Homework 9

1) Let ∇ be a connection on n-dimensional manifold M and $\{R^i{}_{rmn}\}$ be the components of the curvature tensor of a connection ∇ in local coordinates $(x^1, x^2, \dots, x^n)$.

a) For arbitrary vector fields $\mathbf{A}, \mathbf{B}$ and $\mathbf{D}$ calculate the vector field

$$(\nabla_{\mathbf{A}}\nabla_{\mathbf{B}} - \nabla_{\mathbf{B}}\nabla_{\mathbf{A}})\,\mathbf{D} - \nabla_{\mathbf{C}}\mathbf{D}\,,$$

where the vector field $\mathbf{C}$ is a commutator of vector fields $\mathbf{A}$ and $\mathbf{B}$:

$$\mathbf{C} = C^i \frac{\partial}{\partial x^i} = [\mathbf{A}, \mathbf{B}] = \left(A^m \frac{\partial B^i(x)}{\partial x^m} - B^m \frac{\partial A^i(x)}{\partial x^m}\right) \frac{\partial}{\partial x^i}\,.$$

b) Calculate the vector field

$$(\nabla_{\mathbf{A}}\nabla_{\mathbf{B}} - \nabla_{\mathbf{B}}\nabla_{\mathbf{A}})\,\mathbf{D}$$

in the case if for vector fields $\mathbf{A}$ and $\mathbf{B}$ components A^i and B^m are constants (in the local coordinates $(x^1, \dots, x^n)$)

2) Calculate Riemann curvature tensor for the cylindircal surface $x^2 + y^2 = a^2$ in $\mathbf{E}^3$.

3) We know that If R^i_{kmn} is Riemann curvature tensor for Riemannian manifold (M, G) (R^i_{kmn} us curvature tensor for Levi-Civita connection on M) then the following identities hold:

$$R_{ikmn} = -R_{iknm}, \quad R_{ikmn} = -R_{kimn}\,, \qquad R_{ikmn} = R_{mnik}\,.$$

Show that Riemann curvature tensor for 2-dimensional Riemannian manifold (M, G) possesses only one non-trivial component.

4) If (M, G) is surface in $\mathbf{E}^3$ then

$$K = \frac{R}{2} = \frac{R_{1212}}{\det g}\,,$$

where K is Gaussian curvature of the surface, and R^i_{kmn} Riemann curvature tensor with respect to induced metric,

a) Prove by straightforward calculations that $\frac{R}{2} = \frac{R_{1212}}{\det g}$

b*) Prove that $K = \frac{R}{2} = \frac{R_{1212}}{\det g}$. (It is convenient to choose the orthonormal basis $\{\mathbf{e}, \mathbf{f}\}$ and use derivation formula.)