Homework 3

In all exercises we assume by default that Riemannian metric on embedded surfaces is induced by the Euclidean metric.

1 a) Consider the domain D on the cone $x^2 + y^2 - k^2z^2$ defined by the condition $0 < z < H$. Find an area of this domain using induced Riemannian metric. Compare with the answer when using standard formulae.

2 Find an area of the segment of the height h of the sphere of radius R (surface: $x^2 + y^2 + z^2 = R^2, \leq a \leq a + h$ for an arbitrary a: $-R \leq a \leq R - h$)

3 Find an area of 2-dimensional sphere of radius R using explicit formulae for induced Riemannian metric in stereographic coordinates.

4 Show that two spheres of different radii in Euclidean space are not isometric to each other, i.e. there is no an isometry of one sphere on another.

5 In the previous exercise you consider Riemannian manifolds $(\mathbf{R}^2, G^{(1)})$ and $(\mathbf{R}^2, G^{(2)})$, where

$$G^{(1)} = \frac{a(dx^2 + dy^2)}{(1 + x^2 + y^2)^2}\,, \quad \text{and} \quad G^{(2)} = \frac{4R^4(du^2 + dv^2)}{(R^2 + u^2 + v^2)^2}$$

(The second manifold is sphere of radius R without North pole in stereographic coordinates) You proved in fact that in the case if $a = 4R^2$ then under isometry $\begin{cases} u = Rx \\ v = Ry \end{cases}$ these Riemannina manifolds are isometric. Using the result of previous exercise, Prove now that in the case if the condition $a = 4R^2$ is not obeyed, then these manifolds *are not isometric.*

6 Let D be a domain in Lobachevsky plane which is lying between lines $x = a, x = -a$ and outside of the disc $x^2 + y^2 = 1$, $(0 < a < 1)$: $D = \{(x, y)\colon\ |x| < a, x^2 + y^2 > 1\}$,

a) Find the area of this domain.

b*) Find the angles between lines and arc of the circle.

Lobachevsky plane, i.e. hyperbolic plane is the upper half plane with Riemannian metric $\frac{dx^2+dy^2}{y^2}$ *in Cartesian coordinates* x, y $(y > 0)$.

7 Consider the plane $\mathbf{R}^2$ with standard coordinates (x, y) equipped with the Riemannian metric

$$G = \frac{dx^2 + dy^2}{(1 + x^2 + y^2)^2}\ .$$

Calculate the area S_a of the domain $x^2 + y^2 \leq a^2$.

Find the limit S_a when $a \to \infty$.

Show that there is no isometry between the plane with this Riemannian metric and the Euclidean plane $\mathbf{E}^2$.

8† Find a volume of n-dimensional sphere of radius a. (You may use Riemannian metric in stereographic coordinates, or you may do it in other way... You just have to calculate the answer.)

Hint: One way to do it is the following. Denote by σ_n the volume of n-dimensional unit sphere embedded in Euclidean space $\mathbf{E}^{n+1}$. One can see that the volume of n-dimensional sphere of the radius R equals to $\sigma_n R^n$. We need to calculate just σ_n. Consider the following integral:

$$I_k = \int e^{-r^2} dx^1 dx^2 \dots dx^k \,,$$

where $r^2 = (x^1)^2 + (x^2)^2 + \dots + (x^k)^2$. One can see that on one hand

$$I_k = \int e^{-r^2} dx^1 dx^2 \dots dx^k = \left(\int e^{-x^2} dx \right)^n = \pi^{\frac{n}{2}} \,.$$

On the other hand $I_k = \int e^{-r^2} dx^1 dx^2 \dots dx^k = \sigma_{k-1} \int e^{-r^2} r^{k-1} dr$. Comparing these integrals we calculate σ_n.