Homework 7. Solutions

1 *Show that great circles are geodesics on sphere. Do it*

a) using the fact that for geodesic, acceleration is orthogonal to the surface.

b) using straightforwardly equations for geodesics*

c) using the fact that geodesic is shortest.

a) See Lecture notes, subsection 3.2.3

b)*) The straightforward equations for geodesic: $\frac{d^2x^i}{dt^2} + \frac{dx^k}{dt}\Gamma^i_{km}\frac{dx^m}{dt} = 0$ are just equation of motion for free Lagrangian on the Riemannian surface. Hence in the case of sphere they are equations of motion of the Lagrangian of "free" particle on the sphere is $L = \frac{R^2\dot\theta^2 + R^2\sin^2\theta\dot\varphi^2}{2}$. Its equations of motion are second order differential equations

$$\begin{cases} \ddot\theta - \sin\theta\cos\theta\dot\varphi^2 = 0 \\ \ddot\varphi + 2\text{cotan}\,\theta\dot\theta\dot\varphi = 0 \end{cases} \qquad \text{with intitial conditions } \begin{cases} \theta(t)|_{t=0} = \theta_0,\ \dot\theta(t)|_{t=0} = a \\ \varphi(t)|_{t=0} = \varphi_0,\ \dot\varphi(t)|_{t=0} = b \end{cases} \tag{1}$$

for geodesics $\theta(t), \varphi(t)$ starting at the initial point $\mathbf{p} = (\theta_0, \varphi_0)$ with initial velocity $\mathbf{v}_0 = a\frac{\partial}{\partial\theta} + b\frac{\partial}{\partial\varphi}$. (All Christoffel symbols vanish except $\Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta$, and $\Gamma^\varphi_{\varphi\theta} = \Gamma^\varphi_{\theta\varphi} = \text{cotan}\,\theta$.)

This differential equation is not very easy to solve in general case. On the other hand use the fact that rotations are isometries of the sphere. Rotate the sphere in a way such that the initial point transforms to the point the point $\theta_0 = \frac{\pi}{2}, \varphi_0 = 0$ and then rotate the sphere with respect to the axis $0X$ such that θ-component of velocity becomes zero. We come to the same differential equation but with changed initial conditions:

$$\begin{cases} \ddot\theta - \sin\theta\cos\theta\dot\varphi^2 = 0 \\ \ddot\varphi + 2\text{cotan}\,\theta\dot\theta\dot\varphi = 0 \end{cases} \qquad \text{with intitial conditions } \begin{cases} \theta(t)|_{t=0} = \frac{\pi}{2},\ \dot\theta(t)|_{t=0} = 0 \\ \varphi(t)|_{t=0} = 0,\ \dot\varphi(t)|_{t=0} = \Omega_0 \end{cases} \tag{2}$$

where we denote by Ω_0 the magnitude of initial velocity. One can easy check that the functions

$$\begin{cases} \theta(t) = \frac{\pi}{2} \\ \varphi(t) = \Omega_0 t \end{cases}$$

are the solution of the differential equations for geodesic with initial conditions (2). Hence this is geodesic passing through the point $\left(\theta_0 = \frac{\pi}{2}, \varphi = 0\right)$ with initial velocity $\Omega_0\frac{\partial}{\partial\varphi}$. We see that this geodesic is the equator of the sphere. We proved that an arbitrary geodesic after applying the suitable rotation is the great-circle— equator. On the other hand an equator is the great circle (the intersection of the sphere $x^2 + y^2 + z^2 = R^2$ with the plane $z = 0$) and the rotation transforms the equator to the another great circle. Hence all arcs of great circles are geodesics and all geodesics are arcs of great circles.

c) See the lecture notes the subsection 3.4.1 ("Again on geodesics on sphere and on Lobachevsky plane".)

