Homework 8. Solutions

1. *Find coordinate basis vectors, first quadratic form and unit normal vector field for the following surfaces:*

a) sphere of the radius R:

$$\mathbf{r}(\varphi,\theta) \qquad \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases} \qquad (0\leq\varphi<2\pi, 0\leq\theta\leq\pi)\,, \tag{1}$$

b) cylinder

$$\mathbf{r}(h,\varphi) \qquad \begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \\ z = h \end{cases} \qquad (0\leq\varphi<2\pi, -\infty<h<\infty) \tag{2}$$

c) cone $x^2+y^2-k^2z^2=0$,

$$\mathbf{r}(h,\varphi) \qquad \begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases} \qquad (0\leq\varphi<2\pi, -\infty<h<\infty) \tag{2}$$

d) graph of the function $z=F(x,y)$

$$\mathbf{r}(u,,v) \qquad \begin{cases} x = u \\ y = v \\ z = F(u,v) \end{cases} \qquad (-\infty<u<\infty, -\infty<v<\infty) \tag{3}$$

in the case if $F(u,v)=F=Au^2+2Buv+Cv^2$.

Consider coordinate basis vectors, first quadratic form and unit normal vector field at origin, i.e. at the point $u=v=0$.

Put down the special case of saddle when $F=uv$.

a) *sphere* $x^2+y^2+z^2=R^2$ *(of the radius R):*

$$\mathbf{r}(\theta,\varphi) \qquad \begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases} \tag{1}$$

$(0\leq\varphi<2\pi,\ 0\leq\theta\leq\pi)$,

$$\mathbf{r}_\theta = \frac{\partial\mathbf{r}(\varphi,\theta)}{\partial\theta} = \begin{pmatrix} R\cos\theta\cos\varphi \\ R\cos\theta\sin\varphi \\ -R\sin\theta \end{pmatrix},\ \mathbf{r}_\varphi = \frac{\partial\mathbf{r}(\varphi,\theta)}{\partial\varphi} = \begin{pmatrix} -R\sin\theta\sin\varphi \\ R\sin\theta\cos\varphi \\ 0 \end{pmatrix},\ \mathbf{n}(\theta,\varphi) = \frac{\mathbf{r}(\theta,\varphi)}{R} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

(Sometimes we denote by $\mathbf{r}_\theta$ by ∂_θ and $\mathbf{r}_\varphi$ by ∂_φ.)

Check that $\mathbf{n}(\theta,\varphi)$ is indeed unit normal vector (in fact this is obvious from geometric considerations):

$$(\mathbf{n},\mathbf{n}) = \sin^2\theta(\cos^2\varphi+\sin^2\varphi)+\cos^2\theta = 1\,,$$

$$(\mathbf{n},\mathbf{r}_\theta) = R\sin\theta\cos\theta(\cos^2\varphi+\sin^2\varphi) - R\sin\theta\cos\theta = 0,\ \ (\mathbf{n},\mathbf{r}_\varphi) = R\sin^2\theta(-\cos\varphi\sin\varphi+\cos\varphi\sin\varphi) = 0\,.$$

Unit normal vector is defined up to a sign; $-\mathbf{n}$ is unit normal vector too.

Calculate now first quadratic form. $(\mathbf{r}_\theta, \mathbf{r}_\theta) = R^2 \cos^2\theta(\cos^2\varphi + \sin^2\varphi) + R^2\sin^2\theta = R^2$, $(\mathbf{r}_\theta, \mathbf{r}_\varphi) = 0$, $(\mathbf{r}_\varphi, \mathbf{r}_\varphi) = R^2\sin^2\theta(\sin^2\varphi + \cos^2\varphi) = R^2\sin^2\theta$. Thus

