Riemannian Geometry (31082, 41082)

2010

COURSEWORK

Starred questions are for the 15 credit version

This assignment counts for 20% of your marks.

Solutions are due by 26 April

Write solutions in the provided spaces.

STUDENTS'S NAME:

1

(a)

Explain why the positive-definiteness of a Riemannian metric implies its non degeneracy.

(b) Consider $\mathbf{R}^2$ with the metric $G = \frac{\lambda(dx^2+dy^2)}{(1+x^2+y^2)^2}$, where λ is a constant ($\lambda > 0$).

Show that this Riemannian manifold is isometric to a sphere with a point removed, in $\mathbf{E}^3$. Find the radius $R = R(\lambda)$ of this sphere.

Find an arbitrary non-identity transformation of coordinates x, y such that it preserves the metric G.

* *Find a non-linear transformation of coordinates x, y such that it preserves the metric G.*

(c) Find the length of the shortest curve on the cone $x^2 + y^2 - k^2z^2 = 0$, joining the points $A = (kh, 0, h)$ and $B = (-kh, 0, h)$.

2

(a) Write down the formula for the volume element on a Riemannian manifold M with metric $G = g_{ik}dx^i dx^k$.

Show explicitly that for a 3-dimensional Riemannian manifold the volume element is invariant under changing of coordinates.

(b) Calculate the volume of n-dimensional space $\mathbf{R}^n$ equipped with the Riemannian metric

$$G = e^{-u_1^2 - u_2^2 - \ldots - u_n^2} \left((du_1)^2 + (du_2)^2 + \ldots + (du_n)^2 \right) .$$

(c) Evaluate the area of the part of the sphere of radius R between the planes given by equations $4x + 5y + 20z = 1$ and $4x + 5y + 20z = 2$.

3

(**a**) Explain what is meant by Christoffel symbols and write down their transformation law.

Let $\Gamma^{i(1)}_{km}$ be the Christoffel symbols of a connection $\nabla^{(1)}$ and $\Gamma^{i(2)}_{km}$ be the Christoffel symbols of a connection $\nabla^{(2)}$. Show that the linear combinations $f\Gamma^{i(1)}_{km} + g\Gamma^{i(2)}_{km}$, (where f and g are some functions) are Christoffel symbols for some connection if and only if $f + g \equiv 1$.

(b) Let ∇ be the connection on the cone $\mathbf{r}(h, \varphi)$: $\begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases}$ in $\mathbf{E}^3$ induced by canonical flat connection $\nabla^{\text{can.flat}}$ in $\mathbf{E}^3$:

$$\nabla_{\mathbf{X}}\mathbf{Y} = \left(\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}\right)_{\text{tangent}} .$$

Calculate the Christoffel symbols of this connection using this relation.

Calculate again the Christoffel symbols using Lagrangians.

(c) Define two flat connections $\nabla^{(\pm)}$ so that the connection $\nabla^{(+)}$ is defined on the sphere except the north pole and the connection $\nabla^{(-)}$ is defined on the sphere except the south pole.

(You may use stereographic coordinates)

* *Using these connections show the existence of a global connection such that it is flat at all the points of the sphere except neighborhoods of North and South poles.*

(You may use partition of unity arguments. You do not need to use explicit expressions for functions.)

4

(a) Let M be a surface of revolution

$$\mathbf{r}(h,\varphi)\colon \begin{cases} x = f(h)\cos\varphi \\ y = f(h)\sin\varphi \\ z = h \end{cases} \qquad (f(h) > 0)$$

in $\mathbf{E}^3$.

Calculate the Christoffel symbols of the Levi-Civita connection on the surface M.

Find first order differential equations for h and φ which define geodesics.

In the special case of $f(h) = kh$ ($h > 0$), where k is constant (cone) solve the equations for geodesics.

(b) Let C be a geodesic on a manifold M and $x^i = x^i(t)$ some parameterisation of this geodesic. Show that there exists a function $s = s(t)$ such that

$$\frac{d^2x^i(t)}{dt^2} + \frac{ds(t)}{dt}\frac{dx^i(t)}{dt} + \frac{dx^k(t)}{dt}\Gamma^i_{km}(x(t))\frac{dx^m(t)}{dt} = 0\,,$$

where Γ^i_{km} are Christoffel symbols of the connection on M.

Explain the geometrical meaning of this fact.

Express the function $s(t)$ in terms of velocity vector $v^i = \frac{dx^i(t)}{dt}$ in the case if the connection is the Levi-Civita connection of the metric $G = g_{ik}dx^i dx^k$.

Hint; *You may choose substitution* $s(t) = \log a(t)$.

(c)* *Show that the acceleration vector for an arbitrary parameterised geodesic on a surface in* $\mathbf{E}^3$ *is orthogonal to the surface.*

5

(**a**) Consider a closed curve C: $\theta(t) = \theta_0, \varphi(t) = t, 0 \leq t < 2\pi$, on the sphere of radius R in $\mathbf{E}^3$. Find the length of the shortest curve on the sphere which joins points A and B on the curve C.

Consider the parallel transport of an arbitrary tangent vector along the closed curve C. Show that as a result of parallel transport the vector rotates through the angle

$$\Delta\Phi = KS(D),$$

where K equals to $\frac{1}{R^2}$ is the Gaussian curvature of the sphere and $S(D)$ is the area of the part of the sphere above the curve C.

(**b**) On the sphere $x^2 + y^2 + z^2 = R^2$ in $\mathbf{E}^3$ consider points $A = (0, 0, R)$, $B = (R, 0, 0)$ and $C = (R\cos\varphi, R\sin\varphi, 0)$. Consider the isosceles triangle ABC. (Sides of this triangle are the shortest curves joining these points.) Show that:

$$KS(\triangle ABC) = \alpha + \beta + \gamma - \pi, \tag{1}$$

where K is the Gaussian curvature of the sphere, $S(\triangle ABC)$ is the area of the triangle $\triangle ABC$ and α, β, γ are angles of this triangle.

(c) Consider the points $B = (a, \sqrt{1-a^2})$, $C = (-a, \sqrt{1-a^2})$ on the Lobachevsky plane realised as half-plane. Consider vertical lines $x = \pm a$ and the half-circle $x^2 + y^2 = 1$ which pass through the points B and C.

Find the angles β, γ between these lines and the circle.

* *Show that the distance between points $B'_h = (a, h)$ and $C'_h = (-a, h)$ on vertical lines tends to zero if $h \to \infty$.*

* *Find the area $S(D)$ of the domain D delimited by the vertical lines and above half-circle and show that*

$$KS(D) = \beta + \gamma - \pi,$$

where $K = -1$ is Gaussian curvature of Lobachevsky plane. (Compare with the formula (1).)