2 *Consider in* $\mathbf{E}^3$ *a vector* $\mathbf{X} = \frac{\partial}{\partial y}$ *attached at the point* $\mathbf{p}$: $\left(x = \frac{R}{2}, y = 0, z = \frac{\sqrt{3}R}{2}\right)$.

Consider also a sphere $x^2 + y^2 + z^2 = R^2$ *in* $\mathbf{E}^3$ *and the following two curves: a curve* C_1 *which is the intersections of this sphere with plane* $y = 0$ *and a curve* C_1 *which is the intersections of this sphere with the plane* $z = \frac{\sqrt{3}R}{2}$. *Both curves* C_1, C_2 *pass through the point* $\mathbf{p}$.

Show that the vector $\mathbf{X}$ *is tangent to the sphere and express this vector in spherical coordinates.*

* *Describe the parallel transport of the vector* $\mathbf{X}$ *along these closed curves.*

What will be the result of parallel transport of the vector $\mathbf{X}$ *along these closed curves?*

The vector $\mathbf{X}_0 = \frac{\partial}{\partial y}$ attached at the point $\mathbf{p} = \left(x = \frac{R}{2}, y = 0, z = \frac{\sqrt{3}R}{2}\right)$ of the sphere $x^2+y^2+z^2 = R^2$ is tangent to this sphere since $\partial_{\mathbf{X}_0}(x^2+y^2+z^2-R^2) = 0$:

$$\partial_{\mathbf{X}}(x^2+y^2+z^2-R^2)|_{\mathbf{p}} = 0 = \frac{\partial}{\partial y}(x^2+y^2+z^2-R^2)_{\mathbf{p}} = 2y|_{\mathbf{p}} = 0\,.$$

The vector $\mathbf{X}_0 = \frac{\partial}{\partial y}$ attached at the point $\mathbf{p}$ is proportional to the vector $\frac{\partial}{\partial \varphi}$: $\mathbf{X}_0 = b_0\frac{\partial}{\partial \varphi}$. The spherical coordinates of the point $\mathbf{p}$ are $\theta_0 = \frac{\pi}{6}$ ($\cos\theta_0 = \frac{\sqrt{3}}{2}$). Since the metric on the sphere is $G = R^2(d\theta^{+}\sin^2\theta d\varphi^2)$ then calculating the length of the vector $\mathbf{X}_0$ in ambient space in Cartesian coordinates and on the sphere in spherical coordinates we come to

$$|\mathbf{X}_0| = \left|\frac{\partial}{\partial y}\right| = 1 = \sqrt{R^2\sin^2\theta_0 d\varphi^2\left(b\frac{\partial}{\partial\varphi}\right)} = Rb\frac{\sqrt{3}}{2} \Rightarrow b = \frac{1}{R\sin\theta_0} = \frac{2\sqrt{3}}{3R}, \mathbf{X}_0 = \frac{\partial_\varphi}{R\sin\theta_0} = \frac{2\sqrt{3}}{3R}\partial_\varphi\,.$$

Using the formula $\nabla\mathbf{X} = \frac{dX(t)}{dt} + \dot{x}^k\Gamma^i_{km}X^m = 0$ for parallel transport we see that for an arbitrary curve $\theta(t), \varphi(t)$ on the sphere starting at the initial point $\theta(t_0) = \theta_0, \varphi(t_0) = \varphi_0$ the parallel transport $\mathbf{X}(t) = a(t)\partial_\theta + b(t)\partial_\varphi$ of the initial vector $\mathbf{X}|_{t_0} = a_0\partial_\theta + b_0\partial_\varphi$ over the curve C is defined by the first order differential equation:

$$\begin{cases} \dot{a} + \dot{\varphi}\Gamma^\theta_{\varphi\varphi}b = \dot{a} - \dot{\varphi}\sin\theta\cos\theta b = 0 \\ \dot{b} + \dot{\varphi}\Gamma^\varphi_{\varphi\theta}a + \dot{\theta}\Gamma^\varphi_{\theta\varphi}b = \dot{b} + (\dot{\varphi}a + \dot{\theta}b)\mathrm{cotan}\,\theta = 0 \end{cases} \quad \text{with initial conditions} \quad \begin{cases} a(t_0) = a_0 \\ b(t_0) = b_0 \end{cases} \tag{3}$$

(We use here the information about Christoffel symbols of sphere.) Apply this equation for both curves (For the first curve there is another much more clear solution: see below.)

