Discussion of Coursework's solutions

April 2017

Question 1

Answering this question students have problems with subquestion **c)** Many students were trying straightforwardly to calculate the length of the curves C_1', and C_2' instead observing the fact that their lengths are equal to the lengths of the curves C_1, C_2 respectively.

The curves C_1, C_2 are the great circles on the sphere, since they are intersections of the sphere with the planes passing thorugh origin. We see that in stereographic coordinates

$$\begin{cases} u = \frac{Rx}{R-z} \\ v = \frac{Ry}{R-z} \end{cases}, \quad \begin{cases} x = \frac{2uR^2}{u^2+v^2+R^2} \\ y = \frac{2vR^2}{u^2+v^2+R^2} \\ z = \frac{u^2+v^2-R^2}{u^2+v^2+R^2} R \end{cases}.$$

the curve C_1: $\begin{cases} x+y=0 \\ x^2+y^2+z^2=R^2 \end{cases}$ transofrms to the line C_1' described by the equation $u=-v$, and the length of this line has to be equal to $2\pi R$ since stereographic projection is isometry, it preserves the metric.

Respectively curve C_2: $\begin{cases} x+z=0 \\ x^2+y^2+z^2=R^2 \end{cases}$ transforms to the curve

$$x+z = \frac{2uR^2}{R^2+u^2+v^2} + R\frac{u^2+v^2-R^2}{R^2+u^2+v^2} = 0\,.$$

We see that C_2' image of the curve C_2 is defined by equation: $2uR+u^2+v^2-R^2=0$. This is the circle of radius $\sqrt{2}R$: $2uR+u^2+v^2-R^2=(u+R)^2+v^2-2R^2=0$, and its length in the Rimeannian metric induced by stereopgraphic projection is also $2\pi R$, The curve C_1' is straight line, the curve C_2' is the circle, and these both curves have the same length.

Sure one can calculate the length of these curves straightforwardly. For the first curve it is just easy walk, but for the second curve this is little bit tricky exercise.

Show it.

For the curve C_1': $u+v=0$, parameterised by $u=t, v=-t$ if we calculate its length straightforwardly in the Rimeannian metric $G=\frac{4R^4(du^2+dv^2)}{(R^2+u^2+v^2)^2}$ we come to the integral:

$$L(C_1') = \int \sqrt{\left(\frac{4R^4}{(R^2+u^2+v^2)^2}\right)(u_t^2+v_t^2)}dt =$$

$$\int_{-\infty}^{\infty}\sqrt{\left(\frac{4R^4}{(R^2+u^2+u^2)^2}\right)(1+1)du} = 2R\int_{-\infty}^{\infty}\frac{dw}{1+w^2} = 2\pi R\,,$$

where $w = R\sqrt{a^2+1}u$.

For the curve C_2' calculations are not so evident. Consider the parameterisation of the circle C_2': $\begin{cases} u = \sqrt{2}R\cos t - R \\ v = \sqrt{2}R\sin t \end{cases}$ and calculating the length we come to the integral

$$L(C_2') = \int_0^{2\pi}\sqrt{\frac{4R^4(u_t^2+v_t^2)}{(R^2+u^2+v^2)^2}} = \int_0^{2\pi}\frac{2\sqrt{2}R^3}{(4R^2-2\sqrt{2}R^2\cos t)}dt =$$

$$\frac{R\sqrt{2}}{2}\int_0^{2\pi}\frac{dt}{1-\frac{\sqrt{2}}{2}\cos t}dt = 2\pi R$$

One can calculate the last integral using some tricks (see Appendix at the end of discussions where we give two different receipts for caclulation of this integral)

Question 2

Almost all students did calculations of the vertical , horizontal segments and the length fo the arc of the circle.

On the other hand the explanations of the fact that the lenght of the arc $= 2\log\cot\frac{\varphi}{2}$ is less than the length of the horisontal segment $= 2\cot\varphi$ was not satisfactory. Many students proved it, but almost all considerations were not performed carefully. E.g. some students were proving it based on the inequality $x > \log x$, but they did not make the considerations clear . (in fact the length is positive, and $|x|$ is not bigger than $|\log x|$.)

Sure the geometrical meaning of this inequality is that arc of the circle is geodesic, only few students mentioned it.

15 credits students did well additional exercises. The straightforward checking of the fact that inversion is isometry is little bit long. One can simplify considerations using complex variables $z = x + iy$.

Question 3

Some students had trouble to answer the quesiton on finding coordinates u, v for sphere such that area element is equal to $dudv$.

In fact for sphere of radius R, $G = R^2(d\theta^2 + \sin^2\theta d\varphi^2)$ and $\det G = R^4 \sin^2\theta$, $dv = R^2 \sin\theta d\theta d\varphi$ (See lecture notes.) Note that $d\cos\theta = -\sin\theta d\theta$. We have for volume form $\sin\theta d\theta \wedge d\varphi = d(-\cos\theta) \wedge d\varphi$. Thus if we consider $u = -\cos\theta, v = \varphi$ then $du \wedge dv = d(-\cos\theta) \wedge d\varphi = \sin\theta d\theta \wedge d\varphi$.

