EXAM FEEDBACK

INTRODUCTION TO GEOMETRY (Math 20222) . Spring 2014

ANSWER **THREE** OF THE FOUR QUESTIONS
If four questions are answered credit will be given for the best three questions.
Each question is worth 20 marks.

Electronic calculators may not be used

This text is not the text of solutions of exam papers! Here we will discuss the solutions of the exampapers.

1. *Throughout this question* $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ *is an orthonormal basis in* $\mathbf{E}^3$.

(a) *Explain what is meant by saying that two bases in* $\mathbf{E}^3$ *have the same orientation.*

Consider the ordered triple $\{\mathbf{e}+\mathbf{f}, \mathbf{e}-\mathbf{f}, \mathbf{g}\}$.

Show that this triple is a basis.

Show that the bases $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ *and* $\{\mathbf{e}+\mathbf{f}, \mathbf{e}-\mathbf{f}, \mathbf{g}\}$ *have opposite orientations.*

Let an ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ *be a basis in* $\mathbf{E}^3$.

Explain why the bases $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ *and* $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ *have the same orientation or the bases* $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ *and* $\{\mathbf{e}+\mathbf{f}, \mathbf{e}-\mathbf{f}, \mathbf{g}\}$ *have the same orientation.*

`[7 marks]`

(b) *Consider the ordered triple of vectors*

$$\left\{\mathbf{e}\,,\ \frac{3\mathbf{f}+4\mathbf{g}}{5}\,,\ \mathbf{a}\right\}\,,$$

where $\mathbf{a}$ *is such a vector that this triple is an orthonormal basis which has the orientation opposite to the orientation of the basis* $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$.

Find vector $\mathbf{a}$ *(express it as a linear combination of the vectors* $\mathbf{e}, \mathbf{f}, \mathbf{g}$*).*

`[4 marks]`

(c) *Formulate the Euler Theorem about rotations.*

Let P be a linear orthogonal operator on $\mathbf{E}^3$ such that it preserves orientation and

$$P(\mathbf{f}) = \mathbf{g}\,, \quad P(\mathbf{g}) = \mathbf{e}\,.$$

Find $P(\mathbf{e})$.

We know that, due to the Euler Theorem, P is a rotation operator. Find the axis and angle of this rotation.

`[9 marks]`

Discussion of first question

a) Students have no special problems when solving this question. Some students have proved the fact that the ordered triple $\{\mathbf{e}+\mathbf{f}, \mathbf{e}-\mathbf{f}, \mathbf{g}\}$ is a basis checking straightforwardly that these vectors are linear independent. Some students just deduced this fact from the non-degeneracy of transition matrix. Almost all students answered right, that the bases have opposite orientation since the transition matrix has negative determinant.

b) Many students solved this problem in a following way: if $\mathbf{a} = x\mathbf{e} + y\mathbf{f} + z\mathbf{g}$ then using condition of orthonormality they calculated coefficients x, y and z and calculated the vector $\mathbf{a}$ up to a sign: $\mathbf{a} = \pm\left(\frac{3\mathbf{g}-4\mathbf{f}}{5}\right)$. Then the final answer follows from the fact that new basis has opposite orientation. This is right solution. Much more nice solution is based on the fact that orhonormality implies that vector $\mathbf{a}$ must be equal up to a sign to vector product of vectors $\mathbf{e}$ and $\frac{3\mathbf{f}+4\mathbf{g}}{5}$. The orientation arguments imply that

$$\mathbf{a} = -\left(\mathbf{e} \times \frac{3\mathbf{f}+4\mathbf{g}}{5}\right) = \frac{4\mathbf{f}-3\mathbf{g}}{5}\,.$$

Few students solved this question in this way.