First consider the curve C_1. For the curve C_1 in spherical coordinates: $\theta(t) = \theta_0 + t, \varphi(t) = 0$, $\dot{\theta} = 1, \dot{\varphi} = 0$. The differential equation (3) becomes:

$$\begin{cases} \dot{a} - \dot{\varphi}\sin\theta\cos\theta b = \dot{a} = 0 \\ \dot{b} + (\dot{\varphi}a + \dot{\theta}b)\mathrm{cotan}\,\theta = \dot{b} + b\mathrm{cotan}\left(\frac{\pi}{6} + t\right) = 0 \end{cases} \quad \text{with initial conditions} \quad \begin{cases} a(t_0) = 0 \\ b(t_0) = b_0 = \frac{2\sqrt{3}}{3R} \end{cases} \tag{3a}$$

The solution of this equation is $a(t) = 0$, $b(t) = \frac{1}{R\sin\theta(t)}$, $\mathbf{X} = \frac{1}{R\sin\theta(t)}\frac{\partial}{\partial\varphi} = \frac{1}{R\sin\left(t+\frac{\pi}{6}\right)}\frac{\partial}{\partial\varphi}$. This answer is evident without any calculation: Namely the vector $\mathbf{X} = \partial_y$ during transport along the curve C_1 in the ambient space $\mathbf{E}^3$, remains tangent to the sphere. Hence $\mathbf{X}(t) = \frac{\partial}{\partial y}$ at any point of this circle. The vector ∂_y at point $\theta(t)$ equals to $\frac{1}{R\sin\theta(t)}\frac{\partial}{\partial\varphi}$. We come to the answer.

Note that during parallel transport along the curve C_1 the final vector coincides with initial one.

For the curve C_2. This curve in spherical coordinates is $\theta(t) = \theta_0 = \frac{\pi}{6}$, $\varphi(t) = t$, $\dot{\theta} = 0, \dot{\varphi} = 1$. Hence the equation (3) for parallel transport becomes:

$$\begin{cases} \dot{a} - \dot{\varphi}\sin\theta\cos\theta b = \dot{a}(t) - \sin\theta_0\cos\theta_0 b(t) = 0 \\ \dot{b}(t) + a(t)\mathrm{cotan}\,\theta_0 = 0 \end{cases} \quad \text{with initial conditions} \quad \begin{cases} a(t_0) = 0 \\ b(t_0) = b_0 \end{cases} \tag{3}$$

Here we have two first order differential equations. Introduce $b'(t) = \sin\theta_0 b(t)$. We have

$$\begin{cases} \frac{da(t)}{dt} - \cos\theta_0 b'(t) = 0 \\ \frac{db'(t)}{dt} + \cos\theta_0 a(t) = 0 \end{cases}, \quad \text{i.e. } \frac{d}{dt}\begin{pmatrix} a(t) \\ b'(t) \end{pmatrix} = \cos\theta_0\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a(t) \\ b'(t) \end{pmatrix}$$

or in other way

$$\frac{d}{dt}(a + ib') = -i\cos\theta_0(a + ib')$$

The solution is

$$a(t)+ib'(t)=\mathbf{e}^{-it\cos\theta_0}(a_0+ib'_0)=(\cos(t\cos\theta_0)-i\sin(t\cos\theta_0))(a_0+ib'_0)$$

In particular as a result of parallel transport along the closed latitude

$$a(t)+ib'(t)\big|_{t=2\pi}=\mathbf{e}^{-2\pi i\cos\theta_0}(a_0+ib'_0)$$

that is the vector rotates on the angle $\Phi=2\pi\cos\theta_0$.

