Homework 5. Solutions, (part b)

Remark *Here are the solutions of exercises 6 and 7 of homework 5 except the solution of exercise 6 d), which will appear after Coursework handling.*

6 *Consider the surface M in the Euclidean space $\mathbf{E}^n$. Calculate the induced connection in the following cases*

a) $M = S^1$ in $\mathbf{E}^2$,

b) M— parabola $y = x^2$ in $\mathbf{E}^2$,

c) cylinder in $\mathbf{E}^3$.

d) cone in $\mathbf{E}^3$.

e) sphere in $\mathbf{E}^3$.

f) saddle $z = xy$ in $\mathbf{E}^3$

Solution.

a) Consider polar coordinate on S^1, $x = R\cos\varphi, y = R\sin\varphi$. We have to define the connection on S^1 induced by the canonical flat connection on $\mathbf{E}^2$. It suffices to define $\nabla_{\frac{\partial}{\partial\varphi}}\frac{\partial}{\partial\varphi} = \Gamma^\varphi_{\varphi\varphi}\partial_\varphi$.

Recall the general rule. Let $\mathbf{r}(u^\alpha)$: $x^i = x^i(u^\alpha)$ is embedded surface in Euclidean space $\mathbf{E}^n$. The basic vectors $\frac{\partial}{\partial u^\alpha} = \frac{\partial \mathbf{r}(u)}{\partial u^\alpha}$. To take the induced covariant derivative $\nabla_{\mathbf{X}}\mathbf{Y}$ for two tangent vectors $\mathbf{X}, \mathbf{Y}$ we take a usual derivative of vector $\mathbf{Y}$ along vector $\mathbf{X}$ (the derivative with respect to canonical flat connection: in Cartesian coordiantes is just usual derivatives of components) then we take the tangent component of the answer, since in general derivative of vector $\mathbf{Y}$ along vector $\mathbf{X}$ is not tangent to surface:

$$\nabla_{\frac{\partial}{\partial u^\alpha}}\frac{\partial}{\partial u^\beta} = \Gamma^\gamma_{\alpha\beta}\frac{\partial}{\partial u^\gamma} = \left(\nabla^{(\text{canonical})}_{\partial_\alpha}\frac{\partial}{\partial u^\beta}\right)_{\text{tangent}} = \left(\frac{\partial^2\mathbf{r}(u)}{\partial u^\alpha\partial u^\beta}\right)_{\text{tangent}}$$

($\nabla_{\text{canonical }\partial_\alpha}\frac{\partial}{\partial u^\beta}$) is just usual derivative in Euclidean space since for canonical connection all Christoffel symbols vanish.)

In the case of 1-dimensional manifold, curve it is just tangential acceleration!:

$$\nabla_{\frac{\partial}{\partial u}}\frac{\partial}{\partial u} = \Gamma^u_{uu}\frac{\partial}{\partial u} = \left(\nabla^{(\text{canonical})}_{\partial_u}\frac{\partial}{\partial u}\right)_{\text{tangent}} = \left(\frac{d^2\mathbf{r}(u)}{du^2}\right)_{\text{tangent}} = \mathbf{a}_{\text{tangent}}$$

For the circle S^1, ($x = R\cos\varphi, y = R\sin\varphi$), in $\mathbf{E}^2$. We have

$$\mathbf{r}_\varphi = \frac{\partial}{\partial\varphi} = \frac{\partial x}{\partial\varphi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\varphi}\frac{\partial}{\partial y} = -R\sin\varphi\frac{\partial}{\partial x} + R\cos\varphi\frac{\partial}{\partial y},$$

$$\nabla_{\frac{\partial}{\partial\varphi}}\frac{\partial}{\partial\varphi} = \Gamma^\varphi_{\varphi\varphi}\partial_\varphi = \left(\nabla^{(\text{canonic.})}_{\partial_\varphi}\partial_\varphi\right)_{\text{tangent}} = \left(\frac{\partial}{\partial\varphi}\mathbf{r}_\varphi\right)_{\text{tangent}} =$$

