Homework 5. Solutions.

1. *Calculate Levi-Civita connection of the metric $G = a(u,v)du^2 + b(u,v)dv^2$*
a) in the case if functions $a(u,v)$, $b(u,v)$ are constants.
b) In general case*

We know that for Levi-Civita connection

$$\Gamma^i_{mk} = \frac{1}{2} g^{ij} \left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^m} - \frac{\partial g_{mk}}{\partial x^j} \right) . \tag{1}$$

a) We do not need to do any calculations since a and b are constants, and all partial derivatives $\frac{\partial g_{jm}}{\partial x^k}$ for metric $G = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ are equal to zero. Hence all Christoffel symbols vanish.

b) In this case we have to perform calculations:
We have

$$G = a(u,v)du^2 + b(u,v)dv^2, G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} a(u,v) & 0 \\ 0 & b(u,v) \end{pmatrix}, G^{-1} = \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{a(u,v)} & 0 \\ 0 & \frac{1}{b(u,v)} \end{pmatrix}.$$

Hence according to (1)

$$\begin{aligned}
\Gamma^1_{11} = \Gamma^u_{uu} &= \tfrac{1}{2} g^{11} \left(\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11} \right) = \tfrac{1}{2} g^{11} \partial_u g_{uu} = & \tfrac{a_u}{2a} \\
\Gamma^1_{21} = \Gamma^1_{12} = \Gamma^u_{uv} = \Gamma^u_{vu} &= \tfrac{g^{11}}{2} \left(\partial_1 g_{12} + \partial_2 g_{11} - \partial_1 g_{12} \right) = \tfrac{g^{uu}}{2} \partial_v g_{uu} = & \tfrac{a_v}{2a} \\
\Gamma^1_{22} = \Gamma^u_{vv} &= \tfrac{g^{11}}{2} \left(\partial_2 g_{12} + \partial_2 g_{12} - \partial_1 g_{22} \right) = -\tfrac{g^{uu}}{2} \partial_u g_{vv} = & -\tfrac{b_u}{2a} \\
\Gamma^2_{11} = \Gamma^v_{uu} &= \tfrac{g^{22}}{2} \left(\partial_1 g_{12} + \partial_1 g_{12} - \partial_2 g_{11} \right) = -\tfrac{g^{vv}}{2} \partial_v g_{uu} = & -\tfrac{a_v}{2b} \\
\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^v_{uv} = \Gamma^v_{vu} &= \tfrac{g^{22}}{2} \left(\partial_2 g_{21} + \partial_1 g_{22} - \partial_2 g_{21} \right) = \tfrac{g^{vv}}{2} \partial_u g_{vv} = & \tfrac{b_u}{2b} \\
\Gamma^2_{22} = \Gamma^v_{vv} &= \tfrac{g^{22}}{2} \left(\partial_2 g_{22} + \partial_2 g_{22} - \partial_2 g_{22} \right) = \tfrac{g^{vv}}{2} \partial_v g_{vv} = & \tfrac{b_v}{2b}
\end{aligned}$$

(We use notations $(u,v) = (x^1, x^2)$.)

2. *Calculate Levi-Civita connection of the metric $G = adu^2 + bdv^2$ at the point $u = v = 0$ in the case if functions $a(u,v)$, $b(u,v)$ equal to constants at the point $u = v = 0$ up to the second order:*

$$a(u,v) = a_0 + \dots, \quad b(u,v) = b_0 + \dots$$

where dots mean the terms of the second and higher order with respect to u, v.

We see that at the point $u = v = 0$ all the derivatives $\frac{\partial g_{jm}}{\partial x^k}$ are equal to zero. Hence according to (1) all Christoffel symbols vanish at the point $u = v = 0$.

3. *Calculate $\nabla_{\frac{\partial}{\partial u}} \left(u \frac{\partial}{\partial v} \right)$ at the point $u = v = 0$ for the Levi-Civita connection considered in the previous problem.*

$$\nabla_{\frac{\partial}{\partial u}} \left(u \frac{\partial}{\partial v} \right) = \nabla_{\frac{\partial}{\partial u}} (u) \frac{\partial}{\partial v} + u \nabla_{\frac{\partial}{\partial u}} \left(\frac{\partial}{\partial v} \right) = \frac{\partial}{\partial u} + u \left(\Gamma^u_{uv} \partial_u + \Gamma^v_{uv} \partial_v \right) = \partial_u + u \left(\frac{a_v}{2a} \partial_u + \frac{b_u}{2b} \partial_v \right) .$$

