Introduction to Geometry (20222)
COURSEWORK 2008. Solutions
1

a) *Consider the parabola $y = 1 + x^2$ in $\mathbf{E}^2$, a point $M_t = (t, 1 + t^2)$ on the parabola ($t \in (-\infty, \infty)$) and the point $B = (1, 1)$. Denote by l_t the straight line passing through the points B and M_t. Find an equation of the line l_t. Calculate the coordinates of the second point of intersection of the line l_t with the parabola. Find values of parameter t such that the line l_t touches the parabola.*

The equation of the line l_t is

$$\frac{y-1}{x-1} = \frac{(1+t^2)-1}{t-1}, \quad \textit{i.e. } y = 1 + \frac{t^2}{t-1}(x-1) \qquad \text{if } t \neq 1\,.$$

If $x = 1$ then $y = 1$, if $x = t$ then $y = 1 + t^2$. (In the special case if $t = 1$ the line l_t is parallel to y-axis. It is given by an equation $x = 1$).

To find coordinates of points of intersection of the line l_t with the parabola $y = 1 + x^2$ we have to solve simultaneous equations

$$\begin{cases} y = 1 + \frac{t^2}{t-1}(x-1) \\ y = 1 + x^2 \end{cases}, \quad (t \neq 1)$$

We come to the quadratic equation on x, $1 + \frac{t^2}{t-1}(x-1) = 1 + x^2$, i.e. $x^2 - \frac{t^2}{t-1}x + \frac{t^2}{t-1} = 0$.

One root of this equation is equal to $x_1 = t$, because the line l_t intersects the parabola at the point $M_t = (t, 1 + t^2)$. The product of roots is equal to $x_1 x_2 = \frac{t^2}{t-1}$. Hence the second root, i.e. the x-coordinate of the second intersection point is equal to

$$x_2 = \frac{x_1 x_2}{x_2} = \frac{\frac{t^2}{t-1}}{t} = \frac{t}{t-1}.$$

Respectively $y_2 = 1 + x_2^2 = 1 + \frac{t^2}{(t-1)^2}$. The second point of the intersection is the point $(x_2, y_2) = \left(\frac{t}{t-1}, 1 + \frac{t^2}{(t-1)^2}\right)$. (Instead this simple consideration many students apply "brute force" solving straightforwardly the quadratic equation in radicals .)

If $x_1 = x_2$, i.e. $t = \frac{t}{t-1}$, i.e. $t = 0$ or $t = 2$ then these two points coincide and the line l_t touches the parabola.

If $t = 1$, then equation of the line l_t is $x = 1$. This line intersects the parabola only at the point* $(1, 2)$ ■

b) *In the Euclidean space $\mathbf{E}^2$ consider two points $A = (-1, 2)$ and $B = (4, 14)$.Find a unit vector $\mathbf{a}$ attached at the origin $O = (0, 0)$ such that it is collinear to the vector AB. Find an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ in $\mathbf{E}^2$ such that $\mathbf{a} = \mathbf{e}_1$.*

Vector $AB = (4 - (-1), 14 - 2) = (5, 12)$. Its length is equal to $\sqrt{5^2 + 12^2} = 13$. Hence the unit vector $\mathbf{a}$ collinear to AB is equal to $\mathbf{a} = \pm\frac{1}{13}(5, 12) = \pm\left(\frac{5}{13}, \frac{12}{13}\right)$. Vector $\mathbf{e}_2 = (x, y)$ is defined by the conditions $x^2 + y^2 = 1$ (unit vector) and $(\mathbf{e}_2, \mathbf{a}) = 0$, i.e. $5x + 12y = 0$. Hence $\mathbf{e}_2 = \pm\left(\frac{12}{13}, \frac{-5}{13}\right)$. We have two bases which obey these conditions: $(\mathbf{e}_1, \mathbf{e}_2)$ and $(\mathbf{e}_1, -\mathbf{e}_2)$ ■

c) *Consider the system of equations* $\begin{cases} x^2 + y^2 = R^2 \\ |x| + |y| = 1 \end{cases}$ *where R is a parameter. By using a sketch find the number of solutions of this system for different values of R.*

The set of the points $|x| + |y| = 1$ is the square. Its vertices are points $(1, 0), (0, 1)$, $(-1, 0)$ and $(-1, -1)$. Its diagonals are x and y axis. If the radius of the circle $x^2 + y^2 = R^2$ is greater than 1 then the circle and

* The line $x = 1$ does not touch parabola, in spite of the fact that it intersects the parabola only at one point $(1, 2)$. In projective space approach the projective line $x = z$ intersects the projective parabola $yz = z^2 + x^2$ (it corresponds to the line $x = 1$ and parabola $y = 1 + x^2$ in the chart $z = 1$). This projective line intersects the projective parabola in the points $[1 : 2 : 1]$ (which corresponds to the point $((1, 2))$ and the point $[0 : 1 : 0]$– this point is on infinity in the chart $z = 0$.

