

### Homework 9. Solutions.

1. Find coordinate basis vectors, first quadratic form, unit normal vector field, shape operator and Gaussian and mean curvatures for

a) sphere of the radius  $R$ :  $x^2 + y^2 + z^2 = R^2$ ,

$$\mathbf{r}(\theta, \varphi) \quad \begin{cases} x = R \sin \theta \cos \varphi \\ y = R \sin \theta \sin \varphi \\ z = R \cos \theta \end{cases} \quad (0 \leq \varphi < 2\pi, 0 \leq \theta \leq \pi),$$

b) cylinder  $x^2 + y^2 = a^2$ ,

$$\mathbf{r}(h, \varphi) \quad \begin{cases} x = a \cos \varphi \\ y = a \sin \varphi \\ z = h \end{cases} \quad (0 \leq \varphi < 2\pi, -\infty < h < \infty)$$

c) cone  $x^2 + y^2 - k^2 z^2 = 0$ ,

$$\mathbf{r}(h, \varphi) \quad \begin{cases} x = kh \cos \varphi \\ y = kh \sin \varphi \\ z = h \end{cases} \quad (0 \leq \varphi < 2\pi, -\infty < h < \infty)$$

d) graph of the function  $z = F(x, y)$

$$\mathbf{r}(u, v) \quad \begin{cases} x = u \\ y = v \\ z = uv \end{cases} \quad (-\infty < u < \infty, -\infty < v < \infty)$$

in the case if  $F(u, v) = Au^2 + 2Buv + Cv^2$

Put down the special case when  $F(u, v) = uv$  (saddle).

For the case d) you have to calculate First quadratic form, shape operator and curvatures only at origin.

Solution

a) SPHERE (of radius  $R$ )  $x^2 + y^2 + z^2 = R^2$  :

$$\mathbf{r}(\theta, \varphi) \quad \begin{cases} x = R \sin \theta \cos \varphi \\ y = R \sin \theta \sin \varphi \\ z = R \cos \theta \end{cases}$$

$$(0 \leq \varphi < 2\pi, 0 \leq \theta \leq \pi),$$

$$\mathbf{r}_\theta = \frac{\partial \mathbf{r}(\varphi, \theta)}{\partial \theta} = \begin{pmatrix} R \cos \theta \cos \varphi \\ R \cos \theta \sin \varphi \\ -R \sin \theta \end{pmatrix}, \quad \mathbf{r}_\varphi = \frac{\partial \mathbf{r}(\varphi, \theta)}{\partial \varphi} = \begin{pmatrix} -R \sin \theta \sin \varphi \\ R \sin \theta \cos \varphi \\ 0 \end{pmatrix}$$

$$\mathbf{n}(\theta, \varphi) = \frac{\mathbf{r}(\theta, \varphi)}{R} = \begin{pmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{pmatrix} \quad (1)$$

(Sometimes we denote  $\mathbf{r}_\theta$  by  $\partial_\theta$  and  $\mathbf{r}_\varphi$  by  $\partial_\varphi$ .)

Check that  $\mathbf{n}(\theta, \varphi)$  is indeed unit normal vector (in fact this is obvious from geometric considerations):

$$(\mathbf{n}, \mathbf{n}) = \sin^2 \theta (\cos^2 \varphi + \sin^2 \varphi) + \cos^2 \theta = 1,$$

$$(\mathbf{n}, \mathbf{r}_\theta) = R \sin \theta \cos \theta (\cos^2 \varphi + \sin^2 \varphi) - R \sin \theta \cos \theta = 0, \quad (\mathbf{n}, \mathbf{r}_\varphi) = R \sin^2 \theta (-\cos \varphi \sin \varphi + \cos \varphi \sin \varphi) = 0.$$

Unit normal vector is defined up to a sign;  $-\mathbf{n}$  is unit normal vector too.

Calculate now first quadratic form.  $(\mathbf{r}_\theta, \mathbf{r}_\theta) = R^2 \cos^2 \theta (\cos^2 \varphi + \sin^2 \varphi) + R^2 \sin^2 \theta = R^2$ ,  $(\mathbf{r}_\theta, \mathbf{r}_\varphi) = 0$ ,  $(\mathbf{r}_\varphi, \mathbf{r}_\varphi) = R^2 \sin^2 \theta (\sin^2 \varphi + \cos^2 \varphi) = R^2 \sin^2 \theta$ . Thus

