Very important example

In thick morphisms very important role play the case when thick morphism is just usual morphism. In this case

$$S(x,q) = \varphi^i(x) q_i$$

and the Hamiltonian is linear function.

In some sence it is degenerate case.

There is also the following (may be not less important?) example:`free particle`.

Consider the free particle. Its action is

$$S_t(x,y) = \frac{m(y-x)^2}{2t}$$

and

$$\mathcal{S}_t(x,q) = x^i q_i + \frac{q^2 t}{2m}$$

The LHS is the action, but it is in (x,q)-representation, it is the Legendre transfomr of the classical action (see the texts for July):

$$\mathcal{S}_t(x,q) = y^i q_i - S_t(x,y) \text{ with } q_i = \frac{\partial S_t(x,y)}{\partial y^i}\,.$$

It is easy to see that in this case one can easy to calculate the quantum case properly (ot just in stationary limit!)

The classical case is

$$f(x) = (\Phi^* g)(x) = g(x) + x^i q_i + \frac{q^2 t}{2m} - y^i q_i \text{ with } y^i = \frac{\partial \mathcal{S}_t(x,q)}{\partial q_i} = x^i + \frac{q^i t}{m}\,, q_i = \frac{\partial g(y)}{\partial y^i},$$

i.e.

$$f(x) = g\left(x^i + \frac{t}{m} q^i\right) = g\left(x^i + \frac{t}{m}\left(x^i + \frac{t}{m} q^i\right)\right) =$$

$$g\left(x^i + \frac{t}{m}\left(x^i + \frac{t}{m}\left(x^i + \frac{t}{m} q^i\right)\right)\right) = \ldots =$$

$$\lim_{N\to\infty} g\left(x^i + \frac{t}{m}\left(x^i + \frac{t}{m}\left(x^i + \ldots\right)\ldots\right)\right)\,, \text{where } N \text{ is `number' of brackets}\,.$$

This iteration procedure has simple integral representation since it can be described properly using quantum case:

$$\exp\left(\frac{i}{\hbar} f(x)\right) = \int \exp\left(\frac{i}{\hbar}\left(g(y) + \mathcal{S}_t(x,q) - y^i q_i\right)\right) \mathcal{D}q\mathcal{D}y =$$

$$\int \exp\left(\frac{i}{\hbar}\left(g(y) + x^i q_i + \frac{q^2 t}{2m} - y^i q_i\right)\right) \mathcal{D}q\mathcal{D}y$$

For free particle one can calculate this itnegral for an arbitrary $\hbar$ (If $\hbar \to 0$ then this integral is á la Legendre transform, andn this case it describes the pull back in classical case).

We have

$$\exp\left(\frac{i}{\hbar}f\left(x\right)\right) = \int \exp\left(\frac{i}{\hbar}\left(g\left(y\right) + x^i q_i + \frac{q^2 t}{2m} - y^i q_i\right)\right)\mathcal{D}q\mathcal{D}y =$$

$$\int \exp\left(\frac{i}{\hbar}\left(g\left(y\right)\right)\right)K_t(x,y)\mathcal{D}y\,,$$

whee $K_t(x,y)$ is Green function of heat equation:

$$K_t(x,y) = \int \exp\left(\frac{i}{\hbar}\left(x^i q_i + \frac{q^2 t}{2m} - y^i q_i\right)\right)\mathcal{D}q =$$

$$\int \exp\left(\frac{i}{\hbar}\left(x^i q_i + \frac{t}{2m}\left(q^i - \frac{m}{t}\left(y^i - x^i\right)\right)^2 - \frac{m\left(y-x\right)^2}{2t}\right)\right)\mathcal{D}q =$$

$$\frac{C}{\sqrt{t}}\exp\left(\frac{i}{\hbar}\left(-\frac{m\left(y-x\right)^2}{2t}\right)\right)$$