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_\theta, \mathbf{r}_\theta) & (\mathbf{r}_\theta, \mathbf{r}_\varphi) \\ (\mathbf{r}_\varphi, \mathbf{r}_\theta) & (\mathbf{r}_\varphi, \mathbf{r}_\varphi) \end{pmatrix} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix}$$

$$dl^2 = G_{11}d\theta^2 + 2G_{12}d\theta d\varphi + G_{22}d\varphi^2 = R^2 d\theta^2 + R^2\sin^2\theta d\varphi^2$$

The length of the curve $\mathbf{r}(t) = \mathbf{r}(\theta(t), \varphi(t))$ with $\theta = \theta(t), \varphi = \varphi(t)$, $t_1 \le t \le t_2$ is given by the integral:

$$\int_{t_1}^{t_2} |\mathbf{v}(t)| dt = \int_{t_1}^{t_2} \sqrt{G_{11}\dot\theta^2 + 2G_{12}\dot\theta\dot\varphi + G_{22}\dot\varphi^2} dt = \int_{t_1}^{t_2} \sqrt{R^2\dot\theta^2 + R^2\sin^2\theta\dot\varphi^2} dt \qquad (1a)$$

b) cylinder $x^2 + y^2 = R^2$

$$\mathbf{r}(h, \varphi) \qquad \begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \\ z = h \end{cases} \qquad (0 \le \varphi < 2\pi, -\infty < h < \infty) \qquad (2)$$

$$\mathbf{r}_\varphi\big|_{\varphi,h} = \frac{\partial \mathbf{r}(\varphi, h)}{\partial \varphi} = \begin{pmatrix} -R\sin\varphi \\ R\cos\varphi \\ 0 \end{pmatrix}, \ \mathbf{r}_h\big|_{\varphi,h} = \frac{\partial \mathbf{r}(\varphi, h)}{\partial h} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ \mathbf{n}(\varphi, h) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$$

Sometimes we denote $\mathbf{r}_\varphi$ by ∂_φ and $\mathbf{r}_h$ by ∂_h.

Check that $\mathbf{n}(\varphi, h)$ is indeed unit normal vector:

$$(\mathbf{n}, \mathbf{n}) = \cos^2\varphi + \sin^2\varphi = 1, \ (\mathbf{n}, \mathbf{r}_\varphi) = R\cos\varphi\sin\varphi(-1+1) = 0, \ (\mathbf{n}, \mathbf{r}_h) = 0$$

Unit normal vector is defined up to a sign; $-\mathbf{n}$ is unit normal vector too.

Calculate now first quadratic form. $(\mathbf{r}_\varphi, \mathbf{r}_\varphi) = R^2(\sin^2\varphi + \cos^2\varphi) = R^2$, $(\mathbf{r}_\varphi, \mathbf{r}_h) = 0$, $(\mathbf{r}_h, \mathbf{r}_h) = 1$. Thus

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} G_{hh} & G_{h\varphi} \\ G_{\varphi h} & G_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_h, \mathbf{r}_h) & (\mathbf{r}_h, \mathbf{r}_\varphi) \\ (\mathbf{r}_\varphi, \mathbf{r}_\varphi) & (\mathbf{r}_\varphi, \mathbf{r}_\varphi) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & R^2 \end{pmatrix}$$

$$dl^2 = G_{11}dh^2 + 2G_{12}dhd\varphi + G_{22}d\varphi^2 = dh^2 + R^2 d\varphi^2 \,.$$

The length of the curve $\mathbf{r}(t) = \mathbf{r}(\varphi(t), h(t))$ with $\varphi = \varphi(t), h = h(t)$, $t_1 \le t \le t_2$ is given by the integral:

$$\int_{t_1}^{t_2} \sqrt{G_{11}\dot h^2 + 2G_{12}\dot\varphi\dot h + G_{22}\dot\varphi^2} dt = \int_{t_1}^{t_2} \sqrt{\dot h^2 + R^2\dot\varphi^2} dt, \qquad (2a)$$

