

Homework 2. Solutions

1 Consider an upper half-plane ($y > 0$) in \mathbf{R}^2 equipped with Riemannian metric

$$G = \sigma(x, y)(dx^2 + dy^2)$$

a) Show that $\sigma > 0$

b) In the case if $\sigma = \frac{1}{y^2}$ (the Lobachevsky metric) calculate the lengths of vectors $\mathbf{A} = 2\partial_x$ and $\mathbf{B} = 12\partial_x + 5\partial_y$ attached at the point $(x, y) = (1, 2)$.

c) calculate the cosine of the angle between the vectors \mathbf{A} and \mathbf{B} and show that the answer does not depend on the choice of the function $\sigma(x, y)$.

a) $\sigma > 0$ since positive definiteness: e.g. $G(\mathbf{X}, \mathbf{X}) = \sigma(x, y) > 0$ if $\mathbf{X} = \partial_x$.

b)

$$|\mathbf{A}| = \sqrt{G(\mathbf{A}, \mathbf{A})} = \sqrt{\frac{A_x^2 + A_y^2}{y^2}} = \sqrt{\frac{2^2 + 0^2}{2^2}} = 1, |\mathbf{B}| = \sqrt{G(\mathbf{B}, \mathbf{B})} = \sqrt{\frac{B_x^2 + B_y^2}{y^2}} = \sqrt{\frac{12^2 + 5^2}{2^2}} = \frac{13}{2}.$$

c) Calculate the cosine for an arbitrary σ : $\cos(\angle(\mathbf{A}, \mathbf{B})) = \frac{G(\mathbf{A}, \mathbf{B})}{\sqrt{G(\mathbf{A}, \mathbf{A})}\sqrt{G(\mathbf{B}, \mathbf{B})}} = \frac{\langle \mathbf{A}, \mathbf{B} \rangle_G}{|\mathbf{A}||\mathbf{B}|} =$

$$\frac{\sigma(x, y)(A_x B_x + A_y B_y)}{\sqrt{\sigma(x, y)(A_x^2 + A_y^2)}\sqrt{\sigma(x, y)(B_x^2 + B_y^2)}} = \frac{(A_x B_x + A_y B_y)}{\sqrt{(A_x^2 + A_y^2)}\sqrt{(B_x^2 + B_y^2)}} = \frac{2 \cdot 12 + 0 \cdot 5}{1 \cdot 2 \cdot 13} = \frac{12}{13}.$$

It does not depend on $\sigma!!!$

2 a) Write down explicit formulae expressing stereographic coordinates for n -dimensional sphere $(x^1)^2 + \dots + (x^{n+1})^2 = R^2$ of radius R via coordinates x^1, \dots, x^{n+1} and vice versa.

(For simplicity you may consider cases $n = 2, 3$.)

b)[†] Check that for unit sphere S^2 ($x^2 + y^2 + z^2 = 1$) all the points with rational cartesian coordinates x, y, z have rational stereographic coordinates u, v and vice versa.

a) Write down the stereographic projection from the North pole of the sphere—point $N = (0, 0, \dots, R)$ on the plane $x^{n+1} = 0$. Consider the segment ND which intersects the sphere at the point (x^1, \dots, x^{n+1}) ($(x^1)^2 + (x^2)^2 + \dots + (x^{n+1})^2 = R^2$). This segment intersects the plane $x^{n+1} = 0$ at the point D with coordinates $x^i u^i$ for $i = 1, \dots, n$. Then comparing similar triangles we have

$$\frac{R}{R - x^{n+1}} = \frac{u^i}{x^i}, \quad \text{i.e. } u^i = \frac{R x^i}{R - x^{n+1}} \quad (i = 1, \dots, n)$$

and

$$x^i = \frac{u^i(R - x^{n+1})}{R}, \quad (i = 1, \dots, n).$$

Using the fact that $(x^1)^2 + \dots + (x^{n+1})^2 = R^2$ we come to

$$(x^1)^2 + \dots + (x^n)^2 = \frac{(R - (x^{n+1})^2)}{R^2} \sum_{i=1}^n (u^i)^2 = (R - x^{n+1})(R + x^{n+1}).$$

Dividing by $R - x^{n+1}$ ($x^{n+1} \neq R$ since North pole is removed) we come to

$$x^{n+1} = \frac{\sum_{i=1}^n (u^i)^2 - R^2}{\sum_{i=1}^n (u^i)^2 + R^2} R, \quad x^i = \frac{2u^i R^2}{\sum_{i=1}^n (u^i)^2 + R^2} \quad (i = 1, 2, \dots)$$

For projection with centre in South pole we have to change $x^{n+1} \mapsto -x^{n+1}$.

