Action in equation

Let L be a Lagrangian.

Denote by $x^i(\tau)$ the solution of Euler-Lagrange equation such that

$$\begin{cases} x^i(0) = 0 \\ x^i(t) = q^i_{\text{fin}} \end{cases}$$

Let $\tilde{x}^i(\tau) = x^i(\tau) + \varepsilon h^i(\tau)$ be another solution of Euler Lagrange equation such that $h^i(0) = 0$. At the moment $\tau = t + \delta t$

$$\tilde{x}^i(t+\delta t) = x^i(t+\delta t) + \varepsilon h^i(t+\delta t) = x^i(t) + \dot{x}^i(t)\delta t + \varepsilon h^i(t) \Rightarrow \quad \delta q^i = \dot{x}^i(t)\delta t + \varepsilon h^i(t)\,.$$

Calculate the changing of the action:

$$S(0; t+\delta t, q+\delta q) = \int_0^{t+\delta t} L(x^i + \varepsilon h^i, \dot{x} + \varepsilon \dot{h}^i) d\tau =$$

$$\int_0^t L(x^i, \dot{x}) d\tau + \int_t^{t+\delta t} L(x^i, \dot{x}) d\tau + \varepsilon \int_0^t \left(\frac{\partial L}{\partial x^i} h^i(\tau) + \frac{\partial L}{\partial \dot{x}^i} \dot{h}^i(\tau) \right) d\tau =$$

$$S(0,0;t,q) + L(x,\dot{x})\delta t + \varepsilon \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) h^i(t) + \varepsilon \underbrace{\int \left(\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right)\right)}_{\text{vanishes}}$$

Thus we have

$$S(0; t+\delta t, q^i + \delta q^i) = S(0; t+\delta t, q^i + \dot{x}^i \delta t + \varepsilon h^i) =$$

$$S(0;t,q) + L(x,\dot{x})\delta t + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right)\left(\delta q^i - \dot{x}^i \delta t\right)$$

This implies that

$$\frac{\partial S(0;q,t)}{\partial t} = L(x,\dot{x}) - \dot{x}^i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = -H\,,$$

and

$$\frac{\partial S(0;q,t)}{\partial q^i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = p_i\,,$$

`this was waited hundred years`