Solutions of Homework 4

In all exercises we assume by default that Riemannian metric on embedded surfaces is induced by the Euclidean metric.

1 *Consider parallelogram* $\Pi_{\mathbf{a},\mathbf{b}}$ *formed by two vectors in Euclidean space* $\mathbf{E}^2$*:*

$$\Pi_{\mathbf{a},\mathbf{b}} = u\mathbf{a} + v\mathbf{b}\,, \quad 0 \le u \le 1, 0 \le v \le 1\,,$$

$$\mathbf{r}(u,v) = \mathbf{a}u + \mathbf{b}v = \begin{pmatrix} a_x \\ a_y \end{pmatrix} u \begin{pmatrix} b_x \\ b_y \end{pmatrix} v = \begin{cases} x = a_x u + b_x v \\ y = a_y u + b_y v \end{cases}.$$

a) Write down standard Euclidean metric $G = dx^2 + dy^2$ *in coordinates* (u,v)*.*

b) Calculate the area of parallelogram $\Pi_{\mathbf{a},\mathbf{b}}$ *using Riemannian volume form*

c) Compare the answer with standard formula for area of parallelogram (See subsection 1.5.1 "Motiation. Gramm formula for volume of parallellepiped")

a)

$$G = dx^2+dy^2 = (a_x du+b_x dv)^2+(b_x du+b_y dv)^2 = (a_x^2+a_y^2)du^2+2(a_x b_x+a_y b_y)dudv+(b_x^2+b_y^2)dv^2 =$$

$$(\mathbf{a},\mathbf{a})du^2 + 2(\mathbf{a},\mathbf{b})dudv + (\mathbf{b},\mathbf{b})dv^2$$

The matrix of metric G is

in Cartesian coordinates (x,y) $g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, in coordinates (u,v) $g = \begin{pmatrix} (\mathbf{a}, ac) & (\mathbf{a},\mathbf{b}) \\ (\mathbf{a},\mathbf{b}) & (\mathbf{b},\mathbf{b}) \end{pmatrix}.$

Calculate area of parallelogram in coordinates (u,v)

$$\text{Area}(\Pi_{\mathbf{a},\mathbf{b}}) = \int_0^1 dv \int_0^1 du \sqrt{\det g} dudv$$

We have that in coordinates u, v

$$\det g = \det \begin{pmatrix} (\mathbf{a}, ac) & (\mathbf{a},\mathbf{b}) \\ (\mathbf{a},\mathbf{b}) & (\mathbf{b},\mathbf{b}) \end{pmatrix} = (a_x^2+a_y^2)(b_x^2+b_y^2)-(a_x b_x+a_y b_y)^2 = (a_x b_y - a_y b_x)^2 = (\det g)^2$$

This is Gram formula from lecture notes.

Thus we have

$$\text{Area}(\Pi_{\mathbf{a},\mathbf{b}}) = \int_0^1 dv \int_0^1 du \sqrt{\det g} dudv = 1 \cdot \sqrt{\det g} = |\det g| = |a_x b_y - a_y b_x| = |\mathbf{a} \times \mathbf{b}|\,.$$

This is standard formula for area of parallelogram.

2 a) *Consider the domain* D *on the cone* $x^2 + y^2 - k^2 z^2$ *defined by the condition* $0 < z < H$*. Find an area of this domain using induced Riemannian metric. Compare with the answer when using standard formulae.*

We have cone with height H with radius $R = kH$ $(k > 0)$.

First of all standard answer: The area of cone (of surface of cone) is area of the sector with the radius $\sqrt{H^2 + R^2}$ and length of the arc $2\pi R$:

$$S = \frac{1}{2} \cdot \sqrt{R^2 + H^2} \cdot 2\pi R = \pi R \sqrt{H^2 + R^2} = \pi k \sqrt{1 + k^2} H^2 \,.$$

Now calculate this are using Riemannian geometry. Metric of the cone is $G = (k^2+1)dh^2 + k^2h^2d\varphi^2$. We see that

$$G = G = \begin{pmatrix} k^2+1 & 0 \\ 0 & k^2h^2 \end{pmatrix} ,$$

Hence volume form on the cone in coordinates r, φ is equal to

$$d\sigma = \sqrt{\det G} dh \wedge d\varphi = k\sqrt{1 + k^2} dh \wedge d\varphi$$

since $G = \begin{pmatrix} k^2+1 & 0 \\ 0 & k^2h^2 \end{pmatrix}$ Hence

$$S = \int_{0<h<H} \sqrt{\det G} dh \wedge d\varphi = \int_{0<h<H} k\sqrt{1+k^2} dh \wedge d\varphi = 2\pi k \sqrt{1+k^2} \int_0^H h dh = \pi k \sqrt{1+k^2} H^2 \,.$$

(Compare with standard calculations). (Here we use notation $dh \wedge d\varphi$ for area form. If you do not like it use just $drd\varphi$.)

