

## Feedback

RIEMANNIAN GEOMETRY(31082, 41082,61082)

ANSWER ANY THREE OF QUESTIONS 1—4 AND QUESTION 5

All questions are worth 20 marks

For 31082 fifth question is excluded  
Electronic calculators may not be used

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**1.**

**(a)** Explain what is meant by saying that  $G$  is a Riemannian metric on a manifold  $M$ .

Let  $G = g_{ik}(x)dx^i dx^k$ ,  $i, k = 1, 2, \dots, n$  be a Riemannian metric on an  $n$ -dimensional manifold  $M$ .

Show that all diagonal components  $g_{11}(x), g_{22}(x), \dots, g_{nn}(x)$  are positive functions.

Let  $G = cdu^2 + dudv + dv^2$  be Riemannian metric on 2-dimensional manifold  $M$ , where  $c$  is a real constant. Show that  $c > \frac{1}{4}$ .

( *Hint:* You may consider the length of a vector  $\mathbf{X} = \partial_u + t\partial_v$  where  $t$  is an arbitrary real number.)

[8 marks]

**(b)** Explain what is meant by saying that a Riemannian manifold is locally Euclidean.

Show that the surface of the cylinder  $x^2 + y^2 = 4$  in the Euclidean space  $\mathbf{E}^3$  with the induced Riemannian metric is locally Euclidean.

[4 marks]

**(c)** Consider a Riemannian manifold  $M^n$  with a metric  $G = g_{ik}(x)dx^i dx^k$ .

Write down the formula for the volume element on  $M^n$  (area element for  $n = 2$ ).

Consider the plane  $\mathbf{R}^2$  with standard coordinates  $(x, y)$  equipped with the Riemannian metric

$$G = \frac{dx^2 + dy^2}{(1 + x^2 + y^2)^2}.$$

Calculate the area  $S_a$  of the domain  $x^2 + y^2 \leq a^2$ .

Find the limit  $S_a$  when  $a \rightarrow \infty$ .

Show that there is no isometry between the plane with this Riemannian metric and the Euclidean plane  $\mathbf{E}^2$ .

[8 marks]

### Discussion of first question

a) Problems were related with question about metric  $cdu^2 + dudv + dv^2$ . Vector  $\mathbf{X} = \partial_u + t\partial_v \neq 0$  for all  $t$ , hence positive-definiteness condition implies that

$$P(t) = \langle \mathbf{X}, \mathbf{X} \rangle = c + t + t^2 = \left(t - \frac{1}{2}\right)^2 + c - \frac{1}{4} > 0 \text{ for all } t. \text{ Hence } c > \frac{1}{4}.$$

Some students instead considering condition  $P(t) > 0$  considered condition  $\sqrt{P(t)} = \sqrt{c + t + t^2} > 0$ . This is meaningless.

Some students wrote that  $P(t)$  has to be positive, yes this is right, but they could not show that this implies that  $c > 1/4$ .

Some students just checked that determinant is positive. This is necessary condition, but is it sufficient? This has to be proved.

b) This was not difficult question. Riemannian metric on cylinder  $G = 4d\varphi^2 + dh^2$  can be locally rewritten  $G = du^2 + dv^2$  (if  $u = 2\varphi, v = h$ ). This is easy. Some students failed to answer this question, trying instead to answer another question about finding euclidean coordinates on the cone.

c) Almost all students who were answering this question (more than half) have no problem to calculate  $S_a$  and finding limit of  $S_a$  if  $a \rightarrow \infty$  ( $S_a \rightarrow \pi$ ).

But many of them failed to show why this implies that  $\mathbf{R}^2$  with this metric is not isometric to Euclidean plane: (If they are isometric, then they have the same area, but  $\pi \neq \infty$ , Contradiction.) Some students realised that the inequality  $\pi \neq \infty$  is a reason of the fact that these spaces are not isometric but could not perform exact considerations.

## 2.

(a) Explain what is meant by an affine connection on a manifold.

Give the definition of the canonical flat connection on the Euclidean space  $\mathbf{E}^n$ .

