Solutions of Homework 3

In all exercises we assume by default that Riemannian metric on embedded surfaces is induced by the Euclidean metric.

1 a) *Consider the domain D on the cone $x^2+y^2-k^2z^2$ defined by the condition $0<z<H$. Find an area of this domain using induced Riemannian metric. Compare with the answer when using standard formulae.*

We have cone with height H with radius $R=kH$ $(k>0)$.

First of all standard answer: The area of cone (of surface of cone) is area of the sector with the radius $\sqrt{H^2+R^2}$ and length of the arc $2\pi R$:

$$S=\frac{1}{2}\cdot\sqrt{R^2+H^2}\cdot 2\pi R=\pi R\sqrt{H^2+R^2}=\pi k\sqrt{1+k^2}H^2\,.$$

Now calculate this are using Riemannian geometry. It follows from the result of the exercise (2) that volume form on the cone equals

$$d\sigma=\sqrt{\det G}dh\wedge d\varphi=k\sqrt{1+k^2}dh\wedge d\varphi$$

since $G=\begin{pmatrix} k^2+1 & 0\\ 0 & k^2h^2\end{pmatrix}$ Hence

$$S=\int_{0<h<H}\sqrt{\det G}dh\wedge d\varphi=\int_{0<h<H}k\sqrt{1+k^2}dh\wedge d\varphi=2\pi k\sqrt{1+k^2}\int_0^H hdh=\pi k\sqrt{1+k^2}H^2\,.$$

(Compare with standard calculations).

2*Find an area of the segment of the height h of the sphere of radius R (surface: $x^2+y^2+z^2=R^2,\leq a\leq a+h$ for an arbitrary a: $-R\leq a\leq R-h$)*

For solutions see lecture notes.

3 *Find an area of 2-dimensional sphere of radius R using explicit formulae for induced Riemannian metric in stereographic coordinates.*

Riemannian metric for sphere (without point) in stereographic coordinates is $G=\frac{4R^4du^2+4R^4dv^2}{(R^2+u^2+v^2)^2}$. We already know that doing transformation $u\mapsto ru, v\mapsto Rv$ we come to the expression

$$G=\frac{4R^2du^2+4R^2dv^2}{(1+u^2+v^2)^2}$$

(see the exercise 1.)

$$G=\begin{pmatrix}\frac{4R^2}{(1+u^2+v^2)^2} & 0\\ 0 & \frac{4R^2}{(1+u^2+v^2)^2}\end{pmatrix},\ \det G=\frac{16R^4}{(1+u^2+v^2)^4}$$

Hence the volume (area)of the sphere equals to

$$S=\int_{\mathbf{R}^2}\sqrt{\det G}dudv=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{4R^2}{(1+u^2+v^2)^2}\right)dudv$$

Choosing polar coordinates $u=r\cos\varphi, v=r\sin\varphi$ we come to

$$S=\int_0^{\infty}\int_0^{2\pi}\left(\frac{4R^2}{(1+r^2)^2}\right)d\varphi rdr=8\pi R^2\int_0^{\infty}\frac{rdr}{1+r^2}=4\pi R^2\,.$$

4 *Show that two spheres of different radii in Euclidean space are not isometric to each other.*

Suppose that these two spheres of different radii are isometric (globally). This means that their volume is the same. Contradiction. (In fact two spheres of different radii are not isometric even locally, since they have different curvatures.)

5 *In the previous exercise you consider Riemannian manifolds* $(\mathbf{R}^2, G^{(1)})$ *and* $(\mathbf{R}^2, G^{(2)})$*, where*

$$G^{(1)} = \frac{a(dx^2+dy^2)}{(1+x^2+y^2)^2}\,, \quad \text{and} \quad G^{(2)} = \frac{4R^4(du^2+dv^2)}{(R^2+u^2+v^2)^2}$$

(The second manifold is sphere of radius R without North pole in stereographic coordinates) You proved in fact that in the case if $a = 4R^2$ *then under isometry* $\begin{cases} u = Rx \\ v = Ry \end{cases}$ *these Riemannina manifolds are isometric. Using the result of previous exercise, Prove now that in the case if the condition* $a = 4R^2$ *is not obeyed, then these manifolds are not isometric.*

It follows from the exercise from the former Homework, thjat The first manifold is isometric to the sphere of radius R' (without one point) such that $4(R')^2 = a$. If $R' \neq R$, i.e. $a \neq 4R^2$ then these sphered are not isometric since they have different areas.

