Solutions of problem 1 of Homework 6

1

Calculate the derivatives of the functions $f = x^2 + y^2$, $g = y^2 - x^2$ and $h = q\log|r| = q\log\left(\sqrt{x^2+y^2}\right)$ (q is a constant) along vector fields $\mathbf{A} = x\partial_x + y\partial_y$ and $\mathbf{B} = x\partial_y - y\partial_x$,

a) calculating directional derivatives $\partial_\mathbf{A} f, \partial_\mathbf{A} g, \partial_\mathbf{A} h$, $\partial_\mathbf{B} f, \partial_\mathbf{B} g, \partial_\mathbf{B} h$

b) calculating $df(\mathbf{A}), dg(\mathbf{A}), dh(\mathbf{A}), df(\mathbf{B}), dg(\mathbf{B}), dh(\mathbf{B})$.

We can do this exercise or using the formula for directional derivative or using the 1-form, differential of function: $\partial_\mathbf{A} f = df(\mathbf{A})$.

a) First do using directional derivatives:
$\partial_\mathbf{A} f = A_x\frac{\partial f}{\partial x} + A_y\frac{\partial f}{\partial y} = x\cdot 2x + y\cdot 2y = 2(x^2+y^2)$,
$\partial_\mathbf{A} g = A_x\frac{\partial g}{\partial x} + A_y\frac{\partial g}{\partial y} = x\cdot(-2x) + y\cdot 2y = 2(y^2-x^2)$,
$\partial_\mathbf{A} h = x\frac{\partial h}{\partial x} + y\frac{\partial h}{\partial y} = \frac{x^2 q}{x^2+y^2} + \frac{y^2 q}{x^2+y^2} = q$
$\partial_\mathbf{B} f = B_x\frac{\partial f}{\partial x} + B_y\frac{\partial f}{\partial y} = -y\cdot 2x + x\cdot 2y = 0$,
$\partial_\mathbf{B} g = -y\frac{\partial g}{\partial x} + x\frac{\partial g}{\partial y} = -y\cdot(-2x) + x\cdot 2y = 4xy$
$\partial_\mathbf{B} h = -y\frac{\partial h}{\partial x} + x\frac{\partial h}{\partial y} = \frac{-xyq}{x^2+y^2} + \frac{xyq}{x^2+y^2} = 0$

b) Now calculate using 1-form using the fact that $\partial_\mathbf{A} f = df(\mathbf{A})$:

We have that $df = d(x^2+y^2) = 2xdx + 2ydy$, $dg = d\left(y^2 - x^2\right) = g_x dx + g_y dy = (2ydy - 2xdx)$, $dh = d\left(q\log\sqrt{x^2+y^2}\right) = h_x dx + h_y dy = \frac{qxdx+qydy}{x^2+y^2}$.

Hence

$\partial_\mathbf{A} f = df(\mathbf{A}) = (2xdx + 2ydy)(x\partial_x + y\partial_y) = 2x^2 dx(\partial_x) + 2y^2 dy(\partial_y) = 2x^2 + 2y^2$,

$\partial_\mathbf{A} g = dg(\mathbf{A}) = (2ydy - 2xdx)((x\partial_x + y\partial_y)) = 2ydy(y\partial_y) - 2xdx(x\partial_x) = 2y^2 - 2x^2$.

$\partial_\mathbf{A} h = dh(\mathbf{A}) = \frac{qxdx+qydy}{x^2+y^2}(x\partial_x + y\partial_y) = \frac{qxdx(x\partial_x)+qydy(y\partial_y)}{x^2+y^2} = \frac{qx^2+qy^2}{x^2+y^2} = q$

$\partial_\mathbf{B} f = df(\mathbf{A}) = (2xdx + 2ydy)(-y\partial_x + x\partial_y) = -2xydx(\partial_x) + 2xydy(\partial_y) = 0$,

$\partial_\mathbf{B} g = dg(\mathbf{A}) = (2ydy - 2xdx)((x\partial_y - y\partial_x)) = 2ydy(x\partial_y) - 2xdx(-y\partial_x) = 2xy + 2xy = 4xy$.

$\partial_\mathbf{A} h = dh(\mathbf{A}) = \frac{qxdx+qydy}{x^2+y^2}(-y\partial_x + x\partial_y) = \frac{qxdx(-y\partial_x)+qydy(x\partial_y)}{x^2+y^2} = \frac{-qxy+qxy}{x^2+y^2} = 0$.

