Homework 3. Solutions

1 *Let $\{\mathbf{e},\mathbf{f}\}$ be an orthonormal basis in $\mathbf{E}^2$. Consider the following ordered pairs:*
a) $\{\mathbf{f},\mathbf{e}\}$
b) $\{\mathbf{f},-\mathbf{e}\}$
c) $\{\frac{\sqrt{2}}{2}\mathbf{e}+\frac{\sqrt{2}}{2}\mathbf{f},-\frac{\sqrt{2}}{2}\mathbf{e}+\frac{\sqrt{2}}{2}\mathbf{f}\}$
d) $\{\frac{\sqrt{3}}{2}\mathbf{e}+\frac{1}{2}\mathbf{f},\frac{1}{2}\mathbf{e}-\frac{\sqrt{3}}{2}\mathbf{f}\}$
Show that all these ordered pairs are orthonormal bases in $\mathbf{E}^2$.
Find amongst them the bases which have the same orientation as the orientation of the basis $\{\mathbf{e},\mathbf{f}\}$.
Find amongst them the bases which have the orientation opposite to the orientation of the basis $\{\mathbf{e},\mathbf{f}\}$.
Solution:

First check that all the bases are orthonormal. For the bases a) and b) this is obvious: both vectors have unit length and they are orthogonal to each other.

Check orthogonality condition for the basis d). (For the basis a) all calculations are analogous). We have to check that vectors $\mathbf{a}=\frac{\sqrt{3}}{2}\mathbf{e}+\frac{1}{2}\mathbf{f}$ and $\mathbf{b}=\frac{1}{2}\mathbf{e}-\frac{\sqrt{3}}{2}\mathbf{f}$ have both unit length and are orthogonal to each other. For calculations we use the fact that initial basis is orthonormal too, i.e. vectors $\mathbf{e},\mathbf{f}$ have unit length: scalar products $(\mathbf{e},\mathbf{e})$, $(\mathbf{f},\mathbf{f})$ both are equal to 1 and these vectors are orthogonal: scalar product $(\mathbf{e},\mathbf{f})$ is equal to zero. Calculate scalar products:

$$(\mathbf{a},\mathbf{a})=\left(\frac{\sqrt{3}}{2}\mathbf{e}+\frac{1}{2}\mathbf{f},\frac{\sqrt{3}}{2}\mathbf{e}+\frac{1}{2}\mathbf{f}\right)=\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}(\mathbf{e},\mathbf{e})+\frac{1}{2}\cdot\frac{\sqrt{3}}{2}(\mathbf{f},\mathbf{e})+\frac{\sqrt{3}}{2}\cdot\frac{1}{2}(\mathbf{e},\mathbf{f})+\frac{1}{2}\cdot\frac{1}{2}(\mathbf{f},\mathbf{f})=$$

$$\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\cdot 1+\frac{1}{2}\cdot\frac{\sqrt{3}}{2}\cdot 0+\frac{\sqrt{3}}{2}\cdot\frac{1}{2}\cdot 0+\frac{1}{2}\cdot\frac{1}{2}\cdot 1=\frac{3}{4}+\frac{1}{4}=1. \tag{1}$$

We see that scalar product $(\mathbf{a},\mathbf{a})$ is equal to 1. This means that $|\mathbf{a}|=1$.

Analogously we show that the length of the vector $\mathbf{b}$ is equal to 1:

$$(\mathbf{b},\mathbf{b})=\left(\frac{1}{2}\mathbf{e}-\frac{\sqrt{3}}{2}\mathbf{f},\frac{1}{2}\mathbf{e}-\frac{\sqrt{3}}{2}\mathbf{f}\right)=\frac{1}{2}\cdot\frac{1}{2}\cdot 1-\frac{\sqrt{3}}{2}\cdot\frac{1}{2}\cdot 0-\frac{1}{2}\cdot\frac{\sqrt{3}}{2}\cdot 0+\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\cdot 1=1\,. \tag{2}$$

