## Homework 3

*In all exercises we assume by default that Riemannian metric on embedded surfaces is induced by the Euclidean metric.*

**1** Show that surface of the cone $\begin{cases} x^2+y^2-k^2z^2=0 \\ z>0 \end{cases}$ in $\mathbf{E}^3$ is locally Euclidean Riemannian surface, (is locally isometric to Euclidean plane).

**2** a) Consider the conic surface $C$ defined by the equation $x^2+y^2-z^2=0$ in $\mathbf{E}^3$. Consider a part of this conic surface between planes $z=0$ and $z=H>0$, and remove the line $z=-x, y=0$ from this part of conic surface $C$. We come to the surface $D$ defined by the conditions

$$\begin{cases} x^2+y^2-z^2=0 \\ 0<z<H \\ y\neq 0 \quad \text{if } x<0 \end{cases} .$$

Find a domain $D'$ in Euclidean plane such that it is isometric to the surface $D$.

mb) Find a shortest distance between points $A=(1,0,1)$ and $B=(-1,0,1)$, between points $A=(1.0,1)$ and $D=(0,1,1)$, for an ant living on the conic surface $C$.

**3** Consider plane with Riemannian metric given in Cartesian coordinates $(x,y)$ by the formula

$$G=\frac{a\left((dx)^2+(dy)^2\right)}{(1+x^2+y^2)^2} ,$$

and a sphere of the radius $r$ in the Euclidean space $\mathbf{E}^3$. Find a value of the parameter $r$ such that this plane is isometric to the sphere without north pole. Justify you answer. (You may use the formula for Riemannian metric on the sphere in stereographic coordinates.)

**4** Consider catenoid: $x^2+y^2=\cosh^2 z$ and helicoid: $y-x\tan z=0$.

Find induced Riemannian metrics on these surfaces.

Show that under suitable changing of local coordinates metric of catenoid and metric of helicoid have the same appearance.

* This means that there exists an isometry between..... (*finish the sentence*)

**5** a) Consider the domain $D$ on the cone $x^2+y^2-k^2z^2$ defined by the condition $0<z<H$. Find an area of this domain using induced Riemannian metric. Compare with the answer when using standard formulae.

**6** Find an area of 2-dimensional sphere of radius $R$ using explicit formulae for induced Riemannian metric in stereographic coordinates.

**7** Show that two spheres of different radii in Euclidean space are not isometric to each other.

**8** Find new coordinates $u = u(x, y), v = v(x, y)$ in Euclidean space $\mathbf{E}^2$ such that $du^2 + dv^2 = dx^2 + dy^2$. (You may assume that functions $u(x, y), v(x, y)$ are linear: $u = a + bx + cy$, $v = c + dx + fy$, where $a, b, c, d$ are constants.)

Show that the transformation is a composition of translation, rotation and reflection.

* Will answer change if we allow arbitrary (not only linear functions $u(x, y), v(x, y)$)?

**9** Let $D$ be a domain in Lobachevsky plane which is lying between lines $x = a, x = -a$ and outside of the disc $x^2 + y^2 = 1$, $(0 < a < 1)$: $D = \{(x, y)\colon\ |x| < a, x^2 + y^2 > 1\}$,

a) Find the area of this domain.

b*) Find the angles between lines and arc of the circle.

*Lobachevsky plane, i.e. hyperbolic plane is the upper half plane with Riemannian metric* $\frac{dx^2+dy^2}{y^2}$ *in Cartesian coordinates* $x, y$ *(*$y > 0$*).*

**10**$^\dagger$ Find a volume of $n$-dimensional sphere of radius $a$. (You may use Riemannian metric in stereographic coordinates, or you may do it in other way... You just have to calculate the answer.)

Hint: One way to do it is the following. Denote by $\sigma_n$ the volume of $n$-dimensional unit sphere embedded in Euclidean space $\mathbf{E}^{n+1}$. One can see that the volume of $n$-dimensional sphere of the radius $R$ equals to $\sigma_n R^n$. We need to calculate just $\sigma_n$. Consider the following integral:

$$I_k = \int e^{-r^2} dx^1 dx^2 \dots dx^k\,,$$

where $r^2 = (x^1)^2 + (x^2)^2 + \dots + (x^k)^2$. One can see that on one hand

$$I_k = \int e^{-r^2} dx^1 dx^2 \dots dx^k = \left(\int e^{-x^2} dx\right)^n = \pi^{\frac{n}{2}}\,.$$

On the other hand $I_k = \int e^{-r^2} dx^1 dx^2 \dots dx^k = \sigma_{k-1} \int e^{-r^2} r^{k-1} dr$. Comparing these integrals we calculate $\sigma_n$.