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_\theta, \mathbf{r}_\theta) & (\mathbf{r}_\theta, \mathbf{r}_\varphi) \\ (\mathbf{r}_\varphi, \mathbf{r}_\theta) & (\mathbf{r}_\varphi, \mathbf{r}_\varphi) \end{pmatrix} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2 \theta \end{pmatrix}$$

$$dl^2 = G_{11}d\theta^2 + 2G_{12}d\theta d\varphi + G_{22}d\varphi^2 = R^2 d\theta^2 + R^2 \sin^2 \theta d\varphi^2$$

The length of the curve  $\mathbf{r}(t) = \mathbf{r}(\theta(t), \varphi(t))$  with  $\theta = \theta(t), \varphi = \varphi(t), t_1 \leq t \leq t_2$  is given by the integral:

$$\int_{t_1}^{t_2} |\mathbf{v}(t)| dt = \int_{t_1}^{t_2} \sqrt{G_{11}\dot{\theta}^2 + 2G_{12}\dot{\theta}\dot{\varphi} + G_{22}\dot{\varphi}^2} dt = \int_{t_1}^{t_2} \sqrt{R^2\dot{\theta}^2 + R^2 \sin^2 \theta \dot{\varphi}^2} dt \quad (1b)$$

Now calculate shape operator and Gaussian and mean curvatures for sphere:

By the definition (see lecture notes) the action of shape operator on any tangent vector  $\mathbf{v}$  is given by the formula  $S\mathbf{v} = -\partial_{\mathbf{v}}\mathbf{n}$ . We know that for sphere  $\mathbf{n} = \frac{\mathbf{r}}{R}$  (see the equations (1) above). Hence for basis vectors  $\mathbf{r}_\theta = \partial_\theta, \mathbf{r}_\varphi = \partial_\varphi$  we have

$$S\mathbf{r}_\theta = -\partial_\theta \mathbf{n}(\theta, \varphi) = -\partial_\theta \left( \frac{\mathbf{r}(\theta, \varphi)}{R} \right) = - \left( \frac{\partial_\theta \mathbf{r}(\theta, \varphi)}{R} \right) = -\frac{\mathbf{r}_\theta}{R}$$

and

$$S\mathbf{r}_\varphi = -\partial_\varphi \mathbf{n}(\theta, \varphi) = -\partial_\varphi \left( \frac{\mathbf{r}(\theta, \varphi)}{R} \right) = - \left( \frac{\partial_\varphi \mathbf{r}(\theta, \varphi)}{R} \right) = -\frac{\mathbf{r}_\varphi}{R}$$

We see that shape operator is equal to  $S = -\frac{I}{R}$ , where  $I$  is an identity operator. Its matrix in the basis  $\partial_\theta, \partial_\varphi$  is equal to

$$- \begin{pmatrix} \frac{1}{R} & 0 \\ 0 & \frac{1}{R} \end{pmatrix}.$$

In the case if we choose the opposite direction for unit normal vector then we will come to the same answer just with changing the signs: if  $\mathbf{n} \rightarrow -\mathbf{n}$ ,  $S \rightarrow -S$ .

We see that principal curvatures, i.e. eigenvalues of shape operator are the same:

$$\lambda_1 = \lambda_2 = -\frac{1}{R}, \text{ i.e. } \kappa_1 = \kappa_2 = -\frac{1}{R}$$

(if we choose the opposite sign for  $\mathbf{n}$  then  $\kappa_1 = \kappa_2 = \frac{1}{R}$ ). Thus we can calculate Gaussian and mean curvature: Gaussian curvature

$$K = \kappa_1 \cdot \kappa_2 = \det S = \frac{1}{R^2}.$$

Mean curvature

$$H = \kappa_1 + \kappa_2 = \text{Tr } S = -\frac{2}{R}.$$

If we choose the opposite sign for  $\mathbf{n}$  then  $S \rightarrow -S$ , principal curvatures change the sign, Gaussian curvature  $K = \kappa_1 \cdot \kappa_2$  does not change but mean curvature  $H = \kappa_1 + \kappa_2$  will change the sign: if  $\mathbf{n} \rightarrow -\mathbf{n}$  then  $H = \frac{2}{R}$ .

b) *CYLINDER*  $x^2 + y^2 = a^2$

$$\mathbf{r}(h, \varphi) = \begin{cases} x = a \cos \varphi \\ y = a \sin \varphi \\ z = h \end{cases} \quad (0 \leq \varphi < 2\pi, -\infty < h < \infty)$$

$$\mathbf{r}_\varphi|_{\varphi, h} = \frac{\partial \mathbf{r}(\varphi, h)}{\partial \varphi} = \begin{pmatrix} -a \sin \varphi \\ a \cos \varphi \\ 0 \end{pmatrix}, \quad \mathbf{r}_h|_{\varphi, h} = \frac{\partial \mathbf{r}(\varphi, h)}{\partial h} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{n}(\varphi, h) = \begin{pmatrix} \cos \varphi \\ \sin \varphi \\ 0 \end{pmatrix} \quad (2)$$

Sometimes we denote  $\mathbf{r}_\varphi$  by  $\partial_\varphi$  and  $\mathbf{r}_h$  by  $\partial_h$ .