$$\left(\frac{\partial}{\partial\varphi}\left(-R\sin\varphi\right)\frac{\partial}{\partial x}+\frac{\partial}{\partial\varphi}\left(R\cos\varphi\right)\frac{\partial}{\partial y}\right)_{\text{tangent}}=\left(-R\cos\varphi\frac{\partial}{\partial x}-R\sin\varphi\frac{\partial}{\partial y}\right)_{\text{tangent}}=0,$$

since the vector $-R\cos\varphi\frac{\partial}{\partial x}-R\sin\varphi\frac{\partial}{\partial y}$ is orthogonal to the tangent vector $\mathbf{r}_\varphi$. In other words it means that acceleration is centripetal: tangential acceleration equals to zero.

We see that in coordinate φ, $\Gamma^\varphi_{\varphi\varphi}=0$. $\blacksquare$

Additional work: Perform calculation of Christoffel symbol in stereographic coordinate t:

$$x=\frac{2tR^2}{R^2+t^2}, y=\frac{R(t^2-R^2)}{t^2+R^2}\,.$$

In this case

$$\mathbf{r}_t=\frac{\partial}{\partial t}=\frac{\partial x}{\partial t}\frac{\partial}{\partial x}+\frac{\partial y}{\partial t}\frac{\partial}{\partial y}=\frac{2R^2}{(R^2+t^2)^2}\left((R^2-t^2)\frac{\partial}{\partial x}+2tR\frac{\partial}{\partial x}\right),$$

$$\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t}=\Gamma^t_{tt}\partial_t=\left(\nabla^{(\text{canonic.})}_{\partial_t}\partial_t\right)_{\text{tangent}}=\left(\frac{\partial}{\partial t}\mathbf{r}_t\right)_{\text{tangent}}=(\mathbf{r}_{tt})_{\text{tangent}}=$$

$$\left(-\frac{4t}{t^2+R^2}\mathbf{r}_t+\frac{2R^2}{(R^2+t^2)^2}\left(-2t\frac{\partial}{\partial x}+2R\frac{\partial}{\partial y}\right)\right)_{\text{tangent}}$$

In this case $\mathbf{r}_{tt}$ is not orthogonal to velocity: to calculate $(\mathbf{r}_{tt})_{\text{tangent}}$ we need to extract its orthogonal component:

$$(\mathbf{r}_{tt})_{\text{tangent}}=\mathbf{r}_{tt}-\langle\mathbf{r}_{tt},\mathbf{n}_t\rangle\mathbf{n}$$

We have

$$\mathbf{n}_t=\frac{\mathbf{r}}{|\mathbf{r}|}=\frac{1}{R^2+t^2}\left(2tR\partial_x+(t^2-R^2)\partial_y\right),$$

where $\langle\mathbf{r}_t,\mathbf{n}\rangle=0$. Hence $\langle\mathbf{r}_{tt},\mathbf{n}_t\rangle=\frac{-4R^3}{(t^2+R^2)^2}$ and

$$(\mathbf{r}_{tt})_{\text{tangent}}=\mathbf{r}_{tt}-\langle\mathbf{r}_{tt},\mathbf{n}_t\rangle\mathbf{n}=$$

$$\left(-\frac{4t}{t^2+R^2}\mathbf{r}_t+\frac{2R^2}{(R^2+t^2)^2}\left(-2t\frac{\partial}{\partial x}+2R\frac{\partial}{\partial y}\right)\right)+\frac{4R^3}{(t^2+R^2)^2}\cdot\frac{1}{R^2+t^2}\left(2tR\partial_x+(t^2-R^2)\partial_y\right)=\frac{-2}{t^2+}$$

We come to the answer:

$$\nabla_{\partial_t}\partial_t=\frac{-2t}{t^2+R^2}\partial_t,\quad \text{i.e.}\Gamma^{\text{t}}_{\text{tt}}=\frac{-2\text{t}}{\text{t}^2+\text{R}^2}$$

