Homework 6

Christoffel symbols and Lagrangians

1 Consider the Lagrangian of "free" particle $L = \frac{1}{2} g_{ik}\dot{x}^i \dot{x}^k$ for Riemannian manifold with a metric $G = g_{ik} dx^i dx^k$.
Write down Euler-Lagrange equations of motion for this Lagrangian and compare them with differential equations for geodesics on this Riemannian manifold.

In fact show that

$$\underbrace{\frac{\partial L}{\partial x^i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}}_{\text{Euler-Lagrange equations}} \quad \Leftrightarrow \quad \underbrace{\frac{d^2 x^i}{dt^2} = \Gamma^i_{km}\dot{x}^k \dot{x}^m}_{\text{Equations for geodesics}} \quad , \tag{1}$$

where

$$\Gamma^i_{km} = \frac{1}{2} g^{ij}\left(\frac{\partial g_{jk}}{\partial x^m} + \frac{\partial g_{jm}}{\partial x^k} - \frac{\partial g_{km}}{\partial x^j}\right) . \tag{2}$$

2 a) Write down the Lagrangian of "free" particle $L = \frac{1}{2} g_{ik}\dot{x}^i \dot{x}^k$ for Euclidean plane in polar coordinates. Calculate Christoffel symbols for canonical flat connection in polar coordinates using Euler-Lagrange equations for this Lagrangian. Compare with answers which you obtained by the direct use of transformation formulae for Christoffel symbols (see Homework 4 and lecture notes) and with answers which you obtained by direct use of the formula (2) for Levi-Civita connection.

b) Do the same in cylindrical coordinates in $\mathbf{E}^3$: $x = r\cos\varphi, y = r\sin\varphi, z = h$.

3 Calculate Christoffel symbols of Levi-Civita connection for Riemannian metric $G = adu^2 + bdv^2$. Compare with results of the Exercise $1b)$ in the Homework 5.

4 Write down the Lagrangian of of "free" particle $L = \frac{1}{2} g_{ik}\dot{x}^i \dot{x}^k$ and using Euler-Lagrange equations for this Lagrangian calculate Christoffel symbols (Christoffel symbols of Levi-Civita connection) for

a) cylindrical surface of the radius a

b) for the cone $x^2 + y^2 - k^2 z^2 = 0$

c) for the sphere of radius R

d) for Lobachevsky plane

Compare with the results that you obtained using straightforwardly the formula (1) or using formulae for induced connection.