

Solutions of problem 1 of Homework 6

1

Calculate the derivatives of the functions $f = x^2 + y^2$, $g = y^2 - x^2$ and $h = q \log |r| = q \log(\sqrt{x^2 + y^2})$ (q is a constant) along vector fields $\mathbf{A} = x\partial_x + y\partial_y$ and $\mathbf{B} = x\partial_y - y\partial_x$,

a) calculating directional derivatives $\partial_{\mathbf{A}}f, \partial_{\mathbf{A}}g, \partial_{\mathbf{A}}h, \partial_{\mathbf{B}}f, \partial_{\mathbf{B}}g, \partial_{\mathbf{B}}h$

b) calculating $df(\mathbf{A}), dg(\mathbf{A}), dh(\mathbf{A}), df(\mathbf{B}), dg(\mathbf{B}), dh(\mathbf{B})$.

We can do this exercise or using the formula for directional derivative or using the 1-form, differential of function: $\partial_{\mathbf{A}}f = df(\mathbf{A})$.

a) First do using directional derivatives:

$$\partial_{\mathbf{A}}f = A_x \frac{\partial f}{\partial x} + A_y \frac{\partial f}{\partial y} = x \cdot 2x + y \cdot 2y = 2(x^2 + y^2),$$

$$\partial_{\mathbf{A}}g = A_x \frac{\partial g}{\partial x} + A_y \frac{\partial g}{\partial y} = x \cdot (-2x) + y \cdot 2y = 2(y^2 - x^2),$$

$$\partial_{\mathbf{A}}h = x \frac{\partial h}{\partial x} + y \frac{\partial h}{\partial y} = \frac{x^2 q}{x^2 + y^2} + \frac{y^2 q}{x^2 + y^2} = q$$

$$\partial_{\mathbf{B}}f = B_x \frac{\partial f}{\partial x} + B_y \frac{\partial f}{\partial y} = -y \cdot 2x + x \cdot 2y = 0,$$

$$\partial_{\mathbf{B}}g = -y \frac{\partial g}{\partial x} + x \frac{\partial g}{\partial y} = -y \cdot (-2x) + x \cdot 2y = 4xy$$

$$\partial_{\mathbf{B}}h = -y \frac{\partial h}{\partial x} + x \frac{\partial h}{\partial y} = \frac{-xyq}{x^2 + y^2} + \frac{xyq}{x^2 + y^2} = 0$$

b) Now calculate using 1-form using the fact that $\partial_{\mathbf{A}}f = df(\mathbf{A})$:

$$\begin{aligned} \text{We have that } df &= d(x^2 + y^2) = 2xdx + 2ydy, \quad dg = d(y^2 - x^2) = g_x dx + g_y dy = (2ydy - 2xdx), \\ dh &= d\left(q \log \sqrt{x^2 + y^2}\right) = h_x dx + h_y dy = \frac{qxdx + qydy}{x^2 + y^2}. \end{aligned}$$

Hence

$$\partial_{\mathbf{A}}f = df(\mathbf{A}) = (2xdx + 2ydy)(x\partial_x + y\partial_y) = 2x^2 dx(\partial_x) + 2y^2 dy(\partial_y) = 2x^2 + 2y^2,$$

$$\partial_{\mathbf{A}}g = dg(\mathbf{A}) = (2ydy - 2xdx)((x\partial_x + y\partial_y)) = 2ydy(y\partial_y) - 2xdx(x\partial_x) = 2y^2 - 2x^2.$$

$$\partial_{\mathbf{A}}h = dh(\mathbf{A}) = \frac{qxdx + qydy}{x^2 + y^2} (x\partial_x + y\partial_y) = \frac{qxdx(x\partial_x) + qydy(y\partial_y)}{x^2 + y^2} = \frac{qx^2 + qy^2}{x^2 + y^2} = q$$

$$\partial_{\mathbf{B}}f = df(\mathbf{B}) = (2xdx + 2ydy)(-y\partial_x + x\partial_y) = -2xydx(\partial_x) + 2xydy(\partial_y) = 0,$$

$$\partial_{\mathbf{B}}g = dg(\mathbf{B}) = (2ydy - 2xdx)((x\partial_y - y\partial_x)) = 2ydy(x\partial_y) - 2xdx(-y\partial_x) = 2xy + 2xy = 4xy.$$

$$\partial_{\mathbf{B}}h = dh(\mathbf{B}) = \frac{qxdx + qydy}{x^2 + y^2} (-y\partial_x + x\partial_y) = \frac{qxdx(-y\partial_x) + qydy(x\partial_y)}{x^2 + y^2} = \frac{-qxy + qxy}{x^2 + y^2} = 0.$$

2

Perform the calculations of the previous exercise using polar coordinates.

