Riemannian Geometry

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1 Riemannian manifolds

1.1 Manifolds. Tensors. (Recalling)

I recall briefly basics of manifolds and tensor fields on manifolds.

An n-dimensional manifold is a space such that in a vicinity of any point one can consider local coordinates $\{x^1, \dots, x^n\}$ (charts). One can consider different local coordinates. If both coordinates $\{x^1, \dots, x^n\}$, $\{x^{1'}, \dots, x^{n'}\}$ are defined in a vicinity of the given point then they are related by bijective transition functions (functions defined on domains in $\mathbf{R}^n$ and taking values in $\mathbf{R}^n$).

$$\begin{cases} x^{1'} = x^{1'}(x^1, \dots, x^n) \\ x^{2'} = x^{2'}(x^1, \dots, x^n) \\ \dots \\ x^{n-1'} = x^{n-1'}(x^1, \dots, x^n) \\ x^{n'} = x^{n'}(x^1, \dots, x^n) \end{cases} \tag{1.1}$$

We say that n-dimensional manifold is *differentiable* or *smooth* if transition functions are diffeomorphisms, i.e. they are smooth and rank of Jacobian is equal to k, i.e.

$$\det \begin{pmatrix} \frac{\partial x^{1'}}{\partial x^1} & \frac{\partial x^{1'}}{\partial x^2} \dots & \frac{\partial x^{1'}}{\partial x^n} \\ \frac{\partial x^{2'}}{\partial x^1} & \frac{\partial x^{2'}}{\partial x^2} \dots & \frac{\partial x^{2'}}{\partial x^n} \\ \dots & & \\ \frac{\partial x^{n'}}{\partial x^1} & \frac{\partial x^{n'}}{\partial x^2} \dots & \frac{\partial x^{n'}}{\partial x^n} \end{pmatrix} \neq 0\,. \tag{1.2}$$

A good example of manifold is an open domain D in n-dimensional vector space $\mathbf{R}^n$. Cartesian coordinates on $\mathbf{R}^n$ define global coordinates on D. On the other hand one can consider an arbitrary local coordinates in different domains in $\mathbf{R}^n$. E.g. one can consider polar coordinates $\{r, \varphi\}$ in a domain $D = \{x, y \colon y > 0\}$ of $\mathbf{R}^2$ defined by standard formulae:

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}\,, \tag{1.3}$$

$$\det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{pmatrix} = \det \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix} = r \tag{1.4}$$

or one can consider spherical coordinates $\{r, \theta, \varphi\}$ in a domain $D = \{x, y, z \colon x > 0, y > 0, z > 0\}$ of $\mathbf{R}^3$ (or in other domain of $\mathbf{R}^3$) defined by standard formulae

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases},$$

,

$$\det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{pmatrix} = \det\begin{pmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} = r^2\sin\theta \tag{1.5}$$

Choosing domain where polar (spherical) coordinates are well-defined we have to be award that coordinates have to be well-defined and transition functions (1.1) have to obey condition (1.2), i.e. they have to be diffeomorphisms. E.g. for domain D in example (1.3) Jacobian (1.4) does not vanish if and only if $r > 0$ in D.

Examples of manifolds: $\mathbf{R}^n$, Circle S^1, Sphere S^2, in general sphere S^n, torus $S^1 \times S^1$, cylinder, cone, ...

