THE MANGA GUIDE TO CALCULUS

COMICS INSIDE!

HIROYUKI KOJIMA
SHIN TOGAMI
BECOM CO., LTD.

Ohmsha
no starch press

PRAISE FOR THE MANGA GUIDE SERIES

"A fun and fairly painless lesson on what many consider to be a less-than-thrilling subject."
—SCHOOL LIBRARY JOURNAL

"This is really what a good math text should be like. . . . It presents statistics as something fun, and something enlightening."
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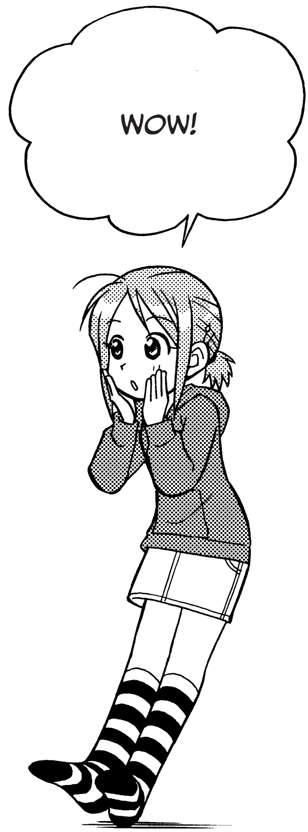

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PREFACE

There are some things that only manga can do.

You have just picked up and opened this book. You must be one of the following types of people.

The first type is someone who just loves manga and thinks, "Calculus illustrated with manga? Awesome!" If you are this type of person, you should immediately take this book to the cashier—you won't regret it. This is a very enjoyable manga title. It's no surprise—Shin Togami, a popular manga artist, drew the manga, and Becom Ltd., a real manga production company, wrote the scenario.

"But, manga that teaches about math has never been very enjoyable," you may argue. That's true. In fact, when an editor at Ohmsha asked me to write this book, I nearly turned down the opportunity. Many of the so-called "manga for education" books are quite disappointing. They may have lots of illustrations and large pictures, but they aren't really manga. But after seeing a sample from Ohmsha (it was *The Manga Guide to Statistics*), I totally changed my mind. Unlike many such manga guides, the sample was enjoyable enough to actually read. The editor told me that my book would be like this, too—so I accepted his offer. In fact, I have often thought that I might be able to teach mathematics better by using manga, so I saw this as a good opportunity to put the idea into practice. I guarantee you that the bigger manga freak you are, the more you will enjoy this book. So, what are you waiting for? Take it up to the cashier and buy it already!

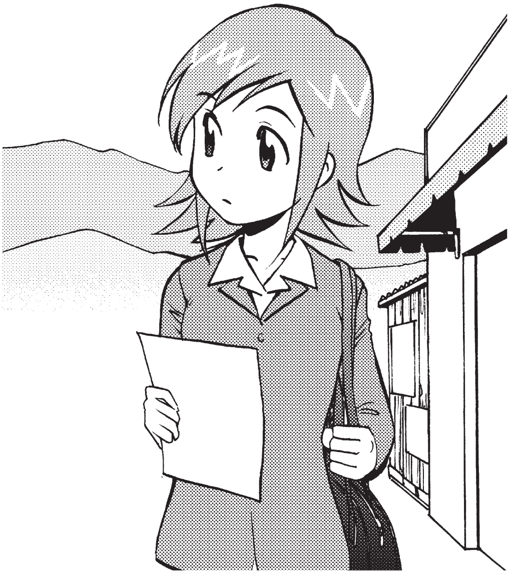
Now, the second type of person is someone who picked up this book thinking, "Although I am terrible at and/or allergic to calculus, manga may help me understand it." If you are this type of person, then this is also the book for you. It is equipped with various rehabilitation methods for those who have been hurt by calculus in the past. Not only does it explain calculus using manga, but the way it explains calculus is fundamentally different from the method used in conventional textbooks. First, the book repeatedly presents the notion of what calculus really does. You will never understand this through the teaching methods that stick to *limits* (or ε-δ logic). Unless you have a clear image of what calculus really does and why it is useful in the world, you will never really understand or use it freely. You will simply fall into a miserable state of memorizing formulas and rules. This book explains all the formulas based on the concept of the *first-order approximation*, helping you to visualize the meaning of formulas and understand them easily. Because of this unique teaching method, you can quickly and easily proceed from differentiation to integration. Furthermore, I have adopted an original method, which is not described in ordinary textbooks, of explaining the differentiation and integration of trigonometric and exponential functions—usually, this is all Greek to many people even after repeated explanations. This book also goes further in depth than existing manga books on calculus do, explaining even Taylor expansions and partial differentiation. Finally, I have invited three regular customers of calculus—physics, statistics, and economics—to be part of this book and presented many examples to show that calculus is truly practical. With all of these devices, you will come to view calculus not as a hardship, but as a useful tool.

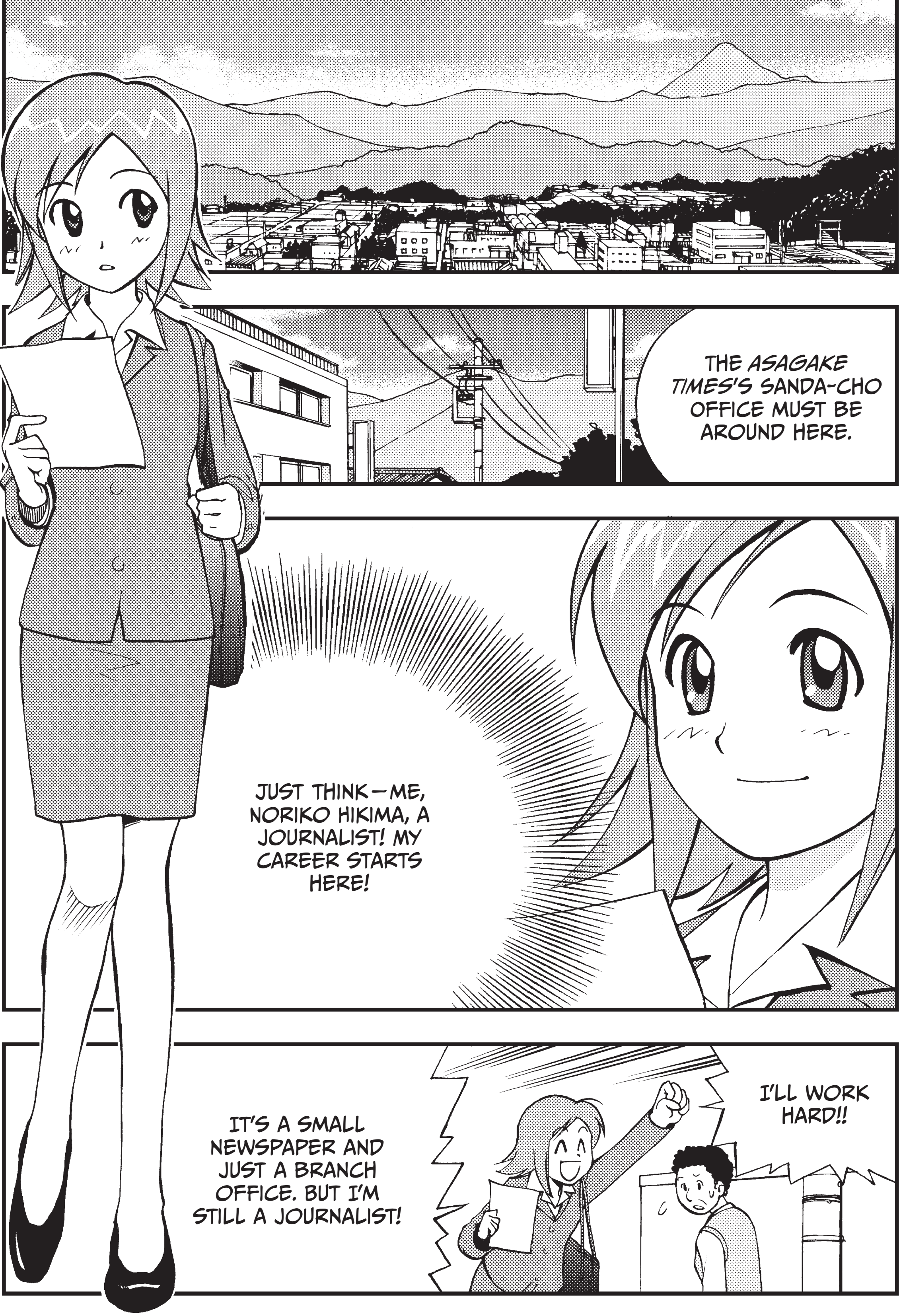
I would like to emphasize again: All of this has been made possible because of manga. Why can you gain more information by reading a manga book than by reading a novel? It is because manga is visual data presented as animation. Calculus is a branch of mathematics that describes dynamic phenomena—thus, calculus is a perfect concept to teach with manga. Now, turn the pages and enjoy a beautiful integration of manga and mathematics.

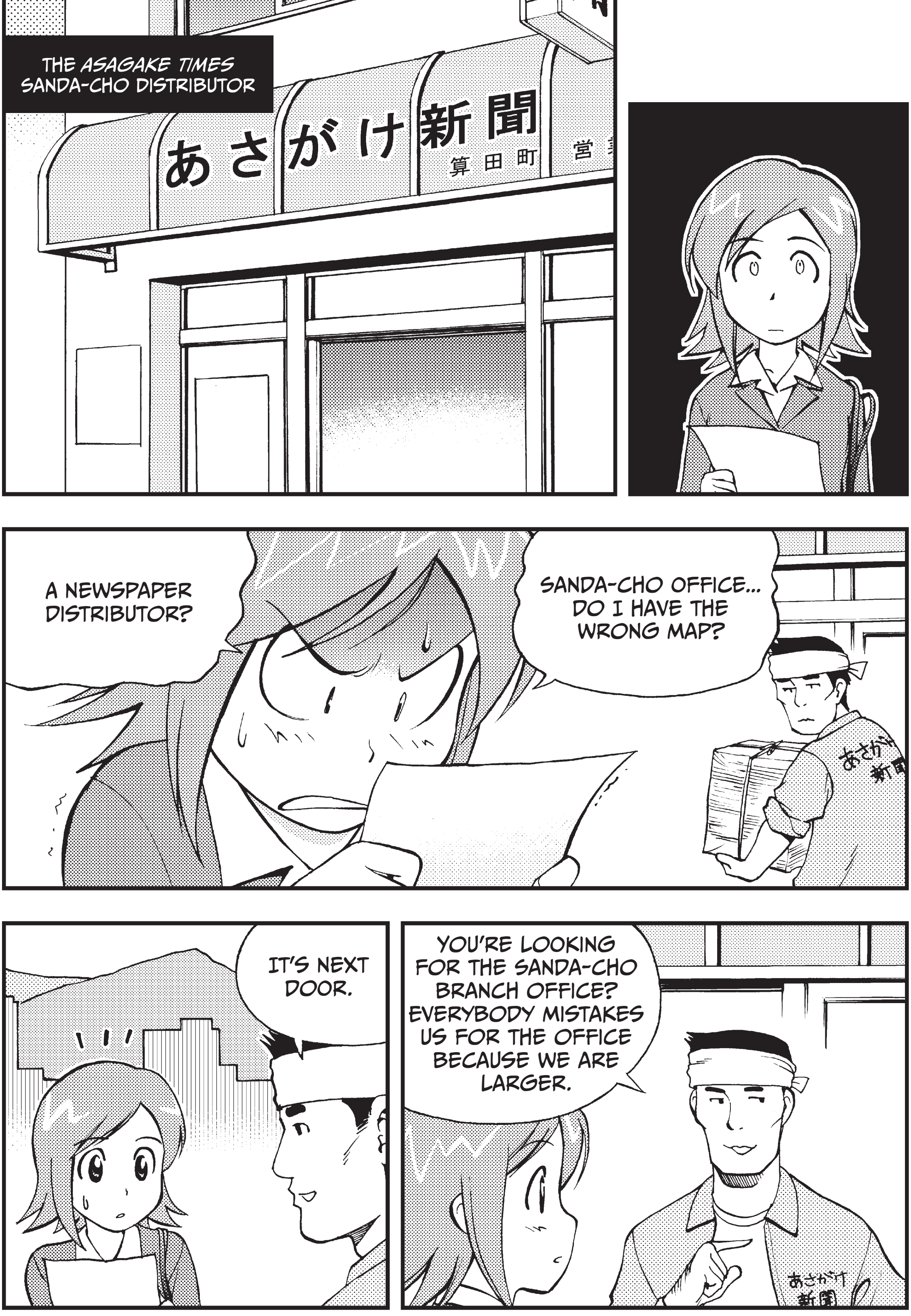
HIROYUKI KOJIMA
NOVEMBER 2005

NOTE: *For ease of understanding, some figures are not drawn to scale.*

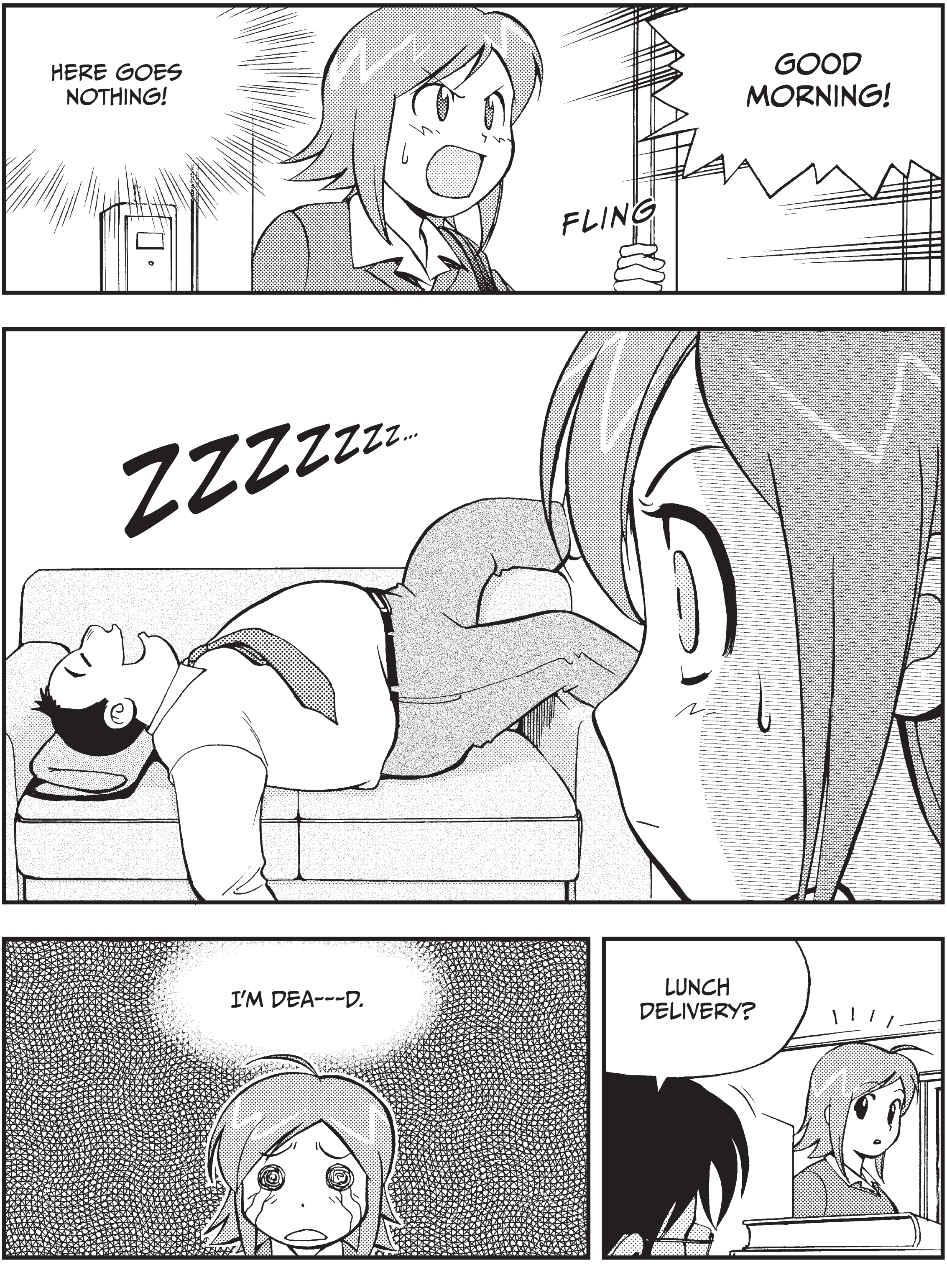
PROLOGUE: WHAT IS A FUNCTION?



