Homework 8

1) Let M be a surface embedded in Euclidean space $\mathbf{E}^3$. We say that the triple of vector fields $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ is adjusted to the surface M if $\mathbf{e}, \mathbf{f}, \mathbf{n}$ be three vector fields defined on the points of this surface such that they form an orthonormal basis at any point, so that the vectors $\mathbf{e}, \mathbf{f}$ are tangent to the surface and the vector $\mathbf{n}$ is orthogonal to the surface.

Consider the derivation formulae for adjusted vector fields $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$:

$$d\begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{f} \\ \mathbf{n} \end{pmatrix}, \tag{1}$$

where a, b, c are 1-forms on the surface M.

Write down the explicit expression for connection, Weingarten operator, the mean curvature and the Gaussian curvature of M in terms of 1-forms a, b, c and vector fields $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$.

2) Show that in derivation formulae $da + b \wedge c$=0.

3) Find explicitly a triple of vector fields $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ adjusted to the surface M if M is
a) cylinder
b) cone
c) sphere

4) Using results of the previous exercise find explicit expression for derivation formulae (1) in the case if the surface M is a) cylinder, b) cone, c) sphere

Deduce from these results the formulae for Gaussian and mean curvature for cylinder, cone and sphere

5) a) Find a triple of vector fields $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ adjusted to the surface M if a Riemannian metric on a surface M is given by formula $G = a(u, v)du^2 + b(u, v)dv^2$.
b*) Calculate 1-form a in derivation formulae in the special case if $a = b = \sigma(u, v)$ (conformal metric). Calculate Gaussian curvature. (It is convenient to use notation $\sigma = e^{\Phi}$).

6*) Let $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ and $\{\tilde{\mathbf{e}}, \tilde{\mathbf{f}}, \tilde{\mathbf{n}}\}$ be two triples of vector fields adjusted to the surface M.

What is the relation between these triples?

How 1-forms a, b, c in derivation formulae will change if we will change $\{\mathbf{e}, \mathbf{f}, \mathbf{n}\}$ to $\{\tilde{\mathbf{e}}, \tilde{\mathbf{f}}, \tilde{\mathbf{n}}\}$?

Show that 2-form da and 2-form $b \wedge c$ are independent on the choice of the adjusted triple.

7 Consider in $\mathbf{E}^3$ a vector $\mathbf{X} = \frac{\partial}{\partial y}$ attached at the point $\mathbf{p}$: $(x = R\cos\theta_0, y = 0, z = R\sin\theta_0)$ of the sphere $x^2 + y^2 + z^2 = R^2$ in $\mathbf{E}^3$. Consider on the sphere the following two curves passing via the point $\mathbf{p}$:

a curve C_1 which is the intersections of this sphere with plane $y = 0$ and a curve C_2 which is the intersections of this sphere with the plane $z = R\sin\theta_0$.

Find the result of parallel transport of the vector $\mathbf{X}$ along these closed curves.

8)* What will be the result of the parallel transport of an arbitrary tangent vector along the closed curve C on the cone?

How this result correlates with the fact that the Gaussian connection of the cone equals to zero.