Homework 0. Solutions

1 *Define on the circle $x^2+y^2=R^2$ the following four local coordinates:*
a) angle coordinate which is not define at the point $(R,0)$
b) polar coordinate which is not define at the point $(-R,0)$
c) stereographic coordinate with respect to North pole
d) stereographic coordinate with respect to South pole
Find transition functions between these coordinates

i) *polar coordinate φ:*

$$\begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \end{cases}, \qquad 0<\varphi<2\pi$$

(this coordinate is defined on all the circle except a point $(R,0)$),

ii) *another polar coordinate φ':*

$$\begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \end{cases}, \qquad -\pi<\varphi<\pi\,,$$

this coordinate is defined on all the circle except a point $(-R,0)$,

Let $N=(0,R)$ be North Pole and A be a point on the circle. Let chorde NA intersects the line $y=0$ at the point P. Let B be a point on the axis OY such that BP is parallel to the axis OX. Consider similar triangles $\triangle NAB$ and $\triangle NPO$, where O is origin. Let $A=(x,y)$ and $P=(t,0)$. We have:

$$\frac{|NB|}{|NO|}=\frac{|AB|}{|OP|}\, i.e. \frac{R-y}{R}=\frac{x}{t}\,, i.e.\,, t=\frac{Rx}{R-y}\,, \text{and}\, x=\frac{t(R-y)}{R}\,.$$

Using $x^2+y^2=\frac{t^2(R-y)^2}{R^2}+y^2=R^2$ we come to $\frac{t^2(R-y)^2}{R^2}=R^2-y^2$. Dividing on $R-y$ we come to $\frac{t^2(R-y)}{R^2}=R+y$. Solving this equation we have

$$\begin{cases} x = \frac{t(R-y)}{R} = \frac{2tR^2}{R^2+t^2} \\ y = \frac{t^2-R^2}{t^2+R^2}R \end{cases}.$$

Analogously for South Pole: Let $S=(0,-R)$ be South Pole and A' a point on the circle. Let chorde SA intersects the line $y=0$ at the point P'. Consider similar triangles $\triangle SAB$ and $\triangle SP'O$. We have:

$$\frac{|SB|}{|SO|}=\frac{|AB|}{|OP'|}\, i.e. \frac{R+y}{R}=\frac{x}{t'}$$

Solving this equations we come to $t'=\frac{Rx}{R+y}$. Respectively:

$$\begin{cases} x = \frac{2t'R^2}{R^2+t'^2} \\ y = \frac{R^2-t'^2}{R^2+t'^2}R \end{cases}.$$

For transition functions see the subsection 1.1 in Lecture notes.

2 *Define on the sphere $x^2+y^2+z^2=a^2$ the following local coordinates:*
a) spherical coordinates,
b) stereographic coordinates with respect to North pole
c) stereographic coordinates with respect to South pole
Find transition functions between these coordinates

For spherical and stereographic coordiantes, and transition functions see lecture notes, subsection 1.1. Here we will just explain how derive formulae for stereographic coordinates, in the way analogous for the circle (see above)

Let $N=(0,0,a)$ be North Pole and A a point on the sphere with coordinates (x,y,z). Let chorde NA intersects the plane $z=0$ at the point P with coordinates $(x=u, y=v)$. Let B be a point on the axis OZ such that BP is parallel to the axis OX. Consider similar triangles, $\triangle NAB$ and $\triangle NPO$, where O is origin. We have:

$$\frac{|NB|}{|NO|}=\frac{|AB|}{|OP|}\, i.e. \frac{a-z}{a}=\frac{x}{u}=\frac{y}{v}\,.$$

Solving this equations we come to $\begin{cases} u=\frac{ax}{a-z} \\ v=\frac{ay}{a-z}\end{cases}$. Respectively: $x=\frac{u(a-z)}{a}, y=\frac{v(a-z)}{a}$, Using

$$x^2+y^2+z^2=\left(\frac{u(a-z)}{a}\right)^2+\left(\frac{v(a-z)}{a}\right)^2+z^2=a^2\Rightarrow \frac{u^2(a-z)^2}{a^2}+\frac{v^2(a-z)^2}{a^2}=a^2-z^2\,.$$