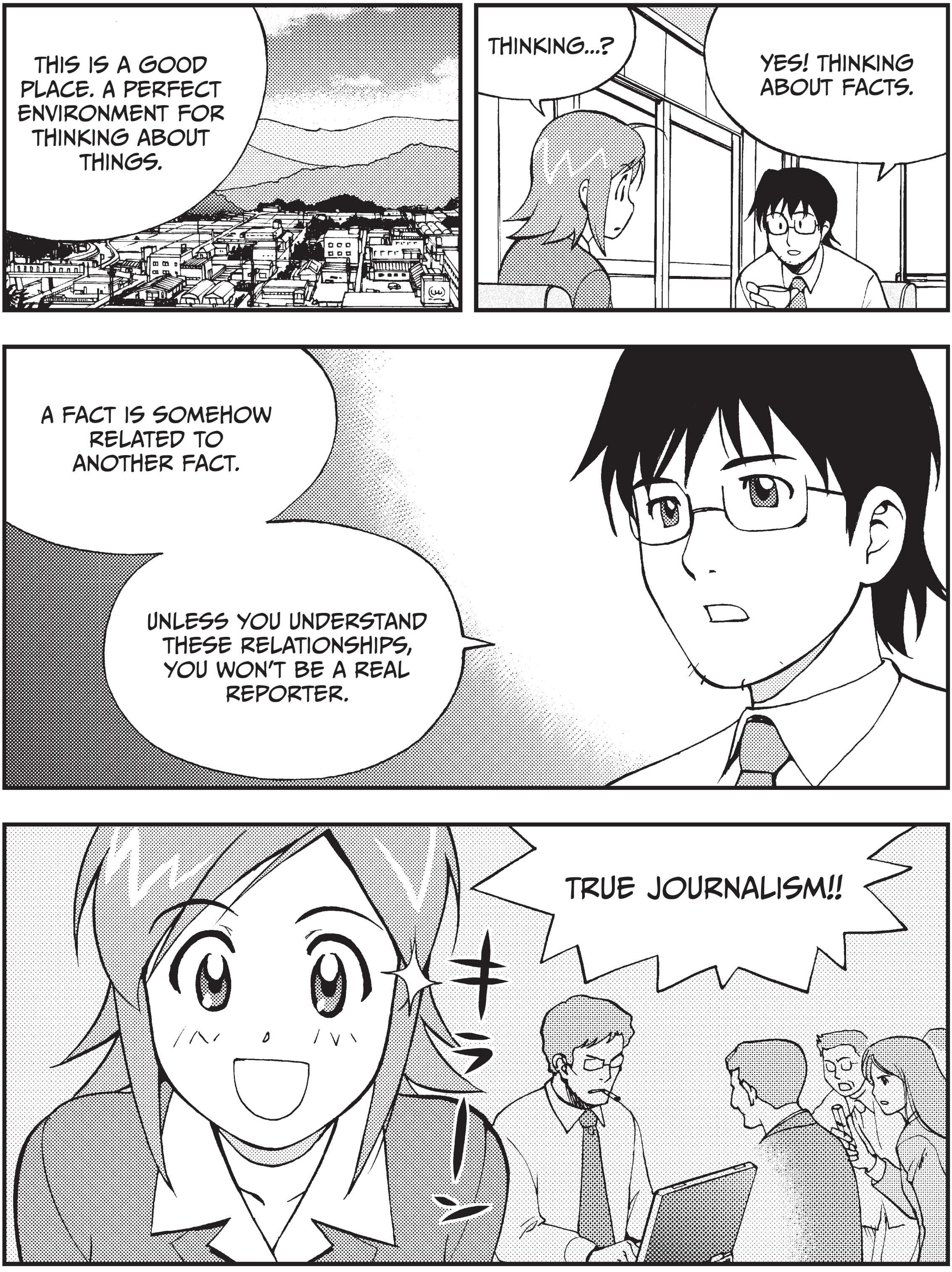

THE *ASAGAKE TIMES*'S SANDA-CHO OFFICE MUST BE AROUND HERE.
JUST THINK—ME, NORIKO HIKIMA, A JOURNALIST! MY CAREER STARTS HERE!
IT'S A SMALL NEWSPAPER AND JUST A BRANCH OFFICE. BUT I'M STILL A JOURNALIST!
I'LL WORK HARD!!

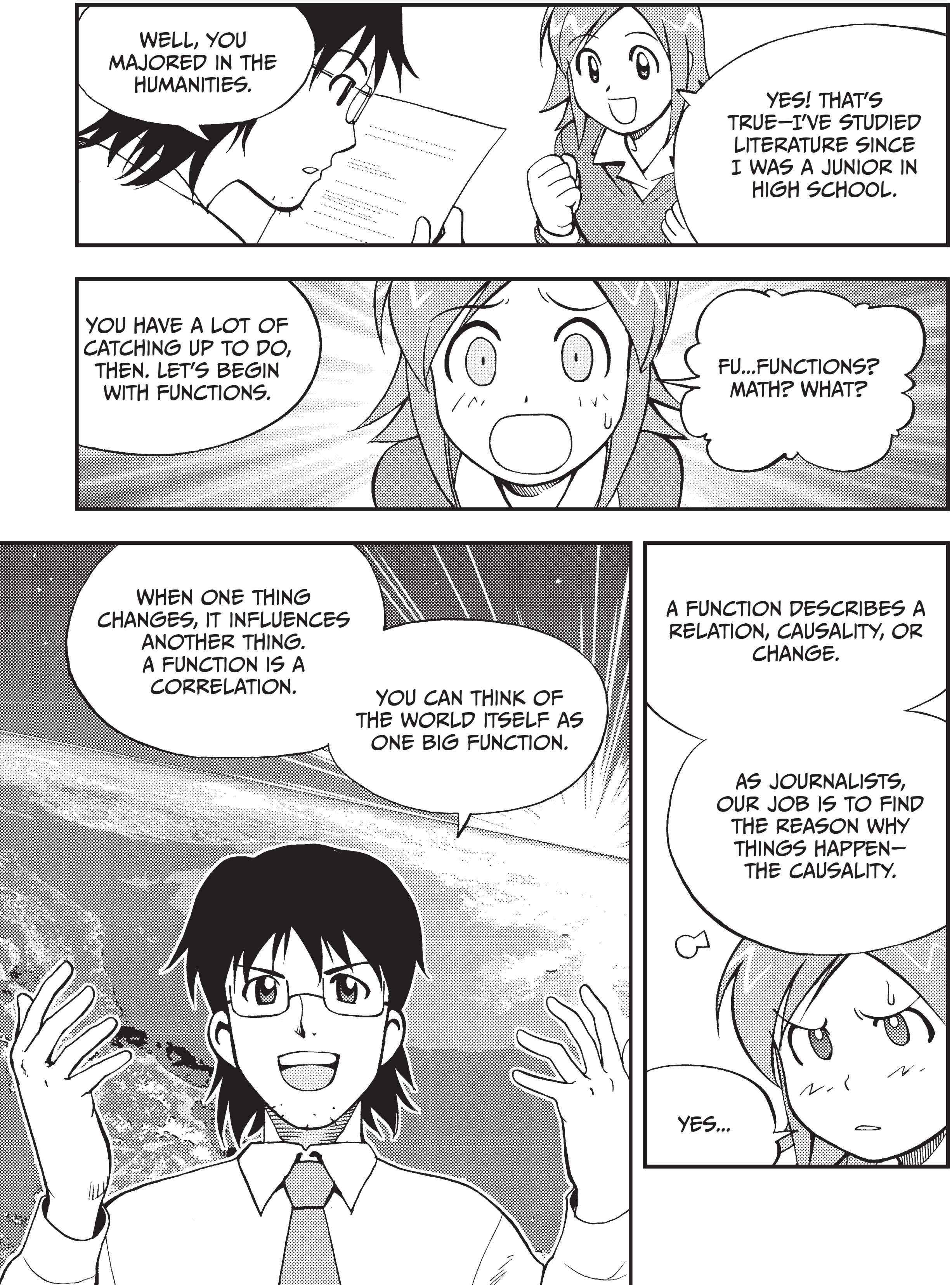

THE *ASAGAKE TIMES* SANDA-CHO DISTRIBUTOR
あさがけ新聞
算田町
営
A NEWSPAPER DISTRIBUTOR?
SANDA-CHO OFFICE... DO I HAVE THE WRONG MAP?
YOU'RE LOOKING FOR THE SANDA-CHO BRANCH OFFICE? EVERYBODY MISTAKES US FOR THE OFFICE BECAUSE WE ARE LARGER.
IT'S NEXT DOOR.

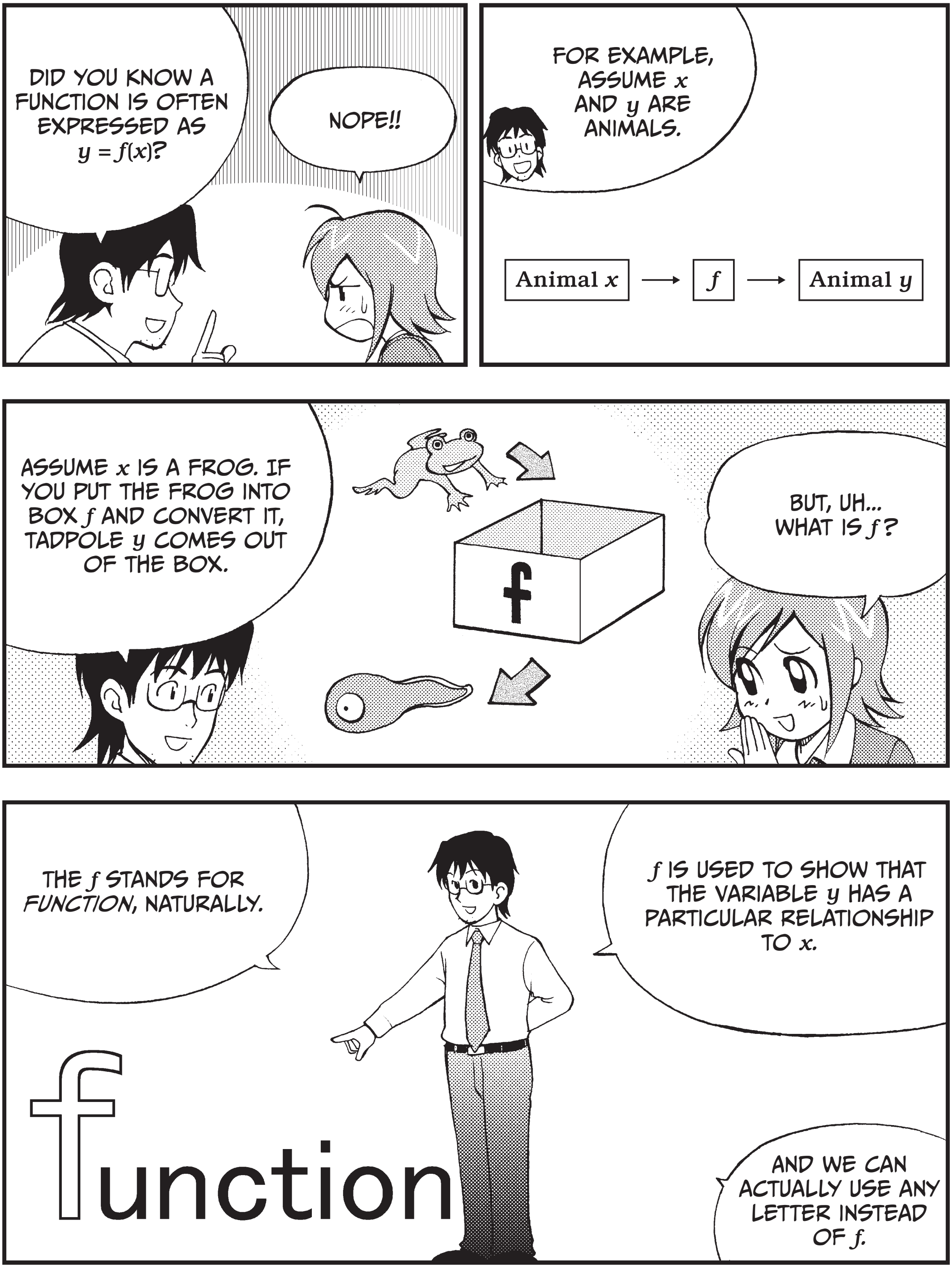

THE *ASAGAKE TIMES*
SANDA-CHO BRANCH OFFICE
あさがけ新聞
算田町支局
WHOOSH
OH, NO!! IT'S A PREFAB!
プレハブじゃん
DON'T...DON'T GET UPSET, NORIKO.
IT'S A BRANCH OFFICE, BUT IT'S STILL THE REAL *ASAGAKE TIMES.*

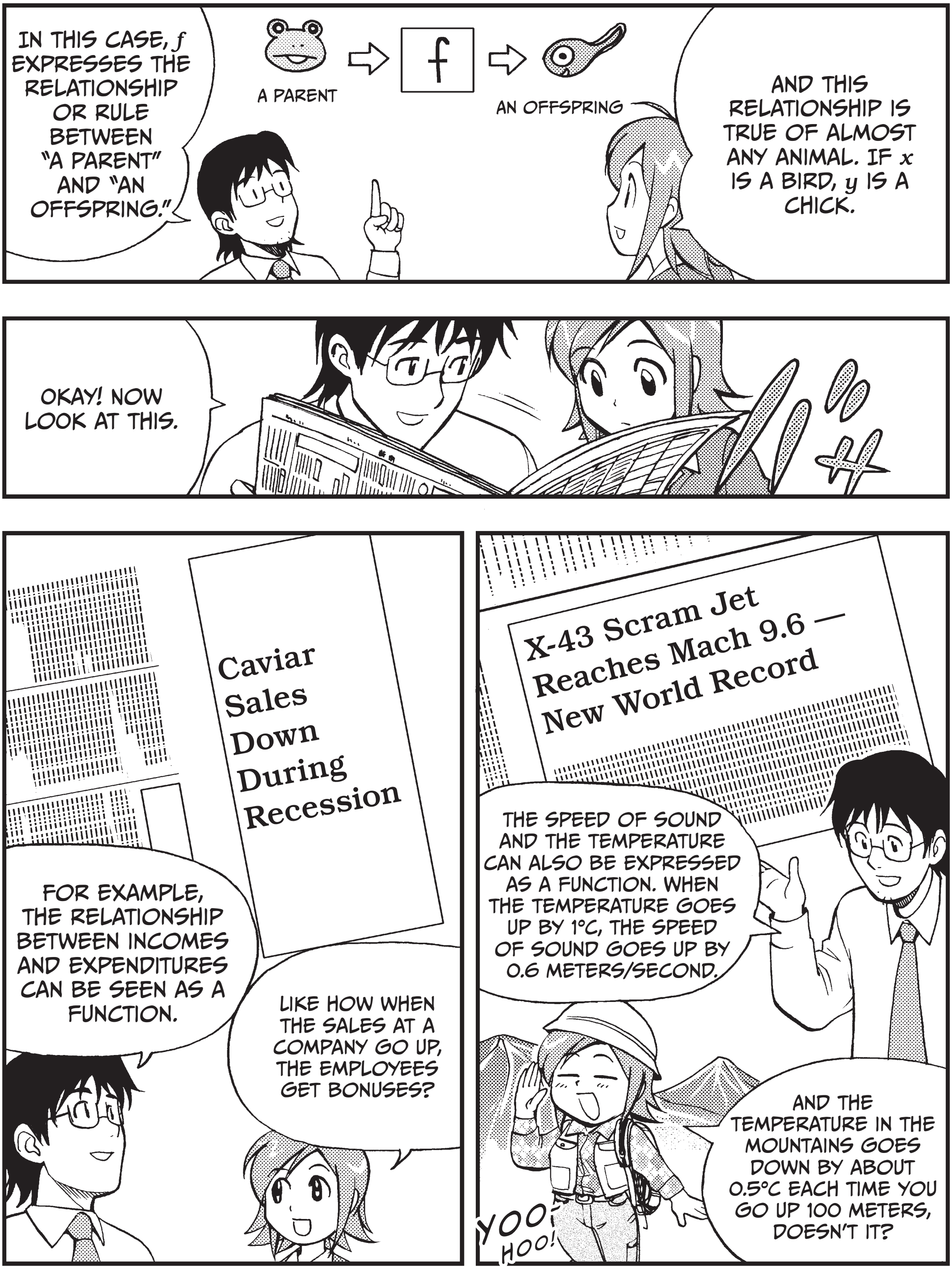

HERE GOES NOTHING!
GOOD MORNING!
FLING
ZZZZZZZ...
I'M DEA---D.
LUNCH DELIVERY?


WILL YOU LEAVE IT, PLEASE?
WAIT, WHAT?
OH, YOU HAVE BEEN ASSIGNED HERE TODAY.
I'M NORIKO HIKIMA.
LONG TRIP, WASN'T IT? I'M KAKERU SEKI, THE HEAD OF THIS OFFICE.
THE BIG GUY THERE IS FUTOSHI MASUI, MY ONLY SOLDIER.
くか～
JUST TWO OF THEM...

