Homework 6

1 Calculate the integrals of the form $\omega = xdy - ydx$ over the following three curves. Compare answers.

$$C_1\colon \mathbf{r}(t) \begin{cases} x = R\cos t \\ y = R\sin t \end{cases},\ 0 < t < \pi, \quad C_2\colon \mathbf{r}(t) \begin{cases} x = R\cos 4t \\ y = R\sin 4t \end{cases},\ 0 < t < \frac{\pi}{4}$$

$$\text{and}\quad C_3\colon \mathbf{r}(t) \begin{cases} x = Rt \\ y = R\sqrt{1-t^2} \end{cases},\ -1 \leq t \leq 1.$$

2 Consider an arc of parabola $x = 2y^2 - 1$, $0 < y < 1$.

Give examples of two different parameterisations of this curve such that these parameterisations have the opposite orientation.

Calculate the integral of the form 1-form $\omega = \sin y dx$ over this curve.

How does the answer depend on a parameterisation?

3 Calculate the integral of the form $\omega = xdy$ over the following curves

a) closed curve $x^2 + y^2 = 12y$

b) arc of the ellipse $x^2 + y^2/9 = 1$ defined by the condition $y \geq 0$.

How does your answer depend on a choice of parameterisation?

Exact forms

4 Calculate the integral $\int_C \omega$ where $\omega = xdx + ydy$ and C is

a) the straight line segment $x = t, y = 1 - t$, $0 \leq t \leq 1$

b) the segment of parabola $x = t$, $y = 1 - t^n$, $0 \leq t \leq 1$, $n = 2, 3, 4, \dots$

c) for **an arbitrary** curve starting at the point $(0,1)$ and ending at the point $((1,0)$.

5 Show that the form 1-form $\omega = 3x^2ydx + x^3dy$ is an exact 1-form.

Calculate integral of this form over the curves considered in exercises 2) and 3).

6. Consider in $\mathbf{E}^2$ 1-forms

a) xdx , b) xdy c) $xdx + ydy$, d)$xdy + ydx$, e) $xdy - ydx$

f) $x^4dy + 4x^3ydx$.

a) Show that 1-forms a), c), d) and f) are exact forms

b) Why are 1-forms b) and e) not exact?

All the exercises below are not compulsory

7† Consider one-form

$$\omega = \frac{xdy - ydx}{x^2 + y^2} \tag{1}$$

This form is defined in $\mathbf{E}^2 \setminus 0$.

Calculate differential of this form.

Write down this form in polar coordinates

Find a (smooth) function f defined for $y > 0$ such that $\omega = df$.

Does there exist a (smooth) function defined in $\mathbf{E}^2 \setminus 0$ such that $\omega = dF$?

8† Calculate the integral of the form $\omega = \frac{xdy - ydx}{x^2+y^2}$ over the curves

a) circle $x^2 + y^2 = 1$

b) circle $(x-3)^2 + y^2 = 1$

c) ellipse $\frac{x^2}{9} + \frac{x^2}{16} = 1$

9† What values can take the integral $\int_C \omega$ if C is an arbitrary curve starting at the point $(0,1)$ and ending at the point $((1,0)$ and $\omega = \frac{xdy - ydx}{x^2+y^2}$.

10† Let $\omega = a(x,y)dx + b(x,y)dy$ be a closed form in $\mathbf{E}^2$, $d\omega = 0$.

Consider the function

$$f(x,y) = x\int_0^1 a(tx,ty)dt + y\int_0^1 b(tx,ty)dt \tag{2}$$

Show that

$$\omega = df\,.$$

This proves that an arbitrary closed form in $\mathbf{E}^2$ is an exact form.

Why we cannot apply the formula (2) to the form ω defined by the expression (1)?