

Homework 6. Solutions.

Christoffel symbols and Lagrangians

1 Consider the Lagrangian of "free" particle $L = \frac{1}{2}g_{ik}\dot{x}^i\dot{x}^k$ for Riemannian manifold with a metric $G = g_{ik}dx^i dx^k$.

Write down Euler-Lagrange equations of motion for this Lagrangian and compare them with differential equations for geodesics on this Riemannian manifold.

In fact show that

$$\underbrace{\frac{\partial L}{\partial x^i} = \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i}}_{\text{Euler-Lagrange equations}} \Leftrightarrow \underbrace{\frac{d^2 x^i}{dt^2} + \Gamma_{km}^i \dot{x}^k \dot{x}^m = 0}_{\text{Equations for geodesics}}, \quad (1)$$

where

$$\Gamma_{km}^i = \frac{1}{2}g^{ij} \left(\frac{\partial g_{jk}}{\partial x^m} + \frac{\partial g_{jm}}{\partial x^k} - \frac{\partial g_{km}}{\partial x^j} \right). \quad (2)$$

Solution: see the lecture notes.

2 a) Write down the Lagrangian of "free" particle $L = \frac{1}{2}g_{ik}\dot{x}^i\dot{x}^k$ for Euclidean plane in polar coordinates. Calculate Christoffel symbols for canonical flat connection in polar coordinates using Euler-Lagrange equations for this Lagrangian. Compare with answers which you obtained by the direct use of the formula (2). b) Do the same for cylindrical coordinates in \mathbf{E}^3 .

Solution. Canonical flat connection is Levi-Civita connection of Euclidean metric $G = dx^2 + dy^2$. Hence we can calculate Christoffel symbols using Lagrangian method.

Euclidean metric in polar coordinates is $dr^2 + r^2 d\varphi^2$. Hence the Lagrangian of the free particle is

$$L = \frac{\dot{r}^2 + r^2 \dot{\varphi}^2}{2}$$

Euler-Lagrange equations:

1) for r :

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) = \ddot{r} = \frac{\partial L}{\partial r} = r\dot{\varphi}^2$$

i.e.

$$\ddot{r} - r\dot{\varphi}^2 = 0 \Rightarrow \Gamma_{rr}^r = \Gamma_{\varphi r}^r = \Gamma_{r\varphi}^r = 0, \Gamma_{\varphi\varphi}^r = -r.$$

2) for φ :

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\varphi}} \right) = \frac{d}{dt} (r^2 \dot{\varphi}) = r^2 \ddot{\varphi} + 2r\dot{r}\dot{\varphi} = \frac{\partial L}{\partial \varphi} = 0,$$

i.e.

$$\ddot{\varphi} + \frac{2}{r}\dot{r}\dot{\varphi} = 0 \Rightarrow \Gamma_{rr}^\varphi = \Gamma_{\varphi\varphi}^\varphi = 0, \Gamma_{r\varphi}^\varphi = \Gamma_{\varphi r}^\varphi = \frac{1}{r}.$$

b) cylindrical coordinates in \mathbf{E}^3 . Calculations almost the same as for polar coordinates in \mathbf{E}^2 . $G = dr^2 + r^2 d\varphi^2 + dh^2$,

$$L = \frac{\dot{r}^2 + r^2 \dot{\varphi}^2 + \dot{h}^2}{2}$$

for r : $\ddot{r} - r\dot{\varphi}^2 = 0 \Rightarrow$

$$\Gamma_{rr}^r = \Gamma_{\varphi r}^r = \Gamma_{r\varphi}^r = \Gamma_{rh}^r = \Gamma_{hr}^r = \Gamma_{h\varphi}^r = \Gamma_{\varphi h}^r = \Gamma_{hh}^r = 0, \Gamma_{\varphi\varphi}^r = -r.$$

for φ , $r^2\ddot{\varphi} + 2r\dot{r}\dot{\varphi} = \frac{\partial L}{\partial \varphi} = 0$, i.e. $\ddot{\varphi} + \frac{2}{r}\dot{r}\dot{\varphi} = 0 \Rightarrow$

$$\Gamma_{rr}^\varphi = \Gamma_{rh}^\varphi = \Gamma_{hr}^\varphi = \Gamma_{\varphi\varphi}^\varphi = \Gamma_{\varphi h}^\varphi = \Gamma_{h\varphi}^\varphi = \Gamma_{hh}^\varphi = 0, \Gamma_{r\varphi}^\varphi = \Gamma_{\varphi r}^\varphi = \frac{1}{r}$$

3) for h , $\ddot{h} = 0$,

$$\Gamma_{rr}^h = \Gamma_{r\varphi}^h = \Gamma_{\varphi r}^h = \Gamma_{rh}^h = \Gamma_{hr}^h = \Gamma_{\varphi\varphi}^h = \Gamma_{\varphi h}^h = \Gamma_{h\varphi}^h = \Gamma_{hh}^h = 0,$$

3

Calculate Christoffel symbols of Levi-Civita connection for Riemannian metric $G = adu^2 + b dv^2$. Compare with results of the Exercise 1b) in the Homework 5.

