# Appendices to lectures

Here I put appendices to the lecture notes on Riemannian geometry
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## Contents

# 1 Examples of surfaces in $\mathbf{E}^3$

## 1.1 Hyperboloids and other quadratic surfaces

*One-sheeted and two-sheeted hyperboloids.*

*These examples were not considered on lectures, but they are interesting for learning purposes.*

Consider surface given by the equation

$$x^2 + y^2 - z^2 = c$$

If $c = 0$ it is a cone. We considered it already above.

If $c > 0$ it is one-sheeted hyperboloid—connected surface in $\mathbf{E}^3$.

If $c < 0$ it is two-sheeted hyperboloid— a surface with two sheets: upper sheet $z > 0$ and another sheet: $z < 0$.

Consider these cases separately.

1) *One-sheeted hyperboloid*: $x^2 + y^2 - z^2 = a^2$. It is ruled surface.

**Exercise**$^\dagger$ Find the lines on two-sheeted hyperboloid

One-sheeted hyperboloid is given by the equation $x^2 + y^2 - z^2 = a^2$. It is convenient to choose parameterisation:

$$\mathbf{r}(\theta, \varphi) : \quad \begin{cases} x = a\cosh\theta\cos\varphi \\ y = a\cosh\theta\sin\varphi \\ z = a\sinh\theta \end{cases} \tag{1.1}$$

$$x^2 + y^2 - z^2 = a^2\cosh^2\theta - a^2\sinh^2\theta = a^2\,.$$

(Compare the calculations with calculations for sphere! We changed functions $\cos, \sin$ on $\cosh, \sinh$.)

Induced Riemannian metric (first quadratic form) is equal to

$$G_{Hyperbol I} = \left(dx^2 + dy^2 + dz^2\right)\big|_{x=a\cosh\theta\cos\varphi, y=a\cosh\theta\sin\varphi, z=a\sinh\theta} =$$

$$(a\sinh\theta\cos\varphi d\theta - a\cosh\theta\sin\varphi d\varphi)^2 + (a\sinh\theta\sin\varphi d\theta + a\cosh\theta\cos\varphi d\varphi)^2 + (a\cosh\theta d\theta)^2 =$$

$$a^2\sinh^2\theta d\theta^2 + a^2\cosh^2\theta d\varphi^2 + a^2\cosh^2\theta d\theta^2 =$$

$$, \qquad = a^2(1+2\sinh^2\theta)d\theta^2 + a^2\cosh^2\theta d\varphi^2\,, \qquad ||g_{\alpha\beta}|| = \begin{pmatrix} a^2(1+2\sinh^2\theta) & 0 \\ 0 & a^2\cosh^2\theta \end{pmatrix}$$

2) *Two-sheeted hyperboloid*: $z^2 - x^2 - y^2 = a^2$. It is not ruled surface!

For two-sheeted hyperboloid calculations will be very similar.

In the same way as for one-sheeted hyperboloid (see equation (1.1)) it is convenient to choose parameterisation:

$$\mathbf{r}(\theta, \varphi) \colon \quad \begin{cases} x = a \sinh\theta \cos\varphi \\ y = a \sinh\theta \sin\varphi \\ z = a \cosh\theta \end{cases} \tag{1.2}$$

$$z^2 - x^2 - y^2 = a^2 \cosh^2\theta - a^2 \sinh^2\theta = a^2$$

(Compare the calculations with calculations for sphere and one-sheeted hyperboloid.

Induced Riemannian metric (first quadratic form) is equal to

$$G_{HyperbolI} = \left(dx^2 + dy^2 + dz^2\right)\big|_{x=a\sinh\theta\cos\varphi, y=a\sinh\theta\sin\varphi, z=a\cosh\theta} =$$

$$(a\cosh\theta\cos\varphi d\theta - a\sinh\theta\sin\varphi d\varphi)^2 + (a\cosh\theta\sin\varphi d\theta + a\sinh\theta\cos\varphi d\varphi)^2 + (a\sinh\theta d\theta)^2 =$$

$$a^2\cosh^2\theta d\theta^2 + a^2\sinh^2\theta d\varphi^2 + a^2\sinh^2\theta d\theta^2 =$$

$$, \qquad = a^2(1+2\sinh^2\theta)d\theta^2 + a^2\sinh^2\theta d\varphi^2\,, \qquad ||g_{\alpha\beta}|| = \begin{pmatrix} a^2(1+2\sinh^2\theta) & 0 \\ 0 & a^2\sinh^2\theta \end{pmatrix} \tag{1.3}$$

We calculated examples of induced Riemannian structure embedded in Euclidean space almost for all quadratic surfaces.

Quadratic surface is a surface defined by the equation

$$Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz + ex + fy + dz + c = 0$$

One can see that any quadratic surface by affine transformation can be transformed to one of these surfaces

- cylinder (elliptic cylinder) $x^2 + y^2 = 1$
- hyperbolic cylinder: $x^2 - y^2 = 1$)
- parabolic cylinder $z = x^2$
- paraboloid $x^2 + y^2 = z$
- hyperbolic paraboloid $x^2 - y^2 = z$
- cone $x^2 + y^2 - z^2 = 0$
- sphere $x^2 + y^2 + z^2 = 1$
- one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$
- two-sheeted hyperboloid $z^2 - x^2 - y^2 = 1$

(We exclude degenerate cases such as "point" $x^2 + y^2 + z^2 = 0$, planes, e.t.c.)

## 1.2 Inversion and metric on circle and sphere in stereographic cooridnates

Formulae (**??**) and (**??**) for metric in stereographic coordinates are very important, look very nice, but everyone who tried to calculate them was forced to do difficult calculations. In this paragraph we will explain how these formulae can be derived almost automatically with use of *inversion*

Let $O$ be an arbitrary point in Euclidean space $\mathbf{E}^n$ (Here we consider just the case $n = 2, 3$ [1])

Let $S_a$ be a sphere of radius $a$ with centre at the point 0.

If $n^i$ are coordinates of the point $O$, then points of the shere are defined by equation $(\sum_{i=1}^n (x^i - n^i)^2 = a^2$. We call this sphere base of inversion.

We define inversion of $\mathbf{E}^n$ with respect to the sphere $S_a$ as a map which maps an arbitrary point $P \neq 0$ in $\mathbf{E}^n$ to the point $P'$ such

- point $P'$ belongs to the ray $OP$
- $$|0P| \cdot ||OP'| = a^2 \tag{1.4}$$

We see that in particular points of the inversion sphere remian fixed under inversion.

It can be proved that transforms lines, $k$-dimensional, planes, circles,.spheres to lines, planes, circles, spheres, and that the inversion does not change angel between tangent vectors.

Stereographic projection is restriction of inversion.

Hence stereographic projection is conformal map. This is why in stereographic coordinates Riemannian metric has conformal appearance.

## 1.3 Induced metric on two-sheeted hyperboloid embedded in pseudo-Euclidean space.

Consider the same two-sheeted hyperboloid $z^2 - x^2 - y^2 = 1$ embedded $\mathbf{R}^3$ (See equation (1.2). For simplicity we assume now that $a = 1$.) Now we consider the ambient space $\mathbf{R}^3$ not as Euclidean space but as *pseudo-Euclidean space*, i.e. in $\mathbf{R}^3$ instead standard scalar product

$$\langle \mathbf{X}, \mathbf{Y} \rangle = X^1Y^1 + X^2Y^2 + X^3Y^3$$

[1]These considerations can be generalised for arbitrary $n$

we consider pseudo-scalar product defined by bilinear form

$$\langle \mathbf{X}, \mathbf{Y} \rangle_{pseud} = X^1Y^1 + X^2Y^2 - X^3Y^3$$

The "pseudoscalar" product is bilinear, symmetric. It is defined by non-degenerate matrix. But it is not positive-definite. E.g. The "pseudo-length" of vectors $\mathbf{X} = (a\cos\varphi, a\sin\varphi, \pm a)$ is equals to zero (such vectors are called null vectors):

$$\mathbf{X} = (a\cos\varphi, a\sin\varphi, \pm a) \Rightarrow \langle \mathbf{X}, \mathbf{X} \rangle_{pseudo} = 0,$$

The corresponding pseudo-Riemannian metric is:

$$G_{pseudo} = dx^2 + dy^2 - dz^2 \tag{1.5}$$

It turns out that the following remarkable fact occurs:

**Proposition** *The pseudo-Riemannian metric* (1.5) *in the ambient* 3*-dimensional pseudo-Euclidean space induces Riemannian metric on two-sheeted hyperboloid* $x^2 + y^2 - z^2 = 1$.