b) cone $x^2 + y^2 - k^2z^2 = 0$

$$\mathbf{r}(h, \varphi) \qquad \begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases} \qquad (0 \le \varphi < 2\pi, -\infty < h < \infty) \qquad (2)$$

$$\mathbf{r}_h\big|_{\varphi,h} = \frac{\partial \mathbf{r}(\varphi, h)}{\partial h} = \begin{pmatrix} k\cos\varphi \\ k\sin\varphi \\ 1 \end{pmatrix}, \ \mathbf{r}_\varphi\big|_{\varphi,h} = \frac{\partial \mathbf{r}(\varphi, h)}{\partial \varphi} = \begin{pmatrix} -kh\sin\varphi \\ kh\cos\varphi \\ 0 \end{pmatrix}.$$

Sometimes we denote $\mathbf{r}_\varphi$ by ∂_φ and $\mathbf{r}_h$ by ∂_h.

To calculate the normal unit vector field $\mathbf{n}(h,\varphi)$ note that the vector $\mathbf{N}(h,\varphi) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix}$ is orthogonal to the surface of the cone: $(\mathbf{N}\, r_h) = (\mathbf{N}, \mathbf{r}_\varphi) = 0$ and its length equals to $|\mathbf{N}| = \sqrt{k^2+1}$. Hence normal unit vector field equals to

$$\mathbf{n}(h,\varphi) = \frac{\mathbf{N}(h,\varphi)}{\sqrt{k^2+1}} = \frac{1}{\sqrt{k^2+1}} \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -k \end{pmatrix}$$

It is indeed normal unit vector field:

$$(\mathbf{n}, \mathbf{n}) = \frac{\cos^2\varphi}{k^2+1} + \frac{\sin^2\varphi}{k^2+1} + \frac{k^2}{k^2+1} = 1,$$

$$(\mathbf{n}, \mathbf{r}_\varphi) = \frac{1}{\sqrt{k^2+1}}(\cos\varphi \cdot (-kh\sin\varphi) + \sin\varphi \cdot (+kh\cos\varphi)) = 0,$$

$$(\mathbf{n}, \mathbf{r}_h) = \frac{1}{\sqrt{k^2+1}}(\cos\varphi \cdot (kh\cos\varphi) + \sin\varphi \cdot k\sin\varphi - k) = 0\,.$$

Unit normal vector is defined up to a sign; $-\mathbf{n}$ is unit normal vector too.

Calculate now first quadratic form. $(\mathbf{r}_h, \mathbf{r}_h) = k^2\cos^2\varphi + k^2\sin^2\varphi + 1 = k^2+1$, $(\mathbf{r}_h, \mathbf{r}_\varphi) = (\mathbf{r}_\varphi, \mathbf{r}_h) = k^2 h\cos\varphi(-\sin\varphi) + k^2 h\sin\varphi\cos\varphi = 0$, $(\mathbf{r}_\varphi, \mathbf{r}_\varphi) = k^2h^2\sin^2\varphi + k^2h^2\cos^2\varphi = k^2h^2$, Thus

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} G_{hh} & G_{h\varphi} \\ G_{\varphi h} & G_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_h, \mathbf{r}_h) & (\mathbf{r}_h, \mathbf{r}_\varphi) \\ (\mathbf{r}_\varphi, \mathbf{r}_h) & (\mathbf{r}_\varphi, \mathbf{r}_\varphi) \end{pmatrix} = \begin{pmatrix} k^2+1 & 0 \\ 0 & k^2h^2 \end{pmatrix}$$

$$dl^2 = G_{hh}dh^2 + 2G_{h\varphi}dhd\varphi + G_{\varphi\varphi}d\varphi^2 = (k^2+1)dh^2 + k^2h^2d\varphi^2 R^2\,,$$