the square is in the interior of the circle. The system has no solutions. If $R = 1$ then there are exactly four solutions, points $(1,0),(0,1)$, $(-1,0)$ and $(-1,-1)$. If $\frac{\sqrt{2}}{2} < R < 1$ there are eight solutions. If $R = \frac{\sqrt{2}}{2}$ there are four solutions: circle touches the sides of the square. If $R < \frac{\sqrt{2}}{2}$ the circle is in the interior of the quadrat, no solutions. We come to

$$\#\text{of solutions} = \begin{cases} 0 \text{ if } R > 1 \\ 4 \text{ if } R = 1 \\ 8 \text{ if } \sqrt{2}/2 < R < 1 \\ 4 \text{ if } R = \sqrt{2}/2 \\ 0 \text{ if } R < \sqrt{2}/2 \end{cases}$$

■

2

a) *Consider vectors* $\mathbf{a} = 2\mathbf{e}_x + \mathbf{e}_y$, $\mathbf{b} = \mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z$ *in* $\mathbf{E}^3$. *Show that these vectors are linearly independent. Find an equation of the plane* α *spanned by vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *attached at the point* $M = (-1,2,3)$. *(Write down an equation in the form* $Ax + By + Cz = D$*).*

Find the distance between the point $K = (1.3.1)$ *and the plane* α*.)*

Suppose these vectors are not linear independent. Hence they are collinear: $\mathbf{a} = \lambda\mathbf{b}$, i.e. $2\mathbf{e}_x + \mathbf{e}_y = \lambda(\mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z)$, i.e. $2 = \lambda, 1 = \lambda, 0 = \lambda$. Contradiction. Hence they are linear independent.

Let $Ax+By+Cz = D$ be the equation of the plane α. We have that $M \in \alpha$. Hence $-A+2B+3C = D$. The vector $\mathbf{N} = (A,B,C)$ is orthogonal to the plane α. On the other hand the vector

$$\mathbf{a} \times \mathbf{b} = (2\mathbf{e}_x + \mathbf{e}_y) \times (\mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z) = \mathbf{e}_x - 2\mathbf{e}_y + \mathbf{e}_z$$

is also orthogonal to the surface. Hence $\mathbf{N}$ is collinear to the vector $\mathbf{a} \times \mathbf{b} = \mathbf{e}_x - 2\mathbf{e}_y + \mathbf{e}_z$, i.e.

$$\mathbf{N} = (A,B,C) = \lambda(1,-2,1), \text{and } D = -A + 2B + 3C = -2\lambda$$

We come to the equation of the plane α, $\lambda x - 2\lambda y + \lambda z = -2\lambda$. Here λ is an arbitrary coefficient $\lambda \neq 0$. E.g. one can take $\lambda = 1$ and we come to equation: $2y - x - z = 2$

The normal equation of the plane is

$$\frac{2y - x - z - 2}{\sqrt{2^2 + 1^2 + 1^2}} = \frac{2y - x - z - 2}{\sqrt{6}} = 0.$$

The distance between the point K and the plane is equal

$$d = \left|\frac{2(y = 3) - (x = 1) - (z = 1) - 2}{\sqrt{6}}\right| = \left|\frac{6 - 1 - 1 - 2}{\sqrt{6}}\right| = \frac{2}{\sqrt{6}}.$$

■

b) *Consider the plane* α *passing through the points* $A = (a,0,0)$, $B = (0,b,0)$ *and* $C = (0,0,c)$, *where* $a,b,c \neq 0$. *Find an equation of the plane* α, *the distance between origin and the plane* α, *and the area of the triangle* ABC. *Hint: You may use the formula for the volume of tetrahedron:* $V = \frac{HS}{3}$.

One can see that equation of the plane is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

The points $(a,0,0),(0,b,0)$ and $(0,0,c)$ evidently satisfy this equation.

Let h be the distance between the origin (point $(0,0,0)$) and the plane ABC. One can calculate the distance using the normal equation of the plane: $\lambda\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) = \lambda$, where $\frac{\lambda^2}{a^2} + \frac{\lambda^2}{b^2} + \frac{\lambda^2}{c^2} = 1$. Hence the distance is equal to $h = \lambda = \frac{abc}{\sqrt{a^2b^2+b^2c^2+a^2c^2}}$. (We suppose here that $a,b,c > 0$)

Let S be an area of the triangle ABC. Consider the volume of the tetrahedron $OABC$. On one hand it is equal to $V = \frac{hS}{3}$. On the other hand consider the side OAB of the tetrahedron. The area of this side is equal to $\frac{ab}{2}$ and the altitude of the tetrahedron on this side is equal to the c. Hence $V = \frac{abc}{6}$. We see that

$$V = \frac{abc}{6} = \frac{hS}{3} = \frac{Sabc}{3\sqrt{a^2b^2 + b^2c^2 + a^2c^2}}\,. \quad \text{Hence } \mathrm{S} = \frac{\sqrt{\mathrm{a^2b^2 + b^2c^2 + a^2c^2}}}{2}\,.$$

(Another way to calculate the area of triangle it is using the formula $S = \frac{1}{2}|\mathbf{a} \times b|$.)

