Riemannian Geometry

COURSEWORK 2019

8 April 2019

Discussions

Here we discuss the solutions of the coursework. (The printed solutions of coursework problems and marked courseworks are distributed via Reception

1

Consider a surface M, the upper sheet of the cone

$$\mathbf{r}(h,\varphi)\colon \begin{cases} x = h\cos\varphi \\ y = h\sin\varphi \\ z = 2h \end{cases}, \qquad 0 \le \varphi < 2\pi\,,\ h > 0\,.$$

Find the length of the shortest curve C which belongs to the surface M, starts at the point $(h_0, 0, 2h_0)$ and ends at the point $(-h_0, 0, 2h_0)$.

This exercise was made by many students. Students calculated right the parameters of the unfolded surface: this is a sector of the circle with radius $R = h_0\sqrt{5}$ and the angle

$$\theta = \frac{\text{circumference of the base circle}}{R} = \frac{2\pi}{\sqrt{5}}\,.$$

However If you take the images of points $A = (h, 0, 2h)$ and $B = (-h, 0, 2h)$ on the unfolded surface, then the angle between these points will be $\theta/2$, not θ. To calculate the length of the segment AB (this is the shortest distance) one may consider the height $OD \perp AB$ of the isocseles $\triangle AOB$, then $\angle DOA = \frac{\theta}{4}$ and

$$|AB| = 2|AO|\sin\frac{\theta}{4}\,, \tag{1.1a}$$

or instead one may consider cosine rule for this this isocseles triangle:

$$|AB| = \sqrt{2|AO|^2 - 2|AO|^2\cos\frac{\theta}{2}}\,. \tag{1.1b}$$

Some students prefered to write the solutions in the form (1.1b). The expression (1.1a) seems to be better if you want to compare the length of the shortest curve with the circumference of the based circle (see the Remark above).

A few students instead unfolding the conical surface, considered new coordinates

$$u, v: \quad \begin{cases} u = \sqrt{1+k^2}h\cos\frac{\varphi}{\sqrt{k^2+1}} \\ v = \sqrt{1+k^2}h\sin\frac{\varphi}{\sqrt{k^2+1}} \end{cases},$$

(here $k = 2$). In these coordinates surface is locally Euclidean: $du^2 + dv^2 = (1+k^2)dh^2 + h^2 d\varphi^2$, and they calculated the distance in these coordinates. This is approach is rigorous and nice, however only one student came to the final solution considering these coordinates.

One or two students instead finding the shortest, just considered the length of the base half-circle, whihc is less than the shortest. In the remark below we study this question

Remark In this remark I would like to consider this problem reintroducing the parameter k: the conic surface now becomes $k^2(x^2+y^2) - z^2 = 0$ instead just $k = 2$

One can see that The length of the shortest curve for arbitrary k is equal to $L = 2h_0\sqrt{1+k^2}\sin\frac{\pi}{2\sqrt{1+k^2}}$.

One can see that L is less than curcumference of the based circle, and in the limit $k \to \infty$ it tends to this answer:

$$L = 2h\sqrt{1+k^2}\sin\frac{\pi}{2\sqrt{1+k^2}} < \pi h\,,\text{since } \frac{\sin x}{x} \leq 1 \text{ for small } x$$

and

$$\lim_{k\to\infty}\left(2h\sqrt{1+k^2}\sin\frac{\pi}{2\sqrt{1+k^2}}\right) = \pi h\,,\ \text{since } \lim_{x\to 0}\frac{\sin x}{x} = 1\,.$$

2

You know that the Riemannian metric on the sphere of radius R in the stereographic coordinates is expressed by the formula

$$G_{\text{stereogr.}} = \frac{4R^4(du^2+dv^2)}{(R^2+u^2+v^2)^2}\,.$$

a) Give an example of a non-identity transformation of coordinates u, v such that it preserves this metric.

b) Give an example of a non-linear transformation of coordinates u, v such that it preserves this metric.

(Hint: You may find this transformation considering transformations of the sphere.)

c) Find the length of the line $v = au$ in $\mathbf{R}^2$ with respect to this metric.

