EXAM FEEDBACK

INTRODUCTION TO GEOMETRY. Spring 2015

ANSWER **THREE** OF THE FOUR QUESTIONS
If four questions are answered credit will be given for the best three questions.
Each question is worth 20 marks.

Electronic calculators may not be used

This text is not the text of solutions of exam papers! Here we will discuss the solutions of the exampapers.

1.

(a) Explain what is meant by saying that two bases in $\mathbf{E}^3$ have the same orientation.

Let $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ be an orthonormal basis.
Consider the ordered triple of vectors

$$\{\mathbf{f}\,,\ \mathbf{g}\,,\ \lambda\mathbf{e} + \mathbf{g}\}\ ,$$

where λ is an arbitrary real number.

Find all values λ such that this triple is a basis which has the orientation opposite to the orientation of the basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$.

Show that the triple $\{\mathbf{f}\,,\ \mathbf{g}\,,\ \lambda\mathbf{e} + \mathbf{g}\}$ is not an orthonormal basis for each value of λ.

[8 marks]

(b) State the Euler Theorem about rotations.

Let P be an orthogonal linear operator on $\mathbf{E}^3$ such that its determinant is equal to 1 and its matrix in some basis has the following appearance:

$$\begin{pmatrix} \frac{2+\sqrt{2}}{4} & * & * \\ * & \frac{\sqrt{2}}{2} & * \\ * & * & \frac{2+\sqrt{2}}{4} \end{pmatrix}\ .$$

Show that P is an operator of rotation on the angle $\pm\frac{\pi}{4}$.

[5 marks]

(c) Let P be a linear operator on $\mathbf{E}^3$ such that

$$P(\mathbf{x}) = 2(\mathbf{n}, \mathbf{x})\mathbf{n} - \mathbf{x}\,.$$

where $\mathbf{n}$ is a unit vector, and $(\ ,\ )$ is scalar product.

Show that P is orthogonal operator preserving orientation.

We know that, due to the Euler Theorem, P is a rotation operator. Find the axis and angle of this rotation.

`[7 marks]`

Discussion of first question

a) Students had no problems to find transition matirx T from initial basis to the set of the vectors $\{\mathbf{e}, \mathbf{f}\ \lambda\mathbf{e} + g\}$ and to see that its determinant is equal to $\det T = -\lambda$. Hence for $\lambda > 0$ the set $\{\mathbf{e}, \mathbf{f}\ \lambda\mathbf{e} + g\}$ is a basis ($\det T \neq 0$) and this basis have orientation opposite to the orientation of initial basis if $\det T < 0 \Leftrightarrow \lambda > 0$.

If $\lambda = 0$ then the triple is not a basis since in this case $\det T = 0$. If $\lambda \neq 0$ then the vector $\lambda\mathbf{e} + \mathbf{g}$ has a length $\sqrt{1 + \lambda^2} \neq 1$. This shows that for ewach value of λ the new triple is not an orthonormal basis.

(The second part of question 1a) confused some students.)

1 (a) The first two parts were usually answered correctly.

There were several possible ways to answer the third part. The easiest one was to note that $(\mathbf{g}, \lambda\mathbf{e} + \mathbf{g}) = 1$, but it would have to be 0 for an orthonormal basis. Another common method was to use that $(\lambda\mathbf{e} + \mathbf{g}, \lambda\mathbf{e} + \mathbf{g}) = \lambda^2 + 1$, which would have to be 1 for an orthonormal basis, so $\lambda = 0$. The correct way to continue is to note that if $\lambda = 0$ then $\lambda\mathbf{e} + \mathbf{g} = \mathbf{g}$, so the vectors are not linearly independent, therefore they do not even form a basis, but a lot of students wrote instead that $\lambda^2 + 1 \neq 1$ since $\lambda < 0$ by the previous part. This is wrong, because it was not stated anywhere that this part of the question was only about those λ for which the orientation is opposite.

(b) Very few students stated the theorem correctly, the most common error was to miss out that it only aplied to 3-dimensional Euclidean space. In the second part the angle of rotation was calculated correctly by almost everybody, but a lot of students failed to justify why P is a rotation operator.

(c) Most students managed to get some partial credit, but there were very few completely correct answers. A typical answer started with "assume $\mathbf{n} = \mathbf{g}$" then went on to calculate $P(\mathbf{e})$, $P(\mathbf{f})$ and $P(\mathbf{g})$ without explaining what $\mathbf{e}$, $\mathbf{f}$ and $\mathbf{g}$ were. There are two problems here: you should not use notation without explaining what it means and you cannot assume that $\mathbf{n}$ is one of the basis vectors. The correct way to do it is to extend $\mathbf{n}$ to an orthonormal basis. The other common error was the failure to prove that P is orthogonal. The fact that its determinant is 1 only proves that it preserves orientation.

b

c)

2.

