Solutions of Homework 6

1

Calculate the derivatives of the functions $f = x^2+y^2$, $g = e^{-(x^2+y^2)}$ *and* $h = q\log|r| = q\log\left(\sqrt{x^2+y^2}\right)$ *(q is a constant) along vector fields* $\mathbf{A} = x\partial_x + y\partial_y$ *and* $\mathbf{B} = x\partial_y - y\partial_x$*, i.e. calculate* $\partial_{\mathbf{A}}f, \partial_{\mathbf{A}}g, \partial_{\mathbf{A}}h, \partial_{\mathbf{B}}f, \partial_{\mathbf{B}}g, \partial_{\mathbf{B}}h$*.*

$\partial_{\mathbf{A}}f = A_x\frac{\partial f}{\partial x} + A_y\frac{\partial f}{\partial y} = x\cdot 2x + y\cdot 2y = 2(x^2+y^2)$,
$\partial_{\mathbf{A}}g = A_x\frac{\partial g}{\partial x} + A_y\frac{\partial g}{\partial y} = x\cdot 2xg - y\cdot 2yg = -2(x^2+y^2)e^{-(x^2+y^2)}$
$\partial_{\mathbf{A}}h = x\frac{\partial h}{\partial x} + y\frac{\partial h}{\partial y} = \frac{x^2q}{x^2+y^2} + \frac{y^2q}{x^2+y^2} = q$
$\partial_{\mathbf{B}}f = B_x\frac{\partial f}{\partial x} + B_y\frac{\partial f}{\partial y} = -y\cdot 2x + x\cdot 2y = 0$,
$\partial_{\mathbf{B}}g = -y\frac{\partial g}{\partial x} + x\frac{\partial g}{\partial y} = -y\cdot 2xg + x\cdot 2yg = 0$
$\partial_{\mathbf{B}}h = -y\frac{\partial h}{\partial x} + x\frac{\partial h}{\partial y} = \frac{-xyq}{x^2+y^2} + \frac{xyq}{x^2+y^2} = 0$

Remark We can do this exercise or using the formula for directional derivative or using the 1-form, differential of function: $\partial_A f = df(\mathbf{A})$. If $\mathbf{A} = A_x\partial_x + A_y\partial_y$ then

$$\partial_A f = (A_x\partial_x + A_y\partial_y)f = A_xf_x + A_yf_y \text{ or } \partial_A f = df(\mathbf{A}) = (f_xdx + f_ydy)(\mathbf{A}) = f_xdx(\mathbf{A}) + f_ydy(\mathbf{A})$$

$$= f_xdx(A_x\partial_x + A_y\partial_y) + f_ydy(A_x\partial_x + A_y\partial_y) = f_xA_x + f_yA_y\,.$$

2

Perform the calculations of the previous exercise using polar coordinates.

For basic fields $\partial_r, \partial_\varphi$ in polar coordinates r,φ ($r = x\cos\varphi, y = r\sin\varphi$) we have that

$$\partial_r = \frac{\partial x}{\partial r}\partial_x + \frac{\partial y}{\partial r}\partial_y = \cos\varphi\partial_x + \sin\varphi\partial_y = \frac{x}{r}\partial_x + \frac{y}{r}\partial_y = \frac{x\partial_x + y\partial_y}{r} = \frac{\mathbf{A}}{r} \Rightarrow \mathbf{A} = r\partial_r$$

and

$$\partial_\varphi = \frac{\partial x}{\partial\varphi}\partial_x + \frac{\partial y}{\partial\varphi}\partial_y = -r\sin\varphi\partial_x + r\cos\varphi\partial_y = -y\partial_x + x\partial_y \Rightarrow \mathbf{B} = \partial_\varphi$$

We see that fields $\mathbf{A}, \mathbf{B}$ have very simple expression in polar coordiantes. Now calculations become almost immediate because in polar coordinates $f = r^2$, $g = e^{-r^2}$ and $h = q\log r$: $\partial_A f = r\partial_r r^2 = 2r^2$, $\partial_A g = r\partial_r e^{-r^2} = -2r^2e^{-r^2}$, $\partial_A h = r\partial_r(q\log r) = q$. For field B it is even easier , because functions f, g, h do not depend on φ; $\partial_{\mathbf{B}}f = \partial_\varphi f = 0$. Analogously $\partial_{\mathbf{B}}g = \partial_{\mathbf{B}}h = 0$.

