Introduction to Geometry (20222)

2018

COURSEWORK

This assignment counts for 20% of your marks.

Solutions are due by 22 March, 3pm

Write solutions in the provided spaces.

STUDENT'S NAME:

Academic Advisor (Tutor):

1

a) Let (x^1, x^2, x^3) be coordinates of the vector $\mathbf{x}$, and (y^1, y^2, y^3) be coordinates of the vector $\mathbf{y}$ in $\mathbf{R}^3$.

Does the formula $(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^3 + x^3y^2$ define a scalar product on $\mathbf{R}^3$? Justify your answer.

b) Consider the matrix $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Calculate the matrix A^2 in the case, if $\theta = \frac{\pi}{4}$.

Calculate the matrix A^{18} in the case, if $\theta = \frac{\pi}{108}$.

Calculate the matrix $A^T \circ A^{2018} \circ A^T$ in the case, if $\theta = \frac{\pi}{14}$ (here A^T is a transposed matrix.)

Find all 2×2 orthogonal matrices A such that

$$2A^3 = \begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix} .$$

c) In oriented Euclidean space $\mathbf{E}^3$ consider the following linear operator

$$A(\mathbf{x}) = \mathbf{x} - \mathbf{a} \times (\mathbf{a} \times \mathbf{x}) ,$$

where the vector $\mathbf{a} = \mathbf{e} + 2\mathbf{f} + 2\mathbf{g}$. Here $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ is an orthonormal basis in $\mathbf{E}^3$ defining orientation, and $\times$ is the vector product.

Find the eigenvectors of operator A. (Describe eigenvectors via the basis vectors $\mathbf{e}, \mathbf{f}, \mathbf{g}$.)

Calculate the trace and determinant of the operator A.

2

a) Consider a vector $\mathbf{a} = 2\mathbf{e} + 3\mathbf{f} + 6\mathbf{g}$, where $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ is an orthonormal basis in $\mathbf{E}^3$.

Show that the angle θ between vectors $\mathbf{a}$ and $\mathbf{g}$ belongs to the interval $\left(\frac{\pi}{6}, \frac{\pi}{4}\right)$.

Find a unit vector $\mathbf{b}$ such that this vector is orthogonal to vectors $\mathbf{a}$ and $\mathbf{g}$, and the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{g}\}$ has the same orientation as the basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$.

Calculate the angle between vectors $\mathbf{b}$ and $\mathbf{e}$.

b) In oriented Euclidean space $\mathbf{E}^3$ consider the following function of three vectors:

$$F(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = (\mathbf{X}, \mathbf{Y} \times \mathbf{Z})\,,$$

where $(\ ,\ )$ is the scalar product and $\mathbf{Y} \times \mathbf{Z}$ is the vector product in $\mathbf{E}^3$.

Show that $F(\mathbf{X}, \mathbf{X}, \mathbf{Z}) = 0$ for arbitrary vectors $\mathbf{X}$ and $\mathbf{Z}$.

Deduce, that $F(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = -F(\mathbf{Y}, \mathbf{X}, \mathbf{Z})$ for arbitrary vectors $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$.

What is the geometrical meaning of the function F?

c) Let $ABCD$ be a rhombus (parallelogram with equal sides) such that

i) vertex A is at the origin

ii) the diagonal AC belongs to the line $y = x$.

iii) vertex B has integer coordinates.

Find the area of this rhombus, if the vertex B has coordinates $(20, 21)$. Justify your answer.

Find all the rhombi, which obey the conditions i), ii) and iii) above, and which have area $S = 25$.

3

We consider in this question a 3-dimensional Euclidean space. We suppose that $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ is an orthonormal basis in this space.

a) Consider an operator P such that

$$P(\mathbf{e}) = \frac{2}{3}\mathbf{e} + \frac{2}{3}\mathbf{f} + \frac{1}{3}\mathbf{g}\,, P(\mathbf{f}) = -\frac{1}{3}\mathbf{e} + \frac{2}{3}\mathbf{f} - \frac{2}{3}\mathbf{g}\,, P(\mathbf{g}) = -\frac{2}{3}\mathbf{e} + \frac{1}{3}\mathbf{f} + \frac{2}{3}\mathbf{g}\,.$$

Show, that it is an orthogonal operator preserving orientation.

Show, that this operator defines rotation, and find the axis and the angle of this rotation.

b) Let P be a linear orthogonal operator acting in $\mathbf{E}^3$, such that it preserves the orientation of $\mathbf{E}^3$ and the following relations hold:

$$P(\mathbf{e}) = \cos\frac{\pi}{5}\ \mathbf{e} + \sin\frac{\pi}{5}\ \mathbf{f}, \quad P(\mathbf{g}) = -\mathbf{g}\,.$$

Write down the matrix of operator P in the basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$.

Show that this operator defines rotation, and find the axis and the angle of this rotation.

c) Orthogonal operator P obeys the condition

$$P \neq I\,, \quad \text{and} \quad P^3 = I\,.$$

Show that P is a rotation operator, and calculate the angle of rotation.