Introduction to Geometry (20222)

2011

COURSEWORK

This assignment counts for 20% of your marks.

Solutions are due by 1-st April

Write solutions in the provided spaces.

STUDENTS'S NAME:

1

a) Let (x^1, x^2, x^3) be coordinates of the vector $\mathbf{x}$, and (y^1, y^2, y^3) be coordinates of the vector $\mathbf{y}$ in $\mathbf{R}^3$.

Does the formula $(\mathbf{x}, \mathbf{y}) = 2x^1y^1 + x^2y^2 + x^1y^3 + x^3y^1 + 2x^3y^3$ define a scalar product on $\mathbf{R}^3$? Justify your answer.

b) Vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ in Euclidean space are pairwise orthogonal to each other and all of them have non-zero length.

Prove that these vectors are linearly independent.

What can you say about a dimension of this Euclidean space?

c) Let $\mathbf{a}, \mathbf{b}$ be two vectors in the Euclidean space $\mathbf{E}^2$ such that the length of the vector $\mathbf{a}$ equals to 3, the length of the vector $\mathbf{b}$ equals to 5 and scalar product of these vectors equals to 9.

Show that these vectors span $\mathbf{E}^2$.

Consider the pair of vectors $\{\mathbf{x}, \mathbf{y}\}$ such that $\{\mathbf{x}, \mathbf{y}\} = \{\mathbf{a}, \mathbf{b}\}T$, where transition matrix T equals to $T = T_1T_2$ and

$$T_1 = \begin{pmatrix} \frac{1}{3} & \frac{1}{4} \\ 0 & -\frac{1}{4} \end{pmatrix}, \; T_2 = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}, \varphi \text{ is an arbitrary angle}.$$

Find lengths of the vectors $\mathbf{x}$ and $\mathbf{y}$ and their scalar product.

2

a) Consider the matrix

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Calculate the matrix A^2 in the case if $\theta = \frac{\pi}{4}$.
Calculate the matrix A^{18} in the case if $\theta = \frac{\pi}{6}$.
Calculate the matrix $A^{2011} - A$ in the case if $\theta = \frac{2\pi}{67}$.

b) Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal basis in 3-dimensional Euclidean space $\mathbf{E}^3$. Let P be a linear operator acting on 3-dimensional Euclidean space $\mathbf{E}^3$, such that $P\mathbf{e}_1 = \mathbf{e}_2$, $P\mathbf{e}_2 = \mathbf{e}_3$ and $P\mathbf{e}_3 = \mathbf{e}_1$.

Show that for arbitrary two vectors $\mathbf{x}, \mathbf{y} \in \mathbf{E}^3$ $(\mathbf{x}, \mathbf{y}) = (P\mathbf{x}, P\mathbf{y})$ where $(\mathbf{x}, \mathbf{y})$ is a scalar product in $\mathbf{E}^3$.

Find a non-zero vector $\mathbf{f}$ such that $P\mathbf{f} = \mathbf{f}$.

What is a geometrical meaning of this vector?

c) On the plane OXY find the horizontal line l: $y = c$ and the point $F = (0, f)$ on the OY axis such that for all the points M on the parabola $y = \frac{x^2-1}{2}$ the distance $|MF|$ equals to the distance between the point M and the line l.

3

a) Consider vector $\mathbf{a} = 2\mathbf{e}_x + 3\mathbf{e}_y + 6\mathbf{e}_z$ in $\mathbf{E}^3$, where $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ is an orthonormal basis for Euclidean space $\mathbf{E}^3$

Show that the angle θ between vectors $\mathbf{a}$ and $\mathbf{e}_z$ belongs to the interval $\left(\frac{\pi}{6}, \frac{\pi}{4}\right)$.

Find a unit vector $\mathbf{b}$ such that it is orthogonal to vectors $\mathbf{a}$ and $\mathbf{e}_z$, and the bases $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and $\{\mathbf{b}, \mathbf{e}_z, \mathbf{a}\}$ have opposite orientation.

b) In oriented Euclidean space $\mathbf{E}^3$ consider the following function of three vectors:

$$F(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = (\mathbf{X}, \mathbf{Y} \times \mathbf{Z}),$$

where $(\ ,\ )$ is the scalar product and $\mathbf{Y} \times \mathbf{Z}$ is the vector product in $\mathbf{E}^3$.

Show that $F(\mathbf{X}, \mathbf{X}, \mathbf{Z}) = 0$ for arbitrary vectors $\mathbf{X}$ and $\mathbf{Z}$.

Deduce that $F(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = -F(\mathbf{Y}, \mathbf{X}, \mathbf{Z})$ for arbitrary vectors $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$.

What is the geometrical meaning of the function F?

c) Consider a triangle $\triangle ABC$ in $\mathbf{E}^3$, formed by the vectors $\mathbf{a} = (85, 48, -36)$ and $\mathbf{b} = (84, 48, -36)$ attached at the point C.

Calculate the area of the triangle $\triangle ABC$.

Calculate the length of the height CM of this triangle. ($CM \perp AB$ and the point M belongs to the line AB.)

Show that the vector $\mathbf{h} = CM$ and the vector $\mathbf{c} = (\mathbf{a} - \mathbf{b}) \times (\mathbf{a} \times \mathbf{b})$ are collinear (proportional).

4

a) Given a vector field $\mathbf{G} = ar\frac{\partial}{\partial r} + b\frac{\partial}{\partial \varphi}$ in polar coordinates express it in Cartesian coordinates ($x = r\cos\varphi$, $y = r\sin\varphi$).

b) Consider the function $f = r^2 \sin 2\varphi$ and the vector fields $\mathbf{A} = x\partial_x + y\partial_y$, $\mathbf{B} = x\partial_y - y\partial_x$.

Calculate $\partial_{\mathbf{A}} f$, $\partial_{\mathbf{B}} f$.

Calculate the value of 1-form df on the vector field $3\mathbf{A} + 2\mathbf{B}$.

Express all answers in polar and in Cartesian coordinates.

(c) Show that for 1-form $\omega = 3(x^2 - y^2)dx - 6xydy$ there exists a function f such that $\omega = df$ (i.e. ω is an exact 1-form).

Express the 1-form ω in polar coordinates.

Calculate the value of the 1-form ω on the vector field $\mathbf{A} = r\cos 3\varphi \frac{\partial}{\partial r} - \sin 3\varphi \frac{\partial}{\partial \varphi}$.