3

Consider in $\mathbf{E}^2$ vector fields $\mathbf{A} = x\partial_x + y\partial_y$, $\mathbf{B} = x\partial_y - y\partial_x$, $\mathbf{C} = \partial_x$, $\mathbf{D} = \partial_y$. Calculate the values of 1-forms df, dg on these vector fields if $f = (x^2+y^2)^n$ and $g = \frac{y}{x}$. For vector fields $\mathbf{A}, \mathbf{B}$ perform calculations also in polar coordinates.

We can perform calculations calculating directional derivative $\partial_{\mathbf{A}}f$ or calculating 1-form df then its value on the vector field $\mathbf{A}$ because $\partial_{\mathbf{A}}f = df(\mathbf{A})$.

$df = 2n(x^2+y^2)^{n-1}(xdx + ydy)$. Hence

$$\partial_{\mathbf{A}}f = df(\mathbf{A}) = 2n(x^2+y^2)^{n-1}(xdx + ydy)(x\partial_x + y\partial_y) = 2n(x^2+y^2)^{n-1}(x^2+y^2) = 2n(x^2+y^2)^n$$

$$\partial_{\mathbf{A}}g = A_x\frac{\partial g}{\partial x} + A_y\frac{\partial g}{\partial y} = x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = x\left(\frac{-y}{x^2}\right) + y\left(\frac{1}{x}\right) = \frac{-xy}{x^2} + \frac{y}{x} = 0\,.$$

$$\partial_{\mathbf{B}}f = B_x\frac{\partial f}{\partial x} + B_y\frac{\partial f}{\partial y} = -y\cdot 2xn(x^2+y^2)^{n-1} + x\cdot 2yn(x^2+y^2)^{n-1} = 0\,.$$

$$\partial_{\mathbf{B}} g = dg(\mathbf{B}) = d\left(\frac{y}{x}\right)(\mathbf{B}) = \frac{dy(\mathbf{B})}{x} - \frac{ydx(\mathbf{B})}{x^2} = \frac{dy(x\partial_y - y\partial_x)}{x} - \frac{ydx(x\partial_y - y\partial_x)}{x^2} = 1 + \frac{y^2}{x^2}$$

$$\partial_{\mathbf{C}} f = f_x, \partial_{\mathbf{D}} f = f_y$$

We performed calculations in cartesian coordinates . Now perform calculations for vector fields $\mathbf{A}$ and $\mathbf{B}$ in polar coordinates using the fact that fields $\mathbf{A}, \mathbf{B}$ have a simple appearance in polar coordinates $\mathbf{A} = r\partial_r$ and $\mathbf{B} = \partial_\varphi$ (see the exercise above). Calculate again df, dg on $\mathbf{A}, \mathbf{B}$ in polar coordinates.

In polar coordinates $f = r^{2n}$ and $g = \tan\varphi$ and

$$df = 2nr^{2n-1}dr, \qquad dg = d\tan\varphi = \frac{d\varphi}{\cos^2\varphi}$$

Calculations become VERY TRANSPARENT:

$\partial_{\mathbf{A}} f = r\partial_r(r^{2n}) = 2nr^{2n}$ or $df(\mathbf{A}) = \partial_{\mathbf{A}} f = 2nr^{2n-1}dr(r\partial_r) = 2nr^{2n}dr$

$df(\mathbf{B}) = \partial_\varphi r^{2n} = 0$ or $df(\mathbf{B}) = \partial_{\mathbf{B}} f = 2nr^{2n-1}dr(\partial_\varphi) = 0$

$dg(\mathbf{A}) = r\partial_r(\tan\varphi) = 0$ or $dg(\mathbf{A}) = \partial_{\mathbf{A}} g = d\tan\varphi(r\partial_r) = \frac{d\varphi}{\cos^2\varphi}(r\partial_r) = 0$

$dg(\mathbf{B}) = \partial_\varphi \tan\varphi = \frac{1}{\cos^2\varphi}$ or $dg = d\tan\varphi = \frac{d\varphi}{\cos^2\varphi}(\partial_\varphi) = \frac{1}{\cos^2\varphi}$

4

Calculate the integrals of the form $\omega = \sin y\, dx$ over the following three curves. Compare answers.