Check that  $\mathbf{n}(\varphi, h)$  is indeed unit normal vector:

$$(\mathbf{n}, \mathbf{n}) = \cos^2 \varphi + \sin^2 \varphi = 1, \quad (\mathbf{n}, \mathbf{r}_\varphi) = a \cos \varphi \sin \varphi (-1 + 1) = 0, \quad (\mathbf{n}, \mathbf{r}_h) = 0$$

Unit normal vector is defined up to a sign;  $-\mathbf{n}$  is unit normal vector too.

Calculate now first quadratic form.  $(\mathbf{r}_\varphi, \mathbf{r}_\varphi) = R^2(\sin^2 \varphi + \cos^2 \varphi) = R^2$ ,  $(\mathbf{r}_\varphi, \mathbf{r}_h) = 0$ ,  $(\mathbf{r}_h, \mathbf{r}_h) = 1$ . Thus

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} G_{hh} & G_{h\varphi} \\ G_{\varphi h} & G_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_h, \mathbf{r}_h) & (\mathbf{r}_h, \mathbf{r}_\varphi) \\ (\mathbf{r}_\varphi, \mathbf{r}_\varphi) & (\mathbf{r}_\varphi, \mathbf{r}_\varphi) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a^2 \end{pmatrix}$$

$$dl^2 = G_{11}dh^2 + 2G_{12}dh d\varphi + G_{22}d\varphi^2 = dh^2 + a^2 d\varphi^2.$$

The length of the curve  $\mathbf{r}(t) = \mathbf{r}(\varphi(t), h(t))$  with  $\varphi = \varphi(t), h = h(t), t_1 \leq t \leq t_2$  is given by the integral:

$$\int_{t_1}^{t_2} \sqrt{G_{11}\dot{h}^2 + 2G_{12}\dot{\varphi}\dot{h} + G_{22}\dot{\varphi}^2} dt = \int_{t_1}^{t_2} \sqrt{\dot{h}^2 + a^2\dot{\varphi}^2} dt, \quad (2b)$$

Now calculate shape operator Gaussian and mean curvatures for cylinder.

To calculate the shape operator for the cylinder we use results of calculations above of vectors  $\mathbf{r}_h, \mathbf{r}_\varphi$  and of unit normal vector  $\mathbf{n}(\varphi, h)$  (see the equations (2) above). By the definition the action of shape operator on any tangent vector  $\mathbf{v}$  is given by the formula  $S\mathbf{v} = -\partial_{\mathbf{v}}\mathbf{n}$ . Hence for basis vectors  $\mathbf{r}_\varphi = \partial_\varphi, \mathbf{r}_h = \partial_h$  we have

$$S\mathbf{r}_h = -\partial_h\mathbf{n}(\varphi, h) = -\partial_h \begin{pmatrix} \cos \varphi \\ \sin \varphi \\ 0 \end{pmatrix} = 0$$

and

$$S\mathbf{r}_\varphi = -\partial_\varphi\mathbf{n}(\varphi, h) = -\partial_\varphi \begin{pmatrix} \cos \varphi \\ \sin \varphi \\ 0 \end{pmatrix} = \begin{pmatrix} \sin \varphi \\ -\cos \varphi \\ 0 \end{pmatrix} = -\frac{\mathbf{r}_\varphi}{a}$$

(Recall that  $\mathbf{n}(h, \varphi) = \begin{pmatrix} \cos \varphi \\ \sin \varphi \\ 0 \end{pmatrix}$  and  $\mathbf{r}_\varphi = \begin{pmatrix} -a \sin \varphi \\ a \cos \varphi \\ 0 \end{pmatrix}$  (See the equations (2) above.)

For an arbitrary tangent vector  $\mathbf{X} = a\mathbf{r}_h + b\mathbf{r}_\varphi$ ,  $S\mathbf{X} = -\frac{b\mathbf{r}_\varphi}{a}$ . Shape operator transforms tangent vectors to tangent vectors. Its matrix in the basis  $\mathbf{r}_h, \mathbf{r}_\varphi$  equals to

$$-\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{a} \end{pmatrix}$$

In the case if we choose the opposite direction for unit normal vector then we will come to the same answer just with changing the signs: if  $\mathbf{n} \rightarrow -\mathbf{n}$ ,  $S \rightarrow -S$ .

