Two hours

THE UNIVERSITY OF MANCHESTER

RIEMANNIAN GEOMETRY

04 June 2018

09:45 – 11:45

Answer **ALL FOUR** questions in Section A (50 marks in total).
Answer **TWO** of the THREE questions in Section B (40 marks in total).
If more than TWO questions in Section B are attempted, the credit will be given for the best TWO answers.

Electronic calculators may not be used.

Feedback

SECTION A

Answer **ALL** FOUR questions

A1.

(a) Explain what is meant by saying that G is a Riemannian metric on a manifold M.

(b) Let $G = g_{ik}(x)dx^i dx^k$, $i, k = 1, \ldots, n$ be a Riemannian metric on an n-dimensional manifold M.
Show that all diagonal components $g_{11}(x), g_{22}(x), \ldots, g_{nn}(x)$ are positive functions.

(c) Consider the plane $\mathbf{R}^2$ with standard coordinates x, y equipped with the Riemannian metric

$$G = \frac{dx^2 + dy^2}{1 + x^2 + y^2}\,.$$

Consider vectors $\mathbf{A} = 2\partial_x + \partial_y$ and $\mathbf{B} = \partial_x + 2\partial_y$ attached at the point $(2, 2)$.
Find the length of these vectors and the cosine of the angle between them (with respect to the metric G).

[10 marks]

Answer on this question is in general alright.

Many students answering the question did not give the full answer on the bookwork subquestion a) about properties of Riemannian metric; in particular they did not emphasize the positive-definitness of Riemannian metric.

Diagonal components $g_{11}, \ldots, g_{nn}$ are positive, since $g_{ii} = \langle \partial_i, \partial_i \rangle > 0$. Many students answered this subquestion.

Surprisingly many mistakes in calculations of length of the vectors, and angles between vectors:

$$|\mathbf{A}| = \sqrt{\langle \mathbf{A}, \mathbf{A} \rangle} = \sqrt{\frac{(A_x^2 + A_y^2)}{(1 + x^2 + y^2)}\Big|_{x=y=2}} = \sqrt{\frac{(2^2 + 1^2)}{(1 + 2^2 + 2^2)}} = \frac{\sqrt{5}}{3}\,,$$

analogously:

$$|\mathbf{B}| = \sqrt{\langle \mathbf{B}, \mathbf{B} \rangle} = \sqrt{\frac{(B_x^2 + B_y^2)}{(1 + x^2 + y^2)}\Big|_{x=y=2}} = \sqrt{\frac{(1^2 + 2^2)}{(1 + 2^2 + 2^2)}} = \frac{\sqrt{5}}{3}\,,$$

To calculate the cosine of the angle between these vectors, notice that metric is conformally Euclidean, hence the answer is the same like for the standard Euclidean metric:

$$\cos\angle(\mathbf{A}, \mathbf{B}) == \frac{(A_x B_x + A_y B_y)}{\sqrt{(A_x^2 + A_y^2)}\sqrt{(B_x^2 + B_y^2)}} = \frac{2 \cdot 1 + 1 \cdot 2}{\sqrt{2^2 + 1}\sqrt{(1^2 + 2^2)}} = \frac{4}{5}\,.$$

We see that calculations of $\cos\angle$ may be simplified using the fact that G is conformally Euclidean. Almost nobody noticed it.

A2.

(a) Explain what is meant by saying that a Riemannian surface is locally Euclidean.

(b) Consider a surface (the upper sheet of a cone) in $\mathbf{E}^3$

$$\mathbf{r}(h,\varphi)\colon \quad \begin{cases} x = h\cos\varphi \\ y = h\sin\varphi \\ z = h \end{cases} \quad , \quad h > 0, 0 \le \varphi < 2\pi\,.$$

Calculate the Riemannian metric on this surface induced by the canonical metric on Euclidean space $\mathbf{E}^3$.

(c) Show that this surface is locally Euclidean.

[10 marks]

Surprisingly many students did not find local coordinates $\begin{cases} u = \sqrt{2}\cos\frac{\varphi}{\sqrt{2}} \\ v = \sqrt{2}\sin\frac{\varphi}{\sqrt{2}} \end{cases}$ (coordinates such that metric becomes locally Euclidean). Few students did wrong attempt to find required local coordinates using substitution $u = \alpha h, v = \beta h\varphi$. This substitution does not work for cone. (The substitution similar to this works for cylindre: local coordinates $u = h$ and $v = r\varphi$ are Euclidean coordinates for the cylindre of the radius R.)