For basic fields $\partial_r, \partial_\varphi$ in polar coordinates r, φ ($r = x \cos \varphi, y = r \sin \varphi$) we have that

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r} \frac{\partial}{\partial x} + \frac{\partial y}{\partial r} \frac{\partial}{\partial y} = \cos \varphi \partial_x + \sin \varphi \partial_y = \frac{x}{r} \partial_x + \frac{y}{r} \partial_y = \frac{x\partial_x + y\partial_y}{r} = \frac{\mathbf{A}}{r} \Rightarrow \mathbf{A} = r\partial_r \quad (1)$$

and

$$\frac{\partial}{\partial \varphi} = \frac{\partial x}{\partial \varphi} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \varphi} \frac{\partial}{\partial y} = -r \sin \varphi \partial_x + r \cos \varphi \partial_y = -y\partial_x + x\partial_y \Rightarrow \mathbf{B} = \partial_\varphi \quad (2)$$

We see that fields \mathbf{A}, \mathbf{B} have very simple expression in polar coordinates. Now calculations become almost immediate because in polar coordinates $f = x^2 + y^2 = r^2$, $g = y^2 - x^2 = r^2(\sin^2 \varphi - \cos^2 \varphi) = -r^2 \cos 2\varphi$ and $h = q \log r$ and

$$\partial_{\mathbf{A}}f = r\partial_r r^2 = 2r^2 = 2(x^2 + y^2),$$

$$\partial_{\mathbf{A}}g = r\partial_r(-r^2 \cos 2\varphi) = -2r^2 \cos 2\varphi = 2(y^2 - x^2), \quad \partial_{\mathbf{A}}h = r\partial_r(q \log r) = q.$$

For field \mathbf{B} we have that: $\partial_{\mathbf{B}} = \partial_\varphi$, hence

$$\partial_{\mathbf{B}}f = \partial_{\mathbf{B}}g = \partial_{\mathbf{B}}h = 0.$$

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since the functions f and h do not depend on φ . For the function $g = y^2 - x^2 = -r^2 \cos 2\varphi$ we have:

$$\partial_{\mathbf{B}}g = \partial_{\varphi}(-r^2 \cos 2\varphi) = 2r^2 \sin 2\varphi = 4r^2 \sin \varphi \cos \varphi = 4r^2 \left(\frac{y}{r}\right) \cdot \left(\frac{x}{r}\right) = 4xy.$$

3

Consider a function $f = x^4 - y^4$.

Calculate the value of 1-form $\omega = df$ on the vector field $\mathbf{B} = x\partial_y - y\partial_x$.

Express this 1-form ω in polar coordinates r, φ ($x = r \cos \varphi, y = r \sin \varphi$).

$$\omega = df = 4x^3 dx - 4y^3 dy, \omega(\mathbf{B}) = 4(x^3 dx - y^3 dy)(x\partial_y - y\partial_x) = 4x^3 dx(-y\partial_y) - 4y^3 dy(x\partial_y) = -4x^3 y - 4y^3 x = -4xy(x^2 + y^2) \text{ since } dx(\partial_x) = dy(\partial_y) = 1 \text{ and } dx(\partial_y) = dy(\partial_x) = 0.$$

One may express differential 1-form $\omega = df = 4x^3 dx - 4y^3 dy$ straightforwardly in polar coordinates. Instead using “brute force” express function f in polar coordinates then calculate $\omega = df$:

$$f = x^4 - y^4 = (x^2 + y^2)(x^2 - y^2) = r^2(r^2 \cos^2 \varphi - r^2 \sin^2 \varphi) = r^4 \cos^2 \varphi - r^4 \sin^2 \varphi = r^4 \cos 2\varphi,$$

$$\text{hence } \omega = df = d(r^4 \cos 2\varphi) = 4r^3 \cos 2\varphi dr - 2r^4 \sin 2\varphi d\varphi.$$

The operation of taking differential can be performed in an arbitrary coordinates in a same way as in Cartesian coordinates.