Of course we could calculate the Christoffel symbol in stereographic coordinates just using the fact that we already know the Christoffel symbol in polar coordinates: $\Gamma^\varphi_{\varphi\varphi}=0$, hence

$$\Gamma^t_{tt}=\frac{dt}{d\varphi}\frac{d\varphi}{dx}\frac{d\varphi}{dx}\Gamma^\varphi_{\varphi\varphi}+\frac{d^2\varphi}{dt^2}\frac{dt}{d\varphi}=\frac{d^2\varphi}{dt^2}\frac{dt}{d\varphi}$$

It is easy to see that $t = R\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)$, i.e. $\varphi = 2\arctan\frac{t}{R}-\frac{\pi}{2}$ and

$$\Gamma^t_{tt} = \frac{d^2\varphi}{dt^2}\frac{dt}{d\varphi} = \frac{\frac{d^2\varphi}{dt^2}}{\frac{d\varphi}{dt}} = -\frac{2t}{t^2+R^2}\,.$$

b) For parabola $x = t, y = t^2$

$$\mathbf{r}_t = \frac{\partial}{\partial t} = \frac{\partial x}{\partial t}\frac{\partial}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial}{\partial y} = \frac{\partial}{\partial x} + 2t\frac{\partial}{\partial y}\,,$$

$$\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t} = \Gamma^t_{tt}\partial_t = \left(\nabla^{(\text{canonic.})}_{\partial_t}\partial_t\right)_{\text{tangent}} = \left(\frac{\partial}{\partial t}\mathbf{r}_t\right)_{\text{tangent}} = (\mathbf{r}_{tt})_{\text{tangent}} = \left(2\frac{\partial}{\partial y}\right)_{\text{tangent}}$$

To calculate $(\mathbf{r}_{tt})_{\text{tangent}}$ we need to extract its orthogonal component: $(\mathbf{r}_{tt})_{\text{tangent}} = \mathbf{r}_{tt} - \langle \mathbf{r}_{tt}, \mathbf{n}_t\rangle\mathbf{n}$, where $\mathbf{n}$ is an orthogonal unit vector: $\langle \mathbf{n}, \mathbf{r}_t\rangle = 0, \langle \mathbf{n}, \mathbf{n}\rangle = 1$:

$$\mathbf{n}_t = \frac{1}{\sqrt{1+4t^2}}\left(-2t\partial_x + \partial_y\right)\,.$$

We have

$$(\mathbf{r}_{tt})_{\text{tangent}} = \mathbf{r}_{tt} - \langle \mathbf{r}_{tt}, \mathbf{n}_t\rangle\mathbf{n} = 2\partial_y - \left\langle 2\partial_y, \frac{1}{\sqrt{1+4t^2}}\left(-2t\partial_x+\partial_y\right)\right\rangle \frac{1}{\sqrt{1+4t^2}}\left(-2t\partial_x+\partial_y\right) =$$

$$\frac{4t}{1+4t^2}\partial_x + \frac{8t^2}{1+4t^2}\partial_y = \frac{4t}{1+4t^2}\left(\partial_x + 2t\partial_y\right) = \frac{4t}{1+4t^2}\partial_t$$

We come to the answer:

$$\nabla_{\partial_t}\partial_t = \frac{4t}{1+4t^2}\partial_t, \quad \text{i.e.}\Gamma^{\mathrm{t}}_{\mathrm{tt}} = \frac{4\mathrm{t}}{1+4\mathrm{t}^2}$$

c) *Cylinder*

$\mathbf{r}(h,\varphi)$: $\begin{cases} x = a\cos\varphi \\ y = a\sin\varphi\,. \\ z = h \end{cases}$

$\partial_h = \mathbf{r}_h = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\partial_\varphi = \mathbf{r}_\varphi = \begin{pmatrix} -a\sin\varphi \\ a\cos\varphi \\ 0 \end{pmatrix}$

