Introduction to Geometry (20222)

2009

COURSEWORK

This assignment counts for 20% of your marks.

Solutions are due by 23 April

Write solutions in the provided spaces.

STUDENTS'S NAME:

1

a) Let (x^1, x^2, x^3) be coordinates of the vector $\mathbf{x}$, and (y^1, y^2, y^3) be coordinates of the vector $\mathbf{y}$ in $\mathbf{R}^3$.

Does the formula $(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^2 + x^2y^3 + x^3y^2 + x^3y^3$ define a scalar product on $\mathbf{R}^3$?

b) Let $\mathbf{x}, \mathbf{y}$ be two arbitrary vectors in $\mathbf{E}^3$. Show that

$$(\mathbf{x} \times \mathbf{y}, \mathbf{x} \times \mathbf{y}) = (\mathbf{x}, \mathbf{x})(\mathbf{y}, \mathbf{y}) - (\mathbf{x}, \mathbf{y})^2 ,$$

where $(\ ,\ )$ is the scalar product in $\mathbf{E}^3$ and $\ldots \times \ldots$ is the vector product in $\mathbf{E}^3$.

c) Let $\mathbf{x}, \mathbf{y}$ be two vectors in the Euclidean space $\mathbf{E}^2$ such that the length of the vector $\mathbf{x}$ is equal to 1, the length of the vector $\mathbf{y}$ is equal to 5 and scalar product of these vectors is equal to 3.

Show that the ordered pair $\{\mathbf{x}, \mathbf{y}\}$ is a basis in $\mathbf{E}^2$. (It suffices to show that vectors $\mathbf{x}, \mathbf{y}$ are linearly independent.)

Find an orthonormal basis $\{\mathbf{e}, \mathbf{f}\}$ in $\mathbf{E}^2$ such that $\mathbf{e} = \mathbf{x}$ and vector $\mathbf{f}$ has an obtuse angle with vector $\mathbf{y}$.

2

a) Consider a point $M_t = (t, 0)$ on x-axis in $\mathbf{E}^2$, where t is an arbitrary parameter ($t \in (-\infty, \infty)$). Consider also the circle $x^2 + y^2 = 2$ in $\mathbf{E}^2$, and the point $B = (1, 1)$ on this circle. Denote by l_t the straight line passing through the points B and M_t.

Find an equation of the line l_t.

Calculate the coordinates of the second point S_t of intersection of the line l_t with the circle.

Find the value of parameter t such that the line l_t is tangent to the circle.

b) In the Euclidean space $\mathbf{E}^2$ consider two points $A = (-4, 3)$ and $B = (16, 24)$.

Find an orthonormal basis $\{\mathbf{a}, \mathbf{b}\}$ in $\mathbf{E}^2$ such that the vector $\mathbf{a}$ is collinear, i.e. proportional to the vector AB.

How many solutions does this problem have?

Find two orthonormal bases obeying the condition above and having the same orientation.

c) Consider the system of equations

$$\begin{cases} x^2 + y^2 = R^2 \\ |x| + |y| = 2 \end{cases}$$

where R is a parameter. By using a sketch find the number of solutions of this system for different values of R.

3

Throughout this question, $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ is an orthonormal basis for Euclidean space $\mathbf{E}^3$.

a) Consider vectors $\mathbf{a} = 2\mathbf{e}_x + \mathbf{e}_y$, $\mathbf{b} = \mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z$ in $\mathbf{E}^3$. Show that these vectors are linearly independent. Find an equation of the plane α spanned by vectors $\mathbf{a}$ and $\mathbf{b}$ attached at the point $M = (-1, 2, 3)$. (Write down an equation in the form $Ax + By + Cz = D$).

b) Consider the plane α in $\mathbf{E}^3$ passing through the points $A = (a, 0, 0)$, $B = (0, b, 0)$ and $C = (0, 0, c)$, where $a, b, c \neq 0$. Find an equation of the plane α, the distance between the origin and the plane α, and the area of the triangle ABC.

Hint: You may use the formula for the volume of tetrahedron: $V = \frac{HS}{3}$.

c) Consider vector $\mathbf{a} = 2\mathbf{e}_x + 3\mathbf{e}_y + 6\mathbf{e}_z$ in $\mathbf{E}^3$.

Show that the angle θ between vectors $\mathbf{a}$ and $\mathbf{e}_z$ belongs to the interval $\left(\frac{\pi}{6}, \frac{\pi}{4}\right)$.

Find a unit vector $\mathbf{b}$ such that it is orthogonal to vectors $\mathbf{a}$ and $\mathbf{e}_z$, and the angle between vectors $\mathbf{b}$ and $\mathbf{e}_x$ is acute.

Show that the ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{e}_z\}$ is a basis and this basis has orientation opposite to the orientation of the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$.

4

a) Given a vector field $\mathbf{G} = r\partial_r + \partial_\varphi$ in polar coordinates express it in cartesian coordinates ($x = r\cos\varphi, y = r\sin\varphi$).

b) Consider the function $f = r^2 \sin 2\varphi$ and the vector fields $\mathbf{A} = x\partial_x + y\partial_y$, $\mathbf{B} = x\partial_y - y\partial_x$. Calculate $\partial_\mathbf{A} f$ and $\partial_\mathbf{B} f$. Perform these calculations both in polar and cartesian coordinates.

Calculate 1-form $\omega = df$ and find the values of this 1-form on the vector fields $\mathbf{A}$, $\mathbf{B}$.

(c) Show that the 1-form $\omega = 2xydx + x^2dy$ is an exact form.

Show that the 1-form $\omega = dx + xdy$ is not an exact form.

Find a function $g = g(y)$ such that 1-form $\omega = g(y)(dx + xdy)$ is an exact form.

5

(**a**) Consider in $\mathbf{E}^2$ the ellipse $\mathbf{r}(t)\colon x = a\cos t, y = b\sin t$, $0 \le t < 2\pi$, $a > b > 0$.

Find the velocity $\mathbf{v} = \frac{d\mathbf{r}(t)}{dt}$ and acceleration $\mathbf{a}(t) = \frac{d^2\mathbf{r}(t)}{dt^2}$ vectors.

Find the points of this curve where speed is increasing.

Find the points of this curve where speed takes maximum value.

(**b**) Consider in $\mathbf{E}^2$ the curve $\mathbf{r}(t)\colon x = t^2 - t, y = 2t$, $0 < t < 1$.

Sketch the image of this curve.

Calculate the integral of the differential form $\omega = xdy + y^2dx$ over this curve.

How does this integral change under the reparameterisation $t = \sin\tau$, $(0 < \tau < \frac{\pi}{2})$?

How does this integral change under the reparameterisation $t = \cos\tau$, $(0 < \tau < \frac{\pi}{2})$?

(**c**) Consider the differential 1-forms $\omega_1 = \cos y\, dx + xdy$ and $\omega_2 = x\sin y\, dy - ydx$ in $\mathbf{E}^2$.

Show that for an arbitrary closed curve C in $\mathbf{E}^2$

$$\int_C \omega_1 = \int_C \omega_2 \,.$$

Hint: You may consider the 1*-form* $\omega = \omega_1 - \omega_2$.