Homework 4. Solutions.

In this exercise we consider three-dimensional oriented Euclidean space $\mathbf{E}^3$, where by default the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ defines the orientation.

1 *Prove that vectors $\mathbf{a}$ and $\mathbf{b}$ are linear independent if and only if $\mathbf{a}\times\mathbf{b}\neq 0$.*

2 *Vectors $\mathbf{a}$ and $\mathbf{b}$ are linear independent. Using the fact that in this case $\mathbf{a}\times\mathbf{b}\neq 0$ (see the previous exercise) prove that the vectors $(\mathbf{a},\mathbf{b},\mathbf{a}\times\mathbf{b})$ are linear independent*

Solution: see the Lemma in subsection 1.11 (Vector product in oriented vector space) of Lecture notes.

3 *Students John and Sarah calculate vector product $\mathbf{a}\times\mathbf{b}$ of two vectors using two different orthonormal bases in the Euclidean space $\mathbf{E}^3$, $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ and $\{\mathbf{e}'_1,\mathbf{e}'_2,\mathbf{e}'_3\}$. John expands the vectors with respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$. Sarah expands the vectors with respect to the basis $\{\mathbf{e}'_1,\mathbf{e}'_2,\mathbf{e}'_3\}$. For two arbitrary vectors $\mathbf{a},\mathbf{b}\in\mathbf{E}^3$*

$$\mathbf{a}=a_1\mathbf{e}_1+a_2\mathbf{e}_2+a_3\mathbf{e}_3=a'_1\mathbf{e}'_1+a'_2\mathbf{e}'_2+a'_3\mathbf{e}'_3\,,$$

$$\mathbf{b}=b_1\mathbf{e}_1+b_2\mathbf{e}_2+b_3\mathbf{e}_3=b'_1\mathbf{e}'_1+b'_2\mathbf{e}'_2+b'_3\mathbf{e}'_3\,.$$

John and Sarah both use so called "determinant" formula. Are their answers the same?

$$\mathbf{a}\times\mathbf{b}=\underbrace{\det\begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{pmatrix}}_{\text{John's calculations}}\overset{?}{=}\underbrace{\det\begin{pmatrix}\mathbf{e}'_1 & \mathbf{e}'_2 & \mathbf{e}'_3\\ a'_1 & a'_2 & a'_3\\ b'_1 & b'_2 & b'_3\end{pmatrix}}_{\text{Sarah's calculations}}$$

Solution: In the case if bases $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ and $\{\mathbf{e}'_1,\mathbf{e}'_2,e'_3\}$ have the same orientation, then answer will be the same. This was proved in detail in the subsection "Vector product in oriented vector space" of lecture notes. (See Proposition 1).

If bases $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ and $\{\mathbf{e}'_1,\mathbf{e}'_2,e'_3\}$ have opposite orientation then bases $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ and $\{-\mathbf{e}'_1,\mathbf{e}'_2,\mathbf{e}'_3\}$ have the same orientation. Consider third student David who performs calculation in the new third basis $\{-\mathbf{e}'_1,\mathbf{e}'_2,\mathbf{e}'_3\}$. We have

$$\mathbf{a}=(-a'_1)(-\mathbf{e}'_1)+a'_2\mathbf{e}'_2+a'_3\mathbf{e}'_3\,,\mathbf{b}=(-b'_1)(-\mathbf{e}'_1)+b'_2\mathbf{e}'_2+b'_3\mathbf{e}'_3$$

and David's calculations coincide with John's calculations since he works in the basis with the same orientation as a basis of John. Hence we have:

$$\underbrace{\det\begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{pmatrix}}_{\text{John's calculations}}=\underbrace{\det\begin{pmatrix}-\mathbf{e}'_1 & \mathbf{e}'_2 & \mathbf{e}'_3\\ -a'_1 & a'_2 & a'_3\\ -b'_1 & b'_2 & b'_3\end{pmatrix}}_{\text{David's calculations}}=\underbrace{-\det\begin{pmatrix}\mathbf{e}'_1 & \mathbf{e}'_2 & \mathbf{e}'_3\\ a'_1 & a'_2 & a'_3\\ b'_1 & b'_2 & b'_3\end{pmatrix}}_{\text{Sarah's calculations}}$$

We see that Sarah and John will come to the answer which differ by the sign: $\mathbf{c}_{\text{John}}=-\mathbf{c}_{\text{Sarah}}$

4 *Calculate the area of parallelograms formed by the vectors $\mathbf{a},\mathbf{b}$ if*

a) $\mathbf{a}=(1,2,3),\mathbf{b}=(1,0,1)$*;*

b) $\mathbf{a}=(2,2,3),\mathbf{b}=(1,1,1)$*;*

c) $\mathbf{a}=(5,8,4),\mathbf{b}=(10,16,8)$.

