Homework 7. Solutions

1 *Find geodesics on sphere and cylinder*

a) using straightforwardly equations for geodesics, or using the fact that for geodesic, acceleration is orthogonal to the surface.

*b *) using the fact that geodesic is shortest.*

a) Geodesics on a surface of cylinder are vertical lines, circles and helixes (see example 2 in section 3.2.1. of lecture notes)

Show here by straightforward calculations that geodesics on sphere are great circles.

The straightforward equations for geodesic: $\frac{d^2x^i}{dt^2} + \frac{dx^k}{dt}\Gamma^i_{km}\frac{dx^m}{dt} = 0$ are just equation of motion for free Lagrangian on the Riemannian surface. Hence in the case of sphere they are equations of motion of the Lagrangian of "free" particle on the sphere is $L = \frac{R^2\dot\theta^2 + R^2\sin^2\theta\dot\varphi^2}{2}$. Its equations of motion are second order differential equations

$$\begin{cases} \ddot\theta - \sin\theta\cos\theta\dot\varphi^2 = 0 \\ \ddot\varphi + 2\mathrm{cotan}\,\theta\dot\theta\dot\varphi = 0 \end{cases} \quad \text{with intitial conditions} \quad \begin{cases} \theta(t)|_{t=0} = \theta_0,\ \dot\theta(t)|_{t=0} = a \\ \varphi(t)|_{t=0} = \varphi_0,\ \dot\varphi(t)|_{t=0} = b \end{cases} \tag{1}$$

for geodesics $\theta(t), \varphi(t)$ starting at the initial point $\mathbf{p} = (\theta_0, \varphi_0)$ with initial velocity $\mathbf{v}_0 = a\frac{\partial}{\partial\theta} + b\frac{\partial}{\partial\varphi}$. (All Christoffel symbols vanish except $\Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta$, and $\Gamma^\varphi_{\varphi\theta} = \Gamma^\varphi_{\theta\varphi} = \mathrm{cotan}\,\theta$.)

This differential equation is not very easy to solve in general case. On the other hand use the fact that rotations are isometries of the sphere. Rotate the sphere in a way such that the initial point transforms to the point the point $\theta_0 = \frac{\pi}{2}, \varphi_0 = 0$ and then rotate the sphere with respect to the axis $0X$ such that θ-component of velocity becomes zero. We come to the same differential equation but with changed initial conditions:

$$\begin{cases} \ddot\theta - \sin\theta\cos\theta\dot\varphi^2 = 0 \\ \ddot\varphi + 2\mathrm{cotan}\,\theta\dot\theta\dot\varphi = 0 \end{cases} \quad \text{with intitial conditions} \quad \begin{cases} \theta(t)|_{t=0} = \frac{\pi}{2},\ \dot\theta(t)|_{t=0} = 0 \\ \varphi(t)|_{t=0} = 0,\ \dot\varphi(t)|_{t=0} = \Omega_0 \end{cases} \tag{2}$$

where we denote by Ω_0 the magnitude of initial velocity. One can easy check that the functions

$$\begin{cases} \theta(t) = \frac{\pi}{2} \\ \varphi(t) = \Omega_0 t \end{cases}$$

are the solution of the differential equations for geodesic with initial conditions (2). Hence this is geodesic passing through the point $\left(\theta_0 = \frac{\pi}{2}, \varphi = 0\right)$ with initial velocity $\Omega_0\frac{\partial}{\partial\varphi}$. We see that this geodesic is the equator of the sphere. We proved that an arbitrary geodesic after applying the suitable rotation is the great-circle—equator. On the other hand an equator is the great circle (the intersection of the sphere $x^2 + y^2 + z^2 = R^2$ with the plane $z = 0$) and the rotation transforms the equator to the another great circle. Hence all arcs of great circles are geodesics and all geodesics are arcs of great circles.

b*) See the lecture notes the section 3.4.1 ("Again on geodesics on sphere and on Lobachevsky plane".)

2) *Consider a sphere $x^2 + y^2 + z^2 = 1$ in $\mathbf{E}^3$ and the curve C which is the intersections of this sphere with plane $y = 0$.*

Consider also in $\mathbf{E}^3$ a vector $\mathbf{X} = \frac{\partial}{\partial z} - \sqrt{3}\frac{\partial}{\partial x}$ attached at the point $\mathbf{p}$: $\left(x = \frac{1}{2}, y = 0, z = \frac{\sqrt{3}}{2}\right)$ and the vector $\mathbf{Y} = \frac{\partial}{\partial y}$ attached at the same point $\mathbf{p}$.