Calculate the Christoffel symbols  $\Gamma_{rr}^r$  and  $\Gamma_{\varphi\varphi}^r$  of the canonical flat connection in the Euclidean space  $\mathbf{E}^2$ , where  $r, \varphi$  are polar coordinates ( $x = r \cos \varphi, y = r \sin \varphi$ ).

[6 marks]

(b) Explain what is meant by the induced connection on a surface in Euclidean space.

Consider the surface of a saddle in the Euclidean space  $\mathbf{E}^3$ :

$$\mathbf{r}(u, v): \begin{cases} x = u \\ y = v \\ z = uv \end{cases}.$$

Calculate the induced connection at the point  $u = v = 0$ .

[6 marks]

(c) Write down the expression for the Christoffel symbols  $\Gamma_{km}^i$  of the Levi-Civita connection in terms of the Riemannian metric  $G = g_{ik}(x)dx^i dx^k$ .

Suppose a symmetric connection  $\nabla$  is the Levi-Civita connection of a Riemannian manifold with metric  $G = du^2 + u^2 dv^2$  in some local coordinates  $u, v$ .

Find  $\Gamma_{vv}^u$ .

Show that there exist other local coordinates  $u', v'$  such that in these coordinates all the Christoffel symbols of the connection  $\nabla$  identically vanish.

[8 marks]

### Discussion of second question

a) Defining canonical flat connection it is important to note that Christoffel symbols vanish in *Cartesian coordinates*. Some students failed to do it.

b) For induced connection

$$\nabla_{\mathbf{X}} \mathbf{Y} = (\nabla_{\mathbf{X}}^{\text{can.flat}} \mathbf{Y})_{\text{tangential}} = \nabla_{\mathbf{X}}^{\text{can.flat}} \mathbf{Y} - \mathbf{n}(\mathbf{n}, \nabla_{\mathbf{X}}^{\text{can.flat}} \mathbf{Y}).$$

Almost all students wrote right answer on this bookwork question. On the other hand when applying this formula to calculate connection at the point  $u = v = 0$  of saddle many students did straightforward brute force calculations and some of them failed in the calculations of finding tangential component of the vector at the given point  $u = v = 0$ . Instead it has to be noticed, that on the final stage of the calculations (when calculating projection of  $\mathbf{r}_{uv} = \nabla_{\frac{\partial}{\partial u}}^{\text{ambient}} \frac{\partial}{\partial v}$  in the ambient space on the surface at the point  $u = v = 0$ ) it is enough to note that normal vector at the point  $u = v = 0$  is proportional to vector  $\mathbf{r}_{uv}$ , hence tangential components vanish, and  $\nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial v} = 0$ . :wq

c) Almost all students calculated right  $\Gamma_{vv}^u$  (modulo some mistakes), but only few students noticed that in coordinates  $u' = u \cos v, v' = u \sin v$  metric becomes Euclidean. One comes to this answer by easy calculations. On the other hand the answer becomes absolutely clear if we will change letters:  $u \mapsto r, v \mapsto \varphi, u' \mapsto x, v' \mapsto y$  we will see that  $G = du^2 + u^2 dv^2 \mapsto dr^2 + r^2 d\varphi^2 = dx^2 + dy^2$  is nothing but  $\mathbf{E}^2$  metric in polar coordinates ( $x = r \cos \varphi, y = r \sin \varphi$ )

**3.**

**(a)** Define a geodesic on a Riemannian manifold as a parameterised curve.

Write down the differential equations for geodesics in terms of the Christoffel symbols.

What are the geodesics of the surface of a cylinder? Justify your answer.

[7 marks]

**(b)** Explain what is meant by the Lagrangian of a free particle on a Riemannian manifold.

Explain the relation between the Lagrangian of a free particle and the differential equations for geodesics.

Calculate the Christoffel symbols of the sphere of radius  $R$  in the Euclidean space  $\mathbf{E}^3$  in spherical coordinates.

(You may use the Lagrangian of a free particle on this sphere  $L = \frac{1}{2} (R^2 \dot{\theta}^2 + R^2 \sin^2 \theta \dot{\varphi}^2)$ .)