Solution

Area of parallelogram formed by the vectors $\mathbf{a},\mathbf{b}$ is equal to the length of the vector $\mathbf{c}=\mathbf{a}\times\mathbf{b}$.

$$\mathbf{c}=\mathbf{a}\times\mathbf{b}=(a_x\mathbf{e}_x+a_y\mathbf{e}_y+a_z\mathbf{e}_z)\times(b_x\mathbf{e}_x+b_y\mathbf{e}_y+b_z\mathbf{e}_z)=a_xb_x\mathbf{e}_x\times\mathbf{e}_x+a_xb_y\mathbf{e}_x\times\mathbf{e}_y+a_xb_z\mathbf{e}_x\times\mathbf{e}_z+$$

$$a_yb_x\mathbf{e}_y\times\mathbf{e}_x+a_yb_y\mathbf{e}_y\times\mathbf{e}_y+a_yb_z\mathbf{e}_y\times\mathbf{e}_z+a_zb_x\mathbf{e}_z\times\mathbf{e}_x+a_zb_y\mathbf{e}_z\times\mathbf{e}_y+a_zb_z\mathbf{e}_z\times\mathbf{e}_z=$$

$$(a_x b_y - a_y b_x)\mathbf{e}_z + (a_y b_z - a_z b_y)\mathbf{e}_x + (a_z b_x - a_x b_z)\mathbf{e}_y$$

$$|\mathbf{c}| = \sqrt{(a_y b_z - a_z b_y)^2 + (a_z b_x - a_x b_z)^2 + (a_x b_y - a_y b_x)^2}$$

a) $S = |\mathbf{a} \times \mathbf{b}| = |-2\mathbf{e}_z + 2\mathbf{e}_x + 2\mathbf{e}_y|$, $S = \sqrt{4+4+4} = 2\sqrt{3}$.

b) $S = |\mathbf{a} \times \mathbf{b}|$. $\mathbf{a} \times \mathbf{b} = -\mathbf{e}_x + \mathbf{e}_y$, $S = \sqrt{1+1} = \sqrt{2}$

c) Vectors $\mathbf{a} = (5, 8, 4), \mathbf{b} = (10, 16, 8)$ are collinear, hence $\mathbf{a} \times \mathbf{b} = 0$, $S = 0$.

5 *Show that for any two vectors* $\mathbf{a}, \mathbf{b} \in \mathbf{E}^3$ *the following identity is satisfied*

$$(\mathbf{a}, \mathbf{a})(\mathbf{b}, \mathbf{b}) = (\mathbf{a}, \mathbf{b})^2 + (\mathbf{a} \times \mathbf{b}, \mathbf{a} \times \mathbf{b})\,.$$

Write down this identity in components.

Compare this identity with CBS inequality . See the problem 7 in the Homework 2).

Solution: This identity can be checked straightforwardly:

$$(a_x^2 + a_y^2 + a_z^2)(b_x^2 + b_y^2 + b_z^2) = (a_x b_x + a_y b_y + a_z b_z)^2 + (a_x b_y - a_y b_x)^2 + (a_y b_z - a_z b_y)^2 + (a_z b_x - a_x b_z)^2 \quad (1)$$

Another solution: Let φ be an angle between vectors $\mathbf{a}, \mathbf{b}$. Then

$$(\mathbf{a}, \mathbf{a})(\mathbf{b}, \mathbf{b}) = |\mathbf{a}|^2 |\mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 (\cos^2\varphi + \sin^2\varphi) = (\mathbf{a}, \mathbf{b})^2 + |\mathbf{a} \times \mathbf{b}|^2$$

Notice that for $n = 2, 3$ the identity (1) is more strong statement than CBS inequality*: $(a_x^2 + a_y^2 + a_z^2)(b_x^2 + b_y^2 + b_z^2) \geq (a_x b_x + a_y b_y + a_z b_z)^2$. CBS inequality $(\mathbf{a}, \mathbf{a})(\mathbf{b}, \mathbf{b}) \geq |\mathbf{a}|^2 |\mathbf{b}|^2$ follows from this identity.

6 *Find a vector* $\mathbf{n}$ *such that the following conditions hold:*

1) It has a unit length

2) It is orthogonal to the vectors $\mathbf{a} = (1, 2, 3)$ *and* $\mathbf{b} = (1, 3, 2)$.