We see that both answers for curves C_1 and C_2 are in coincidence with the formula of rotation of the vector during parallel transport along the closed curve: $\Delta\Phi=\int Kd\sigma$. Indeed for the sphere $K=\frac{1}{R^2}$. For the first curve C_1 the first case $\Delta\Phi=\int Kd\sigma=\frac{1}{R^2}2\pi R^2=2\pi$ and for the curve C_2 $\Delta\Phi=\int Kd\sigma=\frac{1}{R^2}2\pi R^2(1-\cos\theta_0)=2\pi(1-\cos\theta_0)$

3 *Show that vertical lines $x=a$ are geodesics (non-parameterised) on Lobachevsky plane.*

We consider here the realisation of Lobachevsky plane (hyperbolic plane) as upper half of Euclidean plane $\{(x,y)\colon y>0\}$ with the metric $G=\frac{dx^2+dy^2}{y^2}$.

Consider second order differential equations defining geodesics with initial conditions such that "horisontal" velocity equals to zero: (we uses the information from Homework 6 or from Lecture notes about Christoffels for Lobachevsky plane: $\Gamma^x_{xx}=0,\Gamma^x_{xy}=\Gamma^x_{yx}=-\frac{1}{y}$, $\Gamma^x_{yy}=0$, $\Gamma^y_{xx}=\frac{1}{y},\Gamma^y_{xy}=\Gamma^y_{yx}=0,\Gamma^y_{yy}=-\frac{1}{y}$.)

$$\begin{cases}\ddot{x}-\frac{2\dot{x}\dot{y}}{y}=0\\ \ddot{y}+\frac{\dot{x}^2}{y}-\frac{\dot{y}^2}{y}=0\\ x(t)\big|_{t=t_0}=x_0,\dot{x}(t)\big|_{t=t_0}=0\\ y(t)\big|_{t=t_0}=y_0,\dot{y}(t)\big|_{t=t_0}=\dot{y}_0\end{cases}$$

This equation has a solution and it is unique. One can see that if we put $x(t)\equiv 0$, i.e. curve is vertical then we come to the equation $\ddot{y}-\frac{\dot{y}^2}{y}=0$. Solution of these equation gives curve $x=x_0$, $y=y(t)\colon\ddot{y}-\frac{\dot{y}^2}{y}=0$. The image of this curve clearly is vertical ray $x=x_0, y>0$.

4 *Show that the following transformations are isometries of Lobachevsky plane:*

a) horizontal translation $\mathbf{r}\to\mathbf{r}+\mathbf{a}$ where $\mathbf{a}=(a,0)$,

b) homothety: $\mathbf{r}\to\lambda\mathbf{r}$ $(\lambda>0)$,

* *c) inversion with the centre at the points of the line $x=0^*$:*

$$\mathbf{r}\to\mathbf{a}+\frac{\mathbf{r}-\mathbf{a}}{|\mathbf{r}-\mathbf{a}|^2}\ \textit{where}\ \mathbf{a}=(a,0)\colon\quad\begin{cases}x'=a+\frac{x-a}{(x-a)^2+y^2}\\ y'=\frac{y}{(x-a)^2+y^2}\end{cases}.$$

We have to show that Riemannian metric $G=\frac{dx^2+dy^2}{y^2}$ remains invariant under these transformations.

a) horizontal translation. If $x\to x+a, y\to y$ then dx and dy do not change. Hence G is invariant under horisontal translations.

b) Homothety. If $x\to\lambda x$, $y\to\lambda y$ where $\lambda>0$ is a constant then $\frac{dx^2+dy^2}{y^2}\to\frac{\lambda^2dx^2+\lambda^2dy^2}{\lambda^2y^2}=\frac{dx^2+dy^2}{y^2}$ does not change too.

c) * inversion with the centre at the points of the line $y=0$. Since we proved that horisontal translation is isometry it suffices to consider inversion with centre at the point $x=y=0$:

$$\mathbf{r}\to\frac{\mathbf{r}}{|\mathbf{r}|^2}\colon\quad\begin{cases}x'=\frac{x}{x^2+y^2}\\ y'=\frac{y}{x^2+y^2}\end{cases}$$

* *This line is called absolute.*

Now by straightforward calculations one can show that $\frac{dx^2+dy^2}{y^2} = \frac{dx'^2+dy'^2}{y'^2}$.

