Homework 6. Solutions.

Christoffel symbols and Lagrangians

1 *Consider the Lagrangian of "free" particle $L = \frac{1}{2}g_{ik}\dot{x}^i\dot{x}^k$ for Riemannian manifold with a metric $G = g_{ik}dx^i dx^k$.*

Write down Euler-Lagrange equations of motion for this Lagrangian and compare them with differential equations for geodesics on this Riemannian manifold.

In fact show that

$$\underbrace{\frac{\partial L}{\partial x^i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}}_{\textit{Euler-Lagrange equations}} \qquad \Leftrightarrow \qquad \underbrace{\frac{d^2x^i}{dt^2} = \Gamma^i_{km}\dot{x}^k\dot{x}^m}_{\textit{Equations for geodesics}} \quad , \tag{1}$$

where

$$\Gamma^i_{km} = \frac{1}{2}g^{ij}\left(\frac{\partial g_{jk}}{\partial x^m} + \frac{\partial g_{jm}}{\partial x^k} - \frac{\partial g_{km}}{\partial x^j}\right) . \tag{2}$$

Solution: see the lecture notes.

2 *a) Write down the Lagrangian of of "free" particle $L = \frac{1}{2}g_{ik}\dot{x}^i\dot{x}^k$ for Euclidean plane in polar coordinates. Calculate Christoffel symbols for canonical flat connection in polar coordinates using Euler-Lagrange equations for this Lagrangian. Compare with answers which you obtained by the direct use of the formula (2). b) Do the same for cylindrical coordinates in $\mathbf{E}^3$.*

Solution. Canonical flat connection is Levi-Civita connection of Euclidean metric $G = dx^2 + dy^2$. Hence we can calculate Christoffel symbols using Lagrangian method.

Euclidean metric in polar coordinates is $dr^2 + r^2 d\varphi^2$. Hence the Lagrangian of the free particle is

$$L = \frac{\dot{r}^2 + r^2\dot{\varphi}^2}{2}$$

Euler-Lagrange equations:

1) for r:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \ddot{r} = \frac{\partial L}{\partial \varphi} = r\dot{\varphi}^2$$

i.e.

$$\ddot{r} - r\dot{\varphi}^2 = 0 \Rightarrow \Gamma^r_{rr} = \Gamma^r_{\varphi r} = \Gamma^r_{r\varphi} = 0, = \Gamma^r_{\varphi\varphi} = -r.$$

2) for φ:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = \frac{d}{dt}\left(r^2\dot{\varphi}\right) = r^2\ddot{\varphi} + 2r\dot{r}\dot{\varphi} = \frac{\partial L}{\partial \varphi} = 0,$$

i.e.

$$\ddot{\varphi} + \frac{2}{r}\dot{r}\dot{\varphi} = 0 \Rightarrow \Gamma^\varphi_{rr} = \Gamma^\varphi_{\varphi\varphi} = 0, \, \Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = \frac{1}{r}.$$

b) cylindrical coordinates in $\mathbf{E}^3$. Calculations almost the same as for polar coordinates in $\mathbf{E}^2$. $G = dr^2 + r^2 d\varphi^2 + dh^2$,

$$L = \frac{\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{h}^2}{2}$$

for r: $\ddot{r} - r\dot{\varphi}^2 = 0 \Rightarrow$

$$\Gamma^r_{rr} = \Gamma^r_{\varphi r} = \Gamma^r_{r\varphi} = \Gamma^r_{rh} = \Gamma^r_{hr} = \Gamma^r_{h\varphi} = \Gamma^r_{\varphi h} = \Gamma^r_{hh} = 0 \, , \Gamma^r_{\varphi\varphi} = -r.$$

for φ, $r^2\ddot{\varphi}+2r\dot{r}\dot{\varphi}=\frac{\partial L}{\partial \varphi}=0$, i.e. $\ddot{\varphi}+\frac{2}{r}\dot{r}\dot{\varphi}=0 \Rightarrow$

$$\Gamma^{\varphi}_{rr}=\Gamma^{\varphi}_{rh}=\Gamma^{\varphi}_{hr}=\Gamma^{\varphi}_{\varphi\varphi}=\Gamma^{\varphi}_{\varphi h}=\Gamma^{\varphi}_{h\varphi}=\Gamma^{\varphi}_{hh}=0,\ \Gamma^{\varphi}_{r\varphi}=\Gamma^{\varphi}_{\varphi r}=\frac{1}{r}$$