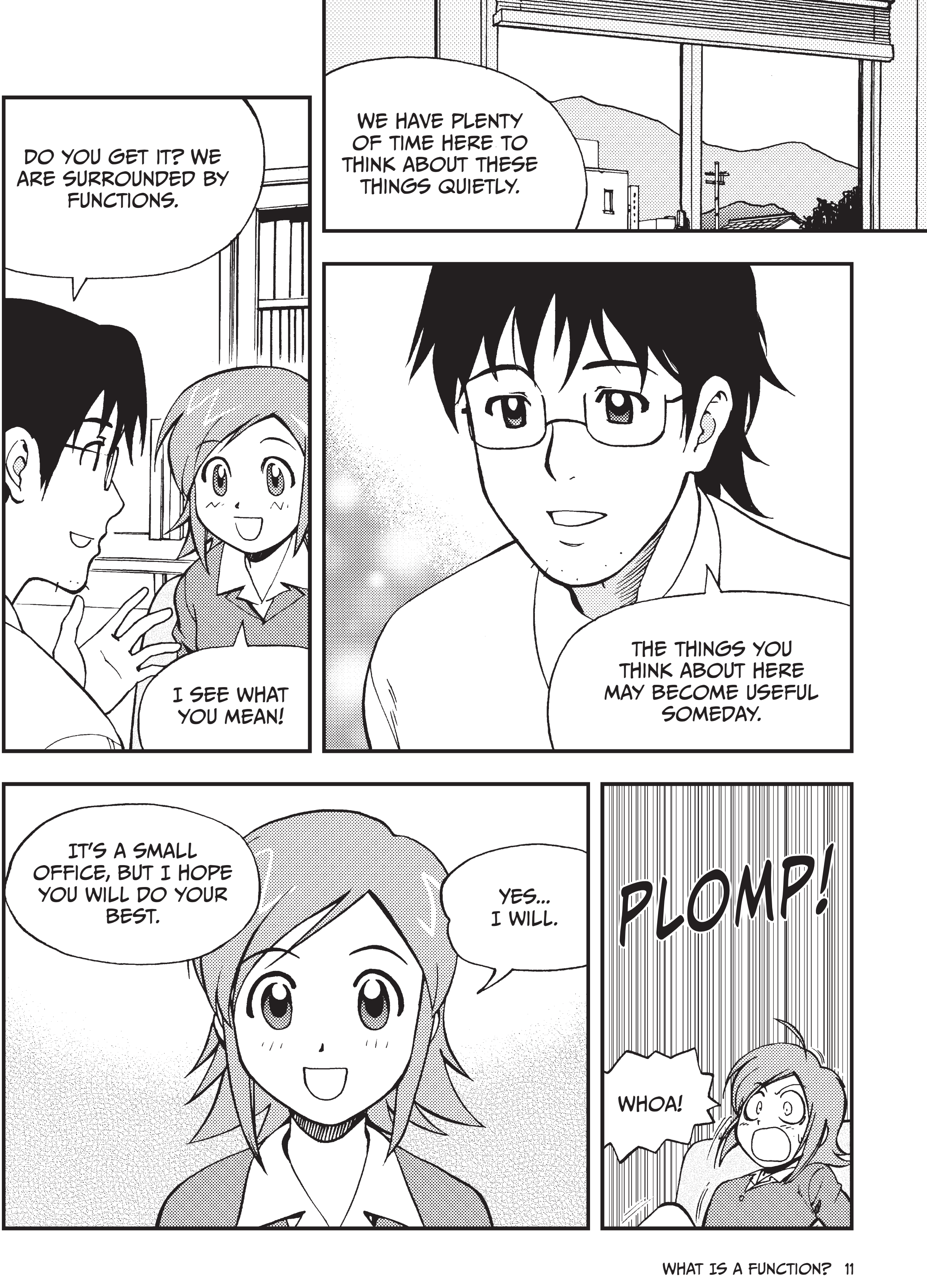

THIS IS A GOOD PLACE. A PERFECT ENVIRONMENT FOR THINKING ABOUT THINGS.
THINKING...?
YES! THINKING ABOUT FACTS.
A FACT IS SOMEHOW RELATED TO ANOTHER FACT.
UNLESS YOU UNDERSTAND THESE RELATIONSHIPS, YOU WON'T BE A REAL REPORTER.
TRUE JOURNALISM!!


WELL, YOU MAJORED IN THE HUMANITIES.
YES! THAT'S TRUE—I'VE STUDIED LITERATURE SINCE I WAS A JUNIOR IN HIGH SCHOOL.
YOU HAVE A LOT OF CATCHING UP TO DO, THEN. LET'S BEGIN WITH FUNCTIONS.
FU...FUNCTIONS? MATH? WHAT?
WHEN ONE THING CHANGES, IT INFLUENCES ANOTHER THING. A FUNCTION IS A CORRELATION.
YOU CAN THINK OF THE WORLD ITSELF AS ONE BIG FUNCTION.
A FUNCTION DESCRIBES A RELATION, CAUSALITY, OR CHANGE.
AS JOURNALISTS, OUR JOB IS TO FIND THE REASON WHY THINGS HAPPEN—THE CAUSALITY.
YES...


DID YOU KNOW A FUNCTION IS OFTEN EXPRESSED AS $y = f(x)$?
NOPE!!
FOR EXAMPLE, ASSUME x AND y ARE ANIMALS.
Animal x → f → Animal y
ASSUME x IS A FROG. IF YOU PUT THE FROG INTO BOX f AND CONVERT IT, TADPOLE y COMES OUT OF THE BOX.
f
BUT, UH... WHAT IS f?
THE f STANDS FOR *FUNCTION*, NATURALLY.
f IS USED TO SHOW THAT THE VARIABLE y HAS A PARTICULAR RELATIONSHIP TO x.
function
AND WE CAN ACTUALLY USE ANY LETTER INSTEAD OF f.



IN THIS CASE, f EXPRESSES THE RELATIONSHIP OR RULE BETWEEN "A PARENT" AND "AN OFFSPRING."
A PARENT
f
AN OFFSPRING
AND THIS RELATIONSHIP IS TRUE OF ALMOST ANY ANIMAL. IF x IS A BIRD, y IS A CHICK.
OKAY! NOW LOOK AT THIS.
Caviar Sales Down During Recession
FOR EXAMPLE, THE RELATIONSHIP BETWEEN INCOMES AND EXPENDITURES CAN BE SEEN AS A FUNCTION.
LIKE HOW WHEN THE SALES AT A COMPANY GO UP, THE EMPLOYEES GET BONUSES?
X-43 Scram Jet Reaches Mach 9.6 — New World Record
THE SPEED OF SOUND AND THE TEMPERATURE CAN ALSO BE EXPRESSED AS A FUNCTION. WHEN THE TEMPERATURE GOES UP BY 1°C, THE SPEED OF SOUND GOES UP BY 0.6 METERS/SECOND.
YOO HOO!
AND THE TEMPERATURE IN THE MOUNTAINS GOES DOWN BY ABOUT 0.5°C EACH TIME YOU GO UP 100 METERS, DOESN'T IT?



WE HAVE PLENTY OF TIME HERE TO THINK ABOUT THESE THINGS QUIETLY.
DO YOU GET IT? WE ARE SURROUNDED BY FUNCTIONS.
THE THINGS YOU THINK ABOUT HERE MAY BECOME USEFUL SOMEDAY.
I SEE WHAT YOU MEAN!
YES... I WILL.
IT'S A SMALL OFFICE, BUT I HOPE YOU WILL DO YOUR BEST.
PLOMP!
WHOA!

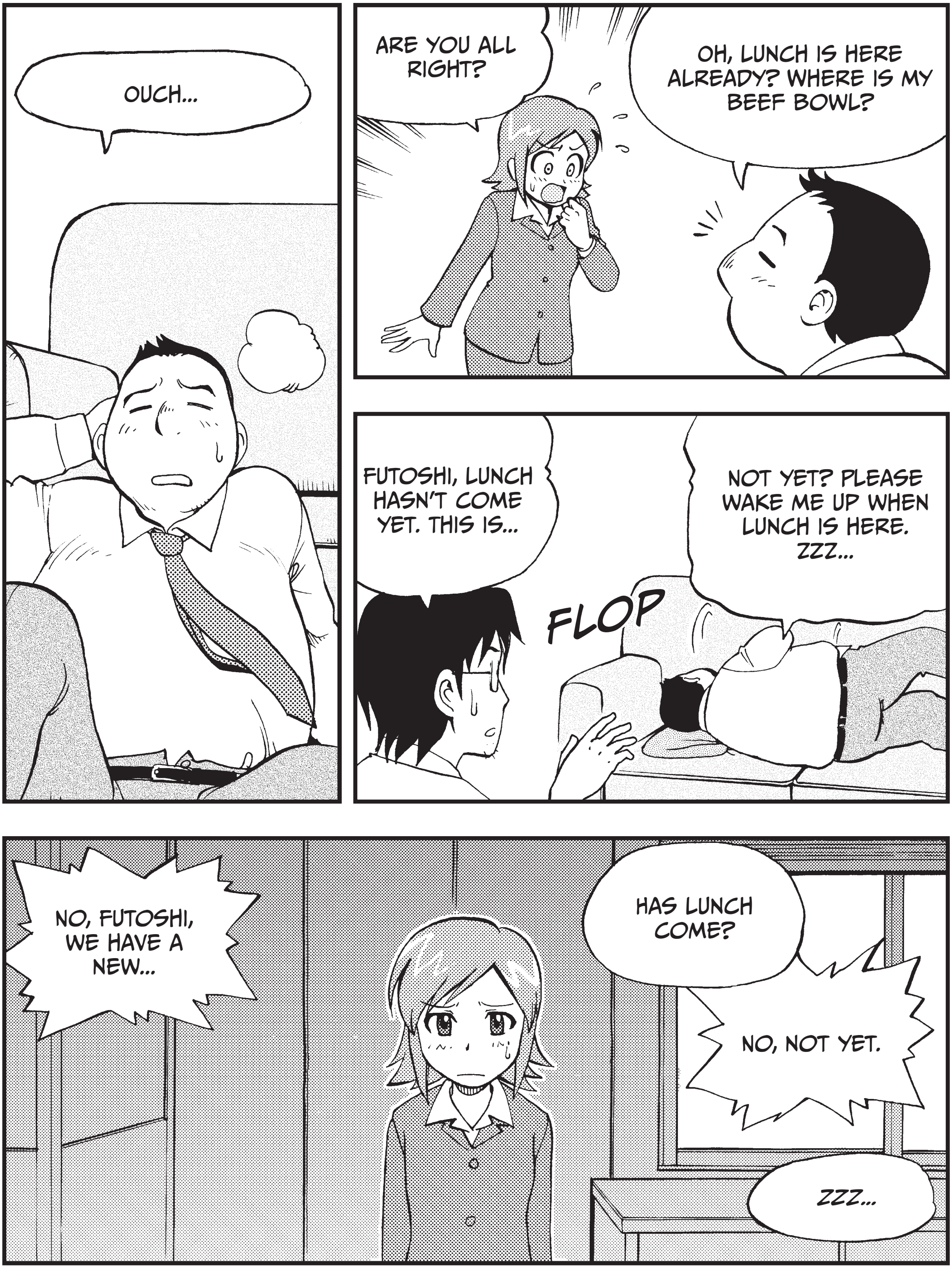
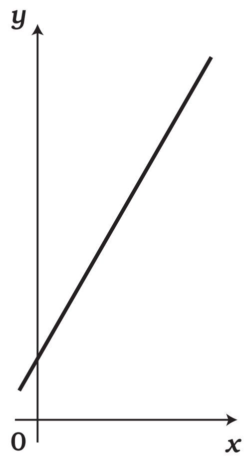
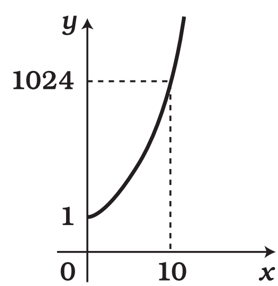
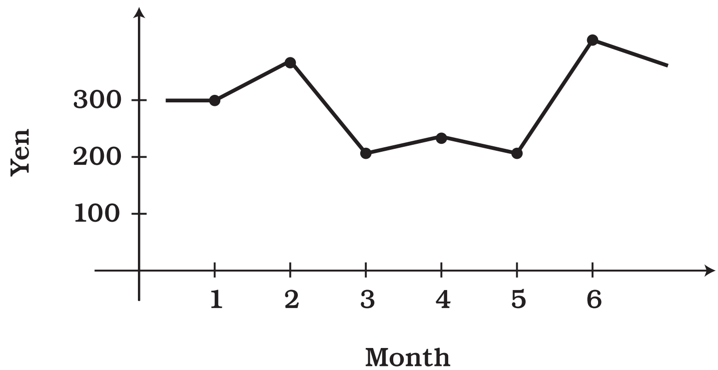

OUCH...
ARE YOU ALL RIGHT?
OH, LUNCH IS HERE ALREADY? WHERE IS MY BEEF BOWL?
FUTOSHI, LUNCH HASN'T COME YET. THIS IS...
NOT YET? PLEASE WAKE ME UP WHEN LUNCH IS HERE. ZZZ...
FLOP
NO, FUTOSHI, WE HAVE A NEW...
HAS LUNCH COME?
NO, NOT YET.
ZZZ...

TABLE 1: CHARACTERISTICS OF FUNCTIONS

<table>
<tr><th>SUBJECT</th><th>CALCULATION</th><th>GRAPH</th></tr>
<tr><td>Causality</td><td>The frequency of a cricket's chirp is determined by temperature. We can express the relationship between y chirps per minute of a cricket at temperature x°C approximately as
$y = g(x) = 7x - 30$
$x = 27°$ $7 \times 27 - 30$
The result is 159 chirps a minute.</td><td rowspan="3">When we graph these functions, the result is a straight line. That's why we call them linear functions.

</td></tr>
<tr><td>Changes</td><td>The speed of sound y in meters per second (m/s) in the air at x°C is expressed as
$y = v(x) = 0.6x + 331$
At 15°C,
$y = v(15) = 0.6 \square 15 + 331 = 340$ m/s
At –5°C,
$y = v(-5) = 0.6 \times (-5) + 331 = 328$ m/s</td></tr>
<tr><td rowspan="2">Unit Conversion</td><td>Converting x degrees Fahrenheit (°F) into y degrees Celsius (°C)
$y = f(x) = \frac{5}{9}(x - 32)$
So now we know 50°F is equivalent to
$\frac{5}{9}(50 - 32) = 10°\text{C}$</td></tr>
<tr><td>Computers store numbers using a binary system (1s and 0s). A binary number with x bits (or binary digits) has the potential to store y numbers.
$y = b(x) = 2^x$
(This is described in more detail on page 131.)</td><td>The graph is an exponential function.

</td></tr>
</table>

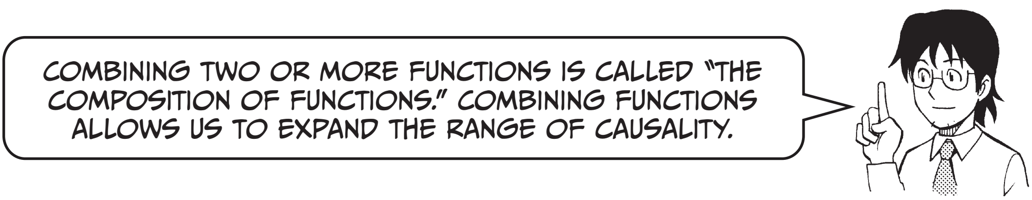
The stock price P of company A in month x in 2009 is

$$y = P(x)$$



$P(x)$ cannot be expressed by a known function, but it is still a function.

If you could find a way to predict $P(7)$, the stock price in July, you could make a big profit.



A composite function of f and g

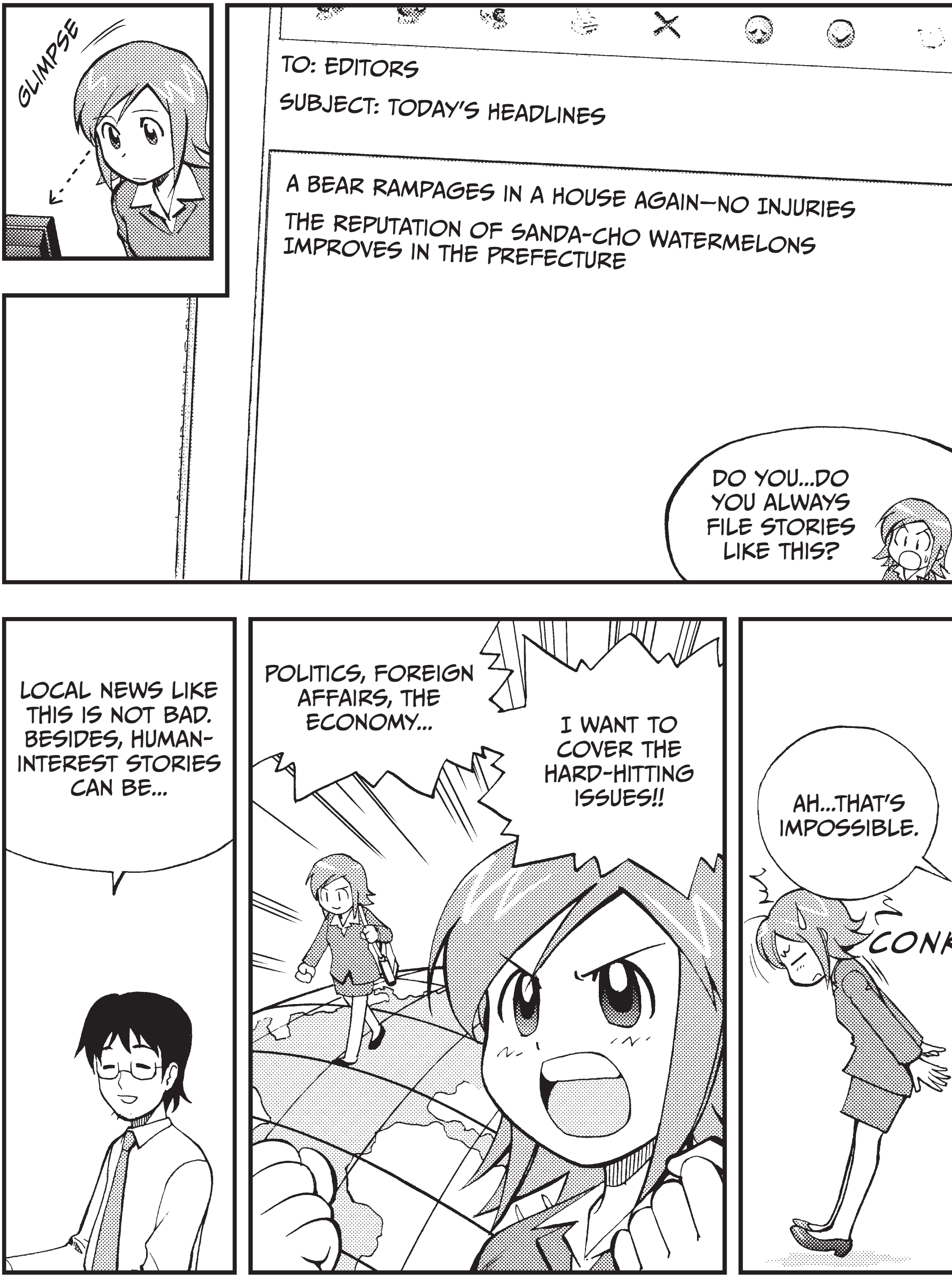
$$x \longrightarrow \boxed{f} \longrightarrow f(x) \longrightarrow \boxed{g} \longrightarrow g(f(x))$$