It remains to show that these vectors are orthogonal , i.e. their scala product is equal to zero:

$$(\mathbf{a},\mathbf{b})=\left(\frac{\sqrt{3}}{2}\mathbf{e}+\frac{1}{2}\mathbf{f},\frac{1}{2}\mathbf{e}-\frac{\sqrt{3}}{2}\mathbf{f}\right)=\frac{\sqrt{3}}{2}\cdot\frac{1}{2}(\mathbf{e},\mathbf{e})+\frac{1}{2}\cdot\frac{1}{2}(\mathbf{f},\mathbf{e})+-\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}(\mathbf{e},\mathbf{f})-\frac{1}{2}\cdot\frac{\sqrt{3}}{2}(\mathbf{f},\mathbf{f})=$$

$$\frac{\sqrt{3}}{2}\cdot\frac{1}{2}\cdot 1+\frac{1}{2}\cdot\frac{1}{2}\cdot 0+-\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\cdot 0-\frac{1}{2}\cdot\frac{\sqrt{3}}{2}\cdot 1=0, \tag{3}$$

i.e. these vectors are orthogonal

Orthonormality conditions for the basis a) could be checked in the same way.

Remark You could ask a question: how comes we call these pairs bases without checking the condition that they are the bases. The point is that if two vectors are not equal to zero and are orthogonal each other (and this was checked) this implies that they are not linearly dependent. (Why?: see the footnote to the lecture notes at the subsection 1.6 or exercise 2a) in Homework 2). Hence the ordered pair of these two vectors form a basis.

Now find orientation of these bases with respect to the basis $\{\mathbf{e},\mathbf{f}\}$. (We already show that all ordered pairs are bases.)

Case a) One can easy see that transition matrix from the basis $\{\mathbf{e},\mathbf{f}\}$ to the basis $\{\mathbf{f},\mathbf{e}\}$ is

$$T=\begin{pmatrix}0&1\\1&0\end{pmatrix} \tag{1a)}$$

Indeed $(\mathbf{f},\mathbf{e}) = (\mathbf{e},\mathbf{f})T$: $(\mathbf{f},\mathbf{e}) = (\mathbf{e},\mathbf{f})\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Calculate determinant of transition matrix $\det T = -1 < 0$. Hence this basis has an orientation opposite to the orientation of the basis $\{\mathbf{e},\mathbf{f}\}$ (The fact that $\det T \neq 0$ makes us to double check that this ordered pair is a basis. Of course in this case it is obvious.).

Case b) Analogously for the case b) one can easy see that transition matrix from the basis $\{\mathbf{e},\mathbf{f}\}$ to the basis $\{\mathbf{f},-\mathbf{e}\}$ is

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Indeed $(\mathbf{f},\mathbf{e}) = (\mathbf{e},\mathbf{f})T$: $(\mathbf{f},\mathbf{e}) = (\mathbf{e},\mathbf{f})\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Calculate determinant of transition matrix $\det T = 1 > 0$. Hence this basis has the same orientation as the basis $\{\mathbf{e},\mathbf{f}\}$. (The fact that $\det T \neq 0$ makes us to double check that this ordered pair is a basis. Of course in this case it is obvious.)

Case c) Analogously for the case c) one can easy see that transition matrix from the basis $\{\mathbf{e},\mathbf{f}\}$ to the basis $\{\frac{\sqrt{2}}{2}\mathbf{e} + \frac{\sqrt{2}}{2}\mathbf{f}, -\frac{\sqrt{2}}{2}\mathbf{e} + \frac{\sqrt{2}}{2}\mathbf{f}\}$ is

$$T = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

Calculate determinant of transition matrix $\det T = 1 > 0$. Hence this basis has the same orientation as the basis $\{\mathbf{e},\mathbf{f}\}$ (The fact that $\det T \neq 0$ makes us to double check that this ordered pair is a basis.)

Case d) and finally for the case d) one can easy see that transition matrix from the basis $\{\mathbf{e},\mathbf{f}\}$ to the basis $\{\frac{\sqrt{3}}{2}\mathbf{e} + \frac{1}{2}\mathbf{f}, \frac{1}{2}\mathbf{e} - \frac{\sqrt{3}}{2}\mathbf{f}\}$ is

$$T = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$$

Calculate determinant of transition matrix $\det T = -1 < 0$. Hence this basis has an orientation opposite to the orientation of the basis $\{\mathbf{e},\mathbf{f}\}$ (The fact that $\det T \neq 0$ makes us to double check that this ordered pair is a basis.)