To avoid the straightforward calculations consider coordinates r, φ: $x = r\cos\varphi, y = r\sin\varphi$ then

$$\frac{dx^2+dy^2}{y^2} = \frac{dr^2+r^2d\varphi^2}{r^2\sin^2\varphi} = \frac{1}{\sin^2\varphi}\frac{dr^2}{r^2} + \frac{d\varphi^2}{\sin^2\varphi} = \frac{1}{\sin^2\varphi}(d\log r)^2 + \frac{d\varphi^2}{\sin^2\varphi}\,.$$

Under transformation of inversion in these coordinates φ does not change, $r \to \frac{1}{r}$, $\log r \to -\log\frac{1}{r}$. It is evident that in coordinates $u = \log r, \varphi$ metric does not change. Hence inversion (with centre at the point $x = y = 0$) is isometry. Hence an inversion with a centre at the arbitrary point $(a, 0)$ and with an arbitrary radius is isometry, since horisontal translation and homothety are isometries.

(In complex cooridinates this is so called Mobius transformation transformation $z \to \frac{1}{\bar{z}}$ which you learned in the course Hyperbolic Geometry.)

5* *Show that upper arcs of semicircles $(x-a)^2+y^2 = R^2, y > 0$ are (non-parametersied) geodesics.*

You may do this exercise solving explicitly differential equations for geodesics, but it is much more nice to use inversion (Mobius) transformation studied in the previous exercise: Consider the inversion of the Lobachevsky plane with the centre at the point $x = a - R, y = 0$ (see the exercise above). This inversion does not change Riemannian metric, it is isometry. Isometry transforms geodesics to geodesics. On the other hand it transforms the semicircle $(x-a)^2+y^2 = R^2, y > 0$ to the vertical ray $x = a - R + \frac{1}{2R}, y > 0$. This can be checked directly. On the other hand the vertical ray is geodesic. Hence the initial curve was the geodesic too.

6 *Let ABC be triangle formed by geodesics on the sphere of the radius R. Express the area of this triangle via its angles.*

Do the previous exercise for the triangle on the Lobachevsky plane.

We use

1) the Theorem from Lecture notes about the rotation of the vector during parallel transport along closed curve $C = \partial D$: $\Delta\Phi = \int_D K d\sigma$ a

2) the fact that velocity vector is covariantly constant along geodesic

3) during parallel transport scalar product and hence the angle between two vectors does not change.

Let ABC be the triangle formed by arcs of geodesics. We denote by α, β, γ angles at the vertices A, B, C.

