration $\mathbf{a}_{\perp}$ of particle which moves along the curve with unit speed: $k = \frac{|\mathbf{a}_{\perp}|}{|\mathbf{v}|}$.

Now we consider *normal acceleration of the curve on the surface.*

Let M: $\mathbf{r} = \mathbf{r}(u,v)$ be a surface and C: $u = u(t), v = v(t)$, i.e. $\mathbf{r}(t) = \mathbf{r}(u(t),v(t))$, be curve on the surface M. Consider an arbitrary point $\mathbf{p} = \mathbf{r}(t) = \mathbf{r}(u(t),v(t))$ on this curve and velocity and acceleration vectors $\mathbf{v} = \frac{d\mathbf{r}(t)}{dt}$, $\mathbf{a} = \frac{d^2\mathbf{r}(t)}{dt^2}$ at this point.

Definition The component of acceleration vector of the curve on the surface orthogonal to the surface is called a normal acceleration of curve on the surface. If $\mathbf{a}$ is acceleration vector then

$$\mathbf{a} = \mathbf{a}_{||} + \mathbf{a}_n, \ , \tag{4.26}$$

where the vector $\mathbf{a}_{||}$ is tangent to the surface and the vector $\mathbf{a}_n$ is orthogonal (perpendicular) to the surface. Calculate vector $\mathbf{a}_n$.

If $\mathbf{n}$ is a normal unit vector to the surface, then vector $\mathbf{a}_n$ is collinear (proportional) to the vector $\mathbf{n}$ and vector $\mathbf{a}_{||}$ is orthogonal to this vector:

$$\mathbf{a}_n = a_n\mathbf{n}, \ (\mathbf{n}, \mathbf{a}_{||}) = 0 \,.$$

Take a scalar product of left and right hand sides of the formula (4.26) on the vector $\mathbf{n}$. We come to:

$$(\mathbf{n}, \mathbf{a}) = (\mathbf{n}, \mathbf{a}_{||} + \mathbf{a}_n) = (\mathbf{n}, \mathbf{a}_{||}) + (\mathbf{n}, \mathbf{a}_n) = 0 + a_n(\mathbf{n}, \mathbf{n}) = a_n.$$

Hence we come to

$$\mathbf{a} = a_n\mathbf{n} = (\mathbf{n}, \mathbf{a})\mathbf{n} \,. \tag{4.27}$$

Avoid confusion! The normal acceleration vector $\mathbf{a}_n$ of the curve on the surface is orthogonal to the surface. The normal acceleration vector of the curve in $\mathbf{E}^3$ $\mathbf{a}_{\perp}$ is orthogonla to the velocity vector of the curve.

Now we are ready give a definition of normal curvature of the curve on the surface.

Definition Let C be a curve on the surface M. Let $\mathbf{v}$, $\mathbf{a}$ be velocity and acceleration vectors at the given point of this curve and $\mathbf{n}$ be normal unit vector at this point. Then

$$\kappa_n = \frac{a_n}{|\mathbf{v}|^2} = \frac{(\mathbf{n}, \mathbf{a})}{(\mathbf{v}, \mathbf{v})} \tag{4.28}$$

is called *normal curvature of the curve* C on the surface M at the point $\mathbf{p}$. Or in other words

$$|\kappa_n| = \frac{|\mathbf{a}_n|}{(\mathbf{v}, \mathbf{v})}, \tag{4.29}$$

i.e. up to a sign normal curvature is equal to modulus of normal acceleration divided on the square of speed (Compare with formula (3.23) for usual curvature.)

Remark Avoid confusion: We know that usual curvature k of the curve is defined by the formula $k = \frac{|\mathbf{a}_\perp|}{|\mathbf{v}|^2}$, where $\mathbf{a}_\perp$ is a magnitude of the acceleration vector orthogonal to the curve (see the formula (3.23)). Normal curvature of the curve on the surface is defined by the analogous formula bunt in terms of normal acceleration $\mathbf{a}_n$ which is orthogonal to the surface, not to the curve!

In fact one can see that $|\mathbf{a}_\perp| \leq |\mathbf{a}_n|$, i.e. modulus of the normal curvature is less or equal to the usual curvature of the curve. (See in details the Appendix "Relations between usual curvature, normal curvature and geodesic curvature")

4.5 Shape operator on the surface

Let $M: \mathbf{r} = \mathbf{r}(u, v)$ be a surface and $\mathbf{L}(u, v)$ be an arbitrary (not necessarily unit normal) vector field at the points of the surface M. We define at every point $\mathbf{p} = \mathbf{r}(u, v)$ a linear operator K_L acting on the vectors tangent to the surface M such that its value is equal to the derivative of vector field $\mathbf{L}(u, v)$ along vector $\boldsymbol{\xi}$

$$K_{\mathbf{L}}\colon\ \boldsymbol{\xi} \in T_{\mathbf{p}}M \mapsto K_{\mathbf{L}}(\boldsymbol{\xi}) = \partial_{\boldsymbol{\xi}} L = \xi_u \frac{\partial \mathbf{L}(u, v)}{\partial u} + \xi_v \frac{\partial \mathbf{L}(u, v)}{\partial v}\,, \tag{4.30}$$

ξ_u, ξ_v are components of vector $\boldsymbol{\xi}$

$$\boldsymbol{\xi} = \xi_u \mathbf{r}_u + \xi_v \mathbf{r}_v \tag{4.31}$$

The vector $K_{\mathbf{L}}\boldsymbol{\xi} \in T_{\mathbf{p}}\mathbf{E}^3$ in general is not a vector tangent to the surface C and $K_{\mathbf{L}}$ is linear operator from the space $T_{\mathbf{p}}M$ in the space $T_{\mathbf{p}}\mathbf{E}^3$ of all vectors in $\mathbf{E}^3$ attached at the point $\mathbf{p}$

It turns out that in the case if vector field $\mathbf{L}(u, v)$ is *a unit normal vector field* then operator $K_{\mathbf{L}}$ takes values in vectors tangent to M and it is very important geometric properties.

Definition-Proposition Let $\mathbf{n}(u,v)$ be a unit normal vector field to the surface M. Then operator

$$S\colon\ S(\mathbf{X}) = \partial_{\mathbf{X}}(-\mathbf{n}) = -X_u\frac{\partial\mathbf{n}(u,v)}{\partial u} - X_v\frac{\partial\mathbf{n}(u,v)}{\partial v} \tag{4.32}$$

maps tangent vectors to the tangent vectors:

$$S\colon T_{\mathbf{p}}M \to T_{\mathbf{p}}M \text{ for every } \mathbf{X} = X_u\mathbf{r}_u + X_v\mathbf{r}_v \in T_{\mathbf{p}}M, \qquad S(\boldsymbol{\xi}) \in T_{\mathbf{p}}M \tag{4.33}$$

This operator is called *shape operator*.

Remark The sign " $-$ " seems to be senseless: if $\mathbf{n}$ is unit normal vector field then $-\mathbf{n}$ is normal vector field too. Later we will see why it is convenient (see the proof of the Proposition below).

Show that the property (4.33) is indeed obeyed, i.e. vector $\mathbf{X}' = S(\boldsymbol{\xi})$ is tangent to surface. Consider derivative of scalar product $(\mathbf{n},\mathbf{n})$ with respect to the vector field $\mathbf{X}$. We have that $(\mathbf{n},\mathbf{n}) = 1$. Hence

$$\partial_{\mathbf{X}}(\mathbf{n},\mathbf{n}) = 0 = \partial_{\mathbf{X}}(\mathbf{n},\mathbf{n}) = (\partial_{\mathbf{X}}\mathbf{n},\mathbf{n}) + (\mathbf{n},\partial_{\mathbf{X}}\mathbf{n}) = 2(\partial_{\mathbf{X}}\mathbf{n},\mathbf{n})\,.$$

Hence $(\partial_{\mathbf{X}}\mathbf{n},\mathbf{n}) = -(S(\mathbf{X}),\mathbf{n}) = -(\mathbf{X}',\mathbf{n}) = 0$, i.e. vector $\partial_{\mathbf{X}}\mathbf{n} = -\mathbf{X}'$ is orthogonal to the vector $\mathbf{n}$. This means that vector $\mathbf{X}'$ is tangent to the surface.

Write down the action of shape operator on coordinate basis $\mathbf{r}_u = \partial_u$, $\partial_v = \mathbf{r}_v$ at the given point $\mathbf{p}$:

$$S(\mathbf{r}_u) = -\partial_{\mathbf{r}_u}\mathbf{n}(u,v) = -\frac{\partial\mathbf{n}(u,v)}{\partial u}, \quad S(\mathbf{r}_v) = -\partial_{\mathbf{r}_v}\mathbf{n}(u,v) = -\frac{\partial\mathbf{n}(u,v)}{\partial v}$$

Since the shape operator transforms tangent vectors to tangent vectors, then

$$\begin{array}{c} S(\mathbf{r}_u) = -\frac{\partial\mathbf{n}(u,v)}{\partial u} = a\,\mathbf{r}_u + c\mathbf{r}_v \\ S(\mathbf{r}_v) = -\frac{\partial\mathbf{n}(u,v)}{\partial v} = b\mathbf{r}_u + d\mathbf{r}_v \end{array},$$

i.e.

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ in the coordinate basis } \mathbf{r}_u, \mathbf{r}_v \tag{4.34}$$

Examples of shape operator see in the subsection above (Shape operator, Gaussian and mean curvature for sphere and cylinder) and in the Homework 9.

Remark. Shape operator as well as normal unit vector is defined up to a sign:

$$\mathbf{n}(u,v) \to -\mathbf{n}(u,v), \quad \text{then} \quad S \to -S\,.$$

We show now that normal acceleration of a curve on the surface and normal curvature are expressed in terms of shape operator.

Let $C\colon \mathbf{r}(t)$ be a curve on the surface M, $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$. Let $\mathbf{v} = \mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt}$, $\mathbf{a} = \mathbf{a}(t) = \frac{d^2r(t)}{dt^2}$ be velocity and acceleration vectors respectively. Recall that

$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \dot{x}\mathbf{e}_x + \dot{y}\mathbf{e}_y + \dot{z}\mathbf{e}_z = \frac{d\mathbf{r}(u(t), v(t))}{dt} = \dot{u}\mathbf{r}_u + \dot{v}\mathbf{r}_v \tag{4.35}$$

be velocity vector; $\dot{u}, \dot{v}$ are internal components of the velocity vector with respect to the basis $\{\mathbf{r}_u = \partial_u, \mathbf{r}_v = \partial_v\}$ and $\dot{x}, \dot{y}, \dot{z}$, are external components velocity vectors with respect to the basis $\{\mathbf{e}_x = \partial_x, \mathbf{e}_y = \partial_y, \mathbf{e}_z = \partial_z\}$. As always we denote by $\mathbf{n}$ normal unit vector.

Proposition *The normal acceleration at an arbitrary point $\mathbf{p} = \mathbf{r}(u(t_0), v(t_0))$ of the curve C on the surface M is defined by the scalar product of the velocity vector $\mathbf{v}$ of the curve at the point $\mathbf{p}$ on the value of the shape operator on the velocity vector:*

$$\mathbf{a}_n = a_n\mathbf{n} = (\mathbf{v}, S\mathbf{v})\,\mathbf{n} \tag{4.36}$$

and normal curvature (4.28) *is equal to*

$$\kappa_n = \frac{(\mathbf{n}, \mathbf{a})}{(\mathbf{v}, \mathbf{v})} = \frac{(\mathbf{v}, S\mathbf{v})}{(\mathbf{v}, \mathbf{v})} \tag{4.37}$$

Proof of the Proposition. According to (5.32) we have

$$\mathbf{a}_n = (\mathbf{n}, \mathbf{a})\mathbf{n} = \mathbf{n}\left(\mathbf{n}, \frac{d}{dt}\mathbf{v}(t)\right)\mathbf{n} = \mathbf{n}\frac{d}{dt}\left(\mathbf{n}, \mathbf{v}(t)\right) - \mathbf{n}\left(\frac{d}{dt}\mathbf{n}(u(t), v(t)), \mathbf{v}(t)\right)$$

$$= 0 + \left(-\partial_{\mathbf{v}}\mathbf{n}, \mathbf{v}\right)\mathbf{n} = (S\mathbf{v}, \mathbf{v})\mathbf{n}$$

This proves Proposition.