A3.

(a) Explain what is meant by an affine connection on a manifold.

(b) Let ∇ be an affine connection on a 2-dimensional manifold M such that in local coordinates (u,v), all Christoffel symbols vanish except $\Gamma^u_{vv} = u$ and $\Gamma^v_{uu} = v$. Calculate the vector field $\nabla_{\mathbf{X}}\mathbf{X}$, where $\mathbf{X} = \frac{\partial}{\partial u} + u\frac{\partial}{\partial v}$.

(c) Give an example of function $f = f(u,v)$ such that f does not vanish identically ($f \not\equiv 0$) and

$$\nabla_{\mathbf{X}}(f\mathbf{X}) = f\nabla_{\mathbf{X}}\mathbf{X}\,,$$

where $\mathbf{X}$ is a vector field considered above, $\mathbf{X} = \frac{\partial}{\partial u} + u\frac{\partial}{\partial v}$.
Justify your answer.

[15 marks]

In general no problems. Just the following remarks

Answering subquestion b) many students did mistakes in calculations. Consider the fragment of calculations were some students did mistakes. (I will write it here little bit more in details)

$$\nabla_{\frac{\partial}{\partial u}}\left(u\frac{\partial}{\partial v}\right)=\left(\nabla_{\frac{\partial}{\partial u}}u\right)\left(\frac{\partial}{\partial v}\right)+u\nabla_{\frac{\partial}{\partial u}}\left(\frac{\partial}{\partial v}\right)=\left(\partial_{\frac{\partial}{\partial u}}u\right)\left(\frac{\partial}{\partial v}\right)+u\nabla_{\frac{\partial}{\partial u}}\left(\frac{\partial}{\partial v}\right)=$$

$$\left(\frac{\partial}{\partial u}u\right)\frac{\partial}{\partial v}+u\left(\Gamma^u_{uv}\frac{\partial}{\partial u}+\Gamma^v_{uv}\frac{\partial}{\partial v}\right)=\frac{\partial}{\partial v}\,,$$

since $\Gamma^u_{uv}=\Gamma^v_{uv}=0$.

Solving subquestion c) about half of students noticed that one has to find a function f such that it obeys differential equation

$$\partial_{\mathbf{X}}f=f_u+uf_v=0\,,\quad (f\neq 0\,. \tag{A3.1}$$

Some students noticed that a function $f=u^2-2v$ obeys these conditions. Many students noticed that one can consider just constant function, $f=c$, with $c\neq 0$.

Remark: A solution of the differential equation (A3.1), $f_u+uf_v=0$ is

$$f(u,v)=\Phi(u^2-2v)\,,$$

where Φ is an arbitrary function. (The question about finding an *arbitrary* solution was not asked on the exam. Students had just to give an example of solution.)

A4.

(a) Define a geodesic on a Riemannian manifold as a parameterised curve.
Write down the differential equation of geodesics in terms of the Christoffel symbols.
Explain what is meant by un-parameterised geodesic.

(b) What are the geodesics of the surface of a cylinder?
Justify your answer.

(c) Explain why the latitude, (curve $\theta=\theta_0$ in spherical coordinates θ,φ) is not a geodesic on the sphere when $\theta_0\neq\frac{\pi}{2}$.
(You may wish use the fact that the acceleration vector of an arbitrary geodesic on a surface is orthogonal to the surface.)

[15 marks]

a) almost all students wrote right, the equation of geodesic, $\nabla_{\mathbf{v}}\mathbf{v}=0$, and many of them wrote right this differential second order equation in terms of Christoffel symbols.

Just one comment: Do not confuse, please the symbol "$\frac{d}{dt}$" of taking derivative with respect to one variable with the symbol "$\frac{\partial}{\partial x}$" of taking partial derivative! You have to write $\frac{d^2x^i(t)}{dt^2}$, NOT $\frac{\partial^2 x^i}{\partial t^2}$.

b) about of geodesics on cylindre: On of the best ways to see how look geodesics on cylindre is the following: in coordinates (h, φ) on cylindre $G = dh^2 + R^2 d\varphi^2$. Entries of matrix of Riemannian metric are constant functions, hence Christoffel symbols in coordinates h, φ vanish. Thus equations of geodesics becomes $\begin{cases} \frac{d^2h}{dt^2} = 0 \\ \frac{d^2\varphi}{dt^2} = 0 \end{cases}$. THis implies that $\begin{cases} h(t) = h_0 + vt \\ \varphi(t) = \varphi_0 + \Omega t \end{cases}$ Hence geodesics is helix. If $\Omega = 0$ it becomes vertical line, if $v = 0$ it becomes circle.

