Solutions of Homework 8

1 *A point moves in $\mathbf{E}^2$ along an ellipse with the law of motion $x = a\cos t$, $y = b\sin t$, $0 \le t < 2\pi$, $(0 < b < a)$. Find the velocity and acceleration vectors. Find the points of the ellipse where the angle between velocity and acceleration vectors is acute. Find the points where speed attains its maximum value.*

Calculate velocity and acceleration vectors

$$\mathbf{v} = \mathbf{r}_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} -a\sin t \\ b\cos t \end{pmatrix}, \quad \mathbf{a} = \mathbf{r}_{tt} = \begin{pmatrix} x_{tt} \\ y_{tt} \end{pmatrix} = \begin{pmatrix} -a\cos t \\ -b\sin t \end{pmatrix}.$$

We see that acceleration is collinear to $\mathbf{r}$: $\mathbf{a} = -\mathbf{r}$.

The scalar product of these vectors is equal to $(\mathbf{v}, \mathbf{a}) = |\mathbf{v}||\mathbf{a}|\cos\alpha = v_x a_x + v_y a_y = (a^2 - b^2)\sin t\cos t$, where α is angle between velocity and acceleration vectors.

Speed is increasing $\Leftrightarrow$ angle α is acute $\Leftrightarrow$ $(\mathbf{v}, \mathbf{a}) > 0 \Leftrightarrow \sin t\cos t > 0 \Leftrightarrow 0 \le t \le \pi/2$ or $\pi < t < 3\pi/2$.

Speed is decreasing $\Leftrightarrow$ angle α is obtuse $\Leftrightarrow$ $(\mathbf{v}, \mathbf{a}) < 0 \Leftrightarrow \sin t\cos t < 0 \Leftrightarrow \pi/2 \le t \le \pi$ or $3\pi/2 < t < 2\pi$.

Speed is attains its maximum when $t = \frac{\pi}{2}, \frac{3\pi}{2}$ and speed attains its minimum when $t = 0, \pi$.

(At these points the acceleration is orthogonal to velocity vector and the scalar product is equal to zero).

2 *Find a natural parameter for the following interval of the straight line*

$$C\colon \begin{cases} x = t \\ y = 2t + 1 \end{cases}, \quad 0 < t < \infty$$

Calculate a curvature of the straight line C.

We know that a natural parameter $s(t)$ measures the length of the arc of the curve between a point $\mathbf{r}(t)$ and initial point. Take the point $t = 0$: $x = 0, y = 1$ as initial point.

$$s(t) = \text{length of the interval of the line between point } (0,1) \text{ and point } (t, 2t+1)$$

If α is the angle between the line and the x-axis then $s(t) = t/\cos\alpha$. $\cos\alpha = \frac{1}{\sqrt{1+2^2}} = \frac{1}{\sqrt{5}}$. Hence $s(t) = t\sqrt{5}$. One comes to the same answer by the straightforward integration:

$$s(t) = \int_0^t \sqrt{x_\tau^2 + y_\tau^2}\,d\tau = \int_0^t \sqrt{1 + 2^2}\,d\tau = t\sqrt{5}\,.$$

If we take another point as the initial point then natural parameter will change by a constant: E.g. if we take the initial point $(1,3)$ $(t = 1)$ then we have the new natural parameter:

$$s'(t) = \int_1^t \sqrt{x_\tau^2 + y_\tau^2}\,d\tau = \int_1^t \sqrt{5}\,d\tau = \sqrt{5}(t-1) = s(t) - \sqrt{5}\,.$$

Usually if a curve $\mathbf{r}(t)$ is given by parameters $t \in [t_1, t_2]$ one takes initial a point $\mathbf{r}(t_1)$ and

$$s(t) = \int_0^t \sqrt{x_\tau^2 + y_\tau^2}\,d\tau\,.$$

The curvature of straight line is equal to zero. This is evident, but one can see it from the formula $k = \frac{|\mathbf{a}_\perp|}{|v|^2}$ since for $C\colon \begin{cases} x = t \\ y = 2t + 1 \end{cases}$ velocity vector $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is constant, hence the acceleration equals zero and the normal component of the acceleration is equal to zero too. One can see it also using the definition: in natural parameterisation $\begin{cases} x = \frac{s}{\sqrt{5}} \\ y = 2\frac{s}{\sqrt{5}} + 1 \end{cases}$. We see that acceleration is equal to zero in a natural parameterisation.

3 *Find a natural parameter for the curve $x^2+y^2=6x+8y$ in $\mathbf{E}^2$.*

Write down the equation of this curve in natural parameterisation.

