In Lobachevsky space one can embeed Eculdian plane, Euclidean sphere and Lobachevsky plane: e.g. if Lobahcevsky space is realised as $\mathbf{H^3}$:

$$G = \frac{dx^2 + dy^2 + dz^2}{z^2}\,,,,,z > 0$$

then

$$M_a = \{x, y, z\colon x + x_0\}$$

is Lobachevsky subspace (any equidistant of Lobachevsky plane is also Lobachevsky plane) equidistant.

Any sphere $S_a = \{\mathbf{r}\colon d(\mathbf{r}, A) = a\}$ is Euclidean sphere, and any horysphere

$$L = \{\mathbf{r}\colon z = a\}$$

is Euclidean plane.

On the other hand one cannot embedd Lobachevsy plane in $\mathbf{E^3}$. This was proved by Hilbert. Idea of this Theorem is that a disc of radius R cannot be the part of the triangle, quadrangle, since the area of triangle is less than equal to π and area of the circle grows with R.

Consider the Eucldean circle $x^2 + (y-a)^2 = R^2$ with $R < a$. Its hyperbolic area is equal to

$$S_{\text{hyperbolic}} = \int_{x^2+(y-a)^2-R^2 \leq 0} \frac{dxdy}{y^2} =$$

$$\int_{-R}^{R} dx \left(\int_{a-\sqrt{R^2-x^2}}^{a+\sqrt{R^2-x^2}} \frac{dy}{y^2} \right) = \int_{-R}^{R} dx \left(\left(-\frac{1}{y} \right) \Big|_{a-\sqrt{R^2-x^2}}^{a+\sqrt{R^2-x^2}} \right) =$$

$$\int_{-R}^{R} \left(\frac{1}{a - \sqrt{R^2 - x^2}} - \frac{1}{a + \sqrt{R^2 - x^2}} \right) dx = 4 \int_0^R \frac{\sqrt{R^2 - x^2}}{a^2 - R^2 + x^2} dx\,.$$

To caluclate this integral use complex variable.

Consider contours C_δ and C_N such that C_δ is closed curve which goes anti-clockwise around points $\pm R$ and this curve is very close to the OX axis, and the curve C_N is a curve of very big radius N around the centre N. (One can think e.g. that C_δ is an ellipse:

$$C_\delta\colon \quad \frac{x^2}{(R+\delta)^2} + \frac{y^2}{\delta^2} = 1$$

with δ avery small positive number, and

$$C_N\colon \quad x^2 + y^2 = N^2$$

with $N \to \infty$

)

One can see that for arbitrary rational function $F(p,x)$ where $p=\sqrt{R^2-x^2}$, function $F(p,z)$ is meromorphic function on the complex plane without interior of the contour C_δ, hence

$$\int_{-R}^{R} F(p,x)dx = \left(\frac{1}{2}\int_{C_\delta} F(p,z)\right)\Big|_{\delta\to 0} = \left(\frac{1}{2}\int_{C_N} F(p,z)\right)\Big|_{N\to 0} + \text{contribution of poles}$$

One can calculate the last integral easily using residues.

For example if $F=p^\alpha$ then the left hand side of this expression is equal to

$$\int_{-R}^{R} p^\alpha dx = 2\int_0^R (R^2-x^2)^\alpha dx = 2R^{2\alpha+1}\int_0^1 (1-t^2)^\alpha dt =$$

and since there are now poles in interior of $C_N \backslash C_\delta$ then

$$2R^{2\alpha+1}\int_0^1 (1-u)^\alpha \frac{du}{2\sqrt{u}} = R^{2\alpha+1} B\left(\alpha+1,\frac{1}{2}\right),$$

The same is for $\alpha=\frac{1}{2}$ an for $\alpha=-\frac{1}{2}$.

If $\alpha=\frac{1}{2}$ then

$$p^{\frac{1}{2}} = \sqrt{R^2-z^2} = iz\sqrt{1-\frac{R^2}{z^2}} = iz - \frac{iR^2}{z} + \dots$$

and integral over C_N is equal to

$$\int_{C_N}\sqrt{R^2-z^2} = \int_{C_N}\left(iz-\frac{iR^2}{z}+\dots\right) = 2\pi i\cdot(-iR^2) = \pi R^2,$$

If $\alpha=-\frac{1}{2}$ then

$$p^{-\frac{1}{2}} = \frac{1}{\sqrt{R^2-z^2}} = \frac{1}{iz} + \dots$$

and integral over C_N is equal to

$$\int_{C_N}\sqrt{R^2-z^2} = \int_{C_N}\left(\frac{1}{iz}+\dots\right) = 2\pi i\cdot(-iR^2) = 2\pi,$$

(`ici il faut fixer le signe`)

Maintenent revenons aux nos moutons....: We have

$$S_{\text{hyperbolic}} = \int_{x^2+(y-a)^2-R^2\le 0}\frac{dxdy}{y^2} =$$

$$2\int_{-R}^{R}\frac{\sqrt{R^2-x^2}}{a^2-R^2+x^2}dx = \int_{C_\delta}\frac{\sqrt{R^2-z^2}}{a^2-R^2+z^2}dx = \int_{C_N}\frac{\sqrt{R^2-z^2}}{a^2-R^2+z^2}dx =$$

We have that

$$\frac{\sqrt{R^2-z^2}}{a^2-R^2+z^2} = \frac{iz\sqrt{1-\frac{R^2}{z^2}}}{z^2\left(1+\frac{a^2-R^2}{z^2}\right)} + \text{contribution of poles}$$