The length of the curve $\mathbf{r}(t) = \mathbf{r}(h(t), \varphi(t))$ with $\varphi = \varphi(t), h = h(t)$, $t_1 \leq t \leq t_2$ is given by the integral:

$$\int_{t_1}^{t_2} \sqrt{G_{11}\dot h^2 + 2G_{12}\dot h\dot\varphi + G_{22}\dot\varphi^2}dt = \int_{t_1}^{t_2} \sqrt{(k^2+1)\dot h^2 + k^2h(t)^2\dot\varphi^2}dt\,. \tag{3a}$$

d) graph of the function $z = F(x,y)$

$$\mathbf{r}(u,v) \qquad \begin{cases} x = u \\ y = v \\ z = F(u,v) \end{cases} \qquad (-\infty < u < \infty, -\infty < v < \infty) \tag{4}$$

in the case if $F(u,v) = Au^2 + 2Buv + Cv^2$

$$\mathbf{r}_u\big|_{u,v} = \frac{\partial\mathbf{r}(u,v)}{\partial u} = \begin{pmatrix} 1 \\ 0 \\ F_u \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2Au+2Bv \end{pmatrix},\ \mathbf{r}_u\big|_{u=v=0} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$\mathbf{r}_v\big|_{u,v} = \frac{\partial\mathbf{r}(u,v)}{\partial v} = \begin{pmatrix} 0 \\ 1 \\ F_v \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2Bu+2Cv \end{pmatrix},\ \mathbf{r}_v\big|_{u=v=0} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

$$\mathbf{n}(u,v) = \frac{1}{\sqrt{1+F_u^2+F_v^2}} \begin{pmatrix} -F_u \\ -F_v \\ 1 \end{pmatrix},\ \mathbf{n}(u,v)\big|_{u=v=0} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Sometimes we denote $\mathbf{r}_u$ by ∂_u and $\mathbf{r}_v$ by ∂_v. The vectors $\mathbf{r}_u\big|_{u=v=0}, \mathbf{r}_v\big|_{u=v=0}$ and $\mathbf{n}\big|_{u=v=0}$ above are the values of tangent vectors and normal unit vector at origin.

Check that $\mathbf{n}(u,v)$ is indeed unit normal vector: $(\mathbf{n},\mathbf{n}) = \frac{1}{1+F_u^2+F_v^2}(F_u^2+F_v^2+1) = 1$, $(\mathbf{n},\mathbf{r}_u) = \frac{1}{\sqrt{1+F_u^2+F_v^2}}(F_u - F_u) = 0$, $(\mathbf{n},\mathbf{r}_v) = \frac{1}{\sqrt{1+F_u^2+F_v^2}}(F_v - F_v) = 0$. Calculate now first quadratic form. $(\mathbf{r}_u,\mathbf{r}_u) = 1+F_u^2$, $(\mathbf{r}_u,\mathbf{r}_v) = F_uF_v$, $(\mathbf{r}_v,\mathbf{r}_v) = 1+F_v^2$. Thus

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_u,\mathbf{r}_u) & (\mathbf{r}_u,\mathbf{r}_v) \\ (\mathbf{r}_v,\mathbf{r}_u) & (\mathbf{r}_v,\mathbf{r}_v) \end{pmatrix} = \begin{pmatrix} 1+F_u^2 & F_uF_v \\ F_uF_v & 1+F_v^2 \end{pmatrix}$$

$$dl^2 = G_{11}d\varphi^2 + 2G_{12}dd\varphi dh + G_{22}dh^2 = (1+F_u^2)du^2 + 2F_uF_vdudv + (1+F_v^2)dv^2$$

At the origin (the point $u = v = 0$), $F_u = F_v = 0$ and First Quadratic form equals to

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_u,\mathbf{r}_u) & (\mathbf{r}_u,\mathbf{r}_v) \\ (\mathbf{r}_v,\mathbf{r}_u) & (\mathbf{r}_v,\mathbf{r}_v) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ dl^2 = du^2 + dv^2$$

