Homework 1. Solutions

1 *Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a basis of the vector space V. Let $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m\}$ be an ordered set of an arbitrary m vectors in this vector space.*

Show that the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m\}$ is linear dependent if $m \geq 4$.

Show that the ordered set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is a basis for V if and only if these three vectors are linear independent.

Show that the ordered set of vectors $\{\mathbf{a}_1, \mathbf{a}_2\}$ is not a basis for V.

Show that the ordered set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is not a basis for V in the case if $\mathbf{a_3} = 0$. *.

First prove that the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m\}$ is linear dependent if $m \geq 4$. The proof below in fact is the proof of the lemma in the subsection 1.2 of lecture notes

We prove this statement just for $m = 4$. This suffices: if $m \geq 4$ then since arbitrary four vectors are linear dependent, hence the set of m vectors is linear dependent too. Consider expansions:

$$\begin{cases} \mathbf{a}_1 = A_{11}\mathbf{e}_1 + A_{12}\mathbf{e}_2 + A_{13}\mathbf{e}_3 \\ \mathbf{a}_2 = A_{21}\mathbf{e}_1 + A_{22}\mathbf{e}_2 + A_{23}\mathbf{e}_3 \\ \mathbf{a}_3 = A_{31}\mathbf{e}_1 + A_{32}\mathbf{e}_2 + A_{33}\mathbf{e}_3 \\ \mathbf{a}_4 = A_{41}\mathbf{e}_1 + A_{42}\mathbf{e}_2 + A_{43}\mathbf{e}_3 \end{cases} \tag{1}$$

Take the first row of this relation. If $\mathbf{a}_1 = 0$, then vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ are obviously linear dependent. Suppose that $\mathbf{a}_1 \neq 0$. Then one of coefficients in the first row is not equal to zero. Without loss of generality suppose that this is the first coefficient: $A_{11} \neq 0$. Hence $\mathbf{e}_1$ cane be expressed as a linear combination of the vectors $\mathbf{a}_1, \mathbf{e}_2$ and $\mathbf{e}_3$:

$$\mathbf{e}_1 = \frac{1}{A_{11}}\mathbf{a}_1 - \frac{A_{12}}{A_{11}}\mathbf{e}_2 - \frac{A_{13}}{A_{11}}\mathbf{e}_3\,. \tag{2}$$

Input this linear expansion of the vector $\mathbf{e}_1$ over the vectors $\mathbf{a}_1, \mathbf{e}_2$ and vector $\mathbf{e}_3$ in the second third and fourth rows of the expansions (1): we will come to the expansions of the vectors $\mathbf{a}_2, \mathbf{a}_3$ and $\mathbf{a}_4$ over the vectors $\mathbf{a}_1, \mathbf{e}_2$ and vector $\mathbf{e}_3$:

$$\begin{cases} \mathbf{a}_2 = B_{21}\mathbf{a}_1 + B_{22}\mathbf{e}_2 + B_{23}\mathbf{e}_3 \\ \mathbf{a}_3 = B_{31}\mathbf{a}_1 + B_{32}\mathbf{e}_2 + B_{33}\mathbf{e}_3 \\ \mathbf{a}_4 = B_{41}\mathbf{a}_1 + B_{42}\mathbf{e}_2 + B_{43}\mathbf{e}_3 \end{cases} \tag{3}$$

Now repeat the previous procedure with the first row of the relation (3). If both coefficients B_{22}, B_{23} are equal to zero, then proof is finished: Vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are linear dependent. Suppose that one of the coefficients B_{22}, B_{23} is not equal to zero. Without loss of generality suppose that $B_{22} \neq 0$. Then we can express $\mathbf{e}_2$ as a linear combination of vectors $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{e}_3$ (Compare with (2)):

$$\mathbf{e}_2 = \frac{1}{B_{22}}\mathbf{a}_2 - \frac{B_{21}}{B_{22}}\mathbf{a}_1 - \frac{B_{23}}{B_{22}}\mathbf{e}_3 \tag{4}$$

Input this expansion for $\mathbf{e}_2$ over the vectors $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{e}_3$ in the second and third rows of the relation (3). We come to the relations:

$$\begin{cases} \mathbf{a}_3 = C_{31}\mathbf{a}_1 + C_{32}\mathbf{a}_2 + C_{33}\mathbf{e}_3 \\ \mathbf{a}_4 = C_{41}\mathbf{a}_1 + C_{42}\mathbf{a}_2 + C_{43}\mathbf{e}_3 \end{cases} \tag{5}$$

