Linear Algebra Review VI

- Nate Olson
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Overview

Covering sections 2.2 and 2.3, which focus on inverse matricies

Section 2.2

Definition of the inverse matrix $\mathbf{A}^{-1}$ when $\mathbf{A}$. $\mathbf{A}\mathbf{A}^{-1} = I$
$\mathbf{A}^{-1}\mathbf{A} = I$

Where I is the identity matrix.

singular matrix non-invertible matrix
nonsingular matrix invertible matrix

Theorem 4

Formula for calculating the inverse of a 2×2 matrix.

Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

If $ad - bc \neq 0$, then $\mathbf{A}$ is invertible and

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ab - bc = 0$, then $\mathbf{A}$ is not invertible.

$ad - bc$ is the *determinant*, chapter 3 focuses on determinants.

Theorem 5

If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$

Theorem 6

Three parts

1. If A is an invertible matrix, then A^{-1} is invertible and $(A^{-1})^{-1} = A$

2. If A and B are $n \times n$ invertible matrices then so is AB, and the inverse of AB is the product of the inverse of A and B in the reverse order. That is, $(AB)^{-1} = B^{-1}A^{-1}$

3. If A is an inversible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is $(A^T)^{-1} = (A^{-1})^T$

Elementary matrix (E) - obtianed by performing a single elementary row operation on an identity matrix.

Three kinds of elementary matricies, thoes that are products of the three elementary row operations replacement, interchange, and scaling (p. 7).

EA is the product of an elementary matrix and matrix A.

Theorem 7

An $n \times n$ matrix A is invertible if and only if A is row equibalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

Algorithm for finding A^{-1}

Row reduce the augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$. IF A is row equivalent to I, then $\begin{bmatrix} A & I \end{bmatrix}$ is row equivalent to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$. Otherwise, A is not invertible.

Section 2.3

Theorem 8

The Invertible Matrix Theorem
Let A be a square $n \times n$ matrix. Then the following statements are equivalent, all statements are *TRUE* or all are *FALSE*.

a. A is an invertible matrix.

b. A is row equivalent to the $n \times n$ identity matrix.

c. A has n pivot positions.

d. The equation $A\mathbf{x} = 0$ has only the trivial solution.

e. The columns of A form a linearly independent set.

f. The linear transformation $x \mapsto A\mathbf{x}$.

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.
i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. There is an $n \times n$ matrix C such that $CA = I$.

k. There is an $n \times n$ matrix D such that $AD = I$.

l. A^T is an invertible matrix.

Continued thoughout the book, p. 179 (parts m - r), 312 (s & t), 479 (u - x)

Proof

See figure 1 page 129 for relationships between individual parts used in proof.

- Circular relationship between a, j, d, c, and b
  - $a \mapsto j$ - by definition of an invertible matrix as $C = A^{-1}$

  - $j \mapsto d$ - from exercise 23 section 2.1
    + $CA = I$ and $Ax = 0$ only has a singular solution.
    + $CAx = C0 = 0$ therefore $I_n x = 0$ and $x = 0$, $Ax = 0$ has not free variables and therefore only has the trivial solution. Trivial solution is $x = 0$, $Ax = 0$ has a nontrivial solution if there is a nonzero vector x that satisfies the equation.

  - $d \mapsto c$ - there are no free variables therefore A has n pivot columns.

  - $c \mapsto b$ - for a square $n \times n$ matrix with n pivot positions the pivot positions must lie on the diagonal and therefore I_n is the reduced echelon form of A.

  - $b \mapsto a$ - by theorem 7 (see above).
- Circular relationship between a,k, and g
  - $a \mapsto k$ - same as $a \mapsto j$

  - $k \mapsto g$ - exercise 24 section 2.1
  - $g \mapsto a$ - exercise 24 section 2.2

- Equivalence g, h, and i (applies for all matrices)
  - Theorem 4 (section 1.4) and theorem 12 (a) section 1.9

- Equivalence d, e, and f (applies for all matrices)
  - Section 1.7 and Theorem 12 (b) section 1.9

- Parts g and d link h, i and e, f to the rest of IMT

- $a \Leftrightarrow a$ Theorem 6 (c) section 2.2.

Theorem 9

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transfromation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unqiue function satisfying (1) and (2).

(1) $S(T)(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$
(2) $T(S)(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$