Homework 7

1 Find geodesics on sphere and cylinder

a) using straightforwardly equations for geodesics, or using the fact that for geodesic, acceleration is orthogonal to the surface.

b *) using the fact that geodesic is shortest.

2 Consider a sphere $x^2 + y^2 + z^2 = 1$ in $\mathbf{E}^3$ and the curve C which is the intersections of this sphere with plane $y = 0$.

Consider also in $\mathbf{E}^3$ a vector $\mathbf{X} = \frac{\partial}{\partial z} - \sqrt{3}\frac{\partial}{\partial x}$ attached at the point $\mathbf{p}$: $\left(x = \frac{1}{2}, y = 0, z = \frac{\sqrt{3}}{2}\right)$ and the vector $\mathbf{Y} = \frac{\partial}{\partial y}$ attached at the same point $\mathbf{p}$.

Show that vectors $\mathbf{X}$ and $\mathbf{Y}$ are tangent to the sphere and express these vector in spherical coordinates.

Describe parallel transport of vectors $\mathbf{X}, \mathbf{Y}$ along the curve C.

3 a) Show that vertical lines $x = a$ are geodesics (un-parameterised) on the Lobachevsky plane [1)].

4 Consider a vertical ray $C\colon x(t) = x_0, y(t) = y_0 + t$, $0 \le t < \infty$, $(y_0 > 0)$ on the Lobachevsky plane. Find the parallel transport $\mathbf{X}(t)$ of the vector $\mathbf{X}_0 = \partial_y$ attached at the initial point (x_0, y_0) along the ray C at an arbitrary point of the ray.

5 Find a parameterisation of vertical lines in the Lobachevsky plane such that they become parameterised geodesics.

6 Show that the following transformations are isometries of Lobachevsky plane (i.e. they do not change the metric)

a) horizontal translation $\mathbf{r} \to \mathbf{r} + \mathbf{a}$ where $\mathbf{a} = (a, 0)$,

b) homothety: $\mathbf{r} \to \lambda\mathbf{r}$ $(\lambda > 0)$,

* c) inversion with the centre at the points of the absolute (the line $x = 0$):

$$\mathbf{r} \to \mathbf{a} + \frac{\mathbf{r} - \mathbf{a}}{|\mathbf{r} - \mathbf{a}|^2} \text{ where } \mathbf{a} = (a, 0): \quad \begin{cases} x' = a + \frac{x-a}{(x-a)^2+y^2} \\ y' = \frac{y}{(x-a)^2+y^2} \end{cases} .$$

7* Show that upper arcs of semicircles $(x-a)^2+y^2 = R^2, y > 0$ are (non-parametersied) geodesics.

8* Let $\mathbf{X}(t)$ be parallel transport of the vector $\mathbf{X}$ along the curve on the surface M embedded in $\mathbf{E}^3$, i.e. $\nabla_{\mathbf{v}}\mathbf{X} = 0$, where $\mathbf{v}$ is a velocity vector of the curve C and ∇ Levi-Civita connection of the metric induced on the surface. Compare the condition $\nabla_{\mathbf{v}}\mathbf{X} = 0$

1) As usual we consider here the realisation of Lobachevsky plane (hyperbolic plane) as upper half of Euclidean plane $\{(x, y)\colon y > 0\}$ with the metric $G = \frac{dx^2+dy^2}{y^2}$. The line $x = 0$ is called *absolute*.

(this is condition of parallel transport for internal observer) with the condition that for the vector $\mathbf{X}(t)$, the derivative $\frac{d\mathbf{X}(t)}{dt}$ is orthogonal to the surface (this is condition of parallel transport for external observer)[2].

Do these two conditions coincide, i.e. do they imply the same parallel transport?

[2] We defined parallel transport in Geometry course using this condition