22 June 2018

Disntance in Lobachevsky geometry

Let $z_1 = x_1 + iy_1$, $z_2 x_2 + iy_2$ be two points in Lobachevsky plane (we consider upper-half plane model).

First consider the case if $z = x + iy$, $w = i$. Transformation

$$z \mapsto \frac{z+a}{1-az}\,, , a \in \mathbf{R}$$

preserves the point $w = i$. Find $a$ such that $z' = \frac{z+a}{1-az}$ is also on $OY$ axis, i.e.

$$a\colon , \mathrm{Re}\,\frac{z+a}{1-az} = 0\,,$$

i.e.

$$\frac{z+a}{1-az} + \frac{\bar z + a}{1-az} = \frac{(z+a)(1-a\bar z) + (\bar z + a)(1-az)}{(1-az(1-a\bar z)} = 0\,.$$

We come to the quadratic equation on $a$

$$(z+a)(1-a\bar z) + (\bar z + a)(1-az) = (z+\bar z) + 2a(1-z\bar z) - a^2(z+\bar z) = 0$$

which defines Mobius transformation which put two points on the axis $0Y$, provided the first point $w = i$. Solving this equation we come to

$$a = \frac{1 - z\bar z \pm \sqrt{(1-z\bar z)^2 + (z+\bar z)^2}}{z+\bar z} = \frac{1 - z\bar z \pm \sqrt{1 + |z|^4 + z^2 + \bar z^2}}{z+\bar z}$$

i.e. Mobius transformation

$$z' = iy' = \frac{z+a}{1-az} = \frac{z + \frac{1-z\bar z+\sqrt{1+|z|^4+z^2+\bar z^2}}{z+\bar z}}{1 - \frac{1-z\bar z+\sqrt{1+|z|^4+z^2+\bar z^2}}{z+\bar z} z} =$$

$$\frac{1 + z^2 + \sqrt{1+|z|^4+z^2+\bar z^2}}{\bar z(1+z^2) - z\sqrt{1+|z|^4+z^2+\bar z^2}} = \frac{1 + z^2 + \sqrt{1+|z|^4+z^2+\bar z^2}}{\bar z - z\left(\sqrt{1+|z|^4+z^2+\bar z^2} - |z|^2\right)}$$

transforms an arbitrary point $z = x + iy$ with $y \neq 0$ to the point on the axis $OY$. (we choose the root with $'+'$, the second root is for the points on $OX$ but we are not interested in these points.)

In other words numerator of the fraction is proportional to denominator multiplied on imaginary number:

$$\left[1 + x^2 - y^2 + \sqrt{1+|z|^4+z^2+\bar z^2} : 2xy\right] =$$

$$\left[y\left(1+\sqrt{1+|z|^4+z^2+\bar z^2}-|z|^2\right) : x\left(1-\sqrt{1+|z|^4+z^2+\bar z^2}+|z|^2\right)\right] ,$$

i.e.

$$\frac{1+x^2-y^2+\sqrt{1+|z|^4+z^2+\bar z^2}}{2xy}=\frac{y\left(1+\sqrt{1+|z|^4+z^2+\bar z^2}-|z|^2\right)}{x\left(1-\sqrt{1+|z|^4+z^2+\bar z^2}+|z|^2\right)}$$

Proverka: $z=1+i$, i.e. $x=y=1$, then

$$\frac{1+x^2-y^2+\sqrt{1+|z|^4+z^2+\bar z^2}}{2xy}=\frac{1+\sqrt5}{2}=\frac{y\left(1+\sqrt{1+|z|^4+z^2+\bar z^2}-|z|^2\right)}{x\left(1-\sqrt{1+|z|^4+z^2+\bar z^2}+|z|^2\right)}=\frac{\sqrt5-1}{3-\sqrt5} .$$

Proverka v obshem sluchae: We have to check that

$$A=\left(1+x^2-y^2+\sqrt{1+|z|^4+z^2+\bar z^2}\right)\cdot\left(x\left(1-\sqrt{1+|z|^4+z^2+\bar z^2}+|z|^2\right)\right)=$$

$$=B=2xy\cdot\left(y\left(1+\sqrt{1+|z|^4+z^2+\bar z^2}-|z|^2\right)\right) ,$$

i.e.

