One beautiful way to calculate one action

To calculate the action of particle in homogeneous field: $H = \frac{p^2}{2m} - mgx$ is not difficult, however calculations are not too enjoyable. We will calculate here this action using just symmetires.

Let

$$W(t_1, x_1; t_2, x_2)$$

be an action of the theory. It obeys the following property:

$$W(t_1, x_1; t_2, x_2) = -W(t_2, x_2; t_1, x_1)\,. \tag{1}$$

We calculated this action straightforwardly (see the blog on July 2019). Calculate here this action using just this symmetry, Hamilton-Jacobi equation and some physical intuition.

Recall that for free particle

$$W_{\text{free}} = \frac{m(x_2 - x_1)^2}{2(t_2 - t_1)}\,.$$

Hence for a particle in homogeneous field,

$$W = W_{free}(t_1, x_1; t_2, x_2) + gW'(t_1, x_1; t_2, x_2)\,,$$

where g is acceleration.

We use the following fact: Let a function has dimension ("razmernostj") of action and

$$F = mg^{\alpha}(x_1 + x_2)^{\beta}(x_2 - x_1)^{\gamma}t^{\delta}\,.$$

Tehn

$$[F] = [mght] = kg \cdot \frac{m}{sec^2} \cdot m \cdot sec\,. \tag{2}$$

The condition (1) implies that

$$\gamma + \delta = 2k + 1$$

and condition (2) implies that

$$[F] = kg \cdot m^{\alpha+\beta+\gamma} \cdot sec^{\delta - 2\alpha} = kg \cdot m^2 \cdot sec^{-1}\,,$$

thus we see that

$$\begin{cases} \gamma + \delta = 2k + 1 \\ \alpha + \beta + \gamma = 2 \\ \delta - 2\alpha = -1 \end{cases}$$

Using this we look for the action of particle in homogeneous filed as

$$W = \underbrace{\frac{m(x_2 - x_1)^2}{2(t_2 - t_1)}}_{\text{free particle}} + c_1 mg(x_1 + x_2)(t_2 - t_1) + c_2 mg^2(t_2 - t_1)^3\,.$$

(This expression transforms in a right way and has right dimension.)

Calculate the constants c_1 and c_2. Use that W obeys Hamilton-Jacobi equation for Hamiltonian $H = \frac{p^2}{2m} - mgx$:

$$0 = \frac{\partial W(x_1,t_1;t_2,x_2)}{\partial t_2} + \left(\frac{\left(\frac{\partial W(x_1,t_1;t_2,x_2)}{\partial x_2}\right)^2}{2m} - mgx_2 \right) =$$

$$-\frac{m(x_2-x_1)^2}{2(t_2-t_1)^2} + c_1 mg(x_1+x_2) + 3c_2 mg^2 t^2 + \frac{\left(\frac{m(x_2-x_1)}{(t_2-t_1)} + c_1 mg(t_2-t_1)\right)^2}{2m} - mgx_2 =$$

$$= 2c_1 mgx_2 - mgx_2 + \left(3c_2 + \frac{c_1^2}{2}\right) mg^2 (t_2-t_1)^2 = 0\,.$$

Thus

$$c_1 = \frac{1}{2}\,, \quad c_2 = -\frac{1}{24}$$

and

$$W = W = \underbrace{\frac{m(x_2-x_1)^2}{2(t_2-t_1)}}_{\text{free particle}} + \frac{1}{2} mg(x_1+x_2)(t_2-t_1) - \frac{1}{24} mg^2 (t_2-t_1)^3\,.$$