6 *Let D be a domain in Lobachevsky plane which is lying between lines* $x = a, x = -a$ *and outside of the disc* $x^2+y^2 = 1$, $(0 < a < 1)$*:* $D = \{(x,y)\colon\ |x| < a, x^2+y^2 > 1\}$,

a) Find the area of this domain.

b^**) Find the angles between lines and arc of the circle.*

Lobachevsky plane, i.e. hyperbolic plane is the upper half plane with Riemannian metric $\frac{dx^2+dy^2}{y^2}$ *in cartesian coordinates* x, y $(y > 0)$.

a) We have $G = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}$. We see that $\sqrt{\det G} = \frac{1}{y^2}$. Hence

$$S = \int_{x^2+y^2\geq 1, -a\leq x\leq a} \sqrt{\det G}dxdy = \int_{x^2+y^2\geq 1, -a\leq x\leq a} \frac{1}{y^2}dxdy = \int_{-a}^{a}\left(\int_{\sqrt{1-x^2}}^{\infty}\frac{dy}{y^2}\right)dx =$$

$$\int_{-a}^{a}\frac{dx}{\sqrt{1-x^2}} = 2\arcsin a\,.$$

This has a deep geometrical meaning*!

Note that if two metrics G, G' are proportional, $G' = \sigma(\mathbf{x})G$, i.e. $g'_{ik} = \sigma(x)g_{ik}$ then the angles calculated with respect to these metrics are the same:

$$\cos \mathbf{t}\angle(\mathbf{X},\mathbf{Y}) = \frac{G'(\mathbf{X},\mathbf{Y})}{\sqrt{\mathbf{t}G(\mathbf{X},\mathbf{X})}\sqrt{\mathbf{t}G(\mathbf{Y},\mathbf{Y})}} = \frac{\sigma G(\mathbf{X},\mathbf{Y})}{\sqrt{\sigma G(\mathbf{X},\mathbf{X})}\sqrt{\sigma G(\mathbf{Y},\mathbf{Y})}} = \frac{\sigma}{\sigma}\frac{G(\mathbf{X},\mathbf{Y})}{\sqrt{G(\mathbf{X},\mathbf{X})}\sqrt{G(\mathbf{Y},\mathbf{Y})}} = \cos\angle(\mathbf{X},\mathbf{Y})$$

(Two proportional metrics are called conformally equivalent).

Notice that Lobachevsky metric $G = \frac{dx^2+dy^2}{y^2} = \frac{1}{y^2}(dx^2+dy^2)$ is proportional to the Euclidean metric dx^2+dy^2, in other words it is conformally Euclidean metric. Hence the angles will be the same as in the Euclidean metric. (The difference is that straigh lines in Euclidean metric are not “straigh lines” in Lobachevsky plane)

---

* there is a remarkable formula that for an arbitrary triangle sum of its angles minus π is equal to integral of curvature over area of triangle: in the case if curvature if constant (this is the case for sphere and hyperbolic plane) it is just proportional to area of triangle.)

7 *Consider the plane $\mathbf{R}^2$ with standard coordinates (x,y) equipped with the Riemannian metric*

$$G = \frac{dx^2+dy^2}{(1+x^2+y^2)^2}\ .$$

Calculate the area S_a of the domain $x^2+y^2\leq a^2$.

Find the limit S_a when $a\to\infty$.

Show that there is no isometry between the plane with this Riemannian metric and the Euclidean plane $\mathbf{E}^2$.