Now look on the first row in the relation (5). If $C_{33} = 0$ then vectors $\mathbf{a}_3, \mathbf{a}_1$ and $\mathbf{a}_2$ are linear dependent and proof is finished. If $C_{33} \neq 0$ then we can express $\mathbf{e}_3$ as a linear combination of vectors $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_3$ (Compare with (2)and (4)):

$$\mathbf{e}_3 = \frac{1}{C_{33}}\mathbf{a}_3 - \frac{C_{31}}{C_{33}}\mathbf{a}_1 - \frac{C_{32}}{C_{33}}\mathbf{a}_3$$

Input this relation into the second row of the relation (5) we come to:

$$\mathbf{a}_4 = C_{41}\mathbf{a}_1 + C_{42}\mathbf{a}_2 + C_{43}\mathbf{e}_3 = \mathbf{a}_4 = C_{41}\mathbf{a}_1 + C_{42}\mathbf{a}_2 + C_{43}\left(\frac{1}{C_{33}}\mathbf{a}_3 - \frac{C_{31}}{C_{33}}\mathbf{a}_1 - \frac{C_{32}}{C_{33}}\mathbf{a}_3\right) =$$

* The questions discussed in exercises 1 and 2 is recalling of the linear algebra stuff

$$= D_{41}\mathbf{a}_1 + D_{42}\mathbf{a}_2 + D_{43}\mathbf{a}_3,$$

i.e. the vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ and $\mathbf{a}_4$ are linear dependent $\blacksquare$.

Remark One can see that the considerations above works for any M vectors in n-dimensional space if $M > n$.

Now show that the ordered set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is a basis of V if and only if these three vectors are linear independent.

It is very easy to prove that if they form a basis they are linear dependent. Indeed take vector $\mathbf{x} = 0$. Its expansion over basis is unique. Hence from $\mathbf{x} = 0 = c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3\mathbf{a}_3$ follows that $c_1 = c_2 = c_3$.

Prove the converse. Suppose that these vectors are linear independent. Prove that they form a basis Take an arbitrary vector $\mathbf{R}$. Consider the set of 4 vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{R}\}$. According to the lemma proved above these vectors are linear dependant:

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3\mathbf{a}_3 + c_4\mathbf{R} = 0,$$

where not all coefficients c_1, c_2, c_3, c_4 are equal to zero. If $c_4 = 0$ then vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ becomes linear dependent. Hence $c_4 \neq 0$. Hence

$$\mathbf{R} = -\frac{c_1}{c_4}\mathbf{a}_1 - \frac{c_2}{c_4}\mathbf{a}_2 - -\frac{c_3}{c_4}\mathbf{a}_3 \,.$$

We proved that an arbitrary vector can be expressed as a linear combination of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$. Prove uniqueness: if for vector $\mathbf{x}$, $\mathbf{x} = m_1\mathbf{a}_1 + m_2\mathbf{a}_2 + m_3\mathbf{a}_3 = m'_1\mathbf{a}_1 + m'_2\mathbf{a}_2 + m'_3\mathbf{a}_3$, then

$$(m_1 - m'_1)\mathbf{a}_1 + (m_2 - m'_2)\mathbf{a}_2 + (m_3 - m'_3)\mathbf{a}_3 = 0$$

Since vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are linear independent, hence $m_1 - m'_1 = m_2 - m'_2 = m_3 - m'_3 = 0$, i.e. $m_1 = m'_1, m_2 = m'_2$ and $m_3 = m'_3$. Uniqueness is proved.

Now prove that the ordered set of vectors $\{\mathbf{a}_1, \mathbf{a}_2\}$ is not a basis for V.

Suppose that it is basis. Then by the lemma (see above) the vectors $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ are linear dependent $(3 > 2)$. Contradiction. Hence the ordered set of vectors $\{\mathbf{a}_1, \mathbf{a}_2\}$ is not a basis for V.

Now prove the last statement: Show that the ordered set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is not a basis for V in the case if $\mathbf{a_3} = 0$.

Suppose that it is a basis: Consider the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2\}$. As it was proved before it is not a basis. Hence there exists a vector $\mathbf{x}$ such that $\mathbf{x}$ cannot be expressed as a linear combination of vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. Hence this vector cannot be expressed as a linear combination of vectors $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_3 = 0$. Contradiction. $\blacksquare$.

