Comments on last year geometry exam

Below you see last year exam. At the end I give short comments about solutions and sometimes discuss typical mistakes. Please, note this is not solutions! Just comments!

MATH 20222

Two hours

THE UNIVERSITY OF MANCHESTER

INTRODUCTION TO GEOMETRY

xx-th May/June 2010
xx:xx

ANSWER **THREE** QUESTIONS
If four questions are answered credit will be given for the best three
All questions are worth 20 marks

Electronic calculators may be used, provided that they cannot store text

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P.T.O.

1.

(a) Explain what is meant by saying that the dimension of a vector space is equal to 2.

Let $\mathbf{a}$ and $\mathbf{b}$ be two linearly independent vectors in a 2-dimensional vector space V.

Show that an arbitrary vector $\mathbf{x} \in V$ can be expressed as a linear combination of the vectors $\mathbf{a}$ and $\mathbf{b}$ in a unique way.

[7 marks]

(b) Let V be a vector space. Explain what is meant by a scalar product on V.

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three linearly independent vectors in 3-dimensional Euclidean space $\mathbf{E}^3$ such that the following conditions hold:

- the lengths of all these vectors are equal to 1,
- the vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal to each other,
- the vector $\mathbf{c}$ is orthogonal to the vectors $\mathbf{a} - \mathbf{b}$ and $\mathbf{a} + \mathbf{b}$.

Show that the ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is an orthonormal basis in $\mathbf{E}^3$.

Show that the ordered triple $\{\mathbf{a} - \mathbf{b}, \mathbf{a} + \mathbf{b}, \mathbf{c}\}$ is not an orthonormal basis in $\mathbf{E}^3$.

[7 marks]

(c)

Let $\mathbf{e}$ and $\mathbf{f}$ be two vectors in a 2-dimensional vector space V and let $B(\mathbf{X}, \mathbf{Y})$ be a bilinear form on V such that

$$B(\mathbf{e}, \mathbf{f}) = B(\mathbf{f}, \mathbf{e}) = 1, \qquad B(\mathbf{e}, \mathbf{e}) = B(\mathbf{f}, \mathbf{f}) = 0\,.$$

Show that the ordered pair $\{\mathbf{e}, \mathbf{f}\}$ is a basis in the vector space V.

Show that the bilinear form $B(\mathbf{X}, \mathbf{Y})$ *does not* define a scalar product on the vector space V.

[6 marks]

P.T.O.

2.

(a) Explain what is meant by the vector product of two vectors in oriented 3-dimensional Euclidean space $\mathbf{E}^3$.

Explain why the vector product of two collinear vectors is equal to zero. [7 marks]

(b) Consider the following two vectors

$$\mathbf{a} = \frac{1}{13}\left(3\mathbf{e}_x + 4\mathbf{e}_y + 12\mathbf{e}_z\right), \quad \mathbf{b} = \frac{1}{13}\left(12\mathbf{e}_x + 3\mathbf{e}_y - 4\mathbf{e}_z\right)$$

in oriented Euclidean space $\mathbf{E}^3$, where $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ is an orthonormal basis in $\mathbf{E}^3$.

Define the vector $\mathbf{c}$ by the relation $\mathbf{c} = \mathbf{a} \times \mathbf{b}$.

Without explicitly calculating the vector $\mathbf{c}$ show that the ordered set of three vectors $\{\mathbf{c}, \mathbf{a}, \mathbf{b}\}$ is an orthonormal basis in $\mathbf{E}^3$.

Show that bases $\{\mathbf{c}, \mathbf{a}, \mathbf{b}\}$ and $\{\mathbf{a}, \mathbf{c}, -\mathbf{b}\}$ have the same orientation. [5 marks]

(c) In oriented Euclidean space $\mathbf{E}^3$ consider the following function of three vectors:

$$F(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = (\mathbf{X}, \mathbf{Y} \times \mathbf{Z}),$$

where $(\ ,\ )$ is the scalar product and $\mathbf{Y} \times \mathbf{Z}$ is the vector product in $\mathbf{E}^3$.

Show that $F(\mathbf{X}, \mathbf{X}, \mathbf{Z}) = 0$ for arbitrary vectors $\mathbf{X}$ and $\mathbf{Z}$.

Deduce that $F(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = -F(\mathbf{Y}, \mathbf{X}, \mathbf{Z})$ for arbitrary vectors $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$.

What is the geometrical meaning of the function F? [8 marks]

P.T.O.

3.

(a) Consider in $\mathbf{E}^2$ a differential 1-form $\omega = \frac{1}{2}(xdy - ydx)$ and an ellipse C defined by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a, b > 0)$.

Choose a parameterisation of the ellipse and calculate $\int_C \omega$.

How does your answer depend on a choice of parameterisation? `[6 marks]`

(b) Show that the integral of a differential 1-form $\sigma = \frac{1}{2}(xdy + ydx)$ over the ellipse C considered in part a) is equal to zero.

Will the answer change if instead of the ellipse we consider an arbitrary closed curve? Justify your answer. `[5 marks]`

(c) Give the definition of the curvature of a curve in $\mathbf{E}^n$.