The length of the curve $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ with $u = u(t), v = v(t)$ can be calculated by the integral:

$$\int_{t_1}^{t_2} \sqrt{G_{11}\dot{u}^2 + 2G_{12}\dot{u}\dot{v} + G_{22}\dot{v}^2}dt = \int_{t_1}^{t_2} \sqrt{(1+F_u^2)\dot{u}^2 + 2F_uF_v\dot{u}\dot{v} + (1+F_v^2)\dot{v}^2}dt \tag{4a}$$

Special case of saddle: In the special case of saddle we just take $F = uv$ in previous formulae. In particular for normal unit vector we have

$$\mathbf{n}(u,v) = \frac{1}{\sqrt{1+v^2+u^2}}\begin{pmatrix} -v \\ -u \\ 1 \end{pmatrix},\ \mathbf{n}(u,v)\big|_{u=v=0} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and first quadratic form is equal to

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_u,\mathbf{r}_u) & (\mathbf{r}_u,\mathbf{r}_v) \\ (\mathbf{r}_v,\mathbf{r}_u) & (\mathbf{r}_v,\mathbf{r}_v) \end{pmatrix} = \begin{pmatrix} 1+F_u^2 & F_uF_v \\ F_uF_v & 1+F_v^2 \end{pmatrix} = \begin{pmatrix} 1+v^2 & vu \\ vu & 1+u^2 \end{pmatrix},$$

$$dl^2 = G_{11}d\varphi^2 + 2G_{12}dd\varphi dh + G_{22}dh^2 = (1+v^2)du^2 + 2uvdudv + (1+u^2)dv^2\,.$$

2. *Consider helix* $\mathbf{r}(t)$: $\begin{cases} x(t) = R\cos t \\ y(t) = R\sin t \\ z(t) = ct \end{cases}$.
Show that this helix belongs to cylinder surface $x^2 + y^2 = R^2$.
Using first quadratic form calculate length of this curve ($0 \le t \le t_0$). *(Compare with problem 4 from Homework 7.)*

This helix belongs to cylinder surface $x^2 + y^2 = R^2$ because $x^2 + y^2 = R^2$ on the points of the helix.

Use First Quadratic form which we obtained in the previous exercise (see equation (2a) and equations above in the solution of exercise 1). For the helix internal coordinates are $\varphi = \varphi(t) = t$ and $h = h(t) = ct$ ($x = R\cos\varphi, y = R\sin\varphi, z = h$)

We come to

$$L = \int_0^{t_0} \sqrt{G_{11}\dot{h}^2 + 2G_{12}\dot{\varphi}\dot{h} + G_{22}\dot{\varphi}^2}dt = \int_0^{t_0} \sqrt{R^2\dot{\varphi}^2 + \dot{h}^2}dt = \int_0^{t_0} \sqrt{R^2+c^2}dt = t\sqrt{R^2+c^2}$$

Of course the answer can be obtained without integration: speed is constant, hence $L = |\mathbf{v}|t = t\sqrt{R^2+c^2}$. This is the calculations of the Internal observer. The external observer will calculate using the coordinates x, y, z: $|\mathbf{v}| = \sqrt{x_t^2 + y_t^2 + z_t^2} = (R^2\cos^2 + R^2\sin^2 t + c^2) = \sqrt{R^2+c^2}$ and will come to the same answer.

3 *Show that the curve $x = t\cos t, y = t\sin t, z = t$ belongs to the cone $x^2 + y^2 - z^2 = 0$. Find the length of this curve (for $0 \le t \le t_0$).*

We have that for the points of the curve $x^2(t) + y^2(t) - z^2(t) = t^2\cos^2 t + t^2\sin^2 t - t^2 = 0$, i.e. this curve belongs to the surface of the cone $x^2 + y^2 - z^2 = 0$.