2 *Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a basis of the vector space V.*

Show that an arbitrary basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}'_m\}$ also possesses three vectors, i.e. if the ordered sets of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \dots, \mathbf{e}_m\}$ in this vector space is also a basis, then $m = 3$.

This statement follows from the lemma that was proved above:

if M vectors $\{\mathbf{a}_1, \dots, \mathbf{a}_M\}$ belong to the span of n vectors $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ and $M > n$ then these M vectors $\{\mathbf{a}_1, \dots, \mathbf{a}_M\}$ are linear dependent.

The proof immediately follows from the lemma. Indeed Let $m > 3$, then it follows form the lemma that $\{\mathbf{e}'_1, \dots, \mathbf{e}'_m\}$ is not a basis, because these vectors are linear dependent.

Let $m < 3$ then suppose that $\{\mathbf{e}'_1, \dots, \mathbf{e}'_m\}$ is a basis. Then vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ belong to the span of the vectors $\mathbf{e}'_1, \dots, \mathbf{e}'_m$. Since $3 > m$ these vectors are linear dependent. Contradiction.

3 *Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a basis of the vector space V.*

Is a set of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ a basis of V in the case if

a) $\mathbf{e}'_1 = \mathbf{e}_2$, $\mathbf{e}'_2 = \mathbf{e}_1$, $\mathbf{e}'_3 = \mathbf{e}_3$;

b) $\mathbf{e}'_1 = \mathbf{e}_1,\ \mathbf{e}'_2 = \mathbf{e}_1 + 3\mathbf{e}_3,\ \mathbf{e}'_3 = \mathbf{e}_3$*;*
c) $\mathbf{e}'_1 = \mathbf{e}_1 - \mathbf{e}_2,\ \mathbf{e}'_2 = 3\mathbf{e}_1 - 3\mathbf{e}_2,\ \mathbf{e}'_3 = \mathbf{e}_3$*;*
d) $\mathbf{e}'_1 = \mathbf{e}_2,\ \mathbf{e}'_2 = \mathbf{e}_1,\ \mathbf{e}'_3 = \mathbf{e}_1 + \mathbf{e}_2 + \lambda\mathbf{e}_3$ *(where λ is an arbitrary coefficient)?*

To analyse the cases we use the fact that 3 vectors in 3-dimensional space form a basis if and only if these vectors are linear independent (See the exercise above.)

Case a) Vectors $\mathbf{e}'_1 = \mathbf{e}_2, \mathbf{e}'_2 = \mathbf{e}_1, \mathbf{e}'_3 = \mathbf{e}_3$ are linear independent, since $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis. Hence $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ is a basis too.

Case b) Vectors $\mathbf{e}'_1 = \mathbf{e}_1, \mathbf{e}'_2 = \mathbf{e}_1 + 3\mathbf{e}_3, \mathbf{e}'_3 = \mathbf{e}_3$ are linear dependent. Indeed

$$\mathbf{e}'_1 - \mathbf{e}'_2 + 3\mathbf{e}'_3 = \mathbf{e}_1 - (\mathbf{e}_1 + 3\mathbf{e}_3) + 3\mathbf{e}_3 = 0\,.$$

Hence it is not a basis.

Case c) First two vectors $\mathbf{e}'_1 = \mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}'_2 = 3\mathbf{e}_1 - 3\mathbf{e}_2$ are already linear dependent. Hence these three vectors do not form a basis.

Case d) Check are vectors linear independent or not. Let $c_1\mathbf{e}'_1 + c_2\mathbf{e}'_2 + c_3\mathbf{e}'_3 = 0$, i.e.

$$c_1\mathbf{e}'_1 + c_2\mathbf{e}'_2 + c_3\mathbf{e}'_3 = c_1\mathbf{e}_2 + c_2\mathbf{e}_1 + c_3(\mathbf{e}_1 + \mathbf{e}_2 + \lambda\mathbf{e}_3) = (c_2 + c_3)\mathbf{e}_1 + (c_1 + c_3)\mathbf{e}_2 + c_3\lambda\mathbf{e}_3 = 0\,.$$

I-st case $\lambda \neq 0$. We have $c_2 + c_3 = c_1 + c_3 = \lambda c_3 = 0$. Hence $c_3 = 0, c_1 = 0, c_2 = 0$. These three vectors are linear independent. This means that ordered triple $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ is a basis.

