Homework 5. Solutions.

1 *Consider the following curves:*

C_1: $\mathbf{r}(t)$ $\begin{cases} x = t \\ y = 2t^2 - 1 \end{cases}$, $0 < t < 1$, C_2: $\mathbf{r}(t)$ $\begin{cases} x = t \\ y = 2t^2 - 1 \end{cases}$, $-1 < t < 1$,

C_3: $\mathbf{r}(t)$ $\begin{cases} x = 2t \\ y = 8t^2 - 1 \end{cases}$, $0 < t < \frac{1}{2}$, C_4: $\mathbf{r}(t)$ $\begin{cases} x = \cos t \\ y = \cos 2t \end{cases}$, $0 < t < \frac{\pi}{2}$,

C_5: $\mathbf{r}(t)$ $\begin{cases} x = t \\ y = 2t - 1 \end{cases}$, $0 < t < 1$, C_6: $\mathbf{r}(t)$ $\begin{cases} x = 1 - t \\ y = 1 - 2t \end{cases}$, $0 < t < 1$,

C_7: $\mathbf{r}(t)$ $\begin{cases} x = \sin^2 t \\ y = -\cos 2t \end{cases}$, $0 < t < \frac{\pi}{2}$, C_8: $\mathbf{r}(t)$ $\begin{cases} x = t \\ y = \sqrt{1 - t^2} \end{cases}$, $-1 < t < 1$,

C_9: $\mathbf{r}(t)$ $\begin{cases} x = \cos t \\ y = \sin t \end{cases}$, $0 < t < \pi$, C_{10}: $\mathbf{r}(t)$ $\begin{cases} x = \cos 2t \\ y = \sin 2t \end{cases}$, $0 < t < \frac{\pi}{2}$,

C_{11}: $\mathbf{r}(t)$ $\begin{cases} x = \cos t \\ y = \sin t \end{cases}$, $0 < t < 2\pi$, C_{12}: $\mathbf{r}(t)$ $\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$, $0 < t < 2\pi$ *(ellipse)*,

Draw the images of these curves.

Write down their velocity vectors.

Indicate parameterised curves which have the same image (equivalent curves).

In each equivalence class of parameterised curves indicate curves with same and opposite orientations.

C_1: $\mathbf{v}(t) = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 1 \\ 4t \end{pmatrix}$, C_2: $\mathbf{v}(t) = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 1 \\ 4t \end{pmatrix}$, C_3: $\mathbf{v}(t) = \begin{pmatrix} 2 \\ 16t \end{pmatrix}$, C_4: $\mathbf{v}(t) = \begin{pmatrix} -\sin t \\ -2\sin 2t \end{pmatrix}$,

C_5: $\mathbf{v}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, C_6: $\mathbf{v}(t) = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$, C_7: $\mathbf{v}(t) = \begin{pmatrix} \sin 2t \\ 2\sin 2t \end{pmatrix}$,

C_8: $\mathbf{v}(t) = \begin{pmatrix} 1 \\ \frac{-t}{\sqrt{1-t^2}} \end{pmatrix}$, C_9: $\mathbf{v}(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$, C_{10}: $\mathbf{v}(t) = \begin{pmatrix} -2\sin 2t \\ 2\cos 2t \end{pmatrix}$

C_{11}: $\mathbf{v}(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$, C_{12}: $\mathbf{v}(t) = \begin{pmatrix} -a\sin t \\ b\cos t \end{pmatrix}$

Curves C_1, C_2, C_3, C_4

Curves C_1, C_3 and C_4 have the same image: it is piece of parabola $y = 2x^2 - 1$ between points $(0, 1)$ and $(1, 1)$. Image of the curve C_2 is piece of the same parabola $y = 2x^2 - 1$ between points $(-1, 1)$ and $(1, 1)$. Image of curve C_1 is a part of the image of the curve C_2.

Curve C_3 can be obtained from the curve C_1 by reparameterisation $t(\tau) = 2\tau$, $\mathbf{r}_3(\tau) = \mathbf{r}_1(t(\tau)) = \mathbf{r}_1(2\tau)$. Respectively $\mathbf{v}_3(\tau) = t'(\tau)\mathbf{v}_1(t(\tau)) = 2\mathbf{v}_1(2\tau)$. Curve C_4 can be obtained from the curve C_1 by reparameterisation $t(\tau) = \cos\tau$, $\mathbf{r}_4(\tau) = \mathbf{r}_1(t(\tau)) = \mathbf{r}_1(\cos\tau)$. Respectively $\mathbf{v}_4(\tau) = \begin{pmatrix} -\sin\tau \\ -2\sin 2\tau \end{pmatrix} = t'(\tau)\mathbf{v}_1(t(\tau)) = -\sin\tau\mathbf{v}_1(\cos\tau) = -\sin\tau \begin{pmatrix} 1 \\ 2\cos\tau \end{pmatrix}$.