4

Calculate the value of 1-form $\omega = xdy - ydx$ on the vector fields $\mathbf{A} = r\partial_r$ and $\mathbf{B} = \partial_{\varphi}$. Perform calculations in Cartesian and polar coordinates).

We know that $r\partial_r = x\partial_x + y\partial_y$ and $\partial_{\varphi} = x\partial_y - y\partial_x$ (see formulae (1,2) in the solution of exercise 2). Hence in Cartesian coordinates $\mathbf{A} = x\partial_x + y\partial_y$ and $\mathbf{B} = x\partial_y - y\partial_x$

$$\omega(\mathbf{A}) = (xdy - ydx)(x\partial_x + y\partial_y) = -xydx(\partial_x) + xydy(\partial_y) = -xy + xy = 0,$$

$$\omega(\mathbf{B}) = (xdy - ydx)(x\partial_y - y\partial_x) = x^2 dx(\partial_x) + y^2 dy(\partial_y) = x^2 + y^2.$$

Now perform calculations in polar coordinates:

$$\omega = xdy - ydx = r \cos \varphi d(r \sin \varphi) - r \sin \varphi d(r \cos \varphi) = r \cos \varphi (\sin \varphi dr + r \cos \varphi d\varphi) - r \sin \varphi (\cos \varphi dr - r \sin \varphi d\varphi) = r^2 (\cos^2 \varphi + \sin^2 \varphi) d\varphi = r^2 d\varphi.$$

Hence in polar coordinates

$$\omega(\mathbf{A}) = r^2 d\varphi(\partial_r) = 0, \quad \omega(\mathbf{B}) = r^2 d\varphi(\partial_{\varphi}) = r^2.$$

$$(dr(\partial_r) = d\varphi(\partial_{\varphi}) = 1, dr(\partial_{\varphi}) = d\varphi(\partial_r) = 0).$$

We see that calculations are much more transparent in polar coordinates.

5

Let f be a function on \mathbf{E}^2 given by $f(r, \varphi) = r^2 \sin 2\varphi$, where r, φ are polar coordinates in \mathbf{E}^2 .

Calculate the 1-form $\omega = df$.

Calculate the value of the 1-form $\omega = df$ on the vector field $\mathbf{X} = r^2 \partial_r + r \partial_{\varphi}$.

Express the 1-form ω in Cartesian coordinates x, y .

$$\omega = 2r \sin 2\varphi dr + 2r^2 \cos 2\varphi d\varphi.$$

The value of the form $\omega = df$ on the vector field $\mathbf{X} = r^2 \partial_r + r \partial_{\varphi}$ is equal to

$$\omega(\mathbf{A}) = (2r \sin 2\varphi dr + 2r^2 \cos 2\varphi d\varphi) (r^2 \partial_r + r \partial_{\varphi}) = 2r^3 \sin 2\varphi dr(\partial_r) + 2r^3 \cos 2\varphi d\varphi(\partial_{\varphi}) = 2r^3 (\sin 2\varphi + \cos 2\varphi). \blacksquare$$

because $dr(\partial_r) = 1, dr(\partial_{\varphi}) = 0$ and $d\varphi(\partial_r) = 0, d\varphi(\partial_{\varphi}) = 1$.

Another solution

$$\omega(\mathbf{X}) = df(\mathbf{X}) = \partial_{\mathbf{X}} f = \left(r^2 \frac{\partial}{\partial r} + r \frac{\partial}{\partial \varphi} \right) (r^2 \sin 2\varphi) = r^2 \cdot 2r \sin 2\varphi + r \cdot 2r^2 \cos 2\varphi = 2r^3 (\sin 2\varphi + \cos 2\varphi).$$

To express the form ω in Cartesian coordinates it is easier to express f in Cartesian coordinates and then to calculate $\omega = df$:

$$f = r^2 \sin 2\varphi = (x^2 + y^2)(2 \cos \varphi \sin \varphi) = (x^2 + y^2)2 \left(\frac{x}{r} \right) \cdot \left(\frac{y}{r} \right) = 2xy.$$

Hence $\omega = d(2xy) = 2ydx + 2xdy$.