Calculate

$$\nabla_{\partial_h}\partial_h = \Gamma^h_{hh}\partial_h + \Gamma^\varphi_{hh}\partial_\varphi = \left(\frac{\partial^2\mathbf{r}}{\partial h^2}\right)_{\text{tangent}} = 0 \,\text{since}\; \mathbf{r}_{hh} = 0.$$

Hence $\Gamma^h_{hh} = \Gamma^\varphi_{hh} = 0$

$$\nabla_{\partial_h}\partial_\varphi = \nabla_{\partial_\varphi}\partial_h = \Gamma^h_{h\varphi}\partial_h + \Gamma^\varphi_{h\varphi}\partial_\varphi = \left(\frac{\partial^2\mathbf{r}}{\partial h\partial\varphi}\right)_{\text{tangent}} = 0 \,\text{since}\; \mathbf{r}_{h\varphi} = 0$$

Hence $\Gamma^h_{h\varphi} = \Gamma^h_{\varphi h} = \Gamma^\varphi_{h\varphi} = \Gamma^\varphi_{\varphi h} = 0$.

$$\nabla_{\partial_\varphi}\partial_\varphi = \Gamma^h_{\varphi\varphi}\partial_h + \Gamma^\varphi_{\varphi\varphi}\partial_\varphi = \left(\frac{\partial^2\mathbf{r}}{\partial\varphi\partial\varphi}\right)_{\text{tangent}} = \left(\begin{pmatrix} -a\cos\varphi \\ -a\sin\varphi \\ 0 \end{pmatrix}\right)_{\text{tangent}} = 0$$

since the vector $\mathbf{r}_{\varphi\varphi} = \begin{pmatrix} -a\cos\varphi \\ -a\sin\varphi \\ 0 \end{pmatrix}$ is orthogonal to the surface of cylinder. Hence $\Gamma^h_{h\varphi} = \Gamma^h_{\varphi h} = \Gamma^\varphi_{h\varphi} = \Gamma^\varphi_{\varphi h} = 0$

We see that for cylinder all Christoffel symbols in cylindrical coordinates vanish. This is not big surprise: in cylindrical coordinates metric equals $dh^2 = a^2 d\varphi^2$. This due to Levi-Civita theorem one can see that Levi-Civita connection which is equal to induced connection vanishes since all coefficients are constants.

d) *Cone* (the solution will appear latter)

e) *Sphere*

For the sphere $\mathbf{r}(\theta,\varphi)$: $\begin{cases} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{cases}$, we have

$$\frac{\partial}{\partial\theta} = \mathbf{r}_\theta = \begin{pmatrix} R\cos\theta\cos\varphi \\ R\cos\theta\sin\varphi \\ -R\sin\theta \end{pmatrix},\ \frac{\partial}{\partial\varphi} = \mathbf{r}_\varphi = \begin{pmatrix} -R\sin\theta\sin\varphi \\ R\sin\theta\cos\varphi \\ 0 \end{pmatrix},\ \mathbf{n} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

Calculate

$$\nabla_{\partial_\theta}\partial_\theta = \Gamma^\theta_{\theta\theta}\partial_\theta + \Gamma^\varphi_{\theta\theta}\partial_\varphi = \left(\frac{\partial^2\mathbf{r}}{\partial\theta^2}\right)_{\text{tangent}} = 0$$

since $\frac{\partial^2\mathbf{r}}{\partial\theta^2} = -R\mathbf{n}$ is orthogonal to the sphere. Hence $\Gamma^\theta_{\theta\theta} = \Gamma^\varphi_{\theta\theta} = 0$.