Example Consider in detail circle S^1. Suppose it is given as $x^2 + y^2 = 1$. One can consider two local polar coordinates: 1) φ, $0 < \varphi < 2\pi$ which covers all points except the point $(1, 0)$ and $\varphi' \colon -\pi < \varphi' < \pi$ which covers all points except point $(-1, 0)$. In a vicinity of any point M on the circle (except these two exceptional points) one can consider both coordinates φ and φ'

We come to another very useful coordinates on a circle using *stereographic projection*. Take north pole of the circle: the point $N = (0, 1)$. Assign to every point $M = (x, y)$ on the circle the point $(t, 0)$ on the x-axis such that the point $(t, 0)$, the point M and the north pole N are on the one line. This can be done for every point of circle except the north pole $(0, 1)$ itself. We come to stereographic projection of circle S^1 without North pole on the line $\mathbf{R}$. In the same way we can define stereographic projection of circle without south pole (the point $(0, -1)$) on the x-axis. We come to coordinate t'. One can see that these coordinates are related by the following simple formula:

$$t' = \frac{1}{t},$$

† One very important property of stereographic projetion which we do not use in this course but it is too beautiful not to mention it: under stereographic projection

all points on the circle $x^2 + y^2 = 1$ with rational coordinates x and y and only these points transform to rational points on line. Thus we come to Pythagorean triples $a^2 + b^2 = c^2$.

Tensors on Manifold

Recall briefly what are tensors on manifold. For every point $\mathbf{p}$ on manifold M one can consider tangent vector space $T_{\mathbf{p}}M$— the space of vectors tangent to the manifold at the point M.

Tangent vector $\mathbf{A}(x) = A^i(x)\frac{\partial}{\partial x^i}$. Under changing of coordinates it transforms as follows:

$$\mathbf{A} = A^i(x)\frac{\partial}{\partial x^i} = A^i(x)\frac{\partial x^{m'}(x)}{\partial x^i}\frac{\partial}{\partial x^{m'}} = A^{m'}(x'(x))\frac{\partial}{\partial x^{m'}}\,.$$

Hence

$$A^{i'}(x') = \frac{\partial x^{i'}(x)}{\partial x^i}A^i(x)\,. \tag{1.6}$$

Consider also cotangent space $T^*_{\mathbf{p}}M$ (for every point $\mathbf{p}$ on manifold M)— space of linear functions on tangent vectors, i.e. space of 1-forms which sometimes are called *covectors.*:

One-form (covector) $\omega = \omega_i(x)dx^i$ transforms as follows

$$\omega = \omega_m(x)dx^m = \omega_m\frac{\partial x^m(x')}{\partial x^{m'}}dx^{m'} = \omega_{m'}(x')dx^{m'}\,.$$

Hence

$$\omega_{m'}(x') = \frac{\partial x^m(x')}{\partial x^{m'}}\omega_m(x)\,. \tag{1.7}$$

Differential form sometimes is called *covector*.

Tensors:

One can consider *contravariant* tensors of the rank p

$$T = T^{i_1 i_2 \dots i_p}(x)\frac{\partial}{\partial x^{i_1}} \otimes \frac{\partial}{\partial x^{i_2}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}}$$

with components $\{T^{i_1 i_2 \dots i_k}\}(x)$. Under changing of coordinates $(x^1, \dots, x^n) \to (x^{1'}, \dots, x^{n'})$ (see (1.1)) they transform as follows:

$$T^{i'_1 i'_2 \dots i'_p}(x') = \frac{\partial x^{i'_1}}{\partial x^{i_1}}\frac{\partial x^{i'_2}}{\partial x^{i_2}}\cdots\frac{\partial x^{i'_p}}{\partial x^{i_p}}T^{i_1 i_2 \dots i_p}(x)\,. \tag{1.8}$$

One can consider *covariant* tensors of the rank q

$$S = S_{j_1 j_2 \dots j_q} dx^{j_1} \otimes dx^{j_2} \otimes \dots dx^{j_q}$$

with components $\{S_{j_1 j_2 \dots j_q}\}$. Under changing of coordinates $(x^1, \dots, x^n) \to (x^{1'}, \dots, x^{n'})$ they transform as follows:

$$S_{j'_1 j'_2 \dots j'_q}(x') = \frac{\partial x^{i_1}}{\partial x^{i'_1}} \frac{\partial x^{i_2}}{\partial x^{i'_2}} \dots \frac{\partial x^{i_p}}{\partial x^{i'_p}} S_{j_1 j_2 \dots j_q}(x) \,. \tag{1.9}$$

