Homework 9. Solutions.

1 *Let ∇ be a connection on n-dimensional manifold M and $\{R^i{}_{rmn}\}$ be the components of the curvature tensor of a connection ∇ in local coordinates $(x^1, x^2, \dots, x^n)$.*

a) For arbitrary vector fields $\mathbf{A}, \mathbf{B}$ and $\mathbf{D}$ calculate the vector field

$$(\nabla_{\mathbf{A}}\nabla_{\mathbf{B}} - \nabla_{\mathbf{B}}\nabla_{\mathbf{A}})\,\mathbf{D} - \nabla_{\mathbf{C}}\mathbf{D}\,,$$

where the vector field $\mathbf{C}$ is a commutator of vector fields $\mathbf{A}$ and $\mathbf{B}$:

$$\mathbf{C} = C^i \frac{\partial}{\partial x^i} = [\mathbf{A}, \mathbf{B}] = \left(A^m \frac{\partial B^i(x)}{\partial x^m} - B^m \frac{\partial A^i(x)}{\partial x^m}\right)\frac{\partial}{\partial x^i}\,. \tag{1.0}$$

b) Calculate the vector field

$$(\nabla_{\mathbf{A}}\nabla_{\mathbf{B}} - \nabla_{\mathbf{B}}\nabla_{\mathbf{A}})\,\mathbf{D}$$

in the case if for vector fields $\mathbf{A}$ and $\mathbf{B}$ components A^i and B^m are constants (in the local coordinates $(x^1, \dots, x^n)$)

(You have to express the answers in terms of components of the vector fields and components of the curvature tensor $R^i{}_{rmn}$.)

a) According to the definition of the curvature tensor for every vector fields $\mathbf{X} = X^m\partial_m, \mathbf{Y} = Y^m\partial_m$ and $\mathbf{Z} = Z^m\partial_m$ we have that $\mathcal{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z} =$

$$\mathcal{R}(X^m\partial_m, Y^n\partial_n)(Z^r\partial_r) = \left(\nabla_{\mathbf{X}}\nabla_{\mathbf{Y}} - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}} - \nabla_{[\mathbf{X},\mathbf{Y}]}\right)\mathbf{Z} = Z^r R^i{}_{rmn} X^m Y^n \partial_i\,.$$

Hence for vector fields $\mathbf{A}, \mathbf{B}, \mathbf{C} = [\mathbf{A}, \mathbf{B}]$ and we have that

$$\left(\nabla_{\mathbf{A}}\nabla_{\mathbf{B}} - \nabla_{\mathbf{B}}\nabla_{\mathbf{A}} - \nabla_{[\mathbf{A},\mathbf{B}]}\right)\mathbf{D} = (\nabla_{\mathbf{A}}\nabla_{\mathbf{B}} - \nabla_{\mathbf{B}}\nabla_{\mathbf{A}})\,\mathbf{D} - \nabla_{\mathbf{C}}\mathbf{D} = \mathcal{R}(\mathbf{A}, \mathbf{B})\mathbf{D} = D^r R^i{}_{rmn} A^m B^n \partial_i\,. \tag{1.1}$$

b) in the case if in the local coordinates $(x^1, \dots, x^n)$ for vector fields $\mathbf{A}$ and $\mathbf{B}$ components A^i and B^m are constants then the commutator of these vector fields $\mathbf{C} = [\mathbf{A}, \mathbf{B}]$ vanishes: $\mathbf{C} = 0$ (see the formula (1.0)). Hence according to the formula (1.1) above we have that

$$(\nabla_{\mathbf{A}}\nabla_{\mathbf{B}} - \nabla_{\mathbf{B}}\nabla_{\mathbf{A}})\,\mathbf{D} - \nabla_{\mathbf{C}}\mathbf{D} = (\nabla_{\mathbf{A}}\nabla_{\mathbf{B}} - \nabla_{\mathbf{B}}\nabla_{\mathbf{A}})\,\mathbf{D} = \mathcal{R}(\mathbf{A}, \mathbf{B})\mathbf{D} = D^r R^i{}_{rmn} A^m B^n \partial_i\,.$$

2) Calculate Riemann curvature tensor for the cylindircal surface $x^2 + y^2 = a^2$ in $\mathbf{E}^3$.