Write down these formulae for cases $n = 1, 2, 3$,

Case $n = 1$: Circle $x^2 + y^2 = R^2$. Stereographic coordinate t . Centre of projection $(0, R)$:

$$t = \frac{Rx}{R-y}, \quad \begin{cases} x = \frac{2tR^2}{R^2+t^2} \\ y = \frac{t^2-R^2}{t^2+R^2}R \end{cases} \quad (1)$$

Case $n = 2$: Sphere $x^2 + y^2 + z^2 = R^2$. Stereographic coordinates u, v . Centre of projection $(0, 0, R)$:

$$\begin{cases} u = \frac{Rx}{R-z} \\ v = \frac{Ry}{R-z} \end{cases}, \quad \begin{cases} x = \frac{2uR^2}{R^2+u^2+v^2} \\ y = \frac{2vR^2}{R^2+u^2+v^2} \\ z = \frac{u^2+v^2-R^2}{u^2+v^2+R^2}R \end{cases} \quad (2)$$

Case $n = 3$: 3-dimensional sphere $x^2 + y^2 + z^2 + t^2 = R^2$. Stereographic coordinates u, v, ω . Centre of projection $(0, 0, 0, R^2)$:

$$\begin{cases} u = \frac{Rx}{R-t} \\ v = \frac{Ry}{R-t} \\ \omega = \frac{Rz}{R-t} \end{cases}, \quad \begin{cases} x = \frac{2uR^2}{R^2+u^2+v^2+\omega^2} \\ y = \frac{2vR^2}{R^2+u^2+v^2+\omega^2} \\ z = \frac{2\omega R^2}{R^2+u^2+v^2+\omega^2} \\ t = \frac{u^2+v^2+\omega^2-R^2}{u^2+v^2+\omega^2+R^2}R \end{cases} \quad (2)$$

b)[†] We see that from explicit formulae. This is rational transformation of conic surfaces.

3 Consider the Riemannian metric on the circle of the radius R induced by the Euclidean metric on the ambient plane.

a) Express it using polar angle as a coordinate on the circle.

b) Express the same metric using stereographic coordinate t obtained by stereographic projection of the circle on the line, passing through its centre.

Riemannian metric of Euclidean space is $G = dx^2 + dy^2$.

a) using the angle: In this case parametric equation of circle is $\begin{cases} x = R \cos \varphi \\ y = R \sin \varphi \end{cases}$. Then

$$G = (dx^2 + dy^2)|_{x=R \cos \varphi, y=R \sin \varphi} = (d \cos \varphi)^2 + (d \sin \varphi)^2 = R^2 d\varphi^2.$$

b) In stereographic coordinate using (1) and the fact that

$$y = R \frac{t^2 - R^2}{t^2 + R^2} = R \left(1 - \frac{2R^2}{t^2 + R^2} \right)$$

we have that

$$\begin{aligned} G = (dx^2 + dy^2)|_{x=x(t), y=y(t)} &= \left(d \left(\frac{2tR^2}{R^2 + t^2} \right) \right)^2 + \left(d \left(\frac{t^2 - R^2}{R^2 + t^2} R \right) \right)^2 = \\ &= \left(\frac{2R^2 dt}{R^2 + t^2} - \frac{4t^2 R^2 dt}{(R^2 + t^2)^2} \right)^2 + \left(-\frac{4R^2 t dt}{(t^2 + R^2)^2} \right)^2 = \frac{4R^4 dt^2}{(R^2 + t^2)^2} \blacksquare \end{aligned}$$

4 Consider the Riemannian metric on the sphere of the radius R induced by the Euclidean metric on the ambient 3-dimensional space.

a) Express it using spherical coordinates on the sphere.

b) Express the same metric using stereographic coordinates u, v obtained by stereographic projection of the sphere on the plane, passing through its centre.