3 *Find an area of the segment of the height h of the sphere of radius R (surface: $x^2 + y^2 + z^2 = R^2, \leq a \leq a + h$ for an arbitrary a: $-R \leq a \leq R - h$)*

For solutions see lecture notes.

4 *Find an area of 2-dimensional sphere of radius R using explicit formulae for induced Riemannian metric in stereographic coordinates.*

Riemannian metric for sphere (without point) in stereographic coordinates is $G = \frac{4R^4du^2 + 4R^4dv^2}{(R^2+u^2+v^2)^2}$. We already know that doing transformation $u \mapsto ru, v \mapsto Rv$ we come to the expression

$$G = \frac{4R^2du^2 + 4R^2dv^2}{(1+u^2+v^2)^2}$$

(see the exercise 4 4 4 4 in the previous homework)

$$G = \begin{pmatrix} \frac{4R^2}{(1+u^2+v^2)^2} & 0 \\ 0 & \frac{4R^2}{(1+u^2+v^2)^2} \end{pmatrix}, \ \det G = \frac{16R^4}{(1+u^2+v^2)^4}$$

Hence the volume (area)of the sphere equals to

$$S = \int_{\mathbf{R}^2} \sqrt{\det G} dudv = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{4R^2}{(1+u^2+v^2)^2} \right) dudv$$

Choosing polar coordinates $u = r\cos\varphi, v = r\sin\varphi$ we come to

$$S = \int_0^\infty \int_0^{2\pi} \left(\frac{4R^2}{(1+r^2)^2}\right) d\varphi r dr = 8\pi R^2 \int_0^\infty \frac{rdr}{1+r^2} = 4\pi R^2\,.$$

5 *Show that two spheres of different radii in Euclidean space are not isometric to each other.*

Suppose that these two spheres of different radii are isometric (globally). This means that their volume is the same. Contradiction. (In fact two spheres of different radii are not isometric even locally, since they have different curvatures.)

6 *In exercise 4 of previous homework you have considered Riemannian manifolds* $(\mathbf{R}^2, G^{(1)})$ *and* $(\mathbf{R}^2, G^{(2)})$, *where*

$$G^{(1)} = \frac{a(dx^2+dy^2)}{(1+x^2+y^2)^2}\,, \quad \text{and} \quad G^{(2)} = \frac{4R^4(du^2+dv^2)}{(R^2+u^2+v^2)^2}$$

(The second manifold is sphere of radius R without North pole in stereographic coordinates) You proved in previous homework that in the case if $a = 4R^2$ *then under isometry* $\begin{cases} u = Rx \\ v = Ry \end{cases}$ *these Riemannina manifolds are isometric. Using the result of previous exercise, prove now that in the case if the condition* $a = 4R^2$ *is not obeyed, then these manifolds are not isometric.*

It follows from the exercise from the former Homework, thjat The first manifold is isometric to the sphere of radius R' (without one point) such that $4(R')^2 = a$. If $R' \neq R$, i.e. $a \neq 4R^2$ then these sphered are not isometric since they have different areas.

7 *Let D be a domain in Lobachevsky plane which is lying between lines* $x = a, x = -a$ *and outside of the disc* $x^2+y^2 = 1$, $(0 < a < 1)$: $D = \{(x,y)\colon\ |x| < a, x^2+y^2 > 1\}$,

a) Find the area of this domain.

b) Find the angles between lines and arc of the circle.*

Lobachevsky plane, i.e. hyperbolic plane is the upper half plane with Riemannian metric $\frac{dx^2+dy^2}{y^2}$ *in cartesian coordinates* x, y $(y > 0)$.