We see that principal curvatures, i.e. eigenvalues of shape operator are:

$$\kappa_1 = 0, \kappa_2 = -\frac{1}{a}$$

(if we choose the opposite sign for  $\mathbf{n}$  then  $\kappa_1 = \kappa_2 = \frac{1}{R}$ ). Thus we can calculate Gaussian and mean curvature: Gaussian curvature

$$K = \kappa_1 \cdot \kappa_2 = \det S = 0.$$

Mean curvature

$$H = \kappa_1 + \kappa_2 = \text{Tr } S = -\frac{1}{a}.$$

If we choose the opposite sign for  $\mathbf{n}$  then  $S \rightarrow -S$ , principal curvatures change the sign, Gaussian curvature  $K = \kappa_1 \cdot \kappa_2$  remains the same but mean curvature  $H = \kappa_1 + \kappa_2$  will change the sign: if  $\mathbf{n} \rightarrow -\mathbf{n}$  then  $H = \frac{1}{a}$ .

$$b) \text{ CONE } x^2 + y^2 - k^2 z^2 = 0$$

$$\mathbf{r}(h, \varphi) = \begin{cases} x = kh \cos \varphi \\ y = kh \sin \varphi \\ z = h \end{cases} \quad (0 \leq \varphi < 2\pi, -\infty < h < \infty) \quad (3)$$

$$\mathbf{r}_h|_{\varphi, h} = \frac{\partial \mathbf{r}(\varphi, h)}{\partial h} = \begin{pmatrix} k \cos \varphi \\ k \sin \varphi \\ 1 \end{pmatrix}, \quad \mathbf{r}_\varphi|_{\varphi, h} = \frac{\partial \mathbf{r}(\varphi, h)}{\partial \varphi} = \begin{pmatrix} -kh \sin \varphi \\ kh \cos \varphi \\ 0 \end{pmatrix}.$$

Sometimes we denote  $\mathbf{r}_\varphi$  by  $\partial_\varphi$  and  $\mathbf{r}_h$  by  $\partial_h$ .

To calculate the normal unit vector field  $\mathbf{n}(h, \varphi)$  note that the vector  $\mathbf{N}(h, \varphi) = \begin{pmatrix} \cos \varphi \\ \sin \varphi \\ -k \end{pmatrix}$  is orthogonal to the surface of the cone:  $(\mathbf{N} r_h) = (\mathbf{N}, \mathbf{r}_\varphi) = 0$  and its length equals to  $|\mathbf{N}| = \sqrt{k^2 + 1}$ . Hence normal unit vector field equals to

$$\mathbf{n}(h, \varphi) = \frac{\mathbf{N}(h, \varphi)}{\sqrt{k^2 + 1}} = \frac{1}{\sqrt{k^2 + 1}} \begin{pmatrix} \cos \varphi \\ \sin \varphi \\ -k \end{pmatrix}$$

It is indeed normal unit vector field:  $(\mathbf{n}, \mathbf{n}) = \frac{\cos^2 \varphi}{k^2 + 1} + \frac{\sin^2 \varphi}{k^2 + 1} + \frac{k^2}{k^2 + 1} = 1$ ,  $(\mathbf{n}, \mathbf{r}_\varphi) = \frac{1}{\sqrt{k^2 + 1}}(\cos \varphi \cdot (-kh \sin \varphi) + \sin \varphi \cdot (kh \cos \varphi)) = 0$ , and  $(\mathbf{n}, \mathbf{r}_h) = \frac{1}{\sqrt{k^2 + 1}}(\cos \varphi \cdot (kh \cos \varphi) + \sin \varphi \cdot k \sin \varphi - k) = 0$ . Unit normal vector is defined up to a sign;  $-\mathbf{n}$  is unit normal vector too.

Calculate now first quadratic form.  $(\mathbf{r}_h, \mathbf{r}_h) = k^2 \cos^2 \varphi + k^2 \sin^2 \varphi + 1 = k^2 + 1$ ,  $(\mathbf{r}_h, \mathbf{r}_\varphi) = (\mathbf{r}_\varphi, \mathbf{r}_h) = k^2 h \cos \varphi (-\sin \varphi) + k^2 h \sin \varphi \cos \varphi = 0$ ,  $(\mathbf{r}_\varphi, \mathbf{r}_\varphi) = k^2 h^2 \sin^2 \varphi + k^2 h^2 \cos^2 \varphi = k^2 h^2$ , Thus

