Homework 8. Solutions

Almost all exercises of this homework are considered in detail in the subsection 4.4 of the lecture notes (see the subsubsections: "Derivation formulae", Gauss condition (structure equations), "Geometrical meaning and Weingarten operator in terms of derivation formulae", "Gaussian and mean curvature in terms of derivation formulae."

1) *Let M be a surface embedded in Euclidean space $\mathbf{E}^3$. We say that the triple of vector fields $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ is adjusted to the surface M if $\mathbf{e}, \mathbf{f}, \mathbf{n}$ be three vector fields defined on the points of this surface such that they form an orthonormal basis at any point, so that the vectors $\mathbf{e}, \mathbf{f}$ are tangent to the surface and the vector $\mathbf{n}$ is orthogonal to the surface.*

Consider the derivation formulae for adjusted vector fields $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$:

$$d\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\\ \mathbf{n}\end{pmatrix} = \begin{pmatrix}0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{pmatrix}\begin{pmatrix}\mathbf{e}\\ \mathbf{f}\\ \mathbf{n}\end{pmatrix}, \tag{1}$$

where a, b, c are 1-forms on the surface M.

Write down the explicit expression for connection, Weingarten operator, the mean curvature and the Gaussian curvature of M in terms of 1-forms a, b, c and vector fields $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$.

Solution (see also lecture notes):

Induced connection

Let ∇ be the connection induced by the canonical flat connection on the surface M. Then according to derivation formulae for every tangent vector $\mathbf{X}$

$$\nabla_{\mathbf{X}}\mathbf{e} = (\partial_{\mathbf{X}}\mathbf{e})_{\text{tangent}} = (d\mathbf{e}(\mathbf{X}))_{\text{tangent}} = (a(\mathbf{X})\mathbf{f} + b(\mathbf{X})\mathbf{n})_{\text{tangent}} = a(\mathbf{X})\mathbf{f}\,.$$

and

$$\nabla_{\mathbf{X}}\mathbf{f} = (\partial_{\mathbf{X}}\mathbf{f})_{\text{tangent}} = (d\mathbf{f}(\mathbf{X}))_{\text{tangent}} = (-a(\mathbf{X})\mathbf{e} + c(\mathbf{X})\mathbf{n})_{\text{tangent}} = -a(\mathbf{X})\mathbf{e}\,.$$

In particular

$$\begin{array}{ll}\nabla_{\mathbf{e}}\mathbf{e} = a(\mathbf{e})\mathbf{f} & \nabla_{\mathbf{f}}\mathbf{e} = a(\mathbf{f})\mathbf{f}\\ \nabla_{\mathbf{e}}\mathbf{f} = -a(\mathbf{e})\mathbf{e} & \nabla_{\mathbf{f}}\mathbf{f} = -a(\mathbf{f})\mathbf{e}\end{array}$$

Weingarten operator

Let S be Weingarten operator: $S\mathbf{X} = -\partial_{\mathbf{X}}\mathbf{n}$. Then

$$S\mathbf{X} = -\partial_{\mathbf{X}}\mathbf{n} = -d\mathbf{n}(\mathbf{X}) = -\left(-b(X)\mathbf{e} - c(\mathbf{X})\mathbf{f}\right) = b(\mathbf{X})\mathbf{e} + c(\mathbf{X})f$$

since $d\mathbf{n} = -b\mathbf{e} - c\mathbf{e}$ due to derivation formulae. In particular

$$S(\mathbf{e}) = b(\mathbf{e})\mathbf{e} + c(\mathbf{e})\mathbf{f}\,, S(\mathbf{f}) = b(\mathbf{f})\mathbf{e} + c(\mathbf{f})\mathbf{f}$$

and the matrix of the Weingarten operator in the basis $\{\mathbf{e}, \mathbf{f}\}$ is

$$S = \begin{pmatrix}b(\mathbf{e}) & c(\mathbf{e})\\ b(\mathbf{f}) & c(\mathbf{f})\end{pmatrix}$$

