Algorithm "nosov" continuous fractions

Here we reproduce the algorithm "vytiagivanija nosov", which I learnt in the book of Arnold.

Let $\mathbf{e}, \mathbf{f}$ be standard basis in $\mathbf{R}^2$. $\mathbf{e} = (1,0)$, and $\mathbf{f} = (0,1)$.

It is useful also consider the module $\mathbf{Z}^2 = \mathbf{Z} \otimes \mathbf{Z}$ over vectors $\mathbf{e}, \mathbf{f}$:

$$\mathbf{Z}^2 = \{(m,n) = m\mathbf{e} + n\mathbf{f} \quad m,n \in \mathbf{Z}\} .$$

We assign to an arbitrary rational number $\frac{p}{q}$ the vector

$$\mathbf{E}\left(\frac{p}{q}\right) = q\mathbf{e} + p\mathbf{f} .$$

which belongs also to $\mathbf{Z} \otimes \mathbf{Z}$. (We assume that p, q are coprime)

Let α be non-negative real number and let $[a_0, a_1, \dots,]$ be its continuous fraction, and $\frac{p_k}{q_k}$ be its k-th approximation:

$$[a_0, \dots, a_k] = \frac{p_k}{q_k}$$

(We assume that p_k, q_k are coprime)

Consider vectors $\{\mathbf{E}_{-2}, \mathbf{E}_{-1}, \mathbf{E}_0 . \mathbf{E}_1, \dots\}$ in $\mathbf{Z}^2$ defined by real number α in the following way:

$$\mathbf{E}_{-2} = \mathbf{e}, \quad \mathbf{E}_{-1} = \mathbf{f} ,$$

Proposition vytiagivanie nosov: for arbitrary k:

$$\mathbf{E}_{k+1} = \mathbf{E}_{k-1} + a_{k+1}\mathbf{E}_k . \qquad (Prop)$$

Proof:

Instead Proposition we will prove the statement that equation (Prop) survives up to collinearity:

for arbitrary k:

$$[\mathbf{E}_{k+1}] = [\mathbf{E}_{k-1} + a_{k+1}\mathbf{E}_k] , \qquad (Statement)$$

where we denote by $[\mathbf{t}]$ the class of vectors collinear to the vector $\mathbf{t}$.

This statement seems to be weaker, but it implies the Proposition. Prove first that statement (*Statement*) implies Proposition (*Prop*), then we will prove statement (*Statement*)

Equation (*Statement*) implies that

$$\mathbf{E}_{k+1} = \lambda\left(\mathbf{E}_{k-1} + a_{k+1}\mathbf{E}_k\right)$$

Now prove the statement.

Consider first small k

For $k=-1$: $\mathbf{E}_0 = q_0\mathbf{e} + p_0\mathbf{f} = \mathbf{e} + a_0\mathbf{f}$ and

$$[\mathbf{E}_0] = [\mathbf{E}_{-2} + a_0\mathbf{E}_{-1}] = [\mathbf{e} + a_0\mathbf{f}]\,.$$

This is true.

For $k=0$:

$$[\mathbf{E}_1] = [q_1\mathbf{e} + p_1\mathbf{f}] = \left[\mathbf{e} + \left(a_0 + \frac{1}{a_1}\right)\mathbf{f}\right] = \left[(\mathbf{e} + a_0\mathbf{f}) + \frac{1}{a_1}\mathbf{f}\right] =$$

$$\left[\mathbf{E}_{-2} + a_0\mathbf{E}_{-1} + \frac{1}{a_1}\mathbf{f}\right] = \left[\mathbf{E}_0 + \frac{1}{a_1}\mathbf{E}_{-1}\right] = [a_1\mathbf{E}_0 + \mathbf{E}_{-1}]\,.$$

This is true.

For $k=1$:

$$[\mathbf{E}_2] = [q_2\mathbf{e} + p_2\mathbf{f}] = \left[\mathbf{e} + \left(a_0 + \frac{1}{a_1 + \frac{1}{a_2}}\right)\mathbf{f}\right] = \left[(\mathbf{e} + a_0\mathbf{f}) + \frac{1}{a_1 + \frac{1}{a_2}}\mathbf{f}\right] =$$

$$\left[\mathbf{E}_{-2} + a_0\mathbf{E}_{-1} + \frac{1}{a_1 + \frac{1}{a_2}}\mathbf{E}_{-1}\right] = \left[\mathbf{E}_0 + \frac{1}{a_1 + \frac{1}{a_2}}\mathbf{E}_{-1}\right] = \left[\left(a_1 + \frac{1}{a_2}\right)\mathbf{E}_0 + \mathbf{E}_{-1}\right] =$$

$$\left[\mathbf{E}_{-1} + a_1\mathbf{E}_0 + \frac{1}{a_2}\mathbf{E}_0\right] = \left[\mathbf{E}_1 + \frac{1}{a_2}\mathbf{E}_0\right] = [a_2\mathbf{E}_1 + \mathbf{E}_0]\,.$$