Calculate the curvature $k(t)$ of the parabola $\mathbf{r}(t)\colon\ x = t, y = at^2$, $(a > 0)$.

Consider the following curve (a helix):

$$\mathbf{r}(t)\colon \quad \begin{cases} x(t) = a\cos t \\ y(t) = a\sin t \\ z(t) = ct \end{cases} \quad -\infty < t < \infty, \ \ a > 0\,.$$

Calculate the curvature of this curve.

Evaluate the curvature and give a geometrical meaning of the answers in the limits $c \to 0$ and $c \to \infty$. `[9 marks]`

P.T.O.

4.

(a)

Explain what is meant by the shape operator for a surface $\mathbf{r} = \mathbf{r}(u, v)$ in $\mathbf{E}^3$ by defining its action on an arbitrary tangent vector to the surface. Explain why the value of the shape operator is also a tangent vector to the surface. `[4 marks]`

(b) Consider a surface (cone)

$$\mathbf{r}(h, \varphi): \quad \begin{cases} x = h\cos\varphi \\ y = h\sin\varphi \\ z = h \end{cases} \quad .$$

Calculate the first quadratic form of this surface.

Calculate a unit normal vector field $\mathbf{n}(h, \varphi)$ at points of this surface.

Calculate the shape operator, Gaussian and mean curvature of this surface. `[9 marks]`

(c)

Consider the curve L_{AB} on the cone considered in part b) joining the point $A = (1, 0, 1)$ and the point $B = (-1, 0, 1)$, and defined by the equations $h = 1, \varphi = t$ $(0 \leq t \leq \pi.)$

Calculate the length of this curve.

Does there exist a curve L'_{AB} on the cone joining the same points such that its length is less than the length of the curve L_{AB}? `[7 marks]`

END OF EXAMINATION PAPER

Comments on the first question

The question a) was the bookwork.

The question b): students had not problem to check that the ordered triple $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is an orthonormal basis in $\mathbf{E}^3$. In fact they had to check only that vector $\mathbf{c}$ is orthogonal to vector $\mathbf{a}$ and $\mathbf{b}$. Since $\mathbf{c}$ is orthogonal to the vectors $\mathbf{a}-\mathbf{b}$ and $\mathbf{a}+\mathbf{b}$ hence scalar products $(\mathbf{c}, \mathbf{a}-\mathbf{b}) = 0$ and $(\mathbf{c}, \mathbf{a}-\mathbf{b}) = 0$. Taking the sum of these relations we come to $(\mathbf{c}, 2\mathbf{a}) = 0$, hence $(\mathbf{c}, \mathbf{a}) = 0$ and vectors $\mathbf{a}$ and $\mathbf{c}$ are orthogonal. Substracting these relations we come to orthogonality of vectors $\mathbf{c}$ and $\mathbf{b}$.

Answer why the ordered triple $\{\mathbf{a}-\mathbf{b}, \mathbf{a}+\mathbf{b}, \mathbf{c}\}$ is not an orthonormal basis in $\mathbf{E}^3$ students had to show that the basis *is not* orthonormal. To show it it suffices just to show that at least one condition of orthonormality is not satisfied. E.g. one can check that vector $\mathbf{a}-\mathbf{b}$ has length $\sqrt{2} \neq 1$.

The question *c*) was difficult question. Since dimension of vector space V equals to 2, then supposing that $\mathbf{e}, \mathbf{f}$ is not basis we come to the fact that they are colinear. One can see that if $B(\mathbf{e}, \mathbf{e}) = B(\mathbf{f}, \mathbf{f}) = 0$ and vectors are proportional then $B(\mathbf{e}, \mathbf{f}) = 0$ too. Contradiction.

The bilinear form $B(\mathbf{X}, \mathbf{Y})$ *does not* define a scalar product on the vector space V since $B(\mathbf{e}, \mathbf{e}) = 0$ but $\mathbf{e} \neq 0$ (the condition of positive-definiteness is not fulfilled.) Or one can consider vector $\mathbf{c} = \mathbf{e} - \mathbf{f}$ and calculate that $B(\mathbf{c}, \mathbf{c}) = -2$ is negative.

Comments on the second question

a) This question was the book work. Students have to give 5 axioms defining vector product. Many students instead it just wrote the formulae for vector product. This was not considered as a full answer. Some students who listed the axioms forget the last one about orientation.

b) Almost all students tried to calculate $\mathbf{c}$ by straightforwar application of the formula for vector product. Many of them did calculations right, some of them did mistakes in calculations, Instead of doing these calculations one can do it in another way: note that the lengths of vectors $\mathbf{a}$ and $\mathbf{b}$ equal to 1 since $3^2+4^2+12^2 = 13^2$ and they are orthogonal each other since $3 \cdot 12 + 4 \cdot 3 + 12 \cdot (-4) = 0$. Hence by the axioms of vector product vector $\mathbf{c}$ MUST have the lenght one, it MUST be orthogonal to vectors $\mathbf{a}, \mathbf{b}$. Hence $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is an orthonormal basis

To show that $\{\mathbf{c}, \mathbf{a}, \mathbf{b}\}$ and $\{\mathbf{a}, \mathbf{c}, -\mathbf{b}\}$ one have to see that transition matrix has positive determinant. Many students did it.

c) This was difficult question, specially its second part. See the solution in Coursewrok solutions. Only few students did this exercise.