Let vector $\mathbf{A}_0$ be velocity vector at the vertex A along the arc AB. Under parallel transport it becomes the vector $\mathbf{A}_1$ which is the velocity vector of the arc AB attached at the vertex B. Take the vector $\mathbf{B}_0$ which is velocity vector at the vertex B along the arc BC. The vector $\mathbf{B}_0$ has the angle β with the vector $\mathbf{A}_1$. Do the parallel transport of two vectors $\mathbf{A}_1$ and $\mathbf{B}_0$ along the arc of geodesic BC. We arrive at the vertex C with the vector $\mathbf{B}_1$ which is the velocity vector along the curve BC at the vertex C and the vector $\mathbf{A}_2$ which has the angle β with the vector $\mathbf{B}_1$ since angles do not change during parallel transport. Now take the vector $\mathbf{C}_0$ which is velocity vector at the vertex C along the arc CA. The vector $\mathbf{C}_0$ has the angle γ with the vector $\mathbf{B}_1$ and it has the angle $\beta+\gamma$ with the vector $\mathbf{A}_2$. Now make the final parallel transport along the arc of geodesic CA of three vectors: vector $\mathbf{C}_0$, the vector $\mathbf{B}_1$ and the vector $\mathbf{A}_2$. We come to three vectors: vectors $\mathbf{C}_1$, vector $\mathbf{B}_2$ and the vector $\mathbf{A}_3$ at the vertex A. The vector $\mathbf{C}_0$ will transform to the vector $\mathbf{C}_1$ which is the velocity vector of the curve CA at the vertex A. The vector $\mathbf{B}_2$ has the angle β with the vector $\mathbf{C}_2$ and the vector $\mathbf{A}_3$ has the angle α with the vector $\mathbf{C}_1$. Hence the vector $\mathbf{A}_3$ has the angle $\beta+\gamma$ with the vector $\mathbf{C}_1$. The vector $\mathbf{C}_1$ has the angle $\pi-\alpha$ with initial velocity vector $\mathbf{A}_0$. We see that after parallel transport along the geodesics the vector $\mathbf{A}_0$ becomes the vector $\mathbf{A}_3$ which has the angle $\alpha+\beta+\gamma-\pi$ with the initial vector $\mathbf{A}_0$. Now using the Theorem we come to

$$\alpha+\beta+\gamma-\pi = \int_{\triangle ABC} K d\sigma$$

For the sphere we have $K = \frac{1}{R^2}$, hence:

$$\alpha + \beta + \gamma - \pi = \int_{\triangle ABC} K d\sigma = \frac{\text{Area of spheric triangle } ABC}{R^2}$$

For the Lobachevsky plane $K = -1$ and

$$\alpha + \beta + \gamma - \pi = \int_{\triangle ABC} K d\sigma = -\left(\text{Area of spheric triangle } ABC\right)$$

7 *Let $\mathbf{X}(t)$ be parallel transport of the vector $\mathbf{X}$ along the curve on the surface M embedded in $\mathbf{E}^3$, i.e. $\nabla_{\mathbf{v}}\mathbf{X} = 0$, where $\mathbf{v}$ is a velocity vector of the curve C and ∇ Levi-Civita connection (induced connection) on the surface. Compare this condition $\nabla_{\mathbf{v}}\mathbf{X} = 0$ (for internal observer) with the condition for external observe that for the vector $\mathbf{X}(t)$ $\frac{d\mathbf{X}(t)}{dt}$ is orthogonal to the surface*[2]*.*

We know that Levi-Civita connection equals to the connection ∇: $\nabla_{\mathbf{X}}\mathbf{Y} = (\nabla^{\text{can.flat}}_{\mathbf{X}}\mathbf{Y})_{\text{tangent}}$. Show that the definitions of External and Internal Observers are equivalent.

Denote by $\mathbf{v}$ the velocity vector of curve C on the surface M. For external observer parallel transport means that $\frac{d\mathbf{X}}{dt} = \partial_{\mathbf{v}}\mathbf{X}$ is colinear to the normal vector to the surface. This is equivalent to the fact that tangential component equals to zero: $\partial_{\mathbf{v}}\mathbf{X}_{\text{tangent}} = 0$. According to definition of induced (Levi-Civita) connection this means that $\nabla_{\mathbf{v}}\mathbf{X} = 0$. Thus we prove that both observers have equivalent definitions.

8* *Let $\mathbf{r} = \mathbf{r}(t)$ be an arbitrary geodesic on the Riemannian manifold. Show that magnitudes $I = g_{ik}\dot{x}^i\dot{x}^k$ is preserved along geodesic.*

Let $\mathbf{r} = \mathbf{r}(t)$ be an arbitrary geodesic on the sphere. Show that magnitudes $I = \sin^2\theta\dot{\varphi}$ and $E = \frac{\sin^2\theta\dot{\varphi}^2 + \dot{\theta}^2}{2}$ are preserved along geodesics.