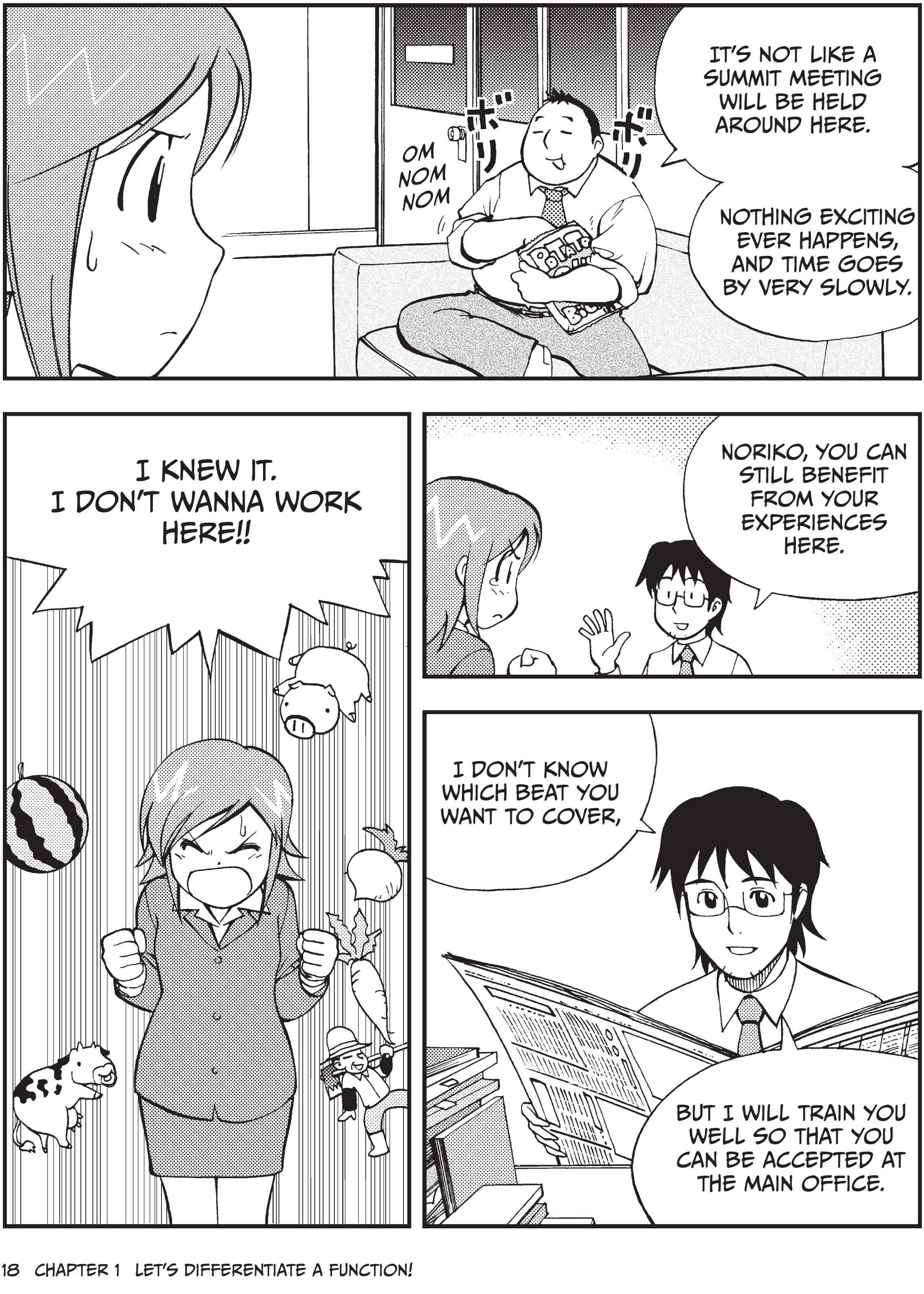
EXERCISE

1. Find an equation that expresses the frequency of z chirps/minute of a cricket at x°F.

1
LET'S DIFFERENTIATE A FUNCTION!

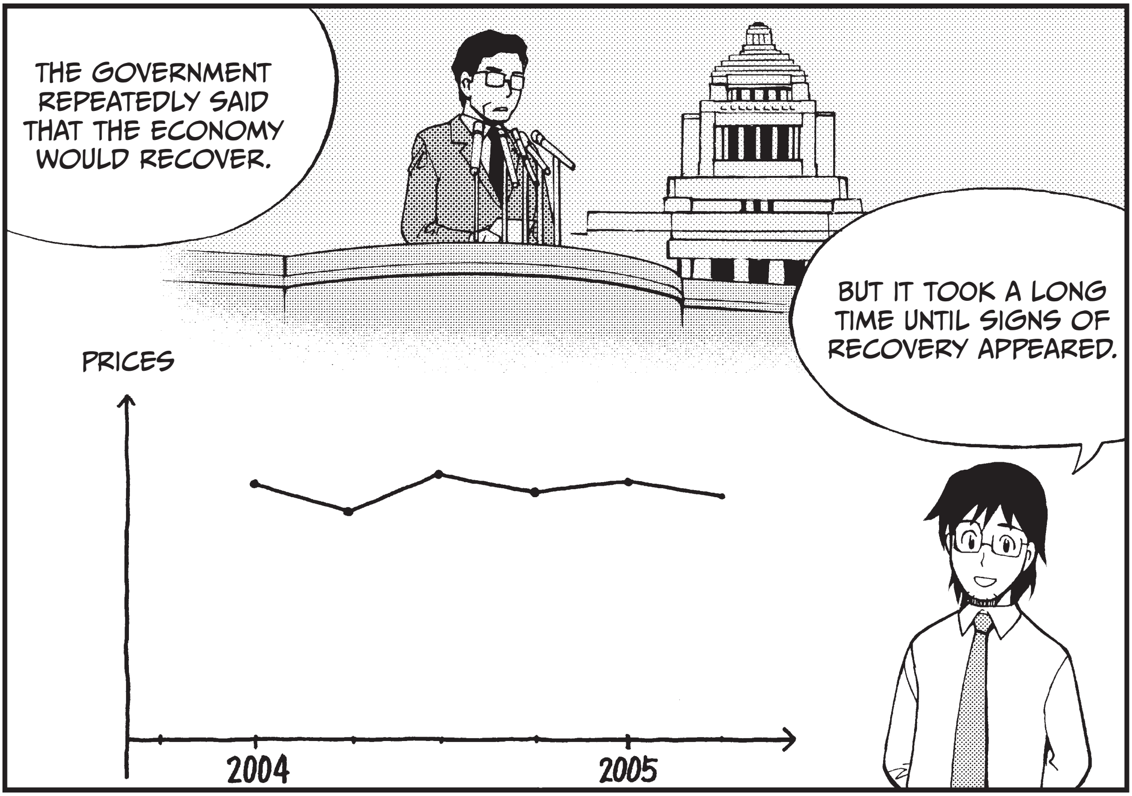
APPROXIMATING WITH FUNCTIONS

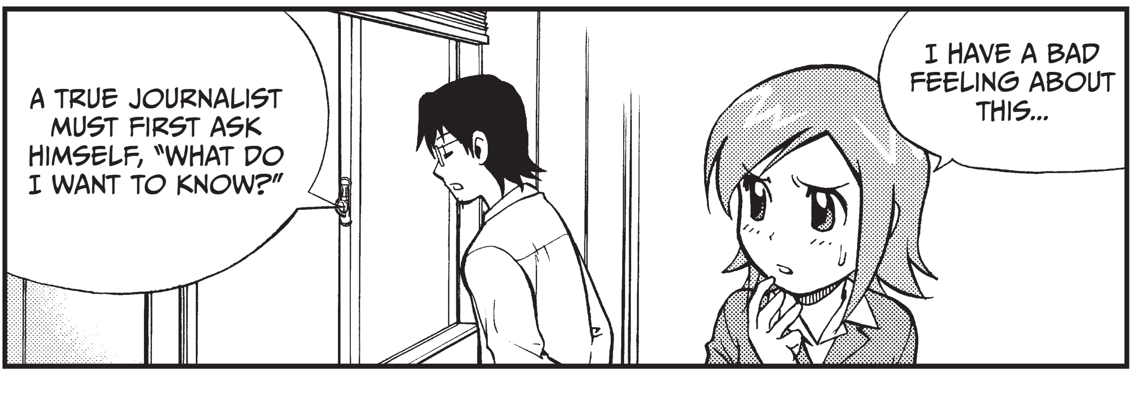

GLIMPSE
TO: EDITORS
SUBJECT: TODAY'S HEADLINES
A BEAR RAMPAGES IN A HOUSE AGAIN—NO INJURIES
THE REPUTATION OF SANDA-CHO WATERMELONS IMPROVES IN THE PREFECTURE
DO YOU...DO YOU ALWAYS FILE STORIES LIKE THIS?
LOCAL NEWS LIKE THIS IS NOT BAD. BESIDES, HUMAN-INTEREST STORIES CAN BE...
POLITICS, FOREIGN AFFAIRS, THE ECONOMY...
I WANT TO COVER THE HARD-HITTING ISSUES!!
AH...THAT'S IMPOSSIBLE.
CONK

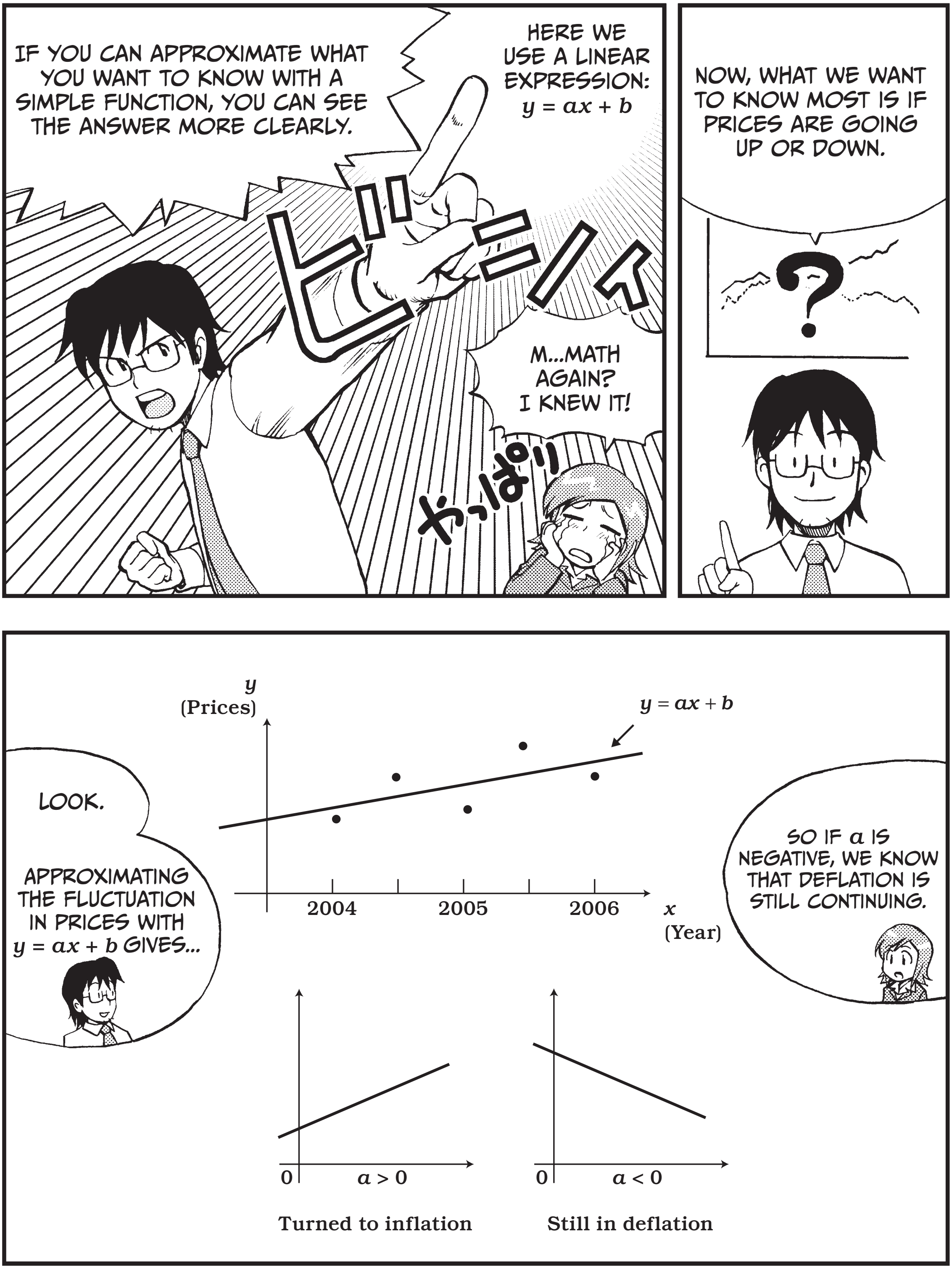


ボリ
ボリ
OM NOM NOM
IT'S NOT LIKE A SUMMIT MEETING WILL BE HELD AROUND HERE.
NOTHING EXCITING EVER HAPPENS, AND TIME GOES BY VERY SLOWLY.
I KNEW IT. I DON'T WANNA WORK HERE!!
NORIKO, YOU CAN STILL BENEFIT FROM YOUR EXPERIENCES HERE.
I DON'T KNOW WHICH BEAT YOU WANT TO COVER,
BUT I WILL TRAIN YOU WELL SO THAT YOU CAN BE ACCEPTED AT THE MAIN OFFICE.

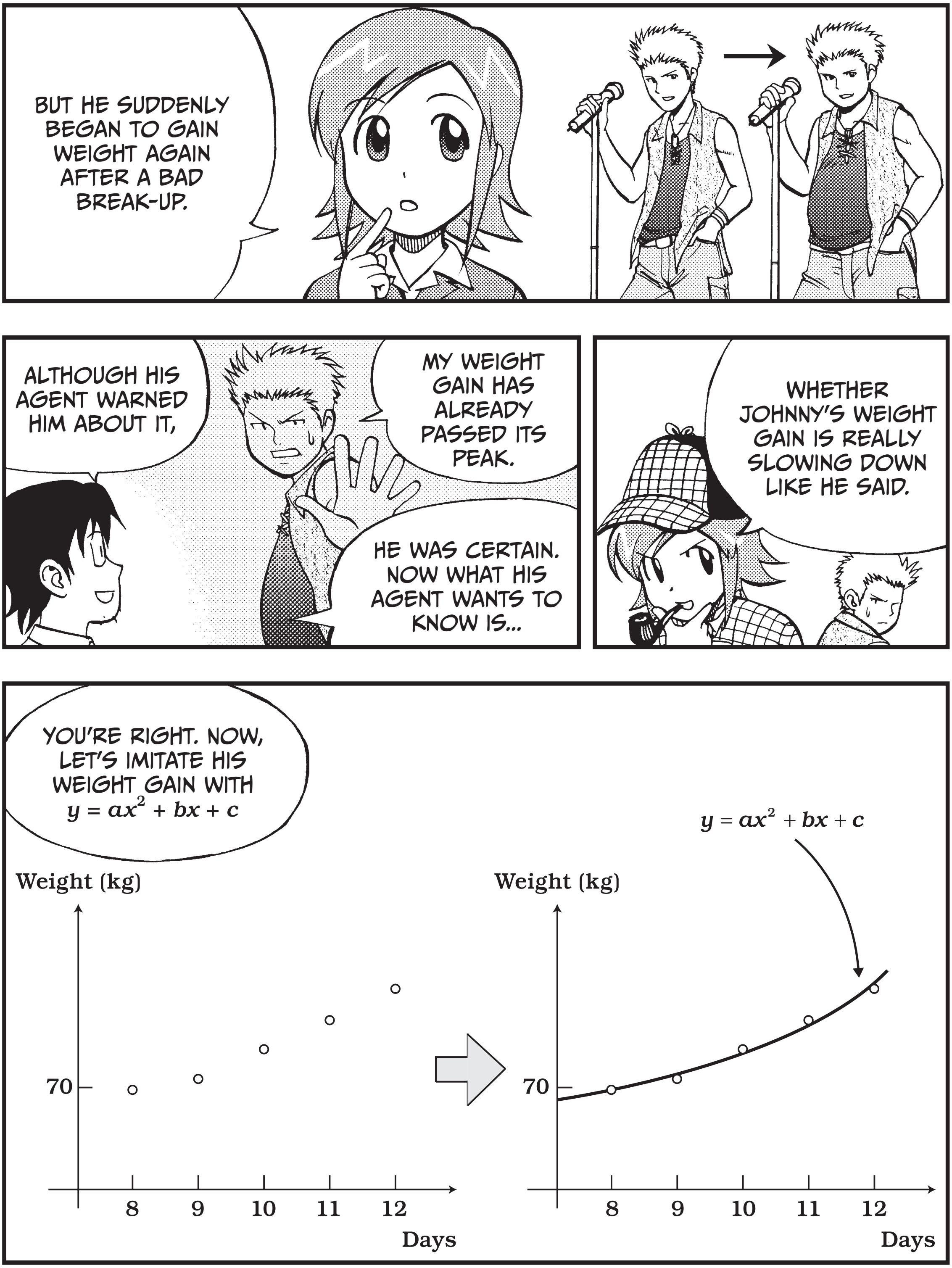

BY THE WAY, DO YOU THINK THE JAPANESE ECONOMY IS STILL EXPERIENCING DEFLATION?
ECONOMIC STIMULUS
I THINK SO. I FEEL IT IN MY DAILY LIFE.