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} G_{hh} & G_{h\varphi} \\ G_{\varphi h} & G_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_h, \mathbf{r}_h) & (\mathbf{r}_h, \mathbf{r}_\varphi) \\ (\mathbf{r}_\varphi, \mathbf{r}_h) & (\mathbf{r}_\varphi, \mathbf{r}_\varphi) \end{pmatrix} = \begin{pmatrix} k^2 + 1 & 0 \\ 0 & k^2 h^2 \end{pmatrix}$$

$$dl^2 = G_{hh} dh^2 + 2G_{h\varphi} dh d\varphi + G_{\varphi\varphi} d\varphi^2 = (k^2 + 1) dh^2 + k^2 h^2 d\varphi^2 R^2,$$

The length of the curve  $\mathbf{r}(t) = \mathbf{r}(h(t), \varphi(t))$  with  $\varphi = \varphi(t)$ ,  $h = h(t)$ ,  $t_1 \leq t \leq t_2$  is given by the integral:

$$\int_{t_1}^{t_2} \sqrt{G_{11} \dot{h}^2 + 2G_{12} \dot{h} \dot{\varphi} + G_{22} \dot{\varphi}^2} dt = \int_{t_1}^{t_2} \sqrt{(k^2 + 1) \dot{h}^2 + k^2 h(t)^2 \dot{\varphi}^2} dt. \quad (3b)$$

To calculate the shape operator for the cone we use the results of calculations of vectors  $\mathbf{r}_h, \mathbf{r}_\varphi$  and of unit normal vector  $\mathbf{n}(\varphi, h)$  (see the equations (3) above.) By the definition the action of shape operator on any tangent vector  $\mathbf{v}$  is given by the formula  $S\mathbf{v} = -\partial_{\mathbf{v}} S$ . Hence for basis vectors  $\mathbf{r}_h = \partial_h, \mathbf{r}_\varphi = \partial_\varphi$

$$S\mathbf{r}_h = -\partial_h \mathbf{n}(\varphi, h) = -\frac{1}{\sqrt{k^2 + 1}} \partial_h \begin{pmatrix} \cos \varphi \\ \sin \varphi \\ -k \end{pmatrix} = 0$$

and

$$S\mathbf{r}_\varphi = -\partial_\varphi \mathbf{n}(\varphi, h) = -\frac{1}{\sqrt{k^2 + 1}} \partial_\varphi \begin{pmatrix} \cos \varphi \\ \sin \varphi \\ -k \end{pmatrix} = \frac{1}{\sqrt{k^2 + 1}} \begin{pmatrix} -\sin \varphi \\ \cos \varphi \\ 0 \end{pmatrix} = -\frac{1}{k\sqrt{k^2 + 1}} \frac{\mathbf{r}_\varphi}{h} \text{ since } \mathbf{r}_\varphi = \begin{pmatrix} -kh \sin \varphi \\ kh \cos \varphi \\ 0 \end{pmatrix}.$$

We see that for an arbitrary tangent vector  $\mathbf{X} = a\mathbf{r}_h + b\mathbf{r}_\varphi$   $S\mathbf{X} = S(a\mathbf{r}_h + b\mathbf{r}_\varphi) = -\frac{b}{kh\sqrt{k^2 + 1}} \mathbf{r}_\varphi$ . Shape operator transforms tangent vectors to tangent vectors. Its matrix in the basis  $\mathbf{r}_h, \mathbf{r}_\varphi$  equals to

$$-\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{kh\sqrt{1+k^2}} \end{pmatrix}$$

In the case if we choose the opposite direction for unit normal vector then we will come to the same answer just with changing the signs: if  $\mathbf{n} \rightarrow -\mathbf{n}$ ,  $S \rightarrow -S$ . We see that principal curvatures, i.e. eigenvalues of shape operator are:

$$\kappa_1 = 0, \kappa_2 = -\frac{1}{hk\sqrt{1+k^2}}$$

(if we choose the opposite sign for  $\mathbf{n}$  then  $\kappa_1 = \kappa_2 = \frac{1}{hk\sqrt{1+k^2}}$ ). Thus we can calculate Gaussian and mean curvature: Gaussian curvature

$$K = \kappa_1 \cdot \kappa_2 = \det S = 0.$$

Mean curvature

$$H = \kappa_1 + \kappa_2 = \text{Tr } S = -\frac{1}{hk\sqrt{1+k^2}}.$$

If we choose the opposite sign for  $\mathbf{n}$  then  $S \rightarrow -S$ , principal curvatures change the sign, Gaussian curvature  $K = \kappa_+ \cdot \kappa_-$  remains the same but mean curvature  $H = \kappa_+ + \kappa_-$  will change the sign: if  $\mathbf{n} \rightarrow -\mathbf{n}$  then  $H = \frac{1}{hk\sqrt{1+k^2}}$ .