**Remark** This is not the fact for one-sheeted hyperboloid (see problem 7 in Homework 2)

Show it. (See also problems 5 and 6 in Homework 2. ) Repeat the calculations above for two-sheeted hyperboloid changing in the ambient space Riemannian metric $G = dx^2 + dy^2 + dz^2$ on pseudo-Riemannian $dx^2 + dy^2 - dz^2$:

Using (1.2) and (1.5) we come now to

$$G = \left(dx^2 + dy^2 - dz^2\right)\big|_{x=a\sinh\theta\cos\varphi, y=a\sinh\theta\sin\varphi, z=a\cosh\theta} =$$

$$(a\cosh\theta\cos\varphi d\theta - a\sinh\theta\sin\varphi d\varphi)^2 + (a\cosh\theta\sin\varphi d\theta + a\sinh\theta\cos\varphi d\varphi)^2 - (a\sinh\theta d\theta)^2 =$$

$$a^2\cosh^2\theta d\theta^2 + a^2\sinh^2\theta d\varphi^2 - a^2\sinh^2\theta d\theta^2$$

$$, \qquad G_L = a^2 d\theta^2 + a^2\sinh^2\theta d\varphi^2\,, \qquad ||g_{\alpha\beta}|| = \begin{pmatrix} 1 & 0 \\ 0 & \sinh^2\theta \end{pmatrix} \tag{1.6}$$

The two-sheeted hyperboloid equipped with this metric is called hyperbolic or Lobachevsky plane.

Now express Riemannian metric in stereographic coordinates. (We did it in detail in homework 2)

Calculations are very similar to the case of stereographic coordinates of 2-sphere $x^2 + y^2 + z^2 = 1$. (See homework 1). Centre of projection $(0, 0, -1)$: For stereographic coordinates $u, v$ we have $\frac{u}{x} = \frac{y}{v} = \frac{1}{1+z}$. We come to

$$\begin{cases} u = \frac{x}{1+z} \\ v = \frac{y}{1+z} \end{cases} \quad , \qquad \begin{cases} x = \frac{2u}{1-u^2-v^2} \\ y = \frac{2v}{1-u^2-v^2} \\ z = \frac{u^2+v^2+1}{1-u^2-v^2} \end{cases} \tag{4}$$

The image of upper-sheet is an open disc $u^2+v^2=1$ since $u^2+v^2=\frac{x^2+y^2}{(1+z)^2}=\frac{z^2-1}{(1+z)^2}=\frac{z-1}{z+1}$. Since for upper sheet $z>1$ then $0\le\frac{z-1}{z+1}<1$.

$$G=(dx^2+dy^2-dz^2)\big|_{x=x(u,v),y=y(u,v),z=z(u,v)}=\left(d\left(\frac{2u}{1-u^2-v^2}\right)\right)^2+$$

$$\left(d\left(\frac{2v}{1-u^2-v^2}\right)\right)^2-\left(d\left(\frac{u^2+v^2+1}{1-u^2-v^2}\right)\right)^2=\frac{4(du)^2+4(dv)^2}{(1-u^2-v^2)^2}\,.$$

These coordinates are very illuminating. One can show that we come to so called hyperbolic plane (see in detail Homework 2)

# 2 Isometries and infinitesimal isometries (Killing vector fields)

Let $\mathbf{X}$ be an arbitrary vector field on Riemannian manifold $M$. It induces infinitesimal diffeomorphism

$$F\colon x^{i'}=x^i+\varepsilon X^i(x),\quad \text{where } \varepsilon^2=0\,.$$

(the condition $\varepsilon^2=0$ reflects the fact that we ignore terms of order $\ge 2$ over $\varepsilon$.) Find a condition which guarantees that infinitesimal diffeomorphism is an isometry. If $x^{i'}=x^i+\varepsilon X^i(x)$, then one can see that the inverse infinitesimal diffeomorphism is defined by the equation $x^i=x^{i'}-\varepsilon X^i(x')$ and equation (**??**) implies that

$$g_{ik}(x)=g_{pq}(x'(x))\frac{\partial x^p(x')}{\partial x^i}\frac{\partial x^q(x')}{\partial x^k}=g_{pq}(x^i+\varepsilon X^i)\left(\delta^p_i+\varepsilon\frac{pX^p(x)}{\partial x^i}\right)\left(\delta^q_k+\varepsilon\frac{pX^q(x)}{\partial x^k}\right)=$$

$$g_{ik}(x)+\varepsilon\left[X^p(x)\frac{\partial g_{ik}(x)}{\partial x^p}+g_{iq}(x)\frac{\partial X^q(x)}{\partial x^k}+g_{pk}(x)\frac{\partial X^p(x)}{\partial x^i}\,,\right]$$

Here we consider only terms of first and zero order over $\varepsilon$ since $\varepsilon^2=0$ (this is related with the fact that transformation is *infinitesimal*). The last relation implies that

$$X^p(x)\frac{\partial g_{ik}(x)}{\partial x^p}+g_{iq}(x)\frac{\partial X^q(x)}{\partial x^k}+g_{pk}(x)\frac{\partial X^p(x)}{\partial x^i}=0\,. \tag{2.1}$$

Left hand side of this relation we denote $\mathcal{L}_{\mathbf{X}}G$— Lie derivative of Riemannian metric along vector field $\mathbf{X}$. Vector field $\mathbf{X}$ induces isometry if Lie derivative of metric along this vector field vanishes. We come to

**Proposition** Vector field $\mathbf{X}$ on Riemannian manifold $(M, G)$ induces infinitesimal isometry if $\mathcal{L}_{\mathbf{X}} G = 0$:

$$\mathcal{L}_{\mathbf{X}} G = X^p(x) \frac{\partial g_{ik}(x)}{\partial x^p} + g_{iq}(x) \frac{\partial X^q(x)}{\partial x^k} + g_{pk}(x) \frac{\partial X^p(x)}{\partial x^i} = 0 \,. \tag{2.2}$$

**Definition)** We call vector field $\mathbf{X}$ *Killing vector field)* if it preserves the metric, i.e. if equation (2.2) is obeyed.

**Example** Consider plane $(x, y)$ with Riemannian metric $G = \sigma(x, y)(dx^2 + dy^2)$. Find differential equation for infinitesimal isometries of this metric, i.e. write down equations (2.2) for this metric.

We have $||g_{ik}(x, y)|| = \begin{pmatrix} \sigma(x, y) & 0 \\ 0 & \sigma(x, y) \end{pmatrix}$.

Let $\mathbf{X} = A(x, y)\partial_x + B(x, y)\partial_y$. Write down equations (2.2) for components $g_11$, $g_12$, $g_{21}$ and $g_{22}$: We will have the following three equations

$$\begin{cases} A(x, y)\frac{\partial \sigma}{\partial x} + B(x, y)\frac{\partial \sigma}{\partial y} + 2\frac{\partial A(x,y)}{\partial x}\sigma = 0 \quad \text{for component } g_{11} \\ A(x, y)\frac{\partial \sigma}{\partial x} + B(x, y)\frac{\partial \sigma}{\partial y} + 2\frac{\partial B(x,y)}{\partial y}\sigma = 0 \quad \text{for component } g_{22} \\ \frac{\partial B(x,y)}{\partial x} + \frac{\partial A(x,y}{\partial y} = 0 \quad \text{for components } g_{12} \text{ and } g_{21} \end{cases} \tag{2.3}$$

Practically for sphere, Lobachevsky plane, e.t.c. it is much easier to find the Killing fields not solving these equations, but considering the usual isometries (see examples in solutions of Coursework and in the Appendix about Killing vector fields for Lobachevsky plane.))

Another simple and interesting exercise: How look Killing vectors for Euclidean space $\mathbf{E}^n$. In this case we come from (2.2) to equation

$$\mathcal{L}_{\mathbf{K}} G = \delta_{iq}(x) \frac{\partial K^q(x)}{\partial x^k} + \delta_{pk}(x) \frac{\partial K^p(x)}{\partial x^i} = 0 \,,$$

i.e.

$$\frac{\partial K^i(x)}{\partial x^k} + \frac{\partial K^k(x)}{\partial x^i} = 0 \,. \tag{2.4}$$

Solve this equation. Differentiating by $x$ we come to

$$\frac{\partial^2 K^i(x)}{\partial x^m \partial x^k} + \frac{\partial K^k(x)}{\partial x^m \partial x^i} = 0$$

Consider tensor field

$$T^i_{mk} = \frac{\partial^2 K^i}{\partial x^m \partial x^k} \tag{2.5}$$

It follows from equation (2.4) that

$$T^i_{mk} = T^i_{km} = -T^m_{ik} \,. \tag{2.6}$$

It is easy to see that this implies that $T^i_{mk} \equiv 0$!!!:

$$T^i_{mk} = -T^m_{ik} = -T^m_{ki} = T^k_{mi} = T^k_{im} = -T^i_{km} = -T^i_{mk} \Rightarrow T^i_{mk} = -T^i_{mk}\,,$$

i.e. $T^i_{mk} = \frac{\partial^2 K^i(x)}{\partial x^m \partial x^k} = 0$. This implies that

$$K^i(x) = C^i + B^i_k x^k$$

We come to

**Theorem** All infinitesimal isometries of $\mathbf{E}^n$ are translations and infinitesimal rotations.

What happens in general case?

