Action and Hamiltonian

Let $H=H(x,p)$ Hamiltonian , and $S_t(x,q)$ be its action:

$$S_0(x,q)=x^i q_i\ ,\quad \frac{\partial S_t(x,q)}{\partial t}=H\left(\frac{\partial S_t(x,q)}{\partial q},q\right)\,, \tag{0a}$$

i.e. $S_t(x,q)$ is its P-exponent. We know (see also the previous blogs) that

$$p_a=\frac{\partial S_t(x,q)}{\partial x^a}\,,\quad y^a=\frac{\partial S_t(x,q)}{\partial q_a}\,,$$

where (x,p) are initial momenta and coordinates and (y,q)–momenta and coordinates at the time t:

$$\begin{pmatrix}x\\p\end{pmatrix}\Rightarrow\begin{pmatrix}y\\q\end{pmatrix}:\quad \begin{cases}y=y(t,x,p)\\ q=q(t,x,p)\end{cases}\ \text{canonical transformation induced by } H \text{ in time } t\ .$$

$$\frac{\partial y^a(t,x,p)}{\partial t}=\frac{\partial H(y,q)}{\partial q_q}\ \frac{\partial q_a(t,x,p)}{\partial t}=-\frac{\partial H(y,q)}{\partial y^q} \tag{0b}$$

One can see that the following differential equations are obeyed

$$\frac{\partial p_a}{\partial p_b}=\delta_a^b\qquad\text{i.e.}\qquad \frac{\partial^2 S_t(x,q)}{\partial x^a\partial q_c}\frac{\partial q_c(t,x,p)}{\partial p_b}=\delta_a^b\,, \tag{1a}$$

$$\frac{\partial p_a}{\partial x^b}=0\qquad\text{i.e.}\qquad \frac{\partial^2 S_t(x,q)}{\partial x^a\partial x^b}+\frac{\partial^2 S_t(x,q)}{\partial x_a\partial q_c}\frac{\partial q_c(t,x,p)}{\partial x^b}=0\,, \tag{1b}$$

$$\frac{\partial p_a}{\partial t}=0\qquad\text{i.e.}\qquad \frac{\partial^2 S_t(x,q)}{\partial x^a\partial t}+\frac{\partial S_t^2(x,q)}{\partial x_a\partial q_c}\frac{\partial q_c(t,x,p)}{\partial t}=\frac{\partial^2 S_t(x,q)}{\partial x_a\partial t}-\frac{\partial S_t^2(x,q)}{\partial x_a\partial q_c}\frac{\partial H(y,q)}{\partial y^c}=0\,, \tag{1c}$$

$$\frac{\partial y^a}{\partial t}=\frac{\partial H(y,q)}{\partial q_a}=\frac{\partial^2 S_t(x,q)}{\partial q_a\partial t}+\frac{\partial^2 S_t(x,q)}{\partial q_a\partial q_c}\frac{\partial q_c(t,x,p)}{\partial t}=\frac{\partial^2 S_t(x,q)}{\partial q_a\partial t}-\frac{\partial^2 S_t(x,q)}{\partial q_a\partial q_c}\frac{\partial H(y,q)}{\partial y^c}\,, \tag{1d}$$

$$\frac{\partial y^a}{\partial x^b}=\frac{\partial^2 S_t(x,q)}{\partial q_a\partial x^b}+\frac{\partial^2 S_t(x,q)}{\partial q_a\partial q_c}\frac{\partial q_c(t,x,p)}{\partial x^b}\,, \tag{1e}$$

$$\frac{\partial y^a}{\partial p_b}=\frac{\partial^2 S_t(x,q)}{\partial q_a\partial q_c}\frac{\partial q_c(t,x,p)}{\partial p_b}\,. \tag{1e}$$

Claim

Let H be quadratic. Then canonical transformtions (0b) are linear and $S(x,q)$ is quadratic.

Example. `Harmonic oscillator`

Let

$$H = \frac{1}{2}(y^2 + q^2)$$

Then action

$$S_t(x, q) = \frac{xq}{\cos t} + \left(\frac{x^2}{2} + \frac{q^2}{2}\right) \operatorname{tg} t \tag{2}$$

action for oscillator defines the group?

