Solutions of Homework 3

In all exercises we assume by default that Riemannian metric on embedded surfaces is induced by the Euclidean metric.

1 a) *Show that surface of the cone* $\begin{cases} x^2+y^2-k^2z^2=0 \\ z>0 \end{cases}$ *in* $\mathbf{E}^3$ *is locally isometric to Euclidean plane.*

Solution.

This means that we have to find local coordinates u, v on the cone such that in these coordinates induced metric $G|_c$ on cone would have the appearance $G|_c = du^2 + dv^2$.

First of all calculate the metric on the cone in coordinates h, φ where

$$\mathbf{r}(h,\varphi)\colon \begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases}.$$

$(x^2 + y^2 - k^2z^2 = k^2h^2\cos^2\varphi + k^2h^2\sin^2\varphi - k^2h^2 = k^2h^2 - k^2h^2 = 0.$

Calculate metric G_c on the cone in coordinates h, φ induced with the Euclidean metric $G = dx^2 + dy^2 + dz^2$:

$$G_c = \left(dx^2 + dy^2 + dz^2\right)\big|_{x=kh\cos\varphi, y=kh\sin\varphi, z=h} = (k\cos\varphi dh - kh\sin\varphi d\varphi)^2 +$$
$$(k\sin\varphi dh + kh\cos\varphi d\varphi)^2 + dh^2 = (k^2+1)dh^2 + k^2h^2d\varphi^2\,.$$

In analogy with polar coordinates try to find new local coordinates u, v such that $\begin{cases} u = \alpha h\cos\beta\varphi \\ v = \alpha h\sin\beta\varphi \end{cases}$, where α, β are parameters. We come to

$$du^2 + dv^2 = (\alpha\cos\beta\varphi dh - \alpha\beta h\sin\beta\varphi d\varphi)^2 + (\alpha\sin\beta\varphi dh + \alpha\beta h\cos\beta\varphi d\varphi)^2 = \alpha^2dh^2 + \alpha^2\beta^2h^2d\varphi^2.$$

Comparing with the metric on the cone $G_c = (1+k^2)dh^2 + k^2h^2d\varphi^2$ we see that if we put $\alpha = k$ and $\beta = \frac{k}{\sqrt{1+k^2}}$ then $du^2 + dv^2 = \alpha^2dh^2 + \alpha^2\beta^2h^2d\varphi^2 = (1+k^2)dh^2 + k^2h^2d\varphi^2$.

Thus in new local coordinates

$$\begin{cases} u = \sqrt{k^2+1}h\cos\frac{k}{\sqrt{k^2+1}}\varphi \\ v = \sqrt{k^2+1}h\sin\frac{k}{\sqrt{k^2+1}}\varphi \end{cases}$$

induced metric on the cone becomes $G|_c = du^2 + dv^2$, i.e. cone locally is isometric to the Euclidean plane ■

2 a) *a) Consider the conic surface C defined by the equation $x^2 + y^2 - z^2 = 0$ in $\mathbf{E}^3$. Consider a part of this conic surface between planes $z = 0$ and $z = H > 0$ and remove the line $z = y, x = 0$ from this part of conic surface C. We come to the surface D defined by the conditions*

$$\begin{cases} x^2 + y^2 - z^2 = 0 \\ 0 < z < H \\ y \neq 0 \quad \text{if } x > 0 \end{cases}.$$

Find a domain D' in Euclidean plane such that it is isometric to the surface D.

b) Find a shortest distance between points $A = (1,0,1)$ and $B = (-1,0,1)$ for an ant living on the conic surface C.

Solution.