To find the length of this curve external observer will use the formula: $L = \int_0^{t_0} \sqrt{x_t^2 + y_t^2 + z_t^2}dt$. We have $x_t = \cos t - t\sin t$, $y_t = \sin t + t\cos t$, $z_t = 1$, hence

$$L = \int_0^{t_0} \sqrt{x_t^2 + y_t^2 + z_t^2}dt = \int_0^{t_0} \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\sin t)^2 + 1}dt = \int_0^{t_0} \sqrt{t^2 + 2}dt\,.$$

The internal observer will use the First quadratic form for the cone obtained in the previous exercise (see the formula (3a) for the length of the curve with use of the First quadratic form for the cone for the case $k = 1$):

$$L = \int_0^{t_0} \sqrt{G_{11}\dot h^2 + 2G_{12}\dot h\dot\varphi + G_{22}\dot\varphi^2}dt = \int_0^{t_0} \sqrt{(k^2 + 1)\dot h^2 + k^2 h(t)^2\dot\varphi^2 dt}\big|_{k=1} = \int_0^{t_0} \sqrt{2\dot h^2 + h(t)^2\dot\varphi^2}dt$$

For the Internal observer $h(t) = t$ and $\varphi(t) = t$, hence $\dot h = 1, \dot\varphi = 1$, hence

$$L = \int_0^{t_0} \sqrt{2\dot h^2 + h(t)^2\dot\varphi^2}dt = \int_0^{t_0} \sqrt{2 \cdot 1 + t^2 \cdot 1}dt = \int_0^{t_0} \sqrt{t^2 + 2}dt\,.$$

We come to the same answer.

4 *On the sphere of the radius R consider two points $\mathbf{r}_A$ with spherical coordinates $\{\theta_A, \varphi_A\}$ and $\mathbf{r}_B$ with spherical coordinates $\{\theta_B, \varphi_B\}$.*

a) In the case if $\varphi_A = \varphi_B$ write down the parametric equation of the arc of the meridian C_{AB} which joins these points and calculate its length.

b) In the case if $\theta_A = \theta_B$ write down the parametric equation of the arc of the latitude which joins these points and calculate tis length.

Is the length of the arc of the great circle joining the points A, B shorter than the length of the arc of latitude? (You may consider only the case if $\varphi_A = 0, \varphi_B = \pi$.)

c†*) Calculate the length of the arc of the great circle joining the points $\mathbf{r}_A = (\theta_A, \varphi_A)$ and $\mathbf{r}_B = (\theta_B, \varphi_B)$.*

a) The parametric equation of the arc of the meridian C_{AB} which joins these points is $\theta(t) = t$, $\varphi(t) = \varphi_a = \varphi_B$, $\theta_A \le t \le \theta_B$ (We assume that $\theta_A \le \theta_B$). It is the arc of the big circle on the sphere. Its radius is equal to R. Hence its length equals to $L = R(\theta_B - \theta_A)$. We will come to the same answer doing calculations with use of First quadratic form (1a): $\dot\theta = 1, \dot\varphi = 0$ hence

$$L = \int_{t_1}^{t_2} \sqrt{R^2\dot\theta^2 + R^2\sin^2\theta\dot\varphi^2}dt = \int_{\theta_A}^{\theta_B} \sqrt{R^2 \cdot 1 + R^2\sin^2\theta \cdot 0}dt = R(\theta_B - \theta_A)\,.$$

b) The parametric equation of the arc of the latitude D_{AB} which joins these points is $\theta(t) = \theta_A = \theta_B = \theta_0$, $\varphi(t) = t$, $\varphi_A \le t \le \varphi_B$ (We assume that $\varphi_A \le \varphi_B$). It is the arc of the circle on the sphere of the radius $r = R\sin\theta_0$. Hence its length equals to $L = r(\varphi_B - \varphi_A) = R\sin\theta_0(\varphi_B - \varphi_A)$.