II-nd case $\lambda = 0$. We have $c_2 + c_3 = c_1 + c_3 = 0c_3 = 0$. Hence c_3 can be an arbitrary number and $c_1 = -c_3, c_2 = -c_3$. c_3 These three vectors are linear dependent. This means that ordered triple $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ is not a basis.

4 *Show that* $(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^2 + x^3y^3$ *is a scalar product in* $\mathbf{R}^3$*.*
Show that $(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^2$ *does not define scalar product in* $\mathbf{R}^3$*.*
Show that $(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^2 - x^3y^3$ *does not define scalar product in* $\mathbf{R}^3$*.*
Show that $(\mathbf{x}, \mathbf{y}) = x^1y^1 + 3x^2y^2 + 5x^3y^3$ *is a scalar product in* $\mathbf{R}^3$*.*

Recall that scalar product on a vector space V is a function $B(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathbf{y})$ on a pair of vectors which takes real values and satisfies the the following conditions:

1) $(\mathbf{x}, \mathbf{y}) = (\mathbf{y}, \mathbf{x})$ (symmetricity condition)
2) $(\lambda\mathbf{x} + \mu\mathbf{y}, \mathbf{z}) = \lambda(\mathbf{x}, \mathbf{z}) + \mu(\mathbf{y}, \mathbf{z})$ (linearity condition (with respect to the first argument))
3) $(\mathbf{x}, \mathbf{x}) \geq 0$, $(\mathbf{x}, \mathbf{x}) = 0 \Leftrightarrow \mathbf{x} = 0$ (positively defined and non-degeneracy condition)
(The linearity condition with respect to the second argument follows from the conditions 2) and 1))

Check all these conditions for $(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^2 + x^3y^3$:
1) $(\mathbf{y}, \mathbf{x}) = y^1x^1 + y^2x^2 + y^3x^3 = x^1y^1 + x^2y^2 + x^3y^3 = (\mathbf{x}, \mathbf{y})$. Hence it is symmetrical.
2) $(\lambda\mathbf{x} + \mu\mathbf{y}, \mathbf{z}) = (\lambda x^1 + \mu y^1)z^1 + (\lambda x^2 + \mu y^2)z^2 + (\lambda x^3 + \mu y^3)z^3 =$
$= \lambda(x^1z^1 + x^2z^2 + x^3z^3) + \mu(y^1z^1 + y^2z^2 + y^3z^3) = \lambda(\mathbf{x}, \mathbf{y}) + \mu(\mathbf{y}, \mathbf{z})$. Hence it is linear.
3) $(\mathbf{x}, \mathbf{x}) = (x^1)^2 + (x^2)^2 + (x^3)^2 \geq 0$. It is non-negative. If $\mathbf{x} = 0$. Then obviously $(\mathbf{x}, \mathbf{x}) = 0$. If $(\mathbf{x}, \mathbf{x}) = (x^1)^2 + (x^2)^2 + (x^3)^2 = 0$, then $x^1 = x^2 = x^3 = 0$. Hence it is non-degenerate.

All conditions are checked. Hence $(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^2 + x^3y^3$ is indeed a scalar product in $\mathbf{R}^3$

Remark Note that x^1, x^2, x^3—are components of the vector, do not be confused with exponents!

Show that $(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^2$ *does not define scalar product* in $\mathbf{R}^3$.

To see that the formula $(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^2$ *does not* define scalar product check the condition of non-degeracy 3): $(\mathbf{x}, \mathbf{x}) = (x^1)^2 + (x^2)^2$ may take zero values for $\mathbf{x} \neq 0$. E.g. if $\mathbf{x} = (0, 0, -1)$ $(\mathbf{x}, \mathbf{x}) = 0$, in spite of the fact that $\mathbf{x} \neq 0$. The condition of non-degeneracy in 3) is not satisfied. Hence it is not scalar product.

Show that $(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^2 - x^3y^3$ *does not define scalar product* in $\mathbf{R}^3$.