Dividing this equation on $a-z$ we have

$$\frac{u^2(a-z)}{a^2}+\frac{v^2(a-z)}{a^2}=a+z\Rightarrow z=\frac{u^2+v^2-a^2}{u^2+v^2+a^2}a\,,\ \frac{a-z}{a}=\frac{2a^2}{u^2+v^2+a^2}\,.$$

Hence:

$$\begin{cases} x=\frac{2ua^2}{u^2+v^2+a^2} \\ y=\frac{2va^2}{u^2+v^2+a^2} \\ z=\frac{u^2+v^2-a^2}{u^2+v^2+a^2}a \end{cases}$$

Analogously for stereographic projection with respect to South Pole.

Let $S=(0,0,-a)$ be South Pole and A a point on the sphere with coordinates (x,y,z). Let chorde NA intersects the plane $z=0$ at the point P' with coordinates $(x=u', y=v')$. Then we come to

$$\begin{cases} u=\frac{ax}{a+z} \\ v=\frac{ay}{a+z}\end{cases}\,,\qquad \begin{cases} x=\frac{2ua^2}{u^2+v^2+a^2} \\ y=\frac{2va^2}{u^2+v^2+a^2} \\ z=\frac{a^2-u^2-v^2}{u^2+v^2+a^2}a \end{cases}\,.$$

3 *Consider on the 2-dimensional manifold, domain $D=\mathbf{R}^2\backslash\mathbf{0}$ new local coordinates (u,v) such that*

$$u=\frac{x}{x^2+y^2}\,,\quad v=\frac{y}{x^2+y^2}\,,\quad (x\neq 0, y\neq 0)\quad (\text{inversion})\,, \tag{1}$$

((x,y) are Cartesian coordinates.)

Calculate Jacobian matrix of changing of local coordinates, and its determinant.

Write down inverse transition functions from coordinates (u,v) to coordinates (x,y).

† *Answer questions above using holomorphic and antiholomorphic functions*

Perform straightfoward calculations

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}=\begin{pmatrix} \frac{1}{r^2}-\frac{2x^2}{r^4} & -\frac{2xy}{r^4} \\ -\frac{2xy}{r^4} & \frac{1}{r^2}-\frac{2y^2}{r^4}\end{pmatrix}\,,\quad (r^2=x^2+y^2)\,.$$

We have that

$$\det\frac{\partial(u,v)}{\partial(x,y)}=\det\begin{pmatrix} \frac{1}{r^2}-\frac{2x^2}{r^4} & -\frac{2xy}{r^4} \\ -\frac{2xy}{r^4} & \frac{1}{r^2}-\frac{2y^2}{r^4}\end{pmatrix}=\frac{1}{r^4}-\frac{2(x^2+y^2)}{r^6}=-\frac{1}{r^2}\,.$$

To caluclate inverse transition function use that

$$\rho^2 = u^2 + v^2 = \frac{x^2}{r^4} + \frac{y^2}{r^4} = \frac{1}{r^2}\,, (r^2 = x^2 + y^2)\,.$$

Hence

$$x = ur^2 = \frac{u}{\rho^2} = \frac{u}{u^2+v^2}\,, y = vr^2 = \frac{v}{\rho^2} = \frac{v}{u^2+v^2}\,.$$

† To use complex calculus note that

$$w = u + iv = \frac{x+iy}{x^2+y^2} = \frac{z}{\bar{z}z} = \frac{1}{\bar{z}}$$

is antiholomorphic function.