Comments on the third question

a) To calculate integral one may consider the parameterisation of the ellipse $x = a\cos t$, $y = b\sin t$. (Few students considered another parameterisation, but did not manage to finish calculations.)

About dependence of the answer on parameterisation: the answer is not changed if new parameterisation has the same orientation, i.e. reparameterisation has positive derivative, it changes the sign if parameterisation changes orientation of the curve.

b) Answering this question it is worth to note that $\omega = \frac{1}{2}d(xy)$ is an exact form.

Some students tried explicitly to calculate the integral to see that it vanishes. It is not good plan especially if you are restricted in time.

To answer the second part of the question you cannot avoid the fact that the form ω is exact.

c)

The definition of curvature and curvature of parabola, this is bookwork. Almost all students did this.

Calculating the curvature of parabola many students used right formula $k = \frac{|v_x a_y - a_x v_y|}{|\mathbf{v}|^{\frac{3}{2}}}$ but did mistakes during calculations.

The best way to calculate the curvature of helix is recall the formula

$$k = \frac{|\mathbf{a}_\perp|}{\mathbf{v}^2}$$

for curvature Then it is very easy to calculate the speed: $v^2 = a^2 + c^2$. Little bit more difficult to see and to justify the fact that the acceleration is orthogonal to $\mathbf{a}\perp\mathbf{v}$: $(\mathbf{a}, \mathbf{v}) = (-a\sin t)(-a\cos t) + (a\cos t)(-a\sin t) = 0$. We come to the answer $k = \frac{a}{a^2+c^2}$

Another way to solve: one can use the formula that curvature equals to area of parallelogram divided on the cube of the speed, this is almost the same solution.

Geometrical meaning: if $c \to 0$ path tends to circle. Curvature $k = \frac{a}{a^2+c^2} \to \frac{1}{a}$, the curvature of the circle of the radius a.

If $c \to \infty$ path tends to the vertical line. Curvature $k = \frac{a}{a^2+c^2} \to 0$, the curvature of the straight line.

This was a difficult question.

Comments on the fourth question

This was not the difficult question. On the other hand many students traditionally escape to choose the last question.

a) This was bookwork. You have just to give the definition of the shape operator using the normal unit vector. To explain that value $\mathbf{A}' = S(\mathbf{A})$ of shape operator is also a tangent vector you have to consider derivative of scalar product $(\mathbf{n}, \mathbf{n}) = 1$ with respect to the vector field $\mathbf{A}$ $(\mathbf{n}, \mathbf{n}) = 1$ and come to the relation $(\mathbf{A}', \mathbf{n}) = 0$, i.e. vector $\mathbf{A}'$ is orthogonal to the vector $\mathbf{n}$. This means that vector $\mathbf{A}'$ is tangent to the surface. (see lecture notes.)

b For the cone under consideration tangent vectors $\mathbf{r}_h, \mathbf{r}_\varphi$ equal to $\mathbf{r}_h = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 1 \end{pmatrix}$.

and $\mathbf{r}_\varphi = h \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}$. To calculate first quadratic form we use that $(\mathbf{r}_h, \mathbf{r}_h) = \cos^2\varphi + \sin^2\varphi + 1 = 2$,

$(\mathbf{r}_h, \mathbf{r}_\varphi) = \cos\varphi \cdot (-h\sin\varphi) + \sin\varphi \cdot (h\cos\varphi) = 0\,, (\mathbf{r}_\varphi, \mathbf{r}_\varphi) = (-h\sin\varphi)^2 + (h\cos\varphi)^2 = h^2\,.$

To calculate normal unit vector $\mathbf{n}(h, \varphi)$ note that the vector $\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -1 \end{pmatrix}$ is orthogonal to the vectors $\mathbf{r}_h$ and $\mathbf{r}_\varphi$ and its length is equal to $\sqrt{2}$. Hence $\mathbf{n} = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -1 \end{pmatrix}$.

Now one can see (doing direct calculations!) that $S\mathbf{r}_h = -\frac{\partial \mathbf{n}}{h} = 0$ and $S\mathbf{r}_\varphi = -\frac{\partial \mathbf{n}}{\varphi} = \frac{\mathbf{r}_\varphi}{h\sqrt{2}}$ Hence remembering the definition of Gaussian and mean curvature we come to $K = 0$ and $H = \frac{1}{\sqrt{2}h}$.

The fact that Gaussian curvature of cone equals to zero is strictly related with the fact that one can construct the cone from the sheet of the paper without "shrinking" it.

c) This question (the second part of it) turned to be the most difficult question on exam. Not because it was difficult question: one can solve this question just using common sense. if we unfold the cone then L will become the arc of the circle, and this is not the shortest curve.