Contact vector field III

Let J^1M be a space of first jets of functions on manifold M. Cooridnates on J^1M are (p_i, q^j, u), where q^j are coordinates on M. Jet of every function $u = u(x)$ has coordinates $\left(p_i = \frac{\partial u}{\partial x q^i}, q^i, u\right)$.

Consider $\mathcal{C}$, the Cartan distribution of $2n$-dimensional planes in J^1M defined by the form $\omega = p_i dq^i - du$

$$\mathcal{C}_{\mathbf{p}} \subset T_{\mathbf{p}} J^1M = \{T_{\mathbf{p}}(J^1M) \ni \mathbf{X}\colon\ \omega(\mathbf{X}) = 0\}\,,$$

Vector field

$$M^i \frac{\partial}{\partial q^i} + N_i \frac{\partial}{\partial p_i} + A \frac{\partial}{\partial u} \text{ belongs to Cartan distribution } \mathcal{C} \text{ if } A = p_i M^i\,.$$

$\mathcal{C}$ is non-integrable distribution.

Consider differential equation,

$$\mathcal{E}\colon F(p, q, u) = 0\,.$$

Differential equation is sumbmanifold of codimension 1.

The Cartan distribution $\mathcal{C}$ of hyperplanes on J^1M defines distribution $\mathcal{C}(\mathcal{E})$ in $T\mathcal{E}$:

$$\mathcal{C}(E) = \mathcal{C} \cap T\mathcal{E}\,.$$

$$\mathbf{X} = M^i \frac{\partial}{\partial q^i} + N_i \frac{\partial}{\partial p_i} + A \frac{\partial}{\partial u} \in \mathcal{C}(\mathcal{E}) \text{ if } A = p_i M^i \& \left(M^i \frac{\partial}{\partial q^i} + N_i \frac{\partial}{\partial p_i} + A \frac{\partial}{\partial u}\right) F(p, q, u)\big|_{F=0} = 0\,.$$

Definition 1 The vector field $\mathbf{K}$ in $2n+1$ is *is an infinitesiaml symmetry* of differential equation $\mathcal{E} = 0$ if it belongs to $\mathcal{C}(\mathcal{E})$:

$$\mathcal{L}_{\mathbf{X}} \mathcal{C}(\mathcal{E}) = 0 \tag{2a}$$

In what follows we consider here mostly an empty differential equation. (We focus the attention on the equation in the next file tomorrow.)

Definition 2 The vector field $\mathbf{K}$ in $2n + 1$ is called *contact vector field* if it is an infinitesimal symmetry of empty differential equation, i.e. if it preserves the Cartan distribution $\mathcal{C}$

$$\mathcal{L}_{\mathbf{X}} \mathcal{C} = 0 \tag{2b}$$

Theorem *There is one-one corrspondence between functions on M and contact vector fields:*

$$C^\infty(M) \ni F = F(p_i, q^j, u) \leftrightarrow \mathbf{X}_F$$

such that

$$F=\omega(\mathbf{X}_F)\,,\ \text{and}\quad \mathbf{X}_F=\frac{\partial F}{\partial p_m}\frac{\partial}{\partial q^m}-\left(\frac{\partial F}{\partial q^m}+p_m\frac{\partial F}{\partial u}\right)\frac{\partial}{\partial p_m}+\left(p_m\frac{\partial F}{\partial p_m}-F\right)\frac{\partial}{\partial u}$$

The proof of the Theorem follows from the

Lemma If $\mathbf{X}$ is contact vector field and $\omega(\mathbf{X})\equiv 0$ then $\mathbf{X}\equiv 0$.

This lemma implies that for every function F there exists at most one contact vector field $\mathbf{X}_F$ such that $\omega(\mathbf{X}_F)=F$.

On the other hand the vecot field (3)

i) is defined for an arbitrary smooth function F

ii) it evidently obyes the condition $\omega(\mathbf{X}_F)=F$

iii) is contact vector field

Conditions ii) and iii) hold evidently. may be checked by direct calculations:

$$\omega\left(\mathbf{X}_F\right)=(p_m dq^m-du)\left(\frac{\partial F}{\partial p_m}\frac{\partial}{\partial q^m}-\left(\frac{\partial F}{\partial q^m}+p_m\frac{\partial F}{\partial u}\right)\frac{\partial}{\partial p_m}+\left(p_m\frac{\partial F}{\partial p_m}-F\right)\frac{\partial}{\partial u}\right)=$$

$$=p_m\frac{\partial F}{\partial p_m}-\left(p_m\frac{\partial F}{\partial p_m}-F\right)=F$$

and

$$\mathcal{L}_{\mathbf{X}_F}=d\left(\omega\rfloor\mathbf{X}_F\right)+d\omega\left(\rfloor\mathbf{X}_F\right)=d\left(\omega\left(\mathbf{X}_F\right)\right)+dp_m\wedge dq^m\left(\rfloor\mathbf{X}_F\right)=$$

$$=dF-\frac{\partial F}{\partial p_m}dp_m-\left(\frac{\partial F}{\partial q^m}+p_m\frac{\partial F}{\partial u}\right)dq^m=\frac{\partial F}{\partial u}\left(du-p_m dq^m\right)=F_u\omega\,,$$

i.e. $\mathbf{X}_F$ preserves the Cartan distribution $\mathcal{C}$.

