Homework 6. Solutions.

1. *Calculate Levi-Civita connection of the metric $G = a(u,v)du^2 + b(u,v)dv^2$*
a) in the case if functions $a(u,v)$, $b(u,v)$ are constants.
b) In general case

We know that for Levi-Civita connection

$$\Gamma^i_{mk} = \frac{1}{2} g^{ij} \left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^m} - \frac{\partial g_{mk}}{\partial x^j} \right) . \tag{1}$$

a) We do not need to do any calculations since a and b are constants, and all partial derivatives $\frac{\partial g_{jm}}{\partial x^k}$ for metric $G = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ are equal to zero. Hence all Christoffel symbols vanish.

b) In this case we have to perform calculations:
We have

$$G = a(u,v)du^2 + b(u,v)dv^2, G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} a(u,v) & 0 \\ 0 & b(u,v) \end{pmatrix}, G^{-1} = \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{a(u,v)} & 0 \\ 0 & \frac{1}{b(u,v)} \end{pmatrix} .$$

Hence according to (1)

$$\begin{aligned}
\Gamma^1_{11} = \Gamma^u_{uu} &= \tfrac{1}{2} g^{11} (\partial_1 g_{11} + \partial_1 g_{11} - \partial_1 g_{11}) = \tfrac{1}{2} g^{11} \partial_u g_{uu} = & \tfrac{a_u}{2a} \\
\Gamma^1_{21} = \Gamma^1_{12} = \Gamma^u_{uv} = \Gamma^u_{vu} &= \tfrac{g^{11}}{2} (\partial_1 g_{12} + \partial_2 g_{11} - \partial_1 g_{12}) = \tfrac{g^{uu}}{2} \partial_v g_{uu} = & \tfrac{a_v}{2a} \\
\Gamma^1_{22} = \Gamma^u_{vv} &= \tfrac{g^{11}}{2} (\partial_2 g_{12} + \partial_2 g_{12} - \partial_1 g_{22}) = -\tfrac{g^{uu}}{2} \partial_u g_{vv} = & -\tfrac{b_u}{2a} \\
\Gamma^2_{11} = \Gamma^v_{uu} &= \tfrac{g^{22}}{2} (\partial_1 g_{12} + \partial_1 g_{12} - \partial_2 g_{11}) = -\tfrac{g^{vv}}{2} \partial_v g_{uu} = & -\tfrac{a_v}{2b} \\
\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^v_{uv} = \Gamma^v_{vu} &= \tfrac{g^{22}}{2} (\partial_2 g_{21} + \partial_1 g_{22} - \partial_2 g_{21}) = \tfrac{g^{vv}}{2} \partial_u g_{vv} = & \tfrac{b_u}{2b} \\
\Gamma^2_{22} = \Gamma^v_{vv} &= \tfrac{g^{22}}{2} (\partial_2 g_{22} + \partial_2 g_{22} - \partial_2 g_{22}) = \tfrac{g^{vv}}{2} \partial_v g_{vv} = & \tfrac{b_v}{2b}
\end{aligned}$$

(We use notations $(u,v) = (x^1, x^2)$.)

2 *Calculate Levi-Civita connection of the Riemannian metric $G = e^{-x^2-y^2}(dx^2 + dy^2)$ at the point $x = y = 0$.*

3. *Calculate Levi-Civita connection of Euclidean metric of a plane in*
a) Cartesian coordinates
b) polar coordinates

In Cartesian coordinates metrics coefficients are constants. All partial derivatives in (1) equal to zero. Hence all Christoffel symbols vanish. The Levi-Civita connection is canonical flat connection.

b) polar coordinates: $G = dr^2 + r^2 d\varphi^2$. We have:

$$G = \begin{pmatrix} g_{rr} & g_{r\varphi} \\ g_{\varphi r} & g_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \qquad G^{-1} = \begin{pmatrix} g^{rr} & g^{r\varphi} \\ g^{\varphi r} & g^{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}$$