4.6 Principal curvatures, Gaussian and mean curvatures and shape operator

Now we introduce on surfaces, principal curvatures, Gaussian curvature and mean curvature.

Let $\mathbf{p}$ be an arbitrary point of the surface M and S be shape operator at this point. S is symmetric operator: $(S\mathbf{a}, \mathbf{b}) = (\mathbf{b}, S\mathbf{a})$. Consider eigenvalues λ_1, λ_2 and eigenvectors $\mathbf{l}_1, \mathbf{l}_2$ of the shape operator S

$$\mathbf{l}_1, \mathbf{l}_2 \in T_{\mathbf{p}}M, \qquad S\,\mathbf{l}_1 = \kappa_1 \mathbf{l}_1, \qquad S\,\mathbf{l}_2 = \kappa_2 \mathbf{l}_2, \tag{4.38}$$

Definition Eigenvalues of shape operator λ_1, λ_2 are called *principal curvatures*:

$$\lambda_1 = \kappa_1, \ \ \lambda_2 = \kappa_2$$

Eigenvectors $\mathbf{l}_1, \mathbf{l}_2$ define the two directions such that curves directed along these vectors have normal curvature equal to the principal curvatures κ_+, κ_-.

These directions are called principal directions

Remark As it was noted above normal unit vector as well as a shape operator are defined up to a sign. Hence principal curvatures, i.e. eigenvalues of shape operator are defined up to a sign too:

$$\mathbf{n} \to -\mathbf{n}, \text{then } S \to -S, \ \text{ then } (\kappa_1, \kappa_2) \to (-\kappa_1, -\kappa_2) \tag{4.39}$$

Remark. Principal directions are well-defined in the case if principal curvatures (eigenvalues of shape operator) are different: $\lambda_1 = \kappa_1 \neq \kappa_2 = \lambda_2$. In the case if eigenvalues $\lambda_1 = \lambda_2 = \lambda$ then $S = \lambda E$ is proportional to unity operator. In this case all vectors are eigenvectors, i.e. all directions are principal directions. (This happens for the shape operator of the sphere: see the Homework 9.)

Remark Do shape operator have always two eigenvectors? Yes, in fact one can prove that it is symmetrical operator: $\langle S\mathbf{a}, \mathbf{b}\rangle = \langle S\mathbf{b}, \mathbf{a}\rangle$ for arbitrary two vectors $\mathbf{a}, \mathbf{b}$, hence it has two eigenvectors. This implies that principal directions are orthogonal to each other. Indeed one can see that $\lambda_2(\mathbf{l}_2, \mathbf{l}_1) = (S\mathbf{l}_2, \mathbf{l}_1) = (\mathbf{l}_2, S\mathbf{l}_1) = \lambda_1(\mathbf{l}_2, \mathbf{l}_1)$. It follows from this relation that eigenvectors are orthogonal ($(\mathbf{l}_-, \mathbf{l}_+) = 0$) if $\lambda_- \neq \lambda_+$ If $\lambda_- = \lambda_+$ then all vectors are eigenvectors. One can choose in this case $\mathbf{l}_-$, $\mathbf{l}_+$ to be orthogonal.

Definition

- *Gaussian curvature* K of the surface M at a point $\mathbf{p}$ is equal to the product of principal curvatures.

$$K = \kappa_1 \kappa_2 \tag{4.40}$$

- *Mean curvature* K of the surface M at a point S is equal to the arithmetic mean of the principal curvatures:

$$H = \kappa_1 + \kappa_2 \tag{4.41}$$

Recall that the product of eigenvalues of a linear operator is determinant of this operator, and the sum of eigenvalues of linear operator is *trace* of this operator. Thus we immediately come to the useful formulae for calculating Gaussian and mean curvatures:

Proposition Let S be a shape operator at the point $\mathbf{p}$ on the surface M. Then

- Gaussian curvature K of the surface M at the point $\mathbf{p}$ is equal to the determinant of the shape operator:

$$K = \kappa_1 \kappa_2 = \det S \tag{4.42}$$

- Mean curvature H of the surface M at the point $\mathbf{p}$ is equal to the trace of the shape operator S:

$$H = \kappa_1 + \kappa_2 = \operatorname{Tr} S \tag{4.43}$$

E.g. if in a given coordinate basis a shape operator is given by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (see e.g. equations (4.33) and(4.36)), then

$$K = \det S = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc, \quad H = \operatorname{Tr} S = \operatorname{Tr} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d \tag{4.44}$$

4.7 Shape operator, Gaussian and mean curvature for sphere and cylinder

Consider now two examples. (These and other examples see in detail in the Homework 8.)

Example Calculate mean an Gaussian curvature for sphere $x^2+y^2+z^2 = R^2$.

For the sphere of radius R in spherical coordinates (see 4.4)

$$\mathbf{r}(\theta, \varphi)\colon \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases} \quad , 0 \le \theta \le \pi, \ 0 \le \varphi \le 2\pi$$

coordinate basis vectors are $\mathbf{r}_\theta = \frac{\partial \mathbf{r}}{\partial \theta} = \begin{pmatrix} R\cos\theta\cos\varphi \\ R\cos\theta\sin\varphi \\ -R\sin\theta \end{pmatrix}$, $\mathbf{r}_\varphi = \frac{\partial \mathbf{r}}{\partial \varphi} = \begin{pmatrix} -R\sin\theta\sin\varphi \\ R\sin\theta\cos\varphi \\ 0 \end{pmatrix}$ and unit normal vector which is orthogonal to sphere equals to $\mathbf{n}(\theta,\varphi) = \frac{\mathbf{r}(\theta,\varphi)}{R} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$. One can see that $\mathbf{n}$ is indeed orthogonal to the sphere. This is evident geometrically and $(\mathbf{n}, \mathbf{r}_\theta) = (\mathbf{n}\,, \mathbf{r}_\varphi) = 0$ and its length equals to 1. Consider shape operator. By definition $S\mathbf{v} = -\partial_{\mathbf{v}}\mathbf{n}$:

$$S\mathbf{r}_\theta = -\frac{\partial \mathbf{n}(\theta,\varphi)}{\partial \theta} = -\frac{\partial}{\partial \theta}\left(\frac{\mathbf{r}(\theta,\varphi)}{R}\right) = -\frac{\mathbf{r}_\theta}{R}$$

and

$$S\mathbf{r}_\varphi = -\frac{\partial \mathbf{n}(\theta,\varphi)}{\partial \varphi} = -\frac{\partial}{\partial \varphi}\left(\frac{\mathbf{r}(\theta,\varphi)}{R}\right) = -\frac{\mathbf{r}_\varphi}{R}$$

Hence in the coordinate basis $\mathbf{r}_\theta, \mathbf{r}_\varphi$ $S = \begin{pmatrix} -\frac{1}{R} & 0 \\ 0 & -\frac{1}{R} \end{pmatrix}$. In the case if we choose the opposite direction for unit normal vector then we will come to the answer just with changing the signs: if $\mathbf{n} \to -\mathbf{n}$, $S \to -S$.

We see that principal curvatures, i.e. eigenvalues of shape operator are the same:

$$\lambda_1 = \lambda_2 = -\frac{1}{R}, \text{i.e. } \kappa_1 = \kappa_2 = -\frac{1}{R}$$

(if we choose the opposite sign for $\mathbf{n}$ then $\kappa_1 = \kappa_2 = \frac{1}{R}$). Thus we can calculate Gaussian and mean curvature: Gaussian curvature

$$K = \kappa_1 \cdot \kappa_2 = \det S = \frac{1}{R^2}\,.$$

Mean curvature

$$H = \kappa_1 + \kappa_2 = \operatorname{Tr} S = -\frac{2}{R}\,.$$

If we choose the opposite sign for $\mathbf{n}$ then $S \to -S$, principal curvatures change the sign, Gaussian curvature $K = \kappa_1 \cdot \kappa_2$ does not change but mean curvature $H = \kappa_1 + \kappa_2$ will change the sign: if $\mathbf{n} \to -\mathbf{n}$ then $H = \frac{2}{R}$.

Example Cylindircal surface $x^2 + y^2 = a^2$

For the cylinder we have (see 4.2)

$$\mathbf{r}(h,\varphi)\colon \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases} ,0 \le \varphi < 2\pi, -\infty < h < \infty\,.$$

Coordinate basis vectors are (see 4.9) $\mathbf{r}_\varphi = \frac{\partial \mathbf{r}}{\partial \varphi} = \begin{pmatrix} -a\sin\varphi \\ a\cos\varphi \\ 0 \end{pmatrix}$, $\mathbf{r}_h = \frac{\partial \mathbf{r}}{\partial h} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and unit normal vector which is orthogonal to cylinder equals to $\mathbf{n}(h,\varphi) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$. One can see that $\mathbf{n}$ is indeed orthogonal to cylinder surface. This is evident geometrically but one can calculate also that $((\mathbf{n},\mathbf{r}_h) = (\mathbf{n}\,,\mathbf{r}_\varphi) = 0)$ and its length equals to 1. Consider shape operator. By definition $S\mathbf{v} = -\partial_\mathbf{v}\mathbf{n}$:

$$S\mathbf{r}_h = -\frac{\partial \mathbf{n}(h,\varphi)}{\partial h} = -\frac{\partial}{\partial h}\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} = 0,$$

and

$$S\mathbf{r}_\varphi = -\frac{\partial \mathbf{n}(\theta,\varphi)}{\partial \varphi} = -\frac{\partial}{\partial \varphi}\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} = -\begin{pmatrix} \sin\varphi \\ -\cos\varphi \\ 0 \end{pmatrix} = \frac{1}{a}\begin{pmatrix} -a\sin\varphi \\ a\cos\varphi \\ 0 \end{pmatrix} = -\frac{\mathbf{r}_\varphi}{a}$$

We see that $\mathbf{r}_h, \mathbf{r}_\varphi$ are eigenvectors of Shape oeprator. In the coordinate basis $\mathbf{r}_h, \mathbf{r}_\varphi$ $S = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{a} \end{pmatrix}$. In the case if we choose the opposite direction for unit normal vector then we will come to the same answer just with changing the signs: if $\mathbf{n} \to -\mathbf{n}$, $S \to -S$.

We see that principal curvatures, i.e. eigenvalues of shape operator are:

$$\lambda_1 = 0, \lambda_2 = -\frac{1}{R}, \text{i.e. } \kappa_1 = 0, \kappa_2 = -\frac{1}{R}$$

(if we choose the opposite sign for $\mathbf{n}$ then $\kappa_2 = \frac{1}{a}$). Thus we can calculate Gaussian and mean curvature: Gaussian curvature

$$K = \kappa_1 \cdot \kappa_2 = \det S = 0\,.$$

Mean curvature

$$H = \kappa_1 + \kappa_2 = \mathrm{Tr}\, S = -\frac{1}{a}\,.$$

If we choose the opposite sign for $\mathbf{n}$ then $S \to -S$, principal curvatures change the sign, Gaussian curvature $K = \kappa_1 \cdot \kappa_2$ does not change but mean curvature $H = \kappa_1 + \kappa_2$ will change the sign: if $\mathbf{n} \to -\mathbf{n}$ then $H = \frac{2}{R}$.

4.8 †Principal curvatures and normal curvature

In this subsection we principal curvatures, eigenvectors of the shape operator by κ_-, κ_+ and respectively eigenvectors by $\mathbf{l}_-, \mathbf{l}_+$.