We will come to the same answer doing calculations with use of First quadratic form (1a): $\dot\varphi = 1, \dot\theta = 0$ hence

$$L = \int_{t_1}^{t_2} \sqrt{R^2\dot\theta^2 + R^2\sin^2\theta\dot\varphi^2}dt = \int_{\varphi_A}^{\varphi_B} \sqrt{R^2 \cdot 0 + R^2\sin^2\theta_0 \cdot 1}dt = \int_{\varphi_A}^{\varphi_B} R\sin\theta_0 dt = R\sin\theta_0(\varphi_B - \varphi_A)\,.$$

† First consider case where $\varphi_A = 0, \varphi_B = \pi$ and suppose that the points A and B have the same latitude $\theta_A = \theta_B = \theta_0$. We assume that the latitude is in the North hemisphere: $0 \leq \theta_0 \leq \frac{\pi}{2}$. Show that the length of the arc of the great circle joining the points A, B is shorter than the length of the arc of latitude. In this case the length of the great circle equals to $2R\theta_0$ (You travel trough North pole.) The length of the arc of latitude equals to $R\pi \sin\ \theta_0$. One can see that $R\pi \sin\ \theta_0 > 2R\theta$ (for $0 \leq \theta \leq \frac{\pi}{2}$)

This is much easier to prove in general case: show that in general case the length of the arc of the big circle joining two arbitrary points A, B on the sphere is the shortest curve. WLOG suppose that these points are on the same meridian $\varphi_A = \varphi_B = 0$ and latitudes θ_A, θ_B. Let $\theta(t), \varphi(t), o \leq t \leq t_0$ be an arbitrary curve joining these points: $\varphi(0) = \varphi(t_0) = 0$. Then

$$L = \int_0^{t_0} \sqrt{R^2\dot{\theta}^2 + R^2 \sin^2\theta \dot{\varphi}^2 dt} \geq \int_0^{t_0} \sqrt{R^2\dot{\theta}^2 dt} = \int_0^{t_0} Rd\theta = R(\theta_B - R\theta_A)\,.$$

We see that the shortest curve is the arc of the great circle and its length equals to $R(\theta_B - R\theta_A)$.

† Now calculate the length of the arc of the great circle joining the points $\mathbf{r}_A = (\theta_A, \varphi_A)$ and $\mathbf{r}_B = (\theta_B, \varphi_B)$.

The length of the arc of the great circle which connects these points equals $R\Psi$ where Ψ is the angle between the vectors $\mathbf{r_A}$ and $\mathbf{r}_B$. Calculate this angle:

$$\cos\Psi = \frac{\langle \mathbf{r}_A, \mathbf{r}_B \rangle}{|\mathbf{r}_A||\mathbf{r}_B|} = \frac{\langle \mathbf{r}_A, \mathbf{r}_B \rangle}{R^2} = \sin\theta_0 \cos\varphi_0 \sin\theta_1 \cos\varphi_1 + \sin\theta_0 \sin\varphi_0 \sin\theta_1 \sin\varphi_1 + \cos\theta_0 \cos\theta_1 =$$

$$\sin\theta_0 \sin\theta_1 (\cos\varphi_0 \cos\varphi_1 + \sin\varphi_0 \sin\varphi_1) + \cos\theta_0 \cos\theta_1 = \sin\theta_0 \sin\theta_1 \cos(\varphi_0 - \varphi_1) + \cos\theta_0 \cos\theta_1 =$$

$$\cos(\theta_0 - \theta_1) - \sin\theta_0 \sin\theta_1 (1 - \cos(\varphi_0 - \varphi_1)) = \cos(\theta_0 - \theta_1) - 2\sin\theta_0 \sin\theta_1 \sin^2\left(\frac{\varphi_0 - \varphi_1}{2}\right),$$

i.e.

$$L_{AB} = R \arccos\left(\cos(\theta_0 - \theta_1) - 2\sin\theta_0 \sin\theta_1 \sin^2\left(\frac{\varphi_0 - \varphi_1}{2}\right)\right).$$