4. *Calculate Levi-Civita connection of the Riemannian metric on the sphere in stereographic coordinates:*

$$G = \frac{4R^4 (du^2 + dv^2)}{(R^2 + u^2 + v^2)^2}$$

a) at the point $u = v = 0$
b) at an arbitrary point.*

Note that in the vicinity of the point $u = v = 0$

$$\frac{4R^4}{(R^2 + u^2 + v^2)^2} = 4 + \text{terms of the order higher than 1 in } u, v$$

Hence according to the problem 2 all Christoffel symbols vanish at the point $u = v = 0$.

b) In this case we have to perform detailed calculations. Use the results of the exercise 1b):

$$\frac{4R^4(du^2 + dv^2)}{(R^2 + u^2 + v^2)^2} = adu^2 + bdv^2 \text{ with } a = b = \frac{4R^4}{(R^2 + u^2 + v^2)^2}.$$

Hence according to the solution for 1b) we have:

$$\Gamma^u_{uu} = \frac{a_u}{2a} = -\frac{2u}{R^2 + u^2 + v^2}, \Gamma^u_{uv} = \Gamma^u_{vu} = \frac{a_v}{2a} = -\frac{2v}{R^2 + u^2 + v^2}, \Gamma^u_{vv} = -\frac{b_u}{2a} = \frac{2u}{R^2 + u^2 + v^2},$$

$$\Gamma^v_{uu} = -\frac{a_v}{2b} = \frac{2v}{R^2 + u^2 + v^2}, \Gamma^v_{uv} = \Gamma^v_{vu} = \frac{b_u}{2b} = -\frac{2v}{R^2 + u^2 + v^2}, \Gamma^v_{vv} = \frac{b_v}{2b} = -\frac{2v}{R^2 + u^2 + v^2}.$$

5. *Calculate Levi-Civita connection of Euclidean metric of a plane in*
a) Cartesian coordinates
b) polar coordinates
Compare with results of previous calculations.

In Cartesian coordinates metrics coefficients are constants. All partial derivatives in (1) equal to zero. Hence all Christoffel symbols vanish. The Levi-Civita connection is canonical flat connection.

b) polar coordinates: $G = dr^2 + r^2 d\varphi^2$. We have:

$$G = \begin{pmatrix} g_{rr} & g_{r\varphi} \\ g_{\varphi r} & g_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \qquad G^{-1} = \begin{pmatrix} g^{rr} & g^{r\varphi} \\ g^{\varphi r} & g^{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}$$

We have:

$$\Gamma^r_{\varphi\varphi} = \frac{1}{2} g^{rr} \left(-\frac{\partial g_{\varphi\varphi}}{\partial r} \right) = \frac{1}{2} (-2r) = -r,$$

$$\Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = \frac{1}{2} g^{\varphi\varphi} \left(\frac{\partial g_{\varphi\varphi}}{\partial r} \right) = \frac{1}{2r^2} (2r) = r$$

all other Christoffel symbols vanish. This is in accordance with calculation of Christoffel symbols in polar coordinates (see Homework 4 and Lecture notes) One can calculate these Christoffel symbols using Lagrangians (see the next homework).

6. *Calculate Levi-Civita connection of the Riemannian metric induced on the surface of a cylinder $x^2 + y^2 = a^2$ (You may use parameterisation:*

$$\mathbf{r}(h, \varphi) \colon \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases}.$$

For surface of cylinder $\mathbf{r}(h, \varphi) \colon \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases}$ the induced Riemannian metric is equal to $G = dh^2 + a^2 d\varphi^2$ (see previous exercises). We see that coefficeints are constants (as in Cartesina coordinates for Euclidean case). Hence Chrsitoffel symbols vanish in coordinates h, φ.

7. *Calculate Levi-Civita connection of the Riemannian metric induced on* *the surface of the cone $x^2 + y^2 - k^2z^2 = 0$. You may use parameterisation:*

$$\mathbf{r}(h,\varphi)\colon \begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases}.$$

Find coordinates on the cone $x^2 + y^2 - k^2z^2 = 0$ such that Christoffel symbols of Levi-Civita connection of induced metric vanish in these coordinates. Is it possible to do this on a sphere?

c) for cone induced metric is $(k^2+1)dh^2 + k^2h^2d\varphi^2$. Calculations will appear after Coursework.