We have:

$$\Gamma^r_{\varphi\varphi} = \frac{1}{2} g^{rr} \left(-\frac{\partial g_{\varphi\varphi}}{\partial r} \right) = \frac{1}{2} (-2r) = -r ,$$

$$\Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = \frac{1}{2} g^{\varphi\varphi} \left(\frac{\partial g_{\varphi\varphi}}{\partial r} \right) = \frac{1}{2r^2} (2r) = \frac{1}{r} ,$$

all other Christoffel symbols vanish. This is in accordance with calculation of Christoffel symbols in polar coordinates (see Lecture notes) One can calculate these Christoffel symbols using Lagrangians (see the question 8a in this homework).

4. *Calculate Levi-Civita connection of the Riemannian metric induced on the surface of a cylinder* $x^2 + y^2 = a^2$ *in coordinates* h, φ*:*

$$\mathbf{r}(h,\varphi)\colon \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases} .$$

For surface of cylinder $\mathbf{r}(h,\varphi)\colon \begin{cases} x = a\cos\varphi \\ y = a\sin\varphi \\ z = h \end{cases}$ the induced Riemannian metric is equal to $G = dh^2 + a^2 d\varphi^2$ (see previous exercises). We see that coefficeints are constants (as in Cartesina coordinates for Euclidean case). Hence Chrsitoffel symbols vanish in coordinates h, φ.

5. *Calculate Levi-Civita connection of the Riemannian metric induced on the surface of the cone* $x^2 + y^2 - k^2 z^2 = 0$. *in coordinates* h, φ*:*

$$\mathbf{r}(h,\varphi)\colon \begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases} .$$

Do there exist coordinates on the cone such that Christoffel symbols of Levi-Civita connection of induced metric vanish in these coordinates?

We have

$$G = (dx^2 + dy^2 + dz^2)\big|_{x=kh\cos\varphi, y=kh\sin\varphi, z=h} = (kdh\cos\varphi - kh\sin\varphi d\varphi)^2 + (kdh\sin\varphi + kh\cos\varphi d\varphi)^2 + dh^2 =$$

$$(k^2+1)dh^2 + k^2h^2 d\varphi^2\,, \quad ||g_{\alpha\beta}|| = \begin{pmatrix} 1+k^2 & \\ 0 & k^2h^2 \end{pmatrix}, \quad ||g^{\alpha\beta}|| = \begin{pmatrix} \frac{1}{1+k^2} & \\ 0 & \frac{1}{k^2h^2} \end{pmatrix} .$$

Now calculate Levi-Civita connection using the formula

$$\Gamma^\alpha_{\beta\rho} = \frac{1}{2} g^{\alpha\pi} \left(\frac{\partial g_{\pi\beta}}{\partial x^\rho} + \frac{\partial g_{\pi\rho}}{\partial x^\beta} - \frac{\partial g_{\rho\beta}}{\partial x^\pi} \right) .$$

Hence

$$\Gamma^h_{hh} = \frac{1}{2} g^{hh} \left(\frac{\partial g_{hh}}{\partial h} \right) = 0\,, \Gamma^h_{h\varphi} = \Gamma^h_{\varphi h} = \frac{1}{2} g^{hh} \left(\frac{\partial g_{hh}}{\partial \varphi} \right) = 0\ , \Gamma^h_{\varphi\varphi} = \frac{1}{2} g^{hh} \left(-\frac{\partial g_{\varphi\varphi}}{\partial h} \right) = \frac{1}{2(1+k^2)} 2k^2 h = -\frac{k^2 h}{1+k^2}\,,$$

$$\Gamma^\varphi_{hh} = \frac{1}{2} g^{\varphi} \left(-\frac{\partial g_{hh}}{\partial \varphi} \right) = 0\,, \Gamma^\varphi_{h\varphi} = \Gamma^\varphi_{\varphi h} = \frac{1}{2} g^{\varphi} \left(\frac{\partial g_{\varphi\varphi}}{\partial h} \right) = \frac{1}{2k^2h^2} 2k^2 h = \frac{1}{h}\,, \Gamma^\varphi_{\varphi\varphi} = \frac{1}{2} g^{\varphi\varphi} \left(-\frac{\partial g_{\varphi\varphi}}{\partial \varphi} \right) = 0\,,$$

