**Homework 2a. Solutions**

**1** *Let $A$ be a linear operator in 2-dimensional vector space $V$ such that for a given basis $\{\mathbf{e},\mathbf{f}\}$,*

$$A(\mathbf{e}) = 27\mathbf{e} + 40\mathbf{f}\,, A(\mathbf{f}) = -16\mathbf{e} - \frac{71}{3}\mathbf{f}\,.$$

*Write down the matrix of the operator $A$ in this basis.*

*Consider the pair of vectors $\{\mathbf{e}',\mathbf{f}'\}$ such that $\mathbf{e}' = 2\mathbf{e} + 3\mathbf{f}$ and $\mathbf{f}' = 3\mathbf{e} + 5\mathbf{f}$.*

*Show that an ordered set of vectors $\{\mathbf{e}',\mathbf{f}'\}$ is also a basis, and find a matrix of the operator $A$ in the new basis.*

*Calculate the determinant and trace of operator $A$ (compare determinants and traces of different matrix representations of this operator. )*

We have that for operator $A$,

$$\{\mathbf{e}',\mathbf{f}'\} = \{\mathbf{e},\mathbf{f}\}\begin{pmatrix} 27 & -16 \\ 40 & -\frac{71}{3} \end{pmatrix}$$

Hence matrix of operator $A$ in the basis $\{\mathbf{e},\mathbf{f}\}$ is $\begin{pmatrix} 27 & -16 \\ 40 & -\frac{71}{3} \end{pmatrix}$. Vectors $\mathbf{e}',\mathbf{f}'$ are linearly independent. Indeed

$$0 = c_1\mathbf{e}' + c_2\mathbf{f}' = c_1(2\mathbf{e} + 3\mathbf{f}) + c_2(3\mathbf{e} + 5\mathbf{f}) = (2c_1 + 3c_2)\mathbf{e} + (3c_1 + 5c_2)\mathbf{f} = 0\,.$$

Hence $2c_1 + 3c_2 = 0, 3c_1 + 5c_2 = 0$, i.e. $c_1 = c_2 = 0$.

Hence $\{\mathbf{e}',\mathbf{f}'\}$ is a basis also. We have that

$$\begin{cases} \mathbf{e}' = 2\mathbf{e} + 3\mathbf{f} \\ \mathbf{f}' = 3\mathbf{e} + 5\mathbf{f} \end{cases} \quad \text{hence} \quad \begin{cases} \mathbf{e} = 5\mathbf{e} - 3\mathbf{f} \\ \mathbf{f} = -3\mathbf{e} + 2\mathbf{f} \end{cases}$$

We have that for basis

$$A(\mathbf{e}') = A(2\mathbf{e} + 3\mathbf{f}) = 2(27\mathbf{e} + 40\mathbf{f}) + 3\left(-16\mathbf{e} - \frac{71}{3}\mathbf{f}\right) = 6\mathbf{e} + 9\mathbf{f} = 3\mathbf{e}'\,,'$$

$$A(\mathbf{f}') = A(3\mathbf{e} + 5\mathbf{f}) = 3(27\mathbf{e} + 40\mathbf{f}) + 5\left(-16\mathbf{e} - \frac{71}{3}\mathbf{f}\right) = \mathbf{e} + \frac{5}{3}\mathbf{f} = \frac{1}{3}\mathbf{f}'\,.'$$

We see that the matrix of operator $A$ in the new basis is $\begin{pmatrix} 3 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$. To calculate tace and determinant of operator $A$ it is convenient to use the representation of this operator my matirx in the second basis, on the other hand it is good to double check the answer in both bases:

$$\det A = \det\begin{pmatrix} 3 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} = 3\cdot\frac{1}{3} = \det\begin{pmatrix} 27 & -16 \\ 40 & -\frac{71}{3} \end{pmatrix} = 27\cdot\left(-\frac{71}{3}\right) - 40\cdot(-16) = -639 + 640 = 1\,,$$

$$\operatorname{Tr} A = \operatorname{Tr}\begin{pmatrix} 3 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} = 3 + \frac{1}{3} = \operatorname{Tr}\begin{pmatrix} 27 & -16 \\ 40 & -\frac{71}{3} \end{pmatrix} = 27 - \frac{71}{3} = \frac{10}{3}\,,$$

**3** *Let $\mathbf{e},\mathbf{f}$ be orthonormal basis in Euclidean space $\mathbf{E}^2$. Consider a vector*

$$\mathbf{n}_\varphi = \mathbf{e}\cos\varphi + \mathbf{f}\sin\varphi\,.$$

*Let $A$ be a linear orthogonal operator acting on the space $\mathbf{E}^2$ such that $A(\mathbf{e}) = \mathbf{n}$.*

