Solutions of Homework 8

Exercises 1 and 2 contain much material which was in the lectures. These exercises are good seprations to solve exercises 3 and 4.

1 *Consider in* $\mathbf{E}^2$ *ellipse*

$$\frac{x^2}{25} + \frac{y^2}{9} = 1\,. \tag{1.1}$$

Find foci of this ellipse

Choose focal polar coordinates for this ellipse and write down the equation of this ellipse in these polar coordinates.

An ellipse is defined analytically in canonical Cartesian coordinates by equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,, \quad \text{canonical equation}\,. \tag{1a}$$

Centre of an ellipse defined by this canonical equation is at the origin. Comparing this equation with equation (1.1) we see that for the ellipse (1.1)

$$a = \text{length of semi-major axis} = 5\,, \quad b = \text{length of semi-minor axis} = 3\,. \tag{1.2}$$

Foci are at the points $(\pm c, 0)$, where $c = \sqrt{a^2 - b^2} = \sqrt{25 - 9} = 4$,

$$F_1 = (-c, 0) = (-4, 0)\,, F_2 = (c, 0) = (4, 0)\,, \quad c = \sqrt{a^2 - b^2}\,. \tag{1.3}$$

Another way to calculate foci. Major axis of the ellipse is equal to $a = 5$ it is on the X axis, hence foci are on the X axis also. Centre of this ellipse is at the origin, hence foci are at the points $F_{2,1} = (\pm c, 0)$. By the geometrical definition of an ellipse we know that the sum of distances from arbitrary point of ellipse to foci is equal to constant. (The ellipse is locus C: $\{K\colon |KF_1| + |KF_2| = \mathit{constant}\}$.) Take two points at the ellipse (1.1), the point $A = (5, 0)$, the vertex on the X axis, and the point $B = (0, 3)$ the upper vertex at the Y axis we have:

$$|A-F_1|+|A-F_2| = |a-c|+|a-(-c)| = 2a = 10 = |B-F_1|+|B-F_2| = 2\sqrt{c^2+b^2}\,. \tag{1.4}$$

This implies that $c = 4$, thus $F_1 = (-4, 0)$ and $F_2 = (4, 0)$.

To choose focal polar coordinates we take the origin of polar coordinates the left focus, the point $F_1 = (-4, 0)$ and the angle φ is the angle between axis OX and the radius-vector $\mathbf{r}$: $\begin{cases} x = -4 + r\cos\varphi \\ y = r\sin\varphi \end{cases}$. We have $|KF_1| = r$ and according to cosine formula for triangle

$$|KF_2| = \sqrt{r^2 - 2r|F_1F_2| + |F_1F_2|^2}\,.$$

The condition that for arbitrary point K of the ellipse $|KF_1|+|KF_2|=2a\equiv 10$, becomes:

$$|KF_1|+|KF+2|=r+\sqrt{r^2-2r|F_1F_2|+|F_1F_2|^2}=r+\sqrt{r^2-2\cdot 8\cdot r\cos\varphi+8^2}=10\Leftrightarrow$$

$$\Leftrightarrow\sqrt{r^2-16r\cos\varphi+64}=10-r\Leftrightarrow \tag{1.5}$$

$$r^2-16\cos\varphi+64=100-20r+r^2\Leftrightarrow r=\frac{36}{20-16\cos\varphi}=\frac{9/5}{1-4/5\cos\varphi}\,.$$

We see that in polar coordinates the equation of the ellipse (1.1) is

$$r=\frac{p}{1-e\cos\varphi}\,,\quad \text{with } p=\frac{a^2-c^2}{a}=\frac{b^2}{a}=\frac{9}{5}\text{ and } e=\frac{c}{a}=\frac{4}{5}\,. \tag{1.6}$$

Remark Note that taking square in equation (1.5) we come to equivalent equation because $10-r$ is non-negative: r is less or equal to the sum of distances from a point to foci, and this sum is equal to 10.

