(Two-level) Logic Synthesis

Implicants and Prime Implicants

Becker/Molitor, Chapter 7.2

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Implicants and Prime implicants

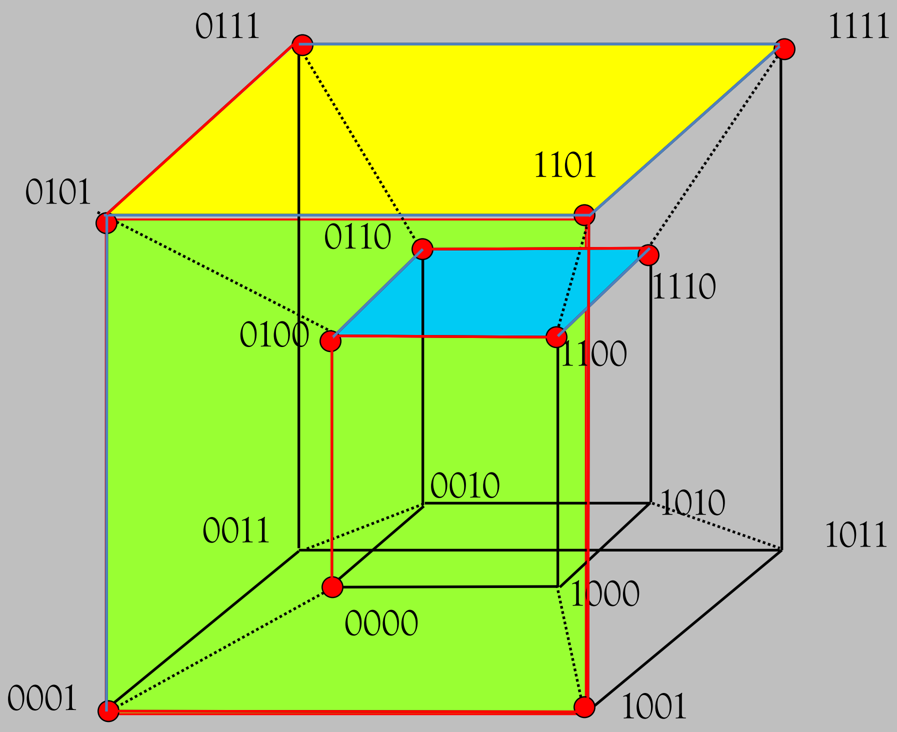
A Boolean function $f \in \mathbf{B}_n$ is *less than or equal to* another Boolean function $g \in \mathbf{B}_n$ ($f \leq g$), if $\forall \alpha \in \mathbf{B}^n$: $f(\alpha) \leq g(\alpha)$. (i.e. if f is 1, then so is g).

Definition:
Let f be a Boolean function with one output.
An **implicant of f** is a monomial q with $\psi(q) \leq f$.
A **prime implicant of f** is a maximal implicant q of f,
i.e., there is no implicant s ($s \neq q$) of f with $\psi(q) \leq \psi(s)$.

Illustration via n-dimensional hypercubes

- An **implicant of f** is a subcube that contains only marked nodes.
- A **prime implicant of f** is a maximal such subcube.

Implicants and Prime implicants



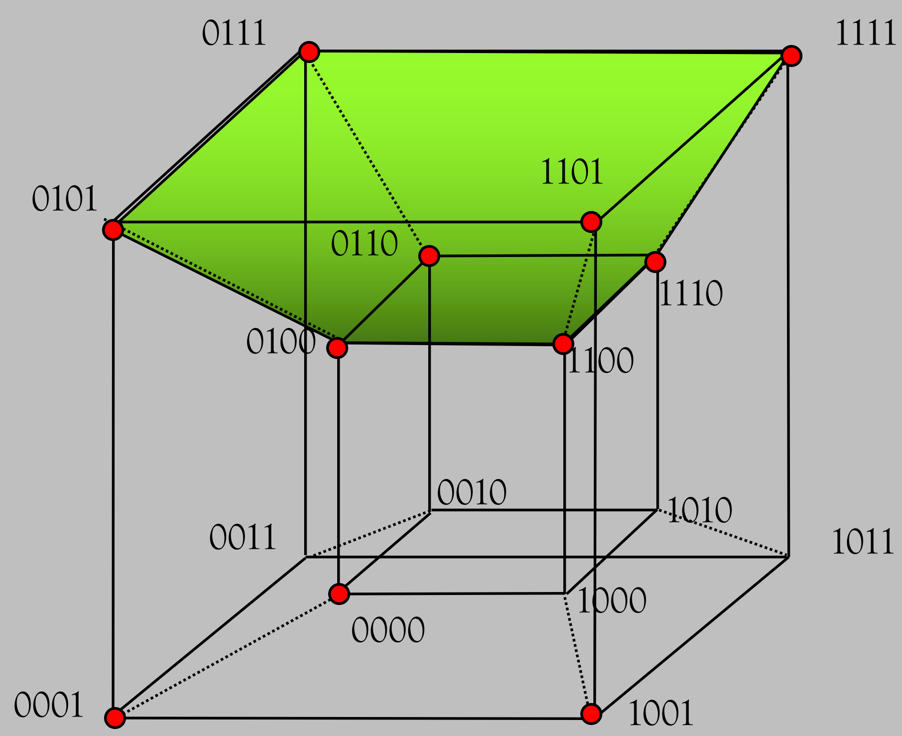
Implicants:

- all marked nodes,
- all edges whose nodes all are marked,
- all surfaces whose nodes all are marked,
- all 3-dimensional subcubes whose nodes all are marked.