*We know that $\det A = \pm 1$ since $A$ is orthogonal operator.*

*In the case if $\det A = 1$, find the image $A(\mathbf{f})$ of vector $\mathbf{f}$ and an image $A(\mathbf{x})$ of arbitrary vector $\mathbf{x} = a\mathbf{e}+b\mathbf{f}$, write down the matrix of operator $A$ in the basis $\mathbf{e},\mathbf{f}$ and explain geometrical meaning of the operator $A$.*

† *How the answer will change if* $\det A = -1$*?*

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be transition matrix of operator $A$ in the orthonormal basis $\{\mathbf{e}, \mathbf{f}\}$:

$$\{\mathbf{e}', \mathbf{f}'\} = \{\mathbf{e}, \mathbf{f}\}$$

New basis is also orthonormal. We have that $\mathbf{e}' = \mathbf{n}_\varphi = \mathbf{e}\cos\varphi + \mathbf{f}\sin\varphi$, hence matrix of the orthonomal operator is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\varphi & b \\ \sin\varphi & d \end{pmatrix}$$

Matrix of orthogonal operator is orthogonal matrix. Hence $\begin{pmatrix} \cos\varphi & b \\ \sin\varphi & d \end{pmatrix}$ is orthogonal matrix, i.e.

$$\begin{cases} b\cos\varphi + d\sin\varphi = 0 \\ b^2 + d^2 = 1 \end{cases}.$$

Put $b = \sin\psi, d = \cos\psi$, then bearing in mind the condition that $\det A = d\cos\varphi - b\sin\varphi = 1$, we come to equations

$$\begin{cases} b\cos\varphi + d\sin\varphi = \sin\psi\cos\varphi + \cos\psi\sin\varphi = \sin(\varphi+\psi) = 0 \\ d\cos\varphi - b\sin\varphi = \cos\psi\cos\varphi - \sin\psi\sin\varphi = \cos(\varphi+\psi) = 1 \end{cases},$$

i.e. we come to $\psi = -\varphi + 2\pi k$. Matrix of operator $A$ in the basis $\{\mathbf{e}, \mathbf{f}\}$ is equal to $\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$. $A$ is operator of rotation on the angle $\varphi$ (see the section 1.9 in lecture notes). $A(\mathbf{f}) = b\mathbf{e} + d\mathbf{f} = -\sin\varphi\mathbf{e} + \cos\varphi\mathbf{f}$. For arbitrary vector $\mathbf{x}$ we have that

$$A(\mathbf{x}) = A(x^1\mathbf{e} + x^2\mathbf{f}) = x^1A(\mathbf{e}) + x^2A(\mathbf{f}) = x^1(\mathbf{e}\cos\varphi + \mathbf{f}\sin\varphi) + x^2(-\mathbf{e}\sin\varphi + \mathbf{f}\cos\varphi) =$$

$$(x^1\cos\varphi - x^2\sin\varphi)\mathbf{e} + (x^1\sin\varphi + x^2\cos\varphi)\mathbf{f},$$

or in the other way: $A(\mathbf{x}) = A(x^1\mathbf{e} + x^2\mathbf{f}) =$

$$= A\left(\{\mathbf{e}, \mathbf{f}\}\begin{pmatrix} x^1 \\ x^2 \end{pmatrix}\right) = \{\mathbf{e}, \mathbf{f}\}\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \{\mathbf{e}, \mathbf{f}\}\begin{pmatrix} x^1\cos\varphi - x^2\sin\varphi \\ x^1\sin\varphi + x^2\cos\varphi \end{pmatrix}.$$

† One can see that in this case this is the operator of reflection with respect to the line directed along the vector $\mathbf{n}_{\frac{\varphi}{2}} = \cos\frac{\varphi}{2}\mathbf{e} + \sin\frac{\varphi}{2}\mathbf{f}$.

**3** *Let* $\mathbf{e}, \mathbf{f}$ *be an orthonormal basis in Euclidean space* $\mathbf{E}^2$*. Consider a vector* $\mathbf{N} = \mathbf{e} + \mathbf{f}$ *in* $\mathbf{E}^2$*.*

*Let* $A$ *be an orthogonal operator acting on the space* $\mathbf{E}^2$ *such that* $A\mathbf{N} = \mathbf{N}$*. (*$\mathbf{N}$ *is eigenvector of* $A$ *with eigenvalue 1.) Suppose that* $A$ *is not identity operator.*

*a) Find an action of operator* $A$ *on the vector* $\mathbf{R} = \mathbf{e} - \mathbf{f}$ *in* $\mathbf{E}^2$*.*

*b) Write down the matrix of operator* $A$ *in the basis* $\mathbf{e}, \mathbf{f}$*.*

*c) Explain geometrical meaning of the operator* $A$*.*

Let $A(\mathbf{R}) = a\mathbf{e} + b\mathbf{f}$. Vectors $\mathbf{N}$ and $\mathbf{R}$ are orthogonal to each other (they both have the length $\sqrt{2}$). Hence the vectors $A(\mathbf{N})$ and $A(\mathbf{R})$ have to be orthogonal to each other also, since orthogonal operator does not change the scalar product.

