Let $H = H(x^i, p_j)$ be a function on T^*M, where (x^i, p_j) standard local coordinates on cotangent bundle ($x^i = x^1, \dots, x^n$ local coordinates on n-dimensional manifold M).

This function defines a vector field $\mathbf{X}_H$ on the infinite-dimnensional space $C^\infty(M)$ of functions:

$$\mathbf{X}_H = \int dx H\left(x, \frac{\partial f}{\partial x}\right) \frac{\delta}{\delta x} \,. \tag{1}$$

It is the vector field which defines at every 'point'—function f the infinitesimal curve $\gamma_{\mathbf{X}_H}$:

$$\gamma_{X_H}\colon \quad f \mapsto f + \varepsilon H\left(x^i, p_i = \frac{\partial f}{\partial x^i}\right) , \quad \varepsilon^2 = 0 \,.$$

The $P \exp \int_0^t \mathbf{X}_H d\tau$ of this vector field acting in the space of function sends any function $f = f_0(x)$ to the function $A(x,t)$ which is the solution of Hamilton-Jacobi differential equation:

$$A(x,t)\colon H\left(x, \frac{\partial A}{\partial t}\right) = 0 \quad \text{with boundary condition } A\big|_{t=0} = f(x) \,.$$

This little bit peculiar point of view on the standard Hamilton Jacobi equation induces the following consequences.

Let M be supermanifold, and let H be an odd function on T^*M such that it is quadratic on fibers:

$$H = H^{ik} p_i p_k \ , \tag{2a}$$

and it obeys equation

$$(H, H) = 0 \,, \tag{2b}$$

where $(\_,\_)$ canonical Poisson bracket on M and H.

Then this Hamiltonian via mechanis of *derived bracket* induces odd Poisson bracket, Schouten bracket on M:

$$\forall f, g \in C^\infty(M) \,, \quad \{f, g\}_H = \frac{1}{2} \left((f, H) \,, g \right) \,, (2c)$$

and condition (2a) provides Jacobi identities*. Note that for even Hamiltonian H in equation (1) condition (2b) is obtained automatically.

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* respectively if we consider instead cotangent bundle, the cotangent bundle ΠT^*M with reversed parity of fibers (Π is parity reversing functor), then instead canonical Poisson bracket $(\_,\_)$ on T^*M we come to canonical odd Poisson bracket (Schouten bracket) on $[\_,\_]$ ΠT^*M. Then the even quadratic Hamiltonian $P = P^{ik}\pi_i\pi_k$ ($p(\pi_i) = p(x^i)+1$) defines usual even Poisson bracket on M via derived bracket construction if condition $[H, H] = 0$ is satisfied.

In the case if Hamiltonian H is an arbitrary odd function on momenta, i.e. only condition (2b) is obeyed, and condition (2a) is not necessarily obeyed, then Hamiltonian $H = H(x, \pi)$ defines homotopy Schouten brackets on M, the series of n-brackets:

$$\{\_\}_H\,, \quad \{\_,\_\}_H\,, \quad \{\_,\_,\_\}_H\,, \dots \tag{3a}$$

where

$$\{f_1, \dots, f_n\}_H = \frac{1}{n!}\,(\dots(H, f_1)\dots f_n)$$

In the case of usual Poisson brackets (i.e. brackets generated by quadratic Hamiltonians) What is a generalisation of Poisson map, i.e. map preserving Poisson brackets for the case of homotopy Poisson brackets.

In the work [1] we attempted to answer this question for special case. Then Ted Voronov developed the notion of thick morphism (see [3.4]) which provides the answer on this question.

The answer is the following: Let M N be two (super)manifolds, and H_M, H_N be an odd function on T^*M and T^*N which generate homotopy Scouten bracket on M and N respectively.

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2. H.M.Khudaverdian, T.Voronov *Thick morphisms, higher Koszul brackets, and L_∞-algebroids*, math-arXiv:180810049

3. T.Voronov *"Nonnlinear pullbacks" of functions and L_∞-morphisms for homotopy Poisson structures.* J. Geom. Phys. 111 (2017), 94-110. arXiv:1409.6475

4. T.Voronov *Microformal geometry and homotopy algebras.* Proc. Steklov Inst. Math. 302 (2018), in press. arXiv:1411.6720 [math.DG]

5. *Thick morphisms of supermanifolds and oscillatory integral operators.* Russian Math. Surveys 71 (4) (2016), 784-786. arXiv:1506.02417