Homework 9–10

1. Find coordinate basis vectors, the First quadratic form, unit normal vector field, shape operator, and Gaussian and mean curvatures for the following surfaces:

a) sphere of the radius R: $x^2 + y^2 + z^2 = R^2$,

$$\mathbf{r}(\theta, \varphi) \qquad \begin{cases} x = R \sin\theta \cos\varphi \\ y = R \sin\theta \sin\varphi \\ z = R \cos\theta \end{cases} \qquad (0 \le \varphi < 2\pi, 0 \le \theta \le \pi)\,,$$

b) cylinder $x^2 + y^2 = R^2$,

$$\mathbf{r}(h, \varphi) \qquad \begin{cases} x = R \cos\varphi \\ y = R \sin\varphi \\ z = h \end{cases} \qquad (0 \le \varphi < 2\pi, -\infty < h < \infty)$$

c) cone $x^2 + y^2 - k^2 z^2 = 0$,

$$\mathbf{r}(h, \varphi) \qquad \begin{cases} x = kh \cos\varphi \\ y = kh \sin\varphi \\ z = h \end{cases} \qquad (0 \le \varphi < 2\pi, -\infty < h < \infty)$$

d) graph of the function $z = F(x, y)$

$$\mathbf{r}(u, , v) \qquad \begin{cases} x = u \\ y = v \\ z = F(u, v) \end{cases} \qquad (-\infty < u < \infty, -\infty < v < \infty)$$

in the case if $F(u, v) = Au^2 + 2Buv + Cv^2$

Put down the special case when $F(u, v) = auv$ (saddle).

For the case d) you have to calculate First quadratic form, shape operator and curvatures only at origin.

2 Consider surface defined by equation $z = Ax^2 - Ay^2 = 0$. (See the exercise 1d) above.) Show that this is a sadle: you have to show that under the rotation on the angle $\varphi = \frac{\pi}{4}$ with respect to z-axis it becomes a surface $z - axy = 0$. Find relation between parameters A and a.

3 Show that there are two straight lines which pass through the point $(3, 4, 12)$ on the saddle $z = xy$ and lie on this saddle.

† Show that this is true for an arbitrary point of the saddle.

4 Consider helix $\mathbf{r}(t)$: $\begin{cases} x(t) = a \cos t \\ y(t) = a \sin t \\ z(t) = ct \end{cases}$. Show that this helix belongs to cylinder surface $x^2 + y^2 = a^2$.

a) Using first quadratic form on the surface of cylindre or in a different way calculate length of the helix ($0 \leq t \leq t_0$).

b)what are relations between principal curvatures of cylinder and curvature of helix?

5 Assume that the action of the shape operator at the tangent coordinate vectors $\mathbf{r}_u = \partial_u$, $\mathbf{r}_v = \partial_v$ at the given point $\mathbf{p}$ of the surface $\mathbf{r} = \mathbf{r}(u, v)$ is defined by the relations: $S(\partial_u) = 2\partial_u + 2\partial_v$ and $S(\partial_v) = -\partial_u + 5\partial_v$. Calculate principal curvatures, Gaussian and mean curvatures of the surface at this point.

6 On the sphere $x^2 + y^2 + z^2 = R^2$ (of radius R) in E^3 consider the triangle ABC with vertices at the North Pole and at Equator: $A = (0, 0, R)$, $B = (R, 0, 0)$ and $C = (R\cos\varphi, R\sin\varphi, 0)$. The edges of this triangle are arcs of the meridians and the arc of the Equator.

Find the result of the parallel transport of vector $\mathbf{X} = \mathbf{e}_x$ attached at the North pole along the edges of the triangle ABC.

Let $\Delta\Phi$ be angle of rotation of vector $\mathbf{X}$ under parallel transport along the triangle ABC. Calculate the ratio

$$\frac{\Delta\Phi}{KS_{\triangle ABC}}$$

where K is Gaussian curvature of the sphere and $S_{\triangle ABC}$ is the area of the spherical triangle ABC.

7† On the sphere $x^2 + y^2 + z^2 = R^2$ in $\mathbf{E}^3$ consider the closed curve $\theta = \theta_0, \varphi = t$, $0 \leq t < 2\pi$ (latitude) Find the result of parallel transport of the vector tangent to the sphere along this curve.