The domain D on the cone can be parameterised as (see the previous exeercise for $k = 1$)

$$\mathbf{r}(h,\varphi)\colon \begin{cases} x = h\cos\varphi \\ y = h\sin\varphi \\ z = h \end{cases} \quad 0 < h < H, 0 < \varphi < 2\pi$$

Using the results of previous exercise for $k = 1$ consider new local coordinates

$$u, v: \quad \begin{cases} u = \sqrt{2}h \cos \frac{\varphi}{\sqrt{2}} \\ v = \sqrt{2}h \sin \frac{\varphi}{\sqrt{2}} \end{cases} \quad 0 < h < H, 0 < \varphi < 2\pi$$

In these coordinates metric $G = du^2 + dv^2$. Consider Euclidean plane with Cartesian coordinates u, v and new polar coordinates

$$(R, \theta): \quad \begin{cases} R = \sqrt{u^2 + v^2} = \sqrt{2}h \\ \theta = \frac{\varphi}{\sqrt{2}} \quad 0 < h < z, 0 < \varphi < 2\pi \end{cases}$$

We come to the sector D' in $\mathbf{E}^2$ with polar coordinates R, θ such that

$$0 < R < \sqrt{2}H, \ , 0 < \theta < \frac{2\pi}{\sqrt{2}} .$$

It is what happens with the cone when we use scissors!: We come to the sector of the circle of the radius $R = H\sqrt{2}$ with the arc $L = 2\pi H$.

To find the shortest distance between points A, B on the cone we again "using scissors" made the isometry with Euclidean plane. We come to the sector of the circle. The points A, B and the origin O make the isosceles triangle $\triangle OAB$ with $OA = OB = \sqrt{2}$, and the angle $\angle A0B = \frac{\pi}{\sqrt{2}}$

The distance $|AB| = 2|0A| \sin \frac{\angle AOB}{2} = 2\sqrt{2} \sin \frac{\pi}{2\sqrt{2}}$

The "naive" distance (trip around the arc of circle) equals to $\pi > 2\sqrt{2} \sin \frac{\pi}{2\sqrt{2}}$.

3 *Consider plane with Riemannian metric given in cartesian coordinates (x, y) by the formula*

$$G = \frac{a\left((dx)^2 + (dy)^2\right)}{(1 + x^2 + y^2)^2} ,$$

and a sphere of the radius r in the Euclidean space $\mathbf{E}^3$. Find r such that this Riemannian manifold is locally isometric to the sphere. Justify you answer. (You may use the formula for Riemannian metric on the sphere in stereographic coordinates.)

Recall that for sphere of radius R in stereographic coordinates $G = 4R^4 \frac{du^2+dv^2}{(R^2+u^2+v^2)^2}$.

If we consider new coordinates $x = Ru, y = Rv$ then

$$G = 4R^4 \frac{du^2 + 4dv^2}{(R^2 + u^2 + v^2)^2} = 4R^4 \frac{R^2dx^2 + R^2dy^2}{(R^2 + R^2x^2 + R^2y^2)^2} = \frac{4R^2dx^2 + 4R^2dy^2}{(1 + x^2 + y^2)^2} .$$

So the answer is clear: The plane with metric $G = \frac{a\left((dx)^2+(dy)^2\right)}{(1+x^2+y^2)^2}$ is locally isometric to the sphere of the radius $R = \frac{\sqrt{a}}{2}$. In fact it is isometric (globally) to this sphere without North pole.

4 *Consider catenoid: $x^2 + y^2 = \cosh^2 z$ and helicoid: $y - x \tan z = 0$.*

Find induced Riemannian metrics on these surfaces.

Show that these surfaces are locally isomorphic.

Write down aprametric equations for catenoid and helicoid.

Catenoid is the surface of resolution:

$$\mathbf{r}(t, \varphi): \quad \begin{cases} x = f(t) \cos \varphi \\ y = f(t) \sin \varphi \\ z = t \end{cases}$$

for $f(t) = cosht$, i.e.

$$\mathbf{r}(t,\varphi)\colon \quad \begin{cases} x = \cosh t \cos\varphi \\ y = \cosh t \sin\varphi \\ z = t \end{cases} \qquad \text{(catenoind)}$$

$(x^2 + y^2 - \cosh^2 z = 0)$.

