Solutions of Homework C2(8)

1 *Let C be the curve defined by the intersection of the plane $\alpha\colon x+2z=2$ with the conic surface M: $x^2+y^2=z^2$.*

Let C_{proj} be the orthogonal projection of this curve onto the plane $z=0$.

Show that the curve C_{proj} is an ellipse.

Explain why the curve C is also an ellipse.

Find the foci of the curve C_{proj}. In particular show that the vertex of the conic surface M is a focus of the ellipse Cproj.

Find the areas of the ellipses C and C_{proj}.

Write down a parameterisation of the ellipse C and of the ellipse C_{proj} (you may choose any parameterisation)

We have that

$$C\colon \quad \begin{cases} x^2+y^2=z^2 \\ z=1-\frac{x}{2} \end{cases}, \tag{1a}$$

and for the orthogonal projection on the horizontal plane:

$$C_{\text{proj}}\colon \quad \begin{cases} x^2+y^2=z(x,y)^2=\left(1-\frac{x}{2}\right)^2 \\ z=0 \end{cases}. \tag{1b}$$

We see that projected curve C_{proj} belongs to the plane $z=0$ and it is described in this plane by the equation $x^2+y^2=\left(1-\frac{x}{2}\right)^2$. Transforming we come to

$$x^2+y^2=\left(1-\frac{x}{2}\right)^2 \Leftrightarrow \frac{3}{4}x^2+x+y^2=1 \Leftrightarrow \frac{3}{4}\left(x^2+\frac{4}{3}x\right)+y^2=1 \Leftrightarrow$$

$$\frac{3}{4}\left(x^2+\frac{4}{3}x+\frac{4}{9}\right)+y^2=1+\frac{1}{3} \Leftrightarrow \frac{3}{4}\left(x+\frac{2}{3}\right)^2+y^2=\frac{4}{3} \Leftrightarrow \frac{9}{16}\left(x+\frac{2}{3}\right)^2+\frac{3}{4}y^2=1\,. \tag{1c}$$

We see that the curve C_{proj} is an ellipse: in Cartesian coordinates $\begin{cases} x'=x+\frac{2}{3} \\ y'=y \end{cases}$ the equation (1c) has the appearance

$$\left(\frac{x'}{a}\right)^2+\left(\frac{y'}{b}\right)^2=1\,, \text{ for } a=\frac{4}{3}, b=\frac{2\sqrt{3}}{3}\,. \tag{1d}$$

The centre of this ellipse is at the point $\left(-\frac{2}{3},0\right)$ (we use ordinary Cartesian coordinates (x,y)). The vertices of the ellipse are the points $A_{1,2}=\left(-\frac{2}{3}\pm\frac{4}{3},0\right)$, on the axis OX (the major axis has the length $L=2a=\frac{8}{3}$), and the points $B_{1,2}=\left(-\frac{2}{3},\pm\frac{2\sqrt{3}}{3}\right)$, (the minor axis has the length $l=2b=\frac{4\sqrt{3}}{3}$). The sum of distances from every point of this ellipse to the foci is equal to $2a=\frac{8}{3}$:

$$C=\left\{K\colon |KF_1|+|KF_2|=2a=\frac{8}{3}\,.\right\}\,. \tag{1e}$$

Find the foci of this ellipse. They are two points on the OX axis symmetric with respect to the centre, the point $\left(-\frac{2}{3}, 0\right)$:

$$F_{1,2} = \left(-\frac{2}{3} \pm c, 0\right) .$$

Note that

$$|A_1F_1| + |A_1F_2| = \left|-2 - \left(-\frac{2}{3} - c\right)\right| + \left|-2 - \left(-\frac{2}{3} + c\right)\right| = 2a = \frac{8}{3} .$$

(Compare with relation (1e).) To find c consider the vertex B_1 of the ellipse:

$$|B_1F_1| + B_1F_2| = 2\sqrt{\left(\frac{2\sqrt{3}}{3}\right)^2 + c^2} \Rightarrow c = \pm\frac{2}{3} .$$

We see that the foci of the ellipse C_{proj} are $F_1 = \left(-\frac{4}{3}, 0\right)$ and $F_2 = (0.0)$. One of the foci of the ellipse C_{proj} is the vertex of the cone: this is a general fact which we know from the lectures: that the vertex of the cone is one of the foci of the orthogonal projection of conic section onto the plane $z = 0$.