3) for h, $\ddot{h}=0$,

$$\Gamma^{h}_{rr}=\Gamma^{h}_{r\varphi}=\Gamma^{h}_{\varphi r}=\Gamma^{h}_{rh}=\Gamma^{h}_{hr}=\Gamma^{h}_{\varphi\varphi}=\Gamma^{h}_{\varphi h}=\Gamma^{h}_{h\varphi}=\Gamma^{h}_{hh}=0,$$

3

Calculate Christoffel symbols of Levi-Civita connection for Riemannian metric $G=adu^2+bdv^2$. Compare with resultls of the Exercise 1*b*) *in the Homework 5.*

Lagrangian of free particle for this metric is

$$L=\frac{a(u,v)\dot{u}^2+b(u,v)\dot{v}^2}{2}\,.$$

Euler-lagrange equations

for u:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}}\right)=\frac{d}{dt}(a\dot{u})=a_u\dot{u}^2+a_v\dot{v}\dot{u}+a\ddot{u}=\frac{\partial L}{\partial u}=\frac{a_u\dot{u}^2+b_u\dot{v}^2}{2}$$

hence

$$\ddot{u}+\frac{1}{2}\frac{a_u}{a}\dot{u}^2+\frac{a_v}{a}\dot{v}\dot{u}-\frac{1}{2}\frac{b_u}{a}\dot{u}^2$$

Comparing with equation

$$\ddot{u}+\Gamma^u_{uu}\dot{u}\dot{u}+\Gamma^u_{uv}\dot{u}\dot{v}+\Gamma^u_{vu}\dot{v}\dot{u}+\Gamma^u_{vv}\dot{v}\dot{v}\ddot{u}+\Gamma^u_{uu}\dot{u}\dot{u}+2\Gamma^u_{uv}\dot{u}\dot{v}+\Gamma^u_{vv}\dot{v}\dot{v}=0$$

we see that

$$\Gamma^u_{uu}=\frac{1}{2}\frac{a_u}{a}\,,\Gamma^u_{uv}=\Gamma^u_{vu}=\frac{1}{2}\frac{a_v}{a},\Gamma^u_{vv}=-\frac{1}{2}\frac{b_u}{a}\,,$$

Analogously v:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{v}}\right)=\frac{d}{dt}(b\dot{v})=b_v\dot{v}^2+b_u\dot{u}\dot{v}+b\ddot{b}=\frac{\partial L}{\partial v}=\frac{a_v\dot{u}^2+b_v\dot{v}^2}{2}$$

hence

$$\ddot{v}+\frac{1}{2}\frac{b_v}{b}\dot{b}^2+\frac{b_u}{b}\dot{u}\dot{v}-\frac{1}{2}\frac{a_v}{b}\dot{v}^2\Rightarrow\Gamma^v_{vv}=\frac{1}{2}\frac{b_v}{b}\,,\Gamma^v_{vu}=\Gamma^v_{uv}=\frac{1}{2}\frac{b_u}{b},\Gamma^v_{uu}=-\frac{1}{2}\frac{a_v}{b}\,.$$

4

Write down the Lagrangian of of "free" particle $L=\frac{1}{2}g_{ik}\dot{x}^i\dot{x}^k$ and using Euler-Lagrange equations for this Lagrangian calculate Christoffel symbols (Christoffel symbols of Levi-Civita connection) for

a) cylindrical surface of the radius R

b) for the cone $x^2+y^2-k^2z^2=0$

c) for the sphere of radius R

d) for Lobachevsky plane

Compare with the results that you obtained using straightforwardly the formula (1) or using formulae for induced connection.