One can consider different curves passing through an arbitrary point $\mathbf{p}$ on the surface M. We know that if $\mathbf{v}$ velocity vector of the curve then normal curvature is equal to $\kappa_n = \frac{(S\mathbf{v},\mathbf{v})}{(\mathbf{v},\mathbf{v})}$ (see (4.37)). What are the relations between normal curvature of curves and principal curvature? The following Proposition establishes these relations.

Proposition

Let κ_-, κ_+, $\kappa_- \leq \kappa_+$ be principal curvatures of the surface M at the point $\mathbf{p}$ (eigenvalues of shape operator S at the point $\mathbf{p}$).

Then normal curvature κ of an arbitrary curve on the surface M at the point $\mathbf{p}$ takes values in the interval (κ_-, κ_+):

$$\kappa_- \leq k_n \leq \kappa_+ \tag{4.45}$$

Example E.g. consider cylinder surface of the radius R. One can calculate that principal curvatures are equal to $\kappa_- = 0, \kappa_+ = \frac{1}{R}$ (see Homework 8). Then for an arbitrary curve on the surface normal curvature κ_n takes values in the interval $(0, \frac{1}{R})$ (up to a sign). (See Homework 8 and appendix "Normal curvature of curves on cylinder surface")

Proof of Proposition: If velocity vector $\mathbf{v}$ of curve is collinear to the eigenvector $\mathbf{l}_+$, $\mathbf{v} = \lambda \mathbf{l}_+$ then normal curvature of the curve C at the point $\mathbf{p}$ according to (4.37) is equal to

$$\kappa_n = \frac{(\mathbf{v}, S\mathbf{v})}{(\mathbf{v},\mathbf{v})} = \frac{(\lambda\mathbf{l}_+, S\lambda\mathbf{l}_+)}{(\lambda\mathbf{l}_+, \lambda\mathbf{l}_+)} = \frac{\lambda^2\,(\mathbf{l}_+, \kappa_+\mathbf{l}_+)}{\lambda^2(\mathbf{l}_+, \mathbf{l}_+)} = \frac{\kappa_+\,(\mathbf{l}_+, \mathbf{l}_+)}{(\mathbf{l}_+, \mathbf{l}_+)} = \kappa_+\,.$$

Analogously if velocity vector $\mathbf{v}$ is collinear to the eigenvector $\mathbf{l}_-$ then normal curvature of the curve C at the point $\mathbf{p}$ is equal to $\kappa_n = \frac{(\mathbf{v},S\mathbf{v})}{(\mathbf{v},\mathbf{v})} = \frac{(\mathbf{l}_-,S\mathbf{l}_-)}{(\mathbf{l}_-,\mathbf{l}_-)} = \frac{(\mathbf{l}_-,\kappa_-\mathbf{l}_-)}{(\mathbf{l}_-,\mathbf{l}_-)} = \kappa_-$.

In the general case if $\mathbf{v} = v_+\mathbf{l}_+ + v_-\mathbf{l}_-$ is expansion of velocity vector with respect to the basis of eigenvectors then we have for normal curvature

$$k_n = \frac{(\mathbf{v}, S\mathbf{v})}{(\mathbf{v},\mathbf{v})} = \frac{(v_+\mathbf{l}_+ + v_-\mathbf{l}_-, \lambda_+v_+\mathbf{l}_+ + \lambda_-v_-\mathbf{l}_-)}{(v_+\mathbf{l}_+ + v_-\mathbf{l}_-, v_+\mathbf{l}_+ + v_-\mathbf{l}_-)} = \frac{\kappa_+v_+^2 + \kappa_-v_-^2}{v_+^2 + v_-^2}\,. \tag{4.46}$$

Hence we come to the conclusion that

$$\kappa_- \leq \kappa_{normal} = \frac{\kappa_+v_+^2 + \kappa_-v_-^2}{v+^2 + v_-^2} \leq \kappa_+ \tag{4.47}$$

Thus we prove that normal curvature varies in the interval (λ_-, λ_+).

Now remember the definition of principal curvatures from the subsection 4.4: we see that λ_-, λ_+ are just principal curvatures.

Summarize all the relations between normal curvature, shape operator and Gaussian and mean curvature.

- *Principal curvatures* κ_-, κ_+ of the surface M at the given point $\mathbf{p}$ are eigenvalues of shape operator S acting at the tangent space $T_{\mathbf{p}}M$ (κ_-, κ_+). Corresponding eigenvectors $\mathbf{l}_+, \mathbf{l}_-$ define directions which are called *principal directions*. Principal directions are orthogonal or can be chosen to be orthogonal if $\kappa_- = \kappa_+$. The normal curvature κ_n for an arbitrary curve on the surface M at the point $\mathbf{p}$ varies in the interval (κ_-, κ_+):
$$\kappa_- \leq \kappa_n \leq \kappa_+ \tag{4.48}$$

- *Gaussian curvature* K of the surface M at a point S is equal to the product of principal curvatures, i.e. determinant of shape operator S:
$$K = \kappa_+ \cdot \kappa_- = \det S \tag{4.49}$$

- *Mean curvature* H of the surface M at a point S is equal to the half-sum of the principal curvatures, i.e. half of the trace of shape operator S:
$$H = \kappa_+ + \kappa_- = \text{Tr } S \tag{4.50}$$

4.9 Geometrical meaning of Gaussian curvature. Theorema Egregium

We know that Gaussian curvature of cylinder cone and plane equals to zero and Gaussian curvature of sphere equals to $\frac{1}{R^2}$ (see the calculations in the end of the subsection 4.7 and Homework 8.)

We know that we can form cylindrical surface and cone surface bending the sheet of paper without "shrinking". On the other hand one can not form a part of sphere from the sheet of the paper without "shrinking" it. How to express mathematically this fact?

Consider on the sheet of the paper two close points A, B and the segment AB. The length of this segment is the shortest distance between points A and B. Any curve starting at the point A and finishing at the point B has the length which is greater or equal than the length of the segment AB. When we form cylindrical (or conic) surface bending the sheet of the paper we *do not distort this property.* The segment AB on the cylindrical surface will become the curve which we will denote also AB, but the length of this curve will be the same and it will be the shortest curve amongst all the curves connecting the points A and B. Internal observer ("ant" mathematician living on the cylindrical surface) observes that the curve AB on the cylinder has the same length as it has before (being the segment on flat sheet of the paper). This is strictly related with the fact that Gaussian curvature of the cylinder surface equals to zero.

Theorem (Theorema Eggregium) *The Gaussian curvature of surface is defined by first quadratic form. If Two surfaces have the same quadratic form then they have the same Gaussian curvature.*

In other words if we measure the length of the curves and angles between them on two surfaces we will come to the same answers, then these surfaces have the same Gaussian curvature.

In particular if a surfaces have vanishing Gaussian curvature then locally one comes to this surface bending the sheet of the paper without "shrinking".

This Theorem explains why sphere even locally cannot be transformed to the plane without distorting.

This remarkable Theorem which belongs to Gauss is the foundation result in differential geometry.

The proof of Theorem will be given in the course of Riemannian geometry.

Note that we calculated Gaussian curvature using Shape operator, i.e. in terms of External observer. The Theorem says that Gaussian curvature depends only on distances on the surface, hence the internal observer can calculate the gaussian curvature. How it can be done?

We will formulate another Theorem which is strictly related with the *Theorema Egregium* and explains how internal observer can calculate Gaussian curvature.

Theorem *Let C be a closed curve on a surface M such that C is a boundary of a compact oriented domain $D \subset M$, then during the parallel transport of an arbitrary tangent vector along the closed curve C the vector rotates through the angle*

$$\Delta\Phi = \angle(\mathbf{X}, \mathbf{R}_C\mathbf{X}) = \int_D K d\sigma\,, \tag{4.51}$$

where K is the Gaussian curvature and $d\sigma = \sqrt{\det g}dudv$ is the area element induced by the First quadratic form on the surface on the surface M, i.e. $d\sigma = \sqrt{\det g}dudv$.

In particular if Gaussian curvature K is constant then

$$\Delta\Phi = \angle(\mathbf{X}, \mathbf{R}_C\mathbf{X}) = KS \tag{4.52}$$

For example consider the sphere $x^2 + y^2 + z^2 = R^2$ and the triangle ABC on it with vertices $A = (0, 0, 1)$, $B = (1, 0, 0)$ and $C = (\cos\varphi, \sin\varphi, 0)$. Then during parallel transport of the vector along the triangle ABC it will rotate on the angle φ (see the Homework 8). On the other hand the area of this triangle equals to $S = R^2\varphi$. We see that

$$\varphi = \frac{S}{R^2} = KS$$

The angle of rotation of tangent vector in fact depends only on the internal geometry of surface. Thus the relation above can be used for proving the Theorema Egregium.

The Theorem above has very interesting

Corollary Let ABC be triangle on the surface M where AB, AC, BC are shortest curves connecting the points A, B, C. Let α, β, γ be angles of this triangle. For usual triangle sum of angles equal to π. It turns out that for triangel on the surface the sum of angles is related with Gaussian curvature:

$$\alpha + \beta + \gamma - \pi = \int_{\triangle ABC} K d\sigma \tag{4.53}$$

Internal observer may use this formula for calculating gaussian curvature at any given point: He draws the triangle calculate the sum of angles and see that

$$K \approx \frac{\alpha + \beta + \gamma - \pi}{S}$$

(See in more detail "A tale on differential geometry" in Appendices.)

5 †Appendices

5.1 Formulae for vector fields and differentials in cylindrical and spherical coordinates

Cylindrical and spherical coordinates

- Cylindrical coordinates in $\mathbf{E}^3$

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = h \end{cases} \qquad (0 \leq \varphi < 2\pi, 0 \leq r < \infty) \tag{5.1}$$

- Spherical coordinates in $\mathbf{E}^3$

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\cos\varphi \\ z = r\cos\theta \end{cases} \quad (0 \leq \varphi < 2\pi, 0 \leq r < \infty) --- \text{cylindrical coordiantes in } \mathbf{E}^3 \tag{5.2}$$

Example (Basis vectors and forms for cylindrical coordinates)

Consider cylindrical coordinates in $\mathbf{E}^3$: $u = r, v = \varphi, w = h$. Then calculating partial derivatives we come to

$$\begin{cases} \partial_r = \frac{\partial x}{\partial r}\partial_x + \frac{\partial y}{\partial r}\partial_y + \frac{\partial z}{\partial r}\partial_z = \cos\varphi\partial_x + \sin\varphi\partial_y \\ \\ \partial_\varphi = \frac{\partial x}{\partial \varphi}\partial_x + \frac{\partial y}{\partial \varphi}\partial_y + \frac{\partial z}{\partial \varphi}\partial_z = -r\sin\varphi\partial_x + r\cos\varphi\partial_y \\ \\ \partial_h = \frac{\partial x}{\partial h}\partial_x + \frac{\partial y}{\partial h}\partial_y + \frac{\partial z}{\partial h}\partial_z = \partial_z \end{cases} \tag{5.3}$$

Basic forms are $dr, d\varphi, dh$ and

$$\begin{gathered} dr(\partial_r) = 1, dr(\partial_\varphi) = 0, dr(\partial_h) = 0 \\ \\ d\varphi(\partial_r) = 0, d\varphi(\partial_\varphi) = 1, d\varphi(\partial_h) = 0 \\ \\ dh(\partial_r) = 0, dh(\partial_\varphi) = 0, dh(\partial_h) = 1 \end{gathered} \tag{5.4}$$

