Solutions of Homework 3

In all exercises we assume by default that Riemannian metric on embedded surfaces is induced by the Euclidean metric.

1 a) *Show that surface of the cone* $\begin{cases} x^2+y^2-k^2z^2=0 \\ z>0 \end{cases}$ *in* $\mathbf{E}^3$ *is locally isometric to Euclidean plane.*

Solution.

This means that we have to find local coordinates u, v on the cone such that in these coordinates induced metric $G|_c$ on cone would have the appearance $G|_c = du^2 + dv^2$.

First of all calculate the metric on cone in natural coordinates h, φ where

$$\mathbf{r}(h,\varphi)\colon \begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases}.$$

$(x^2+y^2-k^2z^2 = k^2h^2\cos^2\varphi + k^2h^2\sin^2\varphi - k^2h^2 = k^2h^2 - k^2h^2 = 0.$

Calculate metric G_c on the cone in coordinates h, φ induced with the Euclidean metric $G = dx^2+dy^2+dz^2$:

$$G_c = \left(dx^2+dy^2+dz^2\right)\big|_{x=kh\cos\varphi, y=kh\sin\varphi, z=h} = (k\cos\varphi dh - kh\sin\varphi d\varphi)^2 +$$

$$(k\sin\varphi dh + kh\cos\varphi d\varphi)^2 + dh^2 = (k^2+1)dh^2 + k^2h^2d\varphi^2\,.$$

In analogy with polar coordinates try to find new local coordinates u, v such that $\begin{cases} u = \alpha h\cos\beta\varphi \\ v = \alpha h\sin\beta\varphi \end{cases}$, where α, β are parameters. We come to

$$du^2+dv^2 = (\alpha\cos\beta\varphi dh - \alpha\beta h\sin\beta\varphi d\varphi)^2 + (\alpha\sin\beta\varphi dh + \alpha\beta h\cos\beta\varphi d\varphi)^2 = \alpha^2dh^2 + \alpha^2\beta^2h^2d\varphi^2.$$

Comparing with the metric on the cone $G_c = (1+k^2)dh^2 + k^2h^2d\varphi^2$ we see that if we put $\alpha = \sqrt{k^2+1}$ and $\beta = \frac{k}{\sqrt{1+k^2}}$ then $du^2+dv^2 = \alpha^2dh^2+\alpha^2\beta^2h^2d\varphi^2 = (1+k^2)dh^2 + k^2h^2d\varphi^2$.

Thus in new local coordinates

$$\begin{cases} u = \sqrt{k^2+1}h\cos\frac{k}{\sqrt{k^2+1}}\varphi \\ v = \sqrt{k^2+1}h\sin\frac{k}{\sqrt{k^2+1}}\varphi \end{cases}$$

induced metric on the cone becomes $G|_c = du^2+dv^2$, i.e. cone locally is isometric to the Euclidean plane ■

2 a) *Consider the domain D on the conic surface $x^2+y^2-z^2$ defined by the conditions*

$$\begin{cases} 0<z<H \\ y\neq 0 \quad \text{if } x>0 \end{cases}.$$

(The second condition means that the line $x=z, y=0$ is removed from the surface of the cone)

Find a domain D' in Euclidean plane such that it is isometric to the surface D.

b) Find a shortest distance between points $A=(0,1,1)$ and $B=(0,-1,1)$ for an ant living on the surface D (we assume that $H>1$).

Solution.

The domain D on the cone can be parameterised as (see the previous exeercise for $k=1$)

$$\mathbf{r}(h,\varphi)\colon \begin{cases} x = h\cos\varphi \\ y = h\sin\varphi \\ z = h \end{cases} \quad 0<h<H, 0<\varphi<2\pi$$

Using the results of previous exercise for $k=1$ consider new local coordinates

$$u, v\colon \quad \begin{cases} u = \sqrt{2}h \cos \frac{\varphi}{\sqrt{2}} \\ v = \sqrt{2}h \sin \frac{\varphi}{\sqrt{2}} \end{cases} \quad 0 < h < H, 0 < \varphi < 2\pi$$

In these coordinates metric $G = du^2 + dv^2$. Consider Euclidean plane with Cartesian coordinates u, v and new polar coordinates

$$(R, \theta)\colon \quad \begin{cases} R = \sqrt{u^2+v^2} = \sqrt{2}h \\ \theta = \frac{\varphi}{\sqrt{2}} \quad 0 < h < z, 0 < \varphi < 2\pi \end{cases}$$

We come to the sector D' in $\mathbf{E}^2$ with polar coordinates R, θ such that

$$0 < R < \sqrt{2}H, \ , 0 < \theta < 2\pi\sqrt{2}\,.$$

(It is what happens with the cone when we use scissors!)