Now calculate

$$\nabla_{\partial_\theta}\partial_\varphi = \Gamma^\theta_{\theta\varphi}\partial_\theta + \Gamma^\varphi_{\theta\varphi}\partial_\varphi = \left(\frac{\partial^2\mathbf{r}}{\partial\theta\partial\varphi}\right)_{\text{tangent}}.$$

We have

$$\frac{\partial^2\mathbf{r}}{\partial\theta\partial\varphi} = \operatorname{cotan}\theta\,\mathbf{r}_\varphi,$$

hence

$$\nabla_{\partial_\theta}\partial_\varphi = \Gamma^\theta_{\theta\varphi}\partial_\theta + \Gamma^\varphi_{\theta\varphi}\partial_\varphi = \left(\frac{\partial^2\mathbf{r}}{\partial\theta\partial\varphi}\right)_{\text{tangent}} = \operatorname{cotan}\theta\,\mathbf{r}_\varphi, i.e.$$

$\Gamma^\theta_{\theta\varphi} = 0, \Gamma^\varphi_{\theta\varphi} = \operatorname{cotan}\theta$

Now calculate

$$\nabla_{\partial_\varphi}\partial_\theta = \Gamma^\theta_{\varphi\theta}\partial_\theta + \Gamma^\varphi_{\varphi\theta}\partial_\varphi = \left(\frac{\partial^2 \mathbf{r}}{\partial\varphi\partial\theta}\right)_{\text{tangent}} .$$

We have

$$\frac{\partial^2 \mathbf{r}}{\partial\varphi\partial\theta} = \operatorname{cotan}\theta \mathbf{r}_\varphi ,$$

hence

$$\nabla_{\partial_\theta}\partial_\varphi = \Gamma^\theta_{\theta\varphi}\partial_\theta + \Gamma^\varphi_{\theta\varphi}\partial_\varphi = \left(\frac{\partial^2 \mathbf{r}}{\partial\theta\partial\varphi}\right)_{\text{tangent}} = \operatorname{cotan}\theta \mathbf{r}_\varphi, i.e.$$

$\Gamma^\theta_{\varphi\theta} = 0, \Gamma^\varphi_{\varphi\theta} = \operatorname{cotan}\theta$. Of course we did not need to perform these calculations: since ∇ is symmetric connection and $\nabla_{\partial_\varphi}\partial_\theta = \nabla_{\partial_\theta}\partial_\varphi$, i.e.

$$\Gamma^\theta_{\varphi\theta} = \Gamma^\theta_{\theta\varphi} = 0 \ \Gamma^\varphi_{\varphi\theta} = \Gamma^\varphi_{\theta\varphi} = \operatorname{cotan}\theta .$$

and finally

$$\nabla_{\partial_\varphi}\partial_\varphi = \Gamma^\theta_{\varphi\varphi}\partial_\theta + \Gamma^\varphi_{\varphi\varphi}\partial_\varphi = \left(\frac{\partial^2 \mathbf{r}}{\partial\varphi^2}\right)_{\text{tangent}} .$$

We have

$$\frac{\partial^2 \mathbf{r}}{\partial\varphi^2} = \mathbf{r}_{\varphi\varphi} = \begin{pmatrix} -R\sin\theta\cos\varphi \\ -R\sin\theta\sin\varphi \\ 0 \end{pmatrix} .$$

The vector $\mathbf{r}_{\varphi\varphi}$ is not proportional to normal vector $\mathbf{n}$, i.e. it is not orthogonal to the sphere; the vector $\mathbf{r}_{\varphi\varphi}$ is not tangent to sphere, i.e. it is not orthogonal to vector $\mathbf{n}$: $0 \neq \langle \mathbf{r}_{\varphi\varphi}, \mathbf{n}\rangle = -R\sin^2\theta$. We decompose the vector $\mathbf{r}_{\varphi\varphi}$ on the sum of tangent vector and orthogonal vector:

$$\mathbf{r}_{\varphi\varphi} = \underbrace{\mathbf{r}_{\varphi\varphi} - \mathbf{n}\langle \mathbf{r}_{\varphi\varphi}, \mathbf{n}\rangle}_{\text{tangent vector}} + \mathbf{n}\langle \mathbf{r}_{\varphi\varphi}, \mathbf{n}\rangle ,$$