Solution

Riemannian metric of Euclidean space is $G = dx^2 + dy^2 + dz^2$.

a) using the spherical coordinates: In this case parametric equation of sphere is $\begin{cases} x = R \sin \theta \cos \varphi \\ y = R \sin \theta \sin \varphi \\ z = R \cos \theta \end{cases}$.

Then

$$G = (dx^2 + dy^2 + dz^2)|_{x=R \sin \theta \cos \varphi, y=R \sin \theta \sin \varphi, z=R \cos \theta} = R^2 ((d \sin \theta \cos \varphi)^2 + R^2 ((d \sin \theta \sin \varphi)^2 + R^2 ((d \cos \theta)^2 = \blacksquare$$

$$R^2 (\cos \theta \cos \varphi d\theta - \sin \theta \sin \varphi d\varphi)^2 + R^2 (\cos \theta \sin \varphi d\theta + \sin \theta \cos \varphi d\varphi)^2 + R^2 (-\sin \theta d\theta)^2 = R^2 (d\theta^2 + \sin^2 \theta d\varphi^2).$$

b) in stereographic coordinates using (2) we have $G = (dx^2 + dy^2 + dz^2)|_{x=x(u,v), y=y(u,v), z=z(u,v)} =$

$$\begin{aligned} & \left(d \left(\frac{2uR^2}{R^2 + u^2 + v^2} \right) \right)^2 + \left(d \left(\frac{2vR^2}{R^2 + u^2 + v^2} \right) \right)^2 + \left(d \left(1 - \frac{2R^2}{R^2 + u^2 + v^2} \right) R \right)^2 = \\ & R^4 \left(\frac{2du}{R^2 + u^2 + v^2} - \frac{2u(2udu + 2vdv)}{(R^2 + u^2 + v^2)^2} \right)^2 + R^4 \left(\frac{2dv}{R^2 + u^2 + v^2} - \frac{2v(2udu + 2vdv)}{(R^2 + u^2 + v^2)^2} \right)^2 + \frac{16R^6(udu + vdv)^2}{(R^2 + u^2 + v^2)^4} = \\ & \frac{4R^4}{(R^2 + u^2 + v^2)^2} \left[\left(du - \frac{2u(udu + vdv)}{R^2 + u^2 + v^2} \right)^2 + \left(dv - \frac{2v(udu + vdv)}{R^2 + u^2 + v^2} \right)^2 + \frac{4R^2(udu + vdv)^2}{(R^2 + u^2 + v^2)^2} \right] = \\ & \frac{4R^4(du^2 + dv^2)}{(R^2 + u^2 + v^2)^2} \blacksquare \end{aligned}$$

Remark

In the case of n -dimensional sphere S^n of radius R in $(n+1)$ -dimensional Euclidean space \mathbf{E}^{n+1} (it can be defined by the equation $(x^1)^2 + \dots + (x^{n+1})^2 = 1$ in Cartesian coordinates x^1, \dots, x^n, x^{n+1}) Riemannian metric on this sphere induced by the Euclidean metric in the ambient space in stereographic coordinates has following appearance:

$$\begin{aligned} G = ((dx^1)^2 + \dots + (dx^{n+1})^2)|_{x^\mu = x^\mu(u^i)} &= \left(\sum_{j=1}^n \left(d \left(\frac{2R^2 u^j}{R^2 + \sum_{i=1}^n (u^i)^2} \right) \right) \right)^2 + \left(d \left(R \frac{\sum_{i=1}^n (u^i)^2 - R^2}{R^2 + \sum_{i=1}^n (u^i)^2} \right) \right)^2 = \blacksquare \\ &= \frac{4R^4 \sum_{i=1}^n (du^i)^2}{(R^2 + \sum_{i=1}^n (u^i)^2)^2} \end{aligned}$$

5 Consider the surface L which is the upper sheet of two-sheeted hyperboloid in \mathbf{R}^3 :

$$L: \quad z^2 - x^2 - y^2 = 1, \quad z > 0.$$

a) Find parametric equation of the surface L using hyperbolic functions \cosh, \sinh following an analogy with spherical coordinates on the sphere.

