Homework 1

1

a) Show that $(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^2 + x^3y^3$ defines a scalar product in $\mathbf{R}^3$.

b) Show that $\langle \mathbf{x}, \mathbf{y} \rangle = x^1y^1 + x^2y^2$ *does not define a scalar product* in $\mathbf{R}^3$.

c) Show that $(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^2 - x^3y^3$ *does not define a scalar product* in $\mathbf{R}^3$.

d) Show that $(\mathbf{x}, \mathbf{y}) = x^1y^1 + 3x^2y^2 + 5x^3y^3$ defines a scalar product in $\mathbf{R}^3$.

e) Show that $(\mathbf{x}, \mathbf{y}) = x^1y^2 + x^2y^1 + x^3y^3$ *does not define a scalar product* in $\mathbf{R}^3$.

f†) Find necessary and sufficient conditions for entries a, b, c of symmetrical matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ such that the formula

$$(\mathbf{x}, \mathbf{y}) = (\, x^1, x^2 \,) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = ax^1y^1 + b(x^1y^2 + x^2y^1) + cx^2y^2$$

defines a scalar product in $\mathbf{R}^2$.

2 a) Let $\mathbf{e}, \mathbf{f}$ and $\mathbf{g}$ be three vectors in 3-dimensional Euclidean space $\mathbf{E}^3$ such that all these vectors have unit length and they are pairwise orthogonal. Show explicitly that the ordered set of these vectors $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ is a basis.

b) Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be three vectors in 3-dimensional Euclidean space $\mathbf{E}^3$ such that vectors $\mathbf{a}$ and $\mathbf{b}$ have unit length, and are orthogonal to each other and vector $\mathbf{c}$ has length $\sqrt{3}$ and it forms an angle $\varphi = \arccos \frac{1}{\sqrt{3}}$ with vectors $\mathbf{a}$ and $\mathbf{b}$.

Show that the ordered set $\{\mathbf{a}, \mathbf{b}, \mathbf{c} - \mathbf{a} - \mathbf{b}\}$ of vectors is an orthonormal basis in $\mathbf{E}^3$.

3 a) Show explicitly that matrix $A_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$ is an orthogonal matrix.

b) Show explicitly that under the transformation $(\mathbf{e}'_1, \mathbf{e}'_2) = (\mathbf{e}_1, \mathbf{e}_2)\, A_\varphi$ an orthonormal basis transforms to an orthonormal one.

c) Show that for orthogonal matrix A_φ defined above the following relations are satisfied:

$$A_\varphi^{-1} = A_\varphi^{T} = A_{-\varphi}\,, \qquad A_\varphi \cdot A_\theta = A_{\varphi+\theta}\ .$$

4 Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal basis of Euclidean space $\mathbf{E}^3$. Consider the ordered set of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ which is expressed via basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as in the exercise 8 of the Homework 1:

a) $\mathbf{e}'_1 = \mathbf{e}_2$, $\mathbf{e}'_2 = \mathbf{e}_1$, $\mathbf{e}'_3 = \mathbf{e}_3$;

b) $\mathbf{e}'_1 = \mathbf{e}_1$, $\mathbf{e}'_2 = \mathbf{e}_1 + 3\mathbf{e}_3$, $\mathbf{e}'_3 = \mathbf{e}_3$;

c) $\mathbf{e}'_1 = \mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{e}'_2 = 3\mathbf{e}_1 - 3\mathbf{e}_2$, $\mathbf{e}'_3 = \mathbf{e}_3$;

d) $\mathbf{e}'_1 = \mathbf{e}_2$, $\mathbf{e}'_2 = \mathbf{e}_1$, $\mathbf{e}'_3 = \mathbf{e}_1 + \mathbf{e}_2 + \lambda\mathbf{e}_3$ (where λ is an arbitrary coefficient)?

Write down explicitly transition matrix which transforms the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to the ordered set of the vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$. What is the rank of this matrix? Is this matrix orthogonal?

Find out is the ordered set of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ a basis in $\mathbf{E}^3$. Is this basis an orthonormal basis of $\mathbf{E}^3$?

(you have to consider all cases a),b) c) and d)).

5† Prove the Cauchy–Bunyakovsky–Schwarz inequality

$$(\mathbf{x}, \mathbf{y})^2 \le (\mathbf{x}, \mathbf{x})(\mathbf{y}, \mathbf{y})\,,$$

where $\mathbf{x}, \mathbf{y}$ are arbitrary two vectors and $(\ ,\ )$ is a scalar product in Euclidean space.

Hint: For any two given vectors $\mathbf{x}, \mathbf{y}$ *consider the quadratic polynomial* $At^2 + 2Bt + C$ *where* $A = (\mathbf{x}, \mathbf{x})$, $B = (\mathbf{x}, \mathbf{y})$, $C = (\mathbf{y}, \mathbf{y})$. *Show that this polynomial has at most one real root and consider its discriminant.*