C_1: $\mathbf{r}(t)$ $\begin{cases} x = 2t^2 - 1 \\ y = t \end{cases}$, $0 < t < 1$, $\qquad C_2$: $\mathbf{r}(t)$ $\begin{cases} x = 8t^2 - 1 \\ y = 2t \end{cases}$, $0 < t < 1/2$,

C_3: $\mathbf{r}(t)$ $\begin{cases} x = \cos 2t \\ y = \cos t \end{cases}$, $0 < t < \frac{\pi}{2}$

For any curve $\mathbf{r}(t), t_1 < t < t_2$

$$\int_C \omega = \int_C \sin y dx = \int_C \sin y dx(\mathbf{v}) = \int_{t_1}^{t_2} \sin y(t)\frac{dx(t)}{dt}dt$$

where $\mathbf{v} = (x_t, y_t)$.

For the first curve $x_t = 4t$ and

$$\int_{C_1} \omega = \int_0^1 4t\sin tdt = 4(-t\cos t + \sin t)\Big|_0^1 = -4\cos 1 + 4\sin 1$$

For the second curve $x_t = 16t$ and

$$\int_{C_2} \omega = \int_0^{1/2} 16t\sin 2tdt = 4(-2t\cos 2t + \sin 2t)\Big|_0^{1/2} = -4\cos 1 + 4\sin 1$$

Answer is the same. Non-surprising. The second curve is reparameterised first curve ($t \mapsto 2t$) and reparameterisation preserves the orientation.

For the third curve $x_t = -2\sin 2tdt$ and

$$\int_{C_3} w = \int_0^{\frac{\pi}{2}} (-2\sin 2t)\sin(\cos t)dt = -4\left(\cos t\cos(\cos t) - \sin(\cos t)\right)\Big|_0^{\pi/2} = 4\cos 1 - 4\sin 1$$

Answer is the same up to a sign: This curve is reparameterised first curve ($t \mapsto \cos t$) and reparameterisation changes the orientation, because $(\cos t)' = -\sin t < 0$ on the interval $(0, \pi/2)$.

Resumé: In these three examples an integral over the same (non-parameteresed) curve was considered. All the answers are the same up to a sign. Sign changes if reparameterisation changes an orientation.

5

Calculate the integral of the form $\omega = e^{-y}dx + \sin x dy$ over the segment of straight line which connects the points $A = (1,1)$, $B = (2,3)$. At what extent an answer depends on a chosen parameterisation?

Choose any parameterisation of this segment, e.g. $x = 1+t, y = 1+2t$, $0 \leq t \leq 1$. Then $\mathbf{v} = (v_x, v_y) = (1,2)$ $(x_t = 1, y_t = 2)$ and

$$\int_C e^{-y}dx + \sin x dy = \int_0^1 \left(e^{-(1+2t)}x_t + \sin(1+t)y_t\right) dt = \int_0^1 \left(e^{-(1+2t)} + 2\sin(1+t)\right) dt\,.$$

What happens if we choose another parameterisation, i.e. consider reparameterisation $t = t(\tau)$. Answer remains the same if the reparameterisation does not change orientation i.e. $t'(\tau) > 0$. This means that starting and ending points of the curve remain the same.

Answer is multiplied on -1 in other case: if the reparameterisation changes orientation i.e. $t'(\tau) < 0$.

6

Calculate the integral of the form $\omega = xdy$ over the upper arc of the unit circle starting at the point $A = (1,0)$ and ending at the point $(0,1)$.

Calculate the integral of the form $w = xdy$ over arc of the unit circle starting at the point $A = (1,0)$ and ending at the point $(0,1)$.

The equation of the arc is $x^2 + y^2 = 1, x \geq 0, y \geq 0$. We know that answer up to a sign does not depend on parameterisation. Choose an arbitrary parameterisation, e.g.

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases}, \quad 0 \leq t \leq \frac{\pi}{2}\,.$$

Then $\mathbf{v} = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$ and

$$\int_C w = \int_0^{\pi/2} w(\mathbf{v})dt = \int_0^{\pi/2} x(t)y_t dt = \int_0^{\pi/2} \cos t \cos t dt = \int_0^{\pi/2} \cos^2 t dt = \pi/4$$

So for an arbitrary parameterisation answer will be $\pm\pi/2$. ($\pi/2$ if orientation is the same and $= \pi/2$ if opposite)

7

Solve the previous problem for the arc of the ellipse $x^2 + y^2/9 = 1$ defined by the condition $y \geq 0$.