# 3 Invariance of volume element under changing of coordinates

Check straighforwardly that volume element is invariant under coordinate transformations, i.e. if $y^1, \dots, y^n$ are new coordinates: $x^1 = x^1(y^1, \dots, y^n), x^2 = x^2(y^1, \dots, y^n)$...,

$$x^i = x^i(y^p), i = 1, \dots, n \quad , p = 1, \dots, n$$

and $\tilde{g}_{pq}(y)$ matrix of the metric in new coordinates:

$$\tilde{g}_{pq}(y) = \frac{\partial x^i}{\partial y^p} g_{ik}(x(y)) \frac{\partial x^k}{\partial y^q}\,. \tag{3.1}$$

Then

$$\sqrt{\det g_{ik}(x)}\, dx^1 dx^2 \dots dx^n = \sqrt{\det \tilde{g}_{pq}(y)}\, dy^1 dy^2 \dots dy^n \tag{3.2}$$

This follows from (3.1). Namely

$$\sqrt{\det g_{ik}(y)}\, dy^1 dy^2 \dots dy^n = \sqrt{\det\left(\frac{\partial x^i}{\partial y^p} g_{ik}(x(y)) \frac{\partial x^k}{\partial y^q}\right)}\, dy^1 dy^2 \dots dy^n$$

Using the fact that $\det(ABC) = \det A \cdot \det B \cdot \det C$ and $\det\left(\frac{\partial x^i}{\partial y^p}\right) = \det\left(\frac{\partial x^k}{\partial y^q}\right)$[2] we see that from the formula above follows:

$$\sqrt{\det g_{ik}(y)}\, dy^1 dy^2 \dots dy^n = \sqrt{\det\left(\frac{\partial x^i}{\partial y^p} g_{ik}(x(y)) \frac{\partial x^k}{\partial y^q}\right)} dy^1 dy^2 \dots dy^n =$$

[2]determinant of matrix does not change if we change the matrix on the adjoint, i.e. change columns on rows.

$$\sqrt{\left(\det\left(\frac{\partial x^i}{\partial y^p}\right)\right)^2}\sqrt{\det g_{ik}(x(y))}dy^1dy^2\dots dy^n =$$

$$\sqrt{\det g_{ik}(x(y))}\det\left(\frac{\partial x^i}{\partial y^p}\right)dy^1dy^2\dots dy^n = \tag{3.3}$$

Now note that

$$\det\left(\frac{\partial x^i}{\partial y^p}\right)dy^1dy^2\dots dy^n = dx^1\dots dx^n$$

according to the formula for changing coordinates in $n$-dimensional integral [3]. Hence

$$\sqrt{\det g_{ik}(x(y))}\det\left(\frac{\partial x^i}{\partial y^p}\right)dy^1dy^2\dots dy^n = \sqrt{\det g_{ik}(x(y))}dx^1dx^2\dots dx^n \tag{3.4}$$

Thus we come to (3.2).

# 4 Connection

## 4.1 Global aspects of existence of connection

We defined connection as an operation on vector fields obeying the special axioms (see the subsubsection 2.1.1). Then we showed that in a given coordinates connection is defined by Christoffel symbols. On the other hand we know that in general coordinates on manifold are not defined globally. (We had not this trouble in Euclidean space where there are globally defined Cartesian coordinates.)

- How to define connection globally using local coordinates?
- Does there exist at least one globally defined connection?
- Does there exist globally defined flat connection?

These questions are not naive questions. Answer on first and second questions is "Yes". It sounds bizzare but answer on the first question is not "Yes" [4]

[3]Determinant of the matrix $\left(\frac{\partial x^i}{\partial y^p}\right)$ of changing of coordinates is called sometimes Jacobian. Here we consider the case if Jacobian is positive. If Jacobian is negative then formulae above remain valid just the symbol of modulus appears.

[4]Topology of the manifold can be an obstruction to existence of global flat connection. E.g. it does not exist on sphere $S^n$ if $n > 1$.

Global definition of connection

The formula (**??**) defines the transformation for Christoffer symbols if we go from one coordinates to another.

Let $\{(x^i_\alpha), U_\alpha\}$ be an atlas of charts on the manifold $M$.

If connection $\nabla$ is defined on the manifold $M$ then it defines in any chart (local coordinates) $(x^i_\alpha)$ Christoffer symbols which we denote by ${}_{(\alpha)}\Gamma^i_{km}$. If $(x^i_\alpha)$, $(x^{i'}_{(\beta)})$ are different local coordinates in a vicinity of a given point then according to (**??**)

$$
{}_{(\beta)}\Gamma^{i'}_{k'm'} = \frac{\partial x^k_{(\alpha)}}{\partial x^{k'}_{(\beta)}} \frac{\partial x^m_{(\alpha)}}{\partial x^{m'}_{(\beta)}} \frac{\partial x^{i'}_{(\beta)}}{\partial x^i_{(\alpha)}}\ {}_{(\beta)}\Gamma^{(\alpha)i}_{mk} + \frac{\partial^2 x^k_{(\alpha)}}{\partial x^{m'}_{(\beta)} \partial x^{k'}_{(\beta)}} \frac{\partial x^{i'}_{(\beta)}}{\partial x^k_{(\alpha)}} \tag{4.1}
$$

**Definition** Let $\{(x^i_\alpha), U_\alpha\}$ be an atlas of charts on the manifold $M$

We say that the collection of Christoffel symbols $\{\Gamma^{(\alpha)i}_{km}\}$ defines globally a connection on the manifold $M$ in this atlas if for every two local coordinates $(x^i_{(\alpha)})$, $(x^i_{(\beta)})$ from this atlas the transformation rules (4.1) are obeyed.

Using partition of unity one can prove the existence of global connection constructing it in explicit way. Let $\{(x^i_\alpha), U_\alpha\}$ $(\alpha = 1, 2, \ldots, N)$ be a finite atlas on the manifold $M$ and let $\{\rho_\alpha\}$ be a partition of unity adjusted to this atlas. Denote by ${}^{(\alpha)}\Gamma^i_{km}$ local connection defined in domain $U_\alpha$ such that its components in these coordinates are equal to zero. Denote by ${}^{(\alpha)}_{(\beta)}\Gamma^i_{km}$ Christoffel symbols of this local connection in coordinates $(x^i_{(\beta)})$ $({}^{(\alpha)}_{(\beta)}\Gamma^i_{km} = 0)$. Now one can define globally the connection by the formula:

$$
{}_{(\beta)}\Gamma^i_{km}(\mathbf{x}) = \sum_\alpha \rho_\alpha(\mathbf{x})\ {}^{(\alpha)}_{(\beta)}\Gamma^i_{km}(\mathbf{x}) = \sum_\alpha \rho_\alpha(\mathbf{x}) \frac{\partial x^i_{(\beta)}}{\partial x^{i'}_{(\alpha)}} \frac{\partial^2 x^{i'}_{(\alpha)}(\mathbf{x})}{\partial x^k_{(\beta)} \partial x^m_{(\beta)}} \,. \tag{4.2}
$$

This connection in general is not flat connection[5]

## 4.2 Killing vectors, antisymmetric operator and antisymmetric bilinear form

We return to Killing vectors.

First consider the following construction.

Let $\mathbf{K}$ be an arbitrary vector field, then consider bilinear form

$$
S_{\mathbf{K}}(\mathbf{X}, \mathbf{Y}) = G\left(\nabla_{\mathbf{X}}\mathbf{K}\,, \mathbf{Y}\right) = \langle \nabla_{\mathbf{X}}\mathbf{K}\,, \mathbf{Y}\rangle = \langle \mathbf{Y}\,, \nabla_{\mathbf{X}}\mathbf{K}\rangle\,, \tag{4.3}
$$

[5]See for detail the text: "Global affine connection on manifold" " in my homepage: "www.maths.mancheser.ac.uk/khudian" in subdirectory Etudes/Geometry

where $\nabla$ is an arbitrary connection, $G$ Riemannian metric, defining scalar product $\langle\ ,\ \rangle$, and $\mathbf{K}$,$\mathbf{X}, \mathbf{Y}$ arbitrary vector fields. One can see that for arbitrary functions $f, g$

$$S(f\mathbf{X}, g\mathbf{Y}) = fgS(\mathbf{X}, \mathbf{Y})$$

This immediately follows from definition of connection (see condition (**??**) in **??**). In local coordinates if $\mathbf{X} = X^m\partial_m$, $\mathbf{Y} = Y^k\partial_k$ then

$$S_{\mathbf{K}}(\mathbf{X}, \mathbf{Y}) = S(\nabla_{\mathbf{X}}\mathbf{K}, \mathbf{Y}) = S(\nabla_{X^m\partial_m}K^i\partial_i, Y^n\partial_n) = X^mY^nS_{mn}\,,$$

where

$$S_{mn} = \langle(\partial_m K^i + \Gamma^i_{mr}K^r)K^r\partial_i, \partial_n\rangle = \langle(\partial_m K^i + \Gamma^i_{mr}K^r)\partial_i, \partial_n\rangle = \left(\partial_m K^i + \Gamma^i_{mr}K^r\right)g_{in}\,. \tag{4.4}$$

We see that this construction defines covariant tensor field.

