Stirling+Fedoruk

I deduce standard Stirling formula using Ted Vornov's comments on Fedoruk paper

This is standard that $n! = \int_0^\infty t^n e^{-t}dt = \int_0^\infty e^{-t+n\log t}dt =$

$$\int_{-1}^{\infty} e^{-n(1+x)+n\log(n(1+x))} n dx = e^{-n} n^{n+1} \int_{-1}^{\infty} e^{-n(\log(1+x)-x)}dx =$$

$$n\left(\frac{n}{e}\right)^n \int_{-1}^{\infty} e^{-n\left(\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}+\dots\right)}dx = n\left(\frac{n}{e}\right)^n \int e^{-n\frac{x^2}{2}} e^{n\left(-\frac{x^3}{6}+\frac{x^4}{24}+\dots\right)}dx\,. \tag{2}$$

Now we use that $\sqrt{2\pi n}e^{\frac{-nx^2}{2}} = \int e^{\frac{-k^2}{2n}} e^{ikx}dk$ and

$$\int e^{-n\frac{x^2}{2}}\varphi(x)dx = \frac{1}{\sqrt{2\pi n}}\int e^{\frac{-k^2}{2n}}e^{ikx}dkdx \int \varphi(p)e^{ipx}dp = \frac{1}{\sqrt{2\pi n}}\int e^{\frac{-k^2}{2n}}e^{ikx}\varphi(p)e^{ipx}dpdkdx =$$

$$\frac{2\pi}{\sqrt{2\pi n}}\int e^{\frac{-k^2}{2n}}\delta(k+p)\varphi(p)dpdk = \int e^{\frac{-k^2}{2n}}\varphi(k)dk = \frac{2\pi}{\sqrt{2\pi n}} e^{-\frac{1}{2n}\left(\frac{d}{dx}\right)^2}\varphi(x)\big|_{x=0}$$

Apply this to eqution (2) *:

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\int_{-1}^{\infty} e^{-n\left(\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}+\dots\right)}dx = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\int e^{-n\frac{x^2}{2}} e^{n\left(-\frac{x^3}{6}+\frac{x^4}{24}+\dots\right)}dx =$$

$$\left(\frac{n}{e}\right)^n \sqrt{2\pi n}\int e^{-n\frac{x^2}{2}} e^{n\left(-\frac{x^3}{6}+\frac{x^4}{24}+\dots\right)} = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\left\{e^{-\frac{1}{2n}\left(\frac{1}{i}\frac{d}{dx}\right)^2}\left[e^{n\left(-\frac{x^3}{6}+\frac{x^4}{24}+\dots\right)}\right]\right\}_{x=0}.$$

... and here the mystery began: since the expression $\left[e^{n\left(-\frac{x^3}{6}+\frac{x^4}{24}+\dots\right)}\right]$ possesses only the terms of the order ≥ 3 in exponent the last integral possess only terms which are proportional to $\frac{1}{n}$!

$$\int e^{-\frac{1}{n}\left(\frac{d}{dx}\right)^2}\left[e^{n\left(-\frac{x^3}{6}+\frac{x^4}{24}+\dots\right)}\right]dx = 1 + \frac{1}{12n} + \frac{1}{288n^2} + \dots$$

Try to show it at least partially:

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\left\{e^{-\frac{1}{2n}\left(\frac{1}{i}\frac{d}{dx}\right)^2}\left[e^{n\left(-\frac{x^3}{6}+\frac{x^4}{24}+\dots\right)}\right]\right\}_{x=0} =$$

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* In fact here we used the following identity (Fedoruk)

$$\int f(x_0-x)u(x)dx = \int \tilde{f}(k)e^{ik(x_0-x)}\tilde{u}(p)e^{ipx}dxdkdp = \int \tilde{f}(k)e^{ikx_0}\tilde{u}(p)e^{i(p-k)x}dxdkdp$$

$$\int \tilde{f}(k)e^{ikx_0}\tilde{u}(p)\delta(p-k)dkdp = \int \tilde{f}(k)e^{ikx_0}\tilde{u}(k)dk = \int f\left(\frac{1}{i}\frac{d}{dx}\right)u(x)\big|_{x=0}.$$

$$\left(\frac{n}{e}\right)^n \sqrt{2\pi n}\left[1-\frac{1}{2n}\left(\frac{1}{i}\frac{d}{dx}\right)^2+\frac{1}{2}\frac{1}{4n^2}\left(\frac{1}{i}\frac{d}{dx}\right)^4-\frac{1}{6}\frac{1}{8n^3}\left(\frac{1}{i}\frac{d}{dx}\right)^6+\frac{1}{24}\frac{1}{16n^4}\left(\frac{1}{i}\frac{d}{dx}\right)^8+\dots\right] \text{ acting}$$

$$\left[1+n\left(\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\frac{x^6}{6}+\frac{x^7}{7}-\frac{x^8}{8}+\dots\right)+\frac{1}{2}n^2\left(\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}+\dots\right)^2+\dots\right]=$$

$$\left(\frac{n}{e}\right)^n \sqrt{2\pi n}\left[1+\frac{1}{2n}\frac{d^2}{dx^2}+\frac{1}{8n^2}\frac{d^4}{dx^4}+\frac{1}{48n^3}\frac{d^6}{dx^6}+\frac{1}{384n^4}\frac{d^8}{dx^8}+\dots\right] \text{ acting on}$$

$$\left[1+n\left(\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\frac{x^6}{6}+\frac{x^7}{7}-\frac{x^8}{8}+\dots\right)+\frac{1}{2}n^2\left(\frac{x^6}{9}-\frac{x^7}{6}+\frac{x^8}{16}+\frac{2x^8}{15}+\dots\right)^2+\dots\right]=$$