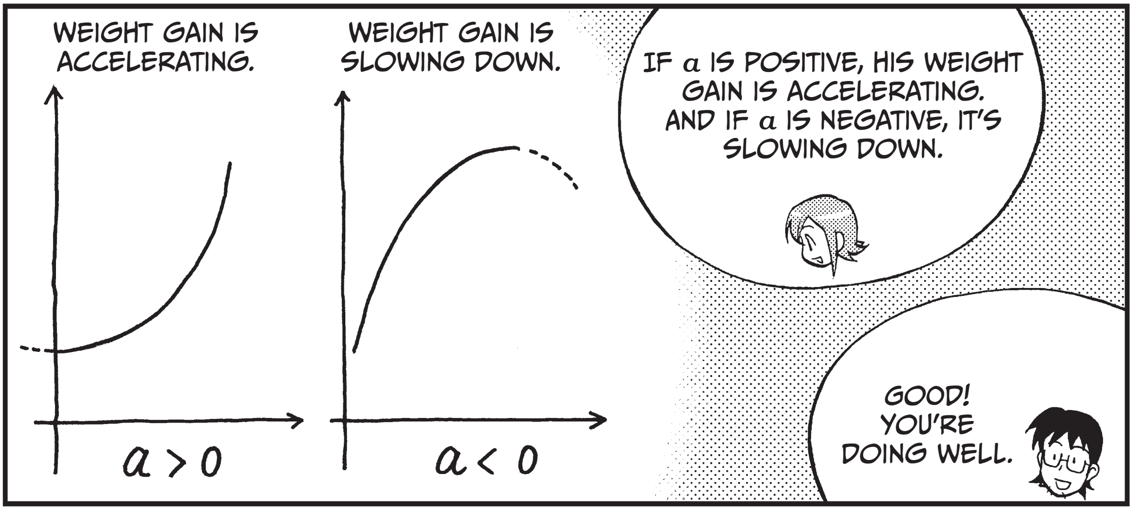

THE GOVERNMENT REPEATEDLY SAID THAT THE ECONOMY WOULD RECOVER.
BUT IT TOOK A LONG TIME UNTIL SIGNS OF RECOVERY APPEARED.
PRICES
2004
2005

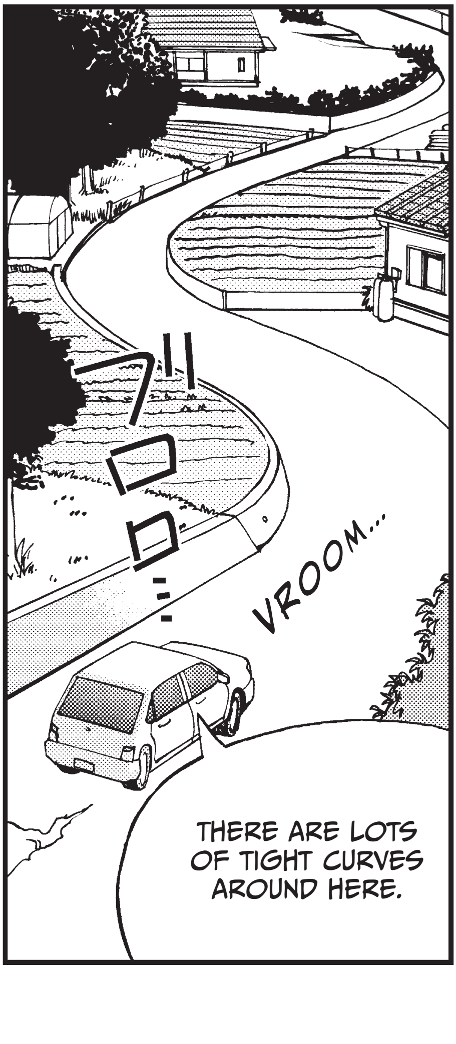

A TRUE JOURNALIST MUST FIRST ASK HIMSELF, "WHAT DO I WANT TO KNOW?"
I HAVE A BAD FEELING ABOUT THIS...

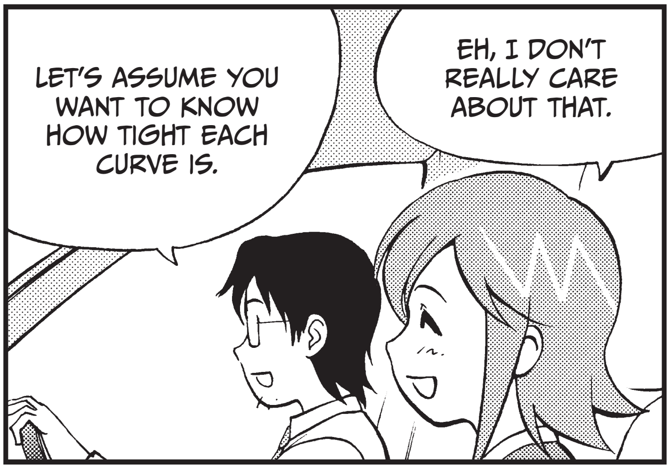

IF YOU CAN APPROXIMATE WHAT YOU WANT TO KNOW WITH A SIMPLE FUNCTION, YOU CAN SEE THE ANSWER MORE CLEARLY.
HERE WE USE A LINEAR EXPRESSION: $y = ax + b$
ビシィ
M...MATH AGAIN? I KNEW IT!
やっぱり
NOW, WHAT WE WANT TO KNOW MOST IS IF PRICES ARE GOING UP OR DOWN.
?
LOOK.
APPROXIMATING THE FLUCTUATION IN PRICES WITH $y = ax + b$ GIVES...
y (Prices)
$y = ax + b$
2004
2005
2006
x (Year)
SO IF a IS NEGATIVE, WE KNOW THAT DEFLATION IS STILL CONTINUING.
0
$a > 0$
0
$a < 0$
Turned to inflation
Still in deflation

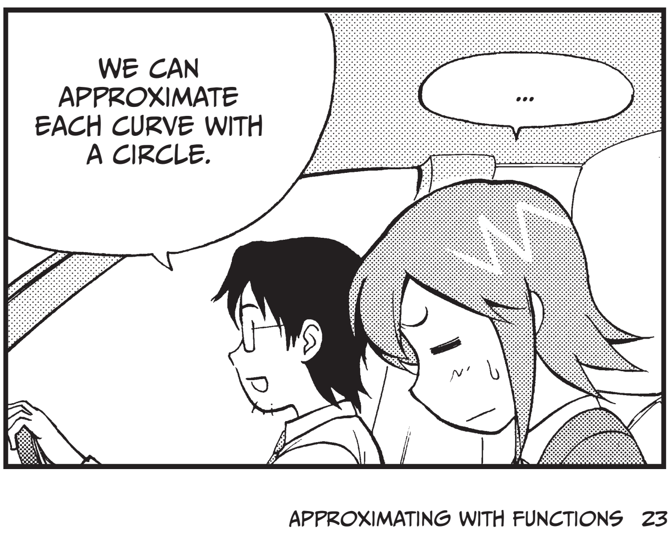

THAT'S RIGHT. YOU ARE A QUICK STUDY.
GROWL
NOW, LET'S DO THE REST AT THE ITALIAN RESTAURANT.
ガタッ
LET'S GET OUTTA HERE!
FUTOSHI, WE'RE LEAVING FOR LUNCH. DON'T EAT TOO MANY SNACKS.
SPEAKING OF SNACKS, DO YOU KNOW ABOUT JOHNNY FANTASTIC, THE ROCKSTAR WHOSE BOOK ON DIETING HAS BECOME A BEST SELLER?
YES.

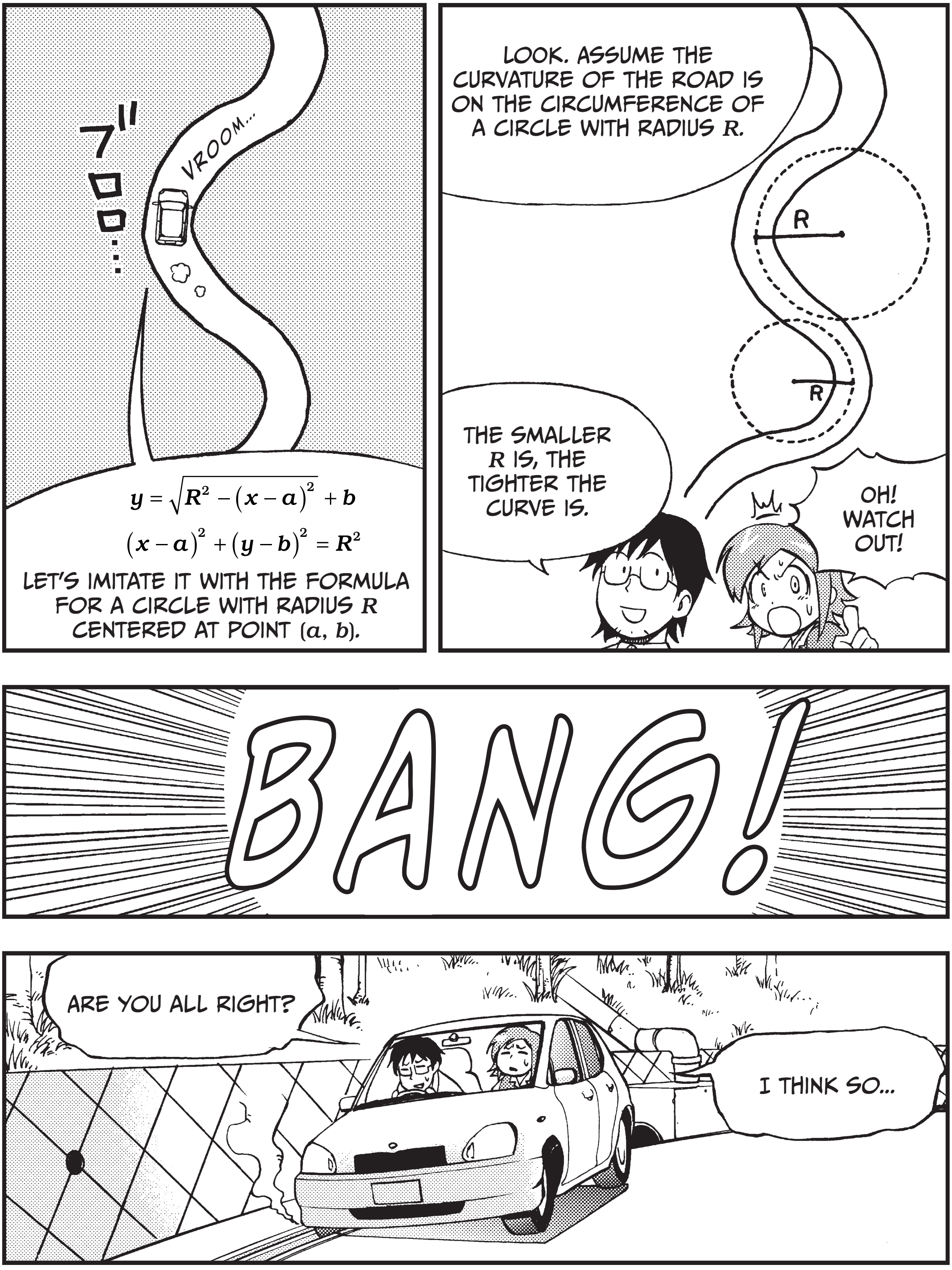

BUT HE SUDDENLY BEGAN TO GAIN WEIGHT AGAIN AFTER A BAD BREAK-UP.
ALTHOUGH HIS AGENT WARNED HIM ABOUT IT,
MY WEIGHT GAIN HAS ALREADY PASSED ITS PEAK.
HE WAS CERTAIN. NOW WHAT HIS AGENT WANTS TO KNOW IS...
WHETHER JOHNNY'S WEIGHT GAIN IS REALLY SLOWING DOWN LIKE HE SAID.
YOU'RE RIGHT. NOW, LET'S IMITATE HIS WEIGHT GAIN WITH $y = ax^2 + bx + c$
Weight (kg)
70
8
9
10
11
12
Days
$y = ax^2 + bx + c$
Weight (kg)
70
8
9
10
11
12
Days


WEIGHT GAIN IS ACCELERATING.
$a > 0$
WEIGHT GAIN IS SLOWING DOWN.
$a < 0$
IF a IS POSITIVE, HIS WEIGHT GAIN IS ACCELERATING. AND IF a IS NEGATIVE, IT'S SLOWING DOWN.
GOOD! YOU'RE DOING WELL.

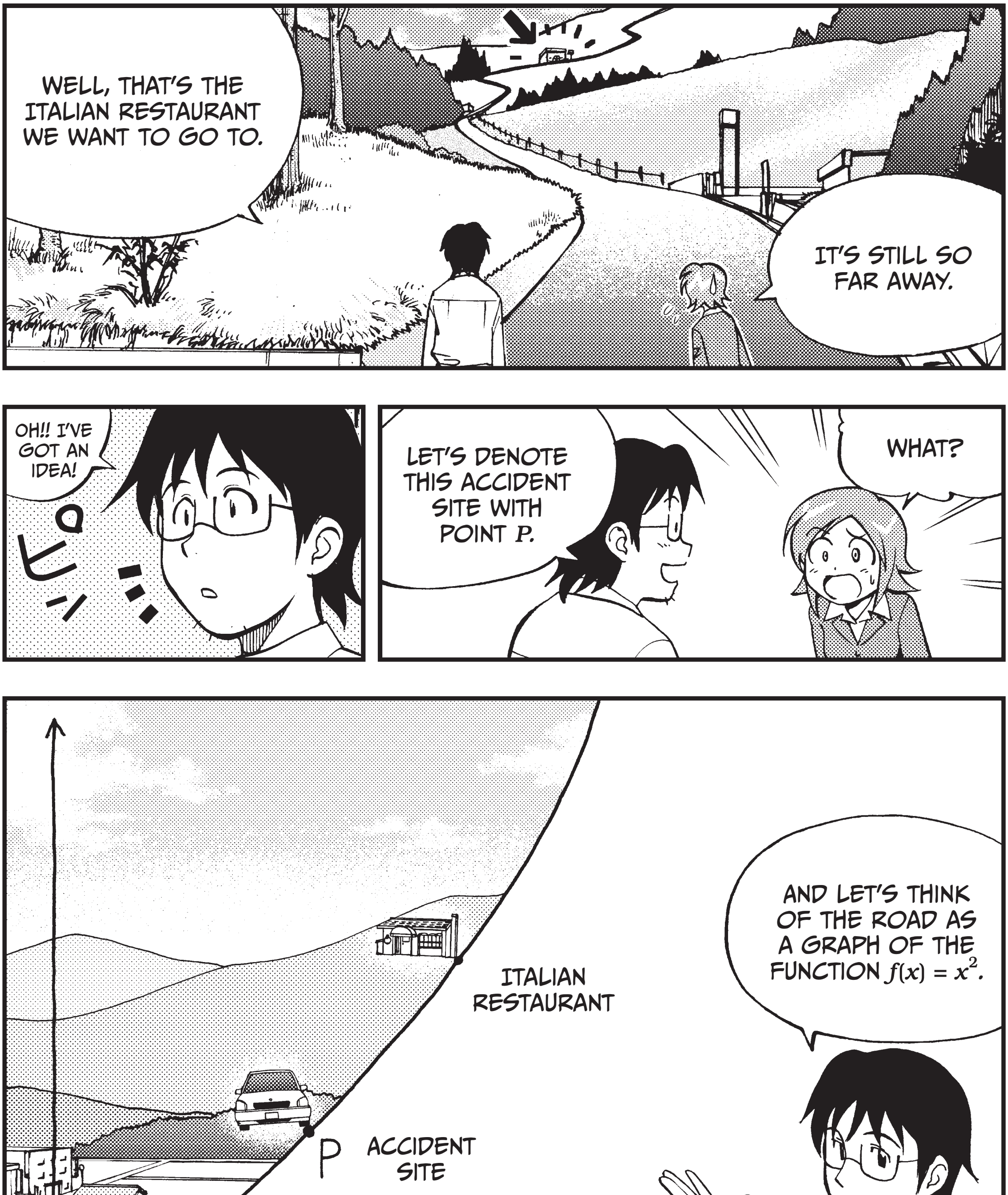

VROOM...
THERE ARE LOTS OF TIGHT CURVES AROUND HERE.