**Theorem** Let $\nabla$ be Levi-Civita connection on Riemannian manifold $(M, G)$. Then vector field $\mathbf{K}$ is a Killing vector field on $M$ if and only if tensor field $S_{\mathbf{K}}(\mathbf{X}, \mathbf{Y})$ is antisymmetric tensor field: $S_{\mathbf{K}}(\mathbf{X}, \mathbf{Y}) = -S_{\mathbf{K}}(\mathbf{X}, \mathbf{Y})$.

*Proof*

First recall properties of Killing vector field.

Let $M$ be Riemannian manifold with Riemannian metric $G$. Recall that a vector field $\mathbf{K}$ is Killing vector field, i.e. it defines infinitesimal isometry, if

$$\mathcal{L}_{\mathbf{K}}G = 0\,,$$

Notice that for an arbitrary vector field $\mathbf{Z}$ and arbitrary vetor fields $\mathbf{X}, \mathbf{Y}$ we have

$$\mathcal{L}_{\mathbf{Z}}G(\mathbf{X}, \mathbf{Y}) = \partial_{\mathbf{Z}}G(\mathbf{X}, \mathbf{Y}) =$$

$$= (\mathcal{L}_{\mathbf{Z}}G)(\mathbf{X}, \mathbf{Y}) + G(\mathcal{L}_{\mathbf{Z}}\mathbf{X}, \mathbf{Y}) + G(\mathbf{X}, \mathcal{L}_{\mathbf{Z}}\mathbf{Y}) = \mathcal{L}_{\mathbf{Z}}G(\mathbf{X}, \mathbf{Y}) + G([\mathbf{Z}, \mathbf{X}], \mathbf{Y}) + G(\mathbf{X}, [\mathbf{Z}, \mathbf{Y}])\,. \tag{4.5}$$

. In the case if $\mathbf{Z} = \mathbf{K}$ is Killing vector field then condition (4.2) reads

$$\partial_{\mathbf{K}}G(\mathbf{X}, \mathbf{Y}) = G(\mathcal{L}_{\mathbf{K}}\mathbf{X}, \mathbf{Y}) + G(\mathbf{X}, \mathcal{L}_{\mathbf{K}}\mathbf{Y}) = G([K, X], Y) + G(\mathbf{X}, [\mathbf{K}, \mathbf{Y}])\,. \tag{4.6}$$

Now let $\nabla$ be Levi-Civita connection of Riemannian metric, i.e.

$$\partial_{\mathbf{Z}}G(\mathbf{X}, \mathbf{Y}) = G(\nabla_{\mathbf{Z}}\mathbf{X}, \mathbf{Y}) + G(\mathbf{X}, \nabla_{\mathbf{Z}}\mathbf{Y})$$

for arbitrary vector fields $\mathbf{Z}, \mathbf{X}$,
$Y$ (see levicivitaconnection1 ). In particular for $\mathbf{Z} = \mathbf{K}$ we have

$$\partial_{\mathbf{K}}G(\mathbf{X}, \mathbf{Y}) = G(\nabla_{\mathbf{K}}\mathbf{X}, \mathbf{Y}) + G(\mathbf{X}, \nabla_{\mathbf{K}}\mathbf{Y})\,.$$

Now transform the relation (4.6) and compare it with this relation. Performing this transformation we will use the symmetricity of Levi-Civita connection, i.e.

$$\nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} = [\mathbf{X}, \mathbf{Y}] . \tag{4.7}$$

(see symmetricconnectioninvariant and Homework 4.) We have:

$$\partial_{\mathbf{K}} G(\mathbf{X}, \mathbf{Y}) = G(\mathcal{L}_{\mathbf{K}}\mathbf{X}, \mathbf{Y}) + G(\mathbf{X}, \mathcal{L}_{\mathbf{K}}\mathbf{Y}) = G([\mathbf{K}, \mathbf{X}], \mathbf{Y}) + G(\mathbf{X}, [\mathbf{K}, \mathbf{Y}]) =$$

$$G(\nabla_{\mathbf{K}}\mathbf{X} - \nabla_{\mathbf{X}}\mathbf{K}, \mathbf{Y}) + G(\mathbf{X}, \nabla_{\mathbf{K}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{K}) =$$

$$\underbrace{G(\nabla_{\mathbf{K}}\mathbf{X}, \mathbf{Y}) + G(\mathbf{X}, \nabla_{\mathbf{K}}\mathbf{Y})}_{\partial_{\mathbf{K}}(\mathbf{X},\mathbf{Y})} - G(\nabla_{\mathbf{X}}\mathbf{K}, \mathbf{Y}) - G(\mathbf{X}, \nabla_{\mathbf{Y}}\mathbf{K})$$

This implies that

$$G(\nabla_{\mathbf{X}}\mathbf{K}, \mathbf{Y}) + G(\mathbf{X}, \nabla_{\mathbf{Y}}\mathbf{K}) = S_{\mathbf{K}}(\mathbf{X}, \mathbf{Y}) + S_{\mathbf{K}}(\mathbf{Y}, \mathbf{X}) = 0 .$$

, i.e. $S_{\mathbf{K}}$ is antisymmetric. One can easy to see that converse implication is also true.

**Remark** Note that the antisymmetric tensor field $S_{\mathbf{K}}$ defines antisymmetric linear operator

$$A\colon \quad \mathbf{X} \mapsto \nabla_{\mathbf{X}}\mathbf{K} .$$

# 5 Geodesics and Lagrangians

## 5.1 Variational principe and Euler-Lagrange equations

Here very briefly we will explain how Euler-Lagrange equations follow from variational principe.

Let $M$ be a manifold (not necessarily Riemannian) and $L = L(x^i, \dot{x}^i)$ be a Lagrangian on it.

Denote my $\mathcal{M}_{\mathbf{x}_1,t_1}^{\mathbf{x}_2,t_2}$ the space of curves (paths) such that they start at the point $\mathbf{x}_1$ at the "time" $t = t_1$ and end at the point $\mathbf{x}_2$ at the "time" $t = t_2$:

$$\mathcal{M}_{\mathbf{x}_1,t_1}^{\mathbf{x}_2,t_2} = \{C\colon\ \mathbf{x}(t), t_1 \leq t \leq t_2,\ \mathbf{x}(t_1) = \mathbf{x}_1, \mathbf{x}(t_2) = \mathbf{x}_2\} . \tag{5.1}$$

Consider the following functional $S$ on the space $\mathcal{M}_{\mathbf{x}_1,t_1}^{\mathbf{x}_2,t_2}$:

$$S[\mathbf{x}(t)] = \int_{t_1}^{t_2} L\left(x^i(t), \dot{x}^i(t)\right) dt . \tag{5.2}$$

for every curve $\mathbf{x}(t) \in \mathcal{M}_{\mathbf{x}_1,t_1}^{\mathbf{x}_2,t_2}$.

This functional is called *action* functional.

**Theorem** *Let functional $S$ attaints the minimal value on the path $\mathbf{x}_0(t) \in \mathcal{M}_{\mathbf{x}_1,t_1}^{\mathbf{x}_2,t_2}$, i.e.*

$$\forall \mathbf{x}(t) \in \mathcal{M}_{\mathbf{x}_1,t_1}^{\mathbf{x}_2,t_2} \quad S[\mathbf{x}_0(t)] \leq S[\mathbf{x}(t)] \,. \tag{5.3}$$

*Then the path $\mathbf{x}_0(t)$ is a solution of Euler-Lagrange equations of the Lagrangian $L$:*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{\partial L}{\partial x^i} \text{ if } \mathbf{x}(t) = \mathbf{x}_0(t) \,. \tag{5.4}$$

**Remark** The path $\mathbf{x}(t)$ sometimes is called *extremal* of the action functional (5.2).

We will use this Theorem to show that the geodesics are in some sense shortest curves [6].

