Solutions of Homework 6

1

Calculate the derivatives of the functions $f = x^2+y^2$, $g = e^{-(x^2+y^2)}$ *and* $h = q\log|r| = q\log\left(\sqrt{x^2+y^2}\right)$ *(q is a constant) along vector fields* $\mathbf{A} = x\partial_x + y\partial_y$ *and* $\mathbf{B} = x\partial_y - y\partial_x$,

a) calculating directional derivatives $\partial_{\mathbf{A}}f, \partial_{\mathbf{A}}g, \partial_{\mathbf{A}}h,\ \partial_{\mathbf{B}}f, \partial_{\mathbf{B}}g, \partial_{\mathbf{B}}h$

b) calculating $df(\mathbf{A}), dg(\mathbf{A}), dh(\mathbf{A}), df(\mathbf{B}), dg(\mathbf{B}), dh(\mathbf{B})$.

$\partial_{\mathbf{A}}f = A_x\frac{\partial f}{\partial x} + A_y\frac{\partial f}{\partial y} = x\cdot 2x + y\cdot 2y = 2(x^2+y^2)$,
$\partial_{\mathbf{A}}g = A_x\frac{\partial g}{\partial x} + A_y\frac{\partial g}{\partial y} = x\cdot 2xg - y\cdot 2yg = -2(x^2+y^2)e^{-(x^2+y^2)}$
$\partial_{\mathbf{A}}h = x\frac{\partial h}{\partial x} + y\frac{\partial h}{\partial y} = \frac{x^2q}{x^2+y^2} + \frac{y^2q}{x^2+y^2} = q$
$\partial_{\mathbf{B}}f = B_x\frac{\partial f}{\partial x} + B_y\frac{\partial f}{\partial y} = -y\cdot 2x + x\cdot 2y = 0$,
$\partial_{\mathbf{B}}g = -y\frac{\partial g}{\partial x} + x\frac{\partial g}{\partial y} = -y\cdot 2xg + x\cdot 2yg = 0$
$\partial_{\mathbf{B}}h = -y\frac{\partial h}{\partial x} + x\frac{\partial h}{\partial y} = \frac{-xyq}{x^2+y^2} + \frac{xyq}{x^2+y^2} = 0$

We can do this exercise or using the formula for directional derivative or using the 1-form, differential of function: $\partial_A f = df(\mathbf{A})$. If $\mathbf{A} = A_x\partial_x + A_y\partial_y$ then

$$\partial_A f = (A_x\partial_x + A_y\partial_y)f = A_x f_x + A_y f_y \text{ or } \partial_A f = df(\mathbf{A}) = (f_x dx + f_y dy)(\mathbf{A}) = f_x dx(\mathbf{A}) + f_y dy(\mathbf{A})$$

$$= f_x dx(A_x\partial_x + A_y\partial_y) + f_y dy(A_x\partial_x + A_y\partial_y) = f_x A_x + f_y A_y\,.$$

E.g.

If $f = x^2+y^2$, then $df = 2xdx+2ydy$ and $df(\mathbf{A}) = 2xdx(\mathbf{A})+2ydy(\mathbf{A}) = 2xdx(x\partial_x+y\partial_y)+2ydy(x\partial_x+y\partial_y) = 2x\cdot x + 2y\cdot y = 2x^2+2y^2$. It is equal to $\partial_{\mathbf{A}}(x^2+y^2)$.

2

Perform the calculations of the previous exercise using polar coordinates.

For basic fields $\partial_r, \partial_\varphi$ in polar coordinates r, φ ($r = x\cos\varphi, y = r\sin\varphi$) we have that

$$\partial_r = \frac{\partial x}{\partial r}\partial_x + \frac{\partial y}{\partial r}\partial_y = \cos\varphi\partial_x + \sin\varphi\partial_y = \frac{x}{r}\partial_x + \frac{y}{r}\partial_y = \frac{x\partial_x + y\partial_y}{r} = \frac{\mathbf{A}}{r} \Rightarrow \mathbf{A} = r\partial_r$$

and

$$\partial_\varphi = \frac{\partial x}{\partial\varphi}\partial_x + \frac{\partial y}{\partial\varphi}\partial_y = -r\sin\varphi\partial_x + r\cos\varphi\partial_y = -y\partial_x + x\partial_y \Rightarrow \mathbf{B} = \partial_\varphi$$

We see that fields $\mathbf{A}, \mathbf{B}$ have very simple expression in polar coordiantes. Now calculations become almost immediate because in polar coordinates $f = r^2$, $g = e^{-r^2}$ and $h = q\log r$: $\partial_A f = r\partial_r r^2 = 2r^2$, $\partial_A g = r\partial_r e^{-r^2} = -2r^2e^{-r^2}$, $\partial_A h = r\partial_r(q\log r) = q$. For field B it is even easier , because functions f, g, h do not depend on φ; $\partial_{\mathbf{B}}f = \partial_\varphi f = 0$. Analogously $\partial_{\mathbf{B}}g = \partial_{\mathbf{B}}h = 0$.