One can also consider mixed tensors:

$$Q = Q^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q} \frac{\partial}{\partial x^{i_1}} \otimes \frac{\partial}{\partial x^{i_2}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes dx^{j_2} \otimes \dots dx^{j_q}$$

with components $\{Q^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q}\}$. We call these tensors *tensors of the type* $\begin{pmatrix} p \\ q \end{pmatrix}$. Tensors of the type $\begin{pmatrix} p \\ 0 \end{pmatrix}$ are called *contravariant tensors of the rank p*. They have p upper indices.

Tensors of the type $\begin{pmatrix} 0 \\ q \end{pmatrix}$ are called *covariant tensors of the rank q*. They have q lower indices.

Having in mind (1.6), (1.7),(1.8) and (1.9) we come to the rule of transformation for tensors which have p upper and q lover indices, tensors of type $\begin{pmatrix} p \\ q \end{pmatrix}$:

$$Q^{i'_1 i'_2 \dots i'_p}_{j'_1 j'_2 \dots j'_q}(x') = \frac{\partial x^{i'_1}}{\partial x^{i_1}} \frac{\partial x^{i'_2}}{\partial x^{i_2}} \dots \frac{\partial x^{i'_p}}{\partial x^{i_p}} \frac{\partial x^{j_1}}{\partial x^{j'_1}} \frac{\partial x^{j_2}}{\partial x^{j'_2}} \dots \frac{\partial x^{j_q}}{\partial x^{j'_q}} Q^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q}(x) \,.$$

E.g. if S_{ik} is a covariant tensor of rank 2 then

$$S_{i'k'}(x') = \frac{\partial x^i(x')}{\partial x^{i'}} \frac{\partial x^k(x')}{\partial x^{k'}} S_{ik}(x) \,. \tag{1.10}$$

If A^i_k is a tensor of rank $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (linear operator on $T_{\mathbf{p}}M$) then

$$A^{i'}_{k'}(x') = \frac{\partial x^{i'}(x)}{\partial x^i} \frac{\partial x^k(x')}{\partial x^{k'}} A^i_k(x) \,.$$

If S^m_{ik} is a tensor of the type $\begin{pmatrix}1\\2\end{pmatrix}$) then

$$S^{m'}_{i'k'} = \frac{\partial x^{m'}}{\partial x^m}\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^k}{\partial x^{k'}} S^m_{ik}(x)\,. \tag{1.11}$$

Transformations formulae (1.6)—(1.11) define vectors, covectors and in generally any tensor fields in components. E.g. covariant tensor (covariant tensor field) of the rank 2 can be defined as matrix S_{ik} (matrix valued function $S_{ik}(x)$) such that under changing of coordinates $\{x^1, x^2, \dots, x^n\} \mapsto \{x^{1'}, x^{2'}, \dots, x^{n'}\}$, (1.1) S_{ik} change by the rule (1.10).

Remark *Einstein summation rules*

In our lectures we always use so called *Einstein summation convention.* it implies that when an index occurs twice in the same expression in upper and in lower postitions, then the expression is implicitly summed over all possible values for that index. Sometimes it is called dummy indices summation rule.

1.2 Riemannian manifold— manifold equipped with Riemannian metric

Definition The Riemannian manifold is a manifold equipped with a Riemannian metric.

The Riemannian metric on the manifold M defines the length of the tangent vectors and the length of the curves.

Definition Riemannian metric G on n-dimensional manifold M^n defines for every point $\mathbf{p} \in M$ the scalar product of tangent vectors in the tangent space $T_{\mathbf{p}}M$ smoothly depending on the point $\mathbf{p}$.

