Homework 5. Solutions.

(*The solution of Exercise 2 goes after the solution of the exercise 3.*)

1 *Consider the following curves:*

$C_1\colon \mathbf{r}(t)\ \begin{cases} x = t \\ y = 2t^2 - 1 \end{cases},\ 0 < t < 1,\qquad C_2\colon \mathbf{r}(t)\ \begin{cases} x = t \\ y = 2t^2 - 1 \end{cases},\ -1 < t < 1,$

$C_3\colon \mathbf{r}(t)\ \begin{cases} x = 2t \\ y = 8t^2 - 1 \end{cases},\ 0 < t < \frac{1}{2},\qquad C_4\colon \mathbf{r}(t)\ \begin{cases} x = \cos t \\ y = \cos 2t \end{cases},\ 0 < t < \frac{\pi}{2},$

$C_5\colon \mathbf{r}(t)\ \begin{cases} x = t \\ y = 2t - 1 \end{cases},\ 0 < t < 1,\qquad C_6\colon \mathbf{r}(t)\ \begin{cases} x = 1 - t \\ y = 1 - 2t \end{cases},\ 0 < t < 1,$

$C_7\colon \mathbf{r}(t)\ \begin{cases} x = \sin^2 t \\ y = -\cos 2t \end{cases},\ 0 < t < \frac{\pi}{2},\qquad C_8\colon \mathbf{r}(t)\ \begin{cases} x = t \\ y = \sqrt{1 - t^2} \end{cases},\ -1 < t < 1,$

$C_9\colon \mathbf{r}(t)\ \begin{cases} x = \cos t \\ y = \sin t \end{cases},\ 0 < t < \pi,\qquad C_{10}\colon \mathbf{r}(t)\ \begin{cases} x = \cos 2t \\ y = \sin 2t \end{cases},\ 0 < t < \frac{\pi}{2},$

$C_{11}\colon \mathbf{r}(t)\ \begin{cases} x = \cos t \\ y = \sin t \end{cases},\ 0 < t < 2\pi,\qquad C_{12}\colon \mathbf{r}(t)\ \begin{cases} x = a\cos t \\ y = b\sin t \end{cases},\ 0 < t < 2\pi$ *(ellipse),*

Draw the images of these curves.

Write down their velocity vectors.

Indicate parameterised curves which have the same image (equivalent curves).

In each equivalence class of parameterised curves indicate curves with same and opposite orientations.

$C_1\colon \mathbf{v}(t) = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 1 \\ 4t \end{pmatrix}$, $C_2\colon \mathbf{v}(t) = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 1 \\ 4t \end{pmatrix}$, $C_3\colon \mathbf{v}(t) = \begin{pmatrix} 2 \\ 16t \end{pmatrix}$, $C_4\colon \mathbf{v}(t) = \begin{pmatrix} -\sin t \\ -2\sin 2t \end{pmatrix}$,

$C_5\colon \mathbf{v}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $C_6\colon \mathbf{v}(t) = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$, $C_7\colon \mathbf{v}(t) = \begin{pmatrix} \sin 2t \\ 2\sin 2t \end{pmatrix}$,

$C_8\colon \mathbf{v}(t) = \begin{pmatrix} 1 \\ \frac{-t}{\sqrt{1-t^2}} \end{pmatrix}$, $C_9\colon \mathbf{v}(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$, $C_{10}\colon \mathbf{v}(t) = \begin{pmatrix} -2\sin 2t \\ 2\cos 2t \end{pmatrix}$

$C_{11}\colon \mathbf{v}(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$, $C_{12}\colon \mathbf{v}(t) = \begin{pmatrix} -a\sin t \\ b\cos t \end{pmatrix}$

Curves C_1, C_2, C_3, C_4

Curves C_1, C_3 and C_4 have the same image: it is piece of parabola $y = 2x^2 - 1$ between points $(0, 1)$ and $(1, 1)$. Image of the curve C_2 is piece of the same parabola $y = 2x^2 - 1$ between points $(-1, 1)$ and $(1, 1)$. Image of curve C_1 is a part of the image of the curve C_2.

