Homework 7

1 Show that vertical lines $x = a$ are geodesics (un-parameterised) on the Lobachevsky plane [1].

* Show that upper arcs of semicircles $(x-a)^2+y^2 = R^2, y > 0$ are (non-parametersied) geodesics.

2 Consider a vertical ray $C\colon x(t) = 1, y(t) = 1 + t$, $0 \leq t < \infty$ on the Lobachevsky plane.

Find the parallel transport $\mathbf{X}(t)$ of the vector $\mathbf{X}_0 = \partial_y$ attached at the initial point $(1, 1)$ along the ray C at an arbitrary point of the ray.

Find the parallel transport $\mathbf{Y}(t)$ of the vector $\mathbf{Y}_0 = \partial_x + \partial_y$ attached at the same initial point $(1, 1)$ along the ray C at an arbitrary point of the ray. (*Exam question, 2013.*)

3 Find a parameterisation of vertical lines in the Lobachevsky plane such that they become parameterised geodesics.

4 Find geodesics on cylinder

a) using straightforwardly equations of geodesics

b) using the properties of acceleration vector for particle movin along geodesic

c) using the fact that geodesic is shortest

In all the cases state clearly, is it parameterised, or un-parameterised geodesic.

5 *Great circle is a geodesic.*

Every geodesic is a great circle.

What curves are these statements about? Parametersied or un-parameterised?

Make these statements precise.

6 Let (M, G) be a Riemannian manifold. Let C be a curve on M starting at the point $\mathbf{pt}_1$ and ending at the point $\mathbf{pt}_2$.

Define an operator $P_C\colon T_{\mathbf{pt}_1}M \to T_{\mathbf{pt}_2}M$.

Explain why the parallel transport P_C is a linear orthogonal operator.

Let the points $\mathbf{pt}_1$ and $\mathbf{pt}_2$ coincide, so that C is a closed curve.

Let $\mathbf{a}$ be a vector attached at the point $\mathbf{pt}_1$, and $\mathbf{b} = P_C(\mathbf{a})$.

Consider operator P_C^2. Suppose that $P_C(\mathbf{a}) = \mathbf{b}$ and $P_C^2(\mathbf{a}) = -\mathbf{a}$. Show that vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal to each other. (*Exam question 2016*)

7 On the unit sphere $x^2 + y^2 + z^2 = 1$ in $\mathbf{E}^3$ consider the curve C defined by the equation $\cos\theta - \sin\theta\sin\varphi = 0$ in spherical coordinates.

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[1] As usual we consider here a realisation of the Lobachevsky plane (hyperbolic plane) as upper half of Euclidean plane $\{(x, y)\colon y > 0\}$ with the metric $G = \frac{dx^2+dy^2}{y^2}$. The line $x = 0$ is called *absolute*.

Show that in the process of parallel transport along the curve C an arbitrary tangent vector to the curve remains tangent to the curve. (*Exam question 2016*)

8 On the sphere $x^2 + y^2 + z^2 = R^2$ of radius R in $\mathbf{E}^3$ consider the following three closed curves.

a) the triangle $\triangle ABC$ with vertices at the points $A = (0,0,1)$, $B = (0,1,0)$ and $C = (1,0,0)$. The edges of triangle are geodesics.

b) the triangle $\triangle ABC$ with vertices at the points $A = (0,0,1)$, $B = (0, \cos\varphi, \sin\varphi)$ and $C = (1,0,0)$, $0 < \varphi < \frac{\pi}{2}$ The edges of triangle are geodesics.

c) the curve $\theta = \theta_0$ (line of constant latitude).

Consider the result of parallel transport of the vectors tangent to sphere over these closed curves.