Homework 1–2. Solutions

1 *Show that the condition of non-degeneracy for a symmetric matrix $||g_{ik}||$ follows from the condition that this matrix is positive-definite.*

Solution Suppose $\det g = 0$, i.e. g is degenerate matrix (rows and columns of the matrix are linear dependent). Then there exists non-zero vector $\mathbf{x} = (x^1, x^2)$ such that $g_{ik}x^k = 0$, hence $g_{ik}x^i x^k = 0$ for $\mathbf{x} \neq 0$. Contradiction to the condition of positive-definiteness.

2 *Let (u, v) be local coordinates on 2-dimensional Riemannian manifold M. Let Riemannian metric be given in these local coordinates by the matrix*

$$||g_{ik}|| = \begin{pmatrix} A(u,v) & B(u,v) \\ C(u,v) & D(u,v) \end{pmatrix},$$

where $A(u,v), B(u,v), C(u,v), D(u,v)$ are smooth functions. Show that the following conditions are fulfilled:

a) $B(u,v) = C(u,v)$,

b) $A(u,v)D(u,v) - B(u,v)C(u,v) \neq 0$,

c) $A(u,v) > 0$,

d) $A(u,v)D(u,v) - B(u,v)C(u,v) > 0$.

e)† Show that conditions a), c) and d) are necessary and sufficient conditions for matrix $||g_{ik}||$ to define locally a Riemannian metric.

f^*) How conditions above will change if the manifold M is pseudo-Riemannian, but not necessarily Riemannian?

Solution

Consider Riemannian scalar product $G(\mathbf{X}, \mathbf{Y}) = g_{ik}X^iY^k$.

a) The condition that $G(\mathbf{X}, \mathbf{Y}) = G(\mathbf{Y}, \mathbf{X})$ means that $g_{ik} = g_{ki}$, i.e. $B(u, v = C(u, v))$.

b) $\det G = A(u,v)D(u,v) - B(u,v)C(u,v) = AD - B^2 \neq 0$ since it is non-degenerate (see the solution of exercise 1)

c) Consider quadratic form $G(\mathbf{x}, \mathbf{x}) = g_{ik}x^ix^k = Ax^2 + 2Bxy + Dy^2$. (We already know that $B = C$) Positive -definiteness means that $G(\mathbf{x}, \mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$. In particular if we put $\mathbf{x} = (1, 0)$ we come to $G(\mathbf{x}, \mathbf{x}) = A > 0$. Thus $A > 0$.

d) Consider quadratic form $G(\mathbf{x}, \mathbf{x}) = g_{ik}x^ix^k = Ax^2 + 2Bxy + Dy^2$. We have an identity

$$G(\mathbf{x}, \mathbf{x}) = g_{ik}x^ix^k = Ax^2 + 2Bxy + Dy^2 = \frac{(Ax + By)^2 + (AD - B^2)y^2}{A}$$

We already know that $A > 0$ (take $\mathbf{x} = (x, 0)$). Now take $\mathbf{x} = (x, y)$: $Ax + By = 0$ (e.g. $\mathbf{x}$=(-B,A)) we come to $G(\mathbf{x}, \mathbf{x}) = \frac{(AD-B^2)y^2}{A} > 0$. Hence $(AD - B^2) = \det G > 0$.

Note This special truck works good for dimension is $n = 2$. We could notice that A and $AD - B^2$ are principal main minors of the matrix G. In the general case (if G is $n \times n$ symmetric matrix) using triangular transformations one can show that quadratic form $A(\mathbf{X}, \mathbf{X}) = a_{ik}x^ix^k$ (and respectively) is positive-definite if and only if all the leading principal minors Δ_k[1] are positive. In this case matrix G_{ik} of bilinear form is transformed to unity matrix.

f) The condition of positive-definiteness can be omitted.

3 Write down explicit formulae expressing stereographic coordinates for n-dimensional sphere $(x^1)^2 + \ldots + (x^{n+1})^2 = 1$ via coordinates $x^1, \ldots, x^{n+1}$ and vice versa.

[1] Leading Principal minor Δ_k of the matrix A is a determinant of the matrix formed by first k columns and first k rows of the matrix A

(For simplicity you may consider cases $n = 2, 3$.)

