## Homework 6

**1** Calculate the integrals of the form $\omega = xdy - ydx$ over the following three curves. Compare answers.

$$C_1\colon \mathbf{r}(t)\ \begin{cases} x = R\cos t \\ y = R\sin t \end{cases},\ 0 < t < \pi, \quad C_2\colon \mathbf{r}(t)\ \begin{cases} x = R\cos 4t \\ y = R\sin 4t \end{cases},\ 0 < t < \frac{\pi}{4}$$

$$\text{and}\quad C_3\colon \mathbf{r}(t)\ \begin{cases} x = Rt \\ y = R\sqrt{1-t^2} \end{cases},\ -1 \le t \le 1.$$

**2** Calculate the integrals of the form $\omega = xdy + ydz + zdx$ over the arc of helix $C$

$$C\colon \mathbf{r}(t)\ \begin{cases} x = R\cos t \\ y = R\sin t \\ z = ct \end{cases},\ 0 \le t < 2\pi\,.$$

**3** Consider an arc of parabola $x = 2y^2 - 1$, $0 < y < 1$.

Give examples of two different parameterisations of this curve such that these parameterisations have the opposite orientation.

Calculate the integral of the form 1-form $\omega = \sin y dx$ over this curve.

How does the answer depend on a parameterisation?

**4** Calculate the integral of the form $\omega = xdy$ over the following curves

a) closed curve $x^2 + y^2 = 12y$

b) arc of the ellipse $x^2 + y^2/9 = 1$ defined by the condition $y \ge 0$.

How does your answer depend on a choice of parameterisation?

Choose two different parameterisations of each of these curves such that integral changes sign under changing of parameterisation.

*Exact forms*

**5** Calculate the integral $\int_C \omega$ where $\omega = xdx + ydy$ and $C$ is

a) the straight line segment $x = t, y = 1 - t$, $0 \le t \le 1$

b) the segment of parabola $x = t$, $y = 1 - t^n$, $0 \le t \le 1$, $n = 2, 3, 4, \dots$

c) for **an arbitrary** curve starting at the point $(0, 1)$ and ending at the point $((1, 0)$.

**6** Show that the form 1-form $\omega = 3x^2ydx + x^3dy$ is an exact 1-form.

Calculate integral of this form over the curves considered in exercises 3) and 4).

**7**. Consider in $\mathbf{E}^2$ 1-forms

a) $xdx$ , b) $xdy$ c) $xdx + ydy$ , d)$xdy + ydx$, e) $xdy - ydx$

f) $x^4dy + 4x^3ydx$ .

Show that 1-forms a), c), d) and f) are exact forms

Why are 1-forms b) and e) not exact?

**8** Conisder in $\mathbf{E}^3$ the following 1-forms.

a) $xdy + ydx + dz$ , b) $xdy - ydx + dz$ , c) $yze^{xy}dx + zxe^{xy}dy + e^{xy}dz$.

Choose the forms which are exact, and the forms which are not exact. Justify your answer.

**9** Consider 1-form $\omega = xdy + aydx$ where $a$ is a constant.

a) Find the integral of this form over a closed curve defined by equation $x^2 + y^2 - 4x - 4y + 7 = 0$.

b) Explain why the form $\omega$ is exact if $a = 1$.

c) Explain why the form $\omega$ is not exact if $a \neq 1$.

**10** * Calculate the integral of the form $\sigma = \frac{xdy-ydx}{x^2}$ over the curve $x^2+y^2-4x-4y+7 = 0$ consdered in the previous exercise.

*All the exercises below are not compulsory*

**11**$^\dagger$ Consider one-form

$$\omega = \frac{xdy - ydx}{x^2 + y^2} \tag{1}$$

This form is defined in $\mathbf{E}^2 \setminus 0$.

Calculate differential of this form.

Write down this form in polar coordinates

Find a function $f$ such that $\omega = df$.

Is this function defined in the same domain as $\omega$?

**12**$^\dagger$ Calculate the integral of the form $\omega = \frac{xdy-ydx}{x^2+y^2}$ over the curves

a) circle $x^2 + y^2 = 1$

b) circle $(x - 3)^2 + y^2 = 1$

c) ellipse $\frac{x^2}{9} + \frac{x^2}{16} = 1$

**13**$^\dagger$ What values can the integral $\int_C \omega$ take if $C$ is an arbitrary curve starting at the point $(0, 1)$ and ending at the point $(1, 0)$ and $\omega = \frac{xdy-ydx}{x^2+y^2}$.

**14**$^\dagger$ Let $\omega = a(x, y)dx + b(x, y)dy$ be a closed form in $\mathbf{E}^2$, $d\omega = 0$.

Consider the function

$$f(x, y) = x \int_0^1 a(tx, ty)dt + y \int_0^1 b(tx, ty)dt \tag{2}$$

Show that

$$\omega = df\ .$$

This proves that an arbitrary closed form in $\mathbf{E}^2$ is an exact form.

Why we cannot apply the formula (2) to the form $\omega$ defined by the expression (1)?