Solution. Cylinder surface in standard coordinates (φ, h) is $\begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases}$ and Riemannian metric $G = dh^2 + a^2 d\varphi^2$. Christoffel symbols Γ^i_{km} of Levi-Civita connection in coordinates (φ, h) vanish since in these coordinates all components of matrix of metric are constants (see Levi-Civita formula!). Hence Riemann curvature tensor

$$R^i{}_{kmn} = \partial_m\Gamma^i_{nk} + \Gamma^i_{mp}\Gamma^p_{nk} - \partial_n\Gamma^i_{mk} - \Gamma^i_{np}\Gamma^p_{mk}$$

vanishes too. Note that the fact that Christoffel symbols vanish in some coordiantes *does not mean* that they vanish in any coordinates. Riemann curvature is a tensor: if it vanishes in some coordinates then it vanishes in any coordinates.

3 *We know that If R^i_{kmn} is Riemann curvature tensor for Riemannian manifold (M, G) (R^i_{kmn} us curvature tensor for Levi-Civita connection on M) then the following identities hold:*

$$R_{ikmn} = -R_{iknm}, \quad R_{ikmn} = -R_{kimn}\,, \qquad R_{ikmn} = R_{mnik} \tag{$*$}$$

a) Show that Riemann curvature tensor for 2-dimensional Riemannian manifold (M, G) possesses only one non-trivial component.

Solution. It follows from identities (*) that all components of curvature tensor may be expressed through component R_{1212}. Indeed if $R_{1212} = a$ then due to identities:

$$R_{1212} = -R_{1221} = -R_{2112} = R_{2121} = a$$

and all other components (for 2-dimensional case) vanish:

$$R_{1111} = R_{1112} = R_{1121} = R_{1122} = R_{1211} = R_{1222} = R_{2111} = R_{2122} = R_{2211} = R_{2212} = R_{2221} = R_{2222} = 0\,.$$

4) *If (M, G) is surface in $\mathbf{E}^3$ then*

$$K = \frac{R}{2} = \frac{R_{1212}}{\det g}\,,$$

where K is Gaussian curvature of the surface, and R^i_{kmn} Riemann curvature tensor with respect to induced metric,

a) Prove by straightforward calculations that $\frac{R}{2} = \frac{R_{1212}}{\det g}$

b) Prove that $K = \frac{R}{2} = \frac{R_{1212}}{\det g}$. (It is convenient to choose the orthonormal basis $\{\mathbf{e}, \mathbf{f}\}$ and use derivation formula.)*

Solution: Show it. Using results of previous exercise we see that

$$R_{11} = R^i{}_{1i1} = R^2{}_{121} = g^{22}R_{2121} + g^{21}R_{1121} = g^{22}R_{1212} = g^{22}a\,,$$
$$R_{22} = R^i{}_{2i2} = R^1{}_{212} = g^{11}R_{1212} + g^{12}R_{2221} = g^{11}R_{1212} = g^{11}a$$
$$R_{12} = R_{21} = R^i{}_{1i2} = R^1{}_{112} = g^{12}R_{2112} = -g^{12}R_{1212} = -g^{12}a$$

Thus

$$R_{ik} = a\begin{pmatrix} g^{22} & -g^{12} \\ -g^{21} & g^{11} \end{pmatrix}\,.$$

Hence

$$R = R^i{}_{kim}g^{km} = R_{km}g^{km} = g^{11}R_{11} + g^{12}R_{12} + g^{21}R_{21} + g^{22}R_{22} =$$
$$2R_{1212}(g^{11}g^{22} - g^{12}g^{12}) = 2a\det g^{-1} = \frac{2a}{\det g}\,,$$

where $\det g = \det\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = g_{11}g_{22} - g_{12}^2$ (matrices $||g_{ik}||$ and $||g_{ik}||$ are inverse). We see that $R = \frac{2a}{\det g} = \frac{2R_{1212}}{\det g}$.

* It remains to prove that Gaussian curvature is equal to $R/2$ or that it is equal to $R_{1212}\det g$.