Write down the stereographic projection from the North pole of the sphere–point $(0, 0, \dots, 1)$ on the plane $x^{n+1} = 0$. Consider the segment ND which intersects the sphere at the point $(x^1, \dots, x^{n+1})$, where D is the point on the plane $z = 0$ with the coordinates $u^i = x^i$ for $i = 1, \dots, n$. Then comparing similar triangles we have

$$\frac{1}{1 - x^{n+1}} = \frac{u^i}{x^i}, \quad \text{i.e. } u^i = \frac{x^i}{1 - x^{n+1}} \ (i = 1, \dots, n)$$

Using the fact that $(x^1)^2 + \dots + (x^{n+1})^2 = 1$ we come to

$$(x^1)^2 + \dots + (x^n)^2 = \sum_{i=1}^{n} \left(u^i (1 - x^{n+1})\right)^2 = (1 - x^{n+1})(1 + x^{n+1}) .$$

Hence

$$x^{n+1} = \frac{\sum_{i=1}^n (u^i)^2 - 1}{\sum_{i=1}^n (u^i)^2 + 1} , \quad x^i = \frac{2u^i}{\sum_{i=1}^n (u^i)^2 + 1} \ (i = 1, 2, \dots)$$

For projection with centre in South pole we have to change $x^{n+1} \mapsto -x^{n+1}$.

Write down these formulae for cases $n = 1, 2, 3$,

Case $n = 1$: Circle $x^2 + y^2 = 1$. Stereographic coordinate t. Centre of projection $(0, 1)$:

$$t = \frac{x}{1 - y}, \qquad \begin{cases} x = \frac{2t}{1+t^2} \\ y = \frac{t^2-1}{t^2+1} \end{cases} \tag{1}$$

Case $n = 2$: Sphere $x^2 + y^2 + z^2 = 1$. Stereographic coordinates u, v. Centre of projection $(0, 0, 1)$:

$$\begin{cases} u = \frac{x}{1-z} \\ v = \frac{y}{1-z} \end{cases}, \qquad \begin{cases} x = \frac{2u}{1+u^2+v^2} \\ y = \frac{2v}{1+u^2+v^2} \\ z = \frac{u^2+v^2-1}{u^2+v^2+1} \end{cases} \tag{2}$$

Case $n = 3$: 3-dimensional sphere $x^2 + y^2 + z^2 + t^2 = 1$. Stereographic coordinates u, v, w. Centre of projection $(0, 0, 0, 1)$:

$$\begin{cases} u = \frac{x}{1-t} \\ v = \frac{y}{1-t} \\ w = \frac{z}{1-t} \end{cases}, \qquad \begin{cases} x = \frac{2u}{1+u^2+v^2+w^2} \\ y = \frac{2v}{1+u^2+v^2+w^2} \\ z = \frac{2w}{1+u^2+v^2+w^2} \\ z = \frac{u^2+v^2+w^2-1}{u^2+v^2+w^2+1} \end{cases} \tag{2}$$

In general case: n-dimensional sphere $(x^1)^2 + (x^2)^2 + \dots + (x^n)^2 + (x^{n+1})^2 = 1$. Stereographic coordinates u^i $(i = 1, \dots, n)$. Centre of projection $(0, \dots, 1)$:

$$u^i = \frac{x^i}{1 - x^{n+1}}, i = 1, \dots, n \qquad \begin{cases} x^i = \frac{2u^i}{1+\sum_{i=1}^n (u^i)^2}, \\ (i = 1, \dots, n) \\ x^{n+1} = \frac{\sum_{i=1}^n (u^i)^2 - 1}{\sum_{i=1}^n (u^i)^2 + 1} \end{cases} \tag{3}$$

4 *Consider the Riemannian metric on the unit circle induced by the Euclidean metric on the ambient plane.*

a) Express it using polar angle as a coordinate on the circle.

b) Express the same metric using stereographic coordinate t obtained by stereographic projection of the circle on the line, passing through its centre.

