SAARLAND UNIVERSITY
COMPUTER SCIENCE

Prof. Dr. Jan Reineke
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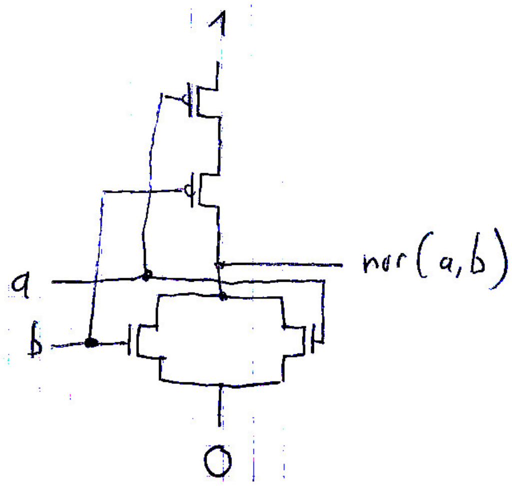
Systemarchitektur SS 2021

Lösungsskizze 3

Aufgabe 3.1: Gatter

1)



2) Wir haben Schaltkreise für AND, OR und NOT. Damit können wir beliebige konjunktive oder disjunktive Normalformen darstellen. Da für jede Boolesche Funktion eine Darstellung in diesen Normalformen existiert, können wir damit jede Boolesche Funktion darstellen.

3) Ja, da wir die in Teil 2 angesprochenen Gatter alle darstellen können.

- $NOT(x) = NAND(x, x)$
- $AND_2(x, y) = NOT(NAND_2(x, y))$
- $OR_2(x, y) = NAND_2(NOT(x), NOT(y))$
- $NOR_2 = NOT(OR_2(x, y))$
- $NAND_2(x, y) = NAND_2(x, y)$
- $XOR_2(x, y) = OR_2(AND_2(x, NOT(y)), AND_2(NOT(x), y))$

Aufgabe 3.2: Ein einfacher Schaltkreis

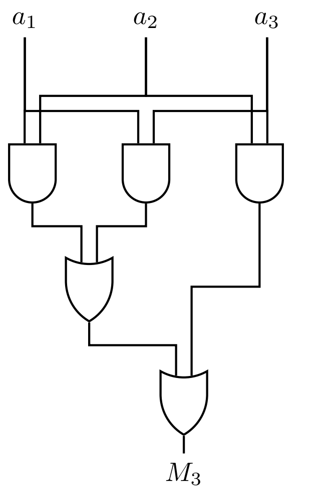
Wir stellen zunächst die Wahrheitstabelle auf:

a_1	a_2	a_3	$M_3(a_1, a_2, a_3)$
0	0	0	0
1	0	0	0
0	1	0	0
1	1	0	1
0	0	1	0
1	0	1	1
0	1	1	1
1	1	1	1

Damit mehr als die Hälfte der drei Parameter von M_3 den Wert 1 haben, muss dies bei mindestens zwei der Parameter erfüllt sein. Ein Boolescher Ausdruck ist also gegeben durch:

$$(a_1 \wedge a_2) \vee (a_1 \wedge a_3) \vee (a_2 \wedge a_3)$$

Daraus ergibt sich der folgende Schaltkreis:



Kosten: 5
Tiefe: 3

Aufgabe 3.3: Binärzahlen

1) $(-1)^1 \cdot (2^{14} + 2^{11} + 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^2 + 2^1 + 2^0) = -20423$

2) $2^{14} + 2^{11} + 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^2 + 2^1 + 2^0 - (2^{15} - 1) = -12344$

3) $-12344 - 1 = -12345$

4) $2^{11} + 2^{10} + 2^7 + 2^6 + 2^5 + 2^4 + 2^3 + 2^2 + 2^{-2} + 2^{-3} + 2^{-4} = 3324,4375$

5) $-2^{11} + 2^{10} + 2^7 + 2^6 + 2^5 + 2^4 + 2^3 + 2^2 + 2^{-2} + 2^{-3} + 2^{-4} = -771,5625$

Aufgabe 3.4: Zweier-Komplement-Darstellung

In den Beweisen ist folgendes Lemma hilfreich (Lemma 1):

$$\forall n \in \mathbb{N}: \sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

Beweis (geometrische Summe):
Für Summen der Form $\sum_{i=0}^{n} q^i$ gilt $\sum_{i=0}^{n} q^i = \frac{q^{n+1}-1}{q-1}$. Damit folgt obige Aussage nach Umformen.