Solution.

For cylindrical surface of the radius a: $x^2 + y^2 = a^2$ $\mathbf{r}(h,\varphi) = \begin{cases} x = a\cos\ \varphi \\ y = a \\ \sin\varphi z = h \end{cases}$ we have that induced metric is $G = dh^2 + a^2 d\varphi^2$ and the Lagrangian of free particle is

$$L = \frac{a^2\dot\varphi^2 + \dot h^2}{2}$$

for φ, Euler-Lagrange equations of motion:

$$\frac{\partial L}{\partial \varphi} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi}\right). \quad \frac{\partial L}{\partial \varphi} = 0,\ \frac{\partial L}{\partial \dot\varphi} = a^2\dot\varphi$$

hence

$$\frac{d}{dt}\left(a^2\dot\varphi\right) = a^2\ddot\varphi = 0\,, \ddot\varphi = 0.$$

Hence all Christoffel symbols $\Gamma^\varphi_{\varphi\varphi}, \Gamma^\varphi_{\varphi h}, \Gamma^\varphi_{h\varphi}$ vanish:

$$\Gamma^\varphi_{\varphi\varphi} = 0, \Gamma^\varphi_{\varphi h} = \Gamma^\varphi_{h\varphi} = 0$$

3) for h, we have the same. Euler-Lagrange equations of motion::

$$\frac{\partial L}{\partial h} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot h}\right). \quad \frac{\partial L}{\partial h} = 0,\ \frac{\partial L}{\partial h} = \dot h$$

hence

$$\frac{d}{dt}\left(\dot h\right) = \ddot h = 0\,,$$

Hence all Christoffel symbols $\Gamma^h_{\varphi\varphi}, \Gamma^h_{\varphi h}, \Gamma^h_{h\varphi}$ vanish:

$$\Gamma^h_{\varphi\varphi} = 0, \Gamma^h_{\varphi h} = \Gamma^h_{h\varphi} = 0$$

We see that on cylindrical surface in coordinates h, φ all Christoffel symbols vanish: this is not surprising, since Riemannian metric $dh^2 + a^2 d\varphi^2$ has constant coefficients.

For the cone: See the coursework.

c) *For the sphere:*

Riemannian metric on sphere in spherical coordinates is $G = R^2 d\theta^2 + R^2 \sin^2\theta d\varphi^2$. Hence the Lagrangian of the free particle is

$$L = \frac{R^2\dot\theta^2 + R^2\sin^2\theta\dot\varphi^2}{2}$$

Euler-Lagrange equations for θ:

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right), \quad \frac{\partial L}{\partial \theta} = R^2\sin\theta\cos\theta\dot\varphi^2,\ \frac{\partial L}{\partial \dot\theta} = R^2\dot\theta$$

Hence

$$\frac{d}{dt}\left(R^2\dot\theta\right) = R^2\sin\theta\cos\theta\dot\varphi^2, R^2\ddot\theta = R^2\sin\theta\cos\theta\dot\varphi^2,$$

hence

$$\ddot\theta - \sin\theta\cos\theta\dot\varphi^2 = 0\,.$$

Comparing with equation for geodesic

$$\ddot\theta + \Gamma^\theta_{\theta\theta}\dot\theta\dot\theta + \Gamma^\theta_{\theta\varphi}\dot\theta\dot\varphi + \Gamma^\theta_{\varphi\theta}\dot\varphi\dot\theta + \Gamma^\theta_{\varphi\varphi}\dot\varphi\dot\varphi = \ddot\theta + \Gamma^\theta_{\theta\theta}\dot\theta\dot\theta + 2\Gamma^\theta_{\theta\varphi}\dot\theta\dot\varphi + \Gamma^\theta_{\varphi\varphi}\dot\varphi\dot\varphi = 0$$

we see that

$$\Gamma^\theta_{\theta\theta} = \Gamma^\theta_{\theta\varphi} = \Gamma^\theta_{\varphi\theta} = 0,\ \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta$$