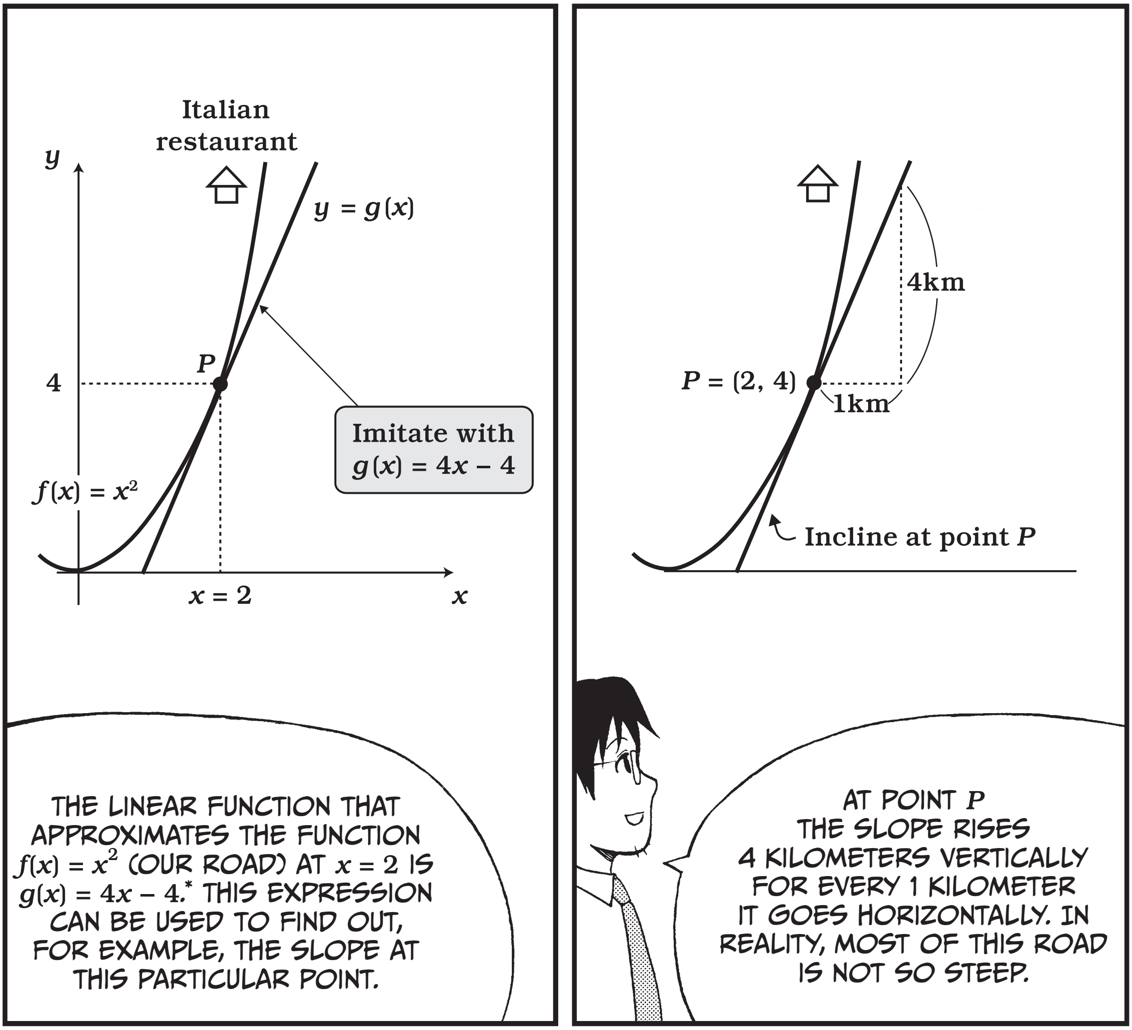

LET'S ASSUME YOU WANT TO KNOW HOW TIGHT EACH CURVE IS.
EH, I DON'T REALLY CARE ABOUT THAT.


WE CAN APPROXIMATE EACH CURVE WITH A CIRCLE.
...

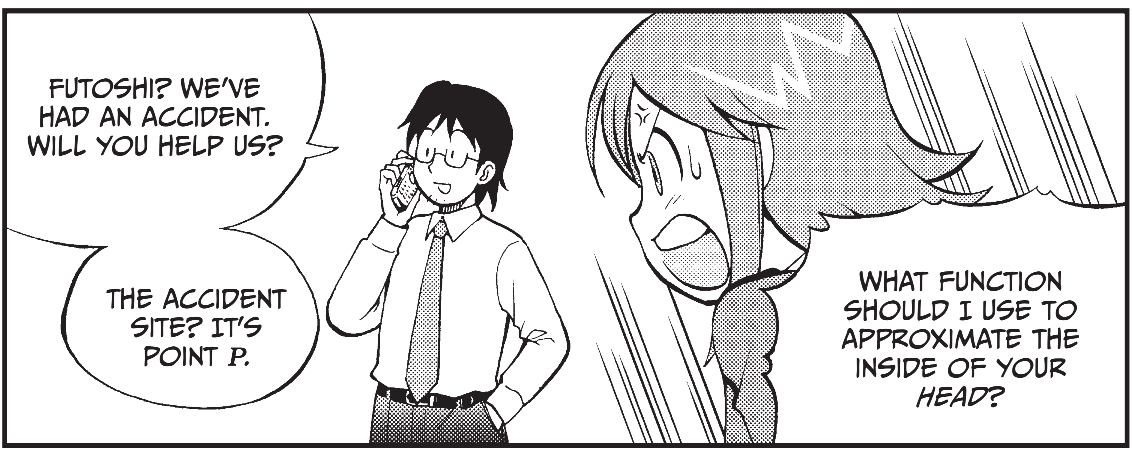


ブロロ…
VROOM…
$y = \sqrt{R^2 - (x-a)^2} + b$
$(x-a)^2 + (y-b)^2 = R^2$
LET'S IMITATE IT WITH THE FORMULA FOR A CIRCLE WITH RADIUS R CENTERED AT POINT (a, b).
LOOK. ASSUME THE CURVATURE OF THE ROAD IS ON THE CIRCUMFERENCE OF A CIRCLE WITH RADIUS R.
R
R
THE SMALLER R IS, THE TIGHTER THE CURVE IS.
OH! WATCH OUT!
BANG!
ARE YOU ALL RIGHT?
I THINK SO...

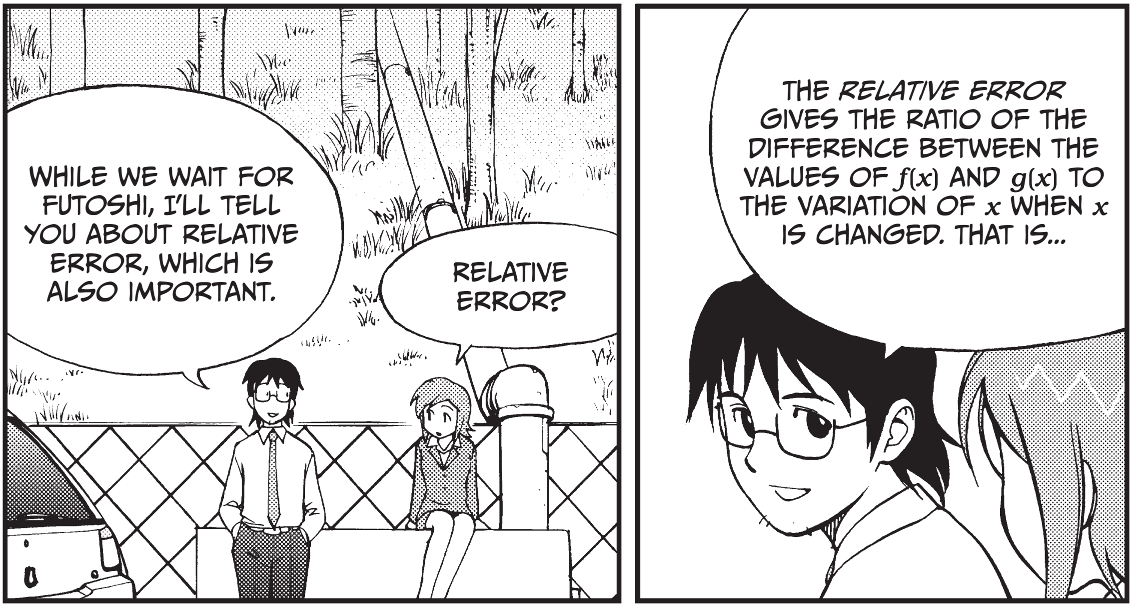

WELL, THAT'S THE ITALIAN RESTAURANT WE WANT TO GO TO.
IT'S STILL SO FAR AWAY.
OH!! I'VE GOT AN IDEA!
LET'S DENOTE THIS ACCIDENT SITE WITH POINT P.
WHAT?
ITALIAN RESTAURANT
P
ACCIDENT SITE
AND LET'S THINK OF THE ROAD AS A GRAPH OF THE FUNCTION $f(x) = x^2$.



* THE REASON IS GIVEN ON PAGE 39.

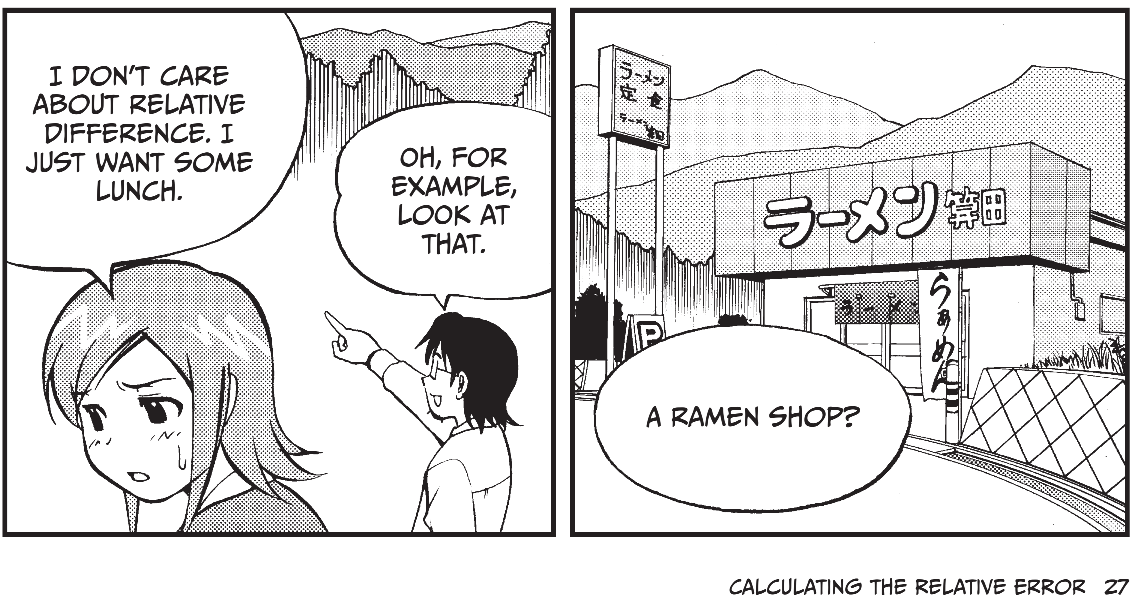


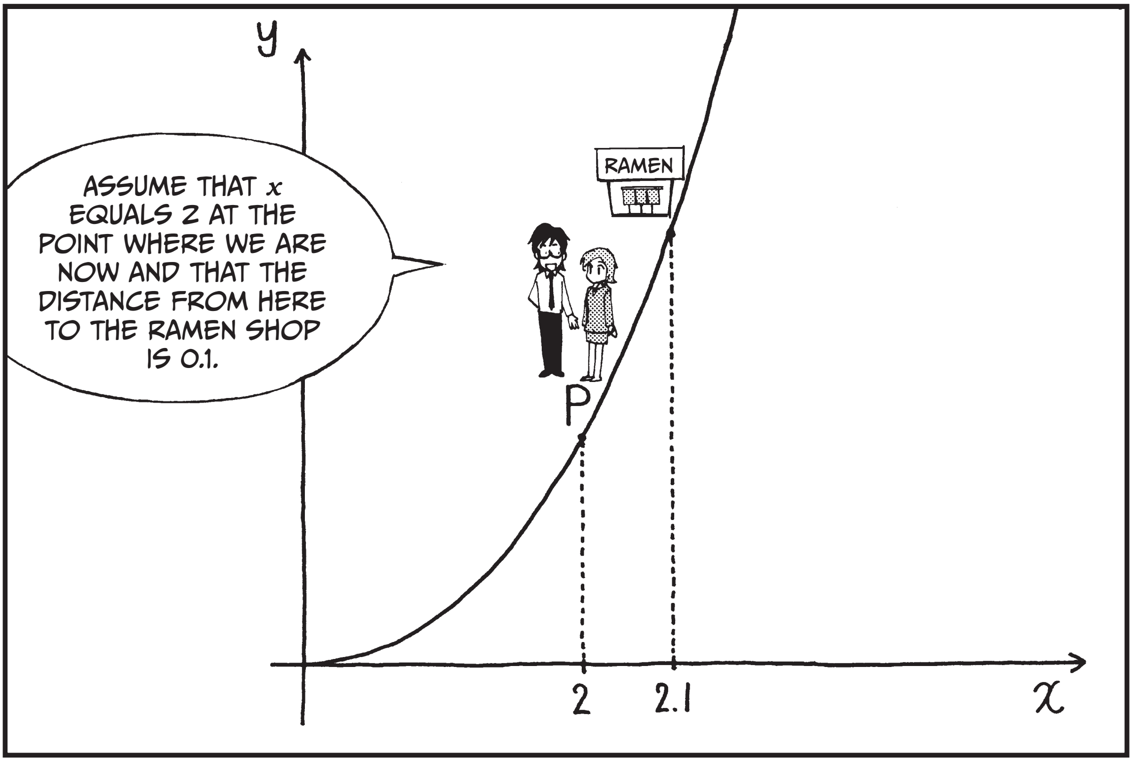
CALCULATING THE RELATIVE ERROR

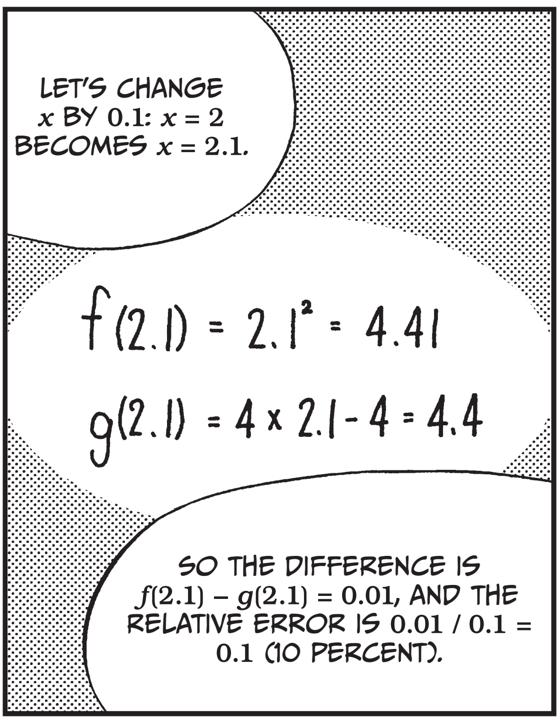


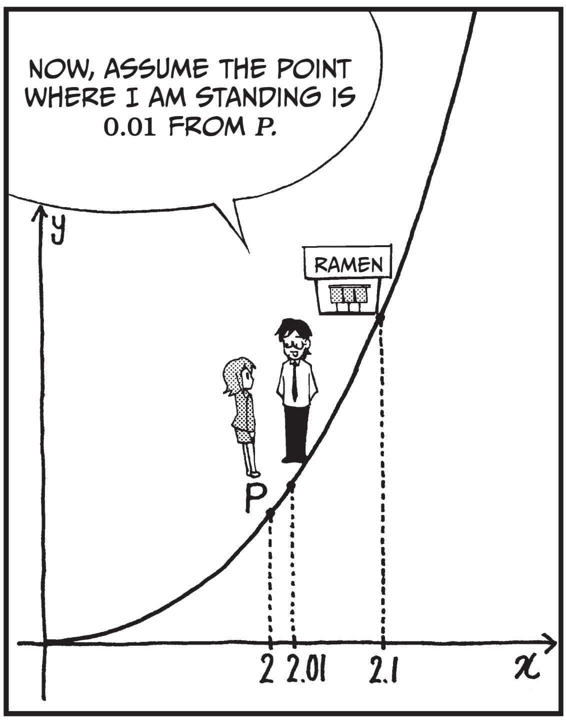
$$\text{Relative error} = \frac{\text{Difference between } \overset{\substack{\text{Our original function} \\ \downarrow}}{f(x)} \text{ and } \overset{\substack{\text{Our approximating function} \\ \downarrow}}{g(x)}}{\text{Change of } x}$$

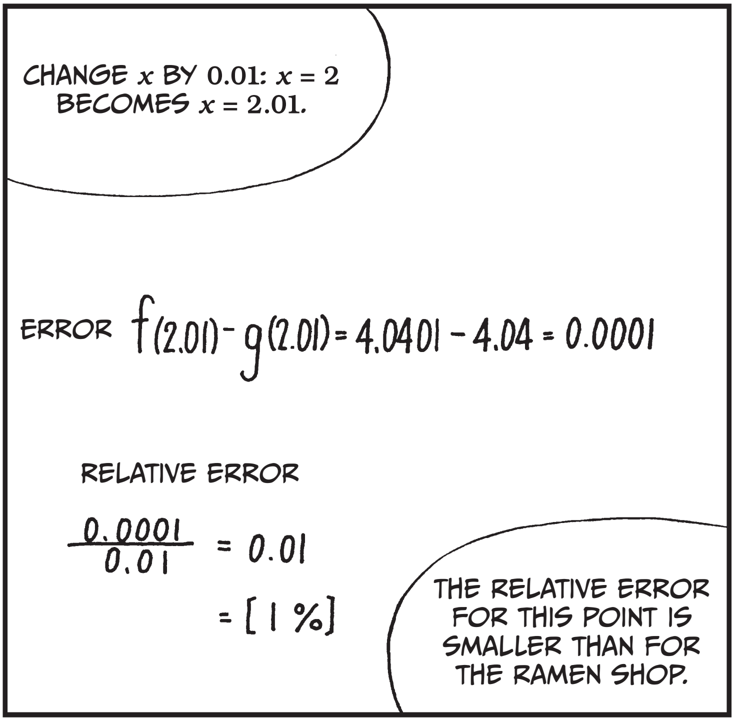
SIMPLE, RIGHT?