Example (Basis vectors for spheric coordinates)
Consider spheric coordinates in $\mathbf{E}^3$: $u = r, v = \theta, w = \varphi$. Then calculating partial derivatives we come to

$$\begin{cases} \partial_r = \frac{\partial x}{\partial r}\partial_x + \frac{\partial y}{\partial r}\partial_y + \frac{\partial z}{\partial r}\partial_z = \sin\theta\cos\varphi\partial_x + \sin\theta\sin\varphi\partial_y + \cos\theta\partial_z \\ \\ \partial_\theta = \frac{\partial x}{\partial \theta}\partial_x + \frac{\partial y}{\partial \theta}\partial_y + \frac{\partial z}{\partial \theta}\partial_z = r\cos\theta\cos\varphi\partial_x + r\cos\theta\sin\varphi\partial_y - r\sin\theta\partial_z \\ \\ \partial_\varphi = \frac{\partial x}{\partial \varphi}\partial_x + \frac{\partial y}{\partial \varphi}\partial_y + \frac{\partial z}{\partial \varphi}\partial_z = -r\cos\theta\sin\varphi\partial_x + r\sin\theta\cos\varphi\partial_y \end{cases} \tag{5.5}$$

Basic forms are $dr, d\theta, d\varphi$ and

$$\begin{gathered} dr(\partial_r) = 1, dr(\partial_\theta) = 0, dr(\partial_\varphi) = 0 \\ \\ d\theta(\partial_r) = 0, d\theta(\partial_\theta) = 1, d\theta(\partial_\varphi) = 0 \\ \\ d\varphi(\partial_r) = 0, d\varphi(\partial_\theta) = 0, d\varphi(\partial_\varphi) = 1 \end{gathered} \tag{5.6}$$

We know that 1-form is a linear function on tangent vectors. If $\mathbf{A}, \mathbf{B}$ are two vectors attached at the point $\mathbf{r}_0$, i.e. tangent to this point and ω, ρ are two 1-forms then one defines the value of $\omega \wedge \rho$ on $\mathbf{A}, B$ by the formula

$$\omega \wedge \rho(\mathbf{A}, \mathbf{B}) = \omega(\mathbf{A})\rho(B) - \omega(B)\rho(A) \tag{5.7}$$

We come to bilinear anisymmetric function on tangent vectors. If $\sigma = a(x, y)dx \wedge dy$ is an arbitrary two form then this form defines bilinear form on pair of tangent vectors: $\sigma(\mathbf{A}, \mathbf{B}) =$

$$a(x, y)dx \wedge dy(A, B) = a(x, y)\left(dx(\mathbf{A})dy(\mathbf{B}) - dx(\mathbf{B})dy(\mathbf{A})\right) = a(x, y)(A_x B_y - A_y B_y) \tag{5.8}$$

One can see that in the case if $a = 1$ then right hand side of this formula is nothing but the area of parallelogram spanned by the vectors $\mathbf{A}, \mathbf{B}$.

This leads to the conception of integral of form over domain.

Let $\omega = a(x)dx \wedge dy$ be a two form and D be a domain in $\mathbf{E}^2$. Then by definition

$$\int_D \omega = \int_D a(x,y)dxdy \tag{5.9}$$

If $\omega = dx \wedge dy$ then

$$\int_D w = \int_D (x,y)dxdy = \text{Area of the domain } D \tag{5.10}$$

The advantage of these formulae is that we do not care about coordinates[12]

Example Let D be a domain defined by the conditions

$$\begin{cases} x^2 + y^2 \leq 1 \\ y \geq 0 \end{cases} \tag{5.12}$$

Calculate $\int_D dx \wedge dy$.

$\int_D dx \wedge dy = \int_D dxdy =$ area of the $D = \frac{\pi}{2}$.

If we consider polar coordinates then according (2.40)

$$dx \wedge dy = rdr \wedge d\varphi$$

Hence $\int_D dx \wedge dy = \int_D rdr \wedge d\varphi = \int_D rdrd\varphi = \int_0^1 \left(\int_0^\pi d\varphi\right) rdr = \pi \int_0^1 rdr = \pi/2$.

Another example

Example Let D be a domain in $\mathbf{E}^2$ defined by the conditions

$$\begin{cases} \frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} \leq 1 \\ y \geq 0 \end{cases} \tag{5.13}$$

D is domain restricted by upper half of the ellipse and x-axis. Ellipse has the centre at the point $(c, 0)$. Its area is equal to $S = \int_D dx \wedge dy$. Consider new variables x', y': $x = c + ax', y = by'$. In new variables domain D becomes the domain from the previous example:

$$\frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} = {x'}^2 + {y'}^2$$

---

[12]If we consider changing of coordinates then jacobian appears: If u, v are new coordinates, $x = x(u,v)$, $y = y(u,v)$ are new coordinates then

$$\int a(x,y)dxdy = \int a(x(u,v), y(u,v)) \det \begin{pmatrix} x_u & x_v \\ x_u & x_v \end{pmatrix} dudv \tag{5.11}$$

In formula(5.9) it appears under as a part of coefficient of differential form.

and $dx \wedge dy = ab dx' \wedge dy'$. Hence

$$S = \int_{\frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} \leq 1, y \geq 0} dx \wedge dy = ab \int_{x'^2 + y'^2 \leq 1, y' \geq 0} dx' \wedge dy' = \frac{\pi ab}{2} \tag{5.14}$$

Theorem 2 (Green formula) Let ω be 2-form such that $\omega = d\omega'$ and D be a domain–interior of the closed curve C. Then

$$\int_D w = \int_C w' \tag{5.15}$$

5.2 Curvature and second order contact (touching) of curves

Let C_1, C_2 be two curves in $\mathbf{E}^2$. For simplicity we here consider only curves in $\mathbf{E}^2$.

Definition Two non-parameterised curves C_1, C_2 have second order contact (touching) at the point $\mathbf{r}_0$ if

- They coincide at the point $\mathbf{r}_0$
- they have the same tangent line at this point
- they have the same curvature at the point $\mathbf{r}_0$

If $\mathbf{r}_1(t), \mathbf{r}_2(t)$ are an arbitrary parameterisations of these curves such that $\mathbf{r}_1(t_0) = \mathbf{r}_2(t_0) = \mathbf{r}_0$ then the condition that they have the same tangent line means that velocity vectors $\mathbf{v}_1(t), \mathbf{v}_2(t)$ are collinear at the point t_0.

(As always we assume that curves under considerations are smooth and regular, i.e. $x(t), y(t)$ are smooth functions and velocity vector $\mathbf{v}(t) \neq 0$.)

Example Consider two curves C_f, C_g—graphs of the functions f_1, f_2. Recall that curvature of the graph of the function f at the point $(x, y = f(x))$ is equal to (see (3.28))

$$k(x) = \frac{f''(x)}{(1 + f'(x))^{\frac{3}{2}}} \tag{5.16}$$

Then condition of the second order touching at the point $\mathbf{r}_0 = (x_0, y_0)$ means that

$$\begin{cases} \text{They coincide at the point } \mathbf{r}_0\text{: } f(x_0) = g(x_0) \\ \text{They have the same tangent line at this point: } f'(x_0) = g'(x_0) \\ \text{They have the same curvature at the point } \mathbf{r}_0\text{:} \frac{f''(x_0)}{(1+f'(x_0))^{\frac{3}{2}}} = \frac{g''(x_0)}{(1+g'(x_0))^{\frac{3}{2}}}, \text{ i.e. } f''(x_0) = g''(x_0) \end{cases}$$

We see that second order touching means that difference of the functions in vicinity of the point x_0 is of order $o((x - x_0)^2)$. Indeed due to Taylor formula

$$\begin{array}{c} f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \dots \\ g(x) = g(x_0) + g'(x_0)(x - x_0) + \frac{1}{2} g''(x_0)(x - x_0)^2 + \dots \end{array} \tag{5.17}$$

where we denote by dots terms which are $o(x-x_0)^2$. (They say that $f(x) = o(x-x_0)^n$ if $\lim_{x\to x_0} \frac{f(x)}{(x-x_0)^n} = 0$).

Hence

$$f(x) - g(x) = o(x-x_0)^2 \tag{5.18}$$

because $f(x_0) = g(x_0), f(x_0) = g(x_0), f'(x_0) = g'(x_0)$ and $f''(x_0) = g''(x_0)$

In general case if two curves have second order contact then in the vicinity of the contact point one can consider these curves as a graphs of the functions $y = f(x)$ (or $x = f(y)$).

To clarify geometrical meaning of second order touching consider the case where one of the curves is a circle. Then second order touching means that curvature of one of these curves is equal to $1/R$, where R is a radius of the circle.

We see that to calculate the radius of the circle which has the second order touching with the given curve at the given point we have to calculate the curvature of this curve at this given point.

Example. Let C_1 be parabola $y = ax^2$ and C_2 be a circle. Suppose these curves have second order contact at the vertex of the parabola: point $0, 0$.

Calculate the curvature of the parabola at the vertex. Curvature at the vertex is equal to $k(t)|_{t=0} = 2a$ (see Homework). Hence the radius of the circle which has second order touching is equal to

$$R = \frac{1}{2a}.$$

To find equation of this circle note that the circle which has second order touching to parabola at the vertex passes trough the vertex (point $(0,0)$) and is tangent to x-axis. The radius of this circle is equal to $R = \frac{1}{2a}$. Hence equation of the circle is

$$(x - R^2) + y^2 = 0, \text{ where } R = \frac{1}{2a}$$

One comes to the same answer by the following detailed analysis:

Consider equation of a circle: $(x-x_0)^2 + (y-y_0)^2 = R^2$. The condition that curves coincide at the point $(0,0)$ means that $x_0^2 + y_0^2 = R^2$. x-axis is tangent to parabola at the vertex. Hence it is tangent to the circle too. Hence $y_0^2 = R^2$ and $x_0 = 0$. We see that an equation of the circle is $x^2 + (y-R)^2 = R^2$. The circle $x^2 + (y-R)^2 = x^2 + y^2 - 2yR = 0$ in the vicinity of the point $(0,0)$ can be considered as a graph of the function $y = R - \sqrt{R^2 - x^2}$.. The condition that functions $y = ax^2$ and $y = R - \sqrt{R^2 - x^2}$ have second order contact means that

$$R - \sqrt{R^2 - x^2} = ax^2 + \text{terms of the order less that } x^2 \,.$$

But

$$R - \sqrt{R^2 - x^2} = R - R\sqrt{1 - \frac{x^2}{2R^2}} = R - R\left(1 - \frac{x^2}{2R^2} + o(x^2)\right) = \frac{x^2}{2R} + o(x^2)\,.$$

Comparing we see that $a = \frac{1}{2R}$ and $\frac{1}{R} = 2a$. But curvature of the parabola at the vertex is equal to $k = 2a$ (if $a > 0$). We see that $k = \frac{1}{R}$.

5.3 Integral of curvature over planar curve.

We consider here the following problem: Let $C = \mathbf{r}(t)$ be a planar curve, i.e. a curve in $\mathbf{E}^2$.

Let $\mathbf{n}(\mathbf{r}(t)$ be a unit normal vector field to the curve, i.e. $\mathbf{n}$ is orthogonal to the curve (velocity vector) and it has unit length.

E.g. if $\mathbf{r}(t) : x(t) = R\cos t, y(t) = R\sin t$, then $\mathbf{n}(\mathbf{r}(t)) = \begin{pmatrix}\cos t\\ \sin t\end{pmatrix}$

If point moves along the curve $\mathbf{r}(t), t_1 \leq t \leq t_2$ then velocity vector and vector field $\mathbf{n}(t)$ rotate on the same angle. It turns out that this angle is expressed via integral of curvature over the curve...

Try to analyze the situation:

Proposition Let C: $\mathbf{r}(t)$ be a curve in $\mathbf{E}^2$, $\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt}$, velocity vector, $k(\mathbf{r}(t))$— curvature and $\mathbf{n}(t)$ unit normal vector field. Denote by $\varphi(t)$ the angle between normal vector $\mathbf{n}(t)$ and x-axis.

