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Chebyshef polynomials

I cannot avoid temptation to explain this

Consider polynomials

$$P_n(x) = \cos n \arccos x$$

Fix n.

Lemma Let $P(x)$ be an arbitrary polynomial such that a polynomial $P(x) - T_n(x)$ is a polynomial of order $\leq n-1$. Then there exists $x_0 \in [-1,1]$ such that

$$P(x_0) > 1\,,$$

i.e. every polynomial which has the same leading term as polynomial $P_n(x)$ has norm greater than 1.

We suppose that

$$||P|| = \max_{x\in[-1,1]} |P(x)|\,.$$

This lemma implies that Chebysev polynomials have minimum norm: i.e. in the space of V_n of polynomials

$$V_n = \{P(x)\colon \quad P(x) = x^n + \text{terms of order }, n\}$$

the norm of Chebyshev polynomial

$$T_n(x) = \frac{P_n(x)}{2^n}$$

is minimal:

$$\text{for every} \quad P \in V_n\,, \quad ||P|| \geq \frac{1}{2^n}\,,$$

and

$$||P|| = \frac{1}{2^n} \Leftrightarrow P = T_n(x)$$

We suppose that

$$||P|| = \max_{x\in[-1,1]} |P(x)|\,.$$

`Proof of the lemma`

Let $P(x)$ be an arbitrary polynomial such that $P(x) - T_n(x)$ has order less than n. Show that this implies that its norm is greater than 1. Suppose that $||P|| \leq 1$. Consider values of polynomial P at the points

$$x_k = \cos\frac{\mathbf{k}\pi}{n}\,, k = 0,1,2,3,\dots,n$$

Note that polynomial $P_n(x)$ takes values ± 1 alternatively at these points:

$$P_n(x_0) = 1, P_n(x_1) = -1, P_n(x_2) = 1\,, \text{and so on}\,.$$

Hence at these points the polynomial $P(x)$ has to be positive or negative alternatively:

$$P_n(x_0) > 0\,, P_n(x_1) < 0\,, P_n(x_2) > 0\,, \text{and so on}\,.$$

since $||P|| < 1$. Hence the polynomial $P(x)$ has at least n roots. On the other hand $P(x) - P_n(x)$ has order $\leq n-1$. Contradiction.