Riemannian Geometry (31082, 41082, 61082)

2016

COURSEWORK

Starred questions are for the 15 credit version

This assignment counts for 20% of your marks.

Solutions are due by 25 April 3pm

Write solutions in the provided spaces.

STUDENTS'S NAME:

(a) You know that the Riemannian metric on the sphere of radius R in the stereographic coordinates is expressed by the formula

$$G_{\text{stereogr.}} = \frac{4R^4(du^2+dv^2)}{(R^2+u^2+v^2)^2}\,. \tag{1}$$

Give an example of a non-identity transformation of coordinates u, v such that it preserves metric (1).

* *Give an example of a non-linear transformation of coordinates u, v such that it preserves metric (1).*

(*Hint: You may find this transformation considering transformations of the sphere.*)

(**b**) Find the length of the line $v = au$ in $\mathbf{R}^2$ with respect to the metric (1).

Explain why the length of this curve does not depend on a.

(c) Consider the stereographic projection of the sphere $x^2+y^2+z^2=1$ (without the north pole) on the plane $\mathbf{R}^2$:

$$\begin{cases} u = \frac{x}{1-z} \\ v = \frac{y}{1-z} \end{cases}, \quad \begin{cases} x = \frac{2u}{u^2+v^2+1} \\ y = \frac{2v}{u^2+v^2+1} \\ z = \frac{u^2+v^2-1}{u^2+v^2+1} \end{cases}.$$

Let C be the intersection of the sphere with the plane $x+z=0$. Let C' be the image of the curve C under the stereographic projection.

Show that C' is a circle.

Find the length of the curve C' with respect to the Riemannian metric, which is induced from the metric of the sphere by the stereographic projection (this is the metric (1) in the case $R=1$).

(d) Consider the Lobachevsky plane: the upper half-plane ($y>0$) with metric $G = \frac{dx^2+dy^2}{y^2}$.

Show that translations $\begin{cases} x' = x+a \\ y' = y \end{cases}$ and homotheties $\begin{cases} x' = \lambda x \\ y' = \lambda y \end{cases}$ ($\lambda > 0$) are isometries of the Lobachevsky plane.

* Show that inversion $\begin{cases} x' = \frac{x}{x^2+y^2} \\ y' = \frac{y}{x^2+y^2} \end{cases}$ is also isometry of the Lobachevsky plane.

* Find infinitesimal isometries (Killing vector fields) corresponding to translations and homotheties.

(a) Consider a Riemannian manifold M with the metric $G = g_{ik}dx^i dx^k$. Write down the formula for the volume element on M.

Write down the volume form on the sphere of radius R in $\mathbf{E}^3$ in spherical coordinates θ, φ.

Find local coordinates u, v such that the volume element on the sphere in new coordinates equals $dudv$.

Express the Riemannian metric on the sphere in these coordinates.

* Show explicitly that the volume element is invariant under a change of coordinates.

(b) Consider the plane $\mathbf{R}^2$ with standard coordinates (x, y) equipped with Riemannian metric

$$G = (1 + x^2 + y^2)e^{-x^2-y^2}\left(dx^2 + dy^2\right) .$$

Calculate the total area of this plane.

(c) Evaluate the area of the part of the sphere of radius $R = 1$ between the planes given by equations $2x + 2y + z = 1$ and $2x + 2y + z = 2$.

(d)* Evalute the area of the interior of the cirle C' considered in the question (1**c**) with respect to the Riemannian metric $G = \frac{4(du^2+dv^2)}{(1+u^2+v^2)^2}$.

(a) Explain shortly what is meant by Christoffel symbols and write down their transformation law.

Let ∇ be an affine connection on the 2-dimensional manifold M such that in local coordinates (u,v), $\nabla_{\frac{\partial}{\partial u}}\left(u^2\frac{\partial}{\partial v}\right) = 3u\frac{\partial}{\partial v} + u\frac{\partial}{\partial u}$.

Calculate the Christoffel symbols Γ^u_{uv} and Γ^v_{uv} of this connection.

Let ∇ be an arbitrary connection on a manifold M. Show that

$$\cos F\nabla_{\mathbf{A}}(\sin F\mathbf{B}) - \sin F\nabla_{\mathbf{A}}(\cos F\mathbf{B}) = (\partial_{\mathbf{A}}F)\mathbf{B}\,,$$

where F is an arbitrary function.

* Let $\Gamma^{i(1)}_{km}$ be the Christoffel symbols of a connection $\nabla^{(1)}$ and $\Gamma^{i(2)}_{km}$ be the Christoffel symbols of a connection $\nabla^{(2)}$. Show that the linear combinations $f\Gamma^{i(1)}_{km} + g\Gamma^{i(2)}_{km}$, (where f and g are some functions) are Christoffel symbols for some connection if $f+g\equiv 1$.

* Explain why $\frac{1}{2}\Gamma^{i(1)}_{km} + \frac{1}{3}\Gamma^{i(2)}_{km}$ is not a connection.

(b) Consider a cone $\mathbf{r}(h,\varphi)$: $\begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases}$ in $\mathbf{E}^3$.

Calculate the induced connection on the cone (the connection induced by canonical flat connection in the ambient Euclidean space: $\nabla_{\mathbf{X}}\mathbf{Y} = \left(\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y}\right)_{\text{tangent}}$.)

Calculate the Riemannian metric on the cone induced by the canonical metric in ambient Euclidean space $\mathbf{E}^3$ and calculate explicitly the Levi-Civita connection of this metric using the Levi-Civita Theorem.

(c) Calculate Levi-Civita connection of the Riemannian metric on the sphere in stereographic coordinates:

$$G = \frac{4R^4(du^2+dv^2)}{(R^2+u^2+v^2)^2}$$

at the point $u=v=0$

* at an arbitrary point.

(d)* Consider a surface M in $\mathbf{E}^3$ defined by the equation $\begin{cases} x = u \\ y = v \\ z = F(u,v) \end{cases}$.

Consider a point $\mathbf{p}$ on M with coordinates $u=x_0, v=y_0$ such that (x_0,y_0) is a point of local extremum for the function F.

Calculate the Christoffel symbols of the Levi-Civita connection at the point $\mathbf{p}$.