The equation of the arc is $x^2 + y^2/9 = 1, y \geq 0$. We know that answer up to a sign does not depend on parameterisation so choose any, e.g. $x = \cos t, y = 3\sin t$, $0 \leq t \leq \pi$. Then $\mathbf{v} = (-\sin t, 3\cos t)$ and

$$\int_C w = \int_0^{\pi} w(\mathbf{v})dt = \int_0^{\pi} x(t)y_t dt = \int_0^{\pi 2} 3\cos t \cos t dt = \int_0^{\pi} 3\cos^2 t dt = 3\pi/2$$

So for an arbitrary parameterisation answer will be $\pm 3\pi/2$, sign depending of parameterisation.

Exact forms

8

Calculate the integral $\int_C \omega$ *where* $\omega = xdx + ydy$ *and* C *is*

a) the straight line segment $x = t, y = 1 - t,\ 0 \leq t \leq 1$

b) the segment of parabola $x = t,\ y = 1 - t^n,\ 0 \leq t \leq 1,\ n = 2, 3, 4, \ldots$

c) the segment of the sinusoid $x = t,\ y = \cos \frac{\pi}{2} t,\ 0 \leq t \leq 1$

d) **an arbitrary** *curve starting at the point* $(0, 1)$ *and ending at the point* $((1, 0)$.

For any of these curves we can perform calculations naively just using definition of integral
E.g. for the curve a)

$$\int_C w = \int_0^1 (x(t)x_t + y(t)y_t)dt = \int_0^1 (t + (1 - t)(-1))dt = \int_0^1 (2t - 1)dt = 0\,,$$

for the curve b) if $n = 2$

$$\int_C w = \int_0^1 (x(t)x_t + y(t)y_t)dt = \int_0^1 (x(t)x_t + y(t)y_t)dt = \int_0^1 (t + (1 - t^2)(-2t))dt = \int_0^1 (2t^3 - 3t^2)dt = 0\,,$$

for the curve b) in general case:

$$\int_C w = \int_0^1 (x(t)x_t + y(t)y_t)dt = \int_0^1 (x(t)x_t + y(t)y_t)dt =$$

$$\int_0^1 (t + (1 - t^n)(-nt^{n-1}))dt = \int_0^1 (t - nt^{n-1} + nt^{2n-1})dt = 0\,.$$

But we immediately come to these results in a clear and elegant way if we use the fact that $w = xdx + ydy$ is an **exact form**, i.e. $w = df$ where $f = \frac{x^2+y^2}{2}$. Indeed using Theorem we see that for an arbitrary curve starting at the point $A = (0, 1)$ and ending at the point $B = (1, 0)$

$$\int_C w = \int_C df = f(x, y)|_A^B = f(1, 0) = f(0, 1) = 0\,.$$

9

Show that the form 1*-form* $\omega = 2xydx + x^2dy$ *is an exact* 1*-form. Calculate integral of this form over the curves considered in exercises 6) and 7) (upper half of the circle and ellipse)*

One can see that $\omega = \omega = 2xydx + x^2dy = d(x^2y)$ $(d(x^2y) = \frac{\partial(x^2y)}{\partial x}dx + \frac{\partial(x^2y)}{\partial y}dy = 2xydx + x^2dy.)$

The integral of this form over over arc of the unit circle starting at the point $A = (1, 0)$ and ending at the point $(0, 1)$ (see the exercise 6) is equal to $\int_C \omega = f|_B^A = f(1, 0) = f(0, 1) = 0$ because $f = x^2y$ and $f(1, 0) = f(0, 1) = 0$. Answer is equal to zero. Hence it does not depend on orientation of the curve.

The integral of this form over ellipse (see exercise 7) is equal to zero: the integral of exact form over an arbitrary closed curve is equal to zero.

10.