2 *Consider a curve in $\mathbf{E}^2$ defined in polar coordinates (r,φ) by the equation*

$$r=\frac{p}{1-e\cos\varphi}\,,\qquad p>0\,. \tag{2.1}$$

a) Write down the equation of this curve in Cartesian coordinates $\begin{cases}x=r\cos\varphi\\ y=r\sin\varphi\end{cases}$ in the case if $p=2, e=\frac{1}{3}$, show that this curve is an ellipse, and find the foci and the centre of this ellipse. Calculate the area of this ellipse.

b) Justify by straighforward calculations that in the case $0\le e<1$ the curve (1) is indeed an ellipse with foci at origin and at the point $(2c,0)$, where $c=\frac{pe}{1-e^2}$, and with semi-major axis $a=\frac{p}{1-e^2}$.

c) How does the curve defined by equation (2.1) look in the case if $e=1$?

a) Consider arbitrary parameter p and e obeying the conditions

$$p>0\,,\quad 0\le e<1 \tag{2.2}$$

Thus we will do simultaneously cases a) and b) of this exercise. We have

$$r-re\cos\varphi=p\Leftrightarrow\sqrt{x^2+y^2}=p+ex \tag{2.3}$$

Take a square of left and right and sides of this equation. (see Remark below):

$$x^2+y^2=p^2+2pex+e^2x^2\Leftrightarrow(1-e^2)x^2-2pex+y^2=p^2\Leftrightarrow(1-e^2)\left(x-\frac{2pe}{1-e^2}+\left(\frac{pe}{1-e^2}\right)^2\right)+y^2=$$

$$= p^2 + \frac{p^2e^2}{1-e^2} \Leftrightarrow (1-e^2)\left(x - \frac{pe}{1-e^2}\right)^2 + y^2 = p^2 + \frac{p^2e^2}{1-e^2} = \frac{p^2}{1-e^2}\,. \qquad (2.4)$$

Hence translating Cartesian coordinates x: $x' = x - \frac{pe}{1-e^2}$ we come to Cartesian coordinates, such that the curve has canonical equation (see equation (1,1a)) in these coordinates:

$$(1-e^2)x'^2 + y^2 = \frac{p^2}{1-e^2} \Leftrightarrow \left(\frac{x'}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1\,, \text{with } a = \frac{p}{1-e^2}\,, b = \frac{p}{\sqrt{1-e^2}} \qquad (2.5)$$

Remark You may be suspicious about equivalence of equation (2.3) and equation (2.4). We come to equation (2.4) taking square of equation (2.3). One can easy to see that indeed these equations are equivalent since $0 \le e < 1$. Indeed equation (2.5) (which is evidently equivalent to equation (2.4)) implies that $|x'| = \left|x - \frac{pe}{1-e^2}\right| \le a = \frac{p}{1-e^2}$. This implies that

$$p + ex > p + \left(\frac{pe}{1-e^2} - \frac{p}{1-e^2}\right)e = \frac{p}{1+e} > 0\,. \qquad (2.3a)$$

One can see it also immediately after taking square of equation (2.3)*

We come in equation (2.5) to analytical definition of ellipse. The foci of this ellipse in canoncial coordinates x', y' are at the points $x' = \pm c, y' = 0$ (see equation (1.3)) hence x, y coordinates of the foci are

$$y = 0\,, x = \pm c + \frac{p}{1-e^2}e = \pm c + a \cdot \left(\frac{c}{a}\right) = c \pm c = 2c \text{ or } 0\,. \qquad (2.6)$$

We see that in coordinates (x, y) one of the foci is at the origin and the second focus is at the point $(2c.0)$.

Centre of this ellipse is at the origin in Cartesian coordinates (x', y), hence in coordinates (x, y) it is at the middle point between foci, at the point $(c, 0)$.

Thus we see that equation (2.1) describes the ellipse with foci at origi and at the point $(0, 2c)$ and with semi-major and semi-minor axis a, b defined by equation (2.5).