2 *Let $\{\mathbf{e},\mathbf{f}\}$ be a basis in two-dimensional linear space V. Consider an ordered pair $\{\mathbf{a},\mathbf{b}\}$ such that*

$$\mathbf{a} = \mathbf{f},\ \mathbf{b} = \gamma\mathbf{e} + \mu\mathbf{f}\,,$$

where γ, μ are arbitrary real numbers.

Find values γ, μ such that an ordered pair $\{\mathbf{a},\mathbf{b}\}$ is a basis and this basis has the same orientation as the basis $\{\mathbf{e},\mathbf{f}\}$.

Solution: Transition matrix T from the basis $\{\mathbf{e},\mathbf{f}\}$ to the ordered pair $\{\mathbf{a},\mathbf{b}\}$ is matrix $T = \begin{pmatrix} 0 & \gamma \\ 1 & \mu \end{pmatrix}$: $\{\mathbf{a},\mathbf{b}\} = \{\mathbf{e},\mathbf{f}\}\begin{pmatrix} 0 & \gamma \\ 1 & \mu \end{pmatrix}$. Its determinant is equal to $-\gamma$. Hence the ordered pair $\{\mathbf{a},\mathbf{b}\}$ is a basis if and only if $\gamma \neq 0$. (This can be done in other way: one can see that if $\gamma = 0$ then vectors $\mathbf{a} = \mathbf{f}$ and $\mathbf{b} = \mu\mathbf{f}$ are linear dependent, and If $\gamma \neq 0$ then vectors $\mathbf{a} = \mathbf{f}$ and $\mathbf{b} = \mu\mathbf{f}$ are linear independent.)

If $\det T = -\gamma > 0$, i.e. $\gamma < 0$ then the bases $\{\mathbf{a},\mathbf{b}\}$ and $\{\mathbf{e}_x,\mathbf{e}_y\}$ have the same orientation.

If $\det T = -\gamma < 0$, i.e. $\gamma > 0$ then the bases $\{\mathbf{a},\mathbf{b}\}$ and $\{\mathbf{e}_x,\mathbf{e}_y\}$ have opposite orientation.

3 *Let $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ be an arbitrary basis in $\mathbf{E}^3$. Show that the basis $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ either has the same orientation as the basis $\{\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z\}$, or the same orientation as the basis $\{\mathbf{e}_y,\mathbf{e}_x,\mathbf{e}_z\}$.*

Solution:. Bases $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$, $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$ have opposite orientation since $(\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z) = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)T$, where $T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\det T = -1 < 0$.

Hence an arbitrary basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ belongs to the equivalence class of of the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ (with respet to orientation relation) or to the equivalence class of of the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$.(See the Proposition in the subsection 1.9 of Lecture notes).

We repeat the proof of the proposition for this special case:

Let T_1 be transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ and T_2 be transition matrix from the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$ to the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$:

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)T_1, \quad (\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z)T_2 \,.$$

We see that $T_1 = T \cdot T_2$:

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z)T_2 = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)TT_2 = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)T_1 \ \textit{i.e.}\ TT_2 = T_1 \,.$$

If $\det T_2 > 0$ then the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ have the same orientation as the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$. If $\det T_2 < 0$ then $\det T_1 = \det(TT_2) = \det T \cdot \det T_2 > 0$ because $\det T = -1 < 0$. Hence in this case the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ have the same orientation as the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$.

In other words bases $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$ have opposite orientations. Hence they belong to different classes of bases (with respect to orientation). There are two classes. Hence the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ belongs to the same equivalence class of the bases to which belongs the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ or the basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ belongs to the same equivalence class of the bases to which belongs the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$.

Arbitrary basis has the same orientation as the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ and the orientation opposite to the orientation of the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$ or vice versa: it has the same orientation as the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$ and the orientation opposite to the orientation of the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$.

