Homework 6

1 Calculate the derivatives of the functions $f = x^2 + y^2$, $g = e^{-(x^2+y^2)}$ and $h = q\log|r| = q\log\left(\sqrt{x^2+y^2}\right)$ (q is a constant) along vector fields $\mathbf{A} = x\partial_x + y\partial_y$ and $\mathbf{B} = x\partial_y - y\partial_x$, i.e. calculate $\partial_{\mathbf{A}} f, \partial_{\mathbf{A}} g, \partial_{\mathbf{A}} h$, $\partial_{\mathbf{B}} f, \partial_{\mathbf{B}} g, \partial_{\mathbf{B}} h$.

2 Perform the calculations of the previous exercise using polar coordinates.

3 Consider in $\mathbf{E}^2$ vector fields $\mathbf{A} = x\partial_x + y\partial_y$, $\mathbf{B} = x\partial_y - y\partial_x$, $\mathbf{C} = \partial_x$, $\mathbf{D} = \partial_y$. Calculate the values of 1-forms df, dg on these vector fields if $f = (x^2+y^2)^n$ and $g = \frac{y}{x}$. For vector fields $\mathbf{A}, \mathbf{B}$ perform calculations also in polar coordinates.

4 Calculate the integrals of the form $\omega = \sin y\, dx$ over the following three curves. Compare answers.

$C_1 \colon \mathbf{r}(t)\ \begin{cases} x = 2t^2 - 1 \\ y = t \end{cases},\ 0 < t < 1, \qquad C_2 \colon \mathbf{r}(t)\ \begin{cases} x = 8t^2 - 1 \\ y = 2t \end{cases},\ 0 < t < 1/2,$

$C_3 \colon \mathbf{r}(t)\ \begin{cases} x = \cos 2t \\ y = \cos t \end{cases},\ 0 < t < \frac{\pi}{2}$

5 Calculate the integral of the form $\omega = e^{-y}dx + \sin x dy$ over the segment of straight line which connects the points $A = (1,1)$, $B = (2,3)$. At what extent an answer depends on a chosen parameterisation?

6 Calculate the integral of the form $\omega = xdy$ over the upper arc of the unit circle starting at the point $A = (1,0)$ and ending at the point $(0,1)$.

7 Solve the previous problem for the arc of the ellipse $x^2 + y^2/9 = 1$ defined by the condition $y \geq 0$.

Exact forms

8 Calculate the integral $\int_C \omega$ where $\omega = xdx + ydy$ and C is

a) the straight line segment $x = t, y = 1 - t$, $0 \leq t \leq 1$

b) the segment of parabola $x = t$, $y = 1 - t^n$, $0 \leq t \leq 1$, $n = 2, 3, 4, \ldots$

c) the segment of the sinusoid $x = t$, $y = \cos\frac{\pi}{2}t$, $0 \leq t \leq 1$

d)Show that 1-form $\omega = xdx + ydy$ is an exact form and for **an arbitrary** curve starting at the point $(0,1)$ and ending at the point $((1,0)$ calculate the integral $\int_C \omega$.

9 Show that the form 1-form $\omega = 2xydx + x^2dy$ is an exact 1-form. Calculate integral of this form over the curves considered in exercises 6) and 7) (upper half of the circle and ellipse)

10. Calculate the differentials of the following 1-forms:

a) xdx , b) xdy c) $xdx + ydy$, d)$xdy + ydx$, e) $xdy - ydx$

f) $x^4dy + 4x^3ydx$, g) $xdy + ydx + dz$, h) $xdy - ydx + dz$.

For each 1-forms listed above find a function f (0-form) such that $df = \omega$, if possible, i.e. if this form is an exact form. If it is not exact form, explain why.

All the exercises below are not compulsory

11† Consider one-form

$$\omega = \frac{xdy - ydx}{x^2 + y^2} \tag{1}$$

This form is defined in $\mathbf{E}^2 \setminus 0$.

Calculate differential of this form.

Write down this form in polar coordinates

Find a function f such that $\omega = df$.

Is this function defined in the same domain as ω?

12† Calculate the integral of the form $\omega = \frac{xdy-ydx}{x^2+y^2}$ over the curves $a), b), c)$ from the previous exercise.

13† What values can take the integral $\int_C \omega$ if C is an arbitrary curve starting at the point $(0, 1)$ and ending at the point $((1, 0)$ and $\omega = \frac{xdy-ydx}{x^2+y^2}$.

14† Let $\omega = a(x, y)dx + b(x, y)dy$ be a closed form in $\mathbf{E}^2$, $d\omega = 0$.

Consider the function

$$f(x, y) = x \int_0^1 a(tx, ty)dt + y \int_0^1 b(tx, ty)dt \tag{2}$$

Show that

$$\omega = df\,.$$

This proves that an arbitrary closed form in $\mathbf{E}^2$ is an exact form.

Why we cannot apply the formula (2) to the form ω defined by the expression (1)?