Homework 2

1 Consider an upper half-plain ($y > 0$) in $\mathbf{R}^2$ equipped with Riemannian metric

$$G = \sigma(x, y)(dx^2 + dy^2)$$

a) Show that $\sigma > 0$

b) In the case if $\sigma = \frac{1}{y^2}$ (the Lobachevsky metric) calculate the lengths of vectors $\mathbf{A} = 2\partial_x$ and $\mathbf{B} = 12\partial_x + 5\partial_y$ attached at the point $(x, y) = (1, 2)$.

c) calculate the cosine of the angle between the vectors $\mathbf{A}$ and $\mathbf{B}$ and show that the answer does not depend on the choice of the function $\sigma(x, y)$.

2 a) Write down explicit formulae expressing stereographic coordinates for n-dimensional sphere $(x^1)^2 + \ldots + (x^{n+1})^2 = R^2$ of radius R via coordinates $x^1, \ldots, x^{n+1}$ and vice versa. (For simplicity you may consider cases $n = 2, 3$.)

b)† Check that for unit sphere S^2 ($x^2 + y^2 + z^2 = 1$) all the points with rational Cartesian coordinates x, y, z have rational stereographic coordinates u, v and vice versa.

3 Consider the Riemannian metric on the circle of the radius R induced by the Euclidean metric on the ambient plane.

a) Express it using polar angle as a coordinate on the circle.

b) Express the same metric using stereographic coordinate t obtained by stereographic projection of the circle on the line, passing through its centre.

4 Consider the Riemannian metric on the sphere of the radius R induced by the Euclidean metric on the ambient 3-dimensional space.

a) Express it using spherical coordinates on the sphere.

b) Express the same metric using stereographic coordinates u, v obtained by stereographic projection of the sphere on the plane, passing through its centre.

5 Consider the surface L which is the upper sheet of two-sheeted hyperboloid in $\mathbf{R}^3$:

$$L: \qquad z^2 - x^2 - y^2 = 1, \quad z > 0\,.$$

a) Find parametric equation of the surface L using hyperbolic functions $\cosh, \sinh$ following an analogy with spherical coordinates on the sphere.

b) Consider the stereographic projection of the surface L on the plane OXY, i.e. the central projection on the plane $z = 0$ with the centre at the point $(0, 0, -1)$.

Show that the image of projection of the surface L is the open disc $x^2 + y^2 < 1$ in the plane OXY.

6* Consider the pseudo-Euclidean metric on $\mathbf{R}^3$ given by the formula

$$ds^2 = dx^2 + dy^2 - dz^2\,. \tag{1}$$

Calculate the induced metric on the surface L considered in the Exercise 5, and show that it is a Riemannian metric (it is positive-definite).

Perform calculations in spherical-like coordinates (see Exercise 5a) above), and in stereographic coordinates (see exercise 5b) above).

Remark *The surface L sometimes is called pseudo-sphere. The Riemannian metric on this surface sometimes is called Lobachevsky (hyperbolic) metric. The surface L with this metric realises Lobachevsky (hyperbolic) geometry, where Euclid's 5-th Axiom fails. This Riemannian manifold (manifold+Riemannian metric) is called Lobachevsky (hyperbolic) plane. In stereographic coordinates Lobachevsky plane is realised as an open disc $u^2+v^2<1$ in $\mathbf{E}^2$. It is so called Poincare model of Lobachevsky geometry. In the exercise 8 below we will consider realisation of Lobachevsky plane as upper half-plane.*

7 * In the exercises 5 and 6 it was shown that pseudo-Euclidean metric (1) in $\mathbf{R}^3$ induces Riemanian metric on two-sheeted hyperboloid $z^2-x^2-y^2=1$. Show that it is not true for one-sheeted hyperboloid: metric on one-sheeted hyperboloid $x^2+y^2-z^2=1$ in $\mathbf{R}^3$ is not Riemannian if it is induced with the pseudo-Euclidean metric (1).

8* In the exercise 6 we realised Lobachevsky plane as a disc $u^2+v^2<1$. Find new coordinates x, y such that in these coordinates Lobachevsky plane (hyperbolic plane) can be considered as an upper half plane $\{(x,y)\colon\ y>0\}$ in $\mathbf{E}^2$ and write down explicitly Riemannian metric in these coordinates.

Hint: *You may use complex coordinates:*

$$z=x+iy, \bar{z}=x-iy, \omega=u+iv, \bar{w}=u-iv$$

and consider a holomorphic transformation:

$$\omega=\frac{1+iz}{1-iz} \Leftrightarrow z=i\frac{1-\omega}{1+\omega},$$

which transforms the open disc $w\bar{w}<1$ onto the upper plane $\mathbf{Im}z>0$.

Remark Later by default we will call "Lobachevsky (hyperbolic) plane" the realisation of Lobachevsky plane as an half-upper plane in $\mathbf{E}^2$ with coordinates x, y $(y>0)$.