4 *Consider on the* 3*-dimensional manifold, domain* $D = \mathbf{R}^3 \backslash \mathbf{0}$ *new local coordinates* (u, v, w) *such that*

$$u = \frac{x}{x^2+y^2+z^2}\,,\quad v = \frac{y}{x^2+y^2+z^2}\,,\quad w = \frac{z}{x^2+y^2+z^2}\,,\qquad (x \neq 0, y \neq 0, z = 0) \quad \text{(inversion)}\,,$$

where (x, y, z) *are Cartesian coordinates.*

Write down inverse transition functions from coordinates (u, v, w) *to coordinates* (x, y, z)*.*

Calculate Jacobian matrix of changing of local coordinates, and its determinant.

Hint To calculate determinant may be easier to perform calculations in general case when dimension n *is an arbitrary number*

In the same way as in the previous problem we have

$$\rho^2 = u^2 + v^2 + w^2 = \frac{x^2}{r^4} + \frac{y^2}{r^4} + \frac{z^2}{r^4} = \frac{1}{r^2}\,, (r^2 = x^2 + y^2 + z^2)\,.$$

Hence

$$x = ur^2 = \frac{u}{\rho^2} = \frac{u}{u^2+v^2+w^2}\,, y = vr^2 = \frac{v}{\rho^2} = \frac{v}{u^2+v^2}\,.z = vr^2 = \frac{w}{\rho^2} = \frac{w}{u^2+v^2+w^2}\,.$$

Straightforward calculations of determinant of the matrix $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ are little bit long. Do it in another way.

In general case we have for inversed coordinates in $\mathbf{E}^n - \mathbf{0}$ the following formula:

$$u^i = \frac{x^i}{r^2} \quad \text{where} \quad r^2 = (x^1)^2 + \ldots + (x^n)^2\,.$$

Entries of Jacobi matrix are

$$\frac{\partial u^i}{\partial x^k} = \frac{\partial\left(\frac{x^i}{r^2}\right)}{\partial x^k} = \frac{1}{r^2}\left(\delta^i_k - \frac{2x^i x^k}{r^2}\right)$$

To calculate determinant of this matrix note that that vector $\mathbf{r} = (x^1, \ldots, x^n)$ is its eigenvector with eigenvalue $-\frac{1}{r^2}$, and all vectors orthogonal to this vector have eigenvalue $\frac{1}{r^2}$:

$$\frac{\partial u^i}{\partial x^k} x^k = \frac{1}{r^2}\left(\delta^i_k - \frac{2x^i x^k}{r^2}\right) x^k = \frac{1}{r^2}\left(x^i - \frac{2x^i r^2}{r^2}\right) = -\frac{1}{r^2} x^i\,,$$

and

$$\frac{\partial u^i}{\partial x^k} b^k = \frac{1}{r^2}\left(\delta^i_k - \frac{2x^i x^k}{r^2}\right) b^k = \frac{1}{r^2}\left(x^i - 0\right) = +\frac{1}{r^2} x^i\,, \qquad \text{if } \mathbf{b} \perp \mathbf{r}, \text{ i.e. } b^k x^k = 0\,.$$

Hence

$$\det\left(\frac{\partial u^i}{\partial x^k}\right) = -\frac{1}{r^2}\cdot\underbrace{\frac{1}{r^2}\cdot\ldots\cdot\frac{1}{r^2}}_{n-1 \text{ times}} = \left(\frac{1}{r^2}\right)^n = -\frac{1}{r^{2n}}\,.$$

For 3-dimensional case it is equal to $-\frac{1}{r^6}$.

5 *Consider the following tensor fields given in local coordinates x^i by equations*

$$\mathbf{A} = x^i\frac{\partial}{\partial x^i}\,, \quad G = g_{ik}(x)dx^i\otimes dx^k\,, \quad Q = F^{ik}(x)\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial x^k}\,.$$

Express these tensor fields in new local coordinates $y^i = 2x^i$.

