Homework 6

1 Calculate the derivatives of the functions $f = x^2 + y^2$, $g = y^2 - x^2$ and $h = q \log |r| = q \log \left(\sqrt{x^2 + y^2} \right)$ (q is a constant) along vector fields $\mathbf{A} = x\partial_x + y\partial_y$ and $\mathbf{B} = x\partial_y - y\partial_x$

a) calculating directional derivatives $\partial_\mathbf{A} f, \partial_\mathbf{A} g, \partial_\mathbf{A} h, \partial_\mathbf{B} f, \partial_\mathbf{B} g, \partial_\mathbf{B} h$

b) calculating $df(\mathbf{A}), dg(\mathbf{A}), dh(\mathbf{A}), df(\mathbf{B}), dg(\mathbf{B}), dh(\mathbf{B})$.

2 Perform the calculations of the previous exercise in polar coordinates.

3 Consider a function $f = x^4 - y^4$.
Calculate the value of 1-form $\omega = df$ on the vector field $\mathbf{B} = x\partial_y - y\partial_x$.
Express this 1-form ω in polar coordinates r, φ ($x = r \cos \varphi, y = r \sin \varphi$).

4 Calculate the value of 1-form $\omega = xdy - ydx$ on the vector fields $\mathbf{A} = r\partial_r$ and $\mathbf{B} = \partial_\varphi$. Perform calculations in Cartesian and polar coordinates.

5 Let f be a function on $\mathbf{E}^2$ given by $f(r, \varphi) = r^2 \sin 2\varphi$, where r, φ are polar coordinates in $\mathbf{E}^2$.
Calculate the 1-form $\omega = df$.
Calculate the value of the 1-form $\omega = df$ on the vector field $\mathbf{X} = r^2 \partial_r + r\partial_\varphi$.
Express the 1-form ω in Cartesian coordinates x, y.

6 Consider 1-forms $\omega = df$ and $\sigma = dg$ such that

$$f(x, y) + ig(x, y) = (x + iy)^3 .$$

Find the values of these 1-forms on vector field $\mathbf{Y} = r\partial_r + \partial_\varphi$.

7 Calculate the integrals of the form $\omega = \sin y \, dx$ over the following three curves. Compare answers.

$$C_1 : \mathbf{r}(t) \begin{cases} x = 2t^2 - 1 \\ y = t \end{cases}, \ 0 < t < 1, \quad C_2 : \mathbf{r}(t) \begin{cases} x = 8t^2 - 1 \\ y = 2t \end{cases}, \ 0 < t < 1/2$$

$$\text{and} \quad C_3 : \mathbf{r}(t) \begin{cases} x = \cos 2t \\ y = \cos t \end{cases}, \ 0 < t < \frac{\pi}{2} .$$

8 Calculate the integrals of the form $\omega = xdy - ydx$ over the following three curves. Compare answers.

$$C_1 : \mathbf{r}(t) \begin{cases} x = R \cos t \\ y = R \sin t \end{cases}, \ 0 < t < \pi, \quad C_2 : \mathbf{r}(t) \begin{cases} x = R \cos 4t \\ y = R \sin 4t \end{cases}, \ 0 < t < \frac{\pi}{4}$$

$$\text{and} \quad C_3 : \mathbf{r}(t) \begin{cases} x = Rt \\ y = R\sqrt{1 - t^2} \end{cases}, \ -1 \leq t \leq 1 .$$