



Lecture 10: Convex Optimization

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Trustworthy AI

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- August 1st – time window 2pm - 5pm
- October 19th – time window 9am - 1pm

Bibliography

- Boyd – Chapters 1-5

1. Convex sets

- Affine and convex sets
- Hyper-planes, half-spaces and polyhedra
- Operations that preserve convexity

2. Convex functions

- Basic properties and examples
- Operations that preserve convexity
- Log-concave and log-convex functions

3. Convex optimization problems

- Optimization problem in standard form
- Convex optimization problems
- Linear Optimization
- Quadratic optimization
- Multicriterion optimization

4. ML examples

- Maximum likelihood estimation
- Logistic regression

5. Duality

- Lagrange dual problem
- Weak and strong duality
- Geometric interpretation
- Optimality (KKT) conditions
 - perturbation and sensitivity analysis
 - examples
 - generalized inequalities

6. Margin-based classification

- Linear classification
- Robust linear classification
- SVM

Mathematical optimization

(mathematical) optimization problem

$$\begin{array}{ll}\text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq b_i, \quad i = 1, \dots, m\end{array}$$

- $x = (x_1, \dots, x_n)$: optimization variables
- $f_0 : \mathbf{R}^n \rightarrow \mathbf{R}$: objective function
- $f_i : \mathbf{R}^n \rightarrow \mathbf{R}, i = 1, \dots, m$: constraint functions

optimal solution x^* has smallest value of f_0 among all vectors that satisfy the constraints

Solving optimization problems

general optimization problem

- very difficult to solve
- methods involve some compromise, *e.g.*, very long computation time, or not always finding the solution

exceptions: certain problem classes can be solved efficiently and reliably

- least-squares problems
- linear programming problems
- convex optimization problems

Least-squares

$$\text{minimize } \|Ax - b\|_2^2$$

solving least-squares problems

- analytical solution: $x^* = (A^T A)^{-1} A^T b$
- reliable and efficient algorithms and software
- computation time proportional to $n^2 k$ ($A \in \mathbf{R}^{k \times n}$); less if structured
- a mature technology

using least-squares

- least-squares problems are easy to recognize
- a few standard techniques increase flexibility (*e.g.*, including weights, adding regularization terms)

Linear programming

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \dots, m\end{array}$$

solving linear programs

- no analytical formula for solution
- reliable and efficient algorithms and software
- computation time proportional to n^2m if $m \geq n$; less with structure
- a mature technology

using linear programming

- not as easy to recognize as least-squares problems
- a few standard tricks used to convert problems into linear programs
(*e.g.*, problems involving ℓ_1 - or ℓ_∞ -norms, piecewise-linear functions)

Convex optimization problem

$$\begin{array}{ll}\text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq b_i, \quad i = 1, \dots, m\end{array}$$

- objective and constraint functions are convex:

$$f_i(\alpha x + \beta y) \leq \alpha f_i(x) + \beta f_i(y)$$

if $\alpha + \beta = 1$, $\alpha \geq 0$, $\beta \geq 0$

- includes least-squares problems and linear programs as special cases

solving convex optimization problems

- no analytical solution
- reliable and efficient algorithms
- computation time (roughly) proportional to $\max\{n^3, n^2m, F\}$, where F is cost of evaluating f_i 's and their first and second derivatives
- almost a technology

using convex optimization

- often difficult to recognize
- many tricks for transforming problems into convex form
- surprisingly many problems can be solved via convex optimization

Nonlinear optimization

traditional techniques for general nonconvex problems involve compromises

local optimization methods (nonlinear programming)

- find a point that minimizes f_0 among feasible points near it
- fast, can handle large problems
- require initial guess
- provide no information about distance to (global) optimum

global optimization methods

- find the (global) solution
- worst-case complexity grows exponentially with problem size

these algorithms are often based on solving convex subproblems

Brief history of convex optimization

theory (convex analysis): ca1900–1970

algorithms

- 1947: simplex algorithm for linear programming (Dantzig)
- 1960s: early interior-point methods (Fiacco & McCormick, Dikin, . . .)
- 1970s: ellipsoid method and other subgradient methods
- 1980s: polynomial-time interior-point methods for linear programming (Karmarkar 1984)
- late 1980s–now: polynomial-time interior-point methods for nonlinear convex optimization (Nesterov & Nemirovski 1994)

applications

- before 1990: mostly in operations research; few in engineering
- since 1990: many new applications in engineering (control, signal processing, communications, circuit design, . . .); new problem classes (semidefinite and second-order cone programming, robust optimization)

Outline

1. Convex Sets
2. Convex Functions
3. Convex Optimization Problems
4. ML examples
5. Duality
6. Margin-based Classification

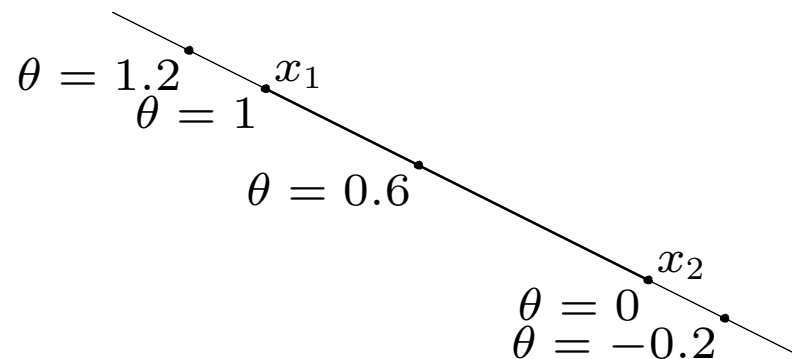
1. Convex sets

- Affine and convex sets
- Hyper-planes, half-spaces and polyhedra
- Operations that preserve convexity

Affine set

line through x_1, x_2 : all points

$$x = \theta x_1 + (1 - \theta)x_2 \quad (\theta \in \mathbf{R})$$



affine set: contains the line through any two distinct points in the set

example: solution set of linear equations $\{x \mid Ax = b\}$

(conversely, every affine set can be expressed as solution set of system of linear equations)

Convex set

line segment between x_1 and x_2 : all points

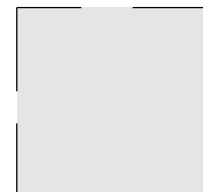
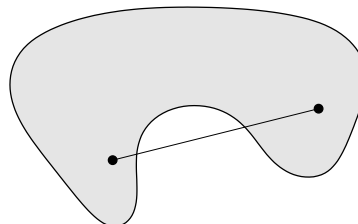
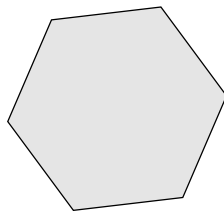
$$x = \theta x_1 + (1 - \theta)x_2$$

with $0 \leq \theta \leq 1$

convex set: contains line segment between any two points in the set

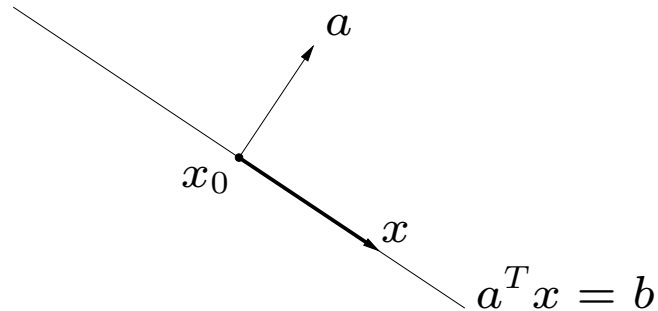
$$x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \implies \quad \theta x_1 + (1 - \theta)x_2 \in C$$

examples (one convex, two nonconvex sets)

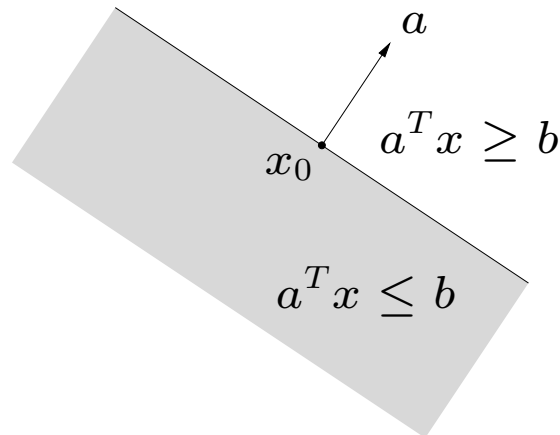


Hyperplanes and halfspaces

hyperplane: set of the form $\{x \mid a^T x = b\}$ ($a \neq 0$)



halfspace: set of the form $\{x \mid a^T x \leq b\}$ ($a \neq 0$)