a) We have $G = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}$. We see that $\sqrt{\det G} = \frac{1}{y^2}$. Hence

$$S = \int_{x^2+y^2\geq 1, -a\leq x\leq a} \sqrt{\det G}dxdy = \int_{x^2+y^2\geq 1, -a\leq x\leq a} \frac{1}{y^2}dxdy = \int_{-a}^{a}\left(\int_{\sqrt{1-x^2}}^\infty \frac{dy}{y^2}\right)dx =$$

$$\int_{-a}^{a} \frac{dx}{\sqrt{1-x^2}} = 2\arcsin a\,.$$

Remark This formula has a deep geometrical meaning Domain D can be considered as a 'triangle' $\triangle ABC$, since vertical rays and arc of the semicircle are geodesics (we will elarn it later.) Notice if we take two points, a point $(-a, y)$ and the point (a, y) on the vertical rays, then distance between these points will tend to zero if $y \to \infty$. We may say that domain D is an isocseles triangle with vertices at the points $A = (-a, \sqrt{1-a^2})$, $B = (a, \sqrt{1-a^2})$ and C—a point at infinity.

Then formula says that the area is equal to difference between π and sum of angles of this triangle.

There is a remarkable formula that for an arbitrary triangle sum of its angles minus π is equal to integral of curvature over area of triangle: in the case if curvature if constant (this is the case for sphere and hyperbolic plane) it is just proportional to area of triangle.)!

Remark Notice that Lobachevsky metric $G = \frac{dx^2+dy^2}{y^2} = \frac{1}{y^2}(dx^2+dy^2)$ is proportional to the Euclidean metric $dx^2 + dy^2$, in other words it is conformally Euclidean metric. Hence the angles will be the same as in the Euclidean metric (see lecture notes) (The difference is that not all straight lines in Euclidean metric are "straight lines" (geodesics) in Lobachevsky plane. We will discuss it later.)

8* *Find a volume of n-dimensional sphere of radius a. (You may use Riemannian metric in stereographic coordinates, or you may do it in other way... You just have to calculate the answer.)*

Denote by σ_n the volume of n-dimensional unit sphere embedded in Euclidean space $\mathbf{E}^n$. Then the volume of n-dimensional sphere of the radius R is equal to $\sigma_n R^n$. Now consider the integral

$$I = \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}.$$

For any integer k consider

$$I^k = \pi^{\frac{k}{2}} = \left(\int_{-\infty}^{\infty} e^{-t^2} dt\right)^k = \int_{\mathbf{E}^k} e^{-x_1^2-x_2^2-\dots-x_k^2} dx_1 dx_2 \dots dx_k \,.$$

Make changing of variables in the volume form $dx_1 \wedge dx_2 \wedge \dots \wedge dx_k$. Since integrand depend only on the radius we can rewrite the integral above as

$$\int_{\mathbf{E}^k} e^{-x_1^2-x_2^2-\dots-x_k^2} dx_1 dx_2 \dots dx_k = \int_{\mathbf{E}^k} e^{-r^2} r^{k-1} \sigma_{k-1} dr \,,$$

where σ_{k-1} is a volume of the unit sphere in dimension $k-1$. (Here is the truck!) We have the identity:

$$\pi^{\frac{k}{2}} = \sigma_{k-1} \int_0^{\infty} e^{-r^2} r^{k-1} dr$$

To calculate this integral consider $r^2 = t$ we come to

$$\int_0^\infty e^{-r^2} r^{k-1} dr = \frac{1}{2}\int_0^\infty e^{-t} t^{\frac{k}{2}-1} dt = \frac{1}{2}\Gamma\left(\frac{k}{2}\right).$$

We come to

$$\pi^{\frac{k}{2}} = \sigma_{k-1}\int_0^\infty e^{-r^2} r^{k-1} dr = \frac{\sigma_{k-1}}{2}\Gamma\left(\frac{k}{2}\right).$$

Thus

$$\sigma_{k-1} = \frac{2\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)}.$$

Recall that $\Gamma(x)$ can be calculated for all $\frac{k}{2}$ using the following recurrent formulae:

1. $\Gamma(n+1) = n!$
2. $\Gamma(x+1) = x\Gamma(x)$
3. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, $(\Gamma(x)\Gamma(1-x) = \pi \sin \pi x)$.
   E.g. the volume of the 15-dimensional unit sphere in $\mathbf{E}^{16}$ equals to $\sigma_{15} = \frac{2\pi^8}{\Gamma(8)} = \frac{2\pi^6}{7!}$