Riemannian metric of Euclidean space is $G = dx^2 + dy^2$.

a) using the angle: In this case parametric equation of circle is $\begin{cases} x = \cos\varphi \\ y = \sin\varphi \end{cases}$. Then

$$G = (dx^2 + dy^2)\big|_{x=\cos\varphi, y=\sin\varphi} = (d\cos\varphi)^2 + (d\sin\varphi)^2 = d\varphi^2$$

b) In stereographic coordinate using (1) we have:

$$G = (dx^2 + dy^2)\big|_{x=x(t), y=y(t)} = \left(d\left(\frac{2t}{1+t^2}\right)\right)^2 + \left(d\left(\frac{t^2-1}{1+t^2}\right)\right)^2 =$$

$$\left(\frac{2dt}{1+t^2} - \frac{4t^2}{(1+t^2)^2}\right)^2 + \left(\frac{2tdt}{1+t^2} - \frac{2t(t^2-1)dt}{(1+t^2)^2}\right)^2 = \left(\frac{2dt}{1+t^2}\right)^2\left(\left(1 - \frac{2t^2}{(1+t^2)}\right)^2 + \left(t - \frac{t(t^2-1)}{(1+t^2)}\right)^2\right)$$

$$= \left(\frac{2dt}{1+t^2}\right)^2\left(\frac{(1-t^2)^2}{(1+t^2)^2} + \frac{4t^2}{(1+t^2)^2}\right) = \left(\frac{2dt}{1+t^2}\right)^2 = \frac{4dt^2}{(1+t^2)^2} \blacksquare$$

5 *Consider the Riemannian metric on the unit sphere induced by the Euclidean metric on the ambient 3-dimensional space.*

a) Express it using spherical coordinates on the sphere.

b) Express the same metric using stereographic coordinates u, v obtained by stereographic projection of the sphere on the plane, passing through its centre.

Solution

Riemannian metric of Euclidean space is $G = dx^2 + dy^2 = dz^2$.

a) using the spherical coordinates: In this case parametric equation of sphere is $\begin{cases} x = \sin\theta\cos\varphi \\ y = \sin\theta\sin\varphi \\ z = \cos\theta \end{cases}$. Then

$$G = (dx^2 + dy^2 + dz^2)\big|_{x=\sin\theta\cos\varphi, y=\sin\theta\sin\varphi, z=\cos\theta} = ((d\sin\theta\cos\varphi))^2 + ((d\sin\theta\sin\varphi))^2 + ((d\cos\theta))^2 =$$

$$(\cos\theta\cos\varphi d\theta - \sin\theta\sin\varphi d\varphi)^2 + (\cos\theta\sin\varphi d\theta + \sin\theta\cos\varphi d\varphi)^2 + (-\sin\theta d\theta)^2 = d\theta^2 + \sin^2\theta d\varphi^2 .$$

b) in stereographic coordinates using (2) we have

$$G = (dx^2 + dy^2 + dz^2)\big|_{x=x(u,v), y=y(u,v), z=z(u,v)} = \left(d\left(\frac{2u}{1+u^2+v^2}\right)\right)^2 + \left(d\left(\frac{2v}{1+u^2+v^2}\right)\right)^2 + \left(d\left(\frac{u^2+v^2-1}{1+u^2+v^2}\right)\right)^2 =$$

$$\left(\frac{2du}{1+u^2+v^2} - \frac{2u(2udu+2vdv)}{(1+u^2+v^2)^2}\right)^2 + \left(\frac{2dv}{1+u^2+v^2} - \frac{2v(2udu+2vdv)}{(1+u^2+v^2)^2}\right)^2 +$$

$$+ \left(\frac{2udu+2vdv}{1+u^2+v^2} - \frac{(u^2+v^2-1)(2udu+2vdv)}{(1+u^2+v^2)^2}\right)^2 = \frac{4(du)^2 + 4(dv)^2}{(1+u^2+v^2)^2}$$

(See detailed calculations for analogous case in the solution of exercise 8.)

6* *Consider the n-dimensional sphere S^n of radius 1 in $(n+1)$-dimensional Euclidean space $\mathbf{E}^{n+1}$. This sphere can be defined by the equation $(x^1)^2 + \ldots + (x^{n+1})^2 = 1$ in Cartesian coordinates $x^1, \ldots, x^n, x^{n+1}$.*

Consider a Riemannian metric on this sphere induced by the Euclidean metric in the ambient space. Write down this metric in stereographic coordinates.