6

Consider 1-forms $\omega = df$ and $\sigma = dg$ such that

$$f(x, y) + ig(x, y) = (x + iy)^3.$$

Find the values of these 1-forms on vector field $\mathbf{Y} = r\partial_r + \partial_\varphi$.

One may try to use the fact that

$$f + ig = (x + iy)^3 = x^3 - 3xy^2 + i(3xy^2 - y^3) \Rightarrow f = x^3 - 3xy^2, g = 3xy^2 - y^3$$

and perform calculations. We come to hard calculations.

There is more nice solution using complex variables. If $z = x + iy = re^{i\varphi}$, then

$$f + ig = z^3 = r^3 e^{3i\varphi} = r^3 \cos 3\varphi + ir^3 \sin 3\varphi \Rightarrow f = r^3 \cos 3\varphi, g = r^3 \sin 3\varphi.$$

Hence

$$\omega(\mathbf{Y}) = df(\mathbf{Y}) = \partial_{\mathbf{Y}}(f) = \left(r \frac{\partial}{\partial r} + \frac{\partial}{\partial \varphi} \right) r^3 \cos 3\varphi = 3r^3 \cos 3\varphi - 3r^3 \sin 3\varphi,$$

and

$$\sigma(\mathbf{Y}) = dg(\mathbf{Y}) = \partial_{\mathbf{Y}}(g) = \left(r \frac{\partial}{\partial r} + \frac{\partial}{\partial \varphi} \right) r^3 \sin 3\varphi = 3r^3 \sin 3\varphi + 3r^3 \cos 3\varphi.$$

7

Calculate the integrals of the form $\omega = \sin y dx$ over the following three curves. Compare answers.

$$C_1: \mathbf{r}(t) \begin{cases} x = 2t^2 - 1 \\ y = t \end{cases}, \quad 0 < t < 1, \quad C_2: \mathbf{r}(t) \begin{cases} x = 8t^2 - 1 \\ y = 2t \end{cases}, \quad 0 < t < 1/2,$$

$$C_3: \mathbf{r}(t) \begin{cases} x = \cos 2t \\ y = \cos t \end{cases}, \quad 0 < t < \frac{\pi}{2}$$

For any curve $\mathbf{r}(t)$, $t_1 < t < t_2$

$$\int_C \omega = \int_C \sin y dx = \int_C \sin y dx(\mathbf{v}) = \int_{t_1}^{t_2} \sin y(t) \frac{dx(t)}{dt} dt$$

where $\mathbf{v} = (x_t, y_t)$.

For the first curve $x_t = 4t$ and

$$\int_{C_1} \omega = \int_0^1 4t \sin t dt = 4(-t \cos t + \sin t) \Big|_0^1 = -4 \cos 1 + 4 \sin 1$$

For the second curve $x_t = 16t$ and

$$\int_{C_2} \omega = \int_0^{1/2} 16t \sin 2t dt = 4(-2t \cos 2t + \sin 2t) \Big|_0^{1/2} = -4 \cos 1 + 4 \sin 1$$

Answer is the same. Non-surprising. The second curve is reparameterised first curve ($t \mapsto 2t$) and reparameterisation preserves the orientation.

For the third curve $x_t = -2 \sin 2t$ and

$$\int_{C_3} w = \int_0^{\pi/2} (-2 \sin 2t) \sin(\cos t) dt = -4(\cos t \cos(\cos t) - \sin(\cos t)) \Big|_0^{\pi/2} = 4 \cos 1 - 4 \sin 1$$

Answer is the same up to a sign: This curve is reparameterised first curve ($t \mapsto \cos t$) and reparameterisation changes the orientation, because $(\cos t)' = -\sin t < 0$ on the interval $(0, \pi/2)$.

