4 August 18

Let $\mathbf{e}, \mathbf{f}$ be standard basis in $\mathbf{R}^2$. $\mathbf{e} = (1, 0)$, and $\mathbf{f} = (0, 1)$.

Let α be an arbitrary non-negative real number and let $[a_0, a_1, \dots,]$ be its continuous fraction; and let $\frac{p_k}{q_k}$ be the k-th approximation corresponding to this fraction:

$$[a_0, \dots, a_k] = \frac{p_k}{q_k} \tag{2b}$$

(We assume that p_k, q_k are coprime)

Consider vectors $\{\mathbf{E}_{-2}, \mathbf{E}_{-1}, \mathbf{E}_0 . \mathbf{E}_1, \dots\}$ in $\mathbf{Z}^2$ defined by real number α in the following way:

$$\mathbf{E}_{-2} = \mathbf{e}, \quad \mathbf{E}_{-1} = \mathbf{f}\,, \tag{3}$$

Proposition "metod vytiagivanie nosov": for arbitrary k:

$$\mathbf{E}_{k+1} = \mathbf{E}_{k-1} + a_{k+1}\mathbf{E}_k\ .(k = -1, 0, 1, 2, \dots) \tag{\textit{Prop}}$$

This Proposition immediately implies the statement about good approximmation

Corollary Area of parallelgrom $\Pi_{\mathbf{E}_k \mathbf{E}_{k+1}}$ is equal to 1:

$$\omega(\mathbf{E}_k, \mathbf{E}_{k+1}) = (-1)^k \omega(\mathbf{e}, \mathbf{f})\,, \quad i.e. \left|\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k}\right| = \frac{1}{q_{k+1}q_k}$$

Indeed by definition of these vectors

$$\omega(\mathbf{E}_k, \mathbf{E}_{k+1}) = \omega(\mathbf{E}_k, \mathbf{E}_{k-1} + a_{k+1}\mathbf{E}_k) = -\omega(\mathbf{E}_{k-1}, \mathbf{E}_{k-1})\,,$$

thus we come by induction to the statement.

(Compare wth proofs in the previous files)

§∈ Convex span for continuous fraction.

The following remarkable property is obeyed.

Convex span of vectors $\{\mathbf{E}_2, \mathbf{E}_0, \mathbf{E}_2, \dots\}$ *in the quadrant* $x > 0, y > 0$ *is the set* $\Pi_-(\alpha)$*, and Convex span of vectors* $\{\mathbf{E}_1, \mathbf{E}_1, \mathbf{E}_3, \dots\}$ *in the quadrant* $x > 0, y > 0$ *is the set* $\Pi_+(\alpha)$.

As usuall we define Π_- as the set of all points in the quadrant $x > 0, y > 0$ which have integer coordinates and they are below the line $y = \alpha x$, respectively Π_- is the set of all points in the quadrant $x > 0, y > 0$ which have integer coordinates and they are over the line $y = \alpha x$, respectively:

$$\Pi_- = \{(m, n) \colon m, n \in \mathbf{Z}\,, \quad 0 \le m \le n\alpha\}\,,$$

$$\Pi_+ = \{(m, n) \colon m, n \in \mathbf{Z}\,, \quad 0 \le n\alpha \le m\}\,.$$