Of course one can find the area of the triangle by "brute force", using Heron formula * for the area in terms of the sides $|AB| = \sqrt{a^2 + b^2}, |AC| = \sqrt{a^2 + c^2}, |BC| = \sqrt{b^2 + c^2}$, but this is not a beautiful solution. ∎

c) *In the Euclidean space $\mathbf{E}^3$ equipped with an orthonormal basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ consider a triple of vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ such that $\mathbf{e}_1 = \cos\varphi\ \mathbf{e}_y + \sin\varphi\ \mathbf{e}_z$, $\mathbf{e}_2 = -\sin\varphi\ \mathbf{e}_y + \cos\varphi\ \mathbf{e}_z$, $\mathbf{e}_3 = \varepsilon \mathbf{e}_x$, where φ is an arbitrary angle and $\varepsilon = \pm 1$. Show that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal basis, and find out if this basis have the same orientation as the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. Define the linear operator $\mathbf{P}\colon \mathbf{E}^3 \to \mathbf{E}^3$ by the condition that $\mathbf{P}\mathbf{e}_x = \mathbf{e}_1$, $\mathbf{P}\mathbf{e}_y = \mathbf{e}_2$, $\mathbf{P}\mathbf{e}_z = \mathbf{e}_3$ for the case $\varphi = 0$ and $\varepsilon = 1$. Find all vectors $\mathbf{N} = N_x\mathbf{e}_x + N_y\mathbf{e}_y + N_z\mathbf{e}_z$ such that $\mathbf{PN} = \mathbf{N}$. Explain the geometrical meaning of these vectors.*

Check that $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is orthonormal basis. Indeed all the vectors are unit vectors, e.g. $(\mathbf{e}_1, \mathbf{e}_1) = (\cos\varphi\ \mathbf{e}_y + \sin\varphi\ \mathbf{e}_z, \cos\varphi\ \mathbf{e}_y + \sin\varphi\ \mathbf{e}_z) = \cos^2\varphi + \sin^2\varphi = 1$ and their scalar product is equal to zero, e.g. $(\mathbf{e}_1, \mathbf{e}_2) = (\cos\varphi\ \mathbf{e}_y + \sin\varphi\ \mathbf{e}_z, -\sin\varphi\ \mathbf{e}_y + \cos\varphi\ \mathbf{e}_z) = -\cos\varphi\sin\varphi + \cos\varphi\sin\varphi = 0$.

The transition matrix from the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is $\begin{pmatrix} 0 & 0 & \varepsilon \\ \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \end{pmatrix}$. Its determinant is equal to ε. Hence the new and old bases have the same orientation if $\varepsilon = 1$ and they have opposite orientation if $\varepsilon = -1$.

We see that linear operator $\mathbf{P}$ preserves (changes) orientation if $\varepsilon = 1$ ($\varepsilon = -1$).

In the case $\varepsilon = 1$ linear operator $\mathbf{P}$ preserves orientation, so the action of this operator is a rotation around axis. To find an axis we have to find an eigenvector with eigenvalue 1.

Find eigenvectors in the case if $\varphi = 0$. For an arbitrary vector $\mathbf{x} = a\mathbf{e}_x + b\mathbf{e}_y + c\mathbf{e}_z$

$$\mathbf{Px} = P(a\mathbf{e}_x + b\mathbf{e}_y + c\mathbf{e}_z) = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3 = a\mathbf{e}_y + b\mathbf{e}_z + c\mathbf{e}_x = \lambda(a\mathbf{e}_x + b\mathbf{e}_y + c\mathbf{e}_z)$$

Hence $a = b = c$ and $\lambda = 1$. The linear orthogonal operator $\mathbf{P}$ has one eigenvector $\mathbf{N} = \mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z$ (defined up to a multiplication to a constant) with eigenvalue 1. The line spanned by the vector $\mathbf{N}$ is the axis of the rotation ∎

3

a) *Given a vector field $\mathbf{G} = a(r,\varphi)\partial_r + b(r,\varphi)\partial_\varphi$ in polar coordinates express it in cartesian coordinates.*
We have $x = r\cos\varphi, y = r\sin\varphi$ and $r = \sqrt{x^2 + y^2}, \varphi = \arctan\frac{y}{x}$.