2

Perform the calculations of the previous exercise using polar coordinates.

For basic fields $\partial_r, \partial_\varphi$ in polar coordinates r, φ ($r = x\cos\varphi, y = r\sin\varphi$) we have that

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\varphi\partial_x + \sin\varphi\partial_y = \frac{x}{r}\partial_x + \frac{y}{r}\partial_y = \frac{x\partial_x + y\partial_y}{r} = \frac{\mathbf{A}}{r} \Rightarrow \mathbf{A} = r\partial_r \tag{1}$$

and

$$\frac{\partial}{\partial\varphi} = \frac{\partial x}{\partial\varphi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\varphi}\frac{\partial}{\partial y} = -r\sin\varphi\partial_x + r\cos\varphi\partial_y = -y\partial_x + x\partial_y \Rightarrow \mathbf{B} = \partial_\varphi \tag{2}$$

We see that fields $\mathbf{A}, \mathbf{B}$ have very simple expression in polar coordinates. Now calculations become almost immediate because in polar coordinates $f = x^2 + y^2 = r^2$, $g = y^2 - x^2 = r^2(\sin^2\varphi - \cos^\varphi) = -r^2\cos 2\varphi$ and $h = q\log r$ and

$$\partial_A f = r\partial_r r^2 = 2r^2 = 2(x^2+y^2)\,,$$

$$\partial_A g = r\partial_r(-r^2\cos 2\varphi) = -2r^2\cos 2\varphi = 2(y^2 - x^2),\ \ \partial_A h = r\partial_r(q\log r) = q\,.$$

For field B we have that: $\partial_\mathbf{B} = \partial_\varphi$, hence

$$\partial_\mathbf{B} f = \partial_\mathbf{B} g = \partial_\mathbf{B} h = 0\,.$$

since the functions f and h do not depend on φ. For the function $g = y^2 - x^2 = -r^2 \cos 2\varphi$ we have:

$$\partial_{\mathbf{B}} g = \partial_\varphi(-r^2 \cos 2\varphi) = 2r^2 \sin 2\varphi = 4r^2 \sin\varphi \cos\varphi = 4r^2 \left(\frac{y}{r}\right) \cdot \left(\frac{x}{r}\right) = 4xy.$$

3

Consider a function $f = x^4 - y^4$.

Calculate the value of 1-form $\omega = df$ on the vector field $\mathbf{B} = x\partial_y - y\partial_x$.

Express this 1-form ω in polar coordinates r, φ ($x = r\cos\varphi, y = r\sin\varphi$).

$\omega = df = 4x^3dx - 4y^3dy$, $\omega(\mathbf{B}) = 4(x^3dx - y^3dy)(x\partial_y - yp_x) = 4x^3dx(-y\partial_y) - 4y^3dy(x\partial_y) = -4x^3y - 4y^3x = -4xy(x^2 + y^2$ since $dx(\partial_x) = dy(\partial_y) = 1$ and $dx(\partial_y) = dy(\partial_x) = 0$.

One may express differential 1-form $\omega = df = 4x^3dx - 4y^3dy$ straightforwardly in polar coordinates. Instead using "brute force" express function f in polar coordinates then calculate $\omega = df$:

$$f = x^4 - y^4 = (x^2 + y^2)(x^2 - y^2) = r^2(r^2\cos^2\varphi - r^2\sin^2\varphi) = r^4\cos^2\varphi - r^2\sin^2\varphi = r^4\cos 2\varphi\,,$$

hence $\omega = df = d(r^4\cos 2\varphi) = 4r^3\cos 2\varphi dr - 2r^4 \sin 2\varphi d\varphi$.

The operation of taking differential can be performed in an arbitrary coordinates in a same way as in Cartesian coordinates.

4

Calculate the value of 1-form $\omega = xdy - ydx$ on the vector fields $\mathbf{A} = r\partial_r$ and $\mathbf{B} = \partial_\varphi$. Perform calculations in Cartesian and polar coordinates).