In general:

Implicants are those subcubes whose nodes all are marked.

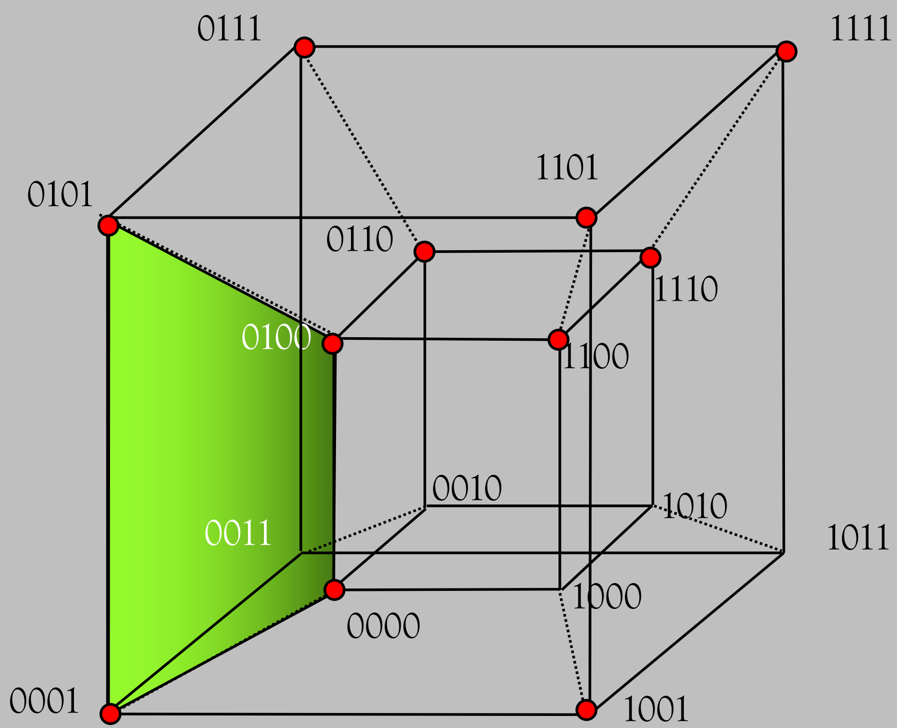
Implicants and Prime implicants



The function defining our hypercube has **3 prime implicants**:

- $\mathbf{x_2}$

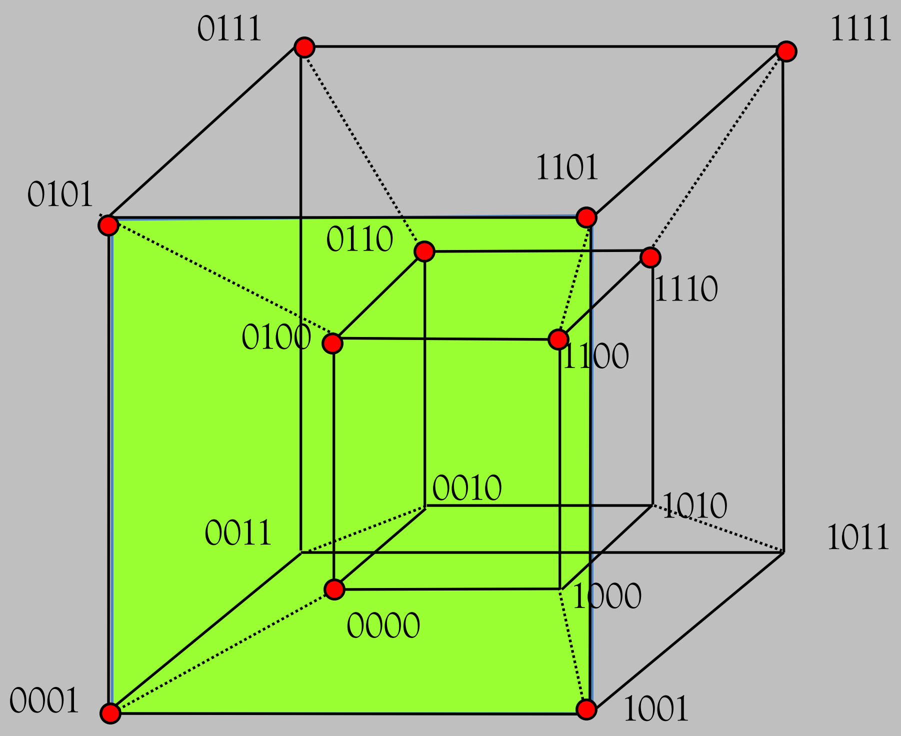
Implicants and Prime implicants



The function defining our hypercube has **3 prime implicants**:

- $\mathbf{x_2}$
- $\mathbf{x_1'x_3'}$

Implicants and Prime implicants



The function defining our hypercube has **3 prime implicants**:

- x_2
- $x_1'x_3'$
- $x_3'x_4$

Polynomials and implicants of a function f

Lemma:
All monomials of a polynomial p of f are implicants of f.