We established isometry between the domain D on the cone and the domain D' on the Euclidean plane. To find the shortest distance between points A, B on the cone we consider images of these points on the domain D' where the shortest distance will be achieved on the straight line.

For point A with coordinates $(0,1,1)$ $h_A = \sqrt{2}, \varphi_A = \frac{\pi}{2}$, and $R_A = \sqrt{2}$, $\theta_A = \frac{\pi}{2\sqrt{2}}$. For the point B with coordinates $(0,-1,1)$ $h_B = \sqrt{2}, \varphi_B = \pi$, and $R_B = \sqrt{2}$, $\theta_B = \frac{\pi}{\sqrt{2}}$. The length of the segment between these points which are on the arc of the circle of the radius $\sqrt{2}$ equals to

$$d = 2\sqrt{2}\sin\left(\frac{\theta_B - \theta_A}{2}\right) = 2\sqrt{2}\sin\left(\frac{\pi}{4\sqrt{2}}\right)$$

Notice that it is shorter that the length of the arc $\frac{\pi}{2}$: $2\sqrt{2}\sin\left(\frac{\pi}{4\sqrt{2}}\right) < \frac{\pi}{2}$.

3 Consider plane with Riemannian metric given in cartesian coordinates (x, y) by the formula

$$G = \frac{4(dx)^2 + 4(dy)^2}{(1+x^2+y^2)^2} \tag{1}$$

Show that this Riemannian manifold is locally isometric to the sphere for an arbitrary.

Problem becomes easy if we just change the names of variables $x \leftrightarrow u$, $y \leftrightarrow v$. Then we immediately recognize the stereographic coordinates u, v for sphere (up to a coefficient). Recall that for unit sphere in stereographic coordinates u, v $G = \frac{4du^2+4dv^2}{1+u^2+v^2}$.

So the answer is clear: The plane with metric (1) is locally isometric to the unit sphere.

(One can write down the explicit transformation of coordinates $x = u, v = y$ to spherical coordinates.)

4 *Consider catenoid: $x^2 + y^2 = \cosh^2 z$ and helicoid: $y - x\tan z = 0$.*

Find induced Riemannian metrics on these surfaces.

Show that these surfaces are locally isomorphic.

Write down apranetric equations for catenoid and helicoid.

Catenoid is the surface of resolution:

$$\mathbf{r}(t, \varphi)\colon \quad \begin{cases} x = f(t)\cos\varphi \\ y = f(t)\sin\varphi \\ z = t \end{cases}$$

for $f(t) = cosht$, i.e.

$$\mathbf{r}(t, \varphi)\colon \quad \begin{cases} x = \cosh t \cos\varphi \\ y = \cosh t \sin\varphi \\ z = t \end{cases} \qquad \text{(catenoind)}$$

$(x^2 + y^2 - \cosh^2 z = 0)$.

We come to helicoid If we rotate the horisontal line and move it in vertical direction with constant speeds *:

$$\mathbf{r}(t,\varphi)\colon \quad \begin{cases} x = t\cos\varphi \\ y = t\sin\varphi \\ z = \varphi \end{cases} \qquad \text{(helicoid)}$$

Calculate induced Riemannian structures:

$$G_{cat} = (dx^2 + dy^2 + dz^2)\big|_{x=\cosh t\cos\varphi, y=\cosh t\sin\varphi, z=t} =$$

$$(\sinh t\cos\varphi dt - \cosh t\sin\varphi d\varphi)^2 + (\sinh t\sin\varphi dt + \cosh t\cos\varphi d\varphi)^2 + dt^2 =$$

$$(1+\sinh^2 t)dty^2 + \cosh^2 t d\varphi^2 = \cosh^2 t(dt^2 + d\varphi^2)\,. \tag{2}$$

$$G_{hel} = (dx^2 + dy^2 + dz^2)\big|_{x=t\cos\varphi, y=t\sin\varphi, z=\varphi} =$$

$$(dt\cos\varphi - t\sin\varphi d\varphi)^2 + (dt\sin\varphi + t\cos\varphi d\varphi)^2 + d\varphi^2 =$$

$$dt^2 + t^2 d\varphi^2 + d\varphi^2 = dt^2 + (1+t^2)d\varphi^2\,. \tag{3}$$

Compare Riemannian metrics (2) and (3). We see that if we consider in (3) $t \mapsto \sinh t$ we come to (2):

$$G_{helicoid} = \left(dt^2 + (1+t^2)d\varphi^2\right)_{t\mapsto\sinh t} = (d\sinh t)^2 + (1+\sinh^2 t)d\varphi^2 = \cosh^2 t(dt^2 + d\varphi^2) = G_{hel}$$

Hence helicoid and catenoid are locally isometric.