Curve C_3 can be obtained from the curve C_1 by reparameterisation $t(\tau) = 2\tau$, $\mathbf{r}_3(\tau) = \mathbf{r}_1(t(\tau)) = \mathbf{r}_1(2\tau)$. Respectively $\mathbf{v}_3(\tau) = t'(\tau)\mathbf{v}_1(t(\tau)) = 2\mathbf{v}_1(2\tau)$. Curve C_4 can be obtained from the curve C_1 by reparameterisation $t(\tau) = \cos\tau$, $\mathbf{r}_4(\tau) = \mathbf{r}_1(t(\tau)) = \mathbf{r}_1(\cos\tau)$. Respectively $\mathbf{v}_4(\tau) = \begin{pmatrix} -\sin\tau \\ -2\sin 2\tau \end{pmatrix} = t'(\tau)\mathbf{v}_1(t(\tau)) = -\sin\tau\mathbf{v}_1(\cos\tau) = -\sin\tau \begin{pmatrix} 1 \\ 2\cos\tau \end{pmatrix}$.

We see that curves C_1, C_3, C_4 are equivalent. They belong to the same equivalence class of non-parameterised curves. Equivalent curves C_1 and C_3 have the same orientation because diffeomorphism $t = 2\tau$ has positive derivative. Equivalent curves C_1 and C_4 (and so C_3 and C_4) have opposite orientation because diffeomorphism $t = \cos\tau$ has negative derivative (for $0 < t < 1$).

Curves C_5, C_6, C_7

Now consider curves C_5, C_6, C_7. It is easy to see that they all have the same image— segment of the line between point $(0, -1)$ and $(1, 1)$. These three curves belong to the same equivalence class of non-parameterised curves. Curve C_6 can be obtained from the curve C_5 by reparameterisation $t(\tau) = 1 - \tau$,

$\mathbf{r}_6(\tau) = \mathbf{r}_5(t(\tau)) = \mathbf{r}_5(1-\tau)$. Respectively $\mathbf{v}_6(\tau) = t'(\tau)\mathbf{v}_5(t(\tau)) = -\mathbf{v}_5(1-\tau)$. (Velocity just changes its direction on opposite.) Curve C_7 can be obtained from the curve C_5 by reparameterisation $t(\tau) = \sin^2\tau$, $\mathbf{r}_7(\tau) = \mathbf{r}_5(t(\tau)) = \mathbf{r}_5(\sin\tau)$. Respectively $\mathbf{v}_7(\tau) = \begin{pmatrix} \sin 2\tau \\ 2\sin 2\tau \end{pmatrix} = t'(\tau)\mathbf{v}_5(t(\tau)) = \sin 2\tau \mathbf{v}_5(\sin\tau) = \sin 2\tau \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Equivalent curves C_5 and C_7 have the same orientation because derivative of diffeomorphism $t = \sin^2\tau$ is positive (on the interval $0 < t < 1$). Curve C_6 has orinetation opposite to the orientation of the curves C_5 and C_6 because derivative of diffeomorphism $t = 1-\tau$ is negative. Or in other words when we go to the curve C_6 starting point becomes ending point and vice versa.

Curves C_8, C_9, C_{10}

Now consider curves C_8, C_9, C_{10}. It is easy to see that they all have the same image— upper part of the circle $x^2 + y^2 = 1$. These three curves belong to the same equivalence class of non-parameterised curves. Curve C_9 can be obtained from the curve C_8 by reparameterisation $t(\tau) = \cos\tau$. Then $\mathbf{r}_9(\tau) = \mathbf{r}_8(t(\tau)) = \mathbf{r}_8(\cos\tau)$. Respectively $\mathbf{v}_9(\tau) = t'(\tau)\mathbf{v}_8(t(\tau)) = -\sin\tau\mathbf{v}_8(\cos\tau)$.

Curve C_{10} can be obtained from the curve C_8 by reparameterisation $t(\tau) = 2\tau$, $\mathbf{r}_{10}(\tau) = \mathbf{r}_8(t(\tau)) = \mathbf{r}_8(2\tau)$. Respectively $\mathbf{v}_{10}(\tau) = t'(\tau)\mathbf{v}_8(t(\tau)) = 2\tau\mathbf{v}_8(2\tau)$.

Equivalent curves C_8 and C_{10} have the same orientation because derivative of diffeomorphism $t = 2\tau$ is positive . Curve C_9 has orinetation opposite to the orientation of the curves C_8 and C_{10} because derivative of diffeomorphism $t = \cos\tau$ on the interval $0 < t < \pi$ is negative.

Curves C_{11}, C_{12}

Image of the curve C_{11} is circle $x^2 + y^2 = 1$.

Image of the curve C_{12} is ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ T

2 *Consider the curves C_1, C_2 given by the parametric equations*

$$C_1\colon \mathbf{r}(\tau) \begin{cases} r(\tau) = \frac{1}{2-\cos\tau} \\ \varphi(\tau) = \tau \end{cases}, \ 0 \le \tau < 2\pi, \ \ C_2\colon \mathbf{r}(t) \begin{cases} x(t) = \frac{2}{3}\cos t + \frac{1}{3} \\ y(t) = \frac{1}{\sqrt{3}}\sin t \end{cases}, \ 0 \le t < 2\pi\,.$$

Here the curve C_1 is defined in polar coordinates r, φ, the curve C_2 is defined in usual cartesian coordinates ($x = r\cos\varphi, y = r\sin\varphi$).