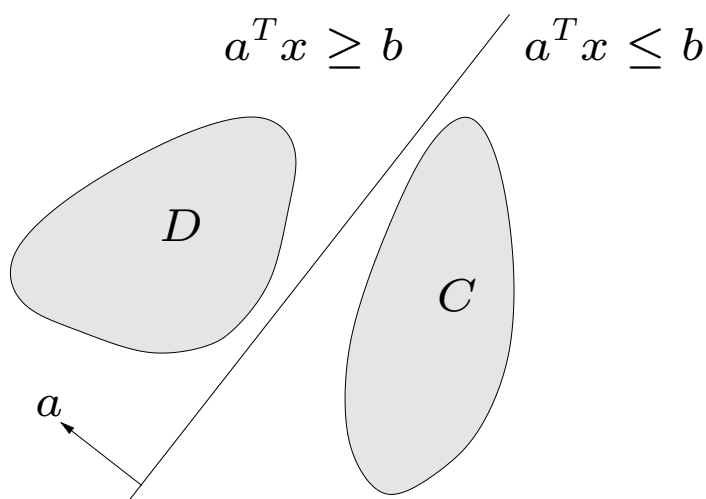


- a is the normal vector
- hyperplanes are affine and convex; halfspaces are convex

Separating hyperplane theorem

if C and D are nonempty disjoint convex sets, there exist $a \neq 0$, b s.t.

$$a^T x \leq b \text{ for } x \in C, \quad a^T x \geq b \text{ for } x \in D$$



the hyperplane $\{x \mid a^T x = b\}$ separates C and D

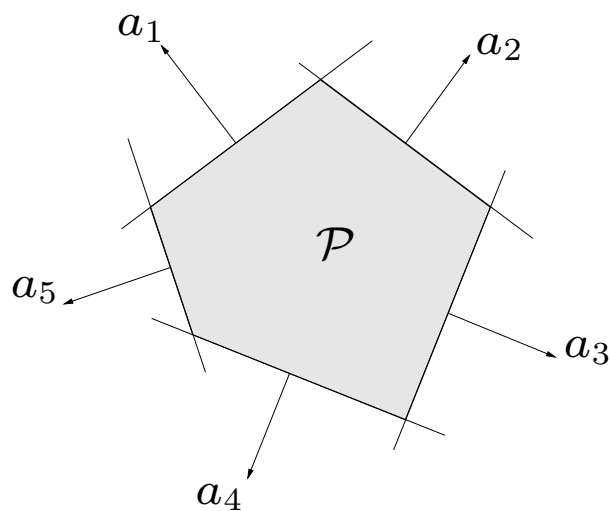
strict separation requires additional assumptions (*e.g.*, C is closed, D is a singleton)

Polyhedra

solution set of finitely many linear inequalities and equalities

$$Ax \preceq b, \quad Cx = d$$

($A \in \mathbf{R}^{m \times n}$, $C \in \mathbf{R}^{p \times n}$, \preceq is componentwise inequality)



polyhedron is intersection of finite number of halfspaces and hyperplanes

Operations that preserve convexity

practical methods for establishing convexity of a set C

1. apply definition

$$x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \implies \quad \theta x_1 + (1 - \theta)x_2 \in C$$

2. show that C is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, . . .) by operations that preserve convexity

- intersection
- affine functions
- perspective function
- linear-fractional functions

Intersection

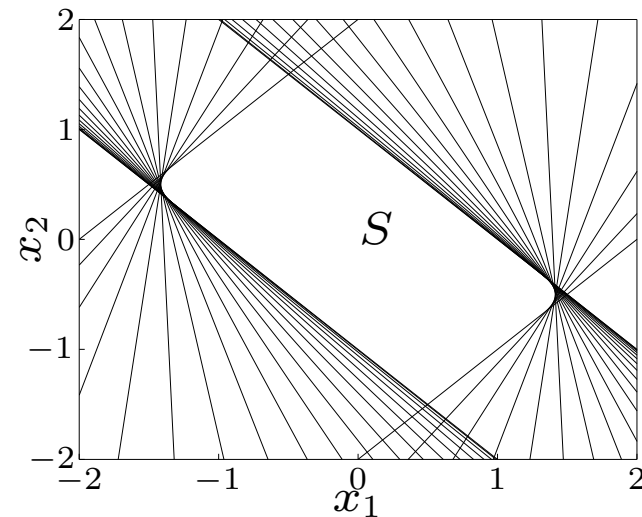
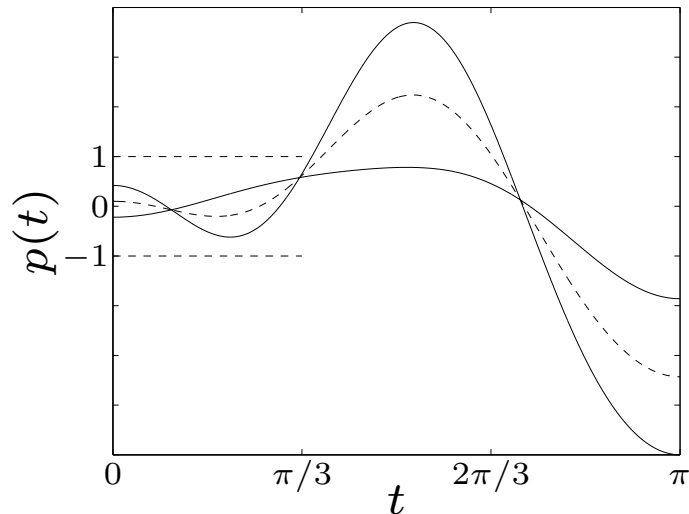
the intersection of (any number of) convex sets is convex

example:

$$S = \{x \in \mathbf{R}^m \mid |p(t)| \leq 1 \text{ for } |t| \leq \pi/3\}$$

where $p(t) = x_1 \cos t + x_2 \cos 2t + \cdots + x_m \cos mt$

for $m = 2$:



Affine function

suppose $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$ is affine ($f(x) = Ax + b$ with $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$)

- the image of a convex set under f is convex

$$S \subseteq \mathbf{R}^n \text{ convex} \implies f(S) = \{f(x) \mid x \in S\} \text{ convex}$$

- the inverse image $f^{-1}(C)$ of a convex set under f is convex

$$C \subseteq \mathbf{R}^m \text{ convex} \implies f^{-1}(C) = \{x \in \mathbf{R}^n \mid f(x) \in C\} \text{ convex}$$

examples

- scaling, translation, projection
- solution set of linear matrix inequality $\{x \mid x_1 A_1 + \cdots + x_m A_m \preceq B\}$
(with $A_i, B \in \mathbf{S}^p$)
- hyperbolic cone $\{x \mid x^T P x \leq (c^T x)^2, c^T x \geq 0\}$ (with $P \in \mathbf{S}_+^n$)

2. Convex functions

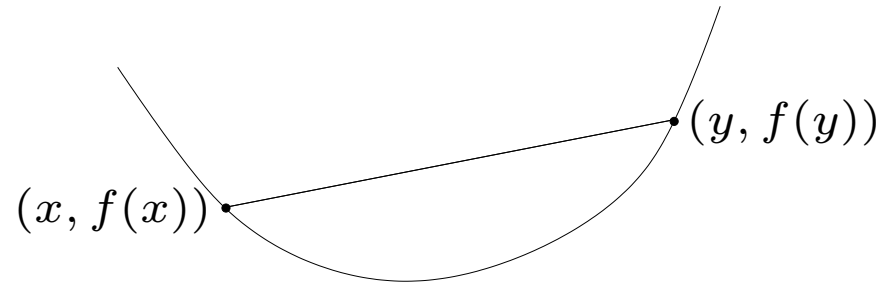
- Basic properties and examples
- Operations that preserve convexity
- Log-concave and log-convex functions

Definition

$f : \mathbf{R}^n \rightarrow \mathbf{R}$ is convex if $\mathbf{dom} f$ is a convex set and

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$$

for all $x, y \in \mathbf{dom} f$, $0 \leq \theta \leq 1$



- f is concave if $-f$ is convex
- f is strictly convex if $\mathbf{dom} f$ is convex and

$$f(\theta x + (1 - \theta)y) < \theta f(x) + (1 - \theta)f(y)$$

for $x, y \in \mathbf{dom} f$, $x \neq y$, $0 < \theta < 1$

Examples on \mathbf{R}

convex:

- affine: $ax + b$ on \mathbf{R} , for any $a, b \in \mathbf{R}$
- exponential: e^{ax} , for any $a \in \mathbf{R}$
- powers: x^α on \mathbf{R}_{++} , for $\alpha \geq 1$ or $\alpha \leq 0$
- powers of absolute value: $|x|^p$ on \mathbf{R} , for $p \geq 1$
- negative entropy: $x \log x$ on \mathbf{R}_{++}

concave:

- affine: $ax + b$ on \mathbf{R} , for any $a, b \in \mathbf{R}$
- powers: x^α on \mathbf{R}_{++} , for $0 \leq \alpha \leq 1$
- logarithm: $\log x$ on \mathbf{R}_{++}

Examples on \mathbf{R}^n and $\mathbf{R}^{m \times n}$

affine functions are convex and concave; all norms are convex

examples on \mathbf{R}^n

- affine function $f(x) = a^T x + b$
- norms: $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $p \geq 1$; $\|x\|_\infty = \max_k |x_k|$

examples on $\mathbf{R}^{m \times n}$ ($m \times n$ matrices)