Hence we have that in coordinates h, φ non-vanishing components of Christoffel symbols are

$$\Gamma^h_{\varphi\varphi} = -\frac{k^2 h}{1+k^2}\,, \Gamma^\varphi_{h\varphi} = \Gamma^\varphi_{\varphi h} = \frac{1}{h} .$$

Yes these coordinates exist. We know that on cone $x^2 + y^2 - k^2 z^2 = 0$ one can find new local coordinates

$$\begin{cases} u = \sqrt{k^2+1} h \cos \frac{k}{\sqrt{k^2+1}} \varphi \\ v = \sqrt{k^2+1} h \sin \frac{k}{\sqrt{k^2+1}} \varphi \end{cases}$$

such that induced metric on the cone becomes $G|_c = du^2 + dv^2$, i.e. cone locally is isometric to the Euclidean plane (see homework 3). In these coordinates according to formula (1) all Christoffel symbols vanish.

6. *Calculate Levi-Civita connection of the metric* $G = R^2(d\theta^2 + \sin^2\theta d\varphi^2)$ *on the sphere.*

We have

$$G = \begin{pmatrix} g_{\theta\theta} & g_{\theta\varphi} \\ g_{\varphi\theta} & g_{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix}, \qquad G^{-1} = \begin{pmatrix} g^{\theta\theta} & g^{\theta\varphi} \\ g^{\varphi\theta} & g^{\varphi\varphi} \end{pmatrix} = \begin{pmatrix} \frac{1}{R^2} & 0 \\ 0 & \frac{1}{R^2\sin^2\theta} \end{pmatrix}$$

We have:

$$\Gamma^\theta_{\varphi\varphi} = \frac{1}{2}g^{\theta\theta}\left(-\frac{\partial g_{\varphi\varphi}}{\partial\theta}\right) = \frac{1}{2}\left(-2\sin\theta\cos\theta\right) = -\sin\theta\cos\theta\,,$$

$$\Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = \frac{1}{2}g^{\varphi\varphi}\left(\frac{\partial g_{\varphi\varphi}}{\partial\theta}\right) = \frac{1}{2sin^2\theta}\left(2\sin\theta\cos\theta\right) = \mathrm{cotan}\,\theta.$$

all other Christoffel symbols vanish. This is in accordance with calculation of Christoffel symbols of the induced connection on the sphere (see Lecture notes the subsubsection 2.2.1) In the next homework we will calculate the Christoffel symbols using Lagrangians.

7 *Consider the Lagrangian of "free" particle* $L = \frac{1}{2}g_{ik}\dot{x}^i\dot{x}^k$ *for Riemannian manifold with a metric* $G = g_{ik}dx^idx^k$.
Write down Euler-Lagrange equations of motion for this Lagrangian and compare them with differential equations for geodesics on this Riemannian manifold.

In fact show that

$$\underbrace{\frac{\partial L}{\partial x^i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}}_{\textit{Euler-Lagrange equations}} \quad \Leftrightarrow \quad \underbrace{\frac{d^2x^i}{dt^2} + \Gamma^i_{km}\dot{x}^k\dot{x}^m = 0}_{\textit{Equations for geodesics}}\,, \tag{1}$$

where

$$\Gamma^i_{km} = \frac{1}{2}g^{ij}\left(\frac{\partial g_{jk}}{\partial x^m} + \frac{\partial g_{jm}}{\partial x^k} - \frac{\partial g_{km}}{\partial x^j}\right)\,. \tag{2}$$

Solution: see the lecture notes.