## 5.2 Geodesics and shortest distance.

Many of you know that geodesics are in some sense shortest curves. We will give an exact meaning to this statement and prove it using variational principe:

Let $M$ be a Riemannian manifold.

**Theorem** *Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two points on $M$. The shortest curve which joins these points is an arc of geodesic.*

*Let $C$ be a geodesic on $M$ and $\mathbf{x}_1 \in C$. Then for an arbitrary point $\mathbf{x}_2 \in C$ which is close to the point $\mathbf{x}_1$ the arc of geodesic joining the points $\mathbf{x}_1, \mathbf{x}_2$ is a shortest curve between these points*[7] .

This Theorem makes a bridge between two different approach to geodesic: the shortest disntance and parallel transport of velocity vector.

Sketch a proof:

[6] The statement of this Theorem is enough for our purposes. In fact in classical mechanics another more useful statement is used: the path $\mathbf{x}_0(t)$ is a solution of Euler-Lagrange equations of the Lagrangian $L$ if and only if it is the stationary "point" of the action functional (5.2), i.e.

$$S[\mathbf{x}_0(t) + \delta\mathbf{x}(t)] - S[\mathbf{x}_0(t) + \delta\mathbf{x}(t)] = 0(\delta\mathbf{x}(t)) \tag{5.5}$$

for an arbitrary infinitesimal variation of the path $\mathbf{x}_0(t)$: $\delta\mathbf{x}_(t_1) = \delta\mathbf{x}_(t_2) = 0$.

[7] More precisely: for every point $\mathbf{x}_1 \in C$ there exists a ball $B_\delta(\mathbf{x}_1)$ such that for an arbitrary point $\mathbf{x}_2 \in C \cap B_\delta(\mathbf{x}_1)$ the arc of geodesic joining the points $\mathbf{x}_1, \mathbf{x}_2$ is a shortest curve between these points.

Consider the following two Lagrangians: Lagrangian of a "free " particle $L_{\text{free}} = \frac{g_{ik}(x)\dot{x}^i\dot{x}^k}{2}$ and the length Lagrangian

$$L_{\text{length}}(x,\dot{x}) = \sqrt{g_{ik}(x)\dot{x}^i\dot{\mathbf{x}}^k} = \sqrt{2L_{\text{free}}}\,.$$

If $C\colon\ x^i(t), t_1 \le t \le t_2$ is a curve on $M$ then

$$\text{Length of the curve } C =$$

$$\int_{t_1}^{t_2} L_{\text{length}}(x^i(t),\dot{x}^i(t))dt = \int_{t_1}^{t_2} \sqrt{g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t)}dt\,. \tag{5.6}$$

The proof of the Theorem follows from the following observation:

*Observation* Euler-Lagrange equations for the length functional (5.6) are equivalent to the Euler-Lagrange equations for action functional (5.2). This means that extremals of the length functional and action functionals coincide.

Indeed it follows from this observation and the variational principe that the shortest curves obey the Euler-Lagrange equations for the action functional. We showed before that Euler-Lagrange equations for action functional (5.2) define geodesics. Hence the shortest curves are geodesics.

One can check the observation by direct calculation: Calculate Euler-Lagrange equations for the Lagrangian $L_{\text{length}} = \sqrt{g_{ik}(x)\dot{x}^i\dot{x}^k} = \sqrt{2L_{\text{free}}}$:

$$\frac{d}{dt}\left(\frac{\partial L_{\text{length}}}{\partial \dot{x}^i}\right) - \frac{\partial L_{\text{length}}}{\partial x^i} = \frac{d}{dt}\left(\frac{1}{\sqrt{g_{ik}\dot{x}^i\dot{x}^k}}g_{ik}\dot{x}^k\right) - \frac{1}{2\sqrt{g_{ik}\dot{x}^i\dot{x}^k}}\frac{\partial g_{km}\dot{x}^k\dot{x}^m}{\partial x^i}$$

$$= \frac{d}{dt}\left(\frac{1}{L_{\text{length}}}\frac{\partial L_{\text{free}}}{\dot{\partial} x^i}\right) - \frac{1}{L_{\text{length}}}\frac{\partial L_{\text{free}}}{\partial x^i} = 0\,. \tag{5.7}$$

To facilitate calculations note that the length functional (5.6) is reparameterisation invariant. Choose the natural parameter $s(t)$ or a parameter proportional to the natural parameter on the curve $x^i(t)$. We come to $L_{\text{length}} = \mathit{const}$ and it follows from (5.7) that

$$\frac{d}{dt}\left(\frac{\partial L_{\text{length}}}{\partial \dot{x}^i}\right) - \frac{\partial L_{\text{length}}}{\partial x^i} = \frac{1}{L_{\text{length}}}\left(\frac{d}{dt}\left(\frac{\partial L_{\text{free}}}{\dot{\partial} x^i}\right) - \frac{\partial L_{\text{free}}}{\partial x^i}\right) = 0\,.$$

We prove that Euler-Lagrange equations for length and action Lagrangians coincide. ■

In the Euclidean space straight lines are the shortest distances between two points. On the other hand their velocity vectors are constant. We realise now that in general Riemannian manifold the role of geodesic is twofold also: they are locally shortest and have covariantly constant velocity vectors.

### 5.2.1 Again geodesics for sphere and Lobachevsky plane

The fact that geodesics are shortest gives us another tool to calculate geodesics.

Consider again examples of sphere and Lobachevsky plane and find geodesics using the fact that they are shortest. The fact that geodesics are locally the shortest curves

Consider again sphere in $\mathbf{E}^3$ with the radius $R$: Coordinates $\theta, \varphi$, induced Riemannian metrics (first quadratic form):

$$G = R^2(d\theta^2 + \sin^2\theta d\varphi^2)\,. \tag{5.8}$$

Consider two arbitrary points $A$ and $B$ on the sphere. Let $(\theta_0, \varphi_0)$ be coordinates of the point $A$ and $(\theta_1, \varphi_1)$ be coordinates of the point $B$

Let $C_{AB}$ be a curve which connects these points: $C_{AB}\colon \theta(t), \varphi(t)$ such that $\theta(t_0) = \theta_0, \theta(t_1) = \theta_1$, $\varphi(t_0) = \theta_0, \theta(t_1) = \theta_1$ then:

$$L_{C_{AB}} = \int R\sqrt{\theta_t^2 + \sin^2\theta(t)\varphi_t^2}dt \tag{5.9}$$

Suppose that points $A$ and $B$ have the same latitude, i.e. if $(\theta_0, \varphi_0)$ are coordinates of the point $A$ and $(\theta_1, \varphi_1)$ are coordinates of the point $B$ then $\varphi_0 = \varphi_1$ (if it is not the fact then we can come to this condition rotating the sphere)

Now it is easy to see that an arc of meridian, the curve $\varphi = \varphi_0$ is geodesics: Indeed consider an arbitrary curve $\theta(t), \varphi(t)$ which connects the points $A, B$: $\theta(t_0) = \theta(t_1) = \theta_0$, $\varphi(t_0) = \varphi(t_1) = \varphi_0$. Compare its length with the length of the meridian which connects the points $A$, $B$:

$$\int_{t_0}^{t_1} R\sqrt{\theta_t^2 + \sin^2\theta\varphi_t^2}dt \geq R\int_{t_0}^{t_1}\sqrt{\theta_t^2}dt = R\int_{t_0}^{t_1}\theta_t dt = R(\theta_1 - \theta_0) \tag{5.10}$$

Thus we see that the great circle joining points $A, B$ is the shortest. *The great circles on sphere are geodesics.* It corresponds to geometrical intuition: The geodesics on the sphere are the circles of intersection of the sphere with the plane which crosses the centre.

*Geodesics on Lobachevsky plane*

Riemannian metric on Lobachevsky plane:

$$G = \frac{dx^2 + dy^2}{y^2} \tag{5.11}$$

The length of the curve $\gamma\colon x = x(t, y = y(t))$ is equal to

$$L = \int\sqrt{\frac{x_t^2 + y_t^2}{y^2(t)}}dt$$

In particularly the length of the vertical interval $[1, \varepsilon]$ tends to infinity if $\varepsilon \to 0$:

$$L = \int \sqrt{\frac{x_t^2 + y_t^2}{y^2(t)}} dt = \int_\varepsilon^1 \sqrt{\frac{1}{t^2}} dt = \log \frac{1}{\varepsilon}$$

One can see that the distance from every point to the line $y = 0$ is equal to infinity. This motivates the fact that the line $y = 0$ is called *absolute*.

Consider two points $A = (x_0, y_0)$, $B = (x_1, y_1)$ on Lobachevsky plane.

