Homework 1

1 Let $G = ||g_{ik}(x)||$ be Riemannian metric on n-dimensional Riemannian manifold M in local coordinates (x^i) $(i = 1, 2, \dots, n)$.

a) Show that

$$g_{11}(x) > 0\,, g_{22}(x) > 0\,, \dots\,, g_{nn}(x) > 0\,.$$

b) show that condition of non-degeneracy for a symmetric matrix $G = ||g_{ik}||$ $(\det g_{ik} \neq 0)$ follows from the condition that this matrix is positive-definite.

2 Let (u, v) be local coordinates on 2-dimensional Riemannian manifold M. Let Riemannian metric be given in these local coordinates by the matrix

$$G = ||g_{ik}|| = \begin{pmatrix} A(u,v) & B(u,v) \\ C(u,v) & D(u,v) \end{pmatrix}\,,$$

where $A(u,v), B(u,v), C(u,v), D(u,v)$ are smooth functions. Show that the following conditions are fulfilled:

a) $B(u,v) = C(u,v)$,

b) $A(u,v)D(u,v) - B(u,v)C(u,v) = A(u,v)D(u,v) - B^2(u,v) \neq 0$,

c) $A(u,v) > 0$,

d)† $A(u,v)D(u,v) - B(u,v)C(u,v) = A(u,v)D(u,v) - B^2(u,v) > 0$.

e)† Show that conditions a), c) and d) are necessary and sufficient conditions for matrix $||g_{ik}||$ to define locally a Riemannian metric.

3 Consider 2-dimensional Euclidean plane with standard Euclidean metric

$$G = dx^2 + dy^2\,.$$

a) How this metric will transform under arbitrary affine coordinates transformation

$$\begin{cases} x = ax' + by' + e \\ y = cx' + dy' + f \end{cases}, \quad (a, b, c, d, e, f \in \mathbf{R})\,. \tag{1}$$

b) Find an affine transformation such that metric has the same appearance in new and old coordinates: $G = dx^2 + dy^2 = (dx')^2 + (dy')^2$.

c) How this metric will transform under coordinates transformation

$$x = \frac{u}{u^2+v^2}\,, \quad y = \frac{v}{u^2+v^2}\,, \quad (u, v \neq 0)\,.$$

d)† Let $x = x(u,v)$, and $y = y(u,v)$ be an arbitrary coordinate transformation such that the metric has the same appearance in new and old coordinates:

$$G = dx^2 + dy^2 = du^2 + dv^2\,.$$

How does this coordinate transformation look?

4 Consider domain in two-dimensional Riemannian manifold with Riemannian metric $G = du^2 + 2bdudv + dv^2$ in local coordinates u, v, where b is a constant.

Show that $b^2 < 1$

5 Riemannian metric G of 3-dmensional manifold M^3 in local coordinates u, v, w in a vicinity of a point $\mathbf{p}$ is given by equation

$$G = du^2 + u^2 dv^2 + dw^2 \,, \quad (u \neq 0) \,.$$

Show that the metric G in the vicinity of the point $\mathbf{p}$ is Euclidean, i.e. there exist coordinates $u', v'.w'$ such that

$$G = (du')^2 + (dv')^2 + (dw')^2$$