Then

$$\frac{d\mathbf{n}(t)}{dt} = \pm k(\mathbf{r}(t))\mathbf{v}(t) \tag{5.19}$$

$$\frac{d\varphi(t)}{dt} = \pm k(\mathbf{r}(t))|\mathbf{v}(t)| \tag{5.20}$$

(Sign depends on the orientation of the pair of vectors $(\mathbf{v}, \mathbf{n})$)

Note that the second statement of the Proposition has a clear geometrical meaning: If C is a circle of the radius R then RHS of (5.20) is equal to $\frac{v}{R}$. It is just angular velocity $d\varphi/dt$.

To prove this Proposition note that $(\mathbf{n}, \mathbf{n}) = 1$. Hence

$$0 = \frac{d}{dt}(\mathbf{n}(t), \mathbf{n}(t)) = 2\left(\frac{d\mathbf{n}(t)}{dt}, \mathbf{n}(t)\right),$$

i.e. vector $\frac{d\mathbf{n}(t)}{dt}$ is orthogonal to the vector $\mathbf{n}$. This means that $\frac{d\mathbf{n}(t)}{dt}$ is collinear to $\mathbf{v}(t)$, because curve is planar. We have $\frac{d\mathbf{n}(t)}{dt} = \kappa(\mathbf{r}(t))\mathbf{v}(t)$ where κ is a coefficient. Show that the coefficient κ is just equal to curvature k (up to a sign). Clearly $(\mathbf{n}, \mathbf{v}) = 0$ because these vectors are orthogonal. Hence

$$0 = \frac{d}{dt}(\mathbf{n}(t), \mathbf{v}(t)) = \left(\frac{d\mathbf{n}(t)}{dt}, \mathbf{v}(t)\right) + \left(\mathbf{n}(t), \frac{d\mathbf{v}(t)}{dt}\right) =$$

$$(\kappa(\mathbf{r}(t))\mathbf{v}(t), \mathbf{v}(t)) + (\mathbf{n}(t), \mathbf{a}(t)) = \kappa(\mathbf{r}(t))|\mathbf{v}(t)|^2 + (\mathbf{n}, \mathbf{a}_\perp),$$

because $(\mathbf{n}(t), \mathbf{a}(t)) = (\mathbf{n}, \mathbf{a}_\perp)$. But $(\mathbf{n}, \mathbf{a}_\perp)$ is just centripetal acceleration: $(\mathbf{n}, \mathbf{a}_\perp) = \pm|\mathbf{a}_\perp|$ and curvature is equal to $|\mathbf{a}_\perp|/|\mathbf{v}|^2$. Hence we come to $\kappa(\mathbf{r}(t)) = \pm\frac{|\mathbf{a}_\perp|}{|\mathbf{v}|^2} = \pm k$. Thus we prove (5.19).

To prove (5.20) consider expansion of vectors $\mathbf{n}(t), \mathbf{v}(t)$ over basis vectors ∂_x, ∂_y. We see that

$$\mathbf{n}(t) = \cos\varphi(t)\partial_x + \sin\varphi(t)\partial_y \text{ and } \mathbf{v}(t) = |\mathbf{v}(t)|\left(-\sin\varphi(t)\partial_x + \cos\varphi(t)\partial_y\right) \tag{5.21}$$

Differentiating $\mathbf{n}(t)$ by t we come to $\frac{d\mathbf{n}(t)}{dt} = \frac{d\varphi(t)}{dt}\left(-\sin\varphi(t)\partial_x + \cos\varphi(t)\partial_y\right) = \frac{d\varphi(t)}{dt}\frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$. Comparing this equation with equation (5.19) we come to (5.20).

The appearance of sign factor in previous formulae related with the fact that normal vector field is defined up to a sign factor $\mathbf{n} \to -\mathbf{n}$.

It is useful to write formulae (5.19), (5.20) in explicit way. Let $\mathbf{r}(t)\colon x(t), y(t)$ be a parameterisation of the curve. Then $\mathbf{v}(t) = \begin{pmatrix} x_t \\ y_t \end{pmatrix}$ velocity vector. One can define normal vector field as

$$\mathbf{n}(t) = \frac{1}{\sqrt{x_t^2 + y_t^2}}\begin{pmatrix} -y_t \\ x_t \end{pmatrix} \tag{5.22}$$

or changing the sign as

$$\mathbf{n}(t) = \frac{1}{\sqrt{x_t^2 + y_t^2}}\begin{pmatrix} y_t \\ -x_t \end{pmatrix} \tag{5.23}$$

If we consider (5.22) for normal vector field then

$$\frac{d\mathbf{n}(t)}{dt} = \frac{x_{tt}y_t - y_{tt}x_t}{(x_t^2 + y_t^2)^{\frac{3}{2}}}\begin{pmatrix} x_t \\ y_t \end{pmatrix} \tag{5.24}$$

Recalling that $k = \frac{|x_{tt}y_t - y_{tt}|x_t}{(x_t^2+y_t^2)^{\frac{3}{2}}}$ we come to (5.19). For the angle we have

$$\frac{d\varphi}{dt} = \frac{x_t y_{tt} - y_t x_{tt}}{(x_t^2 + y_t^2)^{\frac{3}{2}}}\sqrt{x_t^2 + y_t^2} = \frac{x_t y_{tt} - y_t x_{tt}}{(x_t^2 + y_t^2)} \tag{5.25}$$

This follows from the considerations above but it can be also calculated straightforwardly.

Remark Note that last two formulae do not possess indefenity in sign.

This Proposition has very important application. Consider just two examples:

Consider upper half part of the ellipse $x^2/a^2 + y^2/b^2 = 1, y \geq 0$. We already know that curvature at the point $x = a\cos t, y = b\cos t$ of the ellipse is equal to

$$k = \frac{ab}{(a^2 \sin t^2 t + b^2 \cos^2 t)^{3/2}}$$

and speed is equal to $\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}$ Apply formula (5.20) of Proposition. The curvature is not equal to zero at all the point. Hence the sign in the (5.20) is the same for all the points, i.e.

$$\pi = \int_0^\pi d\varphi(t)dt = \pm\int_0^\pi k(\mathbf{r}(t))|\mathbf{v}(t)| = \tag{5.26}$$

$$\int_0^\pi \frac{ab}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}}\sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\,dt = \int_0^\pi \frac{abdt}{a^2 \sin^2 t + b^2 \cos^2 t}\,.$$

We calculated this integral using geometrical considerations: left hand side represents the angle of rotation of normal unit vector and this angle is equal to π. Try to calculate the last integral straightforwardly: it is not easy exercise in calculus.

Another example: Let $\mathbf{r} = \mathbf{r}(t), x = x(t), y = y(t), t_1 \le t \le t_2$ be a closed curve in $\mathbf{E}^2$ ($\mathbf{r}(t_1) = \mathbf{r}(t_2)$.) We suppose that it possesses self-intersections points. We cannot us a formula (5.20) for integration because in general curvature may vanish at some points, but we still can use the formula (5.25). The rotation of the angle φ is equal to $2\pi n$, (n-is called winding number of the curve). Hence according to (5.25) see that

$$\int_{t_1}^{t_2} \frac{x_t y_{tt} - x_t y_{tt}}{x_t^2 + y_t^2} = 2\pi n$$

or

$$\frac{1}{2\pi} \int_{t_1}^{t_2} \frac{x_t y_{tt} - x_t y_{tt}}{x_t^2 + y_t^2} = n \tag{5.27}$$

The integrand us equal to the curvature multiplied by the speed (up to a sign). Left hand side is integral of continuous function divided by transcendent number π. The geometry tells us that the answer must to be equal to integer number.

5.4 Relations between usual curvature normal curvature and geodesic curvature.

Consider at any point $\mathbf{p}$ of the curve the following basis $\{\mathbf{v}, \mathbf{f}, \mathbf{n}\}$, where

- velocity vector $\mathbf{v}$ is tangent to the curve
- the vector $\mathbf{f}$ is the vector tangent to the surface but orthogonal to the vector $\mathbf{v}$.
- $\mathbf{n}$ is the unit normal vector to the surface, i.e.it is orthogonal to vectors $\mathbf{v}$ and $\mathbf{f}$.

Decompose acceleration vector over three directions, i.e. over three one-dimensional spaces spanned by vectors $\mathbf{v}, \mathbf{f}$ and $\mathbf{n}$:

$$\mathbf{a} = \mathbf{a}_{\text{orthogonal to surface}} + \mathbf{a}_{\text{tang.to surf. and orthog. to curve}} + \mathbf{a}_{\text{tangent to curve}} \tag{5.28}$$

The vector $\mathbf{a}_{\text{orthogonal to surface}}$ which is collinear to normal unit vector $\mathbf{n}$, will be called *vector of normal acceleration of the curve on the surface.* We denote it by $\mathbf{a}_n$.

The vector $\mathbf{a}_{\text{tang.to surf. and orthog. to curve}}$, collinear to unit vector $\mathbf{f}_C$ will be called *vector of geodesic acceleration* . We denote it by $\mathbf{a}_{geod}$.

The vector $\mathbf{a}_{\text{tangent to curve}}$, collinear to velocity vector $\mathbf{v}$, is just *vector of tangential acceleration* . We denote it $\mathbf{a}_{tang}$. We can rewrite (5.28) as

$$\mathbf{a} = \mathbf{a}_n + \mathbf{a}_{geod} + \mathbf{a}_{tang} \tag{5.29}$$

Study the expansion (5.29). Both vectors $\mathbf{a}_n$ and $\mathbf{a}_{geod}$ are orthogonal to the curve. The vector $\mathbf{a}_{geod}$ is orthogonal to the curve but it is tangent to the surface. The vector $\mathbf{a}_n$ is orthogonal not only to the curve. It is orthogonal to the surface.

The vector $\mathbf{a}_{geod} + \mathbf{a}_n = \mathbf{a}_\perp$ is orthogonal to the curve. It is the vector of normal acceleration of the curve.

Remark Please note that when we consider the curves on the surface it could arise the confusion between the vector $\mathbf{a}_n$—normal acceleration of the curve on the surface and the vector $\mathbf{a}_\perp$ of normal acceleration of the curve (see (3.2)).

When we decompose in (5.29) the acceleration vector $\mathbf{a}$ in the sum of three vectors $\mathbf{a}_n, \mathbf{a}_{geod}$ and $\mathbf{a}_{tang}$ then the vector $\mathbf{a}_n$, *the normal acceleration of the curve on the surface* is orthogonal to the surface not only to the curve. The vector

$$\mathbf{a}_\perp = \mathbf{a}_n + \mathbf{a}_{geod},$$

is orthogonal only to the curve and in general it is not orthogonal to the surface (if $\mathbf{a}_{geod} \neq 0$). It is the normal acceleration of the curve. It depends only on the curve. The normal acceleration $\mathbf{a}_n$ of the curve on the surface which is orthogonal to the surface depends on the surface where the curve lies.

We know that the curvature of the curve is equal to the magnitude of normal acceleration of the curve divided on the square of the speed (see (3.22)). We have:

$$\text{curvature of the curve } k = \frac{|\mathbf{a}_\perp|}{|\mathbf{v}|^2} = \frac{|\mathbf{a}_n + \mathbf{a}_{geod}|}{|\mathbf{v}|^2}\,.$$

The vectors $\mathbf{a}_n$ and $\mathbf{a}_{geod}$ transform under reparameterisation in the same way as a vector $\mathbf{a}_\perp$ (see (3.11)). If $t \to t(\tau)$ then

$$\mathbf{a}'_\perp(\tau) = t_\tau^2 \mathbf{a}_\perp \ \text{ and } \ \mathbf{a}'_n(\tau) = t_\tau^2 \mathbf{a}_n(t), \quad \mathbf{a}'_{geod}(\tau) = t_\tau^2 \mathbf{a}_{geod}(t) \tag{5.30}$$

where $\mathbf{a}'(\tau) = \frac{d^2}{d\tau^2}\mathbf{r}(t(\tau)) = t_\tau^2 \mathbf{a} + t_{\tau\tau}\mathbf{v}$ (see (3.9), (3.10), (3.8)). Hence the magnitudes

$$\frac{|\mathbf{a}_{geod}|}{|\mathbf{v}|^2} \ \text{and} \ \qquad \frac{|\mathbf{a}_n|}{|\mathbf{v}|^2} \tag{5.31}$$

are reparameterisation invariant as well as magnitude $k = \frac{|\mathbf{a}_\perp|}{|\mathbf{v}|^2} = \frac{|\mathbf{a}_n + \mathbf{a}_{geod}|}{|\mathbf{v}|^2}$.