We see that curves C_1, C_3, C_4 are equivalent. They belong to the same equivalence class of non-parameterised curves. Equivalent curves C_1 and C_3 have the same orientation because diffeomorphism $t = 2\tau$ has positive derivative. Equivalent curves C_1 and C_4 (and so C_3 and C_4) have opposite orientation because diffeomorphism $t = \cos\tau$ has negative derivative (for $0 < t < 1$).

Curves C_5, C_6, C_7

Now consider curves C_5, C_6, C_7. It is easy to see that they all have the same image— segment of the line between point $(0, -1)$ and $(1, 1)$. These three curves belong to the same equivalence class of non-parameterised curves. Curve C_6 can be obtained from the curve C_5 by reparameterisation $t(\tau) = 1 - \tau$, $\mathbf{r}_6(\tau) = \mathbf{r}_5(t(\tau)) = \mathbf{r}_5(1 - \tau)$. Respectively $\mathbf{v}_6(\tau) = t'(\tau)\mathbf{v}_5(t(\tau)) = -\mathbf{v}_5(1 - \tau)$. (Velocity just changes its direction on opposite.) Curve C_7 can be obtained from the curve C_5 by reparameterisation $t(\tau) = \sin^2\tau$, $\mathbf{r}_7(\tau) = \mathbf{r}_5(t(\tau)) = \mathbf{r}_5(\sin\tau)$. Respectively $\mathbf{v}_7(\tau) = \begin{pmatrix} \sin 2\tau \\ 2\sin 2\tau \end{pmatrix} = t'(\tau)\mathbf{v}_5(t(\tau)) = \sin 2\tau \mathbf{v}_5(\sin\tau) = \sin 2\tau \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Equivalent curves C_5 and C_7 have the same orientation because derivative of diffeomorphism $t = \sin^2\tau$ is positive (on the interval $0 < t < 1$). Curve C_6 has orinetation opposite to the orientation of the curves C_5 and C_6 because derivative of diffeomorphism $t = 1 - \tau$ is negative. Or in other words when we go to the curve C_6 starting point becomes ending point and vice versa.

Curves C_8, C_9, C_{10}

Now consider curves C_8, C_9, C_{10}. It is easy to see that they all have the same image— upper part of the circle $x^2 + y^2 = 1$. These three curves belong to the same equivalence class of non-parameterised curves. Curve C_9 can be obtained from the curve C_8 by reparameterisation $t(\tau) = \cos\tau$. Then $\mathbf{r}_9(\tau) = \mathbf{r}_8(t(\tau)) = \mathbf{r}_8(\cos\tau)$. Respectively $\mathbf{v}_9(\tau) = t'(\tau)\mathbf{v}_8(t(\tau)) = -\sin\tau\mathbf{v}_8(\cos\tau)$.

Curve C_{10} can be obtained from the curve C_8 by reparameterisation $t(\tau) = 2\tau$, $\mathbf{r}_{10}(\tau) = \mathbf{r}_8(t(\tau)) = \mathbf{r}_8(2\tau)$. Respectively $\mathbf{v}_{10}(\tau) = t'(\tau)\mathbf{v}_8(t(\tau)) = 2\tau\mathbf{v}_8(2\tau)$.

Equivalent curves C_8 and C_{10} have the same orientation because derivative of diffeomorphism $t = 2\tau$ is positive . Curve C_9 has orinetation opposite to the orientation of the curves C_8 and C_{10} because derivative of diffeomorphism $t = \cos\tau$ on the interval $0 < t < \pi$ is negative.

Curves C_{11}, C_{12}

Image of the curve C_{11} is circle $x^2 + y^2 = 1$.

Image of the curve C_{12} is ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

2 *Consider differential forms $\omega = xdy - ydx$, $\sigma = xdx + ydy$ and vector fields $\mathbf{A} = x\partial_x + y\partial_y$, $\mathbf{B} = x\partial_y - y\partial_x$*

Calculate $\omega(\mathbf{A}), \omega(\mathbf{B})$, $\sigma(\mathbf{A}), \sigma(\mathbf{B})$.