(The parameters, a and c were calculated in equation (2.6)). Area of the ellipse is equal to πab. According equations (2.6) we can express these parameters in terms of parameters p, e:

$$S = \pi ab = \pi \cdot \frac{p}{1-e^2} \cdot \frac{p}{\sqrt{1-e^2}} = \frac{\pi p^2}{(1-e^2)^{\frac{3}{2}}}\,. \qquad (2.6)$$

b) *Another solution* We already did the case b) considering in the case a) arbitrary parameters p, e obeying condition (2.2). We will do it again in other way

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* We have $y^2 = (p+ex)^2 - x^2 = (p+(e+1)x)(p-(1-e)x)$. Thus $(p+(e+1)(p-(1-e)x) > 0$. Bearing in mind that $0 < e < 1$ we come to equation (2.3a).

If r, φ are polar coordinates of a point, then Cartesian coordinates are $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$.

Take the points $F_1 = (0,0)$ and $F_2 = (2c, 0)$. We will prove that for all the points of this curve

$$|KF_1| + |KF_2) = constant \tag{2.7}$$

Thus we will prove that this is an ellipse, according geometrical definition of ellipse (in the solution above we used the analytical definition of ellipse.)
The curve $r = \frac{p}{1-e\cos\varphi}$ passes through the points $A_1 = \left(-\frac{p}{1-e\cos\pi}, 0\right) = \left(-\frac{p}{1+e}, 0\right)$ and the point $A_2 = \left(\left(\frac{p}{1-e\cos 0}\right), 0\right) = \left(\frac{p}{1-e}, 0\right)$ One can see that for these points

$$|A_1F_1| + |A_1F_2| = |A_2F_1| + |A_2||F_2| = |A_1A_2| = \frac{p}{1-e} - \frac{p}{1+e} = \frac{2p}{1-e^2}\,. \tag{2.8}$$

Thus we will try to prove that for arbitrary point $r(\varphi) = \frac{p}{1-e\cos\varphi}$ of the curve relation (2.7) is obeyed with constant which is defined by equation (2.8). We denote this constant by $2a$ (see lecture notes or exercise (1)):

$$|KF_1| + |KF_2| = r(\varphi) + \sqrt{r^2 - 4cr\cos\varphi + 4c^2} = 2a\,. \tag{2.8}$$

Now note that for every point on the curve C, $2a - r > 0$. Indeed since $0 \le e < 1$ $r(\varphi) = \frac{p}{1-e\cos\varphi} < \frac{p}{1-e} < \frac{p}{1-e}\frac{2}{1+e} = \frac{2p}{1-e^2} = 2a$. This is why equation (2.9) is equivalent to equation

$$(\sqrt{r^2 - 4cr\cos\varphi + 4c^2})^2 = (2a - r)^2$$

Now everything is simple: taking squares we see that the last equation is equivalent to equation

$$r = \frac{p}{1 - e\cos\varphi}\,. \tag{2.10}$$

Thus we proved that for every point obeying equation (2.10) belongs to the ellipse with foci at the points F_1, F_2 and with vertices at points A_1, A_2. (In lecture notes we proved the converse implication.) Thus we finished the second way of solving this exercise.

It is useful to focus again on relations between equation (2.10) of ellipse in polar coordinates and equation of ellipse in canonical Cartesina coordinates:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,.$$

(see also (1.1a)).

Relation beetween parametrs a, b (in analytical definition), parametrs p, e (in equation (2.10) in polar coordinates) are the following:

$$\begin{cases} p = \frac{a^2 - c^2}{a} \\ e = \frac{c}{a} \end{cases} \Leftrightarrow \begin{cases} a = \frac{p}{1-e^2} \\ c = \frac{pe}{1-e^2} \end{cases}, \tag{2.11}$$

where $c = \alpha^2 - b^2$, ($2c$ is distance between foci).

