Homework 6

1 Calculate the derivatives of the functions $f = x^2+y^2$, $g = y^2-x^2$ and $h = q\log|r| = q\log\left(\sqrt{x^2+y^2}\right)$ (q is a constant) along vector fields $\mathbf{A} = x\partial_x + y\partial_y$ and $\mathbf{B} = x\partial_y - y\partial_x$

a) calculating directional derivatives $\partial_{\mathbf{A}} f, \partial_{\mathbf{A}} g, \partial_{\mathbf{A}} h$, $\partial_{\mathbf{B}} f, \partial_{\mathbf{B}} g, \partial_{\mathbf{B}} h$

b) calculating $df(\mathbf{A}), dg(\mathbf{A}), dh(\mathbf{A}), df(\mathbf{B}), dg(\mathbf{B}), dh(\mathbf{B})$.

2 Perform the calculations of the previous exercise in polar coordinates.

3 Consider a function $f = x^4 - y^4$.
Calculate the value of 1-form $\omega = df$ on the vector field $\mathbf{B} = x\partial_y - y\partial_x$.
Express this 1-form ω in polar coordinates r, φ ($x = r\cos\varphi, y = r\sin\varphi$).

4 Calculate the value of 1-form $\omega = xdy - ydx$ on the vector fields $\mathbf{A} = r\partial_r$ and $\mathbf{B} = \partial_\varphi$. Perform calculations in Cartesian and polar coordinates.

5 Let f be a function on $\mathbf{E}^2$ given by $f(r,\varphi) = r^2\sin 2\varphi$, where r, φ are polar coordinates in $\mathbf{E}^2$.
Calculate the 1-form $\omega = df$.
Calculate the value of the 1-form $\omega = df$ on the vector field $\mathbf{X} = r^2\partial_r + r\partial_\varphi$.
Express the 1-form ω in Cartesian coordinates x, y.

6 Consider 1-forms $\omega = df$ and $\sigma = dg$ such that

$$f(x,y) + ig(x,y) = (x+iy)^3 .$$

Find the values of these 1-forms on vector field $\mathbf{Y} = r\partial_r + \partial_\varphi$.

7 Calculate the integrals of the form $\omega = \sin y\, dx$ over the following three curves. Compare answers.

$$C_1\colon \mathbf{r}(t) \begin{cases} x = 2t^2 - 1 \\ y = t \end{cases}, \ 0 < t < 1, \quad C_2\colon \mathbf{r}(t) \begin{cases} x = 8t^2 - 1 \\ y = 2t \end{cases}, \ 0 < t < 1/2$$

$$\text{and} \quad C_3\colon \mathbf{r}(t) \begin{cases} x = \cos 2t \\ y = \cos t \end{cases}, \ 0 < t < \frac{\pi}{2} .$$

8 Calculate the integrals of the form $\omega = xdy - ydx$ over the following three curves. Compare answers.

$$C_1\colon \mathbf{r}(t) \begin{cases} x = R\cos t \\ y = R\sin t \end{cases}, \ 0 < t < \pi, \quad C_2\colon \mathbf{r}(t) \begin{cases} x = R\cos 4t \\ y = R\sin 4t \end{cases}, \ 0 < t < \frac{\pi}{4}$$

$$\text{and} \quad C_3\colon \mathbf{r}(t) \begin{cases} x = Rt \\ y = R\sqrt{1-t^2} \end{cases}, \ -1 \le t \le 1.$$