(The surface L sometimes is called pseudo-sphere.)

b) Consider the stereographic projection of the surface L on the plane OXY , i.e. the central projection on the plane $z = 0$ with the centre at the point $(0, 0, -1)$.

Show that the image of projection of the surface L is the open disc $x^2 + y^2 < 1$ in the plane OXY .

a) Parametric equation is $\begin{cases} x = \sinh \theta \cos \varphi \\ y = \sinh \theta \sin \varphi \\ z = \cosh \theta \end{cases}$. We see that the condition $z^2 - x^2 - y^2 = 1$ is fulfilled.

(Compare with equation of sphere in spheric coordinates.)

b) Calculations are very similar to the case of stereographic coordinates for 2-sphere $x^2 + y^2 + z^2 = 1$ of the radius $R = 1$. Stereographic coordinates u, v . Centre of projection $(0, 0, -1)$: We have $\frac{u}{x} = \frac{v}{y} = \frac{1}{1+z}$. Hence $\begin{cases} u = \frac{x}{1+z} \\ v = \frac{y}{1+z} \end{cases}$. Since $x = u(1+z), y = v(1+z)$ then $z^2 - 1 = x^2 + y^2$ and $z^2 - 1 = (u^2 + v^2)(1+z)^2$, i.e. $z = \frac{1+u^2+v^2}{1-u^2-v^2}$. We come to

$$\begin{cases} u = \frac{x}{1+z} \\ v = \frac{y}{1+z} \end{cases}, \quad \begin{cases} x = \frac{2u}{1-u^2-v^2} \\ y = \frac{2v}{1-u^2-v^2} \\ z = \frac{u^2+v^2+1}{1-u^2-v^2} \end{cases}, \quad |u| < 1, |v| < 1. \quad (4)$$

The image of upper-sheet is an open disc $u^2 + v^2 = 1$ since $u^2 + v^2 = \frac{x^2+y^2}{(1+z)^2} = \frac{z^2-1}{(1+z)^2} = \frac{z-1}{z+1}$. Since for upper sheet $z > 1$ then $0 \leq \frac{z-1}{z+1} < 1$.

5 Consider the pseudo-Riemannian, pseudo-Euclidean metric on \mathbf{R}^3 given by the formula

$$ds^2 = dx^2 + dy^2 - dz^2.$$

Calculate the induced metric on the surface L considered in the Exercise 4, and show that it is a Riemannian metric (it is positive-definite).

Perform calculations in spherical-like coordinates (see Exercise 4a) above) and in stereographic coordinates (see exercise 4b) above)

In stereographic coordinates we come to realisation of Lobachevsky plane on the disc in \mathbf{E}^2 . It is so called Poincare model of Lobachevsky geometry.

Solution. The calculations will be very similar to the calculations performed in the exercise 3 above. Just we need consider $\cosh \theta, \sinh \theta$ instead $\cos \theta, \sin \theta$ and sometimes changes the signs.

First of all consider spherical-like coordinates:

Equation of two-sheeted hyperboloid is $\begin{cases} x = \sinh \theta \cos \varphi \\ y = \sinh \theta \sin \varphi \\ z = \cosh \theta \end{cases}$. Then

$$G = (dx^2 + dy^2 - dz^2)|_{x=\sinh \theta \cos \varphi, y=\sinh \theta \sin \varphi, z=\cosh \theta} = ((d \sinh \theta \cos \varphi))^2 + ((d \sinh \theta \sin \varphi))^2 - ((d \cosh \theta))^2 =$$

$$(\cosh \theta \cos \varphi d\theta - \sinh \theta \sin \varphi d\varphi)^2 + (\cosh \theta \sin \varphi d\theta + \sinh \theta \cos \varphi d\varphi)^2 + (\sinh \theta d\theta)^2 = d\theta^2 + \sinh^2 \theta d\varphi^2.$$

matrix of Riemannian metric is $G = \begin{pmatrix} 1 & 0 \\ 0 & \sinh^2 \theta \end{pmatrix}$. In the same way as for sphere these coordinates are well-defined in all points except $z = \pm 1$, where $\sin^2 \theta = 0$.