- affine function

$$f(X) = \mathbf{tr}(A^T X) + b = \sum_{i=1}^m \sum_{j=1}^n A_{ij} X_{ij} + b$$

- spectral (maximum singular value) norm

$$f(X) = \|X\|_2 = \sigma_{\max}(X) = (\lambda_{\max}(X^T X))^{1/2}$$

First-order condition

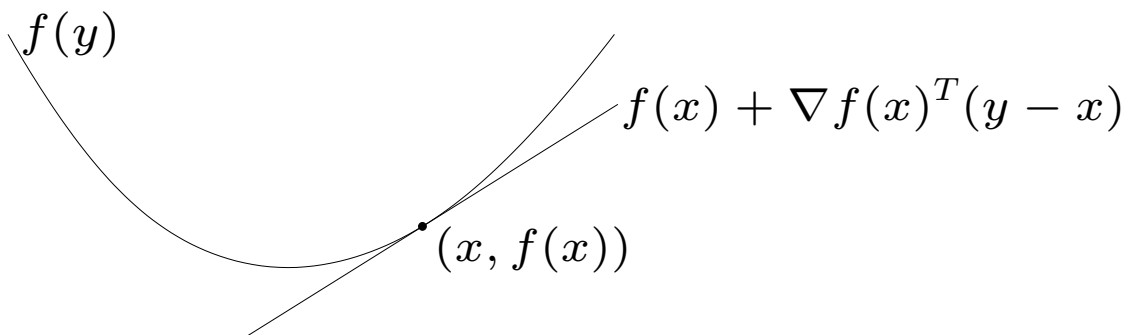
f is **differentiable** if $\text{dom } f$ is open and the gradient

$$\nabla f(x) = \left(\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \dots, \frac{\partial f(x)}{\partial x_n} \right)$$

exists at each $x \in \text{dom } f$

1st-order condition: differentiable f with convex domain is convex iff

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) \quad \text{for all } x, y \in \text{dom } f$$



first-order approximation of f is global underestimator

Second-order conditions

f is **twice differentiable** if $\text{dom } f$ is open and the Hessian $\nabla^2 f(x) \in \mathbf{S}^n$,

$$\nabla^2 f(x)_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}, \quad i, j = 1, \dots, n,$$

exists at each $x \in \text{dom } f$

2nd-order conditions: for twice differentiable f with convex domain

- f is convex if and only if

$$\nabla^2 f(x) \succeq 0 \quad \text{for all } x \in \text{dom } f$$

- if $\nabla^2 f(x) \succ 0$ for all $x \in \text{dom } f$, then f is strictly convex

Examples

quadratic function: $f(x) = (1/2)x^T P x + q^T x + r$ (with $P \in \mathbf{S}^n$)

$$\nabla f(x) = Px + q, \quad \nabla^2 f(x) = P$$

convex if $P \succeq 0$

least-squares objective: $f(x) = \|Ax - b\|_2^2$

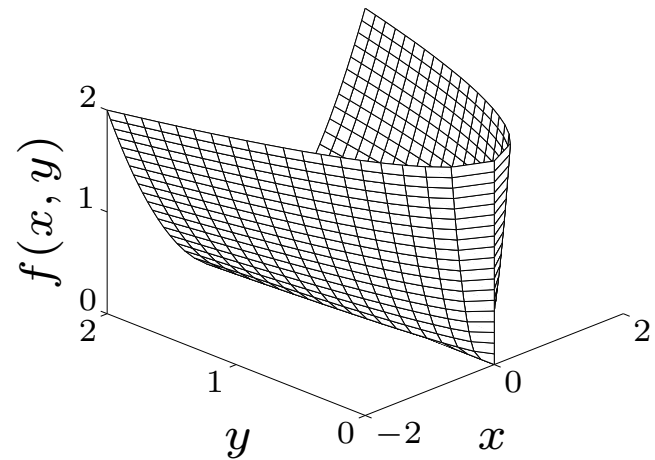
$$\nabla f(x) = 2A^T(Ax - b), \quad \nabla^2 f(x) = 2A^T A$$

convex (for any A)

quadratic-over-linear: $f(x, y) = x^2/y$

$$\nabla^2 f(x, y) = \frac{2}{y^3} \begin{bmatrix} y \\ -x \end{bmatrix} \begin{bmatrix} y \\ -x \end{bmatrix}^T \succeq 0$$

convex for $y > 0$



Jensen's inequality

basic inequality: if f is convex, then for $0 \leq \theta \leq 1$,

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$$

extension: if f is convex, then

$$f(\mathbf{E} z) \leq \mathbf{E} f(z)$$

for any random variable z

basic inequality is special case with discrete distribution

$$\mathbf{prob}(z = x) = \theta, \quad \mathbf{prob}(z = y) = 1 - \theta$$

Operations that preserve convexity

practical methods for establishing convexity of a function

1. verify definition (often simplified by restricting to a line)
2. for twice differentiable functions, show $\nabla^2 f(x) \succeq 0$
3. show that f is obtained from simple convex functions by operations that preserve convexity
 - nonnegative weighted sum
 - composition with affine function
 - pointwise maximum and supremum
 - composition
 - minimization
 - perspective

Positive weighted sum & composition with affine function

nonnegative multiple: αf is convex if f is convex, $\alpha \geq 0$

sum: $f_1 + f_2$ convex if f_1, f_2 convex (extends to infinite sums, integrals)

composition with affine function: $f(Ax + b)$ is convex if f is convex

examples

- log barrier for linear inequalities

$$f(x) = -\sum_{i=1}^m \log(b_i - a_i^T x), \quad \text{dom } f = \{x \mid a_i^T x < b_i, i = 1, \dots, m\}$$

- (any) norm of affine function: $f(x) = \|Ax + b\|$

Pointwise maximum

if f_1, \dots, f_m are convex, then $f(x) = \max\{f_1(x), \dots, f_m(x)\}$ is convex

examples

- piecewise-linear function: $f(x) = \max_{i=1, \dots, m} (a_i^T x + b_i)$ is convex
- sum of r largest components of $x \in \mathbf{R}^n$:

$$f(x) = x_{[1]} + x_{[2]} + \dots + x_{[r]}$$

is convex ($x_{[i]}$ is i th largest component of x)

proof:

$$f(x) = \max\{x_{i_1} + x_{i_2} + \dots + x_{i_r} \mid 1 \leq i_1 < i_2 < \dots < i_r \leq n\}$$

Pointwise supremum

if $f(x, y)$ is convex in x for each $y \in \mathcal{A}$, then

$$g(x) = \sup_{y \in \mathcal{A}} f(x, y)$$

is convex

examples

- support function of a set C : $S_C(x) = \sup_{y \in C} y^T x$ is convex
- distance to farthest point in a set C :

$$f(x) = \sup_{y \in C} \|x - y\|$$

- maximum eigenvalue of symmetric matrix: for $X \in \mathbf{S}^n$,

$$\lambda_{\max}(X) = \sup_{\|y\|_2=1} y^T X y$$

Composition with scalar functions

composition of $g : \mathbf{R}^n \rightarrow \mathbf{R}$ and $h : \mathbf{R} \rightarrow \mathbf{R}$:

$$f(x) = h(g(x))$$

f is convex if $\begin{array}{l} g \text{ convex, } h \text{ convex, } \tilde{h} \text{ nondecreasing} \\ g \text{ concave, } h \text{ convex, } \tilde{h} \text{ nonincreasing} \end{array}$

- proof (for $n = 1$, differentiable g, h)

$$f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x)$$

- note: monotonicity must hold for extended-value extension \tilde{h}

examples

- $\exp g(x)$ is convex if g is convex
- $1/g(x)$ is convex if g is concave and positive

Vector composition

composition of $g : \mathbf{R}^n \rightarrow \mathbf{R}^k$ and $h : \mathbf{R}^k \rightarrow \mathbf{R}$:

$$f(x) = h(g(x)) = h(g_1(x), g_2(x), \dots, g_k(x))$$

f is convex if $\begin{array}{l} g_i \text{ convex, } h \text{ convex, } \tilde{h} \text{ nondecreasing in each argument} \\ g_i \text{ concave, } h \text{ convex, } \tilde{h} \text{ nonincreasing in each argument} \end{array}$

proof (for $n = 1$, differentiable g, h)

$$f''(x) = g'(x)^T \nabla^2 h(g(x)) g'(x) + \nabla h(g(x))^T g''(x)$$

examples

- $\sum_{i=1}^m \log g_i(x)$ is concave if g_i are concave and positive
- $\log \sum_{i=1}^m \exp g_i(x)$ is convex if g_i are convex

Minimization

if $f(x, y)$ is convex in (x, y) and C is a convex set, then

$$g(x) = \inf_{y \in C} f(x, y)$$

is convex

examples

- $f(x, y) = x^T A x + 2x^T B y + y^T C y$ with

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0, \quad C \succ 0$$

minimizing over y gives $g(x) = \inf_y f(x, y) = x^T (A - B C^{-1} B^T) x$

g is convex, hence Schur complement $A - B C^{-1} B^T \succeq 0$

- distance to a set: $\mathbf{dist}(x, S) = \inf_{y \in S} \|x - y\|$ is convex if S is convex