$\omega(\mathbf{A}) = (xdy - ydx)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) =$

$$x^2 dy\left(\frac{\partial}{\partial x}\right) + xydy\left(\frac{\partial}{\partial y}\right) - yxdx\left(\frac{\partial}{\partial x}\right) - y^2 dx\left(\frac{\partial}{\partial y}\right) = x^2\cdot 0 + xy\cdot 1 - yx\cdot 1 - y^2\cdot 0 = 0\,.$$

Later we often denote vector field $\frac{\partial}{\partial x}$ by ∂_x, vector field $\frac{\partial}{\partial y}$ by ∂_y...

$\omega(\mathbf{B}) = (xdy - ydx)(x\partial_y - y\partial_x) = x^2dy(\partial_y) - xydy(\partial_x) - yxdx(\partial_y) + y^2dx(\partial_x) = x^2\cdot 1 - xy\cdot 0 - yx\cdot 0 + y^2\cdot 1 = x^2 + y^2 = r^2$,

$\sigma(\mathbf{A}) = (xdx + ydy)(x\partial_x + y\partial_y) = x^2dx(\partial_x) + xydx(\partial_y) + yxdy(\partial_x) + y^2dy(\partial_y) = x^2\cdot 1 + xy\cdot 0 + yx\cdot 0 + y^2\cdot 1 = x^2 + y^2 = r^2$,

$\sigma(\mathbf{B}) = (xdx + ydy)(x\partial_y - y\partial_x) = x^2dx(\partial_y) - xydx(\partial_x) + yxdy(\partial_y) - y^2dy(\partial_x) = x^2\cdot 0 - xy\cdot 1 + yx\cdot 1 - y^2\cdot 0 = 0$.

3 *Consider a function $f = x^3 - y^3$.*

Calculate the value of 1-form $\omega = df$ on the vector field $\mathbf{B} = x\partial_y - y\partial_x$.

$$df(\mathbf{B}) = \partial_{\mathbf{B}} f = (x\partial_y - y\partial_x)(x^3 - y^3) = -3xy^2 - 3yx^4 = --3xy(x+y)\,.$$

Another solution: $\omega = df = 3x^2dx - 3y^2dy$, thus

$$\omega(\mathbf{B}) = 3x^2dx - 3y^2dy(x\partial_y - y\partial_x) = -3x^2ydx(\partial_x) - 3y^2dy(\partial_y) = -3xy(x+y)\,.$$

4 *Calculate the derivatives of the functions* $f = x^2+y^2$, $g = y^2-x^2$ *and* $h = q\log|r| = q\log\left(\sqrt{x^2+y^2}\right)$ *(q is a constant) along vector fields* $\mathbf{A} = x\partial_x + y\partial_y$ *and* $\mathbf{B} = x\partial_y - y\partial_x$

a) calculating directional derivatives $\partial_{\mathbf{A}} f, \partial_{\mathbf{A}} g, \partial_{\mathbf{A}} h$, $\partial_{\mathbf{B}} f, \partial_{\mathbf{B}} g, \partial_{\mathbf{B}} h$

b) calculating $df(\mathbf{A}), dg(\mathbf{A}), dh(\mathbf{A}), df(\mathbf{B}), dg(\mathbf{B}), dh(\mathbf{B})$.

a) First do using directional derivatives:

$\partial_{\mathbf{A}} f = A_x\frac{\partial f}{\partial x} + A_y\frac{\partial f}{\partial y} = x\cdot 2x + y\cdot 2y = 2(x^2+y^2)$,

$\partial_{\mathbf{A}} g = A_x\frac{\partial g}{\partial x} + A_y\frac{\partial g}{\partial y} = x\cdot(-2x) + y\cdot 2y = 2(y^2-x^2)$,

$\partial_{\mathbf{A}} h = x\frac{\partial h}{\partial x} + y\frac{\partial h}{\partial y} = \frac{x^2q}{x^2+y^2} + \frac{y^2q}{x^2+y^2} = q$

$\partial_{\mathbf{B}} f = B_x\frac{\partial f}{\partial x} + B_y\frac{\partial f}{\partial y} = -y\cdot 2x + x\cdot 2y = 0$,

$\partial_{\mathbf{B}} g = -y\frac{\partial g}{\partial x} + x\frac{\partial g}{\partial y} = -y\cdot(-2x) + x\cdot 2y = 4xy$

$\partial_{\mathbf{B}} h = -y\frac{\partial h}{\partial x} + x\frac{\partial h}{\partial y} = \frac{-xyq}{x^2+y^2} + \frac{xyq}{x^2+y^2} = 0$

b) Now calculate using 1-form using the fact that $\partial_{\mathbf{A}} f = df(\mathbf{A})$:

We have that $df = d(x^2+y^2) = 2xdx + 2ydy$, $dg = d\left(y^2-x^2\right) = g_xdx + g_ydy = (2ydy - 2xdx)$, $dh = d\left(q\log\sqrt{x^2+y^2}\right) = h_xdx + h_ydy = \frac{qxdx+qydy}{x^2+y^2}$.

Hence

$\partial_{\mathbf{A}} f = df(\mathbf{A}) = (2xdx+2ydy)(x\partial_x + y\partial_y) = 2x^2dx(\partial_x) + 2y^2dy(\partial_y) = 2x^2+2y^2$,

$\partial_{\mathbf{A}} g = dg(\mathbf{A}) = (2ydy-2xdx)((x\partial_x+y\partial_y)) = 2ydy(y\partial_y) - 2xdx(x\partial_x) = 2y^2-2x^2$.

$\partial_{\mathbf{A}} h = dh(\mathbf{A}) = \frac{qxdx+qydy}{x^2+y^2}(x\partial_x+y\partial_y) = \frac{qxdx(x\partial_x)+qydy(y\partial_y)}{x^2+y^2} = \frac{qx^2+qy^2}{x^2+y^2} = q$

$\partial_{\mathbf{B}} f = df(\mathbf{A}) = (2xdx+2ydy)(-y\partial_x+x\partial_y) = -2xydx(\partial_x) + 2xydy(\partial_y) = 0$,

$\partial_{\mathbf{B}} g = dg(\mathbf{A}) = (2ydy-2xdx)((x\partial_y-y\partial_x)) = 2ydy(x\partial_y) - 2xdx(-y\partial_x) = 2xy+2xy = 4xy$.

$\partial_{\mathbf{B}} h = dh(\mathbf{A}) = \frac{qxdx+qydy}{x^2+y^2}(-y\partial_x+x\partial_y) = \frac{qxdx(-y\partial_x)+qydy(x\partial_y)}{x^2+y^2} = \frac{-qxy+qxy}{x^2+y^2} = 0$.

5 *Let* f *be a function on* $\mathbf{E}^2$ *given by* $f(r,\varphi) = r^3\cos 3\varphi$, *where* r,φ *are polar coordinates in* $\mathbf{E}^2$.

Calculate the 1*-form* $\omega = df$.

Calculate the value of the 1*-form* $\omega = df$ *on the vector field* $\mathbf{X} = r\partial_r + \partial_\varphi$.

Express the 1*-form* ω *in Cartesian coordinates* x, y[1)]

$\omega = 3r^2cos3\varphi dr - 3r^3\sin 3\varphi d\varphi$.

The value of the form $\omega = df$ on the vector field $\mathbf{X} = r\partial_r + \partial_\varphi$ is equal to

$$\omega(\mathbf{A}) = \left(3r^2\cos 3\varphi dr - 3r^2\sin 3\varphi d\varphi\right)(r\partial_r + \partial_\varphi) = 3r^3\cos 3\varphi dr(\partial_r) - 3r^3\sin 3\varphi d\varphi(\partial_\varphi) = 3r^3(\cos 3\varphi - \sin 3\varphi)\,. \blacksquare$$

because $dr(\partial_r) = 1, dr(\partial_\varphi) = 0$ and $dr(\partial_\varphi) = 0, d\varphi(\partial_\varphi) = 1$.

Another solution

$$\omega(\mathbf{X}) = df(\mathbf{X}) = \partial_{\mathbf{X}} f = \left(r\frac{\partial}{\partial r} + \frac{\partial}{\partial_\varphi}\right)(r^3\cos 3\varphi) = r\cdot 3r^2\cos 3\varphi - 3r^3\sin 3\varphi = 3r^3(\cos 3\varphi - \sin 3\varphi)\,.$$

To express the form ω in Cartesian coordinates it is easier to express f in Cartesian coordinates and then to calculate $\omega = df$:

$$f = r^3\cos 3\varphi = r^3(4\cos^3\varphi - 3\cos\varphi) = 4(r\cos\varphi)^3 - 3r^2(r\cos\varphi) = 4x^3 - 3x(x^2+y^2) = x^3 - 3xy^2$$

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1) *You may use the fact that* $\cos 3\varphi = 4\cos^3\varphi - 3\cos\varphi$.

Hence $\omega = d(x^3 - 3xy^2) = (3x^2 - 3y^2)dx - 5xydy$.

We call 1*-form* ω *exact if there exists a function* F *such that* $\omega = dF$

6 *Show that* 1*-form* $\omega = xdy + ydx$ *is exact.*

Show that 1*-form* $\omega = \sin y dx + x \cos y dy$ *is exact.*

We have $\omega = xdy + ydx = d(xy)$. Hence this is exact form.

We have $\omega = \sin y dx + x \cos y dy = d(x \sin y)$. Hence this is exact form.