Let $\mathbf{r} = \mathbf{r}(t)$ be an arbitrary geodesic on Lobachevsky plane. Show that magnitudes $I = \frac{v_x^2}{y^2}$ and $E = \frac{v_x^2 + v_y^2}{2y^2}$ are preserved along geodesics.

There are two fundamental facts:

1. For Lagrangian $L = L(x^i, \dot{x}^i)$ the magnitude

$$E = \left(\frac{\partial L}{\partial \dot{x}^k}\dot{x}^k - L\right)$$

is preserved on equations of motions. This magnitude is called "energy". More in details this means that for any $x^i = x^i(t)$ which is the solution of Euler-Lagrange differential equations $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right) = \frac{\partial L}{\partial x^k}$ the magnitude

$$E(t) = E|_{x^i(t)} = \left(\frac{\partial L}{\partial \dot{x}^k}\dot{x}^k - L\right)|_{x^i(t)}$$

is a constant. The proof is immediate: using Euler-Lagrange equations of motions we come

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\dot{x}^k - L\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right)\dot{x}^k + \frac{\partial L}{\partial \dot{x}^k}\ddot{x}^k - \frac{dL}{dt} = \frac{\partial L}{\partial x^k}\dot{x}^k + \frac{\partial L}{\partial \dot{x}^k}\ddot{x}^k - \frac{dL}{dt} = \frac{dL}{dt} - \frac{dL}{dt} = 0\,.$$

if $x^i = x^i(t)$ are solutions of Euler-Lagrange equations.

The second fundamental fact is that if Lagrangian $L = L(x^i, \dot{x}^i)$ does no depend on variable x^m ($\frac{\partial L}{\partial x^m} = 0$) then the magnitude

$$p_m = \frac{\partial L}{\partial x^m}$$

[2] *We defined parallel transport in Geometry course using the second condition*

is preserved on equations of motions, i.e. for any $x^i = x^i(t)$ which is the solution of Euler-Lagrange differential equations $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right) = \frac{\partial L}{\partial x^k}$ the magnitude

$$p_m(t) = p_m|_{x^i(t)} = \frac{\partial L}{\partial \dot{x}^m}|_{x^i(t)}$$

is a constant. The proof is even simpler than for the first fact: using Euler-Lagrange equations of motions we come

$$\frac{dp_m(t)}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^m}\right)|_{x^i(t)} = \frac{\partial L}{\partial x^m} = 0\,.$$

The magnitude p_m is called generalised momentum (corresponding to the coordinate x^m)

Now it is very easy to solve the problems for sphere and Lobachevsky plane.

Geodesics are solutions of Euler-Lagrange equations of motion for Lagrangian $L = \frac{g_{pq}\dot{x}^p\dot{x}^q}{2}$. The energy

$$E = \left(\frac{\partial L}{\partial \dot{x}^k}\dot{x}^k - L\right) = 2L - L = L$$

is preserved. Hence the Lagrangian itself preserves on geodesics. We see that for sphere the Lagrangian $L = \frac{g_{pq}\dot{x}^p\dot{x}^q}{2} = \frac{1}{2}(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2)$ is preserved and for Lobachevsky plane the Lagrangian $L = \frac{g_{pq}\dot{x}^p\dot{x}^q}{2} = \frac{\dot{x}^2+\dot{y}^2}{2y^2}$ is preserved.

The Lagrangian of the sphere $L = \frac{1}{2}(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2)$ *does not depend* explicitly on φ, hence $I = \frac{\partial L}{\partial \dot{\varphi}} = \sin^2\theta\dot{\varphi}$ is preserved.

The Lagrangian of the Lobachevsky plane $L = \frac{\dot{x}^2+\dot{y}^2}{2y^2}$ *does not depend* explicitly on x, hence $I = \frac{\dot{x}}{y^2}$ is preserved.