Log-concave and log-convex functions

a positive function f is log-concave if $\log f$ is concave:

$$f(\theta x + (1 - \theta)y) \geq f(x)^\theta f(y)^{1-\theta} \quad \text{for } 0 \leq \theta \leq 1$$

f is log-convex if $\log f$ is convex

- powers: x^a on \mathbf{R}_{++} is log-convex for $a \leq 0$, log-concave for $a \geq 0$
- many common probability densities are log-concave, *e.g.*, normal:

$$f(x) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} e^{-\frac{1}{2}(x-\bar{x})^T \Sigma^{-1}(x-\bar{x})}$$

- cumulative Gaussian distribution function Φ is log-concave

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-u^2/2} du$$

Properties of log-concave functions

- twice differentiable f with convex domain is log-concave if and only if

$$f(x)\nabla^2 f(x) \preceq \nabla f(x)\nabla f(x)^T$$

for all $x \in \text{dom } f$

- product of log-concave functions is log-concave
- sum of log-concave functions is not always log-concave
- integration: if $f : \mathbf{R}^n \times \mathbf{R}^m \rightarrow \mathbf{R}$ is log-concave, then

$$g(x) = \int f(x, y) \, dy$$

is log-concave (not easy to show)

3. Convex optimization problems

- Optimization problem in standard form
- Convex optimization problems
- Linear Optimization
- Quadratic optimization
- Multicriterion optimization

Optimization problem in standard form

$$\begin{array}{ll}\text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p\end{array}$$

- $x \in \mathbf{R}^n$ is the optimization variable
- $f_0 : \mathbf{R}^n \rightarrow \mathbf{R}$ is the objective or cost function
- $f_i : \mathbf{R}^n \rightarrow \mathbf{R}$, $i = 1, \dots, m$, are the inequality constraint functions
- $h_i : \mathbf{R}^n \rightarrow \mathbf{R}$ are the equality constraint functions

optimal value:

$$p^* = \inf \{ f_0(x) \mid f_i(x) \leq 0, \ i = 1, \dots, m, \ h_i(x) = 0, \ i = 1, \dots, p \}$$

- $p^* = \infty$ if problem is infeasible (no x satisfies the constraints)
- $p^* = -\infty$ if problem is unbounded below

Optimal and locally optimal points

x is **feasible** if $x \in \text{dom } f_0$ and it satisfies the constraints

a feasible x is **optimal** if $f_0(x) = p^*$; X_{opt} is the set of optimal points

x is **locally optimal** if there is an $R > 0$ such that x is optimal for

$$\begin{array}{ll} \text{minimize (over } z) & f_0(z) \\ \text{subject to} & f_i(z) \leq 0, \quad i = 1, \dots, m, \quad h_i(z) = 0, \quad i = 1, \dots, p \\ & \|z - x\|_2 \leq R \end{array}$$

examples (with $n = 1$, $m = p = 0$)

- $f_0(x) = 1/x$, $\text{dom } f_0 = \mathbf{R}_{++}$: $p^* = 0$, no optimal point
- $f_0(x) = -\log x$, $\text{dom } f_0 = \mathbf{R}_{++}$: $p^* = -\infty$
- $f_0(x) = x \log x$, $\text{dom } f_0 = \mathbf{R}_{++}$: $p^* = -1/e$, $x = 1/e$ is optimal
- $f_0(x) = x^3 - 3x$, $p^* = -\infty$, local optimum at $x = 1$

Implicit constraints

the standard form optimization problem has an **implicit constraint**

$$x \in \mathcal{D} = \bigcap_{i=0}^m \text{dom } f_i \cap \bigcap_{i=1}^p \text{dom } h_i,$$

- we call \mathcal{D} the **domain** of the problem
- the constraints $f_i(x) \leq 0$, $h_i(x) = 0$ are the explicit constraints
- a problem is **unconstrained** if it has no explicit constraints ($m = p = 0$)

example:

$$\text{minimize } f_0(x) = -\sum_{i=1}^k \log(b_i - a_i^T x)$$

is an unconstrained problem with implicit constraints $a_i^T x < b_i$

Feasibility problem

$$\begin{array}{ll}\text{find} & x \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p\end{array}$$

can be considered a special case of the general problem with $f_0(x) = 0$:

$$\begin{array}{ll}\text{minimize} & 0 \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p\end{array}$$

- $p^* = 0$ if constraints are feasible; any feasible x is optimal
- $p^* = \infty$ if constraints are infeasible

Convex optimization problem

standard form convex optimization problem

$$\begin{array}{ll}\text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & a_i^T x = b_i, \quad i = 1, \dots, p\end{array}$$

- f_0, f_1, \dots, f_m are convex; equality constraints are affine
- problem is *quasiconvex* if f_0 is quasiconvex (and f_1, \dots, f_m convex)

often written as

$$\begin{array}{ll}\text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & Ax = b\end{array}$$

important property: feasible set of a convex optimization problem is convex

example

$$\begin{array}{ll}\text{minimize} & f_0(x) = x_1^2 + x_2^2 \\ \text{subject to} & f_1(x) = x_1/(1 + x_2^2) \leq 0 \\ & h_1(x) = (x_1 + x_2)^2 = 0\end{array}$$

- f_0 is convex; feasible set $\{(x_1, x_2) \mid x_1 = -x_2 \leq 0\}$ is convex
- not a convex problem (according to our definition): f_1 is not convex, h_1 is not affine
- equivalent (but not identical) to the convex problem

$$\begin{array}{ll}\text{minimize} & x_1^2 + x_2^2 \\ \text{subject to} & x_1 \leq 0 \\ & x_1 + x_2 = 0\end{array}$$

Local and global optima

any locally optimal point of a convex problem is (globally) optimal

proof: suppose x is locally optimal, but there exists a feasible y with $f_0(y) < f_0(x)$

x locally optimal means there is an $R > 0$ such that

$$z \text{ feasible, } \|z - x\|_2 \leq R \implies f_0(z) \geq f_0(x)$$

consider $z = \theta y + (1 - \theta)x$ with $\theta = R/(2\|y - x\|_2)$

- $\|y - x\|_2 > R$, so $0 < \theta < 1/2$
- z is a convex combination of two feasible points, hence also feasible
- $\|z - x\|_2 = R/2$ and

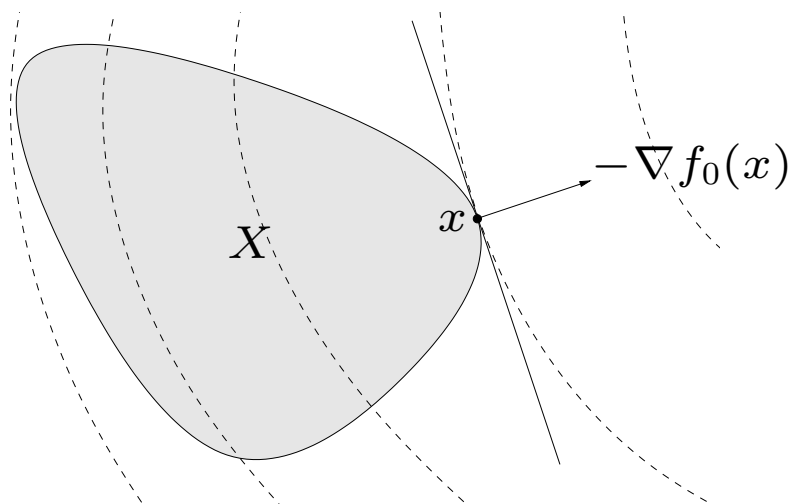
$$f_0(z) \leq \theta f_0(y) + (1 - \theta)f_0(x) < f_0(x)$$

which contradicts our assumption that x is locally optimal

Optimality criterion for differentiable f_0

x is optimal if and only if it is feasible and

$$\nabla f_0(x)^T(y - x) \geq 0 \quad \text{for all feasible } y$$



if nonzero, $\nabla f_0(x)$ defines a supporting hyperplane to feasible set X at x

- **unconstrained problem:** x is optimal if and only if

$$x \in \mathbf{dom} f_0, \quad \nabla f_0(x) = 0$$

- **equality constrained problem**

$$\text{minimize } f_0(x) \quad \text{subject to } Ax = b$$

x is optimal if and only if there exists a ν such that

$$x \in \mathbf{dom} f_0, \quad Ax = b, \quad \nabla f_0(x) + A^T \nu = 0$$

- **minimization over nonnegative orthant**

$$\text{minimize } f_0(x) \quad \text{subject to } x \succeq 0$$

x is optimal if and only if

$$x \in \mathbf{dom} f_0, \quad x \succeq 0, \quad \begin{cases} \nabla f_0(x)_i \geq 0 & x_i = 0 \\ \nabla f_0(x)_i = 0 & x_i > 0 \end{cases}$$