It means that in every coordinate system $(x^1, \dots, x^n)$ a metric $G = g_{ik}dx^i dx^k$ is defined by a matrix valued smooth function $g_{ik}(x)$ ($i = 1, \dots, n; k = 1, \dots n$) such that for any two vectors

$$\mathbf{A} = A^i(x)\frac{\partial}{\partial x^i},\ \mathbf{B} = B^i(x)\frac{\partial}{\partial x^i},$$

tangent to the manifold M at the point $\mathbf{p}$ with coordinates $x = (x^1, x^2, \dots, x^n)$ ($\mathbf{A}, \mathbf{B} \in T_{\mathbf{p}}M$) the scalar product is equal to:

$$\langle \mathbf{A}, \mathbf{B}\rangle_G\big|_{\mathbf{p}} = G(\mathbf{A}, \mathbf{B})\big|_{\mathbf{p}} = A^i(x) g_{ik}(x) B^k(x) =$$

$$
\begin{pmatrix} A^1 \dots A^n \end{pmatrix} \begin{pmatrix} g_{11}(x) & \dots & g_{1n}(x) \\ \dots & \dots & \dots \\ g_{n1}(x) & \dots & g_{nn}(x) \end{pmatrix} \begin{pmatrix} B^1 \\ \cdot \\ \cdot \\ \cdot \\ B^n \end{pmatrix} \tag{1.12}
$$

where

- $G(\mathbf{A}, \mathbf{B}) = G(\mathbf{B}, \mathbf{A})$, i.e. $g_{ik}(x) = g_{ki}(x)$ (symmetricity condition)
- $G(\mathbf{A}, \mathbf{A}) > 0$ if $\mathbf{A} \neq \mathbf{0}$, i.e.
  $g_{ik}(x)u^i u^k \geq 0$, $g_{ik}(x)u^i u^k = 0$ iff $u^1 = \dots = u^n = 0$ (positive-definiteness)
- $G(\mathbf{A}, \mathbf{B})\big|_{\mathbf{p}=x}$, i.e. $g_{ik}(x)$ are smooth functions.

The matrix $||g_{ik}||$ of components of the metric G we also sometimes denote by G.

Now we establish rule of transformation for entries of matrix $g_{ik}(x)$, of metric G.

Notice that an arbitrary matrix entry g_{ik} is nothing but scalar product of vectors ∂_i, ∂_k at the given point:

$$
g_{ik}(x) = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k} \right\rangle , \quad \text{in coordinates } (x^1, \dots, x^n) \tag{1.13}
$$

Use this formula for establishing rule of transformations of $g_{ik}(x)$. In the new coordinates $x^{i'} = (x^{1'}, \dots, x^{n'})$ according this formula we have that

$$
g_{i'k'}(x') = \left\langle \frac{\partial}{\partial x^{i'}}, \frac{\partial}{\partial x^{k'}} \right\rangle , \quad \text{in coordinates } (x^1, \dots, x^n) \,.
$$

Now using chain rule, linearity of scalar product and formula (**??**) we see that

$$
\begin{aligned}
g_{i'k'}(x') &= \left\langle \frac{\partial}{\partial x^{i'}}, \frac{\partial}{\partial x^{k'}} \right\rangle = \left\langle \frac{\partial x^i}{\partial x^{i'}} \frac{\partial}{\partial x^i}, \frac{\partial x^k}{\partial x^{k'}} \frac{\partial}{\partial x^k} \right\rangle \\
&= \frac{\partial x^i}{\partial x^{i'}} \underbrace{\left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k} \right\rangle}_{g_{ik}(x)} \frac{\partial x^k}{\partial x^{k'}} =
\end{aligned}
$$

$$\frac{\partial x^i}{\partial x^{i'}} g_{ik}(x) \frac{\partial x^k}{\partial x^{k'}} \tag{1.14}$$

This transformation law justifies that g_{ik} entries of matrix $||g_{ik}||$ are components of *covariant tensor field $G = g_{ik}dx^i dx^k$ of rank 2*(see equation (1.10)).

