Geometry of diff.equations

§1 Necessary mathematics from Arnold's book

`I began to understand the pages in Arnold on diff. equations.....`

Here it is:

Definition Let ω be 1-form on M which does not vanish. . We say that it is *contact* form if

$$\text{2-form } d\omega \text{ is non-degenerate on the plane } \omega = 0 \text{ in } TM$$

Since $d\omega$ is not degenerate on $\omega = 0$ and $\omega \not\equiv 0$ them $\dim M = 2k + 1$.

Theorem Contact form is defined up to a function (Valya had a talk on it!) If ω is contact form and $f \neq 0$ then $f\alpha$ is contact also.

Let $\mathcal{K}$ be a distributions of $2n$-dimensional planes in TM such that ω vanishes on these planes.

We say that this distribution is a contct structure *

Theorem Let N be a submanifold of M which is an integral submanifold (not necessarily maximal) of contact distribution $\mathcal{K}$, i.e. for every point on N the tangent vectors belong to this distribution. Then $\dim N \leq n$, where $\dim M \leq 2n + 1$

`Proof` Let ω be an arbitrary non-zero form which vanishes at $\mathcal{K}$. Since a form ω vanishes on vectors tangent to the manifold N, the form $d\omega$ vanishes on these vectors also:

$$\iota\colon N \subset M, \qquad d\left(\iota^*\omega\big|_N\right) = \iota^* dw\big|_N = 0\,.$$

Hence two arbitrary vectors are orthogonal to each other with respect to this form. If dimension of tangent plane is bigger than n then there exist at least two vectors which are not orthogonal, since $d\omega$ is not degenerate. Now we apply this mathematics to the differential equations.

§2 Geometry of first order equation

Let J^1M be a space of first jets of functions on manifold M. Cooridnates on J^1M are (p_i, q^j, u), where q^j are coordinates on M. Jet of every function $u = u(x)$ has coordinates $\left(p_i = \frac{\partial u}{\partial x q^i}, q^i, u\right)$.

Consider $\mathcal{C}$, the Cartan distribution of $2n$-dimensional planes in J^1M defined by the form $\omega = p_i dq^i - du$

$$\mathcal{C}_{\mathbf{p}} \subset T_{\mathbf{p}} J^1M = \{T_{\mathbf{p}}(J^1M) \ni \mathbf{X}\colon\ \omega(\mathbf{X}) = 0\}\,,$$

Vector field

$$M^i \frac{\partial}{\partial q^i} + N_i \frac{\partial}{\partial p_i} + A \frac{\partial}{\partial u} \text{ belongs to Cartan distribution } \mathcal{C} \text{ if } A = p_i M^i\,.$$

* One can say that distribution of hyperplanes defines constant structure if an 1-form which vanishes this dstribution is non-degenerated on it

$\mathcal{C}$ is non-integrable distribution. It is a *contact structure* and the form $\omega_C = p_i dq^i = du$ is a contact form since

$$d\omega\big|_{\omega=0} = dp_i \wedge dq^i$$

is non-degenerte form. Consider differential equation,

$$\mathcal{E}\colon F(p,q,u) = 0\,.$$

Differential equation is sumbmanifold of codimension 1 in the space $J^1(M)$.

The Cartan distribution $\mathcal{C}$ of hyperplanes on J^1M defines distribution $\mathcal{C}(\mathcal{E})$ in $T\mathcal{E}$:

$$\mathcal{C}(E) = \mathcal{C} \cap T\mathcal{E}\,.$$

$$\mathbf{X} = M^i \frac{\partial}{\partial q^i} + N_i \frac{\partial}{\partial p_i} + A\frac{\partial}{\partial u} \in \mathcal{C}(\mathcal{E}) \text{ if } A = p_i M^i \,\&\, \left(M^i \frac{\partial}{\partial q^i} + N_i \frac{\partial}{\partial p_i} + A\frac{\partial}{\partial u}\right) F(p,q,u)\big|_{F=0} = 0\,.$$

This distribution is not integrable.

The solution of differential equation (1) is the maximal integral of the distribution.

What is the dimension of N?

Let N be an arbitrary solution. Calculate its dimension. Any tangent plane to N belongs to Cartan distribution and is tangent to $\mathcal{E}$. Consider an arbitrary point $\mathbf{p} \in N$, and consider the plane $\alpha = \alpha_P = T_{\mathbf{p}}N$. Calculate $\dim T_{\mathbf{p}}N$.