Show that vectors $\mathbf{X}$ and $\mathbf{Y}$ are tangent to the sphere and express these vector in spherical coordinates.

Describe parallel transport of vectors $\mathbf{X}, \mathbf{Y}$ along the curve C.

Both vectors $\mathbf{X}$ and $\mathbf{Y}$ attached at the point $\mathbf{p}$ are orthogonal to the radius-vector at this point $\mathbf{r} = \left(\frac{1}{2}, 0, \frac{\sqrt{3}}{2}\right)$ since their scalar product with this vector vanishes. Hence these vectors are tangent to the sphere.

Vector $\mathbf{Y}$ is directed along the horisontal plane $z = \frac{1}{2}$. Hence it is proportional to the vector $\frac{\mathbf{r}_\varphi = \partial}{\partial \varphi}$: $\mathbf{Y} = a\frac{\partial}{\partial \varphi}$. The length of the vector $\mathbf{Y}$ is equal to 1; the length of the vector $\frac{\mathbf{r}_\varphi = \partial}{\partial \varphi}$ is equal to $\sin\theta_0$, since metric on sphere is $G = d\theta^2 + \sin^2\theta d\varphi^2$. (Here θ_0 is a coordinate θ of the point $\mathbf{p}$. Thus $\sin\theta_0 = \frac{\sqrt{3}}{2}$) and we see that

$$|\mathbf{Y}| = 1 = a\left|\frac{\partial}{\partial \varphi}\right| = a\frac{\sqrt{3}}{2}\,. \Rightarrow \mathbf{Y} = \frac{2\sqrt{3}}{3}\frac{\partial}{\partial \varphi}\,.$$

Analogous calculations for vector $\mathbf{X}$. it is equal to $\mathbf{X} = \frac{\partial}{\partial z} - \sqrt{3}\frac{\partial}{\partial x} = b\mathbf{r}_\theta = b\frac{\partial}{\partial \theta}$. Since $\mathbf{r}_\theta = \frac{\partial}{\partial \theta}$ and $\mathbf{X}$ are unit vectors, hence we have

$$\mathbf{X} = \frac{\partial}{\partial z} - \sqrt{3}\frac{\partial}{\partial x} = \mathbf{r}_\theta = b\frac{\partial}{\partial \theta}\,.$$

The curve $C: x^2 + z^2 = 1, y = 0$ is the great circle. It is geodesic on the sphere. We perform parallel transport of these vectors along the geodesics. Moreover vector $\mathbf{X}$ is tangent to the C. Hence during parallel transport this vector remains tangent to C. Moreover its length does not change. Hence at every point of the curve C it is the vector $\mathbf{r}_\theta = \frac{\partial}{\partial \theta}$.

The vector $\mathbf{Y}$ is orthogonal to the vector $\mathbf{X}$ at the initial point $\mathbf{p}$. It will remain orthogonal during parallel transport and its length will remain constant. Hence at every point $\theta = \theta(t)$ of the curve C it will become the vector $\frac{2\sqrt{3}}{3\sin\theta}\frac{\partial}{\partial \varphi}$.

3 *Show that vertical lines $x = a$ are geodesics (non-parameterised) on Lobachevsky plane.*

We consider here the realisation of Lobachevsky plane (hyperbolic plane) as upper half of Euclidean plane $\{(x, y): y > 0\}$ with the metric $G = \frac{dx^2+dy^2}{y^2}$.