It remians to prove the lemma.

Suppose that the vector field $\mathbf{X}=M^i\frac{\partial}{\partial q^i}+N_i\frac{\partial}{\partial p_i}+A\frac{\partial}{\partial u}$ is contact vector field

$$\mathcal{L}_{\mathbf{X}}\omega=\lambda\omega\,, \tag{3a}$$

and

$$\omega(\mathbf{X})=p_iM^i-A=0\,. \tag{3b}$$

Condition (3a) means that

$$\mathcal{L}_{\mathbf{X}}=d\left(\omega\rfloor\mathbf{X}\right)+d\omega\left(\rfloor\mathbf{X}\right)=d\left(\omega\left(\mathbf{X}\right)\right)+dp_m\wedge dq^m\left(\rfloor\mathbf{X}\right)=$$

$$0-M^m dp_m-N_m dq^m=\lambda(pdq^m-du)\,.$$

Thus $\lambda\equiv 0$, and $M^m\equiv 0$, $N_m\equiv 0$ and due to equation (3b), $A\equiv 0$. Hence $\mathbf{X}\equiv 0$ $\blacksquare$.

Poisson brackets on J^1M

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Using the bijection $F \leftrightarrow X_F$ between functions and vector fields one can consider bracket

$$\{F,G\}\colon; \quad \mathbf{X}_{\{F,G\}} = [\mathbf{X}_F, \mathbf{X}_G]$$

Since $F = \omega(\mathbf{X}_F)$ we have

$$\{F,G\} = \omega\left([\mathbf{X}_F, \mathbf{X}_G]\right) \,.$$

On the other hand

$$d\omega\left(\mathbf{X}_F, \mathbf{X}_G\right) = \hat{\mathbf{X}}_F\left(\omega\left(\mathbf{X}_G\right)\right) - \hat{\mathbf{X}}_G\left(\omega\left(\mathbf{X}_F\right)\right) - \omega\left([\mathbf{X}_F, \mathbf{X}_G]\right) = \hat{\mathbf{X}}_F\left(G\right) - \hat{\mathbf{X}}_G\left(F\right) - \{F,G\} \,.$$

Thus we see that

$$\{F,G\} = \hat{\mathbf{X}}_F\left(G\right) - \hat{\mathbf{X}}_G\left(F\right) - d\omega\left(\mathbf{X}_F, \mathbf{X}_G\right) \,.$$

In coordinates using Theorem we have

$$\{F,G\} = \hat{\mathbf{X}}_F\left(G\right) - \hat{\mathbf{X}}_G\left(F\right) - d\omega\left(\mathbf{X}_F, \mathbf{X}_G\right) =$$

$$\left(F^m \partial_m - F_m \partial^m - p_m F_u \partial^m + p_m F^m \partial_u - F \partial_u\right) G - \left(F \leftrightarrow G\right) - dp_m \wedge dq^m (\mathbf{X}_F, \mathbf{X}_G) =$$

$$\left(F^m \partial_m - F_m \partial^m - p_m F_u \partial^m + p_m F^m \partial_u - F \partial_u\right) G - \left(F \leftrightarrow G\right) -$$

$$-dp_m \wedge dq^m$$

$$\left(F^m \partial_m - \left(F_m + p_m F_u\right) \partial^m + \left(p_m F^m - F\right) \partial_u, G^m \partial_m - \left(G_m + p_m G_u\right) \partial^m + \left(p_m G^m - G\right) \partial_u\right) =$$

$$\left(\frac{\partial F}{\partial p^m} \frac{\partial G}{\partial q_m} - \frac{\partial G}{\partial p^m} \frac{\partial F}{\partial q_m}\right) + p_m \left(\frac{\partial F}{\partial p^m} \frac{\partial G}{\partial u} - \frac{\partial F}{\partial p^m} \frac{\partial G}{\partial u}\right) + \left(\frac{\partial F}{\partial u} G - \frac{\partial G}{\partial u} F\right) \quad \blacksquare$$