Using chain rule we have

$$\partial_r = x_r\partial_x + y_r\partial_y = \cos\varphi\partial_x + \sin\varphi\partial_y = \frac{x}{r}\partial_x + \frac{y}{r}\partial_y,$$

$$\partial_\varphi = x_\varphi\partial_x + y_\varphi\partial_y = -r\sin\varphi\partial_x + r\cos\varphi\partial_y = -y\partial_x + x\partial_y\,.$$

Hence $\mathbf{G} = a(r,\varphi)\partial_r + b(r,\varphi)\partial_\varphi = a(r,\varphi)\left(\frac{x\partial_x + y\partial_y}{\sqrt{x^2+y^2}}\right) + b(r,\varphi)\left(-y\partial_x + x\partial_y\right)$

$$= \left(\frac{a(r,\varphi)x}{\sqrt{x^2+y^2}} - b(r,\varphi)y\right)\partial_x + \left(\frac{a(r,\varphi)y}{\sqrt{x^2+y^2}} + b(r,\varphi)x\right)\partial_y\,. \blacksquare$$

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* Heron formula claims that the area of the triangle is equal to $S = \sqrt{p(p-a)(p-b)(p-c)}$, where a, b, c are lengths of the triangle and $p = \frac{a+b+c}{2}$.

b) *Consider the function $f = r^n \cos 2\varphi$ and the vector fields $\mathbf{A} = x\partial_x + y\partial_y$, $\mathbf{B} = x\partial_y - y\partial_x$. Calculate $\partial_{\mathbf{A}} f$ and $\partial_{\mathbf{B}} f$. Perform these calculations both in polar and cartesian coordinates.*

Calculate 1-form $\omega = df$ and find the values of this 1-form on the vector fields $\mathbf{A}$, $\mathbf{B}$.

Note that according previous exercise $\mathbf{A} = x\partial_x + y\partial_y = r\partial_r$, $\mathbf{B} = x\partial_y - y\partial_x = \varphi$. Hence in polar coordinates

$$\partial_{\mathbf{A}} f = r\partial_r(r^n \cos 2\varphi) = nr^n \cos 2\varphi, \quad \partial_{\mathbf{B}} f = \partial_\varphi(r^n \cos 2\varphi) = -2r^n \sin 2\varphi\,.$$

In cartesian coordinates $f = r^n \cos 2\varphi = r^n(2\cos^2\varphi - 1) = (x^2+y^2)^{\frac{n}{2}}\left(\frac{x^2}{x^2+y^2} - 1\right) = (x^2+y^2)^{\frac{n}{2}}\left(\frac{x^2-y^2}{x^2+y^2}\right) = (x^2+y^2)^{\frac{n}{2}-1}(x^2-y^2)$. Hence in cartesian coordinates

$$\partial_{\mathbf{A}} f = (x\partial_x + y\partial_y)\left[(x^2+y^2)^{\frac{n}{2}-1}(x^2-y^2)\right] = \left(\frac{n}{2} - 1\right)(x^2+y^2)^{\frac{n}{2}-2}(2x^2+2y^2)(x^2-y^2) +$$

$$(x^2+y^2)^{\frac{n}{2}-1}(2x^2-2y^2) = (n-2)(x^2+y^2)^{\frac{n}{2}-1}(x^2-y^2) + 2(x^2+y^2)^{\frac{n}{2}-1}(x^2-y^2) = n(x^2+y^2)^{\frac{n}{2}-1}(x^2-y^2)\,,$$

and

$$\partial_{\mathbf{B}} f = (x\partial_y - y\partial_x) = \left[(x^2+y^2)^{\frac{n}{2}-1}(x^2-y^2)\right] =$$

$$\left(\frac{n}{2} - 1\right)(x^2+y^2)^{\frac{n}{2}-2}(2xy - 2yx)(x^2-y^2) + (x^2+y^2)^{\frac{n}{2}-1}(-2xy - 2xy) = -4xy(x^2+y^2)^{\frac{n}{2}-1}$$

The answers are the same.

$\omega = df = nr^{n-1}\cos 2\varphi dr - 2r^n \sin 2\varphi d\varphi$. $\omega(\mathbf{A}) = df(\mathbf{A}) = \partial_A f$, $\omega(\mathbf{B}) = df(\mathbf{B}) = \partial_B f$ ∎

(c) *Consider in $\mathbf{E}^2$ the differential 1-form $\omega = xdy + ydx$. Find a function (0-form) f such that $df = \omega$. Show in detail that, for the 1-form $\omega = xdy - ydx$ in $\mathbf{E}^2$, it is impossible to find a function f such that $df = \omega$.*

One can see that $d(xy) = xdy + ydx$. Hence $df = \omega \Rightarrow f = xy + c$, where c is an arbitrary constant.