3

Calculate the integrals of the form $\omega = \sin y\,dx$ *over the following three curves. Compare answers.*

C_1: $\mathbf{r}(t)$ $\begin{cases} x = 2t^2 - 1 \\ y = t \end{cases}$, $0 < t < 1$, C_2: $\mathbf{r}(t)$ $\begin{cases} x = 8t^2 - 1 \\ y = 2t \end{cases}$, $0 < t < 1/2$,

C_3: $\mathbf{r}(t)$ $\begin{cases} x = \cos 2t \\ y = \cos t \end{cases}$, $0 < t < \frac{\pi}{2}$

For any curve $\mathbf{r}(t), t_1 < t < t_2$

$$\int_C \omega = \int_C \sin y dx = \int_C \sin y dx(\mathbf{v}) = \int_{t_1}^{t_2} \sin y(t)\frac{dx(t)}{dt}dt$$

where $\mathbf{v} = (x_t, y_t)$.

For the first curve $x_t = 4t$ and

$$\int_{C_1} \omega = \int_0^1 4t \sin t dt = 4(-t\cos t + \sin t)\big|_0^1 = -4\cos 1 + 4\sin 1$$

For the second curve $x_t = 16t$ and

$$\int_{C_2} \omega = \int_0^{1/2} 16t \sin 2t dt = 4(-2t\cos 2t + \sin 2t)\big|_0^{1/2} = -4\cos 1 + 4\sin 1$$

Answer is the same. Non-surprising. The second curve is reparameterised first curve ($t \mapsto 2t$) and reparameterisation preserves the orientation.

For the third curve $x_t = -2\sin 2t dt$ and

$$\int_{C_3} w = \int_0^{\frac{\pi}{2}} (-2\sin 2t)\sin(\cos t) dt = -4\left(\cos t\cos(\cos t) - \sin(\cos t)\right)\big|_0^{\pi/2} = 4\cos 1 - 4\sin 1$$

Answer is the same up to a sign: This curve is reparameterised first curve ($t \mapsto \cos t$) and reparameterisation changes the orientation, because $(\cos t)' = -\sin t < 0$ on the interval $(0, \pi/2)$.

Resumé: In these three examples an integral over the same (non-parameteresed) curve was considered. All the answers are the same up to a sign. Sign changes if reparameterisation changes an orientation.

4

Calculate the integral of the form $\omega = e^{-y}dx + \sin x dy$ over the segment of straight line which connects the points $A = (1,1)$, $B = (2,3)$. At what extent an answer depends on a chosen parameterisation?

Choose any parameterisation of this segment, e.g. $x = 1 + t, y = 1 + 2t$, $0 \le t \le 1$. Then $\mathbf{v} = (v_x, v_y) = (1,2)$ $(x_t = 1, y_t = 2)$ and

$$\int_C e^{-y}dx + \sin x dy = \int_0^1 \left(e^{-(1+2t)}x_t + \sin(1+t)y_t\right) dt = \int_0^1 \left(e^{-(1+2t)} + 2\sin(1+t)\right) dt\,.$$

What happens if we choose another parameterisation, i.e. consider reparameterisation $t = t(\tau)$. Answer remains the same if the reparameterisation does not change orientation i.e. $t'(\tau) > 0$. This means that starting and ending points of the curve remain the same.

Answer is multiplied on -1 in other case: if the reparameterisation changes orientation i.e. $t'(\tau) < 0$.

5

Calculate the integral of the form $\omega = x dy$ over the following curves

a) the upper arc of the unit circle starting at the point $A = (1,0)$ and ending at the point $(0,1)$.

b) closed curve $x^2 + y^2 = 2x$

c) arc of the ellipse $x^2 + y^2/9 = 1$ defined by the condition $y \ge 0$.

a) The equation of the arc is $x^2 + y^2 = 1, y \ge 0$. We know that answer up to a sign does not depend on parameterisation. Choose an arbitrary parameterisation, e.g.