Express Riemannian metric in coordinates u, v. $du = \sin\theta d\theta$. Hence $d\theta = \frac{du}{\sin\theta} = \frac{du}{\sqrt{1-u^2}}$ and $d\theta^2 + \sin^2\theta d\varphi^2 = \frac{du^2}{1-u^2} + (1-u^2)dv^2$. $\blacksquare$

Remark One can find another local coordinates such that volume form equals to $du \wedge dv$,e.g. $u = 2\sin^2\frac{\theta}{2}$, $v = \varphi$.

There was some issue with calculating the distance between planes. In fact the normal equations of these planes are $\frac{2}{3}x + \frac{2}{3}x + \frac{2}{3}y + \frac{2}{3}z = \frac{1}{3}$ and $\frac{2}{3}x + \frac{2}{3}y + \frac{2}{3}z = \frac{2}{3}$ The first plane is on the distance $\frac{1}{3}$ of the origin, the second is on the distance $\frac{2}{3}$. They both intersect the sphere. The distance between the planes equals to $h = \frac{1}{3}$. Some of students put it 1 without calculations, and this was wrong.

Some students confused in caclualtions of integral: Total area is equal to $S = \int \sqrt{\det G} dudv = \int_{-\infty}^{\infty} \int_{i\infty}^{i\infty} (1 + u^2 + v^2) e^{-u^2-v^2} dudv = \int_0^{\infty} \int_0^{2\pi} (1 + r^2) e^{-r^2} r dr d\varphi =$
$2\pi \int_0^{\infty} (1 + r^2) e^{-r^2} r dr d = \pi \int_0^{\infty} (1 + z) e^{-z} dz = 2\pi$.

Question 4

Almost all studnets answered this question.

Question 5

Calculating induced connection some students have trouble to caclulcate $(\mathbf{r}_{\varphi\varphi})_{\text{tangent}}$.

Many students come to different answers calculating Christoffel symbols of induced connection and of Levi-Civita connection. The reason of mistake was just a mistake in calculations, but instead finding the mistake, students just have fixed that answers are different. This is wrong:

Christoffel symbols of induced connection and Levi-Civita connection are the same!

This is very important theorem. You have to know it well.

Calculating Christoffel symbols at the point $u = v = 0$ you have to justify why Christoffels vanish:

To solve this it is very important that first derivatives of metric equal to zero at the point $u = v = 0$. Indeed consider Taylor series expansion in the vicinity of the point $u = v = 0$ we come to

$$\frac{4R^4}{(R^2+u^2+v^2)^2} = \frac{4}{\left(1+\frac{u}{R}^2+\frac{v}{R}^2\right)^2} = 4 + \text{terms of the order} \geq 2 \qquad (*)$$

Hence due to Levi-Civita theorem Christoffel symbols vanish:

$$\frac{\partial g_{ik}}{\partial u}|_{u=v=0} = \frac{\partial g_{ik}}{\partial v}|_{u=v=0} = 0 \Rightarrow \Gamma^i_{km}\big|_{u=v=0} = 0\,.$$

Many students wrote the expression (*) for metric without trying to argue why it implies the answer.

Some students did the following mistake. They claim: At the point $u = v = 0$ metric is equal to $G = du^2 + dv^2$, hence due to Levi-Civita Theorem Christoffel symbols vanish. In fact this is an empty statement: every metric is constant at an arbitrary point. This sentence sounds like:

Since the function $y = f(x)$ is equal to constant ($y_0 = f(x_0)$) hence the derivative of this function is equal to zero, which is obviously mistake.

Almost all 15 credits students calculating Chrstoffel symbols at the stationary point of the surface have guessed right that Christoffel symbols vanish at extremum, but their reasoniing was almost always confusing and messy.

In fact one can see that the Riemanian metric at the vicinity of the extremum point is given by quadratic form

$$G = \begin{pmatrix} 1+F_u^2 & F_uF_v \\ F_vF_u & 1+F_v^2 \end{pmatrix}$$

One can see that first derivatvies of metric's components at the extremum point vanish. E.g.

$$\frac{\partial g_{12}}{\partial u}|_{u=v=0} = F_{uu}F_v|_{u=v=0} + F_uF_{vu}|_{u=v=0} = 0$$

since at the extremum point $F_{uu} = F_{uv} = F_{vv} = 0$. Hence according to Levi-Civita formula Christopher symbols vanish at the extremum point (in coordinates u, v.)