Now consider case c):

$$C\colon \quad r = \frac{p}{1-\cos\varphi}\,, \; p > 0\,.$$

Show that this is parabola. We have $r - r\cos\varphi = p$, i.e. $r = p + r\cos\varphi$, i.e. $\sqrt{x^2+y^2} = p + x$, $(p > 0)$. We have

$$\sqrt{x^2+y^2} = (p+x)^2 \Leftrightarrow x^2+y^2 = (p+x)^2 \Leftrightarrow y^2 = (p+2x)p\,.$$

(These equations are equivalent since equation $x^2+y^2 = (p+x)^2$ is equivalent to equation $y^2 = (p+2x)p$, hence $p+2x > 0$ and $p+x > 0$.) We come to

$$y^2 = (p+2x)x = 2px - p^2\,.$$

This is parabola. In Cartesian coordinates $x' = x - \frac{p}{2}$ it has canonical expression

$$y^2 = 2p\left(x - \frac{p}{2}\right) = 2px'\,.$$

Another way to see it is the following: Consider the line $x = -p$, then we see that C is the locus of the points which is on the same distance from the origin and the line $x = -p$:

$$p + r\cos\varphi = p + \frac{p\cos\varphi}{1-\cos\varphi} = \frac{p}{1-\cos\varphi} = r\,.$$

We see that this is the parabola with focus at the origin, and the directrix $x = -p$.

3 *Let C be a curve, intersection of the plane $\alpha\colon 2z - x = 1$ with the conic surface $M\colon x^2+y^2 = z^2$.*

Let C_{proj} be an orthogonal projection of this curve on the plane $z = 0$.

Show that the curve C_{proj} is an ellipse.

Explain why the curve C is also the ellipse.

Find foci of the curve C. In particular show that the origine, vertex of the conic surface M is a focus of the ellipse C.

Find the areas of the ellipses C and C_{pr}.

Write down a parameterisation of ellipse C and of ellipse C_{pr} (you may choose any parameterisation)

We have that

$$C\colon \quad \begin{cases} x^2+y^2 = z^2 \\ z = 1 + \frac{x}{2} \end{cases} \Leftrightarrow C\colon \quad \begin{cases} x^2+y^2 = \left(1+\frac{x}{2}\right)^2 \\ z = 1 + \frac{x}{2} \end{cases}$$

and for orthogonal projection

$$C_{\text{proj}}\colon \quad \begin{cases} x^2+y^2=\left(1+\frac{x}{2}\right)^2 \\ z=0 \end{cases} \tag{3.1}$$

The projected curve C_{proj} belongs to the plane $z=0$ and it is described in this plane by the equation $x^2+y^2=\left(1+\frac{x}{2}\right)^2$. Transforming we come to

$$x^2+y^2=\left(1+\frac{x}{2}\right)^2 \Leftrightarrow \frac{3}{4}x^2-x+y^2=1 \Leftrightarrow \frac{3}{4}\left(x^2-\frac{4}{3}x\right)+y^2=1 \Leftrightarrow$$

$$\frac{3}{4}\left(x^2-\frac{4}{3}x+\frac{4}{9}\right)+y^2=1+\frac{1}{3} \Leftrightarrow \frac{3}{4}\left(x-\frac{2}{3}\right)^2+y^2=\frac{4}{3} \Leftrightarrow \frac{9}{16}\left(x-\frac{2}{3}\right)^2+\frac{3}{4}y^2=1\,.$$

We see that the curve C_{proj} is an ellipse: in Cartesian coordinates $\begin{cases} x'=x-\frac{2}{3} \\ y'=y \end{cases}$ equation (3.1) has the appearance

$$\left(\frac{x'}{a}\right)^2+\left(\frac{y'}{b}\right)^2=1\,,\ \text{for } a=\frac{4}{3}, b=\frac{2\sqrt{3}}{3}\,.$$

The centre of this ellipse is at the point $\left(\frac{2}{3},0\right)$ (we use initial Cartesian coordinates (x,y)). The vertices are at the points $A_{1,2}=\left(\frac{2}{3}\pm a,0\right)=\left(\frac{2}{3}\pm\frac{4}{3},0\right)$, on the axis OX (the major axis has the length $L=2a=\frac{8}{3}$), and the points $B_{1,2}=\left(\frac{2}{3},\pm b\right)=\left(\frac{2}{3},\pm\frac{2\sqrt{3}}{3}\right)$, (the minor axis has the length $l=2b=\frac{4\sqrt{3}}{3}$) *.

The sum of distances from an arbitrary point of this ellipse to foci is equal to $2a=\frac{8}{3}$:

$$C=\left\{K\colon |KF_1|+|KF_2|=2a=\frac{8}{3}\right\}\,.$$

Find foci of this ellipse. We can do it in two different ways.