Calculate the curvature of this curve. This curve is a circle of radius 5 with the centre at the point $(3,4)$. Indeed

$$x^2+y^2=6x+8y \Leftrightarrow (x-3)^2+(y-4)^2=25. \quad \mathbf{r}(t)\colon \begin{cases} x=3+5\cos t \\ y=4+5\sin t \end{cases} \quad 0\le t<2\pi$$

Take a point $(8,4)$ $(t=0)$ as the initial point. t is the angle of the arc. The length of the arc from the initial point to the point $\mathbf{r}(t)$ equal to $s=R\varphi=5t$. Hence $t=\frac{s}{5}$ and we have in natural parameterisation

$$\mathbf{r}(t)\colon \begin{cases} x=3+5\cos\frac{s}{5} \\ y=4+5\sin\frac{s}{5} \end{cases} \quad 0\le s<10\pi=\text{length of the circle}\,.$$

The curvature of the circle equal to $k=\frac{1}{R}$. This is basic. But we can again and again calculate it using the definition:

$$\mathbf{a}(s)=\frac{d^2\mathbf{r}(s)}{ds^2}=\frac{d^2}{ds^2}\begin{pmatrix}3+5\cos\frac{s}{5}\\ 4+5\sin\frac{s}{5}\end{pmatrix}=-\frac{1}{5}\begin{pmatrix}\cos\frac{s}{5}\\ \sin\frac{s}{5}\end{pmatrix}. \quad k=|\mathbf{a}(s)|=\frac{1}{5}\,.$$

4 *Let C: $\mathbf{r}=\mathbf{r}(t)$, $0\le t\le 2$ be a curve in $\mathbf{E}^2$ such that at an arbitrary point of this curve the velocity and acceleration vectors $\mathbf{v}(t)$ and $\mathbf{a}(t)$ are orthogonal to each other and*

$$\mathbf{v}(t)\big|_{t=0}=\begin{pmatrix}3\\4\end{pmatrix}.$$

Find the length of this curve.

The speed is constant because the acceleration is orthogonal to the velocity vector: $\frac{d}{dt}|\mathbf{v}|^2=\frac{d}{dt}(\mathbf{v},\mathbf{v})=2(\mathbf{v},\mathbf{a})=0$. Hence the length of the curve is equal to $|v|t=\sqrt{3^2+4^2}\cdot 2=10$.

5 *Consider the following curve (a helix):*

$$\mathbf{r}(t)\colon \quad \begin{cases} x(t)=R\cos\Omega t \\ y(t)=R\sin\Omega t \\ z(t)=ct \end{cases}.$$

Find velocity and acceleration vector of this curve.

Find the curvature of this helix

Calculate velocity and acceleration vectors:

$$\mathbf{v}(t)=\frac{d\mathbf{r}(t)}{dt}=\begin{pmatrix}-R\Omega\sin t\\ R\Omega\cos t\\ c\end{pmatrix},|\mathbf{v}|=\sqrt{R^2\Omega^2+c^2}\,,\mathbf{a}(t)=\frac{d\mathbf{v}(t)}{dt}=\frac{d^2\mathbf{r}(t)}{dt^2}=\begin{pmatrix}-\Omega^2R\cos t\\ -\Omega^2R\sin t\\ 0\end{pmatrix},|\mathbf{a}|=\Omega^2R\,.$$

The scalar (inner) product of velocity and acceleration vectors is equal to zero: $(\mathbf{v}(t),\mathbf{a}(t))=0$, i.e. these vectors are orthogonal. Hence the projection of the acceleration vector onto the velocity vector (the tangent vector to the curve) is equal to zero. Thus tangential acceleration is equal to zero. (Note that speed $|\mathbf{v}|$ is constant. This also implies that tangential acceleration is equal to zero.)

One can see that helix is contained in the surface of cylinder $x^2+y^2=R^2$ and that the acceleration is orthogonal to surface of the cylinder.