Now express Riemannian metric in stereographic coordinates (4):

$$\begin{aligned} G = (dx^2 + dy^2 - dz^2)|_{x=x(u,v), y=y(u,v), z=z(u,v)} &= \left(d \left(\frac{2u}{1-u^2-v^2} \right) \right)^2 + \left(d \left(\frac{2v}{1-u^2-v^2} \right) \right)^2 - \left(d \left(\frac{2}{1-u^2-v^2} - 1 \right) \right)^2 \\ &= \left(\frac{2du}{1-u^2-v^2} + \frac{2u(2udu + 2vdv)}{(1-u^2-v^2)^2} \right)^2 + \left(\frac{2dv}{1-u^2-v^2} + \frac{2v(2udu + 2vdv)}{(1-u^2-v^2)^2} \right)^2 - \frac{16(udu + vdv)^2}{(1-u^2-v^2)^4} = \\ &= \frac{4(du)^2 + 4(dv)^2}{(1-u^2-v^2)^2}. \end{aligned}$$

(Compare with calculations for sphere $x^2 + y^2 + z^2 = 1$).

Resume: We come to Riemannian metric on the surface L induced by pseudo-Riemannian metric in ambient space.

Remark The surface L sometimes is called pseudo-sphere. The Riemannian metric on this surface sometimes is called Lobachevsky (hyperbolic) metric. The surface L with this metric realises Lobachevsky (hyperbolic) geometry, where Euclid's 5-th Axiom fails. This Riemannian manifold (manifold+Riemannian metric) is called Lobachevsky (hyperbolic) plane. In stereographic coordinates Lobachevsky plane is realised as an open disc $u^2 + v^2 < 1$ in \mathbf{E}^2 . It is so called Poincare model of Lobachevsky geometry. In the exercise 8 below we will consider realisation of Lobachevsky plane as an upper half-plane.

7* In the exercises 5 and 6 it was shown that pseudo-Euclidean metric (1) in \mathbf{R}^3 induces Riemannian metric on two-sheeted hyperboloid $z^2 - x^2 - y^2 = 1$. Show that it is not true for one-sheeted hyperboloid: metric on one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$ in \mathbf{R}^3 is not Riemannian if it is induced with the pseudo-Euclidean metric (1).

Solution. One can perform straightforward calculations in spherical-like coordinates: Equation of one-sheeted hyperboloid is $\begin{cases} x = \cosh \theta \cos \varphi \\ y = \cosh \theta \sin \varphi \\ z = \sinh \theta \end{cases}$. Then

$$G = (dx^2 + dy^2 - dz^2)|_{x=\cosh \theta \cos \varphi, y=\cosh \theta \sin \varphi, z=\sinh \theta} = ((d \cosh \theta \cos \varphi)^2 + (d \cosh \theta \sin \varphi)^2 - (d \sinh \theta)^2 =$$

$$(\sinh \theta \cos \varphi d\theta - \cosh \theta \sin \varphi d\varphi)^2 + (\sinh \theta \sin \varphi d\theta + \cosh \theta \cos \varphi d\varphi)^2 - (\cosh \theta d\theta)^2 = -d\theta^2 + \cosh^2 \theta d\varphi^2.$$

matrix is $G = \begin{pmatrix} -1 & 0 \\ 0 & \cosh^2 \theta \end{pmatrix}$. The condition of positive-definiteness is not fulfilled. This is not Riemannian metric.

Another solution Consider the vectors $\mathbf{e} = \frac{\partial}{\partial y}$ and $\mathbf{f} = \frac{\partial}{\partial z}$ attached at the point $(1, 0, 0)$. One can see that these vectors are tangent to the hyperboloid, but they have the "length" of different sign. (One of these vectors is space-like vector, another time like vector.) We have pseudoriemannian metric at the tangent space spanned by these two vectors.

