

Homework 6

1 Calculate the derivatives of the functions $f = x^2 + y^2$, $g = y^2 - x^2$ and $h = q \log |r| = q \log \left(\sqrt{x^2 + y^2} \right)$ (q is a constant) along vector fields $\mathbf{A} = x\partial_x + y\partial_y$ and $\mathbf{B} = x\partial_y - y\partial_x$

a) calculating directional derivatives $\partial_{\mathbf{A}}f, \partial_{\mathbf{A}}g, \partial_{\mathbf{A}}h, \partial_{\mathbf{B}}f, \partial_{\mathbf{B}}g, \partial_{\mathbf{B}}h$

b) calculating $df(\mathbf{A}), dg(\mathbf{A}), dh(\mathbf{A}), df(\mathbf{B}), dg(\mathbf{B}), dh(\mathbf{B})$.

2 Perform the calculations of the previous exercise in polar coordinates.

3 Calculate the integrals of the form $\omega = \sin y dx$ over the following three curves.

Compare answers.

$$C_1: \mathbf{r}(t) \begin{cases} x = 2t^2 - 1 \\ y = t \end{cases}, \quad 0 < t < 1, \quad C_2: \mathbf{r}(t) \begin{cases} x = 8t^2 - 1 \\ y = 2t \end{cases}, \quad 0 < t < 1/2,$$

$$C_3: \mathbf{r}(t) \begin{cases} x = \cos 2t \\ y = \cos t \end{cases}, \quad 0 < t < \frac{\pi}{2}$$

4 Calculate the integral of the form $\omega = e^{-y}dx + \sin x dy$ over the segment of straight line which connects the points $A = (1, 1)$, $B = (2, 3)$. At what extent an answer depends on a chosen parameterisation?

5 Calculate the integral of the form $\omega = x dy$ over the following curves

a) upper arc of the unit circle starting at the point $A = (1, 0)$ and ending at the point $(0, 1)$.

b) closed curve $x^2 + y^2 = 2x$

c) arc of the ellipse $x^2 + y^2/9 = 1$ defined by the condition $y \geq 0$.

Exact forms

6 Calculate the integral $\int_C \omega$ where $\omega = x dx + y dy$ and C is

a) the straight line segment $x = t, y = 1 - t, 0 \leq t \leq 1$

b) the segment of parabola $x = t, y = 1 - t^n, 0 \leq t \leq 1, n = 2, 3, 4, \dots$

c) for **an arbitrary** curve starting at the point $(0, 1)$ and ending at the point $((1, 0))$.

7 Show that the form 1-form $\omega = 3x^2 y dx + x^3 dy$ is an exact 1-form.

Calculate integral of this form over the curves considered in exercise 5).

8. Consider 1-forms

a) $x dx$, b) $x dy$ c) $x dx + y dy$, d) $x dy + y dx$, e) $x dy - y dx$

f) $x^4 dy + 4x^3 y dx$, g) $x dy + y dx + dz$, h) $x dy - y dx + dz$.

a) Show that 1-forms a), c), d), f) and g) are exact forms

b) Why 1-forms b), e) and h) are not exact?

All the exercises below are not compulsory

9[†] Consider one-form

$$\omega = \frac{xdy - ydx}{x^2 + y^2} \quad (1)$$

This form is defined in $\mathbf{E}^2 \setminus 0$.

Calculate differential of this form.

Write down this form in polar coordinates

Find a function f such that $\omega = df$.

Is this function defined in the same domain as ω ?

10[†] Calculate the integral of the form $\omega = \frac{xdy - ydx}{x^2 + y^2}$ over the curves

a) circle $x^2 + y^2 = 1$

b) circle $(x - 3)^2 + y^2 = 1$

c) ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$

11[†] What values can take the integral $\int_C \omega$ if C is an arbitrary curve starting at the point $(0, 1)$ and ending at the point $((1, 0)$ and $\omega = \frac{xdy - ydx}{x^2 + y^2}$.

12[†] Let $\omega = a(x, y)dx + b(x, y)dy$ be a closed form in \mathbf{E}^2 , $d\omega = 0$.

Consider the function

$$f(x, y) = x \int_0^1 a(tx, ty)dt + y \int_0^1 b(tx, ty)dt \quad (2)$$

Show that

$$\omega = df.$$

This proves that an arbitrary closed form in \mathbf{E}^2 is an exact form.

Why we cannot apply the formula (2) to the form ω defined by the expression (1)?