Why the length of this curve does not depend on the parameter a?

a) Almost all students give the example of non-dentical linear transformation of coordinates u, v. This was a simple question.

b) For an example of non-linear transformatin many students have chosen the inversion of stereographic coordinates u, v: $u = \frac{R^2 u}{u^2+v^2}$, respectively $v = \frac{R^2 v}{u^2+v^2}$.

One can show an example of non-linear transformation without performing straightforward calculations: take an arbitrary orthogonal thransformation of points of sphere, which is not the rotation around Oz axis, e,g, the transformation $\mathbf{r} \to -\mathbf{r}$ or you may take rotation on arbitrary non-zero angle around an arbitrary axis which does not coincide with Oz axis (see the solutions), and you will come to non-linear transformation of stereographic coordiantes preserving the metric[1]).

c) I am very happy that almost all students did this exercise succesfully. Many students did straightforward brute force calculations and came to the right answer $L = 2\pi R$. Usually these students noted that it is the length of the great circle.

3.

Evaluate the area of the part of the sphere of radius $R = 1$ between the planes given by equations $2x + 2y + z = 1$ and $2x + 2y + z = 2$.

No problem arised in this question.

Performing this exercise students had to recall the standard formula on the distance between the origin and the plane, and almost all students did it succesfully.

4

Consider the plane $\mathbf{R}^2$ with standard coordinates (x, y) equipped with Riemannian metric

$$G = (1 + x^2 + y^2)e^{-a^2x^2 - a^2y^2} \left(dx^2 + dy^2\right) .$$

Calculate the total area of this plane.

No problem arised in this question.

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[1]) Helas, it seems to me that nobody did this in spite of the hint to the question.

5

Consider the upper half-plane $y>0$ with the Riemannian metric

$$G=\frac{dx^2+dy^2}{y^2}$$

(the Lobachevsky plane).

In the Lobachevsky plane consider the domain D defined by

$$D=\{x,y:\quad x^2+y^2\geq 1\,,\ -a\leq x\leq a\}\,, \tag{4.2}$$

where a is a parameter such that $0<a<1$.

Find the area of the domain D (with respect to the metric G).

Students did good this exercies. Just I would like to emphasize again: the domain which is considered here is the isocseless triangle with the zero angle at the vertex. Recallung that Lobahevsky plane has curvature $K=-1$ we see why its area is equal to

$$S=K(\alpha+\beta+\gamma-\pi)=(-1)\cdot\left(\left(\frac{\pi}{2}-\text{arcsina}\right)+\left(\frac{\pi}{2}-\text{arcsina}\right)+0-\pi\right)=2\text{arcsina}\,.$$

6

a) Let ∇ be an affine connection on the 2-dimensional manifold M such that in local coordinates (u,v), $\nabla_{\frac{\partial}{\partial u}}\left(u^2\frac{\partial}{\partial v}\right)=3u\frac{\partial}{\partial v}+u\frac{\partial}{\partial u}$.

Calculate the Christoffel symbols Γ^u_{uv} and Γ^v_{uv} of this connection.

b) Let ∇ be an arbitrary connection on a manifold M. Show that

$$\cos F\nabla_{\mathbf{A}}(\sin F\mathbf{B})-\sin F\nabla_{\mathbf{A}}(\cos F\mathbf{B})=(\partial_{\mathbf{A}}F)\mathbf{B}\,,$$

where $\mathbf{A},\mathbf{B}$ are arbitrary vector fields and F is an arbitrary function.

c) Let $\Gamma^{i(1)}_{km}$ be the Christoffel symbols of a connection $\nabla^{(1)}$ and $\Gamma^{i(2)}_{km}$ be the Christoffel symbols of a connection $\nabla^{(2)}$. Show, that the linear combinations $\frac{2}{3}\Gamma^{i(1)}_{km}+\frac{1}{3}\Gamma^{i(2)}_{km}$, are Christoffel symbols for some connection.

Explain, why $\frac{1}{2}\Gamma^{i(1)}_{km}+\frac{1}{3}\Gamma^{i(2)}_{km}$ are not Christoffel symbols for any connection.

Almost all students answered questions a) and b).

Qiestion c) was not easy question, and I am happy that almost all students were doing it 'in right direction.'