(a) Give the definition of a differential 1-form on $\mathbf{E}^n$.

Does there exist on $\mathbf{E}^2$ differential 1-form ω which equals zero on the vectors $\mathbf{A} = \partial_x$ and $\mathbf{B} = \partial_y$ attached at the same point such that its value on the vector $\mathbf{A} + \mathbf{B}$ is equal to 1? Justify your answer.

Let f be a function on $\mathbf{E}^2$ such that $f = x^3 + xy^2$.

Express the 1-form $\omega = df$ in polar coordinates r, φ ($x = r\cos\varphi, y = r\sin\varphi$).

`[6 marks]`

(b) Consider in $\mathbf{E}^2$ two curves

$$C_1 \colon \quad \begin{cases} x = 1 + t \\ y = \sqrt{1 - t^2} \end{cases} \quad -1 < t < 1 \qquad \text{and} \qquad C_2 \colon \quad \begin{cases} x = 1 + \cos\tau \\ y = \sin\tau \end{cases} \quad 0 < \tau < \pi\,.$$

Show that these curves have the same image and opposite orientations.
(You can do this by finding a suitable reparameterisation which transforms the first curve to the second.)

Calculate the integrals $\int_{C_1} \omega$ and $\int_{C_2} \omega$ for 1-form $\omega = 2x dy$. `[6 marks]`

(c) Explain what is meant by saying that a differential 1-form is exact.

Show that the 1-form $\omega = 2xydx + x^2dy$ is exact.

State the theorem about the integral of an exact 1-form over a curve C in $\mathbf{E}^n$.

Let f be a function on $\mathbf{E}^2$. Consider the 1-form $\sigma = fdx + df$.
Show that the integral of the 1-form $\omega = e^x\sigma$ over an arbitrary closed curve in $\mathbf{E}^2$ is equal to zero.

`[8 marks]`

Discussion of second question

a) Almost all students wrote that differential form is a function on tangent vectors which is linear. Yes, it is. This condition (linearity) immediately applies that

$$\omega(\mathbf{A} + \mathbf{B}) = \omega(\mathbf{A}) + \omega(\mathbf{B}) = 0 + 0 = 0 \neq 1.$$

Only two three students used this short and strong argument which follows immediately from the definition. Many students (about 50%) solved this problem in a right way considering 1-form $\omega = adx + bdy$ and coming to contradiction: $\omega(\mathbf{A}) = a = 0$ since $\mathbf{A} = \partial_x$ and $\omega(\mathbf{B}) = b = 0$ since $\mathbf{b} = \partial_y$ hence $a + b = 0$, this is in contradiction with condition that $\omega(\mathbf{A} + \mathbf{B}) = a + b = 1$.

Trying to write down 1-form df in polar coordinates many students realised that it is much, much easier to write down first f in polar coordinates $f = x^3 + xy^2 = r^3 \cos\varphi$, then take its differential. Some students calculated df in Cartesian coordinates, then tried to transform 1-form df from Cartesian to polar coordinates. Only few of them did not fail to perform these calculations.

b) The fact that these two curves have oppostie orientation was carefully worked out by almost all students.

(Many students did hard work and consume much time to caculate explicitly the integral over first curve. It was much much easier not to do these calculations but instead calculate only the integral over curve C_2. (it is equal to π and caclualtions are simple). Then without any additional calculation we come to the fact that the integral over first curve is equal to $-\pi$ since these two curves have opposite orientation.

c) The hard part of this subquestion was to establish that $\int_C e^x(df + f dx) = 0$ if C is closed curve. Many students realised that this follows from the fact that 1-form $e^x(df + f dx)$ is exact. Only about 15-20% of students realised that this form is exact since it is equal to dF where $F = e^x f$.

There were about 5-8 works where students realised that function $e^x f$ us imoprtant for considerations, but they denoted this function by the same letter $f = e^x f$?????. This was sourse of confusion.

3.

(a) Describe what is meant by a natural parameter on a curve in $\mathbf{E}^n$.

Explain why, if a curve is given in natural parameterisation, then the velocity and acceleration vectors are orthogonal to each other at any given point of the curve.

Write down a natural parameterisation for the circle $x^2 + y^2 = R^2$ in $\mathbf{E}^2$.

`[6 marks]`

(b) Give the definition of the curvature of a curve in $\mathbf{E}^n$.

Consider the ellipse $x = 2\cos t$, $y = \sin t$, $0 \leq t < 2\pi$ in $\mathbf{E}^2$.

Calculate the curvature $k(t)$ at an arbitrary point of this ellipse.

Let a point move along this ellipse with constant speed $v = 1$.
Consider parallelogram $\Pi(t)$, formed by velocity and acceleration vectors of the point at arbitrary point $\mathbf{r}(t)$ of the ellipse.
At what points of the ellipse does the area of this parallelogram attain maximum and minimum values?
Calculate maximum and minimum values of area of the parallelogram.