3) An ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{n}\}$ *has an orientation opposite to the orientation of the orthonormal basis* $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ *which defines the orientation of the Euclidean space.*

Solution: Consider a vector $\mathbf{N} = \mathbf{a} \times \mathbf{b}$ and a vector $\frac{\mathbf{N}}{|\mathbf{N}|}$. The vector $\mathbf{N}$ is orthogonal to vectors $\mathbf{a}, \mathbf{b}$ (vector product) and a vector $\frac{\mathbf{N}}{|\mathbf{N}|}$ is a unit vector. It remains to solve the problem of orientation. Both vectors $\pm\frac{\mathbf{N}}{|\mathbf{N}|}$ are unit vectors which are orthogonal to vectors $\mathbf{a}, \mathbf{b}$. On the other hand the ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{N}\}$ is a basis and this basis has the same orientation as a basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. This follows from the axioms defining the vector product and the fact that vectors $\mathbf{N} \neq 0$, i.e. the ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{N}\}$ is a basis. Hence the ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{n}\}$ where $\mathbf{n} = -\frac{\mathbf{N}}{|\mathbf{N}|}$ has an orientation opposite to the orientation of the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$.

The vector

$$\mathbf{n} = -\frac{\mathbf{N}}{|\mathbf{N}|} = -\frac{(\mathbf{e}_x + 2\mathbf{e}_y + 3\mathbf{e}_z) \times (\mathbf{e}_x + 3\mathbf{e}_y + 2\mathbf{e}_z)}{||\mathbf{N}|} = \frac{5\mathbf{e}_x - \mathbf{e}_y - \mathbf{e}_z}{3\sqrt{3}}\,.$$

7 *Volume of parallelepiped* $V(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{a}, \mathbf{b} \times \mathbf{c})$, *formed by the vectors* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *equals to zero if and only if these vectors are linearly dependent. Prove it.*

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* One can generalise the identity (1) on a general case:

$$\left(\sum_{i=1}^n a_i^2\right) \cdot \left(\sum_{i=1}^n b_i^2\right) = \sum_{1 \leq i < j \leq n} (a_i b_j - a_j b_i)^2 + \sum_{i=1}^n (a_i b_i)^2$$

Solution: Let vectors $\{\mathbf{a},\mathbf{b}\,c\}$ are linearly dependent, i.e. $\lambda\mathbf{a}+\mu\mathbf{b}+\tau\mathbf{c}=0$. If $\lambda\neq 0$, then $\mathbf{a}$ belongs to the span of vectors $\mathbf{c},\mathbf{b}$: $\mathbf{a}=\lambda'\mathbf{b}+\mu'\mathbf{c}$ and

$$V(\mathbf{a},\mathbf{b},\mathbf{c})=(\mathbf{a},\mathbf{b}\times\mathbf{c})=(\lambda'\mathbf{b}+\mu'\mathbf{c},\mathbf{b}\times\mathbf{c})=\lambda'(\mathbf{b},\mathbf{b}\times\mathbf{c})+\mu'(\mathbf{c},\mathbf{b}\times\mathbf{c})=0\,.$$

Scalar product $(\mathbf{b},\mathbf{b}\times\mathbf{c})$ equals to zero since $\mathbf{b}\perp\mathbf{b}\times\mathbf{c}$. Analogously scalar product $(\mathbf{c},\mathbf{b}\times\mathbf{c})$ equals to zero.

If $\lambda=0$ in the relation $\lambda\mathbf{a}+\mu\mathbf{b}+\tau\mathbf{c}=0$, then linear dependence of vectors $\mathbf{a},\mathbf{b}$ and $\mathbf{c}$ means that vectors $\mathbf{c}$ and $\mathbf{b}$ are proportional to each other. Hence $\mathbf{b}\times\mathbf{c}=0$ and $V(\mathbf{a},\mathbf{b},\mathbf{c})=(\mathbf{a},\mathbf{b}\times\mathbf{c})=(\mathbf{a},0)=0$.

We proved that linear dependence of vectors $\mathbf{a},\mathbf{b}$ and $\mathbf{c}$ implies that $V(\mathbf{a},\mathbf{b},\mathbf{c})=0$.

Now prove the converse implication.