We know that $r\partial_r = x\partial_x + y\partial_y$ and $\partial_\varphi = x\partial_y - y\partial_x$ (see formulae (1,2) in the solution of exercise 2). Hence in Cartesian coordinates $\mathbf{A} = x\partial_x + y\partial_y$ and $\mathbf{B} = x\partial_y - y\partial_x$

$$\omega(\mathbf{A}) = (xdy - ydx)(x\partial_x + y\partial_y) = -xydx(\partial_x) + xydy(\partial_y) = -xy + xy = 0\,,$$

$$\omega(\mathbf{B}) = (xdy - ydx)(x\partial_y - y\partial_x) = x^2dx(\partial_x) + y^2dy(\partial_y) = x^2 + y^2\,.$$

Now perform calculations in polar coordinates:

$$\omega = xdy - ydx = r\cos\varphi d(r\sin\varphi) - r\sin\varphi d(r\cos\varphi) = r\cos\varphi(\sin\varphi dr + r\cos\varphi d\varphi) - r\sin\varphi(\cos\varphi dr - r\sin\varphi d\varphi) = r^2(\cos^2\varphi + \sin^2$$

Hence in polar coordinates

$$\omega(\mathbf{A}) = r^2 d\varphi(\partial_r) = 0, \quad \omega(\mathbf{B}) = r^2 d\varphi(\partial_\varphi) = r^2\,.$$

$(dr(\partial_r) = d\varphi(\partial_\varphi) = 1, dr(\partial_\varphi) = d\varphi(\partial_r) = 0)$.

We see that calculations are much more transparent in polar coordinates.

5

Let f be a function on $\mathbf{E}^2$ given by $f(r, \varphi) = r^2\cos 2\varphi$, where r, φ are polar coordinates in $\mathbf{E}^2$.

Calculate the 1-form $\omega = df$.

Calculate the value of the 1-form $\omega = df$ on the vector field $\mathbf{X} = r^2\partial_r + r\partial_\varphi$.

Express the 1-form ω in Cartesian coordinates x, y.

$\omega = 2r\cos 2\varphi dr - 2r^2\sin 2\varphi d\varphi$.

The value of the form $\omega = df$ on the vector field $\mathbf{X} = r^2\partial_r + r\partial_\varphi$ is equal to

$$\omega(\mathbf{A}) = \left(2r\cos 2\varphi dr - 2r^2\sin 2\varphi d\varphi\right)\left(r^2\partial_r + r\partial_\varphi\right) = 2r^3\cos 2\varphi dr(\partial_r) - 2r^3\sin 2\varphi d\varphi(\partial_\varphi) = 2r^3(\cos 2\varphi - \sin 2\varphi)\,.\blacksquare$$

because $dr(\partial_r) = 1, dr(\partial_\varphi) = 0$ and $dr(\partial_\varphi) = 0, d\varphi(\partial_\varphi) = 1$.

Another solution

$$\omega(\mathbf{X}) = df(\mathbf{X}) = \partial_{\mathbf{X}} f = \left(r^2 \frac{\partial}{\partial r} + r \frac{\partial}{\partial_\varphi}\right)(r^2 \cos 2\varphi) = r^2 \cdot 2r \cos 2\varphi - r \cdot 2r^2 \sin 2\varphi = 2r^3(\cos 2\varphi - \sin 2\varphi)\,.$$

To express the form ω in Cartesian coordinates it is easier to express f in Cartesian coordinates and then to calculate $\omega = df$:

$$f = r^2 \cos 2\varphi = (x^2+y^2)(\cos^2\varphi - \sin^2\varphi) = (x^2+y^2)2\left(\frac{x^2}{r^2}\right) - \left(\frac{y^2}{r^2}\right) = x^2 - y^2\,.$$

Hence $\omega = d(x^2 - y^2) = 2xdx - 2ydy$.

6

Consider 1-forms $\omega = df$ and $\sigma = dg$ such that

$$f(x,y) + ig(x,y) = (x+iy)^n\,.$$

Find the values of these 1-forms on vector field $\mathbf{Y} = r\partial_r + \partial_\varphi$.

Solution of this problem will be put on the web after finishing your coursework...