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Write down the Lagrangian of free particle $L = \frac{1}{2}g_{ik}\dot{x}^i\dot{x}^k$ *and using Euler-Lagrange equations for this Lagrangian calculate Christoffel symbols (Christoffel symbols of Levi-Civita connection) for*

a) Eulcidean plane in polar coordinates

b) for the sphere of radius R

c) for Lobachevsky plane

Compare with the results that you obtained using straightforwardly the formula (1) or using formulae for induced connection.

Solution.

a) *for Euclidean plane in polar coordinates*

Riemannian metric on the plane $\mathbf{E}^2$ in polar spherical coordinates is $G = dr^2 + r^2d\varphi^2$. Hence the Lagrangian of the free particle is

$$L = \frac{\dot{r}^2 + r^2\dot{\varphi}^2}{2}$$

Euler-Lagrange equations for r:

$$\frac{\partial L}{\partial r} = r\dot{\varphi}^2\,, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{h}}\right) = \frac{d}{dt}\dot{r} = \ddot{r}\,, \quad \text{i.e. } \ddot{r} - r\dot{\varphi}^2 = 0\,,$$

Comparing with equation for geodesic

$$\ddot{r} + \Gamma^r_{rr}\dot{r}\dot{r} + \Gamma^r_{r\varphi}\dot{r}\dot{\varphi} + \Gamma^r_{\varphi r}\dot{\varphi}\dot{\theta} + \Gamma^r_{\varphi\varphi}\dot{\varphi}\dot{\varphi} = \ddot{r} + \Gamma^r_{rr}\dot{r}\dot{r} + 2\Gamma^r_{r\varphi}\dot{r}\dot{\varphi} + \Gamma^r_{\varphi\varphi}\dot{\varphi}\dot{\varphi} = 0$$

we see that

$$\Gamma^r_{rr} = \Gamma^r_{r\varphi} = \Gamma^r_{\varphi r} = 0,\ \Gamma^r_{\varphi\varphi} = -r\,,$$

Analogously Euler-Lagrange equations for φ:

$$\frac{\partial L}{\partial \varphi} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi}\right),\quad \frac{\partial L}{\partial \varphi} = 0,\ \frac{\partial L}{\partial \dot\varphi} = r^2\dot\varphi\,,\text{hence } \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi}\right) = \frac{d}{dt}\left(r^2\dot\varphi\right) = r^2\ddot\varphi + 2r\dot r\dot\varphi = 0\,,$$

i.e.

$$\ddot\varphi + 2\frac{1}{r}\dot r\dot\varphi = 0\,.$$

Comparing with equation for geodesic

$$\ddot\varphi + \Gamma^\varphi_{rr}\dot r\dot r + \Gamma^\varphi_{r\varphi}\dot r\dot\varphi + \Gamma^\varphi_{\varphi r}\dot\varphi\dot r + \Gamma^\varphi_{\varphi\varphi}\dot\varphi\dot\varphi = \ddot\theta + \Gamma^\varphi_{rr}\dot r\dot r + 2\Gamma^\varphi_{r\varphi}\dot r\dot\varphi + \Gamma^\varphi_{\varphi\varphi}\dot\varphi\dot\varphi = 0\,.$$

Comparing with Euler-Lagrange equations for geodesics we see that

$$\Gamma^\varphi_{rr} = \Gamma^\varphi_{\varphi\varphi} = 0\,,\Gamma^\varphi_{\varphi r} = \Gamma^\varphi_{r\varphi} = \frac{1}{r}\,.$$

Hence we see that Christoffel symbols of Euclidean plane in polar coordinates all vanish except

$$\Gamma^r_{\varphi\varphi} = -r\,,\quad \Gamma^\varphi_{\varphi r} = \Gamma^\varphi_{r\varphi} = \frac{1}{r}\,.$$

b) *For the sphere:*

Riemannian metric on sphere in spherical coordinates is $G = R^2 d\theta^2 + R^2\sin^2\theta d\varphi^2$. Hence the Lagrangian of the free particle is

$$L = \frac{R^2\dot\theta^2 + R^2\sin^2\theta\dot\varphi^2}{2}$$

Euler-Lagrange equations for θ:

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right),\quad \frac{\partial L}{\partial \theta} = R^2\sin\theta\cos\theta\dot\varphi^2,\ \frac{\partial L}{\partial \dot\theta} = R^2\dot\theta$$

Hence

$$\frac{d}{dt}\left(R^2\dot\theta\right) = R^2\sin\theta\cos\theta\dot\varphi^2, R^2\ddot\theta = R^2\sin\theta\cos\theta\dot\varphi^2,$$

hence

$$\ddot\theta - \sin\theta\cos\theta\dot\varphi^2 = 0\,.$$

Comparing with equation for geodesic

$$\ddot\theta + \Gamma^\theta_{\theta\theta}\dot\theta\dot\theta + \Gamma^\theta_{\theta\varphi}\dot\theta\dot\varphi + \Gamma^\theta_{\varphi\theta}\dot\varphi\dot\theta + \Gamma^\theta_{\varphi\varphi}\dot\varphi\dot\varphi = \ddot\theta + \Gamma^\theta_{\theta\theta}\dot\theta\dot\theta + 2\Gamma^\theta_{\theta\varphi}\dot\theta\dot\varphi + \Gamma^\theta_{\varphi\varphi}\dot\varphi\dot\varphi = 0$$

we see that

$$\Gamma^\theta_{\theta\theta} = \Gamma^\theta_{\theta\varphi} = \Gamma^\theta_{\varphi\theta} = 0,\ \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta$$

Analogously Euler-Lagrange equations for φ:

$$\frac{\partial L}{\partial \varphi} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi}\right),\quad \frac{\partial L}{\partial \varphi} = 0,\ \frac{\partial L}{\partial \dot\varphi} = R^2\sin^2\theta\dot\varphi\,.$$

Hence

$$\frac{d}{dt}\left(R^2\sin^2\theta\dot\varphi\right) = 0,\ R^2\sin^2\theta\ddot\varphi + 2R^2\sin\theta\cos\theta\dot\theta\dot\varphi = 0,$$

hence

$$\ddot{\theta} + \text{cotan}\,\theta\dot{\theta}\dot{\varphi} = 0,$$

Comparing with equation for geodesic

$$\ddot{\varphi} + \Gamma^{\varphi}_{\theta\theta}\dot{\theta}\dot{\theta} + \Gamma^{\varphi}_{\theta\varphi}\dot{\theta}\dot{\varphi} + \Gamma^{\varphi}_{\varphi\theta}\dot{\varphi}\dot{\theta} + \Gamma^{\varphi}_{\varphi\varphi}\dot{\varphi}\dot{\varphi} = \ddot{\theta} + \Gamma^{\varphi}_{\theta\theta}\dot{\theta}\dot{\theta} + 2\Gamma^{\varphi}_{\theta\varphi}\dot{\theta}\dot{\varphi} + \Gamma^{\varphi}_{\varphi\varphi}\dot{\varphi}\dot{\varphi} = 0$$

we see that

$$\Gamma^{\varphi}_{\theta\theta} = \Gamma^{\varphi}_{\varphi\varphi} = 0\,, \Gamma^{\varphi}_{\varphi\theta} = \Gamma^{\varphi}_{\theta\varphi} = \text{cotan}\,\theta\,.$$

c) *For Lobachevsky plane*:
Lagrangian of "free" particle on the Lobachevsky plane with metric $G = \frac{dx^2+dy^2}{y^2}$ is

