Homework 1

1 Let $G = ||g_{ik}(x)||$ be Riemannian metric on n-dimensional Riemannian manifold M in local coordinates (x^i) $(i = 1, 2, \ldots, n)$.

a) Show that

$$g_{11}(x) > 0\,, g_{22}(x) > 0\,, \ldots g_{nn}(x) > 0\,.$$

b) show that condition of non-degeneracy for a symmetric matrix $G = ||g_{ik}||$ $(\det g_{ik} \neq 0)$ follows from the condition that this matrix is positive-definite.

2 Let (u, v) be local coordinates on 2-dimensional Riemannian manifold M. Let Riemannian metric be given in these local coordinates by the matrix

$$G = ||g_{ik}|| = \begin{pmatrix} A(u,v) & B(u,v) \\ C(u,v) & D(u,v) \end{pmatrix}\,,$$

where $A(u,v), B(u,v), C(u,v), D(u,v)$ are smooth functions. Show that the following conditions are fulfilled:

a) $B(u,v) = C(u,v)$,

b) $A(u,v)D(u,v) - B(u,v)C(u,v) = A(u,v)D(u,v) - B^2(u,v) \neq 0$,

c) $A(u,v) > 0$,

d)† $A(u,v)D(u,v) - B(u,v)C(u,v) = A(u,v)D(u,v) - B^2(u,v) > 0$.

e)† Show that conditions a), c) and d) are necessary and sufficient conditions for matrix $||g_{ik}||$ to define locally a Riemannian metric.

3 Consider 2-dimensional Euclidean plane with standard Euclidean metric

$$G = dx^2 + dy^2\,.$$

a) How this metric will transform under arbitrary affine coordinates transformation

$$\begin{cases} x = ax' + by' + e \\ y = cx' + dy' + f \end{cases}, \quad (a, b, c, d, e, f \in \mathbf{R})\,. \tag{1}$$

b) Find an affine transformation such that metric has the same appearance in new and old coordinates: $G = dx^2 + dy^2 = (dx')^2 + (dy')^2$.

c) How this metric will transform under coordinates transformation

$$x = \frac{u}{u^2+v^2}\,, \quad y = \frac{v}{u^2+v^2}\,, \quad (u, v \neq 0)\,.$$

d)† Let $x = x(u,v)$, and $y = y(u,v)$ be an arbitrary coordinate transformation such that the metric has the same appearance in new and old coordinates:

$$G = dx^2 + dy^2 = du^2 + dv^2\,.$$

How does this coordinate transformation look?

4 Consider domain in two-dimensional Riemannian manifold with Riemannian metric $G = du^2 + 2bdudv + dv^2$ in local coordinates u, v, where b is a constant.

Show that $b^2 < 1$

5 Let $G = cdu^2 + dudv + dv^2$ be Riemannian metric on 2-dimensional manifold M , where c is a real constant. Show that $c > \frac{1}{4}$.

(Hint: You may consider the length of a vector $\mathbf{X} = \frac{\partial}{\partial u} + t\frac{\partial}{\partial v}$ *where* t *is an arbitrary real number.)*