

Riemannian Geometry

2020

COURSEWORK

This assignment counts for 20% of your marks.

Solutions are due by 26 March 4pm

Write solutions in the provided spaces.

STUDENT'S NAME:

1

Consider a surface M , the upper sheet of the cone

$$\mathbf{r}(h, \varphi): \begin{cases} x = h \cos \varphi \\ y = h \sin \varphi \\ z = 2h \end{cases}, \quad 0 \leq \varphi < 2\pi, h > 0. \quad (1)$$

Calculate the Riemannian metric G on this surface induced by the Euclidean metric in \mathbf{E}^3 in coordinates (h, φ) .

Show that this surface is locally Euclidean by giving an example of local coordinates (u, v) , which are Euclidean coordinates.

Find the length of the shortest curve which belongs to the surface M , starts at the point $(h_0, 0, 2h_0)$ and ends at the point $(-h_0, 0, 2h_0)$.

[3 marks]

2

Consider a sphere S^2 of the radius a in spherical coordinates

$$\mathbf{r}(\theta, \varphi): \begin{cases} x = a \sin \theta \cos \varphi \\ y = a \sin \theta \sin \varphi \\ z = a \cos \theta \end{cases}, \quad 0 \leq \varphi < 2\pi, 0 < \theta < \pi. \quad (1)$$

Calculate the Riemannian metric G on this surface induced by the Euclidean metric in \mathbf{E}^3 in spherical coordinates (θ, φ) .

Give an example of non-identical transformation which is the isometry of the sphere.

Consider two points $A = (a \sin \theta_0, 0, a \cos \theta_0)$ $B = (-a \sin \theta_0, 0, a \cos \theta_0)$ on this sphere.

Calculate the length of the arc of the latitude $\{\varphi = t\theta = \theta_0\}$ which connects this points.

Explain why this is not the shortest curve on the sphere which connects points A and

B .

Give an argument explaining why sphere is not locally Euclidean.

[3 marks]

3

Recall that the Riemannian metric on the sphere of radius R in the stereographic coordinates is expressed by the formula

$$G_{\text{stereogr.}} = \frac{4R^4(du^2 + dv^2)}{(R^2 + u^2 + v^2)^2}.$$

(a) Give an example of non-identical isometry, i.e. a non-identical transformation of coordinates u, v which preserves this metric.

(b) Give an example of a non-linear transformation of coordinates u, v which preserves this metric.

(Hint: You may find this transformation considering transformations of the sphere.)

(c) Find the length of the line $v = au$ in \mathbf{R}^2 with respect to this metric.

Explain why the length of this curve does not depend on a .

[4 marks]

4

Evaluate the area of the part of the sphere of radius $R = 1$ between the planes given by equations $2x + 2y + z = 1$ and $2x + 2y + z = 2$.

[2 marks]

5

Consider the plane \mathbf{R}^2 with standard coordinates (x, y) equipped with Riemannian metric

$$G = (1 + x^2 + y^2)e^{-a^2x^2 - a^2y^2} (dx^2 + dy^2) .$$

Calculate the total area of this plane.

[1 marks]

6

Consider the upper half-plane $y > 0$ with the Riemannian metric

$$G = \frac{dx^2 + dy^2}{y^2}$$

(the Lobachevsky plane).

Consider in the Lobachevsky plane the domain D defined by

$$D = \{x, y: \quad x^2 + y^2 \geq 1, \quad -a \leq x \leq a\},$$

where a is a parameter ($0 < a < 1$).

Find the area of the domain D (with respect to the metric G).

Consider the points $A_t = (-a, t)$, $B_t = (a, t)$ on the vertical lines delimiting the domain D . Show that the distance between these points tends to 0 if $t \rightarrow \infty$.

[3 marks]

7

(a) Let ∇ be an affine connection on the 2-dimensional manifold M such that in local coordinates (u, v) , $\nabla_{\frac{\partial}{\partial u}} \left(u^2 \frac{\partial}{\partial v} \right) = 3u \frac{\partial}{\partial v} + u \frac{\partial}{\partial u}$.

Calculate the Christoffel symbols Γ_{uv}^u and Γ_{uv}^v of this connection.

[1 marks]

8

a) Let ∇ be an arbitrary connection on n -dimensional manifold M and let $\{\Gamma_{km}^i(x)\}$ be the Christoffel symbols of this connection. Let $\omega = \omega_i(x)dx^i$ be a differential form on M . Show that

$$\tilde{\Gamma}_{km}^i = \Gamma_{km}^i + \delta_k^i \omega_m$$

are Christoffel symbols of the new connection.

(b) Let $\Gamma_{km}^{i(1)}$ be the Christoffel symbols of a connection $\nabla^{(1)}$ and $\Gamma_{km}^{i(2)}$ be the Christoffel symbols of a connection $\nabla^{(2)}$.

Show that the linear combinations $\frac{2}{3}\Gamma_{km}^{i(1)} + \frac{1}{3}\Gamma_{km}^{i(2)}$, are Christoffel symbols for some new connection.

Explain why $\frac{1}{2}\Gamma_{km}^{i(1)} + \frac{1}{3}\Gamma_{km}^{i(2)}$ are not Christoffel symbols for some connection.

[3 marks]