We come to helicoid If we rotate the horisontal line and move it in vertical direction with constant speeds *:

$$\mathbf{r}(t,\varphi)\colon \quad \begin{cases} x = t \cos\varphi \\ y = t \sin\varphi \\ z = \varphi \end{cases} \qquad \text{(helicoid)}$$

Calculate induced Riemannian structures:

$$G_{cat} = (dx^2 + dy^2 + dz^2)\big|_{x=\cosh t\cos\varphi, y=\cosh t\sin\varphi, z=t} =$$

$$(\sinh t\cos\varphi dt - \cosh t\sin\varphi d\varphi)^2 + (\sinh t\sin\varphi dt + \cosh t\cos\varphi d\varphi)^2 + dt^2 =$$

$$(1 + \sinh^2 t)dt^2 + \cosh^2 t d\varphi^2 = \cosh^2 t(dt^2 + d\varphi^2)\,. \tag{2}$$

$$G_{hel} = (dx^2 + dy^2 + dz^2)\big|_{x=t\cos\varphi, y=t\sin\varphi, z=\varphi} =$$

$$(dt\cos\varphi - t\sin\varphi d\varphi)^2 + (dt\sin\varphi + t\cos\varphi d\varphi)^2 + d\varphi^2 =$$

$$dt^2 + t^2 d\varphi^2 + d\varphi^2 = dt^2 + (1 + t^2)d\varphi^2\,. \tag{3}$$

Compare Riemannian metrics (2) and (3). We see that if we consider in (3) $t \mapsto \sinh t$ we come to (2):

$$G_{helicoid} = \left(dt^2 + (1+t^2)d\varphi^2\right)_{t\mapsto\sinh t} = (d\sinh t)^2 + (1 + \sinh^2 t)d\varphi^2 = \cosh^2 t(dt^2 + d\varphi^2) = G_{hel}$$

Hence helicoid and catenoid are locally isometric.

You could find very beautiful picture how helicoid isometrically can be transformd to catenoid (see Wikipedia).

5 a) *Consider the domain D on the cone $x^2 + y^2 - k^2 z^2$ defined by the condition $0 < z < H$. Find an area of this domain using induced Riemannian metric. Compare with the answer when using standard formulae.*

We have cone with height H with radius $R = kH$ $(k > 0)$.

First of all standard answer: The area of cone (of surface of cone) is area of the sector with the radius $\sqrt{H^2 + R^2}$ and length of the arc $2\pi R$:

$$S = \frac{1}{2} \cdot \sqrt{R^2 + H^2} \cdot 2\pi R = \pi R\sqrt{H^2 + R^2} = \pi k\sqrt{1 + k^2}H^2\,.$$

Now calculate this are using Riemannian geometry. It follows from the result of the exercise (2) that volume form on the cone equals

$$d\sigma = \sqrt{\det G} dh \wedge d\varphi = k\sqrt{1 + k^2} dh \wedge d\varphi$$

since $G = \begin{pmatrix} k^2 + 1 & 0 \\ 0 & k^2 h^2 \end{pmatrix}$ Hence

$$S = \int_{0<h<H} \sqrt{\det G} dh \wedge d\varphi = \int_{0<h<H} k\sqrt{1 + k^2} dh \wedge d\varphi = 2\pi k\sqrt{1 + k^2}\int_0^H h dh = \pi k\sqrt{1 + k^2}H^2\,.$$

* look in Wikipedia for detail

(Compare with standard calculations).

6 *Find an area of 2-dimensional sphere of radius R using explicit formulae for induced Riemannian metric in stereographic coordinates.*

Riemannian metric for sphere (without point) in stereographic coordinates is $G = \frac{4R^4du^2+4R^4dv^2}{(R^2+u^2+v^2)^2}$. We already know that doing transformation $u \mapsto ru, v \mapsto Rv$ we come to the expression

$$G = \frac{4R^2du^2 + 4R^2dv^2}{(1+u^2+v^2)^2}$$

(see the exercise 3.)

$$G = \begin{pmatrix} \frac{4R^2}{(1+u^2+v^2)^2} & 0 \\ 0 & \frac{4R^2}{(1+u^2+v^2)^2} \end{pmatrix}, \ \det G = \frac{16R^4}{(1+u^2+v^2)^4}$$

Hence the volume (area)of the sphere equals to

$$S = \int_{\mathbf{R}^2} \sqrt{\det G} du dv = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{4R^2}{(1+u^2+v^2)^2}\right) du dv$$

Choosing polar coordinates $u = r\cos\varphi, v = r\sin\varphi$ we come to

$$S = \int_0^{\infty}\int_0^{2\pi} \left(\frac{4R^2}{(1+r^2)^2}\right) d\varphi r dr = 8\pi R^2 \int_0^{\infty} \frac{rdr}{1+r^2} = 4\pi R^2 .$$

7 *Show that two spheres of different radii in Euclidean space are not isometric to each other.*

Suppose that these two spheres of different radii are isometric (globally). This means that their volume is the same. Contradiction. (In fact two spheres of different radii are not isometric even locally, since they have different curvatures.)

