Homework 1–2

1 Show that the condition of non-degeneracy for a symmetric matrix $||g_{ik}||$ follows from the condition that this matrix is positive-definite.

2 Let (u, v) be local coordinates on 2-dimensional Riemannian manifold M. Let Riemannian metric be given in these local coordinates by the matrix

$$||g_{ik}|| = \begin{pmatrix} A(u,v) & B(u,v) \\ C(u,v) & D(u,v) \end{pmatrix},$$

where $A(u,v), B(u,v), C(u,v), D(u,v)$ are smooth functions. Show that the following conditions are fulfilled:

a) $B(u,v) = C(u,v)$,

b) $A(u,v)C(u,v) - B(u,v)D(u,v) \neq 0$,

c) $A(u,v) > 0$,

d) $A(u,v)D(u,v) - B(u,v)C(u,v) > 0$.

e)† Show that conditions a), c) and d) are necessary and sufficient conditions for matrix $||g_{ik}||$ to define locally a Riemannian metric.

f^*) How conditions above will change if the manifold M is pseudo-Riemannian, but not necessarily Riemannian?

3 Write down explicit formulae expressing stereographic coordinates for n-dimensional sphere $(x^1)^2 + \ldots + (x^{n+1})^2 = 1$ via coordinates $x^1, \ldots, x^{n+1}$ and vice versa.
(For simplicity you may consider cases $n = 2, 3$.)

4 Consider the Riemannian metric on the unit circle induced by the Euclidean metric on the ambient plane.

a) Express it using polar angle as a coordinate on the circle.

b) Express the same metric using stereographic coordinate t obtained by stereographic projection of the circle on the line, passing through its centre.

5 Consider the Riemannian metric on the unit sphere induced by the Euclidean metric on the ambient 3-dimensional space.

a) Express it using spherical coordinates on the sphere.

b) Express the same metric using stereographic coordinates u, v obtained by stereographic projection of the sphere on the plane, passing through its centre.

6* Consider the n-dimensional sphere S^n of radius 1 in $(n+1)$-dimensional Euclidean space $\mathbf{E}^{n+1}$. This sphere can be defined by the equation $(x^1)^2 + \ldots + (x^{n+1})^2 = 1$ in Cartesian coordinates $x^1, \ldots, x^n, x^{n+1}$.

Consider a Riemannian metric on this sphere induced by the Euclidean metric in the ambient space.

Write down this metric in stereographic coordinates.

7 Consider the surface L which is the upper sheet of one-sheeted hyperboloid in $\mathbf{R}^3$:

$$L: \qquad z^2 - x^2 - y^2 = 1, \quad z > 0\,.$$

a) Find parametric equation of the surface L using hyperbolic functions $\cosh, \sinh$ following an analogy with spherical coordinates on the sphere.

b) Consider the stereographic projection of the surface L on the plane OXY, i.e. the central projection on the plane $z = 0$ with the centre at the point $(0, 0, -1)$.

Show that the image of projection of the surface L is the open disc $x^2 + y^2 < 1$ in the plane OXY.

8* Consider the pseudo-Euclidean metric on $\mathbf{R}^3$ given by the formula

$$ds^2 = dx^2 + dy^2 - dz^2\,. \tag{1}$$

Calculate the induced metric on the surface L considered in the Exercise 7, and show that it is a Riemannian metric (it is positive-definite).

Perform calculations in spherical-like coordinates (see Exercise 7a) above) and in stereographic coordinates (see exercise 7b) above)

Remark The surface L sometimes is called pseudo-sphere. The Riemannian metric on this surface sometimes is called Lobachevsky (hyperbolic) metric.

The surface L with this metric realises Lobachevsky (hyperbolic) geometry, where Euclid's 5-th Axiom fails. This Riemannian manifold (manifold+Riemannian metric) we call Lobachevsky (hyperbolic) plane.

In stereographic coordinates we come to realisation of Lobachevsky plane on the disc in $\mathbf{E}^2$. It is so called Poincare model of Lobachevsky geometry.

9 † Consider the metric induced on one-sheeted hyperboloid $x^2+y^2-z^2 = 1$ embedded in $\mathbf{R}^3$ with the pseudo-Euclidean metric (1). Show that this metric *is not* Riemannian one.

10* Find new coordinates x, y such that in these coordinates Lobachevsky plane (hyperbolic plane) can be considered as an upper half plane $\{x \in \mathbf{R}, y > 0\}$ in $\mathbf{E}^2$ and write down explicitly Riemannian metric in these coordinates.

Hint: *You may use complex coordinates:*

$$z = x + iy, \bar{z} = x - iy, w = u + iv, \bar{w} = u - iv$$

and find an holomorphic transformation $w = w(z)$ *of the open disc* $w\bar{w} < 1$ *onto the upper plane* $\mathbf{Im} z > 0$.

Remark Later by default we will call "Lobachevsky (hyperbolic) plane" the realisation of Lobachevsky plane as an half-upper plane in $\mathbf{E}^2$ with coordinates x, y $(y > 0)$.