It is easy to see that vertical lines are geodesics of Lobachevsky plane.

Namely let points $A, B$ are on the ray $x = x_0$. Let $C_{AB}$ be an arc of the ray $x = x_0$ which joins these points: $C_{AB} \colon x = x_0, y = y_0 + t$ Then it is easy to see that the length of the curve $C_{AB}$ is less or equal than the length of the arbitrary curve $x = x(t), y = y(t)$ which joins these points: $x(t)\big|_{t=0} = x_0, y(t)\big|_{t=0} = y_0$, $x(t)\big|_{t=t_1} = x_0, y(t)\big|_{t=t_1} = y_1$:

$$\int_0^t \sqrt{\frac{x_t^2 + y_t^2}{y^2(t)}} dt \geq \int_0^t \sqrt{\frac{y_t^2}{y^2(t)}} dt = \int_{y_0}^{y_1} \frac{dt}{t} dt = \log \frac{y_1}{y_0} = \text{length of } C_{AB}$$

Hence $C_{AB}$ is shortest. We prove that vertical rays are geodesics.

Consider now the case if $x_0 \neq x_1$. Find geodesics which connects two points $A, B$ which are not on the same vertical ray. Consider semicircle which passes these two points such that its centre is on the absolute. We prove that it is a geodesic.

*Proof* Let coordinates of the centre of the circle are $(a, 0)$. Then consider polar coordinates $(r, \varphi)$:

$$x = a + r\cos\varphi, y = r\sin\varphi \tag{5.12}$$

In these polar coordinates $r$-coordinate of the semicircle is constant.

Find Lobachevsky metric in these coordinates: $dx = -r\sin\varphi d\varphi + \cos\varphi dr$, $dy = r\cos\varphi d\varphi + \sin\varphi dr$, $dx^2 + dy^2 = dr^2 + r^2 d\varphi^2$. Hence:

$$G = \frac{dx^2 + dy^2}{y^2} = \frac{dr^2 + r^2 d\varphi^2}{r^2 \sin^2\varphi} == \frac{d\varphi^2}{\sin^2\varphi} + \frac{dr^2}{r^2\sin^2\varphi} \tag{5.13}$$

We see that the length of the arbitrary curve which connects points $A, B$ is greater or equal to the length of the arc of the circle:

$$L_{AB} = \int_{t_0}^{t_1} \sqrt{\frac{\varphi_t^2}{\sin^2\varphi} + \frac{r_t^2}{r^2\sin^2\varphi}} dt \geq \int_{t_0}^{t_1} \sqrt{\frac{\varphi_t^2}{\sin^2\varphi}} dt = \tag{5.14}$$

$$\int_{t_0}^{t_1} \frac{\varphi_t}{\sin\varphi} dt = \int_{\varphi_0}^{\varphi_1} \frac{d\varphi}{\sin\varphi} = \log \frac{\tan\varphi_1}{\tan\varphi_1}$$

The proof is finished.

## 5.3 Integrals of motions and geodesics.

### 5.3.1 Magnitudes preserving along geodesics—Integrals of motion

It is very useful to find magnitudes which are preserved along geodesics, functions $F = F(x, \dot{x})$ such that for geodesic $C\colon x^i = x^i(t)$ the magnitude

$$I(t) = F(x, \dot{x})\big|_{-}x^i = x^i(t) \quad \text{is preserved along geodesic } x^i(t), \quad \frac{dI(t)}{dt} = 0\,. \tag{5.15}$$

Geodesics are solutions of equations of motions for the Lagrangian of a free particle $L = \frac{g_{ik}(x)\dot{x}^i\dot{x}^k}{2}$. One can consider such functions $F = F(x, \dot{x})$ for an arbitrary Lagrangian $L$. In this case $x^i(t)$ is a solution of the Lagrangian $L$.

These magnitudes which are preserved along solutions of equations of motion (in particular along geodesics in the case if $L$ is the Lagrangian of a free particle) are called *integrals of motion* (See in detail about integrals of motion in Appendix to this lectures).

There is the following very useful criterion to find magnitudes, which are preserved on equations of motions, i.e. integrals fo motion.

**Proposition** *Let Lagrangian $L(x^i, \dot{x}^i)$ in coordinates $\{x^i\}$ does not depend, say on the coordinate $x^1$. $L = L(x^2, \dots, x^n, \dot{x}^1, \dot{x}^2, \dots, \dot{x}^n)$. Then the function*

$$F_1(x, \dot{x}) = \frac{\partial L}{\partial \dot{x}^1}$$

*is integral of motion. (In the case if $L(x^i, \dot{x}^i)$ does not depend on the coordinate $x^i$. the function $F_i(x, \dot{x}) = \frac{\partial L}{\partial \dot{x}^i}$ will be integral of motion.)*

Proof is simple. Check the condition (5.15): Euler-Lagrange equations of motion are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0 \quad (i = 1, 2, \dots, n)$$

In particular for first coordinate $x^1$, $\frac{\partial L}{\partial x^1} = 0$ and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^1}\right) - \frac{\partial L}{\partial x^1} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^1}\right) = 0\,,$$

i.e. the magnitude $I(t) = F(x, \dot{x})$ is preserved if $F = \frac{\partial L}{\partial \dot{x}^1}$. We see that exactly first equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^1}\right) = \frac{d}{dt} F_1(q, \dot{q}) = 0 \quad \text{since } \tfrac{\partial L}{\partial x^1} = 0,.$$

(if $L(x^i, \dot{x}^i)$ does not depend on the coordinate $x^i$ then the function $F_i(x, \dot{x}) = \frac{\partial L}{\dot{x}^1}$ is integral of motion since $i - th$ equation is exactly the condition $\dot{F}_i = 0$.)

The integral of motion $F_i = \frac{\partial L}{\partial \dot{x}^i}$ is called sometimes *generalised momentum.*

Consider examples of calculation of preserved mangnitudes along geodesics.

**Example (sphere)**

Sphere of the radius $R$ in $\mathbf{E}^3$. Riemannian metric: $G = Rd\theta^2 + R^2 \sin^2\theta d\varphi^2$ and $L_{\text{free}} = \frac{1}{2}\left(R^2\dot{\theta}^2 + R^2\sin^2\theta\dot{\varphi}^2\right)$. Lagrangian does not depend explicitly on coordinate $\varphi$. The integral of motion is

$$F = \frac{\partial L_{\text{free}}}{\partial \dot{\varphi}} = R^2 \sin^2\theta \dot{\varphi}\,.$$

It is preserved along geodesics, i.e. along great circles.

**Example (cone)**

Consider cone $\begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases}$. Riemannian metric:

$$G = d(kh\cos\varphi)^2 + d(kh\sin\varphi)^2 + (dh)^2 = (k^2+1)dh^2 + k^2h^2 d\varphi^2\,.$$

and free Lagrangian

$$L_{\text{free}} = \frac{(k^2+1)\dot{h}^2 + k^2h^2\dot{\varphi}^2}{2}\,.$$

Lagrangian does not depend explicitly on coordinate $h$. The integral of motion is

$$F = \frac{\partial L_{\text{free}}}{\partial \dot{\varphi}} = k^2h^2\dot{\varphi}.$$

It is preserved along geodesics.

**Remark** One has to note that for the Lagrangian of a free particle $F = L = g_{ik}\dot{x}^i\dot{x}^k$, kinetik energy, is integral of motion preserved along geodesic:it is nothing that square of the length of velocity vector which is preserved along the geodesic.

### 5.3.2 Using integral of motions to calculate geodesics

Integrals of motions may be very useful to calculate geodesics. The equations for geodesics are second order differential equations. If we know integrals of motions they help us to solve these equations. Consider just an example.

For Lobachevsky plane the free Lagrangian $L = \frac{\dot{x}^2+\dot{y}^2}{2y^2}$. We already calculated geodesics in the subsection 3.3.4. Geodesics are solutions of second order Euler-Lagrange equations for the Lagrangian $L = \frac{\dot{x}^2+\dot{y}^2}{2y^2}$ (see the subsection 3.3.4)

$$\begin{cases} \ddot{x} - \frac{2\dot{x}\dot{y}}{y} = 0 \\ \ddot{y} + \frac{\dot{x}^2}{y} - \frac{\dot{y}^2}{y} = 0 \end{cases}$$

It is not so easy to solve these differential equations.