c) $P(\mathbf{e})$ has to have unit length and it is orthogonal to vectors $P(\mathbf{f}) = \mathbf{g}$ and $P(\mathbf{g}) = \mathbf{e}$ since P is orthogonal operator. Hence $P(\mathbf{e}) = \pm\mathbf{f}$. Orientation arguments imply that $P(\mathbf{e}) = \mathbf{f}$. The same can be deduced in matrix language: The matrix of operator P in the orthonormal basis $\mathbf{e}, \mathbf{f}\, \mathbf{g}$ is $\begin{pmatrix} x & 0 & 1 \\ y & 0 & 0 \\ z & 1 & 0 \end{pmatrix}$ Analysing orthogonality condition of matrix we come to $x = z = 0$ and $y = \pm 1$. Then we see that $y = 1$ since determinant is equal to 1 (preservation of orinetation) The standard application of Euler Theorem implies that axis of rotation is directed along the vector $\mathbf{e} + \mathbf{f} + \mathbf{g}$ (this is eigenvector) and angle $\phi = \frac{2\pi}{3}$.

Problems:

As usual standard mistake was confusing the notion of orthonormal operator and operator with determinant 1 (or in the language of matrices confusion between orthogonal matrices and matrices with determinant 1.)

Many (too many students!!!???) could not caluclate angle ϕ such that $\cos\phi = -\frac{1}{2}$ (without help of calculator)

There is really very beautiful way to calculate angle ϕ which follows from the fact that P^3 is identity matrix (why?), i.e. $3\phi = 2\pi$. Helas, nobody on exam noticed this.

2.

(a) *Give the definition of a differential* 1*-form on* $\mathbf{E}^n$*.*

Let f *be a function on* $\mathbf{E}^2$ *given by* $f(r,\varphi) = r^2 \sin 2\varphi$*, where* r, φ *are polar coordinates in* $\mathbf{E}^2$*.*

Calculate the value of the 1*-form* $\omega = df$ *on the vector field* $\mathbf{A} = \partial_\varphi$*.*

Express the 1*-form* ω *in Cartesian coordinates* x, y $(x = r\cos\varphi, y = r\sin\varphi)$*.*

Give an example of a non-zero vector field $\mathbf{B}$ *such that the* 1*-form* ω *vanishes on the vector field* $\mathbf{B}$ *at all points of* $\mathbf{E}^2$*.*

`[9 marks]`

(b) *Consider in* $\mathbf{E}^2$ *the differential* 1*-form* $\omega = xdy - ydx$ *and the upper-half of the circle* C *defined by equations* $x^2 + y^2 = 4x$ *and* $y \geq 0$*.*

Give examples of two different parameterisations of the curve C *such that these parameterisations have opposite orientations.*

Calculate the integral $\int_C \omega$*.*

How does the answer depend on a parameterisation?

`[7 marks]`

(c) *Show that the integral of the differential form* $\omega = xdy + ydx$ *over the circle* $x^2 + y^2 = 1$ *is equal to zero.*

Will the answer change if instead of this circle we consider another closed curve in $\mathbf{E}^2$*. Justify your answer.*

`[4 marks]`

Discussion of second question

a) Not a difficult question. Just again I would like to focus attention on the fact that it is much easier to calculate $\omega = df$ in Cartesian coordinates in the following way: first to calculate $f = r^2 \sin 2\varphi = 2r^2 \sin\varphi \cos\varphi = 2xy$ in Cartesian coorinates. Then we come immediately to the answer $\omega = df = d(2xy) = 2xdy + 2ydx$. Some students have chosen more difficult way: they first calculated ω in polar coordinates then have tried to express this 1-form in Cartesian coordinates. Many of them failed in these calculations which need much more time and experience.

b) Students did good this question. Just a remark: many students wrote two parameterisations with different orientations but did not justify the answer: you need to show that velocity changes the sign (i.e. reparameterisation has negative derivative) or explicitly show that during motion the starting point becomes final and vice versa.

When writing parameterisation do not forget to fix the interval of parameter changing:

c) Students who have guessed that the form $\omega = xdy + ydx$ is exact ($\omega = d(xy)$) have no any problem to answer this question: exactness implies that integral over arbitrary closed curve vanishes. Unfortunately many students did not notice the exactness of the form. Much more worse was that some students have claimed the wrong statement that for an arbitrary form integral over closed curve is equal to zero.