Fact $\dim T_{\mathbf{p}}N \leq n$. We proved it above, but we repeat the considerations again. Consider the form ω_C on and its differential the form $d\omega_C$ on TN. The form ω_C vanishes on TN, hence the form $d\omega_C$ vanishes on TN also. On the other hand if $\dim N = p > n$, then this is not true, since form $d\omega_C$ has rank 2. In coordinates: in the vicinity of the point $\mathbf{p}$ one can choose local coordinates $\xi^1, \ldots, \xi^{2n+1}$ such that in these coordinates surface N is given by equations $\xi^{p+1} = \ldots = \xi^{2n+1} = 0$ and $d\omega_C = d\xi^1 \wedge d\xi^2 + \ldots + d\xi^{2n-1} \wedge d\xi^{2n}$ (generalised Darboux Theorem). This contradicts to the condition $d\omega_C\big|_N \equiv 0$ on N. (In the case if dimension N is maximal possible it is Lagrangian)

Now we consider some properties of hypersurfces, (recall that hypersurface is differential equation)

DEFINITION Let $\mathcal{E}$ be an arbitrary hypersurface in M: $\dim \mathcal{E} = 2n$). (Hypersurface $\mathcal{E}$ may define differential equation $F(q,u,p) = 0$ ($\mathcal{E}\colon F = 0$)) We say that the hypersurface $\mathcal{E}$ is *non-characteristic hypersurface* if at every point $\mathbf{p}$ the contact hyperplane (hyperplane of distribution $\mathcal{C}$) and the tangent hyperplane are transversal:

$$\mathcal{C}_{\mathbf{p}} \oplus T_p t M' = T_{\mathbf{p}} M \Leftrightarrow \dim\,(\mathcal{C}_{\mathbf{p}} \cap M'\mathbf{p}) = 2n-1\,.$$

DEFINITION Let $\mathcal{E}$ be an arbitrary non-characteristic hypersurface in $J^1(M)$.

For an arbitrary point $\mathbf{p} \in E$ consider the $2n-1$ dimensional subspace $\Pi_{\mathbf{p}}$ of tangent vectors which belong to contact space:

$$\Pi_{\mathbf{p}} = \mathcal{C}_{\mathbf{p}} \cap T_{\mathbf{p}} E \,, \qquad , \dim \Pi_{\mathbf{p}} = 2n-1$$

Thus we define the distribution of $2n-1$-dimensional planes on non-characteristic differential equation $F=0$ (this equation defines the surface $\mathcal{E}$: $\mathcal{E}$: $F=0$).

Now: `VERY IMPORTANT STEP:` On every characteristic plane $\Pi_{\mathbf{p}} = \mathcal{C}_{\mathbf{p}} \cap T_{\mathbf{p}} E$, $d\omega_C = \sum_{i=1}^n dp_i \wedge dq^i$, and since the dimension of $\Pi_{\mathbf{p}}$ is equal to $2n-1$ then one can define the direction, such that dw vanishes along this direction, and it is unique!

Problem Let $\sigma in V^*$ be covector in V^* and let M_σ, me hyperplane in V which is orthogonal to the covector $\mathbf{a}$:

$$M_{\mathbf{a}} = \{\mathbf{x} \in V \colon \quad \sigma(\mathbf{x}) = 0\}$$

Proposition *Subspaces M_{σ_1} and M_{s_2} are transversal if and only if vectors σ_1, σ_2 qre linearly independent, (not proportional to each other)*

Theorem *Let $F = F(u, p_i, q^i) = 0$ defines hypersurface $\mathcal{E} = \mathcal{E}_F$, differential equation. Let covector dF defined by equation, and covector ω_C of contact distribution, are not proportional* Then*

a) hypersurface $F=0$ is non-charcteristical

b) the charcteristic direction l is defined by the vector $\mathbf{X}_F$

Remark *If we change $F \to \lambda F$ where a function λ is not-degenerate on the surface $F=0$, then the condition of linear (in)dependence of covectors dF and ω_C, and the characteristic direction $\mathbf{X}_F$ will not change.*

* *this condition does not depend on choice of function F defining the surface $\mathcal{E}$*