Almost all students wrote that helices are geodesics on cylindre, on the other hand many students did not justify the answer.

c) Suppose that latitude $\theta = \theta_0$ ($\theta_0 \neq \frac{\pi}{2}$) is a geodesic, i.e. under the suitable parameterisation, the acceleration vector of this curve is orthogonal to the sphere, on the other hand the curve is in the plane $z = \cos\theta_0$, hence acceleration vector has to be in the same plane. Contradiction (in the case if $\theta_0 \neq 0$).

This is the solution, and many students did it succesfully. The subquestion c) becomes not so difficult may be because of the hint.

SECTION B

Answer **TWO** of the THREE questions

B5.

(a) Explain what is meant by saying that F is an isometry between two Riemannian manifolds.

(b) Consider the plane $\mathbf{R}^2$ with standard coordinates x, y and with the Riemannian metric

$$G_{(1)} = e^{-a(x^2+y^2)}(dx^2 + dy^2)\,,$$

and consider the same plane $\mathbf{R}^2$ with another Riemannian metric

$$G_{(2)} = be^{-u^2-v^2}(du^2 + dv^2)\,,$$

where a, b are parameters $a > 0$ and $b > 0$.
(We denote standard coordinates x, y in $\mathbf{R}^2$ in second formula by other letters u, v).
Show that the map $F\colon \begin{cases} u = \sqrt{a}x \\ v = \sqrt{a}y \end{cases}$ between these two Riemannian manifolds is an isometry in the case of $b = \frac{1}{a}$.

(c) Calculate the total area of the plane $\mathbf{R}^2$ with respect to the metric $G_{(1)}$, and the total area of the plane $\mathbf{R}^2$ with respect to the metric $G_{(2)}$.
(You may use the formula $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$.)
Deduce why, in the case where $b \neq \frac{1}{a}$, there is no isometry between these Riemannian manifolds.

[20 marks]

Consider first the solution of this question: *Solution.*

A map F between two Riemannian manifolds $(M_1, G_{(1)})$, $(M_2, G_{(2)})$ is isometry if F is a diffeomorphims (one-one smooth map with smooth inverse) of manifold M_1 on manifold M_2 such that it preserves the metrics, i.e. $G_{(1)}$ is pull-back of $G_{(2)}$: $F^*G_{(2)} = G_{(1)}$. In local coordinates

$$g_{(1)ik}(x) = \frac{\partial y^a(x)}{\partial x^i} g_{(2)ab}(y(x)) \frac{\partial y^b(x)}{\partial x^k}\,, \tag{B5.1}$$

where $y^a = y^a(x)$ is local expression for diffeomorphism F. b) The map $\begin{cases} u = \sqrt{a}x \\ v = \sqrt{a}y \end{cases}$ $(a > 0)$ is obviously diffeomorphism and

$$F^*G_{(2)} = be^{-u^2-v^2}(du^2 + dv^2)\big|_{u=\sqrt{a}x, v=\sqrt{a}y} = bae^{-ax^2-ay^2}(dx^2 + dy^2) = G_{(1)} \text{ if } ba = 1\,. \tag{B5.2}$$

c) Calculate areas. For the first metric $\det G_{(1)} = e^{-2a(x^2+y^2)}$, *and for the second metric* $\det G_{(2)} =$

$be^{-2(u^2+v^2)}$. We have that area of $\mathbf{R}^2$ with respect to the first metric is equal to

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{\det G_{(1)}}dxdy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-a(x^2+y^2)}dxdy=\left(\int_{-\infty}^{\infty}e^{-ax^2}dx\right)^2=\frac{\pi}{a}\,, \qquad (B5.3)$$

since under substitution $z=\sqrt{a}x$ we come to $\int_{-\infty}^{\infty}e^{-ax^2}dx=\int_{-\infty}^{\infty}e^{-z^2}\frac{dz}{\sqrt{a}}=\sqrt{\frac{\pi}{a}}$. Analogously area of $\mathbf{R}^2$ with respect to the second metric is equal to

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{\det G_{(2)}}dudv=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}be^{-u^2-v^2}dudv=b\left(\int_{-\infty}^{\infty}e^{-u^2}du\right)^2=b\pi\,. \qquad (B5.4)$$

We see that areas of these two Riemannian manifolds are different if $b\pi\neq\frac{\pi}{a}$, i.e. if $ba\neq 1$. Hence in the case if $ba\neq 1$ there is no isometry between them.