General problem: still many students confuse notation for differential d and vector field ∂: The expression $a\partial_x + b\partial_y$ is for vector field, the expression $adx + bdy$– it is 1-form not a vector field!!!

3.

(a) *Describe what is meant by a natural parameter on a curve in* $\mathbf{E}^n$*.*

Find a natural parameter for the following interval of the straight line:

$$C\colon \quad \begin{cases} x = t \\ y = 3t + 2 \end{cases} \quad , \quad 0 < t < \infty \,.$$

Explain why if a curve is given in natural parameterisation, then velocity and acceleration vectors are orthogonal to each other at any given point of the curve.

`[6 marks]`

(b) *Give the definition of the curvature of a curve in* $\mathbf{E}^n$*.*

For a curve in $\mathbf{E}^2$*, write down the formula for the curvature in terms of an arbitrary parameterisation.*

Calculate the curvature $k(t)$ of the parabola $\mathbf{r}(t)\colon\ x = t, y = t^2$*.*

Calculate the curvature of the parabola $y = x^2 + px + q$ at the vertex of this parabola.

`[9 marks]`

(c)

On the sphere $x^2+y^2+z^2=1$ in $\mathbf{E}^3$ find at least two curves C_1 and C_2 such that curvature of the curve C_1 is equal to 1 at all points of this curve and curvature of the curve C_2 is equal to 1000 at all points of this curve.

`[5 marks]`

Discussion of third question

a) No specific problems with this question.

b) Here I would like to discuss how to find curvature of parabola $y = x^2 + px + q$ at vertex. Students (almost all) calculted in the previous subsection the curvature of parabola $x = t, y = t^2$; it is equal to $k(t) = \frac{2}{(1+4t^2)^{3/2}}$. For parabola $y = x^2$ the vertex is at the point $x = y = 0$ and curvature at this point is equal to 2. Now notice that translations $y \to y + a, x \to x + b$ transform parabola $y = x^2$ into the parabola $y = x^2 + px + q$ and they transform vertex to the vertex. iIt is easy to see that translations do not change the curvature (why?). Hence the curvature of the parabola $y = x^2 + px + q$ at the vertex is the same that the curvature of parabola $y = x^2$ at the vertex. It is equal to 2. This is the shortest and nicest solution of the question. Unfortunately nobody came to this solution. Many students calculated curvature of an arbitrary point of parabola $y = x^2 + px + q$, noticed that vertex is at the point $x = -\frac{p}{2}$ and after some calculations some students (but not many) came to the right answer.

c) This question was not considered as an easy question. First of all the solution: The circle of raidus r has the curvature $k = \frac{1}{r}$. So we have to find two circles C_1 and C_2 on the sphere $x^2 + y^2 + z^2 = 1$ with radii r_1 and r_2 such that $r_1 = 1$ and $r_2 = \frac{1}{1000}$. For curve C_1 we can take any great circle on the sphere e.g. intersection of the plane $z = 0$ with the sphere $x^2 + y^2 + z^2 = 1$, i.e. the curve $\begin{cases} x^2 + y^2 = 1 \\ z = 0 \end{cases}$, or intersection of the plane $x = 0$ with the sphere, i.e. the curve $\begin{cases} z^2 + y^2 = 1 \\ x = 0 \end{cases}$ (In fact you can take an intersection of the sphere with any plane $ax + by + cz = 0$ ($abc \neq 0$) passing via origin.) For curve C_2 you can take intersection of sphere with an arbitrary plane which is on the distance $h = \sqrt{1 - r_2^2} = \sqrt{1 - 10^{-6}}$ from the origin. E.g. you can take the plane $z = \sqrt{1 - 10^{-6}}$, i.e. C_2: $\begin{cases} x^2 + y^2 = 10^{-6} \\ z = \sqrt{1 - 10^{-6}} \end{cases}$.

Many students tried to attack this question. Some students noticed that they have to find a circle of the appropriate radius. Many of these students found the curve C_1 and only few students found the curve C_2 on the sphere.