The curve C is also an ellipse since its orthogonal projection onto the plane $z = 0$ is an ellipse.

The plane $x + 2z = 2$ intersects the plane $z = 0$ under the angle θ: $\tan\theta = \frac{1}{2}$, i.e. $\cos\theta = 2\sqrt{5}$. The minor axis of the ellipse C has the same length $l = 2b$ as minor axis of the ellipse C_{proj}; the major axis of the ellipse C has the length $\frac{2a}{\cos\theta}$.

Area S of the ellipse C_{proj} is equal to

$$S(C_{\text{proj}}) = \pi ab = \pi \cdot \frac{4}{3} \cdot \frac{2\sqrt{3}}{3} = \frac{8\sqrt{3}\pi}{9},$$

area S of the ellipse C is equal to

$$S(C) = \pi\frac{a}{\cos\theta}b = \frac{S(C_{\text{proj}})}{\cos\theta} = \frac{8\sqrt{3}\pi}{9}\frac{\sqrt{5}}{2} .$$

Finally write down a parameterisation for curves C and C_{proj}. Using equation (1c, 1d) of this curve we come to the parameterisation of the curve:

$$C_{\text{proj}}: \quad \begin{cases} x(t) = a\cos t - \frac{2}{3} = \frac{4}{3}\cos t - \frac{2}{3} \\ y(t) = b\sin t = \frac{2\sqrt{3}}{3}\sin t \\ z(t) = 0 \end{cases}, \quad 0 \le t < 2\pi ,$$

and for the conic section C

$$C \quad \begin{cases} x(t) = a\cos t - \frac{2}{3} = \frac{4}{3}\cos t - \frac{2}{3} \\ y(t) = b\sin t = \frac{2\sqrt{3}}{3}\sin t \\ z(t) = 1 - \frac{x}{2} = 1 - \frac{2}{3}\cos t + \frac{1}{3} = \frac{4}{3} - \frac{2}{3}\cos t \end{cases} \quad , \quad 0 \le t < 2\pi\,.$$

This type of parameterisation is convenient for the purposes of integrating differential forms over conic sections. (see the exercises 3, 4 below)

2 *Let C be the curve defined by the intersection of the plane $\alpha\colon kx + z = 1$ (where k is a real parameter) with the conic surface $M\colon 2x^2 + 2y^2 = z^2$.*

Let C_{proj} be the orthogonal projection of this curve onto the plane $z = 0$.

Find the values of k such that the curve C and the curve C_{proj} are ellipses.

Find the values of k such that the curve C and the curve C_{proj} are hyperbolas.

Show that for $k = \pm\sqrt{2}$ the curves C and C_{proj} are parabolas.

Show that the vertex of the conic surface M, the origin, is the focus of the parabola C_{proj} and that the intersection of the plane α and the horizontal plane ($z = 0$) is the directrix of this parabola.

It is enough to analyse the curve C_{proj} since we know that the curve C is an ellipse, parabola or hyperbola if the curve C_{proj} is an ellipse, parabola or hyperbola, respectively.

We see that $z = 1 - kx$. The equation of the curve C_{proj} is

$$C_{\text{proj}}\colon \quad \begin{cases} 2x^2 + 2y^2 = z(x,y)^2 = (1-kx)^2 \\ z = 0 \end{cases} \tag{1b}$$

We see that the projected curve C_{proj} belongs to the plane $z = 0$ and it is described in this plane by the equation $2x^2 + 2y^2 = (1-kx)^2$. Transforming we come to

$$2x^2 + 2y^2 = (1-kx)^2 \Leftrightarrow (2-k^2)x^2 + 2kx + 2y^2 = 1 \tag{2a}$$

If $k = \pm\sqrt{2}$, i.e. $k^2 = 2$, then we see that C_{proj} is parabola $kx + y^2 = \frac{1}{2}$, $(z = 0)$.