To see that the formula $(\mathbf{x},\mathbf{y}) = x^1y^1 + x^2y^2 - x^3y^3$ *does not* define scalar product check the condition 3): $(\mathbf{x},\mathbf{x}) = (x^1)^2 + (x^2)^2 - (x^3)^2$ may take negative values. E.g. if $\mathbf{x} = (0,0,-1)$ $(\mathbf{x},\mathbf{x}) = -1 < 0$. The condition 3) is not satisfied. Hence it is not scalar product.

Now show that $(\mathbf{x},\mathbf{y}) = x^1y^1 + 3x^2y^2 + 5x^3y^3$ is a scalar product in $\mathbf{R}^3$.
We need to check all the conditions above for scalar product for $(\mathbf{x},\mathbf{y}) = x^1y^1 + 3x^2y^2 + 5x^3y^3$:
1) $(\mathbf{y},\mathbf{x}) = y^1x^1 + 3y^2x^2 + 5y^3x^3 = x^1y^1 + 3x^2y^2 + 5x^3y^3 = (\mathbf{x},\mathbf{y})$. Hence it is symmetrical.
2) $(\lambda\mathbf{x} + \mu\mathbf{y},\mathbf{z}) = (\lambda x^1 + \mu y^1)z^1 + 3(\lambda x^2 + \mu y^2)z^2 + 5(\lambda x^3 + \mu y^3)z^3 =$
$= \lambda(x^1z^1 + 3x^2z^2 + 5x^3z^3) + \mu(y^1z^1 + 3y^2z^2 + 5y^3z^3) = \lambda(\mathbf{x},\mathbf{y}) + \mu(\mathbf{y},\mathbf{z})$. Hence it is linear.
3) $(\mathbf{x},\mathbf{x}) = (x^1)^2 + 3(x^2)^2 + 5(x^3)^2 \geq 0$. It is non-negative. If $\mathbf{x} = 0$. Then obviously $(\mathbf{x},\mathbf{x}) = 0$. If $(\mathbf{x},\mathbf{x}) = (x^1)^2 + 3(x^2)^2 + 5(x^3)^2 = 0$, then $x^1 = x^2 = x^3 = 0$. Hence it is non-degenerate.
All conditions are checked. Hence $(\mathbf{x},\mathbf{y}) = x^1y^1 + 3x^2y^2 + 5x^3y^3$ is indeed a scalar product in $\mathbf{R}^3$

5 *The matrix* $T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ *obeys the conditions* $T^tT = I$. *Show that*
a) $\det T = \pm 1$
b) if $\det T = 1$ *then there exists an angle* $\varphi : 0 \leq \varphi < 2\pi$ *such that* $T = T_\varphi$ *where*

$$T_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \quad \textit{(rotation matrix)}$$

c) if $\det T = -1$ *then then there exists an angle* $\varphi : 0 \leq \varphi < 2\pi$ *such that* $T = T_\varphi R$, *where* $R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *(a reflection matrix).*

a) We know that $\det T^t = \det T$. Hence

$$\det(T^tT) = \det(TT) = (\det T)^2 = \det I = 1 \Rightarrow \det T = \pm 1.$$

The answers on a) and b) see in Lecture notes (subsection 1.7).

6 *Show that for matrix* T_φ *defined in the previous exercise the following relations are satisfied:*

$$T_\varphi^{-1} = T_\varphi^t = T_{-\varphi}\,, \qquad T_{\varphi+\theta} = T_\varphi \cdot T_\theta\ .$$

We know (see lecture notes, subsection 1.7) that$T_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$. Then calculate inverse matrix T_φ^{-1}. One can see that $T_\varphi^t = T_\varphi^{-1} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$, because $T_\varphi^tT_\varphi = I$. On the other hand But $\cos\varphi = \cos(-\varphi)$ and $\sin\varphi = -\sin(-\varphi)$. Hence

$$T_\varphi^t = T_\varphi^{-1} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} = \begin{pmatrix} \cos(-\varphi) & -\sin(-\varphi) \\ \sin(-\varphi) & \cos(-\varphi) \end{pmatrix} = T_{-\varphi}\,.$$

Now prove that $T_{\varphi+\theta} = T_\varphi \cdot T_\theta$:

$$T_\varphi \cdot T_\theta = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} (\cos\varphi\cos\theta - \sin\varphi\sin\theta) & -(\cos\varphi\sin\theta + \sin\varphi\cos\theta) \\ (\cos\varphi\sin\theta + \sin\varphi\cos\theta) & (\cos\varphi\cos\theta - \sin\varphi\sin\theta) \end{pmatrix} =$$

$$\begin{pmatrix} \cos(\varphi+\theta) & -\sin(\varphi+\theta) \\ \sin(\varphi+\theta) & \cos(\varphi+\theta) \end{pmatrix} = T_{\varphi+\theta}$$