Choose three unit vector fields $\mathbf{e}, \mathbf{f}, \mathbf{g}$ such that $\mathbf{e}, \mathbf{f}$ form orthonormal basis on points of surface M and $\mathbf{n} = \mathbf{e}\times\mathbf{f}$. According to the definition of curvature calculate

$$R(\mathbf{e}, \mathbf{f})\mathbf{e} = \nabla_{\mathbf{e}}\nabla_{\mathbf{f}}\mathbf{e} - \nabla_{\mathbf{f}}\nabla_{\mathbf{e}}\mathbf{e} - \nabla_{[\mathbf{e},\mathbf{f}]}\mathbf{e}\,. \qquad (**)$$

Using derivation formulae one can calculate this expression and come to the following answer [1]:

$$R(\mathbf{e}, \mathbf{f})\mathbf{e} = \nabla_{\mathbf{e}}\nabla_{\mathbf{f}}\mathbf{e} - \nabla_{\mathbf{f}}\nabla_{\mathbf{e}}\mathbf{e} - \nabla_{[\mathbf{e},\mathbf{f}]}\mathbf{e} = da(\mathbf{e}, \mathbf{f})\mathbf{f}\,.$$

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[1] These calculations are little bit difficult. They are following: note that since the induced connection is symmetrical connection then: $\nabla_{\mathbf{e}}\mathbf{f} - \nabla_{\mathbf{f}}\mathbf{e} - [\mathbf{e}, \mathbf{f}] = 0$ hence $[\mathbf{e}, \mathbf{f}] = \nabla_{\mathbf{e}}\mathbf{f} - \nabla_{\mathbf{f}}\mathbf{e} = -a(\mathbf{e})\mathbf{e} - a(\mathbf{f})\mathbf{f}$. Thus we see that $R(\mathbf{e}, \mathbf{f})\mathbf{e} =$

$$\nabla_{\mathbf{e}}\nabla_{\mathbf{f}}\mathbf{e} - \nabla_{\mathbf{f}}\nabla_{\mathbf{e}}\mathbf{e} - \nabla_{[\mathbf{e},\mathbf{f}]}\mathbf{e} = \nabla_{\mathbf{e}}\left(a(\mathbf{f})\mathbf{f}\right) - \nabla_{\mathbf{f}}\left(a(\mathbf{e})\mathbf{f}\right) + \nabla_{a(\mathbf{e})\mathbf{e}+a(\mathbf{f})\mathbf{f}}\mathbf{e} =$$
$$\partial_{\mathbf{e}}a(\mathbf{f})\mathbf{f} + a(\mathbf{f})\nabla_{\mathbf{e}}\mathbf{f} - \partial_{\mathbf{f}}a(\mathbf{e})\mathbf{f} - a(\mathbf{e})\nabla_{\mathbf{f}}\mathbf{f} + a(\mathbf{e})\nabla_{\mathbf{e}}\mathbf{e} + a(\mathbf{f})\nabla_{\mathbf{f}}\mathbf{e} =$$
$$\partial_{\mathbf{e}}a(\mathbf{f})\mathbf{f} - a(\mathbf{f})a(\mathbf{e})\mathbf{e} - \partial_{\mathbf{f}}a(\mathbf{e})\mathbf{f} + a(\mathbf{e})a(\mathbf{f})\mathbf{e} + a(\mathbf{e})a(\mathbf{e})\mathbf{f} + a(\mathbf{f})a(\mathbf{f})\mathbf{f} =$$
$$\left[\partial_{\mathbf{e}}a(\mathbf{f})\mathbf{f} - \partial_{\mathbf{f}}a(\mathbf{e})\mathbf{f} - a\left[-a(\mathbf{e})\mathbf{e} - a(\mathbf{f})\mathbf{f}\right]\right]\mathbf{f} =$$
$$= \left[\partial_{\mathbf{e}}a(\mathbf{f})\mathbf{f} - \partial_{\mathbf{f}}a(\mathbf{e})\mathbf{f} - a\left([\mathbf{e}, \mathbf{f}]\right)\right]\mathbf{f} = da(\mathbf{e}, \mathbf{f})\mathbf{f}\,.$$

Recall that we established that for Gaussian curvature $K = b \wedge c(\mathbf{e}, \mathbf{f}) = -da(\mathbf{e}, \mathbf{f})$. Hence we come to the relation:

$$R(\mathbf{e}, \mathbf{f})\mathbf{e} = da(\mathbf{e}, \mathbf{f}) = -K\mathbf{f}\,.$$

This means that

$$R^2_{112} = -K$$

Note that in the basis $\mathbf{e}, \mathbf{f}$ Riemannian metric is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence $R^2_{112} = R_{2112} = -R_{1212}$, $\det g = 1$. Thus we come to the relation

$$K = \frac{R_{1212}}{\det g} = \frac{R}{2}\,.$$