Using (3) we have that

$$G = \left((dx^1)^2 + \ldots + (dx^{n+1})^2\right)\big|_{x^\mu = x^i(u^i)} = \left(\sum_{j=1}^n \left(d\left(\frac{2u^j}{1+\sum_{i=1}^n (u^i)^2}\right)\right)\right)^2 + \left(d\left(\frac{\sum_{i=1}^n (u^i)^2 - 1}{1+\sum_{i=1}^n (u^i)^2}\right)\right)^2 =$$

$$\left(\frac{2du^j}{1+\sum_{i=1}^n (u^i)^2} - \frac{2u(2udu+2vdv)}{(1+u^2+v^2)^2}\right)^2 + \left(\frac{2dv}{1+u^2+v^2} - \frac{2v(2udu+2vdv)}{(1+u^2+v^2)^2}\right)^2 +$$
$$+\left(\frac{2udu+2vdv}{1+u^2+v^2} - \frac{(u^2+v^2-1)(2udu+2vdv)}{(1+u^2+v^2)^2}\right)^2 = \frac{4(du)^2+4(dv)^2}{(1+\sum_{i=1}^n (u^i)^2)^2}$$

(See detailed calculations for analogous case in the solution of exercise 8.)

7 *Consider the surface L which is the upper sheet of one-sheeted hyperboloid in $\mathbf{R}^3$:*

$$L: \qquad z^2 - x^2 - y^2 = 1, \quad z > 0\,.$$

a) Find parametric equation of the surface L using hyperbolic functions $\cosh, \sinh$ *following an analogy with spherical coordinates on the sphere.*

(The surface L sometimes is called pseudo-sphere.)

b) Consider the stereographic projection of the surface L on the plane OXY, i.e. the central projection on the plane $z=0$ with the centre at the point $(0,0,-1)$.

Show that the image of projection of the surface L is the open disc $x^2+y^2<1$ in the plane OXY.

Solution. Calculations are very similar to the case of stereographic coordinates of 2-sphere $x^2+y^2+z^2 = 1$. Stereographic coordinates u, v. Centre of projection $(0,0,-1)$: We have $\frac{u}{x} = \frac{y}{v} = \frac{1}{1+z}$. Hence $\begin{cases} u = \frac{x}{1+z} \\ v = \frac{y}{1+z} \end{cases}$. Since $x = u(1+z), y = v(1+z)$ then $z^2-1 = x^2+y^2$ and $z^2-1 = (u^2+v^2)(1+z)^2$, i.e. $z = \frac{1+u^2+v^2}{1-u^2-v^2}$. We come to

$$\begin{cases} u = \frac{x}{1+z} \\ v = \frac{y}{1+z} \end{cases}, \qquad \begin{cases} x = \frac{2u}{1-u^2-v^2} \\ y = \frac{2v}{1-u^2-v^2} \\ z = \frac{u^2+v^2+1}{1-u^2-v^2} \end{cases} \tag{4}$$

The image of upper-sheet is an open disc $u^2+v^2=1$ since $u^2+v^2 = \frac{x^2+y^2}{(1+z)^2} = \frac{z^2-1}{(1+z)^2} = \frac{z-1}{z+1}$. Since for upper sheet $z>1$ then $0 \le \frac{z-1}{z+1} < 1$.

8* *Consider the pseudo-Euclidean metric on $\mathbf{R}^3$ given by the formula*

$$ds^2 = dx^2 + dy^2 - dz^2\,. \tag{1}$$

Calculate the induced metric on the surface L considered in the Exercise 7, and show that it is a Riemannian metric (it is positive-definite).

Perform calculations in spherical-like coordinates (see Exercise 7a) above) and in stereographic coordinates (see exercise 7b) above)

Remark The surface L sometimes is called pseudosphere. The Riemannian metric on this surface sometimes is called Lobachevsky (hyperbolic) metric.