ASSUME THAT x EQUALS 2 AT THE POINT WHERE WE ARE NOW AND THAT THE DISTANCE FROM HERE TO THE RAMEN SHOP IS 0.1.
RAMEN
y
P
2
2.1
x


LET'S CHANGE x BY 0.1: $x = 2$ BECOMES $x = 2.1$.
$f(2.1) = 2.1^2 = 4.41$
$g(2.1) = 4 \times 2.1 - 4 = 4.4$
SO THE DIFFERENCE IS $f(2.1) - g(2.1) = 0.01$, AND THE RELATIVE ERROR IS $0.01 / 0.1 = 0.1$ (10 PERCENT).


NOW, ASSUME THE POINT WHERE I AM STANDING IS 0.01 FROM P.
RAMEN
y
P
2
2.01
2.1
x



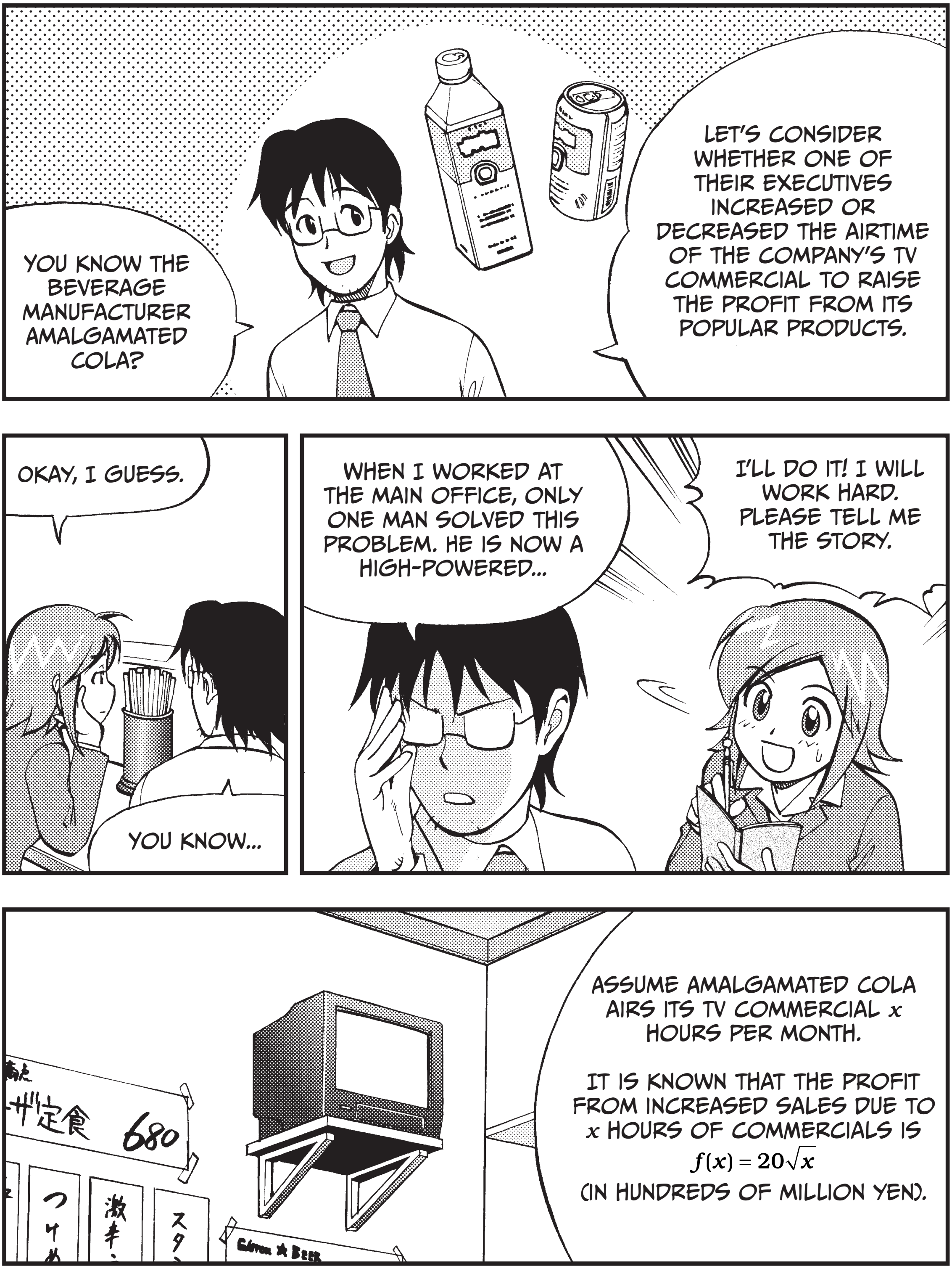
As the variation approaches 0, the relative error also approaches 0.

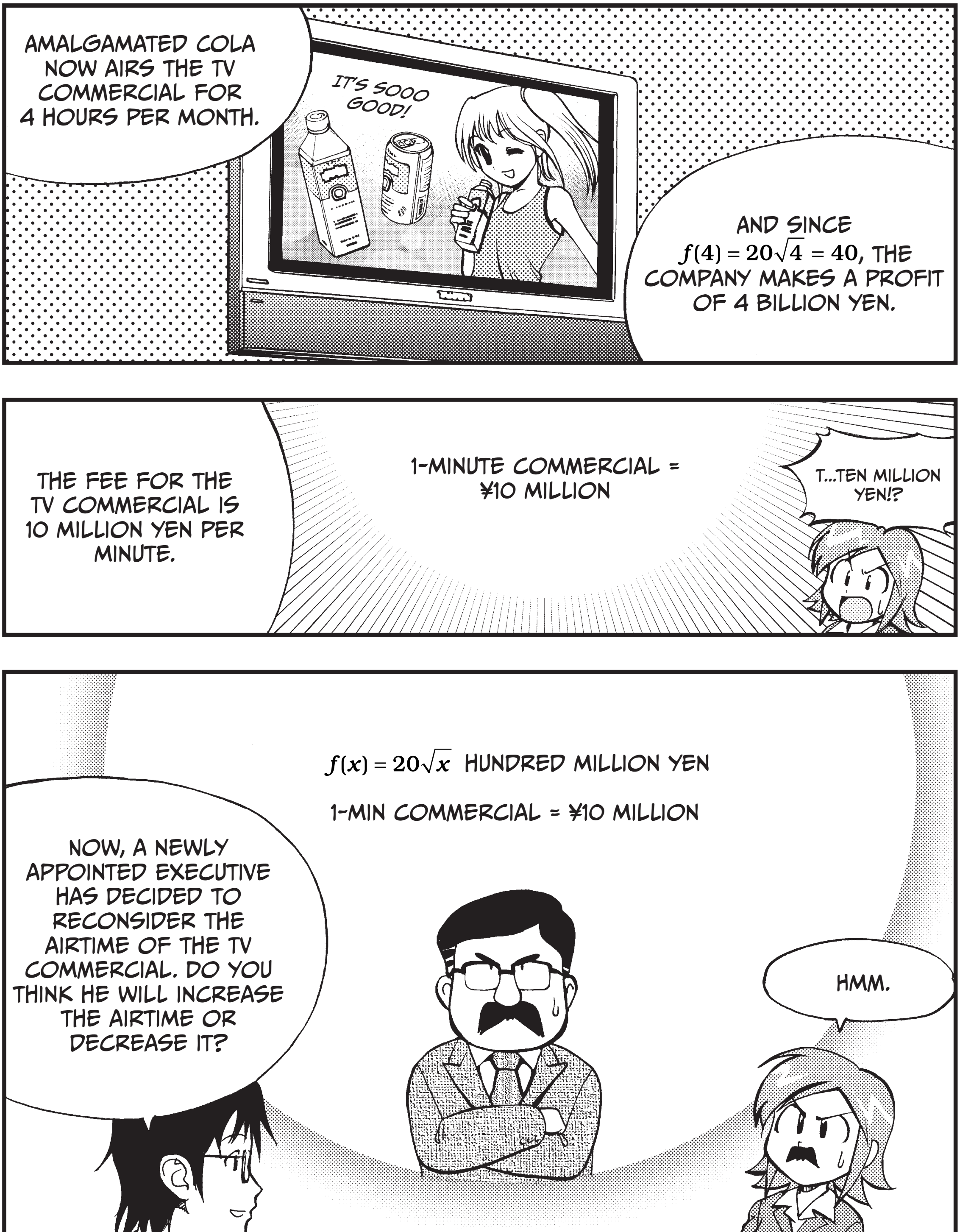
Variation of x from 2	$f(x)$	$g(x)$	Error	Relative error
1	9	8	1	100.0%
0.1	4.41	4.4	0.01	10.0%
0.01	4.0401	4.04	0.0001	1.0%
0.001	4.004001	4.004	0.000001	0.1%
↓				↓
0				0


THAT'S NOT SO SURPRISING, IS IT?
GREAT! YOU ALREADY UNDERSTAND DERIVATIVES.
SO, THE RESTAURANT HAVING THE SMALLEST RELATIVE ERROR IS...
ラーメン
THE RAMEN SHOP.
BE STRAIGHT WITH ME! WE'RE GONNA EAT AT THE RAMEN SHOP, AREN'T WE?
YES. TODAY WE WILL EAT AT THE RAMEN SHOP, WHICH IS CLOSER TO POINT P.
THE APPROXIMATE LINEAR FUNCTION IS SUCH THAT ITS RELATIVE ERROR WITH RESPECT TO THE ORIGINAL FUNCTION IS LOCALLY ZERO.
SO, AS LONG AS LOCAL PROPERTIES ARE CONCERNED, WE CAN DERIVE THE CORRECT RESULT BY USING THE APPROXIMATE LINEAR FUNCTION FOR THE ORIGINAL FUNCTION.
SEE PAGE 39 FOR THE DETAILED CALCULATION.

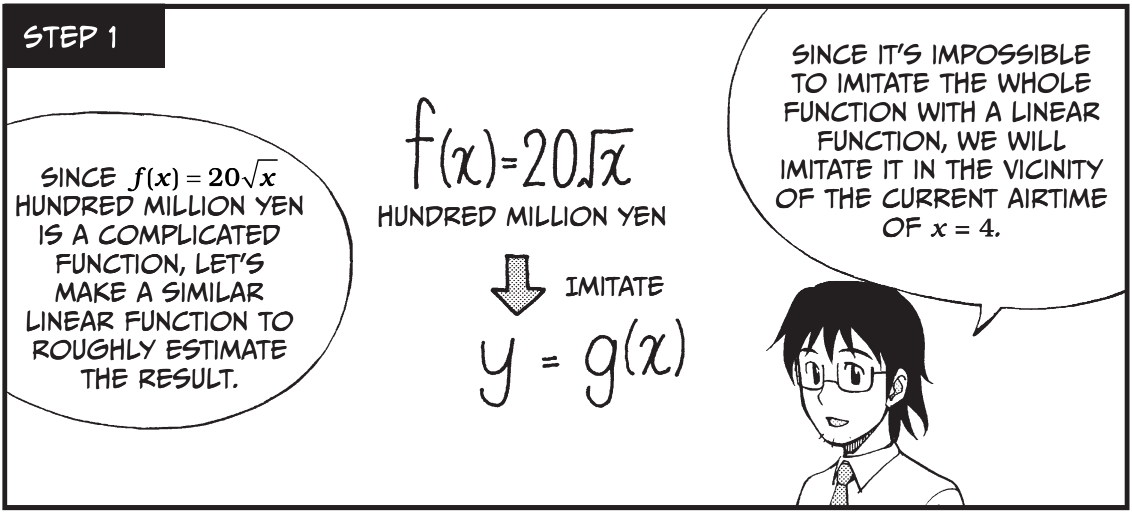

RAMEN SANDA
ラーメン
SLURP
WHY IS FUTOSHI EATING SO MUCH? HE JUST CAME TO RESCUE US.
ボー…
SIGH. I LIKE RAMEN, BUT I WANTED TO EAT ITALIAN FOOD.
NORIKO, WE CAN ALSO ESTIMATE THE COST-EFFECTIVENESS OF TV COMMERCIALS USING APPROXIMATE FUNCTIONS.
REALLY?

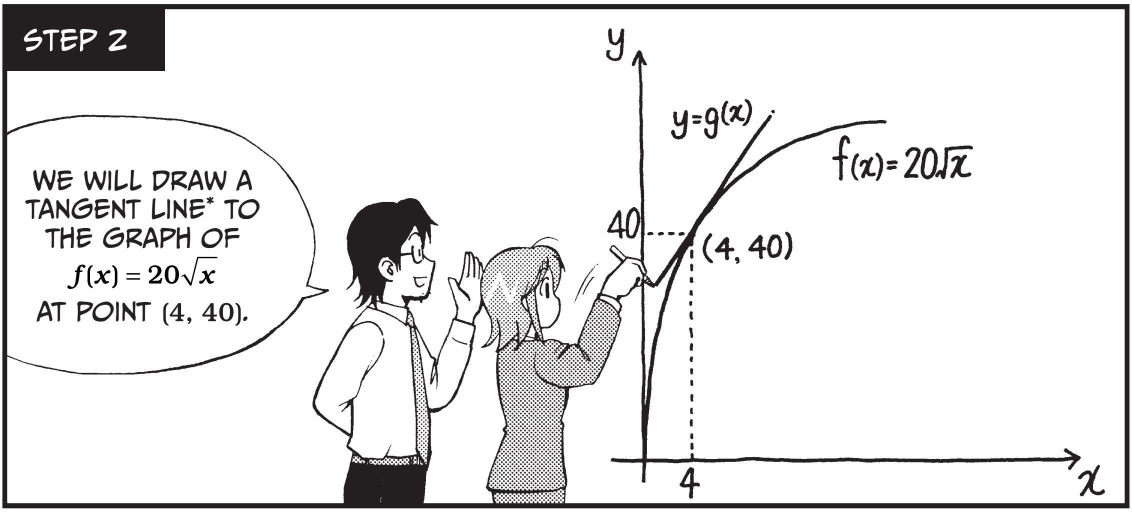
THE DERIVATIVE IN ACTION!





AMALGAMATED COLA NOW AIRS THE TV COMMERCIAL FOR 4 HOURS PER MONTH.
IT'S SOOO GOOD!
AND SINCE $f(4) = 20\sqrt{4} = 40$, THE COMPANY MAKES A PROFIT OF 4 BILLION YEN.
THE FEE FOR THE TV COMMERCIAL IS 10 MILLION YEN PER MINUTE.
1-MINUTE COMMERCIAL = ¥10 MILLION
T...TEN MILLION YEN!?
$f(x) = 20\sqrt{x}$ HUNDRED MILLION YEN
1-MIN COMMERCIAL = ¥10 MILLION
NOW, A NEWLY APPOINTED EXECUTIVE HAS DECIDED TO RECONSIDER THE AIRTIME OF THE TV COMMERCIAL. DO YOU THINK HE WILL INCREASE THE AIRTIME OR DECREASE IT?
HMM.





* Here is the calculation of the tangent line. (See also the explanation of the derivative on page 39.)

For $f(x) = 20\sqrt{x}$, $f'(4)$ is given as follows.