The shortest solution:

$$f + ig = (x+iy)^n = (re^{i\varphi})^n = r^n e^{in\varphi} = r^n \cos n\varphi + r^n \sin n\varphi\,, \quad f = r^n \cos n\varphi, g = r^n \sin n\varphi\,,$$

$$df(\mathbf{Y}) = \partial_{\mathbf{Y}} f = (r\partial_r + \partial_\varphi) r^n \cos n\varphi = nr^n(\cos n\varphi - \sin n\varphi)\,,$$

$$dg(\mathbf{Y}) = \partial_{\mathbf{Y}} f = (r\partial_r + \partial_\varphi) r^n \sin n\varphi = nr^n(\cos n\varphi + \sin n\varphi)\,.$$

7

Calculate the integrals of the form $\omega = xdy - ydx$ over the following three curves. Compare answers.

$$C_1 \colon \mathbf{r}(t) \begin{cases} x = R\cos t \\ y = R\sin t \end{cases}, \ 0 < t < \pi, \quad C_2 \colon \mathbf{r}(t) \begin{cases} x = R\cos 4t \\ y = R\sin 4t \end{cases}, \ 0 < t < \frac{\pi}{4}$$

$$\textit{and} \quad C_3 \colon \mathbf{r}(t) \begin{cases} x = Rt \\ y = R\sqrt{1-t^2} \end{cases}, \ -1 \le t \le 1.$$

(For solutions see also ecture notes the end of subsection 2.5)

We have that

$$\int_C \omega = \int_{t_1}^{t_2} \omega(\mathbf{v}(t))dt = \int_{t_1}^{t_2} (xdy - ydx)(x_t\partial_x + y_t\partial_y)dt = \int_{t_1}^{t_2} (-y(t)x_t(t) + x(t)y_t(t))dt\,,$$

where $\mathbf{v} = (x_t, y_t)$ is velocity vector: $dx(\partial_x) = dy(\partial_y) = 1,\ dx(\partial_y) = dy(\partial_x) = 0.$)

For the first curve C_1 we have $\mathbf{v}(t) = (-R\sin t, R\cos t)$ and $\int_{C_1}\omega = \int_0^\pi (xdy - ydx)(-R\sin t\partial_x + R\cos t\partial_y) =$

$$\int_0^\pi (R\cos t dy - R\sin t dx)(-R\sin t\partial_x + R\cos t\partial_y) = \int_0^\pi (R^2\cos^2 t + R^2\sin^2 t)dt = \int_0^\pi R^2 \cdot dt = \pi R^2\,.$$

For the second curve C_2 we have $\mathbf{v}(t) = (-4R\sin 4t, 4R\cos 4t)$ and $\int_{C_2}\omega = \int_0^{\frac{\pi}{4}} (xdy - ydx)(-4R\sin 4t\partial_x + 4R\cos 4t\partial_y) =$

$$\int_0^{\frac{\pi}{4}} (R\cos 4t dy - R\sin 4t dx)(-4R\sin 4t\partial_x + 4R\cos 4t\partial_y) = \int_0^{\frac{\pi}{4}} (4R^2\cos^2 4t + 4R^2\sin^2 4t)dt = \int_0^{\frac{\pi}{4}} 4R^2 \cdot dt = \pi R^2\,.$$

Answer is the same. The second curve is reparameterised first curve ($t \mapsto 4t$) and reparameterisation preserves the orientation: $(4t)' = 4 > 0$.

For the third curve C_3 we have $\mathbf{v}(t) = \left(-R, -\frac{Rt}{\sqrt{1-t^2}}\right)$ and $\omega(\mathbf{v}(t)) = (xdy - ydx)(v_x\partial_x + v + y\partial_y) =$

$$= \left(Rtdy - R\sqrt{1-t^2}dx\right)\left(R\partial_x - \frac{Rt}{\sqrt{1-t^2}}\partial_y\right) = -R^2\sqrt{1-t^2} - \frac{R^2t^2}{\sqrt{1-t^2}} = -\frac{R^2}{1-t^2}\,.$$

Hence

$$\int_{C_3} \omega = \int_0^1 \omega(\mathbf{v}(t)) = \int_0^1 \left(-\frac{R^2}{\sqrt{1-t^2}}\right) dt = -R^2 \int_0^1 \frac{dt}{\sqrt{1-t^2}} = -\pi R^2\,.$$

Answer is the same up to a sign: This curve is reparameterised first curve. If we put $t = \cos\tau$ then third curve C_3 will transform to the first curve C_1. This reparameterisation changes the orientation, because $(\cos t)' = -\sin t < 0$ on the interval $(0, \pi/2)$.

Resumé: In these three examples was considered an integral over the same (non-parameteresed) half-circle. All the answers are the same up to a sign. Sign changes if reparameterisation changes an orientation.

8

8 *Consider an arc of parabola $x = 2y^2 - 1$, $0 < y < 1$.*

Give examples of two different parameterisations of this curve such that these parameterisations have the opposite orientation.