One can say that Riemannian metric is defined by symmetric covariant smooth tensor field G of the rank 2 which defines scalar product in the tangent spaces $T_{\mathbf{p}}M$ smoothly depending on the point $\mathbf{p}$. Components of tensor field G in coordinate system are matrix valued functions $g_{ik}(x)$:

$$G = g_{ik}(x) dx^i \otimes dx^k \,. \tag{1.15}$$

In practice it is more convenient to perform transformation of metric G under changing of coordinates in the following way:

$$G = g_{ik} dx^i \otimes dx^k = g_{ik} \left(\frac{\partial x^i}{\partial x^{i'}} dx^{i'} \right) \otimes \left(\frac{\partial x^k}{\partial x^{k'}} dx^{k'} \right) =$$

$$\frac{\partial x^i}{\partial x^{i'}} g_{ik} \frac{\partial x^k}{\partial x^{k'}} dx^{i'} \otimes dx^{k'} = g_{i'k'} dx^{i'} \otimes dx^{k'}$$

Hence

$$g_{i'k'} = \frac{\partial x^i}{\partial x^{i'}} g_{ik} \frac{\partial x^k}{\partial x^{k'}} \,. \tag{1.16}$$

We come to transfomation rule (1.14).

Later by some abuse of notations we sometimes omit the sign of tensor product and write a metric just as

$$G = g_{ik}(x) dx^i dx^k \,.$$

Examples

- $\mathbf{R}^n$ with canonical coordinates $\{x^i\}$ and with metric

  $$G = (dx^1)^2 + (dx^2)^2 + \cdots + (dx^n)^2$$

  $G = ||g_{ik}|| = \text{diag}\ [1, 1, \ldots, 1]$

  Recall that this is a basis example of n-dimensional Euclidean space, where scalar product is defined by the formula:

  $$G(\mathbf{X}, \mathbf{Y}) = \langle \mathbf{X}, \mathbf{Y} \rangle = g_{ik} X^i Y^k = X^1 Y^1 + X^2 Y^2 + \cdots + X^n Y^n \,.$$

In the general case if $G = ||g_{ik}||$ is an arbitrary symmetric positive-definite metric then $G(\mathbf{X}, \mathbf{Y}) = \langle \mathbf{X}, \mathbf{Y} \rangle = g_{ik} X^i Y^k$. One can show that there exists a new basis $\{\mathbf{e}_i\}$ such that in this basis $G(\mathbf{e}_i, \mathbf{e}_k) = \delta_{ik}$. This basis is called orthonormal basis. (See the Lecture notes in Geometry)

Scalar product in vector space defines the *same* scalar product at all the points. In general case for Riemannian manifold scalar product depends on a point. In Riemannian manifold we consider arbitrary transformations from local coordinates to new local coordinates.

- $\mathbf{R}^2$ with polar coordinates in the domain $y > 0$ ($x = r\cos\varphi, y = r\sin\varphi$):

  $dx = \cos\varphi dr - r\sin\varphi d\varphi, dy = \sin\varphi dr + r\cos\varphi d\varphi$. In new coordinates the Riemannian metric $G = dx^2 + dy^2$ will have the following appearance:

  $$G = (dx)^2 + (dy)^2 = (\cos\varphi dr - r\sin\varphi d\varphi)^2 + (\sin\varphi dr + r\cos\varphi d\varphi)^2 = dr^2 + r^2 (d\varphi)^2$$

  We see that for matrix $G = ||g_{ik}||$

  $$\underbrace{G = \begin{pmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},}_{\text{in Cartesian coordinates}} \qquad \underbrace{G = \begin{pmatrix} g_{rr} & g_{r\varphi} \\ g_{\varphi r} & g_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}}_{\text{in polar coordinates}}$$