Analogously Euler-Lagrange equations for φ:

$$\frac{\partial L}{\partial\varphi} = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\varphi}\right), \quad \frac{\partial L}{\partial\varphi} = 0,\ \frac{\partial L}{\partial\dot\varphi} = R^2\sin^2\theta\dot\varphi\,.$$

Hence

$$\frac{d}{dt}\left(R^2\sin^2\theta\dot\varphi\right) = 0,\ R^2\sin^2\theta\ddot\varphi + 2R^2\sin\theta\cos\theta\dot\theta\dot\varphi = 0,$$

hence

$$\ddot\theta + \text{cotan}\ \theta\dot\theta\dot\varphi = 0,$$

Comparing with equation for geodesic

$$\ddot\varphi + \Gamma^\varphi_{\theta\theta}\dot\theta\dot\theta + \Gamma^\varphi_{\theta\varphi}\dot\theta\dot\varphi + \Gamma^\varphi_{\varphi\theta}\dot\varphi\dot\theta + \Gamma^\varphi_{\varphi\varphi}\dot\varphi\dot\varphi = \ddot\theta + \Gamma^\varphi_{\theta\theta}\dot\theta\dot\theta + 2\Gamma^\varphi_{\theta\varphi}\dot\theta\dot\varphi + \Gamma^\varphi_{\varphi\varphi}\dot\varphi\dot\varphi = 0$$

we see that

$$\Gamma^\varphi_{\theta\theta} = \Gamma^\varphi_{\varphi\varphi} = 0\,, \Gamma^\varphi_{\varphi\theta} = \Gamma^\varphi_{\theta\varphi} = \text{cotan}\,\theta\,.$$

d) *For Lobachevsky plane*:
Lagrangian of "free" particle on the Lobachevsky plane with metric $G = \frac{dx^2+dy^2}{y^2}$ is

$$L = \frac{1}{2}\frac{\dot x^2 + \dot y^2}{y^2}.$$

Euler-Lagrange equations are

$$\frac{\partial L}{\partial x} = 0 = \frac{d}{dt}\frac{\partial L}{\partial\dot x} = \frac{d}{dt}\left(\frac{\dot x}{y^2}\right) = \frac{\ddot x}{y^2} - \frac{2\dot x\dot y}{y^3}, \text{i.e.} \quad \ddot x - \frac{2\dot x\dot y}{y} = 0\,,$$

$$\frac{\partial L}{\partial y} = -\frac{\dot x^2 + \dot y^2}{y^3} = \frac{d}{dt}\frac{\partial L}{\partial\dot y} = \frac{d}{dt}\left(\frac{\dot y}{y^2}\right) = \frac{\ddot y}{y^2} - \frac{2\dot y^2}{y^3}, \text{i.e.} \quad \ddot y + \frac{\dot x^2}{y} - \frac{\dot y^2}{y} = 0\,.$$

Comparing these equations with equations for geodesics: $\ddot x^i - \dot x^k\Gamma^i_{km}\dot x^m = 0$ $(i = 1, 2,\ x = x^1, y = x^2)$ we come to

$$\Gamma^x_{xx} = 0, \Gamma^x_{xy} = \Gamma^x_{yx} = -\frac{1}{y},\ \Gamma^x_{yy} = 0,\ \Gamma^y_{xx} = \frac{1}{y}, \Gamma^y_{xy} = \Gamma^y_{yx} = 0, \Gamma^y_{yy} = -\frac{1}{y}\,.\ \blacksquare$$

The answers are the same as calculated with other methods. We see that Lagrangians give us the nice and quick way to calculate Christoffel symbols.