7 *Show that under the transformation* $(\mathbf{e}'_1, \mathbf{e}'_2) = (\mathbf{e}_1, \mathbf{e}_2)\, T_\varphi$ *an orthonormal basis transforms to an orthonormal one.*

How coordinates of vectors change if we rotate the orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2)$ *on the angle* $\varphi = \frac{\pi}{3}$ *anticlockwise?*

We have to check that scalar products $(\mathbf{e}'_1, \mathbf{e}'_1) = (\mathbf{e}'_2, \mathbf{e}'_2) = 1$ and $(\mathbf{e}'_1, \mathbf{e}'_2) = 0$. Calculations show that $(\mathbf{e}'_1, \mathbf{e}'_1) = (\cos\varphi\mathbf{e}_1 + \sin\varphi\mathbf{e}_2, \cos\varphi\mathbf{e}_1 + \sin\varphi\mathbf{e}_2) = \cos^2\varphi(\mathbf{e}_1, \mathbf{e}_1) + 2\cos\varphi\sin\varphi(\mathbf{e}_1, \mathbf{e}_2) + \sin^2\varphi(\mathbf{e}_2, \mathbf{e}_2) = 1$, $(\mathbf{e}'_2, \mathbf{e}'_2) = (-\sin\varphi\mathbf{e}_1 + \cos\varphi\mathbf{e}_2, -\sin\varphi\mathbf{e}_1 + \cos\varphi\mathbf{e}_2) = 1$, $(\mathbf{e}'_1, \mathbf{e}'_2) = (\cos\varphi\mathbf{e}_1 + \sin\varphi\mathbf{e}_2, -\sin\varphi\mathbf{e}_1 + \cos\varphi\mathbf{e}_2) = 0$.

Now answer the second question.

If $\mathbf{a} = x\mathbf{e}_x + y\mathbf{e}_2 = x'\mathbf{e}'_x + y'\mathbf{e}'_2$ and $T_\varphi = T_{\frac{\pi}{3}} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ is the matrix of bases transformation then we have:

$$\mathbf{a} = (\mathbf{e}_x, \mathbf{e}_y)\begin{pmatrix} x \\ y \end{pmatrix} = (\mathbf{e}'_x, \mathbf{e}'_y)\begin{pmatrix} x' \\ y' \end{pmatrix} = (\mathbf{e}_x, \mathbf{e}_y)T_{\frac{\pi}{3}}\begin{pmatrix} x' \\ y' \end{pmatrix} = (\mathbf{e}_x, \mathbf{e}_y)\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

Hence

$$\begin{pmatrix} x \\ y \end{pmatrix} = T_{\frac{\pi}{3}}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

and

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = T^{-1}_{\frac{\pi}{3}}\begin{pmatrix} x \\ y \end{pmatrix} = T_{-\frac{\pi}{3}}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

8 *Let* $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ *be an orthonormal basis of Euclidean space* $\mathbf{E}^3$. *Consider the ordered set of vectors* $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ *which is expressed via basis* $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ *as in the exercise 3.*

Find out is the ordered set of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ *an orthonormal basis of* $\mathbf{E}^3$.

Write down explicitly transition matrix from the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ *to the ordered set of the vectors* $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$.

What is the rank of this matrix?

Is this matrix orthogonal?

(you have to consider all cases a),b) c) and d)).

Case a) The ordered set $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\} = \{\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_3\}$ is evidently orthonormal basis. Transition matrix $T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $(\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3) = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)T$. This matrix is non-degenerate, its rank is equal to 3. It is orthogonal because both bases are orthonormal.

Case b) The ordered set $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\} = \{\mathbf{e}_1, \mathbf{e}_1 + 3\mathbf{e}_3, \mathbf{e}_3\}$ is not a basis because vectors are linear dependent: $\mathbf{e}'_1 - \mathbf{e}'_2 + 3\mathbf{e}'_3 = 0$. Transition matrix $T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 1 \end{pmatrix}$, $(\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3) = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)T$. This matrix is degenerate, its rank ≤ 2, because vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ are linear dependent. On the other hand vectors $\{\mathbf{e}'_1, \mathbf{e}'_2\}$ are linear independent. Hence rank of the matrix T is equal to 2. This matrix is not orthogonal.