d) GRAPH OF THE FUNCTION  $z = F(x, y)$

$$\mathbf{r}(u, v) = \begin{cases} x = u \\ y = v \\ z = F(u, v) \end{cases} \quad (-\infty < u < \infty, -\infty < v < \infty) \quad (4)$$

in the case if  $F(u, v) = Au^2 + 2Buv + Cv^2$

$$\begin{aligned} \mathbf{r}_u|_{u,v} &= \frac{\partial \mathbf{r}(u, v)}{\partial u} = \begin{pmatrix} 1 \\ 0 \\ F_u \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2Au + 2Bv \end{pmatrix}, \quad \mathbf{r}_u|_{u=v=0} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \\ \mathbf{r}_v|_{u,v} &= \frac{\partial \mathbf{r}(u, v)}{\partial v} = \begin{pmatrix} 0 \\ 1 \\ F_v \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2Bu + 2Cv \end{pmatrix}, \quad \mathbf{r}_v|_{u=v=0} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \\ \mathbf{n}(u, v) &= \frac{1}{\sqrt{1+F_u^2+F_v^2}} \begin{pmatrix} -F_u \\ -F_v \\ 1 \end{pmatrix}, \quad \mathbf{n}(u, v)|_{u=v=0} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

Sometimes we denote  $\mathbf{r}_u$  by  $\partial_u$  and  $\mathbf{r}_v$  by  $\partial_v$ .

Check that  $\mathbf{n}(u, v)$  is indeed unit normal vector:  $(\mathbf{n}, \mathbf{n}) = \frac{1}{1+F_u^2+F_v^2}(F_u^2 + F_v^2 + 1) = 1$ ,  $(\mathbf{n}, \mathbf{r}_u) = \frac{1}{\sqrt{1+F_u^2+F_v^2}}(F_u - F_u) = 0$ ,  $(\mathbf{n}, \mathbf{r}_v) = \frac{1}{\sqrt{1+F_u^2+F_v^2}}(F_v - F_v) = 0$ . Calculate now first quadratic form.  $(\mathbf{r}_u, \mathbf{r}_u) = 1 + F_u^2$ ,  $(\mathbf{r}_u, \mathbf{r}_v) = F_u F_v$ ,  $(\mathbf{r}_v, \mathbf{r}_v) = 1 + F_v^2$ . Thus

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_u, \mathbf{r}_u) & (\mathbf{r}_u, \mathbf{r}_v) \\ (\mathbf{r}_v, \mathbf{r}_u) & (\mathbf{r}_v, \mathbf{r}_v) \end{pmatrix} = \begin{pmatrix} 1 + F_u^2 & F_u F_v \\ F_u F_v & 1 + F_v^2 \end{pmatrix}$$

$$dl^2 = G_{11}d\varphi^2 + 2G_{12}d\varphi dh + G_{22}dh^2 = (1 + F_u^2)du^2 + 2F_u F_v dudv + (1 + F_v^2)dv^2$$

At the point  $u = v = 0$ ,  $F_u = F_v = 0$  and

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_u, \mathbf{r}_u) & (\mathbf{r}_u, \mathbf{r}_v) \\ (\mathbf{r}_v, \mathbf{r}_u) & (\mathbf{r}_v, \mathbf{r}_v) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad dl^2 = du^2 + dv^2$$

The length of the curve  $\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$  with  $u = u(t), v = v(t)$  can be calculated by the integral:

$$\int_{t_1}^{t_2} \sqrt{G_{11}\dot{u}^2 + 2G_{12}\dot{u}\dot{v} + G_{22}\dot{v}^2} dt = \int_{t_1}^{t_2} \sqrt{(1 + F_u^2)\dot{u}^2 + 2F_u F_v \dot{u}\dot{v} + (1 + F_v^2)\dot{v}^2} dt \quad (4a)$$

Special case of saddle: In the special case of saddle we just take  $F = auv$  in previous formulae. In particular normal for normal unit vector we have

$$\mathbf{n}(u, v) = \frac{1}{\sqrt{1 + a^2 v^2 + a^2 u^2}} \begin{pmatrix} -av \\ -au \\ 1 \end{pmatrix}, \quad \mathbf{n}(u, v)|_{u=v=0} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and first quadratic form is equal to