In the case if $k \neq \pm\sqrt{2}$ we continue the transformation (2a) of the curve C_{proj}:

$$C_{\text{proj}}\colon \begin{cases} (2-k^2)x^2 + 2kx + 2y^2 = 1 \\ z = 0 \end{cases} \Leftrightarrow \begin{cases} (2-k^2)\left(x + \frac{k}{2-k^2}\right)^2 + y^2 = 1 + \frac{k^2}{2-k^2} \\ z = 0 \end{cases}. \tag{2c}$$

We see that if $k^2 < 2$ then the curve C_{proj} (and respectively the curve C) is an ellipse, and if $k^2 > 2$ then the curve C_{proj} (and respectively the curve C) will be a hyperbola.

Indeed we have from last equation that if $k^2 \neq 2$ then

$$C_{\text{proj}}\colon \begin{cases} (2-k^2)\left(x + \frac{k}{2-k^2}\right)^2 + 2y^2 = \frac{2}{2-k^2} \\ z = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{(2-k^2)^2}{2}\left(x + \frac{k}{2-k^2}\right)^2 + (2-k^2)y^2 = 1 \\ z = 0 \end{cases}.$$

We see that in Cartesian coordinates $\begin{cases} x' = x + \frac{k}{k^2-2} \\ y' = y \end{cases}$ in the case if $2 < k^2$, the curve $C_{\rm proj}$ in the horizontal plane $z = 0$ has the appearance

$$\left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 = 1\,,\ \text{for } a = \frac{\sqrt{2}}{2-k^2}, b = \frac{1}{\sqrt{2-k^2}}\,, (z=0)\,,$$

and in the case if $2 > k^2$, it has the appearance

$$\left(\frac{x'}{a}\right)^2 - \left(\frac{y'}{b}\right)^2 = 1\,,\ \text{for } a = \frac{\sqrt{2}}{2-k^2}\,, b = \frac{1}{\sqrt{k^2-2}}\,, (z=0)\,.$$

Now return to the case of the parabola: in the case if $k = \pm\sqrt{2}$ we see that the curve is the parabola $kx + y^2 = \frac{1}{2}$ (see equation (2a) above). The intersection of the plane α with horizontal plane $z = 0$ is $kx = 1$, $x = \frac{1}{k}$ $(k = \pm\sqrt{2})$. Consider the distance from the arbitrary point $K = (x, y)$ on this parabola and origin. We have

$$|KO| = \sqrt{x^2+y^2} = \sqrt{x^2 + \frac{1}{2} - kx} = \sqrt{\left(x - \frac{1}{k}\right)^2} = \left|x - \frac{1}{k}\right|\,, (k = \pm\sqrt{2})\,.$$

We see that the focus of the parabola $C_{\rm proj}$ is the vertex of the cone, and the directrix is the line $x = \frac{1}{k}$.

3 *Let C be an ellipse in $\mathbf{E}^2$ with foci $F_1 = (0,0)$, $F_2 = (6.0)$ which passes through the point $B = (0,8)$. Write down the equation of this ellipse.*

Choose a parameterisation of this ellipse and calculate $\int_C xdy - ydx$.

To what extent does this integral depend on the choice of parameterisation?

The foci of this ellipse belong to the axis OX. Hence the equation of the ellipse is

$$\frac{(x-k)^2}{a^2} + \frac{y^2}{b^2} = 1\,. \tag{3}$$

with $a > b$. Now we will find parameters a, b and k. We know that $2a$, the length of the major axis, is the sum of the distances from an arbitrary point of the ellipse to the foci. We have for the point B that

$$|BF_1| + |BF_2| = 8 + \sqrt{8^2+6^2} = 18\,. \Rightarrow a = 9$$

If the ellipse intersects the axis OX at the point $A = (c, 0)$ then $|AF_1|+|AF_2| = |c|+|c-6| = 18$, i.e. $c = -6$ or $c = 12$. Hence the centre of the ellipse is at the point $\frac{-6+12}{2} = 3$, i.e. $k = 3$. To find parameter b in equation (3) put in the coordinates of the point B:

$$\frac{(x-k)^2}{a^2} + \frac{y^2}{b^2} = \frac{(0-3)^2}{9^2} + \frac{8^2}{b^2} = 1 \Rightarrow b^2 = 72.$$