Suppose that there exists a function f such that $df = \omega = xdy - ydx$. We have

$$d\omega = d(xdy - ydx) = 2dx \wedge dy = d(df) = d^2 f = 0. \Rightarrow 2dx \wedge dy = 0. \textit{ Contradiction.}$$

Hence it is impossible to find a function f such that $df = \omega = xdy - ydx$. ∎

Another solution One can solve the problem without using differential forms calculus:

Suppose that there exist f such that $\omega = xdy - ydx = df = f_x dx + f_y dy$. Hence $f_x = -y$, $f_y = x$. Integrating we come to $f = \int(-y)dx = -xy + a(y)$ or $f = \int xdy = xy + b(x)$. This means that $-xy + a(y) = xy + b(x)$, i.e. $2xy = a(y) - b(x)$. It is easy to see that this is impossible. e.g.

$$\frac{\partial^2}{\partial x\,\partial y}(2xy) = 2 = \frac{\partial^2}{\partial x\,\partial y}(a(y) - b(x)) = 0.\text{Contradiction.}$$

4

(a) *Consider in $\mathbf{E}^2$ the curve $\mathbf{r}(t)$: $x = t, y = \sin t$, $0 \le t \le \pi$.*
Find the velocity $\mathbf{v} = \frac{d\mathbf{r}(t)}{dt}$ and acceleration $\mathbf{a}(t) = \frac{d^2\mathbf{r}(t)}{dt^2}$ vectors.
Find the points of this curve where speed is decreasing.

$\mathbf{v} = \begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} 1 \\ \cos t \end{pmatrix}$, $\mathbf{a} = \begin{pmatrix} x_{tt} \\ y_{tt} \end{pmatrix} = \begin{pmatrix} 0 \\ -\sin t \end{pmatrix}$. Calculate scalar product: $(\mathbf{v}, \mathbf{a}) = |\mathbf{v}||\mathbf{a}|\cos\alpha = v_x a_x + v_y a_y = -\sin t \cos t$, where α is an angle between velocity and acceleration. Speed is decreasing $\Leftrightarrow$ the angle is obtuse $\Leftrightarrow$ scalar product $(\mathbf{v}, \mathbf{a})$ is negative: $(\mathbf{v}, \mathbf{a}) = -\sin t \cos t < 0$ on the interval $(0, \pi)$ if $t \in (0, \pi/2)$. Hence speed is decreasing for $0 < t < \pi/2$.

Another solution: The speed is equal to $\sqrt{v_x^2 + \mathbf{v}_y^2} = \sqrt{1 + \cos^2 t} = \sqrt{\cos 2t/2}$. Drawing the graph we see that this function is decreasing for $0 < t < \pi/2$ and it is increasing for $\pi/2 < t < \pi$. ∎

(b) *Consider in $\mathbf{E}^2$ the curve $\mathbf{r}(t)$: $x = 1 + t^2, y = t$, $0 < t < 1$.*

Sketch the image of this curve.
Calculate the integral of the differential form $\omega = xdy + y^2dx$ over this curve.
How does this integral change under the reparameterisation $t = \sin\tau$, $(0 < \tau < \frac{\pi}{2})$?
How does this integral change under the reparameterisation $t = \cos\tau$, $(0 < \tau < \frac{\pi}{2})$?

The velocity vector $\mathbf{v} = \begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} 2t \\ 1 \end{pmatrix}$. The value of differential form $\omega = xdy + y^2dx$ on the velocity vector is equal to

$$\omega(\mathbf{v}) = x(t)dy(\mathbf{v}(t)) + y^2(t)dx(\mathbf{v}(t)) = x(t)v_x(t) + y^2(t)v_y(t) = (1+t^2)\cdot 1 + t^2\cdot 2t = 1 + t^2 + 2t^3 .$$

$$\int_C \omega = \int_0^1 \omega(\mathbf{v}(t)) = \int_0^1 (1 + t^2 + 2t^3)dt = 1 + \frac{1}{3} + \frac{2}{4} = \frac{11}{6} .$$

This integral does no change under the reparameterisation $t = \sin\tau$, $(0 < \tau < \frac{\pi}{2})$ because this reparameterisation does not change the the orientation of the curve: $\frac{dt}{d\tau} = \cos\tau > 0$ for $0 < \tau < \frac{\pi}{2}$.

This integral does change the sign under the reparameterisation $t = \cos\tau$, $(0 < \tau < \frac{\pi}{2})$ ($\int_C \omega$=-13/6) because this reparameterisation changes the the orientation of the curve: $\frac{dt}{d\tau} = -\sin\tau < 0$ for $0 < \tau < \frac{\pi}{2}$ ■.

(c) *Consider in $\mathbf{E}^3$ two curves. A helix*

$$C_1: \quad \mathbf{r}(t) \begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases} , 0 \le t \le t_0 ,$$

and an interval of straight line

$$C_2: \quad \mathbf{r}(t) \begin{cases} x = 1 + at \\ y = bt \\ z = t \end{cases} , 0 \le t \le t_0 .$$

Note that starting points of these curves coincide.
Find values of parameters a, b such that ending points of these curves coincide too.
For chosen values of parameters a and b calculate integrals $\int_{C_1}\omega_1$, $\int_{C_2}\omega_1$, $\int_{C_1}\omega_2$ and $\int_{C_2}\omega_2$ of differential 1-forms $\omega_1 = xdy + ydx + dz$ and $\omega_2 = xdy - ydx + dz$ over these curves.