We have for vector fild $\mathbf{A}$

$$\mathbf{A} = x^i\frac{\partial}{\partial x^i} = x^i(y)\frac{\partial y^m}{\partial x^i}\frac{\partial}{\partial y^m} = \frac{1}{2}y^i\cdot 2\delta_i^m\frac{\partial}{\partial y^m} = y^i\frac{\partial}{\partial y^i}$$

Vector field has the same appearance in coordinates y^i.

Respectively

$$G = g_{ik}(x)dx^i\otimes dx^k = g_{ik}(x(y))\left(\frac{\partial x^i}{\partial y^p}dy^p\right)\otimes\left(\frac{\partial x^k}{\partial y^q}dy^q\right) =$$

$$g_{ik}\left(\frac{y}{2}\right)\left(\frac{1}{2}dy^i\right)\otimes\left(\frac{1}{2}dy^k\right) = \frac{1}{4}g_{ik}\left(\frac{y}{2}\right)dy^i\otimes dy^k\,,$$

and

$$Q = F^{ik}(x)\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial x^k} = F^{ik}\left(\frac{y}{2}\right)\left(\frac{\partial y^p}{\partial x^i}\frac{\partial}{\partial y^p}\right)\otimes\left(\frac{\partial y^q}{\partial x^k}\frac{\partial}{\partial y^q}\right) = 4F^{ik}\left(\frac{y}{2}\right)\frac{\partial}{\partial y^i}\otimes\frac{\partial}{\partial y^k} =$$

6 *Consider in manifold $\mathbf{E}^2\backslash\mathbf{0}$ vector field $\mathbf{K} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ and differential 1-form $\omega = \frac{xdy-ydx}{x^2+y^2}$ Express these objects in inversed local coordinates (u,v) defined in equation (1).*

Perform straightforwardcalculations:

$$K = x\partial_y - y\partial_x = x\left(v_y\partial_v + u_y\partial_u\right) - y\left(v_x\partial_v + u_x\partial_u\right) =$$

$$x\left(\left(\frac{1}{r^2}-\frac{2y^2}{r^4}\right)\partial_v + \left(\frac{-2xy}{r^4}\right)\partial_u\right) - y\left(\left(-\frac{2xy}{r^4}\right)\partial_v + \left(\frac{1}{r^2}-\frac{2x^2}{r^4}\right)\partial_u\right) = \frac{x}{r^2}\frac{\partial}{\partial v} - \frac{y}{r^2}\frac{\partial}{\partial u} = u\frac{\partial}{\partial v} - v\frac{\partial}{\partial u}\,.$$

The field has the same appearance.

Now for 1-form.

We have that

$$\begin{cases} x = \frac{u}{u^2+v^2} = \frac{u}{\rho^2} \\ y = \frac{v}{u^2+v^2} = \frac{v}{\rho^2} \end{cases}, \quad \rho^2 = u^2+v^2\,.$$

(see Exercise 3). Hence

$$dx = \frac{du}{\rho^2} - \frac{2u^2du}{\rho^4} - \frac{2uvdv}{\rho^4}\,, \quad dy = \frac{dv}{\rho^2} - \frac{2v^2dv}{\rho^4} - \frac{2uvdu}{\rho^4}\,,$$

and

$$\omega = \frac{xdy-ydx}{x^2+y^2} = \frac{x}{x^2+y^2}\left(\frac{dv}{\rho^2} - \frac{2v^2dv}{\rho^4} - \frac{2uvdu}{\rho^4}\right) - \frac{y}{x^2+y^2}\left(\frac{du}{\rho^2} - \frac{2u^2du}{\rho^4} - \frac{2uvdv}{\rho^4}\right) =$$

$$= u\left(\frac{dv}{\rho^2} - \frac{2v^2dv}{\rho^4} - \frac{2uvdu}{\rho^4}\right) - v\left(\frac{du}{\rho^2} - \frac{2u^2du}{\rho^4} - \frac{2uvdv}{\rho^4}\right) = \frac{udv-vdu}{u^2+v^2}\,.$$