Lagrangian of free particle for this metric is

$$L = \frac{a(u,v)\dot{u}^2 + b(u,v)\dot{v}^2}{2}.$$

Euler-lagrange equations

for u :

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{u}} \right) = \frac{d}{dt}(a\dot{u}) = a_u\dot{u}^2 + a_v\dot{v}\dot{u} + a\ddot{u} = \frac{\partial L}{\partial u} = \frac{a_u\dot{u}^2 + b_u\dot{v}^2}{2}$$

hence

$$\ddot{u} + \frac{1}{2} \frac{a_u}{a} \dot{u}^2 + \frac{a_v}{a} \dot{v}\dot{u} - \frac{1}{2} \frac{b_u}{a} \dot{v}^2 = 0$$

Comparing with equation

$$\ddot{u} + \Gamma_{uu}^u \dot{u}\dot{u} + \Gamma_{uv}^u \dot{u}\dot{v} + \Gamma_{vu}^u \dot{v}\dot{u} + \Gamma_{vv}^u \dot{v}\dot{v} = 0$$

we see that

$$\Gamma_{uu}^u = \frac{1}{2} \frac{a_u}{a}, \Gamma_{uv}^u = \Gamma_{vu}^u = \frac{1}{2} \frac{a_v}{a}, \Gamma_{vv}^u = -\frac{1}{2} \frac{b_u}{a},$$

Analogously v :

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{v}} \right) = \frac{d}{dt}(b\dot{v}) = b_v\dot{v}^2 + b_u\dot{u}\dot{v} + b\ddot{v} = \frac{\partial L}{\partial v} = \frac{a_v\dot{u}^2 + b_v\dot{v}^2}{2}$$

hence

$$\ddot{v} + \frac{1}{2} \frac{b_v}{b} \dot{v}^2 + \frac{b_u}{b} \dot{u}\dot{v} - \frac{1}{2} \frac{a_v}{b} \dot{u}^2 = 0 \Rightarrow \Gamma_{vv}^v = \frac{1}{2} \frac{b_v}{b}, \Gamma_{vu}^v = \Gamma_{uv}^v = \frac{1}{2} \frac{b_u}{b}, \Gamma_{uu}^v = -\frac{1}{2} \frac{a_v}{b}.$$

4

Write down the Lagrangian of "free" particle $L = \frac{1}{2} g_{ik} \dot{x}^i \dot{x}^k$ and using Euler-Lagrange equations for this Lagrangian calculate Christoffel symbols (Christoffel symbols of Levi-Civita connection) for

- cylindrical surface of the radius R
- for the cone $x^2 + y^2 - k^2 z^2 = 0$
- for the sphere of radius R
- for Lobachevsky plane

Compare with the results that you obtained using straightforwardly the formula (1) or using formulae for induced connection.

Solution.

For cylindrical surface of the radius a : $x^2 + y^2 = a^2$ $\mathbf{r}(h, \varphi) = \begin{cases} x = a \cos \varphi \\ y = a \sin \varphi \\ z = h \end{cases}$ we have that induced metric is $G = dh^2 + a^2 d\varphi^2$ and the Lagrangian of free particle is

$$L = \frac{a^2 \dot{\varphi}^2 + \dot{h}^2}{2}$$

for φ , Euler-Lagrange equations of motion:

$$\frac{\partial L}{\partial \varphi} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\varphi}} \right) \cdot \quad \frac{\partial L}{\partial \varphi} = 0, \quad \frac{\partial L}{\partial \dot{\varphi}} = a^2 \dot{\varphi}$$

hence

$$\frac{d}{dt} (a^2 \dot{\varphi}) = a^2 \ddot{\varphi} = 0, \quad \ddot{\varphi} = 0.$$

