31 October 2018

We calculate here the free action for particle in homogeneous field:

$$L = \frac{mv^2}{2} - U = L = \frac{mv^2}{2} - mgx = , \quad H = \frac{p^2}{2m} + mgx$$

$S = S(x_0, t_0, x, t)$ obeys the Hamilton-Jacobi equation

$$\frac{S_x^2}{2m} + U = -S_t \, .$$

The Legendre transform

$$\mathcal{S}(x, E) = S(x, t) + Et \, , \quad \text{where } t\text{: } S_t + E = 0$$

obeys the equation $\frac{\mathcal{S}_x^2}{2m} + mgx = E$, i.e.

$$\mathcal{S}(x, E) = \int_{x_0}^{x} \sqrt{2m(E - U(x')} dx' = \int_{x_0}^{x} \sqrt{2m(E - mgx')} dx' =$$

(we use here the special initial condition)

$$-\frac{1}{3m^2 g} \left(2m \left(E - mgx\right)\right)^{\frac{3}{2}} \Big|_{x_0}^{x} \, .$$

Now find $S(x, t)$ which is the Legendre transform of $\mathcal{S}$:

$$S(x, t) = -\frac{1}{3m^2 g} \left(2m \left(E - mgx\right)\right)^{\frac{3}{2}} \Big|_{x_0}^{x} - Et \, ,$$

where $t = t(x, E)$ is defined by the condition $\mathcal{S}_E - t = 0$:

$$\mathcal{S}_E - t = -\frac{2m}{3m^2 g} \sqrt{2m \left(E - mgx\right)} - t$$