You could find very beautiful picture how helicoid isometrically can be transformd to catenoid (see Wikipedia).

5 a) *Consider the domain D on the cone $x^2 + y^2 - k^2 z^2$ defined by the condition $0 < z < H$. Find an area of this domain using induced Riemannian metric. Compare with the answer when using standard formulae.*

We have cone with height H with radius $R = kH$ $(k > 0)$.

First of all standard answer: The area of cone is area of the sector with the radius $\sqrt{H^2 + R^2}$ and length of the arc $2\pi R$:

$$S = \frac{1}{2}\cdot\sqrt{R^2 + H^2}\cdot 2\pi R = \pi R\sqrt{H^2 + R^2} = \pi k\sqrt{1+k^2}H^2\,.$$

Now calculate this are using Riemannian geometry. It follows from the result of the exercise (2) that volume form on the cone equals

$$d\sigma = \sqrt{\det G}dh\wedge d\varphi = k\sqrt{1+k^2}dh\wedge d\varphi$$

since $G = \begin{pmatrix} k^2+1 & 0 \\ 0 & k^2h^2 \end{pmatrix}$ Hence

$$S = \int_{0<h<H} \sqrt{\det G}dh\wedge d\varphi = \int_{0<h<H} k\sqrt{1+k^2}dh\wedge d\varphi = 2\pi k\sqrt{1+k^2}\int_0^H hdh = \pi k\sqrt{1+k^2}H^2\,.$$

(Compare with standard calculations).

6 a) *Find an area of 2-dimensional sphere of radius R using explicit formulae for induced Riemannian metric in stereographic coordinates.*

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* look in Wikipedia for detail

b)† *Find a volume of n-dimensional sphere of radius a. (You may use Riemannian metric in stereographic coordinates, or you may do it in other way... You just have to calculate the answer.)*

a) In the case of the sphere of the radius R the relations between stereographic coordinates u, v and Cartesian coordinates in ambient space are very similar to those we established in Homework 1 for unit sphere:

Sphere $x^2 + y^2 + z^2 = R^2$. Stereographic coordinates u, v. Centre of projection $(0, 0, R)$:

$$\begin{cases} u = \frac{Rx}{R-z} \\ v = \frac{Ry}{R-z} \end{cases}, \qquad \begin{cases} x = \frac{2R^2u}{R^2+u^2+v^2} \\ y = \frac{2R^2v}{R^2+u^2+v^2} \\ z = \frac{R(u^2+v^2-R^2)}{u^2+v^2+R^2} \end{cases} \tag{2}$$

Sure using brute force we can repeat the calculations for differential using (2). This is little bit boring since calculations for the case $R = 1$ were not very quick. Try to escape the straightforward calculations.

Consider homothetic transformation of ambient space and of the space with coordinates u, v:

$$x = R\tilde{x},\ y = R\tilde{y},\ z = R\tilde{z},\ u = R\tilde{u},\ v = R\tilde{v}\,.$$

We see that

$$\begin{cases} \tilde{u} = \frac{u}{R} = \frac{x}{R-z} = \frac{R\tilde{x}}{R-R\tilde{z}} = \frac{\tilde{x}}{1-\tilde{z}} \\ \tilde{v} = \frac{v}{R} = \frac{y}{R-z} = \frac{R\tilde{y}}{R-R\tilde{z}} = \frac{\tilde{y}}{1-\tilde{z}} \end{cases},$$

coordinates $\tilde{u}, \tilde{v}$ are related with coordinates $\tilde{x}, \tilde{y}, \tilde{z}$ in the same way as stereographic coordinates u, v are related with coordinates x, y, z in the case if $R = 1$. For unit sphere we already know the expression of metric in stereographic coordinates:

$$G = \left(dx^2 + dy^2 + dz^2\right)\big|_{x^2+y^2+z^2=1} = \frac{4du^2 + 4dv^2}{(1 + u^2 + v^2)^2}$$