`I-st way`: using the statement of Theorem we see that one of the foci of this ellipse is origin, the vertex of the ellipse, the point $(0,0)$: $F_1=(0,0)$. (In exercise 2a) we also calcualted explicitly foci of ellipse.) Consider equation of the projection C_{proj}

$$C_{\text{proj}}\colon \quad x^2+y^2=\left(1+\frac{x}{2}\right)^2\,, \quad z=0 \tag{1b}$$

(see equation(1b)) in polar coordinates $\begin{cases} x=r\cos\varphi \\ y=r\sin\varphi \end{cases}$. We come to

$$\sqrt{x^2+y^2}=r=\left(1+\frac{x}{2}\right)=\left(1+\frac{r\cos\varphi}{2}\right)\,,$$

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* One may calculate the coordinates of vertices $A_{1,2}$ in another way also: The vertices of the ellipse on OX axis are $A_{1,2}=(x_{1,2},0)$. Numbers $x_{1,2}$ are roots of equation $x^2=\left(1+\frac{x}{2}^2\right)$ (this is equation of ellipse at $y=0$). Hence $x_1=-\frac{2}{3}, x_2=1$.

i.e.

$$r = \frac{1}{1 - \frac{1}{2}\cos\varphi}$$

This is equation of ellipse in polar coordinates (see lecture notes or equation (2.11) above) with parameters

$$\begin{cases} p = \frac{a^2-c^2}{a} = 1 \\ e = \frac{c}{a} = \frac{1}{2} \end{cases}$$

Solving this equation we come to $c = \frac{2\sqrt{3}}{3}$ and $a = \frac{4}{9}$ (compare with answers to Exercise 2.).

One focus is at origin $F_1 = (0, 0)$, the other focus is at the $F_2 = (2c, 0) = \left(\frac{4\sqrt{3}}{3}, 0\right)$.

Remark *Compare these calculations to calculations of foci for ellipse in exercise 2a) above,*

`II-nd way` We can avoid the calculations in polar coordinates. We know already that the centre of the ellipse C is at the point $O = \left(\frac{2}{3}, 0\right)$, and the ellipse intersects OX axis at the points $A_{1,2} = \left(\frac{2}{3} \pm \frac{4}{3}, 0\right)$, hence foci are at the points $F_{1,2} = \left(\frac{2}{3} \pm c, 0\right)$. To caluclate parameter c we can use formula $c = \sqrt{a^2 - b^2}$ (see exercise 1):

$$c = \sqrt{\left(\frac{4}{3}\right)^2 - \left(\frac{2\sqrt{3}}{3}\right)^2} = \frac{2}{3}\,.$$

Another way to calculate c it is to use the fact that for vertex B

$$2a = \frac{8}{3} = |BF_1| + |BF_2| = 2|BF_1| = 2a = 2\sqrt{c^2 + \left(\frac{2\sqrt{3}}{3}\right)^2}\,,$$

$F_{1,2} = \left(\frac{2}{3} \pm c, 0\right) =.\ \left(\frac{2}{3} \pm \frac{2}{3}, 0\right)$,i.e. $F_1 = (0, 0)$ and $F_2 = \left(\frac{4}{3}, 0\right)$. The answer is the same.

One of the foci of the ellipse C_{proj} is the vertex of the cone: this is general fact which we know from the lectures: vertex of the cone is one of the foci of the orthogonal projection of any conic section on the plane $z = 0$.

Now return to the curve C.

The curve C is also an ellipse since its orthogonal projection on the plane $z = 0$ is the ellipse.

The plane $2z = 2 + x$ intersects the plane $z = 0$ under the angle θ: $\tan\theta = \frac{1}{2}$, i.e. $\cos\theta = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$. Hence the Cartesian coordinates on the plane of this ellipse are

$$\tilde{x} = \frac{x}{\cos\theta} = \frac{\sqrt{5}}{2}x\,, \quad \tilde{y} = y,.$$

The minor axis of the ellipse C has the same length $l = 2b$ as the minor axis of the projected ellipse C_{proj}; the major axis of the ellipse C has the length $\frac{2a}{\cos\theta}$, where a is the length of the major axis of the projected ellipse C_{proj}.