Multiply left and right hand sides of the equation (5.29) on unit normal vector $\mathbf{n}$. Then $(\mathbf{a}_{tang}, \mathbf{n}) = (\mathbf{a}_{geod}, \mathbf{n}) = 0$ because vectors $\mathbf{a}_{geod}$ and $\mathbf{a}_{tang}$ are orthogonal to the vector $\mathbf{n}$. We come to the relation

$$\mathbf{a}_n = (\mathbf{n}, \mathbf{a})\mathbf{n} \ \text{ and}\,, |\mathbf{a}_n| = |(\mathbf{a}, \mathbf{n})|\,. \tag{5.32}$$

Or in other words scalar product $(\mathbf{n}, \mathbf{a})$ is equal to $|\mathbf{a}_n|$ (up to a sign).

Compare the formula

$$\kappa_n = \frac{(\mathbf{n}, \mathbf{a})}{(\mathbf{v}, \mathbf{v})} \tag{5.33}$$

(see (4.28)) for normal curvature with the formula

$$k = \frac{|\mathbf{a}_\perp|}{(\mathbf{v}, \mathbf{v})}$$

for usual curvature (see (3.22)).

It follows from (5.30), (5.31) and (4.28) (or (5.33)) that for any curve on the surface the modulus of the normal curvature is less or equal than usual curvature.

$$|\kappa_n| \leq k \tag{5.34}$$

Indeed we have for usual curvature

$$k = \frac{|\mathbf{a}_\perp|}{|\mathbf{v}|^2} = \frac{|\mathbf{a}_{geod} + \mathbf{a}_{normal}|}{|\mathbf{v}|^2} = \sqrt{\frac{\mathbf{a}_{geod}^2 + \mathbf{a}_{norm}^2}{|\mathbf{v}|^2}} \geq \frac{|\mathbf{a}_{normal}|}{|\mathbf{v}|^2} = |\kappa_n| \tag{5.35}$$

Normal curvature is a positive or negative real number. (Usual curvature is non-negative real number). Normal curvature changes a sign if $\mathbf{n} \to -\mathbf{n}$.

Remark We obtained in (5.31) that the magnitude $\frac{|\mathbf{a}_{geod}|}{|\mathbf{v}|^2}$ is reparameterisation invariant. It defines so called *geodesic curvature* $\kappa_{geod} = \frac{|\mathbf{a}_{geod}|}{|\mathbf{v}|^2}$. We see that usual curvature k, normal curvature κ and geodesic curvature κ_{geod} are related by the formula

$$k^2 = \kappa_{geod}^2 + \kappa_{normal}^2 \tag{5.36}$$

5.5 Normal curvature of curves on cylinder surface.

Example Consider an arbitrary curve $C\colon h = h(t), \varphi = \varphi(t)$ on the cylinder

$$\mathbf{r}(\varphi, h)\colon \begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \\ z = h \end{cases}$$

Pick any point $\mathbf{p}$ on this curve and find normal acceleration vector at this point of this curve.

Without loss of generality suppose that point $\mathbf{p}$ is just a point $(R, 0, 0)$. Note that vector $\mathbf{e}_x$ attached at the point $(R, 0, 0)$ is unit vector orthogonal to the surface of cylinder, i.e. $\mathbf{e}_x = -\mathbf{n}$ at the point $\mathbf{p} = (R, 0, 0)$.

RemarkUnit vector, as well as normal curvature is defined up to a sign. It is convenient for us to choose $\mathbf{n} = -\mathbf{e}_x$, not $\mathbf{n} = \mathbf{e}_x$.

Vectors $\mathbf{e}_y, \mathbf{e}_z$ are tangent to the surface of cylinder. At the point $\mathbf{p} = (R, 0, 0)$ $\varphi = 0, h = 0$.

We have

$$\mathbf{v} = \frac{d\mathbf{r}(t)}{dt} = \frac{dx(t)}{dt}\mathbf{e}_x + \frac{dy(t)}{dt}\mathbf{e}_y + \frac{dz(t)}{dt}\mathbf{e}_z =$$

$$R\frac{d\cos\varphi(t)}{dt}\mathbf{e}_x + R\frac{d\sin\varphi(t)}{dt}\mathbf{e}_y + \frac{dh(t)}{dt}\mathbf{e}_z = -R\sin\varphi\dot{\varphi}\mathbf{e}_x + R\cos\varphi\dot{\varphi}\mathbf{e}_y + \dot{h}\mathbf{e}_z$$

$$\text{Thus } \mathbf{v} = R\dot{\varphi}\mathbf{e}_y + \dot{h}\mathbf{e}_z \text{ at the point } \mathbf{p} = (R, 0, 0)\,. \tag{5.37}$$

For acceleration vector

$$\mathbf{a} = \frac{d^2\mathbf{r}(t)}{dt^2} = \frac{d^2x(t)}{dt^2}\mathbf{e}_x + \frac{d^2y(t)}{dt^2}\mathbf{e}_y + \frac{d^2z(t)}{dt^2}\mathbf{e}_z = R\frac{d^2\cos\varphi(t)}{dt^2}\mathbf{e}_x + R\frac{d^2\sin\varphi(t)}{dt^2}\mathbf{e}_y + \frac{d^2h(t)}{dt^2}\mathbf{e}_z =$$

$R\left(-(\dot{\varphi})^2\cos\varphi - \ddot{\varphi}\sin\varphi\right)\mathbf{e}_x + R\left(-(\dot{\varphi})^2\sin\varphi + \ddot{\varphi}\cos\varphi\right)\mathbf{e}_y + \ddot{h}\mathbf{e}_z = \ddot{\varphi}R\mathbf{e}_y + \ddot{h}\mathbf{e}_z - (\dot{\varphi})^2R\mathbf{e}_x$ at the point $\mathbf{p} = (R, 0, 0)$ where $\cos\varphi = 0, \sin\varphi = 1$. We see that

$$\mathbf{a} = \underbrace{\ddot{\varphi}R\mathbf{e}_y + \ddot{h}\mathbf{e}_z}_{\text{tangent to the surface}} - \underbrace{(\dot{\varphi})^2R\mathbf{e}_x}_{\text{normal to the surface}} \tag{5.38}$$

We see that $\mathbf{a}_n = (\dot{\varphi})^2R\mathbf{e}_x$. Comparing with velocity vector (5.37) we see that

$$\mathbf{a}_n = \frac{\mathbf{v}^2_{horizontal}}{R}\mathbf{n} \tag{5.39}$$

We see that for any curve on the cylinder $x^2 + y^2 = R^2$ the normal curvature $\frac{(\mathbf{a}_n, \mathbf{n})}{|\mathbf{v}|^2}$ (see (4.28)) is equal to

$$\frac{(\mathbf{a}_n, \mathbf{n})}{|\mathbf{v}|^2} = \frac{R\dot{\varphi}^2}{R^2\dot{\varphi}^2 + \dot{h}^2} \tag{5.40}$$

and it obeys relations

$$0 \le \kappa_{normal} \le \frac{1}{R}$$

depending of the curve. E.g. if the curve on the cylinder is a straight line $x = x_0, y = y_0, z = t$ then $\mathbf{a} = 0$ and normal curvature of this curve is equal to the naught as well as usual curvature.

If the curve is circle $x = R\cos t, y = R\sin t, z = z_0$ then normal curvature of this curve as well as usual curvature is equal to $\frac{1}{R}$.

Remark Very important conclusion from this example is
normal curvature of the cylinder of the radius R takes values in the interval $\left(0, \frac{1}{R}\right)$. *It cannot be greater than* $\frac{1}{R}$
Note that we can consider on cylinder very curly curve of very big curvature. The normal curvature at the points of this curve will be still less than $\frac{1}{R}$.

At any point of the surface normal curvature in general depends on the curve but it takes values in the restricted interval.

E.g. for the sphere of radius R one can see that normal curvature at any point is equal to $\frac{1}{R}$ independent of curve. In spite of this fact the usual curvature of curve can be very big [13]. If we consider the circle of very small radius r on the sphere then its usual curvature is equal to $k = \frac{1}{r}$ and $k \to \infty$ if $r \to 0$ So we see that one can define curvature of surface in terms of normal curvature.

[13]It is the geodesic curvature of the curves which characterises its curvature with respect to the curve. The relation between usual geodesic and normal curvature is given by the formula (5.36).

5.6 Concept of parallel transport of the vector tangent to the surface

Parallel transport of the vectors is one of the fundamental concept of differential geometry. Here we just give some preliminary ideas and formulate the concept of parallel transport for surfaces embedded in Euclidean space.

Let M be a surface $\mathbf{r} = \mathbf{r}(u, v)$ in $\mathbf{E}^3$ and $C\colon \mathbf{r}(t) = \mathbf{r}(u(t), v(t))$, $t_1 \le t \le t_2$ be a curve on this surface.

Let $\mathbf{X}_1$ be a vector tangent to the surface at the initial point $p = \mathbf{r}(t_1)$ of the curve $\mathbf{r}(t)$ on the surface: $\mathbf{X}_1 \in T_pM$. Note that $\mathbf{X}_1$ is a vector tangent to the surface, not necessarily to the curve. We define now parallel transport of the vector along the curve C:

Definition Let $\mathbf{X}(t)$ be a family of vectors depending on the parameter t ($t_1 \le t \le t_2$) such that following conditions hold

- For every $t \in [t_1, t_2]$ vector $\mathbf{X}(t)$ is a vector tangent to the surface M at the point $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ of the curve C.
- $\mathbf{X}(t) = \mathbf{X}_1$ for $t = t_1$
- $\frac{d\mathbf{X}(t)}{dt}$ is orthogonal to the surface, i.e.

$$\frac{d\mathbf{X}(t)}{dt} \text{ is collinear to the normal vector } \mathbf{n}(t),\ \frac{d\mathbf{X}(t)}{dt} = \lambda(t)\mathbf{n}(t) \tag{5.41}$$

  Recall that normal vector $\mathbf{n}(t)$ is a vector attached to the point $\mathbf{r}(t)$ of the curve $C\colon \mathbf{r}(t)$. This vector is orthogonal to the surface M.

  The condition (5.41) means that only orthogonal component of vector field $\mathbf{X}(t)$ can be changed.

We say that a family $\mathbf{X}(t)$ is a parallel transport of the vector $\mathbf{X}_1$ along a curve $C\colon \mathbf{r}(t)$ on the surface M. The final vector $\mathbf{X}_2 = \mathbf{X}(t_2)$ is the image of the vector $\mathbf{X}_1$ under the parallel transport along the curve C.