Consider second order differential equations defining geodesics with initial conditions such that "horisontal" velocity equals to zero: (we uses the information from Homework 6 or from Lecture notes about Christoffels for Lobachevsky plane: $\Gamma^x_{xx} = 0, \Gamma^x_{xy} = \Gamma^x_{yx} = -\frac{1}{y}$, $\Gamma^x_{yy} = 0$, $\Gamma^y_{xx} = \frac{1}{y}, \Gamma^y_{xy} = \Gamma^y_{yx} = 0, \Gamma^y_{yy} = -\frac{1}{y}$.)

$$\begin{cases} \ddot{x} - \frac{2\dot{x}\dot{y}}{y} = 0 \\ \ddot{y} + \frac{\dot{x}^2}{y} - \frac{\dot{y}^2}{y} = 0 \\ x(t)\big|_{t=t_0} = x_0, \dot{x}(t)\big|_{t=t_0} = 0 \\ y(t)\big|_{t=t_0} = y_0, \dot{y}(t)\big|_{t=t_0} = \dot{y}_0 \end{cases}$$

This equation has a solution and it is unique. One can see that if we put $x(t) \equiv 0$, i.e. curve is vertical then we come to the equation $\ddot{y} - \frac{\dot{y}^2}{y} = 0$. Solution of these equation gives curve $x = x_0$, $y = y(t)$: $\ddot{y} - \frac{\dot{y}^2}{y} = 0$. The image of this curve clearly is vertical ray $x = x_0, y > 0$.

4 *Consider a vertical ray $C: x(t) = x_0, y(t) = y_0 + t$, $0 \le t < \infty$, $(y_0 > 0)$ on the Lobachevsky plane. Find the parallel transport $\mathbf{X}(t)$ of the vector $\mathbf{X}_0 = \partial_y$ attached at the initial point (x_0, y_0) along the ray C at an arbitrary point of the ray.*

The vertical ray is geodesic hence the tangent vector remains tangent during parallel transport. Hence $\mathbf{X}(t) = a(t)\partial_t$, where $a(0) = 1$. On the other hand during parallel transport the length of the vector does not change, because parallel transport is defined via Levi-Civita connection. Hence $|\mathbf{X}(t)|$ is constant in time. The metric on Lobachevsky plane is $G = \frac{dx^2+dy^2}{y^2}$, the length of initial vector equals to $\sqrt{\langle \partial_t, \partial_t \rangle} = \sqrt{\frac{1}{y_0^2}} = \frac{1}{y_0}$.

We have $|\mathbf{X}(t)| = \sqrt{\langle \mathbf{X}(t), \mathbf{X}(t) \rangle} = \sqrt{a(t)^2(t)\frac{1}{y^2(t)}} = \frac{a(t)}{y_0+t} = \frac{1}{y_0}$. Hence $a(t) = \frac{y_0+t}{y_0}$.

5 *Find a parameterisation of vertical lines in the Lobachevsky plane such that they become parameterised geodesics.*

We know also that vertical line is geodesic. Let $x = x_0, y = f(t)$ is right parameterisation, i.e. parameterisation such that veclotiy vector remains velocity vector during parallel transport. Velocity vector $\mathbf{v}(t) = \begin{pmatrix} 0 \\ f_t \end{pmatrix}$. Its length is equal to $\sqrt{\frac{x_t^2+y_t^2}{y^2}} = \sqrt{\frac{0+f_t^2}{f^2}} = \frac{f_t}{f}$ and it has remain the same. Hence $\frac{f_t}{f} = c$, i.e. $f(t) = Ae^{ct}$. We see that $x = x_0, y = ae^{ct}$ is parameterised geodesic. (On can see that differential equation of geodesics are obeyed (see the exercise 3)).

6 *Show that the following transformations are isometries of Lobachevsky plane:*
a) horizontal translation $\mathbf{r} \to \mathbf{r} + \mathbf{a}$ *where* $\mathbf{a} = (a, 0)$*,*
b) homothety (dilation): $\mathbf{r} \to \lambda\mathbf{r}$ $(\lambda > 0)$*,*
* *c) inversion with the centre at the points of the line* $x = 0^*$*:*

$$\mathbf{r} \to \mathbf{a} + \frac{\mathbf{r} - \mathbf{a}}{|\mathbf{r} - \mathbf{a}|^2} \text{ where } \mathbf{a} = (a, 0): \quad \begin{cases} x' = a + \frac{x-a}{(x-a)^2+y^2} \\ y' = \frac{y}{(x-a)^2+y^2} \end{cases} .$$

We have to show that Riemannian metric $G = \frac{dx^2+dy^2}{y^2}$ remains invariant under these transformations.

a) horizontal translation. If $x \to x + a, y \to y$ then dx and dy do not change. Hence G is invariant under horisontal translations.

