**No go result?**

*Two days ago I was very happy to calculate straightforwadly that gradient of action is momentum. (see the text on 26-th Juky and also the Appendix to this text.) I was thinking about generalisation of this resuilt. however I realised that this hypothese is wrong: Finally I realised that one my hypothese is wrong: see abouve TEMPTATION OF WRONG DEFINITION*

In fact let $L = L(x, \dot{x}, t)$ be Lagrangian of theory, then consider the function

$$S(t_1, Q_1; t_2, Q_2) = \int_{t_1}^{t_2} L(x(\tau), \dot{x}(\tau)) d\tau \,, \tag{1}$$

where $x^i(\tau)$ obeys Euler Lagrange equations and boundary conditions:

$$x^i(\tau)\colon \quad \begin{cases} \frac{\partial L}{\partial x^i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) \\ x^i(t_1) = Q_1 \\ x^i(t_2) = Q_2 \end{cases} \tag{1$a$}$$

One can consider momentum and Hamiltonian:

$$p_i = \frac{\partial L}{\partial \dot{x}^i} \,, \quad H(p, Q, t) = p_i \dot{Q}^i - L(Q, \dot{Q}, t) \tag{2}$$

($H$ is Legendre transform, in particular on eq. of motions it does not depend on $\dot{Q}$.)

This is the text-book fact that

$$\frac{\partial S}{\partial Q_2^i} = p_i \,, \tag{3$a$}$$

$$\frac{\partial S}{\partial Q_1^i} = -p_i \,, \tag{3$b$}$$

$$\frac{\partial S}{\partial t_1} = -\frac{\partial S}{\partial t_2} = -H \,. \tag{3$c$}$$

This can be calculated straightfowardly, by brute force, and I am happy that I did these caluclations. (see my 'blog' on 26 July of this month or calculations in `Appendix`.

One can consider Legendre transform of this action: We will denote the action above by $S(t_1, x, t_2, y)$ and we will consider its Legendre transform

$$\Sigma(t_1, x; t_2, q)\colon \quad e^{\frac{i}{\hbar}\Sigma_t(x,q)} \approx e^{\frac{i}{\hbar}S_t(x,y)} \,.$$

It is funnny to write down differential equations for these both actions.

$$S_t(x, y)\colon \begin{cases} \text{dif.equation } \frac{\partial S_t(x,y)}{\partial t} + H\left(\frac{\partial S_t(x,y)}{\partial y}, y\right) = 0 \\ \text{boundary conditions} \quad e^{\frac{i}{\hbar}S_t(x,y)} \approx \delta(x-y) \end{cases} \,, \tag{0$a$}$$

$$\Sigma_t(x,q)\colon \begin{cases} \text{dif.equation } \frac{\partial \Sigma_t(x,q)}{\partial t} + H\left(q, \frac{\partial \Sigma_t(x,y)}{\partial q}, y\right) = 0 \\ \text{boundary conditions} \quad \Sigma_t(x,q)\big|_{t=0} = xq, \text{ i.e. } e^{\frac{i}{\hbar}\Sigma_t(x,q)} \approx \text{Fourrier of } \delta \end{cases}, \qquad (0b)$$

Define $\tilde{\Sigma}_t(x,q)$ as magnitude which is equal to the integral of Lagrangian $L(x,\dot{x})$ over trajectory $x(\tau)$ such that $x(\tau)$ obeys Euler-Lagrange equations, it begins at the point $x$, and at the moment $\tau = t$ it ends at the point $y$ such that the momentum is equal to $q$:

$$y = y(q,t) \qquad (1)$$

`TEMPTATION` $\tilde{\Sigma} = \Sigma$. THIS IS WRONG!!!!

**RIGHT STATEMENT** $\Sigma_t(x,q) = qy(q,t) - \tilde{\Sigma}$, where a function $y = y(q,t)$ is defined by (1).

One can say that due to (3) equation (1) reveals Legendre....

**Appendix**

I will recall the calculations. (3).

Let $x^i(\tau)$ is an arbitrary solution of Euler-Lagrange equations which begins at $Q_1$ and ends at $Q$, (see (1)). Let $h^i(\tau)$ be its arbitrary variation.

First prove (3a), Consider solution of Euler-Lagrange equaion $\tilde{x}(\tau) = x(\tau) + h(\tau)$ which is infinitesimally close to the initial solution. To calculate $\frac{\partial S}{\partial Q_2^i}$ we choose the new solution such that for infnitesimal small function $h^i(\tau)$, $h^i(t_1) = 0$, and

$$Q_2' = Q_2 + h^i(t_2)\,, \text{ i.e. in this case } \delta Q_2 = h^i(t_2)$$

Thus we have

$$S(Q_1,t_1,Q_2',t_2) = S(Q_1,t_1,Q_2+h(t_2),t_2) = \int_{t_1}^{t_2} L\left(x^i(\tau)+h^i(\tau), \dot{x}^i(\tau)+\dot{h}^i(\tau)\right)d\tau =$$

$$\underbrace{\int_{t_1}^{t_2} L\left(x^i(\tau), \dot{x}^i(\tau)(\tau)\right)d\tau}_{S(t_1,Q_1;t_2,Q_2)} + \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial x^i}h^i(\tau) + \frac{\partial L}{\partial \dot{x}^i}\dot{h}^i(\tau)\right).$$