We come to the equation of the ellipse:

$$\frac{(x-3)^2}{81} + \frac{y^2}{72} = 1\,.$$

To calculate the integral $\int_C xdy - ydx$ we choose a convenient parameterisation:

$$C\colon \begin{cases} x = 3 + \sqrt{81}\cos t = 3 + 9\cos t \\ y = \sqrt{72}\sin t = 6\sqrt{2}\sin t \end{cases}, \quad 0 \le t < 2\pi\,.$$

Now calculating the integral. $\mathbf{v} = \begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} -9\sin t \\ 6\sqrt{2}\cos t \end{pmatrix}$. Hence:

$$\int_C xdy - ydx = \int_0^{2\pi} \omega(\mathbf{v})dt = \int_0^{2\pi} (xy_t - yx_t)\, dt =$$

$$\int_0^{2\pi} \Big((3 + 9\cos t)6\sqrt{2}\cos t - 6\sqrt{2}\sin t(-9\sin t)\Big)\, dt =$$

$$\int_0^{2\pi} \Big(54\sqrt{2}(\cos^2 t + \sin^2 t) + 18\sqrt{2}\cos t\Big)\, dt = 2\pi \cdot 54\sqrt{2} = 108\sqrt{2}\pi\,.$$

(Here we use the fact that $\int_0^{2\pi} \cos t dt = 0$.) If we choose another parameterisation the integral will remain the same if this new parameterisation has the same orientation, and it will change the sign if the new parameterisation has the opposite orientation.

4 *Let C be the curve defined by the intersection of the plane $\alpha\colon 2x + z = 1$ with the conic surface $M\colon 5x^2 + 5y^2 = z^2$*

Choose a parameterisation of this conic surface and calculate the integral of the 1-form $\omega = xdy - ydx + dz$ over this conic section.

To what extent does this integral depend on the choice of parameterisation?

b) The curve C is

$$\begin{cases} 5x^2 + 5y^2 = z^2 \\ z = 1 - 2x \end{cases} \Leftrightarrow \begin{cases} 5x^2 + 5y^2 = (1-2x)^2 \\ z = 1 - 2x \end{cases} \Leftrightarrow \begin{cases} x^2 + 4x + 5y^2 = 1 \\ z = 1 - 2x \end{cases} \Leftrightarrow$$

$$\begin{cases} (x+2)^2 + 5y^2 = 5 \\ z = 1 - 2x \end{cases} \Leftrightarrow \begin{cases} \frac{(x+2)^2}{5} + y^2 = 1 \\ z = 1 - 2x \end{cases} \Leftrightarrow$$

We see that the orthogonal projection of the conic section onto the horizontal plane $z = 0$ is

$$C_{\mathrm{proj}}\colon \quad \frac{(x+2)^2}{5} + y^2 = 1\,, (z = 0)\,.$$

It is an ellipse, hence the curve C is also an ellipse. We choose a convenient parameterisation of the ellipse C:

$$\begin{cases} x(t) = \sqrt{5}\cos t - 2 \\ y(t) = \sin t \\ z(t) = 1 - 2x(t) = 1 - 2(\sqrt{5}\cos t - 2) = 5 - 2\sqrt{5}\cos t \end{cases}, \quad 0 \le t < 2\pi\,.$$

Now we calculate the integral $\int_C xdy - ydx + dz$.