The area S of the projected ellipse C_{proj} is equal to

$$S(C_{\text{proj}}) = \pi ab = \pi \cdot \frac{4}{3} \cdot \frac{2\sqrt{3}}{3} = \frac{8\sqrt{3}\pi}{9}\,,$$

and the area S of the ellipse C is equal to

$$S(C) = \pi \frac{ab}{\cos\theta} = \frac{S(C_{\text{proj}})}{\cos\theta} = \frac{8\sqrt{3}\pi}{9}\frac{\sqrt{5}}{2}\,.$$

4 *Let C be a curve, intersection of the plane α: $z = 1 + kx$ (k is real parameter) with the conic surface M: $2x^2 + 2y^2 = 9z^2$.*

Let C_{proj} be an orthogonal projection of this curve on the plane $z = 0$.

Find the values of k such that the curve C and the curve C_{pr} are ellipses.

Find the values of k such that the curve C and curve C_{pr} are parabolas.

In the case if a curve C (and a curve C_{proj}) are parabolas,show that the vertex of the conic surface M, the origin, is the focus of this parabola C_{proj}

Find the directrix of this parabola.

The equations of the curve C and of the curve C_{proj} which is its orthogonal projection on the plane $z = 0$ are

$$C\colon\quad \begin{cases} 2x^2 + 2y^2 = 9z^2 \\ z = 1 + kx \end{cases}, \quad C_{\text{proj}}\colon\quad \begin{cases} 2x^2 + 2y^2 = 9(1 + kx)^2 \\ z = 0 \end{cases} \tag{4.1}$$

It is enough to analyze the curve C_{proj}. We know that the curve C is ellipse, parabola or hyperbola if the curve C_{proj} is ellipse, parabola or hyperbola, respectively.

The projected curve C_{proj} belongs to the plane $z = 0$ and it is described in this plane by the equation $2x^2 + 2y^2 = 9\,(1 + kx)^2$.

Transforming we come to

$$2x^2 + 2y^2 = 9\,(1 + kx)^2 \Leftrightarrow (2 - 9k^2)x^2 - 18kx + 2y^2 = 9\,.$$

If $k = \pm\frac{\sqrt{2}}{3}$ then we see that C_{proj} is parabola

$$C_{\text{proj}}\colon\quad y^2 = \frac{9}{2} + 9kx = \frac{9}{2} \pm 3\sqrt{2}x \quad (z = 0)\,. \tag{4.2}$$

In the case if $k \neq \pm\frac{\sqrt{2}}{3}$ we continue the transformation (2a) of the curve C_{proj}

$$C_{\text{proj}}\colon\ (2-9k^2)x^2-18kx+2y^2 = 9 \leftrightarrow (2-9k^2)\left(x - \frac{9k}{2 - 9k^2}\right)^2 + 2y^2 = 9 + \frac{81k^2}{2 - 9k^2}\,. \tag{4.3}$$

We see that if $2-9k^2>0$ then the curve C_{proj} (and respectively the curve C) are ellipses, and if $2-9k^2<0$ then the curve C_{proj} (and respectively the curve C) are hyperbolas. (If $2-9k^2=0$ it is parabola.)

Consider in detail the case of ellipse, $(2-9k^2>0)$.

Notice that in (equation (4.3) the right hand side is positive: $9+\frac{81k^2}{2-9k^2}>9$ if $2-9k^2>0$. Denote it by H^2, $H^2=9+\frac{81k^2}{2-9k^2}$. We can rewrite equation (4.3) in the following way:

$$(2-9k^2)\left(x-\frac{9k}{2-9k^2}\right)^2+2y^2=H^2\,.$$

We see that in Cartesian coordinates $\begin{cases} x'=x-\frac{9k}{2-9k^2} \\ y'=y \end{cases}$ the curve C_{proj} has appearance

$$\left(\frac{x'}{a}\right)^2+\left(\frac{y'}{b}\right)^2=1\,,\ \text{where } a^2=\frac{H^2}{2-9k^2}\,,\, b^2=\frac{H^2}{2}\,,\, H^2=9+\frac{81k^2}{2-9k^2}\,,\,(z=0)\,.$$

We see that if $k\colon 2-9k^2>0$ then the curve C_{proj} and the curve C are ellipses.

