Again about contact vector field

Let $\mathbf{X}$ be a vector field which preserves the contact structure:

$$\mathbf{X} = R^m(q,p,u)\frac{\partial}{\partial q^m} + T_m(q,p,u)\frac{\partial}{\partial p_m} + F(q,p,u)\frac{\partial}{\partial u} \tag{1}$$

such that

$$\mathcal{L}_{\mathbf{X}}\alpha(p_m dq^m - du) = \lambda(p_m, q^m, u)(p_m dq^m - du)\,. \tag{2}$$

Consider the 'Hamiltonian'

$$H_{\mathbf{X}} = \alpha(\mathbf{X}) = p_m R^m(q,p,u) - F(q,p,u)\,. \tag{3}$$

Then it follows from equation (2) that the 1-form

$$\mathcal{L}_{\mathbf{X}}\alpha = d(\iota_{\mathbf{X}} \circ \alpha) + \iota_{\mathbf{X}} \circ d\alpha = d(H_{\mathbf{X}}) + \iota_{\mathbf{X}} \circ (dp_m \wedge dq^m) =$$

$$\frac{\partial H_{\mathbf{X}}}{\partial p_m}dp_m + \frac{\partial H_{\mathbf{X}}}{\partial q^m}dq^m + \frac{\partial H_{\mathbf{X}}}{\partial u}du + T_m dq^m - R^m dp_m = \lambda(p_m, q^m, u)(p_m dq^m - du)\,.$$

Comparing the left and right hand sides of this equation we come to

$$\frac{\partial H_{\mathbf{X}}}{\partial p_m} - R^m = 0\,, \quad \lambda = -\frac{\partial H_{\mathbf{X}}}{\partial u}\,, \frac{\partial H_{\mathbf{X}}}{\partial q^m} + T_m = \lambda p_m = -\frac{\partial H_{\mathbf{X}}}{\partial u}p_m\,,$$

i.e.

$$R^m = \frac{\partial H_{\mathbf{X}}}{\partial p_m}\,, \quad T_m = -\left(\frac{\partial H_{\mathbf{X}}}{\partial q^m} + \frac{\partial H_{\mathbf{X}}}{\partial u}p_m\right)\,, \quad F = p_m R^m(q,p,u) - H_{\mathbf{X}}\,.$$

i.e.

$$\mathbf{X} = \frac{\partial H_{\mathbf{X}}}{\partial p_m}\frac{\partial}{\partial q^m} - \left(\frac{\partial H_{\mathbf{X}}}{\partial q^m} + \frac{\partial H_{\mathbf{X}}}{\partial u}p_m\right)\frac{\partial}{\partial p_m} + \left(p_m\frac{\partial H_{\mathbf{X}}}{\partial p_m} - H_{\mathbf{X}}\right)\frac{\partial}{\partial u}\,. \tag{4}$$

We see that if vector field $\mathbf{X}$ preserves contact structure, then equation (4) is obeyed.

On the base of these considerations prove the following Theorem.

Let $\mathcal{X}(M)$ be the space of all vector fields on M

Consider maps

$$\mathcal{F}\colon \quad \mathcal{X}(M) \to C(M)\,, \quad \mathcal{S}\colon \quad C(M) \to \mathcal{X}(M)\,,$$

such that

$$\mathcal{F}(\mathbf{X}) = \iota_{\mathbf{X}} \circ \alpha = H_{\mathbf{X}}\,, \quad \mathcal{S}(H) = \frac{\partial H}{\partial p_m}\frac{\partial}{\partial q^m} - \left(\frac{\partial H}{\partial q^m} + \frac{\partial H}{\partial u}p_m\right)\frac{\partial}{\partial p_m} + \left(p_m\frac{\partial H}{\partial p_m} - H\right)\frac{\partial}{\partial u}\,.$$

One can see that on the subspace $\mathcal{X}_{\text{contact}}(M)$

$$\mathcal{S} \circ \mathcal{F} = \mathbf{id}\,, \quad \text{and}\,, \mathcal{F} \circ \mathcal{S} = \mathbf{id}$$

This means that there is one-one correspondence between contact vector fields and Hamiltonians.

Segodnia russkaja paskha, a zavtra u menia operatsija.