Homework 6

1 Calculate the integrals of the form $\omega = xdy - ydx$ over the following three curves. Compare answers.

$$C_1\colon \mathbf{r}(t) \begin{cases} x = R\cos t \\ y = R\sin t \end{cases}, \ 0 < t < \pi, \quad C_2\colon \mathbf{r}(t) \begin{cases} x = R\cos 4t \\ y = R\sin 4t \end{cases}, \ 0 < t < \frac{\pi}{4}$$

$$\text{and} \quad C_3\colon \mathbf{r}(t) \begin{cases} x = Rt \\ y = R\sqrt{1-t^2} \end{cases}, \ -1 \leq t \leq 1.$$

(this exercise was done during the XIV-th lecture (Tuesday, VII-th week): see the last example in subsection 2.4 "Integration of differential forms over curves" in Lecture notes)

2 Consider an arc of parabola $x = 2y^2 - 1$, $0 < y < 1$.

Give examples of two different parameterisations of this curve such that these parameterisations have the opposite orientation.

Calculate the integral of the form 1-form $\omega = \sin y dx$ over this curve.

How does the answer depend on a parameterisation?

3 Calculate the integral of the form $\omega = xdy$ over the following curves

a) closed curve $x^2 + y^2 = 12y$

b) arc of the ellipse $x^2 + y^2/9 = 1$ defined by the condition $y \geq 0$.

How does your answer depend on a choice of parameterisation?

4 a) Calculate the integrals $\int_{C_1} \omega$ and $\int_{C_2} \omega$ of the 1-form $\omega = xdy - ydx$ over the curves C_1: $x^2 + y^2 = 9$ and C_2: $x^2 + y^2 = 6y$.

b) Perform the calculations of integrals $\int_{C_1} \omega$ and $\int_{C_2} \omega$ in polar coordinates.

Hint Performing the calculations for the curve C_2 one may use the polar coordinates r', φ' with the centre at the point (a, b): $\begin{cases} x = a + r'\cos\varphi' \\ y = b + r'\sin\varphi' \end{cases}$.

Exact forms

5 Calculate the integral $\int_C \omega$ where $\omega = xdx + ydy$ and C is

a) the straight line segment $x = t, y = 1 - t$, $0 \leq t \leq 1$

b) the segment of parabola $x = t$, $y = 1 - t^n$, $0 \leq t \leq 1$, $n = 2, 3, 4, \ldots$

c) for **an arbitrary** curve starting at the point $(0, 1)$ and ending at the point $((1, 0)$.

6 Show that the form 1-form $\omega = 3x^2ydx + x^3dy$ is an exact 1-form.

Calculate integral of this form over the curves considered in exercises 2) and 3).

7. Consider in $\mathbf{E}^2$ 1-forms

a) xdx, b) xdy c) $xdx + ydy$, d)$xdy + ydx$, e) $xdy - ydx$

f) $x^4dy + 4x^3ydx$.

a) Show that 1-forms a), c), d) and f) are exact forms

b) Why are 1-forms b) and e) not exact?

8 Consider 1-form

$$\omega = \frac{xdy - ydx}{x^2 + y^2} \tag{1}$$

This form is defined in $\mathbf{E}^2 \setminus 0$, i.e. in all the points except origin: $x^2 + y^2 \neq 0$.

Write down this form in polar coordinates

9† Consider again 1-form defined in eqation (1).

a) † Does there exist a (smooth) function defined in $\mathbf{E}^2 \setminus 0$ such that $\omega = dF$?

This proves that ω is a closed form.

b) † What values can take the integral $\int_C \omega$ for the 1-form ω considered in equation (1) if C is an arbitrary curve starting at the point $(0, 1)$ and ending at the point $((1, 0)$ (we suppose that the curve C does not pass trough the origin)

10† Let $\omega = a(x, y)dx + b(x, y)dy$ be a closed form in $\mathbf{E}^2$, $d\omega = 0$.

Consider the function

$$f(x, y) = x \int_0^1 a(tx, ty)dt + y \int_0^1 b(tx, ty)dt \tag{2}$$

$\dagger$ Show that

$$\omega = df\,.$$

(*This proves that an arbitrary closed form in $\mathbf{E}^2$ is an exact form.*

$\dagger$ Why we cannot apply the formula (2) to the form ω defined by the expression (1)?