Riemannian Geometry

it is a draft of Lecture Notes of H.M. Khudaverdian.
Manchester, 2 February, 2019

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1 Riemannian manifolds

1.1 Manifolds. Tensors. (Recollection)

1.1.1 Manifolds

I recall briefly basics of manifolds and tensor fields on manifolds.

An n-dimensional manifold $M = M^n$ is a space[1]

such that in a vicinity of an arbitrary point one can consider local coordinates $\{x^1, \dots, x^n\}$. (We say that in a vicinity of this point a manifold M is covered by local coordinates $\{x^1, \dots, x^n\}$). One can consider different local coordinates. If coordinates $\{x^1, \dots, x^n\}$ and $\{x^{1'}, \dots, x^{n'}\}$ both are defined in a vicinity of the given point then they are related by *bijective transition functions* which are defined on domains in $\mathbf{R}^n$ and taking values also in $\mathbf{R}^n$:

$$\begin{cases} x^{1'} = x^{1'}(x^1, \dots, x^n) \\ x^{2'} = x^{2'}(x^1, \dots, x^n) \\ \dots \\ x^{n-1'} = x^{n-1'}(x^1, \dots, x^n) \\ x^{n'} = x^{n'}(x^1, \dots, x^n) \end{cases} \tag{1.1}$$

We say that n-dimensional manifold is *differentiable* or *smooth* if all transition functions are diffeomorphisms, i.e. they are smooth. Invertability implies that Jacobian matrix is non-degenerate:

$$\det \begin{pmatrix} \frac{\partial x^{1'}}{\partial x^1} & \frac{\partial x^{1'}}{\partial x^2} \dots & \frac{\partial x^{1'}}{\partial x^n} \\ \frac{\partial x^{2'}}{\partial x^1} & \frac{\partial x^{2'}}{\partial x^2} \dots & \frac{\partial x^{2'}}{\partial x^n} \\ \dots & & \\ \frac{\partial x^{n'}}{\partial x^1} & \frac{\partial x^{n'}}{\partial x^2} \dots & \frac{\partial x^{n'}}{\partial x^n} \end{pmatrix} \neq 0\,. \tag{1.2}$$

(If bijective function $x^{i'} = x^{i'}(x^i)$ is smooth function, and its inverse, the transition function $x^i = x^i(x^i)$ is also smooth function, then matrices $||\frac{\partial x^{i'}}{\partial x^i}||$ and $||\frac{\partial x^i}{\partial x^{i'}}||$ are both well defined, hence condition (1.2) is obeyed.

[1] A space M is a topological space, i.e. it is covered by a collection $\mathcal{F}$ of sets, which are called *open* sets. This collection obeys the following axioms

i) the union of an arbitrary set of open sets is an open set

ii) the intersection of finite number of open sets is an open set

iii) the whole space M and the empty set $\emptyset$ are open sets

Example

open domain in $\mathbf{E}^n$

A good example of manifold is an open domain D in n-dimensional vector space $\mathbf{R}^n$. Cartesian coordinates on $\mathbf{R}^n$ define global coordinates on D. On the other hand one can consider an arbitrary local coordinates in different domains in $\mathbf{R}^n$. E.g. one can consider polar coordinates $\{r, \varphi\}$ in a domain $D = \{x, y \colon y > 0\}$ of $\mathbf{R}^2$ defined by standard formulae:

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}, \tag{1.3}$$

$$\det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{pmatrix} = \det\begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix} = r \tag{1.4}$$

or one can consider spherical coordinates $\{r, \theta, \varphi\}$ in a domain $D = \{x, y, z \colon x > 0, y > 0, z > 0\}$ of $\mathbf{R}^3$ (or in other domain of $\mathbf{R}^3$) defined by standard formulae

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases},$$

,

$$\det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{pmatrix} = \det\begin{pmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} = r^2\sin\theta \tag{1.5}$$

Choosing domain where polar (spherical) coordinates are well-defined we have to be aware that coordinates have to be well-defined and transition functions (1.1) have to obey condition (1.2), i.e. they have to be diffeomorphisms. E.g. for domain D in example (1.3) Jacobian (1.4) does not vanish if and only if $r > 0$ in D.