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} (\mathbf{r}_u, \mathbf{r}_u) & (\mathbf{r}_u, \mathbf{r}_v) \\ (\mathbf{r}_v, \mathbf{r}_u) & (\mathbf{r}_v, \mathbf{r}_v) \end{pmatrix} = \begin{pmatrix} 1 + F_u^2 & F_u F_v \\ F_u F_v & 1 + F_v^2 \end{pmatrix} = \begin{pmatrix} 1 + v^2 & vu \\ vu & 1 + u^2 \end{pmatrix},$$

$$dl^2 = G_{11}d\varphi^2 + 2G_{12}d\varphi dh + G_{22}dh^2 = (1 + a^2 v^2)du^2 + 2a^2 uv du dv + (1 + a^2 u^2)dv^2.$$

To calculate shape operator we use results of calculations vectors  $\mathbf{r}_u, \mathbf{r}_v$  and for unit normal vector  $\mathbf{n}(u, v)$ . We do calculations only at origin. For basic vectors  $\mathbf{r}_u = \partial_u, \mathbf{r}_v = \partial_v$  we have

$$S\mathbf{r}_u = -\frac{\partial \mathbf{n}(u, v)}{\partial u} \Big|_{u=v=0} = -\partial_u \left( \frac{1}{\sqrt{1 + F_u^2 + F_v^2}} \begin{pmatrix} -F_u \\ -F_v \\ 1 \end{pmatrix} \right) \Big|_{u=v=0} =$$

$$\left( \frac{1}{\sqrt{1 + F_u^2 + F_v^2}} \right) \Big|_{u=v=0} \begin{pmatrix} F_{uu} \\ F_{uv} \\ 1 \end{pmatrix} \Big|_{u=v=0} = \begin{pmatrix} 2A \\ 2B \\ 0 \end{pmatrix} = 2A\mathbf{r}_u + 2B\mathbf{r}_v$$

and  $S\mathbf{r}_v = -\partial_v (\mathbf{n}(u, v))|_{u=v=0} =$

$$-\partial_v \left( \frac{1}{\sqrt{1 + F_u^2 + F_v^2}} \begin{pmatrix} -F_u \\ -F_v \\ 1 \end{pmatrix} \right) \Big|_{u=v=0} = \left( \frac{1}{\sqrt{1 + F_u^2 + F_v^2}} \right) \Big|_{u=v=0} \begin{pmatrix} F_{vu} \\ F_{vv} \\ 1 \end{pmatrix} \Big|_{u=v=0} = \begin{pmatrix} 2B \\ 2C \\ 0 \end{pmatrix} = 2B\mathbf{r}_u + 2C\mathbf{r}_v$$

The matrix of the shape operator in the basis  $\mathbf{r}_u, \mathbf{r}_v$  is  $\begin{pmatrix} 2A & 2B \\ 2B & 2C \end{pmatrix}$ . Gaussian curvature at origin is equal to  $\det S = 4AC - 4B^2$  and mean curvature is equal to  $H = \text{tr} S = 2A + 2C$ . (Mean curvature as always is defined up to a sign.)

In the case of saddle  $F = auv$ , i.e.  $A = C = 0, B = \frac{a}{2}$ . The shape operator at the point  $u = v = 0$  equals to

$$\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}$$

Gaussian curvature  $K = k_- \cdot k_+ = \det S = -a^2$  and mean curvature  $H = (k_- + k_+) = \text{Tr} S = 0$ .

**2** Consider surface defined by equation  $z - Ax^2 - Ay^2 = 0$ . (See the exercise 1d) above.) Show that this is a saddle: you have to show that under the rotation on the angle  $\varphi = \frac{\pi}{4}$  with respect to  $z$ -axis it becomes a surface  $z - axy = 0$ . Find relation between parameters  $A$  and  $a$ .

Matrix of rotation on the angle  $\frac{\pi}{4}$  (in the basis of vectors  $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ ) directed along  $x, y$  and  $z$  axis is

$$P = \begin{pmatrix} \cos \frac{\pi}{4} & -\sin \frac{\pi}{4} & 0 \\ \sin \frac{\pi}{4} & \cos \frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Under the rotation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow P \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{x-y}{\sqrt{2}} \\ \frac{x+y}{\sqrt{2}} \\ z \end{pmatrix}$$

Hence  $Ax^2 - Ay^2 \rightarrow A\left(\frac{x-y}{\sqrt{2}}\right)^2 - A\left(\frac{x+y}{\sqrt{2}}\right)^2 = -2Axy$ ,  $a = -2A$ .

**3** Show that there are two straight lines which pass through the point  $(3, 4, 12)$  on the saddle  $z = xy$  and lie on this saddle.

<sup>†</sup> Show that this is true for an arbitrary point of the saddle.

