22 December 2018

**Characteristics**

*I am studying Courant Robbins book. Here is my attemt to study §3 page 41*

**First order linear dif.equation**

Consider equation

$$A^i \frac{\partial u(x^1, \dots x^n)}{\partial x^i} = F(x^1, \dots x^n)\,, \tag{1}$$

and

$$A^i \frac{\partial u(x^1, \dots x^n)}{\partial x^i} = 0\,. \tag{1'}$$

its characteristics: the system of ODE:

$$\frac{dx^i}{ds} = A^i(x^1, \dots, x^n)\,, \quad i = 1, \dots, n\,. \tag{2}$$

The solution of this system:

$$x^i = x^i(s, c_a)$$

depends on $n$ constants, but one constant is "additive": $s \to s + \tau$. In fact it depends on $n-1$, consitants, the surface of codimension 1 which is transversal to the exponent of the vector field $\mathbf{A} = \mathbf{A^i}\partial_\mathbf{i}$. One can see that

$u = u(x^1, \dots, x^n)$ is the solution of (1') $\Leftrightarrow$ Integral of equation (2)

Indeed let $u = g(x^1, \dots, x^n)$ be solution of (1'), and let $x^i(s) = f^i(s)$ be an arbitrary solution of (2). Then

$$\frac{d}{ds}\left( g(x^1, \dots, x^n)\big|_{x^i = f^i(s)} \right) = 0 \Rightarrow \text{the function } g \text{ is preserved on the solution}$$

On the other hand if a function $g$ is preserved over an arbitrary solution of ODE (2) then $g$ is the solution of equation (1')

Show that (2) indeed has $n-1$ integrals. Let $x^i = x^i(s, c_a)$ $(a = 1, \dots N)$, i.e. $N = n-1$. Choose as a new parameter, some $x_i$. WLOG suppose that it is $x_n$:

$$x^n = x^n(s, c_1, \dots, c_N) \Rightarrow s = s(x^n, c_a)\,,$$

and

$$x^\mu = x^\mu\left(s\left(x^n, c_1, \dots, c_N\right), c_1, \dots, c_N\right) \quad \mu = 1, \dots, n-1\,.$$

Thus

$$x^\mu = x^\mu(c_1, \dots c_N; x_n)\,, \ \mu = 1, \dots, n-1\,.$$

We see that $n-1$ variables $x^1, x^3, \ldots, x^{n-1}$ depend on $N$ constants $c_1, c_2, \ldots, c_N$, $N = n-1$.

These $n-1$ variables play the role of initial data, and $N = n-1$.

Expressing $c_a$ as functions of $x^a$ we come to $N = n-1$ functions $c_a = c_a(x^i)$ , and solution of (1') is

$$u = W\left(c_1\left(x^2, \ldots, x^n\right), \ldots, c_{N-1}\left(x^2, \ldots, x^n\right)\right) ,$$

where $W$ is an arbitrary function on $n-1$ variables.

Now how to solve (1)? We formaly introduce the new variable $z = x^0$ and consider instead (1) equation

**Example**

Cosnider equation

$$u(x,y)\colon\ (x+y)u_x + yu_y = y^2$$

and consider its lifting, the equation

$$U(x,y,z)\colon \quad (x+y)U_x + yU_y - y^2 U_z$$

The characteristic euqation for (E2) is

$$\begin{cases} x_t = x + y \\ y_t = y \\ z_t = y^2 \end{cases} \qquad \text{solution} \begin{cases} x(t) = x_0 e^t + y_0 t e^t \\ y(t) = y_0 e^t \\ z(t) = z_0 + y_0^2 \frac{e^{2t}-1}{2} \end{cases}$$

The solution passes throught the point $(x_0, y_0, z_0)$ at the point $t+0$. Choose the moment $T$ when it passes trhough the plane $y = 1$:

$$T\colon \begin{cases} x(T) = x_0 e^T + y_0 T e^T = \frac{x_0}{y_0} - \log y_0 \\ y(T) = y_0 e^T = 1 \\ z(T) = z_0 + y_0^2 \frac{e^{2T}-1}{2} = z_0 + \frac{1}{2} - \frac{y_0^2}{2} \end{cases}$$

Thus $c_1 = \frac{x}{y} - \log y$ and $c_2 = z - \frac{y^2}{2}$ are integrals. The solution is

$$U(x,y,z) = W(c_1, c_2) = W\left(\frac{x}{y} - \log y, z - \frac{y^2}{2}\right) .$$

(see also the next text.)