**Two hours**

**THE UNIVERSITY OF MANCHESTER**

RIEMANNIAN GEOMETRY

XX May-XX June 2018

XX:00 – XX:00

Answer **ALL FOUR** questions in Section A (60 marks in total).
Answer **TWO** of the THREE questions in Section B (40 marks in total).
If more than TWO questions in Section B are attempted, the credit will be given for the best TWO answers.

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Electronic calculators may not be used.

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Throughout the paper, where the index notation is used, the Einstein summation convention over repeated indices is applied if it is not explicitly stated otherwise.

# SECTION A

Answer **ALL** FOUR questions

**A1.**

(a) Explain what is meant by saying that $G$ is a Riemannian metric on a manifold $M$.

Let $G = g_{ik}(x)dx^i dx^k$, $i, k = 1, \dots, n$ be a Riemannian metric on $n$-dimensional manifold $M$. Show that all diagonal components $g_{11}(x), g_{22}(x), \dots, g_{nn}(x)$ are positive functions.

(b) Consider the plane $\mathbf{R}^2$ with standard coordinates $x, y$ equipped with the Riemannian metric

$$G = (1 + x^2 + y^2)(dx^2 + dy^2)\,.$$

Consider vectors $\mathbf{A} = 2\partial_x + \partial_y$ and $\mathbf{B} = \partial_x + 2\partial_y$ attached at the point $(1, 1)$. Find the length of these vectors, and the cosine of the angle between the vectors.

[10 marks]

**A2.**

(a) Explain what is meant that a Riemannian surface is locally Euclidean.

(b) Consider a surface (the upper sheet of a cone) in $\mathbf{E}^3$

$$\mathbf{r}(h, \varphi)\colon \quad \begin{cases} x = h\cos\varphi \\ y = h\sin\varphi \\ z = h \end{cases} \quad , \quad h > 0, 0 \le \varphi < 2\pi\,.$$

Calculate the Riemannian metric on this surface induced by the canonical metric on Euclidean space $\mathbf{E}^3$.

Show that this surface is locally Euclidean.

[10 marks]

**A3.**

(a) Explain what is meant by an affine connection on a manifold.

(b) Let $\nabla$ be an affine connection on a 2-dimensional manifold $M$ such that in local coordinates $(u, v)$ all Christoffel symbols vanish except $\Gamma^u_{vv} = u$ and $\Gamma^v_{uu} = v$. Calculate the vector field $\nabla_{\mathbf{X}}\mathbf{X}$, where $\mathbf{X} = \frac{\partial}{\partial u} + u\frac{\partial}{\partial v}$.

(c) Give an example of function $f = f(u, v)$ such that $f$ does not vanish identically ($f \not\equiv 0$) and

$$\nabla_{\mathbf{X}}(f\mathbf{X}) = f\nabla_{\mathbf{X}}\mathbf{X} .$$

Justify the answeri.

[15 marks]

**A4.**

(a) Define a geodesic on a Riemannian manifold as a parameterised curve
Write down the differential equation of geodesics in terms of the Christoffel symbols.
What are the geodescis of the surface of cylindre?

(b) Explain why the acceleration vector of an arbitrary parameterised geodesic on a surface $M$ is orthogonal to the surface.
Explain why the latitude, (curve $\theta = \theta_0$ in spherical coordinates $\theta, \varphi$) is not a geodesic on the sphere in the case if $\theta_0 \neq \frac{\pi}{2}$.

[15 marks]

## SECTION B

Answer **TWO** of the THREE questions

**B5.**

(a) Explain what is meant by saying that $F$ is an isometry between two Riemannian manifolds.
Consider the plane $\mathbf{R}^2$ with coordinates $(x, y)$ and with the Riemannian metric

$$G_{(1)} = e^{-a^2(x^2+y^2)}(dx^2 + dy^2)\,,$$

and consider the same plane $\mathbf{R}^2$ with another Riemannian metric

$$G_{(2)} = be^{-u^2-v^2}(du^2 + dv^2)\,, \quad (b > 0)\,,$$

(we denote the standard coordinates by another letters $u, v$.)
Show that the map $F\colon \begin{cases} u = ax \\ v = ay \end{cases}$ between these two Riemannian manifolds is isometry in the case of $b = \frac{1}{a^2}$.

(b) Calculate the total area of the plane with respect to the metric $G_{(1)}$, and the total area of the plane with respect to the metric $G_{(2)}$.
(You may use the formula $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.)
Deduce why, in the case where $b \neq \frac{1}{a^2}$, there is no isometry between these Riemannian manifolds.

[20 marks]

**B6.**

(a) Give a detailed formulation of the Levi-Civita Theorem. In particular write down the expression for the Christoffel symbols $\Gamma^i_{km}$ of the Levi-Civita connection in terms of the Riemannian metric $G = g_{ik}(x)dx^i dx^k$.
Consider the upper half-plane, $y > 0$ in $\mathbf{R}^2$ equipped with the Riemannian metric

$$G = \frac{dx^2 + dy^2}{y^2}\,,$$

(Lobachevsky plane).
Calculate Cristoffel symbols of the Levi-Civita connection of this Riemannian manifold.

(b) Let $\nabla'$ be a symmetric connection on Lobachevsky plane such that all Christoffel symbols of this connection in coordinates $(x, y)$ vanish identically.
Explain why this connection does not preserve the Riemannian metric of Lobachevsky plane.

[20 marks]

**B7.**

(a)

(b) curvature

(c)

[20 marks]

**END OF EXAMINATION PAPER**