- Circle

  Interval $[0, 2\pi)$ in the line $0 \leq x < 2\pi$ with Riemannian metric

  $$G = a^2 dx^2 \tag{1.17}$$

  Renaming $x \mapsto \varphi$ we come to habitual formula for metric for circle of the radius a: $x^2 + y^2 = a^2$ embedded in the Euclidean space $\mathbf{E}^2$:

  $$G = a^2 d\varphi^2 \qquad \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \end{cases}, 0 \leq \varphi < 2\pi, \tag{1.18}$$

- Domain in $\mathbf{R}^2$ with metric $G = du^2 + u^2 dv^2$

  (Compare with $\mathbf{R}^2$ with polar coordinates).

- Cylinder surface

  Domain in $\mathbf{R}^2$ $D = \{(x, y) : ,\ 0 \leq x < 2\pi$ with Riemannian metric

$$G = a^2 dx^2 + dy^2 \tag{1.19}$$

  We see that renaming variables $x \mapsto \varphi$, $y \mapsto h$ we come to habitual, familiar formulae for metric in standard polar coordinates for cylinder surface of the radius a embedded in the Euclidean space $\mathbf{E}^3$:

$$G = a^2 d\varphi^2 + dh^2 \qquad \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases} ,0 \leq \varphi < 2\pi, -\infty < h < \infty \tag{1.20}$$

- Sphere

  Domain in $\mathbf{R}^2$, $0 < x < 2\pi$, $0 < y < \pi$ with metric $G = dy^2 + \sin^2 y dx^2$ We see that renaming variables $x \mapsto \varphi$, $y \mapsto h$ we come to habitual, familiar formulae for metric in standard spherical coordinates for sphere $x^2 + y^2 + z^2 = a^2$ of the radius a embedded in the Euclidean space $\mathbf{E}^3$:

$$G = a^2 d\theta^2 + a^2 \sin^2\theta d\varphi^2 \qquad \begin{cases} x = a\sin\theta\cos\varphi \\ y = a\sin\theta\sin\varphi z = a\cos\theta \end{cases} ,0 \leq \varphi < 2\pi, -\infty < h < \infty \tag{1.21}$$

(See examples also in the Homeworks.)

1.2.1 * Pseudoriemannian manifold

If we omit the condition of positive-definiteness for Riemannian metric we come to so called Pseudoriemannian metric. Manifold equipped with pseudoriemannian metric is called pseudoriemannian manifold. Pseudoriemannian manifolds appear in applications in the special and general relativity theory.

In pseudoriemanninan space scalar product $(\mathbf{X}, \mathbf{X})$ may take an arbitrary real values: it can be positive, negative, it can be equal to zero. Vectors $\mathbf{X}$ such that $(\mathbf{X}, \mathbf{X}) = 0$ are called null-vectors. (See the problem 6 in Homework 1).

Example Consider $n+1$-dimensional linear space $\mathbf{R}^{n+1}$ with pseudometric [1]

$$G = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - \cdots - (dx^n)^2 \,.$$

For an arbitrary vector $\mathbf{X} = (a^0, a^1, a^2, \dots, a^n)$ scalar product $(\mathbf{X}, \mathbf{X})$ is positive if $(a^0)^2 > (a_1)^2 + (a_2)^2 + \cdots + (a_n)^2$, it is negative if $(a^0)^2 < (a_1)^2 + (a_2)^2 + \cdots + (a_n)^2$, and $\mathbf{X}$ is null-vector if $(a^0)^2 = (a_1)^2 + (a_2)^2 + \cdots + (a_n)^2$.

[1] In the case $n = 3$ it is so called Minkovski space. The coordinate x^0 plays a role of the time: $x^0 = ct$, where c is the value of the speed of the light. Vectors $\mathbf{X}$ such that $(\mathbf{X}, \mathbf{X}) > 0$ are called time-like vectors and they called space-like vectors if $(\mathbf{X}, \mathbf{X}) < 0$