In this exercise we have to calculate the curvature of the curve in three-dimensional Euclidean space. So we need to use the formula

$$k(t)=\frac{\text{Area of parallelogram formed by vectors }\mathbf{v},\mathbf{a}}{|\mathbf{v}|^3}=\frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$$

We already calculated the velocity and acceleration vectors for the helix (see exercise 3)

We already noticed that acceleration is orthogonal to velocity vector, since their scalar product is equal to zero. Hence

$$|\mathbf{v}\times\mathbf{a}| = |\mathbf{v}|\cdot|\mathbf{a}| = \Omega^2 R\sqrt{\Omega^2R^2+c^2}\,.$$

and the curvature is equal to

$$k = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{|\mathbf{v}||\mathbf{a}|}{|\mathbf{v}|^3} = \frac{|\mathbf{a}|}{|\mathbf{v}|^2} = \frac{\Omega^2R}{\Omega^2R^2+c^2} \qquad (*).$$

Another solution we could calculate curvature using the formula $k = \frac{|\mathbf{a}_\perp|}{|\mathbf{v}|^2}$. We already know that tangential acceleration is equal to zero, hence $\mathbf{a} = \mathbf{a}_{norm}$ and

$$k = \frac{|\mathbf{a}_\perp|}{|\mathbf{v}|^2} = k = \frac{|\mathbf{a}|}{|\mathbf{v}|^2}$$

We come to the formula (*).

6 *Calculate the curvature of the parabola $x = t, y = mt^2$ $(m > 0)$ at an arbitrary point. Let s be a natural parameter on this parabola. Show that the integral $\int_0^\infty k(s)ds = \int_0^\infty k(t)|\mathbf{v}(t)|dt$ and calculate this integral.*

It is not practical to use the definition of curvature for calculations. It is much more practical to use the formula for curvature in an Tarbitrary parameterisation:

$$k(t) = \frac{|x_ty_{tt}-y_tx_{tt}|}{(x_t^2+y_t^2)^{3/2}} = \frac{|1\cdot 2m - 2mt\cdot 0|}{(1^2+(2mt)^2)^{3/2}} = \frac{2m}{(1+4m^2t^2)^{3/2}},\ \ (m>0)\,.$$

We see that the curvature at the point (t, mt^2) is equal to $k(t) = \frac{2m}{(1+m^2t^2)^{3/2}}$ $(a > 0)$.

(Curvature is positive by definition. If $m < 0$, then $k(t) = \frac{-2m}{(1+4m^2t^2)^{3/2}}$).

To show that $\int k(s)ds = \int_0^\infty k(t)|\mathbf{v}(t)|dt$, where s is natural parameter, use the fact that $\frac{ds(t)}{dt} = |\mathbf{v}(t)|$. Hence

$$\int k(s)ds = \int k(s(t))\frac{ds(t)}{dt}dt = \int k(t)|\mathbf{v}(t)|dt$$

To calculate the integral $\int_0^\infty k(t)|\mathbf{v}(t)|$ use the results of the previous exercise:

$$\int_0^\infty k(t)|\mathbf{v}(t)| = \int_0^\infty \frac{|x_tty_t - y_{tt}x_t|}{(x_t^2+y_t^2)^{\frac{3}{2}}}\sqrt{x_t^2+y_t^2}dt =$$

$$\int_0^\infty \frac{|x_tty_t - y_{tt}x_t|}{x_t^2+y_t^2}dt = \int_0^\infty \frac{2m}{(1+4m^2t^2)}dt = \int_0^\infty \frac{du}{(1+u^2)}du = \arctan u|_0^\infty = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

Another solution:

In fact the answer depends only on the "boundaries" of the curve: One can see that

$$k(t)|\mathbf{v}(t)| = \frac{d}{dt}\varphi(t),$$

where $\varphi(t)$ is the angle between the velocity vector and a given direction . One can see this also by straightforward calculation:

$$\pm k(t)|\mathbf{v}(t)| = \frac{x_ty_{tt}-x_{tt}y_t}{x_t^2+y_t^2} = \frac{d}{dt}\arctan\frac{y_t}{x_t}$$

Hence $\int k(s)ds = \varphi|_0^{+\infty} = \pi/2$. (see in detail appendix to lecture notes)

7 Consider the parabola

$$\mathbf{r}(t)\colon \begin{cases} x = v_x t \\ y = v_y t - \frac{gt^2}{2} \end{cases} .$$

(It is the path of a point moving under gravity with initial velocity $\mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$.) Calculate the curvature at the vertex of this parabola.

To calculate the curvature one has to perform the same calculations as in the exercise 5:

$$k(t) = \frac{|x_t y_{tt} - y_t x_{tt}|}{(x_t^2 + y_t^2)^{3/2}} = \frac{|v_x \cdot (-g)|}{\left((v_x^2 + (v_y - gt)^2)\right)^{3/2}}$$

At the vertex of this parabola, the vertical component of velocity is equal to zero. Hence curvature at the vertex is equal to

$$k = \frac{|v_x \cdot (-g)|}{\left((v_x^2 + (v_y - gt)^2)\right)^{3/2}}|_{v_y = gt} = \frac{g}{v_x^2} .$$

Another solution: The curvature at any point is equal to the ratio of the normal acceleration to the square of the velocity: $k = \frac{|\mathbf{a}_\perp|}{v^2}$. The normal acceleration at the vertex is equal to g. Hence $k = \frac{g}{v_x^2}$.