Consider another examples of manifolds, and local coordinates on manifolds.

Example

Circle S^1 in $\mathbf{E}^2$

Consider circle $x^2 + y^2 = R^2$ of radius R in $\mathbf{E}^2$.

One can consider on the circle different local coordinates

i) *polar coordinate* φ:

$$\begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \end{cases} , \qquad 0 < \varphi < 2\pi$$

(this coordinate is defined on all the circle except a point $(R, 0)$),

ii) *another polar coordinate* φ':

$$\begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \end{cases} , \qquad -\pi < \varphi < \pi\,,$$

this coordinate is defined on all the circle except a point $(-R, 0)$,

iii) stereographic coordinate t with respect to north pole of the circle

$$\begin{cases} x = \frac{2R^2 t}{t^2+R^2} \\ y = R\frac{t^2-R^2}{t^2+R^2} \end{cases} , \quad t = \frac{Rx}{R-y}\,, \tag{1.6}$$

this coordinate is defined at all the circle except the north pole,

iiii) stereographic coordinate t' with respect to south pole of the circle

$$\begin{cases} x = \frac{2R^2 t'}{t'^2+R^2} \\ z = R\frac{R^2-t'^2}{t'^2+R^2} \end{cases} , \quad t' = \frac{Rx}{R+y}\,,$$

this coordinate is defined at all the points except the south pole.

We considered four different local coordinates on the circle S^1. Write down some transition functions (1.1) between these coordinates

- polar coordinate φ coincide with polar coordinate φ' in the domain $x^2 + y^2 > 0$, and in the domain $x^2 + y^2 < 0$ $\varphi' = \varphi - 2\pi$.
- Transition function from polar coordinate φ to stereographic coordinates t is $t = \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)$,
- transition function from stereographic coordinate t to stereographic coordinate t' is

$$t' = \frac{R^2}{t}\,,$$

(see Homework 0.)

Example

Sphere S^2 in $\mathbf{E}^3$

Consider sphere $x^2+y^2+z^2=R^2$ of radius a in $\mathbf{E}^3$.
One can consider on the sphere different local coordinates
i) *spherical coordinates on domain of sphere θ,φ:*

$$\begin{cases} x = a\sin\theta\cos\varphi \\ y = a\sin\theta\sin\varphi \\ z = a\cos\theta \end{cases}, \quad 0<\theta<\pi\,, -\pi<\varphi<\pi$$

ii) stereographic coordinates u,v with respect to north pole of the sphere

$$\begin{cases} x = \frac{2a^2u}{a^2+u^2+v^2} \\ y = \frac{2a^2v}{a^2+u^2+v^2} \\ z = a\frac{u^2+v^2-a^2}{a^2+u^2+v^2} \end{cases}, \quad \frac{x}{u}=\frac{y}{v}=\frac{a-z}{a}\,, \quad \begin{cases} u = \frac{ax}{a-z} \\ v = \frac{ay}{a-z} \end{cases}.$$

iii) stereographic coordinates u',v' with respect to south pole of the sphere

$$\begin{cases} x = \frac{2a^2u'}{a^2+u'^2+v'^2} \\ y = \frac{2a^2v}{a^2+u'^2+v'^2} \\ z = a\frac{a^2-u'^2-v'^2}{a^2+u^2+v^2} \end{cases}, \quad \frac{x}{u'}=\frac{y}{v'}=\frac{a+z}{a}\,, \quad \begin{cases} u' = \frac{ax}{a+z} \\ v' = \frac{ay}{a+z} \end{cases}.$$

(see also Homework 0)

Spherical coordinates are defined elsewhere except poles and the meridians $y=0, x\leq 0$.

Stereographical coordinates (u,v) are defined elsewhere except north pole; stereographic coordinates (u',v') are defined elsewhere except south pole.