We see that

$$\left(\frac{\partial^2 \mathbf{r}}{\partial\varphi^2}\right)_{\text{tangent}} = \mathbf{r}_{\varphi\varphi} - \mathbf{n}\langle \mathbf{r}_{\varphi\varphi}, \mathbf{n}\rangle = \mathbf{r}_{\varphi\varphi} + R\sin^2\theta\mathbf{n} = \begin{pmatrix} -R\sin\theta\cos\varphi \\ -R\sin\theta\sin\varphi \\ 0 \end{pmatrix} + R\sin^2\theta \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} =$$

$$\begin{pmatrix} -R\cos^2\theta\sin\theta\cos\varphi \\ -R\cos^2\theta\sin\theta\sin\varphi \\ R\sin^2\theta\cos\theta \end{pmatrix} = -\sin\theta\cos\theta \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} = -\sin\theta\cos\theta \mathbf{r}_\theta .$$

We have

$$\nabla_{\partial_\varphi}\partial_\varphi = \Gamma^\theta_{\varphi\varphi}\partial_\theta + \Gamma^\varphi_{\varphi\varphi}\partial_\varphi = \left(\frac{\partial^2 \mathbf{r}}{\partial\varphi\partial\varphi}\right)_{\text{tangent}} = -\sin\theta\cos\theta \mathbf{r}_\theta,\ i.e.$$

$\Gamma^{\theta}_{\varphi\varphi} = -\sin\theta\cos\theta, \Gamma^{\varphi}_{\varphi\varphi} = 0$.

f) *Saddle*

For saddle $z = xy$: We have $\mathbf{r}(u,v)$: $\begin{cases} x = u \\ y = v \\ z = uv \end{cases}$, $\partial_u = \mathbf{r}_u = \begin{pmatrix} 1 \\ 0 \\ v \end{pmatrix}$, $\partial_v = \mathbf{r}_v = \begin{pmatrix} 0 \\ 1 \\ u \end{pmatrix}$ It will be useful also to use the normal unit vector $\mathbf{n} = \frac{1}{\sqrt{1+u^2+v^2}} \begin{pmatrix} -v \\ -u \\ 1 \end{pmatrix}$.

Calculate:

$$\nabla_{\partial_u}\partial_u = \Gamma^u_{uu}\partial_u + \Gamma^v_{uu}\partial_v = \left(\frac{\partial^2\mathbf{r}}{\partial u^2}\right)_{\text{tangent}} = (\mathbf{r}_{uu})_{\text{tangent}} = 0 \,\text{since}\; \mathbf{r}_{uu} = 0.$$

Hence $\Gamma^u_{uu} = \Gamma^v_{uu} = 0$.

Analogously $\Gamma^u_{vv} = \Gamma^v_{vv} = 0$ since $\mathbf{r}_{vv} = 0$.

Now calculate $\Gamma^u_{uv}, \Gamma^v_{uv}, \Gamma^u_{vu}, \Gamma^v_{vu}$:

$$\nabla_{\partial_u}\partial_v = \nabla_{\partial_v}\partial_u = \Gamma^u_{uv}\partial_u + \Gamma^v_{uv}\partial_v = (\mathbf{r}_{uv})_{\text{tangent}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{\text{tangent}}$$

Using normal unit vector $\mathbf{n}$ we have: $(\mathbf{r}_{uv})_{\text{tangent}} = \mathbf{r}_{uv} - \langle \mathbf{r}_{uv}, \mathbf{n}\rangle \mathbf{n} = \Gamma^u_{uv}\partial_u + \Gamma^v_{uv}\partial_v =$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{\text{tangent}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{1+u^2+v^2}} \begin{pmatrix} -v \\ -u \\ 1 \end{pmatrix} \right\rangle \frac{1}{\sqrt{1+u^2+v^2}} \begin{pmatrix} -v \\ -u \\ 1 \end{pmatrix} =$$

$$\frac{1}{1+u^2+v^2} \begin{pmatrix} v \\ u \\ u^2+v^2 \end{pmatrix} = \frac{v}{1+u^2+v^2} \begin{pmatrix} 1 \\ 0 \\ v \end{pmatrix} + \frac{u}{1+u^2+v^2} \begin{pmatrix} 0 \\ u \\ u \end{pmatrix} = \frac{v\mathbf{r}_u + u\mathbf{r}_v}{1+u^2+v^2}.$$