Resumé: In these three examples an integral over the same (non-parameterised) curve was considered. All the answers are the same up to a sign. Sign changes if reparameterisation changes an orientation.

8

Calculate the integrals of the form $\omega = xdy - ydx$ over the following three curves. Compare answers.

$$C_1: \mathbf{r}(t) \begin{cases} x = R \cos t \\ y = R \sin t \end{cases}, \quad 0 < t < \pi, \quad C_2: \mathbf{r}(t) \begin{cases} x = R \cos 4t \\ y = R \sin 4t \end{cases}, \quad 0 < t < \frac{\pi}{4}$$

$$\text{and } C_3: \mathbf{r}(t) \begin{cases} x = Rt \\ y = R\sqrt{1-t^2} \end{cases}, \quad -1 \leq t \leq 1.$$

In the same way like for the previous integral

$$\int_C \omega = \int_{t_1}^{t_2} \omega(\mathbf{v}(t)) dt = \int_{t_1}^{t_2} (xdy - ydx)(x_t \partial_x + y_t \partial_y) dt = \int_{t_1}^{t_2} (-y(t)x_t(t) + x(t)y_t(t)) dt,$$

where $\mathbf{v} = (x_t, y_t)$ is velocity vector: $dx(\partial_x) = dy(\partial_y) = 1$, $dx(\partial_y) = dy(\partial_x) = 0$.

For the first curve C_1 we have $\mathbf{v}(t) = (-R \sin t, R \cos t)$ and $\int_{C_1} \omega = \int_0^\pi (xdy - ydx)(-R \sin t \partial_x + R \cos t \partial_y) =$

$$\int_0^\pi (R \cos t dy - R \sin t dx)(-R \sin t \partial_x + R \cos t \partial_y) = \int_0^\pi (R^2 \cos^2 t + R^2 \sin^2 t) dt = \int_0^\pi R^2 \cdot dt = \pi R^2.$$

For the second curve C_2 we have $\mathbf{v}(t) = (-4R \sin 4t, 4R \cos 4t)$ and $\int_{C_2} \omega = \int_0^{\pi/4} (xdy - ydx)(-4R \sin 4t \partial_x + 4R \cos 4t \partial_y) =$

$$\int_0^{\pi/4} (R \cos 4t dy - R \sin 4t dx)(-4R \sin 4t \partial_x + 4R \cos 4t \partial_y) = \int_0^{\pi/4} (4R^2 \cos^2 4t + 4R^2 \sin^2 4t) dt = \int_0^{\pi/4} 4R^2 \cdot dt = \pi R^2.$$

Answer is the same. The second curve is reparameterised first curve ($t \mapsto 4t$) and reparameterisation preserves the orientation: $(4t)' = 4 > 0$.

For the third curve C_3 we have $\mathbf{v}(t) = \left(-R, -\frac{Rt}{\sqrt{1-t^2}}\right)$ and $\omega(\mathbf{v}(t)) = (xdy - ydx)(v_x \partial_x + v_y \partial_y) =$

$$= \left(Rt dy - R\sqrt{1-t^2} dx\right) \left(R \partial_x - \frac{Rt}{\sqrt{1-t^2}} \partial_y\right) = -R^2 \sqrt{1-t^2} - \frac{R^2 t^2}{\sqrt{1-t^2}} = -\frac{R^2}{1-t^2}.$$

Hence

$$\int_{C_3} \omega = \int_0^1 \omega(\mathbf{v}(t)) dt = \int_0^1 \left(-\frac{R^2}{\sqrt{1-t^2}}\right) dt = -R^2 \int_0^1 \frac{dt}{\sqrt{1-t^2}} = -\pi R^2.$$

Answer is the same up to a sign: This curve is reparameterised first curve. If we put $t = \cos \tau$ then third curve C_3 will transform to the first curve C_1 . This reparameterisation changes the orientation, because $(\cos t)' = -\sin t < 0$ on the interval $(0, \pi/2)$.

Resumé: In these three examples was considered an integral over the same (non-parameterised) half-circle. All the answers are the same up to a sign. Sign changes if reparameterisation changes an orientation.