8 *Find new local coordinates $u = u(x,y), v = v(x,y)$ be new local in Euclidean space $\mathbf{E}^2$ such that $du^2 + dv^2 = dx^2 + dy^2$ (transformation is linear: $u = a + bx + cy$, $v = e + dx + fy$)*

† *Will answer change if we allow arbitrary (not only linear transformations?)*

If $u = a + bx + cy$, $v = c + dx + fy$ then

$$du^2 + dv^2 = (bdx + cdy)^2 + (ddx + fdy)^2 = (b^2 + d^2)dx^2 + 2(bc + df)dxdy + (c^2 + f^2)dy^2 = dx^2 + dy^2$$

This means that $b^2 + d^2 = c^2 + f^2 = 1$ and $bc + df = 0$, i.e. for matrix $A = \begin{pmatrix} b & d \\ c & f \end{pmatrix}$ rows have length 1 and they are orthogonal, i.e. the matrix A is orthogonal: $AA^T = I$. We come to the answer: In the class of linear transformations the transformation that preserves the Euclidean metric is a translation and orthogonal transformation, i.e. translations, rotations and reflections.

† It is very interesting to answer the question: how look general transformations which preserve the metric? Answer: any transformation preserving Euclidean metric is linear transformation, i.e. there is no rotation "depending on point." There are many different and beautiful and illuminating proofs of this fact which is true for any dimensions. We consider here not the best one:

Let $u = u(x,y), v = v(x,y)$ be transformation such that $du^2 + dv^2 = dx^2 + dy^2$. Then we come to the condition that

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} \cos\Psi(x,y) & -\sin\Psi(x,y) \\ \sin\Psi(x,y) & \cos\Psi(x,y) \end{pmatrix}$$

thus the function $F(x,y) = u(x,y) + iv(x,y)$ is holomorphic function (Cauchy-Riemann conditions $u_x = v_y$, $u_y + v_x = 0$). The condition $u_x^2 + v_x^2 = 1$ means that the modulus of the analytical function $F' = \frac{\partial F}{\partial z}$ equals to 1. Thus $F' = const$, This means that $\Psi(x,y) = constant$, i.e. it is global orthogonal transformation. (There is no differential rotation!)

9 *Let D be a domain in Lobachevsky plane which is lying between lines $x = a, x = -a$ and outside of the disc $x^2 + y^2 = 1$, $(0 < a < 1)$: $D = \{(x,y)\colon\ |x| < a, x^2 + y^2 > 1\}$,*

a) Find the area of this domain.

b) Find the angles between lines and arc of the circle.*

Lobachevsky plane, i.e. hyperbolic plane is the upper half plane with Riemannian metric $\frac{dx^2+dy^2}{y^2}$ in cartesian coordinates x, y $(y > 0)$.

a) We have $G = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \\ \frac{1}{y^2} & \end{pmatrix}$. $\sqrt{\det G} = \frac{1}{y^2}$. Hence

$$S = \int_{x^2+y^2\geq 1, -a\leq x\leq a} \sqrt{\det G}\,dxdy = \int_{x^2+y^2\geq 1, -a\leq x\leq a} \frac{1}{y^2}dxdy = \int_{-a}^{a}\left(\int_{\sqrt{1-x^2}}^{\infty}\frac{dy}{y^2}\right)dx =$$

$$\int_{-a}^{a}\frac{dx}{\sqrt{1-x^2}} = 2\arccos a$$

This has a deep geometrical meaning!

