Homework 3.

1 Let $\{\mathbf{e},\mathbf{f}\}$ be an orthonormal basis in $\mathbf{E}^2$. Consider the following ordered pairs:
a) $\{\mathbf{f},\mathbf{e}\}$,
b) $\{\mathbf{f},-\mathbf{e}\}$,
c) $\{\frac{\sqrt{2}}{2}\mathbf{e}+\frac{\sqrt{2}}{2}\mathbf{f}, -\frac{\sqrt{2}}{2}\mathbf{e}+\frac{\sqrt{2}}{2}\mathbf{f}\}$,
d) $\{\frac{\sqrt{3}}{2}\mathbf{e}+\frac{1}{2}\mathbf{f}, \frac{1}{2}\mathbf{e}-\frac{\sqrt{3}}{2}\mathbf{f}\}$.
Show that all these ordered pairs are orthonormal bases in $\mathbf{E}^2$.
Find amongst them the bases which have the same orientation as the orientation of the basis $\{\mathbf{e},\mathbf{f}\}$.
Find amongst them the bases which have the orientation opposite to the orientation of the basis $\{\mathbf{e},\mathbf{f}\}$.

2 Let $\{\mathbf{e},\mathbf{f}\}$ be a basis in two-dimensional linear space V. Consider an ordered pair $\{\mathbf{a},\mathbf{b}\}$ such that

$$\mathbf{a}=\mathbf{f},\ \mathbf{b}=\gamma\mathbf{e}+\mu\mathbf{f}\,,$$

where γ,μ are arbitrary real numbers.

Find values γ,μ such that an ordered pair $\{\mathbf{a},\mathbf{b}\}$ is a basis and this basis has the same orientation as the basis $\{\mathbf{e},\mathbf{f}\}$.

3 Let $\{\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z\}$ be an orthonormal basis in $\mathbf{E}^3$ and let $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ be an arbitrary basis in $\mathbf{E}^3$. Show that the basis $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ either has the same orientation as the basis $\{\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z\}$, or the same orientation as the basis $\{\mathbf{e}_y,\mathbf{e}_x,\mathbf{e}_z\}$.

4 Let $\{\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z\}$ be an orthonormal basis in $\mathbf{E}^3$. Consider the following ordered triples:
a) $\{\mathbf{e}_x,\mathbf{e}_x+2\mathbf{e}_y,5\mathbf{e}_z\}$,
b) $\{\mathbf{e}_y,\mathbf{e}_x,5\mathbf{e}_z\}$,
c) $\{\mathbf{e}_y,\mathbf{e}_x,-5\mathbf{e}_z\}$,
d) $\{\frac{\sqrt{3}}{2}\mathbf{e}_x+\frac{1}{2}\mathbf{e}_y, -\frac{1}{2}\mathbf{e}_x+\frac{\sqrt{3}}{2}\mathbf{e}_y,\mathbf{e}_z\}$,
e) $\{\mathbf{e}_y,\mathbf{e}_x,\mathbf{e}_z\}$,
f) $\{\mathbf{e}_y,\mathbf{e}_x,-\mathbf{e}_z\}$.

Show that all ordered triples a),b),c),d),e),f) are bases.
Show that the bases a), c), d) and f) have the same orientation as the basis $\{\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z\}$, and the bases b) and e) have the orientation opposite to the orientation of the basis $\{\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z\}$.
Show that bases d), e) and f) are orthonormal bases and bases a), b) and c) are not orthonormal bases.

5 Let $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$ be a basis in vector space V. Show that ordered triples $\{\mathbf{f},\mathbf{e}+2\mathbf{f},3\mathbf{g}\}$ and $\{\mathbf{e},\mathbf{f},2\mathbf{f}+3\mathbf{g}\}$ are bases and these bases have opposite orientations.

6 Let $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$ be an orthonormal basis in Euclidean space $\mathbf{E}^3$. Consider a linear operator P in $\mathbf{E}^3$ such that

$$\mathbf{e}'=P(\mathbf{e})=\mathbf{e},\quad \mathbf{f}'=P(\mathbf{f})=\frac{\sqrt{2}}{2}\mathbf{f}+\frac{\sqrt{2}}{2}\mathbf{g},\quad \mathbf{g}'=P(\mathbf{g})=-\frac{\sqrt{2}}{2}\mathbf{f}+\frac{\sqrt{2}}{2}\mathbf{g}\,.$$

Write down the transition matrix from the basis $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$ to the ordered triple $\{\mathbf{e}',\mathbf{f}',\mathbf{g}'\}$.
Show that P is an orthogonal operator.
Show that orthogonal operator P preserves the orientation of $\mathbf{E}^3$.
Find an axis of the rotation and the angle of the rotation.

7 Consider a linear operator P_1 in $\mathbf{E}^3$ such that it transforms the orthonormal basis $\{\mathbf{e},\mathbf{f},\mathbf{g}\}$ into the orthonormal basis $\{\mathbf{f},\mathbf{e},\mathbf{g}\}$. Consider also a linear operator P_2 such that it is the reflection operator with respect to the plane spanned by vectors $\mathbf{e}$ and $\mathbf{f}$.
Is the operator P_1 a rotation or reflection operator?
Do operators P_1, P_2 preserve orientation?
Show that operator $P=P_2\circ P_1$ is a rotation operator.
Find an angle and the axis of this rotation.