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases}, \quad 0 \le t \le \pi\,.$$

Then $\mathbf{v} = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$ and

$$\int_C w = \int_0^\pi w(\mathbf{v}) dt = \int_0^\pi x(t) y_t dt = \int_0^\pi \cos t \cos t dt = \int_0^\pi \cos^2 t dt = \pi/2$$

So for an arbitrary parameterisation answer will be $\pm\frac{\pi}{2}$. ($\frac{\pi}{2}$ if orientation is the same and $-\frac{\pi}{2}$ if opposite)

b) Consider now closed curve $x^2 + y^2 = 2x$. We have

$$0 = x^2 - 2x + y^2 = (x-1)^2 + y^2 - 1\,.$$

That is this curve is a circle of the radius 1 with a centre at the point $(1, 0)$. The parametric equation of this circle is

$$\begin{cases} x = 1 + \cos t \\ y = \sin t \end{cases}, \quad 0 \le t \le 2\pi\,.$$

In the same way
$\mathbf{v} = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$ and

$$\int_C w = \int_0^{2\pi} w(\mathbf{v})dt = \int_0^{2\pi} x(t)y_t dt = \int_0^{2\pi} \cos t \cos t dt = \int_0^{2\pi} \cos^2 t dt = \pi$$

So for an arbitrary parameterisation answer will be $\pm\pi$. (π if orientation is the same and $-\pi$ if opposite)

c) For the the arc of the ellipse $x^2 + y^2/9 = 1, y \ge 0$ choose a parameterisation: $x = \cos t, y = 3\sin t$, $0 \le t \le \pi$. Then $\mathbf{v} = (-\sin t, 3\cos t)$ and

$$\int_C w = \int_0^{\pi} w(\mathbf{v})dt = \int_0^{\pi} x(t)y_t dt = \int_0^{\pi 2} 3\cos t \cos t dt = \int_0^{\pi} 3\cos^2 t dt = 3\pi/2$$

So for an arbitrary parameterisation answer will be $\pm 3\pi/2$, sign depending of parameterisation.

Exact forms

6

Calculate the integral $\int_C \omega$ where $\omega = xdx + ydy$ and C is
a) the straight line segment $x = t, y = 1 - t$, $0 \le t \le 1$
b) the segment of parabola $x = t$, $y = 1 - t^n$, $0 \le t \le 1$, $n = 2, 3, 4, \dots$
c) **an arbitrary** *curve starting at the point $(0, 1)$ and ending at the point $((1, 0)$.*

For any of these curves we can perform calculations naively just using definition of integral
E.g. for the curve a)

$$\int_C \omega = \int_0^1 (x(t)x_t + y(t)y_t)dt = \int_0^1 (t + (1-t)(-1))dt = \int_0^1 (2t-1)dt = 0\,,$$

for the curve b) if $n = 2$

$$\int_C \omega = \int_0^1 (x(t)x_t + y(t)y_t)dt = \int_0^1 (x(t)x_t + y(t)y_t)dt = \int_0^1 (t + (1-t^2)(-2t))dt = \int_0^1 (2t^3 - 3t^2)dt = 0\,,$$

for the curve b) in general case:

$$\int_C \omega = \int_0^1 (x(t)x_t + y(t)y_t)dt = \int_0^1 (x(t)x_t + y(t)y_t)dt =$$

$$\int_0^1 (t + (1-t^n)(-nt^{n-1}))dt = \int_0^1 (t - nt^{n-1} + nt^{2n-1})dt = 0\,.$$

But we immediately come to these results in a clear and elegant way if we use the fact that $\omega = xdx+ydy$ is an **exact form**, i.e. $\omega = df$ where $f = \frac{x^2+y^2}{2}$. Indeed using Theorem we see that for an arbitrary curve starting at the point $A = (0,1)$ and ending at the point $B = (1,0)$

$$\int_C \omega = \int_C df = f(x,y)|_A^B = f(1,0) = f(0,1) = 0\,.$$

7

Show that the form 1-form $\omega = 3xydx + x^3dy$ is an exact 1-form. Calculate integral of this form over the curves considered in exercise 5).

One can see that $\omega = \omega = 3xydx + x^3dy = d(x^3y)$ $(d(x^3y) = \frac{\partial(x^3y)}{\partial x}dx + \frac{\partial(x^3y)}{\partial y}dy = 3xydx + x^3dy.)$

It is an exact form.