Beweis (durch Induktion):
Wir zeigen:

$$\forall n \in \mathbb{N}: \sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

Beweis durch Induktion über n.
IA: n = 0

$$\sum_{i=0}^{n} 2^i = 1 = 2 - 1 = 2^1 - 1$$

IV:

$$\text{Für beliebiges aber festes } n \in \mathbb{N}: \sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

IS: n $\to$ n+1

$$\sum_{i=0}^{n+1} 2^i = 2^{n+1} + \sum_{i=0}^{n} 2^i \overset{IV}{=} 2^{n+1} + 2^{n+1} - 1 = 2^{n+2} - 1$$

■

(a)

$$\begin{aligned}
[0a] = & \sum_{i=0}^{n-1} \delta(a_i)2^i - 0 \cdot 2^n && \text{Definition } [\cdot] \\
= & \sum_{i=0}^{n-1} \delta(a_i)2^i \\
= & \langle a \rangle && \text{Definition } \langle\cdot\rangle
\end{aligned}$$

■

(b)

$$\begin{aligned}
[a] * 2 = & (\sum_{i=0}^{n-2} \delta(a_i)2^i - \delta(a_{n-1})2^{n-1}) * 2 && \text{Definition } [\cdot] \\
= & (\sum_{i=0}^{n-2} \delta(a_i)2^i) * 2 - \delta(a_{n-1})2^n \\
= & \sum_{i=0}^{n-2} \delta(a_i)2^{i+1} - \delta(a_{n-1})2^n \\
= & \sum_{i=0}^{n-2} \delta(a_i)2^{i+1} + 0 * 2^0 - \delta(a_{n-1})2^n \\
= & [a0]
\end{aligned}$$

■

(c) Die Aussage gilt nicht. Beweis durch Gegenbeispiel. Sei a = 1

$$[10] = -2$$
$$< 1 > *2 = 2$$

Da $2 \neq -2$, ist die Aussage widerlegt.

(d) Diese Aussage gilt nicht. Beweis durch Gegenbeispiel. Sei a = 10

$$[10] = -2$$
$$[100] = -4$$

Da $-2 \neq -4$, ist die Aussage widerlegt.

(e)

$$\begin{aligned}
[a] &= \sum_{i=0}^{n-2} \delta(a_i)2^i - \delta(a_{n-1})2^{n-1} && \text{Definition } [\cdot] \\
&= \sum_{i=0}^{n-2} \delta(a_i)2^i - 2\delta(a_{n-1})2^{n-1} + \delta(a_{n-1})2^{n-1} \\
&= \sum_{i=0}^{n-2} \delta(a_i)2^i - \delta(a_{n-1})2^{n} + \delta(a_{n-1})2^{n-1} \\
&= \sum_{i=0}^{n-1} \delta(a_i)2^i - \delta(a_{n-1})2^{n} \\
&= [a_{n-1}a]
\end{aligned}$$

■

(f)

$$\begin{aligned}
[\bar{a}] + 1 &= (\sum_{i=0}^{n-2} \delta(\bar{a}_i)2^i) - \delta(\bar{a}_{n-1})2^{n-1} + 1 && \text{Definition } [\cdot] \\
&= (\sum_{i=0}^{n-2} ((1-\delta(a_i))2^i)) - (1-\delta(a_{n-1}))2^{n-1} + 1 && \forall x \in \mathbb{B}.\ \bar{x} = 1 - x \\
&= (\sum_{i=0}^{n-2} (2^i - \delta(a_i)2^i)) - 2^{n-1} + \delta(a_{n-1})2^{n-1} + 1 \\
&= (\sum_{i=0}^{n-2} 2^i) - (\sum_{i=0}^{n-2} \delta(a_i)2^i) - 2^{n-1} + \delta(a_{n-1})2^{n-1} + 1 \\
&= 2^{n-1} - 1 - (\sum_{i=0}^{n-2} \delta(a_i)2^i) - 2^{n-1} + \delta(a_{n-1})2^{n-1} + 1 && \text{Lemma 1} \\
&= -(\sum_{i=0}^{n-2} \delta(a_i)2^i) + \delta(a_{n-1})2^{n-1} \\
&= -((\sum_{i=0}^{n-2} \delta(a_i)2^i) - \delta(a_{n-1})2^{n-1}) \\
&= -[a] && \text{Definition } [\cdot]
\end{aligned}$$