Equivalent convex problems

two problems are (informally) **equivalent** if the solution of one is readily obtained from the solution of the other, and vice-versa

some common transformations that preserve convexity:

- **eliminating equality constraints**

$$\begin{array}{ll}\text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & Ax = b\end{array}$$

is equivalent to

$$\begin{array}{ll}\text{minimize (over } z) & f_0(Fz + x_0) \\ \text{subject to} & f_i(Fz + x_0) \leq 0, \quad i = 1, \dots, m\end{array}$$

where F and x_0 are such that

$$Ax = b \iff x = Fz + x_0 \text{ for some } z$$

- **introducing equality constraints**

$$\begin{array}{ll}\text{minimize} & f_0(A_0x + b_0) \\ \text{subject to} & f_i(A_ix + b_i) \leq 0, \quad i = 1, \dots, m\end{array}$$

is equivalent to

$$\begin{array}{ll}\text{minimize (over } x, y_i) & f_0(y_0) \\ \text{subject to} & f_i(y_i) \leq 0, \quad i = 1, \dots, m \\ & y_i = A_ix + b_i, \quad i = 0, 1, \dots, m\end{array}$$

- **introducing slack variables for linear inequalities**

$$\begin{array}{ll}\text{minimize} & f_0(x) \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \dots, m\end{array}$$

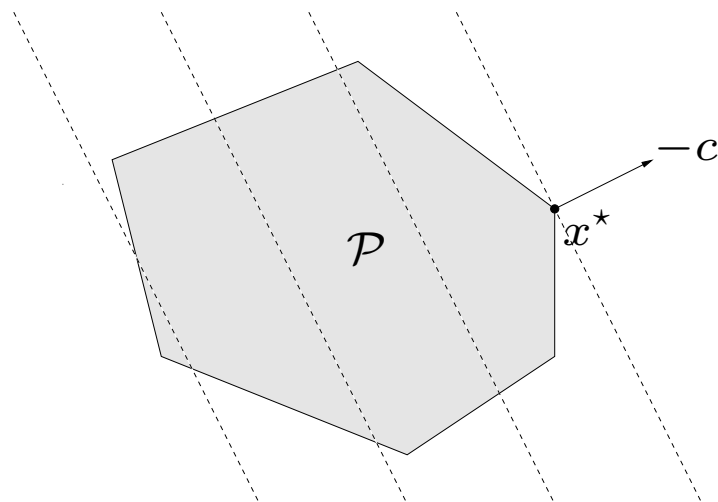
is equivalent to

$$\begin{array}{ll}\text{minimize (over } x, s) & f_0(x) \\ \text{subject to} & a_i^T x + s_i = b_i, \quad i = 1, \dots, m \\ & s_i \geq 0, \quad i = 1, \dots, m\end{array}$$

Linear program (LP)

$$\begin{array}{ll}\text{minimize} & c^T x + d \\ \text{subject to} & Gx \preceq h \\ & Ax = b\end{array}$$

- convex problem with affine objective and constraint functions
- feasible set is a polyhedron



Examples

diet problem: choose quantities x_1, \dots, x_n of n foods

- one unit of food j costs c_j , contains amount a_{ij} of nutrient i
- healthy diet requires nutrient i in quantity at least b_i

to find cheapest healthy diet,

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \succeq b, \quad x \succeq 0\end{array}$$

piecewise-linear minimization

$$\text{minimize} \quad \max_{i=1,\dots,m} (a_i^T x + b_i)$$

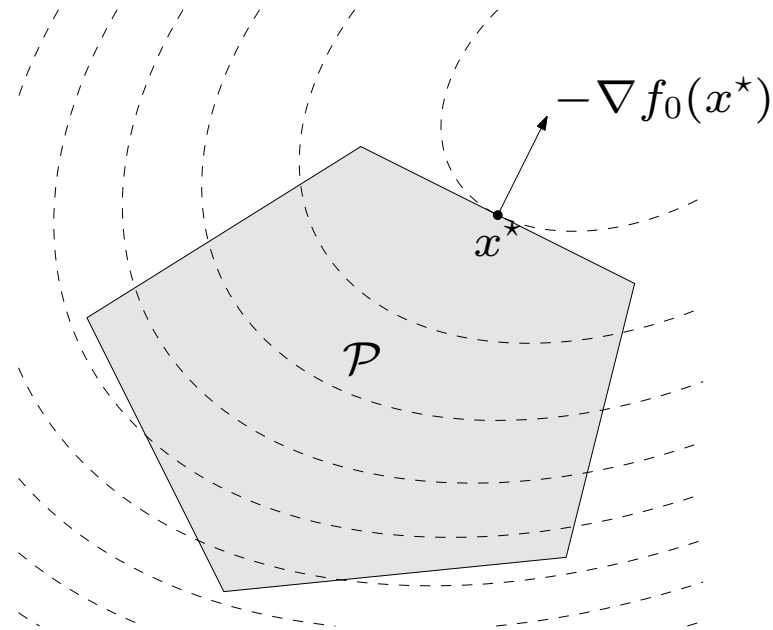
equivalent to an LP

$$\begin{array}{ll}\text{minimize} & t \\ \text{subject to} & a_i^T x + b_i \leq t, \quad i = 1, \dots, m\end{array}$$

Quadratic program (QP)

$$\begin{array}{ll}\text{minimize} & (1/2)x^T P x + q^T x + r \\ \text{subject to} & Gx \preceq h \\ & Ax = b\end{array}$$

- $P \in \mathbf{S}_{+}^n$, so objective is convex quadratic
- minimize a convex quadratic function over a polyhedron



Examples

least-squares

$$\text{minimize} \quad \|Ax - b\|_2^2$$

- analytical solution $x^* = A^\dagger b$ (A^\dagger is pseudo-inverse)
- can add linear constraints, *e.g.*, $l \preceq x \preceq u$

linear program with random cost

$$\begin{aligned} &\text{minimize} && \bar{c}^T x + \gamma x^T \Sigma x = \mathbf{E} c^T x + \gamma \mathbf{var}(c^T x) \\ &\text{subject to} && Gx \preceq h, \quad Ax = b \end{aligned}$$

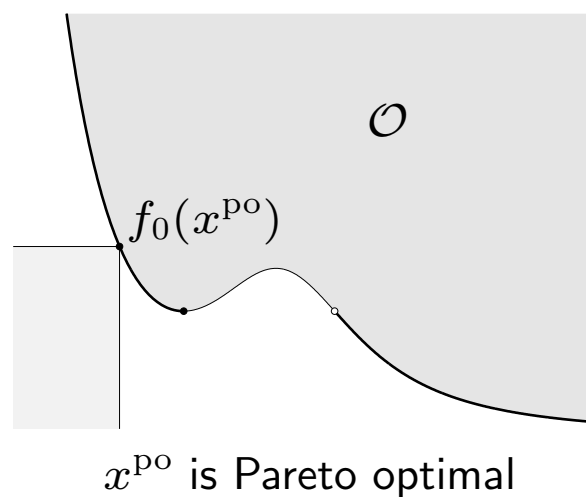
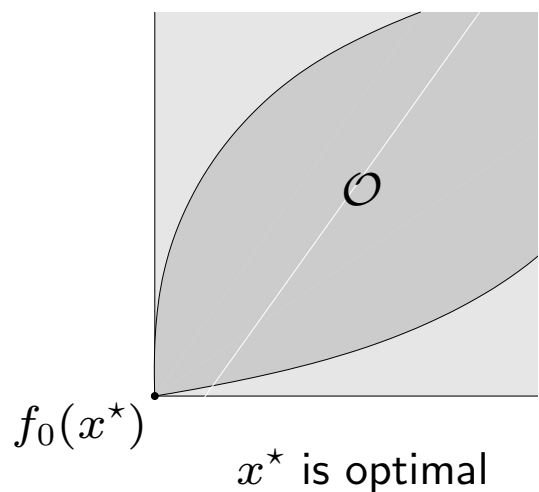
- c is random vector with mean \bar{c} and covariance Σ
- hence, $c^T x$ is random variable with mean $\bar{c}^T x$ and variance $x^T \Sigma x$
- $\gamma > 0$ is risk aversion parameter; controls the trade-off between expected cost and variance (risk)

Optimal and Pareto optimal points

set of achievable objective values

$$\mathcal{O} = \{f_0(x) \mid x \text{ feasible}\}$$

- feasible x is **optimal** if $f_0(x)$ is the minimum value of \mathcal{O}
- feasible x is **Pareto optimal** if $f_0(x)$ is a minimal value of \mathcal{O}