Contribution is gven by the terms of order $4, 6$ and 8. More in detail: consider the action of the terms $\left[\frac{1}{2n}\left(\frac{1}{i}\frac{d}{dx}\right)^2\right]^\lambda$ which act on the monoms

$$F_{p_1}F_{p_2}\dots F_{p_r},$$

where every F_{p_k} is a monom of the order p_k, $(p_i \geq 3)$ which belong to the term

$$n\left(\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\frac{x^6}{6}+\dots\right).$$

The condition that the action of the operator $\left[\frac{1}{2n}\left(\frac{1}{i}\frac{d}{dx}\right)^2\right]^\lambda$ on the monom $F_{p_1}F_{p_2}\dots F_{p_r}$, is not zero is the following:

$$2\lambda = p_1 + p_2 + \dots + p_r\,,$$

and the power of n corresponding to this monom is equal to

$$\left(\frac{1}{n}\right)^\lambda \cdot n^r = n^{r-\lambda}\,.$$

Use this formula.

First of all note that all $p_i \geq 3$, hence $2\lambda \geq 3r$, hence

$$r-\lambda \leq r-\frac{3r}{2} < 0\,.$$

Thus we have proved that in the expansion of $n!$ the contribution is given by negative powrts of n.

Now calculate the contribution of power $1n^k$ for $k = 1, 2, 3 \dots$.

We have that

$$\begin{cases} 2\lambda = p_1 + \dots + p_r \\ r-\lambda = -k \end{cases}, \quad (p_i \geq 3)\,.$$

Hence

$$\begin{cases} 2\lambda = p_1 + \ldots + p_r \geq 3r \\ \lambda = r + k \end{cases} \Rightarrow 2r + 2k \geq 3r, \text{ i.e. } r \leq 2k$$

Thus we see that contribution to terms of order $\frac{1}{n^k}$ is given by action of $\exp -\frac{1}{n}\left(\frac{1}{i}\frac{d}{dx}\right)^2$ on terms which possess not more than $2k$ monoms

I Calculate contribution to $\frac{1}{n}$, $k = 1$:

$$\begin{cases} 2\lambda = p_1 + \ldots + p_r \geq 3r \\ \lambda = r + 1 \end{cases} \Rightarrow r = 1, 2\,.$$

$$a) r = 1\,, \begin{cases} 2\lambda = p_1 \\ \lambda = r + 1 = 2 \end{cases}, p_1 = 4\,, \quad b) r = 2\,, \begin{cases} 2\lambda = p_1 + p_2 \geq 3r \\ \lambda = r + 1 = 3 \end{cases}, p_1 = p_2 = 3\,.$$

I Calculate contribution to $\frac{1}{n^2}$, $k = 2$:

$$\begin{cases} 2\lambda = p_1 + \ldots + p_r \geq 3r \\ \lambda = r + 2 \end{cases} \Rightarrow r = 1, 2, 3, 4\,.$$

$$a) r = 1\,, \begin{cases} 2\lambda = p_1 \\ \lambda = r + 2 = 3 \end{cases}, p_1 = 6\,,$$

$$b) r = 2\,, \begin{cases} 2\lambda = p_1 + p_2 \geq 3r \\ \lambda = r + 2 = 4 \end{cases}, 2\lambda = 8,\ p_1 = 3, p_2 = 5 \text{ or } p_1 = p_2 = 4\,,$$

$$c) r = 3\,, \begin{cases} 2\lambda = p_1 + p_2 + p_3 \geq 3r \\ \lambda = r + 2 = 5 \end{cases}, 2\lambda = 10\,, p_1 = p_2 = 3, p_3 = 4\,,$$

$$d) r = 4\,, \begin{cases} 2\lambda = p_1 + p_2 + p_3 + p_4 \\ \lambda = r + 2 = 6 \end{cases}, 2\lambda = 12\,, p_1 = p_2 = p_3 = p_4 = 3\,,$$

On the base of these considerations calculate $n!$ up to the terms $\frac{1}{n}$. We have according previous considerations that

$$n! = \exp\left(-\frac{1}{2n}\left(\frac{1}{i}\frac{d}{dx}\right)^2\right)\exp\left(n\left(\frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \ldots\right)\right)_{x=0} =$$

$$\left(\frac{n}{e}\right)^n \sqrt{2\pi n}\left[1 + \frac{1}{8n^2}\frac{d^4}{dx^4}\left(-n\frac{x^4}{4}\right) + \frac{1}{48n^3}\frac{d^6}{dx^6}\left(+\frac{1}{2}n^2\frac{x^6}{9}\right) + \ldots\right] =$$