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We check here Fock considerations (see Dirack book) for the free partcle

Consider for the free particle

$$\begin{cases} i\hbar\frac{\partial\Psi}{\partial t} = \frac{\hat{p}^2}{2m}\Psi \\ \Psi(x,t)\big|_{t=0} = \delta(x-x_0) = \frac{1}{2\pi\hbar}\int e^{\frac{i}{\hbar}p(x-x_0)}dp \end{cases}$$

We have

$$\Psi(x,t) = \frac{1}{2\pi\hbar}\int\left(e^{-\frac{i}{\hbar}\frac{p^2}{2m}t+\frac{i}{\hbar}p(x-x_0)}\right)dp =$$

$$\frac{1}{2\pi\hbar}\int\exp\left[-\frac{i}{\hbar}\frac{p^2}{2m}t+\frac{i}{\hbar}p(x-x_0)\right]dp =$$

$$\frac{1}{2\pi\hbar}\int\exp\left[-\frac{i}{\hbar}\left(p\sqrt{\frac{t}{2m}}-\frac{x-x_0}{2\sqrt{\frac{t}{2m}}}\right)^2-\frac{i}{\hbar}\frac{m(x-x_0)^2}{2t}\right]dp =$$

$$\underbrace{\sqrt{\frac{m}{2\pi\hbar t}}}_{\text{semidensity}}\exp\left[-\frac{i}{\hbar}\underbrace{\frac{m(x-x_0)^2}{2t}}_{\text{action of free particle}}\right]$$

Show that unitarity is preserved in time:

$$\int\Psi^*(x,x_0)\Psi(x,x_1)dx =$$

$$\int\left(\sqrt{\frac{m}{2\pi\hbar t}}\exp\left[\frac{i}{\hbar}\frac{m(x-x_1)^2}{2t}\right]\right)\left(\sqrt{\frac{m}{2\pi\hbar t}}\exp\left[-\frac{i}{\hbar}\frac{m(x-x_0)^2}{2t}\right]\right)dx =$$

$$\frac{m}{2\pi\hbar t}\int\exp\left[\frac{i}{\hbar}\frac{m}{2t}\left((x-x_1)^2-(x-x_0)^2\right)\right]dx = \frac{m}{2\pi\hbar t}\int\exp\left[\frac{i}{\hbar}\frac{m}{2t}(x_0^2-x_1^2+2x(x_0-x_1))\right]dx$$

$$= e^{\frac{i}{\hbar}\frac{m}{2t}(x_0^2-x_1^2)}\frac{m}{2\pi\hbar t}\int\exp\left[\frac{i}{\hbar}\frac{m}{t}(x(x_0-x_1))\right]dx = e^{\frac{i}{\hbar}\frac{m}{2t}(x_0^2-x_1^2)}\delta(x_0-x_1) = \delta(x_0-x_1)\,.$$