Second variation of thick functional

Let $\Phi^* = \Phi^*_S$ be non-linear pull-back of functions generated by thick morphism Φ with generating function $S = S(x,l)$.

$$C(N) \ni g\,, \quad \Phi^*(g) = g(y) + S(x,l) - y^a q_a\,, \quad y^a_g(x) = \frac{\partial S(x,l)}{\partial l_a}\Big|_{l_a = \frac{\partial g(y)}{\partial y^a}}\Big|_{y=y_g} \tag{0}$$

We will calculate :

$$\Phi^*_S(g + tG + \varepsilon H) - \Phi^*_S(g)\,,$$

where H, G are arbitrary functions on N, t, ε are nilpotents such that

$$t^2 = \varepsilon^2 = 0\,, \quad \text{but} \quad t\varepsilon \neq 0\,. \tag{1}$$

We know that

$$\Phi^*_S(g + \varepsilon H) - \Phi^*_S(g) = \varepsilon H(y_g)\,,$$

where y_g is defined by equation (0). Let

$$y_{g+tG} = y_g + t\frac{\delta y_g}{\delta g} * G\,,$$

here $*$ is in general convolution, but we will show that this is local operator.

According (0) we have:

$$y^a_{g+tG}(x) = y^a_g(x) + t\left[\frac{\delta y^a_g}{\delta g} * G\right](x) =$$

$$= \frac{\partial S(x,l)}{\partial l_a}\Big|_{l_a = \frac{\partial(g+tG)(y)}{\partial y^a}}\Big|_{y^a(x) = y^a_{g+tG}}(x) =$$

$$\frac{\partial S(x,l)}{\partial l_a}\Big|_{l_a = \frac{\partial(g+tG)}{\partial y^a}}\Big|_{y^a = y^a_g(x) + t\left[\frac{\delta y^a_g}{\delta g} * G\right](x)} =$$

$$\frac{\partial S(x,l)}{\partial l_a}\Big|_{l_a = \frac{\partial g}{\partial y^a}\big|_{y^a = y^a_g + t\left[\frac{\delta y^a_g}{\delta g} * G\right](x)} + t\frac{\partial G(y)}{\partial y^a}\big|_{y^a = y^a_g(x)}} =$$

$$\frac{\partial S(x,l)}{\partial l_a}\Big|_{l_a = \frac{\partial g}{\partial y^a} + t\frac{\partial^2 g}{\partial y^a \partial y^b}\left[\frac{\delta y^b_g}{\delta g} * G\right](x) + t\frac{\partial G(y)}{\partial y^a}}\Big|_{y^a = y^a_g(x)} =$$

$$\frac{\partial S(x,l)}{\partial l_a}\Big|_{l_a = \frac{\partial g}{\partial y^a}}\Big|_{y^a = y^a_g(x)} +$$

$$t\frac{\partial^2 S(x,l)}{\partial l_a \partial l_d}\frac{\partial^2 g}{\partial y^d \partial y^b}\left[\frac{\delta y_g^b}{\delta g} * G\right](x) + t\frac{\partial G(y)}{\partial y^d}\Big|_{y^a=y^a_g, l_a=\frac{\partial g}{\partial y^a}} =$$

$$y_g^a + t\frac{\partial^2 S(x,l)}{\partial l_a \partial l_d}\frac{\partial^2 g}{\partial y^d \partial y^b}\left[\frac{\delta y_g^b}{\delta g} * G\right](x) + t\frac{\partial G(y)}{\partial y^d}\Big|_{y^a=y^a_g, l_a=\frac{\partial g}{\partial y^a}} .$$

Comparing the beginning and the end of this formula we come to

$$\left[\frac{\delta y_g^a}{\delta g} * G\right](x) = \left(\frac{\partial^2 S(x,l)}{\partial l_a \partial l_d}\frac{\partial^2 g}{\partial y^d \partial y^b}\left[\frac{\delta y_g^b}{\delta g} * G\right](x) + \frac{\partial G(y)}{\partial y^d}\right)\Big|_{l_a=\frac{\partial g}{\partial y},\, y^a=y^a_g} ,$$

i.e.

$$y^a_{g+tG} - y^a_g = t\left[\frac{\delta y_g^a}{\delta g} * G\right] = t\frac{\frac{\partial^2 S(x,L)}{\partial l_a \partial l_d}\frac{\partial G(y)}{\partial y^d}}{\delta^a_b - \frac{\partial^2 S(x,L)}{\partial l_a \partial l_d}\frac{\partial^2 g(y)}{\partial y^d \partial y^b}}\Big|_{l_a=\frac{\partial g}{\partial y}, y^a=y^a_g} ,$$

the denominator means that we have infinite geometric progression over $\frac{\partial^2 S(x,L)}{\partial l_a \partial l_d}\frac{\partial^2 g(y)}{\partial y^d \partial y^b}$. For example in one-dimensional case and if

$$S = \varphi(x) + \frac{1}{2}A(x)l^2$$

we have

$$y_{g+tG} = y_g + \frac{A(x)G'(y)}{1 - A(x)g''(y)}y_g + AG'(1 + Ag'' + A^2(g'')^2 + \ldots)$$

Finally calculate (1):

$$\Phi^*_S(g+tG+\varepsilon H) = \Phi^*_S(g+tG)+\varepsilon H(y_{g+tG}(x)) = \Phi^*_S(g)+tG(y_g(x))+\varepsilon H\left(y_g + t\frac{\delta y_g}{\delta g} * G\right) =$$

$$\Phi^*_S(g) + tG(y_g(x)) + \varepsilon H(y_g) + t\varepsilon\frac{\frac{\partial H}{\partial y^a}\frac{\partial^2 S(x,L)}{\partial l_a \partial l_d}\frac{\partial G(y)}{\partial y^d}}{\delta^a_b - \frac{\partial^2 S(x,L)}{\partial l_a \partial l_d}\frac{\partial^2 g(y)}{\partial y^d \partial y^b}}$$

and it is symmetric....

Consider examples