4 *Let $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ be an orthonormal basis in $\mathbf{E}^3$. Consider the following ordered triples:*
a) $\{\mathbf{e}_x, \mathbf{e}_x + 2\mathbf{e}_y, 5\mathbf{e}_z\}$,
b) $\{\mathbf{e}_y, \mathbf{e}_x, 5\mathbf{e}_z\}$,
c) $\{\mathbf{e}_y, \mathbf{e}_x, -5\mathbf{e}_z\}$,
d) $\{\frac{\sqrt{3}}{2}\mathbf{e}_x + \frac{1}{2}\mathbf{e}_y, -\frac{1}{2}\mathbf{e}_x + \frac{\sqrt{3}}{2}\mathbf{e}_y, \mathbf{e}_z\}$,
e) $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$,
f) $\{\mathbf{e}_y, \mathbf{e}_x, -\mathbf{e}_z\}$.
Show that all these ordered triples a),b),c),d),e),f) are bases.
Show that the bases a), c), d) and f) have the same orientation as the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$, and the bases b) and e) have the orientation opposite to the orientation of the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$.

Show that bases d), e) and f) are orthonormal bases and bases a), b) and c) are not orthonormal bases.

Solution:

Recall that to check that is an ordered triple a basis or no, and to find an orientation of this basis we have to find transition matrix T. If this matrix is non-degenerate, i.e. $\det T \neq 0$ then it transforms basis to a basis. If determinant of transition matrix is positive, then these two bases have the same orientation. If determinant of transition matrix is negative , then these two bases have opposite orientation.

To show that basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal basis we have to show that all basis vectors have the unit length and they are orthogonal each other, i.e. we have to check that following relations are satisfied:

$$(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \tag{4.1}$$

(Another way to check orthonormality of new basis: one have to check that transition matrix is orthogonal, i.e. it satisfies the condition $TT^t = I$ (I is identity matrix), i.e. it is a orthogonal matrix.)

Case a). One can easy see that transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the ordered triples $\{\mathbf{e}_x, \mathbf{e}_x + 2\mathbf{e}_y, 5\mathbf{e}_z\}$ is

$$T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix} \tag{1a)}$$

Indeed $\{\mathbf{e}_x, \mathbf{e}_x + 2\mathbf{e}_y, 5\mathbf{e}_z\} = \{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}T$: $\{\mathbf{e}_x, \mathbf{e}_x + 2\mathbf{e}_y, 5\mathbf{e}_z\} = \{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.
Calculate determinant of transition matrix: $\det T = 10 \neq 0$. Transition matrix is non-degenerate. Hence the ordered triple $\{\mathbf{e}_x, \mathbf{e}_x + 2\mathbf{e}_y, 5\mathbf{e}_z\}$ is a basis. $\det T = 10 > 0$. Hence this new basis has the same orientation as the initial basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. This new basis is not orthogonal. One can see it e.g. checking that the length of the third vector is equal to $5 \neq 1$.

Case b). In this case analogously: transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the ordered triple $\{\mathbf{e}_y, \mathbf{e}_x, 5\mathbf{e}_z\}$ is

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \text{because } \{\mathbf{e}_y, \mathbf{e}_x, 5\mathbf{e}_z\} = \{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix},$$

Calculate determinant of transition matrix: $\det T = -5 \neq 0$. Transition matrix is non-degenerate. Hence the ordered triple $\{\mathbf{e}_y, \mathbf{e}_x, 5\mathbf{e}_z\}$ is a basis. $\det T = -5 < 0$. Hence this new basis has the orientation opposite to the orientation of the initial basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. This new basis is not orthogonal. One can see it e.g. checking that the length of the third vector is equal to $5 \neq 1$.

Case c) Analogously: transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the basis $\{\mathbf{e}_y, \mathbf{e}_x, -5\mathbf{e}_z\}$ is

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -5 \end{pmatrix}, \text{because } \{\mathbf{e}_y, \mathbf{e}_x, -5\mathbf{e}_z\} = \{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -5 \end{pmatrix},$$

Calculate determinant of transition matrix: $\det T = 5 \neq 0$. Transition matrix is non-degenerate. Hence the ordered triple $\{\mathbf{e}_y, \mathbf{e}_x, -5\mathbf{e}_z\}$ is a basis. $\det T = 5 > 0$. Hence this new basis has the same orientation as the initial basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. This new basis is not orthogonal. One can see it e.g. checking that the length of the third vector is equal to $5 \neq 1$.