(See homework 1). Now using the fact that coordinates $\tilde{u}, \tilde{v}$ are related with coordinates $\tilde{x}, \tilde{y}, \tilde{z}$ in the same way as coordinates u, v with coordinates x, y, z for unit sphere we have for a sphere for an arbitrary radius:

$$G = \left(dx^2 + dy^2 + dz^2\right)\big|_{x^2+y^2+z^2=R^2} = R^2\left(d\tilde{x}^2 + d\tilde{y}^2 + d\tilde{z}^2\right)\big|_{\tilde{x}^2+\tilde{y}^2+\tilde{z}^2=1} = R^2\frac{4d\tilde{u}^2 + 4d\tilde{v}^2}{1 + \tilde{u}^2 + \tilde{v}^2} =$$

$$R^2\frac{4\left(\frac{du}{R}\right)^2 + 4\left(\frac{dv}{R}\right)^2}{\left(1 + \left(\frac{u}{R}\right)^2 + \left(\frac{v}{R}\right)^2\right)^2} = \frac{4R^4du^2 + 4R^4dv^2}{(R^2 + u^2 + v^2)^2}$$

To calculate the volume (area) of 2-sphere we note that $\det G = \frac{4R^4}{(R^2+u^2+v^2)^2}$, thus

$$A = \int \sqrt{\det G}\,dudv = \int \left(\frac{4R^4}{(R^2 + u^2 + v^2)^2}\right) dudv$$

Consider homothety $u = R\tilde{u}, v = R\tilde{v}$ we come to

$$A = \int \left(\frac{4R^4}{(R^2 + u^2 + v^2)^2}\right) dudv = \int \left(\frac{4R^6}{(R^2 + R^2\tilde{u}^2 + R^2\tilde{v}^2)^2}\right) d\tilde{u}d\tilde{v} = \mathbf{R}^2 \int \frac{4d\tilde{u}d\tilde{v}}{(1 + \tilde{u}^2 + \tilde{v}^2)^2}$$

We come to the desired formula $A(R) = R^2A(1)$.

Taking polar coordinates $u = r\cos\theta, v = r\sin\theta$ we come to

$$A = R^2 \int \frac{4d\tilde{u}d\tilde{v}}{(1 + \tilde{u}^2 + \tilde{v}^2)^2} = R^2 \int \frac{4rdrd\varphi}{(1 + r^2)^2} = 4\pi R^2 \int_0^\infty \frac{du}{(1 + u)^2} = 4\pi R^2\,.$$

b†) Denote by σ_n the volume of n-dimensional unit sphere embedded in Euclidean space $\mathbf{E}^{n+1}$.

One can see that the volume of n-dimensional sphere of the radius R equals to $\sigma_n R^{n+1}$. Now consider the magnitude

$$I = \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}.$$

For any integer k consider

$$I^k = \pi^{\frac{k}{2}} = \left(\int_{-\infty}^{\infty} e^{-t^2} dt\right)^k = \int_{\mathbf{E}^k} e^{-x_1^2-x_2^2-\ldots-x_k^2} dx_1 dx_2 \ldots dx_k$$

make changing of variables in the volume form $dx_1 \wedge dx_2 \wedge \ldots \wedge dx_k$.

Since integrand depend only on the radius we can rewrite the integral above as

$$\int_{\mathbf{E}^k} e^{-x_1^2-x_2^2-\ldots-x_k^2} dx_1 dx_2 \ldots dx_k = \int_{\mathbf{E}^k} e^{-r^2} r^{k-1} \sigma_{k-1} dr\,,$$

where σ_{k-1} is a volume of the unit sphere in dimension $k-1$. (Here is the truck!)

Now we have the identity:

$$\pi^{\frac{k}{2}} = \sigma_{k-1} \int_0^{\infty} e^{-r^2} r^{k-1} dr$$

To calculate this integral consider $r^2 = t$ we come to

$$\int_0^{\infty} e^{-r^2} r^{k-1} dr = \frac{1}{2} \int_0^{\infty} e^{-t} t^{\frac{k}{2}-1} dt = \frac{1}{2} \Gamma\left(\frac{k}{2}\right)\,.$$

We come to

$$\pi^{\frac{k}{2}} = \sigma_{k-1} \int_0^{\infty} e^{-r^2} r^{k-1} dr = \frac{\sigma_{k-1}}{2} \Gamma\left(\frac{k}{2}\right)\,.$$

Thus

$$\sigma_{k-1} = \frac{2\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)}.$$

Recall that $\Gamma(x)$ can be calculated for all $\frac{k}{2}$ using the following recurrent formulae:

1. $\Gamma(n+1) = n!$
2. $\Gamma(x+1) = x\Gamma(x)$
3. $\frac{\Gamma(1}{2)=\sqrt{\pi}}$ $(\Gamma(x)\Gamma(1-x) = \pi \sin \pi x)$.

E.g. the volume of the 15-dimensional unit sphere in $\mathbf{E}^{16}$ equals to $\sigma_{15} = \frac{2\pi^8}{\Gamma(8)} = \frac{2\pi^6}{7!}$