The answer in fact immediately follows from considerations of classical mechanics: If curvature in the vertex is equal to k then radius of the circle which has second order touching is equal to $R = \frac{1}{k}$ and centripetal acceleration is equal to $a = \frac{v_x^2}{R}$. On the other hand $a = g$. Hence $R = \frac{v_x^2}{g}$ and $k = \frac{g}{v_x^2}$.

Remark Note that $v_x = \sqrt{\frac{g}{k}} = \sqrt{Rg}$. if we take $R \approx 6400km$ (radius of the Earth) then $v_x \approx 8km\,\mathrm{sec}$— if a point has this velocity then it will become satellite of the Earth (we ignore resistance of the atmosphere).

8 *Consider the ellipse* $x = a\cos t, y = b\sin t$ $(a, b > 0,\ 0 \le t < 2\pi)$ *in* $\mathbf{E}^2$. *Calculate the curvature* $k(t)$ *at an arbitrary point of this ellipse.*

Find the radius of a circle which has second order touching with the ellipse at the point $(0, b)$.

† *Calculate* $\int k(s)ds$ *over the ellipse where* s *is a natural parameter.*

For the ellipse $\mathbf{r}(t)\colon x = a\cos t, y = b\sin t$ velocity vector $\mathbf{v}(t) = (-a\sin t, b\cos t)$, acceleration vector $\mathbf{a}(t) = (-a\cos t, -b\sin t)$ and for curvature

$$k(t) = \frac{|v_x a_y - v_y a_x|}{(v_x^2 + v_y^2)^{3/2}} = \frac{ab\sin^2 t + ab\cos^2 t}{(a^2\sin^2 t + b^2\cos^2 t)^{\frac{3}{2}}} = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{\frac{3}{2}}} .$$

The value of parameter t at the point $(0, b)$ is $t = \frac{\pi}{2}$. The curvature of the ellipse at the point $(0, b)$ is equal to $k(t)|_{t=\frac{\pi}{2}} = \frac{ab}{(a^2)^{\frac{3}{2}}} = \frac{b}{a^2}$. The circle has the same curvature $k = \frac{1}{R}$. Hence its radius is equal to $\frac{a^2}{b}$.

† It follows from the previous exercise that $\int k(s)ds = \int k(t)|\mathbf{v}(t)|dt$. One can calculate this integral using explicit formulae for curvature and velocity. On the other hand we already know that

$$\int_C k(s)ds = \int_C k(t)|v(t)|dt = \int_C \frac{d}{dt}\arctan\frac{y_t}{x_t} = \Delta\varphi = 2\pi.$$

9 *Calculate the curvature of the following curve (latitude on the sphere)*

$$\begin{cases} x = R\sin\theta_0\cos\varphi(t) \\ y = R\sin\theta_0\sin\varphi(t) \\ z = R\cos\theta_0 \end{cases}, \text{ where } \varphi(t) = t, 0 \le t < 2\pi .$$

The curve under consideration is the circle of the radius $r = R\sin\theta_0$. Hence its curvature is equal to $k = \frac{1}{R\sin\theta_0}$.

10† *Show that the curvature of an arbitrary curve on the sphere of radius R is greater than or equal to $\frac{1}{R}$.*

Let $\mathbf{r}(s)$ be a curve on the sphere of the radius R in natural parameterisat ion. We have that the curve is on the sphere. Hence $\langle \mathbf{r}(s), \mathbf{r}(s) \rangle = R^2$. Differentiate it with respect to s we come to $\left\langle \frac{d\mathbf{r}(s)}{ds}, \mathbf{r}(s) \right\rangle = 0$. Differentiate it again over s we come to

$$0 = \frac{d}{ds}\left(\left\langle \frac{d\mathbf{r}(s)}{ds}, \mathbf{r}(s) \right\rangle\right) = \left\langle \frac{d^2\mathbf{r}(s)}{ds^2}, \mathbf{r}(s) \right\rangle + \left\langle \frac{d\mathbf{r}(s)}{ds}, \frac{d\mathbf{r}(s)}{ds} \right\rangle = kR\cos\Psi + 1 = 0 \Rightarrow k \geq \frac{1}{R}.$$

(Here Ψ is the angle between the acceleration vector and the vector $\mathbf{r}$.)