Consider an arbitrary geodesic  $\mathbf{r} = \mathbf{r}(t)$  on the sphere.

Show that the magnitude  $I(t) = \sin^2 \theta(t) \dot{\varphi}(t)$  is preserved along the geodesic.

[8 marks]

**(c)** Consider the Lobachevsky plane as an upper half-plane,  $y > 0$  in  $\mathbf{R}^2$  equipped with the metric  $G = \frac{dx^2 + dy^2}{y^2}$ .

Consider a semicircle  $C: x^2 + y^2 = 4x$  (i.e.  $(x - 2)^2 + y^2 = 4$ ),  $y > 0$  in the Lobachevsky plane.

Consider two points  $A = (2, 2)$  and  $B = (3, \sqrt{3})$  on this semicircle.

Find the parallel transport of the vector  $\mathbf{X} = \partial_x$  from the initial point  $A$  to the point  $B$  along the semicircle  $C$ .

(You may use the fact that the semicircle  $C$  is a geodesic in the Lobachevsky plane.)

[5 marks]

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### Discussion of third question

**a)** No special problems.

**b)** No special problems.

**c)** The parallel transport along semicircle was performed properly only by few students. To solve this question one has to note that since semicircle  $C$  is geodesic and vector

$\mathbf{X}_0 = \partial_x$  is tangent to  $C$ , i.e. it is proportional to velocity vector at the point  $A$ , then during parallel transport vector  $\mathbf{X}(t)$  remains proportional to velocity vector. Velocity vector at the point  $B$  is orthogonal to radius-vector  $(1, \sqrt{3})$ . Hence it is proportional to vector  $\sqrt{3}\partial_x - \partial_y$  (Lobachevsky metric is conformally Euclidean hence orthogonality is the same). We see that vector  $X_1$  attached at the point  $B$  is proportional to vector  $\sqrt{3}\partial_x - \partial_y$ ,  $\mathbf{X}_1 = k(\sqrt{3}, -1)$ , ( $k > 0$ ). On the other hand during parallel transport its length is not changed, since the connection is Levi-Civita connection. We have

$$\langle \mathbf{X}_0, \mathbf{X}_0 \rangle_A = \langle \partial_x, \partial_x \rangle_A = \frac{1}{4}$$

$$\langle \mathbf{X}_1, \mathbf{X}_1 \rangle_B = \langle k\sqrt{3}\partial_x - k\partial_y, k\sqrt{3}\partial_x - k\partial_y \rangle_B = \frac{3k^2 + k^2}{3} = \frac{4k^2}{3}$$

We have  $\frac{3k^2}{4} = \frac{1}{4}$ , hence  $k = \frac{\sqrt{3}}{4}$  and  $\mathbf{X}_1 = \frac{\sqrt{3}}{4}(\sqrt{3}\partial_x - \partial_y)$ .

On one hand many students tried to solve this problem and beginning of solutions usually was encouraging. Students noted that since semicircle  $x^2 + y^2 = 4x$  is geodesic, then tangent vectors will remain tangent. But on the other hand when trying to make these calculations they did many mistakes. In particular to work with tangent vectors it is very important to note that they are proportional to velocity vectors. (Some students instead tangent vectors considered just horizontal vectors.)

#### 4.

(a) Consider the conic surface in the Euclidean space  $\mathbf{E}^3$

$$\mathbf{r}(h, \varphi): \begin{cases} x = 2h \cos \varphi \\ y = 2h \sin \varphi \\ z = h \end{cases}.$$

Let  $\mathbf{e}, \mathbf{f}$  be the unit vectors in the directions of the vectors  $\mathbf{r}_h = \frac{\partial \mathbf{r}}{\partial h}$  and  $\mathbf{r}_\varphi = \frac{\partial \mathbf{r}}{\partial \varphi}$ , and  $\mathbf{n}$  be a unit normal vector to the cone.

Express these vectors explicitly.

For the obtained orthonormal basis  $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$  calculate the 1-forms  $a, b$  and  $c$  in the derivation formula

$$d \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix}.$$

Deduce from these calculations the mean curvature and the Gaussian curvature of the cone.

