one differential equation

We consider the solution of the equation

$$S(x,t)\colon \quad \begin{cases} \frac{S_x^2}{2m} + U(x) + S_t = 0 \\ \\ S(x,t)\big|_{t=0} = f(x) \end{cases} \tag{1}$$

Consider coordinates (t,x) in 2-dimensional space M, and (p,x,ρ,t) in cotangent bundle T^*M (coordinate p is conjugate to the coordinate x and) coordinate ρ is conjugate to the coordinate t.

The initial conditinion: $S(x,t)\big|_{t=0} = f(x)$ define the one dimensional curve B_{S_0}

$$B_{S_0} =$$

Let $S = S(x,t)$ be a solution of equation (1), and let $\Lambda_S = \text{graph} S \subset T^*M$ bue Lagrangian surface corresponding to S:

$$\Lambda_S = \{(p,x,\rho,t)\colon \frac{p^2}{2m} + U(x) + \rho = 0\,, \quad p = S_x\,, \rho = S_t\}\} \tag{2}$$

The boundary condition means that the curve

$$B_{S_0}\colon \quad \begin{cases} p(\xi) = f_\xi(\xi) \\ x(\xi) = \xi \\ \rho(\xi) = -\frac{p^2}{2m} = -\frac{f_\xi^2}{2m} \\ t(\xi) = 0 \end{cases} \tag{3}$$

belongs to the Lagrangian surface Λ_S.

To construct the surface Λ_S iwe do the following: emit from the points of the curve B_{S_0} the Hamiltonian vector field

$$\mathbf{D}_H = \frac{\partial H}{\partial p}\frac{\partial}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial}{\partial p} + \frac{\partial H}{\partial \rho}\frac{\partial}{\partial t} - \frac{\partial H}{\partial t}\frac{\partial}{\partial \rho} \quad \text{of Hamiltonian } H = \frac{p^2}{2m} + U(x) + \rho\,. \tag{4}$$

We see that this surface is Lagrangian surface and it obeys the equation (2). since $dH(\mathbf{D}_H) = 0$.

Do this:

$$Ł_S\colon \begin{pmatrix} p(\xi,\eta) \\ x(\xi,\eta) \\ \rho(\xi,\eta) \\ t(\xi,\eta) \end{pmatrix} \quad \begin{cases} p_\eta = -\frac{\partial H}{\partial x} = -U_x(x) \\ x_\eta = \frac{\partial H}{\partial p} = p \\ \rho_\eta = -\frac{\partial H}{\partial t} = 0 \\ x_\eta = \frac{\partial H}{\partial \rho} = 1 \end{cases} \tag{5a}$$

with boundary condition (3):

$$B_{S_0} \subset \Lambda_S\,, \text{i.e.} \quad \begin{pmatrix} p(\xi,\eta) \\ x(\xi,\eta) \\ \rho(\xi,\eta) \\ t(\xi,\eta) \end{pmatrix}\Big|_{\eta=0} = \begin{pmatrix} p(\xi,0) \\ x(\xi,0) \\ \rho(\xi,0) \\ t(\xi,0) \end{pmatrix} = \begin{pmatrix} f_\xi(\xi) \\ \xi \\ -\frac{f_\xi^2}{2m} \\ 0 \end{pmatrix} \tag{5b}$$

These are just ODE.

First case, $U = 0$

Then $H = \frac{p^2}{2m} + \rho$ and solution of equation (5) will be

$$\Lambda_S \quad \begin{cases} p(\xi,\eta) = f_\xi(\xi) \\ x(\xi,\eta) = \xi + \eta p = \xi + \frac{\eta}{m} f_\xi(\xi) \\ \rho(\xi,\eta) = -\frac{f_\xi^2}{2m} \\ t(\xi,\eta) = \eta \end{cases} \tag{6a}$$

Consider example $f = c\xi$, then we have

$$\Lambda_S \quad \begin{cases} p(\xi,\eta) = c \\ x(\xi,\eta) = \xi + \eta p = \xi + \frac{c\eta p}{m} \\ \rho(\xi,\eta) = -\frac{c^2}{2m} \\ t(\xi,\eta) = 0 \end{cases} \tag{7a}$$

then we have that according to (7a)

$$\begin{cases} \xi = x - pt = x - \frac{cp}{m} t \\ \eta = t \end{cases} \Rightarrow \begin{cases} S_x = p = f_\xi(\xi) = c \\ S_t = \rho = -\frac{c^2}{2m} \end{cases} \Rightarrow S(x,t) = cx - \frac{c^2}{2m} t + constant$$

Another example $f = \frac{k\xi^2}{2}$, then

$$\Lambda_S \quad \begin{cases} p(\xi,\eta) = k\xi \\ x(\xi,\eta) = \xi + \eta p = \xi + \frac{k\xi\eta}{2m} \\ \rho(\xi,\eta) = -\frac{k^2\xi^2}{2m} \\ t(\xi,\eta) = \eta \end{cases} \tag{8a}$$

and

$$\begin{cases} \xi = x - pt = \frac{mx}{m+kt} \\ \eta = t \end{cases} \Rightarrow \begin{cases} S_x = p = f_\xi(\xi) = k\xi = \frac{kmx}{m+kt} \\ S_t = \rho = -\frac{k^2\xi^2}{2m} = -\frac{k^2}{2m}\left(\frac{mx}{m+kt}\right)^2 = -\frac{k^2mx^2}{2(m+kt)^2} \end{cases}$$

Of course the integration condition is obeyed: $\left(\frac{kmx}{m+kt}\right)_t = \left(-\frac{k^2mx^2}{2(m+kt)^2}\right)_x$ since the surface is Lagrnagian and we come to the solution

$$\Rightarrow S(x,t) = \frac{kmx^2}{2(m+kt)} + constant$$

Now consider the interesting example: $f(x) = \delta(x-a)$. Then we come to