Calculate the differentials of the following 1*-forms:*

a) xdx *, b)* xdy *c)* $xdx + ydy$ *, d)*$xdy + ydx$*, e)* $xdy - ydx$

f) $x^4dy + 4x^3ydx$ *, g)* $xdy + ydx + dz$ *, h)* $xdy - ydx + dz$ *.*

For each 1-forms listed above find a function f *(*0*-form) such that* $df = \omega$*, if possible. If it is not possible, explain why.*

General remark: if $d\omega \neq 0$ then the equation $\omega = df$ has no solution, because if $\omega = df$ then $dw = d(df) = 0$. In other words exact form necessarily has to be closed. The inverse implication is true if the form is defined on the whole Euclidean space.

a) $d(xdx) = dx \wedge dx = 0$. $xdx = df$ where $f = \frac{x^2}{2} + c$, where c is a constant.

b) $d(xdy) = dx \wedge dy \neq 0$. Hence this form is not exact.

c) $d(xdx + ydy) = dx \wedge dx + dy \wedge dy = 0$. $xdx + ydy = d\left(\frac{x^2+y^2}{2} + c\right)$, ($c$ is a constant).

d) $d(xdy + ydx) = dx \wedge dy + dy \wedge dx = dx \wedge dy - dx \wedge dy = 0$. $xdy + ydx = d(xy + c)$,where c is a constant.

e) $d(xdy - ydx) = dx \wedge dy - dy \wedge dx = 2dx \wedge dy \neq 0$. Hence this form is not exact.

f) $d(x^4dy + 4x^3ydx) = 4x^3dx \wedge dy + 4x^3dy \wedge dx = 4x^3(dx \wedge dy + dy \wedge dx) = 0$. $x^4dy + 4x^3ydx = d(x^4y + c)$, where c is a constant.

g) $d(xdy + ydx + dz) = dx \wedge dy + dy \wedge dx + ddz = 0$. $xdy + ydx + dz = d(xy + z + c)$, where c is a constant.

h) $d(xdy - ydx + dz) = dx \wedge dy - dy \wedge dx = 2dx \wedge dy \neq 0$. The form is not exact.

All the exercises below are not compulsory

11†

Consider one-form

$$\omega = \frac{xdy - ydx}{x^2 + y^2} \tag{1}$$

This form is defined in $\mathbf{E}^2 \setminus 0$.

Calculate differential of this form.

Write down this form in polar coordinates

Find a function f such that $\omega = df$.

Is this function defined in the same domain as ω?

First calculate differential in cartesian coordinates with "brute force"

$$d\omega = d\left(\frac{xdy - ydx}{x^2 + y^2}\right) = \frac{d(xdy - ydx)}{x^2 + y^2} - (xdy - ydx) \wedge d\left(\frac{1}{x^2 + y^2}\right) = \frac{2dx \wedge dy}{x^2 + y^2} +$$

$$\frac{(xdy - ydx) \wedge d(x^2 + y^2)}{(x^2 + y^2)^2} = \frac{2dx \wedge dy}{x^2 + y^2} + \frac{(xdy - ydx) \wedge (2xdx + 2ydy)}{(x^2 + y^2)^2} =$$

$$\frac{2dx \wedge dy}{x^2 + y^2} + \frac{2x^2dy \wedge dx + 2y^2dy \wedge dx}{(x^2 + y^2)^2} = 0\,.$$

Much more illuminating to write down this form in polar coordinates then calculate its differential. We know already that $xdy - ydx = r^2d\varphi$. Indeed

$dx = d(r\cos\varphi) = \cos\varphi dr - r\sin\varphi d\varphi = \frac{x}{r}dr - yd\varphi$ and $dy = d(r\sin\varphi) = \sin\varphi dr + r\cos\varphi d\varphi = \frac{y}{r}dr + xd\varphi$. Hence

$$xdy - ydx = x\left(\frac{y}{r}dr + xd\varphi\right) - y\left(\frac{x}{r}dr - yd\varphi\right) = (x^2 + y^2)d\varphi \text{ and } \frac{xdy - ydx}{x^2 + y^2} = d\varphi$$

Hence the form is closed.

For the form $\omega = \frac{xdy - ydx}{x^2+y^2}$ one can consider the function $f = \varphi = \arctan\frac{y}{x}$, such that $\omega = df$, but the function f is not well-defined on whole $\mathbf{E}^2$. It is well-defined e.g. we remove the ray $(-\infty, 0]$.

Note that ω is defined in $\mathbf{E}^2 \setminus 0$, but f is defined on $\mathbf{E}^2 \setminus (-\infty, 0]$.

