Action in gravity

Here we will calculate the action in homogeneous gravity field

$$L = \frac{m\dot{q}^2}{2} - mgq\,,$$

$x(t) = v_0 t - \frac{gt^2}{2}$ and at $t = T$

$$x = v_0 T - \frac{gT^2}{2} = y \Rightarrow v_0 = \frac{y}{T} + \frac{gT}{2}\,.$$

We assume that at $t = 0$ $x = 0$. Hence we have

$$S_T(0; y) = \int_0^T L dt = \frac{m}{2}\int_0^T \left(\frac{y}{T} + \frac{gT}{2} - gt\right)^2 dt - mg\int_0^T \left(\frac{y}{T}t + \frac{gT}{2}t - \frac{gt^2}{2}\right)^2 dt =$$

$$\frac{my^2}{2T} - \frac{mgyT}{2} - \frac{mg^2T^3}{24}\,.$$

One can double checlk that this action obeys Hamilton-Jacobi equation. Hamiltonina is equal to $H(p,q) = \frac{p^2}{2m} + mgy$ and

$$\frac{\partial S(y,T)}{\partial T} + H(q,y)\big|_{q=\frac{\partial S(y,T)}{\partial y}} = \frac{\partial S(y,T)}{\partial T} + \frac{1}{2m}\left(\frac{\partial S(y,T)}{\partial y}\right)^2 + mgy =$$

$$-\frac{my^2}{2T^2} - \frac{mgy}{2} - \frac{mg^2T^2}{8} + \frac{1}{2m}\left(\frac{my}{T} - \frac{mgT}{2}\right)^2 + mgy = 0\,.$$

Now calculate $\mathcal{S}(q,T)$. It is Legendre transform of $S(x,T)$:

$$\mathcal{S}(q,T) = yq - S(y,T)\,,\ \text{with}\quad q = \frac{\partial S}{\partial y} = \frac{my}{T} - \frac{mgT}{2}\,,\text{i.e. } y = \frac{T}{m}\left(q + \frac{mgT}{2}\right) = \frac{qT}{m} + \frac{gT^2}{2}\,,$$

thus

$$\mathcal{S}(q,T) = (yq - S(y,T))\big|_{y=\frac{qT}{m}+\frac{gT^2}{2}} =$$

$$\left(yq - \frac{my^2}{2T} + \frac{mgyT}{2} + \frac{mg^2T^3}{24}\right)\big|_{y=\frac{qT}{m}+\frac{gT^2}{2}} = \frac{q^2T}{2m} + \frac{gT^2q}{2} + \frac{mg^2T^3}{6}\,.$$

We have that

$$q = \frac{\partial S(y,T)}{\partial y}\,,\quad y = \frac{\partial \mathcal{S}(q,T)}{\partial q}$$

and $\mathcal{S}(q,t)$ also obeys Hamilton Jacobi (its Legendre):

$$\frac{\partial \mathcal{S}(q,T)}{\partial T} + H(q,y)\big|_{y=\frac{\partial \mathcal{S}(q,T)}{\partial q}} = \frac{\partial \mathcal{S}(q,T)}{\partial T} + \frac{q^2}{2m} + mgy\big|_{y=\frac{\partial \mathcal{S}(q,T)}{\partial q}}$$

$$= 0\,.$$