**Homework 1. Solutions**

**1** *Show that the set of vectors $\{\mathbf{a}_1, \mathbf{a}_2 \ldots, \mathbf{a}_m\}$ in vector space $V$ is linear dependent if at least one of these vectors is equal to zero.*

WLOG suppose that $\mathbf{a}_1 = 0$. Then

$$\lambda \mathbf{a}_1 + 0 \cdot \mathbf{a}_2 + \ldots + 0 \cdot \mathbf{a}_n = 0$$

where $\lambda$ is an arbitrary real number. We see that there exists a linear combinations of vectors $\{\mathbf{a}_1, \mathbf{a}_2 \ldots, \mathbf{a}_m\}$ which is equal to zero and one of the coefficients $\{\lambda, 0, \ldots, 0\}$ could be equal to non-zero. Hence vectors $\{\mathbf{a}_1, \mathbf{a}_2 \ldots, \mathbf{a}_m\}$ are linear dependent.

**2** *Show that any three vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ in $\mathbf{R}^2$ are linear dependent.* We will show it straightforwardly here.

Let three vectors

$$\begin{array}{c} \mathbf{x}_1 = (a^1, a^2) \\ \mathbf{x}_2 = (b^1, b^2) \\ \mathbf{x}_3 = (c^1, c^2) \end{array}$$

be linear independent. If vector $\mathbf{x}_1 = (a_1, a_2) = 0$ then nothing to prove.(See exercise 1). Let $\mathbf{x}_1 \neq 0$.WLOG suppose $a_1 \neq 0$. Consider

$$\begin{array}{c} \mathbf{x}'_2 = \mathbf{x}_2 - \frac{b_1}{a_1}\mathbf{x}_1 = (b^1, b^2) - \frac{b_1}{a_1}(a_1, a_2) = (0, b'_2) \\ \mathbf{x}'_3 = \mathbf{x}_3 - \frac{c_1}{a_1}\mathbf{x}_1 = (c^1, c^2) - \frac{c_1}{a_1}(a_1, a_2) = (0, c'_2) \end{array}$$

We see that vectors $\mathbf{x}'_2, \mathbf{x}'_3$ are proportional—i.e. they are linear dependent: there exist $\mu_2 \neq 0$ or $\mu_3 \neq 0$ such that $\mu_2 \mathbf{x}'_2 + \mu_3 \mathbf{x}'_3 = 0$ E.g. we can take $\mu_2 = c'_2$, $\mu_3 = -b'_2$ if $c'_2 \neq 0$ or $b'_2 \neq 0$ (if $c'_2 = b'_2 \neq 0$ then we can take coefficients $\mu_1, \mu_2$ any real numbers. ) We have:

$$0 = \mu_2 \mathbf{x}'_2 + \mu_3 \mathbf{x}'_3 = \mu_2 \left(\mathbf{x}_3 - \frac{c_1}{a_1}\mathbf{x}_1\right) + \mu_3 \left(\mathbf{x}_3 - \frac{c_1}{a_1}\mathbf{x}_1\right) = \mu_2 \mathbf{x}_2 + \mu_3 \mathbf{x}_3 - \left(\frac{\mu_2 b_1}{a_1} + \frac{\mu_3 c_1}{a_1}\right)\mathbf{x}_1 = 0,$$

where $\mu_2 \neq 0$ or $\mu_3 \neq 0$. Hence vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linear dependent. $\blacksquare$

(Compare with the solution of general statement in the next exercise.)

**3** *Let* 3 *vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ in vector space $V$ can be expressed as a linear combination of* 2 *vectors $\{\mathbf{a}, \mathbf{b}\}$ of this vector space, i.e. 3 vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ belong to the span of 2 vectors $\{\mathbf{a}, \mathbf{b}\}$. Prove that three vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ are linear dependent.*

Let

$$\begin{cases} \mathbf{x}_1 = \lambda_1 \mathbf{a} + \mu_1 \mathbf{b} \\ \mathbf{x}_2 = \lambda_2 \mathbf{a} + \mu_2 \mathbf{b} \\ \mathbf{x}_3 = \lambda_3 \mathbf{a} + \mu_3 \mathbf{b} \end{cases} \tag{1}$$

If one of vectors is equal to zero then nothing to prove (See previous exercise).

