Solutions of Homework C3(9)

1 *Four points A, B, C, D are given on the projective line $\mathbf{RP}^1$. Their homogeneous coordinates are*

$$A = [2:2]\,, B = [1:5]\,, C = [3:7]\,, D = [2:1]\,.$$

Calculate the affine coordinate u of these points, ($u = \frac{x}{y}$) and calculate cross-ratio of these points.

$$u_A = \frac{2}{2} = 1\,, u_B = \frac{1}{5}\,, u_C = \frac{3}{7}\,, u_D = \frac{2}{1} = 2\,.$$

Calculate the cross-ratio (A, B, C, D). All points have finite coordinate (they are not at infinity). Thus we can calculate the cross-ratio using affine coordinate u.

$$(A, B, C, D) = \frac{(u_A - u_C)(u_B - u_D)}{(u_A - u_D)(u_B - u_C)} = \tag{1a}$$

$$= \frac{\left(1 - \frac{3}{7}\right)\left(\frac{1}{5} - 2\right)}{(1-2)\left(\frac{1}{5} - \frac{3}{7}\right)} = -\frac{9}{4}\,.$$

Sure one can calculate the cross-ratio using homogeneous coordinates. (This works for any four points not only for finite points):

$$(A, B, C, D) = \frac{\det\begin{pmatrix} x_A & x_C \\ y_A & y_C \end{pmatrix} \det\begin{pmatrix} x_B & x_D \\ y_B & y_D \end{pmatrix}}{\det\begin{pmatrix} x_A & x_D \\ y_A & y_D \end{pmatrix} \det\begin{pmatrix} x_B & x_C \\ y_B & y_C \end{pmatrix}} = \tag{1b}$$

$$= \frac{\det\begin{pmatrix} 2 & 3 \\ 2 & 7 \end{pmatrix} \det\begin{pmatrix} 1 & 2 \\ 5 & 1 \end{pmatrix}}{\det\begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} \det\begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix}} = \frac{8 \cdot (-9)}{(-2)\cdot(-8)} = -\frac{9}{2}\,.$$

2 *As usual denote by (A, B, C, D) the cross-ratio of the four points A, B, C, D on the projective line.*

a) Does the cross-ratio change if we change the order of these points?

b) Let $(A, B, C, D) = \lambda$.

Calculate the cross-ratios (B, A, C, D), (A, B, D, C) and (B, A, D, C).

* *Calculate cross- ratio (A, C, B, D).*

* *Calculate cross-ratio of arbitrary permutation of the points A, B, C, D.*

We see from definition of cross-ratio that it depends on the order of the points.

Show that

$$(B, A, C, D) = \frac{1}{\lambda} \quad \text{if} \quad (A, B, C, D) = \lambda\,, \tag{2a}$$

and the same is for permutation (A, B, D, C)

$$(A, B, D, C) = \frac{1}{\lambda} \quad \text{if} \quad (A, B, C, D) = \lambda\,, \tag{2a'}$$

and

$$(A, B, C, D) = (C, D, B, A)\,. \tag{2a''}$$

First prove relation (2a$''$)

It follows from equation (1a) that

$$(C, D, A, B) = \frac{(u_C - u_A)\,(u_D - u_B)}{(u_C - u_B)\,(u_D - u_A)} = \frac{(u_A - u_C)\,(u_B - u_D)}{(u_A - u_D)\,(u_B - u_C)} = (A, B, C, D)\,.$$

Now prove relation (2a)

It follows from equation (1a) that

$$(B, A, C, D) = \frac{(u_B - u_C)\,(u_A - u_D)}{(u_B - u_D)\,(u_A - u_C)} = \frac{1}{\frac{(u_A-u_C)(u_B-u_D)}{(u_A-u_D)(u_B-u_C)}} = \frac{1}{(A, B, C, D)}\,.$$

Relation (2a') can be proved analogously.

Remark 1 Sure relations (2a) and (2a') inply relation (2a$''$).

Remark 2 Strictly speaking one cannot use formula (1a) for affine coordinates in the case if one of points is at infinity. On the other hand one can see using continuity arguments that formulae (2a,2a') are valid for arbitrary points if they are valid for finite points (see also the second solution of next exercise). (Sure one can straightforwardly apply formulae (1b) for homogeneus coordinates.)