Note that dz is an exact form and so that $\int_C dz = 0$, since C is a closed curve. Hence $\int_C xdy - ydx + dz = \int_C xdy - ydx$. Calculating this integral we have that in fact it is not necessary to choose a parameterisation of $z(t)$ since differential form does not depend on dz)

$$\int_C xdy - ydx = \int \left(x(t)dy(\mathbf{v}_y(t)) - y(t)dx(\mathbf{v}_x(t))\right) dt.$$

For the components of velocity vector we have

$$v_x(t) = \frac{dx(t)}{dt} = -\sqrt{5}\sin t\,, v_y(t) = \frac{dy(t)}{dt} = \cos t\,, (\mathbf{v} = v_x\partial_x + v_y\partial_y + v_z\partial_z)$$

and $\int_C xdy - ydx =$

$$\int_0^{2\pi} \left(\left(-2 + \sqrt{5}\cos t\right)\cos t - (\sin t)\left(-\sqrt{5}\sin t\right)\right) dt = \int_0^{2\pi} (\sqrt{5} - 2\cos t)dt = 2\pi\sqrt{5}\,.$$

Under changing the parameterisation the integral does not change (if orientation of new parameterisation is the same), or it changes the sign, if new parameterisation has opposite orientation.

Remark One has to note that if instead the form $\omega = xdy - ydx + dz$ we will integrate the form $\sigma = xdy + ydx$ (see Mock examination) then one can see without any calculation that this integral vanishes

$$\int_C xdy + ydx = 0\,,$$

since the form $\sigma = xdy + ydx = d(xy)$ is the exact 1-form.

5 *Let C be the curve in $\mathbf{E}^3$, defined by the intersection of the conic surface $x^2+y^2 = z^2$ with the plane $kx + z = 1$, and let C_{proj} be the orthogonal projection of the curve C onto the plane $z = 0$.*

Show that if $|k| < 1$ then the curve C is an ellipse.

Show that the curve C_{proj} is a parabola in the case if $k = 1$, and find focus and directrix of this parabola.

We have

$$C\colon \quad \begin{cases} x^2 + y^2 = z^2 \\ z = 1 - kx \end{cases} \Leftrightarrow \begin{cases} x^2 + y^2 = (1 - kx)^2 \\ z = 1 - kx \end{cases} \Leftrightarrow \begin{cases} (1 - k^2)x^2 + 2kx + y^2 = 1 \\ z = 1 - kx \end{cases}.$$

If $k \neq \pm 1$ then

$$(1-k^2)x^2 + 2k = (1-k^2)\left(x + \frac{k}{1-k^2}\right)^2 - \frac{k^2}{1-k^2}\,.$$

and

$$C\colon \quad \begin{cases} (1-k^2)(x+\frac{k}{1-k^2})^2 + y^2 = \frac{1}{1-k^2} \\ z = 1 - kx \end{cases}.$$

We see that if $k^2 < 1$ then this curve is an ellipse, and if $k^2 > 1$ this curve is a hyperbola since the projection is an ellipse or a hyperbola respectively. Indeed

$$C_{\text{proj}}\colon \quad \begin{cases} (1-k^2)(x+\frac{k}{1-k^2})^2 + y^2 = \frac{1}{1-k^2} \\ z = 0 \end{cases}.$$

and in the plane $z = 0$ it is an ellipse with centre at the point $x = -\frac{k}{1-k^2}, y = 0$.

Consider in detail the case $k = 1$. In this case

$$C\colon \quad \begin{cases} 2x + y^2 = 1 \\ z = 1 - x \end{cases}, \quad C_{\text{proj}}\colon \quad \begin{cases} 2x + y^2 = 1 \\ z = 0 \end{cases}.$$

Consider the line l on the intersection of the plane $x + z = 1$ (this plane's section is a parabola) with the horizontal plane $z = 0$. We see that $l\colon x = 1, z = 0$.

One can see that for an arbitrary point (x, y) of the parabola C_{proj} $(z = 0)$ we have

$$\sqrt{x^2 + y^2} = \sqrt{x^2 + 1 - 2x} = |x - 1|\,,$$

i.e. the distance between the parabola C_{proj} and the vertex of conic surface is equal to the distance to the line l. Hence the vertex of conic surface is the focus of the parabola C_{proj} and the line l is the directrix of C_{proj}.

Remark Note that we know the general fact that the vertex of the cone is one of the foci of the projection of the conic section, In this exercise we have proved it explicitly.