$$L = \frac{1}{2}\frac{\dot{x}^2 + \dot{y}^2}{y^2}.$$

Euler-Lagrange equations are

$$\frac{\partial L}{\partial x} = 0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{d}{dt}\left(\frac{\dot{x}}{y^2}\right) = \frac{\ddot{x}}{y^2} - \frac{2\dot{x}\dot{y}}{y^3}, \text{i.e.} \quad \ddot{x} - \frac{2\dot{x}\dot{y}}{y} = 0\,,$$

$$\frac{\partial L}{\partial y} = -\frac{\dot{x}^2 + \dot{y}^2}{y^3} = \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = \frac{d}{dt}\left(\frac{\dot{y}}{y^2}\right) = \frac{\ddot{y}}{y^2} - \frac{2\dot{y}^2}{y^3}, \text{i.e.} \quad \ddot{y} + \frac{\dot{x}^2}{y} - \frac{\dot{y}^2}{y} = 0\,.$$

Comparing these equations with equations for geodesics: $\ddot{x}^i - \dot{x}^k\Gamma^i_{km}\dot{x}^m = 0$ $(i = 1, 2,\ x = x^1, y = x^2)$ we come to

$$\Gamma^x_{xx} = 0, \Gamma^x_{xy} = \Gamma^x_{yx} = -\frac{1}{y},\ \Gamma^x_{yy} = 0,\ \Gamma^y_{xx} = \frac{1}{y}, \Gamma^y_{xy} = \Gamma^y_{yx} = 0, \Gamma^y_{yy} = -\frac{1}{y}\,.\ \blacksquare$$

The answers are the same as calculated with other methods. We see that Lagrangians give us the nice and quick way to calculate Christoffel symbols.

9 *Let $\mathbf{E}^2$ be the Euclidean plane with the standard Euclidean metric $G_{\text{Eucl.}} = dx^2 + dy^2$.*

You know that for the Levi-Civita connection of this metric the Christoffel symbols vanish in the Cartesian coordinates x, y. (Why?)

Let ∇ be a symmetric connection on the Euclidean plane $\mathbf{E}^2$ such that its Christoffel symbols satisfy the condition $\Gamma^y_{xy} = \Gamma^y_{yx} \neq 0$.

Show that for vector fields $\mathbf{A} = \partial_x$ and $\mathbf{B} = \partial_y$, $\partial_{\mathbf{A}}\langle \mathbf{B}, \mathbf{B}\rangle \neq 2\langle\nabla_{\mathbf{A}}\mathbf{B}, \mathbf{B}\rangle$, i.e. the connection ∇ does not preserve the Euclidean scalar product $\langle\ \ ,\ \ \rangle$.

For Euclidean metric all componets of metric $G = dx^2 + dy^2$ are constants: $||g_{ik}|| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ Due to the formula above all derivatives vanish. Hence all Christoffel symbols vanish.

Consider vector field $\mathbf{A} = \partial_x$ and vector field $\mathbf{B} = \partial_y$. Scalar product of the vector field $\mathbf{B}$ on itself is equal to 1 and $\nabla_{\mathbf{A}}(\mathbf{B}, \mathbf{B}) = \partial_{\mathbf{A}}1 = 0$. On the other hand $\nabla_{\mathbf{A}}\mathbf{B} = \nabla_{\partial_x}\partial_y = \Gamma^x_{xy}\partial_x + \Gamma^y_{xy}\partial_y$ and the scalar product $\langle\nabla_{\mathbf{A}}\mathbf{B}, \mathbf{Y}\rangle$ is equal to

$$\langle\nabla_{\mathbf{A}}\mathbf{B}, \mathbf{B}\rangle = \langle\Gamma^x_{xy}\partial_x + \Gamma^y_{xy}\partial_y, \partial_y\rangle = \Gamma^y_{xy} \neq 0\,.$$

Hence we see that $\nabla_{\mathbf{A}}\langle\mathbf{B}, \mathbf{B}\rangle = 0 \neq 2\langle\nabla_{\mathbf{A}}\mathbf{B}, \mathbf{B}\rangle$.