$$A=(1+x^2-y^2+S)x(1+x^2+y^2-S)=2xy^2(1-x^2-y^2+S)=B .$$

Check it:

$$A=\left(1+x^2-y^2+\sqrt{1+|z|^4+z^2+\bar z^2}\right)\cdot\left(x\left(1-\sqrt{1+|z|^4+z^2+\bar z^2}+|z|^2\right)\right)=$$

$$\left(1+x^2-y^2+S\right)\left(x-xS+x^3+xy^2\right)=$$

$$(x+x^3+xy^2-xS)+(x^3+x^5+x^3y^2-x^3S)+(-xy^2-x^3y^2-xy^4+xy^2S)+(xS+x^3S+xy^2S-xS^2)=$$

$$x+2x^3+x^5-xy^4+2xy^2S-x\left(1+|z|^4+z^2+\bar z^2\right)=$$

$$x+2x^3+x^5-xy^4+2xy^2S-x\left(1+x^4+2x^2y^2+y^4+2x^2-2y^2\right)=$$

$$-2xy^4+2xy^2S-2x^3y^2+2xy^2=2xy^2(-y^2-x^2+1+S)=$$

$$=2xy\cdot\left(y\left(1+\sqrt{1+|z|^4+z^2+\bar z^2}-|z|^2\right)\right)=B$$

Thus we checked that the expression is right.

*Resumé*

Let $ac=bd$, then

$$z'=iy'=\frac{a+bi}{c-di}=\frac{i(b-ia)}{c-di}=i\frac{b}{c}=i\frac{a}{d}$$

According to this identity return to calculations:

$$z' = iy' = \frac{1+z^2+\sqrt{1+|z|^4+z^2+\bar{z}^2}}{\bar{z}-z\left(\sqrt{1+|z|^4+z^2+\bar{z}^2}-|z|^2\right)} =$$

$$\frac{\overbrace{(1+x^2-y^2+S)}^{a}+\overbrace{2xy}^{b}i}{\underbrace{x(1+x^2+y^2-S)}_{c}-i\underbrace{y(1+S-x^2-y^2)}_{d}} = \frac{1+x^2-y^2+S}{y(1-x^2-y^2+S)}i = \frac{2y}{1+x^2+y^2-S}i\,,$$

where

$$S = \sqrt{1+x^4+2x^2y^2+y^4+2x^2-2y^2}\,.$$

Now play with the last formula:

$$y' = \frac{1+x^2-y^2+S}{y(1-x^2-y^2+S)} = \frac{1+x^2-y^2+\sqrt{(1+x^2-y^2)^2+4x^2y^2}}{y\left(1-x^2-y^2+\sqrt{(1-x^2-y^2)^2+4x^2}\right)} =$$

$$\frac{2xy\left(\left(\frac{1+x^2-y^2}{2xy}\right)+\sqrt{\left(\frac{1+x^2-y^2}{2xy}\right)^2+1}\right)}{2xy\left(\left(\frac{1-x^2-y^2}{2x}\right)+\sqrt{\left(\frac{1-x^2-y^2}{2x}\right)^2+1}\right)} = e^{\operatorname{arcsh}\frac{1+x^2-y^2}{2xy}-\operatorname{arcsh}\frac{1-x^2-y^2}{2x}}$$

Quickly:

$$ch\,(a-b) = \sqrt{1+a^2}\sqrt{1+b^2}-ab$$

$$ch\left(\operatorname{arcsh}\frac{1+x^2-y^2}{2xy}-\operatorname{arcsh}\frac{1-x^2-y^2}{2x}\right) = \frac{x^2+y^2+1}{2y}\,.$$

Hence we come to

$$d(z,w) = d(x_1+iy_1, x_2+iy_2) = d\,(x_1-x_2+iy_1, iy_2) = d\left(\frac{x_1-x_2+iy_1}{y_2}, i\right) =$$

$$\operatorname{arcch}\left(\frac{x^2+y^2+1}{2y}\right)\Big|_{x\mapsto\frac{x_1-x_2}{y_2},\,y\mapsto\frac{y_1}{y_2}} = \operatorname{arcch}\left(\frac{\left(\frac{x_1-x_2}{y_2}\right)^2+\left(\frac{y_1}{y_2}\right)^2+1}{2\left(\frac{y_1}{y_2}\right)}\right) =$$

$$\operatorname{arcch}\left(\frac{(x_1-x_2)^2+y_1^2+y_2^2}{2y_1y_2}\right) = \operatorname{arcch}\left(1+\frac{(x_1-x_2)^2+(y_1-y_2)^2}{2y_1y_2}\right) =$$