Homework 3

In all exercises we assume by default that Riemannian metric on embedded surfaces is induced by the Euclidean metric.

1 Consider plane $\mathbf{R}^2$ with Riemannian metric given in Cartesian coordinates (x, y) by the formula

$$G_{\mathbf{R}^2} = \frac{a\left((dx)^2 + (dy)^2\right)}{(1 + x^2 + y^2)^2}\,, \quad (a > 0)\,, \tag{1}$$

and a sphere S_r $x^2 + y^2 + z^2 = r^2$ (of the radius r) in the Euclidean space $\mathbf{E}^3$.

Consider the following map F from the plane $\mathbf{R}^2$ to the sphere

$$F(x, y)\colon \begin{cases} u = rx \\ v = ry \end{cases},$$

where (u, v) are stereographic coordinates of the sphere ($u = \frac{rx}{r-z}$, $v = \frac{ry}{r-z}$).

The map F is a diffeomorphism of $\mathbf{R}^2$ on the sphere without North pole (the point N with coordinates $x = 0, y = 0, z = r$), F: $\mathbf{R}^2 \to S_r \backslash N$

a) Write down the Riemannian metric G_S on the sphere in stereographic coordinates.

b) Write down the metric on the plane $\mathbf{R}^2$, the pull-back F^*G_S of the metric on the sphere.

c) Find parameter a such that F is isometry of the plane $\mathbf{R}^2$ equipped with Riemannian metric (1) and $S_r \backslash N$, i.e. $G_{\mathbf{R}^2} = F^*G_{S_r}$

2 Show that surface of the cone $\begin{cases} x^2 + y^2 - k^2z^2 = 0 \\ z > 0 \end{cases}$ in $\mathbf{E}^3$ is locally Euclidean Riemannian surface, (is locally isometric to Euclidean plane).

3 a) Consider the conic surface C defined by the equation $x^2 + y^2 - z^2 = 0$ in $\mathbf{E}^3$. Consider a part of this conic surface between planes $z = 0$ and $z = H > 0$, and remove the line $z = -x, y = 0$ from this part of conic surface C. We come to the surface D defined by the conditions

$$D\colon \quad \begin{cases} x^2 + y^2 - z^2 = 0 \\ 0 < z < H \\ y \neq 0 \quad \text{if } x < 0 \end{cases}.$$

Find a domain D' in Euclidean plane such that it is isometric to the surface D, that is there exists isometry $F\colon D \to D'$.

b) Find a shortest distance between points $A = (1, 0, 1)$ and $B = (-1, 0, 1)$, between points $A = (1.0, 1)$ and $E = (0, 1, 1)$, for an ant living on the conic surface C.

4 Find a diffeomorphism F: $\begin{cases} u = u(x, y) \\ v = v(x, y) \end{cases}$ of Euclidean plane on itslelf, such that it is an isometry, in other words $du^2 + dv^2 = dx^2 + dy^2$. (You may assume that functions $u(x, y), v(x, y)$ are linear: $u = a + bx + cy$, $v = c + dx + fy$, where a, b, c, d are constants.)

Show that the transformation is a composition of translation, rotation and reflection.

* Will the answer change if we allow arbitrary (not only linear functions) $u(x, y), v(x, y)$?$\blacksquare$

5 Let $\mathbf{K} = K^i(x)\frac{\partial}{\partial x^i}$ be a Killing vector field on Euclidean plane, i.e. a vector field such that it induces infinitesimal isometry of Euclidean space.

a) Show that

$$\frac{\partial K^i(x)}{\partial x^j} + \frac{\partial K^j(x)}{\partial x^i} = 0\,,$$

b) Find all Killing vector fields of Euclidean plane $\mathbf{E}^2$

c)*) Find all Killing vector fields of $\mathbf{E}^n$ and compare the answer with 4* for $n = 2$.)