For a surface of cylinder we also found these coordinates: $G = dh^2 + a^2d\varphi^2$, $u = h, v = a\varphi$. Now for cone. We know that on cone $x^2 + y^2 - k^2z^2 = 0$ one can find new local coordinates

$$\begin{cases} u = \sqrt{k^2+1}h\cos\frac{k}{\sqrt{k^2+1}}\varphi \\ v = \sqrt{k^2+1}h\sin\frac{k}{\sqrt{k^2+1}}\varphi \end{cases}$$

such that induced metric on the cone becomes $G|_c = du^2 + dv^2$, i.e. cone locally is isometric to the Euclidean plane (see homework 3). In these coordinates according to formula (1) all Christoffel symbols vanish.

8. *Calculate Levi-Civita connection of the metric $G = R^2(d\theta^2 + \sin^2\theta d\varphi^2)$ on the sphere. Compare with results of previous calculations.*

We have

$$G = \begin{pmatrix} g_{\theta\theta} & g_{\theta\varphi} \\ g_{\varphi\theta} & g_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix}, \qquad G^{-1} = \begin{pmatrix} g^{\theta\theta} & g^{\theta\varphi} \\ g^{\varphi\theta} & g^{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} 1\mathbf{R}^2 & 0 \\ 0 & \frac{1}{R^2\sin^2\theta} \end{pmatrix}$$

We have:

$$\Gamma^{\theta}_{\varphi\varphi} = \frac{1}{2}g^{\theta\theta}\left(-\frac{\partial g_{\varphi\varphi}}{\partial\theta}\right) = \frac{1}{2}\left(-2\sin\theta\cos\theta\right) = -\sin\theta\cos\theta\,,$$

$$\Gamma^{\varphi}_{\theta\varphi} = \Gamma^{\varphi}_{\varphi\theta} = \frac{1}{2}g^{\varphi\varphi}\left(\frac{\partial g_{\varphi\varphi}}{\partial\theta}\right) = \frac{1}{2sin^2\theta}\left(2\sin\theta\cos\theta\right) = \mathrm{cotan}\,\theta.$$

all other Christoffel symbols vanish. This is in accordance with calculation of Christoffel symbols of the induced connection on the sphere (see Lecture notes the subsubsection 2.2.1) In the next homework we will calculate the Christoffel symbols using Lagrangians.

9 *Let $\mathbf{E}^2$ be the Euclidean plane with the standard Euclidean metric $G_{\text{Eucl.}} = dx^2 + dy^2$.*

You know that for the Levi-Civita connection of this metric the Christoffel symbols vanish in the Cartesian coordinates x, y. (Why?)

Let ∇ be a symmetric connection on the Euclidean plane $\mathbf{E}^2$ such that its Christoffel symbols satisfy the condition $\Gamma^y_{xy} = \Gamma^y_{yx} \neq 0$.

Show that for vector fields $\mathbf{A} = \partial_x$ and $\mathbf{B} = \partial_y$, $\partial_{\mathbf{A}}\langle\mathbf{B},\mathbf{B}\rangle \neq 2\langle\nabla_{\mathbf{A}}\mathbf{B},\mathbf{B}\rangle$, i.e. the connection ∇ does not preserve the Euclidean scalar product $\langle\ ,\ \rangle$.

For Euclidean metric all componets of metric $G = dx^2 + dy^2$ are constants: $||g_{ik}|| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ Due to the formula above all derivatives vanish. Hence all Christoffel symbols vanish.

Consider vector field $\mathbf{A} = \partial_x$ and vector field $\mathbf{B} = \partial_y$. Scalar product of the vector field $\mathbf{B}$ on itself is equal to 1 and $\nabla_{\mathbf{A}}(\mathbf{B},\mathbf{B}) = \partial_{\mathbf{A}}1 = 0$. On the other hand $\nabla_{\mathbf{A}}\mathbf{B} = \nabla_{\partial_x}\partial_y = \Gamma^x_{xy}\partial_x + \Gamma^y_{xy}\partial_y$ and the scalar product $\langle\nabla_{\mathbf{A}}\mathbf{B},\mathbf{Y}\rangle$ is equal to

$$\langle\nabla_{\mathbf{A}}\mathbf{B},\mathbf{B}\rangle = \langle\Gamma^x_{xy}\partial_x + \Gamma^y_{xy}\partial_y, \partial_y\rangle = \Gamma^y_{xy} \neq 0\,.$$

Hence we see that $\nabla_{\mathbf{A}}\langle\mathbf{B},\mathbf{B}\rangle = 0 \neq 2\langle\nabla_{\mathbf{A}}\mathbf{B},\mathbf{B}\rangle$.