For Lobachevsky plane we know two integrals of motions:

$$E = L = \frac{\dot{x}^2 + \dot{y}^2}{2y^2}, \quad \text{and } F = \frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{y^2} \,. \tag{5.16}$$

These both integrals are preserved in time: if $x(t), y(t)$ is geodesics then

$$\begin{cases} F = \frac{\dot{x}(t)}{y(t)^2} \\ E = \frac{\dot{x}(t)^2 + \dot{y}(t)^2}{2y(t)^2} = C_2 \end{cases} \Rightarrow \begin{cases} \dot{x} = C_1 y^2 \\ \dot{y} = \pm\sqrt{2C_2 y^2 - C_1^2 y^4} \end{cases} \tag{5.17}$$

These are first order differential equations. It is much easier to solve these equations in general case than initial second order differential equations.

We see how useful in Riemannian geometry to use the Lagrangian approach.

To solve and study solutions of Lagrangian equations (in particular geodesics which are solutions of Euler-Lagrange equations for Lagrangian of free particle) it is very useful to use integrals of motion

### 5.3.3 Integral of motion for arbitrary Lagrangian $L(x, \dot{x})$

Let $L = L(x, \dot{x})$ be a Lagrangian, the function of point and velocity vectors on manifold $M$ (the function on tangent bundle $TM$).

**Definition** We say that the function $F = F(q, \dot{q})$ on $TM$ is *integral of motion* for Lagrangian $L = L(x, \dot{x})$ if for any curve $q = q(t)$ which is the solution of Euler-Lagrange equations of motions the magnitude $I(t) = F(x(t), \dot{x}(t))$ is preserved along this curve:

$$F(x(t), \dot{x}(t)) = const \text{ if } x(t) \text{ is a solution of Euler-Lagrange equations(\textbf{??})}. \tag{5.18}$$

In other words

$$\frac{d}{dt}\left(F(x(t), \dot{x}(t))\right) = 0 \text{ if } x^i(t)\colon \ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0 \,. \tag{5.19}$$

### 5.3.4 Basic examples of Integrals of motion: Generalised momentum and Energy

Let $L(x^i, \dot{x}^i)$ does not depend on the coordinate $x^1$. $L = L(x^2, \dots, x^n, \dot{x}^1, \dot{x}^2, \dots, \dot{x}^n)$. Then the function

$$F_1(x, \dot{x}) = \frac{\partial L}{\partial \dot{x}^1}$$

is integral of motion. (In the case if $L(x^i, \dot{x}^i)$ does not depend on the coordinate $x^i$. the function $F_i(x, \dot{x}) = \frac{\partial L}{\partial \dot{x}^i}$ will be integral of motion.)

Proof is simple. Check the condition (5.19): Euler-Lagrange equations of motion are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0 \quad (i = 1, 2, \dots, n)$$

We see that exactly first equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^1}\right) = \frac{d}{dt}F_1(q,\dot{q}) = 0 \quad \text{since } \tfrac{\partial L}{\partial x^1} = 0, .$$

(if $L(x^i, \dot{x}^i)$ does not depend on the coordinate $x^i$ then the function $F_i(x,\dot{x}) = \frac{\partial L}{\dot{x}^1}$ is integral of motion since $i-th$ equation is exactly the condition $\dot{F}_i = 0$.)

The integral of motion $F_i = \frac{\partial L}{\partial \dot{x}^i}$ is called sometimes *generalised momentum.*

Another very important example of integral of motion is: energy.

$$E(x,\dot{x}) = \dot{x}^i \frac{\partial L}{\partial \dot{x}^i} - L\,. \tag{5.20}$$

One can check by direct calculation that it is indeed integral of motion. Using Euler Lagrange equations $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i}$ we have:

$$\frac{d}{dt}E(x(t),\dot{x}(t)) = \frac{d}{dt}\left(\dot{x}^i \frac{\partial L}{\partial \dot{x}^i} - L\right) = \frac{\partial L}{\partial \dot{x}^i}\frac{d\dot{x}^i}{dt} + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right)\dot{x}^i - \frac{dL}{dt} =$$

$$\frac{\partial L}{\partial \dot{x}^i}\frac{d\dot{x}^i}{dt} + \frac{\partial L}{\partial x^i}\frac{dx^i}{dt} - \frac{dL(x,\dot{x})}{dt} = \frac{dL(x,\dot{x})}{dt} - \frac{dL(x,\dot{x})}{dt} = 0\,.$$

### 5.3.5 Integrals of motion for geodesics

Apply the integral of motions for studying geodesics.

The Lagrangian of "free" particle $L_{\text{free}} = \frac{g_{ik}(x)\dot{x}^i\dot{x}^k}{2}$. For Lagrangian of free particle solution of Euler-Lagrange equations of motions are geodesics.

If $F = F(x,\dot{x})$ is the integral of motion of the free Lagrangian $L_{\text{free}} = \frac{g_{ik}(x)\dot{x}^i\dot{x}^k}{2}$ then the condition (5.18) means that the magnitude $I(t) = F(x^i(t), \dot{x}^i(t))$ is preserved along the geodesics:

$$I(t) = F(x^i(t), \dot{x}^i(t)) = const, \text{i.e. } \frac{d}{dt}I(t) = 0 \text{ if } x^i(t) \text{ is geodesic.} \tag{5.21}$$

Consider examples of integrals of motion for free Lagrangian, i.e. magnitudes which preserve along the geodesics:

**Example 1** Note that for an arbitrary "free" Lagrangian Energy integral (5.20) is an integral of motion:

$$E = \dot{x}^i \frac{\partial L}{\partial \dot{x}^i} - L = \dot{x}^i \frac{\partial\left(\frac{g_{pq}(x)\dot{x}^p\dot{x}^q}{2}\right)}{\partial \dot{x}^i} - \frac{g_{ik}(x)\dot{x}^i\dot{x}^k}{2} =$$

$$\dot{x}^i g_{iq}(x)\dot{x}^q - \frac{g_{ik}(x)\dot{x}^i\dot{x}^k}{2} = \frac{g_{ik}(x)\dot{x}^i\dot{x}^k}{2}\,. \tag{5.22}$$

This is an integral of motion for an arbitrary Riemannian metric. It is preserved on an arbitrary geodesic

$$\frac{dE(t)}{dt} = \frac{d}{dt}\left(\frac{1}{2}g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t)\right) = 0\,.$$

In fact we already know this integral of motion: Energy (5.22) is proportional to the square of the length of velocity vector:

$$|\mathbf{v}| = \sqrt{g_{ik}(x)\dot{x}^i\dot{x}^k} = \sqrt{2E}\,. \tag{5.23}$$

We already proved that velocity vector is preserved along the geodesic (see the Proposition in the subsection 3.2.1 and its proof (**??**).)

**Example 2** Consider Riemannian metric $G = adu^2 + bdv^2$ (see also calculations in subsection 2.3.3) in the case if $a = a(u)$, $b = b(u)$, i.e. coefficients do not depend on the second coordinate $v$:

$$G = a(u)du^2 + b(u)dv^2,\ L_{\text{free}} = \frac{1}{2}\left(a(u)\dot{u}^2 + b(u)\dot{v}^2\right) \tag{5.24}$$

We see that Lagrangian does not depend on the second coordinate $v$ hence the magnitude

$$F = \frac{\partial L_{\text{free}}}{\partial \dot{v}} = b(u)\dot{v} \tag{5.25}$$

is preserved along geodesic. It is integral of motion because Euler-Lagrange equation for coordinate $v$ is

$$\frac{d}{dt}\frac{\partial L_{\text{free}}}{\partial \dot{v}} - \frac{\partial L_{\text{free}}}{\partial v} = \frac{d}{dt}\frac{\partial L_{\text{free}}}{\partial \dot{v}} = \frac{d}{dt}F = 0 \ \text{ since } \frac{\partial L_{\text{free}}}{\partial v} = 0.\,.$$

In fact all revolution surfaces which we consider here (cylinder, cone, sphere,...) have Riemannian metric of this type. Indeed consider further examples.

**Example (sphere)**

Sphere of the radius $R$ in $\mathbf{E}^3$. Riemannian metric: $G = Rd\theta^2 + R^2\sin^2\theta d\varphi^2$ and $L_{\text{free}} = \frac{1}{2}\left(R^2\dot{\theta}^2 + R^2\sin^2\theta\dot{\varphi}^2\right)$ It is the case (5.24) for $u = \theta, v = \varphi$, $b(u) = R^2\sin^2\theta$ The integral of motion is

$$F = \frac{\partial L_{\text{free}}}{\partial \dot{\varphi}} = R^2\sin^2\theta\dot{\varphi}$$

**Example (cone)**

Consider cone $\begin{cases} x = ah\cos\varphi \\ y = ah\sin\varphi \\ z = bh \end{cases}$. Riemannian metric:

$$G = d(ah\cos\varphi)^2 + d(ah\sin\varphi)^2 + (dbh)^2 = (a^2+b^2)dh^2 + a^2h^2d\varphi^2\,.$$

and free Lagrangian

$$L_{\text{free}} = \frac{(a^2+b^2)\dot{h}^2 + a^2h^2\dot{\varphi}^2}{2}\,.$$

The integral of motion is

$$F = \frac{\partial L_{\text{free}}}{\partial \dot{\varphi}} = a^2h^2\dot{\varphi}.$$

**Example (general surface of revolution)**
Consider a surface of revolution in $\mathbf{E}^3$:

$$\mathbf{r}(h,\varphi)\colon \begin{cases} x = f(h)\cos\varphi \\ y = f(h)\sin\varphi \\ z = h \end{cases} \qquad (f(h)>0) \tag{5.26}$$

(In the case $f(h) = R$ it is cylinder, in the case $f(h) = kh$ it is a cone, in the case $f(h) = \sqrt{R^2 - h^2}$ it is a sphere, in the case $f(h) = \sqrt{R^2 + h^2}$ it is one-sheeted hyperboloid, in the case $z = \cos h$ it is catenoid,...)

