One approach

Denote by $g_{\mathbf{k}}(y)$ a function $g = g_{\mathbf{k}} = \mathbf{k} \cdot \mathbf{y} = k_a y^a$

If $\Phi = \Phi_S$ is a thick morphism, then

$$\Phi^*(g_{\mathbf{k}})(x) = S(x, \mathbf{k})$$

and

$$y^a_{\mathbf{k}}(x) = \frac{\partial S(x, l)}{\partial l^a}\Big|_{\mathbf{l}=\mathbf{k}}$$

Let L be an arbitrary map such that

$$L(g + tH) - L(G) = tH(y_g(x))$$

Study the relation between symmetricity of second variation and the condition

$$\frac{\partial y^a_{\mathbf{k}}}{\partial k_b} - \frac{\partial y^b_{\mathbf{k}}}{\partial k_a} = 0 \qquad (1)$$

If we show that symmetricity implies (1) then we can integrate $y^a_{\mathbf{k}}$.