

Homework 4

1 Calculate the Christoffel symbols of the canonical flat connection in \mathbf{E}^3 in

a) cylindrical coordinates $(x = r \cos \varphi, y = r \sin \varphi, z = h)$,

b) spherical coordinates.

(For the case b) try to make calculations at least for components $\Gamma_{rr}^r, \Gamma_{r\theta}^r, \Gamma_{r\varphi}^r, \Gamma_{\theta\theta}^r, \dots, \Gamma_{\varphi\varphi}^r$)

2 Let ∇ be an affine connection on a 2-dimensional manifold M such that in local coordinates (u, v) it is given that $\Gamma_{uv}^u = v, \Gamma_{uv}^v = 0$.

Calculate the vector field $\nabla_{\frac{\partial}{\partial u}} \left(u \frac{\partial}{\partial v} \right)$.

3 Let ∇ be an affine connection on the 2-dimensional manifold M such that in local coordinates (u, v)

$$\nabla_{\frac{\partial}{\partial u}} \left(u \frac{\partial}{\partial v} \right) = (1 + u^2) \frac{\partial}{\partial v} + u \frac{\partial}{\partial u}.$$

Calculate the Christoffel symbols Γ_{uv}^u and Γ_{uv}^v of this connection.

4 a) Consider a connection such that its Christoffel symbols are symmetric in a given coordinate system: $\Gamma_{km}^i = \Gamma_{mk}^i$.

Show that they are symmetric in an arbitrary coordinate system.

b)* Show that the Christoffel symbols of connection ∇ are symmetric (in any coordinate system) if and only if

$$\nabla_{\mathbf{X}} \mathbf{Y} - \nabla_{\mathbf{Y}} \mathbf{X} - [\mathbf{X}, \mathbf{Y}] = 0,$$

for arbitrary vector fields \mathbf{X}, \mathbf{Y} .

c)* Consider for an arbitrary connection the following operation on the vector fields:

$$S(\mathbf{X}, \mathbf{Y}) = \nabla_{\mathbf{X}} \mathbf{Y} - \nabla_{\mathbf{Y}} \mathbf{X} - [\mathbf{X}, \mathbf{Y}]$$

and find its properties.

Exercises 4b) and 4c) are for students who know commutator of vector fields. ■

5 Let ∇_1, ∇_2 be two different connections. Let ${}^{(1)}\Gamma_{km}^i$ and ${}^{(2)}\Gamma_{km}^i$ be the Christoffel symbols of connections ∇_1 and ∇_2 respectively.

a) Find the transformation law for the object : $T_{km}^i = {}^{(1)}\Gamma_{km}^i - {}^{(2)}\Gamma_{km}^i$ under a change of coordinates. Show that it is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ tensor.

b) Consider an operation $\nabla_1 - \nabla_2$ on vector fields and find its properties.

6 * a) Consider $t_m = \Gamma_{im}^i$. Show that the transformation law for t_m is

$$t_{m'} = \frac{\partial x^m}{\partial x^{m'}} t_m + \frac{\partial^2 x^r}{\partial x^{m'} \partial x^{k'}} \frac{\partial x^{k'}}{\partial x^r}.$$

b) [†] Show that this law can be written as

$$t_{m'} = \frac{\partial x^m}{\partial x^{m'}} t_m + \frac{\partial}{\partial x^{m'}} \left(\log \det \left(\frac{\partial x}{\partial x'} \right) \right) .$$

The exercise 7 is for the week 8

7 Calculate Christoffel symbols of the connection induced on the surface M in \mathbf{E}^n equipped with canonical flat connection.

- a) $M = S^1$ in \mathbf{E}^2
- b) M — parabola $y = x^2$ in \mathbf{E}^2
- c) cylinder, cone and sphere in \mathbf{E}^3 .
- d) saddle $z = xy$ in \mathbf{E}^3