Riemannian Geometry

2018

COURSEWORK

This assignment counts for 20% of your marks.

Solutions are due by 18 April 3pm

Write solutions in the provided spaces.

STUDENT'S NAME:

1

Consider a surface M, the upper sheet of the cone

$$\mathbf{r}(h,\varphi)\colon \begin{cases} x = h\cos\varphi \\ y = h\sin\varphi \\ z = kh \end{cases}, \qquad 0 \le \varphi < 2\pi\,,\ h > 0\,.$$

Find the length of the shortest curve C which belongs to the surface M, starts at the point $(h, 0, kh)$ and ends at the point $(-h, 0, kh)$.

2

You know that the Riemannian metric on the sphere of radius R in the stereographic coordinates is expressed by the formula

$$G_{\text{stereogr.}} = \frac{4R^4(du^2+dv^2)}{(R^2+u^2+v^2)^2}\,.$$

a) Give an example of a non-identity transformation of coordinates u, v such that it preserves this metric.

b) Give an example of a non-linear transformation of coordinates u, v such that it preserves this metric.
(*Hint: You may find this transformation considering transformations of the sphere.*)

c) Find the length of the line $v = au$ in $\mathbf{R}^2$ with respect to this metric.

Why the length of this curve does not depend on the parameter a?

3.

a) Evaluate the area of the part of the sphere of radius $R = 1$ between the planes given by equations $2x + 2y + z = 1$ and $2x + 2y + z = 2$.

b) Consider the plane $\mathbf{R}^2$ with standard coordinates (x, y) equipped with Riemannian metric

$$G = (1 + x^2 + y^2)e^{-a^2x^2 - a^2y^2} \left(dx^2 + dy^2\right) .$$

Calculate the total area of this plane.

4

a) Consider the points $A = (0, 0, R)$, $B = (R, 0, 0)$ and $C = (R\cos\varphi, R\sin\varphi, 0)$ on the sphere $x^2 + y^2 + z^2 = R^2$ in $\mathbf{E}^3$ $(0 < \varphi < \pi)$. Consider the isosceles triangle ABC on this sphere. (Sides of this triangle are the arcs of great circles joining these points.) Show that:

$$\frac{\text{Area } (\triangle ABC)}{R^2} = \alpha + \beta + \gamma - \pi \,,$$

where α, β, γ are angles of this triangle.

b) Consider the upper half-plane $y > 0$ with the Riemannian metric

$$G = \frac{dx^2 + dy^2}{y^2}$$

(the Lobachevsky plane).

In the Lobachevsky plane consider the domain D defined by

$$D = \{x, y\colon \quad x^2 + y^2 \geq 1\,,\ -a \leq x \leq a\}\,,$$

where a is a parameter such that $0 < a < 1$.

Find the area of the domain D (with respect to the metric G).

Show that

$$\text{Area of the domain } D = \pi - \beta - \gamma\,,$$

where β, γ are angles between the vertical lines $x = \pm a$ and arc of the circle delimiting the domain D.

Consider the points $A_t = (-a, t)$ and $B_t = (a, t)$ on the vertical rays $x = \pm a$ delimiting the domain D. Show that the distance between these points tends to 0 if $t \to \infty$.

Explain why the domain D can be considered as an isosceles triangle.

Why it can be said that the third angle of this triangle vanishes.

5

a) Let ∇ be an affine connection on the 2-dimensional manifold M such that in local coordinates (u, v), $\nabla_{\frac{\partial}{\partial u}} \left(u^2 \frac{\partial}{\partial v}\right) = 3u \frac{\partial}{\partial v} + u \frac{\partial}{\partial u}$.

Calculate the Christoffel symbols Γ^u_{uv} and Γ^v_{uv} of this connection.

b) Let ∇ be an arbitrary connection on a manifold M. Show that

$$\cos F \nabla_{\mathbf{A}} (\sin F \mathbf{B}) - \sin F \nabla_{\mathbf{A}} (\cos F \mathbf{B}) = (\partial_{\mathbf{A}} F) \mathbf{B} \,,$$

where $\mathbf{A}, \mathbf{B}$ are arbitrary vector fields and F is an arbitrary function.

c) Let $\Gamma^{i(1)}_{km}$ be the Christoffel symbols of a connection $\nabla^{(1)}$ and $\Gamma^{i(2)}_{km}$ be the Christoffel symbols of a connection $\nabla^{(2)}$. Show, that the linear combinations $\frac{2}{3}\Gamma^{i(1)}_{km} + \frac{1}{3}\Gamma^{i(2)}_{km}$, are Christoffel symbols for some connection.

Explain, why $\frac{1}{2}\Gamma^{i(1)}_{km} + \frac{1}{3}\Gamma^{i(2)}_{km}$ are not Christoffel symbols for any connection.

6

Let M be a surface considered in question **1** (the upper sheet of a cone),

a) Calculate the induced connection on this surface (the connection induced by the canonical flat connection in the ambient Euclidean space: $\nabla_{\mathbf{X}} \mathbf{Y} = \left(\nabla^{\text{can.flat}}_{\mathbf{X}} \mathbf{Y}\right)_{\text{tangent}}$).

b) Calculate the Riemannian metric on the cone induced by the canonical metric in ambient Euclidean space $\mathbf{E}^3$ and calculate explicitly the Levi-Civita connection of this metric using the Levi-Civita Theorem.

c) Calculate the Christoffel symbols of Levi–Civita connection on the cone using Lagrangian of free particle.