Remark 3 Cross-ratio cannot take value 1.

* Relations (2a,2a') and (2a$''$) induce natural question. How transform cross-ration under arbitrary permutation of four points. One can see by direct calculation that

$$(A, B, C, D) + (A, C, B, D) = 1\,. \tag{2b}$$

Relations (2a,2a',2a$''$ and 2b) define action of permutation group S_4 on cross-ratio:

$$(A, C, D, B) = \frac{1}{(A, C, B, D)} = \frac{1}{1 - (A, B, C, D)} = \frac{1}{1 - \lambda}\,,$$

$$(B, C, A, D) = 1 - (B, A, C, D) = 1 - \frac{1}{(A, B, C, D)} = 1 - \frac{1}{\lambda}\,,$$

and so on.

3 *Four points A, B, C, D sre given on the projective line. Show that the cross-ratio*

$$(A, B, C, D) = \frac{u_A - u_C}{u_B - u_C}$$

in the case if point D is at infinity.

Choose homogeneous coordinates of these points:

$$A = [x_A : y_A] = [u_A : 1]\,, B = [x_B : y_B] = [u_B : 1]\,, C = [x_C : y_C] = [u_C : 1]\,, \text{and} D = [1 : 0]$$

since the point D is at infinity.

According formula (1b) we have: $(A, B, C, D) =$

$$\frac{\det\begin{pmatrix} x_A & x_C \\ y_A & y_C \end{pmatrix} \det\begin{pmatrix} x_B & x_D \\ y_B & y_D \end{pmatrix}}{\det\begin{pmatrix} x_A & x_D \\ y_A & y_D \end{pmatrix} \det\begin{pmatrix} x_B & x_C \\ y_B & y_C \end{pmatrix}} = \frac{\det\begin{pmatrix} u_A & u_C \\ 1 & 1 \end{pmatrix} \det\begin{pmatrix} u_B & 1 \\ 1 & 0 \end{pmatrix}}{\det\begin{pmatrix} u_A & 1 \\ 1 & 0 \end{pmatrix} \det\begin{pmatrix} u_B & u_C \\ 1 & 1 \end{pmatrix}} = \frac{u_A - u_C}{u_B - u_C}$$

Second solution In spite of the fact that point D is at infinity one can still apply formula (1a) using continuity considerations: instead coordinate $u_D = \infty$ we will put a big N, then consider a limit $N \to \infty$:

$$(A, B, C, D) = \frac{(u_A - u_C)(u_B - u_D)}{(u_A - u_D)(u_B - u_C)} = \lim_{N \to infty} \frac{(u_A - u_C)(u_B - N)}{(u_A - N)(u_B - u_C)} = \frac{u_A - u_C}{u_B - u_D}$$

since $\lim_{N \to infty} \frac{u_B - N}{u_A - N} = 1$.

4 *Four points $A, B, C, D \in \mathbf{RP}^2$ are given in homogeneous coordinates by*

$$A = [1 : -1 : 1]\,, \quad B = [10 : -15 : 5]\,, \quad C = \left[1 : -\frac{9}{5} : \frac{1}{5}\right]\,, \quad D = [1 : 0 : 2]\,.$$

Show that these points are collinear.

Calculate their cross-ratio.

First show that these points are collinear. We will do it in two different ways

1-st solution All these four points are not at infinity. Calculate coordinates (u, v) of these points on the projective plane ($u = \frac{x}{z}, v = \frac{y}{z}$)

$$u_A = \frac{x_A}{z_A} = 1\,, v_A = \frac{y_A}{z_A} = -1\,, u_B = \frac{x_B}{z_B} = 2\,, v_B = \frac{y_B}{z_B} = -3\,,$$

$$u_C = \frac{x_C}{z_C} = 5\,, v_C = \frac{y_C}{z_C} = -9\,, u_D = \frac{x_D}{z_D} = \frac{1}{2}\,, v_D = \frac{y_D}{z_D} = 0\,. \tag{3}$$

One can see that all these points belong to the same line $2u + v = 1$:

$$2u_A + v_A = 2u_B + v_B = 2u_C + v_C = 2u_D + v_D = 1\,,$$

i.e. all these points belong to the same line, and they are collinear.