Let $V(\mathbf{a},\mathbf{b},\mathbf{c})=0$. Assume that vectors $\mathbf{a},\mathbf{b}$ and $\mathbf{c}$ are linear independent and come to contradiction. In particularly this means that vectors $\mathbf{b}$ and $\mathbf{c}$ are linear independent. Hence due to the Lemma (see the Lemma in the subsection "Vector product in oriented Euclidean space $\mathbf{E}^3$") vectors $\{\mathbf{b},\mathbf{c},\mathbf{b}\times\mathbf{c}\}$ form a basis. Expand the vector $\mathbf{a}$ with respect to this basis:

$$\mathbf{a}=\lambda(\mathbf{b}\times\mathbf{c})+\mu\mathbf{b}+\tau\mathbf{c}$$

$\lambda\neq 0$ since vectors $\mathbf{a},\mathbf{b}$ and $\mathbf{c}$ are linear independent. Hence we have

$$V=V(\mathbf{a},\mathbf{b},\mathbf{c})=(\lambda(\mathbf{b}\times\mathbf{c})+\mu\mathbf{b}+\tau\mathbf{c},\mathbf{b}\times\mathbf{c})=\lambda((\mathbf{b}\times\mathbf{c}),\mathbf{b}\times\mathbf{c})=\lambda|\mathbf{b}\times\mathbf{c}|^2\neq 0.\text{Contradiction.}$$

8 *Vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *are orthogonal unit vectors. Calculate the length of the vector* $\mathbf{c}\times(\mathbf{a}\times\mathbf{b})$*, where* $\mathbf{c}=\mathbf{a}+\mathbf{b}$.

Vector $\mathbf{a}$ and $\mathbf{b}$ are orthogonal to the vector $\mathbf{a}\times\mathbf{b}$. Hence the vector $\mathbf{c}=\mathbf{a}+\mathbf{b}$ is orthogonal to the vector $\mathbf{a}\times\mathbf{b}$. Due to axioms of vector product it means that the length of the vector $\mathbf{c}\times(\mathbf{a}\times\mathbf{b})$ equals to the product of the length of the vector $\mathbf{c}$ on the length of the vector $\mathbf{a}\times\mathbf{b}$. By the same arguments the length of the vector $\mathbf{a}\times\mathbf{b}$ equals to the product of the length of the vector $\mathbf{a}$ on the length of the vector $\mathbf{b}$. We come to

$$|\mathbf{c}\times(\mathbf{a}\times\mathbf{b})|=|\mathbf{c}|\cdot|\mathbf{a}\times\mathbf{b}|=|\mathbf{c}|\cdot|\mathbf{a}|\cdot|\mathbf{b}|=1\cdot 1\cdot 1=1\,.$$

Another solution: Just choose the new basis $\{\mathbf{e}'_x,\mathbf{e}'_y,\mathbf{e}'_z\}$ where $\mathbf{e}'_x=\mathbf{a},\mathbf{e}'_y=\mathbf{b}$ and $\mathbf{e}'_z=\mathbf{e}'_x\times\mathbf{e}'_y$. This basis has the same orientation as an initial one (Why?) In this basis calculations become very simple.

9 a) *Show that in general* $\mathbf{a}\times\mathbf{b}\times\mathbf{c}\neq(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$. *(Associativity law is not obeyed)*

Consider e.g. $\mathbf{a}=\mathbf{e}_y,\mathbf{b}=\mathbf{e}_y,\mathbf{c}=\mathbf{e}_z$. Then

$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\mathbf{e}_y\times(\mathbf{e}_y\times\mathbf{e}_z)=\mathbf{e}_y\times\mathbf{e}_x=-\mathbf{e}_z$, and

$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=(\mathbf{e}_y\times\mathbf{e}_y)\times\mathbf{e}_z=0\times\mathbf{e}_z=0\neq-\mathbf{e}_z$.

† b) Show that $\mathbf{a}\times\mathbf{b}\times\mathbf{c}=\mathbf{b}(\mathbf{a},\mathbf{c})-\mathbf{c}(\mathbf{a},\mathbf{b})$ and

$$\mathbf{a}\times\mathbf{b}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}\times\mathbf{a}+\mathbf{c}\times\mathbf{a}\times\mathbf{b}=0\quad\text{(Jacobi identity)}\,.$$

Solution: e.g. straightforward calculations. They could be little bit simplified if you choose an adjusted basis such that $\mathbf{e}_x$ is proportional to one of the vectors.

The oriented vector space $\mathbf{E}^3$ equipped with vector product becomes an algebra where multiplication $\times$ is anticommutative and not-associative but obeying the Jacobi identity. These algebras are called Lie algebras. The Lie algebra of vectors with cross product is Lie algebra of the Lie group $SO(3)$. It is may be the simplest example of non-trivial Lie algebra**.

** See also in my home-page an etude "Jacobi Identity and intersection of heights of triangle" in subdirectory Etudes/Teaching/Geometry .