Hence vector $A(\mathbf{R})$ has to be proportional to the vector $\mathbf{R}$ also, i.e. $A(\mathbf{R}) = a\mathbf{R}$. The length of the vector iss not changed under othogonal transformation, hence $a = \pm 1$. If $a = 1$, i.e. $A(\mathbf{R}) = \mathbf{R}$ we see that operator $A$ is identical on two linear independent vectors, hence it is identical on their span, i.e. $A = \mathbf{id}$.

On the other hand we know that $A$ is not identity operator. Hence $a = -1$. We come to the conclusion that $A(\mathbf{R}) = -\mathbf{R}$.

Operator $A$ is reflection operator with respect to the line directed along the vector $\mathbf{N}$. (if $A(\mathbf{R}) = a\mathbf{e}+b\mathbf{f}$ then $(A(\mathbf{R}), A(\mathbf{N})) = (a\mathbf{e} + b\mathbf{f}, \mathbf{N}) = (ae + b\mathbf{f}, \mathbf{e} - \mathbf{f}) = a - b = 0$)

Matrix of the operator $A$ in the basis $\{\mathbf{e}, \mathbf{f}\}$ is the transition matrix of this operator for this basis. We have that $\mathbf{e} = \frac{\mathbf{N}+\mathbf{R}}{2}$ and $\mathbf{f} = \frac{\mathbf{N}-\mathbf{R}}{2}$. Hence $A(\mathbf{e}) = \frac{\mathbf{N}-\mathbf{R}}{2} = \mathbf{f}$ and $A(\mathbf{f}) = \frac{\mathbf{N}+\mathbf{R}}{2} = \mathbf{e}$. We see that the matrix of operator $A$ in the bases $\{\mathbf{e}, \mathbf{f}\}$ is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. ■

**4** *Let $V$ be a space of functions, which are solutions of differential equation*

$$\frac{d^2y(x)}{dx^2} + p\frac{dy(x)}{dx} + qy(x) = 0\,, \tag{1}$$

*where parameters $p, q$ are equal to*

$$p = -7\,, q = 12\,.$$

*Show that $V$ is 2-dimensional vector space.*

*Find a basis in this vector space, and write down the operator $A$ in this basis.*

*Differentiation $A = \frac{d}{dx}$ is linear operator on space $V$ which transforms every vector from $V$ to another vector on $V$. Check it.*

*Find determinant and trace of this operator.*

This is linear differential equation. Linear combination of solutions is a solution. Hence space of solutions is a vector space.

If $y(x)$ is a solution of differential equation (1), then obviously $Ay(x) = \frac{d}{dx}y(x)$ is a solution also. Hence $A$ is an operator on space of solutions.

One can see that an arbitrary solution of this equation is

$$y(x) = c_1e^{3x} + c_2e^{4x}\,,$$

where functions $e^{3x}$, $e^{4x}$ are eigenvectors of the operator $A$ with eigenvalues 3 and 4 respectively. Space of solutions is a span of eignevectors $e^{3x}, e^{4x}$.

These vectors (functions) form a basis $\{\mathbf{e}, \mathbf{f}\}$ in the vector space $V$, $\mathbf{e} = e^{3x}$, $\mathbf{f} = e^{4x}$.

$$A(\mathbf{e}) = \frac{d}{dx}e^{3x} = 3e^{3x} = 3\mathbf{e} \quad A(\mathbf{f}) = \frac{d}{dx}e^{4x} = 4e^{3x} = 4\mathbf{f}$$

matrix of the operator $A$ in this basis is $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$. We have that $\det A = 12$ and $\operatorname{Tr} A = 7$.

**5**$^\dagger$ *Solve the problem 2 in the case if parameters $p, q$ are equal to $p = -6$, $q = 9$.*

In this case solution of this equation are

$$y(x) = c_1e^{4x} + c_2xe^{4x}\,.$$

i.e. space of solutions is a span of functions $e^4x, xe^{4x}$.

These functions form a basis $\{\mathbf{e}, \mathbf{f}\}$ in the vector space $V$, $\mathbf{e} = e^{4x}$, $\mathbf{f} = xe^{4x}$.

$$A(\mathbf{e}) = \frac{d}{dx}e^{4x} = 4e^{3x} = 4\mathbf{e} \quad A(\mathbf{f}) = \frac{d}{dx}(xe^{4x}) = e^{4x} + 4xe^{4x} = \mathbf{e} + 4\mathbf{f}$$

matrix of the operator $A$ in this basis is $\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}$. This is Jordan cell. It cannot be diagonalized. We have that $\det A = 16$ and $\operatorname{Tr} A = 8$.