Second way to see why these points are collinear:

One can do it also using homogeneous coordinates: One can see that

$$A = [1 : -1 : 1]\, B = [10 : -15 : 5]\,, C = \left[1 : -\frac{9}{5} : \frac{1}{5}\right] = [5 : -9 : 1]\,, D = [1 : 0 : 2]\,,$$

and both matrices T_{ABC}, T_{ABD} corresponding to triples of points (A, B, C) and (B, C, D) are degenerate

$$\det T_{ACD} = \det \begin{pmatrix} 1 & 5 & 1 \\ -1 & -9 & 0 \\ 1 & 1 & 2 \end{pmatrix} = 0\,, \quad \det T_{ABD} = \det \begin{pmatrix} 1 & 2 & 1 \\ -1 & -3 & 0 \\ 1 & 1 & 2 \end{pmatrix} = 0\,.$$

This means that points A, B, D are collinear and points A, C, D are collienar also. We see that all these points belong to the line l_{AD}; they all are collinear.

Now calculate cross-ratio of points A, B, C, D. We see that all these points have distinct coordinate u (see equation (3)). Calculate cross-ratio using this affine coordinate:

$$(A, B, C, D) = \frac{(u_A - u_C)\,(u_B - u_D)}{(u_A - u_D)\,(u_B - u_C)} = \frac{(1-5)\left(2 - \frac{1}{2}\right)}{\left(1 - \frac{1}{2}\right)(2-5)} = 4\,.$$

One can calculate cross-ratio using instead affine coordinate u, another affine coordinate v:

$$(A, B, C, D) = \frac{(v_A - v_C)\,(v_B - v_D)}{(v_A - v_D)\,(v_B - v_C)} = \frac{(-1+9)\,(-3-0)}{(-1-0)\,(-3-(-9))} = 4\,.$$

5 *Two points $A, B \in \mathbf{RP}^2$ are given in homogeneous coordinates, $A = [2 : 2 : 4]$, $B = [3 : 7 : 2]$. Consider the projective line AB passing through the points A, B.*

Show that the point $C = [1 : 2 : 1]$ belongs to the line AB, i.e. the points A, B, C are collinear.

Show that a point $E_{\lambda,\mu} = [2\lambda + 3\mu : 2\lambda + 7\mu : 4\lambda + 2\mu]$ where λ, μ are arbitrary real numbers belongs to the line AB

Show that the point $K = [2 : 0 : 1]$ does not belong to the line AB, i.e. the points A, B, K are not collinear.

Consider a point $D = [1 : 3 : 0]$ which is at infinity. Show that this point is collinear with the points A and B, i.e. it also belongs to the projective line AB.

* *Calculate the cross-ratio (A, B, C, D).*

Let $X = [a : b : c]$ be an arbitrary point on the projective plane $\mathbf{RP}^2$. Consider the matrix

$$T_{ABX} = \begin{pmatrix} 2 & 3 & a \\ 2 & 7 & b \\ 4 & 2 & c \end{pmatrix}\,. \qquad (5)$$

We know that a point $X = [a : b : c]$ belongs to the line AB which passes through points $A = [2 : 2 : 4]$ and $B = [3 : 7 : 2]$, i.e. points A, B and X are collinear if and only if matrix

T_{ABX} is degenerate, i.e. $\det T_{ABX} = 0$. Recall the geometrical meaning of this condition. The point $X \in \mathbf{RP}^2$ is represented by the line in $\mathbf{R}^3$ which passes trhough origin and the point $(a, b, c) \in \mathbf{R}^3$, i.e. the line directed along the vector $\mathbf{r}_{(a,b,c)}$ attached at the origin. (This line represents the point $X = [a : b : c]$.) The line $AB \in \mathbf{RP}^2$ on the projective plane is represented by the plane in $\mathbf{R}^3$ which passes through origin and the points $(2, 2, 4) \in \mathbf{R}^3$, $(3, 7, 2) \in \mathbf{R}^3$, i.e. the plane which is spanned by the linear combination of the vectors $\mathbf{r}_{(2,2,4)}$ and $\mathbf{r}_{(3,7,4)}$. Hence $X \in l_{AB}$ means that the vector $\mathbf{r}_{(a.b,c)}$ is linear combination of vectors $\mathbf{r}_{(2,2,4)}$ and $\mathbf{r}_{(3,7,4)}$. In other words the third column of the matrix T_{ABX} in equation (5) is linear combination of first and second column, i.e. rank of this matrix is equal to 2, or in other words this matrix is degenerate. (It is easy to see that inverse implication is also true.) Now check the condition $\det T_{ABX} = 0$ for (5) for points $C, E_{\lambda,\mu}$ and a point K:

1) point C:

$$\det T_{ABC} = \det \begin{pmatrix} 2 & 3 & 1 \\ 2 & 7 & 2 \\ 4 & 2 & 1 \end{pmatrix} = 0\,.$$

(One can see it without of explicit calculations of the determinant: the third column is linear combination of the first and second: $8 \cdot III = I + 2 \cdot II$, hence matrix is degenerate.) Hence the points A, B, C are collinear.

2) point $E_{\lambda,\mu}$

$$\det T_{ABE_{\lambda,\mu}} = \det \begin{pmatrix} 2 & 3 & 2\lambda + 3\mu \\ 2 & 7 & 2\lambda + 7\mu \\ 4 & 2 & 4\lambda + 2\mu \end{pmatrix} = 0\,.$$

(One can see it also without explicit calculations of determinant: the third column is linear combination of the first and second, hence matrix is degenerate.) Hence the points $A, B, E_{\lambda,\mu}$ are collinear.

3) the point K

$$\det T_{ABK} = \det \begin{pmatrix} 2 & 3 & 2 \\ 2 & 7 & 0 \\ 4 & 2 & 1 \end{pmatrix} \neq 0\,.$$

Hence the points A, B, K are not collinear.

Finally consider the point $D = [1 : 3 : 0]$. This is a point at infinity, and

$$\det T_{ABD} = \det \begin{pmatrix} 2 & 3 & 3 \\ 2 & 7 & 1 \\ 4 & 2 & 0 \end{pmatrix} = 0\,.$$

Points A, B, D are collinear*.

* The point D at infinity it is a point where projetive line l_{AB} meets projective line at infinty.

* We see that points A, B, C, D are collinear, and a point D belongs to infinity. Calculate their cross-ratio. Choose affine coordinate $u = \frac{x}{z}$ for these points:

$$u_A = \frac{x_A}{z_A} = \frac{1}{2}\,, u_B = \frac{x_B}{z_B} = \frac{3}{2}\,, u_C = \frac{x_C}{z_C} = 1\,.$$

The point D is at infinity. Using exercise 3 calculate cross-ratio:

$$(A, B, C, D) = \frac{u_A - u_C}{u_B - u_C} = \frac{\frac{1}{2} - 1}{\frac{3}{2} - 1} = -1\,.$$

Remark Notice that cross-ratio is equal to -1, i.e. it is harmonic ratio, and the point C is in the middle of the segment $[AB]$

6* *Let $\triangle ABC$ be a triangle in the Euclidean plane $\mathbf{E}^2$.*

Consider the line l_{AB} passing through the vertices A and B of this triangle.

Let P, Q be such points on the line l_{AB} such that the segment CP is the bisectrix of the angle ACB, and the segment CQ is the bisectrix of the external angle ACB. (We suppose that $|AC| \neq |BC|$. In the case if $|AC| = |BC|$ then the bisectrix of the external angle is parallel to the line l_{AB}.)

Calculate the cross-ratio (A, B, P, Q).

One can include $\mathbf{E}^2$ in projective plane $\mathbf{RP}^2$. What happens with triangle ABC if we will perform the projective transformation of $\mathbf{RP}^2$ which send ths point Q to infinity?

According to the well-known theorem on properties of bisectirix we have that bissectrix of the triangle divides the opposie side of the trinagle on two parts which are proportional to the lengths of sides of the triangle:

$$\frac{|PA|}{|PB|} = \frac{|CA|}{|CB|}\,, \tag{6a}$$

$$\frac{|QA|}{|QB|} = \frac{|CA|}{|CB|}\,, \tag{6a}$$

It follows from these two relations that cross ratio (A, P, P, Q) is harmonic ratio:

$$(A, P, B, Q) = -1\,.$$

Consider projective transformation of $\mathbf{E}^2$ (this transformation destroys Euclidean structure of E^2) which send the point Q to infinity. The relation (6) will not change. Under this transformation the triangle ABC will become the isocseless triangle.