On one hand the average mark for this question is not low, on the other hand answering this question some students had essential problems.

Almost no student answering this question [1] gave the detailed formulation of what is it isometry. (An answer like: "Isometry is a map that preserves distance.." is very far to be considered as a full answer.)

Two very important points were usually missed:

a) isometry F is the diffeomorphism,

b) almost nobody wrote explicitly the formula (B5.1) for isometry F.

However in spite the average poor answer on the subquestion a) students answered good on the subquestion b). Almost all these students showed that the map F in equation B5.2 in fact is isometry.

Answering the subquestion on calculating integrals some students demonstrated a low culture of the calculations. One can calculate these integrals using the hint (see the solution above) or in the following very simple way in polar coordinates:

$$\int e^{-ax^2-ay^2}dxdy=\int_{\mathbf{E}^2}e^{-ar^2}rdrd\varphi=\int_0^{\infty}rdr\left(\int_0^{2\pi}d\varphi e^{-ar^2}\right)=2\pi\int_0^{\infty}e^{-ar^2}rdr\,.$$

Now consider new variable $z=r^2$ we come to:

$$\int e^{-ax^2-ay^2}dxdy=2\pi\int_0^{\infty}e^{-ar^2}rdr=\pi\int_0^{\infty}e^{-az}dz=\pi\left(-\frac{e^{-az}}{a}\right)\Big|_0^{\infty}=\frac{\pi}{a}\,.$$

Almost all students (answering this question) answered right the question about non-existence of isometry in the case if $ba\neq 1$.

B6.

[1] This was question by choice, and not all students have to answer it.

(a) Give a detailed formulation of the Levi-Civita Theorem. In particular write down the expression for the Christoffel symbols Γ^i_{km} of the Levi-Civita connection in terms of the Riemannian metric $G = g_{ik}(x)dx^i dx^k$.

(b) Consider the upper half-plane, $y > 0$ in $\mathbf{R}^2$ equipped with the Riemannian metric

$$G = \frac{dx^2 + dy^2}{y^2}\,, \qquad (B6.1)$$

(the Lobachevsky plane).
Calculate the Christoffel symbols of the Levi-Civita connection of this Riemannian manifold.

(c) Let ∇' be a symmetric connection on the Lobachevsky plane such that all the Christoffel symbols of this connection in coordinates (x, y) vanish identically.
Explain why this connection does not preserve the Riemannian metric of the Lobachevsky plane.

[20 marks]

Answering the subquestion about Levi-Civita Theorem almost all students (except two) wrote properly the formula

$$\Gamma^m_{ik}(x) = \frac{1}{2} g^{mn}(x) \left(\frac{\partial g_{in}(x)}{\partial x^k} + \frac{\partial g_{kn}(x)}{\partial x^i} - \frac{\partial g_{ik}(x)}{\partial x^n} \right) . \qquad (B6.2)$$

for the Christoffel symbols of the Levi-Civita connection. Almost all students except (two or three) wrote right the definition of the Levi-Civita connection [2], however some students did not state its uniqueness. This is very important, and this is important also for solving the last subquestion.

Answering subquestion b) almost all students (except three of four) calculated right Levi-Civita connection in Lobachevsky plane. Almost all students (except two or three) who calculated Levi-Civita connection of Riemannian metric (B6.1) did it using equation (B6.2). This is alright[3].

The answer on subquestion c) can be done in two ways. This can be deduced as the corollary from the Levi-Civita Theorem, or the counterexample has to be explicitly constructed.

