Homework 9

1 Let M be a surface embedded in Euclidean space $\mathbf{E}^3$. We say that the triple of vector fields $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ is adjusted to the surface M if $\mathbf{e}, \mathbf{f}, \mathbf{n}$ be three vector fields defined on the points of this surface such that they form an orthonormal basis at any point, so that the vectors $\mathbf{e}, \mathbf{f}$ are tangent to the surface and the vector $\mathbf{n}$ is orthogonal to the surface.

Consider the derivation formulae for adjusted vector fields $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$:

$$d\begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix}, \tag{1}$$

where a, b, c are 1-forms on the surface M.

In terms of 1-forms a, b, c and vector fields $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$. write down the explicit expression for

Weingarten operator, (shape operator),

the mean curvature and the Gaussian curvature of M

* connection,

2* Show that in derivation formulae $\begin{cases} da + b \wedge c = 0 \\ db + c \wedge a = 0 \\ dc + a \wedge b = 0 \end{cases}$.

3 Find explicitly a triple of vector fields $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ adjusted to the surface M if M is a) cylinder, b) cone c) sphere.

4 Using results of the previous exercise find explicit expression for derivation formulae (1) in the case if the surface M is a) cylinder,

b) cone,

c) sphere, and deduce from these results the formulae for Gaussian and mean curvature for cylinder, cone and sphere

5 a) Find a triple of vector fields $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ adjusted to the surface M if a Riemannian metric on a surface M is given by the formula $G = \sigma(u, v)(du^2 + dv^2)$, i.e. u, v are conformal coordinates on the surface.

b)†) Using derivation formulae calculate Gaussian curvature for surface given in conformal coordinates. Show that it is expressed by the formula:

$$K = -\frac{1}{2\sigma}\left(\frac{\partial^2 \sigma(u, v)}{\partial u^2} + \frac{\partial^2 \sigma(u, v)}{\partial v^2}\right).$$