One can consider transition function between these different coordinates.E.g. transition functions from spherical coordinates i) to stereographic coordinates (u,v) are

$$\begin{cases} u = \frac{ax}{a-z} = \frac{a\sin\theta\cos\varphi}{1-\cos\theta} = a\mathrm{cotan}\,\frac{\theta}{2}\cos\varphi \\ v = \frac{ay}{a-z} = \frac{a\sin\theta\sin\varphi}{1-\cos\theta} = a\mathrm{cotan}\,\frac{\theta}{2}\sin\varphi \end{cases},$$

and transition function from stereographic coordinates u,v to stereographic coordinates (u',v') are

$$\begin{cases} u' = \frac{a^2u}{u^2+v^2} \\ v' = \frac{a^2v}{u^2+v^2} \end{cases},$$

(see Homework 0.)

Remark

† One very important property of stereographic projetion which we do not use in this course but it is too beautiful not to mention it: under stereographic projection all points of the circle of radius $R = 1$ with rational coordinates x and y and only these points transform to rational points on line. Thus we come to Pythagorean triples $a^2 + b^2 = c^2$. The same is for unit sphere: the stereographic projection establishes one-one correspondence between points on the unit sphere with rational coordinates and rational points on the plane.

1.1.2 Tensors on Manifold

tangent vector and tangent vector space

Tangent vector at the given point can be considered as a derivation of function at this point. For an arbitrary (smooth) function f defined in a vicinity of a given point $\mathbf{p}$ a tangent vector $\mathbf{A}(x) = A^i(x)\frac{\partial}{\partial x^i}$ defines the directional derivative of this function

$$\mathbf{A}\colon \quad f \mapsto \partial_{\mathbf{A}} f\big|_{\mathbf{p}} = A^i(x)\frac{\partial f}{\partial x^i}\big|_{\mathbf{p}}\,.$$

Using the chain rule one can see that under changing of coordinates it transforms as follows:

$$\mathbf{A} = A^i(x)\frac{\partial}{\partial x^i} = A^i(x)\frac{\partial x^{i'}(x)}{\partial x^i}\frac{\partial}{\partial x^{i'}} = A^{i'}(x'(x))\frac{\partial}{\partial x^{i'}}\,,$$

i.e.

$$A^{i'}(x') = \frac{\partial x^{i'}}{\partial x^i}A^i(x)\,. \tag{1.7}$$

This leads as to the following equivalent definition of the tangent vector.

Definition Let $M = M^n$ be n-dimensional manifold, and $\mathbf{p}$ the point on it. To define a vector $\mathbf{A}$ tangent to the manifold at the point $\mathbf{p}$ we assign to an arbitrary given local coordinates $\{x^i\}$ the array $\{A^i\}$ $(i = 1, \dots, n)$ of numbers (components) such that under changing of local coordinates this array transforms according to equation (1.7):

$$\begin{array}{ccc} \texttt{coordinates} & & \texttt{components of vector} \\ \{x^i\} & \rightarrow & \{A^i\} \\ \{x^{i'}\} & \rightarrow & \{A^{i'}\} \end{array} \qquad \text{such that } A^{i'} = \frac{\partial x^{i'}(x)}{\partial x^i}\big|_{\mathbf{p}} A^i\,. \tag{1.8}$$

Tangent vector space $T_{\mathbf{p}}M$ at the point $\mathbf{p}$ is the space of vectors tangent to the manifold at the point M.

We use in all formulae Einstein summation rules.

Remark *Einstein summation rules*

In our lectures we always use so called *Einstein summation convention.* it implies that when an index occurs twice in the same expression in upper and in lower postitions, then the expression is implicitly summed over all possible values for that index. Sometimes it is called dummy indices summation rule.

Using Einstein summation rules we avoid to write bulky expressions. Later we will see that these notations are really very effective. E.g. equation (1.7) in 'standard' notations will appear as

$$\text{for every } i' = 1, \dots, n \ A^{i'}(x') = \sum_{i=1}^{n} \frac{\partial x^{i'}}{\partial x^i} A^i(x)\,.$$