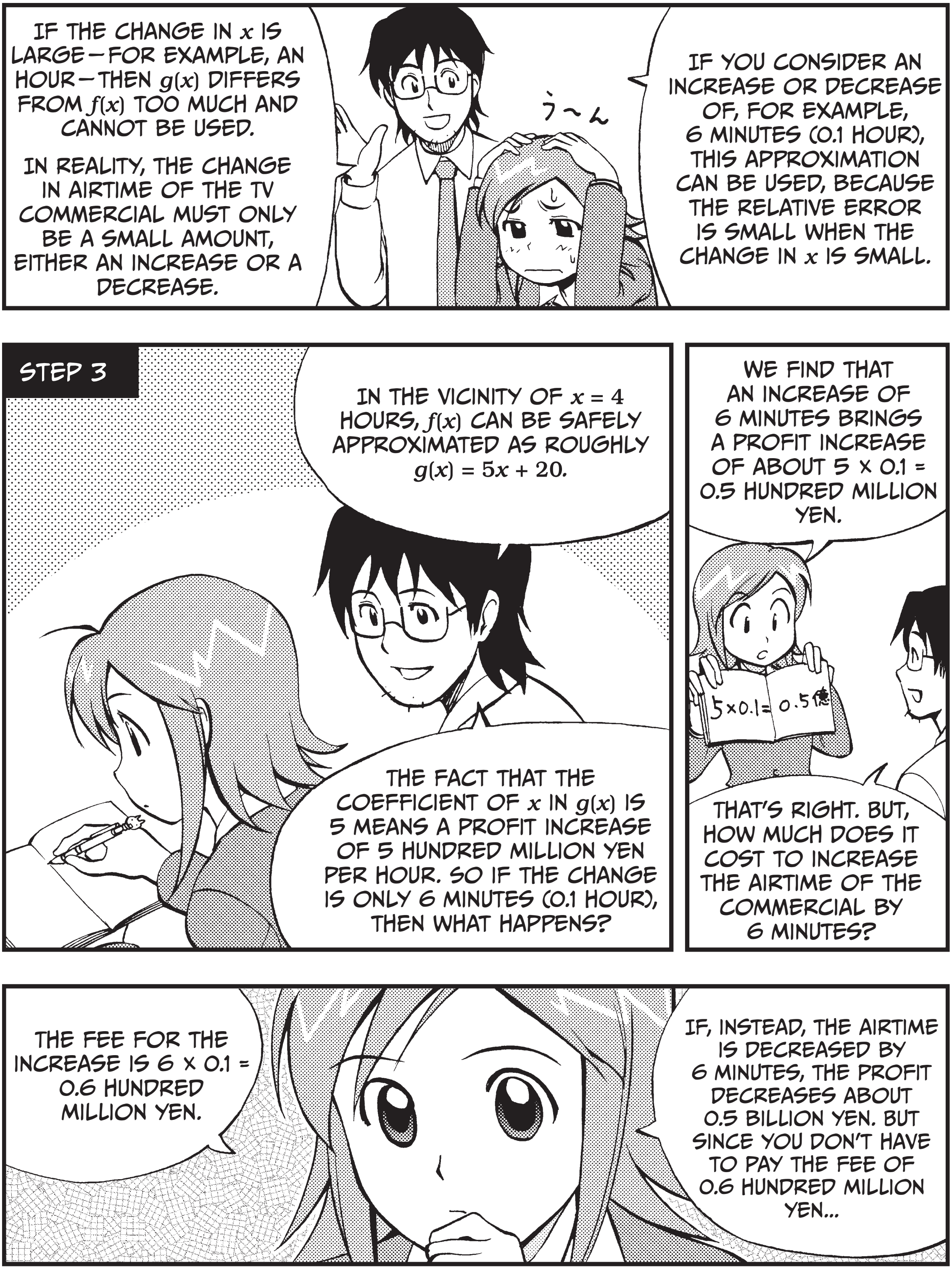
$$\frac{f(4+\varepsilon)-f(4)}{\varepsilon} = \frac{20\sqrt{4+\varepsilon}-20\times 2}{\varepsilon} = 20\,\frac{\left(\sqrt{4+\varepsilon}-2\right)\times\left(\sqrt{4+\varepsilon}+2\right)}{\varepsilon\times\left(\sqrt{4+\varepsilon}+2\right)}$$

$$= 20\,\frac{4+\varepsilon-4}{\varepsilon\left(\sqrt{4+\varepsilon}+2\right)} = \frac{20}{\sqrt{4+\varepsilon}+2} \quad ❶$$

When ε approaches 0, the denominator of ❶ $\sqrt{4+\varepsilon}+2 \to 4$.

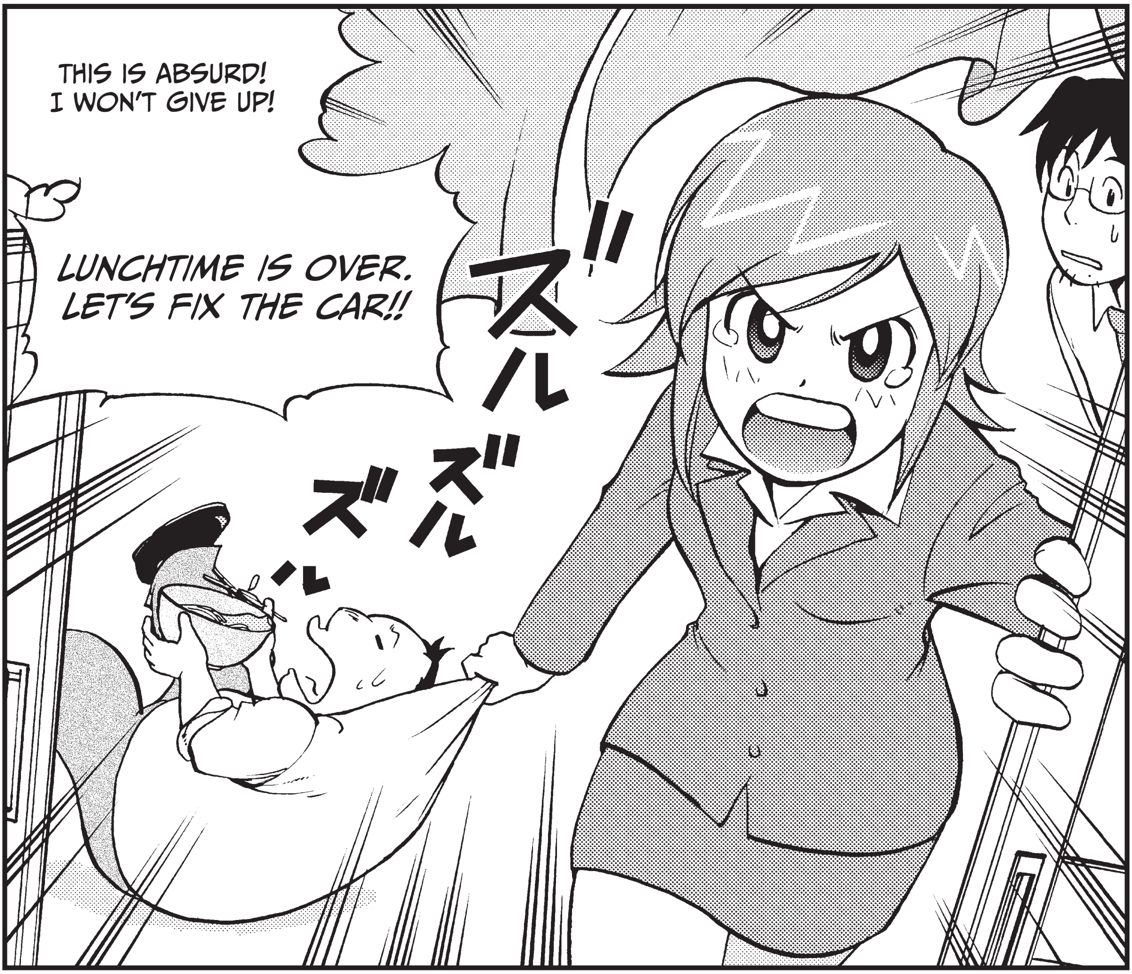
Therefore, ❶ $\to 20 \div 4 = 5$.

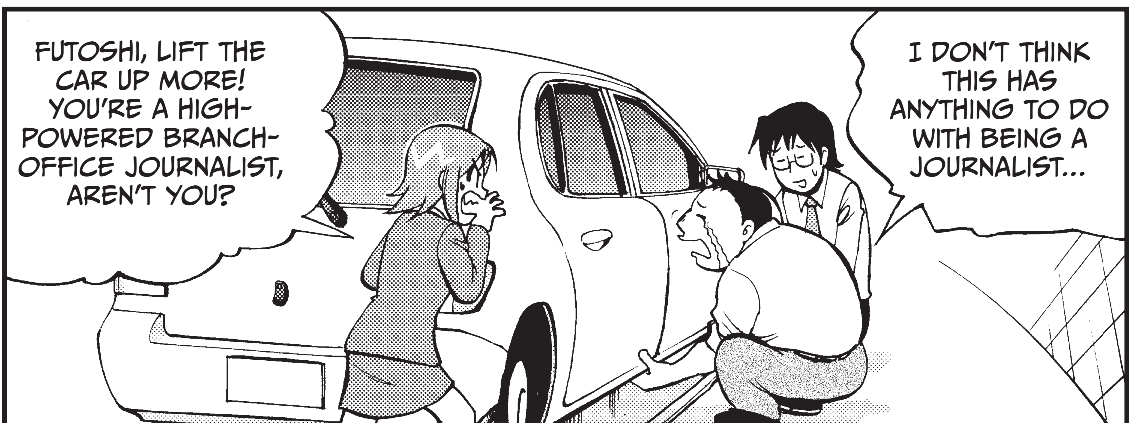
Thus, the approximate linear function $g(x) = 5(x-4)+40 = 5x+20$


IF THE CHANGE IN x IS LARGE—FOR EXAMPLE, AN HOUR—THEN $g(x)$ DIFFERS FROM $f(x)$ TOO MUCH AND CANNOT BE USED.
IN REALITY, THE CHANGE IN AIRTIME OF THE TV COMMERCIAL MUST ONLY BE A SMALL AMOUNT, EITHER AN INCREASE OR A DECREASE.
う〜ん
IF YOU CONSIDER AN INCREASE OR DECREASE OF, FOR EXAMPLE, 6 MINUTES (0.1 HOUR), THIS APPROXIMATION CAN BE USED, BECAUSE THE RELATIVE ERROR IS SMALL WHEN THE CHANGE IN x IS SMALL.
STEP 3
IN THE VICINITY OF $x = 4$ HOURS, $f(x)$ CAN BE SAFELY APPROXIMATED AS ROUGHLY $g(x) = 5x + 20$.
THE FACT THAT THE COEFFICIENT OF x IN $g(x)$ IS 5 MEANS A PROFIT INCREASE OF 5 HUNDRED MILLION YEN PER HOUR. SO IF THE CHANGE IS ONLY 6 MINUTES (0.1 HOUR), THEN WHAT HAPPENS?
WE FIND THAT AN INCREASE OF 6 MINUTES BRINGS A PROFIT INCREASE OF ABOUT 5 × 0.1 = 0.5 HUNDRED MILLION YEN.
5×0.1= 0.5億
THAT'S RIGHT. BUT, HOW MUCH DOES IT COST TO INCREASE THE AIRTIME OF THE COMMERCIAL BY 6 MINUTES?
THE FEE FOR THE INCREASE IS 6 × 0.1 = 0.6 HUNDRED MILLION YEN.
IF, INSTEAD, THE AIRTIME IS DECREASED BY 6 MINUTES, THE PROFIT DECREASES ABOUT 0.5 BILLION YEN. BUT SINCE YOU DON'T HAVE TO PAY THE FEE OF 0.6 HUNDRED MILLION YEN...


THE ANSWER IS...THE COMPANY DECIDED TO DECREASE THE COMMERCIAL TIME!
CORRECT!
PEOPLE USE FUNCTIONS TO SOLVE PROBLEMS IN BUSINESS AND LIFE IN THE REAL WORLD.
THAT'S TRUE WHETHER THEY ARE CONSCIOUS OF FUNCTIONS OR NOT.
BY THE WAY, WHO IS THE MAN THAT SOLVED THIS PROBLEM?


OH, IT WAS FUTOSHI.
ズズズ
SLURP
YOU'RE KIDDING!
BUT YOU SAID HE WAS HIGH-POWERED, DIDN'T YOU?
HE IS A HIGH-POWERED BRANCH-OFFICE JOURNALIST.
AS I EXPECTED...SOLVING MATH PROBLEMS HAS NOTHING TO DO WITH BEING A HIGH-POWERED JOURNALIST.
はぁ…
YANK!
!?


THIS IS ABSURD! I WON'T GIVE UP!
LUNCHTIME IS OVER. LET'S FIX THE CAR!!
ズルズルズル


I DON'T THINK THIS HAS ANYTHING TO DO WITH BEING A JOURNALIST...
FUTOSHI, LIFT THE CAR UP MORE! YOU'RE A HIGH-POWERED BRANCH-OFFICE JOURNALIST, AREN'T YOU?

Calculating the Derivative

Let's find the imitating linear function $g(x) = kx + l$ of function $f(x)$ at $x = a$.

We need to find slope k.

❶ $g(x) = k(x - a) + f(a)$ ($g(x)$ coincides with $f(a)$ when $x = a$.)

Now, let's calculate the relative error when x changes from $x = a$ to $x = a + \varepsilon$.

$$\text{Relative error} = \frac{\text{Difference between } f \text{ and } g \text{ after } x \text{ has changed}}{\text{Change of } x \text{ from } x = a}$$

$$= \frac{f(a+\varepsilon) - g(a+\varepsilon)}{\varepsilon}$$

$$= \frac{f(a+\varepsilon) - (k\varepsilon + f(a))}{\varepsilon}$$

$g(a+\varepsilon) = k(a+\varepsilon-a) + f(a)$
$= k\varepsilon + f(a)$

$$= \frac{f(a+\varepsilon) - f(a)}{\varepsilon} - k \xrightarrow[\varepsilon \to 0]{} 0$$

When ε approaches 0, the relative error also approaches 0.

$$k = \lim_{\varepsilon \to 0} \frac{f(a+\varepsilon) - f(a)}{\varepsilon}$$

$\frac{f(a+\varepsilon) - f(a)}{\varepsilon}$ approaches k when $\varepsilon \to 0$.

(The *lim* notation expresses the operation that obtains the value when ε approaches 0.)

Linear function ❶, or $g(x)$, with this k, is an approximate function of $f(x)$.

k is called the *differential coefficient* of $f(x)$ at $x = a$.

$$\lim_{\varepsilon \to 0} \frac{f(a+\varepsilon) - f(a)}{\varepsilon}$$

Slope of the line tangent to $y = f(x)$ at any point $(a, f(a))$.

We make symbol f' by attaching a prime to f.

$$f'(a) = \lim_{\varepsilon \to 0} \frac{f(a+\varepsilon) - f(a)}{\varepsilon}$$

$f'(a)$ is the slope of the line tangent to $y = f(x)$ at $x = a$.

Letter a can be replaced with x.

Since f' can been seen as a function of x, it is called "the function derived from function f," or the *derivative* of function f.

Calculating the Derivative of a Constant, Linear, or Quadratic Function

1. Let's find the derivative of constant function $f(x) = \alpha$. The differential coefficient of $f(x)$ at $x = a$ is

$$\lim_{\varepsilon \to 0} \frac{f(a+\varepsilon) - f(a)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\alpha - \alpha}{\varepsilon} = \lim_{\varepsilon \to 0} 0 = 0$$

Thus, the derivative of $f(x)$ is $f'(x) = 0$. This makes sense, since our function is constant—the rate of change is 0.

NOTE The *differential coefficient* of $f(x)$ at $x = a$ is often simply called the derivative of $f(x)$ at $x = a$, or just $f'(a)$.

2. Let's calculate the derivative of linear function $f(x) = \alpha x + \beta$. The derivative of $f(x)$ at $x = \alpha$ is

$$\lim_{\varepsilon \to 0} \frac{f(\alpha+\varepsilon) - f(a)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\alpha(a+\varepsilon) + \beta - (\alpha a + \beta)}{\varepsilon} = \lim_{\varepsilon \to 0} \alpha = \alpha$$

Thus, the derivative of $f(x)$ is $f'(x) = \alpha$, a constant value. This result should also be intuitive—linear functions have a constant rate of change by definition.

3. Let's find the derivative of $f(x) = x^2$, which appeared in the story. The differential coefficient of $f(x)$ at $x = a$ is

$$\lim_{\varepsilon \to 0} \frac{f(a+\varepsilon) - f(a)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{(a+\varepsilon)^2 - a^2}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{2a\varepsilon + \varepsilon^2}{\varepsilon} = \lim_{\varepsilon \to 0} (2a + \varepsilon) = 2a$$

Thus, the differential coefficient of $f(x)$ at $x = a$ is $2a$, or $f'(a) = 2a$. Therefore, the derivative of $f(x)$ is $f'(x) = 2x$.

Summary

- The calculation of a limit that appears in calculus is simply a formula calculating an error.
- A limit is used to obtain a derivative.
- The derivative is the slope of the tangent line at a given point.
- The derivative is nothing but the rate of change.

The derivative of $f(x)$ at $x = a$ is calculated by

$$\lim_{\varepsilon \to 0} \frac{f(a+\varepsilon) - f(a)}{\varepsilon}$$

$g(x) = f'(a)\,(x - a) + f(a)$ is then the *approximate linear function* of $f(x)$.

$f'(x)$, which expresses the slope of the line tangent to $f(x)$ at the point $(x, f(x))$, is called the *derivative* of $f(x)$, because it is derived from $f(x)$.

Other than $f'(x)$, the following symbols are also used to denote the derivative of $y = f(x)$.

$$y', \quad \frac{dy}{dx}, \quad \frac{df}{dx}, \quad \frac{d}{dx} f(x)$$

EXERCISES

1. We have function $f(x)$ and linear function $g(x) = 8x + 10$. It is known that the relative error of the two functions approaches 0 when x approaches 5.
   A. Obtain $f(5)$.
   B. Obtain $f'(5)$.
2. For $f(x) = x^3$, obtain its derivative $f'(x)$.