`[9 marks]`

(c) Consider the following curve in $\mathbf{E}^3$ (a helix):

$$\mathbf{r}(t): \qquad \begin{cases} x = 2\cos t \\ y = 2\sin t \\ z = ct \end{cases} \qquad -\infty < t < \infty\,,$$

where c is a constant parameter.

Calculate the curvature of this curve.

Find how the curvature behaves if $c \to \infty$ and if $c \to 0$.
Explain the geometrical meaning of the answer.

`[5 marks]`

Discussion of third question

a) This question was bookwork question. To my surprise many students could not explain why in natural parameterisation acceleration and velocity vectors are orthogonal to each other. The most awfull mistake was when students confused velocity and speed and deduce that acceleration vanishes since speed is constant, but constant speed does not mean that acceleration vanishes! This is tangential acceleration which vanishes (if speed is constant)!

b) The question about calculating area of parallelogram turns out to be the most difficult question in exam. The rifght solution is:

$$\text{Area of parallelogram}\Pi_{\mathbf{a}}(t) = k(t)|\mathbf{v}|^3(t) = k(t) = \frac{2}{(4\sin^2 t + \cos^2 t)^{3/2}}\,, \quad \text{since } |\mathbf{v}| = 1\,.$$

Then it is easy to calculate the extrema points of this function.

Many students realisng that area of parallelogram is related with curvature instead using the formula above they calculated area of parallelogram using the fact that speed is constant, and on the other hand calculated speed from the equation $\begin{cases} x = 2\cos t \\ y = \sin t \end{cases}$. This is wrong: in the case if speed is constant, the curve has another (natural) aprameterisation!

c Many students (about 50of helix using the right formula that $k = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$. They jsut calculated straightforwardly with a use of brute force. More that half of them failed in calculations. Sure there is much easier way to calculate the curvature: it is not difficlt that velocity and acceleration vectors are orthogonal to each other thus you do not need to calculated straighforwardly the vector product: curvature is equal to modulus of acceleration vector divided on square of speed.

Explaining geometrical meaning it was important to show that curve tends to circle of radius 2 and to straight line if $c \to 0$ or $c \to \infty$.

4.

(a) Explain how to find a unit normal vector $\mathbf{n}(u, v)$ at an arbitrary point of the surface $\mathbf{r} = \mathbf{r}(u, v)$ in $\mathbf{E}^3$.

Consider the surface of the cone $x^2 + y^2 - k^2z^2 = 0$ in $\mathbf{E}^3$:

$$\mathbf{r}(h, \varphi): \quad \begin{cases} x = kh\cos\varphi \\ y = kh\sin\varphi \\ z = h \end{cases} .$$

Find a unit normal vector to this surface at an arbitary point $\mathbf{r} = \mathbf{r}(h, \varphi)$, $(h \neq 0)$.

[7 marks]

(b) Write down explicitly the shape operator for the cylindrical surface in $\mathbf{E}^3$ $x^2+y^2 = a^2$, $(a \neq 0)$.

Consider two vectors $\mathbf{A}$ and $\mathbf{B}$ tangent to this cylindrical surface and attached at the point $(a, 0, 0)$, such that vector $\mathbf{A}$ is parallel to the axis Oy, and vector $\mathbf{B}$ is parallel to the axis Oz.
Write down the action of the shape operator on the vectors $\mathbf{A}$ and $\mathbf{B}$.
(You do not need to check that vectors $\mathbf{A}$ and $\mathbf{B}$ are tangent to the surface of cylinder.)

Show that the Gaussian curvature of the cylindrical surface equals zero.
What is the geometrical meaning of this fact?

[8 marks]

(c) Define the parallel transport of a tangent vector along a curve in a surface in $\mathbf{E}^3$.

Show that the length of a tangent vector does not change during parallel transport.

[5 marks]

Discussion of fourth question

a Not many students tired to do this question, but students had no special problems with it.

b Vector $\mathbf{A}$ is touching cylindre in the horisontal plane $z = 0$ and vector $\mathbf{B}$ is vertical, hence $S(\mathbf{A}) = -\frac{\mathbf{A}}{a}$ and $S(\mathbf{B}) = 0$. This follows immediately from the calculation of shape operator. About 30 students tried to answer the subquestion **b)**, almost all have no problem to calculate the shape operator, but only two (yes, two!) students manage to answer the question about action of shape operator on the vectors $\mathbf{A}, \mathbf{B}$.

All students who asnwered this question showed in a right way that Gaussian curvaatre vanishes. On the other hand trying to explain geometrical meaning they rightly suggsted arguments about "not shrinking paper" but only two students firmly wrote that "not shrinking paper" is related with preservation of length. It is here where referring Theorema Egregium would be very nice.

c) Not many students tired to do this question, but students had no special problems with it.