The integral of this form over arc of the unit circle starting at the point $A = (1,0)$ and ending at the point $(0,1)$ (see the exercise 5) is equal to $\int_C \omega = f|_B^A = f(1,0) = f(0,1) = 0$ because $f = x^2y$ and $f(1,0) = f(0,1) = 0$. Answer is equal to zero. Hence it does not depend on orientation of the curve.

Integral of this exact form over the closed circle $x^2 + +y^2 = 2x$ equals to zero, since starting and ending points coincide.

Integral of this exact form over the arc of the ellipse $x^2 + y/9 = 1, y \geq 0$ is the same as an integral over arc of the unit circle $x^2 + y^2 = 0, y > 0$. It is equal to zero, since starting and ending points coincide.

The integral of this form over ellipse (see exercise 7) is equal to zero: the integral of exact form over an arbitrary closed curve is equal to zero.

8.

Calculate the differentials of the following 1-forms:

a) xdx , b) xdy c) $xdx + ydy$, d)$xdy + ydx$, e) $xdy - ydx$

f) $x^4dy + 4x^3ydx$, g) $xdy + ydx + dz$, h) $xdy - ydx + dz$.

a) Show that 1-forms a), c), d), f) and g) are exact forms

b)Why 1-forms b), e) and and h) are not exact?

a) It is an exact form since $xdx = df$ where $f = \frac{x^2}{2} + c$, where c is a constant.

b) Suppose $\omega = xdy$ is an exact form: $\omega = df = f_xdx + f_ydy$. Hence $f_x = 0, f_y = x$. We see that $f_{xy} = \frac{\partial}{\partial x}f_y = 1$. On the other hand $f_{yx} = \frac{\partial}{\partial y}f_x = f_{xy} = 0$. Contradiction.

c) It is an exact form since $xdx + ydy = d\left(\frac{x^2+y^2}{2} + c\right)$, ($c$ is a constant).

d)It is an exact form since $xdy + ydx = d(xy + c)$,where c is a constant.

e)Suppose $\omega = xdy - ydx$ is an exact form: $\omega = df = f_xdx + f_ydy$. Hence $f_x = -y, f_y = x$. We see that $f_{xy} = 1$. On the other hand $f_{yx} = f_{xy} = -1$. Contradiction.

f) It is an exact form since $x^4dy + 4x^3ydx = d(x^4y + c)$, where c is a constant.

g) It is an exact form since $xdy + ydx + dz = d(xy + z + c)$, where c is a constant.

h) Suppose $\omega = xdy - ydx + dz$ is an exact form: $\omega = df = f_xdx + f_ydy + f_zdz$. Hence $f_x = -y, f_y = x, f_z = 1$. We see that $f_{xy} = 1$. On the other hand $f_{yx} = f_{xy} = -1$. Contradiction.

All the exercises below are not compulsory

9†

Consider one-form

$$\omega = \frac{xdy - ydx}{x^2 + y^2} \tag{1}$$

This form is defined in $\mathbf{E}^2 \setminus 0$.

Calculate differential of this form.
Write down this form in polar coordinates
Find a function f such that $\omega = df$.
Is this function defined in the same domain as ω?

First calculate differential in cartesian coordinates with "brute force"

$$d\omega = d\left(\frac{xdy - ydx}{x^2+y^2}\right) = \frac{d(xdy - ydx)}{x^2+y^2} - (xdy - ydx) \wedge d\left(\frac{1}{x^2+y^2}\right) = \frac{2dx \wedge dy}{x^2+y^2} +$$

$$\frac{(xdy - ydx) \wedge d(x^2+y^2)}{(x^2+y^2)^2} = \frac{2dx \wedge dy}{x^2+y^2} + \frac{(xdy - ydx) \wedge (2xdx + 2ydy)}{(x^2+y^2)^2} =$$

$$\frac{2dx \wedge dy}{x^2+y^2} + \frac{2x^2dy \wedge dx + 2y^2dy \wedge dx}{(x^2+y^2)^2} = 0\,.$$

Much more illuminating to write down this form in polar coordinates then calculate its differential. We know already that $xdy - ydx = r^2d\varphi$. Indeed

$dx = d(r\cos\varphi) = \cos\varphi dr - r\sin\varphi d\varphi = \frac{x}{r}dr - yd\varphi$ and $dy = d(r\sin\varphi) = \sin\varphi dr + r\cos\varphi d\varphi = \frac{y}{r}dr + xd\varphi$.
Hence

$$xdy - ydx = x\left(\frac{y}{r}dr + xd\varphi\right) - y\left(\frac{x}{r}dr - yd\varphi\right) = (x^2+y^2)d\varphi \text{ and } \frac{xdy - ydx}{x^2+y^2} = d\varphi$$

Hence the form is closed.