The surface L with this metric realises Lobachevsky (hyperbolic) geometry, where Euclid's 5-th Axiom fails. This Riemannian manifold (manifold+Riemannian metric) we call Lobachevsky (hyperbolic) plane.

In stereographic coordinates we come to realisation of Lobachevsky plane on the disc in $\mathbf{E}^2$. It is so called Poincare model of Lobachevsky geometry.

Solution. The calculations will be very similar to the calculations performed in the exercise 5 above. Just we need consider $\cosh\theta, \sinh\theta$ instead $\cos\theta, \sin\theta$ and and sometimes changes the signs.

First of all consider spherical-lime coordinates:

Equation of one-sheeted hyperboloid is $\begin{cases} x = \sinh\theta\cos\varphi \\ y = \sinh\theta\sin\varphi \\ z = \cosh\theta \end{cases}$. Then

$$G = (dx^2+dy^2-dz^2)\big|_{x=\sinh\theta\cos\varphi, y=\sinh\theta\sin\varphi, z=\cosh\theta} = ((d\sinh\theta\cos\varphi))^2 + ((d\sinh\theta\sin\varphi))^2 - ((d\cosh\theta))^2 =$$

$$(\cosh\theta\cos\varphi d\theta - \sinh\theta\sin\varphi d\varphi)^2 + (\cosh\theta\sin\varphi d\theta + \sinh\theta\cos\varphi d\varphi)^2 + (\sinh\theta d\theta)^2 = d\theta^2 + \sinh^2\theta d\varphi^2 \,.$$

matrix of Riemannian metric is $G = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$. In the same way as for sphere these coordinates are well-defined in all points except $z = \pm 1$, where $\sin^2\theta = 0$.

Now express Riemannian metric in stereographic coordinates (4):

$$G = (dx^2 + dy^2 - dz^2)\big|_{x=x(u,v),y=y(u,v),z=z(u,v)} = \left(d\left(\frac{2u}{1-u^2-v^2}\right)\right)^2 + \left(d\left(\frac{2v}{1-u^2-v^2}\right)\right)^2 -- \left(d\left(\frac{u^2+v^2+1}{1-u^2-v^2}\right)\right)^2 =$$

(Compare with calculations for sphere $x62 + y^2 + z^2 = 1$). We have $G =$

$$\left(\frac{2du}{1-u^2-v^2} + \frac{2u(2udu+2vdv)}{(1-u^2-v^2)^2}\right)^2 + \left(\frac{2dv}{1-u^2-v^2} + \frac{2v(2udu+2vdv)}{(1-u^2-v^2)^2}\right)^2$$

$$-\left(\frac{2udu+2vdv}{1-u^2-v^2} + \frac{(u^2+v^2+1)(2udu+2vdv)}{(1-u^2-v^2)^2}\right)^2 = \frac{4(du)^2+4(dv)^2}{(1+u^2+v^2)^2}$$

To continue calculations it is convenient to denote by $s = 1 - u^2 - v^2$. We come to

$$G = \frac{4}{s^4}\left[\left((1+u^2-v^2)du + 2uvdv\right)^2 + \left((1+v^2-u^2)du + 2uvdu\right)^2 - 4(udu+vdv)^2\right] =$$

$$\frac{4}{s^4}\left[\left(\left(1+u^2-v^2\right)^2 + 4u^2v^2 - 4u^2\right)du^2 + (u \leftrightarrow v) + \left(4uv\left(1+u^2-v^2\right) + 4uv\left(1+v^2-u^2\right) - 8uv\right)dudv\right] =$$

$$\frac{4}{s^4}\left[s^2du^2 + s^2dv^2 + 0\right] = \frac{4du^2+4dv^2}{s^2} = \frac{4du^2+4dv^2}{(1-u^2-v^2)^2}\,.$$

(One could perform the analogous calculations for the sphere in the Exercise 5.)

9 † *Consider the metric induced on one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$ embedded in $\mathbf{R}^3$ with the pseudo-Euclidean metric (1). Show that this metric is not Riemannian one.*

Solution. Consider the vectors $\mathbf{e} = \frac{\partial}{\partial y}$ and $\mathbf{f} = \frac{\partial}{\partial z}$ attached at the point $(1,0,0)$. One can see that these vectors are tangent to the hyperboloid, but they have the "length" of different signe. (One of these vectors is space-like vector, another time like vector.) We have pseudoriemannian metric at the tangent space spanned by these two vectors.

