Homework 7

An exercise 8 and all the exercises on the second page are not compulsory. They are based on the material of subsections 2.8 and 2.9 of lecture notes.

1 Calculate the integral of the form $\omega = e^{-y}dx + \sin x dy$ over the segment of straight line which connects the points $A = (1, 1)$, $B = (2, 3)$. How does your answer depend on a choice of parameterisation?

2 Calculate the integral of the form $\omega = xdy$ over the following curves

a) closed curve $x^2 + y^2 = 12y$

b) arc of the ellipse $x^2 + y^2/9 = 1$ defined by the condition $y \geq 0$.

How does your answer depend on a choice of parameterisation?

Choose two different parameterisations of each of these curves such that integral changes sign under changing of parameterisation.

3 Calculate the integral of the form $\omega = 5xdy + 4ydx$ over the upper arc of the unit circle which passes through the point $A = (4, 0)$ and the point $B = (2, 0)$.

Exact forms

4 Calculate the integral $\int_C \omega$ where $\omega = xdx + ydy$ and C is

a) the straight line segment $x = t, y = 1 - t$, $0 \leq t \leq 1$

b) the segment of parabola $x = t$, $y = 1 - t^n$, $0 \leq t \leq 1$, $n = 2, 3, 4, \ldots$

c) for **an arbitrary** curve starting at the point $(0, 1)$ and ending at the point $((1, 0)$.

5 Show that the form 1-form $\omega = 3x^2ydx + x^3dy$ is an exact 1-form.

a) Calculate integral of this form over the curves considered in exercises 2) and 3).

b) Write down the 1-form ω in polar coordinates.

6. Consider 1-forms

a) xdx , b) xdy c) $xdx + ydy$, d)$xdy + ydx$, e) $xdy - ydx$

f) $x^4dy + 4x^3ydx$, g) $xdy + ydx + dz$, h) $xdy - ydx + dz$.

a) Show that 1-forms a), c), d), f) and g) are exact forms

b) Why are 1-forms b), e) and h) not exact?

7 Consider 1-form $\omega = xdy + aydx$ where a is a constant.

a) Find the integral of this form over a closed curve defined by equation $x^2 + y^2 - 4x - 4y + 7 = 0$.

b) Explain why the form ω is exact if $a = 1$.

c) Explain why the form ω is not exact if $a \neq 1$.

8* Calculate the integral of the form $\sigma = \frac{xdy - ydx}{x^2}$ over the curve $x^2+y^2-4x-4y+7 = 0$ consdered in the previous exercise.

All the exercises below are not compulsory

9† Consider one-form

$$\omega = \frac{xdy - ydx}{x^2 + y^2} \tag{1}$$

This form is defined in $\mathbf{E}^2 \setminus 0$.

Calculate differential of this form.

Write down this form in polar coordinates

Find a function f such that $\omega = df$.

Is this function defined in the same domain as ω?

10† Calculate the integral of the form $\omega = \frac{xdy-ydx}{x^2+y^2}$ over the curves

a) circle $x^2 + y^2 = 1$

b) circle $(x - 3)^2 + y^2 = 1$

c) ellipse $\frac{x^2}{9} + \frac{x^2}{16} = 1$

11† What values can take the integral $\int_C \omega$ if C is an arbitrary curve starting at the point $(0, 1)$ and ending at the point $((1, 0)$ and $\omega = \frac{xdy-ydx}{x^2+y^2}$.

12† Let $\omega = a(x, y)dx + b(x, y)dy$ be a closed form in $\mathbf{E}^2$, $d\omega = 0$.

Consider the function

$$f(x, y) = x \int_0^1 a(tx, ty)dt + y \int_0^1 b(tx, ty)dt \tag{2}$$

Show that

$$\omega = df\,.$$

This proves that an arbitrary closed form in $\mathbf{E}^2$ is an exact form.

Why we cannot apply the formula (2) to the form ω defined by the expression (1)?