MATH 20222

Two hours

THE UNIVERSITY OF MANCHESTER

INTRODUCTION TO GEOMETRY

xx-th May/June 2010
xx:xx

ANSWER **THREE** QUESTIONS
If four questions are answered credit will be given for the best three
All questions are worth 20 marks

Electronic calculators may be used, provided that they cannot store text

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P.T.O.

1.

(a) Explain what is meant by saying that the dimension of a vector space is equal to 2.

Let $\mathbf{a}$ and $\mathbf{b}$ be two linearly independent vectors in a 2-dimensional vector space V.

Show that an arbitrary vector $\mathbf{x} \in V$ can be expressed as a linear combination of the vectors $\mathbf{a}$ and $\mathbf{b}$ in a unique way.

[7 marks]

(b) Let V be a vector space. Explain what is meant by a scalar product on V.

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three linearly independent vectors in 3-dimensional Euclidean space $\mathbf{E}^3$ such that the following conditions hold:

- the lengths of all these vectors are equal to 1,
- the vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal to each other,
- the vector $\mathbf{c}$ is orthogonal to the vectors $\mathbf{a} - \mathbf{b}$ and $\mathbf{a} + \mathbf{b}$.

Show that the ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is an orthonormal basis in $\mathbf{E}^3$.

Show that the ordered triple $\{\mathbf{a} - \mathbf{b}, \mathbf{a} + \mathbf{b}, \mathbf{c}\}$ is not an orthonormal basis in $\mathbf{E}^3$.

[7 marks]

(c)

Let $\mathbf{e}$ and $\mathbf{f}$ be two vectors in a 2-dimensional vector space V and let $B(\mathbf{X}, \mathbf{Y})$ be a bilinear form on V such that

$$B(\mathbf{e}, \mathbf{f}) = B(\mathbf{f}, \mathbf{e}) = 1, \qquad B(\mathbf{e}, \mathbf{e}) = B(\mathbf{f}, \mathbf{f}) = 0\,.$$

Show that the ordered pair $\{\mathbf{e}, \mathbf{f}\}$ is a basis in the vector space V.

Show that the bilinear form $B(\mathbf{X}, \mathbf{Y})$ *does not* define a scalar product on the vector space V.

[6 marks]

P.T.O.

2.

(a) Explain what is meant by the vector product of two vectors in oriented 3-dimensional Euclidean space $\mathbf{E}^3$.

Explain why the vector product of two collinear vectors is equal to zero. [7 marks]

(b) Consider the following two vectors

$$\mathbf{a} = \frac{1}{13}\left(3\mathbf{e}_x + 4\mathbf{e}_y + 12\mathbf{e}_z\right)\,, \quad \mathbf{b} = \frac{1}{13}\left(12\mathbf{e}_x + 3\mathbf{e}_y - 4\mathbf{e}_z\right)$$

in oriented Euclidean space $\mathbf{E}^3$, where $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ is an orthonormal basis in $\mathbf{E}^3$.

Define the vector $\mathbf{c}$ by the relation $\mathbf{c} = \mathbf{a} \times \mathbf{b}$.

Without explicitly calculating the vector $\mathbf{c}$ show that the ordered set of three vectors $\{\mathbf{c}, \mathbf{a}, \mathbf{b}\}$ is an orthonormal basis in $\mathbf{E}^3$.

Show that bases $\{\mathbf{c}, \mathbf{a}, \mathbf{b}\}$ and $\{\mathbf{a}, \mathbf{c}, -\mathbf{b}\}$ have the same orientation. [5 marks]

(c) In oriented Euclidean space $\mathbf{E}^3$ consider the following function of three vectors:

$$F(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = (\mathbf{X}, \mathbf{Y} \times \mathbf{Z})\,,$$

where $(\ ,\ )$ is the scalar product and $\mathbf{Y} \times \mathbf{Z}$ is the vector product in $\mathbf{E}^3$.

Show that $F(\mathbf{X}, \mathbf{X}, \mathbf{Z}) = 0$ for arbitrary vectors $\mathbf{X}$ and $\mathbf{Z}$.

Deduce that $F(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = -F(\mathbf{Y}, \mathbf{X}, \mathbf{Z})$ for arbitrary vectors $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$.

What is the geometrical meaning of the function F? [8 marks]

P.T.O.

3.

(a) Consider in $\mathbf{E}^2$ a differential 1-form $\omega = \frac{1}{2}(xdy - ydx)$ and an ellipse C defined by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a, b > 0)$.

Choose a parameterisation of the ellipse and calculate $\int_C \omega$.

How does your answer depend on a choice of parameterisation? `[6 marks]`

(b) Show that the integral of a differential 1-form $\sigma = \frac{1}{2}(xdy + ydx)$ over the ellipse C considered in part a) is equal to zero.

Will the answer change if instead of the ellipse we consider an arbitrary closed curve? Justify your answer. `[5 marks]`

(c) Give the definition of the curvature of a curve in $\mathbf{E}^n$.

Calculate the curvature $k(t)$ of the parabola $\mathbf{r}(t)\colon\ x = t, y = at^2$, $(a > 0)$.

Consider the following curve (a helix):

$$\mathbf{r}(t)\colon \quad \begin{cases} x(t) = a\cos t \\ y(t) = a\sin t \\ z(t) = ct \end{cases} \qquad -\infty < t < \infty, \ \ a > 0\,.$$

Calculate the curvature of this curve.

Evaluate the curvature and give a geometrical meaning of the answers in the limits $c \to 0$ and $c \to \infty$. `[9 marks]`

P.T.O.

4.

(a)

Explain what is meant by the shape operator for a surface $\mathbf{r} = \mathbf{r}(u, v)$ in $\mathbf{E}^3$ by defining its action on an arbitrary tangent vector to the surface. Explain why the value of the shape operator is also a tangent vector to the surface. `[4 marks]`

(b) Consider a surface (cone)

$$\mathbf{r}(h, \varphi): \quad \begin{cases} x = h\cos\varphi \\ y = h\sin\varphi \\ z = h \end{cases} .$$

Calculate the first quadratic form of this surface.

Calculate a unit normal vector field $\mathbf{n}(h, \varphi)$ at points of this surface.

Calculate the shape operator, Gaussian and mean curvature of this surface. `[9 marks]`

(c)

Consider the curve L_{AB} on the cone considered in part b) joining the point $A = (1, 0, 1)$ and the point $B = (-1, 0, 1)$, and defined by the equations $h = 1, \varphi = t$ $(0 \leq t \leq \pi.)$

Calculate the length of this curve.

Does there exist a curve L'_{AB} on the cone joining the same points such that its length is less than the length of the curve L_{AB}? `[7 marks]`

END OF EXAMINATION PAPER