Homework 1

1 a) Let $x^2 + y^2 = R^2$ be a circle in $\mathbf{E}^2$. Write down explicitly formulae for stereographic projections with respect to the North pole (the point $(0, 1)$) and South pole (the point $(0, -1)$).

b) Do the same exercise for the sphere $x^2 + y^2 + z^2 = R^2$ in $\mathbf{E}^3$. (North pole (the point $(0, 0, 1)$) and South pole (the point $(0, 0, -1)$).

2 Consider in $\mathbf{E}^n$ the transformation:

$$\mathbf{r}' = \varphi(\mathbf{r}) = \frac{\mathbf{r}}{|\mathbf{r}|^2}\,, \tag{1}$$

inversion with the centre at origin. (Strictly speaking this transformation is defined on $\mathbf{E}^n \backslash 0$).

Analyze its geometrical meaning.

Show that this trasformation is an involution: $\varphi(\varphi(\mathbf{r})) = \mathbf{r}$.

Let $\mathbf{a}$ be an arbitrary vector attached at the arbitrary point $\mathbf{r}$ of $\mathbf{E}^n$ ($\mathbf{r} \neq 0$). Let $\mathbf{a}'$ be a vector attached at the point $\mathbf{r}' = \varphi(\mathbf{r})$, such that $\mathbf{a}' = \varphi_* \mathbf{a}$ is the image of vector $\mathbf{a}$ under inversion (1), Show that

$$\mathbf{a}' = \frac{\mathbf{a}r^2 - 2\mathbf{r}(\mathbf{r}, \mathbf{a})}{|\mathbf{r}|^4}\,, \tag{2}$$

where $(\ ,\ )$ is the scalar product in Euclidean space $\mathbf{E}^n$.

Using this equation show that inversion preserves angles between vectors.

Find the image of the hyperplane $x^n = a$ under the inversion (1). ($(x^1, \dots, x^n)$ are standard Cartesian coordinates) [1)].

For simplicity you may consider just cases $n = 2, 3$

3* Consider in $\mathbf{E}^3$ the transformation:

$$\varphi_{N}(\mathbf{r}) = \mathbf{N} + \frac{C\mathbf{r}}{|\mathbf{r} - \mathbf{N}|^2}\,, \quad C \neq 0\,, \tag{3}$$

where $\mathbf{N}$ is an arbitrary vector.

(Strictly speaking this transformation is defined on $\mathbf{E}^n \backslash N$.

Analyze geometrical meaning of this transformation: in particular analyze the relation of this transformation with the transformation (1).

Find the image $\mathbf{a}' = (\varphi_{N})_* \, \mathbf{a}$ of an arbitrary vector $\mathbf{a}$ (Compare with (2)).

Show that this transformation preserves angles.

[1)] Show that this is a sphere of radius $\frac{1}{2|a|}$ with the centre at the point $x^1 = \ldots = x^{n-1} = 0, x^n = \frac{1}{2a}$.

Find a transformation inverse to transformation (3).

Find an image of a plane under the transformation (3).

4* Consider a stereographic projection of unit sphere $x^2+y^2+z^2=1$ in $\mathbf{E}^3$ on the plane $z=0$ with respect to the North pole $N=(0,0,1)$. Find a transformation (3) of $\mathbf{E}^3$ such that its restriction on the sphere is this stereographic projection.

5* Using the result of previous exercise explain why stereographic projection of unit sphere $x^2+y^2+z^2=1$ establsihes bijection between points with rational coordinates on the unit sphere with points with rational coordinates on the plane $z=0$.

6 Show that for an arbitrary n-dimensional Riemannian manifold the condition of non-degeneracy for a symmetric matrix $G=||g_{ik}||$ follows from the condition that this matrix is positive-definite.

7 Let (u,v) be local coordinates on 2-dimensional Riemannian manifold M. Let Riemannian metric be given in these local coordinates by the matrix

$$||g_{ik}|| = \begin{pmatrix} A(u,v) & B(u,v) \\ C(u,v) & D(u,v) \end{pmatrix},$$

where $A(u,v), B(u,v), C(u,v), D(u,v)$ are smooth functions. Show that the following conditions are fulfilled:

a) $B(u,v)=C(u,v)$,

b) $A(u,v)D(u,v)-B(u,v)C(u,v)=A(u,v)D(u,v)-B^2(u,v)\neq 0$,

c) $A(u,v)>0$,

d*) $A(u,v)D(u,v)-B(u,v)C(u,v)=A(u,v)D(u,v)-B^2(u,v)>0$.

e)* Show that conditions a), c) and d) are necessary and sufficient conditions for matrix $||g_{ik}||$ to define locally a Riemannian metric.

8 Consider two-dimensional Riemannian manifold with Euclidean metric $G=dx^2+dy^2$. How this metric will transform under arbitrary affine transformation $\begin{cases} x=ax'+by'+e \\ y=cx'+dy'+f \end{cases}$?

9 Consider domain in two-dimensional Riemannian manifold with Riemannian metric $G=du^2+2bdudv+dv^2$ in local coordinates u,v, where b is a constant.

Show that $b^2<1$

10 Show that $G=dx^2+dy^2+cdz^2$ in $\mathbf{R}^3$ defines Riemannian metric iff $c>0$.

* Find null-vectors (isotropic vectors) of pseudo-Riemannian metric G if $c<0$.