$\mathbf{x}_1 \neq 0$. WLOG suppose that $\lambda_1 \neq 0$. Thus vector $\mathbf{a}$ can be expressed as a linear combination of vectors $\mathbf{x}_1$ and $\mathbf{b}$:

$$\mathbf{a} = \frac{1}{\lambda_1}\mathbf{x}_1 - \frac{\mu_1}{\lambda_1}\mathbf{b} \tag{2}$$

. (If $\lambda_1 = 0$ then $\mu \neq 0$ and we express the vector $b$ as a linear combination of vectors $\mathbf{x}_1$ and $\mathbf{a}$). Then using the relation (2) we express vector $\mathbf{x}_2$ as linear combinations of vectors $\mathbf{a}$ and $\mathbf{x}_1$:

$$\mathbf{x}_2 = \lambda_2 \left(\frac{1}{\lambda_1}\mathbf{x}_1 - \frac{\mu_1}{\lambda_1}\mathbf{b}\right) + \mu_2 \mathbf{b} = \lambda'_2 \mathbf{x}_1 + \mu'_2 \mathbf{b} \tag{3}$$

If $\mu'_2 = 0$ then everything is proved: vectors $\mathbf{x}_1, \mathbf{x}_2$ are linear dependent, hence vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linear dependent too. If $\mu'_2 \neq 0$ we express vector $\mathbf{b}$ via vectors $\mathbf{x}_1$ and $\mathbf{x}_2$:

$$\mathbf{b} = -\frac{1}{\mu_1}\mathbf{x}_2 - \frac{\lambda'}{\mu'}\mathbf{x}_1 \tag{4}$$

and using relations (4) and (2) we express vector $\mathbf{x}_3$ in (1) as a linear combinations of vectors $\mathbf{x}_1$ and $\mathbf{x}_2$, thus proving that vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ are linear dependent.

$$\mathbf{x}_3 = \lambda_3 \mathbf{a} + \mu_3 \mathbf{b} = \lambda_3 \left( -\frac{1}{\lambda_1}\mathbf{x}_1 - \frac{\mu_1}{\lambda_1}\mathbf{b} \right) + \mu_3 \left( \frac{1}{\mu_1}\mathbf{x}_2 - \frac{\lambda'}{\mu'}\mathbf{x}_1 \right) =$$

$$\lambda_3 \left( -\frac{1}{\lambda_1}\mathbf{x}_1 - \frac{\mu_1}{\lambda_1} \left( \frac{1}{\mu_1}\mathbf{x}_2 - \frac{\lambda'}{\mu'}\mathbf{x}_1 \right) \right) + \mu_3 \left( \frac{1}{\mu_1}\mathbf{x}_2 - \frac{\lambda'}{\mu'}\mathbf{x}_1 \right) = \lambda_3'' \mathbf{x}_1 + \mu_3'' \mathbf{x}_2$$

Vector $\mathbf{x}_3$ is a linear combination of vectors $\mathbf{x}_2, \mathbf{x}_3$. Hence vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linear dependent.

$^\dagger$ In a similar way one can prove that any $m+1$ vectors are linear dependent if they belong to the span of $m$ vectors (See the lemma and its proof in the subsection 1.3 of Lecture notes).

**4** *Let $\{\mathbf{a}, \mathbf{b}\}$ be two vectors in the linear space $V$ such that*
*i) these vectors are linear independent*
*ii) for an arbitrary vector $\mathbf{x} \in V$ vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{x}\}$ are linear dependent.*
*What is a dimension of the vector space $V$?*
*Is an ordered set $\{\mathbf{a}, \mathbf{b}\}$ a basis in the vector space $V$?*

Recall that the dimension of vector space $V$ is equal to $n$ if there exist $n$ linear independent vectors and any $n+1$ vectors are linear dependent.

Show that the dimension of the vector space under consideration is equal to 2.

On one hand there exist two linear dependent vectors $\mathbf{a}$ and $\mathbf{b}$. This means that dimension of $V$ is greater or equal than 2: $\dim V \geq 2$.