Note that if for two metrics $G, \tilde{G}$ are proportional, $\tilde{G} = \sigma(\mathbf{x})G$, i.e. $\tilde{g}_{ik} = \sigma(x)g_{ik}$ then the angles calculated with respect to these metrics are the same:

$$\cos\tilde{\angle}(\mathbf{X},\mathbf{Y}) = \frac{\tilde{G}(\mathbf{X},\mathbf{Y})}{\sqrt{\tilde{G}(\mathbf{X},\mathbf{X})}\sqrt{\tilde{G}(\mathbf{Y},\mathbf{Y})}} = \frac{\sigma G(\mathbf{X},\mathbf{Y})}{\sqrt{\sigma G(\mathbf{X},\mathbf{X})}\sqrt{\sigma G(\mathbf{Y},\mathbf{Y})}} = \frac{\sigma}{\sigma}\frac{G(\mathbf{X},\mathbf{Y})}{\sqrt{G(\mathbf{X},\mathbf{X})}\sqrt{G(\mathbf{Y},\mathbf{Y})}} = \cos\angle(\mathbf{X},\mathbf{Y})$$

(Two proportional metrics are called conformally equivalent).

Notice that Lobachevsky metric $G = \frac{dx^2+dy^2}{y^2} = \frac{1}{y^2}(dx^2 + dy^2)$ is proportional to the Euclidean metric $dx^2 + dy^2$. Hence the angles will be the same as in the Euclidean metric.

10† *Find a volume of n-dimensional sphere of radius a. (You may use Riemannian metric in stereographic coordinates, or you may do it in other way... You just have to calculate the answer.)*

Denote by σ_n the volume of n-dimensional unit sphere embedded in Euclidean space $\mathbf{E}^{n+1}$.

One can see that the volume of n-dimensional sphere of the radius R equals to $\sigma_n R^{n+1}$. Now consider the magnitude

$$I = \int_{-\infty}^{\infty} e^{-t^2}dt = \sqrt{\pi}.$$

For any integer k consider

$$I^k = \pi^{\frac{k}{2}} = \left(\int_{-\infty}^{\infty} e^{-t^2}dt\right)^k = \int_{\mathbf{E}^k} e^{-x_1^2-x_2^2-\ldots-x_k^2}dx_1dx_2\ldots dx_k$$

make changing of variables in the volume form $dx_1 \wedge dx_2 \wedge \ldots \wedge dx_k$.

Since integrand depend only on the radius we can rewrite the integral above as

$$\int_{\mathbf{E}^k} e^{-x_1^2-x_2^2-\ldots-x_k^2}dx_1dx_2\ldots dx_k = \int_{\mathbf{E}^k} e^{-r^2}r^{k-1}\sigma_{k-1}dr\,,$$

where σ_{k-1} is a volume of the unit sphere in dimension $k-1$. (Here is the truck!)

Now we have the identity:

$$\pi^{\frac{k}{2}} = \sigma_{k-1} \int_0^\infty e^{-r^2} r^{k-1} dr$$

To calculate this integral consider $r^2 = t$ we come to

$$\int_0^\infty e^{-r^2} r^{k-1} dr = \frac{1}{2} \int_0^\infty e^{-t} t^{\frac{k}{2}-1} dt = \frac{1}{2} \Gamma\left(\frac{k}{2}\right) .$$

We come to

$$\pi^{\frac{k}{2}} = \sigma_{k-1} \int_0^\infty e^{-r^2} r^{k-1} dr = \frac{\sigma_{k-1}}{2} \Gamma\left(\frac{k}{2}\right) .$$

Thus

$$\sigma_{k-1} = \frac{2\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)}.$$

Recall that $\Gamma(x)$ can be calculated for all integers and half-integers using the following recurrent formulae:

1. $\Gamma(n+1) = n!$
2. $\Gamma(x+1) = x\Gamma(x)$
3. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, $(\Gamma(x)\Gamma(1-x) = \pi \sin \pi x)$.

E.g. the volume of the 15-dimensional unit sphere in $\mathbf{E}^{16}$ equals to $\sigma_{15} = \frac{2\pi^8}{\Gamma(8)} = \frac{2\pi^6}{7!}$