Show that the images of both curves are ellipses.

Check that these ellipses coincide.

† *Find foci of this ellipse.*

Just to recall the definition of the ellipse (see in a details the Appendix at the end of this text.).

Definition. The locus of points in the plane such that sum of the distances to two fixed points is constant:

$$\{\mathbf{r}\colon |\mathbf{r} - F_1| + |\mathbf{r} - F_2| = constant\}\,. \tag{2.1}$$

F_1, F_2 are called foci of the ellipse.

One can show that in suitable cartesian coordinates the equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,. \tag{2.2}$$

(See the sketch of the proof in Appendix at the end of the solutions.) The inverse is also true: any curve which is defined by (2.2) is an ellipse.

The ellipse (2.2) can be defined by parametric equation

$$\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}, 0 \le t \le 2\pi \tag{2.3}$$

where a, b are arbitrary parameters $a \neq 0, b \neq 0$.

Now return to our problem. One can easy see that the second curve is an ellipse:
$x - \frac{1}{3} = \frac{2}{3}\cos t,\ y = \frac{1}{\sqrt{3}}\sin t$. Hence

$$\frac{9}{4}\left(x - \frac{1}{3}\right)^2 + 3y^2 = \cos^2 t + \sin^2 t = 1\,. \tag{2.4}$$

if we translate coordinate $x \mapsto x - \frac{1}{3}$ we come to the equation (2.2) with $a = \frac{2}{3},\ b = \frac{1}{\sqrt{3}}$.

The first curve defines the set of points $r(2 - \cos\varphi) = 1$ (in polar coordinates). Hence $2r = 1 + r\cos\varphi$, i.e. $2\sqrt{x^2 + y^2} = 1 + r\cos\varphi = 1 + x$). Taking squares we come to $4x^2 + 4y^2 = (1 + x)^2$ (condition, $1 + x \geq 0$). Hence $4x^2 + 4y^2 = 1 + 2x + x^2$, i.e. $3x^2 - 2x + 4y^2 = 1$.

$$3\left(x^2 - \frac{2x}{3}\right) + 4y^2 = 1 \Leftrightarrow 3\left(x - \frac{1}{3}\right)^2 + 4y^2 = 1 + \frac{1}{3} \Leftrightarrow \frac{9}{4}\left(x - \frac{1}{3}\right)^2 + 3y^2 = 1$$

We see that this equation coincides with equation (2.4). Ellipses coincide.

Now find the foci of this ellipse and check the condition (2.1).

Consider the two points $F_1 = (0, 0)$ and $F_2 = (0, f)$. Take an arbitrary point P on the ellipse $r = \frac{1}{2 - \cos\varphi}$. (We prefer to work in polar coordinates.) Denote by $l(P)$ the sum of the distances from the point P on the ellipse to two points F_1, F_2

$$l(P) = |P - F_1| + |P - F_2|$$

Show that one can choose f such that the sum $l(P)$ is constant for an arbitrary point P: $r(1 - 2\cos\varphi) = 1$.

Considering the triangle F_1F_2P we see that

$$|PF_1|^2 + |F_1F_2|^2 - 2|PF_1||F_1F_2|\cos\varphi = |PF_2|^2\,.$$

Thus if (r, φ) are polar coordinates of the point P then

$$r^2 + f^2 - 2rf\cos\varphi = (l - r)^2 \Leftrightarrow f^2 - 2fr\cos\varphi = l^2 - 2lr \Leftrightarrow r = \frac{l^2 - f^2}{2l - 2f\cos\varphi}$$

On the other hand $r = \frac{1}{2 - \cos\varphi}$. Hence

$$r = \frac{1}{2 - \cos\varphi} = \frac{l^2 - f^2}{2l - 2f\cos\varphi} \Leftrightarrow 2(l^2 - f^2 - l) = (l^2 - f^2 - 2f)\cos\varphi$$

This equation is valid for an arbitrary φ. Hence $l^2 - f^2 - l = 0$ and $l^2 - f^2 - 2f = 0$, i.e. $f = \frac{2}{3}, l = \frac{4}{3}$. We proved that the foci of the ellipse are at the points $F_1 = (0, 0)$ and $F_2 = \left(\frac{2}{3}, 0\right)$. The sum of the distances from any point on the ellipse to foci is equal to $\frac{4}{3}$

3 *Consider the following curve (helix):* $\mathbf{r}(t)$: $\begin{cases} x(t) = R\cos\Omega t \\ y(t) = R\sin\Omega t \\ z(t) = ct \end{cases},\qquad 0 \leq t \leq t_0.$

Show that the image of this curve belongs to the surface of cylinder $x^2 + y^2 = a^2$.

