Homework 1

1 Show that the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2 \ldots, \mathbf{a}_m\}$ in vector space V is linear dependent if at least one of these vectors is equal to zero.

2 Show that any three vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ in $\mathbf{R}^2$ are linear dependent.

3 Let 3 vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ be expressible as a linear combination of 2 vectors $\{\mathbf{a}, \mathbf{b}\}$, i.e. 3 vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ belong to the span of 2 vectors $\{\mathbf{a}, \mathbf{b}\}$. (All vectors $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{a}, \mathbf{b})$ belong to the vector space V.)

Prove that the three vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ are linear dependent.

† Prove that $m+1$ vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{m+1}\}$ in V are linear dependent if they belong to the span of m vectors $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$.

4 Let $\{\mathbf{a}, \mathbf{b}\}$ be two vectors in the vector space V such that

i) these vectors are linear independent

ii) for an arbitrary vector $\mathbf{x} \in V$ vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{x}\}$ are linear dependent.

What is a dimension of the vector space V?

Is an ordered set $\{\mathbf{a}, \mathbf{b}\}$ a basis in the vector space V?

5 Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a basis in 3-dimensional vector space V. Show that

a) all vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are not equal to zero.

b) an arbitrary vector $\mathbf{x} \in V$ can be expressed as a linear combination of the basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in a unique way, i.e. if

$$\mathbf{x} = a^1\mathbf{e}_1 + a^2\mathbf{e}_2 + a^3\mathbf{e}_3 = a'^1\mathbf{e}_1 + a'^2\mathbf{e}_2 + a'^3\mathbf{e}_3 \text{ then } a^1 = a'^1, a^2 = a'^2, a^3 = a'^3$$

c)† Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be an ordered set of vectors in the vector space V such that an arbitrary vector $\mathbf{x} \in V$ can be expressed as a linear combination of the vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ in a unique way. Show that V is an n-dimensional vector space and an ordered set $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis in V.

(Try to prove it first for $n = 2, 3$.)

6 Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a basis in 3-dimensional vector space. Show that it is a maximal set of linear independent vectors in V, i.e. every set of vectors which contains base vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a set of linear dependent vectors.

7 Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a basis of 3-dimensional vector space V.

Is a set of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ a basis of V in the case if

a) $\mathbf{e}'_1 = \mathbf{e}_2$, $\mathbf{e}'_2 = \mathbf{e}_1$, $\mathbf{e}'_3 = \mathbf{e}_3$;

b) $\mathbf{e}'_1 = \mathbf{e}_1$, $\mathbf{e}'_2 = \mathbf{e}_1 + 3\mathbf{e}_3$, $\mathbf{e}'_3 = \mathbf{e}_3$;

c) $\mathbf{e}'_1 = \mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{e}'_2 = 3\mathbf{e}_1 - 3\mathbf{e}_2$, $\mathbf{e}'_3 = \mathbf{e}_3$;

d) $\mathbf{e}'_1 = \mathbf{e}_2$, $\mathbf{e}'_2 = \mathbf{e}_1$, $\mathbf{e}'_3 = \mathbf{e}_1 + \mathbf{e}_2 + \lambda\mathbf{e}_3$ (where λ is an arbitrary coefficient)?