On the other hand it is well defined in any domain where we can define one-valued continuous function $f = \varphi$, i.e. the domain does not contain a loop which rotates around origin. (The function $f = \varphi$ is multi-valued function in the domain $\mathbf{R}^2 \setminus 0$ which contains loops rotating around origin). E.g. one can see that for an arbitrary convex domain which does not contain the origin, or for an arbitrary domain which does not contain a ray $[-\infty, 0]$ a function $f = \varphi$ is well defined one-valued function.

12†

Calculate the integral of the form $\omega = \frac{xdy - ydx}{x^2+y^2}$ *over the curves* $a), b), c)$ *from the previous exercise 8.*

As it follows from the previous exercise one can consider a domain D such that curves $a), b), c)$ (exercise 8) belong to this domain and a function function $f = \varphi$ is well defined. E.g. one can consider $D = \{x \geq 0, y \geq 0\} \backslash \{x^2 + y^2 \leq \varepsilon\}$. Hence for all these curves

$$\int_C \frac{xdy - ydx}{x^2+y^2} = \varphi|_{\mathbf{r}_1}^{\mathbf{r}_2} = \pm \left((\frac{\pi}{2} - 0 \right) = \pm \frac{\pi}{2} \,.$$

The sign depends on orientation.

13†

What values can take the integral $\int_C \omega$ *if* C *is an arbitrary curve starting at the point* $(0,1)$ *and ending at the point* $((1,0)$ *and* $\omega = \frac{xdy - ydx}{x^2+y^2}$.

Answer is the same as in previous exercise: if the curve **does not pass the origin** then the integral is well-defined, It is equal $\frac{\pi}{2}$ if starting point of the curve is $(1,0)$ and ending point is $(1,0)$. It is equal $\frac{-\pi}{2}$ if vice versa.

Remark Please, note that the form $\omega = \frac{xdy - ydx}{x^2+y^2}$ strictly speaking is not exact, because it is not defined for all points (it is not defined at origin) and moreover its "antiderivative" $f = \varphi$ ($\omega = df$) is not well-defined function.

In the next exercise we show that for 1-forms which are defined in the whole $\mathbf{E}^2$ the exactness coincide with closeness.

14†

Let $\omega = a(x,y)dx + b(x,y)dy$ *be a closed form in* $\mathbf{E}^2$, $d\omega = 0$.

Consider the function

$$f(x,y) = x \int_0^1 a(tx,ty)dt + y \int_0^1 b(tx,ty)dt \tag{2}$$

Show that

$$\omega = df \,.$$

This proves that an arbitrary closed form in $\mathbf{E}^2$ *is an exact form. (Converse implication is always true.)*

Why we cannot apply the formula (2) to the form ω *defined by the expression (1)?*

Perform the calculations: $df = f_x d + f_y dy$.

$$f_x = \int_0^1 a(tx,ty)dt + x \int_0^1 a_x(tx,ty)tdt + y \int_0^1 b_x(tx,ty)tdt \,.$$

and

$$f_y = \int_0^1 b(tx,ty)dt + x \int_0^1 a_y(tx,ty)tdt + y \int_0^1 b_y(tx,ty)tdt \,.$$

On the other hand $d\omega = d(adx + bdy) = (b_x - a_y)dx \wedge dy = 0$. Hence $b_x = a_y$ and

$$f_x = \int_0^1 a(tx,ty)dt + x \int_0^1 a_x(tx,ty)tdt + y \int_0^1 a_y(tx,ty)tdt = \int_0^1 \left(\frac{d}{dt} \left(ta(tx,ty) \right) \right) = ta(tx,ty)\big|_0^1 = a(x,y) \,,$$

because

$$\frac{d}{dt} \left(ta(tx,ty) \right) = a(tx,ty) + xta_x(tx,ty) + yta_y(tx,ty).$$

Analogously

$$f_y = \int_0^1 b(tx,ty)dt + x\int_0^1 b_x(tx,ty)tdt + y\int_0^1 b_y(tx,ty)tdt = \int_0^1 \left(\frac{d}{dt}\left(tb(tx,ty)\right)\right) = tb(tx,ty)\big|_0^1 = b(x,y)\,,$$

We see that $f_x = a(x,y)$ and $f_y = b(x,y)$, i.e. $df = adx + bdy$ $\blacksquare$