Find the velocity vector of this curve.

Find the length of this curve.

Finish the following sentence:

After developing the surface of cylinder to the plane the curve will develop to the...

For any point of this curve $x^2(t) + y^2(t) = a^2$. Hence all points belong to the surface of cylinder $x^2 + y^2 = a^2$.

Calculate velocity vector at the point $\mathbf{r}(t)$: $\begin{cases} x(t) = R\cos\Omega t \\ y(t) = R\sin\Omega t \\ z(t) = ct \end{cases}$ of the helix:

$$\mathbf{v}(t) = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = \begin{pmatrix} -\omega R\sin\Omega t \\ \omega R\cos\Omega t \\ c \end{pmatrix} \tag{3.1}$$

and

$$|\mathbf{v}|^2 = \mathbf{v}^2 = v_x^2 + v_y^2 + v_z^2 = \Omega^2 R^2 \sin^2 \Omega t + \Omega^2 R^2 \cos^2 \Omega t + c^2 = \Omega^2 R^2 + c^2$$

The speed is constant, hence length is equal to $L = |\mathbf{v}|t_0 = t_0\sqrt{\Omega^2 R^2 + c^2}$.

Consider surface of cylinder $\mathbf{r}(\varphi, h)$: $x = R\cos\varphi, y = R\sin\varphi, z = h$. Any point (φ, h) after developing on the plane will have the coordinates $L(\varphi) = R\varphi$ (length of the arc) and $z = h$. For points of the helix an angle $\varphi = \Omega t$, $h = ct$. Hence for these points $L(t) = R\Omega t$ and $z(t) = ct$, i.e. $z = \frac{c}{R\Omega}L$. It is a line.

4 *Consider differential forms $\omega = xdy - ydx$, $\sigma = xdx + ydy$ and vector fields $\mathbf{A} = x\partial_x + y\partial_y$, $\mathbf{B} = x\partial_y - y\partial_x$*

a) Calculate $\omega(\mathbf{A}), \omega(\mathbf{B})$, $\sigma(\mathbf{A}), \sigma(\mathbf{B})$.

b) Calculate differential forms ω and σ in polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$

a) $\omega(\mathbf{A}) = (xdy - ydx)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) =$

$$x^2 dy\left(\frac{\partial}{\partial x}\right) + xydy\left(\frac{\partial}{\partial y}\right) - yxdx\left(\frac{\partial}{\partial x}\right) - y^2 dx\left(\frac{\partial}{\partial y}\right) = x^2 \cdot 0 + xy \cdot 1 - yx \cdot 1 - y^2 \cdot 0 = 0\,.$$

Later we often denote vector field $\frac{\partial}{\partial x}$ by ∂_x, vector field $\frac{\partial}{\partial y}$ by ∂_y...
$\omega(\mathbf{B}) = (xdy - ydx)(x\partial_y - y\partial_x) = x^2 dy(\partial_y) - xydy(\partial_x) - yxdx(\partial_y) + y^2 dx(\partial_x) = x^2 \cdot 1 - xy \cdot 0 - yx \cdot 0 + y^2 \cdot 1 = x^2 + y^2 = r^2$,
$\sigma(\mathbf{A}) = (xdx + ydy)(x\partial_x + y\partial_y) = x^2 dx(\partial_x) + xydx(\partial_y) + yxdy(\partial_x) + y^2 dy(\partial_y) = x^2 \cdot 1 + xy \cdot 0 + yx \cdot 0 + y^2 \cdot 1 = x^2 + y^2 = r^2$,
$\sigma(\mathbf{B}) = (xdx + ydy)(x\partial_y - y\partial_x) = x^2 dx(\partial_y) - xydx(\partial_x) + yxdy(\partial_y) - y^2 dy(\partial_x) = x^2 \cdot 0 - xy \cdot 1 + yx \cdot 1 - y^2 \cdot 0 = 0$.

b) In polar coordinates $\omega = xdy - ydx = r\cos\varphi d(r\sin\varphi) - r\sin\varphi d(r\cos\varphi) =$

$$r\cos\varphi(r\cos\varphi d\varphi + \sin\varphi dr) - r\sin\varphi(-r\sin\varphi d\varphi + \cos\varphi dr) = r^2(\cos^2\varphi + \sin^2\varphi)d\varphi = r^2 d\varphi\,.$$

