Homework 7

1. Find coordinate basis vectors, first quadratic form and unit normal vector field for the following surfaces:

a) sphere of the radius R:

$$\mathbf{r}(\varphi,\theta) \qquad \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases} \tag{1}$$

$$(0 \leq \varphi < 2\pi,\ 0 \leq \theta \leq \pi),$$

b) cylinder

$$\mathbf{r}(\varphi,h) \qquad \begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \\ z = h \end{cases} \qquad (0 \leq \varphi < 2\pi, -\infty < h < \infty) \tag{2}$$

c) graph of the function $z = F(x,y)$

$$\mathbf{r}(u,,v) \qquad \begin{cases} x = u \\ y = v \\ z = F(u,v) \end{cases} \qquad (-\infty < u < \infty, -\infty < v < \infty) \tag{3}$$

in the case if $F(u,v) = F = Au^2 + 2Buv + Cv^2$.

2 Show that there are two straight lines which pass through the point $(3,4,12)$ on the saddle $z = xy$ and lie on this saddle.

Show that this is true for an arbitrary point of the saddle.

3. Consider on the sphere (1) the following curves:

$$C_1\colon\ \mathbf{r} = \mathbf{r}(\theta(t),\varphi(t)),\ 0 \leq t \leq 2\pi,\ \text{where } \theta(t) = \theta_0, \varphi(t) = t,\ \ \text{(circle)}$$

$$C_2\colon\ \mathbf{r} = \mathbf{r}(\theta(t),\varphi(t)),\ 0 \leq t \leq \pi,\ \text{where } \theta(t) = t, \varphi(t) = \varphi_0,\ \ \text{(semicircle)}$$

Sketch these curves.

Calculate length of these curves considering them in the ambient Euclidean space. Calculate length of these curves using first quadratic form.

3a* Take arbitrary two points A, B on the curve C_2. Show that the arc of the curve C_2 is the shortest curve on the sphere between these points, i.e. for an arbitrary curve C on the sphere which starts at the point A and ends at the point B the length of C is greater or equal than the length of this arc of C_2.

How to find the shortest curve between two arbitrary points on the sphere?

4. Consider on the sphere (1) the following circles:
C_1: $x = R\cos t,\ y = R\sin t,\ z = 0$ (Equator),
C_2: $x = R\cos t,\ y = 0,\ z = R\sin t$ ("Greenwich" Meridian),
C_3: $x = R\sin\theta_0\cos t,\ y = R\sin\theta_0\sin t,\ z = \cos\theta_0$ (Circle of constant latitude)

$$(0 \le t < 2\pi)$$

Calculate normal curvatures at points of these circles.

Let C be an arbitrary curve on the sphere. What values can take the normal curvature at points of this curve?

5. Consider on the cylinder (2) the following curves:
C_1: $x = R\cos t,\ y = R\sin t,\ z = h_0$ (circle),
C_2: $x = R\cos t,\ y = R\sin t,\ z = vt$ (helix),
C_3: $x = R\cos\varphi_0,\ y = R\sin\varphi_0,\ z = t$ (straight line).
Calculate normal curvatures at points of these curves.

Let C be an arbitrary curve on the cylinder. What values can take the normal curvature at points of this curve?

6. Calculate shape operator for an arbitrary point of the sphere (1).

7. Calculate shape operator for an arbitrary point of the cylinder (2).

8. Calculate shape operator for the surface (3) at the point $u = v = 0$.

9 Calculate principal curvatures, Gaussian and mean curvature at the points of the sphere (1) using results of exercise 4 or using results of the exercise 6.

10 Calculate principal curvatures, Gaussian and mean curvature at the points of the cylinder (2) using results of exercise 5 or using results of the exercise 7.

11 Calculate Gaussian and mean curvature of the surface (3) at the point $u = v = 0$.

12 Assume that the action of shape operator at the tangent coordinate vectors ∂_u, ∂_v at the given point $\mathbf{p}$ of the surface $\mathbf{r} = \mathbf{r}(u, v)$ is defined by the relations: $S(\partial_u) = 2\partial_u + 2\partial_v$ and $S(\partial_v) = -\partial_u + 5\partial_v$. Calculate principal curvatures, Gaussian and mean curvatures of the surface at this point.