Calculate the integral of the form 1-form $\omega = \sin y dx$ over this curve.

How does the answer depend on a parameterisation?

One can consider parameterisation:

C_1: $\mathbf{r}(t)$ $\begin{cases} x = 2t^2 - 1 \\ y = t \end{cases}$, $0 < t < 1$,

To consider a different parameterisation we may take an arbitrary number $n \neq 0$ and consider

C_n: $\mathbf{r}(t)$ $\begin{cases} x = 2n^2t^2 - 1 \\ y = nt \end{cases}$, $0 < t < 1/n$,

These two different parameterisation are related with the reparameterisation $t' = nt$. If $n > 0$, then reparameterisation preserves orientation, If $n < 0$, then reparameterisation changes orientation of the curve. For example if we take $n = 2$ then we will come to the curve

C_2: $\mathbf{r}(t)$ $\begin{cases} x = 8t^2 - 1 \\ y = 2t \end{cases}$, $0 < t < 1/2$, , with the same orientation as initial curve and if we will take $n = -2$ we will come to the curve

C_2': $\mathbf{r}(t)$ $\begin{cases} x = 8t^2 - 1 \\ y = -2t \end{cases}$, $-\frac{1}{2} < t < 0$, ,

we will come to the curve with orientation opposite to the orientation opposite to the orientation of the initial curve.

Sure we can change parametersiation in a different way. E.g. we may consider

C_3: $\mathbf{r}(t)$ $\begin{cases} x = 2\cos^2 t - 1 \cos 2t \\ y = \cos t \end{cases}$, $0 < t < \frac{\pi}{2}$

Curve C_3 has orientation opposite to the orientation of the curves C, C_2 and the same orienation with the curve C_2' since reparameterisation $t' = \cos t$ changes orienation ($\frac{dt'}{dt} = -\sin t < 0$ for $0 \leq t \leq \frac{\pi}{2}$).

Now we calculate integrals for all these curves. (Sure we do not need to do it, it suffices to calculate the integral jsut for one curve and then using orientation arguments to find integrals for other curves, but jsut for exercise we will do all examples.)

For any curve $\mathbf{r}(t), t_1 < t < t_2$

$$\int_C \omega = \int_C \sin y dx = \int_C \sin y dx(\mathbf{v}) = \int_{t_1}^{t_2} \sin y(t) \frac{dx(t)}{dt} dt$$

where $\mathbf{v} = (x_t, y_t)$.

For the first curve C_1 $x_t = 4t$ and

$$\int_{C_1} \omega = \int_0^1 4t \sin t dt = 4(-t\cos t + \sin t)\big|_0^1 = -4\cos 1 + 4\sin 1$$

For the second curve C_2 $x_t = 16t$ and

$$\int_{C_2} \omega = \int_0^{1/2} 16t \sin 2t dt = 4(-2t\cos 2t + \sin 2t)\big|_0^{1/2} = -4\cos 1 + 4\sin 1\,.$$

Answer is the same. Non-surprising. The second curve is reparameterised first curve ($t \mapsto 2t$) and reparameterisation preserves the orientation.

For the third curve C_2' $x_t = 16t$ and

$$\int_{C_2} \omega = \int_{-1/2}^0 16t \sin(-2t) dt = -4(-2t\cos 2t + \sin 2t)\big|_{-1/2}^0 = 4\cos 1 - 4\sin 1\,.$$

Answer is the same up to a sign Non-surprising. This curve is reparameterised first curve ($t \mapsto -2t$) and reparameterisation changes the orientation.

For the last curve $x_t = -2\sin 2t dt$ and

$$\int_{C_3} w = \int_0^{\frac{\pi}{2}} (-2\sin 2t)\sin(\cos t) dt = -4\left(\cos t\cos(\cos t) - \sin(\cos t)\right)\big|_{-1/2}^{\pi/2} = 4\cos 1 - 4\sin 1$$

Answer is the same as for the previous curve: This curve is reparameterised first curve with opposite orientation ($t \mapsto \cos t$) and reparameterisation changes the orientation, because $(\cos t)' = -\sin t < 0$ on the interval $(0, \pi/2)$. hence the fourst integral is equal to the third on and it has a sign opposite to the second and first one.

Resumé: In these three examples an integral over the same (non-parameteresed) curve was considered. All the answers are the same up to a sign. Sign changes if reparameterisation changes an orientation.