Hence $\Gamma^u_{uv} = \Gamma^u_{vu} = \frac{v}{1+u^2+v^2}$ and $\Gamma^v_{uv} = \Gamma^v_{vu} = \frac{u}{1+u^2+v^2}$. $\blacksquare$

Sure one may calculate this connection as Levi-Civita connction of the induced Riemannian metric using explicit Levi-Civita formula or using method of Lagrangian of free particle.

7 *Let ∇_1, ∇_2 be two different connections. Let ${}^{(1)}\Gamma^i_{km}$ and ${}^{(2)}\Gamma^i_{km}$ be the Christoffel symbols of connections ∇_1 and ∇_2 respectively.*

a) Find the transformation law for the object : $T^i_{km} = {}^{(1)}\Gamma^i_{km} - {}^{(2)}\Gamma^i_{km}$ under a change of coordinates. Show that it is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ tensor.

b)$^{*?}$ Consider an operation $\nabla_1 - \nabla_2$ on vector fields and find its properties.

Christoffel symbols of both connections transform according the law (1). The second term is the same. Hence it vanishes for their difference:

$$T^{i'}_{k'm'} = {}^{(1)}\Gamma^{i'}_{k'm'} - {}^{(2)}\Gamma^{i'}_{k'm'} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\left({}^{(1)}\Gamma^i_{km} -{}^{(2)}\Gamma^i_{km}\right) = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}T^i_{km}$$

We see that $T^{i'}_{'km'}$ transforms as a tensor of the type $\begin{pmatrix}1\\2\end{pmatrix}$.

b) One can do it in invariant way. Using axioms of connection study $T = \nabla_1 - \nabla_2$ is a vector field. Consider

$$T(\mathbf{X}, \mathbf{Y}) = \nabla_{1\mathbf{X}}\mathbf{Y} - \nabla_{2\mathbf{X}}\mathbf{Y}$$

Show that $T(f\mathbf{X}, \mathbf{Y}) = fT(\mathbf{X}, \mathbf{Y})$ for an arbitrary (smooth) function, i.e. it does not possesses derivatives:

$$T(f\mathbf{X}, \mathbf{Y}) = \nabla_{1fX}\mathbf{Y} - \nabla_{2fX}\mathbf{Y} = (\partial_{\mathbf{X}}f)\mathbf{Y} + f\nabla_{1\mathbf{X}}\mathbf{Y} - (\partial_{\mathbf{X}}f)\mathbf{Y} - f\nabla_{2\mathbf{X}}\mathbf{Y} = fT(\mathbf{X}, \mathbf{Y}).$$

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Let $\mathbf{K}$, $\mathbf{X}$ be vector fields on manifold M, and ∇ connection. Consider the operation

$$\mathbf{K}\,, \mathbf{X} \mapsto A_{\mathbf{K}}(\mathbf{X}) = \nabla_{\mathbf{K}}\mathbf{X} - \mathcal{L}_{\mathbf{K}}\mathbf{X}\,, (\mathcal{L} \text{ is a Lie derivative}, \quad \mathcal{L}_{\mathbf{X}}\mathbf{Y} = [\mathbf{X}, Y]) \qquad (1)$$

$$a) \text{ Show that for an arbitrary function } f\,, A_{\mathbf{K}}(f\mathbf{X}) = f\mathbf{A}_{\mathbf{K}}(\mathbf{X})\,. \qquad (2)$$