Multicriterion optimization

vector optimization problem with $K = \mathbf{R}_+^q$

$$f_0(x) = (F_1(x), \dots, F_q(x))$$

- q different objectives F_i ; roughly speaking we want all F_i 's to be small
- feasible x^* is optimal if

$$y \text{ feasible} \implies f_0(x^*) \preceq f_0(y)$$

if there exists an optimal point, the objectives are noncompeting

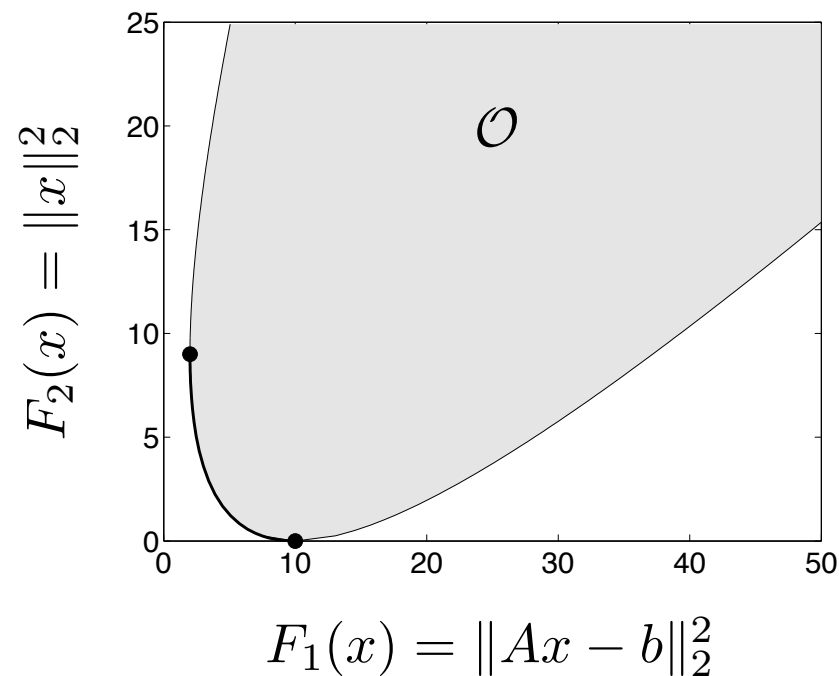
- feasible x^{po} is Pareto optimal if

$$y \text{ feasible, } f_0(y) \preceq f_0(x^{\text{po}}) \implies f_0(x^{\text{po}}) = f_0(y)$$

if there are multiple Pareto optimal values, there is a trade-off between the objectives

Regularized least-squares

$$\text{minimize (w.r.t. } \mathbf{R}_+^2) \quad (\|Ax - b\|_2^2, \|x\|_2^2)$$



example for $A \in \mathbf{R}^{100 \times 10}$; heavy line is formed by Pareto optimal points

4. ML examples

- Maximum likelihood estimation
- Logistic regression

Parametric distribution estimation

- distribution estimation problem: estimate probability density $p(y)$ of a random variable from observed values
- parametric distribution estimation: choose from a family of densities $p_x(y)$, indexed by a parameter x

maximum likelihood estimation

$$\text{maximize (over } x) \quad \log p_x(y)$$

- y is observed value
- $l(x) = \log p_x(y)$ is called log-likelihood function
- can add constraints $x \in C$ explicitly, or define $p_x(y) = 0$ for $x \notin C$
- a convex optimization problem if $\log p_x(y)$ is concave in x for fixed y

Linear measurements with IID noise

linear measurement model

$$y_i = a_i^T x + v_i, \quad i = 1, \dots, m$$

- $x \in \mathbf{R}^n$ is vector of unknown parameters
- v_i is IID measurement noise, with density $p(z)$
- y_i is measurement: $y \in \mathbf{R}^m$ has density $p_x(y) = \prod_{i=1}^m p(y_i - a_i^T x)$

maximum likelihood estimate: any solution x of

$$\text{maximize } l(x) = \sum_{i=1}^m \log p(y_i - a_i^T x)$$

(y is observed value)

examples

- Gaussian noise $\mathcal{N}(0, \sigma^2)$: $p(z) = (2\pi\sigma^2)^{-1/2} e^{-z^2/(2\sigma^2)}$,

$$l(x) = -\frac{m}{2} \log(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^m (a_i^T x - y_i)^2$$

ML estimate is LS solution

- Laplacian noise: $p(z) = (1/(2a)) e^{-|z|/a}$,

$$l(x) = -m \log(2a) - \frac{1}{a} \sum_{i=1}^m |a_i^T x - y_i|$$

ML estimate is ℓ_1 -norm solution

- uniform noise on $[-a, a]$:

$$l(x) = \begin{cases} -m \log(2a) & |a_i^T x - y_i| \leq a, \quad i = 1, \dots, m \\ -\infty & \text{otherwise} \end{cases}$$

ML estimate is any x with $|a_i^T x - y_i| \leq a$

Logistic regression

random variable $y \in \{0, 1\}$ with distribution

$$p = \mathbf{prob}(y = 1) = \frac{\exp(a^T u + b)}{1 + \exp(a^T u + b)}$$

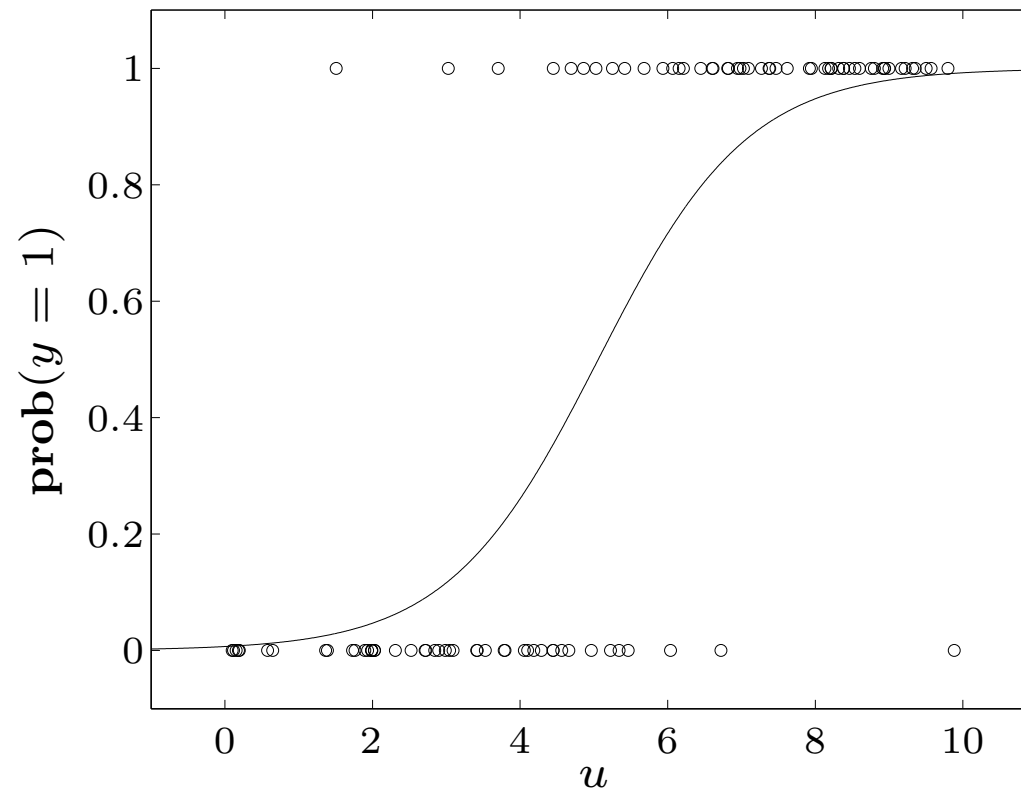
- a, b are parameters; $u \in \mathbf{R}^n$ are (observable) explanatory variables
- estimation problem: estimate a, b from m observations (u_i, y_i)

log-likelihood function (for $y_1 = \dots = y_k = 1, y_{k+1} = \dots = y_m = 0$):

$$\begin{aligned} l(a, b) &= \log \left(\prod_{i=1}^k \frac{\exp(a^T u_i + b)}{1 + \exp(a^T u_i + b)} \prod_{i=k+1}^m \frac{1}{1 + \exp(a^T u_i + b)} \right) \\ &= \sum_{i=1}^k (a^T u_i + b) - \sum_{i=1}^m \log(1 + \exp(a^T u_i + b)) \end{aligned}$$

concave in a, b

example ($n = 1$, $m = 50$ measurements)



- circles show 50 points (u_i, y_i)
- solid curve is ML estimate of $p = \exp(au + b) / (1 + \exp(au + b))$

5. Duality

- Lagrange dual problem
- Weak and strong duality
- Geometric interpretation
- Optimality (KKT) conditions
- Duality and problem reformulations

Constraint optimization problem:

$$\max_{x,y} f(x,y)$$

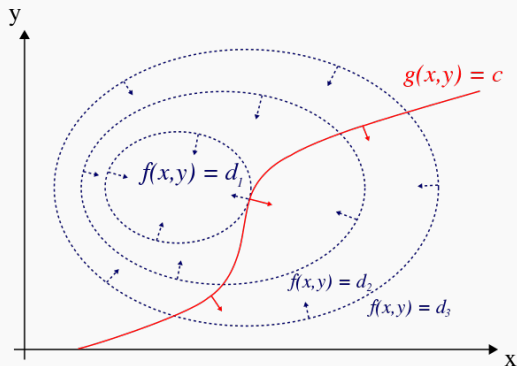
$$\text{subject to } g(x,y) = 0$$

Lagrangian:

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda g(x, y)$$

$$\nabla_{x,y} \mathcal{L}(x, y, \lambda) = 0 \rightarrow \nabla_{x,y} f = \lambda \nabla_{x,y} g$$

$$\nabla_{\lambda} \mathcal{L}(x, y, \lambda) = 0 \rightarrow g(x, y) = 0$$



Lagrangian

standard form problem (not necessarily convex)

$$\begin{array}{ll}\text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p\end{array}$$

variable $x \in \mathbf{R}^n$, domain \mathcal{D} , optimal value p^*

Lagrangian: $L : \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^p \rightarrow \mathbf{R}$, with $\text{dom } L = \mathcal{D} \times \mathbf{R}^m \times \mathbf{R}^p$,

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \nu_i h_i(x)$$

- weighted sum of objective and constraint functions
- λ_i is Lagrange multiplier associated with $f_i(x) \leq 0$
- ν_i is Lagrange multiplier associated with $h_i(x) = 0$

Lagrange dual function

Lagrange dual function: $g : \mathbf{R}^m \times \mathbf{R}^p \rightarrow \mathbf{R}$,

$$\begin{aligned} g(\lambda, \nu) &= \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) \\ &= \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \nu_i h_i(x) \right) \end{aligned}$$

g is concave, can be $-\infty$ for some λ, ν

lower bound property: if $\lambda \succeq 0$, then $g(\lambda, \nu) \leq p^*$

proof: if \tilde{x} is feasible and $\lambda \succeq 0$, then

$$f_0(\tilde{x}) \geq L(\tilde{x}, \lambda, \nu) \geq \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) = g(\lambda, \nu)$$

minimizing over all feasible \tilde{x} gives $p^* \geq g(\lambda, \nu)$

Least-norm solution of linear equations

$$\begin{array}{ll}\text{minimize} & x^T x \\ \text{subject to} & Ax = b\end{array}$$

dual function

- Lagrangian is $L(x, \nu) = x^T x + \nu^T (Ax - b)$
- to minimize L over x , set gradient equal to zero:

$$\nabla_x L(x, \nu) = 2x + A^T \nu = 0 \quad \implies \quad x = -(1/2)A^T \nu$$

- plug in in L to obtain g :

$$g(\nu) = L((-1/2)A^T \nu, \nu) = -\frac{1}{4}\nu^T A A^T \nu - b^T \nu$$

a concave function of ν

lower bound property: $p^* \geq -(1/4)\nu^T A A^T \nu - b^T \nu$ for all ν

Standard form LP

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax = b, \quad x \succeq 0\end{array}$$

dual function

- Lagrangian is

$$\begin{aligned}L(x, \lambda, \nu) &= c^T x + \nu^T (Ax - b) - \lambda^T x \\ &= -b^T \nu + (c + A^T \nu - \lambda)^T x\end{aligned}$$

- L is affine in x , hence

$$g(\lambda, \nu) = \inf_x L(x, \lambda, \nu) = \begin{cases} -b^T \nu & A^T \nu - \lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases}$$

g is linear on affine domain $\{(\lambda, \nu) \mid A^T \nu - \lambda + c = 0\}$, hence concave

lower bound property: $p^* \geq -b^T \nu$ if $A^T \nu + c \succeq 0$

The dual problem

Lagrange dual problem

$$\begin{array}{ll}\text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0\end{array}$$

- finds best lower bound on p^* , obtained from Lagrange dual function
- a convex optimization problem; optimal value denoted d^*
- λ, ν are dual feasible if $\lambda \succeq 0, (\lambda, \nu) \in \mathbf{dom} g$
- often simplified by making implicit constraint $(\lambda, \nu) \in \mathbf{dom} g$ explicit

example: standard form LP and its dual (page 5–5)

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \succeq 0\end{array}$$

$$\begin{array}{ll}\text{maximize} & -b^T \nu \\ \text{subject to} & A^T \nu + c \succeq 0\end{array}$$

Weak and strong duality

weak duality: $d^* \leq p^*$

- always holds (for convex and nonconvex problems)
- can be used to find nontrivial lower bounds for difficult problems
for example, solving the SDP

$$\begin{array}{ll}\text{maximize} & -\mathbf{1}^T \nu \\ \text{subject to} & W + \mathbf{diag}(\nu) \succeq 0\end{array}$$

gives a lower bound for the two-way partitioning problem on page 5–7

strong duality: $d^* = p^*$

- does not hold in general
- (usually) holds for convex problems
- conditions that guarantee strong duality in convex problems are called **constraint qualifications**

Slater's constraint qualification

strong duality holds for a convex problem

$$\begin{array}{ll}\text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & Ax = b\end{array}$$

if it is strictly feasible, *i.e.*,

$$\exists x \in \mathbf{int} \mathcal{D} : \quad f_i(x) < 0, \quad i = 1, \dots, m, \quad Ax = b$$

- also guarantees that the dual optimum is attained (if $p^* > -\infty$)
- can be sharpened: *e.g.*, can replace $\mathbf{int} \mathcal{D}$ with $\mathbf{relint} \mathcal{D}$ (interior relative to affine hull); linear inequalities do not need to hold with strict inequality, . . .
- there exist many other types of constraint qualifications

Inequality form LP

primal problem

$$\begin{array}{ll}\text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b\end{array}$$

dual function

$$g(\lambda) = \inf_x ((c + A^T \lambda)^T x - b^T \lambda) = \begin{cases} -b^T \lambda & A^T \lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases}$$

dual problem

$$\begin{array}{ll}\text{maximize} & -b^T \lambda \\ \text{subject to} & A^T \lambda + c = 0, \quad \lambda \succeq 0\end{array}$$

- from Slater's condition: $p^* = d^*$ if $A\tilde{x} \prec b$ for some \tilde{x}
- in fact, $p^* = d^*$ except when primal and dual are infeasible

Quadratic program

primal problem (assume $P \in \mathbf{S}_{++}^n$)

$$\begin{array}{ll}\text{minimize} & x^T P x \\ \text{subject to} & Ax \preceq b\end{array}$$

dual function

$$g(\lambda) = \inf_x (x^T P x + \lambda^T (Ax - b)) = -\frac{1}{4} \lambda^T A P^{-1} A^T \lambda - b^T \lambda$$

dual problem

$$\begin{array}{ll}\text{maximize} & -(1/4) \lambda^T A P^{-1} A^T \lambda - b^T \lambda \\ \text{subject to} & \lambda \succeq 0\end{array}$$

- from Slater's condition: $p^* = d^*$ if $A\tilde{x} \prec b$ for some \tilde{x}
- in fact, $p^* = d^*$ always

Complementary slackness

assume strong duality holds, x^* is primal optimal, (λ^*, ν^*) is dual optimal

$$\begin{aligned} f_0(x^*) = g(\lambda^*, \nu^*) &= \inf_x \left(f_0(x) + \sum_{i=1}^m \lambda_i^* f_i(x) + \sum_{i=1}^p \nu_i^* h_i(x) \right) \\ &\leq f_0(x^*) + \sum_{i=1}^m \lambda_i^* f_i(x^*) + \sum_{i=1}^p \nu_i^* h_i(x^*) \\ &\leq f_0(x^*) \end{aligned}$$

hence, the two inequalities hold with equality

- x^* minimizes $L(x, \lambda^*, \nu^*)$
- $\lambda_i^* f_i(x^*) = 0$ for $i = 1, \dots, m$ (known as complementary slackness):

$$\lambda_i^* > 0 \implies f_i(x^*) = 0, \quad f_i(x^*) < 0 \implies \lambda_i^* = 0$$

Karush-Kuhn-Tucker (KKT) conditions

the following four conditions are called KKT conditions (for a problem with differentiable f_i, h_i):

1. primal constraints: $f_i(x) \leq 0, i = 1, \dots, m, h_i(x) = 0, i = 1, \dots, p$
2. dual constraints: $\lambda \succeq 0$
3. complementary slackness: $\lambda_i f_i(x) = 0, i = 1, \dots, m$
4. gradient of Lagrangian with respect to x vanishes:

$$\nabla f_0(x) + \sum_{i=1}^m \lambda_i \nabla f_i(x) + \sum_{i=1}^p \nu_i \nabla h_i(x) = 0$$

from page 5–17: if strong duality holds and x, λ, ν are optimal, then they must satisfy the KKT conditions

KKT conditions for convex problem

if \tilde{x} , $\tilde{\lambda}$, $\tilde{\nu}$ satisfy KKT for a convex problem, then they are optimal:

- from complementary slackness: $f_0(\tilde{x}) = L(\tilde{x}, \tilde{\lambda}, \tilde{\nu})$
- from 4th condition (and convexity): $g(\tilde{\lambda}, \tilde{\nu}) = L(\tilde{x}, \tilde{\lambda}, \tilde{\nu})$

hence, $f_0(\tilde{x}) = g(\tilde{\lambda}, \tilde{\nu})$

if **Slater's condition** is satisfied:

x is optimal if and only if there exist λ , ν that satisfy KKT conditions

- recall that Slater implies strong duality, and dual optimum is attained
- generalizes optimality condition $\nabla f_0(x) = 0$ for unconstrained problem

Duality and problem reformulations

- equivalent formulations of a problem can lead to very different duals
- reformulating the primal problem can be useful when the dual is difficult to derive, or uninteresting

common reformulations

- introduce new variables and equality constraints
- make explicit constraints implicit or vice-versa
- transform objective or constraint functions
e.g., replace $f_0(x)$ by $\phi(f_0(x))$ with ϕ convex, increasing

Introducing new variables and equality constraints

$$\text{minimize } f_0(Ax + b)$$

- dual function is constant: $g = \inf_x L(x) = \inf_x f_0(Ax + b) = p^*$
- we have strong duality, but dual is quite useless

reformulated problem and its dual

$$\begin{array}{ll} \text{minimize} & f_0(y) \\ \text{subject to} & Ax + b - y = 0 \end{array} \qquad \begin{array}{ll} \text{maximize} & b^T \nu - f_0^*(\nu) \\ \text{subject to} & A^T \nu = 0 \end{array}$$

dual function follows from

$$\begin{aligned} g(\nu) &= \inf_{x,y} (f_0(y) - \nu^T y + \nu^T Ax + b^T \nu) \\ &= \begin{cases} -f_0^*(\nu) + b^T \nu & A^T \nu = 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

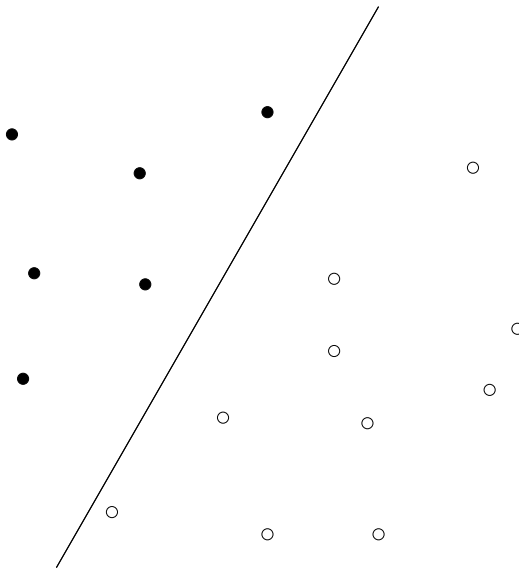
6. Margin-based classification

- Linear classification
- Robust linear classification
- SVM

Linear discrimination

separate two sets of points $\{x_1, \dots, x_N\}$, $\{y_1, \dots, y_M\}$ by a hyperplane:

$$a^T x_i + b > 0, \quad i = 1, \dots, N, \quad a^T y_i + b < 0, \quad i = 1, \dots, M$$



homogeneous in a , b , hence equivalent to

$$a^T x_i + b \geq 1, \quad i = 1, \dots, N, \quad a^T y_i + b \leq -1, \quad i = 1, \dots, M$$

a set of linear inequalities in a , b

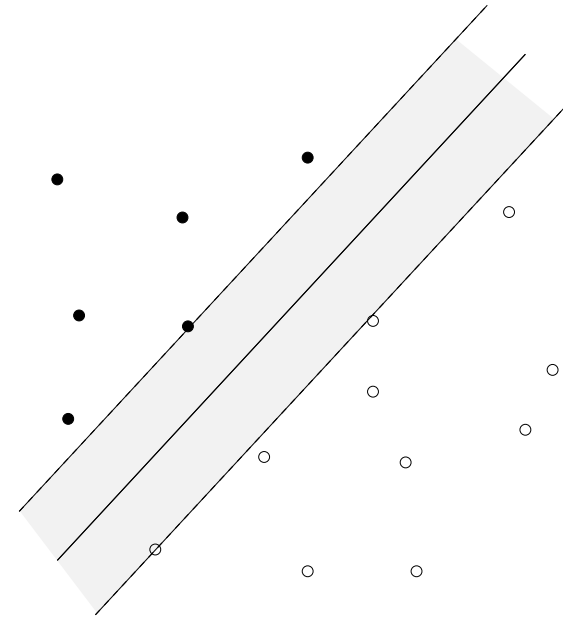
Robust linear discrimination

(Euclidean) distance between hyperplanes

$$\mathcal{H}_1 = \{z \mid a^T z + b = 1\}$$

$$\mathcal{H}_2 = \{z \mid a^T z + b = -1\}$$

is $\text{dist}(\mathcal{H}_1, \mathcal{H}_2) = 2/\|a\|_2$



to separate two sets of points by maximum margin,

$$\begin{aligned} & \text{minimize} && (1/2)\|a\|_2 \\ & \text{subject to} && a^T x_i + b \geq 1, \quad i = 1, \dots, N \\ & && a^T y_i + b \leq -1, \quad i = 1, \dots, M \end{aligned} \tag{1}$$

(after squaring objective) a QP in a, b

Lagrange dual of maximum margin separation problem (1)

$$\begin{aligned} & \text{maximize} && \mathbf{1}^T \lambda + \mathbf{1}^T \mu \\ & \text{subject to} && 2 \left\| \sum_{i=1}^N \lambda_i x_i - \sum_{i=1}^M \mu_i y_i \right\|_2 \leq 1 \\ & && \mathbf{1}^T \lambda = \mathbf{1}^T \mu, \quad \lambda \succeq 0, \quad \mu \succeq 0 \end{aligned} \tag{2}$$

from duality, optimal value is inverse of maximum margin of separation

interpretation

- change variables to $\theta_i = \lambda_i / \mathbf{1}^T \lambda$, $\gamma_i = \mu_i / \mathbf{1}^T \mu$, $t = 1 / (\mathbf{1}^T \lambda + \mathbf{1}^T \mu)$
- invert objective to minimize $1 / (\mathbf{1}^T \lambda + \mathbf{1}^T \mu) = t$

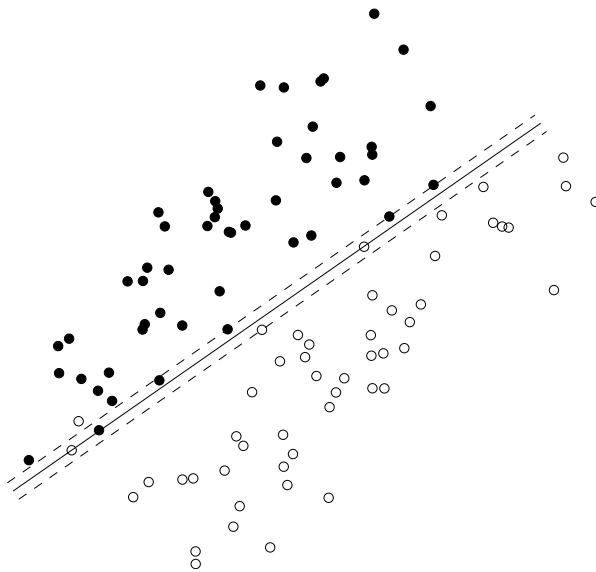
$$\begin{aligned} & \text{minimize} && t \\ & \text{subject to} && \left\| \sum_{i=1}^N \theta_i x_i - \sum_{i=1}^M \gamma_i y_i \right\|_2 \leq t \\ & && \theta \succeq 0, \quad \mathbf{1}^T \theta = 1, \quad \gamma \succeq 0, \quad \mathbf{1}^T \gamma = 1 \end{aligned}$$

optimal value is distance between convex hulls

Approximate linear separation of non-separable sets

$$\begin{array}{ll}\text{minimize} & \mathbf{1}^T u + \mathbf{1}^T v \\ \text{subject to} & a^T x_i + b \geq 1 - u_i, \quad i = 1, \dots, N \\ & a^T y_i + b \leq -1 + v_i, \quad i = 1, \dots, M \\ & u \succeq 0, \quad v \succeq 0\end{array}$$

- an LP in a, b, u, v
- at optimum, $u_i = \max\{0, 1 - a^T x_i - b\}$, $v_i = \max\{0, 1 + a^T y_i + b\}$
- can be interpreted as a heuristic for minimizing #misclassified points



Support vector classifier

$$\begin{aligned} &\text{minimize} && \|a\|_2 + \gamma(\mathbf{1}^T u + \mathbf{1}^T v) \\ &\text{subject to} && a^T x_i + b \geq 1 - u_i, \quad i = 1, \dots, N \\ & && a^T y_i + b \leq -1 + v_i, \quad i = 1, \dots, M \\ & && u \succeq 0, \quad v \succeq 0 \end{aligned}$$

produces point on trade-off curve between inverse of margin $2/\|a\|_2$ and classification error, measured by total slack $\mathbf{1}^T u + \mathbf{1}^T v$

same example as previous page,
with $\gamma = 0.1$:

