Homework 1

1 Show that for an arbitrary n-dimensional Riemannian manifold the condition of non-degeneracy for a symmetric matrix $G = ||g_{ik}||$ follows from the condition that this matrix is positive-definite.

2 Let (u, v) be local coordinates on 2-dimensional Riemannian manifold M. Let Riemannian metric be given in these local coordinates by the matrix

$$||g_{ik}|| = \begin{pmatrix} A(u,v) & B(u,v) \\ C(u,v) & D(u,v) \end{pmatrix} ,$$

where $A(u,v), B(u,v), C(u,v), D(u,v)$ are smooth functions. Show that the following conditions are fulfilled:

a) $B(u,v) = C(u,v)$,

b) $A(u,v)D(u,v) - B(u,v)C(u,v) = A(u,v)D(u,v) - B^2(u,v) \neq 0$,

c) $A(u,v) > 0$,

d) $A(u,v)D(u,v) - B(u,v)C(u,v) = A(u,v)D(u,v) - B^2(u,v) > 0$.

e)† Show that conditions a), c) and d) are necessary and sufficient conditions for matrix $||g_{ik}||$ to define locally a Riemannian metric.

f^*) How conditions above will change if the manifold M is pseudo-Riemannian, but not necessarily Riemannian?

3 Consider two-dimensional Riemannian manifold with Euclidean metric $G = dx^2 + dy^2$. How this metric will transform under arbitrary linear transformation $\begin{cases} x = ax' + by' \\ y = cx' + dy' \end{cases}$?

4 Consider two-dimensional Riemannian manifold with Riemannian metric $G = du^2 + 2bdudv + dv^2$, where b is a constant.

a) Show that $b^2 < 1$

b) Find new coordinates x, y such that under a "triangular" linear transformation $\begin{cases} x = u + \beta v \\ y = \delta v \end{cases}$ metric G transforms to the Euclidean metric $dx^2 + dy^2$. (Find parameters β, δ of this linear transformation)

c) Write down the metric $G = du^2 + 2bdudv + dv^2$ in new coordinates r, θ where $\begin{cases} u = r\cos\theta \\ y = r\sin\theta \end{cases}$

5 Let γ be a curve in Riemannian manifold given in local coordinates by parametric equation $x^i = x^i(t)$, $t_1 \leq t \leq t_2$. Show that the length of this curve

$$L = \int_{t_1}^{t_2} \sqrt{g_{ik}(x(t))\dot{x}^i(t)\dot{x}^k(t)}dt$$

does not change under arbitrary reparameterisation $t = t(\tau)$.