This is true.

and so on:

$$[\mathbf{E}_m] = [q_m\mathbf{e} + p_m\mathbf{f}] = [\mathbf{e} + [a_0, \ldots, a_m]\,\mathbf{f}] = \left[\mathbf{e} + \left(a_0 + \frac{1}{[a_1, \ldots, a_m]}\right)\mathbf{f}\right] =$$

$$\left[(\mathbf{e} + a_0\mathbf{f}) + \frac{1}{[a_1, \ldots, a_m]}\mathbf{f}\right] = \left[\mathbf{E}_0 + \frac{1}{[a_1, \ldots, a_m]}\mathbf{E}_{-1}\right] = [[a_1, \ldots, a_m]\,\mathbf{E}_0 + \mathbf{E}_{-1}] =$$

$$\left[\left(a_1+\frac{1}{[a_2,\ldots,a_m]}\right)\mathbf{E}_0+\mathbf{E}_{-1}\right]=\left[a_1\mathbf{E}_0+\mathbf{E}_{-1}+\frac{1}{[a_2,\ldots,a_m]}\mathbf{E}_0\right]=$$

$$\left[\mathbf{E}_1+\frac{1}{[a_2,\ldots,a_m]}\mathbf{E}_0\right]=\left[[a_2,\ldots,a_m]\,\mathbf{E}_1+\mathbf{E}_0\right]=$$

$$\left[\left(a_2+\frac{1}{[a_3,\ldots,a_m]}\right)\mathbf{E}_1+\mathbf{E}_0\right]=\left[a_2\mathbf{E}_1+\mathbf{E}_0+\frac{1}{[a_3,\ldots,a_m]}\mathbf{E}_1\right]=$$

$$\left[\mathbf{E}_2+\frac{1}{[a_3,\ldots,a_m]}\mathbf{E}_1\right]=\left[[a_3,\ldots,a_m]\,\mathbf{E}_2+\mathbf{E}_1\right]=$$

$$\left[\left(a_3+\frac{1}{[a_4,\ldots,a_m]}\right)\mathbf{E}_2+\mathbf{E}_1\right]=\left[a_3\mathbf{E}_2+\mathbf{E}_1+\frac{1}{[a_4,\ldots,a_m]}\mathbf{E}_2\right]=$$

$$\left[\mathbf{E}_3+\frac{1}{[a_4,\ldots,a_m]}\mathbf{E}_2\right]=\left[[a_4,\ldots,a_m]\,\mathbf{E}_3+\mathbf{E}_2\right]=$$

$$\left[\left(a_4+\frac{1}{[a_5,\ldots,a_m]}\right)\mathbf{E}_3+\mathbf{E}_2\right]=\left[a_4\mathbf{E}_3+\mathbf{E}_2+\frac{1}{[a_5,\ldots,a_m]}\mathbf{E}_3\right]=$$

$$\left[\mathbf{E}_4+\frac{1}{[a_5,\ldots,a_m]}\mathbf{E}_3\right]=\left[[a_5,\ldots,a_m]\,\mathbf{E}_4+\mathbf{E}_3\right]=\ldots$$

$$\left[\mathbf{E}_{m-2}+\frac{1}{[a_{m-1},a_m]}\mathbf{E}_{m-3}\right]=\left[[a_{m-1},a_m]\,\mathbf{E}_{m-2}+\mathbf{E}_{m-3}\right]=$$

$$\left[\left(a_{m-1}+\frac{1}{a_m}\right)\mathbf{E}_{m-2}+\mathbf{E}_{m-3}\right]=\left[a_{m-1}\mathbf{E}_{m-2}+\mathbf{E}_{m-3}+\frac{1}{a_m}\mathbf{E}_{m-2}\right]=$$

$$\left[\mathbf{E}_{m-1}+\frac{1}{a_m}\mathbf{E}_{m-2}\right]=\left[a_m\mathbf{E}_{m-1}+\mathbf{E}_{m-2}\right]\,.$$