Gramm decomposition

Let $A \in GL(n, \mathbf{R})$. Let $\{\mathbf{e}_i\}$ be a standard basis in $\mathbf{R}^n$ Consider

$$\mathbf{a}_k = \mathbf{e}_r A_{rk} \,.$$

Now consider

$$\mathbf{b}_1 = \mathbf{a}_1, \quad \mathbf{f}_1 = \frac{\mathbf{b}_1}{|\mathbf{b}_1|} \,,$$

$$\mathbf{b}_2 = \mathbf{a}_2 - \mathbf{f}_1 \langle \mathbf{f}_1, \mathbf{a}_2 \rangle \,, \quad \mathbf{f}_2 = \frac{\mathbf{b}_2}{|\mathbf{b}_2|}$$

and so on:

$$\mathbf{b}_3 = \mathbf{a}_3 - \mathbf{f}_1 \langle \mathbf{f}_1, \mathbf{a}_3 \rangle - \mathbf{f}_2 \langle \mathbf{f}_2, \mathbf{a}_3 \rangle \,, \quad \mathbf{f}_3 = \frac{\mathbf{b}_3}{|\mathbf{b}_3|} \,,$$

$$\mathbf{b}_4 = \mathbf{a}_4 - \mathbf{f}_1 \langle \mathbf{f}_1, \mathbf{a}_4 \rangle - \mathbf{f}_2 \langle \mathbf{f}_2, \mathbf{a}_4 \rangle - \mathbf{f}_3 \langle \mathbf{f}_3, \mathbf{a}_4 \rangle \,, \quad \mathbf{f}_4 = \frac{\mathbf{b}_4}{|\mathbf{b}_4|} \,,$$

$$\mathbf{b}_5 = \mathbf{a}_5 - \mathbf{f}_1 \langle \mathbf{f}_1, \mathbf{a}_5 \rangle - \mathbf{f}_2 \langle \mathbf{f}_2, \mathbf{a}_5 \rangle - \mathbf{f}_3 \langle \mathbf{f}_3, \mathbf{a}_5 \rangle - \mathbf{f}_4 \langle \mathbf{f}_3, \mathbf{a}_5 \rangle \,, \quad \mathbf{f}_5 = \frac{\mathbf{b}_5}{|\mathbf{b}_5|} \,,$$

$$\dots$$

We see that

$$\mathbf{f}_i = \sum_{r \leq i} \mathbf{a}_r T_{ri} \,,$$

i.e. T is an upper-triangular matrix, and $\{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ is also orthonormal basis: $\langle \mathbf{f}_i, \mathbf{f}_j \rangle = \delta_{ij}$ hence:

$$\mathbf{f}_i = \mathbf{e}_k P_{ki},$$

where P is orthogonal matrix. Thus we have that

$$\mathbf{f}_i = \mathbf{a}_r T_{ri} = \mathbf{e}_m A_{mr} T_{ri} = \mathbf{e}_m P_{mi} \,,$$

i.e.

$$A = P \circ T^{-1}$$

We come to decomposition:

$$GL(n, \mathbf{R}) = O(n, \mathbf{R}) \times T(n, \mathbf{R}) \,.$$