To prove that $\dim V = 2$ it remains to prove that any three vectors are linear dependent.

Show that arbitrary vector $\mathbf{x} \in V$ can be expressed via vectors $\mathbf{a}, \mathbf{b}$. Indeed vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{x}\}$ are linear dependent, hence

$$\mu_1 \mathbf{a} + \mu_2 \mathbf{b} + \mu_3 \mathbf{x} = 0, \quad \text{where } \mu_1 \neq 0, \text{ or } \mu_2 \neq 0 \text{ or } \mu_3 \neq 0$$

If $\mu_3 = 0$ then $\mu_1 \neq 0$, or $\mu_2 \neq 0$ and $\mu_1 \mathbf{a} + \mu_2 \mathbf{b} = 0$, i.e. vectors $\mathbf{a}, \mathbf{b}$ are linear dependent. Contradiction. Hence $\mu_3 \neq 0$, that is a vector $\mathbf{x}$ can be expressed as a linear combination of vectors $\mathbf{a}, \mathbf{b}$, i.e. it belongs to the span of the vectors $(\mathbf{a}, \mathbf{b})$.

Let $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ be a set of arbitrary 3 vectors. We just proved that any of these vectors belong to the span of the vectors $\{\mathbf{a}, \mathbf{b}\}$. Hence according to previous exercise these three vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ are linear dependent. Thus we proved that any three vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ are linear dependent.

Hence the dimension of the space $V$ is equal to 2.

The vectors $\{\mathbf{a}, \mathbf{b}\}$ are two linear independent vectors in 2-dimensional vector space $V$. Hence it is a basis.

**5** *Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a basis in 3-dimensional vector space $V$. Show that*
*a) all vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are not equal to zero.*
*b) an arbitrary vector $\mathbf{x} \in V$ can be expressed as a linear combination of the basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in a unique way, i.e. if*

$$\mathbf{x} = a^1 \mathbf{e}_1 + a^2 \mathbf{e}_2 + a^3 \mathbf{e}_3 = a'^1 \mathbf{e}_1 + a'^2 \mathbf{e}_2 + a'^3 \mathbf{e}_3 \ \text{ then } \ a_1 = a_1', a_2 = a_2', a_3 = a_3' \tag{5}$$

a) Suppose one of these vectors is equal to zero: $\mathbf{e}_1 = 0$. Then the vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are linear dependent. (See the exercise 1).

b) First prove the uniqueness of expansion (5) then the existence. Let $\mathbf{x}$ be an arbitrary vector in $V$. Suppose

$$\mathbf{x} = a^1 \mathbf{e}_1 + a^2 \mathbf{e}_2 + a^3 \mathbf{e}_3 = a'^1 \mathbf{e}_1 + a'^2 \mathbf{e}_2 + a'^3 \mathbf{e}_3 \, .$$

Then

$$0 = \mathbf{x} - \mathbf{x} = (a^1 - a'^1)\mathbf{e}_1 + (a^2 - a'^2)\mathbf{e}_2 + (a^3 - a'^3)\mathbf{e}_3$$

On the other hand vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are linear independent. Hence all coefficients $(a^1 - a'^1), (a^2 - a'^2), (a^3 - a'^3)$ are equal to zero:

$$a^1 - a'^1 = a^2 - a'^2 = a^3 - a'^3, \text{ i.e. } a^1 = a'^1, a^2 = a'^2, a^3 = a'^3$$

According to definition of basis 4 vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{x}\}$ are linear dependent. Hence vector $\mathbf{x}$ can be expressed via the vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Indeed there exist coefficients $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that

$$\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 + \lambda_3 \mathbf{e}_3 + \lambda_4 \mathbf{x} = 0 \tag{6}$$

and at least one of these coefficients is not equal to zero.

