Introduction to Geometry (20222)

2008

COURSEWORK

This assignment counts for 20% of your marks.

Solutions are due by 10 April

Write solutions in the provided spaces.

STUDENTS'S NAME:

1

a) Consider the parabola $y = 1 + x^2$ in $\mathbf{E}^2$, a point $M_t = (t, 1 + t^2)$ on the parabola ($t \in (-\infty, \infty)$) and the point $B = (1, 1)$. Denote by l_t the straight line passing through the points B and M_t.

Find an equation of the line l_t. Calculate the coordinates of the second point of intersection of the line l_t with the parabola.

Find values of parameter t such that the line l_t touches the parabola.

b) In the Euclidean space $\mathbf{E}^2$ consider two points $A = (-1, 2)$ and $B = (4, 14)$.

Find a unit vector $\mathbf{a}$ attached at the origin $O = (0, 0)$ such that it is collinear to the vector AB. Find an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ in $\mathbf{E}^2$ such that $\mathbf{a} = \mathbf{e}_1$.

c) Consider the system of equations

$$\begin{cases} x^2 + y^2 = R^2 \\ |x| + |y| = 1 \end{cases}$$

where R is a parameter. By using a sketch find the number of solutions of this system for different values of R.

2

a) Consider vectors $\mathbf{a} = 2\mathbf{e}_x + \mathbf{e}_y$, $\mathbf{b} = \mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z$ in $\mathbf{E}^3$. Show that these vectors are linearly independent. Find an equation of the plane α spanned by vectors $\mathbf{a}$ and $\mathbf{b}$ attached at the point $M = (-1, 2, 3)$. (Write down an equation in the form $Ax + By + Cz = D$).

Find the distance between the point $K = (1, 3, 1)$ and the plane α.

b) Consider the plane α passing through the points $A = (a, 0, 0)$, $B = (0, b, 0)$ and $C = (0, 0, c)$, where $a, b, c \neq 0$. Find an equation of the plane α, the distance between origin and the plane α, and the area of the triangle ABC.

Hint: You may use the formula for the volume of tetrahedron: $V = \frac{HS}{3}$.

c) In the Euclidean space $\mathbf{E}^3$ equipped with an orthonormal basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ consider a triple of vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ such that $\mathbf{e}_1 = \cos\varphi\ \mathbf{e}_y + \sin\varphi\ \mathbf{e}_z$, $\mathbf{e}_2 = -\sin\varphi\ \mathbf{e}_y + \cos\varphi\ \mathbf{e}_z$, $\mathbf{e}_3 = \varepsilon\mathbf{e}_x$, where φ is an arbitrary angle and $\varepsilon = \pm 1$.

Show that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal basis, and find out if this basis have the same orientation as the basis $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$.

Define the linear operator $P\colon \mathbf{E}^3 \to \mathbf{E}^3$ by the condition that $P(\mathbf{e}_x) = \mathbf{e}_1$, $P(\mathbf{e}_y) = \mathbf{e}_2$, $P(\mathbf{e}_z) = \mathbf{e}_3$ for the case $\varphi = 0$ and $\varepsilon = 1$.

Find all vectors $\mathbf{N} = N_x\mathbf{e}_x + N_y\mathbf{e}_y + N_z\mathbf{e}_z$ such that $P\mathbf{N} = \mathbf{N}$.

Explain the geometrical meaning of these vectors.

3

a) Given a vector field $\mathbf{G} = a(r, \varphi)\partial_r + b(r, \varphi)\partial_\varphi$ in polar coordinates express it in cartesian coordinates ($x = r\cos\varphi, y = r\sin\varphi$).

b) Consider the function $f = r^n \cos 2\varphi$ and the vector fields $\mathbf{A} = x\partial_x + y\partial_y$, $\mathbf{B} = x\partial_y - y\partial_x$. Calculate $\partial_{\mathbf{A}} f$ and $\partial_{\mathbf{B}} f$. Perform these calculations both in polar and cartesian coordinates.