One can solve this problem in another much more elegant way:

Another solution One can see that Christoffel symbols of induced connection (which is equals to Levi-Civita) connection vanish. Indeed it is evident that at the extremum point $\mathbf{p}$ normal unit vector $\mathbf{n} = (0,0,1)$ since basic vectors $\mathbf{r}_u = (1,0,0)$ and $\mathbf{r}_v = (0,1,0)$.

$$\nabla^M_{\partial u_\alpha} \partial_{u_\beta} = \left(\nabla^{\text{canonicalflat}}_{\partial u_\alpha} \partial_{u_\beta}\right)_{\text{tangent}} = \left(\partial_{\partial u_\alpha} \partial_{u_\beta}\right)_{\text{tangent}} = (\mathbf{r}_{\alpha\beta})_{\text{tangent}} = 0$$

since $\mathbf{r}_{\alpha\beta} = (0,0,z_{\alpha\beta})$ is colinear to normal unit vector.

Two students did it!

Appendix

Straightforward calculations of the length of the curve C_2' lead to the following integral:

$$I(z) = \int_0^{2\pi} \frac{d\varphi}{1 - z\cos\varphi}\,,\,,\,(|z| < 1)$$

(for $z = \frac{\sqrt{2}}{2}$). Here I present two different ways to calculate this integral.

First way

This integral can be calculated explicitly, the answer is beautiful:

$$I(z) = \int_0^{2\pi} \frac{d\varphi}{1 - z\cos\varphi} = \frac{2\pi}{\sqrt{1 - z^2}}\,.$$

Do it. One can see that for $|z| < 1$,

$$I(z) = \int_0^{2\pi} \frac{d\varphi}{1 - z\cos\varphi} = \int_0^{2\pi} \left(1 + z\cos\varphi + z^2\cos^2\varphi + \ldots\right) d\varphi =$$

$$\int_0^{2\pi} \left(\sum z^n \cos^n\varphi\right) d\varphi = \sum_{n=0}^{\infty} c_n z^n\,, \text{where } c_n = \int_0^{2\pi} \cos^n \varphi d\varphi\,.$$

Calculate c_n:

$$c_n = \int_0^{2\pi} \cos^n \varphi d\varphi = \int_0^{2\pi} \left(\frac{e^{i\varphi} + e^{-i\varphi}}{2}\right)^n d\varphi = \begin{cases} 2\pi \frac{C_{2k}^k}{2^k} \text{for } n = 2k \\ 0 \ \ \text{for } n = 2k+1 \end{cases}\,.$$

since $(a+b)^n = \sum_j C_n^j a^j b^{n-j}$, and $\int_0^{2\pi} e^{ik\varphi} d\varphi = 0$ if $k \neq 0$. Hence we have that

$$I(z) = \int_0^{2\pi} \frac{d\varphi}{1 - z\cos\varphi} = 2\pi \sum_{n=0}^{\infty} c_n z^n = \sum_{n=0}^{\infty} z^n = \sum \frac{C_{2n}^n z^{2n}}{2^n} = \frac{2\pi}{\sqrt{1 - z^2}}\,.$$

Second way

$$I(z) = \int_0^{2\pi} \frac{d\varphi}{1 - z\cos\varphi} = \int_0^{2\pi} \frac{\sin\varphi d\varphi}{\sin\varphi(1 - z\cos\varphi)} = \int_0^{2\pi} \frac{d\cos\varphi}{(\sqrt{1-\cos^2\varphi})(1 - z\cos\varphi)} =$$

$$\int_0^1 \frac{dw}{(\sqrt{1-w^2})(1-zw)} = \frac{1}{2}\int_0^1 \frac{dw}{(\sqrt{1-w^2})(1-zw)}\,.$$

Now consider an integrand, the function $F(w) = \frac{1}{\sqrt{1-w^2}(1-zw)}$ in the plane excluding the neigborhood of the interval $[-1,1]$ which connects the branching points of this function. We take the following branch $F'(w)$ of this function such that it is holomorphic function in the plane without neighborhood of the segment [1] Now we note that the integral of function over the great circle tends to zero. The function $F'(w)$ has a pole at the point $w = \frac{1}{z}$. Hence if $|z| < 1$ $F(z')$ is holomorphic function in plane without noiborhood of interval AB. We have:

$$0 = \int_{C_1} F'(w)dw + \int_{C_2} F(w)dw = I(z) - \frac{1}{z}\int_{C_2} \frac{1}{i\sqrt{w^2-1}\left(w - \frac{1}{z}\right)} =$$

$$I(z) - \frac{2\pi i}{z}\left(\frac{1}{i\sqrt{w^2-1}}\right)\Big|_{w=\frac{1}{z}} = I(z) - \frac{2\pi}{\sqrt{1-z^2}} \Rightarrow I(z) = \frac{2\pi}{\sqrt{1-z^2}}\,.$$

where we denote by C_1 the closed curve around the interval AB, and C_2 the circle of small radius around

[1] If $P = w = u + iv$ is an arbitrary point of complex plane, $A = -1$ and $B = 1$, and φ is an angle between AB and AP (anti-clock wise), and ψ is an angle between BA and BP (anti-clock wise), then $F' = \sqrt{|AP||BP|}e^{i\frac{\phi+\psi}{2}}$. In particular $F(w) = i\sqrt{w^2-1}$ if w is a real number which is greater than 1.