Case d) Transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the basis $\{\frac{\sqrt{3}}{2}\mathbf{e}_x + \frac{1}{2}\mathbf{e}_y, -\frac{1}{2}\mathbf{e}_x + \frac{\sqrt{3}}{2}\mathbf{e}_y, \mathbf{e}_z\}$ is

$$T = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{because } \left\{\frac{\sqrt{3}}{2}\mathbf{e}_x + \frac{1}{2}\mathbf{e}_y, -\frac{1}{2}\mathbf{e}_x + \frac{\sqrt{3}}{2}\mathbf{e}_y, \mathbf{e}_z\right\} = \{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\} \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

Calculate determinant of transition matrix: $\det T = 1 \neq 0$. Transition matrix is non-degenerate. Hence the ordered triple $\{\frac{\sqrt{3}}{2}\mathbf{e}_x + \frac{1}{2}\mathbf{e}_y, -\frac{1}{2}\mathbf{e}_x + \frac{\sqrt{3}}{2}\mathbf{e}_y, \mathbf{e}_z\}$ is a basis. $\det T = 1 > 0$. Hence this new basis has the same orientation as the initial basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. This new basis orthonormal basis. Indeed the first two vectors have the length 1 and they are orthogonal to each other (See equations (1,2,3) in the previous exercise). The third vector $\mathbf{e}_z$ has length one and it is obviously is orthogonal to first two vectors.

Case e) for the case e) transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the basis $\{\mathbf{e}_y, \mathbf{e}_x, -\mathbf{e}_z\}$ is

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \text{because } \{\mathbf{e}_y, \mathbf{e}_x, -\mathbf{e}_z\} = \{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

Calculate determinant of transition matrix: $\det T = 1 \neq 0$. Transition matrix is non-degenerate. Hence the ordered triple $\{\mathbf{e}_y, \mathbf{e}_x, -\mathbf{e}_z\}$ is a basis. $\det T = 1 > 0$. Hence this new basis has the same orientation as the initial basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. This new basis is obviously orthonormal basis because all the vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ have unit length and they are orthogonal to each other.

Case f) for the case f) transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$ is

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{because } (\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z) = (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

Calculate determinant of transition matrix: $\det T = 1 \neq 0$. Transition matrix is non-degenerate. Hence the ordered triple $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$ is a basis. $\det T = -1 > 0$. Hence this new basis has the orientation opposite to the orientation of the initial basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. (Another way to see it: we come to basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$ from the basis $\{\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z\}$ just by swapping the vectors $\mathbf{e}_x$ and $\mathbf{e}_y$.) This new basis is obviously orthonormal basis because all the vectors $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ have unit length and they are orthogonal to each other.

5 *Let $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ be a basis in linear three-dimensional space V.*
Consider the following ordered triples:

$$\{\mathbf{f}, \mathbf{e} + 2\mathbf{f}, 3\mathbf{g}\}, \quad \{\mathbf{e}, \mathbf{f}, 2\mathbf{f} + 3\mathbf{g}\}$$

Show that these ordered triples are bases and these bases have opposite orientations.

To write transition matrix from the basis $\{\mathbf{f}, \mathbf{e} + 2\mathbf{f}, 3\mathbf{g}\}$ to the basis $\{\mathbf{e}, \mathbf{f}, 2\mathbf{f} + 3\mathbf{g}\}$? This is little bit long exercise. We do it in another way:

Consider transition matrices;

T_1—transition matrix from the initial basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ to the basis $\{\mathbf{f}, \mathbf{e} + 2\mathbf{f}, 3\mathbf{g}\}$.