Remark Two years ago on 2012 on exam it was suggested the following problem: to find the curve C on the cylindre $x^2 + y^2 = 1$ such that the curvature of this curve is equal to $k = \frac{1}{100}$. One can consider as a curve C a helix. Unfortunately too many students instead of solving the exam problem and find a circle on the sphere (the simplest possible curve with non-zero curvature) tried desperately to use the solution of former exam problem (helix on the cylinder) and tried to find in vain 'helix' on the sphere.

4.

(a) *Explain what is meant by the shape operator for a surface* $\mathbf{r} = \mathbf{r}(u, v)$ *in* $\mathbf{E}^3$ *by defining its action on an arbitrary tangent vector to the surface.*

Explain why the value of the shape operator is also a tangent vector to the surface.

`[6 marks]`

(b) *Consider a sphere of radius* R *in* $\mathbf{E}^3$*:*

$$\mathbf{r}(\theta, \varphi): \quad \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases} \quad .$$

Show that $\mathbf{n}(\theta, \varphi) = \frac{\mathbf{r}(\theta,\varphi)}{R}$ *is a normal unit vector to the sphere at the point* $\mathbf{r}(\theta, \varphi)$*.*

Let $\mathbf{p}$ *be an arbitrary point on this sphere and* $\mathbf{X}$ *be an arbitrary tangent vector at the point* $\mathbf{p}$*. Calculate the action of shape operator on the vector* $\mathbf{X}$*.*

Calculate the Gaussian and mean curvatures of the sphere.

`[9 marks]`

(c) *On the sphere in part* (**b**) *consider three points* A, B, C *with Cartesian coordinates* $A - (0, 0, R)$, $B - (R, 0, 0)$ *and* $C - (R\cos\varphi, R\sin\varphi, 0)$, *where* φ *is an angle such that* $0 < \varphi < \frac{\pi}{2}$*. Consider the 'isosceles' triangle* ABC *formed by arcs of great circles which connect these points.*

Calculate the integral of the Gaussian curvature of the sphere over the interior of this triangle.

`[5 marks]`

Discussion of fourth question

a) Students have not specific problems to answer this question.

b) Many students who solved this problem checked by straightforward calcualtions the conditions that $\mathbf{n}$ is a unit vector and it is orthogonal to the surface, i.e. $(\mathbf{n}, \mathbf{n}) = 1$ and $(\mathbf{n}, \mathbf{r}_\theta) = (\mathbf{n}, \mathbf{r}_\varphi) = 0$. Sure this is right solution. On the other hand there is is another nicer way to deal with this question (its idea was discussed during tutorials) which based on the relation

$$(\mathbf{r}, \mathbf{r}) = R^2 \,.$$

for points of sphere. Differentiating this relation along θ and φ we come to the conditions $(\mathbf{r}, \mathbf{r}_\theta) = (\mathbf{r}, \mathbf{r}_\varphi) = 0$, This implies the answer.

We see that $\frac{\partial \mathbf{n}_\theta}{\partial \theta} = \frac{\mathbf{r}_\theta}{R}$ and $\frac{\partial \mathbf{n}_\theta}{\partial \varphi} = \frac{\mathbf{r}_\varphi}{R}$, i.e. shape operator S is proportional to identity operator, $S\mathbf{X} = \frac{1}{R}\mathbf{X}$ (up to a sign). Almost all students who were solving this problem calculated the matrix of shape operator but many of them did not write the action of operator on arbitrary vector.

c) We see that

$$\frac{\text{Area of triangle } ABC}{\text{Area of hemisphere}} = \frac{\varphi}{2\pi R^2} \Rightarrow S_{\triangle ABC} = R^2\varphi \Rightarrow \int_{S_{\triangle ABC}} K = R^2\varphi \cdot \frac{1}{R^2} = \varphi \,.$$

This is all. I was really surprised that almost nobody solved this problem. Again as in the previous problem many students instead solving the problem were trying to 'revivre' a different problem from the quesiont 4 of exam paper in 2012 year which sound similar.