For Riemannian metric $G=\frac{dx^2+dy^2}{(1+x^2+y^2)^2}$, $\det G = \det\begin{pmatrix}\frac{1}{(1+x^2+y^2)^2} & 0\\ 0 & \frac{1}{(1+x^2+y^2)^2}\end{pmatrix} = \frac{1}{(1+x^2+y^2)^4}$, Hence the area of a domain is equal to

$$\int_{x^2+y^2\leq a^2}\sqrt{\det G}dxdy = \int_{x^2+y^2\leq a^2}\frac{1}{(1+x^2+y^2)^2}dxdy =$$

$$\int_{r\leq a}\int_0^{2\pi}\frac{1}{(1+r^2)^2}rdrd\varphi = 2\pi\int_0^a r\leq a\frac{1}{(1+r^2)^2}rdr = \pi\int_0^{a^2}\frac{1}{(1+u)^2}du = -\pi\frac{1}{1+u}\Big|_0^{a^2} =$$

$\pi\left(1-\frac{1}{1+a^2}\right)$. Taking $a\to\infty$ we see that $S_a=\pi\left(1-\frac{1}{1+a}\right)\to\pi$:

Area of all the plane is equal to π. On the other hand the area of Euclidean plane with standard Euclidean metric is equal to infinity. Hence they are not isometric.

8[†] *Find a volume of n-dimensional sphere of radius a. (You may use Riemannian metric in stereographic coordinates, or you may do it in other way... You just have to calculate the answer.)*

Denote by σ_n the volume of n-dimensional unit sphere embedded in Euclidean space $\mathbf{E}^n$. Then the volume of n-dimensional sphere of the radius R is equal to $\sigma_n R^n$. Now consider the integral

$$I=\int_{-\infty}^{\infty}e^{-t^2}dt=\sqrt{\pi}.$$

For any integer k consider

$$I^k=\pi^{\frac{k}{2}}=\left(\int_{-\infty}^{\infty}e^{-t^2}dt\right)^k=\int_{\mathbf{E}^k}e^{-x_1^2-x_2^2-\dots-x_k^2}dx_1dx_2\dots dx_k\,.$$

Make changing of variables in the volume form $dx_1\wedge dx_2\wedge\dots\wedge dx_k$. Since integrand depend only on the radius we can rewrite the integral above as

$$\int_{\mathbf{E}^k}e^{-x_1^2-x_2^2-\dots-x_k^2}dx_1dx_2\dots dx_k=\int_{\mathbf{E}^k}e^{-r^2}r^{k-1}\sigma_{k-1}dr\,,$$

where σ_{k-1} is a volume of the unit sphere in dimension $k-1$. (Here is the truck!) We have the identity:

$$\pi^{\frac{k}{2}}=\sigma_{k-1}\int_0^{\infty}e^{-r^2}r^{k-1}dr$$

To calculate this integral consider $r^2=t$ we come to

$$\int_0^{\infty}e^{-r^2}r^{k-1}dr=\frac{1}{2}\int_0^{\infty}e^{-t}t^{\frac{k}{2}-1}dt=\frac{1}{2}\Gamma\left(\frac{k}{2}\right)\,.$$

We come to

$$\pi^{\frac{k}{2}} = \sigma_{k-1} \int_0^\infty e^{-r^2} r^{k-1} dr = \frac{\sigma_{k-1}}{2} \Gamma\left(\frac{k}{2}\right) .$$

Thus

$$\sigma_{k-1} = \frac{2\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)}.$$

Recall that $\Gamma(x)$ can be calculated for all $\frac{k}{2}$ using the following recurrent formulae:

1. $\Gamma(n+1) = n!$
2. $\Gamma(x+1) = x\Gamma(x)$
3. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, $(\Gamma(x)\Gamma(1-x) = \pi \sin \pi x)$.

   E.g. the volume of the 15-dimensional unit sphere in $\mathbf{E}^{16}$ equals to $\sigma_{15} = \frac{2\pi^8}{\Gamma(8)} = \frac{2\pi^6}{7!}$