10* *Lobachevsky plane (hyperbolic plane) L in stereographic coordinates can be considered as an open disc $u^2 + v^2 < 1$ in the plane. In the Exercise 8 in particularly we calculated Riemannian metric of L in these coordinates.*

Find new coordinates x, y such that in these coordinates Lobachevsky plane (hyperbolic plane) can be considered as an upper half plane $\mathbf{x} \in \mathbf{R}, y > 0\}$ and write down explicitly Riemannian metric in these coordinates.

Hint: You may use complex coordinates:

$$z = x + iy, \bar{z} = x - iy, w = u + iv, \bar{w} = u - iv$$

and find an holomorphic transformation $w = w(z)$ of the open disc $w\bar{w} < 1$ onto the upper plane $\mathbf{Im}z > 0$.

Solution.

Recall that in stereographic coordinates $u, v, u^2 + v^2, 1$ expression for Lobachevsky metric is $G = \frac{4du^2+4dv^2}{(1-u^2-v^2)^2}$ (see the exercise 8). (It was realisation of Lobachevsky plane on the Euclidean disc. Sometimes it called Poincare model of Lobachevsky (hyperbolic) geometry.)

In complex coordinates $w = u + iv, \bar{w} = u - iv$ the metric $G = \frac{4du^2+4dv^2}{(1-u^2-v^2)^2}$ obtained in the exercise 8 can be rewritten $G = \frac{4dwd\bar{w}^2}{(1-w\bar{w})}$. Indeed

$$G = \frac{4dwd\bar{w}}{(1-w\bar{w})^2} = G = \frac{4d(u+iv)d(u-iv)}{(1-(u+iv)(u-iv))^2} = \frac{4du^2+4dv^2}{(1-u^2-v^2)^2}\,.$$

It is a beautiful problem in complex analysis: find Mobius transformation $w = \frac{az+b}{cz+d}$ transformation which transforms the interior of circle $w\bar{w} = 11$ into upper half plane $Imz > 0$. One can see that

$$w = \frac{1+iz}{1-iz}, \qquad z = i\frac{1-w}{1+w}$$

is the transformation which we need (Can you find all Mobius transformations which transform upper half plane to the interior of unit circle?.)

Now calculate G in coordinates $z, \bar{z}$. i.e. in coordinates (x, y):

$$G = \frac{4du^2+4dv^2}{(1-u^2-v^2)^2} = \frac{4dwd\bar{w}}{(1-w\bar{w})^2}$$

We have

$$dw = d\left(\frac{1+iz}{1-iz}\right) = \frac{2idz}{(1-iz)^2}, d\bar{w} = \frac{-2id\bar{z}}{(1+i\bar{z})^2},$$

$$1 - w\bar{w} = 1 - \frac{1+iz}{1-iz}\frac{1-i\bar{z}}{1+i\bar{z}} = \frac{2i(\bar{z}-z)}{(1-iz)(1+i\bar{z})}$$

Hence

$$G = \frac{4dwd\bar{w}}{(1-w\bar{w})^2} = \frac{4\left(\frac{2idz}{(1-iz)^2}\right)\left(\frac{-2id\bar{z}}{(1+i\bar{z})^2}\right)}{\frac{-4(\bar{z}-z)^2}{(1-iz)^2(1+i\bar{z})^2}} = \frac{-4dd\bar{z}}{(\bar{z}-z)^2} = \frac{dx^2+dy^2}{y^2}$$

since $z = x + iy$ and $\bar{z} - z = -2iy$.

We come to the very useful interpretation of hyperbolic geometry: upper half plane in $\mathbf{E}^2$ with metric $G = \frac{dx^2+dy^2}{y^2}$. Later by default we will call "Lobachevsky (hyperbolic) plane" the realisation of Lobachevsky plane as an half-upper plane in $\mathbf{E}^2$ with these coordinates x, y $(y > 0)$ with metric $G = \frac{dx^2+dy^2}{y^2}$.