Explain why $\int_{C_1}\omega_1 = \int_{C_2}\omega_1$.

The starting points of these curves have coordinates $(x = 1, y = 0, z = 0)$.
The ending point of the curve C_1 has coordinates $(x = \cos t_0, y = \sin t_0, z = t_0)$.
The ending point of the curve C_2 has coordinates $(x = 1 + at_0, y = bt_0, z = t_0)$.
These points coincide if $\cos t_0 = 1 + at_0$ and $\sin t_0 = bt_0$. Thus if $a = \frac{\cos t_0 - 1}{t_0}$ and $b = \frac{\sin t_0}{t_0}$. then ending points of curves C_1 and C_2 coincide too.

To calculate integral of ω_1 note that this form is exact $\omega_1 = d(xy + z)$. Hence the integral of this form on both curves is equal to the difference of the function $f = xy + z$ on ending and starting points of curves C_1, C_2:

$$\int_{C_1} \omega = \int_{C_2} \omega = (xy + z)\big|_{\mathbf{r}_{starting}}^{r_{ending}} = \cos t_0 \sin t_0 + t_0 .$$

The form ω_2 is not exact. We have to calculate both integrals straightforwardly. For the curve C_1 $\int_{C_1}(xdy - ydx + dz) =$

$$\int_0^{t_0} (x(t)dy(\mathbf{v}) - y(t)dx(\mathbf{v}) + dz(\mathbf{v}))dt = \int_0^{t_0} (\cos t v_y(t) - \sin t v_x(t) + \mathbf{v}_z(t))dt = \int_0^{t_0} (\cos^2 + \sin^2 t + 1)dt = 2t_0$$

and for the curve C_2 $\int_{C_2}(xdy - ydx + dz) = \int_0^{t_0}(x(t)dy(\mathbf{v}) - y(t)dx(\mathbf{v}) + dz(\mathbf{v}))dt = \int_0^{t_0}(x(t)dy(\mathbf{v}) - y(t)dx(\mathbf{v}) + dz(\mathbf{v}))dt =$

$$\int_0^{t_0} ((1 + at)v_y(t) - btv_x(t) + \mathbf{v}_z(t))dt = \int_0^{t_0} ((1 + at)b - abt + 1)\, dt = (b + 1)t_0 = \sin t_0 + t_0 .$$

The form ω_1 is exact form. Hence its integral over all curves with the same ending and starting points must be the same. $\blacksquare$

5

(a) *Let C be the upper half of the circle with centre at the point $(R, 0)$ which is tangent to the y-axis. Write down an equation of this circle. Choose any parameterisation of this curve and calculate the integral $\int_C \omega$ if i) $\omega = x^2 dy$ and ii) $\omega = x^2 dy + 2xydx$.*

Does answers depend on the chosen parameterisation?

The circle touches y-axis and has the centre at the point $(R, 0)$. Hence its radius is equal to R. The equation is $(x - R)^2 + y^2 = R^2$.

Choose any parameterisation of the upper half of the circle, e.g. $\begin{cases} x = R + R\cos t \\ y = R\sin t \end{cases}, 0 \le t \le \pi$. Calculate $\int_C \omega$ if if $\omega = x^2 dy$ using this parameterisation:

$$\int_C x^2 dy = \int_0^\pi x^2(t)dy(\mathbf{v})dt = \int_0^\pi x^2(t)v_y(t)dt = \int_0^\pi R(R+R\cos t)^2 \cos t dt = \int_0^\pi R^3(\cos t+2\cos^2 t+\cos^3 t)dt$$

Note that $\int_0^\pi \cos t dt = 0$, $\int_0^\pi \cos^2 t dt = \frac{\pi}{2}$ and $\int_0^\pi \cos^3 t dt = 0$ because $\cos^3 t = \frac{\cos 3t + 3\cos t}{4}$. Hence

$$\int_C \omega = R^3 \int_0^\pi (\cos t + 2\cos^2 t + \cos^3 t)dt = \pi R^3 \quad \blacksquare.$$

For the second form $\omega = x^2 dy + 2xydx$ one can avoid straightforward calculations because $\omega = x^2 dy + 2xydx = d(x^2 y)$ is an exact form. Hence the integral over an arbitrary curve is defined by values of this functions at the starting and ending points of the curve:

$$\int_C (x^2 dy + 2xydx) = \int_C d(x^2 y) = x^2 y\big|_{\mathbf{r}(t_2)} - x^2 y\big|_{\mathbf{r}(t_1)}$$

Starting and ending point of the curve are $(2R, 0)$ and $(0, 0)$.