In polar coordinates $\sigma = xdx + ydy = r\cos\varphi d(r\cos\varphi) + r\sin\varphi d(r\sin\varphi) =$

$$r\cos\varphi(-r\sin\varphi d\varphi + \cos\varphi dr) + r\sin\varphi(r\cos\varphi d\varphi + \sin\varphi dr) = r(\cos^2\varphi + \sin^2\varphi)dr = rdr\,.$$

5 *Consider differential forms $\omega = xdy - dx$ and $\sigma = xdx + ydy + zdz$ in $\mathbf{E}^3$. Calculate $\omega(\mathbf{v})$ and $\sigma(\mathbf{v})$ on the velocity vectors of helix considered in question 3).*

Recall that at the point $\mathbf{r}(t) = \begin{pmatrix} R\cos\Omega t \\ R\sin\Omega t \\ ct \end{pmatrix}$ of the helix the velocity vector equals $\mathbf{v}(t) = \begin{pmatrix} -\Omega R\sin\Omega t \\ \Omega R\cos\Omega t \\ c \end{pmatrix}$ (see equation (3.2) above), i.e. $\mathbf{v}(t) = v_x\mathbf{e}_x + v_y\mathbf{e}_y + v_z\mathbf{e}_z = -\Omega a\sin\Omega t\mathbf{e}_x + \Omega R\cos\Omega t\mathbf{e}_y c\mathbf{e}_z =$

$$v_x\partial_x + v_y\partial_y + v_z\partial_z = -\Omega R\sin\Omega t\partial_x + \Omega R\cos\Omega t\partial_y + c\partial_z\,.$$

We have

$$\omega(\mathbf{r}(t),\mathbf{v}(t)) = (xdy - ydx)(\mathbf{v}) = (x(t)dy - y(t)dx)(\mathbf{v}_x(t)\partial_x + v_y(t)\partial_y + v_z(t)\partial_z) = x(t)v_y - v_x(t)y(t)$$

since $dx(\partial_x) = dy(\partial_y) = 1$ and $dx(\partial_y) = dy(\partial_x) = dz(\partial_x) = dz(\partial_y) = 0$. Finally we have that $\omega(\mathbf{r}(t),\mathbf{v}(t)) = x(t)v_y - y(t)v_x(t) = R\cos\Omega t R(\Omega R\sin\Omega t) - R\sin\Omega t(-\Omega R\sin\Omega t) = \Omega R^2$.

Now calculate $\sigma(\mathbf{r}(t),\mathbf{v}(t))$. We have $\sigma(\mathbf{r}(t),\mathbf{v}(t)) =$

$$\sigma(\mathbf{r}(t),\mathbf{v}(t)) = (xdx+ydy+zdz)(\mathbf{v}) = (x(t)dx+y(t)dy+z(t)dz)(\mathbf{v}_x(t)\partial_x+v_y(t)\partial_y+v_z(t)\partial_z) = x(t)v_x+v_x(t)y(t)+z(t)v_z =$$

$$R\cos\Omega t(-\Omega R\sin\Omega t) + R\sin\Omega t(\Omega R\cos\Omega t) + ct(c) = c^2 t\,.$$

Appendix†

Ellipse can be defined as a locus of points in a plane such that the sum of the distances to two fixed points is a constant. These two fixed points are called foci.

Consider an ellipse with foci at the points F_1, F_2. Sum of the distances from any point of the ellipse to foci equals l:

$$\text{ellipse} = \{P\colon |P - F_1| + |P - F_2| = l\}\,. \tag{A1}$$

Put an origin of coordinate frame at the point F_1 and x-axis along the ray F_1F_2.

Denote by f the length of the interval F_1F_2, $f = |F_1F_2|$. Let point P is at the distance r from the focus F_1. Let φ be an angle between the rays F_1P and F_1F_2. Then it follows from (A1) that

$$r^2 - 2rf\cos\varphi + f^2 = (l - r)^2$$

Opening brackets we come to $r(2l - 2f\cos\varphi) = l^2 - f^2$. r, φ are polar coordinates: $x = r\cos\varphi$, $r^2 = x^2 + y^2$. We see that $2lr - 2fx = l^2 - f^2$. Hence $4l^2r^2 = 4l^2(x^2 + y^2) = (l^2 - f^2 + 2fx)^2$. This last equation is in cartesian coordinates. It is easy to see that by translation $x \mapsto x - s$ we come to the canonical equation (2.2).