Curvatures We have that Gaussian curvature

$$K = \det S = \det\begin{pmatrix}b(\mathbf{e}) & c(\mathbf{e})\\ b(\mathbf{f}) & c(\mathbf{f})\end{pmatrix} = b(\mathbf{e})c(\mathbf{f}) - b(\mathbf{f})c(\mathbf{e}),$$

and Mean curvature

$$H = \operatorname{Tr} S = \operatorname{Tr} \begin{pmatrix} b(\mathbf{e}) & c(\mathbf{e}) \\ b(\mathbf{f}) & c(\mathbf{f}) \end{pmatrix} = b(\mathbf{e}) + c(\mathbf{f}),$$

Remark* Note that Gaussian curvature $K = b(\mathbf{e})c(\mathbf{f}) - b(\mathbf{f})c(\mathbf{e}) = b \wedge c(\mathbf{e}, \mathbf{f}) = da(\mathbf{e}, \mathbf{f})$ due to Gauss condition. This is very important to deduce the formula of rotation of the vector during parallel transport along the closed curve.

2) *Show that in derivation formulae $da + b \wedge c$=0.*

Solution

Recall that a, b, c are 1-forms, $\mathbf{e}, f, \mathbf{n}$ are vector valued functions (0-forms) and $d\mathbf{e}, d\mathbf{f}, d\mathbf{n}$ are vector valued 1-forms. (We use the simple identity that $ddf = 0$ and the fact that for 1-form $\omega \wedge \omega = 0$.) We have from derivation formulae that

$$d^2\mathbf{e} = 0 = d(a\mathbf{f} + b\mathbf{n}) = da\mathbf{f} - a \wedge d\mathbf{f} + db\mathbf{n} - b \wedge d\mathbf{n} =$$

$$da\,\mathbf{f} - a \wedge (-a\mathbf{e} + c\mathbf{n}) + db\,\mathbf{n} - b \wedge (-b\mathbf{e} - c\mathbf{f}) =$$

$$(da + b \wedge c)\mathbf{f} + (a \wedge a + b \wedge b)\mathbf{e} + (db - a \wedge c)\mathbf{n} = (da + b \wedge c)\mathbf{f} + (db + c \wedge a)\mathbf{n} = 0\,.$$

We see that $(da + b \wedge c)\mathbf{f} + (db + c \wedge a)\mathbf{n} = 0$. Hence components of the left hand side equal to zero: $(da + b \wedge c) = 0\,,\ (db + c \wedge a) = 0$. In particular $da + b \wedge c = 0$ ■.

3) *Find explicitly a triple of vector fields $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ adjusted to the surface M if M is a) cylinder, b) cone, c) sphere*

Solution.

See the detailed solution of this and of the next exercise in the Lecture Notes (section 4.4).

One can consider different adjusted triples. In this solution we just consider an example of the adjusted triple

4) *Using results of the previous exercise find explicit expression for derivation formulae (1) in the case if the surface M is a) cylinder, b) cone, c) sphere*

Deduce from these results the formulae for Gaussian and mean curvature for cylinder, cone and sphere

See the detailed solution of this and of the previous exercise in the lecture notes (section 4.4).

5) *Consider surface M which is given by equation*

$$\mathbf{r}(u, v)\colon \quad \begin{cases} x = u \\ y = v \\ z = F(u, v) \end{cases}$$

Find explicitly a triple of vector fields $\mathbf{e}, \mathbf{f}, \mathbf{n}$ adjusted to the surface M.

Suppose the origin (point $u = v = 0$) is a stationary point of the function $F(u, v)$, i.e. $F_u = F_v = 0$ at $u = v = 0$.

Calculate in this case vector 1-forms $d\mathbf{e}, d\mathbf{f}, d\mathbf{n}$, 1-forms a, b, c at origin, and calculate Gaussian and mean curvature at origin.

See the detailed solution of this exercise in the lecture notes (section 4.5).

