`Area of sphere (delicate calculations)`

Denote by σ_n the rea of unit sphere $x_1^2+x_2^2+\ldots+x_{n+1}^2=1$ in $\mathbf{E}^{n+1}$. The standard way to calculate σ_n is the following: Consider

$$I_n=\int_{\mathbf{E}^{n+1}}\exp-\left(x_1^2+x_2^2+\ldots+x_{n+1}^2\right)dx_1dx_2\ldots dx_n\,. \tag{01}$$

On one hand

$$I_n=\left(\int_{\infty}^{\infty}\exp\left(-x^2\right)dx\right)^{n+1}=\left(\sqrt{\pi}\right)^{\frac{n+1}{2}}\,, \tag{02}$$

and on the other hand

$$I_n=\int_0^\infty\exp\left(-r^2\right)\sigma_n r^n dr\,. \tag{03}$$

since the area of the sphere S^n of radius r is equal to $r^n\sigma_n$.

Comparing these integrals we come to

$$I_n=\int_0^\infty\exp\left(-r^2\right)\sigma_n r^n dr=\sigma_n\int_0^\infty e^{-u}u^{\frac{n}{2}}\frac{du}{2\sqrt{u}}=\sigma_n\frac{1}{2}\Gamma\left(\frac{n+1}{2}\right)=\pi^{\frac{n+1}{2}}\,, \tag{04}$$

thus:

$$\sigma_n=\frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)\,.} \tag{05}$$

Present now more 'delicate considerations' to calculate the area

Let ω be n-form in $\mathbf{E}^{n+1}$ and let C be an n-dimensional hypersurface $C=C_F$ defined by equation $F=0$:

$$C\colon\ \{\mathbf{x},F(\mathbf{x})=0\}\,.$$

Then

$$\int_C w=\pm\int\delta(F)dF\wedge\omega \tag{1.1}$$

THe right hand side is the $n+1$-form $dF\wedge\omega$

Exercise: Prove this formula

it Note what happens if you change $F\to gF$???

Now consider the following n-form in $\mathbf{E}^{n+1}$

$$\sigma=\frac{1}{r}\left(x_1dx_2\wedge dx_3\wedge\ldots\wedge dx_{n+1}-x_2dx_1\wedge dx_3\wedge\ldots\wedge dx_{n+1}+\right.$$

$$\left.+x_3dx_1\wedge dx_2\wedge dx_4\ldots\wedge dx_{n+1}+(-1)^n x_{n+1}dx_1\wedge dx_2\wedge dx_3\ldots\wedge dx_n)\right) \tag{1.2}$$

E.g. for $n=2$

$$\sigma=\frac{1}{r}\left(xdy\wedge dz+ydz\wedge dx+zdx\wedge dy+\right)$$

Exercise: Show that for every domain D in sphere

$$\int_D \sigma = \text{Area of this domain}$$

Idea of solution: first note that the form σ is invariant with respect to group $SO(n+1)$ and to calculate the coefficient check its value on the "vertical" tangent plane.

Remark Please do not think that there exist s a form which gives area of an arbitrary hypersurface.

Why?

Now continue calculations

Consider the sphere of radius a:

$$F(x,y,z) = x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = a^2 \,.$$

Use (1.1) we come to

$$\text{Area of sphere in } \mathbf{E}^{n+1} \text{ is equal to} = \int \delta(F - a^2) dF \wedge \sigma$$

We have

$$dF \wedge \sigma = 2r(dx_1 \wedge dx_2 \ldots dx_n \wedge dx_{n+1}) \,.$$

Hence we come to

$$\sigma_n = 2 \int \delta(F - 1) dv \,.$$

where $dv = dx_1 \wedge dx_2 \ldots dx_n \wedge dx_{n+1}$

To calculate the last integral, we shake it: Consider

$$L(t) = \sigma_n = \int \delta(F - t) dv \,.$$

and calculate $\int_{-\infty}^{\infty} L(t) e^{-xt}$