[8 marks]

(b) Give a definition of the curvature tensor for a manifold equipped with a connection.

Consider the surface of a cylinder in the Euclidean space  $\mathbf{E}^3$  with the induced Riemannian metric. Explain why the Riemann curvature tensor of the Levi-Civita connection of this surface vanishes.

[6 marks]

(c) State the theorem about the result of parallel transport along a closed curve on a surface in the Euclidean space  $\mathbf{E}^3$ .

On the sphere  $x^2 + y^2 + z^2 = R^2$  in  $\mathbf{E}^3$  consider a circle  $C$  which is the intersection of the sphere with the plane  $z = R - h$ ,  $0 < h < R$ .

Let  $\mathbf{X}$  be an arbitrary vector tangent to the sphere at a point of  $C$ .

Find the angle between  $\mathbf{X}$  and the result of parallel transport of  $\mathbf{X}$  along  $C$ . [6 marks]

### Discussion of fourth question

a) No special problems

b) Many students who tried to define curvature failed to do it,

Riemannian metric on cylinder is locally Euclidean, hence in coordinates  $u = h, v = r\varphi$  ( $du^2 + dv^2 = dh^2 + R^2 d\varphi^2$ ). Hence *in these coordinates* Christoffel symbols of Levi-Civita connection vanish. Hence in these coordinates tensor  $R_{kmn}^i = 0$ . This is a tensor. Hence it vanishes in all coordinates. This is a clear and quick consideration. Unfortunately only few students did it properly.

c) The area of domain  $D$  of the sphere with  $R - h < z < R$  ( $\partial D = C$ ) is equal to

$$S = 2\pi Rh \quad (*)$$

Hence angle of rotation is equal to  $S/K = 2\pi Rh/R^2 = 2\pi h/R$ .

Students who try to solve this problem had problems to use equation (\*).

The following question is compulsory.

5.

(a) Explain what is meant by saying that a vector field  $\mathbf{K}$  is an infinitesimal isometry (Killing vector field) of a Riemannian manifold  $M$ .

Write down explicitly in local coordinates the condition that Killing vector field

$\mathbf{K} = K^i(x) \frac{\partial}{\partial x^i}$  preserves the Riemannian metric  $G = g_{ik}(x) dx^i dx^k$ .

Let  $\nabla$  be the Levi-Civita connection of a Riemannian metric  $G$ . Show that the condition that  $\mathbf{K}$  is a Killing vector field is equivalent to the condition that the linear operator  $A_{\mathbf{K}}$  on tangent vectors defined by the relation  $A_{\mathbf{K}}(\mathbf{X}) = \nabla_{\mathbf{X}}\mathbf{K}$  is antisymmetric.

[10 marks]

**(b)** Find all Killing vector fields of the Euclidean space  $\mathbf{E}^n$ .

Denote by  $\kappa(M)$  the dimension of the vector space of the Killing vector fields on a Riemannian manifold  $M$ .

Calculate  $\kappa(M)$  in the case if  $M$  is the Euclidean space  $\mathbf{E}^n$ .

Give an examples of 2-dimensional Riemannian manifolds  $M_1$  and  $M_2$  such that  $M_1$  has a non-vanishing Gaussian curvature, but  $\kappa(M_1) = 3$ , and  $M_2$  has zero Gaussian curvature, but  $\kappa(M_2) = 2$ . Justify your answer.

[10 marks]

### Discussion of fifth question

Students who answered this question ahve no problems with its bookwork part. In particular they wrote Killing vector fields for  $\mathbf{E}^n$ :

$$\mathbf{K} = (T^i + x^k B_k^i) \partial_i$$

where  $B_k^i$  is an arbitrary antisymmetric matrix.

On the other hand calculating  $\kappa()$  almost all students failed to perform elementary considerations which show that vector fields

$$\underbrace{\partial_i}_{i=1,\dots,n}, \quad \underbrace{x^j \partial_i - x^i \partial_j}_{i < j}$$

form basis in the space of Killing vector fields (It has to be shown that these fields are linearly independent. Unfortunately almost all students failed to do this linear algebra exercise.)