Thus

$$S(Q_1,t_1,Q_2+\delta Q_2,t_2) - S(Q_1,t_1,Q_2,t_2) =$$

$$\int_{t_1}^{t_2}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}h^i(\tau)\right) + \left(\underbrace{\frac{\partial L}{\partial x^i} - \frac{\partial L}{\partial \dot{x}^i}}_{\text{Euler-Lagrange equat.}}\right)h^i(\tau)\right)d\tau = \frac{\partial L}{\partial \dot{x}^i}\delta Q_2^i(\tau)\,. \qquad (4)$$

This proves (3a). We prove (3b) analogously just choosing $h^i(t)$ such that $h^i(t_2) = 0$.

Now prove (3c). We do it little bit more carefully.

Consider again arbitrary new solution $\tilde{x}^i(\tau) = x^i(\tau) + h^i(t)$ which is infinitesimally close to the initial solution $x^i(t)$. Later we will impose the condition that under this variation $\delta Q_1$ and $\delta Q_2$ vanish:

$$\delta Q_1 = \delta Q_2 = 0\,. \tag{i5a}$$

however just repeat the calculations of (4) for an it arbitrary infnitesimal transformation, and temprorarly we will forget about condition (5a) On the otehr hand during calculations we will use that $\delta t_1$, $\delta t_2$ and $h(t)$ are nilpotents.

So let us begin:

$$S(t_1 + \delta t_1, Q_1'; t_2 + \delta t_2 Q_2') = S(t_1 + \delta t_1, Q_1 + h(t_1); t_2 + \delta t_2, Q_2 + h(t_2)) =$$

$$\int_{t_1+\delta t_1}^{t_2+\delta t_2} L\left(x^i(\tau) + h^i(\tau), \dot{x}^i(\tau) + \dot{h}^i(\tau)\right) d\tau =$$

$$\underbrace{\int_{t_1}^{t_2} L\left(x^i(\tau), \dot{x}^i(\tau)(\tau)\right) d\tau}_{S(t_1,Q_1;t_2,Q_2)} + \int_{t_1+\delta t_1}^{t_2} L(x,\dot{x}) d\tau + \int_{t_2}^{t_2+\delta t_2} L(x,\dot{x}) d\tau$$

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x^i} h^i(\tau) + \frac{\partial L}{\partial \dot{x}^i} \dot{h}^i(\tau)\right)\,.$$

Thus

$$S(t_1 + \delta t_1, Q_1 + \delta Q_1; t_2 + \delta t_2, Q_2 + \delta Q_2) - S(t_1, Q_1; t_2, Q_2) =$$

$$\int_{t_1+\delta t_1}^{t_2} L(x,\dot{x}) d\tau + \int_{t_2}^{t_2+\delta t_2} L(x,\dot{x}) d\tau +$$

$$\int_{t_1+\delta t_1}^{t_2+\delta t_2} \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i} h^i(\tau)\right) + \left(\underbrace{\frac{\partial L}{\partial x^i} - \frac{\partial L}{\partial \dot{x}^i}}_{\text{Euler-Lagrange equat.}}\right) h^i(\tau)\right) d\tau = \tag{5}$$

$$L(x(t_2), x(t_2))\delta t_2 - L(x(t_1), x(t_1))\delta t_1 + \left(\frac{\partial L}{\partial \dot{x}^i} h^i(\tau)\right)\Big|_{t_1+\delta t_1}^{t_2+\delta t_2} =$$

$$L(x_2, \dot{x}_2)\delta t_2 - L(x_1, \dot{x}_1)\delta t_1 + p_i(t_2) h^i(t_2) - p_i(t_2) h^i(t_2)\,.(5c)$$

s is general formula, now fix the formulae for $\delta Q_2$ and $\delta Q_1$

$$\delta Q_2 = x^i(t + \delta t_2) + h^i(t_2 + \delta t_2) - x^i(t) = \dot{x}^i(t)\delta t_2 + h^i(t_2)\,,$$

and analogously $\delta Q_1 = \dot{x}^i(t)\delta t_1 + h^i(t_1)$. Hence the condition (5a) may be rewritten as

$$\delta Q_1 = h^i(t_1) + \dot{x}^i(t_1)\delta t_1 = 0\,, \quad \text{and} \quad \delta Q_2 = h^i(t_2) + \dot{x}^i(t_2)\delta t_2 = 0\,,$$

, i.e.

$$h^i(t_1) = -\dot{x}^i(t_1)\delta t_1\,, \quad h^i(t_2) = -\dot{x}^i(t_2)\delta t_2\,.$$

and

$$S(t_1 + \delta t_1, Q_1; t_2 + \delta t_2, Q_2) - S(t_1, Q_1; t_2, Q_2) =$$

$$L(x_2, \dot{x}_2)\delta t_2 - L(x_1, \dot{x}_1)\delta t_1 - p_i(t_2)\dot{x}^i(t_2)\delta t_2 + p_i(t_1)\dot{x}^i(t_1)\delta t_2 = H_1\delta t_1 - H_2\delta t_2\,.$$

This finishes the proof of euations (3) $\blacksquare$