Homework 4

In this exercise we consider three-dimensional oriented Euclidean space $\mathbf{E}^3$, where by default the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ defines the orientation.

1 Prove that vectors $\mathbf{a}$ and $\mathbf{b}$ are linear independent if and only if $\mathbf{a} \times \mathbf{b} \neq 0$.

2 Vectors $\mathbf{a}$ and $\mathbf{b}$ are linear independent. Using the fact that in this case $\mathbf{a} \times \mathbf{b} \neq 0$ (see the previous exercise) prove that the vectors $(\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b})$ are linear independent

3 Students John and Sarah calculate vector product $\mathbf{a} \times \mathbf{b}$ of two vectors using two different orthonormal bases in the Euclidean space $\mathbf{E}^3$, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$. John expands the vectors with respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Sarah expands the vectors with respect to the basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$. For two arbitrary vectors $\mathbf{a}, \mathbf{b} \in \mathbf{E}^3$

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = a'_1\mathbf{e}'_1 + a'_2\mathbf{e}'_2 + a'_3\mathbf{e}'_3\,,$$

$$\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 = b'_1\mathbf{e}'_1 + b'_2\mathbf{e}'_2 + b'_3\mathbf{e}'_3\,.$$

John and Sarah both use so called "determinant" formula. Are their answers the same?

$$\mathbf{a} \times \mathbf{b} = \underbrace{\det \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}}_{\text{John's calculations}} \stackrel{?}{=} \underbrace{\det \begin{pmatrix} \mathbf{e}'_1 & \mathbf{e}'_2 & \mathbf{e}'_3 \\ a'_1 & a'_2 & a'_3 \\ b'_1 & b'_2 & b'_3 \end{pmatrix}}_{\text{Sarah's calculations}}$$

4 Calculate the area of parallelograms formed by the vectors $\mathbf{a}, \mathbf{b}$ if

a) $\mathbf{a} = (1, 2, 3), \mathbf{b} = (1, 0, 1)$;
b) $\mathbf{a} = (2, 2, 3), \mathbf{b} = (1, 1, 1)$;
c) $\mathbf{a} = (5, 8, 4), \mathbf{b} = (10, 16, 8)$.

5 Show that for any two vectors $\mathbf{a}, \mathbf{b} \in \mathbf{E}^3$ the following identity is satisfied

$$(\mathbf{a}, \mathbf{a})(\mathbf{b}, \mathbf{b}) = (\mathbf{a}, \mathbf{b})^2 + (\mathbf{a} \times \mathbf{b}, \mathbf{a} \times \mathbf{b})\,.$$

Write down this identity in components.
Compare this identity with CBS inequality . See the problem 7 in the Homework 2).

6 Find a vector $\mathbf{n}$ such that the following conditions hold:
1) It has a unit length
2) It is orthogonal to the vectors $\mathbf{a} = (1, 2, 3)$ and $\mathbf{b} = (1, 3, 2)$.
3) An ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{n}\}$ has an orientation opposite to the orientation of the orthonormal basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ which defines the orientation of the Euclidean space.

7 Volume of parallelepiped $V(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{a}, \mathbf{b} \times \mathbf{c})$, formed by the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ equals to zero if and only if these vectors are linearly dependent. Prove it.

8 Vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal unit vectors. Calculate the length of the vector $\mathbf{c} \times (\mathbf{a} \times \mathbf{b})$, where $\mathbf{c} = \mathbf{a} + \mathbf{b}$.

9 a) Show that in general $\mathbf{a} \times \mathbf{b} \times \mathbf{c} \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$. (Associativity law is not obeyed)
† b) Show that $\mathbf{a} \times \mathbf{b} \times \mathbf{c} = \mathbf{b}(\mathbf{a}, \mathbf{c}) - \mathbf{c}(\mathbf{a}, \mathbf{b})$ and

$$\mathbf{a} \times \mathbf{b} \times \mathbf{c} + \mathbf{b} \times \mathbf{c} \times \mathbf{a} + \mathbf{c} \times \mathbf{a} \times \mathbf{b} = 0 \quad \text{(Jacobi identity)}\,.$$