Let

$$H = H(p, q) = U_{ik} x^i x^k + L_i^k x^i p_k + T^{ik} p_i p_k \tag{3}$$

be quadratic Hamiltonian.

Our aim is to define its action $S(x, q)$.

`The claim says that it is quadratic also`

One can see it just straightforwardly

First return to oscilator:

Look for

$$S_t(x, q) = A(t)xq + \frac{1}{2}B(t)x^2 + \frac{1}{2}C(t)q^2 +$$

then Hamilton-Jacobi equation (0) give that

$$y = \frac{\partial S}{\partial q} = A(t)x + C(t)q$$

and

$$\frac{\partial S_t(x, q)}{\partial t} = \frac{dA(t)}{dt}xq + \frac{1}{2}\frac{dB(t)}{dt}x^2 + \frac{1}{2}\frac{dC(t)}{dt}q^2 = H\left(\frac{\partial S_t(x, q)}{\partial q}, q\right) =$$

$$\frac{1}{2}\left(A(t)x + C(t)q\right)^2 + q^2 = A(t)C(t)xq + \frac{1}{2}A^2(t)x^2 + \frac{1}{2}\left(C^2 + 1\right)q^2,$$

thus we come to differential equations

$$\begin{cases} \frac{dC(t)}{dt} = 1 + C^2 \\ \frac{dA(t)}{dt} = A(t)C(t) \\ \frac{dB(t)}{dt} = A^2(t) \end{cases} \text{with bundary conditions} \quad \begin{cases} C(t)\big|_{t=0} = 0 \\ A(t)\big|_{t=0} = 1 \\ B(t)\big|_{t=0} = 0 \end{cases}$$

thus we come to solution

$$\begin{cases} C(t) = \operatorname{tg}(t + C) \\ A(t) = 1\cos(t + C) \\ B(t) = \operatorname{tg}(t + C) \end{cases}$$

This is the answer (compare with example (2))

Now consider the general quadratic case

Now retunr to general case (3)

$$H = H(p,q) = \frac{1}{2}U_{ik}x^i x^k + L_i^k x^i p_k + \frac{1}{2}T^{ik}p_i p_k$$

Look for

$$S(x,q) = q_i A_k^i(t)x^k + \frac{1}{2}B_{ik}(t)x^i x^k + \frac{1}{2}C^{ik}(t)(t)q_i q_k\,, \quad \text{with} \begin{cases} A_k^i(t)\big|_{t=0} = \delta_k^i \\ B_{ik}(t)\big|_{t=0} = 0 \\ C^{ik}(t)\big|_{t=0} = 0 \end{cases} . \qquad (5)$$

One can find $S(x,a)$ in (5) just solving equation (0):

$$y^i = \frac{\partial S}{\partial q_i} = A_k^i(t)x^k + C^{ik}(t)(t)q_k\,,$$

and

$$\frac{\partial S}{\partial t} = q_i \frac{dA_k^i(t)}{dt}x^k + \frac{1}{2}\frac{dB_{ik}(t)}{dt}x^i x^k + \frac{1}{2}\frac{dC^{ik}(t)(t)}{dt}q_i q_k =$$

$$\left(\frac{1}{2}U_{ik}y^i y^k + L_i^k y^i q_k + \frac{1}{2}T^{ik}q_i q_k\right)\Big|_{y^i = \frac{\partial S}{\partial q_i} = A_k^i(t)x^k + C^{ik}(t)(t)q_k} =$$

$$\frac{1}{2}U_{ij}\left(A_k^i(t)x^k + C^{ik}(t)(t)q_k\right)\left(A_r^j(t)x^r + C^{jr}(t)(t)q_r\right) + L_i^k\left(A_k^i(t)x^k + C^{ik}(t)(t)q_k\right)q_k + \frac{1}{2}T^{ik}q_i q_k$$

Comparing tensors with coefficients $x^i x^k$, $x^i q_k$ and $q_i q_k$ we come to first order differential equations. E.g. for $x^i x^k$ we have equation

$$\frac{dB(t)}{dt} = A^+(t)UA(t)\,, \quad B(t)\big|_{t=0} = 0\,.$$

Another way to do it, it is solve equations (1) just step by step, and in this case we will also find the linear canonical transformations.