Hence all Christoffel symbols $\Gamma_{\varphi\varphi}^\varphi, \Gamma_{\varphi h}^\varphi, \Gamma_{h\varphi}^\varphi$ vanish:

$$\Gamma_{\varphi\varphi}^\varphi = 0, \Gamma_{\varphi h}^\varphi = \Gamma_{h\varphi}^\varphi = 0$$

for h , we have the same. Euler-Lagrange equations of motion::

$$\frac{\partial L}{\partial h} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{h}} \right) \cdot \quad \frac{\partial L}{\partial h} = 0, \quad \frac{\partial L}{\partial \dot{h}} = \dot{h}$$

hence

$$\frac{d}{dt} (\dot{h}) = \ddot{h} = 0,$$

Hence all Christoffel symbols $\Gamma_{\varphi\varphi}^h, \Gamma_{\varphi h}^h, \Gamma_{h\varphi}^h$ vanish:

$$\Gamma_{\varphi\varphi}^h = 0, \Gamma_{\varphi h}^h = \Gamma_{h\varphi}^h = 0$$

We see that on cylindrical surface in coordinates h, φ all Christoffel symbols vanish: this is not surprising, since Riemannian metric $dh^2 + a^2 d\varphi^2$ has constant coefficients.

For cone: $x^2 + y^2 = k^2 z^2$ we have $\mathbf{r}(h, \varphi) = \begin{cases} x = kh \cos \varphi \\ y = kh \sin \varphi \\ z = h \end{cases}$ and induced metric is $G = (k^2 + 1)dh^2 + k^2 h^2 d\varphi^2$ (see previous Homeworks) and the Lagrangian of free particle is

$$L = \frac{k^2 h^2 \dot{\varphi}^2 + (k^2 + 1) \dot{h}^2}{2}$$

for φ , Euler-Lagrange equations of motion:

$$\frac{\partial L}{\partial \varphi} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\varphi}} \right) \cdot \quad \frac{\partial L}{\partial \varphi} = 0, \quad \frac{\partial L}{\partial \dot{\varphi}} = k^2 h^2 \dot{\varphi}$$

hence

$$\frac{d}{dt} (k^2 h^2 \dot{\varphi}) = k^2 h^2 \ddot{\varphi} + 2k^2 h \dot{h} \dot{\varphi} = 0,$$

i.e.

$$\ddot{\varphi} + \frac{2}{h} \dot{h} \dot{\varphi} = 0,$$

i.e. for Christoffel symbols Γ_{\dots}^φ we have:

$$\Gamma_{\varphi h}^\varphi = \Gamma_{h\varphi}^\varphi = \frac{1}{h}$$

and $\Gamma_{\varphi\varphi}^\varphi, \Gamma_{hh}^\varphi$ vanish.

For h , we have Euler-Lagrange equations of motion::

$$\frac{\partial L}{\partial h} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{h}} \right), \quad \frac{\partial L}{\partial h} = k^2 h \dot{\varphi}^2, \quad \frac{\partial L}{\partial \dot{h}} = (k^2 + 1) \dot{h},$$

hence

$$\frac{d}{dt} \left((k^2 + 1) \dot{h} \right) = (k^2 + 1) \ddot{h} = k^2 h \dot{\varphi}^2, \text{ i.e. } \ddot{h} = \frac{k^2 h}{k^2 + 1} \dot{\varphi}^2$$

Hence we have that $\Gamma_{\varphi\varphi}^h = -\frac{k^2 h}{k^2 + 1}$ and $\Gamma_{\varphi h}^h = \Gamma_{h\varphi}^h = \Gamma_{hh}^h = 0$.

c) *For the sphere:*

Riemannian metric on sphere in spherical coordinates is $G = R^2 d\theta^2 + R^2 \sin^2 \theta d\varphi^2$. Hence the Lagrangian of the free particle is

$$L = \frac{R^2 \dot{\theta}^2 + R^2 \sin^2 \theta \dot{\varphi}^2}{2}$$

Euler-Lagrange equations for θ :

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right), \quad \frac{\partial L}{\partial \theta} = R^2 \sin \theta \cos \theta \dot{\varphi}^2, \quad \frac{\partial L}{\partial \dot{\theta}} = R^2 \dot{\theta}$$