■

(g)

$$\begin{aligned}
[a] &= \sum_{i=0}^{n-2} \delta(a_i)2^i - \delta(a_{n-1})2^{n-1} && \text{Definition } [\cdot] \\
&\leq \sum_{i=0}^{n-2} 2^i - \delta(a_{n-1})2^{n-1} && \forall i \leq n-1. \delta(a_i) \leq 1 \\
&= 2^{n-1} - 1 - \delta(a_{n-1})2^{n-1} && \text{Lemma 1} \\
\delta(a_{n-1}) = 1 \Leftrightarrow & \quad [a] \leq 2^{n-1} - 1 - 2^{n-1} = -1 < 0
\end{aligned}$$

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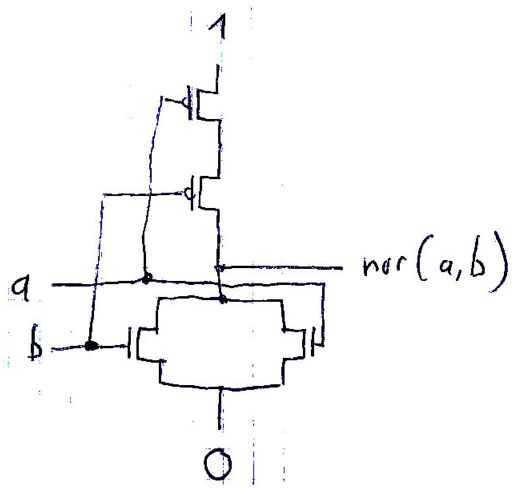
System Architecture SS 2021

Solution Sketch 3

Problem 3.1: Gates

1)



2) We have circuits for AND, OR, and NOT. With these, we can implement arbitrary conjunctive or disjunctive normal forms. Since for every Boolean function there is a representation with these normal forms, we can implement any Boolean function.

3) Yes, because we can implement all the gates mentioned in part 2.

- $NOT(x) = NAND(x, x)$
- $AND_2(x, y) = NOT(NAND_2(x, y))$
- $OR_2(x, y) = NAND_2(NOT(x), NOT(y))$
- $NOR_2 = NOT(OR_2(x, y))$
- $NAND_2(x, y) = NAND_2(x, y)$
- $XOR_2(x, y) = OR_2(AND_2(x, NOT(y)), AND_2(NOT(x), y))$

Problem 3.2: A Simple Circuit

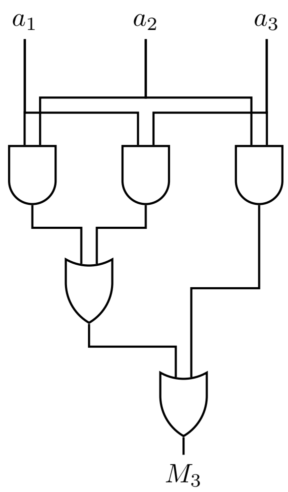
We first construct the truth table:

a_1	a_2	a_3	$M_3(a_1, a_2, a_3)$
0	0	0	0
1	0	0	0
0	1	0	0
1	1	0	1
0	0	1	0
1	0	1	1
0	1	1	1
1	1	1	1

More than half of the three parameters of M_3 have the value 1 if this is satisfied for at least two of the parameters. A Boolean expression is thus given by:

$$(a_1 \wedge a_2) \vee (a_1 \wedge a_3) \vee (a_2 \wedge a_3)$$

This results in the following circuit:



Cost: 5
Depth: 3

Problem 3.3: Binary Numbers

1) $(-1)^1 \cdot (2^{14} + 2^{11} + 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^2 + 2^1 + 2^0) = -20423$

2) $2^{14} + 2^{11} + 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^2 + 2^1 + 2^0 - (2^{15} - 1) = -12344$

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Problem 3.4: Two's Complement Representation

For the proofs, the following lemma is helpful (Lemma 1):

$$\forall n \in \mathbb{N}: \sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

Proof (geometric sum):
For sums of the form $\sum_{i=0}^{n} q^i$, we have $\sum_{i=0}^{n} q^i = \frac{q^{n+1}-1}{q-1}$. The above claim can be obtained via transforming.