In the first case the full answer is something like this: Suppose ∇' is symetric connection which preserves the Riemannian metric (B6.1). Then *by Levi- Civita Theorem* this connection is unique, hence its Christoffel symbols *has to coincide* with Christofel symbols of the Levi-Civita connection which are calculated above. On the other hand Christoffel symbols of connection ∇' vanish in coordinates

[2] that it is a symmetric connection on M (its Christophel symbols Γ^m_{ik} satisfies the condition: $\Gamma^m_{ik} = \Gamma^m_{ki}$) and it preserves scalar product, i.e. for arbitrary vectors $\mathbf{X}$, $\mathbf{Y}, \mathbf{Z}$ at an arbitrary point

$$\partial_{\mathbf{X}} < \mathbf{Y}, \mathbf{Z} > = < \nabla_{\mathbf{X}} (\mathbf{Y}) , \mathbf{Z} > + < \mathbf{Y}, \nabla_{\mathbf{X}} (\mathbf{Z}) > \ .$$

[3] there is another shorter way to calulate Levi-Civita connection: use Lagrangian $L = \frac{\dot{x}^2+\dot{y}^2}{2y^2}$ of free particle on Lobachevsky plane. Only two (or three students) have chosen this time saving approach

(x, y), and some of Christoffel symbols of Riemannian metric (B6.1) do not vanish identically, (e.g. $\Gamma^x_{xy} = -\frac{1}{y} \not\equiv 0$. Contradiction. Hence the connection ∇' **does not** preserve the Riemannian metric (B6.1).

`Another way to solve this subquestion is the following:`

one has to construct counterexample— vector fields such that the connection ∇' does not preserve the scalar product of these vector fields. E.g. consider vector fields $\mathbf{A} = \partial_y$ and $\mathbf{B} = \partial_x$ $\langle \mathbf{B}, \mathbf{B} \rangle = \frac{1}{y^2}$, $\partial_{\mathbf{A}} \langle \mathbf{B}, \mathbf{B} \rangle = -\frac{1}{y^3}$, and obviously $\langle \nabla'_{\mathbf{A}} \mathbf{B}, \mathbf{B} \rangle = \langle \mathbf{B}, \nabla'_{\mathbf{A}} \mathbf{B} \rangle = 0$, hence

$$\partial_{\mathbf{A}} \langle \mathbf{B}, \mathbf{B} \rangle \neq \langle \nabla'_{\mathbf{A}} \mathbf{B}, \mathbf{B} \rangle + \langle \mathbf{B}, \nabla'_{\mathbf{A}} \mathbf{B} \rangle$$

i.e. this connection is not compatible with the metric.

B7.

(a) State the theorem about the result of parallel transport along a closed curve on a surface in the Euclidean space $\mathbf{E}^3$.

(b) Deduce that the Gaussian curvature of the surface depends only on the induced Riemannian metric on the surface, i.e. it is invariant under isometries (*Gauß Theorema Egregium*).

(c) Consider the sphere of radius R in $\mathbf{E}^3$. How does the Weingarten (shape) operator look for this sphere?
Justify your answer.
Calculate the Gaussian curvature of this sphere.
For an arbitrary point $\mathbf{p}$ of this sphere find local coordinates (u, v) such that the induced Riemannian metric on the sphere at the point $\mathbf{p}$ in these local coordinates is equal to

$$G\big|_{\mathbf{p}} = du^2 + dv^2 \,. \qquad (B7.1)$$

Explain why it is impossible to find coordinates such that this condition holds not only at the point $\mathbf{p}$ but in the vicinity of this point also.

[20 marks]

Almost all students answered properly the subquestion a),

The Theorema Egregium is the corollary of this Theorem. Many students who answered the subquestion a), answered also the subquetion b). One student instead *deducing the Theorema Egregium* from the Theorem on parallel transport came to this Theorem using relation for surfaces in $\mathbf{E}^3$ between Gaussian curvature and Riemann curvature tensor.

Answering subquestion c) students in general had no problems with calculation of Weingarten (shape) operator, but many students could not define local coordinates on the sphere such that condition (B7.1) holds *for arbitrary given point.*

Many students who tried to construct example of metric (B7.1) did it in the right way for the points on equator, where $\theta = \pi/2$. Only two or three students did it properly for arbitrary point. Sure the answer for point on equator implies the answer for arbitrary point. This is obvious from symmetry considerations (one can always find coordinates such that a point $\mathbf{p}$ is an equatorial point). Unfortunately many students did not realise that. Moreover one student did the following: he(she) first wrote the right example of metric which is (B7.1)-like at the equator then he (she) become to explain why it is impossibe to do this for arbitrary point. Sure this is non-sence. All points on the sphere are on an equal footing.

I like that many students who graduated this course understand well that there are no local coordinates on the sphere such that metric looks (B7.1)-like not only at the given point $\mathbf{p}$ but also in a vicinity of this point, i.e. there are no *Euclidean* coordinates on the sphere.

Finally I would like to tell that any student who would like to discuss the questions above is welcomed to contact with me.

END OF EXAMINATION PAPER