Homework 8

1) Show that vertical lines $x = a$ are geodesics (un-parameterised) on the Lobachevsky plane [1)].

* Show that upper arcs of semicircles $(x-a)^2+y^2 = R^2, y > 0$ are (non-parametersied) geodesics.

2 Consider a vertical ray $C: x(t) = 1, y(t) = 1 + t,\ 0 \leq t < \infty$ on the Lobachevsky plane.

Find the parallel transport $\mathbf{X}(t)$ of the vector $\mathbf{X}_0 = \partial_y$ attached at the initial point $(1, 1)$ along the ray C at an arbitrary point of the ray.

Find the parallel transport $\mathbf{Y}(t)$ of the vector $\mathbf{Y}_0 = \partial_x + \partial_y$ attached at the same initial point $(1, 1)$ along the ray C at an arbitrary point of the ray. (*Exam question, 2013.*)

3 Find a parameterisation of vertical lines in the Lobachevsky plane such that they become parameterised geodesics.

4 Consider the plane $\mathbf{R}^2$ with Cartesian coordinates and with Riemannian metric

$$G = \frac{4R^2(du^2 + dv^2)}{(R^2 + u^2 + v^2)^2)}.$$

Show that all lines passing through the origin ($u = v = 0$) and only these lines are geodesics of the Levi-Civita connection of this metric.

Give examples of other geodesics.

† Find all geodesics of this metric.

(*You may use the fact that this Rimeannian manifold is isometric to the sphere without North pole.*)

5* Let $\mathbf{X}(t)$ be parallel transport of the vector $\mathbf{X}$ along the curve on the surface M embedded in $\mathbf{E}^3$, i.e. $\nabla_{\mathbf{v}}\mathbf{X} = 0$, where $\mathbf{v}$ is a velocity vector of the curve C and ∇ Levi-Civita connection of the metric induced on the surface. Compare the condition $\nabla_{\mathbf{v}}\mathbf{X} = 0$ (this is condition of parallel transport for internal observer) with the condition that for the vector $\mathbf{X}(t)$, the derivative $\frac{d\mathbf{X}(t)}{dt}$ is orthogonal to the surface (this is condition of parallel transport for external observer)[2)].

Do these two conditions coincide, i.e. do they imply the same parallel transport?

6 On the sphere $x^2 + y^2 + z^2 = R^2$ of radius R in $\mathbf{E}^3$ consider the following three closed curves.

[1)] As usual we consider here a realisation of the Lobachevsky plane (hyperbolic plane) as upper half of Euclidean plane $\{(x, y) : y > 0\}$ with the metric $G = \frac{dx^2+dy^2}{y^2}$. The line $x = 0$ is called *absolute*.

[2)] We defined parallel transport in Geometry course using this condition

a) the triangle $\triangle ABC$ with vertices at the points $A = (0,0,1)$, $B = (0,1,0)$ and $C = (1,0,0)$. The edges of triangle are geodesics.

b) the triangle $\triangle ABC$ with vertices at the points $A = (0,0,1)$, $B = (0, \cos\varphi, \sin\varphi)$ and $C = (1,0,0)$, $0 < \varphi < \frac{\pi}{2}$ The edges of triangle are geodesics.

c) the curve $\theta = \theta_0$ (line of constant latitude).

Consider the result of parallel transport of the vectors tangent to sphere over these closed curves.