Proof (by induction):
To show:

$$\forall n \in \mathbb{N}: \sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

Proof by induction on n.
Base case: n = 0

$$\sum_{i=0}^{n} 2^i = 1 = 2 - 1 = 2^1 - 1$$

Induction hypothesis (IH):

$$\text{For an arbitrary but fixed } n \in \mathbb{N}: \sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

Induction step: n $\rightarrow$ n+1

$$\sum_{i=0}^{n+1} 2^i = 2^{n+1} + \sum_{i=0}^{n} 2^i \overset{IH}{=} 2^{n+1} + 2^{n+1} - 1 = 2^{n+2} - 1$$

■

(a)

$$\begin{aligned}
[0a] = &\sum_{i=0}^{n-1} \delta(a_i)2^i - 0 \cdot 2^n && \text{Definition } [\cdot] \\
= &\sum_{i=0}^{n-1} \delta(a_i)2^i \\
= &\langle a \rangle && \text{Definition } \langle\cdot\rangle
\end{aligned}$$

■

(b)

$$\begin{aligned}
[a] * 2 = &\left(\sum_{i=0}^{n-2} \delta(a_i)2^i - \delta(a_{n-1})2^{n-1}\right) * 2 && \text{Definition } [\cdot] \\
= &\left(\sum_{i=0}^{n-2} \delta(a_i)2^i\right) * 2 - \delta(a_{n-1})2^n \\
= &\sum_{i=0}^{n-2} \delta(a_i)2^{i+1} - \delta(a_{n-1})2^n \\
= &\sum_{i=0}^{n-2} \delta(a_i)2^{i+1} + 0 * 2^0 - \delta(a_{n-1})2^n \\
= &[a0]
\end{aligned}$$

■

(c) The property does not hold. Proof by counterexample. Let a = 1

$$[10] = -2$$
$$< 1 > *2 = 2$$

As $2 \neq -2$, the statement is disproved.

(d) The property does not hold. Proof by counterexample. Let a = 10

$$[10] = -2$$
$$[100] = -4$$

As $-2 \neq -4$, the statement is disproved.

(e)

$$\begin{aligned}
[a] = \;& \sum_{i=0}^{n-2} \delta(a_i)2^i - \delta(a_{n-1})2^{n-1} && \text{Definition } [\cdot] \\
= \;& \sum_{i=0}^{n-2} \delta(a_i)2^i - 2\delta(a_{n-1})2^{n-1} + \delta(a_{n-1})2^{n-1} \\
= \;& \sum_{i=0}^{n-2} \delta(a_i)2^i - \delta(a_{n-1})2^{n} + \delta(a_{n-1})2^{n-1} \\
= \;& \sum_{i=0}^{n-1} \delta(a_i)2^i - \delta(a_{n-1})2^{n} \\
= \;& [a_{n-1}a]
\end{aligned}$$

■

(f)

$$\begin{aligned}
[\bar{a}] + 1 = \;& (\sum_{i=0}^{n-2} \delta(\bar{a}_i)2^i) - \delta(\bar{a}_{n-1})2^{n-1} + 1 && \text{Definition } [\cdot] \\
= \;& (\sum_{i=0}^{n-2} ((1 - \delta(a_i))2^i)) - (1 - \delta(a_{n-1}))2^{n-1} + 1 && \forall x \in \mathbb{B}.\ \bar{x} = 1 - x \\
= \;& (\sum_{i=0}^{n-2} (2^i - \delta(a_i)2^i)) - 2^{n-1} + \delta(a_{n-1})2^{n-1} + 1 \\
= \;& (\sum_{i=0}^{n-2} 2^i) - (\sum_{i=0}^{n-2} \delta(a_i)2^i) - 2^{n-1} + \delta(a_{n-1})2^{n-1} + 1 \\
= \;& 2^{n-1} - 1 - (\sum_{i=0}^{n-2} \delta(a_i)2^i) - 2^{n-1} + \delta(a_{n-1})2^{n-1} + 1 && \text{Lemma 1} \\
= \;& -(\sum_{i=0}^{n-2} \delta(a_i)2^i) + \delta(a_{n-1})2^{n-1} \\
= \;& -((\sum_{i=0}^{n-2} \delta(a_i)2^i) - \delta(a_{n-1})2^{n-1}) \\
= \;& -[a] && \text{Definition } [\cdot]
\end{aligned}$$

■

(g)

$$\begin{aligned}
[a] = \;& \sum_{i=0}^{n-2} \delta(a_i)2^i - \delta(a_{n-1})2^{n-1} && \text{Definition } [\cdot] \\
\leq \;& \sum_{i=0}^{n-2} 2^i - \delta(a_{n-1})2^{n-1} && \forall i \leq n-1. \delta(a_i) \leq 1 \\
= \;& 2^{n-1} - 1 - \delta(a_{n-1})2^{n-1} && \text{Lemma 1} \\
\delta(a_{n-1}) = 1 \Leftrightarrow \;& \text{[a]} \leq 2^{n-1} - 1 - 2^{n-1} = -1 < 0
\end{aligned}$$

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