Hence $\int_C (x^2 dy + 2xydx) = \int_C d(x^2 y) = 4R^2 \cdot 0 - 0 \cdot 0 = 0$. We see that answer is equal to zero. If we change starting point becomes an ending point and ending point becomes the starting point,and integral has change the sign. In this the answer remains the same because $-0 = 0$

Of course one can come to the same answer doing the detailed calculations. But we do not need to perform them because $\omega = df$. For the first form $\omega = x^2 dy$ this trick does not work: the relation $\omega = df$ is impossible because ω is not a closed form: $d\omega = d(x^2 dy) = 2xdx \wedge dy \neq 0$, On the other hand if $\omega = df$ then $d\omega = 0$. Contradiction $\blacksquare$.

b) *Consider the curve in $\mathbf{E}^2$ defined by the equation $r(2 - \cos\varphi) = 3$ in polar coordinates.*

Show that the sum of the distances between the points $F_1 = (0, 0)$ and $F_2 = (2, 0)$, and an arbitrary point of this curve is constant, i.e. the curve is an ellipse and points F_1, F_2 are its foci..

Find the integral of the 1-form $\omega = ydx$ over the part of this curve where $0 \le \varphi \le \pi$. Here as usual x, y are cartesian coordinates $x = r\cos\varphi, y = r\sin\varphi$.

Hint: Write down the equation of the ellipse in cartesian coordinates.

First find a sum of distances between points F_1 and F_2 and an arbitrary point on this curve. Denote this sum by $l(\mathbf{r})$: $l(\mathbf{r}) = d(F_1, \mathbf{r}) + d(F_2, \mathbf{r}) =$. It is conevenient to calculate this distance in polar coordinates.The distance $d(F_1, \mathbf{r})$ is equal to r. Considering the triangle formed by the points $\mathbf{r}, F_1$ and F_2 we see that $d(F_2, \mathbf{r}) = \sqrt{r^2 + 4 - 4r\cos\varphi}$. We have $d(F_2, r) = l(\mathbf{r}) - r = \sqrt{r^2 + 4 - 4r\cos\varphi}$. Taking the square we come to the equation $l^2 - 2lr = 4 - 4r\cos\varphi$, i.e.

$$r(2l - 4\cos\varphi) = l^2 - 4 \,. \qquad \text{On the other hand} \qquad r(2 - \cos\varphi) = 3 \,. \qquad (*)$$

Comparing these two equations we come to $l \equiv 4$. Thus sum of the distances between points F_1 and F_2 and an arbitrary point on this curve is equal to 4. This curve is ellipse, and points F_1, F_2 are the foci of this ellipse.

Now write down the equation (*) in cartesian coordinates.

$r = \sqrt{x^2+y^2}$, $r\cos\varphi = x$. Hence we have $2\sqrt{x^2+y^2} - x = 3$, i.e. $4(x^2+y^2) = (3+x)^2 \Rightarrow 3x^2 - 6x + 4y^2 = 9$. But $3x^2 - 6x = 3(x-1)^2 - 3$. We see that the curve $r(2-\cos\varphi) = 3$ is

$$3(x-1)^2 + 4y^2 = 12 \Leftrightarrow \frac{(x-1)^2}{4} + \frac{y^2}{3} = 1$$

Changing $x \mapsto x-1$ we see that this is an ellipse with the centre at the point $(1,0)$. (The equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Here $a = 2, b = \sqrt{3}$) One can consider the following parameterisation of this ellipse:

$$\mathbf{r}(t)\colon \begin{cases} x - 1 = 2\cos t \\ y = \sqrt{3}\sin t \end{cases} \text{ or } \mathbf{r}(t)\colon \begin{cases} x = 1 + 2\cos t \\ y = \sqrt{3}\sin t \end{cases}, \qquad 0 \le t \le \pi \ (0 \le t \le \pi) \tag{**}$$

The parameter t is in the interval $(0,\pi)$ because polar angle φ is positive (we consider upper half of the ellipse)

Now one can calculate the integral $\int_C \omega$ using parameterisation (**):

$$\int_C ydx = \int_0^\pi ydx(\mathbf{v})dt = \int_0^\pi y(t)dx(-2\sin t\partial_x + \sqrt{3}\cos t\partial_y)dt = \int_0^\pi -2\sqrt{3}\sin^2 tdt = -\pi\sqrt{3}$$

If we consider an arbitrary parameterisation then integral will be $\pm\pi\sqrt{3}$. (It will be equal to $-\pi\sqrt{3}$ for all parameterisations with the same orientation as a parametersiation (**) and it will be equal to $\pi\sqrt{3}$ for all parameterisations with the orientation opposite to orientation of parametersiation (**).)

(Note that up to the sign it is equal just to the area of the upper half of the ellipse because $d(ydx) = dy \wedge dx$).

c) Consider 1-form $\omega = \frac{xdy - ydx}{(x^2+y^2)^\alpha}$, where α is an arbitrary parameter.