For the surface of revolution (5.26)

$$G = d(f(h)\cos\varphi)^2 + d(f(h)\sin\varphi)^2 + (dh)^2 = (f'(h)\cos\varphi dh - f(h)\sin\varphi d\varphi)^2 +$$
$$(f'(h)\sin\varphi dh + f(h)\cos\varphi d\varphi)^2 + dh^2 = (1 + f'^2(h))dh^2 + f^2(h)d\varphi^2\,.$$

The "free" Lagrangian of the surface of revolution is

$$L_{\text{free}} = \frac{\left(1 + f'^2(h)\right)\dot h^2 + f^2(h)\dot\varphi^2}{2}\,.$$

and the integral of motion is

$$F = \frac{\partial L_{\text{free}}}{\partial\dot\varphi} = f^2(h)\dot\varphi.$$

### 5.3.6 Using integral of motions to calculate geodesics

Integrals of motions may be very useful to calculate geodesics. The equations for geodesics are second order differential equations. If we know integrals of motions they help us to solve these equations. Consider just an example.

For Lobachevsky plane the free Lagrangian $L = \frac{\dot x^2 + \dot y^2}{2y^2}$. We already calculated geodesics in the subsection 3.3.4. Geodesics are solutions of second order Euler-Lagrange equations for the Lagrangian $L = \frac{\dot x^2 + \dot y^2}{2y^2}$ (see the subsection 3.3.4)

$$\begin{cases} \ddot x - \frac{2\dot x\dot y}{y} = 0 \\ \ddot y + \frac{\dot x^2}{y} - \frac{\dot y^2}{y} = 0 \end{cases}$$

It is not so easy to solve these differential equations.

For Lobachevsky plane we know two integrals of motions:

$$E = L = \frac{\dot x^2 + \dot y^2}{2y^2}, \quad \text{and } F = \frac{\partial L}{\partial\dot x} = \frac{\dot x}{y^2}\,. \tag{5.27}$$

These both integrals preserve in time: if $x(t), y(t)$ is geodesics then

$$\begin{cases} F = \frac{\dot x(t)}{y(t)^2} \\ E = \frac{\dot x(t)^2 + \dot y(t)^2}{2y(t)^2} = C_2 \end{cases} \Rightarrow \begin{cases} \dot x = C_1 y^2 \\ \dot y = \pm\sqrt{2C_2y^2 - C_1^2y^4} \end{cases}$$

These are first order differential equations. It is much easier to solve these equations in general case than initial second order differential equations.

### 5.3.7 Killing vectors of Lobachevsky plane and geodesics

Killing vector field of Rimeannina manifold $(M, G)$ is an infinitesimal isometry of the Rimeannian metric $G$: under infinitesimal transform $x \to x + \varepsilon\mathbf{K}, x^i \to x^i + \varepsilon K^I(x)$ ($\varepsilon^2 = 0$) metric does not change:

$$g_{ik}(x)dx^i dx^k = g_{ik}(x^r + \varepsilon K^i)(dx^i + \partial_m K^i dx^m)(dx^k + \partial_n K^k dx^n)\,. \tag{1}$$

Expanding this formula by $\varepsilon$ and using the fact that $\varepsilon^2 = 0$ we come to

$$K^i \partial_i g_{km} + \partial_k K^r g_{rm} + \partial_m K^r g_{rk} = 0\,, \tag{1a}$$

(i.e. Lie derivative $\mathcal{L}_{\mathbf{K}} G = 0$.)

Examples: Killings of plane, sphere, cylindre, Lobachevsky plane.....

**Theorem** Let $V$ be a vector space of all Killing vector fields of Riemannian manifold $M$. Then the dimension of $V$ is less or equal than $\frac{n(n+1}{2)}$.

It means that for surfaces the number of independent Killing vector fields is less or equal to 3.

One can prove that it is only for plane, sphere and Lobachevsky plane that number of independent Killing vector fiels is equal to 3.

We calculate here Killing vector fields for Lobachevsky plane and use them for finding geodesics.

**Theorem** Let $\mathbf{K}$ be Killing tor field on Riemannian manifold $(M, G)$ , and $L = \frac{g_{kp}\dot{x}^k \dot{x}^p}{2}$ Lagragian of 'free' particle on $M$. We know that geodesics are solutions of its equations of motions.

The magnitude

$$I = I_{\mathbf{K}} = K^i(x)\frac{\partial L(x, \dot{x})}{\partial \dot{x}^i}$$

is an integral of motion, i.e. it is preserved along geodesics.

The proof of the Theorem is obvious. The condition that $\mathbf{K}$ is Killing vector field means that

$$L(x^i + \varepsilon K^i, \dot{x}^i + \varepsilon \dot{K}^i)\,, \tag{2}$$

i.e.

$$K^i(x)\frac{\partial L}{\partial x^i} + \frac{dK^i}{dt}\frac{\partial L}{\partial \dot{x}^i} = 0\,. \tag{2a}$$

Hence

$$\frac{d}{dt} I_{\mathbf{K}} = \frac{d}{dt}\left(K^i(x)\frac{\partial L(x, \dot{x})}{\partial \dot{x}^i}\right) = \frac{dK^i}{dt}\frac{\partial L(x, \dot{x})}{\partial \dot{x}^i} + K^i \frac{d}{dt}\left(\frac{\partial L(x, \dot{x})}{\partial \dot{x}^i}\right) =$$

$$\underbrace{K^i(x)\frac{\partial L}{\partial x^i} + \frac{dK^i}{dt}\frac{\partial L}{\partial \dot{x}^i}}_{\text{condition that } \mathbf{K} \text{ is Killing}} + K^i \underbrace{\left(\frac{\partial L(x, \dot{x})}{\partial x^i} - \frac{\partial L(x, \dot{x})}{\partial \dot{x}^i}\right)}_{\text{equations of motion}} = 0\,.$$

Use this Theorem to find geodesics.

First find Killing vector fields, i.e. infinitesimal isometries.

Since the dimension is equal 2, the dimension of space of Killing vector fields is $\leq 3$. We will find three independent Killing vector fields.

There are two evident Killing vectors: Metric $G = \frac{dx^2+dy^2}{y^2}$ is evidently invariant with respect to translations $x \to x + a$ and homothety: $\begin{cases} x \to \lambda x \\ y \mapsto \lambda y \end{cases} : \frac{d(\lambda x)^2+d(\lambda y)^2}{(\lambda y)^2} = \frac{dx^2+dy^2}{y^2}$.

Infinitesimal translation is $x' = x + \varepsilon, y' = y$, the vector field $D_1 = \frac{\partial}{\partial x}$. Infinitesimal homothety is $x' = x + \varepsilon x, y' = y + \varepsilon y$, the vector field $D_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$.

Now most interesting: find the third Killing vector field. Use the fact that inversion $\mathbf{O}\colon (x, y) \mapsto \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$ preserves the metric. Consider the infinitesimal transformation $L_\varepsilon = \mathbf{O} \circ T_\varepsilon \circ \mathbf{O}$ ($L_0 = \mathbf{id}$):

$$L_\varepsilon\colon \quad \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \frac{x}{x^2+y^2} \\ \frac{y}{x^2+y^2} \end{pmatrix} \to \begin{pmatrix} \frac{x}{x^2+y^2} + \varepsilon \\ \frac{y}{x^2+y^2} \end{pmatrix} \mapsto \begin{pmatrix} \frac{\frac{x}{x^2+y^2}+\varepsilon}{\left(\frac{x}{x^2+y^2}+\varepsilon\right)^2+\left(\frac{y}{x^2+y^2}\right)^2} \\ \frac{\frac{y}{x^2+y^2}}{\left(\frac{x}{x^2+y^2}+\varepsilon\right)^2+\left(\frac{y}{x^2+y^2}\right)^2} \end{pmatrix}.$$