Again on distance in Grassmanian

In Septembre we wrote the function "`distance`" in Grassmanian. Recall this construction. Let M, N be two k-planes in $\mathbf{R}^N$, i.e. points in the Grassmanian $G_{k,N}$. We suppose that $\mathbf{R}^N$ is equipped with a Euclidean structure. Then define the "`distance`" between these points in the following way:

$$\texttt{"distance"}(M,N) = \sqrt{\mathrm{Tr}\,\left(\langle \mathbf{m}_i, \mathbf{m}_k\rangle\langle \mathbf{n}_k, \mathbf{n}_j\rangle - \langle \mathbf{m}_i, \mathbf{n}_k\rangle\langle \mathbf{m}_k, \mathbf{n}_j\rangle\right)} =$$

$$\sqrt{\mathrm{Tr}\,\left(I_{k\times k} - \langle \mathbf{m}_i, \mathbf{n}_k\rangle\langle \mathbf{m}_k, \mathbf{n}_j\rangle\right)} = \sqrt{\left(k - \langle \mathbf{m}_i, \mathbf{n}_k\rangle\langle \mathbf{m}_k, \mathbf{n}_i\rangle\right)}\,,$$

where $\{\mathbf{m}_i\}$ is an arbitrary orthonormal basis in the plane M, and $\{\mathbf{n}_i\}$ is an arbitrary orthonormal basis in the plane N. One can see that the function "`distance`"(M,N) is well-defined, i.e. it does not depend on a choice of the orthonormal bases.

`QUESTION` What are properties of this function?

For given arbitrary orthonormal bases $\{\mathbf{m}_i\}$ in M and $\{\mathbf{n}_j\}$ in N, consider the matrix

$$C_{ij} = \langle \mathbf{m}_i, \mathbf{n}_j\rangle\,, e$$

and consider two vectors $\mathbf{x}, \mathbf{y} \in \mathbf{R}^k$, and the bilinear form

$$B(\mathbf{x},\mathbf{y}) = \langle x^i\mathbf{m}_i + y^i\mathbf{n}_i, x^k\mathbf{m}_k + y^k\mathbf{n}_k\rangle = \mathbf{x}^2 + \mathbf{y}^2 + 2C(\mathbf{x},\mathbf{y}) =$$

$$(\mathbf{x} + C(\mathbf{y}))^2 + \left(1_{k\times k} - C^+C\right)_{ij} y^i y^j\,.$$

We come to the following Cachy-Bunyakovsky-Schwarz like inequality:

Proposition CBS inequality for matrices.

Let $\{\mathbf{a}_1, \ldots . \mathbf{a}_k\}$ and $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ be two sets of vectors in $\mathcal{E}^N$ such that

$$\langle \mathbf{a}_i, \mathbf{a}_j\rangle = \delta_{ij}\,, \quad \langle \mathbf{b}_i, \mathbf{b}_j\rangle = \delta_{ij}\,,$$

then the matrix

$$B_{ij} = \delta_{ij} - \sum_k \langle \mathbf{a}_i, \mathbf{b}_k\rangle\langle \mathbf{a}_j, \mathbf{b}_k\rangle$$

is weakly positive definite:

$$B_{ij} y^i y^j \geq 0\,.$$