Students stated that the linear combinations $\frac{2}{3}\Gamma^{i(1)}_{km} + \frac{1}{3}\Gamma^{i(2)}_{km}$, are Christoffel symbols for some connection, and on the other hand that the linear combinations $\frac{1}{2}\Gamma^{i(1)}_{km} + \frac{1}{3}\Gamma^{i(2)}_{km}$, are *not* the Christoffel symbols for any connection, because $1/3 + 2/3 = 1$ and $1/2 + 1/3 \neq 1$; a part of these students were trying to explain why.

One solution of this question is to check how transform under changing of coordinates the linear combinations $\gamma\Gamma^{i(1)}_{km} + \mu\Gamma^{i(2)}_{km}$. It is easy to see that this combination transforms as Christoffel symbol of some connection if and only if $\lambda + \mu = 1$ (see the solutions).

Few students were trying to do it in the following way: they consider symbols

$$\mathcal{G}^i_{km} = \lambda\Gamma^{i(1)}_{km} + \mu\Gamma^{i(2)}_{km}$$

and the operation:

$$S_{\mathbf{X}}(\mathbf{Y}) = \lambda\nabla^{(1)}_{\mathbf{X}}\mathbf{Y} + \mu\nabla^{(2)}_{\mathbf{X}}\mathbf{Y}$$

One can see that this operation obeys axioms of connection: if and only if $\lambda + \mu = 1$. In particular for Leibnitz rule

$$S_{\mathbf{X}}(f\mathbf{Y}) = \lambda\nabla^{(1)}_{\mathbf{X}}(f\mathbf{Y}) + \mu\nabla^{(2)}_{\mathbf{X}}(f\mathbf{Y}) = \lambda f\nabla^{(1)}_{\mathbf{X}}(\mathbf{Y}) + \mu f\nabla^{(2)}_{\mathbf{X}}(\mathbf{Y}) + (\lambda+\mu)\partial_{\mathbf{X}} f\mathbf{Y} =$$

$$f S_{\mathbf{X}}\mathbf{Y} + \partial_{\mathbf{X}} f\mathbf{Y}\,, \quad \text{if and only if } \lambda + \mu = 1\,.$$

This is a nice approach.

Students who tried this method received credits.

7

Let ∇ be a connection in $\mathbf{E}^3$ such that Christoffel symbols of this connection in Cartesian coordinates are the following:

$$\Gamma^i_{km} = \begin{cases} 0 \quad \textit{if at least two of indices coincide: } i=k \textit{ or } k=m \textit{ or } i=m \\ +1 \quad \textit{if } \{ikm\} \textit{ is an even permutation of indices } \{123\} \\ : wq - 1 \quad \textit{if } ikm \textit{ is an odd permutation of indices } \{123\} \end{cases},$$

e.g. $\Gamma^1_{13} = \Gamma^3_{22} = \Gamma^2_{12} = 0$, $\Gamma^1_{23} = \Gamma^3_{12} = 1$, *and* $\Gamma^1_{32} = \Gamma^2_{13} = -1$.

Show that this connection preserves the Euclidean scalar product.

[3 marks]

This exercise is not difficult (see the solution), however it is very important exrcise, since it shows an example of connection which is not Levi-Civita connection, in spite of the fact that metric is invariant with respect to this connection. It focuses the attention on the fact how imporant is the condition of symmetricity of Levi-Civita connection.

8

Let M be a surface considered in question **1** *(the upper sheet of a cone),*

a) Calculate the induced connection on this surface (the connection induced by the canonical flat connection in the ambient Euclidean space: $\nabla_{\mathbf{X}}\mathbf{Y} = \left(\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}\right)_{\text{tangent}}$).

b) Calculate the Riemannian metric on the cone induced by the canonical metric in ambient Euclidean space $\mathbf{E}^3$ and calculate explicitly the Levi-Civita connection of this metric using the Levi-Civita Theorem.

In this exercise students have to calculate Christoffel symbols on the surface of cone in $\mathbf{E}^3$. of induced connection, and of Levi-Civita connection with use of Levi-Civita standard formula or Lagrangin of free particle.

This exercise is almost bookwork, on the other hand some calculations (e.g. calculations for components of $\Gamma^h_{\varphi\varphi}$ of inudced connection need a time.)

It is easy to make a mistake in calculations, but you have know firmly that

`Levi-Civita connection of induced Riemannian metric = induced connection`

In this exercise you have calculated the same Christoffel symbols using different methods.