This condition implies that equation (1) defines linear operaration on tangent vectors, i.e. it is well defined on tangent vectors (not vector fields) and it is linear. In other words for a given vector $\mathbf{X}_0$ tangent to manifold M at the given point $\mathbf{p}$, $\mathbf{X}_0 \in T_{\mathbf{p}}M$, consider an arbitrary vector field $\mathbf{X}$ passing via this vector, i,e, such that value of vector field at the given point $\mathbf{p}$ coincides with the vector $\mathbf{X}_0$,$\mathbf{X}\big|_{\mathbf{p}} = \mathbf{X}_0$. Condition (**) tells that the answer at the point $\mathbf{p}$ does not depend on a choice of vector field passing through vector $\mathbf{X}_0$. It depends only on the value of this vector field at the point $\mathbf{p}_0$. Indeed let two vector fields $\mathbf{X}, \tilde{\mathbf{X}}$ coincide at the point $\mathbf{p}$, i.e. the vector field $\tilde{\mathbf{X}} - \mathbf{X}$ vanishes at the point $\mathbf{p}$. Moreover Hadamard lemma [1)]. tells that in this case vector field $\tilde{\mathbf{X}} - \mathbf{X}$ is linear combination of vector fields with coefficients vanishing at the point $\mathbf{p}$: $\tilde{\mathbf{X}} - \mathbf{X} = \sum_a h_a(x)\mathbf{T}_\mathbf{a}$, where all $h_a(x)$ vanish at the point $\mathbf{p}$. Hence due to (2)

$$A_{\mathbf{K}}(\tilde{X} - \mathbf{X})|_{\mathbf{p}} = A_{\mathbf{K}}\left(\sum_a h_a(x)\mathbf{T}_\mathbf{a}\right)|_{\mathbf{p}} = \sum_a h_a(x)|_{\mathbf{p}}A_{\mathbf{K}}(\mathbf{T}_\mathbf{a}) = \mathbf{0}\,.$$

[1)] *if smooth function g vanishes at the origin, then $g = \sum_i x^i h_i(x)$, where $h_i(x)$ are also smooth.*

b) Show that linear operator $A_{\mathbf{K}}$ in equation (*) is equal to

$$A_{\mathbf{K}}(\mathbf{X}) = \nabla_{\mathbf{X}}\mathbf{K} + S(\mathbf{K},\mathbf{X})\,.$$

c) How does it look this operator for symmetric connection?

Solution:

a) Using definition of connection and properties of commutator we see that

$$A_{\mathbf{K}}(f\mathbf{X}) = \nabla_{\mathbf{K}}(f\mathbf{X}) - \mathcal{L}_{\mathbf{K}}(f\mathbf{X}) = (\partial_{\mathbf{K}}f\mathbf{X} + f\nabla_{\mathbf{K}}\mathbf{X}) - [\mathbf{K}, f\mathbf{X}] =$$

$$(\partial_{\mathbf{K}}f\mathbf{X} + f\nabla_{\mathbf{K}}\mathbf{X}) - f[\mathbf{K},\mathbf{X}] - (\partial_{\mathbf{K}}f)\mathbf{X} = f(\nabla_{\mathbf{K}}\mathbf{X} - [\mathbf{K},\mathbf{X}]) = fA_{\mathbf{K}}(\mathbf{X})\,.$$

b) Now prove the relation (2):

$$A_{\mathbf{K}}(f\mathbf{X}) = \nabla_{\mathbf{K}}(f\mathbf{X}) - \mathcal{L}_{\mathbf{K}}(f\mathbf{X}) = (\nabla_{\mathbf{K}}\mathbf{X} - \nabla_{\mathbf{X}}\mathbf{K} + [\mathbf{K},\mathbf{X}]) + (\nabla_{\mathbf{X}}\mathbf{K} = \nabla_{\mathbf{X}}\mathbf{K} + S(\mathbf{K},\mathbf{X})$$

c) In the case of symmetric connection $S \equiv 0$, hence

$$A_{\mathbf{K}}(\mathbf{X}) = \nabla_{\mathbf{X}}\mathbf{K}\,.$$