b) Homothety. If $x \to \lambda x$, $y \to \lambda y$ where $\lambda > 0$ is a constant then $\frac{dx^2+dy^2}{y^2} \to \frac{\lambda^2 dx^2+\lambda^2 dy^2}{\lambda^2 y^2} = \frac{dx^2+dy^2}{y^2}$ does not change too.

c) * inversion with the centre at the points of the line $y = 0$. Since we proved that horisontal translation is isometry it suffices to consider inversion with centre at the point $x = y = 0$:

$$\mathbf{r} \to \frac{\mathbf{r}}{|\mathbf{r}|^2}: \quad \begin{cases} x' = \frac{x}{x^2+y^2} \\ y' = \frac{y}{x^2+y^2} \end{cases}$$

Now by straightforward calculations one can show that $\frac{dx^2+dy^2}{y^2} = \frac{dx'^2+dy'^2}{y'^2}$.

To avoid the straightforward calculations consider coordinates r, φ: $x = r\cos\varphi, y = r\sin\varphi$ then

$$\frac{dx^2 + dy^2}{y^2} = \frac{dr^2 + r^2 d\varphi^2}{r^2 \sin^2\varphi} = \frac{1}{\sin^2\varphi}\frac{dr^2}{r^2} + \frac{d\varphi^2}{\sin^2\varphi} = \frac{1}{\sin^2\varphi}(d\log r)^2 + \frac{d\varphi^2}{\sin^2\varphi} .$$

Under transformation of inversion in these coordinates φ does not change, $r \to \frac{1}{r}$, $\log r \to -\log\frac{1}{r}$. It is evident that in coordinates $u = \log r, \varphi$ metric does not change. Hence inversion (with centre at the point $x = y = 0$) is isometry. Hence an inversion with a centre at the arbitrary point $(a, 0)$ and with an arbitrary radius is isometry, since horisontal translation and homothety are isometries.

(In complex cooridinates this is so called Mobius transformation transformation $z \to \frac{1}{\bar{z}}$ which you learned in the course Hyperbolic Geometry.)

7* *Show that upper arcs of semicircles* $(x - a)^2 + y^2 = R^2, y > 0$ *are (non-parametersied) geodesics.*

You may do this exercise solving explicitly differential equations for geodesics, but it is much more nice to use inversion (Mobius) transformation studied in the previous exercise: Consider the inversion of the Lobachevsky plane with the centre at the point $x = a - R, y = 0$ (see the exercise above). This inversion does not change Riemannian metric, it is isometry. Isometry transforms geodesics to geodesics. On the other hand it transforms the semicircle $(x - a)^2 + y^2 = R^2, y > 0$ to the vertical ray $x = a - R + \frac{1}{2R}, y > 0$. This can be checked directly. On the other hand the vertical ray is geodesic. Hence the initial curve was the geodesic too.

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* *This line is called absolute.*

8* *Let* $\mathbf{X}(t)$ *be parallel transport of the vector* $\mathbf{X}$ *along the curve on the surface* M *embedded in* $\mathbf{E}^3$*, i.e.* $\nabla_{\mathbf{v}}\mathbf{X} = 0$*, where* $\mathbf{v}$ *is a velocity vector of the curve* C *and* ∇ *Levi-Civita connection of the metric induced on the surface. Compare the condition* $\nabla_{\mathbf{v}}\mathbf{X} = 0$ *(this is condition of parallel transport for internal observer) with the condition that for the vector* $\mathbf{X}(t)$*, the derivative* $\frac{d\mathbf{X}(t)}{dt}$ *is orthogonal to the surface (this is condition of parallel transport for external observer)*[2]*.*

Do these two conditions coincide, i.e. do they imply the same parallel transport?

We know that Levi-Civita connection on surfaces coicides with induced conenction. We have by definition of induced connection that

$$\nabla_{\mathbf{v}}\mathbf{X} = \left(\nabla^{\text{can. flat}}_{\mathbf{v}}\mathbf{X}\right) = \left(\partial_{\mathbf{v}}\mathbf{X}\right)_{\text{tangent}} = \left(\frac{d\mathbf{X}(t)}{dt}\right)_{\text{tangent}}$$

Hence $\nabla_{\mathbf{v}}\mathbf{X} = 0$ if and only if the derivative $\frac{d\mathbf{X}(t)}{dt}$ is orthogonal to the surface.

[2] *We defined parallel transport in Geometry course using this condition*