Calculate 1-form $\omega = df$ and find the values of this 1-form on the vector fields $\mathbf{A}$, $\mathbf{B}$.

(c) Consider in $\mathbf{E}^2$ the differential 1-form $\omega = xdy + ydx$. Find a function (0-form) f such that $df = \omega$.

Show in detail that, for the 1-form $\omega = xdy - ydx$ in $\mathbf{E}^2$, it is impossible to find a function f such that $df = \omega$.

4

($\mathbf{a}$) Consider in $\mathbf{E}^2$ the curve $\mathbf{r}(t)$: $x = t, y = \sin t$, $0 \leq t \leq \pi$.

Find the velocity $\mathbf{v} = \frac{d\mathbf{r}(t)}{dt}$ and acceleration $\mathbf{a}(t) = \frac{d^2\mathbf{r}(t)}{dt^2}$ vectors.

Find the points of this curve where speed is decreasing.

($\mathbf{b}$) Consider in $\mathbf{E}^2$ the curve $\mathbf{r}(t)$: $x = 1 + t^2, y = t$, $0 < t < 1$.

Sketch the image of this curve.

Calculate the integral of the differential form $\omega = xdy + y^2dx$ over this curve.

How does this integral change under the reparameterisation $t = \sin \tau$, $(0 < \tau < \frac{\pi}{2})$?

How does this integral change under the reparameterisation $t = \cos \tau$, $(0 < \tau < \frac{\pi}{2})$?

($\mathbf{c}$) Consider in $\mathbf{E}^3$ two curves. A helix

$$C_1: \quad \mathbf{r}(t) \begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases}, 0 \leq t \leq t_0 \,,$$

and an interval of straight line

$$C_2: \quad \mathbf{r}(t) \begin{cases} x = 1 + at \\ y = bt \\ z = t \end{cases}, 0 \leq t \leq t_0 \,.$$

Note that starting points of these curves coincide.

Find values of parameters a, b such that ending points of these curves coincide too.

For chosen values of parameters a and b calculate integrals $\int_{C_1} \omega_1$, $\int_{C_2} \omega_1$, $\int_{C_1} \omega_2$ and $\int_{C_2} \omega_2$ of differential 1-forms $\omega_1 = xdy + ydx + dz$ and $\omega_2 = xdy - ydx + dz$ over these curves.

Explain why $\int_{C_1} \omega_1 = \int_{C_2} \omega_1$.

5

(**a**) Let C be the upper half of the circle with centre at the point $(R, 0)$ which is tangent to the y-axis. Write down an equation of this circle. Choose any parameterisation of this curve and calculate the integral $\int_C \omega$ if i) $\omega = x^2 dy$ and ii) $\omega = x^2 dy + 2xydx$.

Does answers depend on the chosen parameterisation?

b) Consider the curve in $\mathbf{E}^2$ defined by the equation $r(2 - \cos\varphi) = 3$ in polar coordinates.

Show that the sum of the distances between the points $F_1 = (0, 0)$ and $F_2 = (2, 0)$, and an arbitrary point of this curve is constant, i.e. the curve is an ellipse and points F_1, F_2 are its foci..

Find the integral of the 1-form $\omega = ydx$ over the part of this curve where $0 \leq \varphi \leq \pi$. Here as usual x, y are cartesian coordinates $x = r\cos\varphi, y = r\sin\varphi$.

Hint: Write down the equation of the ellipse in cartesian coordinates.

c) Consider 1-form $\omega = \frac{xdy - ydx}{(x^2+y^2)^\alpha}$, where α is an arbitrary parameter.

Find the integral of the form ω over the upper half of the circle with centre at the origin and with radius R.

For $\alpha = 1$, find the integral of the form ω over upper half of the ellipse defined by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

How the previous answer will change if we translate the upper half of the ellipse along the x-axis, i.e. if we consider the curve $\frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} = 1, y \geq 0$.

Hint: It is useful to write down the form ω in polar coordinates.