Using the relation (5.41) it is easy to see that the scalar product of two vectors remains invariant under parallel transport. In particularly it means that length of the vector does not change. If $\mathbf{X}(t)$, $\mathbf{Y}(t)$ are parallel transports of vectors $\mathbf{X}_1, \mathbf{Y}_1$ then

$$\frac{d}{dt}\left(\mathbf{X}(t), \mathbf{Y}(t)\right) = \left(\frac{d\mathbf{X}(t)}{dt}, \mathbf{Y}(t)\right) + \left(\mathbf{X}(t), \frac{d\mathbf{Y}(t)}{dt}\right) = 0$$

because vector $\frac{d\mathbf{X}(t)}{dt}$ is orthogonal to the vector $\mathbf{Y}(t)$ and vector $\frac{d\mathbf{Y}(t)}{dt}$ is orthogonal to the vector $\mathbf{X}(t)$. In particularly length does not change:

$$\frac{d}{dt}|\mathbf{X}(t)|^2 = \frac{d}{dt}(\mathbf{X}(t), \mathbf{X}(t)) = 2\left(\frac{d\mathbf{X}(t)}{dt}, \mathbf{X}(t)\right) = 2(\lambda(t)\mathbf{n}(t), \mathbf{X}(t)) = 0 \tag{5.42}$$

Remark The relation (5.41) shows how the surface is engaged in the parallel transport. Note that it is non-sense to put the right hand side of the equation (5.41) equal to zero:

In general a tangent vector ceased to be tangent to the surface if it is not changed! (E.g. consider the vector which transports along the great circle on the sphere)

Example

In the case if surface is a plane then everything is easy. If vector $\mathbf{X}_1$ is tangent to the plane at the given point, it is tangent at all the points. Vector does not change under parallel transport $\mathbf{X}(t) \equiv \mathbf{X}$, $\frac{d\mathbf{X}(t)}{dt} = 0$.

Example

Consider the vector $\mathbf{e}_x = \frac{\partial}{\partial x}$ attached at the point $(0, 0, R)$. It is tangent vector to the sphere $x^2 + y^2 + z^2 = R^2$ at the North Pole. Define parallel transport of this vector along the meridian $\varphi = 0, \theta = t$: $\mathbf{r}(t)\colon x = R \sin t, y = 0, z = R \cos t$.

Consider the vector field $\mathbf{X}(t) = \begin{pmatrix} \cos t \\ 0 \\ -\sin t \end{pmatrix}$ attached at the point $\mathbf{r}(t)$ of the meridian.

One can see that $\mathbf{X}(t)|_{t=0} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is the initial vector attached at the North pole and

$$\frac{d\mathbf{X}(t)}{dt} = \frac{d}{dt} \begin{pmatrix} \cos t \\ 0 \\ -\sin t \end{pmatrix} = \begin{pmatrix} -\sin t \\ 0 \\ -\cos t \end{pmatrix} = -\frac{\mathbf{r}(t)}{R}$$

is orthogonal to the surface of the sphere. Hence $\mathbf{X}(t)$ is the parallel transport of the initial vector along the meridian on the sphere.

We consider other more sophisticated examples of parallel transport of vectors along curves on surfaces in next sections.

5.7 Parallel transport of vectors tangent to the sphere.

1. Consider now in a more detail the parallel transport along curves on sphere.

In the case if surface is a plane then everything is easy. If vector $\mathbf{X}_1$ is tangent to the plane at the given point, it is tangent at all the points. Vector does not change under parallel transport $\mathbf{X}(t) \equiv \mathbf{X}$.

Consider a case of parallel transport along curves on the sphere.

Consider on the sphere $x^2 + y^2 + z^2 = a^2$ (a is a radius) tangent vectors:

$$\mathbf{r}_\theta = \begin{pmatrix} a \cos\theta \cos\varphi \\ a \cos\theta \sin\varphi \\ -a \sin\theta \end{pmatrix} \quad \mathbf{r}_\varphi = \begin{pmatrix} -a \sin\theta \sin\varphi \\ a \sin\theta \cos\varphi \\ 0 \end{pmatrix} \tag{5.43}$$

attached at the point $\mathbf{r}(\theta, \varphi) = \begin{pmatrix} a \sin\theta \cos\varphi \\ a \sin\theta \sin\varphi \\ a \cos\theta \end{pmatrix}$. One can see that

$$(\mathbf{r}_\theta, \mathbf{r}_\theta) = a, \quad (\mathbf{r}_h, \mathbf{r}_\varphi) = 0, \quad (\mathbf{r}_\varphi, \mathbf{r}_\varphi) = a^2 \sin^2\theta$$

It is convenient to introduce vectors which are parallel to these vectors but have unit length:

$$\mathbf{e}_\theta = \frac{\mathbf{r}_\theta}{a}, \quad \mathbf{e}_\varphi = \frac{\mathbf{r}_\varphi}{a \sin\theta} \qquad (\mathbf{e}_\theta, \mathbf{e}_\theta) = 1, (\mathbf{e}_\theta, \mathbf{e}_\varphi) = 0, (\mathbf{e}_\varphi, \mathbf{e}_\varphi) = 1\,. \tag{5.44}$$

How these vectors change if we move along parallel (i.e. what is the value of $\frac{\partial \mathbf{e}_\theta}{\partial \varphi}$, $\frac{\partial \mathbf{e}_\varphi}{\partial \varphi}$); how these vectors change if we move along meridians (i.e. what is the value of $\frac{\partial \mathbf{e}_\theta}{\partial \theta}$, $\frac{\partial \mathbf{e}_\varphi}{\partial \theta}$). First of all recall that unit normal vector to the sphere at the point θ, φ is equal to $\frac{\mathbf{r}(\theta,\varphi)}{a}$:

$$\mathbf{n}(\theta,\varphi) = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

Now calculate:

$$\frac{\partial \mathbf{e}_\theta}{\partial \theta} = \frac{\partial}{\partial \theta} \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} = \begin{pmatrix} -\sin\theta\cos\varphi \\ -\sin\theta\sin\varphi \\ -\cos\theta \end{pmatrix} = -\mathbf{n} \tag{5.45}$$

,

$$\frac{\partial \mathbf{e}_\theta}{\partial \varphi} = \frac{\partial}{\partial \varphi} \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} = \begin{pmatrix} -\cos\theta\sin\varphi \\ \cos\theta\cos\varphi \\ 0 \end{pmatrix} = \cos\theta \mathbf{e}_\varphi, \tag{5.46}$$

,

$$\frac{\partial \mathbf{e}_\varphi}{\partial \theta} = \frac{\partial}{\partial \theta} \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} -\cos\theta\sin\varphi \\ \cos\theta\cos\varphi \\ 0 \end{pmatrix} = 0, \tag{5.47}$$

$$\frac{\partial \mathbf{e}_\varphi}{\partial \varphi} = \frac{\partial}{\partial \varphi} \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} = \begin{pmatrix} -\cos\varphi \\ -\sin\varphi \\ 0 \end{pmatrix} = -\sin\theta\mathbf{n} - \cos\theta\mathbf{e}_\theta, \tag{5.48}$$

Some of these formulaes are intuitively evident: For example formula (5.45) which means that family of the vectors $\mathbf{e}_\theta(\theta)$ is just parallel transport along meridian, because its derivation is equal to $-\mathbf{n}$.

Another intuitively evident example: consider the meridian $\theta(t) = t$, $\varphi(t) = \varphi_0$, $0 \le t \le \pi$. It is easy to see that the vector field

$$\mathbf{X}(t) = \mathbf{e}_\theta(\theta(t), \varphi_0) = \begin{pmatrix} \cos\theta(t)\cos\varphi_0 \\ \cos\theta(t)\sin\varphi_0 \\ -\sin\theta(t) \end{pmatrix}$$

attached at the point $(\theta(t), \varphi_0)$ is a parallel transport because for family of vectors $\mathbf{X}(t)$ all the conditions of parallel transport are satisfied. In particular according to (5.45)

$$\frac{d\mathbf{X}(t)}{dt} = \frac{d\theta(t)}{dt}\frac{\partial}{\partial\theta} \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} = -\mathbf{n}(\theta(t), \varphi_0)$$

Now consider an example which is intuitively not-evident.

Example. Calculate parallel transport of the vector $\mathbf{e}_\varphi$ along the parallel. On the sphere of the radius a consider the parallel

$$\theta(t) = \theta_0, \varphi(t) = t, \quad 0 \leq t \leq 2\pi \tag{5.49}$$

In cartesian coordinates equation of parallel will be:

$$\mathbf{r}(t) = \begin{pmatrix} a\sin\theta(t)\cos\varphi(t) \\ a\sin\theta(t)\sin\varphi(t) \\ -a\cos\theta(t) \end{pmatrix} = \begin{pmatrix} a\sin\theta_0\cos t \\ a\sin\theta(t)\sin t \\ -a\cos\theta_0 \end{pmatrix}, \quad 0 \leq t \leq 2\pi \tag{5.50}$$

It is easy to see that the family of the vectors $\mathbf{e}_\varphi(\theta_0, \varphi(t))$ on parallel, is not parallel transport! because $\frac{d\mathbf{e}_\varphi(\theta_0,\varphi(t))}{dt} = \frac{d\mathbf{e}_\varphi(\theta_0,\varphi)}{d\varphi}$ is not equal to zero (see (5.48) above). Let a family of vectors $\mathbf{X}(t)$ be a parallel transport of the vector $\mathbf{e}_\varphi$ along the parallel (5.49): $\mathbf{X}(t) = a(t)\mathbf{e}_\theta(t) + b(t)\mathbf{e}_\varphi(t)$ where $a(t), b(t)$ are components of the tangent vector $\mathbf{X}(t)$ with respect to the basis $\mathbf{e}_\theta, \mathbf{e}_\varphi$ at the point $\theta = \theta_0, \varphi = t$ on the sphere. Initial conditions for coefficients are $a(t)|_{t=0} = 0, b(t)|_{t=0} = 1$ According to the definition of parallel transport and formulae (5.45)—(5.48) we have:

$$\frac{d\mathbf{X}(t)}{dt} = \frac{d\left(a(t)\mathbf{e}_\theta(t) + b(t)\mathbf{e}_\varphi(t)\right)}{dt} = \left(\frac{da(t)}{dt}\right)\mathbf{e}_\theta + a(t)\cos\theta_0\mathbf{e}_\varphi + \frac{db(t)}{dt}\mathbf{e}_\varphi +$$

$$b(t)\left(-\sin\theta_0\mathbf{n} - \cos\theta\mathbf{e}_\theta\right) =$$

$$= \left(\frac{da(t)}{dt} - b(t)\cos\theta_0\right)\mathbf{e}_\theta + \left(\frac{db(t)}{dt} + a(t)\cos\theta_0\right)\mathbf{e}_\varphi - b(t)\sin\theta_0\mathbf{n} \tag{5.51}$$

Under parallel transport only orthogonal component of the vector changes. Hence we come to differential equations

$$\begin{cases} \frac{da(t)}{dt} - wb(t) = 0 \\ \frac{db(t)}{dt} + wa(t) \end{cases} \qquad a(0) = 0, b(0) = 0, w = \cos\theta_0 \tag{5.52}$$

The solution of these equations is $a(t) = \sin wt, b(t) = \cos wt$. We come to the following answer: parallel transport along parallel $\theta = \theta_0$ of the initial vector $\mathbf{e}_\varphi$ is the family

$$\mathbf{X}(t) = \sin wt\,\mathbf{e}_\theta + \cos wt\,\mathbf{e}_\varphi, w = \cos\theta_0 \tag{5.53}$$

During traveling along the parallel $\theta = \theta_0$ the $\mathbf{e}_\theta$ component becomes non-zero At the end of the traveling the initial vector $\mathbf{X}(t)|_{t=0} = \mathbf{e}_\varphi$ becomes $\mathbf{X}(t)|_{t=2\pi} = \sin 2\pi w\mathbf{e}_\theta + \cos 2\pi w\mathbf{e}_\varphi$: **the vector $\mathbf{e}_\varphi$ after woldtrip traveling along the parallel $\theta = \theta_0$ transforms to the vector $\sin(2\pi\cos\theta_0)\mathbf{e}_\theta + \cos(2\pi\cos\theta_0)\mathbf{e}_\varphi$. In particularly this means that the vector $\mathbf{e}_\varphi$ after parallel transport will rotate on the angle**

$$\text{angle of rotation} = 2\pi\cos\theta_0$$

Compare the angle of rotation with the area of the segment of the sphere above the parallel $\theta = \theta_0$. According to the formula (**??**) area of this segment is equal to $S = 2\pi ah = 2\pi a^2(1 - \cos\theta_0)$. On the other hand Gaussian curvature of the sphere is equal to $\frac{1}{a^2}$. Hence we see that up to the sign angle of rotation is equal to area of the seqment divided on the Gaussian curvature:

$$\Delta\varphi = \pm\frac{S}{K} = \pm 2\pi\cos\theta_0 \tag{5.54}$$

5.8 Parallel transport along a closed curve on arbitrary surface.

The formula above for the parallel transport along parallel on the sphere keeps in the general case.

