Killing metric and Killing

Killing form on algebra $\mathcal{G}$ is

$$K(X,Y) = \mathrm{Tr}(ad_X ad_Y) \tag{1}$$

Killing metric on algebra $\mathcal{G}$ is a metric, i.e. symmetric non-degenerate bilinear form on algebra $\mathcal{G}$, such that algebra acts on this form by antisymmetric operators, i.e. a form (,) such that for an arbitrary vector $Z \in \mathcal{G}$

$$(ad_Z X, Y) = -(X, ad_Z Y) \tag{2}$$

Killing form obeys condition (2) and for semisimple Lie algebra a Killing metric is proportional to Killing form.

On Lie groj G left invarint vector fields define canonical flat connection, but it si not symmetric connection. What about Levi-Civita for Killing metric?

Killing metric (arbitrary) on Lie algebra $\mathcal{G}$ define (pseudo)Riemannian metric on group via left-invariant vector fields:

$$< L_X, L_Y >= (X,Y)$$

How to define connection compatible with this metric?

Answer: connection ∇ such that

$$\nabla_{L_X} L_Y = \frac{1}{2} L_[X,Y]$$

is Levi-Civita connection of Killing metric.