T_2—transition matrix from the initial basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ to the basis $\{\mathbf{e}, \mathbf{f}, 2\mathbf{f} + 3\mathbf{g}\}$ It is easy to see that

$$T_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \qquad \text{and} \qquad T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

We see that determinant of the transition matrix T_1 is negative and the determinant of the transition matrix T_2 is positive. This means that the both ordered triples $\{\mathbf{f}, \mathbf{e} + 2\mathbf{f}, 3\mathbf{g}\}$ and $\{\mathbf{e}, \mathbf{f}, 2\mathbf{f} + 3\mathbf{g}\}$ are bases. The first basis has an orientation opposite to the orientation of the initial basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ and the second basis has the same orientation as the initial basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$. Hence these both bases have opposite orientation $\blacksquare$.

6 *Let $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ be an orthonormal basis in Euclidean space $\mathbf{E}^3$. Consider a linear operator P in $\mathbf{E}^3$ such that*

$$\mathbf{e}' = P(\mathbf{e}) = \mathbf{e}, \quad \mathbf{f}' = P(\mathbf{f}) = \frac{\sqrt{2}}{2}\mathbf{f} + \frac{\sqrt{2}}{2}\mathbf{g}, \quad \mathbf{g}' = P(\mathbf{g}) = -\frac{\sqrt{2}}{2}\mathbf{f} + \frac{\sqrt{2}}{2}\mathbf{g}.$$

Write down the transition matrix from the basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ to the ordered triple $\{\mathbf{e}', \mathbf{f}', \mathbf{g}'\}$.

Show that P is an orthogonal operator.

Show that orthogonal operator P preserves the orientation of $\mathbf{E}^3$.

Find an axis of the rotation and the angle of the rotation.

Write down transition matrix T: $\{\mathbf{e}', \mathbf{f}', \mathbf{g}'\} = \{\mathbf{e}, \mathbf{f}, \mathbf{g}\}T$:

$$\{\mathbf{e}', \mathbf{f}', \mathbf{g}'\} = \{P(\mathbf{e}), P(\mathbf{f}), P(\mathbf{g})\} = \left\{\mathbf{e}, \frac{\sqrt{2}}{2}\mathbf{f} + \frac{\sqrt{2}}{2}\mathbf{g}, -\frac{\sqrt{2}}{2}\mathbf{f} + \frac{\sqrt{2}}{2}\mathbf{g}\right\} =$$

$$\{\mathbf{e}, \mathbf{f}, \mathbf{g}\} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

One can see that the matrix is invertible. The triple $\{\mathbf{e}', \mathbf{f}', \mathbf{g}'\}$ is a basis. It is easy to see that the new basis $\{\mathbf{e}', \mathbf{f}', \mathbf{g}'\}$ is orthonormal basis since the former basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ is orthonormal one: $(\mathbf{e}', \mathbf{e}') = (\mathbf{f}', \mathbf{f}') = (\mathbf{g}', \mathbf{g}') = 1$ and $(\mathbf{e}', \mathbf{f}') = (\mathbf{e}', \mathbf{g}') = (\mathbf{f}', \mathbf{g}') = 0$. (Compare with exercise 1c). Hence transtion matrix is orthogonal matrix and linear operator is orthogonal operator. (The condition of orthogonality can be checked straightforwardly:

$$P^T \cdot P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Determinant of transition matrix equal 1. Hence the linear operator P does not change orientation.

According to the Euler Theorem P performs the rotation of $\mathbf{E}^3$. We have that $P(\mathbf{e}) = \mathbf{e}$. Vector $\mathbf{e}$ is an eigenvector of rotation with eigenvalue 1, i.e. the axis of rotation is along the vector $\mathbf{e}$. It is the rotation on the angle $\frac{\pi}{4}$ since

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ 0 & \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix}$$

Remark To be more precise one have to consider orientation of axis and orientation in space $\mathbf{E}^3$. This will change sign of the angle of rotation.

7 *Consider a linear operator P_1 in $\mathbf{E}^3$ such that it transforms the orthonormal basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ into the orthonormal basis $\{\mathbf{f}, \mathbf{e}, \mathbf{g}\}$. Consider also a linear operator P_2 such that it is the reflection operator with respect to the plane spanned by vectors $\mathbf{e}$ and $\mathbf{f}$.*

Is the operator P_1 a rotation or reflection operator?