Theorem Let M be a surface in $\mathbf{E}^3$. Let $\mathbf{r}(t)\colon \mathbf{r}(t), t_1 \leq t \leq t_2, \mathbf{r}(t_1) = \mathbf{r}(t_2)$ be a closed curve on the surface M such that it is a boundary of domain D of the surface M. (We suppose that the domain D is bounded an orientate.) Let $\mathbf{X}(t)$ be a parallel transport of the arbitrary tangent vector along this closed curve. Consider initial and final vectors $\mathbf{X}(t_1), \mathbf{X}(t_2)$. They have the same length according to (5.42).

Theorem *The angle $\Delta\varphi$ between these vectors is equal to the integral of Gaussian curvature over the domain D:*

$$\Delta\varphi = \pm \int_D K d\sigma \tag{5.55}$$

where we denote by $d\sigma$ the element of the area of surface of M.

The calculations above for traveling along the parallel are just example of this Theorem. The integral of Gaussian curvature over the domain above parallel $\theta = \theta_0$ is equal to $K \cdot 2\pi a(1 - \cos\theta_0) = \frac{1}{a^2} \cdot 2\pi a^2(1 - \cos\theta_0) = 2\pi(1 - \cos\theta_0)$. This is equal to the angle of rotation $2\pi\cos\theta_0$ (up to a sign and modulo 2π). Another simple

Example. Consider on the sphere $x^2 + y^2 + z^2 = a^2$ points $A = (0,0,1)$, $B = (1,0,0)$ and $C = (0,1,0)$. Consider arcs of great circles which connect these points. Consider the vector $\mathbf{e}_x$ attached at the point A. This vector is tangent to the sphere. It is easy to see that under parallel transport along the arc AB it will transform at the point B to the vector $-\mathbf{e}_z$. The vector $-\mathbf{e}_z$ under parallel transport along the arc BC will remain the same vector $-\mathbf{e}_z$. And finally under parallel transport along the arc CA the vector $-\mathbf{e}_z$ will transform at the point A to the vector $-\mathbf{e}_y$. We see that under traveling along the curvilinear triangle ABC vector $\mathbf{e}_x$ becomes the vector $-\mathbf{e}_y$, i.e. it rotates on the angle $\frac{\pi}{2}$. It is just the integral of the curvature $\frac{1}{a^2}$ over the triangle ABC: $K \cdot S = \frac{1}{a^2} \cdot \frac{4\pi a^2}{8} = \frac{\pi}{2}$.

We know that for planar triangles sum of the angles is equal to π. It turns out that

Corollary Let ABC be a triangle on the surface formed by geodesics. Then

$$\angle A + \angle B + \angle C = \pi + \int_{\triangle ABC} K ds \tag{5.56}$$

The Gaussian curvature measures the difference of π and sum of angles.

The corollary evidently follows form the Theorem. It is of great importance: It gives us tool to measure curvature. (See the tale about ant.)

5.9 Gauss Bonnet Theorem

Consider the integral of curvature over whole closed surface M. According to the Theorem above the answer has to be equal to 0 (modulo 2π), i.e. $2\pi N$ where N is an integer, because this integral is a limit when we consider very small curve. We come to the formula:

$$\int_D K d\sigma = 2\pi N$$

(Compare this formula with formula (5.27)).

What is the value of integer N?

We present now one remarkable Theorem which answers this question and prove this Theorem using the formula (5.56).

Let M be a closed orientable surface.[14] All these surfaces can be classified up to a diffeomorphism. Namely arbitrary closed oriented surface M is diffeomorphic either to sphere (zero holes), or torus (one hole), or pretzel (two holes),... "Number k" of holes is intuitively evident characteristic of the surface. It is related with very important characteristic—Euler characteristic $\chi(M)$ by the following formula:

$$\chi(M) = 2(1 - g(M)), \quad \text{where } g \text{ is number of holes} \tag{5.57}$$

Remark What we have called here "holes" in a surface is often referred to as "handles" attached o the sphere, so that the sphere itself does not have any handles, the torus has one handle, the pretzel has two handles and so on. The number of handles is also called genus.

Euler characteristic appears in many different way. The simplest appearance is the following:

Consider on the surface M an arbitrary set of points (vertices) connected with edges (graph on the surface) such that surface is divided on polygons with (curvilinear sides)—plaquets. ("Map of world")

Denote by P number of plaquets (countries of the map)

Denote by E number of edges (boundaries between countries)

Denote by V number of vertices.

Then it turns out that

$$P - E + V = \chi(M) \tag{5.58}$$

It does not depend on the graph, it depends only on how much holes has surface.

E.g. for every graph on M, $P - E + V = 2$ if M is diffeomorphic to sphere. For every graph on M $P - E + V = 0$ if M is diffeomorphic to torus.

Now we formulate Gauß-Bonnet Theorem.

Let M be closed oriented surface in $\mathbf{E}^3$.

Let $K(p)$ be Gaussian curvature at any point p of this surface.

Recall that sign of Gaussian curvature does not depend on the orientation. If we change direction of normal vector $\mathbf{n} \to -\mathbf{n}$ then both principal curvatures change the sign and Gaussian curvature $K = \det A / \det G$ does not change the sign [15].

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[14] Closed means compact surface without boundaries. Intuitively orientability means that one can define out and inner side of the surface. In terms of normal vectors orientability means that one can define the continuous field of normal vectors at all the points of M. The direction of normal vectors at any point defines outward direction. Orientable surface is called oriented if the direction of normal vector is chosen.

[15] For an arbitrary point p of the surface M one can always choose cartesian coordinates (x, y, z) such that surface in a vicinity of this spoint is defined by the equation $z = ax^2 + bx^2 + \dots$, where dots means terms of the order higher than 2. Then Gaussian

Theorem (Gauß -Bonnet) The integral of Gaussian curvature over the closed compact oriented surface M is equal to 2π multiplied by the Euler characteristic of the surface M

$$\frac{1}{2\pi}\int_M K d\sigma = \chi(M) = 2(1 - \text{number of holes}) \tag{5.59}$$

In particular for the surface M diffeomorphic to the sphere $\kappa(M) = 2$, for the surface diffeomorphic to the torus it is equal to 0.

The value of the integral does not change under continuous deformations of surface! It is integer number (up to the factor π) which characterises topology of the surface.

E.g. consider surface M which is diffeomorphic to the sphere. If it is sphere of the radius R then curvature is equal to $\frac{1}{R^2}$, area of the sphere is equal to $4\pi R^2$ and left hand side is equal to $\frac{4\pi}{2\pi} = 2$.

If surface M is an arbitrary surface diffeomorphic to M then metrics and curvature depend from point to the point, Gauß-Bonnet states that integral nevertheless remains unchanged.

Very simple but impressive corollary:

Let M be surface diffeomorphic to sphere in $\mathbf{E}^3$. Then there exists at least one point where Gaussian curvature is positive.

Proof: Suppose it is not right. Then $\int_M K\sqrt{\det g}dudv \leq 0$. On the other hand according to the Theorem it is equal to 4π. Contradiction.

In the first section in the subsection "Integrals of curvature along the plane curve" we proved that the integral of curvature over closed convex curve is equal to 2π. This Theorem seems to be "ancestor" of Gauß-Bonnet Theorem[16].

Proof of Gauß-Bonet Theorem

Consider triangulation of the surface M. Suppose M is covered by N triangles. Then number of edges will be $3N/over2$. If V number of vertices then according to Euler Theorem

$$N - \frac{3N}{2} + V = V - \frac{N}{2} = \chi(M).$$

Calculate the sum of the angles of all triangles. On the one hand it is equal to $2\pi V$. On the other hand according the formula (5.56) it is equal to

$$\sum_{i=1}^{N}\left(\pi + \int_{\triangle_i} K d\sigma\right) = \pi N + \sum_{i=1}^{N}\left(\int_{\triangle_i} K d\sigma\right) = N\pi + \int_M K d\sigma$$

---

curvature at this point will be equal to ab. If a, b have the same sign then a surfaces looks as paraboloid in the vicinity of the point p. If If a, b have different signs then a surfaces looks as saddle in the vicinity of the point p. Gaussian curvature is positive if $ab > 0$ (case of paraboloid) and negative if $ab < 0$ saddle

[16]Note that there is a following deep difference: Gaussian curvature is internal property of the surface: it does not depend on isometries of surface. Curvature of curve depends on the position of the curve in ambient space.

We see that $2\pi V = N\pi + \int_M K d\sigma$, i.e.

$$\int_M K d\sigma = \pi \left(2V - \frac{N}{2} \right) = 2\pi\chi(M) \blacksquare$$

5.10 A Tale on Differential Geometry

Once upon a time there was an ant living on a sphere of radius R. One day he asked himself some questions: What is the structure of the Universe (surface) where he lives? Is it a sphere? Is it a torus? Or may be something more sophisticated, e.g. pretzel (a surface with two holes)

Three-dimensional human beings do not need to be mathematicians to distinguish between a sphere torus or pretzel. They just have to look on the surface. But the ant living on two-dimensional surface cannot fly. He cannot look on the surface from outside. How can he judge about what surface he lives on [17]This is not very far from reality: For us human beings it is impossible to have a global look on three-dimensional manifold. We need to develop local methods to understand global properties of our Universe. *Differential Geometry* allows to study global properties of manifold with local tools.?

Our ant loved mathematics and in particular *Differential Geometry.* He liked to draw various triangles, calculate their angles α, β, γ, area $S(\Delta)$. He knew from geometry books that the sum of the angles of a triangle equals π, but for triangles which he drew it was not right!!!!

Finally he understood that the following formula is true: For every triangle

$$\frac{(\alpha + \beta + \gamma - \pi)}{S(\Delta)} = c \tag{1}$$

A constant in the right hand side depended neither on size of triangle nor the triangles location. After hard research he came to conclusion that its Universe can be considered as a sphere embedded in three-dimensional Euclidean space and a constant c is related with radius of this sphere by the relation

$$c = \frac{1}{R^2} \tag{2}$$

...Centuries passed. Men have deformed the sphere of our old ant. They smashed it. It seized to be round, but the ant civilisation survived. Moreover old books survived. New ant mathematicians try to understand the structure of their Universe. They see that formula (1) of the Ancient Ant mathematician is not true. For triangles at different places the right hand side of the formula above is different. Why? If ants could fly and look on the surface from the cosmos they could see how much the sphere has been damaged by humans beings, how much it has been deformed, But the ants cannot fly. On the other hand they adore mathematics and in particular *Differential Geometry.* One day considering for every point very small triangles they introduce so called curvature for every point P as a limit

[17]*

of right hand side of the formula (1) for small triangles:

$$K(P) = \lim_{S(\Delta)\to 0} \frac{(\alpha+\beta+\gamma-\pi)}{S(\Delta)}$$

Ants realise that curvature which can be calculated in every point gives a way to decide where they live on sphere, torus, pretzel... They come to following formula [18] : integral of curvature over the whole Universe (the sphere) has to equal 4π, for torus it must equal 0, for pretzel it equalts -4π...

$$\frac{1}{2\pi}\int K(P)dP = 2\,(1 - \text{number of holes})$$

[18] In human civilisation this formula is called Gauß-Bonet formula. The right hand side of this formula is called Euler characteristics of the surface.