For the form $\omega = \frac{xdy-ydx}{x^2+y^2}$ one can consider the function $f = \varphi = \arctan\frac{y}{x}$, such that $\omega = df$, but the function f is not well-defined on whole $\mathbf{E}^2$. It is well-defined e.g. we remove the ray $(-\infty, 0]$.

Note that ω is defined in $\mathbf{E}^2 \setminus 0$, but f is defined on $\mathbf{E}^2 \setminus (-\infty, 0]$.

On the other hand it is well defined in any domain where we can define one-valued continuous function $f = \varphi$, i.e. the domain does not contain a loop which rotates around origin. (The function $f = \varphi$ is multi-valued function in the domain $\mathbf{R}^2 \setminus 0$ which contains loops rotating around origin). E.g. one can see that for an arbitrary convex domain which does not contain the origin, or for an arbitrary domain which does not contain a ray $[-\infty, 0]$ a function $f = \varphi$ is well defined one-valued function.

10†

Calculate the integral of the form $\omega = \frac{xdy-ydx}{x^2+y^2}$ over the curves
a) circle $x^2 + y^2 = 1$
b) circle $(x-3)^2 + y^2 = 1$
c) ellipse $\frac{x^2}{9} + \frac{x^2}{16} = 1$

As it follows from the previous exercise answer equals to $\pm 2\pi$ for the first curve and third curves and it is equal to zero for the second curve.

11†

What values can take the integral $\int_C \omega$ if C is an arbitrary curve starting at the point $(0,1)$ and ending at the point $((1,0)$ and $\omega = \frac{xdy-ydx}{x^2+y^2}$.

Answer is the same as in previous exercise: if the curve **does not pass the origin** then the integral is well-defined, It is equal $\frac{\pi}{2} + 2\pi n$ if starting point of the curve is $(1,0)$ and ending point is

The integer n depends on the curve.

Remark Please, note that the form $\omega = \frac{xdy-ydx}{x^2+y^2}$ strictly speaking is not exact, because it is not defined for all points (it is not defined at origin) and moreover its "antiderivative" $f = \varphi$ ($\omega = df$) is not well-defined function.

In the next exercise we show that for 1-forms which are defined in the whole $\mathbf{E}^2$ the exactness coincide with closeness.

12†

Let $\omega = a(x,y)dx + b(x,y)dy$ be a closed form in $\mathbf{E}^2$, $d\omega = 0$.
Consider the function

$$f(x,y) = x\int_0^1 a(tx,ty)dt + y\int_0^1 b(tx,ty)dt \tag{2}$$

Show that

$$\omega = df\,.$$

This proves that an arbitrary closed form in $\mathbf{E}^2$ is an exact form. (Converse implication is always true.)
Why we cannot apply the formula (2) to the form ω defined by the expression (1)?

Perform the calculations: $df = f_x d + f_y dy$.

$$f_x = \int_0^1 a(tx,ty)dt + x\int_0^1 a_x(tx,ty)tdt + y\int_0^1 b_x(tx,ty)tdt\,.$$

and

$$f_y = \int_0^1 b(tx,ty)dt + x\int_0^1 a_y(tx,ty)tdt + y\int_0^1 b_y(tx,ty)tdt\,.$$

On the other hand $d\omega = d(adx + bdy) = (b_x - a_y)dx \wedge dy = 0$. Hence $b_x = a_y$ and

$$f_x = \int_0^1 a(tx,ty)dt + x\int_0^1 a_x(tx,ty)tdt + y\int_0^1 a_y(tx,ty)tdt = \int_0^1 \left(\frac{d}{dt}\left(ta(tx,ty)\right)\right) = ta(tx,ty)\big|_0^1 = a(x,y)\,,$$

because

$$\frac{d}{dt}\left(ta(tx,ty)\right) = a(tx,ty) + xta_x(tx,ty) + yta_y(tx,ty).$$

Analogously

$$f_y = \int_0^1 b(tx,ty)dt + x\int_0^1 b_x(tx,ty)tdt + y\int_0^1 b_y(tx,ty)tdt = \int_0^1 \left(\frac{d}{dt}\left(tb(tx,ty)\right)\right) = tb(tx,ty)\big|_0^1 = b(x,y)\,,$$

We see that $f_x = a(x,y)$ and $f_y = b(x,y)$, i.e. $df = adx + bdy$ ■