Prove that $\lambda_4 \neq 0$. Suppose $\lambda_4 = 0$. Then it follows from (6) that vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are linear dependent. Contradiction. Hence $\lambda_4 \neq 0$ and $\mathbf{x}$ can be expressed via $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$:

$$\mathbf{x} = -\frac{\lambda_1}{\lambda_4}\mathbf{e}_1 - \frac{\lambda_2}{\lambda_4}\mathbf{e}_2 - \frac{\lambda_3}{\lambda_4}\mathbf{e}_3$$

**6** † *Let $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ be an ordered set of vectors in the vector space $V$ such that an arbitrary vector $\mathbf{x} \in V$ can be expressed as a linear combination of the vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in a unique way. Show that $V$ is $n$-dimensional space and an ordered set $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is a basis in $V$.*

(See the proof of the Proposition 2 in the subsection 1.3)

**7** Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a basis of 3-dimensional vector space $V$.

Is a set of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ a basis of $V$ in the case if

a) $\mathbf{e}'_1 = \mathbf{e}_2$, $\mathbf{e}'_2 = \mathbf{e}_1$, $\mathbf{e}'_3 = \mathbf{e}_3$;

b) $\mathbf{e}'_1 = \mathbf{e}_1$, $\mathbf{e}'_2 = \mathbf{e}_1 + 3\mathbf{e}_3$, $\mathbf{e}'_3 = \mathbf{e}_3$;

c) $\mathbf{e}'_1 = \mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{e}'_2 = 3\mathbf{e}_1 - 3\mathbf{e}_2$, $\mathbf{e}'_3 = \mathbf{e}_3$;

d) $\mathbf{e}'_1 = \mathbf{e}_2$, $\mathbf{e}'_2 = \mathbf{e}_1$, $\mathbf{e}'_3 = \mathbf{e}_1 + \mathbf{e}_2 + \lambda\mathbf{e}_3$ (where $\lambda$ is an arbitrary coefficient)?

To analyse the cases we use the definition of basis: 3 vectors in 3-dimensional space form a basis if and only if these vectors are linear independent.

Case a) Vectors $\mathbf{e}'_1 = \mathbf{e}_2, \mathbf{e}'_2 = \mathbf{e}_1, \mathbf{e}'_3 = \mathbf{e}_3$ are linear independent, since $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis. Hence $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ is a basis too.

Case b) Vectors $\mathbf{e}'_1 = \mathbf{e}_1, \mathbf{e}'_2 = \mathbf{e}_1 + 3\mathbf{e}_3, \mathbf{e}'_3 = \mathbf{e}_3$ are linear dependent. Indeed

$$\mathbf{e}'_1 - \mathbf{e}'_2 + 3\mathbf{e}'_3 = \mathbf{e}_1 - (\mathbf{e}_1 + 3\mathbf{e}_3) + 3\mathbf{e}_3 = 0\,.$$

Hence it is not a basis.

Case c) First two vectors $\mathbf{e}'_1 = \mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}'_2 = 3\mathbf{e}_1 - 3\mathbf{e}_2$ are already linear dependent. Hence these three vectors do not form a basis.

Case d) Check are vectors linear independent or not. Let $c_1\mathbf{e}'_1 + c_2\mathbf{e}'_2 + c_3\mathbf{e}'_3 = 0$, i.e.

$$c_1\mathbf{e}'_1 + c_2\mathbf{e}'_2 + c_3\mathbf{e}'_3 = c_1\mathbf{e}_2 + c_2\mathbf{e}_1 + c_3(\mathbf{e}_1 + \mathbf{e}_2 + \lambda\mathbf{e}_3) = (c_2 + c_3)\mathbf{e}_1 + (c_1 + c_3)\mathbf{e}_2 + c_3\lambda\mathbf{e}_3 = 0\,.$$

I-st case $\lambda \neq 0$. We have $c_2 + c_3 = c_1 + c_3 = \lambda c_3 = 0$. Hence $c_3 = 0, c_1 = 0, c_2 = 0$. These three vectors are linear independent. This means that ordered triple $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ is a basis.

II-nd case $\lambda = 0$. We have $c_2 + c_3 = c_1 + c_3 = 0c_3 = 0$. Hence $c_3$ can be an arbitrary number and $c_1 = -c_3, c_2 = -c_3$. $c_3$ These three vectors are linear dependent. This means that ordered triple $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ is not a basis.