Find the integral of the form ω over the upper half of the circle with centre at the origin and with radius R.

For $\alpha = 1$, find the integral of the form ω over upper half of the ellipse defined by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

How the previous answer will change if we translate the upper half of the ellipse along the x-axis, i.e. if we consider the curve $\frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} = 1, y \ge 0$.

Hint: It is useful to write down the form ω in polar coordinates.

Write down the form ω in polar coordinates:

$$\omega = \frac{xdy - ydx}{(x^2+y^2)^\alpha} = \frac{r\cos\varphi d(r\sin\varphi) - r\sin\varphi d(r\cos\varphi)}{r^{2\alpha}} = r^{2-2\alpha}d\varphi\,. \tag{***}$$

The parametric equation of the upper half of the circle in polar coordinates is $\begin{cases} \mathbf{r}(t) = R_0 \\ \varphi(t) = t \end{cases}, 0 \le t \le \pi$. The velocity vector (tangent vector to the curve) is equal to $\mathbf{v} = \begin{pmatrix} r_t \\ \varphi_t \end{pmatrix} = \begin{pmatrix} r_t \\ \varphi_t \end{pmatrix}$. $\omega(\mathbf{v}) = r^{2-2\alpha}d\varphi(\mathbf{v}) = r^{2-2\alpha}v_\varphi = R^{2-2\alpha}$. Hence

$$\int_C \omega = \int_0^\pi r^{2-2\alpha}d\varphi(\mathbf{v})dt = \int_0^\pi R^{2-2\alpha}dt = \pi R^{2-2\alpha}\,.$$

In the case if $\alpha = 1$ this integral does not depend on the radius of the circle. Moreover in this case the form (***) is closed form: $d\omega = 0$. Is it exact? It is temptation to say: Yes, it is, $\omega = df$, where $f = \varphi$. But the function $f = \varphi$ is not well-defined function on whole $\mathbf{E}^{2n}$. If we make rotation around origin $\varphi \to \varphi + 2\pi$, i.e. it is multivalued function. On the other hand if the curve C belongs to the domain where the function φ can be defined as one-valued differentiable function, then we can apply the Theorem about the integral of exact form($\int_C \omega = df\big|_{\partial C}$) for this domain.

Having this in mind consider the integral of the form $\omega = \frac{xdy - ydx}{(x^2+y^2)}$ over upper half of the ellipse. Consider the following domain D: take upper half plane and remove the small disc of the radius ε:

$$D = \{(x, y) : y > 0,\ x^2 + y^2 > \varepsilon\} \tag{$****$}$$

The function φ is well-defined one-valued function in the domain D and the upper half of the ellipse belongs to this domain. Now we calculate integral:

$$\int_C \frac{xdy - ydx}{x^2 + y^2} = \varphi\big|_{\partial C} = \pm\pi\,.$$

The sign depends on orientation. ∎

Remark One can calculate this integral using brute force: The parametric equation of ellipse is $x = a\cos t, y = b\sin t, 0 \leq t \leq \pi$

$$\int_C \frac{xdy - ydx}{x^2 + y^2} = \int_0^\pi \frac{a\cos t d(b\sin t) - b\sin t d(a\cos t)}{a^2\cos^2 t + b^2\sin^2 t} = \int_0^\pi \frac{ab}{a^2\cos^2 + b^2\sin^2 t} dt$$

To calculate this integral use the identity for the integrand: $\frac{ab}{a^2\cos^2 + b^2\sin^2 t} = \frac{d}{dt}\left(\arctan\frac{b\sin t}{a\cos t}\right)$. It follows from this "miraculous" identity that $\int_0^\pi \frac{ab}{a^2\cos^2 + b^2\sin^2 t} dt = \pi$ The geometrical considerations above reveal the meaning of this miraculous identity: $\arctan\frac{b\sin t}{a\cos t} = \varphi$!

Remark You may ask: How to prove that 1-form $\omega = \omega = \frac{xdy - ydx}{(x^2+y^2)}$ is not exact? Yes we see that $\omega = d\varphi$ is not well-defined, but may be there exists another well defined function such that $\omega = df$. The proof that *this function does not exist* is following. The integral of the form ω over closed circle with the centre in the origin is equal to 2π. It does not equal to zero. Hence the relation $\omega = df$ cannot be well-defined.

Now consider the ellipse $\frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} = 1$.

If $a - c > 0$ then all upper ellipse is in the positive quadrant $x > 0, y > 0$. If $a - c < 0$ then upper ellipse intersects y axis.

Now considering the domain D defined by the equation (****) and choosing ε enough small we come to the answer:

$$\int_{\frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} = 1, y \geq 0} \frac{xdy - ydx}{x^2 + y^2} = \begin{cases} 0 \text{ if } a - c > 0 \\ \pm\pi \text{ if } a - c < 0 \end{cases}$$