<u>**Two hours**</u>

THE UNIVERSITY OF MANCHESTER

INTRODUCTION TO GEOMETRY. **MOCK EXAMINATION**

XX May-XX June 2017

XX:00 – XX:00

Answer **ALL FIVE** questions in Section A (50 marks in total).
Answer **TWO** of the THREE questions in Section B (30 marks in total).
If more than TWO questions in Section B are attempted, the credit will be given for the best TWO answers.

Electronic calculators may <u>not</u> be used.

SECTION A

Answer **ALL** FIVE questions

A1.

(a) Explain what is meant by saying that two bases in $\mathbf{E}^3$ have the opposite orientation.

(b) Let $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ be a basis in $\mathbf{E}^3$.

Consider the ordered triple $\{\mathbf{e}+\mathbf{f}+\mathbf{g}, \mathbf{e}+\mathbf{f}, \mathbf{e}\}$.

Show that this triple is a basis.

Show that the bases $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ and $\{\mathbf{e}+\mathbf{f}+\mathbf{g}, \mathbf{e}+\mathbf{f}, \mathbf{e}\}$ have opposite orientations.

(c) Suppose that the basis $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ considered above is an orthonormal basis. Explain why the basis $\{\mathbf{e}+\mathbf{f}+\mathbf{g}, \mathbf{e}+\mathbf{f}, \mathbf{e}\}$ is not an orthonormal basis.

[10 marks]

A2.

(a) State the Euler Theorem about rotations.

(b) Let P_1 be a linear operator such that

$$P_1(\mathbf{e}) = \mathbf{f}\,, P_1(\mathbf{f}) = \mathbf{e}\,, P(\mathbf{g}) = \mathbf{g}\,,$$

where $\{\mathbf{e}, \mathbf{f}, \mathbf{g}\}$ is an orthonormal basis in $\mathbf{E}^3$. Show that P_1 is an orthogonal operator.

Does this operator preserve orientation?

(c) Show that an operator $P_2 = -P_1$ is a rotation operator.

[10 marks]

A3.

(a) Give a definition of a differential 1-form in $\mathbf{E}^n$.

(b) Let f be a function on $\mathbf{E}^2$ given by $f(x,y) = x^3 - y^3$.

Let ω be 1-form such that $\omega = df$, and let $\mathbf{A}$ be a vector field such that $\mathbf{A} = x\partial_x + y\partial_y$.

Show that $\omega(\mathbf{A}) = 3f$.

(c) Explain why an 1- form $\sigma = xdy$ is not an exact form.

[10 marks]

A4.

(a) Give the definition of a parabola with focus at the given point F and directrix l.

(b) Let C be an ellipse in the plane $\mathbf{E}^2$ such that it has foci $F_1 = (0,2)$ and $F_2 = (0,6)$, and it passes through the point $(3,2)$.

Show that this ellipse passes through origin.

(c) Find the area of this ellipse.

[10 marks]

A5.

(a) Explain what is meant by the cross-ratio of four collinear points on the projective plane $\mathbf{RP}^2$.

(b) Four points $A, B, C, D \in \mathbf{RP}^2$ are given in homogeneous coordinates by

$$A = [1 : -1 : 1]\,, \quad B = [10 : -15 : 5]\,, \quad C = \left[1 : -\frac{9}{5} : \frac{1}{5}\right]\,, \quad D = [1 : 0 : 2]\,.$$

Show that these points are collinear.

(c) Calculate their cross-ratio.

[10 marks]

SECTION B

Answer **TWO** of the THREE questions

B6.

(a) Let P be a linear operator on $\mathbf{E}^3$ such that

$$P(\mathbf{x}) = 2(\mathbf{n}, \mathbf{x})\mathbf{n} - \mathbf{x}\,.$$

where $\mathbf{n}$ is a unit vector, and $(\, , \,)$ is scalar product.

Show that P is orthogonal operator preserving orientation.

(b) We know that, due to the Euler Theorem, P is a rotation operator. Find the axis and angle of this rotation.

[15 marks]

B7.

(a) Let C be an ellipse in $\mathbf{E}^2$ with foci $F_1 = (0,0)$, $F_2 = (6,0)$ which passes through the point $B = (0,8)$. Write down the equation of this ellipse

(b) Calculate the integrals $\int_C x dy - y dx$ and $\int_C x dy + y dx$.

To what extent do these integrals depend on the choice of parameterisation?

[15 marks]

B8.

(a) Let C be a curve in $\mathbf{E}^3$, defined by the intersection of the conic surface $4x^2 + 4y^2 - z^2 = 0$ with the plane $z + kx = 1$, and let C_{proj} be the orthogonal projection of the curve C onto the plane $z = 0$.

Show that if $|k| < 2$ then the curve C is an ellipse.

(b) Show that the curve C_{proj} is a parabola in the case if $k = 2$, and find focus and directrix of this parabola.

[15 marks]

END OF EXAMINATION PAPER