Do operators P_1, P_2 preserve orientation?

Show that operator $P = P_2 \circ P_1$ is a rotation operator.

Find an angle and the axis of this rotation. P is orthogonal opertor since it transforms orthonormal basis to ornthonormal one. We have that

$$P_1(\mathbf{e}) = \mathbf{f},\ P_1(\mathbf{f}) = \mathbf{e},\ P(\mathbf{g}) = \mathbf{g}\,. \tag{7.1}$$

The transition matirx of basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ to basis $\{\mathbf{f}, \mathbf{e}, \mathbf{g}\}$ is a matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ since } \{\mathbf{f}, \mathbf{e}, \mathbf{g}\} = \{\mathbf{e}, \mathbf{f}, \mathbf{g}\} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Its determinant equals $-1 < 0$. Hence linear operator P_1 changes orientation. It is reflection operator (with respect to the plan spanned by vectors $\mathbf{e} + \mathbf{f}$ and $\mathbf{g}$, These vectors and their arbitrary linear combinations are eigenvalues of this operator:

$$P(\mathbf{e} + \mathbf{f}) = \mathbf{f} + \mathbf{e} = \mathbf{e} + \mathbf{f}, \qquad P(\mathbf{g}) = \mathbf{g}\,, P(\lambda(\mathbf{e} + \mathbf{f}) + \mu\mathbf{g}) = \lambda(\mathbf{e} + \mathbf{f}) + \mu\mathbf{g}\,.$$

Now consider orthogonal operator P_2. The plane spanned by vectors $\mathbf{e}, \mathbf{f}$ remains intact, hence $P_2(\mathbf{e}) = \mathbf{e}$ and $P_2(\mathbf{f}) = \mathbf{f}$. Vector $\mathbf{g}$ transforms to vector $-\mathbf{g}$ since it is orthogonal to vectors $\mathbf{e}$ and $\mathbf{f}$. We have

$$P_2(\mathbf{e}) = \mathbf{e},\ P_2(\mathbf{f}) = \mathbf{f},\ P_2(\mathbf{g}) = -\mathbf{g}\,. \tag{7.2}$$

Vectors $\mathbf{e}, \mathbf{f}$ and $\mathbf{g}$ are eigenvectors with eigenvalues $1, 1, -1$ respectively. Determinant of operator P_2 is equal to product of eigenvalues: $\det P = 1 \cdot 1 \cdot (-1) = 1$. This orthogonal operator as well as orthogonal operator P_1 does not preserve orientation. Using equations (7.1) and (7.2) we have that for operator $P = P_2 \circ P_1$

$$P(\mathbf{e}) = P_2 \circ P_1(\mathbf{e}) = P_2(\mathbf{f}) = \mathbf{f},\ P(\mathbf{f}) = P_2 \circ P_1(\mathbf{f}) = P_2(\mathbf{e}) = \mathbf{e},\ P(\mathbf{g}) = P_2 \circ P_1(\mathbf{g}) = P_2(\mathbf{g}) = -\mathbf{g},$$

$\det P = \det(P_2 \circ P_1) = \det P_2 \cdot \det P_1 = (-1)(-1) = 1$. P is orthogonal matrix which preserves orientation.

Consider the vector $\mathbf{N} = \mathbf{e} + \mathbf{f}$. This is eigenvector of operator P:

$$P(\mathbf{N}) = P(\mathbf{e} + \mathbf{f}) = \mathbf{f} + \mathbf{e} = \mathbf{N}$$

We see that $\mathbf{N}$ is an eigenvector of non-identical orthogonal operator preserving orientation. Thus axis of rotation is along the vector $\mathbf{N}$. To calculate the angle of rotation notice that vector $\mathbf{g}$ transforms to vector $-g$. Hence the rotation is on the angle π [1]. (See in more detail the example in the end of subsection 1.10 of Lecture notes.)

[1] One can see that an arbitrary vector $\mathbf{a}$ orthogonal to vector $\mathbf{N}$ ("axis") changes to vector $-\mathbf{a}$.