8* In the exercise 6 we realised Lobachevsky plane as a disc $u^2 + v^2 < 1$. Find new coordinates x, y such that in these coordinates Lobachevsky plane (hyperbolic plane) can be considered as an upper half plane $\{(x, y): y > 0\}$ in \mathbf{E}^2 and write down explicitly Riemannian metric in these coordinates.

Hint: You may use complex coordinates:

$$z = x + iy, \bar{z} = x - iy, \omega = u + iv, \bar{\omega} = u - iv$$

and consider a holomorphic transformation:

$$\omega = \frac{1 + iz}{1 - iz} \Leftrightarrow z = i \frac{1 - \omega}{1 + \omega},$$

which transforms the open disc $w\bar{w} < 1$ onto the upper plane $\text{Im} z > 0$.

Solution.

Recall that in the previous exercise we calculated expression for Lobachevsky metric in stereographic coordinates $u, v, u^2 + v^2 < 1$. We come to the answer: $G = \frac{4du^2 + 4dv^2}{(1 - u^2 - v^2)^2}$ (see exercise 6). (It was realisation of Lobachevsky plane on the Euclidean disc, so called Poincare model of Lobachevsky (hyperbolic) geometry.)

In complex coordinates $\omega = u + iv, \bar{\omega} = u - iv$ the metric $G = \frac{4du^2 + 4dv^2}{(1 - u^2 - v^2)^2}$ obtained in the exercise 6 can be rewritten $G = \frac{4d\omega d\bar{\omega}}{(1 - w\bar{w})^2}$. Indeed

$$G = \frac{4d\omega d\bar{\omega}}{(1 - w\bar{w})^2} = G = \frac{4d(u + iv)d(u - iv)}{(1 - (u + iv)(u - iv))^2} = \frac{4du^2 + 4dv^2}{(1 - u^2 - v^2)^2}.$$

Now consider Mobius transformation $\omega = \frac{1+iz}{1-iz}$, which transforms the disc, interior of circle $\omega\bar{\omega} = 1$ onto upper half plane $\text{Im} z > 0$. One can see that

$$\omega = \frac{1+iz}{1-iz}, \quad z = i \frac{1-\omega}{1+\omega}$$

(Can you find all Mobius transformations which transform upper half plane to the interior of unit circle?.)

Now calculate G in coordinates z, \bar{z} . i.e. in coordinates (x, y) :

$$G = \frac{4du^2 + 4dv^2}{(1-u^2-v^2)^2} = \frac{4d\omega d\bar{\omega}}{(1-\omega\bar{\omega})^2}.$$

We have

$$d\omega = d\left(\frac{1+iz}{1-iz}\right) = \frac{2idz}{(1-iz)^2}, \quad d\bar{\omega} = \frac{-2id\bar{z}}{(1+i\bar{z})^2},$$

$$1 - \omega\bar{\omega} = 1 - \frac{1+iz}{1-iz} \frac{1-i\bar{z}}{1+i\bar{z}} = \frac{2i(\bar{z}-z)}{(1-iz)(1+i\bar{z})}$$

Hence

$$G = \frac{4d\omega d\bar{\omega}}{(1-\omega\bar{\omega})^2} = \frac{4\left(\frac{2idz}{(1-iz)^2}\right)\left(\frac{-2id\bar{z}}{(1+i\bar{z})^2}\right)}{\frac{-4(\bar{z}-z)^2}{(1-iz)^2(1+i\bar{z})^2}} = \frac{-4dd\bar{z}}{(\bar{z}-z)^2} = \frac{dx^2 + dy^2}{y^2},$$

since $z = x + iy$ and $\bar{z} - z = -2iy$.

We come to the very useful and nice interpretation of hyperbolic geometry: upper half plane in \mathbf{E}^2 with metric $G = \frac{dx^2 + dy^2}{y^2}$. Later by default we will call "Lobachevsky (hyperbolic) plane" the realisation of Lobachevsky plane as an half-upper plane in \mathbf{E}^2 with these coordinates x, y ($y > 0$) with metric $G = \frac{dx^2 + dy^2}{y^2}$.

Remark What will happen if we consider another Mobius transformation of disc $\omega\bar{\omega} < 1$ onto plane $\text{Im} z > 0$?