Proof (by contradiction)

Let p be a polynomial of f and let m be a monomial of p.

Assume for a contradiction that m is not an implicant of f, i.e., $\psi(m) \leq f$ does not hold.

Thus, there must be a valuation $(\alpha_1,\ldots,\alpha_n)$ of the variables $(x_1,\ldots,x_n)$ with

- $f(\alpha_1,\ldots,\alpha_n) = 0$, but
- $\psi(m)(\alpha_1,\ldots,\alpha_n) = 1$, and so also $\psi(p)(\alpha_1,\ldots,\alpha_n) = 1$.

However, by assumption p is a polynomial of f, and so $\psi(p)(\alpha_1,\ldots,\alpha_n) = f(\alpha_1,\ldots,\alpha_n)$. Contradiction!

Cheapest covering of all marked nodes

We are searching for a polynomial f of minimal cost, i.e., we are searching for a so-called minimal polynomial:

Definition:

A **minimal polynomial** p of a Boolean function f is a polynomial of f of minimal cost, i.e., a polynomial of f, s.t., $cost(p) \leq cost(p')$ for all (other) polynomials p' of f.

Prime Implicant Theorem of Quine

Theorem (Quine):
Every minimal polynomial p of a Boolean function f consists only of prime implicants of f.

Willard Quine (1928-2000)

Proof (by contradiction)

Assume that p contains an implicant of f that is not prime.
Thus, m is covered by a prime implicant m' of f. In other words, it is contained in m'.

By definition of cost, we have cost(m') < cost(m).

Replacing the implicant m by the prime implicant m', we obtain another polynomial p', that is still a polynomial of f, s.t. cost(p') < cost(p).

This contradicts the assumption that p is a minimal polynomial!

→ To construct a minimal polynomial of a Boolean function f, we should first find its prime implicants!

Computation of Implicants (1/2)

Lemma 1:
If m is an implicant of f, then so are $m \cdot x$ and $m \cdot x'$ for every variable x that occurs neither as positive or negative literal in m.

Proof

More formally: By assumption, m is an implicant of f, i.e.: $\psi(m) \leq f$.

$$\begin{aligned}
\psi(m) &= \psi(m)\cdot(\psi(x)+\neg\psi(x)) && \text{(Complements)}\\
&= \psi(m)\cdot(\psi(x)+\psi(x')) && \text{(Definition of } \psi\text{)}\\
&= \psi(m)\cdot\psi(x)+\psi(m)\cdot\psi(x') && \text{(Distributivity)}\\
&= \psi(m\cdot x)+\psi(m\cdot x') \geq \psi(m\cdot x),\ \psi(m\cdot x') && \text{(Definition of } \psi\text{)}
\end{aligned}$$

So we have: $\psi(m\cdot x), \psi(m\cdot x') \leq f$, i.e. $m\cdot x$ and $m\cdot x'$ are also implicants of f.

Computation of Implicants (2/2)

Lemma 2:
If $m \cdot x$ and $m \cdot x'$ are implicants of f, then so is m.

Proof
As $m \cdot x$ and $m \cdot x'$ are implicants of f, by definition of implicants, we have $f \geq \psi(m \cdot x)$ and $f \geq \psi(m \cdot x')$.

Thus, we also have $f \geq \psi(m \cdot x)+\psi(m \cdot x')$

$$
\begin{aligned}
f &\geq \psi(m \cdot x)+\psi(m \cdot x') \\
&= \psi(m) \cdot \psi(x)+\psi(m) \cdot \psi(x') && \text{(Definition of } \psi\text{)} \\
&= \psi(m) \cdot (\psi(x)+\psi(x')) && \text{(Distributivity)} \\
&= \psi(m) \cdot (\psi(x)+\neg\psi(x)) && \text{(Definition of } \psi\text{)} \\
&= \psi(m) && \text{(Complements)}
\end{aligned}
$$

Characterization of Implicants

Theorem (Implicants):

A monomial m is an implicant of f if and only if, either

- m is a minterm of f, or
- $m \cdot x$ and $m \cdot x'$ are implicants of f
  for a variable x that does not occur in m.

Thus:

$$m \in \text{Implicant}(f) \Leftrightarrow$$
$$[m \in \text{Minterm}(f)] \vee [m \cdot x,\ m \cdot x' \in \text{Implicant}(f)]$$

Proof

Follows directly from Lemma 1 and Lemma 2.