Homework 9

1 On the sphere $x^2 + y^2 + z^2 = R^2$ in $\mathbf{E}^3$ consider a circle C which is the intersection of the sphere with the plane $z = R - h$, $0 < h < R$

Let $\mathbf{X}$ be an arbitrary vector tangent to the sphere at a point of C.

Find the angle between $\mathbf{X}$ and the result of parallel transport of $\mathbf{X}$ along C.

2 Write down components of curvature tensor R^i_{kmn} in terms of Christoffel symbols Γ^i_{km} and its derivatives.

3 Let ∇ be a connection on n-dimensional manifold M and $\{R^i{}_{rmn}\}$ be the components of the curvature tensor of a connection ∇ in local coordinates $(x^1, x^2, \ldots, x^n)$.

a) For arbitrary vector fields $\mathbf{A}, \mathbf{B}$ and $\mathbf{D}$ calculate the vector field

$$(\nabla_{\mathbf{A}}\nabla_{\mathbf{B}} - \nabla_{\mathbf{B}}\nabla_{\mathbf{A}})\,\mathbf{D} - \nabla_{\mathbf{C}}\mathbf{D}\,,$$

where the vector field $\mathbf{C}$ is a commutator of vector fields $\mathbf{A}$ and $\mathbf{B}$:

$$\mathbf{C} = C^i \frac{\partial}{\partial x^i} = [\mathbf{A}, \mathbf{B}] = \left(A^m \frac{\partial B^i(x)}{\partial x^m} - B^m \frac{\partial A^i(x)}{\partial x^m}\right) \frac{\partial}{\partial x^i}\,.$$

b) Calculate the vector field

$$(\nabla_{\mathbf{A}}\nabla_{\mathbf{B}} - \nabla_{\mathbf{B}}\nabla_{\mathbf{A}})\,\mathbf{D}$$

in the case if for vector fields $\mathbf{A}$ and $\mathbf{B}$ components A^i and B^m are constants (in the local coordinates $(x^1, \ldots, x^n)$)

(*You have to express the answers in terms of components of the vector fields and components of the curvature tensor $R^i{}_{rmn}$.*)

4 *For every of the statements below prove it or show that it is wrong considering counterexample.*

a) If there exist coordinates u, v such that Riemannian metric G at the given point $\mathbf{p}$ is equal to $G = du^2 + dv^2$ in these coordinates, then curvature of Levi-Civita connection at the point $\mathbf{p}$ vanishes.

b) If all first derivatives of components of Riemannian metric in coordinates u, v vanish at the given point with coordinates (u_0, v_0):

$$\frac{\partial g_{ik}(u,v)}{\partial u}\Big|_{u=u_0, v=v_0} = \frac{\partial g_{ik}(u,v)}{\partial v}\Big|_{u=u_0, v=v_0} = 0\,,$$

then curvature of Levi-Civita connection also vanishes at this point.

c) If all first and second derivatives of components of Riemannian metric

$$\frac{\partial g_{ik}(u,v)}{\partial u}, \frac{\partial g_{ik}(u,v)}{\partial v}, \frac{\partial^2 g_{ik}(u,v)}{\partial u^2}, \frac{\partial^2 g_{ik}(u,v)}{\partial u \partial v}, \frac{\partial^2 g_{ik}(u,v)}{\partial v^2},$$

vanish at the given point then curvature of Levi-Civita connection also vanishes at this point.

5 Using relation between Gaussian curvature and Riemann curvature tensor for Levi-Civita connection, write down all components $\{R_{ikmn}\}$ of Riemann curvature tensor for sphere of radius R in spherical coordinates.

6 * Consider a surface M in $\mathbf{E}^3$ defined by the equation

$$\begin{cases} x = u \\ y = v \\ z = F(u,v) \end{cases} \quad (1) .$$

Calculate explicitly the component R_{1212} of the Riemannian curvature tensor at the point with coordinates $u = v = 0$ in the case if $F(u,v) = \frac{1}{2}(au^2 + 2buv + bv^2)$, where a, b, c are parameters.

Consider a point $\mathbf{p}$ on the surface M with coordinates $u = x_0, v = y_0$ such that (x_0, y_0) is a point of local extremum for the function F.

Using the results of previous exercise calculate the component R_{1212} of the Riemannian curvature tensor at the point $\mathbf{p}$.