Let  $\mathbf{r}_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$  be an arbitrary point on the saddle:  $z_0 = x_0 y_0$ . Consider the following two lines:

the line  $\mathbf{r} = \mathbf{r}_0 + t\mathbf{a}$ , where  $\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ y_0 \end{pmatrix}$ , i.e.  $\begin{cases} x = x_0 + t \\ y = y_0 \\ z = z_0 + y_0 t \end{cases}$

and the line  $\mathbf{r} = \mathbf{r}_0 + t\mathbf{b}$ , where  $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ x_0 \end{pmatrix}$ , i.e.  $\begin{cases} x = x_0 \\ y = y_0 + t \\ z = z_0 + x_0 t \end{cases}$

It is easy to check that these both lines belong to the saddle:  $xy = (x_0 + t)y_0 = z_0 + ty_0 = z$  and  $xy = x_0(y_0 + t) = z_0 + x_0 t$ . On the other hand it is easy to see that it is all: If  $\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$  is an arbitrary straight line on the saddle passing through the point  $(x_0, y_0, z_0)$ ,  $x_0 y_0 = z_0$  then

$$xy = (x_0 + at)(y_0 + bt) = z = z_0 + ct \text{ for all } t$$

Hence  $ab = 0$ . Thus  $a = 0$  or  $b = 0$ . We see that through the an arbitrary point on the saddle pass exactly two straight lines.

**4** Consider helix  $\mathbf{r}(t)$ :  $\begin{cases} x(t) = a \cos t \\ y(t) = a \sin t \\ z(t) = ct \end{cases}$ . Show that this helix belongs to cylinder surface  $x^2 + y^2 = a^2$ .

Using first quadratic form on the surface of cylindre or in a different way a) calculate length of the helix ( $0 \leq t \leq t_0$ ).

b) what are relations between principal curvatures of cylinder and curvature of helix?

*Solution* This helix belongs to cylinder surface  $x^2 + y^2 = a^2$  because  $x^2 + y^2 = a^2$  on the points of the helix.

For the helix internal coordinates are  $\varphi = \varphi(t) = t$  and  $h = h(t) = ct$  ( $x = R \cos \varphi, y = R \sin \varphi, z = h$ ). Use First Quadratic form which we obtained in the previous exercise (see equation (2b)). We come to

$$L = \int_0^{t_0} \sqrt{G_{11}\dot{h}^2 + 2G_{12}\dot{\varphi}\dot{h} + G_{22}\dot{\varphi}^2} dt = \int_0^{t_0} \sqrt{a^2\dot{\varphi}^2 + \dot{h}^2} dt = \int_0^{t_0} \sqrt{a^2 + c^2} dt = t_0 \sqrt{a^2 + c^2}$$

Of course the answer can be obtained without integration: speed is constant, hence  $L = |\mathbf{v}|t = t\sqrt{a^2 + c^2}$ . This is the calculations of the Internal observer. The external observer will calculate using the coordinates  $x, y, z$ :  $|\mathbf{v}| = \sqrt{x_t^2 + y_t^2 + z_t^2} = (a^2 \cos^2 t + a^2 \sin^2 t + c^2) = \sqrt{a^2 + c^2}$  and will come to the same answer.

b) Note that curvature of the helix equals to  $k = \frac{a}{a^2 + c^2}$ . Curvatur of helix varies between zero and  $\frac{1}{a}$ , i.e. principal curvatures of cylinder surface:

$$0 \leq \frac{a}{a^2 + c^2} \leq \frac{1}{a}.$$

**5** Assume that the action of the shape operator at the tangent coordinate vectors  $\mathbf{r}_u = \partial_u$ ,  $\mathbf{r}_v = \partial_v$  at the given point  $\mathbf{p}$  of the surface  $\mathbf{r} = \mathbf{r}(u, v)$  is defined by the relations:  $S(\partial_u) = 2\partial_u + 2\partial_v$  and  $S(\partial_v) = -\partial_u + 5\partial_v$ . Calculate principal curvatures, Gaussian and mean curvatures of the surface at this point.

*Solution* We see that the matrix of the shape operator in the basis  $\partial_u, \partial_v$  is equal to

$$S = \begin{pmatrix} 2 & -1 \\ 2 & 5 \end{pmatrix}$$

Hence Gaussian curvature  $K = \det S = 12$  and mean curvature  $H = \text{Tr } S = 7$ . To calculate principal curvatures  $k_1, k_2$  note that

$$\begin{cases} k_1 + k_2 = H = 7 \\ k_1 \cdot k_2 = K = 12 \end{cases}$$

Hence  $k_1 = 3, k_2 = 4$ ;  $k_1, k_2$  are eigenvalues of the shape operator.

(We assume that  $a \geq 0$ ).