Hence

$$\frac{d}{dt} (R^2 \dot{\theta}) = R^2 \sin \theta \cos \theta \dot{\varphi}^2, \quad R^2 \ddot{\theta} = R^2 \sin \theta \cos \theta \dot{\varphi}^2,$$

hence

$$\ddot{\theta} - \sin \theta \cos \theta \dot{\varphi}^2 = 0.$$

Comparing with equation for geodesic

$$\ddot{\theta} + \Gamma_{\theta\theta}^\theta \dot{\theta} \dot{\theta} + \Gamma_{\theta\varphi}^\theta \dot{\theta} \dot{\varphi} + \Gamma_{\varphi\theta}^\theta \dot{\varphi} \dot{\theta} + \Gamma_{\varphi\varphi}^\theta \dot{\varphi} \dot{\varphi} = \ddot{\theta} + \Gamma_{\theta\theta}^\theta \dot{\theta} \dot{\theta} + 2\Gamma_{\theta\varphi}^\theta \dot{\theta} \dot{\varphi} + \Gamma_{\varphi\varphi}^\theta \dot{\varphi} \dot{\varphi} = 0$$

we see that

$$\Gamma_{\theta\theta}^\theta = \Gamma_{\theta\varphi}^\theta = \Gamma_{\varphi\theta}^\theta = 0, \quad \Gamma_{\varphi\varphi}^\theta = -\sin \theta \cos \theta$$

Analogously Euler-Lagrange equations for φ :

$$\frac{\partial L}{\partial \varphi} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\varphi}} \right), \quad \frac{\partial L}{\partial \varphi} = 0, \quad \frac{\partial L}{\partial \dot{\varphi}} = R^2 \sin^2 \theta \dot{\varphi}.$$

Hence

$$\frac{d}{dt} (R^2 \sin^2 \theta \dot{\varphi}) = 0, \quad R^2 \sin^2 \theta \ddot{\varphi} + 2R^2 \sin \theta \cos \theta \dot{\theta} \dot{\varphi} = 0,$$

hence

$$\ddot{\theta} + \cotan \theta \dot{\theta} \dot{\varphi} = 0,$$

Comparing with equation for geodesic

$$\ddot{\varphi} + \Gamma_{\theta\theta}^\varphi \dot{\theta} \dot{\theta} + \Gamma_{\theta\varphi}^\varphi \dot{\theta} \dot{\varphi} + \Gamma_{\varphi\theta}^\varphi \dot{\varphi} \dot{\theta} + \Gamma_{\varphi\varphi}^\varphi \dot{\varphi} \dot{\varphi} = \ddot{\varphi} + \Gamma_{\theta\theta}^\varphi \dot{\theta} \dot{\theta} + 2\Gamma_{\theta\varphi}^\varphi \dot{\theta} \dot{\varphi} + \Gamma_{\varphi\varphi}^\varphi \dot{\varphi} \dot{\varphi} = 0$$

we see that

$$\Gamma_{\theta\theta}^\varphi = \Gamma_{\varphi\varphi}^\varphi = 0, \quad \Gamma_{\varphi\theta}^\varphi = \Gamma_{\theta\varphi}^\varphi = \cotan \theta.$$

d) For Lobachevsky plane:

Lagrangian of "free" particle on the Lobachevsky plane with metric $G = \frac{dx^2+dy^2}{y^2}$ is

$$L = \frac{1}{2} \frac{\dot{x}^2 + \dot{y}^2}{y^2}.$$

Euler-Lagrange equations are

$$\begin{aligned} \frac{\partial L}{\partial x} = 0 &= \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{d}{dt} \left(\frac{\dot{x}}{y^2} \right) = \frac{\ddot{x}}{y^2} - \frac{2\dot{x}\dot{y}}{y^3}, \text{ i.e. } \ddot{x} - \frac{2\dot{x}\dot{y}}{y} = 0, \\ \frac{\partial L}{\partial y} &= -\frac{\dot{x}^2 + \dot{y}^2}{y^3} = \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} = \frac{d}{dt} \left(\frac{\dot{y}}{y^2} \right) = \frac{\ddot{y}}{y^2} - \frac{2\dot{y}^2}{y^3}, \text{ i.e. } \ddot{y} + \frac{\dot{x}^2}{y} - \frac{\dot{y}^2}{y} = 0. \end{aligned}$$

Comparing these equations with equations for geodesics: $\ddot{x}^i - \dot{x}^k \Gamma_{km}^i \dot{x}^m = 0$ ($i = 1, 2, x = x^1, y = x^2$) we come to

$$\Gamma_{xx}^x = 0, \Gamma_{xy}^x = \Gamma_{yx}^x = -\frac{1}{y}, \Gamma_{yy}^x = 0, \Gamma_{xx}^y = \frac{1}{y}, \Gamma_{xy}^y = \Gamma_{yx}^y = 0, \Gamma_{yy}^y = -\frac{1}{y}. \blacksquare$$

The answers are the same as calculated with other methods. We see that Lagrangians give us the nice and quick way to calculate Christoffel symbols.