6) *a) Find explicitly a triple of vector fields $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ adjusted to the surface M if a Riemannian metric on a surface M is given by formula $G = a(u, v)du^2 + b(u, v)dv^2$.*

b) Calculate 1-form a in derivation formulae in the special case if $a = b = \sigma(u, v)$ (conformal metric). Calculate Gaussian curvature. (It is convenient to use notation $\sigma = e^{\Phi}$).*

Solution.

a) E.g. one may take

$$\mathbf{e} = \frac{1}{\sqrt{a}}\frac{\partial}{\partial u}\,, \mathbf{e} = \frac{1}{\sqrt{b}}\frac{\partial}{\partial u}$$

and $\mathbf{n}$ its vector product. It is evident that these vectors form orthonormal basis. Of course one may consider another examples (see for detail exercise 6.)

b)* In the case $a = b = \sigma$ we come to isothermal coordinates. See detailed calculations in Lecture Notes (section 4).

7 *Calculate Gaussian curvature for surface M if induced Riemannian metric is equal to*

$$G = (u^2 + y^2)(dx^2 + dv^2)$$

* *Show that this surface is locally Euclidean: find coordinates p, q, $p = p(u, v), q = q(u, v)$ such that $G = dp^2 + dq^2$ in these coordinates.* One can see that $\Delta \log(u^2 + v^2) = 0$, hence $K = 0$.

We see that $G = (u^2+v^2)(du^2+dv^2) = z\bar{z}dzd\bar{z}$. Hence consider $w = p+iq = \frac{z^2}{2} = \frac{(u+iv)^2}{2} = \frac{u^2-v^2}{2}+iuv$, i.e. $\left\{ p = \frac{u^2-v^2}{2} \right.$. We have

$$dp^2 + dq^2 = \left(d\left(\frac{u^2 - v^2}{2}\right)\right)^2 + (d(uv))^2 = (udu - vdv)^2 + (udv + vdu)^2 = (u^2 + v^2)(du^2 + dv^2)\,.$$

8 *Choose conformal coordinates on sphere of radius R and calculate the curvature of sphere.*

Deduce that sphere is not locally Euclidean surface, i.e. there are no local coordinates on sphere such that induced metric in these coordinates is equal to $G = du^2 + dv^2$.

One can choose standard stereographic coordinates (u, v), then metric is equalt to $G = \sigma(du^2 + dv^2)$ with $\sigma = \frac{4R^4}{(R^2+u^2+v^2)^2}$. Calculate curvature:

$$K = -\frac{1}{2\sigma}\Delta \log \sigma = -\frac{(R^2 + u^2 + v^2)^2}{8R^4}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\log\left(\frac{4R^4}{(R^2 + u^2 + v^2)^2}\right) =$$

$$\frac{(R^2 + u^2 + v^2)^2}{4R^4}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\log\left(R^2 + u^2 + v^2)^2\right) = \frac{(R^2 + u^2 + v^2)^2}{4R^4}\left(\frac{\partial}{\partial u}\left(\frac{2u}{u^2 + v^2 + R^2}\right) + \frac{\partial}{\partial v}\left(\frac{2v}{u^2 + v^2 + R^2}\right)\right) =$$

$$\frac{(R^2 + u^2 + v^2)^2}{4R^4}\left(\frac{2v^2 - 2u^2 + 2R^2}{(u^2 + v^2 + R^2)^2} + \frac{2u^2 - 2v^2 + 2R^2}{(u^2 + v^2 + R^2)^2}\right) = \frac{4R^2}{4R^4} = \frac{1}{R^2}\,.$$

Show that spjere is not locally Euclidean surface. If it is locally Euclidean, then tehre exist coordinates x, y such that $G = dx^2 + dy^2$, It is obvious that in these coordinates Gaussian curvature is equal to 0. Contradiction.

9* *Let u, v be coordinates on the locally Euclidean surface M such that $G = du^2 + dv^2$. Let p, q be another coordinates such that*

$$w = p + iq = F(z)\big|_{z=x+iy},$$

where $F = F(z)$ is a holomorphic function. Show that u, v are conformal coordinates also in the case if

a) $F = e^z$.

b) F is an arbitrary (non-zero) holomorphic function.*

This is a result of straightforward calculations