Homework 2

1

a) Show that $(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^2 + x^3y^3$ defines a scalar product in $\mathbf{R}^3$.

b) Show that $\langle \mathbf{x}, \mathbf{y} \rangle = x^1y^1 + x^2y^2$ *does not define a scalar product* in $\mathbf{R}^3$.

c) Show that $(\mathbf{x}, \mathbf{y}) = x^1y^1 + x^2y^2 - x^3y^3$ *does not define a scalar product* in $\mathbf{R}^3$.

d) Show that $(\mathbf{x}, \mathbf{y}) = x^1y^1 + 3x^2y^2 + 5x^3y^3$ defines a scalar product in $\mathbf{R}^3$.

e†) Find necessary and sufficient conditions for entries a, b, c of symmetrical matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ such that the formula

$$(\mathbf{x}, \mathbf{y}) = (\, x^1, x^2 \,) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = ax^1y^1 + b(x^1y^2 + x^2y^1) + cx^2y^2$$

defines scalar product in $\mathbf{R}^2$.

2 The matrix $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ obeys the conditions $A^{T} A = I$. Show that

a) $\det A = \pm 1$

b) if $\det A = 1$ then there exists an angle $\varphi : 0 \le \varphi < 2\pi$ such that $A = A_\varphi$ where

$$A_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \text{ (rotation matrix)}$$

c) if $\det A = -1$ then then there exists an angle $\varphi : 0 \le \varphi < 2\pi$ such that $A = A_\varphi R$, where $R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (a reflection matrix).

3 Show that for matrix A_φ defined in the previous exercise the following relations are satisfied:

$$A_\varphi^{-1} = A_\varphi^{T} = A_{-\varphi}\,, \qquad A_{\varphi+\theta} = A_\varphi \cdot A_\theta\ .$$

4 Show that under the transformation $(\mathbf{e}'_1, \mathbf{e}'_2) = (\mathbf{e}_1, \mathbf{e}_2)\, A_\varphi$ an orthonormal basis transforms to an orthonormal one.

How coordinates of vectors change if we rotate the orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2)$ on the angle $\varphi = \frac{\pi}{3}$ anticlockwise?

5 Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal basis of Euclidean space $\mathbf{E}^3$. Consider the ordered set of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ which is expressed via basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as in the exercise 6 of the Homework 1.

Write down explicitly transition matrix from the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to the ordered set of the vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$. What is the rank of this matrix? Is this matrix orthogonal?

Find out is the ordered set of vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ a basis in $\mathbf{E}^3$. Is this basis an orthonormal basis of $\mathbf{E}^3$?

(you have to consider all cases a),b) c) and d)).

6†. Show that an arbitrary orthogonal transformation of two-dimensional Euclidean space can be considered as a composition of reflections.

7† Prove the Cauchy–Bunyakovsky–Schwarz inequality

$$(\mathbf{x}, \mathbf{y})^2 \le (\mathbf{x}, \mathbf{x})(\mathbf{y}, \mathbf{y})\,,$$

where $\mathbf{x}, \mathbf{y}$ are arbitrary two vectors and (,) is a scalar product in Euclidean space.

Hint: For any two given vectors $\mathbf{x}, \mathbf{y}$ *consider the quadratic polynomial* $At^2 + 2Bt + C$ *where* $A = (\mathbf{x}, \mathbf{x})$, $B = (\mathbf{x}, \mathbf{y})$, $C = (\mathbf{y}, \mathbf{y})$. *Show that this polynomial has at most one real root and consider its discriminant.*