Case c) The ordered set $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\} = \{\mathbf{e}_1 - \mathbf{e}_2, 3\mathbf{e}_1 - 3\mathbf{e}_2, \mathbf{e}_3\}$ is not a basis because vectors are linear dependent: $3\mathbf{e}'_1 - \mathbf{e}'_2 = 0$. Transition matrix $T = \begin{pmatrix} 1 & 3 & 0 \\ -1 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $(\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3) = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)T$. This matrix is degenerate, its rank ≤ 2, because vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ are linear dependent. On the other hand vectors $\{\mathbf{e}'_1, \mathbf{e}'_3\}$ are linear independent. Hence rank of the matrix T is equal to 2. This matrix is not orthogonal.

Case d)

The transition matrix from the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to the ordered triple $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\} = \{\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_1+\mathbf{e}_2+\lambda\mathbf{e}_3\}$ is $T = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & \lambda \end{pmatrix}$, $(\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3) = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)T$

I-st case. $\lambda \neq 0$. The ordered set $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ is a basis because vectors are linear independent (see the exercise 3), This basis is not orthogonal, because the length vector is not equal to 1 ($(\mathbf{e}'_3, \mathbf{e}'_3) = |\mathbf{e}'_3|^2 = 2+\lambda^2$). This matrix is not orthogonal, because the new basis is not orthonormal.

II-nd case $\lambda = 0$. The ordered set $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ is not a basis because vectors are linear independent: $\mathbf{e}'_1 + \mathbf{e}'_2 - \mathbf{e}'_3 = 0$. The transition matrix T has rank less or equal to 2, because vectors are linear dependent. On the other hand vectors $\mathbf{e}'_1, \mathbf{e}'_2$ are linear independent. Hence the rank of the matrix is equal to 2.

8† (not compulsory). *Show that an arbitrary orthogonal transformation of two-dimensional Euclidean space can be considered as a composition of reflections.*

Consider two cases. If the determinant of orthogonal transformation is equal to -1 then

$$T = \tilde{T}_\varphi = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}$$

One can see that this is reflection with respect to the axis which have an angle $\varphi/2$ with Ox axis.

If the determinant of orthogonal transformation is equal to 1 then

$$T = T_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \tilde{T}_\varphi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

i.e. rotation on the angle φ is a composition of two reflections.

9†(not compulsory). Prove the Cauchy–Bunyakovsky–Schwarz inequality

$$(\mathbf{x}, \mathbf{y})^2 \leq (\mathbf{x}, \mathbf{x})(\mathbf{y}, \mathbf{y}),$$

where $\mathbf{x}, \mathbf{y}$ are arbitrary two vectors and $(\ , \ )$ is a scalar product in Euclidean space.

Hint: For any two given vectors $\mathbf{x}, \mathbf{y}$ *consider the quadratic polynomial* $At^2 + 2Bt + C$ *where* $A = (\mathbf{x}, \mathbf{x})$, $B = (\mathbf{x}, \mathbf{y})$, $C = (\mathbf{y}, \mathbf{y})^2$. *Show that this polynomial has at most one real root and consider its discriminant.*

Consider quadratic polynomial $P(t) = \sum_{i=1}^n (tx^i + y^i)^2 = At^2 + 2Bt + C$, where $A = \sum_{i=1}^n (x^i)^2 = (\mathbf{x}, \mathbf{x})$, $B = \sum_{i=1}^n (x^i y^i) = (\mathbf{x}, \mathbf{y})$, $C = \sum_{i=1}^n (y^i)^2 = (\mathbf{y}, \mathbf{y})$. We see that equation $P(t) = 0$ has at most one root (and this is the case if only vector $\mathbf{x}$ is collinear to the vector $\mathbf{y}$). This means that discriminant of this equation is less or equal to zero. But discriminant of this equation is equal to $4B^2 - 4AC$. Hence $B^2 \leq AC$. It is just CBS inequality. ($(\mathbf{x}, \mathbf{y})^2 = (\mathbf{x}, \mathbf{x})(\mathbf{y}, \mathbf{y})$, i.e. discriminant is equal to zero $\Leftrightarrow$ vectors $\mathbf{x}$, $\mathbf{y}$ are colinear.