The semi-major axis of the ellipse C_{proj} has length $a=\frac{H}{\sqrt{2-9k^2}}$, the semi-minor axis of the ellipse C_{proj} has length $b=\frac{H}{\sqrt{2}}=\frac{H\sqrt{2}}{2}$.

Now return to the case of parabola (see equation (4.2)):

In fact we have two parabolas, but trasformation $x\mapsto -x$ transform them to each other. We consider only one of these parabola: for $k=\frac{\sqrt{2}}{3}$:

$$C_{\text{proj}}\colon\quad y^2=\frac{9}{2}+3\sqrt{2}x\ (z=0)\,. \tag{4.4}$$

How to find the focus of this parabola? Do it in two ways:

`I-st way` We know that the focus of the parabola $y^2=\frac{9}{2}+3\sqrt{2}x$ is the vertex of conical surface: $F=(0,0)$.

Find the vertex of the parabola: if $y=0$ then $x=-\frac{3\sqrt{2}}{4}$. Hence the vertex of the parabola is at the point $\left(-\frac{3\sqrt{2}}{4},0\right)$. The vetex is at the middle point between the intersection of directrix with OX axis and the focus. Hence the directrix is $x=-\frac{3\sqrt{2}}{2}$.

`II-nd way` Choose canonical Cartesian coordinates for this parabola. In canonical Cartesian coordinates this parabola looks like $y'^2=2px'$. Equation (4.4) for this parabola states that

$$y^2=3\sqrt{2}\left(x+\frac{3\sqrt{2}}{4}\right)\,.$$

We see that in the new Cartesian coordinates $\begin{cases} x'=x+\frac{3\sqrt{3}}{4} \\ y'=y \end{cases}$ this parabola looks canonically:

$$y'^2=2px'\,,\quad p=\frac{3\sqrt{2}}{2}$$

Hence in the canonical Cartesian coordinates (x', y') the focus of the parabola is at the point $x' = \frac{p}{2} = \frac{3\sqrt{2}}{4}$, $y' = 0$ and the directrix is $x' = -\frac{p}{2} = -\frac{3\sqrt{2}}{4}$. Hence in initial Cartesian coordinates we have for the focus of parabola

$$F = (x_F, y_F) = (x'_F - \frac{3\sqrt{2}}{4}, 0) = (0, 0)\,.$$

This is in the accordance with the statement that the vertex of the cone is the focus of the parabola; and for directrix we have

$$x = x' - \frac{3\sqrt{2}}{4} = -\frac{3\sqrt{2}}{2}\,.$$

5 *Find foci and directrix of the parabola $y = ax^2$, $(a > 0)$.*

Choose focal polar coordinates and write down the equation of this parabola in these polar coordinates

One way to do this exercise is to find canonical coordinates. We can find focus of parabola straightforwardly. Focus of parabola $y = ax^2$ belongs to OY axis, $F = (0, f)$ and directrix is $y = -c$. Since for every point of parabola the distances coincide we have:

$$\sqrt{x^2 + (y - f)^2} = y + c\,.$$

Since $y = ax^2 > 0$ this equation is equivalent to the equation:

$$x^2 + y^2 - 2fy + f^2 = y^2 + 2yc + c^2\,, \quad \text{for } y = ax^2\,,$$

i.e.

$$x^2 = 2y(c + f) + c^2 - f^2 = 2ax^2(c + f) + c^2 - f^2$$

This relation holds for all x. Hence $c = f$ (directrix is $y = -c$) and $4af = 1$. We come to answer $F = \left(0, \frac{1}{4a}\right)$.

To find focal coordinates adjusted to this parabola we have to take the origin at the focal point $(0, 1/4a)$ and then $\begin{cases} x = r \sin\varphi \\ y = \frac{1}{4a} + r\cos\varphi \end{cases}$.